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On Model Structures Relating to Spectral Sequences

Submitted for the degree of Doctor of Philosophy of Mathematics
on Monday, 26th September 2022

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Abstract

In [CELW19] Cirici, Egas Santander, Livernet and Whitehouse define model structures on filtered chain complexes and bicomplexes whose weak equivalences are the r-weak equivalences, i.e. isomorphisms on the $(r+1)$-pages of the associated spectral sequences. In this thesis we study and generalise these model structures. These generalisations $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$ for fixed such r are indexed by subsets S of $\{0,1,\ldots,r\}$ containing r in the former case and 0 and r in the latter and are finitely cofibrantly generated.

We show each of these model structures is a left (and right) proper, cellular and stable model category. We construct a left adjoint $\mathcal{L}$ to the product totalisation functor and show, by means of Greenlees and Shipley's cellularization principle, that it is a Quillen equivalence for suitable indexing sets S. As a consequence all the model categories considered thus far have equivalent homotopy categories induced via a zig-zag of Quillen equivalences given by compositions of the $\mathcal{L}$-product totalisation, identity-identity and shift-décalage adjunctions. The model structures with r-weak equivalences are shown to have no left Bousfield localisation to a model structure with $(r+1)$-weak equivalences. We also derive existence of various bounded variants of the model structures $(f\mathcal{C})_S$.

We then focus on the model structures on filtered chain complexes, give a classification of their cofibrant objects and cofibrations with a boundedness restriction on their filtrations and show the $(f\mathcal{C})_S$ satisfy the unit and pushout-product axioms thereby giving monoidal model categories. Furthermore the $(f\mathcal{C})_S$ satisfy the monoid axiom of Schwede and Shipley yielding model structures on modules and algebras enhancing the homotopy theory of Halperin and Tanré on filtered differential graded algebras to a model category structure.

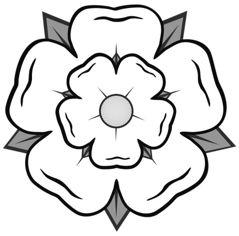
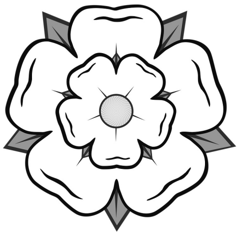


Table of Notation

Notation	Description	Page List
F_p	p^{th} filtration piece	5
$f\mathcal{C}$	category of filtered chain complexes over a ring R	5
$R^n_{(p)}$	a filtered chain complex R concentrated in filtration degree p and cohomological degree n	5
$\otimes$	tensor product of filtered chain complexes or bicomplexes	6, 9
$R^{(n)+(m)}_{(p)+(q)}$	the tensor product of the filtered chain complexes $R^n_{(p)}$ and $R^m_{(q)}$	6
$\underline{\mathrm{Hom}}_{f\mathcal{C}}$	internal hom object of filtered chain complexes	6
Σ^r	the r-suspension functor of filtered chain complexes or bicomplexes	8, 10
Ω^r	the r-loops functor of filtered chain complexes or bicomplexes	8, 10
$b\mathcal{C}$	category of bicomplexes over a ring R	9
Tot^{Π}	the product totalisation functor from filtered chain complexes to bicomplexes	9
$\mathrm{Tot}^{\oplus}$	the coproduct totalisation functor from filtered chain complexes to bicomplexes	9
$E_r^{*,*}$	r-page of a spectral sequence	10, 11
$Z_r^{*,*}$	r-cycles of a filtered chain complex	11
$\mathcal{Z}_r^{*,*}$	representing object for r-cycles of filtered chain complexes	11
$B_r^{*,*}$	r-boundaries of a filtered chain complex	11
$\mathcal{B}_r^{*,*}$	representing object for r-boundaries of filtered chain complexes	11
S^r	r composites of the shift functor	11
Dec^r	r composites of the décalage functor	12
$\mathcal{M}$	a model category	12
$\mathcal{W}$, Fib and Cof	weak equivalences, fibrations and cofibrations respectively	12
$\rightarrowtail$	a cofibration	13
$\twoheadrightarrow$	a fibration	13
$\xrightarrow{\sim}$	a weak equivalence	13
$\mathrm{Ho}(\mathcal{M})$	homotopy category of the model category $\mathcal{M}$	13
I-`Inj`	morphisms with the right lifting property with repsect to I	14
I-`Proj`	morphisms with the left lifting property with respect to I	14
I-`Cof`	morphisms with the left lifting property with respect to I-`Inj`	14
I-`Cell`	relative I-cellular morphisms	14

Notation	**Description**	**Page List**
I	generating cofibrations of a model category (possibly adorned with subscripts)	15
J	generating acyclic cofibrations of a model category (possibly adorned with subscripts)	15
$\lrcorner_h$	homotopy pullback	19
$\ulcorner_h$	homotopy pushout	20
Σ	suspension functor on the homotopy category	20
Ω	loop functor on the homotopy category	20
map	homotopy function complex	23
$L_{\mathcal{C}}$	left Bousfield localisation at a subclass of morphisms $\mathcal{C}$	24
$R_{\mathcal{C}}$	right Bousfield localisation at a subclass of morphisms $\mathcal{C}$	24
$\mathcal{K}\text{-}\mathtt{cell}\text{-}\mathcal{M}$	right Bousfield localisation at the $\mathcal{K}$-colocal equivalences	24
I_r and J_r	generating cofibrations and acyclic cofibrations for the r-model structures on filtered chain complexes and bicomplexes	25, 28
I'_r and J'_r	generating cofibrations and acyclic cofibrations for the r'-model structures on filtered chain complexes and bicomplexes	25, 28
C_r	the r-cone functor of filtered chain complexes or bicomplexes	26, 29
$ZW_r^{*,*}$	witness r-cycles of a bicomplex A	27
$\mathcal{ZW}_r^{*,*}$	representing object for witness r-cycles of bicomplexes	27
$BW_r^{*,*}$	witness r-boundaries of a bicomplex A	27
$\mathcal{BW}_r^{*,*}$	representing object for witness r-boundaries of bicomplexes	27
$\mathcal{L}$	left adjoint to the product totalisation functor	33
$\mathcal{R}$	right adjoint to the coproduct totalisation functor	37
$(f\mathcal{C})_S$	the S-model structure on filtered chain complexes	43, 45
$(b\mathcal{C})_S$	the S-model structure on bicomplexes	43, 47
I_S and J_S	generating cofibrations and acyclic cofibrations for the S-model structures on filtered chain complexes and bicomplexes	45, 47
$Q_r I$	S-cofibrant replacement for the unit in $(f\mathcal{C})_S$	89

Contents



Introduction

The primary aim of this thesis is to investigate various model structures relating to spectral sequences which were introduced by Cirici, Egas Santander, Livernet and Whitehouse in [CELW19] and establish what properties of model categories commonly sought these model structures satisfy. These model structures, on the categories of filtered chain complexes and bicomplexes, have as their weak equivalences those morphisms of filtered chain complexes or bicomplexes inducing a quasi-isomorphism on the r-page of the associated spectral sequences, equivalently isomorphisms on the $(r+1)$-pages. Thus their weak equivalences are determined at a finite stage of the associated spectral sequences. Fibrations were determined by various surjectivity conditions on k-cycles for $k \leq r$ used in constructing the k-pages of the spectral sequences, this can be seen to be in analogy with the surjectivity conditions of the projective model structure on chain complexes whose fibrations are degreewise surjections.

A list of the results of this thesis appears towards the end of this introduction. We firstly give some motivation for model categories and spectral sequences.

Motivation for model categories

Model categories were introduced by Quillen in [Qui67] as a framework for studying the homotopy category $\mathrm{Ho}(\mathcal{C})$ associated to some category $\mathcal{C}$ with a notion of weak equivalence $\mathcal{W}$. A major problem with studying homotopy theory prior to model categories is a lack of control over $\mathrm{Ho}(\mathcal{C})$. Indeed the model for $\mathrm{Ho}(\mathcal{C})$ as being the localisation of $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ at $\mathcal{W}$ has set theoretic issues; the collection of homotopy classes of morphisms between any two objects may form a proper class instead of a set. Model categories circumvent this issue by exhibiting a model of $\mathrm{Ho}(\mathcal{C})$ by firstly taking a full subcategory of $\mathcal{C}$ and then quotienting the sets of morphisms by an equivalence relation thus ensuring we still have a set of morphisms. The equivalence of categories between $\mathcal{C}[\mathcal{W}^{-1}]$ and the latter construction is justified by the Whitehead theorem [Hov99, Proposition 1.2.8].

Model categories also axiomatise various constructions and properties frequently used in and common to lots of homotopy theories; (co)fibrant objects, cylinder and path objects, function complex objects, etc., they provide a means by which one can ‘compare’ model categories via *Quillen functors* and a notion of deriving functors which generalises those found in homological algebra to a non-abelian setting.

This general framework allows model category structures to be constructed in many different contexts (listed shortly) both algebraic and topological. The model structures considered in this thesis will be *cofibrantly generated* and so verification of a model category structure is made much simpler by the *small object argument* of Quillen, Theorem 1.4.2.4.

Other models for homotopy theories

There are many other axiomatic frameworks for homotopy theories: *relative categories* [DK80] later studied further in [BK12], *homotopical categories* [DHKS04] and *categories with weak equivalences* have the notion of weak equivalences; *categories of fibrant objects* (also called Brown categories) [Bro73], *partial Brown categories* [Hor16] and *almost Brown categories* [LW22] extend these by introducing a subcategory of fibrations with various additional axioms; dually there are *cofibration categories* [Bau89] and *Waldhausen categories* [Wal85] (the latter introduced for the study of K-theory) which instead add cofibrations; *Cartan-Eilenberg categories* [GNPR10] only introduce the notion of *cofibrant models* and *strong* and *weak equivalences*; *model categories* [Qui67] (already mentioned) and *infinity categories* [Cis19] and their many models capturing higher homotopical information.

Examples of model categories

To demonstrate the pervasiveness and applicability of model categories in algebraic topology (and more widespread) we list some examples.

The archetypal example of a model category is the *Quillen model structure* on simplicial sets introduced in [Qui67] whose weak equivalences are the π_*-isomorphisms. This model structure has many nice properties and many other model categories have simplicial structure in some sense compatible with the Quillen model structure (or such simplicial structure is introduced via *Reedy model structures* [Hov99, Theorem 5.2.5]); such model categories have particularly useful constructions, e.g. *homotopy function complexes* which encode the 'higher dimensional' information of the set of homotopy classes of maps between two objects. Of more relevance and comparable to the model structures considered in this thesis is the *projective model structure* on chain complexes. Weak equivalences are the quasi-isomorphisms and fibrations the degreewise surjections. Many model categories on more *algebraic* categories use similar weak equivalences and fibrations, often these model structures are defined via a *transfer theorem* from the projective model structure. E.g the model structure of differential graded algebras is transferred from the projective model structure along the free-forgetful adjunction.

Model categories have more generally found use in more specialised branches of homotopy theory. In the stable setting, where the suspension functor on spaces has been inverted so as to obtain a triangulated category, model category structures have been defined in the guise of sequential spectra [BF78], symmetric spectra [HSS00], orthogonal spectra [MMSS01] and many more similar. Equivariant topological spaces, where spaces are now equipped with group actions, form a model category [DK85] and there is a stable analogue [HHR21]. Categories of pro-objects, i.e. formal completions of a category, can be equipped with various model categories [EH76, Isa01, Isa04] discussed later in Question C.3. In the setting of dendroidal sets, [MW07], there is a model structure, [CM11], whose fibrant objects are the ∞-operads. Bergner, [Ber07], defines a model structure on simplicial categories whose weak equivalences are the Dwyer-Kan equivalences, i.e. those simplicial functors $F\colon \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Hom}_{sSet}(c_1, c_2) \to \mathrm{Hom}_{sSet}(Fc_1, Fc_2)$ is a weak equivalence for all $c_1, c_2 \in \mathcal{C}$. The geometric realisation-singular simplices adjunction provides a Quillen equivalence to a model structure on topological categories. There are many model structures on simplicial presheaves some of which are collected in [Bla01] including model categories in the motivic setting $Sh\left((Sm/k)_{Nis}, \mathbb{A}^1\right)$.

Many more examples can be found in, for example, [Bal21].

Motivation for spectral sequences

Spectral sequences arose as a computational tool from work of Leray, [Ler46a, Ler46b], as a means of calculating homology groups of a chain complex by a series of approximations. For a chain complex whose homology one wishes to compute, one begins by equipping it with a *filtration*. In topology this most often comes from some sort of geometric data. Using the filtration a series of approximations are obtained by taking some notion of 'r-homology' denoted E_r, that is we work with *r-cycles* instead of the kernel of the differential (those elements of the chain complex whose differential is r-degrees lower in filtration) and similarly a notion of *r-boundary* in place of the image of the differential. Each of these *r-pages* E_r forms a collection of chain complexes where the differential is some appropriate restriction (in a very loose sense of the word) of the differential, which compute the next page E_{r+1}, so that a spectral sequence is an infinite series of computations.

As a computational tool one hopes (or better expects) that the spectral sequence *collapses* at some stage, meaning there are isomorphisms of bigraded modules $E_k^{*,*} \cong E_{k+1}^{*,*} \cong \ldots \cong E_r^{*,*}$ for all $r \geq k$ for some k, thereby giving some ∞-page $E_\infty^{*,*}$. Under good convergence criteria this E_∞-page along a diagonal gives the *graded pieces* of a filtration of the homology of the chain complex from which one can recover, up to isomorphism and extension problems, the homology of the chain complex.

Examples of spectral sequences

As demonstration of their computational worth and significance in homotopy theory we give some examples of spectral sequences appearing frequently in the literature. The grading conventions in these examples is not consistent.

- *The Leray-Serre spectral sequence, [Ser50],* which given a fibration $F \to E \to B$ of spaces computes the homology of the total space E given the data of the homology of the base B with coefficients in the homology of the fibre F (we assume for simplicity here that the fundamental group of B acts trivially on the fibre):

  $$E_2^{p,q} := H^p\left(B; H^q(F)\right) \Rightarrow H^{p+q}\left(E\right)$$

  whose use is immediately apparent by considering fibrations such as the loop-path space fibration $\Omega S^n \to * \to S^n$. This spectral sequence then gives a way to compute the homology of loop spaces.

- *The Eilenberg-Moore spectral sequence, [EM66, Smi69],* which (over a field) computes the homology of a pullback of a fibration (again we assume for simplicity the action of the fundamental group is trivial). I.e. given a pullback square

$$\begin{tikzcd}
E_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads] \\
X \arrow[r] & B
\end{tikzcd}$$

there is a spectral sequence:

$$E_2^{p,q} := \mathrm{Tor}^{*,*}_{H^*(B)}\left(H^*(X);H^*(E)\right) \Rightarrow H^{p+q}\left(E_f\right) \ .$$

- *The Adams spectral sequence, [Ada58, Nov67],* (and its many generalisations) which for bounded below spectra of finite type X and Y computes the stable homotopy class of maps from X to the p-completion $Y^{\wedge}_{(p)}$. The spectral sequence is given by:
$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{\mathcal{A}}\left(H^*Y, H^*X\right) \Rightarrow [X, Y^{\wedge}_{(p)}]$$
where $H = H\mathbb{F}_p$ here is the Eilenberg-Mac Lane spectrum with $\mathbb{F}_p$ coefficients, i.e. mod p cohomology, and $\mathcal{A}$ is the Steenrod algebra H^*H. The special case with $X = Y = \mathbb{S}$ the sphere spectrum reduces to a spectral sequence computing the p-primary part of the stable homotopy groups of spheres:
$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{\mathcal{A}}\left(H^*\mathbb{S}, H^*\mathbb{S}\right) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}\left(\mathbb{F}_p, \mathbb{F}_p\right) \Rightarrow [\mathbb{S}, \mathbb{S}^{\wedge}_{(p)}] \cong \left(\pi_*(\mathbb{S})\right)_p \ .$$

- *The Atiyah-Hirzebruch spectral sequence, [AH61, Mau63],* used to compute some generalised cohomology $E^*(X)$ given knowledge of the ordinary cohomology $H^*(X)$ and $E^*(*)$:
$$E_2^{p,q} := H^p\left(X; E^q(*)\right) \Rightarrow E^{p+q}\left(X\right) \ ,$$
for example complex topological K-theory KU has $KU(*)$ being a Laurent polynomial ring generated by the Bott element β in degree 2 and the Atiyah-Hirzebruch spectral sequence now gives a way of computing the topological K-theory $KU^*(X)$.

Literature review

Shortly we will discuss our motivation for the work of this thesis, first however we discuss work already carried out in this area. Model structures, or homotopy theories, relating to spectral sequences have already been considered in [HT90, Cir12, CELW19, MR19, FGLW22, LW22]. We give a brief overview of their results here in chronological order.

The homotopy theory of Halperin & Tanré on filtered commutative differential graded algebras

With motivation from rational homotopy theory, see [AH86] for a survey article or [FHT01] for a reference book, Halperin and Tanré construct a homotopy theory on the category of filtered commutative differential graded algebras with many good properties when the underlying ring contains the field $\mathbb{Q}$, although they stop short of showing whether it is a fully fledged model category — they only consider morphisms they term (R,r)*-extensions*, [HT90, Definition 2.2 (v)], instead of cofibrations in general. Here R is the commutative ground ring the algebras are taken over, viewed as being concentrated in degree 0. These extensions of an algebra A take the form of a (completed) tensor product $A\hat{\otimes}\Lambda Y$ where Λ is the free commutative differential graded algebra on some graded module Y, c.f. the notion of an I-`Cell` object for I a generating set of cofibrations. They do however define in generality a notion of quasi-isomorphism and fibration, [HT90, Definitions 2.2 (iii) & (iv)], termed (R,r)*-quasi-isomorphism* and (R,r)*-fibration* respectively. These are morphisms inducing isomorphisms on the $(r+1)$-page of the associated spectral sequence and surjections on the r-cycles respectively which are a common feature of the subsequent homotopy theories as well.

We mention some of the good properties this homotopy theory has. [HT90, Propositions 3.4 & 3.5] show that pushouts preserve (R,r)-extensions (resp. pullbacks preserve (R,r)-fibrations) as well as preserving (R,r)-extensions that are also (R,r)-quasi-isomorphisms (resp. (R,r)-fibrations that are also (R,r)-quasi-isomorphisms). They show, [HT90, Theorem 4.2], that all morphisms $\varphi\colon A \to A'$ admit a *model*, i.e. a factorisation into an (R,r)-extension followed by an (R,r)-quasi-isomorphism. They say of their technique for showing this:

> «L'idée de la démonstration est de construire d'abord un modèle "classique" pour le morphism $E_r(\varphi)$, et ensuite de le "perturber" afin d'arriver au modèle voulu de φ.»

That is they construct a model on the r-page of the spectral sequence in the usual sense of the rational homotopy theory of Quillen, [Qui69], and perturb it to obtain a model on the level of the algebras.

[HT90, §5] shows the existence of lifts one would expect if it were a model category, e.g. lifts of (R, r)-extensions against morphisms that are (R, r)-fibrations and (R, r)-quasi-isomorphisms, [HT90, Theorem 5.1]. In [HT90, §6] they construct cylinder objects, use them to define a homotopy relation and show homotopic morphisms induce the same morphism on the $(r + 1)$-page of the associated spectral sequence.

[HT90, §7] restricts to the case $\mathbb{Q} \subseteq R$ and shows existence of lifts of morphisms that are both (R, r)-extensions and (R, r)-quasi-isomorphisms against (R, r)-fibrations and [HT90, §8] addresses uniqueness of *minimal models*. Tanré later makes use of this homotopy theory in [Tan94].

The P-categories of Cirici on filtered differential graded algebras

In their PhD. thesis, [Cir12], Cirici considers the category of filtered (homological and non-negatively graded) differential graded algebras over a field $\Bbbk$. They introduce a notion of a *P-category with cofibrant models*, similar to that of a Brown category, with two distinguished classes of morphisms *fibrations* and *weak equivalences* along with a *functorial path construction* required to satisfy a list of axioms [Cir12, Definition 1.2.18]. In their category of filtered differential graded algebras (fdgas) they define a morphism of fdgas $f\colon A \to B$ to be a filtered fibration, [Cir12, Definition 4.2.4], if the induced morphism on graded pieces $Gr^p f\colon Gr^p A \to Gr^p B$ is a surjection for all $p \in \mathbb{Z}$ and a weak equivalence if $H^n(Gr^p f)\colon H^n(Gr^p A) \to H^n(Gr^p B)$ is an isomorphism for all $p \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. With these notions of fibrations and weak equivalences they show that the category of filtered differential graded algebras over a field k is a P-category, [Cir12, Proposition 4.2.9]. Forgetting the algebra structure these fibrations and weak equivalences further agree with those of the $r = 0$ model structure of [CELW19] on the category of (unbounded) filtered chain complexes to be discussed shortly. They further introduce, [Cir12, Definition 4.2.13], the notion of a *filtered KS-extension* of degree n and weight p of an fdga A by a filtered graded module V, use this to define *filtered cofibrant dgas*, [Cir12, Definition 4.2.14], and show that these lift against those morphisms that are both weak equivalences and fibrations, thereby deserving the name cofibrant, [Cir12, Proposition 4.2.15]. These cofibrant filtered dgas C behave well with respect to the homotopy theory in that the functor $[C, -]$ sending an fdga to the class of maps defined by filtered homotopy equivalence sends weak equivalences to bijections, [Cir12, Corollary 4.2.16].

They then generalise these definition to E_r-fibrations, those morphisms surjective on the r-page of the associated spectral sequence, and similarly to E_r-quasi isomorphisms, along with an r-path object [Cir12, Definitions 4.3.1, 4.3.2 & 4.3.8] and with these notions show the category of fdgas has a P-category structure, [Cir12, Proposition 4.3.12]. The notion of a filtered cofibrant fdga generalises to [Cir12, Definition 4.3.13] with similar analogous results [Cir12, Propositions 4.3.17 & 4.3.18]. Note whilst the E_r-quasi isomorphisms are the weak equivalences of the r-model structures considered later in [CELW19] the fibrations do not agree.

These P-category structres are suitably comparable in the sense that the décalage functor of Deligne, [Del71], induces equivalences of homotopy categories from the P-category with E_{r+1}-quasi isomorphisms to the P-category with E_r-quasi isomorphisms, [Cir12, Theorem 4.3.7]. Lastly of relevance to this thesis is the left adjoint to the décalage functor which is shown to be inverse on the subcategories of cofibrant objects, [Cir12, Lemma 4.3.16].

The model categories of Cirici, Egas Santander, Livernet and Whitehouse on filtered complexes and bicomplexes

We give here an overview of the results of our principal reference, [CELW19], whose model structure we investigate in this thesis. From a filtered chain complex we can extract an associated spectral sequence, and likewise for bicomplexes via first applying a (product) totalisation functor. The r-page of the spectral sequence associated to a filtered chain complex A is given as a quotient of the *r-cycles*, denoted $Z_r(A)$, of A by the *r-boundaries*, denoted $B_r(A)$. These can be viewed as approximations to the kernel and image of the differential respectively and the r-cycle and r-boundary functors are representable by filtered chain complexes $\mathcal{Z}_r^{*,*}$ and $\mathcal{B}_r^{*,*}$. The authors use these representing objects to construct, for each r, two cofibrantly generated model structures on $f\mathcal{C}$, [CELW19, Theorems 3.14 and 3.16]. Setting $I_r = \{\mathcal{Z}_{r+1} \to \mathcal{B}_{r+1}\}$, where we supress any bidegrees, and similarly $J_r = \{0 \to \mathcal{Z}_r\}$ the two model structures denoted $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$ are defined by:

model category	generating cofibrations	generating acyclic cofibrations
$(f\mathcal{C})_r$	I_r	J_r
$(f\mathcal{C})_{r'}$	$I_r \cup \bigcup_{k=0}^{r-1} J_r$	$\bigcup_{k=0}^{r} J_r$

which agree for $r = 0$. These are right proper, since every object is fibrant, and for a fixed r *Quillen equivalent*, [CELW19, Remark 3.17]. They consider the *shift-décalage adjunction*, $S \dashv \mathrm{Dec}$ of Deligne. These are endomorphisms on the category of filtered chain complexes and on the pages of the associated spectral sequences have the effect of shifting the pages back and forth, i.e. $E_{k+1}\mathrm{Dec}A \cong E_{k+2}A$, with some shift of indices. The shift-décalage adjunction is shown to give a Quillen equivalence $S\colon \ (f\mathcal{C})_r \underset{\perp}{\rightleftarrows} (f\mathcal{C})_{r+l} \ :\mathrm{Dec}$ as well as $S\colon \ (f\mathcal{C})_{r'} \underset{\perp}{\rightleftarrows} (f\mathcal{C})_{(r+l)'} \ :\mathrm{Dec}$. Thus the homotopy categories of all these model structures are equivalent. Similar model structures are also constructed on bicomplexes with the caveat that some care is needed to construct appropriate representing objects for r-cycles and r-boundaries. These they term *witness r-cycles* and *witness r-boundaries* denoted $\mathcal{ZW}_r$ and $\mathcal{BW}_r$. Defining $I_r = \{\mathcal{ZW}_{r+1} \to \mathcal{BW}_{r+1}\}$ and $J_r = \{0 \to \mathcal{ZW}_r\}$ as before they constuct two right proper cofibrantly generated model structures on $b\mathcal{C}$, [CELW19, Theorems 4.37 and 4.39] determined by:

model category	generating cofibrations	generating acyclic cofibrations
$(b\mathcal{C})_r$	I_r	$J_0 \cup J_r$
$(b\mathcal{C})_{r'}$	$I_r \cup \bigcup_{k=0}^{r-1} J_r$	$\bigcup_{k=0}^{r} J_r$

which again agree when $r = 0$.

The model structures of Muro and Roitzheim on bicomplexes and multicomplexes

In [MR19] Muro and Roitzheim define two model structures on the category of (right plane) bicomplexes with horizontal differential $d_h\colon X_{p,*} \to X_{p-1,*}$. The first model structure, [MR19, Theorem 3.1], is cofibrantly generated and has weak equivalences detected by the totalisation functor and fibrations $f\colon X \to Y$ the bidegree-wise surjections and isomorphisms on the vertical homology $H^v_{p,*}(f)\colon H^v_{p,*}(X) \xrightarrow{\cong} H^v_{p,*}(Y)$. This model structure is shown to be monoidal with cofibrant unit satisfying the monoid axiom, [MR19, Proposition 3.3], and there is in fact a strong symmetric monoidal Quillen equivalence with the projective model structure on (unbounded) chain complexes, [MR19, Proposition 3.4], whose left Quillen adjoint is inclusion of a chain complex into bicomplexes in horizontal degree 0. Their second model structure, [MR19, Theorem 4.1], on bicomplexes they name the *Cartan-Eilenberg model structure* or the *E^2-model structure*; the former since a cofibrant resolution of a chain complex concentrated in degree 0 is a *Cartan-Eilenberg resolution* and the latter since the weak equivalences are those morphisms inducing isomorphisms on the 2-page of the associated spectral sequence. The fibrations of the Cartan-Eilenberg model structure are those morphisms inducing 1) bidegree-wise surjections, 2) are surjections on $\ker d^b_{p,*}$ for $p > 0$ and 3) $H^h_{p,q}(f)$ is an isomorphism for all p and q. The Cartan-Eilenberg model structure is also shown to be monoidal with cofibrant unit satisfying the monoid axiom, [MR19, Proposition 4.2]. Lastly they generalise the total model structure on bicomplexes to a total model structure on (bounded) *multicomplexes* also known as *twisted complexes*. Such structures have not just horizontal and vertical differentials but *off diagonal differentials* $d_i\colon X_{p,q} \to X_{p-i,q+i-1}$ for $i \geq 0$ with their sign conventions required to satisfy $\sum_{i+j=n} d_i d_j = 0$ for $n \geq 0$. An analogous (coproduct) totalisation functor is defined [MR19, Definition 5.2] and a cofibrantly generated model structure, [MR19, Theorem 5.13], referred to as the *total model structure* on multicomplexes, is constructed with equivalences those becoming isomorphisms after applying the totalisation functor, and whose fibrations are also bidegree-wise surjective morphisms that induce isomorphisms on vertical homology. It is similarly shown to be monoidal with cofibrant unit and satisfying the monoid axiom (for an appropriate tensor product) in [MR19, Proposition 5.15] and they show that there is a strong symmetric monoidal Quillen equivalence with the projective model structure on chain complexes, [MR19, Proposition 5.16].

The model categories of Fu, Guan, Livernet and Whitehouse on multicomplexes

In [FGLW22] the authors construct model structures on the category of (truncated) multicomplexes in a similar way to [CELW19]. They define an *n-truncated multicomplex*, whose category is denoted $n - m\mathcal{C}_R$, as a multicomplex with the differentials $d_i = 0$ for $i \geq n$. The difference here is that the analogous witness cycles and boundaries for the model structures on n-multicomplexes are inductively defined via iterated pushouts, [FGLW22, Definitions 3.15 and 3.17]. Similar generating sets of cofibrations and acyclic cofibrations as for bicomplexes in [CELW19] are defined and [FGLW22, Theorem 3.28] provides existence of a right proper cofibrantly generated model structure on n-multicomplexes for $2 \leq n \leq \infty$ whose weak equivalences are the r-quasi isomorphisms on the associated spectral sequences, here $\infty - m\mathcal{C}_R$ denotes the category of multicomplexes. There is also a model structure analogous to that of $(b\mathcal{C})_{r'}$ given by [FGLW22, Theorem 3.30]. For each $r \geq 0$ the authors also demonstrate Quillen equivalences in [FGLW22, Theorem 4.5]:

$$2 - m\mathcal{C}_R \underset{\perp}{\rightleftarrows} 3 - m\mathcal{C}_R \underset{\perp}{\rightleftarrows} 4 - m\mathcal{C}_R \underset{\perp}{\rightleftarrows} \ldots \underset{\perp}{\rightleftarrows} n - m\mathcal{C}_R \underset{\perp}{\rightleftarrows} \infty - m\mathcal{C}_R\ .$$

They also define model structures on the left half plane (truncated) multicomplexes via transfer [FGLW22, Proposition 5.11] and consider cofibrancy of objects of the model structures [FGLW22, Theorem 3.30] demonstrating that the unit, a copy of R concentrated in a single bidegree, is not cofibrant and that the various *infinite witness cycles* $\mathcal{ZW}^n_\infty(*,*)$ are cofibrant replacements for the unit, [FGLW22, Proposition 6.7].

The almost Brown category of Livernet and Whitehouse on spectral sequences

More recently in the preprint [LW22] Livernet and Whitehouse investigate existence of homotopy theories directly on a category of spectral sequences SpSe_R; objects are a family (A_r, ψ_r) of bigraded modules where A_r is an r-bigraded module, i.e. has a differential $\delta_r \colon A_r^{p,q} \to A_r^{p-r,q-r+1}$ and ψ_r is an isomorphism $\psi_r \colon H_*(A_r) \xrightarrow{\cong} A_{r+1}$ of bigraded modules. Morphisms of such objects $f \colon (A,\psi) \to (B,\varphi)$ are appropriately defined and in fact are deteremined by the 0-page $f_0 \colon A_0 \to B_0$. The immediate problem with this category is that it is neither complete nor cocomplete, [LW22, §3.2], so a well behaved model category cannot be constructed on this category (the situation is similar if one tries to work in a category of Cartan-Eilenberg systems [CE56, Chapter XV §7]). Instead they introduce the notion of an *almost Brown category* [LW22, Definition 4.1.1], (c.f. *Brown categories*, [Bro73], and *partial Brown categories*, [Hor16]). This has notions of weak equivalences and fibrations with all finite products and satisfying some model category like axioms. Defining as before, for a fixed r, the weak equivalences to be E_r-quasi isomorphisms and fibrations to be surjections on the first r-pages they show that the category of spectral sequences admits an almost Brown structure for each $r \geq 0$. The fibrations and acyclic fibrations are also shown to be detected by generating sets analogous to generating cofibrations and generating acyclic cofibrations, [LW22, Propositions 5.4.2 and 5.4.3]. Analogues on the level of these spectral sequences of the shift-décalage functors of Deligne, [Del71], are also constructed and shown to preserve sufficient homotopy information referred to as *left exactness* preserving weak equivalences, acyclic fibrations and pullbacks of acyclic fibrations. I.e. the shift functor $S \colon (\mathrm{SpSe}_R)_r \to (\mathrm{SpSe}_R)_{r+1}$ is left exact for all $r \geq 0$ and the décalage functor $\mathrm{Dec} \colon (\mathrm{SpSe}_R)_r \to (\mathrm{SpSe}_R)_{r-1}$ is left exact for all $r \geq 1$, [LW22, Proposition 5.3.3 and 5.3.4]. They also note similar to the previous homotopy theories introduced that there are nested inclusions of weak equivalences and fibrations $\mathcal{W}_r \subset \mathcal{W}_{r+1}$ and $\mathrm{Fib}_{r+1} \subset \mathrm{Fib}_r$. One can interpret the model structures $(f\mathcal{C})_r$ as almost Brown categories and the obvious spectral sequence functor $E \colon (f\mathcal{C})_r \to (\mathrm{SpSe}_R)_r$ is shown to be a left exact functor [LW22, Proposition 6.1.1]. They note however that the spectral sequence functor E does not induce an equivalence on homotopy categories after inverting the weak equivalences since the associated graded functors lose information, [LW22, Proposition 6.1.3]. Similarly they show the spectral sequence functor from the r-model category of n-multicomplexes to $(\mathrm{SpSe}_R)_r$ is a left exact functor.

Other appearences of spectral sequences in model categories

Here we make note of other appearences in the literature of model structures closely related to spectral sequences.

Dwyer, Kan and Stover defined in [DKS93] a model structure on the category of *simplicial objects in pointed spaces*. In their model structure a morphism of simplicial pointed spaces $X \to Y$ is a weak equivalence, [DKS93, §3.2], if the morphism of simplicial groups $\pi_i X \to \pi_j Y$ is a weak equivalence of the underlying simplicial sets or equivalently there are induced isomorphisms $\pi_j\pi_i X \cong \pi_j\pi_i Y$ for all $j \geq 0$. They refer to these weak equivalences as E_2-weak equivalences and justify this teminology as follows.

Given a simplicial pointed space X (which is also *Reedy fibrant*) there is a first quadrant spectral sequence, known as the *Quillen-Bousfield-Friedlander* spectral sequence, whose E_2-page is:

$$E^2_{p,q} \coloneqq \pi_p\pi_{q+1}X \Rightarrow \pi_{p+q}\mathrm{map}(S^1, X) \ ,$$

given in [DKS93, §3.6] (see also [BF78, Theorem 4.5]). A weak equivalence in this model structure then corresponds to a morphism inducing an isomorphism between the E_2-pages of the Quillen-Bousfield-Friedlander spectral sequences of X and Y.

[DKS93, §5] relaxes the assumptions somewhat to more general simplical object categories sC_* that are pointed, closed model categories with all colimits, all objects being cofibrant and a choice of cofibrant co-grouplike object. The weak equivalences in this generalised model structure are also known as E_2-weak equivalences and justifying this there is a corresponding spectral sequence; for an $X \in sC_*$ (which is Reedy fibrant) there is a first quadrant spectral sequence whose E_2-page is:

$$E^2_{p,q} \coloneqq \pi_p[\Sigma^q M, X] \Rightarrow \pi_{p+q}\mathrm{map}(M, X) \ ,$$

where M is the co-grouplike object, [DKS93, §5.7].

Motivation for the study of model categories relating to spectral sequences

We provide here some motivation for studying model structures on objects yielding a spectral sequence whose weak equivalences are those morphisms inducing an isomorphism on the $(r+1)$-page.

We have already discussed the work of Halperin and Tanré, [HT90], who consider filtered commutative differential graded algebras. Their motivation for considering such objects comes from the viewpoint of rational homotopy theory where there is an equivalence of categories between *rational homotopy types* and commutative differential graded algebras, see for example [AH86, Theorem 1.7] for a precise statement. Rational homotopy theory has proved an important area of research owing to this equivalence. Halperin and Tanré's work allowed a generalisation to models for fibrations where the usual restriction of the fundamental group of the base acting nilpotently [AH86, Theorem 3.5] is removed. They stop short of showing a model category structure on such objects however. Constructing a model structure on such objects is then a desireable goal to elevate their homotopy theory fully to a model category.

A second motivation comes from the study of *A_∞-algebras*. We firstly introduce some background. A_∞-algebras can be thought of as *homotopy associative* replacements for differential graded algebras; the operad encoding them is in fact a cofibrant replacement of the associative operad in a relevant model category [BM03] and is obtained by a Boardman-Vogt or cobar-bar resolution construction, [BV73, BM06]. Over a field a result of Kadeishvili, [Kad80], asserts that every differential graded algebra A is quasi-isomorphic to a *minimal A_∞-algebra* given by an A_∞-algebra structure on the homology H_*A. This can also be seen as a special case of the *homotopy transfer theorem*, [LV12, Theorem 10.3.3]. These minimial models classify isomorphism classes of differential graded algebras up to quasi-isomorphism. The use of the field here is in constructing a cycle selection map so one can apply a similar result with sufficient projectivity assumptions.

Sagave considers what modifications to this theory can be made if one removes the field/projective assumptions. In [Sag10] Sagave introduces the notion of a *derived A_∞-algebra*, denoted dA_∞, which is now a bigraded object and introduced so as to allow *projective resolutions* of an A_∞-algebra. In this setting any differential graded algebra A over a ring k admits a *k-projective minimal dA_∞-algebra model E* well defined (up to *E_2-equivalence*) together with an *E_2-equivalence $E \to A$*. Here the E_2-equivalence refers to an isomorphism on the E_2-page of the spectral sequence associated to the totalisation of the bigraded dA_∞-algebra E. These minimal dA_∞-algebras classify differential graded algebras up to quasi-isomorphism. A model structure on such objects whose weak equivalences are the isomorphisms on the E_2-page could then assist with determining whether any two minimal models are E_2-equivalent. Similarly generalisations to higher r where weak equivalences are now isomorphisms on the $(r+1)$-page of the spectral sequence associated to some object could help in situations where an object is defined up to an E_{r+1}-equivalence. In such a model structure if it indeed exists the k-projective minimal models of Sagave just discussed may be cofibrant replacements for an A_∞-algebra viewed as a dA_∞-algebra.

In [LH03], Lefèvre-Hasegawa equips the category of A_∞-algebras (with ∞-morphisms) over a field with a model structure (without limits) whose weak equivalences are *A_∞-quasi-isomorphisms*. The homotopy theory of A_∞-algebras has also been studied in e.g. [Gra99], dA_∞-algebras have been studied in [LH03, CELW18] and closely related notions of the latter, called $D_\infty^{(s)}$-differential A_∞-algebras, in [Lap02].

With regard to tentative model structures on dA_∞-algebras the authors of [CELW18] remark:

> "We expect that both the new descriptions of derived A_∞-algebras and the properties of homotopies developed here will allow us to endow the category of derived A_∞-algebras with the structure of a model category without limits in the future, with weak equivalences being E_r-quasi-isomorphisms."

The model structures on filtered chain complexes and bicomplexes constructed in [CELW19] and considered in this thesis are a starting point for such a model structure in the simpler cases where we do not consider the homotopy notions of D_∞, i.e. the operad encoding multicomplexes, or A_∞. A model structure on D_∞-structures with E_r-weak equivalences is constructed in [FGLW22] and has already been discussed. Note too that one of the equivalent formulations of dA_∞-algebras given in [CELW18] is as *split A_∞-algebras in filtered chain complexes*.

One could view the model structures considered on filtered chain complexes or bicomplexes as natural generalisations of the projective model structure on chain complexes; the 0-model structure on bicomplexes is really just the projective model structure on (vertical) chain complexes of (horizontal) chain complexes. Further varying r there are inclusions of weak equivalences (and fibrations for certain model structures) and the interaction between the model structures is of interest in its own right. In particular one question the author would like to understand better is to what extent (with regard to the model category structure) is a model structure on either filtered chain complexes or bicomplexes with $(r+1)$-weak equivalences a localisation of a model structure with r-weak equivalences. We will see however that these model structures are already Quillen equivalent via a shift-décalage adjunction.

This feature of multiple closely related cofibrantly generated model structures on the same underlying category could prove important as a 'testing ground' for new results; indeed it has already led to the detection of an error in a draft copy of [HHR21] on building new model structures from multiple existing ones which has since been corrected.

With more consideration to spectral sequences themselves the knowledge of when a morphism of spectral sequences induces an isomorphism on the infinity page is useful information for comparing their homologies. Whilst we do not directly consider the infinity page in the main work of this thesis for those spectral sequences that collapse these model structures may prove useful in this regard. In Question C.3 we consider how one might equip an appropriate category with a model structure whose weak equivalences are the isomorphisms on the ∞-page.

Filtrations appear naturally in many areas, as evidenced by the abundance of spectral sequences, so having a good homotopy theory of filtered objects could prove very useful. This and the examples given previously provide the author with plenty of motivation for further investigating these model structures.

Areas of further work

There are many questions that arose from the work constituting this thesis some of which have been listed in Appendix C and left unanswered as direction of potential future work. Two major considerations worth stating now and not studied in this thesis beyond short remarks are firstly issues relating to the convergence properties of any of the associated spectral sequences and secondly, and related to the first, is existence of model structures whose weak equivalences are given by isomorphisms on the ∞-page of the associated spectral sequence. Whilst we do consider the model structures on filtered chain complexes' interaction with a tensor product we do not use a completed tensor product as in [HT90].

Structure of the document and summary of new results

We detail now the new results of this thesis. As stated earlier many of the results are establishing various properties one would like to have of model categories.

Chapter 1

This chapter contains the necessary background, references, conventions, etc. for the subsequent chapters.

Chapter 2

Firstly work in the existing literature has not considered any comparison between the model categories of filtered chains and bicomplexes. We construct then a left adjoint to the product totalisation functor with the aim of later showing a Quillen equivalence between the model categories.

Proposition 2.1.0.2. *There is an adjunction of categories* $\mathcal{L}\colon f\mathcal{C} \underset{\perp}{\rightleftarrows} b\mathcal{C} : \mathrm{Tot}^{\Pi}$.

Similarly we show there is an adjunction involving the coproduct totalisation functor:

Proposition 2.3.0.2. *There is an adjunction of categories* $\mathrm{Tot}^{\oplus}\colon b\mathcal{C} \underset{\perp}{\rightleftarrows} f\mathcal{C} : \mathcal{R}$.

However we make no use of this adjunction. We also compute $\mathcal{L}$ and Tot^{Π} applied to r-cycles and r-witness cycles respectively and additionally show the following theorem asserting the unit map is an s-weak equivalence on s-cycles.

Proposition 2.2.1.2. *For* $s \geq 1$ *the unit of the adjunction applied to an* s*-cycle,* $\mathcal{Z}_s(p, p+n) \to \mathrm{Tot}^{\Pi}\mathcal{L}\mathcal{Z}_s(p, p+n)$*, is an isomorphism on the* s*-page.*

Lastly we Kan transfer a model structure to obtain a total model structure.

Corollary 2.4.0.3. *There is a total model structure on bicomplexes cofibrantly generated by generating cofibrations* $I := \{\mathcal{ZW}_{\infty,-\infty}(n) \to \mathcal{BW}_{\infty,-\infty}(n)\}$ *and generating acyclic cofibrations* $J := \{0 \to \mathcal{BW}_{\infty,-\infty}\}$ *in which:*

1. *weak equivalences are those morphisms* f *of bicomplexes such that* $H\mathrm{Tot}^{\Pi}$ *is an isomorphism,*
2. *fibrations are those morphisms* f *of bicomplexes such that* Tot^{Π} *is (homologically) degreewise surjective, i.e.* f *is bidegreewise surjective.*

Chapter 3

We then generalise the model structures of [CELW19] by introducing model structures 'in between' those of $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$ and similarly for bicomplexes and obtain various results concerning this collection of model categories. These follow easily from the work of [CELW19]. We define $I_S := I_r \cup \bigcup_{s\in S\setminus\{r\}} J_s$ and $J_S := \bigcup_{s\in S} J_s$.

Theorem 3.1.0.2. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r, the category $f\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(f\mathcal{C})_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $Z_s(f)$ is bidegree-wise surjective for each $s \in S$, and*
3. *I_S and J_S are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Further $(f\mathcal{C})_S$ is a finitely generated model category.

The cases $S = \{r\}$ and $S = \{0, 1, 2, \ldots, r\}$ give the original model structures $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$ of [CELW19]. Similarly for bicomplexes we define $I_S := I_r \cup \bigcup_{s\in S\setminus\{0,r\}} J_s$ and $J_S := \bigcup_{s\in S} J_s$ and we also have the following theorem.

Theorem 3.2.0.2. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including both 0 and r, the category $b\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(b\mathcal{C})_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $ZW_s(f)$ is bidegree-wise surjective for each $s \in S$, and*
3. *I_S and J_S are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Further $(b\mathcal{C})_S$ is a finitely generated model category.

The cases $S = \{0, r\}$ and $S = \{0, 1, 2, \ldots, r\}$ give the original model structures $(b\mathcal{C})_r$ and $(b\mathcal{C})_{r'}$ of [CELW19]. Thus to the tables of the model structures of [CELW19] we gave earlier we add the model structures $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$ generalising the previous. For filtered chain complexes we have

model category	generating cofibrations	generating acyclic cofibrations
$(f\mathcal{C})_r$	I_r	J_r
$(f\mathcal{C})_{r'}$	$I_r \cup \bigcup_{k=0}^{r-1} J_r$	$\bigcup_{k=0}^{r} J_r$
$(f\mathcal{C})_S$	I_S	J_S

and for bicomplexes we have

model category	generating cofibrations	generating acyclic cofibrations
$(b\mathcal{C})_r$	I_r	J_r
$(b\mathcal{C})_{r'}$	$I_r \cup \bigcup_{k=0}^{r-1} J_r$	$\bigcup_{k=0}^{r} J_r$
$(b\mathcal{C})_S$	I_S	J_S

Fixing an r we show all the model structures on filtered chain complexes indexed by an S with $\max S = r$ are Quillen equivalent via identity-identity adjunctions.

Proposition 3.1.0.6. *For a fixed r and subsets $S' \subseteq S \subseteq \{0, 1, \ldots, r\}$ both containing r there is a Quillen equivalence:*

$$id\colon (f\mathcal{C})_{S'} \underset{\perp}{\rightleftarrows} (f\mathcal{C})_S :id \quad .$$

Similarly fixing an r the model structures on bicomplexes indexed by an S with $\max S = r$ are Quillen equivalent via the identity-identity adjunction.

Proposition 3.2.0.6. *For a fixed* r *and subsets* $S' \subseteq S \subseteq \{0, 1, \ldots, r\}$ *containing* 0 *and* r *there is a Quillen equivalence*

$$id \colon (b\mathcal{C})_{S'} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (b\mathcal{C})_S : id \quad .$$

We show in Corollaries 3.3.0.4 and 3.3.0.9 that all these model category structures are distinct. We then show there is a homotopical comparison between these model structures on filtered chain complexes and bicomplexes by showing the $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ is a Quillen adjunction for appropriate indexing sets S.

Proposition 3.4.0.2. *For* S *a subset of* $\{0, 1, \ldots, r\}$ *containing both* 0 *and* r *there is a Quillen adjunction*

$$\mathcal{L} \colon (f\mathcal{C})_S \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (b\mathcal{C})_S : \mathrm{Tot}^{\Pi} \quad .$$

Writing $S + 1$ for the set $\{s + 1 \mid s \in S\}$ we show that the shift-décalage adjunctions generalise as follows.

Proposition 3.5.0.2. *There are Quillen equivalences given by the shift-décalage adjunction:*

$$S \colon (fC)_S \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (fC)_{S+1} : \mathrm{Dec} \quad .$$

We further explain the poset given on the model structures $(f\mathcal{C})_S$ where $S' < S$ if there is a left Quillen functor from $(f\mathcal{C})_{S'}$ to $(f\mathcal{C})_S$ obtained by composing some of the identity-identity and shift-décalage adjunctions. The model structures $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$ are shown to be left proper in Theorems 3.7.1.7 and 3.7.2.8, cellular in Propositions 3.8.1.5 and 3.8.2.2 and stable in Propositions 3.9.1.2 and 3.9.2.4. We then show there is a Quillen equivalence between the model categories of filtered chain complexes and bicomplexes.

Theorem 3.10.0.4. *For* S *contatining both* 0 *and* r *there is a Quillen equivalence between the* S*-model structure on filtered chain complexes and the* S*-model structure on bicomplexes given by the* $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ *adjunction:*

$$\mathcal{L} \colon (f\mathcal{C})_S \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (b\mathcal{C})_S : \mathrm{Tot}^{\Pi} \quad .$$

In Proposition 3.11.0.1 we show for the model categories of the above with weak equivalences the r-equivalences there are no left Bousfield localisations with weak equivalances the $(r+1)$-equivalences. We then briefly explain constructions of various bounded variants of the model structures $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$.

Corollary 3.12.1.2. *There is a cofibrantly generated model structure denoted* $\left(f\mathcal{C}^{\geq}\right)_S$ *on* $f\mathcal{C}^{\geq}$ *whose weak equivalences are the* r*-quasi isomorphisms and with generating cofibrantions* τI_S *and generating acyclic cofibrations* τJ_S*.*

Theorem 3.12.2.1. *For every* $r \geq 0$ *and every subset* $S \subseteq \{0, 1, \ldots, r\}$ *including* r*, the category* $f\mathcal{C}^{\leq}$ *admits a right proper cofibrantly generated model structure, which we denote* $\left(f\mathcal{C}^{\leq}\right)_S$*, where:*

1. *weak equivalences are* E_r*-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes* $f \colon A \to B$ *such that* $Z_s^{p,p+n}(f)$ *is bidegree-wise surjective for* $n \leq -1$ *and* $s \in S$*, and*
3. κI_S *and* κJ_S *are the sets of generating cofibrations and generating acyclic cofibrations respectively.*

Furthermore $\left(f\mathcal{C}^{\leq}\right)_S$ *is a finitely generated model category.*

Theorem 3.12.3.15. *For every subset* $S \subseteq \{0, 1, 2, \ldots, r\}$ *containing* r *the category* $f_{\geq}\mathcal{C}$ *admits a right proper cofibrantly generated model structures, which we denote* $(f_{\geq}\mathcal{C})_S$*, whose:*

1. *weak equivalences are the* E_r*-quasi-isomorphisms,*
2. *fibrations are morphisms that for all* $s \in S$ *are* $Z_s^{p,p+n}$*-surjective for* $p \geq s$ *and all* n*, and*
3. *generating cofibrations and generating acyclic cofibrations are given by* $I_S^{\geq}$ *and* $J_S^{\geq}$ *respectively.*

Furthermore $(f_{\geq}\mathcal{C})_S$ *is a finitely generated model category.*

Chapter 4

After recalling a classification of cofibrations in the projective model structure on unbounded chain complexes we begin a partial classification of cofibrant objects and cofibrations in $(f\mathcal{C})_r$. We show that cofibrant objects must satisfy a list of conditions incorporating both usual projective conditions and conditions relating to r-homotopy. These conditions, given in the following lemma, are shown by a sequence of lemmas within.

Lemma 4.1.0.1. *A cofibrant filtered chain complex A in the r-model structure on $f\mathcal{C}$ satisfies the following conditions:*

1. $\frac{A^n}{F_pA^n}$ *is a projective R-module for all $p, n \in \mathbb{Z}$,*
2. A^n *is a projective R-module for all $n \in \mathbb{Z}$,*
3. *the filtration on A is exhaustive, and*
4. *for an element $a \in F_pA^n$ we have $da \in F_{p-r}A^{n+1}$.*

Via another sequence of lemmas we give a nice interpretation of how cofibrant objects differ in the model categories $(f\mathcal{C})_S$ where we vary S but keep $\max S = r$ fixed. The following proposition interprets the cofibrant objects in $(f\mathcal{C})_S$ as those where the pages of the spectral sequence beneath r are allowed to change from the s-page to the $(s+1)$-page only if $s \in S$.

Proposition 4.1.0.14. *Let A be a cofibrant object of $(f\mathcal{C})_S$. Then for $k < r$ and $k \notin S$ the k-page differential d_k of A is* 0.

We then give our partial classification for the cofibrant objects in $(f\mathcal{C})_r$. These satisfy the previous list of conditions in addition to a boundedness condition on the filtration given in the following as the final condition.

Proposition 4.1.0.16. *Given a filtered chain complex A such that the following conditions hold*

1. *the graded pieces Gr_pA^n are projective for all $p, n \in \mathbb{Z}$,*
2. *for a pure element $a \in F_pA^n$ we have $da \in F_{p-r}A^{n+1}$ for all $p, n \in \mathbb{Z}$,*
3. *the filtration on A is exhaustive, and*
4. *whenever we have an r-acyclic filtered chain complex K and a morphism $A \to \Sigma^r K$ there is a lift in the following diagram:*

$$\begin{tikzcd} & C_r(K) \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & \Sigma^r K \end{tikzcd},$$

5. *and further such that for all n there is a $p(n) \in \mathbb{Z}$ such that $F_{p(n)}A^n = 0$ (i.e. the filtration is bounded below but not necessarily uniformly),*

then A is cofibrant in the r-model structure on $f\mathcal{C}$.

With some cofibrant objects understood we classify those cofibrations whose cokernel are cofibrant objects of this bounded form. As for cofibrations in chain complexes we have the following lemma.

Lemma 4.2.0.3. *An r-cofibration $i\colon A \to B$ is such that B is isomorphic to a twisted direct sum of A and the cokernel of i as filtered chain complexes.*

In the above then we have $B \cong A \oplus_\tau C$ for C the cokernel of i and a *twist differential* τ. We define such a twisted filtered chain complex to be surpressive if the twist map τ *supresses filtration by* r and using this terminology show the following partial classification of cofibrations in $(f\mathcal{C})_r$ again with the same boundedness assumption on the filtration of the cokernel.

Lemma 4.2.0.6. *An r-supressive inclusion $i\colon A \longrightarrow B$ whose cokernel C is cofibrant and such that for any n there is a $p(n)$ with $F_{p(n)}C^n = 0$ is an r-cofibration.*

Restricting to the subcategory of those objects whose differential is l-supressive we show that the shift-décalage adjunction induces an equivalence of categories between such l-supressive objects and $(l+1)$-supressive objects and show too that décalage preserves cofibrancy.

Lemma 4.3.0.5. *Let B be a cofibrant object of $(f\mathcal{C})_{S+l}$, then $\mathrm{Dec}^l B$ is a cofibrant object in $(f\mathcal{C})_S$.*

Chapter 5

We next consider the interaction of the monoidal product in $f\mathcal{C}$ with the model structures. We construct a cofibrant replacement for the unit. We let Q_rI denote the filtered chain complex given by

$$Q_rI \coloneqq \left(\bigoplus_{i=0}^{\infty} R^0_{(-i)} \longrightarrow \bigoplus_{j=1}^{\infty} R^1_{(-r-j)} \right) ,$$

where the differential is given by mapping each $R^0_{(-i)}$ diagonally onto the copies of R indexed as $R^1_{(-r-i)}$ and $R^1_{(-r-i-1)}$ for $i \geq 1$ and by the identity map from $R^0_{(0)}$ to $R^1_{(-r-1)}$.

Corollary 5.1.0.6. *The filtered chain complex Q_rI is an S-cofibrant replacement of the unit.*

Using this cofibrant replacement for the unit we verify the unit axiom for a monoidal model structure on filtered chain complexes.

Proposition 5.2.0.2. *The composite function $Q_rI \otimes A \longrightarrow I \otimes A \longrightarrow A$ is an r-weak equivalence for all (not necessarily cofibrant) A.*

And show too that the $(f\mathcal{C})_S$ satisfy the pushout-product axiom.

Lemma 5.3.2.1 and Corollary 5.3.1.4. *The pushout product of generating cofibrations i and j is a cofibration which is additionally acyclic if either i or j is.*

A consequence of these results is that the model categories $(f\mathcal{C})_S$ are monoidal so that there is an induced monoidal structure on the homotopy category.

Theorem 5.3.2.2. *Each of the model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 is a monoidal model category.*

Additionally we show that these monoidal model structures satisfy Greenlees and Shipley's monoid axiom.

Corollary 5.4.0.4. *The model categories $(f\mathcal{C})_S$ satisfy the monoid axiom.*

As a consequence we have model categories indexed by S on various categories of algebras or modules.

Theorems 5.4.0.5 to 5.4.0.7. *Fix r and let S be a subset of $\{0, 1, \ldots, r-1, r\}$ containing r. Let A be a filtered differential graded algebra. Then there are cofibrantly generated model structures, whose weak equivalences are the r-quasi isomorphisms and fibrations those morphisms that are surjective on all s-cycles for $s \in S$, on the categories of left A-modules, and when A is graded-commutative, on the categories of A-modules and A-algebras.*

Lastly in Corollaries 5.5.0.6 and 5.5.0.9 we use a result of [Mur15] to adapt the S-model structures on filtered chain complexes to ones where the unit of the tensor product is a cofibrant object and where all shifts of the unit are additionally cofibrant. These new model structures are additionally monoidal satisfying the monoid axiom by the same result of [Mur15] so one can immediately deduce existence of S-model structures on modules and algebras of $f\mathcal{C}$ whose unit is now cofibrant.

Appendix A

We consider in this appendix the question of finding a *cylinder object* on an S-cofibrant filtered chain complex. We also discuss a notion of flatness for filtered chain complexes.

Appendix B

This appendix proves the functor $\mathcal{R}$ is indeed a right adjoint to the coproduct totalisation functor $\mathrm{Tot}^{\oplus}$. The argument is very much dual to that for $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$.

Appendix C

We detail in this appendix questions that have arisen in the work of this thesis and that remain unanswered in this document. They are potential future directions of work.

Acknowledgements

I am deeply grateful to Sarah Whitehouse for all the encouragement, support and guidance given to me during my time in Sheffield. The enthusiasm and interest shown towards supervising this project has been nothing short of fantastic.

* * *

The Topology Group in Sheffield, including all the seminars and reading groups we've arranged, has been a fantastic font of knowledge for me to learn from and included amongst them I owe many thanks to Daniel Graves, Luca Pol, Callum Reader and Jordan Williamson for all the helpful conversations and suggestions they've contributed over the years regarding my work.

* * *

My time here has been made that much more enjoyable by everyone in the department and the friends I've made. Between all the walks, pub outings and quizzes, and crossword sessions they have helped keep me sane, in particular Poppy Jeffries has provided me many distractions (and very much vice versa) with the latter for when working didn't feel like an option.

* * *

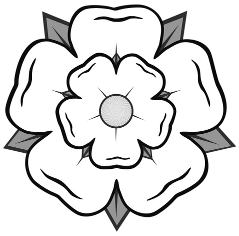
I'm also eternally thankful to my family: Michael, Frances, Anna and Sophie for putting up with me and my pacing around these past four years.

* * *

I am grateful to my examiners Frank Neumann and Markus Szymik who provided many corrections and suggested improvements to this thesis.

* * *

Lastly due are thanks for the support of the University of Sheffield for when I was working, to UCU when I wasn't (and also when I was), and to EPSRC who funded my studies.



CHAPTER 1

Background and Conventions

In this chapter we collect all the background material and conventions we will use in the following chapters. We assume some level of familiarity with regard to commonly used machinery within algebraic topology, e.g. definitions of categories and their (co)limits will be left unstated as will chain complexes and their homologies. We will however recall some terminology and definitions regarding ordinals within set theory for the purpose of the small object argument in constructing cofibrantly generated model categories.

Filtered chain complexes and bicomplexes will be defined and some conventions established with regard to them along with various adjunctions involving one or both categories. We define spectral sequences, give definitions for r-cycle and r-boundaries as well as their representing objects and state how to obtain a spectral sequence from a filtered chain complex. We also define the shift-décalage adjunction of Deligne.

Our principal focus of this thesis being model categories, we define and explain in detail the framework of a model category along with some important properties and results. We also include the motivation for (and a particular case of) homotopy (co)limits.

We recall the projective model structure on chain complexes as it serves as frequent motivation for constructions or proof methods in the filtered setting we consider. We also recall the r and r'-model structures of [CELW19] on filtered chain complexes and bicomplexes along with their constructions and other pertinent constructions. In particular they are cofibrantly generated model structures whose generating sets are analogous to the sphere and disc inclusions of chain complexes. The representing objects of r-cycles and r-boundaries are the analogues of the spheres and discs in these setttings.

We end this chapter with a short summary of conventions frequently used throughout this document.

1.1 Set theory: cardinals, ordinals and smallness

This section details the prerequisite material on set theory necessary for cofibrantly generated model categories as well as the notion of smallness. We use notation and conventions that can be found in [Hov99, §2.1.1], [Hir03, §10.1] or [Jec03, §2].

Definition 1.1.0.1. We make the following definitions which are standard in set theory.

1. A set T is *well ordered* if it is totally ordered and such that every non-empty subset of T has a minimal element.
2. A set T is *transitive* if every element of T is a subset of T.
3. A set α is an *ordinal* if it is well ordered by the membership relation $\in$ and is transitive.
4. For α an ordinal we define the *succesor ordinal* to be $\mathrm{Succ}(\alpha) := \alpha \cup \{\alpha\}$. The set $\mathrm{Succ}(\alpha)$ is an ordinal whenever α is.
5. If α is an ordinal but not a successor ordinal it is of the form $\alpha = \sup\{\beta \mid \beta < \alpha\}$ and said to be a *limit ordinal* (we include $\{\}$ as being a limit ordinal).

The following theorem yields all ordinals.

Theorem 1.1.0.2 (Transfinite Induction, [Jec03, Theorem 2.14]). *Let C be a class of ordinals such that:*

1. $\{\} \in C$,
2. *if* $\alpha \in C$ *then so too is* $\mathrm{Succ}(\alpha)$*, and*
3. *if* $\alpha \neq \{\}$ *is a limit ordinal and* $\beta \in C$ *for all* $\beta < \alpha$ *then* $\alpha \in C$,

then C *is the class of all ordinals.*

Remark 1.1.0.3. We will view an ordinal λ as a category with objects the elements of λ and for all $\alpha, \beta \in \lambda$ there is exactly one morphism $\alpha \to \beta$ whenever $\alpha < \beta$. Note that in this interpretation a $\gamma \in \lambda$ such that γ is a limit ordinal is the colimit in the category of λ over the full subcategory of elements of λ which are less than γ or equivalently admit morphisms to λ.

Definition 1.1.0.4. For $\mathcal{C}$ a cocomplete category:

1. if λ is an ordinal, then a *λ-sequence* is a colimit preserving functor $X\colon \lambda \to \mathcal{C}$ (most helpfully viewed as a sequence of composed morphisms in $\mathcal{C}$:
$$(X_0 \to X_1 \to X_2 \to \ldots)$$
where $\mathrm{colim}_{\alpha<\gamma} X_\alpha \to X_\gamma$ is an isomorphism for all limit ordinals γ), and
2. the *composition* of the λ-sequence is the morphism
$$X_0 \to \operatorname*{colim}_{\beta\in\lambda} X_\beta \ .$$

Definition 1.1.0.5. A *cardinal* is an ordinal of greater cardinality than any lesser ordinal.

Definition 1.1.0.6. The *cardinal of a set T* is the unique cardinal in bijection with T.

Definition 1.1.0.7. A cardinal γ is *regular* if for any set A whose cardinal is less than γ and sets S_a for each $a \in A$ also whose cardinals are less than γ, we have the cardinal of $\bigcup_{a\in A} S_a$ is less than γ.

Definition 1.1.0.8. Let α be an ordinal and γ a cardinal, we say that α is *γ-filtered* if it is a limit ordinal and for any $A \subseteq \alpha$ with the cardinal of A less than or equal to γ, then $\sup A < \alpha$.

Definition 1.1.0.9. Let $\mathcal{C}$ be a cocomplete category, M a subclass of the morphisms of $\mathcal{C}$, $C \in \mathcal{C}$ and κ a regular cardinal. Then C is said to be *κ-small relative to M* if for all κ-filtered ordinals λ and λ-sequences $X\colon \lambda \to \mathcal{C}$ with each morphism $X_\alpha \to X_{\mathrm{Succ}(\alpha)}$ in M, then:
$$\operatorname*{colim}_{\alpha<\lambda} \mathrm{Hom}_{\mathcal{C}}(C, X_\alpha) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, \operatorname*{colim}_{\alpha<\gamma} X_\alpha)$$
is an isomorphism. Further C is *small relative to M* if it is κ-small relative to M for some cardinal κ, and C is *small* if it is small relative to $\mathcal{C}$.

Remark 1.1.0.10. If κ is a finite cardinal in Definition 1.1.0.9 we also use the terminology *finite* in place of *small*.

Recall M is a finitely presented R-module if there is a short exact sequence of the form $\bigoplus_m R \to \bigoplus_n R \to M \to 0$, i.e. there are finitely many generators given by the basis of $\bigoplus_n R$ and finitely many relations given by those of $\bigoplus_m R$.

Examples 1.1.0.11. Examples of small and finite objects given in [Hov99, Examples 2.1.4, 2.1.6 & Lemma 2.3.2] include:

1. in the category of sets every set is small, and finite sets are precisely the finite objects,
2. in the category of R-modules every R-module is small and the finitely presented R-modules are the finite objects,
3. in the category of chain complexes over a ring R every chain complex is small and the bounded (above and below) chain complexes of finitely presented R-modules are the finite objects.

1.2 Category theory

1.2.1 Category of filtered chain complexes

Throughout this thesis R will denote a commutative unital ring.

Definition 1.2.1.1. A *filtered object* X in a category $\mathcal{C}$ is an object X with an *increasing filtration*, i.e. subobjects $F_pX \subseteq X$ for all $p \in \mathbb{Z}$ such that $F_pX \subseteq F_{p+1}X$.

Example 1.2.1.2. We will make use of the following two filtered objects.

1. A *filtered R-module M* consists of an R-module M with submodules $F_pM \subseteq M$ for each $p \in \mathbb{Z}$ with inclusions $F_pM \subseteq F_{p+1}M$.

2. A *filtered chain complex* of R-modules C consists of a (cohomologically graded) chain complex C and subchain complexes F_pC of C for each $p \in \mathbb{Z}$. In particular the differentials of C preserve filtration, i.e. for $c \in C^n$ with $c \in F_pC^n$ then $dc \in F_pC^{n+1}$.

Definition 1.2.1.3. A *morphism of filtered objects* $f\colon X \longrightarrow Y$ in a category $\mathcal{C}$ is a morphism of the underlying objects that preserves the filtrations, i.e. $f(F_pX) \subseteq F_pY$, for all $p \in \mathbb{Z}$.

Notation 1.2.1.4. The *category of filtered chain complexes over a ring* R will be denoted $f\mathcal{C}$.

Notation 1.2.1.5. The filtered chain complex A with one copy of R in cohomological degree n and such that $0 = F_{p-1}A \subset F_pA = A$ will be denoted $R^n_{(p)}$. We will frequently abuse notation and also use $R^n_{(p)}$ to denote a subobject of another filtered chain complex, to build larger ones from these pieces.

Definition 1.2.1.6. An element of a filtered chain complex $a \in A$ is said to be of *pure filtration degree p* if $a \in F_pA$ and $a \notin F_{p-1}A$, i.e. p is the first filtration indexing where a appears.

The previous notation $R^n_{(p)}$ could then be described as the chain complex with a copy of R in degree n of pure degree p.

Definition 1.2.1.7. A morphism of filtered chain complexes $f\colon A \longrightarrow B$ is *strict* if whenever $a \in A^n$ is such that $f(a) \in F_pB^n$ then we have $a \in F_pA^n$.

We are principally interested in this category of filtered chain complexes. In defining model category structures on this category we will need to know that $f\mathcal{C}$ admits all small (co)limits. This was established in [CELW19, Remark 2.6]. Limits are computed as one might naïvely think, but the same is not true for colimits. We will need the process of constructing colimits later for verifying the pushout-product of generating (acyclic) cofibrations is also a cofibration (which is acyclic if either cofibration is) so we recall a construction here.

Notation 1.2.1.8. We denote by $\mathbb{Z}_\infty$ the category with objects $\mathbb{Z} \cup \{\infty\}$ thought of as a category with a unique morphism $n \to m$ whenever $n \leq m$ in $\mathbb{Z}$ and a unique morphism $n \to \infty$ for each $n \in \mathbb{Z}$, so that $\mathbb{Z}_\infty$ is the category $\mathbb{Z}$ with a terminal object adjoined.

Lemma 1.2.1.9 ([CELW19, Remark 2.6])**.** *For the category $f\mathcal{C}$ we can compute*

- *limits levelwise, i.e. for a diagram $D\colon I \longrightarrow f\mathcal{C}$ the underlying chain complex of the limit of D is $\lim_I D(i)$ where we forget filtration, and the p-filtered part of the limit is given by $F_p \lim_I D(i) = \lim_I F_pD(i)$, and*

- *colimits can be computed by viewing $f\mathcal{C}$ as a reflective subcategory of $\mathcal{C}_R^{\mathbb{Z}_\infty}$, i.e. there is an adjunction $\rho\colon \mathcal{C}_R^{\mathbb{Z}_\infty} \underset{\perp}{\rightleftarrows} f\mathcal{C} : i$ where the left adjoint ρ sends a $\mathbb{Z}_\infty$ indexed chain complex $E\colon \mathbb{Z}_\infty \longrightarrow \mathcal{C}_R$ to the filtered chain complex with underlying chain complex $E(\infty)$ and p-filtered part $F_p\rho E = \operatorname{im}(E(p) \longrightarrow E(\infty))$. The colimit of a diagram $D\colon J \to f\mathcal{C}$ can then be computed as $\operatorname{colim}_J D = r \operatorname{colim}_J iD$, so that the underlying chain complex is the colimit of the underlying chain complexes of the $D(j)$ and the p-filtered part is given by*

$$F_p \operatorname*{colim}_J D = \operatorname{im}\left(\operatorname*{colim}_J D(j)(p) \longrightarrow \operatorname*{colim}_J D(j)(\infty)\right) .$$

We consider a specific case of the colimit construction for filtered chain complexes for those morphisms $f\colon A \longrightarrow B$ of $f\mathcal{C}$ with the property that if an element $a \in A^n$ is such that $f(a) \in F_pB^n$ then $a \in F_pA^n$, i.e. that f is a strict morphism of filtered chain complexes. We consider the cokernel of the morphism f which we denote by B/A. This is computable according to Lemma 1.2.1.9 and have underlying chain complex B/A (where we forget filtrations) and filtration given by:

$$F_p\left(B/A\right)^n = \operatorname{im}\left(\frac{F_pB^n}{F_pA^n} \longrightarrow \frac{B^n}{A^n}\right) \ .$$

For the given morphism f if an element of F_pB^n/F_pA^n becomes 0 in B^n/A^n then it was of the form $\overline{f(a)}$ for some $a \in A^n$ and where $f(a) \in F_pB^n$, however the assumption on f implies that $a \in F_pA^n$ hence in F_pB^n/F_pA^n the element $\overline{f(a)}$ is 0. This gives the map of the image is an inclusion and so we can compute the cokernel homological degree-wise and filtration degree-wise.

Lemma 1.2.1.10. *For a strict morphism of filtered chain complexes the cokernel can be computed filtration degree-wise.*

The category of chain complexes can be equipped with a monoidal product, the tensor product, for which $- \otimes A$ is left adjoint to the enriched $\underline{\mathrm{Hom}}$ functor $\underline{\mathrm{Hom}}_{\mathcal{C}_R}(A, -)$, (in this paper we use an underline to denote an enriched Hom functor). We show here that this can be extended to the filtered setting we work in.

Definition 1.2.1.11. The *tensor product* of $X, Y \in f\mathcal{C}$ has underlying chain complex the usual one:

$$(X \otimes Y)^n := \bigoplus_{i+j=n} X^i \otimes Y^j \ ,$$

and is given in filtration p by:

$$F_p\left(X \otimes Y\right) := \sum_{i+j=p} \operatorname{im}\left(F_iX \otimes F_jY \to X \otimes Y\right) \ ,$$

where as usual the differential is given according to the Koszul sign rule, $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ with $|x|$ being the cohomological degree of x.

Remark 1.2.1.12. Note that frequently in the literature one wants to take a completed tensor product when the tensor components carry some topological information. This is used for example in [HT90]. We do not consider issues relating to convergence in this thesis however and it is advantageous to have a tensor-hom adjunction so we use an uncompleted form.

Notation 1.2.1.13. We denote by $R^{(n)+(m)}_{(p)+(q)}$ the tensor product $R^n_{(p)} \otimes R^m_{(q)}$ making judicious use of bracketing for clarity.

Definition 1.2.1.14. The *internal hom object* of $X, Y \in f\mathcal{C}$ has underlying chain complex the usual one:

$$\underline{\mathrm{Hom}}_{f\mathcal{C}}(X, Y)^n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}(X^i, Y^{i+n}) \ ,$$

and is given in filtration p by:

$$F_p\underline{\mathrm{Hom}}_{f\mathcal{C}}\left(X, Y\right)^n := \prod_{i \in \mathbb{Z}} F_p\mathrm{Hom}_R\left(X^i, Y^{i+n}\right)$$

where $F_p\mathrm{Hom}_R(X^i, Y^{i+n}) := \left\{f \mid f(F_aX^i) \subseteq F_{a+p}Y^{i+n} \ \forall a\right\}$. The differential on an element $f = (f_i)_i \in \underline{\mathrm{Hom}}_{f\mathcal{C}}(X, Y)^n$ is given by:

$$\left(f_i\right)_i \mapsto \left(d^Y \circ f_i - (-1)^n f_{i+1} \circ d^X\right)_i \ .$$

Lemma 1.2.1.15. *For any $X \in f\mathcal{C}$ there is an adjunction pair $(- \otimes Y) \dashv \underline{\mathrm{Hom}}_{f\mathcal{C}}(Y, -)$ on the category of filtered chain complexes.*

$$\mathrm{Hom}_{f\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{f\mathcal{C}}(X, \underline{\mathrm{Hom}}_{f\mathcal{C}}(Y, Z))$$

Proof. There is already a closed monoidal structure on the level of non-filtered chain complexes which is the underlying bijection in the filtered case, we need only check that the bijection sends filtered morphisms to filtered morphisms. One direction of the bijections takes the form:

$$\begin{aligned}\varphi\colon \operatorname{Hom}_{f\mathcal{C}}(X\otimes Y, Z) &\longrightarrow \operatorname{Hom}_{f\mathcal{C}}(X, \underline{\operatorname{Hom}}_{f\mathcal{C}}(Y,Z))\\ f &\longmapsto \left(x \longmapsto \left(\tilde{f}(x)_i\colon y \mapsto f(x\otimes y)\right)_i\right)\ ,\end{aligned}$$

where $y \in Y^i$. Given then a morphism $f\colon X\otimes Y \longrightarrow Z$ we need to verify the adjoint morphism $\varphi(f)$ just defined is a morphism of filtered chains. Take then an $x \in F_pX^n$, we need to check that the morphism $\tilde{f}(x)_i$ satisfies the condition imposed in Definition 1.2.1.14, i.e. that $\tilde{f}(x)_i(F_aY^i) \subseteq F_{a+p}Y^{i+n}$. Given then $y \in F_aY^i$ we have that $\tilde{f}(x)_iy = f(x\otimes y)$ where $x\otimes y \in F_{p+a}Y^{i+n}$ and so $\tilde{f}(x)_i(F_aY^i) \subseteq F_{p+a}Y^{i+n}$, showing that $\varphi(f)$ is too a morphism of filtered chains when f is. The inverse of the bijection is given by:

$$\begin{aligned}\psi\colon \operatorname{Hom}_{f\mathcal{C}}(X, \underline{\operatorname{Hom}}_{f\mathcal{C}}(Y,Z)) &\longrightarrow \operatorname{Hom}_{f\mathcal{C}}(X\otimes Y, Z)\\ g &\longmapsto (x\otimes y \mapsto g(x)_i(y))\ .\end{aligned}$$

Suppose now g is a morphism of filtered chains, for an element of $F_p(X\otimes Y)^n$ in the image of $F_jX^k\otimes F_{p-j}Y^{n-k}$, say from $x\otimes y$, we have $(g(x)_i)_i \in F_j\underline{\operatorname{Hom}}_{f\mathcal{C}}(Y,Z)^k$ and so by definition of the filtration on the $\underline{\operatorname{Hom}}_{f\mathcal{C}}$ we have $g(x)_{n-k}(F_{p-j}Y^{n-k}) \subseteq F_{p-j+j}Z^{n-k+k} = F_pZ^n$, i.e. $g(x)_{n-k}(y) \in F_pZ^n$ and $\psi(g)$ preserves the filtration.

Lemma 1.2.1.16. *A filtered chain complex A is a finite object of the category $f\mathcal{C}$ if and only if it satisfies the following conditions:*

1. *F_pA^n is finitely presented for all p and n,*
2. *$A^n = 0$ for all $n \leq n_1$ for some n_1,*
3. *$A^n = 0$ for all $n \geq n_2$ for some n_2,*
4. *$F_{p_1}A = 0$ for some finite p_1, and*
5. *$F_{p_2}A = A$ for some finite p_2.*

In the presence of conditions 2 and 3, the conditions 4 and 5 are equivalent to the following: for each n there are $p_1(n)$ and $p_2(n)$ such that $F_{p_1(n)}A^n = 0$ and $F_{p_2(n)}A^n = A^n$

Such a filtered chain complex can be visualised, via the inclusion functor $i\colon f\mathcal{C} \longrightarrow \mathcal{C}_R^{\mathbb{Z}_\infty}$, to be bounded within a box such that left, right and down of the box are 0 modules, within the box are finitely presented R-modules and above the box the morphisms of iX induced from $\mathbb{Z}_\infty$ are identities. In preparation for the proof consider a λ-sequence of filtered chain complexes $(X_0 \to X_1 \to \ldots)$. We give an expressions for the colimit of this λ-sequence. The underlying chain complex of the colimit is just the colimit of the underlying chain complexes. We can compute the p-filtered part using the reflector-inclusion adjunction of Lemma 1.2.1.9 as follows:

$$\begin{aligned}F_p\left(\operatorname{colim} X_\beta\right)^n &= F_p\left(r\operatorname{colim} iX_\beta\right)^n\\ &= r\operatorname{colim}(iX_\beta)_p^n\\ &= \operatorname{im}\left(\operatorname{colim}(iX_\beta)_p^n \longrightarrow \operatorname{colim}(iX_\beta)_\infty^n\right)\ . \end{aligned}\tag{1.1}$$

One can easily show that relaxing any of the conditions of Lemma 1.2.1.16 yields a non-finite filtered chain complex, we are then left with proving the other direction.

Proof. To show A is finite we must show that for any limit ordinal λ and any λ-sequence

$$(X_0 \to X_1 \to X_2 \to \ldots \to X_\beta \to \ldots)$$

the following set map is an isomorphism

$$\operatorname*{colim}_{\beta<\lambda} \operatorname{Hom}_{f\mathcal{C}}(A, X_\beta) \longrightarrow \operatorname{Hom}_{f\mathcal{C}}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)\ . \tag{1.2}$$

Surjectivity: Consider a morphism $f \in \operatorname{Hom}_{f\mathcal{C}}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$. This gives morphisms of R-modules on the p-filtered parts

$$F_pf^n\colon F_pA^n \longrightarrow \operatorname{im}\left(\operatorname{colim}(iX_\beta)_p^n \longrightarrow \operatorname{colim}(iX_\beta)_\infty^n\right) \tag{1.3}$$

using Equation (1.1). Consider a set of R-module generators of the F_pA^n for $p_1 \leq p \leq p_2$ and $n_1 \leq n \leq n_2$ of the theorem . There are finitely many such generators $\{a_i\}$ as each is finitely presented and our range of p and n is finite. For an a_i to map into $\operatorname{colim} X_\beta$ it must land in the image of Equation (1.3) so that there is some element of $x_i \in \operatorname{colim}(iX_\beta)^n_p$ whose image in $\operatorname{colim}(iX_\beta)^n_\infty$ is the same as that of a_i. There is then an $x_i^{\beta(i)}$ in some $F_pX^n_{\beta(i)}$ equal to x_i in the colimit. Constructing these for all generators gives a set map from our chosen set of generators into various X_β. This need not give a morphism of filtered chains yet. We consider now, for those chosen generators of each F_pA^n in the range $p_1 \leq p \leq p_2$ and $n_1 \leq n \leq n_2$ the relations $\{r_j\}$ between them which again is a finite set as these R-modules are finitely presented and our range of p and n is finite. Each relation is a sum of some of the generators a_i which is 0 in A however the sum of the corresponding lifts $x_i^{\beta(i)}$ need not be 0 in $X_{max(\beta(i))}$. The relation is however 0 in $\operatorname{colim} X_\beta$ so there is some X_α where the relation in terms of the $x_i^{\beta(i)}$ are satisfied. Doing so for all generators and relations between them and taking the maximum of all $\beta(i)$ and α constructed thus far (which really is a maximum as we only have finitely many such) gives R-module morphisms of the filtered parts F_pA^n into $\operatorname{colim} X_\beta$ which factorises via some $X_\bullet$. These do not yet assemble to a morphism of filtered chain complexes. One can however perform a similar trick to ensure that differentials commute with the constructed map. We have finitely many generators which have lifts into some $X_\bullet$, we find lifts too of the da_i into some $X_\bullet$ but the differential of the lifts of the generators need not agree with the lifts of the da_i however do eventually, again we take a maximum of the indices of the $X_\bullet$ to obtain a map of R-modules which commutes with the differentials. Lastly we ensure that for $F_pA^n \subseteq F_{p+1}A^n$ in our range that the morphisms into the $X_\bullet$ agree, again we can do so on generators (of which there are finitely many of them) and finally take a maximum over all indices of the $X_\bullet$ which we have factorised via, say into X_μ. This is then a morphism of filtered chain complexes $A \longrightarrow X_\mu$ since we have ensured it is a morphism of R-modules on the filtered parts, that it commutes with differentials, and such that morphism firstly restricted to $F_{p+1}A^n$ and then to F_pA^n agrees with that constructed by restricting first to F_pA^n. The composite $A \longrightarrow X_\mu \longrightarrow \operatorname{colim} X_\beta$ is a factorisation of $A \longrightarrow \operatorname{colim} X_\beta$ and so the morphism Equation (1.2) is indeed surjective.

Injectivity: Consider two morphisms f, g in $\operatorname{colim} \operatorname{Hom}_{f\mathcal{C}}(A, X_\beta)$ which under the morphism Equation (1.2) become equal. Represent them by some $f, g \colon A \longrightarrow X_\gamma$. Via similar arguments for surjectivity we can then, for each generator, find $X_\bullet$ in which the two maps agree on that generator after post composition $A \longrightarrow X_\gamma \longrightarrow X_\bullet$. Taking the maximum over the indices of each such $X_\bullet$ gives a filtered chain complex, say X_δ, in which the composite of f and g with $X_\gamma \longrightarrow X_\delta$ agree on all generators and so f and g are equal in $\operatorname{colim} \operatorname{Hom}_{f\mathcal{C}}(A, X_\beta)$.

A similar proof shows that all filtered chain complexes are small objects.

Lemma 1.2.1.17. *All objects of the category $f\mathcal{C}$ are small.*

The following definition is a simple renaming of [CELW19, Definition 3.5] of the r-translation functor T_r and its inverse. We introduce this new notation to agree with the usual stable notation.

Definition 1.2.1.18. For a filtered chain complex A the *r-suspension* $\Sigma^r A$ and *r-loops* $\Omega^r A$ of A have underlying chain complex the usual suspension and desuspension of A and whose underlying filtrations are given by:

$$F_p\Sigma^r A^n = F_{p-r}A^{n+1} \ ,$$
$$F_p\Omega^r A^n = F_{p+r}A^{n-1} \ ,$$

and whose differentials are $d^{\Omega^r A} = d^{\Sigma^r A} = -d^A$.

Lemma 1.2.1.19. *The Σ^r functor is isomorphic to tensoring on the left by $R^{-1}_{(r)}$ and the Ω^r functor is isomorphic to tensoring on the left by $R^1_{(-r)}$.*

Remark 1.2.1.20. Whilst the category of chain complexes is an abelian category the category of filtered chain complexes is not. A consequence of the abelian axioms is that a morphism $f \colon A \to B$ is an isomorphism if and only if its kernel and cokernel are both 0. However one can define a morphism of filtered R-modules which has 0 kernel and cokernel but is not an isomorphism, [GM03, Chapter II §5.17].

Lastly we make a remark on linear maps for an R-module compatible with a filtration. Consider a direct sum of R-modules $A := R_{(p_1)} \oplus R_{(p_2)} \oplus \ldots \oplus R_{(p_k)}$ where the p_i denote the pure filtration degree of the copy of R and are such that $p_1 \geq p_2 \geq \ldots p_k$. For a matrix representing a linear map of chain complexes $A \to A$ with the obvious basis to be a map of filtered chain complexes it must be lower triangular. We will later make use of this regarding change of bases compatible with a filtration.

1.2.2 Category of bicomplexes

Definition 1.2.2.1. A *bicomplex* A over a ring R is a collection of bigraded modules $A^{i,j}$ for $i, j \in \mathbb{Z}$ with differentials $d_0\colon A^{i,j} \longrightarrow A^{i,j+1}$ and $d_1\colon A^{i,j} \longrightarrow A^{i-1,j}$ called the vertical and horizontal differentials respectively such that $d_0 \circ d_0 = 0$ and $d_1 \circ d_1 = 0$, and such that they commute $d_0 \circ d_1 = d_1 \circ d_0$.

Definition 1.2.2.2. A *morphism of bicomplexes* $f\colon A \longrightarrow B$ is a collection of R-module morphisms $f^{i,j}\colon A^{i,j} \longrightarrow B^{i,j}$ for $i, j \in \mathbb{Z}$ such that they commute with the differentials, $f \circ d_0 = d_0 \circ f$ and $f \circ d_1 = d_1 \circ f$.

Notation 1.2.2.3. The *category of bicomplexes over a ring R* will be denoted $b\mathcal{C}$.

Remark 1.2.2.4. The category of bicomplexes is clearly isomorphic to the category $\mathcal{C}^v(\mathcal{C}^h)$ of chain complexes of chain complexes over R where the v and h superscripts refer to the direction of the differential. There is also an isomorphism of categories from $b\mathcal{C}$ to $\mathcal{C}^h(\mathcal{C}^v)$. We will make use of the former in interpreting the $r = 0$ model structure of [CELW19] on bicomplexes.

Definition 1.2.2.5. The *product totalisation functor* $\mathrm{Tot}^\Pi\colon b\mathcal{C} \to f\mathcal{C}$ is defined on a bicomplex K by:

$$\mathrm{Tot}^\Pi(K)^n := \prod_{i\in\mathbb{Z}} K^{i,i+n} \ ,$$

with filtration given by the column filtration:

$$F_p\mathrm{Tot}^\Pi(K)^n := \prod_{i\leq p} K^{i,i+n} \ ,$$

and with differential given on an element $(k_i)_i \in \prod_i K^{i,i+n}$ by:

$$d\colon \ (k_i)_i \mapsto (d_0 k_i + (-1)^n d_1 k_{i+1})_i \ .$$

On a morphism of bicomplexes $f\colon K \to J$ the functor Tot^Π is given by $\mathrm{Tot}^\Pi(f)^n := \prod_i f^{i,i+n}$. Note that this preserves filtration and differentials.

There is also a coproduct totalisation functor but we won't make so much use of it and so where we omit the adornments and write Tot we mean Tot^Π.

Definition 1.2.2.6. The *coproduct totalisation functor* $\mathrm{Tot}^\oplus\colon b\mathcal{C} \to f\mathcal{C}$ is defined on a bicomplex K by:

$$\mathrm{Tot}^\oplus(K)^n := \bigoplus_{i\in\mathbb{Z}} K^{i,i+n} \ ,$$

with filtration given by the column filtration:

$$F_p\mathrm{Tot}^\oplus(K)^n := \bigoplus_{i\leq p} K^{i,i+n} \ ,$$

and with differential given on an element $(k_i)_i \in \bigoplus_i K^{i,i+n}$ by:

$$d\colon \ (k_i)_i \mapsto (d_0 k_i + (-1)^n d_1 k_{i+1})_i \ .$$

On a morphism of bicomplexes $f\colon K \to J$ the functor $\mathrm{Tot}^\oplus$ is given by $\mathrm{Tot}^\oplus(f)^n := \bigoplus_i f^{i,i+n}$. Note that this preserves filtration and differentials.

There is a symmetric monoidal tensor product on bicomplexes given in the same way as for chain complexes.

Definition 1.2.2.7. For bicomplexes A and B we define the tensor product of A and B denoted $A \otimes B$ to be such that

$$(A \otimes B)^{p,q} := \bigoplus_{\substack{n_1+n_2=p \\ m_1+m_2=q}} A^{n_1,m_1} \otimes B^{n_2,m_2}$$

with differentials $d_0^{A\otimes B}$ and $d_1^{A\otimes B}$ given on an element $a \otimes b \in A^{n_1,m_1} \otimes B^{n_2,m_2} \subseteq (A \otimes B)^{p,q}$ by:

$$d_0^{A\otimes B}(a \otimes b) = d_0^A a \otimes b + a \otimes (-1)^{|a|} d_0^B b \ ,$$
$$d_1^{A\otimes B}(a \otimes b) = d_1^A a \otimes b + a \otimes (-1)^{|a|} d_1^B b \ ,$$

where $|a| = n_1 + m_1$.

Definition 1.2.2.8. For a bicomplex A the *r-suspension* $\Sigma^r A$ and *r-loops* $\Omega^r A$ of A are given by:

$$\begin{aligned}(\Sigma^r A)^{p,q} &= A^{p-r,q-r+1} \ , \\ (\Omega^r A)^{p,q} &= A^{p+r,q+r-1} \ ,\end{aligned}$$

and whose differentials are $d_0^{\Omega^r A} = d_0^{\Sigma^r A} = (-1)^{r+1} d_0^A$ and $d_1^{\Omega^r A} = d_1^{\Sigma^r A} = (-1)^r d_1^A$.

1.2.3 Locally presentable categories

Definition 1.2.3.1. A category $\mathcal{I}$ is *filtered* if it is non-empty and the following two conditions hold

1. for all $i, j \in \mathcal{I}$ there exists a $k \in \mathcal{I}$ and morphisms $i \to k$ and $j \to k$, and
2. for all $i, j \in \mathcal{C}$ and every pair $f, g \colon i \rightrightarrows j$ there exists a $k \in \mathcal{I}$ and $h \colon j \to k$ with $hf = hg$.

Definition 1.2.3.2. A category $\mathcal{C}$ is a *locally presentable category* if there is a regular cardinal λ such that

1. $\mathcal{C}$ is *locally small*, i.e. the collection of morphisms between two objects is a set,
2. $\mathcal{C}$ has all small colimits, and
3. there is a set of objects $\mathcal{S}$ such that any object of $\mathcal{C}$ is a λ-filtered colimit of objects of $\mathcal{S}$.

Lemma 1.2.3.3. *The category $f\mathcal{C}$ of filtered chain complexes is a locally presentable category.*

Proof. Note that $\mathcal{C}_R$ is a locally presentable category and since $\mathbb{Z}_\infty$ is a small category the functor category $\mathcal{C}_R^{\mathbb{Z}_\infty}$ is locally presentable. Using the adjunction $\rho\colon \mathcal{C}_R^{\mathbb{Z}_\infty} \underset{\perp}{\rightleftarrows} f\mathcal{C} : i$ we can then exhibit the generation condition for locally presentable categories. If S is a set of λ-small objects generating $\mathcal{C}_R^{\mathbb{Z}_\infty}$ under λ-filtered colimits then the set ρS generates $f\mathcal{C}$; for an object $A \in f\mathcal{C}$ we have that $iA = \operatorname{colim}_I F$ for some λ-filtered functor $F \colon I \to \mathcal{C}_R^{\mathbb{Z}_\infty}$ with image of objects in S and so $A = \rho i A = \rho \operatorname{colim}_I F = \operatorname{colim}_I \rho F$ since left adjoints commute with colimits.

1.3 Spectral sequences

Spectral sequences are tools used to aid in homology (and homotopy) calculations by constructing a sequence of successive approximations. The end result being a bigraded collection of algebraic data from which one may need to solve extension problems to correctly construct the homology.

1.3.1 Definition

Our definition of a spectral sequence is near identical to that of McCleary [McC01, Definition 2.2] except we begin indexing of our pages at 0 instead of 1 and the bidegrees of the differentials are a mix of McCleary's homological and cohomological grading, the latter chosen to agree with the setup in [CELW19].

Definition 1.3.1.1. A *spectral sequence* is a sequence of bigraded differential modules $\{E_r^{*,*}, d_r\}$ for $r \geq 0$ with d_r of bidegree $(-r, 1-r)$ such that

$$E_{r+1}^{p,q} \cong H^{p,q}\left(E_r^{*,*}, d_r\right) \cong \frac{\ker\left(d_r \colon E_r^{p,q} \to E_r^{p-r,q+1-r}\right)}{\operatorname{im}\left(d_r \colon E_r^{p+r,q-1+r} \to E_r^{p,q}\right)}$$

for each $r \geq 0$. The bigraded module $E_r^{*,*}$ will be referred to as the *r-page*, and the differential d_r as the *r-differential.*

Definition 1.3.1.2. A *morphism of spectral sequences* $f \colon \{E_r^{*,*}, d_r\} \longrightarrow \{\tilde{E}_r^{*,*}, \tilde{d}_r\}$ is a sequence of morphisms of bigraded differential modules $f_r \colon E_r^{*,*} \longrightarrow \tilde{E}_r^{*,*}$ such that f_r induces the morphism f_{r+1} for each $r \geq 0$.

Lemma 1.3.1.3 ([McC01, Theorem 3.4]). *For a morphism $f \colon \{E_r^{*,*}, d_r\} \longrightarrow \{\tilde{E}_r^{*,*}, \tilde{d}_r\}$ of spectral sequences, if f_s is an isomorphism of bigraded differential modules for some s then all subsequent f_r, $r \geq s$, are isomorphisms too.*

1.3.2 Construction

We will construct our spectral sequence via r-cycle and r-boundary objects of filtered chain complexes as in [McC01, §2.2] however with slightly different notation to agree with that of [CELW19]. There are alternate setups, e.g. exact couples [McC01, §2.2] and Cartan-Eilenberg Systems [McC01, Exercise 2.2] or [CE56, Chapter XV]. The construction we take here is in fact equivalent to that of exact couples, see [McC01, Proposition 2.11].

Definition 1.3.2.1. For $r \geq 0$, the *r-cycles* $Z_r^{*,*}(A)$ of a filtered chain complex A are given in bidegree $(p, n+p)$ by

$$Z_r^{p,n+p}(A) := F_pA^n \cap d^{-1}F_{p-r}A^{n+1} \ .$$

Definition 1.3.2.2. The *representing object for the r-cycles* of filtered chain complexes, denoted $\mathcal{Z}_r^{*,*}$, is given by

$$\mathcal{Z}_r(p, n+p) := \left(R^n_{(p)} \xrightarrow{1} R^{n+1}_{(p-r)}\right) \ .$$

Definition 1.3.2.3. For $r \geq 1$, the *r-boundaries* $B_r^{*,*}(A)$ of a filtered chain complex A is given in bidegree $(p, n+p)$ by

$$B_r^{p,n+p}(A) := dZ_{r-1}^{p+r-1,p+r-1+n-1}(A) + Z_{r-1}^{p-1,p-1+n}(A) \ ,$$

and for $r = 0$ is given by

$$B_0^{p,n+p}(A) := Z_0^{p-1,p-1+n}(A) \ .$$

Definition 1.3.2.4. The *representing object for the r-boundaries* of filtered chain complexes, denoted $\mathcal{B}_r^{*,*}$, is given for $r \geq 1$ by

$$\mathcal{B}_r(p, n+p) := \mathcal{Z}_{r-1}(p+r-1, p+r-1+n-1) \oplus \mathcal{Z}_{r-1}(p-1, p-1+n) \ .$$

Note 1.3.2.5. McCleary uses $B_r^{*,*}$ to denote just $dZ_r^{*,*}$ whereas we include the extra r-cycle component. This is done to agree with the notation of [CELW19].

Notation 1.3.2.6. We will also occasionally make use of objects we denote $\mathcal{Z}_r(p, p+n)(N)$ for some R-module N which we take to be $\left(N^n_{(p)} \to N^{n+1}_{(p-r)}\right)$.

Given a filtered chain complex A we can now define a spectral sequence. The notation differs slightly from [CELW19] here in that we use d_r for the r-differential instead of their δ_r.

Definition 1.3.2.7. The *r-page* $E_r^{*,*}(A)$ of a filtered chain complex (A, d) for $r \geq 0$ is given in bidegree $(p, p+n)$ by

$$E_r^{p,p+n}(A) := \frac{Z_r^{p,p+n}(A)}{B_r^{p,p+n}(A)} \ ,$$

and for an element $[a] \in E_r^{p,p+n}(A)$ represented by $a \in Z_r^{p,p+n}(A)$ we define its r-differential by $d_r[a] := [da] \in E_r^{p-r,p-r+1}(A)$.

Lemma 1.3.2.8 ([McC01, Theorem 2.6])**.** *The r-pages and r-differentials of Definition 1.3.2.7 define the structure of a spectral sequence.*

Definition 1.3.2.9. A morphism f of filtered chain complexes will be called an *r-weak equivalence*, *r-quasi-isomorphism* or E_r-quasi-isomorphism if it induces an isomorphism between the $(r+1)$-pages of the associated spectral sequence.

1.3.3 Shift-décalage adjunction

On the category of filtered chain complexes there is the shift-décalage adjunction, $S^1 \dashv \mathrm{Dec}$, of Deligne [Del71, Definition 1.3.3]. with the property that $(S^r)^{-1}(\mathcal{E}_{k+r}) = \mathcal{E}_k$ and $(\mathrm{Dec}^r)^{-1}(\mathcal{E}_k) = \mathcal{E}_{k+r}$.

Definition 1.3.3.1. Let $r \geq 0$ and A be a filtered chain complex. On the category of filtered chain complexes we define the endofunctors:

1. *r-shift* of A, denoted S^rA, with the same underlying chain complex and filtration given by:

$$F_pS^rA^n := F_{p+rn}A^n \ ,$$

2. *r-décalage* of A, denoted $\mathrm{Dec}^r A$, with the same underlying chain complex and filtration given by:

$$F_p \mathrm{Dec}^r A^n := F_{p-rn} A^n \cap d^{-1}\left(F_{p-rn-r} A^{n+1}\right) = Z_r^{p-rn, p-rn+n}(A) \; .$$

All differentials being induced from the filtered chain complex A.

One can easily check these do indeed define functors, that $S^r = S^1 \circ \ldots \circ S^1$, $\mathrm{Dec}^r = \mathrm{Dec}^1 \circ \ldots \circ \mathrm{Dec}^1$, $id = \mathrm{Dec}^0 = S^0$, and finally that we have the following lemma.

Lemma 1.3.3.2 ([CG16, Proposition 2.16]). *For each $r \geq 0$ there is an adjunction pair $S^r \vdash \mathrm{Dec}^r$ for which the unit of the adjunction, $\eta \colon id \Longrightarrow \mathrm{Dec}^r \circ S^r$, is the identity morphism.*

1.4 Model categories

A model category is a particulary nice framework in which to study homotopy theory and general enough to encompass the standard homotopy theories on spaces, chain complexes, spectra and many more. One of the main problems with a homotopy theory is the construction of its homotopy category, i.e. inverting the weak equivalences. As an example there are homotopically poorly behaved spaces which whilst being weakly equivalent to a second space are not homotopic to it. A model category makes this process of inverting the weak equivalences comparatively tractible by providing the notions of cofibrant and fibrant objects. These have the nice property that mapping out of a cofibrant object is homotopically well behaved and similarly for mapping into fibrant objects. If one then restricts to the category of fibrant-cofibrant objects, i.e. those that are both fibrant and cofibrant, the process of constructing the homotopy category is as simple as inverting the homotopy equivalences, a more easily manageable class of morphisms.

1.4.1 Definition

Definition 1.4.1.1. For morphisms f and g in a category $\mathcal{C}$ we say that f is a *retract* of g if there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1"'] & D \arrow[r] & B \end{tikzcd} \; .$$

Definition 1.4.1.2. A *model category* $\mathcal{M}$ is a category with three subclasses of the morphisms called *weak equivalences*, *fibrations* and *cofibrations*, and denoted respectively by $\mathcal{W}$, Fib and Cof with functorial factorisations (α, β) and (γ, δ) satisfying the following:

1. *(Co)completeness:* The underlying category has all small colimits and all small limits.
2. *2-out-of-3:* For a commutative triangle

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & C \\ & B \arrow[ur] & \end{tikzcd}$$

   in $\mathcal{M}$ if any two of the three morphisms are in $\mathcal{W}$ so too is the third.
3. *Retracts:* For f and g morphisms of $\mathcal{C}$ with f a retract of g, then f is in $\mathcal{W}$ (resp. Fib, Cof) whenever g is in $\mathcal{W}$ (resp. Fib, Cof).
4. *Liftings:* For a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "h"] & X \end{tikzcd} \tag{1.4}$$

   in $\mathcal{C}$ in which we either have:

- i is a cofibration and p is both a weak equivalence and fibration, or
- i is both a weak equivalence and cofibration, and p is a fibration,

then there exists a, not necessarily unique, morphism h which makes the diagram commute.

5. *Factorisations:* For a morphism f we can factor f as either:
   - $\beta(f) \circ \alpha(f)$ where $\alpha(f)$ is a cofibration, and $\beta(f)$ a weak equivalence and fibration, or
   - $\delta(f) \circ \gamma(f)$ where $\gamma(f)$ is a weak equivalence and cofibration, and $\delta(f)$ a fibration.

Remarks 1.4.1.3. We note some remarks and immediate consequences regarding the definition.

1. Quillen's original defintion, [Qui67], lacked the functorial aspect of the factorisations however this can prove useful and the model categories we consider will all be constucted in such a way that we are handed functorial factorisations so we have included it in our definition as has become common. The original definition also only required finite limits and colimits.
2. The category being (co)complete means there exist initial and terminal objects, we denote these by 0 and 1 respectively.

We list some standard nomenclature for referring to morphisms and their properties in model categories.

Definition 1.4.1.4. We will call a morphism that is both a weak equivalence and fibration an *acylic fibration*, and a morphism that is both a weak equviialence and cofibration an *acyclic cofibration*. These are also commonly called *trivial fibrations* and *trivial cofibrations*.

Notation 1.4.1.5. We will denote the property that a morphism $A \to B$ is:

1. a cofibration by $A \rightarrowtail B$,
2. a fibration by $A \twoheadrightarrow B$, and
3. a weak equivalence by $A \xrightarrow{\sim} B$,

and combine these when appropriate.

Definition 1.4.1.6. For an object A of a model category $\mathcal{M}$ we say A is:

1. *cofibrant* if the morphism $0 \to A$ is a cofibration, and
2. *fibrant* if the morphism $A \to 1$ is a fibration.

Definition 1.4.1.7. A morphism i is said to have the *left lifting property* with respect to p, alternatively p has the *right lifting property* with respect to i, if there is a lift h in any commutative diagram of the form of Diagram 1.4.

Remark 1.4.1.8. In particular Definitions 1.4.1.4 and 1.4.1.7 say that the acyclic cofibrations have the left lifting property with respect to the fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations.

Remark 1.4.1.9. Definition 1.4.1.2 contains redundant information in that any two of the sub-classes of morphisms $\mathcal{W}$, Fib and Cof determine the third. Part of this claim is proven in [Hov99, Lemma 1.1.10].

The following is one definition of the homotopy category. We will shortly, Definition 1.4.1.14, introduce a second more tractable definition of the homotopy category which is equivalent to this one as categories. This definition can be found in [Hov99, Definition 1.2.1].

Definition 1.4.1.10. The *homotopy category* of a model category $\mathcal{M}$, denoted $\mathrm{Ho}(\mathcal{M})$, is the category obtained by inverting the class of weak equivalences in $\mathcal{M}$. That is the objects are those of $\mathcal{M}$. For the morphisms we first take strings $(f_1, f_2, \ldots, f_n)$ of composable morphisms of $\mathcal{M}$ with each f_i either a morphism of $\mathcal{M}$ or the reversal w^{-1} of a morphism of $\mathcal{W}$. The morphisms of $\mathrm{Ho}(\mathcal{M})$ are then this class with the identifications $(f, g) = (g \circ f)$ for f, g composable morphisms of $\mathcal{M}$, $(w, w^{-1}) = id$, $(w^{-1}, w) = id$ for the identity on the domain and codomain of w respectively and where the identity on an object X is given by (1_X).

Definition 1.4.1.11. Let A and X be objects of a model category $\mathcal{C}$

1. A *cylinder object* for A is a factorisation of the fold morphism $\nabla\colon A\coprod A \to A$ by a cofibration followed by a weak equivalence:
$$A\coprod A \to I\times A \to A\ ,$$
we denote the inclusions of the first map of either component of A by i_1 and i_2.

2. A *path object* for X is a factorisation of the diagonal morphism $\Delta\colon X \to X\times X$ by a weak equivalence followed by a fibration:
$$X\to X^I \to X\times X\ .$$
we denote the projections of the second map onto either component by π_1 and π_2.

Remarks 1.4.1.12. The following are worth noting with regard to these objects.

1. Despite what the notation might suggest the cylinder object $I\times A$ need not be a product (categorical or otherwise) of an object I with A. Similarly the path object X^I need not be an object of functions. However both of these are common ways of forming such cylinder and path objects.

2. Observe that in a model category we always have at least one way of doing this using the functorial factorisations and that the weak equivalences in the definition will be in addition fibrations and cofibrations respectively.

Definition 1.4.1.13. For A and X objects of a model category $\mathcal{M}$ with two morphisms $f, g\colon A \to X$ we define:

1. a *left homotopy* to be a morphism $h\colon I\times A \to X$ from a cylinder object of A to X such that $hi_1 = f$ and $hi_2 = g$,

2. a *right homotopy* to be a morphism $k\colon A \to X^I$ from A to a path object on X such that $\pi_1 k = f$ and $\pi_2 k = g$.

Each of these notions of homotopy generate an equivalence relation on $\mathrm{Hom}(A, X)$ and for A cofibrant and Y fibrant these equivalence relations, which we denote by $\simeq$, coincide, [Hov99, Corollary 1.2.6]. Further by [Hov99, Theorem 1.2.10] we can take the following as the definition of the homotopy category of $\mathcal{M}$. It is equivalent as a category to that previously given in Definition 1.4.1.10.

Definition 1.4.1.14. The *homotopy category* of a model category $\mathcal{M}$ denoted $\mathrm{Ho}(\mathcal{M})$ has objects the fibrant-cofibrant objects of $\mathcal{M}$ and morphisms $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(A, X) = \mathrm{Hom}_{\mathcal{M}}(A, X)/\simeq$.

1.4.2 Cofibrantly generated model categories

A recurring problem in constructing model categories is that whilst some of the three defining classes may be simple to describe one of them may have a less simple classification, e.g. the projective model structure on unbounded chain complexes has simple descriptions for the weak equivalences and fibrations but a more complicated form for the cofibrations explained at the start of Chapter 4. We can however take advantage of the lifting axioms, more particularly Remark 1.4.1.9, to specify only two of the three classes. And going one step further we can potentially choose subsets of the cofibrations (resp. acyclic cofibrations) such that those morphisms with the right lifting property with respect to these subsets are the acyclic fibrations (resp. fibrations). Having then determined the (acyclic) fibrations these, by the lifting axioms, determine the (acyclic) cofibrations and so determine the three subclasses of weak equivalences, fibrations and cofibrations. We will describe here cofibrantly generated model categories via the small object argument which has the added advantage of providing for us the factorisation axioms required of a model category.

Definition 1.4.2.1. For I a set of morphisms of a category we denote by

1. I-`Inj` the set of morphisms with the right lifting property with respect to I, and call these morphisms *I-injective*,

2. I-`Proj` the set of morphisms with the left lifting property with respect to I, can call these morphisms *I-projective,*

3. I-`Cof` the set of morphisms with the left lifting property with respect to I-`Inj`, i.e. I-`Cof` $= (I$-`Inj`$)$ -`Proj`, and call these morphisms *I-cofibrations*.

Definition 1.4.2.2. Given a set of morphisms I of a category, the *relative I-cells* are those morphisms obtained from an object by a transfinite composition of pushouts of maps in I. These are denoted by I-`Cell`.

Notation 1.4.2.3. The morphisms I-`Cell` are sometime referred to as *I-regular cofibrations* and denoted $I\text{-}\mathtt{Cof}_{\mathtt{reg}}$, see for example [SS00].

Theorem 1.4.2.4 (Small Object Argument, [Hov99, Theorem 2.1.14]). *For $\mathcal{C}$ a category with all small colimits and I a set of morphisms of $\mathcal{C}$, for which the domains of I are small relative to $I\text{-}\mathtt{Cell}$ there are functorial factorisations (α, β) of any morphism f into $\beta(f) \circ \alpha(f)$ with $\alpha(f) \in I\text{-}\mathtt{Cell}$ and $\beta(f) \in I\text{-}\mathtt{Inj}$.*

Lemma 1.4.2.5 ([Hov99, Lemma 2.1.10]). *For I a set of morphisms of a category we have $I\text{-}\mathtt{Cell} \subseteq I\text{-}\mathtt{Cof}$.*

Definition 1.4.2.6. A *cofibrantly generated model category* is a model category $\mathcal{M}$ with two set of morphisms I and J such that:

1. the domains of I (resp. J) are small relative to $I\text{-}\mathtt{Cell}$ (resp. $J\text{-}\mathtt{Cell}$),
2. the class of trivial fibrations (resp. fibrations) is $I\text{-}\mathtt{Inj}$ (resp. $J\text{-}\mathtt{Inj}$).

The set I is called the *generating cofibrations* and J the *generating acyclic cofibrations*.

Note 1.4.2.7. When referring to a cofibrantly generated model category the (possibly adorned) letter I will always refer to the generating cofibrations, and J the generating acyclic cofibrations.

Definition 1.4.2.8. A cofibrantly generated model structure such that the domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite relative to the cofibrations will be refrred to as a *finitely cofibrantly generated model category.*

The following theorem allows us to check when sets I and J do indeed determine a model category structure and the model structures of [CELW19] are constucted using it.

Theorem 1.4.2.9 ([Hov99, Theorem 2.1.19]). *For $\mathcal{C}$ a category closed under all small (co)limits, with $\mathcal{W}$ a subclass of the morphisms, and I and J sets of morphisms. Then I and J determine a cofibrantly generated model category with weak equivalences $\mathcal{W}$ if and only if*

1. *$\mathcal{W}$ satisfies the two out of three property,*
2. *the domains of I (resp. J) are small relative to $I\text{-}\mathtt{Cell}$ (resp. $J\text{-}\mathtt{Cell}$),*
3. *$J\text{-}\mathtt{Cell} \subseteq \mathcal{W} \cap I\text{-}\mathtt{Cof}$,*
4. *$I\text{-}\mathtt{Inj} \subseteq \mathcal{W} \cap J\text{-}\mathtt{Inj}$, and*
5. *either $\mathcal{W} \cap I\text{-}\mathtt{Cof} \subseteq J\text{-}\mathtt{Cof}$ or $\mathcal{W} \cap J\text{-}\mathtt{Inj} \subseteq I\text{-}\mathtt{Inj}$.*

A useful (classification) result of cofibrations in a cofibrantly generated model category is the following which gives a morphism is a cofibration if and only if it is the retract of an $I\text{-}\mathtt{Cell}$ morphism. A similar proof shows that acyclic cofibrations are precisely the retracts of the $J\text{-}\mathtt{Cell}$ morphisms.

Proposition 1.4.2.10. *For a cofibrantly generated model category $\mathcal{M}$ with generating cofibration I any cofibration is a retract of an I-cell morphism.*

Proof. For a cofibration $f\colon A \to B$ factorise it via the small object argument into an I-cell morphism $i\colon A \to X$ followed by an acyclic fibration $p\colon X \to B$. There is then a lift h in the lifting problem

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] & X \arrow[d, two heads, "p"', "\sim"] \\
B \arrow[r, "id"'] \arrow[ur, dotted, "h"] & B
\end{tikzcd}$$

and we can form the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "id"'] \arrow[rr, bend left, "id"] \arrow[d, tail, "f"'] & A \arrow[r, "id"'] \arrow[d, tail, "i"] & A \arrow[d, tail, "f"] \\
B \arrow[r, "h"] \arrow[rr, bend right, "id"'] & X \arrow[r, "p"] & B
\end{tikzcd}$$

which exhibits f as a retract of the I-cell i.

1.4.3 Quillen adjunctions and equivalences

Lemma 1.4.3.1 ([Hov99, Lemma 1.3.4]). *For an adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ of model categories the following are equivalent:*

1. *F preserves cofibrations and acyclic cofibrations, and*
2. *G preserves fibrations and acyclic fibrations.*

Definition 1.4.3.2. For $\mathcal{M}$ and $\mathcal{N}$ model categories, an adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ is a *Quillen adjunction* if one (and therefore both) of the conditions of Lemma 1.4.3.1 is satisfied.

Definition 1.4.3.3. For a Quillen adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ we define:

1. the *total left derived functor* $\mathbb{L}F$ of F is the composite of the cofibrant replacement Q followed by F on the homotopy categories
$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathcal{M}_c) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathcal{N}) \quad ,$$
2. the *total right derived functor* $\mathbb{R}G$ of G is the composite of the fibrant replacement R followed by G on the homotopy categories
$$\mathrm{Ho}(\mathcal{N}) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathcal{N}_f) \xrightarrow{\mathrm{Ho}(G)} \mathrm{Ho}(\mathcal{M}) \quad .$$

Lemma 1.4.3.4 ([Hov99, Lemma 1.3.10]). *For a Quillen adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ the total left and derived functors give an adjunction, called the derived adjunction, on the homotopy categories $\mathbb{L}F\colon \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) \colon \mathbb{R}G$.*

Definition 1.4.3.5. A Quillen adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ is a *Quillen equivalence* if for a cofibrant object $m \in \mathcal{M}$ and fibrant object $n \in \mathcal{N}$ a morphism $f\colon Fm \longrightarrow n$ is a weak equivalence in $\mathcal{N}$ if and only if its adjunct morphism $\tilde{f}\colon m \longrightarrow Gn$ is a weak equivalence in $\mathcal{M}$.

Lemma 1.4.3.6 ([Hov99, Proposition 1.3.13]). *For a Quillen equivalence $F\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ the derived adjunction is an equivalence on the homotopy categories.*

Example 1.4.3.7. There is an adjunction $i\colon \mathcal{C}_R^{\leq} \rightleftarrows \mathcal{C}_R \colon \tau$ between the categories of non-positively graded chain complexes and the category of unbounded chain complexes. The left adjoint i is inclusion and the right adjoint τ is truncation which is the identity in negative degrees, 0 in positive degrees, and in degree 0 gives the kernel of the differential. It can be shown that τ preserves both fibrations and acyclic fibrations and so forms a Quillen adjunction by Definition 1.4.3.2. This is not however a Quillen equivalence; take in Definition 1.4.3.5 c to be the chain complex 0 and d to be the chain complex with R in degree 1 and 0 otherwise. Then we have a Quillen equivalence $c \xrightarrow{\sim} \tau d = 0$ but the morphism $ic \longrightarrow d$ is not a weak equivalence.

We will later, Section 3.12, use adjunctions of this form to infer bounded variants of the model structures considered on filtered chain complexes and bicomplexes.

1.4.4 Transfer theorems of model structures

Theorem 1.4.4.1 (Kan Transfer Theorem, [Hir03, Theorem 11.3.2]). *Let $\mathcal{M}$ be a model category cofibrantly generated by I and J, $\mathcal{C}$ a complete and cocomplete category, and an adjunction pair $F\colon \mathcal{M} \rightleftarrows \mathcal{C} \colon U$. Then there exists a cofibrantly generated model structure on $\mathcal{C}$ with generating cofibrations FI, generating acyclic cofibrations FJ and weak equivalences $U^{-1}\mathcal{W}$ if*

1. *FI and FJ admit the small object argument, and*
2. *U takes relative FJ-cell complexes to weak equivalences.*

Example 1.4.4.2. The free-forgetful adjunction $F\colon sSets \rightleftarrows sAb \colon U$ gives a way of transferring the cofibrantly generated Kan model structure on simplicial sets to a model structure on simplicial Abelian groups via the Kan transfer theorem. The model structure obtained is in fact Quillen equivalent to the standard projective model structure on non-negatively (homologically) graded chain complexes via the Dold-Kan adjunction.

Other results on transferring model structure exist and usually rely on checking an *acyclicity condition*, see for example [BHK$^+$15].

1.4.5 Monoidal model categories

Suppose we have both a model category structure and a monoidal category structure on the same category $\mathcal{C}$. We state here standard conditions and results for existence of an induced monoidal structure on the homotopy category.

Definition 1.4.5.1. The *pushout-product* $i \boxtimes j$ of two morphisms $i\colon A \longrightarrow B$ and $j\colon C \longrightarrow D$ is given by

$$i \boxtimes j\colon A \otimes D \coprod_{A \otimes C} B \otimes C \longrightarrow B \otimes D\ .$$

The following definition of a monoidal model category has two conditions, the first ensuring that we do indeed have an induced product on the homotopy category and the second ensures we have a unit in the homotopy category.

Definition 1.4.5.2. A *monoidal model category* is a model category equipped with the structure of a closed symmetric monoidal category such that:

1. the pushout-product $i \boxtimes j$ is a cofibration whenever both i and j are, and additionally is acyclic if either i or j is,

2. there exists a cofibrant replacement of the tensor unit I, i.e. weak equivalence $QI \xrightarrow{\sim} I$ with QI cofibrant, such that for all cofibrant X the morphism $QI \otimes X \longrightarrow X$ is a weak equivalence.

Remark 1.4.5.3. Note that when the unit is already cofibrant the second condition is redundant, this is true for many common examples of monoidal model categories including $sSets_*$ and $\mathcal{C}_R$. An example where the unit is not cofibrant are symmetric spectra, [Mur15, Example 2]. We will see that $f\mathcal{C}$ is a monoidal model category in which the unit is not cofibrant providing a new example.

The major result of having a monoidal model category structure is the following theorem.

Theorem 1.4.5.4 ([Hov99, Theorem 4.3.2]). *For $\mathcal{M}$ a monoidal model category there is a closed symmetric monoidal structure on $\mathrm{Ho}(\mathcal{M})$ with tensor product the left derived functor $- \otimes^{\mathbb{L}} -$.* ❂

The following result allows us to check the pushout-product axiom only for the generating (acyclic) cofibrations.

Lemma 1.4.5.5 ([Hov99, Lemma 4.2.4]). *For I and J the generating cofibrations and acyclic cofibrations respectively of a model category further equipped with the structure of a closed symmetric monoidal category satisfying $I \boxtimes I \subseteq I\text{-}\mathit{Cof}$, $I \boxtimes J \subseteq J\text{-}\mathit{Cof}$, and $J \boxtimes I \subseteq J\text{-}\mathit{Cof}$, then the pushout-product axiom holds.* ❂

1.4.6 Monoid axiom

The monoid axiom, defined by Schwede and Shipley in [SS00] provides a condition from which, along with a cofibratnly generated assumption, we can infer a cofibrantly generated model structure on the category of left R-modules for R a monoid or R-modules and R-algebras for R a commutative monoid.

Having a good understanding of the cofibrations of a cofibrantly generated model category $\mathcal{C}$ does not however necessarily give a good understanding of the cofibrations in the model structures on left R-modules, R-modules or R-algebras. For example, the projective model structure of bounded chain complexes has well understood cofibrations: they are the degreewise monomorphisms with cofibrant cokernel (i.e. degreewise projective), however the cofibrations of R-algebras are not so easily classified. In particularly nice cases, e.g. in rational homotopy theory, cofibrations in model categories of commutative differential graded algebras can be described in terms of retracts of relative Sullivan algebras.

The following is Schwede and Shipley's definition of the monoid axiom, a condition required to obtain the above model structures, see [SS00, Definition 3.3]. We will make use of Definition 1.4.6.1 and Theorem 1.4.6.3 to show there are S-model structures of filtered differential graded algebras in Section 5.4.

Definition 1.4.6.1 (Monoid Axiom). A monoidal model category $\mathcal{M}$ is said to satisfy the *monoid axiom* if the relative cells obtained from the monoid product of the acyclic cofibrations with $\mathcal{M}$ are weak equivalences, i.e :

$$((\mathcal{W} \cap \mathrm{Cof}) \otimes \mathcal{M})\texttt{-Cell} \subseteq \mathcal{W}\ .$$

It suffices to check the monoid axiom on the generating acyclic cofibrations as shown in [SS00].

Lemma 1.4.6.2 ([SS00, Lemma 3.5 (2)])**.** *Let $\mathcal{M}$ be a cofibrantly generated model category with closed symmetric monoidal structure and generating acyclic cofibrations J. If every map of $(J \otimes \mathcal{M})\text{-}\mathtt{Cell}$ is a weak equivalence then the monoid axiom holds.*

Theorem 1.4.6.3 ([SS00, Theorem 4.1])**.** *For a cofibrantly generated, monoidal model category $\mathcal{M}$ satisfying the monoid axiom and such that every object of $\mathcal{M}$ is small then:*

1. *for R a monoid in $\mathcal{M}$, the category of left R-modules is a cofibrantly generated model category,*
2. *for R in addition commutative, the category of R-modules is a cofibrantly generated, monoidal model category satisfying the monoid axiom, and*
3. *for R a commutative monoid, the category of R-algebras is a cofibrantly generated model category.*

Remark 1.4.6.4. The weak equivalences and fibrations in the model categories of Theorem 1.4.6.3 are precisely those of the underlying model category $\mathcal{M}$.

1.4.7 Left and right properness

Recall that the (co)base change of an isomorphism is an isomorphism. One might ask for a similar condition for weak equivalences however this need not be true, instead we have the following definitions.

Definition 1.4.7.1. A model category $\mathcal{M}$ is said to be:

1. *left proper* if cobase changes along cofibrations preserve weak equivalences,
2. *right proper* if base changes along fibrations preserve weak equivalences, and
3. *proper* if it is both left and right proper.

Remark 1.4.7.2. A model category in which every object is cofibrant (resp. fibrant) is automatically left (resp. right) proper.

Example 1.4.7.3. The model category of projective unbounded chain complexes is proper (despite not all objects being cofibrant) and the Quillen model structure on $sSets$ is proper (despite not all objects being fibrant).

Left and right properness have some nice consequences for the model category. For example pushouts (resp. pullbacks) along cofibrations (resp. fibrations) are automatically homotopy pushouts (resp. pullbacks) in left (resp. right) proper model categories, Section 1.4.8. A model category which is both left proper and cellular is guaranteed to have Bousfield localisations at any set of morphisms, Section 1.4.12. A result of Dugger, [Dug01, Proposition A.5], also asserts that a left proper cofibrantly generated model category has a set of cofibrant objects $\{C_i\}$ detecting weak equivalences in the sense that $Y \to X$ is a weak equivalence if and only if the function complexes $\operatorname{map}(C_i, Y) \to \operatorname{map}(C_i, X)$ are weak equivalences for all C_i.

1.4.8 Homotopy pullbacks and pushouts

The definition of homotopy (co)limits repairs a fundamental flaw in the lack of homotopy invariance in (co)limits in a model category. I.e. given a general model category $\mathcal{M}$, an indexing category I and two functor $F, G\colon I \to \mathcal{M}$ with natural transformation $\alpha\colon F \Rightarrow G$ satisfying for each object $i \in I$ the morphism $\alpha_i\colon F(i) \to G(i)$ is a weak equivalence it is not necessarily true that there is a weak equivalence $\operatorname{colim} F \to \operatorname{colim} G$ or $\lim F \to \lim G$. Indeed the following provides a standard counterexample for the case of the colimit.

Counterexample 1.4.8.1. We consider pushouts in the Quillen model category of pointed topological spaces. In the following two pushouts we have a weak equivalence between the corresponding objects of the three corners of the pushout however the induced map on the pushout $* \to S^2$ is not a weak equivalence.

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & * \arrow[d, dotted] \\
* \arrow[r, dotted] & * \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & D^2 \arrow[d, dotted] \\
D^2 \arrow[r, dotted] & S^2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We will recall here a homotopical correction for a particular case of this failing of (co)limits. There are very general constructions producing good notions of homotopy (co)limits invariant under weak equivalence of diagrams as above however we will only need a very basic case of this involving pullbacks so we limit ourselves to discussing homotopy pullbacks – the notion of homotopy pushout is dual. The general method for constructing homotopy (co)limits in an arbitraty (i.e. not necessarily simplicial) model category $\mathcal{M}$ involves simplicial framings and are given as [Hir03, Definitions 19.1.2 & 19.1.5]. In a right proper model category the pullback of fibrant objects is naturally weakly equivalent to such a general homotopy limit by [Hir03, Proposition 19.5.3]. The definition of a homotopy pullback we then use is given in the following definition which replaces the morphisms of a pullback by fibrations. It can be found as [Hir03, Definition 13.3.2]. We use E to denote a functorial factorisation of a morphism $f\colon X \to Z$ into a trivial cofibration $i_f\colon X \to E(f)$ followed by a fibration $p_f\colon E(f) \to Z$.

Definition 1.4.8.2. Suppose $\mathcal{M}$ is a right proper model category. The *homotopy pullback* of a diagram

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

which will be denoted diagramatically by

$$\begin{tikzcd}
P \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner h", very near start] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

in which P is given by the actual pullback diagram

$$\begin{tikzcd}
P \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & E(f) \arrow[d, two heads, "p_g"] \\
E(g) \arrow[r, two heads, "p_f"'] & C
\end{tikzcd} .$$

This construction really is homotopy invariant in a right proper model category.

Proposition 1.4.8.3 ([Hir03, Proposition 13.3.4])**.** *Suppose $\mathcal{M}$ is a right proper model category in the diagram*

$$\begin{tikzcd}
X_1 \arrow[r, two heads, "f_1"] \arrow[d, "\sim"', "w_X"] & Z_1 \arrow[d, "\sim"', "w_Z"] & Y_1 \arrow[l, two heads, "g_1"'] \arrow[d, "\sim"', "w_Y"] \\
X_2 \arrow[r, two heads, "f_2"'] & Z_2 & Y_2 \arrow[l, two heads, "g_2"]
\end{tikzcd}$$

that the morphisms f_1, f_2, g_1 and g_2 are fibrations and the morphisms w_X, w_Y and w_Z are weak equivalences. Then the induced map on the pullbacks is a weak equivalence.

As a corollary replacing either X or Y by a weakly equivalent object yields a weakly equivalent homotopy pullback, [Hir03, Corollary 13.3.5]. We will use the following form of computing homotopy pullbacks which asserts that one need only replace either the morphism f or g by a fibration in Definition 1.4.8.2.

Corollary 1.4.8.4 ([Hir03, Corollary 13.3.8])**.** *Suppose $\mathcal{M}$ is a right proper model category in which either f or g is a fibration in the diagram*

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd} ,$$

then the pullback is weakly equivalent to the homotopy pullback.

Dually one has the following definition and results about homotopy pushouts. Now let E denote a functorial factorisation of morphism $f\colon Z \to X$ into a cofibration $i_f\colon Z \to E(X)$ followed by an acyclic fibration $p_f\colon E(X) \to X$.

Definition 1.4.8.5. Suppose $\mathcal{M}$ is a left proper model category. The *homotopy pushout* of a diagram

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \\
Y &
\end{tikzcd}$$

which will be denoted diagramatically by

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, dotted] \\
Y \arrow[r, dotted] & P \arrow[ul, phantom, "\ulcorner h", very near start]
\end{tikzcd}$$

in which P is given by the actual pushout diagram

$$\begin{tikzcd}
Z \arrow[r, tail, "i_f"] \arrow[d, tail, "i_g"] & E(X) \arrow[d, dotted] \\
E(Y) \arrow[r, dotted] & P \arrow[ul, phantom, "\ulcorner h", very near start]
\end{tikzcd} .$$

Which is similarly homotopy invariant by a dual argument.

Proposition 1.4.8.6 ([Hir03, Proposition 13.5.3])**.** *Suppose $\mathcal{M}$ is a left proper model category in the diagram*

$$\begin{tikzcd}
X_1 \arrow[d, "w_X", "\sim"'] & Z_1 \arrow[l, tail, "i_1"'] \arrow[r, tail, "j_1"] \arrow[d, "w_Z", "\sim"'] & Y_1 \arrow[d, "w_Y", "\sim"'] \\
X_1 & Z_1 \arrow[l, tail, "i_2"'] \arrow[r, tail, "j_2"] & Y_2
\end{tikzcd}$$

that the morphisms i_1, i_2, j_1 and j_2 are fibrations and the morphisms w_X, w_Y and w_Z are weak equivalences. Then the induced map on the pushouts is a weak equivalence.

And similarly it can be computed by replacing only one of f and g by a cofibration.

Corollary 1.4.8.7. *Suppose $\mathcal{M}$ is a left proper model category in which either f or g is a cofibration in the diagram*

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \\
X &
\end{tikzcd} ,$$

then the pushout is weakly equivalent to the homotopy pushout.

1.4.9 Stable model categories

Recall in a pointed category the initial and terminal objects are isomorphic and in this setting we denote such an object by $*$.

Definition 1.4.9.1. For a pointed model category $\mathcal{M}$ we define

1. The *suspension functor* Σ on an object $X \in \mathrm{Ho}(\mathcal{M})$ to be the homotopy pushout

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d, dotted] \\
* \arrow[r, dotted] & \Sigma X \arrow[ul, phantom, "\ulcorner h", very near start]
\end{tikzcd} ,$$

2. The *loop functor* Ω on an object $X \in \mathrm{Ho}(\mathcal{M})$ to be the homotopy pullback

$$\begin{tikzcd}
\Omega X \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner h", very near start] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd} .$$

Quillen's original definition makes use of cylinder and path objects on an object X [Qui67, Chapter 1 §2]. Hovey, [Hov99], alternatively defines the suspension and loop functor via the functors $-\wedge^{L} S^1$ and $R\mathrm{Hom}(S^1, -)$ one obtains from a simplicial framing. The discussion following [Hov99, Definition 6.1.1] justifies our use of the above definition. We have that on the homotopy category the suspension and loop functors form an adjoint pair $\Sigma\colon \mathrm{Ho}(\mathcal{M}) \underset{\perp}{\rightleftarrows} \mathrm{Ho}(\mathcal{M}) : \Omega$ [Qui67, Chapter 1 §2] or [Hov99, Corollary 3.1.6].

Note that by Corollary 1.4.8.4 in a right proper model category we have the following proposition.

Lemma 1.4.9.2. *For a pullback diagram in a right proper pointed model category where f is a fibration and Y is acyclic a pullback diagram of the following form is a homotopy pullback and therefore $Z \simeq \Omega X$:*

$$\begin{tikzcd}
Z \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, two heads, "f"] \\
* \arrow[r] & X
\end{tikzcd} .$$

Definition 1.4.9.3. A pointed model category $\mathcal{M}$ is a *stable model category* if the suspension-loop adjoint pair is an equivalence of categories of the homotopy category $\mathrm{Ho}(\mathcal{M})$.

Examples 1.4.9.4. The following are the two standard examples of stable model categories for which the suspension and loop functors can in fact be realised on the model category level.

1. Unbounded chain complexes with the suspension and loop functors shifting the degree by ± 1, and
2. Spectra which are constructed specifically to be a stabilisation of the model category of spaces. The suspension and loop functors here correspond to smashing with the spectrum S^1 and taking the function complex $\mathrm{Hom}(S^1, -)$.

1.4.10 Cellular model categories

Definition 1.4.10.1. A cofibrantly generated model category $\mathcal{M}$ with sets I and J of generating (acyclic) cofibrations is said to be *cellular* if the following hold:

1. the domains and codomains of I are small,
2. the domains of J are small relative to I, and
3. the cofibrations are *effective monomorphisms*, i.e that any cofibration $i\colon X \to Y$ is the equaliser of

$$\begin{tikzcd}
Y \arrow[r, shift left] \arrow[r, shift right] & Y \coprod_X Y
\end{tikzcd} .$$

Definition 1.4.10.2. A monomorphism $i\colon A \longrightarrow B$ is a *regular monomorphism* if it is the equaliser of some pair of morphisms $B \longrightarrow C$,

$$\begin{tikzcd}
A \arrow[r] & B \arrow[r, shift left] \arrow[r, shift right] & C
\end{tikzcd} .$$

Proposition 1.4.10.3. *In a category with equalisers and cokernel pairs the class of regular monomorphisms coincides with the class of effective monomorphisms.*

Proof. The proof is dual to that of [Bor08, Proposition 2.5.7].

So in particular in a model category $\mathcal{M}$ to show a monomorphism is an effective monomorphism we need only show it is a regular monomorphism.

1.4.11 Homotopy function complexes

The notion of a *homotopy function complex* (or mapping complex) from X to Y is a simplicial generalisation of the homotopy set of maps between X and Y taking account of the 'higher dimensional information'. In a simplicial model category, [Hir03, Definition 9.1.6], the homotopy function complex $\mathrm{map}(X, Y)$ is easily defined, see [Hov99, Remark 5.2.10], however in model categories lacking the simplicial structure the construction is much more involved as the simplicial structure must be introduced in a homotopically coherent manner. We do not make extensive use of homotopy function complexes in this thesis so we mainly reference constructions from [Hov99]. Their use in this thesis will be in defining Bousfield localisations.

Definition 1.4.11.1. A *Reedy category* is a category $\mathcal{D}$ with two subcategories $\mathcal{D}_-$ and $\mathcal{D}_+$ and a *degree function* $deg\colon \mathcal{D} \to \lambda$ for some ordinal λ such that

- every non-identity morphism of $\mathcal{D}_-$ lowers degree,
- every non-identity morphism of $\mathcal{D}_+$ raised degree, and
- every morphism of $\mathcal{D}$ factors uniquely into a morphism of $\mathcal{D}_-$ followed by a morphism of $\mathcal{D}_+$.

Example 1.4.11.2. The simplex category Δ is an example with Δ_- the subcategory of surjective order-preserving maps and Δ_+ the subcategory of injective order-preserving maps. Similarly Δ^{op} is a Reedy category with $(\Delta^{op})_- = (\Delta_+)^{op}$ and $(\Delta^{op})_+ = (\Delta_-)^{op}$.

For $\mathcal{C}$ a category and $\mathcal{D}$ a Reedy category, we have a subcategory of the slice category denoted $(\mathcal{D}_+)_i$ whose objects are the non-identity morphisms of $\mathcal{D}_+$ with codomain i, $f\colon j \to i$ and whose morphisms from $f\colon j \to i$ to $g\colon k \to i$ are commutative triangles h:

$$\begin{tikzcd}
j \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & k \arrow[dl, "g"] \\
& i &
\end{tikzcd} ,$$

with h a morphism of $\mathcal{D}_+$. Note for an object X of $\mathcal{C}^{\mathcal{D}}$ there is a functor from this category to $\mathcal{C}$ which assigns to an object $f\colon j \to i$ the object X_j. Similarly we have a subcategory of the coslice category denoted $(\mathcal{D}_-)^i$ whose objects are the non-identity morphisms of $\mathcal{D}_-$ with domain i, $f\colon i \to j$ and whose morphisms from $f\colon i \to j$ to $g\colon i \to k$ are commutative triangles h:

$$\begin{tikzcd}
& i \arrow[dl, "f"'] \arrow[dr, "g"] & \\
j \arrow[rr, dashed, "h"] & & k
\end{tikzcd} ,$$

with h a morphism of $\mathcal{D}_-$, which again for an object X of $\mathcal{C}^{\mathcal{D}}$ there is a functor sending an object $f\colon i \to j$ to X_j.

Definition 1.4.11.3. For a complete and cocomplete category $\mathcal{C}$, Reedy category $\mathcal{D}$ and $X \in \mathcal{C}^{\mathcal{D}}$ we define:

1. the *latching space object* L_iX given by

$$L_iX \coloneqq \operatorname*{colim}_{(j \to i) \in (\mathcal{D}_+)_i} X_j \ ,$$

2. and the *matching space object* M_iX given by

$$M_iX \coloneqq \lim_{(i \to j) \in (\mathcal{D}_-)^i} X_j \ .$$

Theorem 1.4.11.4 ([Hov99, Theorem 5.2.5])**.** *For $\mathcal{M}$ a model category and $\mathcal{D}$ a Reedy category there is a model category structure, called the Reedy model category, on the category $\mathcal{M}^{\mathcal{D}}$ where a morphism $f\colon X \to Y$ of $\mathcal{M}^{\mathcal{D}}$, i.e. natural transformation of functors $X \Rightarrow Y$ is:*

1. *a weak equivalence if $f_i\colon X_i \to Y_i$ is a weak equivalence for each $i \in \mathcal{D}$,*
2. *an (acyclic) fibration if $X_i \coprod_{L_iX} L_iY \to Y_i$ is an (acyclic) fibration for each $i \in \mathcal{D}$, and*
3. *an (acyclic) cofibration if $X_i \to M_iX \prod_{M_iY} Y_i$ is an (acyclic) cofibration for each $i \in \mathcal{D}$.*

In particular we have Reedy model category structures on the category of cosimplicial and simplicial objects in any model category $\mathcal{M}$. For an $X \in \mathcal{M}$ we denote by:

- ℓ^*X the cosimplicial object whose n^{th} object is $\coprod_{n+1} X$ and whose coface and codegeneracy maps are inclusions and fold maps,
- r^*X the cosimplicial objects whose n^{th} object is X and whose coface and codegeneracy maps are identities,
- ℓ_*X the simplicial object whose n^{th} object is X and whose face and degeneracy maps are identities, and

- r_*X the simplicial object whose n^{th} object is $\prod_{n+1} X$ and whose face and degeneracy maps are projections and inclusions.

Definition 1.4.11.5. For $\mathcal{M}$ a model category and X, Y objects of $\mathcal{M}$ we define:

1. a *cosimplicial frame* of X to be a factorisation, in the Reedy model category on $\mathcal{M}^{\Delta}$, of the natural map $\ell^*X \to r^*X$ into a cofibration followed by a weak equivalence which are isomorphisms in degree 0:
$$\ell^*X \rightarrowtail X^* \xrightarrow{\sim} r^*X \ .$$
2. a *simplicial frame* of Y to be a factorisation, in the Reedy model category on $\mathcal{M}^{\Delta^{op}}$, of the natural map $\ell_*Y \to R_*Y$ into a weak equivalence followed by a fibration which are isomorphisms in degree 0:
$$\ell_*Y \xrightarrow{\sim} Y_* \twoheadrightarrow R_*Y \ .$$

[Hov99, Theorem 5.2.8] give functorial cosimplicial and simplicial framings in $\mathcal{M}$. We can now define left and right homotopy function complexes.

Definition 1.4.11.6. For $\mathcal{M}$ a model category, X, Y objects of $\mathcal{M}$ and X^*, Y_* cosimplicial and simplicial frames of X and Y respectively we define:

1. the *left homotopy function complex* $\mathrm{map}_l(X, Y)$ to be the simplicial set whose n^{th} object is $\mathrm{Hom}_{\mathcal{M}}(X^*[n], Y)$, and
2. the *right homotopy function complex* $\mathrm{map}_r(X, Y)$ to be the simplicial set whose n^{th} object is $\mathrm{Hom}_{\mathcal{M}}(X, Y_*[n])$

both equiped with the obvious induced simplicial structure from X^* and Y_*.

If in addition we require that X is cofibrant and Y fibrant in $\mathcal{M}$ then the left and right homotopy functions are homotopy equivalent. This follows by relating both to the bisimplicial set with $n \times m^{\text{th}}$ object $\mathrm{Hom}_{\mathcal{M}}(X^*[n], Y_*[m])$, we denote either of map_r or map_l simply by map in this case.

Proposition 1.4.11.7 ([Hov99, Theorem 5.4.7])**.** *For $\mathcal{M}$ a model category, X a cofibrant object of $\mathcal{M}$ and Y a fibrant object of $\mathcal{M}$ then there are weak equivalences*
$$\mathrm{map}_l(X, Y) \to \mathrm{map}(X, Y) \leftarrow \mathrm{map}_r(X, Y) \ .$$

Henceforth we shall only consider homotopy function complexes with a cofibrant domain and fibrant codomain so we simply write map.

1.4.12 Bousfield localisations

A localised model category is a modification of some model category so as to expand the class of weak equivalences whilst retaining the structure of a model category. Such a construction achieves a new category with an appropriate universal property outlined below in it's original construction. Often this is done so as to localise at (or away from) a prime to focus on p-torsion information or to pass to a rational homotopy theory. As a result one of (or both) the classes of fibrations and cofibrations must also change since any two defining subclasses of morphisms of a model category determines the third. Bousfield constructed the first such localised homotopy theories in [Bou75] where, for an arbitrary homology theory h_*, a localisation functor on the category of simplicial sets is constructed which induces a h_*-localisation functor on the homotopy category, i.e. a functor $E\colon \mathrm{Ho}(sSets) \to \mathrm{Ho}(sSets)$ and natural transformation $\eta_X : 1 \Rightarrow E$ such that:

1. $\eta_X : X \to EX$ induces $h_*(X) \cong h_*(EX)$, and
2. whenever $f\colon X \to Y$ (in the homotopy category) induces a h_* isomorphism there is a unique factorisation
$$\begin{tikzcd} X \arrow[r, "f_*"] \arrow[dr, "\eta_X"'] & Y \arrow[d, dashed] \\ & EX \end{tikzcd} .$$

Bousfield later made a similar definition and construction for localisation of spectra with respect to homology in [Bou79]. Later notions of localisations with respect to a class of morphisms in an arbitrary model category were considered, see [Hir03] for a textbook account.

We consider here localisations known as *left Bousfield localisations* and *right Bousfield localisations*. These have the nice property that their existence are known, under reasonable assumptions, and that they only alter two of the three defining classes of morphisms, the weak equivalences and fibrations for left Bousfield localisations, and the weak equivalences and cofibrations for the right Bousfield localisations. The reference for these localisations which we follow is [Hir03].

Definition 1.4.12.1. For a model category $\mathcal{M}$ with subclass of morphisms $\mathcal{C}$ we say:

1. an object W of $\mathcal{M}$ is *$\mathcal{C}$-local* if it is fibrant in $\mathcal{M}$ and for all $f\colon A \to B$ of $\mathcal{C}$ the induced maps on homotopy function complexes, $\mathrm{map}(f, W)\colon \mathrm{map}(B, W) \to \mathrm{map}(A, W)$, are all weak equivalences, and

2. a morphism $g\colon X \to Y$ of $\mathcal{M}$ is a *$\mathcal{C}$-local equivalence* if for every *$\mathcal{C}$-local* object W of $\mathcal{M}$ the induced maps on homotopy function complexes, $\mathrm{map}(g, W)\colon \mathrm{map}(Y, W) \to \mathrm{map}(X, W)$, are all weak equivalences.

Definition 1.4.12.2. The left Bousfield localisation of a model category $\mathcal{M}$ with respect to a subclass of morphisms $\mathcal{C}$ is, if it exits, the model category, denoted $L_{\mathcal{C}}\mathcal{M}$, with:

1. weak equivalences of $L_{\mathcal{C}}\mathcal{M}$ being the $\mathcal{C}$-local equivalences, and

2. cofibrations of $L_{\mathcal{C}}\mathcal{M}$ being the cofibrations of $\mathcal{M}$.

The following gives existence of left Bousfield localisations under reasonsable assumptions.

Theorem 1.4.12.3 ([Hir03, Theorem 4.1.1]). *For $\mathcal{M}$ a left proper and cellular model category with a subset $\mathcal{C}$ of morphisms the left Bousfield localisation $L_{\mathcal{C}}\mathcal{M}$ exists and is also left proper and cellular.*

Example 1.4.12.4. Standard model categories which are left proper and cellular, and therefore satisfy the conditions of Theorem 1.4.12.3 include the Quillen model category of (pointed) simplicial sets and the projective model structure on chain complexes.

The existence theorem for right Bousfield localisations we state here and use later is a localisation at a set of objects instead of set of maps. It also goes by the name of *cellularization*. The following definition appears as (part of) [Hir03, Definition 3.1.8].

Definition 1.4.12.5. Let $\mathcal{M}$ be a model category and $\mathcal{K}$ a set of objects of $\mathcal{M}$. A morphism $g\colon X \to Y$ of $\mathcal{M}$ is said to be a *$\mathcal{K}$-colocal equivalence* or a *$\mathcal{K}$-cellular equivalence* if for all $A \in \mathcal{K}$ the induced map of homotopy function complexes $\mathrm{map}(A, g)\colon \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$ is a weak equivalence.

Definition 1.4.12.6. The *right Bousfield localisation* of a model category $\mathcal{M}$ with respect to a subclass of morphisms $\mathcal{C}$ is, if it exists, the model category, denoted $R_{\mathcal{C}}\mathcal{M}$, with:

1. weak equivalences of $R_{\mathcal{C}}\mathcal{M}$ being the $\mathcal{C}$-colocal equivalences, and

2. fibrations of $R_{\mathcal{C}}\mathcal{M}$ being the fibrations of $\mathcal{M}$.

The following statement of existence of the right Bousfield localisation at $\mathcal{K}$ includes corrections from the errata of [Hir03].

Theorem 1.4.12.7 (Existence of Right Bousfield Localisations,[Hir03, Theorem 5.1.1]). *For $\mathcal{M}$ a right proper and cellular model cateogry and subset of objects $\mathcal{K}$ of $\mathcal{M}$ with $\mathcal{C}$ the class of $\mathcal{K}$-colocal equivalences the right Bousfield localisation $R_{\mathcal{C}}\mathcal{M}$ exists and is also right proper and cellular.* ۞

Definition 1.4.12.8. Right Bousfield localisation of $\mathcal{M}$ at a set $\mathcal{K}$ as above is often called the *$\mathcal{K}$-cellularization* of $\mathcal{M}$ at $\mathcal{K}$ and the resulting localisation $R_{\mathcal{C}}\mathcal{M}$ will be denoted $\mathcal{K}$-`cell`-$\mathcal{M}$.

1.5 The projective model category of chain complexes

The category of (cohomologically) graded chain complexes can be equipped with a model category structure commonly know as the projective model structure in which the weak equivalences are the quasi-isomorphisms and the fibrations the degree-wise surjections. This model structure will serve as frequent motivation and intuition for various constructions and definitions to come so we recall here the projective model structure in detail.

Definition 1.5.0.1. The sphere S^n and disc D^n objects in the category of chain complexes are given by:

$$S^n := (\ldots \to 0 \to R^n \to 0 \to \ldots) \ ,$$
$$D^{n-1} := \big(\ldots \to 0 \to R^{n-1} \to R^n \to 0 \to \ldots\big) \ .$$

with inclusion $i_n \colon S^n \to D^{n-1}$ given by the identity in degree n.

These are representing objects for the kernel and image of the differentials. The following can be found proved in [Hov99, Theorem 2.3.11] or in much greater generality in [CH02, Theorem 2.2] and is commonly referred to as the projective model structure on (unbounded) chain complexes.

Theorem 1.5.0.2. *There is a cofibrantly generated projective model structure on $\mathcal{C}_R$ where:*

1. *weak equivalences are the quasi-isomorphisms,*
2. *fibrations are the degreewise surjections, and*
3. *generating cofibrations are given by $I = \{i_n \colon S^n \to D^{n-1}\}_{n\in\mathbb{Z}}$ and generating acyclic cofibrations by $J = \{0 \to D^n\}_{n\in\mathbb{Z}}$.*

In the introduction to Chapter 4 we explain how one classifies the cofibrations. This model structure has many nice properties including being finitely cofibrantly generated, left and right proper, cellular and stable.

1.6 r-Model categories of filtered chain complexes

The following results are those of [CELW19] and establish two model structures on $f\mathcal{C}$ for each $r \geq 0$ (although these agree for $r = 0$).

Definition 1.6.0.1. The morphism $\varphi_r \colon \mathcal{Z}_r(p, n) \to \mathcal{B}_r(p, n)$ is given bidegree-wise by the diagonal whenever possible or otherwise the identity.

$$\begin{tikzcd}
& \big(R^n_{(p)} \arrow[r] \arrow[d, "\varphi^n"'] & R^{n+1}_{(p-r-1)}\big) \arrow[d, "\varphi^{n+1}"] \\
\big(R^{n-1}_{(p+r)} \arrow[r, "\binom{1}{0}"'] & R^n_{(p)} \oplus R^n_{(p-1)} \arrow[r, "(0\ 1)"'] & R^{n+1}_{(p-r-1)}\big)
\end{tikzcd}$$

Definition 1.6.0.2. Denote by $w_r \colon B_r^{*,*}(A) \longrightarrow Z_r^{*,*}(A)$ the morphism of filtered chain complexes obtained by precomposing an element of $B_r^{*,*}(A)$ thought of as a map $B_r(*, *) \to A$ by $\varphi_r \colon Z_r(*, *) \to B_r(*, *)$.

The following definitions provides the generating (acylic) cofibrations.

Definition 1.6.0.3. Let I_r and J_r be the sets of morphisms of $f\mathcal{C}$ given by

$$I_r := \{\mathcal{Z}_{r+1}(p, n) \longrightarrow \mathcal{B}_{r+1}(p, n)\}_{p,n\in\mathbb{Z}} \ ,$$
$$J_r := \{0 \longrightarrow \mathcal{Z}_r(p, n)\}_{p,n\in\mathbb{Z}} \ .$$

We henceforth refer to these as the *generating r-cofibrations* and *generating r-acyclic cofibrations* respectively.

Definition 1.6.0.4. Let I'_r and J'_r be the sets of morphisms of $f\mathcal{C}$ given by

$$I'_r := I_r \cup \bigcup_{k=0}^{r-1} J_k \ ,$$

$$J'_r := \bigcup_{k=0}^{r} J_k \ .$$

We henceforth refer to these as the *generating r'-cofibrations* and *generating r'-acyclic cofibrations* respectively.

Theorem 1.6.0.5 ([CELW19, Theorem 3.14]). *There is a right proper cofibrantly generated model structure on $f\mathcal{C}$, which we denote by $(f\mathcal{C})_r$, with generating cofibrations I_r and generating acyclic cofibrations J_r. These give the following classifications of morphisms:*

1. *weak equivalences are the E_r-quasi-isomorphisms,*
2. *fibrations are those morphisms f with $Z_r(f)$ bidegree-wise surjective.*

Theorem 1.6.0.6 ([CELW19, Theorem 3.16]). *There is a right proper cofibrantly generated model structure on $f\mathcal{C}$, which we denote by $(f\mathcal{C})_{r'}$, with generating cofibrations I'_r and generating acyclic cofibrations J'_r. These give the following classifications of morphisms:*

1. *weak equivalences are the E_r-quasi-isomorphisms,*
2. *fibrations are those morphisms f with $Z_k(f)$ bidegree-wise surjective for all $0 \leq k \leq r$.*

Note one has the pushout of $\mathcal{Z}_{r+1}(p, p+n) \to \mathcal{B}_{r+1}(p, p+n)$ by 0 is also an $(r+1)$-cycle as observed in [CELW19, Lemma 3.2] and so since pushouts of cofibrations are cofibrations the following lemma is immediate.

Lemma 1.6.0.7. *The morphisms $0 \to \mathcal{Z}_{r+1}(p, p+n)$ are cofibrations in both $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$.*

We will later extend these results and show there are model structures *in between* the $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$ model structures. We state here the sequence of lemmas and propositions that prove the existence of the model structure $(f\mathcal{C})_r$ as shown in [CELW19]. The result for $(f\mathcal{C})_{r'}$ is a consequnce of the former.

Proposition 1.6.0.8 ([CELW19, Proposition 3.12]). *We have I_r-$\mathtt{Inj} = \mathcal{E}_r \cap J_r$-$\mathtt{Inj}$.*

Proposition 1.6.0.9 ([CELW19, Proposition 3.13]). *For all $r \geq 0$ and all $0 \leq k \leq r$ we have J_k-$\mathtt{Cof} \subseteq \mathcal{E}_r$.*

Theorem 1.6.0.5 now follows from these by an application of Theorem 1.4.2.9. Further the shift-décalage adjunctions of Lemma 1.3.3.2 induce Quillen equivalences given as the following theorem.

Theorem 1.6.0.10 ([CELW19, Theorem 3.22]). *For all $r, l \geq 0$ we have a Quillen equivalence:*

$$S^l\colon (f\mathcal{C})_r \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (f\mathcal{C})_{r+l} \colon Dec^l \quad .$$

There is a similar result for the r'-model structure. The following is a useful surjectivity result.

Lemma 1.6.0.11 ([CELW19, Lemma 2.8]). *For $f\colon A \to B$ a morphism of filtered chain complexes and $r \geq 0$ the following are equivalent:*

1. *the maps $Z_r(f)$ and $ZW_{r-1}(f)$ are surjective,*
2. *the maps $E_r(f)$ and $ZW_{r-1}(f)$ are surjective.*

We define the r-cone of a filtered chain complex with a different sign convention then that of [CELW19, Definition 3.5]. It has useful properties analogous to that of the cone object for chain complexes.

Definition 1.6.0.12. The *r-cone* $C_r(A)$ of a morphisms $f\colon A \to B$ of filtered chain complexs has underlying filtered graded modules that of $\Sigma^r A \oplus B$ with filtration given by

$$F_p C_r(f)^n := F_{p-r} A^{n+1} \oplus F_p B^n$$

and differential given by $d\colon (a, b) \mapsto (-da, fa + db)$ or in matrix notation $d = \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}$. We further denote the r-cone of the identity morphism $id\colon A \to A$ by $C_r(A)$.

Lemma 1.6.0.13 ([CELW19, Remark 3.7]). *A morphism $f\colon A \to B$ is an r-weak equivalence if and only if the r-bigraded complex $E_r(C_r(f))$ is acyclic, i.e. the r-cone of f is r-acyclic.*

Note there are inclusion and projection morphisms $i_2\colon B \to C_r(f)$ and $\pi\colon C_r(f) \to \Sigma^r A$.

Lemma 1.6.0.14 ([CELW19, Notation 3.7]). *The r-cone is r-acyclic and the projection $\pi\colon C_r(A) \longrightarrow \Sigma^r A$ is Z_s-bidegree-wise surjective for all $0 \leq s \leq r$.*

Definition 1.6.0.15. We say the differential d of a filtered chain complex A *supresses the filtration by r* if for all p and n we have $dF_pA^n \subseteq F_{p-r}A^{n+1}$. Equivalently we say the object A is r-supressive.

1.7 r-Model categories of bicomplexes

Note given a bicomplex we can apply the totalisation functor of Definition 1.2.2.5 to obtain a filtered chain complex.

Definition 1.7.0.1. A morphism f of bicomplexes will be called an *r-weak equivalence* or *r-quasi-isomorphism* if it induces an isomorphism between the $(r+1)$-pages of the associated spectral sequence after applying the product totalisation functor of Definition 1.2.2.5.

The following results are those of [CELW19] and establish two model structures on $b\mathcal{C}$ for each $r \geq 0$ (although these agree for $r = 0$ and also for $r = 1$). The following definitions provide the generating cofibrations (and the generating acyclic cofibrations after a small modification). Note our definition of ZW_0 and $\mathcal{ZW}_0$ (and these ones only) are shifted vertically by 1 in contrast with those of [CELW19]. In the figures of this section which represent bicomplexes, the symbol • denotes a copy of the R-module R in some bidegree. Bidegrees of certain R-modules • have been specified.

Definition 1.7.0.2. For $r \geq 1$, the *r-cycles* $ZW_r^{*,*}(A)$ of a filtered chain complex A are given in bidegree $(p, p+n)$ by sequences of elements $(a_0, a_1, \ldots, a_{r-1})$ with $a_i \in A^{p-i,p-i+n}$ and such that $d_0 a_0 = 0$ and $d_0 a_{i+1} = d_1 a_i$ for $0 \leq i \leq r-2$. For $r = 0$ the *r-cycles* $ZW_0^{*,*}(A)$ of a bicomplex A are given in bidegree $p, p+n$ by $A^{p,p+n}$.

Definition 1.7.0.3. The *representing object for the r-cycles* of bicomplexes, denoted $\mathcal{ZW}_r^{*,*}$, is given in bidegree $(p, p+n)$ for $r \geq 1$ by the bicomplex with a copy of R in bidegrees $(p-i, p-i+n)$ for $0 \leq i \leq r-1$ and a copy of R in bidegrees $(p-1, p-1+n+1)$ for $0 \leq i \leq r-1$ where differentials are the identity whenever possible, Figure 1.1. For $r = 0$ the bicomplex $\mathcal{ZW}_0^{*,*}$ is given in bidegree $(p, p+n)$ with a copy of R in bidegrees $(p, p+n)$, $(p-1, p-1+n+1)$, $(p, p+n+1)$ and $(p-1, p-1+n+2)$ with identity differentials whenever possible, Figure 1.2.

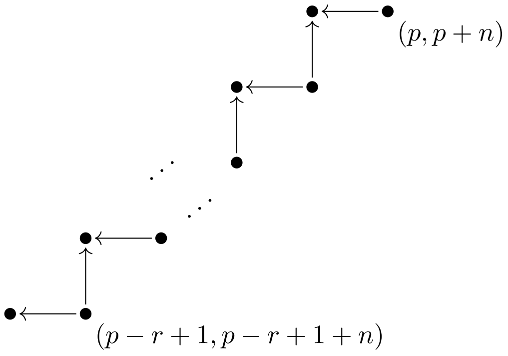


Figure 1.1: The bicomplex $\mathcal{ZW}_r(p, p+n)$

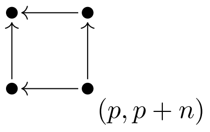


Figure 1.2: The bicomplex $\mathcal{ZW}_0(p, p+n)$

Definition 1.7.0.4. For $r \geq 2$ the *r-boundary* $BW_r^{*,*}(A)$ of a bicomplex A is given in bidegree $(p, p+n)$ by $ZW_{r-1}^{p+r-1,p+r+n-2}(A) \oplus A^{p,p+n-1} \oplus ZW_{r-1}^{p-1,p-1+n-1}(A)$. For $r = 1$ the *r-boundaries* $BW_0^{*,*}(A)$ is $A^{p,p+n-1}$. For $r = 0$ the *r-boundaries* $BW_0^{*,*}(A)$ is 0.

Definition 1.7.0.5. The *representing object for the r-boundaries* of a bicomplex, denoted $\mathcal{BW}_r^{*,*}$, is given in bidegree $(p, p+n)$ for $r \geq 2$ by $\mathcal{ZW}_r(p+r-1, p+r-1+n-1) \oplus \mathcal{ZW}_0(p, p+n-1) \oplus \mathcal{ZW}_r(p-1, p-1+n)$, for $r = 1$ by $\mathcal{ZW}_0(p, p+n-1)$ and for $r = 0$ by 0, Figure 1.3.

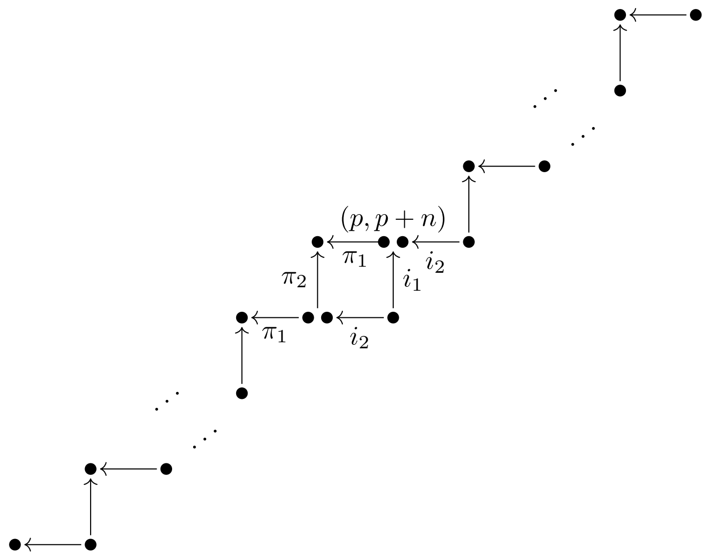


Figure 1.3: The bicomplex $\mathcal{BW}_r(p, p+n)$

Definition 1.7.0.6. The morphism $\varphi_r \colon \mathcal{ZW}_r(*,*) \to \mathcal{BW}_r(*,*)$ is given component-wise by the diagonal whenever possible or otherwise the identity.

Definition 1.7.0.7. Denote by $w_r \colon BW_r^{*,*}(A) \longrightarrow ZW_r^{*,*}(A)$ the morphism of bicomplexes of [CELW19, Definition 4.1] obtained by precomposing an element of $BW_r^{*,*}(A)$ thought of as a map $\mathcal{BW}_r(*,*) \to A$ by $\varphi_r \colon \mathcal{ZW}_r(*,*) \to \mathcal{BW}_r(*,*)$.

Definition 1.7.0.8. Let I_r and J_r be the sets of morphisms of $f\mathcal{C}$ given by

$$\begin{aligned} I_r &:= \{\mathcal{ZW}_{r+1}(p,n) \longrightarrow \mathcal{BW}_{r+1}(p,n)\}_{p,n\in\mathbb{Z}} \ , \\ J_r &:= \{0 \longrightarrow \mathcal{ZW}_r(p,n)\}_{p,n\in\mathbb{Z}} \ . \end{aligned}$$

We henceforth refer to I_r as the *generating r-cofibrations* and $J_0 \cup J_r$ as the *generating r-acyclic cofibrations* respectively.

Definition 1.7.0.9. Let I'_r and J'_r be the sets of morphisms of $f\mathcal{C}$ given by

$$\begin{aligned} I'_r &:= I_r \cup \bigcup_{k=0}^{r-1} J_k \ , \\ J'_r &:= \bigcup_{k=0}^{r} J_k \ . \end{aligned}$$

We henceforth refer to $I_r \cup \bigcup_{k=0}^{r-1} J_k$ as the *generating r'-cofibrations* and $\bigcup_{k=0}^{r} J_k$ as the *generating r'-acyclic cofibrations* respectively.

Theorem 1.7.0.10 ([CELW19, Theorem 4.37])**.** *There is a right proper cofibrantly generated model structure on $b\mathcal{C}$, which we denote by $(b\mathcal{C})_r$, with generating cofibrations I_r and generating acyclic cofibrations $J_0 \cup J_r$. These give the following classifications of morphisms:*

1. *weak equivalences are the E_r-quasi-isomorphisms,*
2. *fibrations are those morphisms f with $ZW_0(f)$ and $ZW_r(f)$ bidegree-wise surjective.*

Theorem 1.7.0.11 ([CELW19, Theorem 4.39])**.** *There is a right proper cofibrantly generated model structure on $b\mathcal{C}$, which we denote by $(b\mathcal{C})_{r'}$, with generating cofibrations I'_r and generating acyclic cofibrations J'_r. These give the following classifications of morphisms:*

1. *weak equivalences are the E_r-quasi-isomorphisms,*

2. *fibrations are those morphisms f with $ZW_k(f)$ bidegree-wise surjective for all $0 \leq k \leq r$.*

Analogous results hold for bicomplexes as did for filtered chains proving existence of these model categories however note in this setting there is no analogous shift-décalage adjunction. The following is a useful surjectivity result.

Lemma 1.7.0.12 ([CELW19, Remark 4.5]). *For $f\colon A \longrightarrow B$ a morphism of bicomplexes and $r \geq 1$ the following are equivalent:*

1. *the maps $ZW_r(f)$, $ZW_{r-1}(f)$, and f are surjective,*
2. *the maps $E_r(f)$, $ZW_{r-1}(f)$, and f are surjective.*

Remark 1.7.0.13. Note that the 0-model structure on bicomplexes is such that its weak equivalences are those morphisms inducing an isomorphism on vertical homology and whose fibrations are bidegree-wise surjective. Identifying the category of bicomplexes with the category of (vertical) chain complexes of (horizontal) chain complexes we see that the 0-model structure on bicomplexes is the projective model structure of (vertical) chain complexes of (horizontal) chain complexes of for example [CH02]. To explain further their result, [CH02, Theorem 2.2], states that for a *projective class*, [CH02, Definition 1.1], in an abelian category $\mathcal{A}$ there is a model structure on unbounded chain complexes in $\mathcal{A}$ with the weak equivalences and fibrations (and therefore so too the cofibrations) determined by the projective class. Here we take $\mathcal{A} = \mathcal{C}_R$ to be the category of (horizontal) chain complexes and the projective class to consist of the data of the acyclic cofibrant objects in $\mathcal{A} = \mathcal{C}_R$ (with its usual projective model structure) and the collection of maps to be the degreewise surjections. The obtained model structure on chain complexes of chain complexes has fibrations being the bidegreewise surjections and weak equivalences the vertical homology isomorphisms. In addition [CH02, Theorem 5.7] gives precisely the same set of generating cofibrations and acyclic cofibrations as above.

The r-cone of a bicomplex is defined by the same means as in [CELW19, Remark 4.27].

Definition 1.7.0.14. The r-cone, for $r \geq 1$, of a bicomplex A is denoted by $C_r(A)$ and given by the tensor product $\mathcal{ZW}_r(r, r-1) \otimes A$.

Note there is a projection morphism $\psi_r\colon C_r(A) \to \Sigma^r A$.

Lemma 1.7.0.15 ([CELW19, Propositions 4.29 & 4.32]). *The r-cone is r-acyclic and the projection $\psi_r\colon C_r(A) \to \Sigma^r A$ is ZW_s-bidegree-wise surjective for all $0 \leq r \leq s$.*

Lemma 1.7.0.16. *A bicomplex A is a finite object if and only if it is bounded and each $A^{i,j}$ is a finitely presented R-module.*

Proof. The proof is similar to that of [Hov99, Lemma 2.3.2].

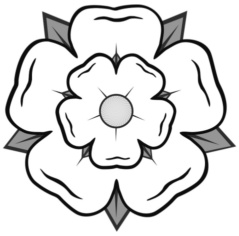
1.8 Conventions

- Throughout R will denote a fixed commutative unital ring.
- Unless otherwise stated chain complexes are taken to be cohomologically graded.
- We will ocasionally have need to specify a name for the generator 1 of the R-module R. We will do so by writing $R\{a\}$ to mean the free R-module R on one generator a thought of as the element 1. For the filtered R-modules $R^n_{(p)}$ we will do so similarly by denoting by $R^n_{(p)}\{a\}$ the filtered R-module $R^n_{(p)}$ with generator a thought of as the element 1. Where multiple such identifications arise, e.g. $R\{a\}$ and $R\{b\}$ and we have need for change of bases we may abuse notation somewhat and write $R\{a+b\}$ for another copy of the R-module R on one generator given by the sum $a+b$.
- In commutative diagrams we frequently denote some R-modules with maps between them form a (filtered) chain complex by enclosing them in brackets.
- The tensor product $\otimes$ will denote either the symmetric monoidal tensor product of filtered chain complexes or bicomplexes.

- The morphisms Δ, ∇, Δ^- and ∇^- will denote the following morphisms:

$$\begin{aligned} \Delta\colon A &\to A \oplus A & \nabla\colon A \oplus A &\to A \\ a &\mapsto (a, a) & (a, b) &\mapsto a + b \\ \Delta^-\colon A &\to A \oplus A & \nabla^-\colon A \oplus A &\to A \\ a &\mapsto (a, -a) & (a, b) &\mapsto a - b \end{aligned}$$

- The maps i_1 and i_2 will denote inclusions into the first and second components and similarly π_1 and π_2 the projections onto the first and second components.
- In diagrams we usually omit a label for an identity morphism between copies of R, or denote it either by 1 or id.
- The class of morphisms of filtered chain complexes (or bicomplexes) consisting of r-weak equivalences will be denoted $\mathcal{E}_r$.



CHAPTER 2

Adjoints to Totalisation Functors

We construct an adjoint to the product and coproduct totalisation functors, although we only make use of the former. That is we have the following propositions.

Proposition 2.1.0.2. *There is an adjunction of categories* $\mathcal{L}\colon f\mathcal{C} \underset{\perp}{\rightleftarrows} b\mathcal{C} : \mathrm{Tot}^{\Pi}$.

We give descriptions of the functors $\mathcal{L}$ and Tot^{Π} applied to representing (witness) cycle and boundary objects and show that the unit of the adjunction $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ on an s-cycle is an s-equivalence.

Proposition 2.2.1.2. *For* $s \geq 1$ *the unit of the adjunction applied to an* s*-cycle,* $\mathcal{Z}_s(p, p+n) \to \mathrm{Tot}^{\Pi}\mathcal{L}\mathcal{Z}_s(p, p+n)$*, is an isomorphism on the* s*-page.*

We will later make use of this s-equivalence of the unit map to show that Proposition 2.1.0.2 is a Quillen equivalence with appropriate model categories yet to be defined. Lastly we show existence of a total model structure on $b\mathcal{C}$ induced by the Kan transfer.

Corollary 2.4.0.3. *There is a total model structure on bicomplexes cofibrantly generated by generating cofibrations* $I := \{\mathcal{ZW}_{\infty,-\infty}(n) \to \mathcal{BW}_{\infty,-\infty}(n)\}$ *and generating acyclic cofibrations* $J := \{0 \to \mathcal{BW}_{\infty,-\infty}\}$ *in which*

1. *weak equivalences are those morphisms* f *of bicomplexes such that* $H^*\mathrm{Tot}^{\Pi}$ *is an isomorphism,*

2. *fibrations are those morphisms* f *such that* Tot^{Π} *is (homologically) degreewise surjective, i.e.* f *is bidegreewise surjective.*

2.1 Left adjoint to product totalisation

We construct a left adjoint, denoted $\mathcal{L}$, to the totalisation functor Tot^{Π} of Definition 1.2.2.5. In private communications between the authors of [CELW19] an adjunction between (non-filtered) chain complexes and bicomplexes is established with the aim of extending this to the filtered setting. This unfortunately fails due to non-naturality of split short exact sequences arising from the graded pieces $\mathrm{Gr}_i C := F_i C / F_{i-1} C$ and so reconstructions of maps are not guaranteed to be compatible with the differential. We correct for this here by instead using the quotient objects $C/F_{i-1}C$ from which we can reconstruct a map defined on the whole of C.

Definition 2.1.0.1. The functor $\mathcal{L}\colon f\mathcal{C} \to b\mathcal{C}$ is defined on a filtered chain complex C by:

$$\mathcal{L}(C)^{i,i+n} := \frac{C^n}{F_{i-1}C^n} \oplus \frac{C^{n-1}}{F_i C^{n-1}} \ ,$$

where the differentials d_0 and d_1 are given on an $(x, y) \in \mathcal{L}(C)^{i,i+n}$ by:

$$\begin{aligned} d_0 &\colon (x, y) \mapsto (dx, x - dy) \ , \\ d_1 &\colon (x, y) \mapsto \big(0, (-1)^{n+1}x\big) \ . \end{aligned}$$

On a morphism of filtered chain complexes, $g\colon C \to D$, the functor $\mathcal{L}$ is given by $\mathcal{L}(g)^{i,i+n} := \bar{g}_i^n \oplus \bar{g}_{i+1}^{n-1}$ where $\bar{g}_i^n$ denotes the map $C^n/F_{i-1}C^n \to D^n/F_{i-1}D^n$ induced by g, noting that this map is defined as g preserves filtration and commutes with differentials since g does.

It is a straightforward check that the differentials define a bicomplex structure on the bigraded modules $\mathcal{L}(C)^{*,*}$ and that $\mathcal{L}(g)$ is indeed a morphism of bicomplexes.

Note that the x written in the image of d_0 should perhaps more properly be written as the class $[x]$ of $x \in C^n/F_{i-1}C^n$ in the module C^n/F_iC^n, we will omit such notation most of the time. Similarly the d appearing in d_0 is also used to denote the induced differential on the quotient modules.

We need to then describe natural maps between the hom sets $\mathrm{Hom}_{b\mathcal{C}}\left(\mathcal{L}(C), K\right)$ and $\mathrm{Hom}_{f\mathcal{C}}\left(C, \mathrm{Tot}^{\Pi}(K)\right)$ which are bijections. Given a map $f\colon \mathcal{L}(C) \to K$ of bicomplexes we obtain a map of filtered chain complexes $\tilde{f}\colon C \to \mathrm{Tot}^{\Pi}(K)$ from the following diagram:

$$\begin{tikzcd}
C^n \arrow[rr, dotted, "\tilde{f}^n"] \arrow[dr] & & \mathrm{Tot}^{\Pi}(K)^n = \prod_i K^{i,i+n} \\
& \prod_i \frac{C^n}{F_{i-1}C^n} \arrow[ur, "\left(f^{i,i+n}(-,0)\right)_i"'] &
\end{tikzcd}$$

where the map from C^n to $\prod_i C^n/F_{i-1}C^n$ is an infinite diagonal followed by a product of quotient maps.

We check the map $\tilde{f}$ as defined is a map of filtered chain complexes, i.e. commutes with differentials — it clearly respects filtration. Take an element $c \in C^n$. In the following series of equalities we write $\bar{c}_i$ to denote the class of c in the quotient group $C^n/F_{i-1}C^n$ and note that the class $[\bar{c}_i]$ of $\bar{c}_i$ in C^n/F_iC^n is equal to $\bar{c}_{i+1}$. We then have in the following the first equality follows by definition of $\tilde{f}$, the second by definition of the differential on Tot^{Π}, the third commutes d_0 and d_1 past the bicomplex map f, the fourth applies the definition of d_0 and d_1 in $\mathcal{L}(C)$ (noting here we are careful about the class $[c_{i+1}]$), the fifth cancels signs and uses the class $[\bar{c}_i] \in C^n/F_iC^n$ of $\bar{c}_i \in C^n/F_{i-1}C^n$ is just $\bar{c}_{i+1}$, the sixth cancels terms of opposite sign in the second component and the last again uses the definition of $\tilde{f}$.

$$\begin{aligned}
d\tilde{f}(c) &= d\left(f^{i,i+n}(\bar{c}_i, 0)\right)_i \\
&= \left(d_0 f^{i,i+n}(\bar{c}_i, 0) + (-1)^n d_1 f^{i+1,i+1+n}(\bar{c}_{i+1}, 0)\right)_i \\
&= \left(f^{i,i+n+1} d_0(\bar{c}_i, 0) + (-1)^n f^{i,i+n+1} d_1(\bar{c}_{i+1}, 0)\right)_i \\
&= \left(f^{i,i+n+1}(d\bar{c}_i, [\bar{c}_i]) + (-1)^n f^{i,i+n+1}(-1)^{n+1}(0, \bar{c}_{i+1})\right)_i \\
&= \left(f^{i,i+n+1}(d\bar{c}_i, \bar{c}_{i+1}) - f^{i,i+n+1}(0, \bar{c}_{i+1})\right)_i \\
&= \left(f^{i,i+n+1}(d\bar{c}_i, 0)\right)_i \\
&= \tilde{f}(dc)
\end{aligned}$$

Now suppose we have a map $g\colon C \to \mathrm{Tot}^{\Pi}(K)$ of filtered chain complexes, we define a map $\hat{g}\colon \mathcal{L}(C) \to K$ of bicomplexes bidegreewise as follows. Let $g^{i,i+n}$ be the composite of $g^n\colon C^n \to \mathrm{Tot}^{\Pi}(K)^n$ with the projection $\pi^i\colon \mathrm{Tot}^{\Pi}(K)^n \to K^{i,i+n}$. Now we define:

$$\begin{aligned}
\hat{g}^{i,i+n}\colon \mathcal{L}(C)^{i,i+n} &\to K^{i,i+n} \\
(\bar{x}, \bar{y}) &\mapsto g^{i,i+n}x + (-1)^n d_1 g^{i+1,i+1+n-1}y
\end{aligned}$$

where $x \in C^n$ is a choice of representative of $\bar{x} \in C^n/F_{i-1}C^n$, and $y \in C^{n-1}$ a choice of representative of $\bar{y} \in C^{n-1}/F_iC^{n-1}$. Note however that since g maps $F_{i-1}C^n$ into $\prod_{j\leq i-1} K^{j,j+n}$ and similarly F_iC^{n-1} into $\prod_{j\leq i} K^{j,j+n-1}$, we have the image of $\hat{g}^{i,i+n}$ is well defined, i.e. does not depend on choice of representing elements.

Note that we have the following relation between projection maps π^i and differentials d, d_0, and d_1:

$$\pi^i d(k_j)_j = \pi^i(d_0 k_j + (-1)^n d_1 k_{j+1})_j = d_0 k_i + (-1)^n d_1 k_{i+1} \tag{2.1}$$

when the degree of $(k_j)_j$ is n.

We now check that the maps $\hat{g}^{i,i+n}$ together define a map of bicomplexes, i.e. commute with the differentials d_0 and d_1. Writing $\bar{d}$ for the induced differential on quotients appearing in the definition of $\mathcal{L}$, note below that $\bar{d}\bar{x} = \bar{dx}$.

The first equality follows by definition of d_0, the second by definition of $\hat{g}$, the third rewrites $g^{*,*}$ using the projections, the fourth commutes d and g, the fifth uses Equation (2.1), the sixth cancels terms of opposite sign and one involving $d_1 d_1$ as well as commuting a d_0 and d_1, and lastly we apply the definition of $\hat{g}$.

$$\begin{aligned}
\hat{g}^{i,i+n+1}d_0(\bar{x},\bar{y}) &= \hat{g}^{i,i+n+1}(\bar{dx}, \bar{x}-\bar{dy}) \\
&= g^{i,i+n+1}(dx) + (-1)^{n+1}d_1 g^{i+1,i+1+n}(x-dy) \\
&= \pi^i g dx + (-1)^{n+1}d_1\pi^{i+1}gx - (-1)^{n+1}d_1\pi^{i+1}gdy \\
&= \pi^i dgx + (-1)^{n+1}d_1\pi^{i+1}gx - (-1)^{n+1}d_1\pi^{i+1}dgy \\
&= d_0 g^{i,i+n}x + (-1)^n d_1 g^{i+1,i+1+n}x + (-1)^{n+1}d_1 g^{i+1,i+1+n}x \\
&\quad - (-1)^{n+1}d_1\left(d_0 g^{i+1,i+1+n-1}y + (-1)^{n-1}d_1 g^{i+2,i+2+n-1}y\right) \\
&= d_0 g^{i,i+n}x + (-1)^n d_0 d_1 g^{i+1,i+1+n-1}y \\
&= d_0\hat{g}^{i,i+n}(\bar{x},\bar{y})
\end{aligned}$$

In the following the first equality follows by definition of d_1, the second by definition of $\hat{g}$, the third by $d_1 d_1 = 0$, and the last again by definition of $\hat{g}$.

$$\begin{aligned}
\hat{g}^{i-1,i-1+n+1}d_1(\bar{x},\bar{y}) &= \hat{g}^{i-1,i-1+n+1}(0,(-1)^{n+1}\bar{x}) \\
&= (-1)^{n+1}(-1)^{n+1}d_1 g^{i,i+n}x \\
&= d_1\left(g^{i,i+n}x + (-1)^n d_1 g^{i+1,i+1+n-1}y\right) \\
&= d_1\hat{g}^{i,i+n}(\bar{x},\bar{y})
\end{aligned}$$

Thus $\hat{g}$ computes with both differentials. We now verify that $\hat{\tilde{f}} = f\colon \mathcal{L}(C) \to K$. In what follows the first equality follows by definition of $\hat{g}$, the second rewrites $\tilde{f}^{*,*}$ using the projections π_*, the third by definition of $\tilde{f}$ noting here $\bar{x}_j$ is the class of our choice x in the quotient group $C^n/F_{j-1}C^n$ and in particular $\bar{x}_i = \bar{x}$, similarly for $\bar{y}_j$. The fourth applies the projections and uses $\bar{x}_i = \bar{x}$, the fifth commutes differentials, the sixth applies d_1, and lastly we cancel signs and use linearity.

$$\begin{aligned}
\hat{\tilde{f}}^{i,i+n}(\bar{x},\bar{y}) &= \tilde{f}^{i,i+n}x + (-1)^n d_1\tilde{f}^{i+1,i+1+n-1}y \\
&= \pi^i\tilde{f}x + (-1)^n d_1\pi^{i+1}\tilde{f}y \\
&= \pi^i\left(f^{j,j+n}(\bar{x}_j,0)\right)_j + (-1)^n d_1\pi^{i+1}\left(f^{j,j+n-1}(\bar{y}_j,0)\right)_j \\
&= f^{i,i+n}(\bar{x},0) + (-1)^n d_1 f^{i+1,i+1+n-1}(\bar{y},0) \\
&= f^{i,i+n}(\bar{x},0) + (-1)^n f^{i,i+n}d_1(\bar{y},0) \\
&= f^{i,i+n}(\bar{x},0) + (-1)^n f^{i,i+n}(0,(-1)^n\bar{y}) \\
&= f^{i,i+n}(\bar{x},\bar{y})
\end{aligned}$$

Finally we check that $\tilde{\hat{g}} = g\colon C \to \mathrm{Tot}^\Pi(K)$. The first equality does nothing other than remind us $\hat{g}$ is a map of bicomplexes, the second follows by definition of $\tilde{f}$, the third by definition of $\hat{g}$ noting that a choice of representative of the class $\bar{c}_i$ of c is c itself, the fourth by removing the zero term, and lastly by definition of $g^{i,i+n}$.

$$\begin{aligned}
\tilde{\hat{g}}^n(c) &= (\widetilde{\hat{g}^{i,i+n}(-,-)})(c) \\
&= \left(\hat{g}^{i,i+n}(-,0)\right)_i(c) \\
&= \left(g^{i,i+n}c + (-1)^n d_1 g^{i+1,i+1+n-1}0\right)_i \\
&= \left(g^{i,i+n}c\right)_i \\
&= g^n(c)
\end{aligned}$$

The bijections between the hom sets $\mathrm{Hom}_{b\mathcal{C}}(\mathcal{L}A, B)$ and $\mathrm{Hom}_{f\mathcal{C}}\left(A, \mathrm{Tot}^\Pi B\right)$ are easily seen to be natural by construction. We have then proved the following result.

Proposition 2.1.0.2. *There is an adjunction of categories* $\mathcal{L}\colon f\mathcal{C} \overset{\perp}{\rightleftarrows} b\mathcal{C} : \mathrm{Tot}^\Pi$.

2.2 The adjoints applied to representing cycle and boundary objects

We consider the effect of applying the left adjoint $\mathcal{L}$ to s-cycles $\mathcal{Z}_s$ and s-boundaries $\mathcal{B}_s$, as well as the effect of Tot^{Π} on s-witness cycles $\mathcal{ZW}_s$ and s-witness boundaries $\mathcal{BW}_s$. Since $\mathcal{L}$ is a left adjoint it commutes with coproducts and so the effect on boundaries is known from $\mathcal{B}_s = \mathcal{Z}_{s-1} \oplus \mathcal{Z}_{s-1}$ via $\mathcal{LB}_s \cong \mathcal{L}(\mathcal{Z}_{s-1} \oplus \mathcal{Z}_{s-1}) \cong \mathcal{LZ}_{s-1} \oplus \mathcal{LZ}_{s-1}$. It suffices then to analyse the bicomplexes $\mathcal{LZ}_s$.

2.2.1 $\mathcal{L}$ applied to $\mathcal{Z}_s$

We fix an s-cycle $\mathcal{Z}_s(p, p+n)$ to which we will apply $\mathcal{L}$. The definition of $\mathcal{L}$ was given on a filtered chain complex C by:

$$\mathcal{L}(C)^{i,i+n} := \frac{C^n}{F_{i-1}C^n} \oplus \frac{C^{n-1}}{F_iC^{n-1}} \ ,$$

so applying $\mathcal{L}$ to our s-cycle we have non-zero entries of the bicomplex only on diagonals n, $n+1$ and $n+2$. For $C = \mathcal{Z}_s(p, p+n)$ the quotients of the direct sum of the $\mathcal{L}C^{i,i+n}$ are either a copy of R or 0. We temporarily introduce the notation $M = R^n_{(p)}$ and $N = R^{n+1}_{(p-s)}$ to help keep track of each summand. The bicomplex $\mathcal{LZ}_s(p, p+n)$ is then depicted in Figure 2.1 where the modules $M \oplus 0$ and $N \oplus M$ demarcated by dashed boxes are in bidegrees $(p, p+n)$ and $(p-s, p-s+n+1)$ respectively. The differentials of Figure 2.1 are also obtained from the definition of $\mathcal{L}$.

We now show that up to isomorphism the bicomplex $\mathcal{LZ}_s(p, p+n)$ is isomorphic to a direct sum of a witness s-cycle and an infinite number of witness 0-cycles. More precisely:

$$\mathcal{LZ}_s(p, p+n) \cong \mathcal{ZW}_s(p, p+n) \oplus \bigoplus_{k \geq 0} \mathcal{ZW}_0(p-s-k, p-s-k+n) \ . \tag{2.2}$$

To see this we make a change of basis of the R-modules of Figure 2.1 of the form $N \oplus M$. We now write e and g for the element 1 of $M = R$ and f and h for $1 \in N = R$, so that $M = R\{e\} = R\{g\}$ and $N = R\{f\} = R\{h\}$ in the notation of Section 1.8. Consider one of the subdiagrams of Figure 2.1 given by the following:

$$\begin{tikzcd}
& R\{f\} & R\{f\} \oplus R\{e\} \arrow[l, "(-1)^{n+1}\pi_1"'] & R\{e\} \arrow[l, "(-1)^n i_2"] \\
R\{h\} & R\{h\} \oplus R\{g\} \arrow[l, "(-1)^{n+1}\pi_1"] \arrow[u, "\nabla^-"] & R\{g\} \arrow[l, "(-1)^n i_2"] \arrow[u, "\Delta"']
\end{tikzcd} \ . \tag{2.3}$$

We change basis on the modules given by $R\{f\} \oplus R\{e\}$ (and similarly $R\{h\} \oplus R\{g\}$) as follows:

$$\begin{aligned} \theta \colon R\{f\} \oplus R\{e\} &\longrightarrow R\{f+e\} \oplus R\{e\} \\ (a, b) &\longmapsto (a, b-a) \end{aligned}$$

to obtain an isomorphic diagram given by:

$$\begin{tikzcd}
& R\{f\} & R\{f+e\} \oplus R\{e\} \arrow[l, "(-1)^{n+1}\pi_1"'] & R\{e\} \arrow[l, "(-1)^n i_2"] \\
R\{h\} & R\{h+g\} \oplus R\{g\} \arrow[l, "(-1)^{n+1}\pi_1"] \arrow[u, "\begin{pmatrix} 0 & -1 \end{pmatrix}"] & R\{g\} \arrow[l, "(-1)^n i_2"] \arrow[u, "i_1"']
\end{tikzcd} \ , \tag{2.4}$$

where the isomorphism is given by θ on $R\{f\} \oplus R\{e\}$ and $R\{h\} \oplus R\{g\}$, and by the identity elsewhere. This is easily verified to commute with all differentials. Applying this isomorphism to all such modules of the bicomplex $\mathcal{LZ}_s(p, p+n)$ we obtain the bicomplex of Figure 2.2 which, up to signs, is of the form Equation (2.2). Writing N' for the R-modules $R\{f+e\}$ and $R\{h+g\}$ under this change of basis the new structure of the bicomplex $\mathcal{LZ}_s(p, p+n)$ is given in Figure 2.2. We have then shown:

Lemma 2.2.1.1. *There is an isomorphism of bicomplexes:*

$$\mathcal{LZ}_s(p, p+n) \cong \mathcal{ZW}_s(p, p+n) \oplus \bigoplus_{k \geq 0} \mathcal{ZW}_0(p-s-k, p-s-k+n) \ .$$

We consider now the unit of the $\mathcal{L} \dashv \mathrm{Tot}^\Pi$ adjunction on cycles $\mathcal{Z}_s(p, p+n) \to \mathrm{Tot}^\Pi \mathcal{L}\mathcal{Z}_s(p, p+n)$. We will show such a morphism is an s-equivalence. The unit is the adjoint morphism to the identity $\mathcal{L}\mathcal{Z}_s(p, p+n) \to \mathcal{L}\mathcal{Z}_s(p, p+n)$. Recall the adjoint to a morphism $f \colon \mathcal{L}(C) \to K$ is given in degree n by $\tilde{f}$ as follows:

$$\begin{tikzcd}
C^n \arrow[rr, dotted, "\tilde{f}^n"] \arrow[dr] & & \mathrm{Tot}^\Pi(K)^n = \prod_i K^{i,i+n} \\
& \prod_i \frac{C^n}{F_{i-1}C^n} \arrow[ur, "\left(f^{i,i+n}(-,0)\right)_i"'] &
\end{tikzcd} \tag{2.5}$$

where the map from C^n to $\prod_i C^n/F_{i-1}C^n$ is an infinite diagonal followed by a product of quotient maps. We then apply this construction with $C = \mathcal{Z}_s(p, p+n)$ and $K = \mathcal{L}\mathcal{Z}_s(p, p+n)$. We compute the images of the generator 1 of $R^n_{(p)}$ and 1 of $R^{n+1}_{(p-s)}$ in the filtered chain complex $\mathcal{Z}_s(p, p+n)$. From the description of Equation (2.5) the image of $1 \in R^n_{(p)}$ is the diagonal of all generators 1 of the $M \oplus 0$ on the n-diagonal of Figure 2.1, and the image of $1 \in R^{n+1}_{(p-s)}$ is the diagonal of the generators of the N in the $(n+1)$-diagonal summands $N \oplus M$.

From the second description of $\mathcal{L}\mathcal{Z}_s(p, p+n)$ given in Figure 2.2 it now follows that the unit of the adjunction $\mathcal{Z}_s(p, p+n) \to \mathrm{Tot}^\Pi \mathcal{L}\mathcal{Z}_s(p, p+n)$ is an s-weak equivalence: under the change of basis this map still maps the generator $1 \in R^n_{(p)}$ to the same diagonal of generators of M, however now the image of the generator $1 \in R^{n+1}_{(p-s)}$ is given by the diagonal on $(1, -1)$ in the modules of the form $N' \oplus M$. The former can be rewritten as the sum $(1, 1, 1, \ldots, 1, 0, 0, \ldots) + (0, \ldots, 0, 1, 1, \ldots)$ where the first element has s components being 1 and the rest 0, and the second s components being 0 and the rest 1. We denote these by z and b respectively. Both the infinite diagonal on 1 and z are elements of $Z_s^{p,p+n}\mathrm{Tot}^\Pi \mathcal{L}\mathcal{Z}_s(p, p+n)$ and differ by b which is an element of $B_s^{p,p+n}\mathrm{Tot}^\Pi \mathcal{L}\mathcal{Z}_s(p, p+n)$ and so represent the same element of the s-page. The 0-page through to the s-page for both $\mathcal{Z}_s(p, p+n)$ and $\mathrm{Tot}^\Pi \mathcal{L}\mathcal{Z}_s(p, p+n)$ consists of two copies of R, one in bidegree $(p, p+n)$ and the other in bidegree $(p-s-1, p-s+n)$ with an identity differential appearing on the s-page. Our description above shows that up to a boundary element the element $(1, 1, \ldots)$ has image z so that we have an isomorphism of s-pages. We have then shown the following result.

Proposition 2.2.1.2. *For $s \geq 1$ the unit of the adjunction applied to an s-cycle, $\mathcal{Z}_s(p, p+n) \to \mathrm{Tot}^\Pi \mathcal{L}\mathcal{Z}_s(p, p+n)$, is an isomorphism on the s-page.*

2.2.2 Tot^Π applied to $\mathcal{Z}\mathcal{W}_s$ and $\mathcal{B}\mathcal{W}_s$

We have less use for these results but sketch the answers for completeness. One can make a similar change of basis (compatible with the filtration) such that for $s \geq 1$ the filtered chain complex $\mathrm{Tot}^\Pi \mathcal{Z}\mathcal{W}_s$ becomes the direct sum of an s-cycle and $(s-1)$ 0-cycles. More precisely we have:

$$\mathrm{Tot}^\Pi \mathcal{Z}\mathcal{W}_s(p, p+n) \cong \mathcal{Z}_s(p, p+n) \oplus \bigoplus_{i=1}^{s-1} \mathcal{Z}_0(p-i, p-i+n)$$

where the direct sum is empty for $s = 1$. For $s = 0$ one also has

$$\mathrm{Tot}^\Pi \mathcal{Z}\mathcal{W}_0(p, p+n) \cong \mathcal{Z}_0(p, p+n) \oplus \mathcal{Z}_0(p-1, p+n) \ .$$

Note then that $\mathrm{Tot}^\Pi \mathcal{B}\mathcal{W}_s$ can be similarly expressed since the totalisation functor commutes with finite direct sums.

2.3 Right adjoint to coproduct totalisation

There is an analogous right adjoint to the coproduct totalisation functor $\mathrm{Tot}^\oplus$ of Definition 1.2.2.6 which we define here and whose proof that it is indeed a right adjoint we defer to Appendix B as we have no use for it in our work. It is defined similarly to Tot^Π instead using subobjects rather than quotient objects and with minor changes to signs.

Definition 2.3.0.1. The functor $\mathcal{R} \colon f\mathcal{C} \to b\mathcal{C}$ is defined on a filtered chain complex C by:

$$\mathcal{R}(C)^{i,i+n} := F_{i-1}C^{n+1} \oplus F_i C^n \ ,$$

where the differentials d_0 and d_1 are given on an $(x, y) \in \mathcal{R}(C)^{i,i+n}$ by:

$$\begin{aligned} d_0 &\colon (x, y) \mapsto (-dx, dy + x) \ , \\ d_1 &\colon (x, y) \mapsto \big(0, (-1)^{n+1}x\big) \ . \end{aligned}$$

$$\begin{tikzcd}
 & & & & & & & & 0 \oplus M & \boxed{M \oplus 0} \arrow[l, "(-1)^{n+1}"'] \\
 & & & & & & & 0 \oplus M & M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \arrow[u, "1"'] & \\
 & & & & & & 0 \oplus M & M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \arrow[u, "1"'] & & \\
 & & & & & \iddots & \iddots \qquad \iddots \arrow[u, "1"'] & & & \\
 & & & & 0 \oplus M & M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \arrow[u, "1"'] & & & & \\
 & & 0 \oplus N & \boxed{N \oplus M} \arrow[l, "(-1)^n \pi_1"'] & M \oplus 0 \arrow[l, "(-1)^{n+1} i_2"] \arrow[u, "1"'] & & & & & \\
 & 0 \oplus N & N \oplus M \arrow[l, "(-1)^n \pi_1"'] \arrow[u, "\nabla^-"] & M \oplus 0 \arrow[l, "(-1)^{n+1} i_2"] \arrow[u, "\Delta"'] & & & & & & \\
0 \oplus N & N \oplus M \arrow[l, "(-1)^n \pi_1"'] \arrow[u, "\nabla^-"] & M \oplus 0 \arrow[l, "(-1)^{n+1} i_2"] \arrow[u, "\Delta"'] & & & & & & & \\
\iddots \arrow[u, "\nabla^-"] & \iddots \arrow[u, "\Delta"'] & & & & & & & &
\end{tikzcd}$$

Figure 2.1: The bicomplex $\mathcal{L}\mathcal{Z}_s(p, p+n)$

$$\begin{tikzcd}
 & & & & & & & & 0 \oplus M & |[draw, dashed]| M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \\
 & & & & & & & 0 \oplus M & M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \arrow[u, "1"'] & \\
 & & & & & & 0 \oplus M & M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \arrow[u, "1"'] & & \\
 & & & & & \iddots \arrow[r, phantom, "\iddots"] & \iddots \arrow[u, "1"'] & & & \\
 & & & & 0 \oplus M & M \oplus 0 \arrow[l, "(-1)^{n+1}"'] \arrow[u, "1"'] & & & & \\
 & & 0 \oplus N & |[draw, dashed]| N' \oplus M \arrow[l, "(-1)^n \pi_1"'] & M \oplus 0 \arrow[l, "(-1)^{n+1} i_2"] \arrow[u, "1"'] & & & & & \\
 & 0 \oplus N & N' \oplus M \arrow[l, "(-1)^n \pi_1"'] \arrow[u, "\begin{pmatrix} 0 & -1 \end{pmatrix}"] & M \oplus 0 \arrow[l, "(-1)^{n+1} i_2"] \arrow[u, "i_1"'] & & & & & & \\
0 \oplus N & N' \oplus M \arrow[l, "(-1)^n \pi_1"'] \arrow[u, "\begin{pmatrix} 0 & -1 \end{pmatrix}"] & M \oplus 0 \arrow[l, "(-1)^{n+1} i_2"] \arrow[u, "i_1"'] & & & & & & & \\
\iddots \arrow[u, "\begin{pmatrix} 0 & -1 \end{pmatrix}"] & \iddots \arrow[u, "i_1"'] & & & & & & & &
\end{tikzcd}$$

Figure 2.2: Change of basis of the bicomplex $\mathcal{L}\mathcal{Z}_s(p, p+n)$

On a morphism of filtered chain complexes, $g\colon C \to D$, the functor $\mathcal{R}$ is given by $\mathcal{R}(g)^{i,i+n} := g^{n+1}_{|i-1} \oplus g^{n}_{|i}$ where $g^{n}_{|i}$ denotes the map $F_iC^n \to F_iD^n$ given by restricting g^n, noting that this map is defined as g preserves filtration and commutes with differentials since g does.

It is a straightforward check that the differentials define a bicomplex structure on the bigraded modules $\mathcal{R}(C)^{*,*}$ and that $\mathcal{R}(g)$ is indeed a morphism of bicomplexes.

Proposition 2.3.0.2. *There is an adjunction of categories* $\mathrm{Tot}^{\oplus}\colon b\mathcal{C} \overset{\perp}{\rightleftarrows} f\mathcal{C} :\mathcal{R}$.

Proof. The proof is Appendix B.

2.4 Total model structure

Muro and Roitzheim construct a total model structure on bicomplexes using their coproduct totalisation functor from bounded bicomplexes to bounded chain complexes whose weak equivalences are those morphisms such that $H\mathrm{Tot}^{\oplus}$ are isomorphisms, [MR19, Theorem 3.1]. They also generalise this to a total model structure on twisted chain complexes, [MR19, Theorem 5.13]. We briefly sketch a total model structure on unbounded chain complexes using the product totalisation functor and obtained via the Kan transfer theorem.

We have an adjoint pair from chain complexes to bicomplexes obtained by composing the adjunction of Proposition 2.1.0.2 with an adjunction between chain complexes and filtered chain complexes; we denote this adjunction $G\colon \mathcal{C}_R \overset{\perp}{\rightleftarrows} f\mathcal{C} :U$ where the functor U forgets filtration and the functor G equips a chain complex A with a filtration such that $F_pGA = 0$ for all $p \in \mathbb{Z}$.

Using the Kan transfer theorem and this composite adjunction we can then equip the category of bicomplexes with a model structure in which the weak equivalences (resp. fibrations) are those morphisms which are weak equivalences (resp. fibrations) after applying the composite functor. This will be cofibrantly generated by applying the Kan transfer theorem, Theorem 1.4.4.1.

Lemma 2.4.0.1. *The adjunction* $\mathcal{L}\colon \mathcal{C}_R \overset{\perp}{\rightleftarrows} b\mathcal{C} :\mathrm{Tot}^{\Pi}$ *satisfies the conditions of the Kan transfer theorem where we equip* $\mathcal{C}_R$ *with the projective model structure, i.e :*

1. $\mathcal{L}(I)$ *and* $\mathcal{L}(J)$ *admit the small object argument, and*
2. Tot^{Π} *takes relative* $\mathcal{L}(J)$*-cell complexes to weak equivalences.*

Proof. Condition 1 follows since every bicomplex is small. For Condition 2 a relative $\mathcal{L}(J)$-cell complex is a transfinite pushout of a bicomplex A by elements of $\mathcal{L}(J)$. The elements of $\mathcal{L}(J)$ consists of morphisms from the 0 bicomplex into an infinite direct sum of discs $\mathcal{ZW}_0$, so the transfinite pushout is of the form $A \longrightarrow A \oplus \bigoplus \mathcal{ZW}_0$ which is taken by Tot^{Π} to a weak equivalence.

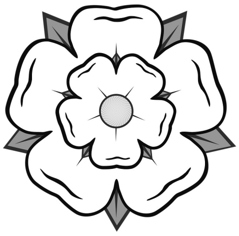
Definition 2.4.0.2. We denote by $\mathcal{ZW}_{\infty,-\infty}(n)$ the bicomplex given by the limit

$$\mathcal{ZW}_{\infty,-\infty}(n) := \lim_{p} \left(\mathcal{ZW}_{\infty}(p, p+n)\right) \ .$$

Applying the functor $\mathcal{L}$ to a sphere object S^n in $\mathcal{C}_R$ gives the bi-infinite staircase $\mathcal{ZW}_{\infty,-\infty}(n)$, and as noted above applied to a disc object D^n gives an infinite direct sum of bicomplex discs $\mathcal{ZW}_0$ which we'll denote $\mathcal{BW}_{\infty,-\infty}(n)$.

Corollary 2.4.0.3. *There is a total model structure on bicomplexes cofibrantly generated by generating cofibrations* $I := \{\mathcal{ZW}_{\infty,-\infty}(n) \to \mathcal{BW}_{\infty,-\infty}(n)\}$ *and generating acyclic cofibrations* $J := \{0 \to \mathcal{BW}_{\infty,-\infty}\}$ *in which:*

1. *weak equivalences are those morphisms* f *of bicomplexes such that* $H\mathrm{Tot}^{\Pi}$ *is an isomorphism,*
2. *fibrations are those morphisms* f *of bicomplexes such that* Tot^{Π} *is (homologically) degreewise surjective, i.e.* f *is bidegreewise surjective.*



CHAPTER 3

Poset of Model Structures

In this chapter we establish some generalisations of the model structures $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$ on filtered chain complexes of Theorems 1.6.0.5 and 1.6.0.6 and $(b\mathcal{C})_r$ and $(b\mathcal{C})_{r'}$ on bicomplexes of Theorems 1.7.0.10 and 1.7.0.11 and establish some properties these model structures have in addition to existence of Quillen equivalences between them, the effect of certain Bousfield localisations and derive existence of various bounded variants.

The new model structures are established in the following theorems whose proofs follow in much the same way as in [CELW19]. For a set S with $\max S = r$ we begin by defining generating sets I_S and J_S which are the usual I_r and J_r with extra the acyclic morphisms J_s added to both for each $s \in S$.

Theorem 3.1.0.2. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \dots, r\}$ including r, the category $f\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(f\mathcal{C})_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $Z_s(f)$ is bidegree-wise surjective for each $s \in S$, and*
3. *I_S and J_S are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Further $(f\mathcal{C})_S$ is a finitely generated model category.

I_S and J_S are similarly defined for bicomplexes, noting however that J_S must contain the morphisms of J_0.

Theorem 3.2.0.2. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \dots, r\}$ including both 0 and r, the category $b\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(b\mathcal{C})_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $ZW_s(f)$ is bidegree-wise surjective for each $s \in S$, and*
3. *I_S and J_S are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Further $(b\mathcal{C})_S$ is a finitely generated model category.

We show that these model structures are indeed distinct in Corollaries 3.3.0.4 and 3.3.0.9 by constructing morphisms with specific cycle surjectivity properties.

Propositions 3.1.0.6 and 3.5.0.2 establish some Quillen equivalences between all of the S-model categories on $f\mathcal{C}$, Proposition 3.2.0.6 establishes Quillen equivalences between some of the S-model structures on $b\mathcal{C}$, and Proposition 3.4.0.2 establishes a Quillen adjunction between the S-model structures on $f\mathcal{C}$ and $b\mathcal{C}$ when $\{0, r\} \subseteq S$.

We then consider the poset denoted $\mathcal{N}$ whose elements are the S-model structures on $f\mathcal{C}$ and with the $\leq$ relation given by existence of a left adjoint constructed by identity maps and the shift functor. We show this poset has a distributive lattice structure whose join and meet operations are given by the *initial model structure amongst the $(f\mathcal{C})_S$ admitting left adjoints of these forms* and the *terminal model structure amongst the $(f\mathcal{C})_S$ admitting left adjoints of these forms.*

Corollary 3.6.0.16. *The lattice structure on $\mathcal{N}$ is a distributive lattice.*

We then turn to demonstrating properties of an individual $(f\mathcal{C})_S$ or $(b\mathcal{C})_S$. These model structures are right proper automatically since all objects are fibrant. For left properness we make use of a proof technique of a theorem of Lack, [Lac02, Theorem 6.3], which on finitely cofibrantly generated model categories. We study s-cycles in the model structures and prove the following theorems.

Theorem 3.7.1.7. *The model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 are left proper.*

Theorem 3.7.2.8. *The model categories $(b\mathcal{C})_S$ of Theorem 3.2.0.2 are left proper.*

We also demonstrate that they are cellular model categories:

Proposition 3.8.1.5. *The model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 are cellular.*

Proposition 3.8.2.2. *The model categories $(b\mathcal{C})_S$ of Theorem 3.2.0.2 are cellular.*

And by computing pullbacks of fibrations from an acyclic object (the r-loops on the r-cone) to an A along the 0 morphism $0 \to A$ give a description of the loop and suspension functors and that the model categories are stable.

Proposition 3.9.1.2. *The model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 are stable model categories whose loops and suspension functors are given by Ω^r and Σ^r.*

Proposition 3.9.2.4. *The model categories $(b\mathcal{C})_S$ of Theorem 3.2.0.2 are stable model categories whose loops and suspension functors are given by Ω^r and Σ^r.*

Using the adjunction $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ of Proposition 2.1.0.2 and the cellularization principle of Greenlees and Shipley, Theorem 3.10.0.3, we show that the adjunction is in fact a Quillen equivalence.

Theorem 3.10.0.4. *For S containing both 0 and r there is a Quillen equivalence between the S-model structure on filtered chain complexes and the S-model structure on bicomplexes given by the $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ adjunction:*

$$\begin{tikzcd} \mathcal{L}\colon (f\mathcal{C})_S \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (b\mathcal{C})_S \arrow[l, shift left] \colon \mathrm{Tot}^{\Pi} \end{tikzcd} .$$

We show that one cannot left Bousfield localise from an S-model structure on either $f\mathcal{C}$ or $b\mathcal{C}$ with weak equivalences the r-weak equivalences and obtain a model category with weak equivalences the $(r+1)$-weak equivalences.

Proposition 3.11.0.1. *Let $\mathcal{M}_S$ be one of the model structures of either poset, where $\mathcal{M}$ is either $f\mathcal{C}$ or $b\mathcal{C}$ whose weak equivalences are the r-weak equivalences. Then there is no left Bousfield localisation $\mathcal{M}_{new}$ of $\mathcal{M}_S$ whose weak equivalences are the $(r+1)$-weak equivalences.*

Lastly we construct some bounded model structures on $f\mathcal{C}$. The categories $f\mathcal{C}^{\geq}$ and $f\mathcal{C}^{\leq}$ denote the categories of filtered non-negatively graded chain complexes and filtered non-positively graded chain complexes respectively.

Corollary 3.12.1.2. *There is a cofibrantly generated model structure denoted $\left(f\mathcal{C}^{\geq}\right)_S$ on $f\mathcal{C}^{\geq}$ whose weak equivalences are the r-quasi isomorphisms and with generating cofibrations τI_S and generating acyclic cofibrations τJ_S.*

Theorem 3.12.2.1. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r, the category $f\mathcal{C}^{\leq}$ admits a right proper cofibrantly generated model structure, which we denote $\left(f\mathcal{C}^{\leq}\right)_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $Z_s^{*,*+n}(f)$ is bidegree-wise surjective for $n \leq -1$ and $s \in S$, and*
3. *κI_S and κJ_S are the sets of generating cofibrations and generating acyclic cofibrations respectively.*

Furthermore $\left(f\mathcal{C}^{\leq}\right)_S$ is a finitely generated model category.

Lastly we define S-model structures on the category of non-negatively filtered chain complexes $f_{\geq}\mathcal{C}$.

Theorem 3.12.3.15. *For every subset $S \subseteq \{0, 1, 2, \ldots, r\}$ containing r the category $f_{\geq}\mathcal{C}$ admits a right proper cofibrantly generated model structures, which we denote $(f_{\geq}\mathcal{C})_S$, whose:*

1. *weak equivalences are the E_r-quasi-isomorphisms,*
2. *fibrations are morphisms that for all $s \in S$ are $Z_s^{p,p+n}$-surjective for $p \geq s$ and all n, and*
3. *generating cofibrations and generating acyclic cofibrations are given by $I_S^{\geq}$ and $J_S^{\geq}$ respectively.*

Furthermore $(f_{\geq}\mathcal{C})_S$ is a finitely generated model category.

Due to the difference in representing objects used in generating cofibrations and acyclic cofibrations between the 0-model structure and the r-model structure ($r \geq 1$) on bicomplexes, Theorem 1.7.0.10, many of the proofs of this chapter do not work as written for the 0-model structure. However due to Remark 1.7.0.13 many of the results of this chapter regarding the 0-model structure are already known from [CH02], in particular stability [CH02, Lemma 2.16], properness [CH02, Proposition 2.18] and (finite) cofibrant generation [CH02, Theorem 5.7]. Henceforth for bicomplexes whilst we write our results *for all* $r \geq 0$ our proof may only apply for $r \geq 1$ and we rely on the above for the case $r = 0$.

3.1 Construction of new model structures on filtered chain complexes

The original model structures of [CELW19] on either bicomplexes or filtered chain complexes for a fixed r can be seen to differ by the descriptions of their fibrations. For convenience we use the descriptions for filtered chain complexes in the following discussion. In the first model structure we only require surjections on the r-cycles and in the second we require surjections on all s-cycles for $0 \leq s \leq r$. There should be *intermediate model structures* with r-weak equivalences and where fibrations are characterised as being surjective on all s-cycles for a fixed subset $S \subseteq \{0, 1, \ldots, r\}$ containing r and indeed this section proves existence of such model structures. Originally the author attempted to show existence of these model structures by *mixing* the model structures $(f\mathcal{C})_s$ for each $s \in S$. There are various results in the literature on mixing model structures.

Cole in [Col06, Theorem 2.1] shows that for two model structures on $\mathcal{C}$ with weak equivalences $\mathcal{W}_1$ and $\mathcal{W}_2$, and fibrations Fib_1 and Fib_2 respectively such that $\mathcal{W}_1 \subseteq \mathcal{W}_2$ and $\mathrm{Fib}_1 \subseteq \mathrm{Fib}_2$ then there is a *mixed model structure* on $\mathcal{C}$ with weak equivalences $\mathcal{W}_2$ and fibrations Fib_1. There is a dual result replacing the inclusion of fibrations by an inclusion of cofibrations. The r-model structures of [CELW19] do not satisfy the fibration or cofibration inclusions along with the weak equivalence inclusion so we cannot apply this result here. One could also attempt to use [HHR21, Proposition 5.2.34] however their conditions are not satisfied in our setting either.

The author in fact shows existence of such intermediate model structures by explicitly giving the generating cofirbations and generating cofibrations from which the model structures follows easily from results already established in [CELW19]. These new model structures on $f\mathcal{C}$ and $b\mathcal{C}$ are in fact examples of *intermediate model structures* of [Bal21, Proposition 4.9.4], although for this proposition one still needs to exhibit a *weak factorisation system* (see [Bal21, Definition 2.1.12]) to apply the result which we obtain from the small object argument.

Definition 3.1.0.1. We fix an $r \geq 0$, and denote by S a subset of $\{0, 1, \ldots, r\}$ which must include r. We define I_S and J_S as follows:

$$I_S \coloneqq I_r \cup \bigcup_{s \in S \setminus \{r\}} J_s \ ,$$

$$J_S \coloneqq \bigcup_{s \in S} J_s \ .$$

Theorem 3.1.0.2. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r, the category $f\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(f\mathcal{C})_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f \colon A \to B$ such that $Z_s(f)$ is bidegree-wise surjective for each $s \in S$, and*
3. *I_S and J_S are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Further $(f\mathcal{C})_S$ is a finitely generated model category.

The cases $S = \{r\}$ and $S = \{0, 1, \ldots, r\}$ are the model structures constructed in [CELW19]. The proof proceeds by the same method as in [CELW19], i.e. we verify conditions 3,4 and 5 of Theorem 1.4.2.9.

Proposition 3.1.0.3. *We have $I_S\text{-}\mathtt{Inj} = \mathcal{E}_r \cap J_S\text{-}\mathtt{Inj}$.*

Proof. The third equality in the following is the result $I_r\text{-}\mathtt{Inj} = \mathcal{E}_r \cap J_r\text{-}\mathtt{Inj}$ for the model structure of Theorem 1.6.0.5.

$$\begin{aligned} I_S\text{-}\mathtt{Inj} &= \left(I_r \cup \bigcup_{s \in S\setminus\{r\}} J_s \right)\text{-}\mathtt{Inj} \\ &= I_r\text{-}\mathtt{Inj} \cap \bigcap_{s\in S\setminus\{r\}} J_s\text{-}\mathtt{Inj} \\ &= (\mathcal{E}_r \cap J_r\text{-}\mathtt{Inj}) \cap \bigcap_{s\in S\setminus\{r\}} J_s\text{-}\mathtt{Inj} \\ &= \mathcal{E}_r \cap J_S\text{-}\mathtt{Inj}\ . \end{aligned}$$

The proof of the following is also identical to its counterpart in [CELW19]. Recall the r-loops functor Ω^r of Definition 1.2.1.18 and r-cone functor C_r of Definition 1.6.0.12.

Proposition 3.1.0.4. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r we have $J_S\text{-}\mathtt{Cof} \subseteq \mathcal{E}_r$.*

Proof. Let $f\colon A \to B$ be a J_S-$\mathtt{Cof}$ so that it has the left lifting property with respect to those morphisms g that are $Z_s(g)$ bidegree-wise surjective for all $s \in S$. Consider then the lifting problem:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A \oplus \Omega^r C_r(B) \arrow[d, "{(f,\pi_1)}"] \\ B \arrow[r, "id"'] \arrow[ur, dashed] & B \end{tikzcd} .$$

The morphism (f, π_1) is a Z_s-surjection for all $s \in S$, by Lemma 1.6.0.14 so there exists a lift. Since $\Omega^r C_r(B)$ is r-acyclic, again by Lemma 1.6.0.14, applying E_{r+1} to the diagram gives $f \in \mathcal{E}_r$.

Proof of Theorem 3.1.0.2. It remains to prove conditions 3, 4 and 5 of Theorem 1.4.2.9 hold. Conditions 4 and 5 are Proposition 3.1.0.3. The inclusion $J\text{-}\mathtt{Cell} \subseteq \mathcal{W}$ of condition 3 follows from Proposition 3.1.0.4 and the inclusion $J\text{-}\mathtt{Cell} \subseteq I\text{-}\mathtt{Cof}$ as follows: by Proposition 3.1.0.3 we have $I_S\text{-}\mathtt{Inj} \subseteq J_S\text{-}\mathtt{Inj}$, hence $J_S\text{-}\mathtt{Cof} \subseteq I_S\text{-}\mathtt{Cof}$. Right properness follows since every object is fibrant. Lastly it is finitely generated since the domains and codomains of I_S and J_S are finite relative to the cofibrations (in fact relative to the entire category) by Lemma 1.2.1.16.

Theorem 3.1.0.2 then gives for a fixed r, 2^r cofibrantly generated model structures indexed by the powerset of $\{0, 1, \ldots, r-1\}$ or alternatively as described above by those subsets of $\{0, 1, \ldots, r\}$ including r.

Notation 3.1.0.5. For $S \subseteq \{0, 1, \ldots, r\}$ containing r we write $(f\mathcal{C})_S$ for the model structure given by Theorem 3.1.0.2. The special cases of $S = \{r\}$ and $S = \{0, 1, \ldots, r\}$ will be denoted by $(f\mathcal{C})_r$ and $(f\mathcal{C})_{r'}$ respectively in agreement with [CELW19]. We also refer to the cofibrations of the S-model structure as S-cofibrations and similarly refer to the fibrations as the S-fibrations.

Proposition 3.1.0.6. *For a fixed r and subsets $S' \subseteq S \subseteq \{0, 1, \ldots, r\}$ both containing r there is a Quillen equivalence:*

$$id\colon (f\mathcal{C})_{S'} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} (f\mathcal{C})_S :id\ \ .$$

Proof. We check the right adjoint sends (acyclic) fibrations to (acyclic) fibrations. A fibration on the right hands side is a morphism f with $Z_s(f)$ bidegree-wise surjective for all $s \in S$. This also then satisfies $Z_s(g)$ bidegree-wise surjectivity for all $s \in S'$ since $S' \subseteq S$, hence the right adjoint preserves fibrations. Preserving acyclic fibrations follows since weak equivalences are the same on both sides and the right adjoint is an identity functor. This gives the identity-identity adjunction is a Quillen adjunction. Quillen equivalence follows since the functors are the identity and weak equivalences are the same on both sides.

3.2 Construction of new model structures on bicomplexes

Definition 3.2.0.1. We fix an $r \geq 0$, and denote by S a subset of $\{0, 1, \ldots, r\}$ which must include 0 and r. We define I_S and J_S as follows:

$$I_S \coloneqq I_r \cup \bigcup_{s \in S \setminus \{0,r\}} J_s \ ,$$
$$J_S \coloneqq \bigcup_{s \in S} J_s \ .$$

Theorem 3.2.0.2. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including both 0 and r, the category $b\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(b\mathcal{C})_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f \colon A \to B$ such that $ZW_s(f)$ is bidegree-wise surjective for each $s \in S$, and*
3. *I_S and J_S are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Further $(b\mathcal{C})_S$ is a finitely generated model category.

The proof proceeds by the same method as in [CELW19], i.e. we verify conditions 3, 4 and 5 of Theorem 1.4.2.9.

Proposition 3.2.0.3. *We have I_S-$\textit{Inj} = \mathcal{E}_r \cap J_S$-$\textit{Inj}$.*

Proof. The third equality in the following is the result $I_r\text{-}\texttt{Inj} = \mathcal{E}_r \cap J_0 \inf \cap J_r\text{-}\texttt{Inj}$ for the model structure of Theorem 1.7.0.10.

$$\begin{aligned} I_S\text{-}\texttt{Inj} &= \left(I_r \cup \bigcup_{s \in S \setminus \{0,r\}} J_s \right)\text{-}\texttt{Inj} \\ &= I_r\text{-}\texttt{Inj} \cap \bigcap_{s \in S \setminus \{0,r\}} J_s\text{-}\texttt{Inj} \\ &= (\mathcal{E}_r \cap J_0\text{-}\texttt{Inj} \cap J_r\text{-}\texttt{Inj}) \cap \bigcap_{s \in S \setminus \{0,r\}} J_s\text{-}\texttt{Inj} \\ &= \mathcal{E}_r \cap J_S\text{-}\texttt{Inj} \ . \end{aligned}$$

∎

Recall the r-loops functor Ω^r of Definition 1.2.2.8 and r-cone functor C_r of Definition 1.7.0.14.

Proposition 3.2.0.4. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including 0 and r we have J_S-$\textit{Cof} \subseteq \mathcal{E}_r$.*

Proof. Let $f \colon A \to B$ be a J_S-$\texttt{Cof}$ so that it has the left lifting property with respect to those morphisms g that are $ZW_s(g)$ bidegree-wise surjective for all $s \in S$. Consider then the lifting problem:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A \oplus \Omega^r C_r(B) \arrow[d, "{(f,\psi_r)}"] \\ B \arrow[r, "id"'] \arrow[ur, dashed] & B \end{tikzcd} .$$

The morphism ψ_r is Ω^r applied to the morphism of Lemma 1.7.0.15 so that (f, ψ_r) is a ZW_s-bidegree-wise surjection for all $s \in S$ so there exists a lift. Since $\Omega^r C_r(B)$ is r-acyclic by Lemma 1.7.0.15, applying E_{r+1} to the diagram gives $f \in \mathcal{E}_r$. ∎

Proof of Theorem 3.2.0.2. This is the same as for filtered chains noting that finite objects in $(b\mathcal{C})_S$ are the bounded bicomplexes which in each bidegree are finitely presented R-modules, Lemma 1.7.0.16. ∎

Notation 3.2.0.5. For $S \subseteq \{0, 1, \ldots, r\}$ containing both 0 and r we write $(b\mathcal{C})_S$ for the model structure given by Theorem 3.2.0.2. The special cases of $S = \{0, r\}$ and $S = \{0, 1, \ldots, r\}$ will be denoted by $(b\mathcal{C})_r$ and $(b\mathcal{C})_{r'}$ respectively in agreement with [CELW19]. Again we call the cofibrations and fibrations of the S-model structure the S-cofibrations and S-fibrations.

Proposition 3.2.0.6. *For a fixed r and subsets $S' \subseteq S \subseteq \{0, 1, \ldots, r\}$ containing 0 and r there is a Quillen equivalence*

$$id \colon (b\mathcal{C})_{S'} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} (b\mathcal{C})_S : id \quad .$$

3.3 Distinctness of model structures

We verify that all these model structures defined in Theorems 3.1.0.2 and 3.2.0.2 are indeed distinct model structures. We do so by showing that the fibrations are different for which we show existence of morphisms satisfying specific Z_k or ZW_k surjectivity conditions.

Definition 3.3.0.1. The morphisms $\alpha_s^{p,p+n}$ and $\beta_s^{p,p+n}$ of filtered chain complexes are given by:

- $\alpha_s^{p,p+n} \colon Z_{s+1}(p+1, p+1+n) \longrightarrow Z_s(p, p+n)$ whose underlying maps of R-modules are the identity wherever possible:

$$\begin{tikzcd} \big(R^n_{p+1} \arrow[r] \arrow[d] & R^{n+1}_{p-s}\big) \arrow[d] \\ \big(R^n_p \arrow[r] & R^{n+1}_{p-s}\big) \end{tikzcd} .$$

- $\beta_s^{p,p+n} \colon Z_{s-1}(p, p+n) \oplus R^{n+1}_{(p-s)} \longrightarrow Z_s(p, p+n)$ whose underlying maps of R-modules are the identity or fold maps wherever possible:

$$\begin{tikzcd} \Big(R^n_p \arrow[r] \arrow[d] & R^{n+1}_{p-s+1} \oplus R^{n+1}_{(p-s)}\Big) \arrow[d, "\nabla"] \\ \big(R^n_p \arrow[r] & R^{n+1}_{p-s}\big) \end{tikzcd} .$$

Lemma 3.3.0.2. *We have the following surjectivity results:*

- *the morphisms $\alpha_s^{*,*}$ are Z_k-surjective for all $k \geq s+1$ and not Z_k-surjective otherwise,*
- *the morphisms $\beta_s^{*,*}$ are Z_k-surjective for all $k \leq s-1$ and not Z_k-surjective otherwise.*

Remark 3.3.0.3. We can easily form a morphism of filtered chain complexes that is Z_k-surjective for all $k \neq s$ by taking the composition of the direct sum of the α and β with the fold map:

$$\gamma_s^{p,p+n} := \nabla \circ \left(\alpha_s^{p,p+n} \oplus \beta_s^{p,p+n}\right) \ .$$

Existence of such morphisms then immediately proves the following distinctness result by the classification of the fibrations.

Corollary 3.3.0.4. *The model structures of Theorem 3.1.0.2 are all distinct.*

We construct similar morphisms δ, ϵ and ζ for bicomplexes to show distinctness for the bicomplex model structures. Firstly we define a corner bicomplex.

Definition 3.3.0.5. The bicomplex $CW(p, p+n)$ has a copy of R in bidegrees $(p, p+n)$, $(p-1, p-1+n+1)$ and $(p, p+n+1)$ whose differentials are the identity morphism whenever possible.

In Figures 3.1 and 3.2 of the following definition all $\bullet$ denote a copy of R with a bidegree indicated and all differentials are identities except the ones labelled i_1 and i_2 which are inclusions into either the first or second copy R of the R-module $R \oplus R$ denoted by $\bullet\bullet$.

Definition 3.3.0.6. The morphisms $\delta_s^{p,p+n}$ and $\epsilon_s^{p,p+n}$ of bicomplexes are given by:

- $\delta_s^{p,p+n} \colon ZW_{s+1}(p+1, p+1+n) \longrightarrow ZW_s(p, p+n)$ whose underlying maps of R-modules are the identity whenever possible and is depicted in Figure 3.1.

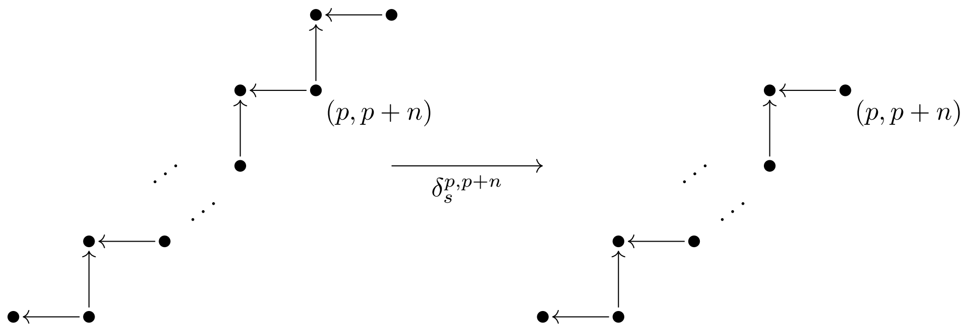


Figure 3.1: The morphism $\delta_s^{p,p+n}$

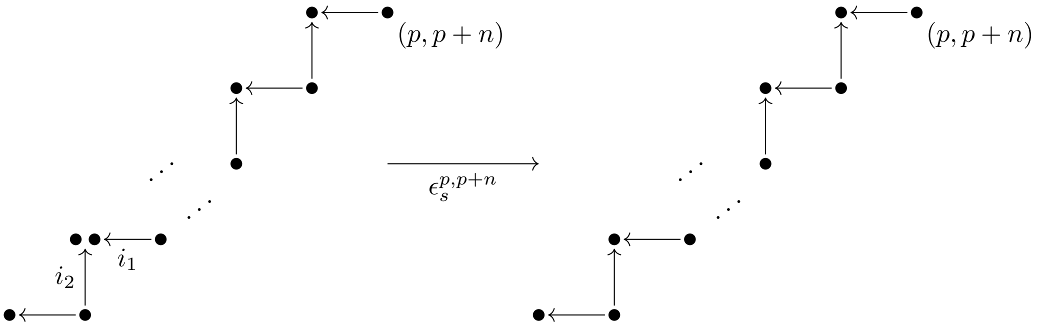


Figure 3.2: The morphism $\epsilon_s^{p,p+n}$

- $\epsilon_s^{p,p+n}\colon ZW_{s-1}(p, p+n) \oplus CW(p-s+1, p-s+1+n) \longrightarrow ZW_s(p, p+n)$ whose underlying maps of R-modules are the identity or fold maps whenever possible and is depicted in Figure 3.2.

Lemma 3.3.0.7. *We have the following surjectivity results:*

- *the morphisms $\delta_s^{*,*}$ are ZW_k-surjective for all $k \geq s+1$ or $k = 0$ and not ZW_k-surjective otherwise,*
- *the morphisms $\epsilon_s^{*,*}$ are ZW_k-surjective for all $k \leq s-1$ and not ZW_k surjective otherwise.*

Remark 3.3.0.8. We can also form a morphism of bicomplexes that is ZW_k-surjective for all $k \neq s$ by taking the composition of the direct sum of the δ and ϵ with the fold map:

$$\zeta_s^{p,p+n} := \nabla \circ \left(\delta_s^{p,p+n} \oplus \epsilon_s^{p,p+n}\right) .$$

Existence of these morphisms proves the following distinctness result.

Corollary 3.3.0.9. *The model structures of Theorem 3.2.0.2 are all distinct.*

3.4 Quillen adjunctions between $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$

We show that for S a subset of $\{0, 1, \ldots, r\}$ containing 0 and r there is a Quillen adjunction

$$\mathcal{L}\colon (f\mathcal{C})_S \underset{\perp}{\rightleftarrows} (b\mathcal{C})_S \colon \mathrm{Tot}^{\Pi} \quad ,$$

by showing that Tot^{Π} is a right adjoint. By definition of the r-weak equivalences of $(b\mathcal{C})_S$ the functor Tot^{Π} already preserves weak equivalences. We just need to check then that Tot^{Π} sends fibrations to fibrations.

Lemma 3.4.0.1. *Let $f\colon Y \to X$ be a morphism of bicomplexes that is both ZW_0-surjective and ZW_s-surjective, then $\mathrm{Tot}^{\Pi}(f)$ is Z_s-surjective.*

Proof. Let $(x_i)_{i\leq p} \in F_p\mathrm{Tot}^\Pi X^n$ where $x_i \in X^{i,i+n}$ and such that $d^{\mathrm{Tot}^\Pi X}(x_i)_{i\leq p} \in F_{p-s}X^n$. Recall the differential $d^{\mathrm{Tot}^\Pi X}\colon (k_i)_i \mapsto (d_0k_i + (-1)^n d_1 k_{i+1})_i$ so that $d_0x_p = 0$ and $d_0x_{p-i-1} = -(-1)^n d_1 x_{p-i}$ for $0 \leq i \leq s-2$. The sequence $\big(x_p, (-1)^{n+1}x_{p-1}, x_{p-2}, (-1)^{n+1}x_{p-3}, \ldots, \pm x_{p-s+1}\big)$ is then an element of $ZW_s^{p,p+n}X$ and since f is ZW_s-surjective we can find a lift $(y_p, \ldots, y_{p-s+1})$. By ZW_0-surjectivity of f we can also find lifts y_{p-i} of x_{p-i} for $i \geq s+2$. The element $\big(y_p, (-1)^{n+1}y_{p-1}, y_{p-2}, (-1)^{n+1}y_{p-3}, \ldots, \pm y_{p-s+1}, y_{p-s}, y_{p-s-1}, \ldots\big)$ is then a Z_s-lift as required.

The lemma shows that Tot^Π sends S-fibrations of $(b\mathcal{C})_S$ to S-fibrations of $(f\mathcal{C})_S$. Hence we have shown the following.

Proposition 3.4.0.2. *For S a subset of $\{0, 1, \ldots, r\}$ containing both 0 and r there is a Quillen adjunction*

$$\begin{tikzcd} \mathcal{L}\colon (f\mathcal{C})_S \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (b\mathcal{C})_S \colon \mathrm{Tot}^\Pi \arrow[l, shift left] \end{tikzcd} .$$

In fact the model structures on $(b\mathcal{C})_S$ is a right transferred model structure along this adjunction since its weak equivalences and fibrations are determined by the functor Tot^Π.

3.5 Quillen equivalences between the $(f\mathcal{C})_S$

For $S' \subseteq S$ with $\max S' = \max S = r$ there were the identity-identity Quillen equivalences of Propositions 3.1.0.6 and 3.2.0.6 for filtered chains and bicomplexes respectively. For the category $f\mathcal{C}$ there are in fact more Quillen equivalences. Recall the shift-décalage adjunction of Lemma 1.3.3.2 which gave Quillen equivalences of the form

$$\begin{tikzcd} S^k\colon (f\mathcal{C})_r \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (f\mathcal{C})_{r+k} \colon \mathrm{Dec}^k \arrow[l, shift left] \end{tikzcd} .$$

Notation 3.5.0.1. For a set S and $l \in \mathbb{N}$ we denote by $S + l$ the set $\{s + l \mid s \in S\}$.

Proposition 3.5.0.2. *There are Quillen equivalences given by the shift-décalage adjunction:*

$$\begin{tikzcd} S\colon (f\mathcal{C})_S \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (f\mathcal{C})_{S+1} \colon \mathrm{Dec} \arrow[l, shift left] \end{tikzcd} .$$

Proof. This follows from the proof in the case $S^k\colon\ (f\mathcal{C})_r \rightleftarrows (f\mathcal{C})_{r+k}\ \colon \mathrm{Dec}^k$. See [CELW19, Theorem 3.22].

3.6 Quillen zig-zags and a distributive lattice

Remark 3.6.0.1. The shift-décalage and identity-identity adjunctions do not give that all model structures in the posets are Quillen equivalent, only that there are zig-zags of Quillen equivalences between any two. Consider the following diagram

$$\begin{tikzcd} (f\mathcal{C})_{\{3\}} \arrow[d, shift left, "\mathrm{Dec}"] & \\ (f\mathcal{C})_{\{2\}} \arrow[u, shift left, "S"] \arrow[r, shift left, "id"] & (f\mathcal{C})_{\{1,2\}} \arrow[l, shift left, "id"] \end{tikzcd} .$$

There is a Quillen equivalence between $(f\mathcal{C})_{\{2\}}$ and the other two model categories but not between $(f\mathcal{C})_{\{3\}}$ and $(f\mathcal{C})_{\{1,2\}}$ (at least if we only use the shift-décalage and identity-identity adjunctions).

We now assemble for all r the posets of model structures of filtered chain complexes together into one larger poset. The underlying set of the poset is then finite non-empty subsets of the power set of $\mathbb{N}_0$ and we generate the partial order $<$ as follows: for two model structures indexed by T and S we have $T < S$ if either

1. $T \subset S$, with $\max T = \max S$,
2. $S = T + 1$.

These generating inequalities of the poset are respectively the left adjoints of the identity-identity adjunctions and the shift-décalage adjunctions.

Definition 3.6.0.2. Denote by $\mathcal{N}$ the poset with the above definition for $<$.

$$\begin{tikzcd}
 & & & & & \{0,1,2,3\} & & \\
 & & \{0,1,2\} \arrow[rr] & & \{1,2,3\} \arrow[ur] & & \{0,2,3\} \arrow[ul] & \\
 & & & & & \{0,1,3\} \arrow[uu] & & \\
 & \{0,1\} \arrow[rr] & & \{1,2\} \arrow[rr] \arrow[uul] & & \{2,3\} \arrow[uul] \arrow[uur] & & \cdots \\
 & & \{0,2\} \arrow[rr] \arrow[uuu] & & \{1,3\} \arrow[uuu] \arrow[uur] & & \{0,3\} \arrow[uul] \arrow[uuu] & \\
\{0\} \arrow[r] & \{1\} \arrow[rr] \arrow[uu] & & \{2\} \arrow[rr] \arrow[ul] \arrow[uu] & & \{3\} \arrow[ul] \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$

Figure 3.3: The poset $\mathcal{N}$

For those elements of $\mathcal{N}$ whose maxima are 3 or less the poset is displayed in Figure 3.3 with an arrow $a \to b$ denoting the relation $a \leq b$.

Given that not all these model structures are Quillen equivalent via composites of these two adjunctions it is worth finding for any two model structures indexed by T and S the 'terminal model structure admitting a left Quillen functor to these' and the 'initial model structure admitting a left Quillen functor from these'. We will show such operations give the join and meet operations respectively for a distributive lattice structure on $\mathcal{N}$. We recall now the definition of a distributive lattice. It can be found for instance as [DP02, Definition 4.4].

Definition 3.6.0.3. A *lattice* $(\Lambda, \vee, \wedge)$ is a partially ordered set Λ with binary operations $\vee$, called join, and $\wedge$, called meet, on its elements such that

1. $a \leq a \vee b = b \vee a$, and
2. $a \wedge b = b \wedge a \leq a$.

It is further a distributive lattice if for all $a, b, c \in \Lambda$ we have:

1. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, and
2. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

We will abuse notation and denote a poset and distributive lattice over that poset by the same symbol. The meet and join operations we will define on $\mathcal{N}$ will form the structure of a distributive lattice which will not be easy to write down a proof of, so we instead prove this indirectly by showing such structure is isomorphic to another distributive lattice by virtue of Birkhoff's Representation Theorem 3.6.0.6.

Definition 3.6.0.4. Given a distributive lattice Λ an element $a \in \Lambda$ is said to be *join-irreducible* if it is neither:

1. the least element of the lattice, nor
2. the join of two smaller elements.

Definition 3.6.0.5. A *lower set* L of a lattice Λ is a subset $L \subset \Lambda$ such that if $a \leq l$ for all $l \in L$ then $a \in L$ too.

Theorem 3.6.0.6 (Birkhoff's Representation Theorem [DP02, Theorem 5.12])**.** *Any finite distributive lattice Λ is isomorphic to the distributive lattice on the set of lower sets of the partial order on the join-irreducible elements with meet and join operations usual set theoretic intersection and union.* ✪

The correspondence between elements of the two lattices of Theorem 3.6.0.6 is given by sending an element $\lambda \in \Lambda$ to the set of join-irreducible elements of Λ less than or equal to λ and by sending a lower set L to the join of all elements of L (note this is a finite join since Λ is finite). Our proof of a distributive lattice structure on $\mathcal{N}$ will procede as follows:

1. define the meet and join operations on $\mathcal{N}$,

2. find the join-irreducible elements of $\mathcal{N}$ (we do not need a distributive lattice structure to define these elements),
3. restricting to those elements of $\mathcal{N}$ with largest element less than or equal to some n show that our meet and join elements we defined on $\mathcal{N}$ correspond to the union and intersection operations on the set of lower sets of the partial order of the join-irreducible elements.

Definition 3.6.0.7. Given two elements S and T of $\mathcal{N}$ we define the join and meet operations as follows:

- $S \vee T = (S + \max\{S \cup T\} - \max S) \cup (T + \max\{S \cup T\} - \max T)$, and
- $S \wedge T = (S - \max\{S \cup T\} + \max T) \cap (T - \max\{S \cup T\} + \max T)$.

The intuition behind these join and meet operations with respect to two indexing sets S and T for the model structures is as follows: for join, we apply repeated shift functors to the set with the smaller maximum so that the maxima of the two sets agree and then we take the union, for meet we apply décalage repeatedly to the set with the larger maximum so the maxima agree and then take intersections. The unions and intersections here are encoding repeated applications of either the left or right Quillen identity functors. An alternate way of viewing this is the meet operation gives the 'largest' or 'terminal' model structure in the poset admitting left adjoints (of the form composites of identity-identity and shift-décalage) to S and T, and the join operation the 'smallest' or 'initial' model structure admitting left adjoint from S and T to it.

Lemma 3.6.0.8. *The join-irreducible elements of $\mathcal{N}$ equipped with the join operation of Definition 3.6.0.7 are those of the form $\{n\}$ or $\{0, n\}$ where $n \geq 1$.*

Proof. Note $\{0\}$ is not a join-irreducible element since it is the least element of the poset $\mathcal{N}$. Suppose U is not of the form of $\{n\}$ or $\{0, n\}$ with $n \geq 0$ so that there is a second greatest element of U which is non-zero. Say $\{m, n\} \subset U$ with $n = \max U$ and $m = \max(U \setminus \{n\})$. Then we have that $U = \{m-1, n-1\} \vee (U \setminus \{m\})$. Note that we have both $\{m-1, n-1\} < U$ and $U \setminus \{m\} < U$ and that $m - 1 \geq 0$ so that U cannot be join-irreducible.

Now consider an element of the form $\{n\}$ with $n \geq 1$, it is clearly join-irreducible since the join is defined as the union of two sets and so for $S \vee T = \{n\}$ we must have either (or both) $(S + \max\{S \cup T\} - \max S) = \{n\}$ and $(T + \max\{S \cup T\} - \max T) = \{n\}$ so that at least one of S or T is $\{n\}$. But this does not exhibit $\{n\}$ as the join of two smaller elements, hence $\{n\}$ is join-irreducible.

Lastly consider an element of the form $\{0, n\}$ with $n \geq 1$. Again either we have one of the two sets $(S + \max\{S \cup T\} - \max S) = \{n\}$ and $(T + \max\{S \cup T\} - \max T) = \{n\}$ is $\{0, n\}$ which doesn't exhibit $\{0, n\}$ as the join of two smaller sets, or we have one of $(S + \max\{S \cup T\} - \max S) = \{n\}$ and $(T + \max\{S \cup T\} - \max T) = \{n\}$ is $\{0\}$ and the other $\{n\}$. But one of these sets has been shifted so that their maxima agree which is not the case since $0 < n$ hence $\{0, n\}$ is join-irreducible.

The poset of join-irreducible elements of $\mathcal{N}$ is depicted in Figure 3.4. For the proof that $\mathcal{N}$ is a distributive lattice

$$\begin{tikzcd}
\{0,1\} & \{0,2\} & \{0,3\} & \{0,4\} & \{0,5\} & \{0,6\} & \cdots \\
\{1\} \arrow[u] \arrow[r] & \{2\} \arrow[u] \arrow[r] & \{3\} \arrow[u] \arrow[r] & \{4\} \arrow[u] \arrow[r] & \{5\} \arrow[u] \arrow[r] & \{6\} \arrow[u] \arrow[r] & \cdots \arrow[u]
\end{tikzcd}$$

Figure 3.4: The poset of join-irreducibles of $\mathcal{N}$

we will restrict to the sub-lattices $\mathcal{N}_r$ consisting of those sets whose maximum element is at most r. The poset of join-irreducibles of $\mathcal{N}_r$ is then the obvious truncation of Figure 3.4 consisting of $\{n\}$ and $\{0, n\}$ with $n \leq r$.

Definition 3.6.0.9. The poset of join-irreducible elements of $\mathcal{N}$ will be denoted $\mathcal{JIN}$ and the poset of join-irreducibles of the truncation $\mathcal{N}_r$ by $\mathcal{JIN}_r$.

Example 3.6.0.10. The poset $\mathcal{JIN}_3$ of join-irreducible elements of $\mathcal{N}_3$ is depicted in Figure 3.5. The set of lower sets of this poset are

$$\begin{array}{lll}
\{\emptyset\} & \{\{1\},\{0,1\},\{2\}\} & \{\{1\},\{0,1\},\{2\},\{0,2\}\} \\
\{\{1\}\} & \{\{1\},\{0,1\},\{2\},\{3\}\} & \{\{1\},\{0,1\},\{2\},\{0,2\},\{3\}\} \\
\{\{1\},\{2\}\} & \{\{1\},\{2\},\{0,2\}\} & \{\{1\},\{0,1\},\{2\},\{3\},\{0,3\}\} \\
\{\{1\},\{2\},\{3\}\} & \{\{1\},\{2\},\{0,2\},\{3\}\} & \{\{1\},\{2\},\{0,2\},\{3\},\{0,3\}\} \\
\{\{1\},\{0,1\}\} & \{\{1\},\{2\},\{3\},\{0,3\}\} & \{\{1\},\{0,1\},\{2\},\{0,2\},\{3\},\{0,3\}\}
\end{array}$$

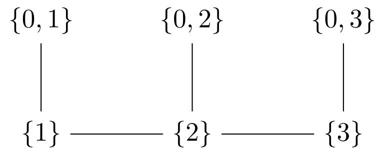


Figure 3.5: The poset of join-irreducibles of $\mathcal{N}_3$

The particularly simple structure of the posets $\mathcal{JIN}_r$ make it easy to classify the set of lower sets of the join-irreducible elements of $\mathcal{N}_r$. One can see the following easily from the structure of $\mathcal{N}_r$.

Lemma 3.6.0.11. *The lower sets of the join-irreducible elements of $\mathcal{N}_r$ are of the form $\emptyset$, or*

$$\{\{1\},\{2\},\dots,\{s\}\} \cup \{\{0,t_1\},\{0,t_2\},\dots,\{0,t_k\}\}$$

where the first set contains all elements i for $1 \le i \le s$ and $\max_i t_i \le s$. ❁

We now show a bijection between the set of lower sets of the partial order on the join-irreducible elements of $\mathcal{N}_r$ and elements of $\mathcal{N}_r$.

Lemma 3.6.0.12. *There is a set bijection from $\mathcal{N}_r$ to the set of lower sets of join-irreducible elements of $\mathcal{N}_r$.*

Proof. Lower set to $\mathcal{N}_r$: We send a lower set to the join of its elements in $\mathcal{N}_r$. This maps $\emptyset$ to the element $\{0\}$. Given a non-empty lower set of the form

$$\{\{1\},\{2\},\dots,\{s\}\} \cup \{\{0,t_1\},\{0,t_2\},\dots,\{0,t_k\}\}$$

where $s \le r$ we send it to the join of its elements. Accordingly we have to shift all the subsets up so that their maxima agree and take union by the definition of the join operation. This gives the set

$$\{s-t_1, s-t_2, \dots, s-t_k, s\} \ .$$

$\mathcal{N}_r$ to lower set: We send an element of $\mathcal{N}_r$ to the lower set of all join-irreducible elements of $\mathcal{N}_r$ less than or equal to it. This maps $\{0\}$ to $\emptyset$. Given an element $S = \{t_1, t_2, \dots, t_k, s\}$ the singleton join-irreducibles less than S are $\{\{0\},\{1\},\{2\},\dots,\{s\}\}$. For a join-ireducible of the form $\{0,a\}$ this is less than S if and only if $\{0,a\} \wedge S = \{0,a\}$. This holds if and only if $a \le s$ and $t_i = s - a$ for some $t_i \in S$.

These operations are easily seen to be inverse to each other. ❁

Definition 3.6.0.13. Denote by α the set morphism sending an element of $\mathcal{N}_r$ to a lower set of the bijection of Lemma 3.6.0.12 and β its inverse.

Lemma 3.6.0.14. *We have the following identities:*

$$\begin{aligned} \alpha(A \vee B) &= (\alpha A) \cup (\alpha B) \\ \alpha(A \wedge B) &= (\alpha A) \cap (\alpha B) \ , \end{aligned}$$

i.e. α and β preserve the join and meet operations.

Proof. Let the elements A and B of $\mathcal{N}_r$ be given by:

$$\begin{aligned} A &= \{s_1, s_2, \dots, s_n, s\} \\ B &= \{t_1, t_2, \dots, t_m, t\} \end{aligned}$$

where without loss of generality $t \le s$ and $s_1 < s_2 < \dots s_n < s$ and similarly for the t_i. We then have

$$\begin{aligned} \alpha(A \vee B) &= \alpha\left(\{s_1, s_2, \dots, s_n, s\} \cup \{t_1 + s - t, t_2 + s - t, \dots, t_m + s - t, s\}\right) \\ &= \alpha\left(\{s_1, s_2, \dots, s_n, t_1 + s - t, t_2 + s - t, \dots, t_m + s - t, s\}\right) \\ &= \{\{1\},\{2\},\dots,\{s\}\} \cup \{\{0, s - s_i\} \mid 1 \le i \le n\} \cup \{\{0, t - t_i\} \mid 1 \le i \le m\} \\ &= (\alpha A) \cup (\alpha B) \end{aligned}$$

where we have used the definition of α from Lemma 3.6.0.12. Similarly for compatibility with the meet operations we have:

$$\begin{aligned}\alpha(A \wedge B) &= \alpha\left(\{s_1 + t - s, s_2 + t - s, \ldots, s_n + t - s, t\} \cap \{t_1, t_2, \ldots, t_m, t\}\right) \\ &= \{\{1\}, \ldots, \{t\}\} \cup \{\{0, a\} \mid a = t - t_i \text{ for some } 1 \leq i \leq m, \\ &\qquad\qquad \text{and } a = t - (s_j + t - s) = s - s_j \text{ for some } 1 \leq j \leq n\}\end{aligned}$$

$$\begin{aligned}\alpha(A) \cap \alpha(B) &= \{\{1\}, \{2\}, \ldots, \{s\}\} \cap \{\{0, s - s_i\} \mid 1 \leq i \leq n\} \\ &\quad \cap \{\{1\}, \{2\}, \ldots, \{t\}\} \cap \{\{0, t - t_i\} \mid 1 \leq i \leq m\}\end{aligned}$$

These two descriptions of $\alpha(A \wedge B)$ and $\alpha(A) \cap \alpha(B)$ are equal and so we have shown α commutes with the meet and join operations. By replacing A and B in the equations of Lemma 3.6.0.14 with $\beta(C)$ and $\beta(D)$ and applying β to the same equations we obtain β preserves the meet and join operations too.

Corollary 3.6.0.15. *The lattice structure on $\mathcal{N}_r$ is a distributive lattice.*

Proof. Since $\mathcal{JIN}_r$ is a distributive lattice since its join and meet operations are union and intersection so too then is $\mathcal{N}_r$ by Lemma 3.6.0.14.

Corollary 3.6.0.16. *The lattice structure on $\mathcal{N}$ is a distributive lattice.*

Proof. The meet and join operations on S and T can be computed in $\mathcal{N}_r$ where r is the larger of the maxima of S and T. The distributive equations hold in $\mathcal{N}_r$ hence they hold too in $\mathcal{N}$.

3.7 Left properness

In this section we prove all model structures $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$ constructed are left proper, i.e. the cobase change of a weak equivalence along a cofibration is a weak equivalence. The author learnt the following technique from [Lac02] where a model structure on the category of (small) 2-categories with morphisms the 2-functors is constructed and shown to be left (and right) proper. We reproduce it here adding a couple of details. Recall the notion of a finitely cofibrantly generated model category Definition 1.4.2.8.

Proposition 3.7.0.1. *Let $\mathcal{M}$ be a finitely cofibrantly generated model category with generating cofibrations I such that whenever we have a double pushout diagram of the form*

$$\begin{tikzcd} S \arrow[r] \arrow[d, "i"'] & A \arrow[r, two heads, "\sim", "p"'] \arrow[d, "f"] & B \arrow[d] \\ D \arrow[r] \arrow[ur, phantom, "\ulcorner", very near start] & C \arrow[r, "p'"'] \arrow[ur, phantom, "\ulcorner", very near start] & P \end{tikzcd} \tag{3.1}$$

with i a generating cofibration and p a trivial fibration then p' is a trivial fibration too, then $\mathcal{M}$ is a left proper model category.

Proof. We follow the proof given in [Lac02] which proceeds in three steps. We consider diagrams of the form

$$\begin{tikzcd} A \arrow[r, "\sim", "p"'] \arrow[d, "f"] & B \arrow[d] \\ C \arrow[r, "p'"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{3.2}$$

starting with the assumptions of the proposition and proceed as follows: firstly we upgrade f to be a relative I-`Cell` morphism, secondly we upgrade the relative I-`Cell` complex to a cofibration, and lastly we remove the requirement that p be a fibration. As noted in [Lac02] the first of these steps requires the finitely generated assumption and the second & third steps are general facts about model categories.

Step 1: Suppose in Equation (3.2) that f is a relative I-`Cell` complex so that it is a transfinite composition of pushouts of generating cofibrations and that p is a trivial fibration. We want to show p' is also a trivial fibration.

Suppose the transfinite composition is indexed by a limit ordinal λ and that by induction the result holds for all smaller ordinals α, then each vertical map with domain an A_α is a trivial fibration.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "p"', "\sim"] \arrow[rrrr, bend left, "f"] \arrow[dr, phantom, "\ulcorner", very near end] & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & A_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \ldots \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & C \arrow[d, "p'"] \\
B \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \ldots \arrow[r] & P
\end{tikzcd}$$

Then by [Hov99, Lemma 7.4.1] the colimit is a trivial fibration since $\mathcal{M}$ is finitely cofibrantly generated and the $A_\alpha \to B_\alpha$ are trivial fibrations. The case of λ being a successor ordinal is taken care of by the assumptions on Diagram 3.1.

Step 2: Suppose now f is a cofibration in Equation (3.2) and p a trivial fibration. Factorise f using the small object argument into a relative I-`Cell` complex $u\colon A \rightarrowtail Q$ followed by a trivial fibration $v\colon Q \overset{\sim}{\twoheadrightarrow} C$, so $f = v \circ u$. We then have the following diagram:

$$\begin{tikzcd}
A \arrow[r, tail, "u"] \arrow[d, tail, "f"'] & Q \arrow[d, two heads, "v", "\sim"'] \\
C \arrow[r, "id"'] \arrow[ur, dotted, "h"] & C
\end{tikzcd}$$

which admits a lift h. We can now form the following diagram:

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, two heads, "p"', "\sim"] \arrow[dr, phantom, "\ulcorner", very near end] & C \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & Q \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & C \arrow[d, "p'"] \\
B \arrow[r] & P \arrow[r] & P_1 \arrow[r] & P
\end{tikzcd}$$

noting that $v \circ h = id$ so too is the pushout composite $P \longrightarrow P_1 \longrightarrow P$. The composite $h \circ f = u$ and is therefore a relative I-`Cell` complex, then by step 1 the pushout of p along $h \circ f$ is a trivial fibration. Hence the morphism $Q \to P_1$ is a trivial fibration. But since $v \circ h = id$ and so too its pushout we have that $C \longrightarrow P$ is a retract of the trivial fibration $Q \longrightarrow P_1$ hence $p'\colon C \longrightarrow P$ is a trivial fibration.

Step 3: Suppose now f is a cofibration and p merely a weak equivalence. We factorise p into a relative I-`Cell` complex $s\colon A \rightarrowtail K$ followed by a trivial fibration $t\colon K \overset{\sim}{\twoheadrightarrow} B$, so that $p = t \circ s$ and note that by the two out of three property s is also a weak equivalence. We then have the following diagram in which the composite of the top row is p:

$$\begin{tikzcd}
A \arrow[r, tail, "s", "\sim"'] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\ulcorner", very near end] & K \arrow[r, two heads, "t", "\sim"'] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d] \\
C \arrow[r] & P_2 \arrow[r] & P
\end{tikzcd} .$$

Pushouts of (trivial) cofibrations are (trivial) cofibrations so $C \longrightarrow P_2$ is a trivial cofibration and $K \longrightarrow P_2$ a cofibration. But since the right hand square is a pushout, t is a trivial fibration and $K \longrightarrow P_2$ a cofibration, by step 2 we have that $P_2 \longrightarrow P$ is a trivial fibration. Hence the composite $C \longrightarrow P_2 \longrightarrow P$ is the composite of a trivial cofibration followed by a trivial fibration and so is a weak equivalence.

Our next aim is then to use Proposition 3.7.0.1 to verify that the model categories $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$ satisfy the conditions of Proposition 3.7.0.1, i.e. that they are finitely generated model categories and that in the diagram of Diagram 3.1 the morphism p' is a trivial fibration, and therefore that they are all left proper.

3.7.1 Left properness of $(f\mathcal{C})_S$

Recall the notation $R\{a\}$ of Section 1.8 denoting the generator $1 \in R$ of the R-module $R\{a\} := R$ by a. We also abuse notation and use it to denote the image of the generator under a map $R\{a\} \cong R \to A$.

We first calculate, for a filtered chain complex A, the pushout of A with a generating cofibration $\mathcal{Z}_{r+1}(p,n) \longrightarrow \mathcal{B}_{r+1}(p,n)$.

$$\begin{tikzcd}
 & \Big(R^n_{(p)}\{a\} \arrow[r] \arrow[d, "\Delta"'] & R^{n+1}_{(p-r-1)}\Big) \arrow[r] \arrow[d] & A \arrow[d] \\
\Big(R^{n-1}_{(p+r)}\{\gamma\} \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & R^n_{(p)} \oplus R^n_{(p-1)}\{\alpha\} \arrow[r, "\begin{pmatrix}0 & 1\end{pmatrix}"] & R^{n+1}_{(p-r-1)}\Big) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A'
\end{tikzcd} \tag{3.3}$$

Recall that one computes colimits in $(f\mathcal{C})_S$ by first passing to $\mathcal{C}_R^{\mathbb{Z}_+}$ via the inclusion functor, computing the colimit there and then passing back via the reflector, Lemma 1.2.1.9. The pushout can then be described as follows. The underlying chain complex of A' is A with an extra summand $R\{\alpha\}$ in homological degree n and an extra summand $R\{\gamma\}$ in homological degree $n-1$, with filtration given by:

$$\begin{aligned}
F_q(A')^m &= F_qA^m & m \neq n-1, n, \\
F_q(A')^n &= \begin{cases} F_qA^n & q < p-1, \\ F_qA^n \oplus R\{\alpha\} & q \geq p-1, \end{cases} \\
F_q(A')^{n-1} &= \begin{cases} F_qA^{n-1} & q < p+r, \\ F_qA^{n-1} \oplus R\{\gamma\} & q \geq p+r. \end{cases}
\end{aligned}$$

and where the differentials of the two new elements α and γ are given by $d\alpha = da$ and $d\gamma = a - \alpha$. We now compute the s-cycles in A' for $s \leq r$. Note that $Z_s^{q,q+m}(A')$ will be equal to $Z_s^{q,q+m}(A)$ when either $m \neq n, n-1$ or if $m = n$ when $q < p-1$, or lastly if $m = n-1$ when $q < p+r$. The proof of the following lemma concerns the remaining cases.

Lemma 3.7.1.1. *For the filtered chain complex A' of the pushout Equation* (3.3), *and $s \leq r$ the following describes the s-cycles:*

$$\begin{aligned}
Z_s^{q,q+n}(A') &= \begin{cases} Z_s^{q,q+n}(A) & q < p-1, \\ Z_s^{q,q+n}(A) \oplus R\{\alpha\} & q \geq p-1, \end{cases} \\
Z_s^{q,q+n-1}(A') &= \begin{cases} Z_s^{q,q+n-1}(A) & q < p+r, \\ Z_s^{q,q+n-1}(A) \oplus R\{\gamma\} & q \geq p+r, \end{cases} \\
Z_s^{q,q+m}(A') &= Z_s^{q,q+m}(A) \qquad m \neq n, n-1.
\end{aligned}$$

Proof. In homological degree n with $q \geq p-1$, we have an element $(x, k\alpha) \in F_qA^n \oplus R\{\alpha\} = F_q(A')^n$, for some $k \in R$, is in $Z_s^{q,q+n}(A')$ if and only if $dx + kd\alpha \in F_{q-s}(A')^n$ or equivalently when $dx + kda \in F_{q-s}A^n$ since $d\alpha = da$ and $a \in A \subset A'$. But if $q \geq p-1$ and $s \leq r$ then since $kda \in F_{p-r-1}A^n$ we have

$$kda \in F_{p-r-1}A^n \subseteq F_{q-r}A^n \subseteq F_{q-s}A^n ,$$

so that in fact $dx \in F_{q-s}A^n$ and then that $x \in Z_s^{q,q+n}(A)$. Hence for $q \geq p-1$ the s-cycles in question are given by:

$$Z_s^{q,q+n}(A') = Z_s^{q,q+n}(A) \oplus R\{\alpha\} .$$

In homological degree $n-1$ with $q \geq p+r$ we have an element $(x, k\gamma) \in F_qA^{n-1} \oplus R\{\gamma\} = F_q(A')^{n-1}$ is in $Z_s^{q,q+n-1}(A')$ if and only if $dx + kd\gamma \in F_{q-s}(A')^{n-1}$ or equivalently when $dx + ka - k\alpha \in F_{q-s}(A')$ since $d\gamma = a - \alpha$. Since $s \leq r$ we have $q - s \geq p + r - s \geq p$ so that ka and $k\alpha$ are both in $F_{q-s}(A')$ so that $dx \in F_{q-s}(A)$ and then that $x \in Z_s^{q,q+n-1}(A)$. Hence for $q \geq p+r$ the s-cycles for $s \leq r$ are given by:

$$Z_s^{q,q+n-1}(A') = Z_s^{q,q+n-1}(A) \oplus R\{\gamma\} .$$ ۞

We also have need for a description of the $(r+1)$-cycles of the pushout A'. For most $(r+1)$-cycles we have a similar classification as for the s-cycles with $s \leq r$, the exceptions being $Z_{r+1}^{p-1,p-1+n}(A')$ and $Z_{r+1}^{p+r,p+r+n-1}(A')$.

Lemma 3.7.1.2. *For the filtered chain complex A' of the pushout of Equation* (3.3)*, the following describe the* $(r+1)$*-cycles:*

$$Z_{r+1}^{q,q+n}(A') = \begin{cases} Z_{r+1}^{q,q+n}(A) & q < p-1, \\ \{(x, k\alpha) \mid dx + kda \in F_{p-r-2}A^{n+1}\} & q = p-1, \\ Z_{r+1}^{q,q+n}(A) \oplus R\{\alpha\} & q \geq p, \end{cases}$$

$$Z_{r+1}^{q,q+n-1}(A') = \begin{cases} Z_{r+1}^{q,q+n-1}(A) & q < p+r, \\ \{(x, k\gamma) \mid dx + ka - k\alpha \in F_{p-1}(A')^n\} & q = p+r, \\ Z_{r+1}^{q,q+n-1}(A) \oplus R\{\gamma\} & q \geq p+r+1, \end{cases}$$

$$Z_{r+1}^{q,q+m}(A') = Z_s^{q,q+m}(A) \qquad m \neq n, n-1.$$

Proof. The first case in each cohomological degree is clear, the third cases in cohomological degrees n and $n-1$ are similar to the proof of Lemma 3.7.1.1. We calculate then the second cases in degrees n and $n-1$.

In homological degree n with $q = p-1$ we have an element $(x, k\alpha) \in F_qA^n \oplus R\{\alpha\} = F_q(A')^n$ is in $Z_{r+1}^{q,q+n}(A') = Z_{r+1}^{p-1,p-1+n}(A')$ if and only if $dx + kd\alpha \in F_{q-r-1}(A')^{n+1} = F_{p-r-2}(A')^{n+1}$, or equivalently $dx + kda \in F_{p-r-2}A^{n+1}$. The case for homological degree $n-1$ with $q = p+r$ is similar.

Note if we further have a morphism $A \longrightarrow B$ and form the iterated pushout

$$\begin{tikzcd} \mathcal{Z}_{r+1}(p,n) \arrow[r] \arrow[d] & A \arrow[r, "p"] \arrow[d] & B \arrow[d] \\ \mathcal{B}_{r+1}(p,n) \arrow[r] & A' \arrow[r, "p'"'] \arrow[ul, phantom, "\ulcorner", very near start] & B' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

then B' takes a similar form to A' in that since each square is a pushout so too is the composite square so B' is obtained from the pushout square

$$\begin{tikzcd} \mathcal{Z}_{r+1}(p,n) \arrow[r] \arrow[d] & B \arrow[d] \\ \mathcal{B}_{r+1}(p,n) \arrow[r] & B' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the top horizontal morphism is the composite of $\mathcal{Z}_{r+1}(p,n) \longrightarrow A$ with p. We denote the new elements by β (instead of α) and δ (instead of γ), which appear in the same filtration and homological degrees as α and γ respectively, and whose differentials are $d\beta = db$ where $b = pa$, and $d\delta = b - \beta$. Note too that $p'\alpha = \beta$ and $p'\gamma = \delta$.

Proposition 3.7.1.3. *For a morphism of filtered chain complexes $p\colon A \longrightarrow B$ we have the following surjectivity results on cycles:*

1. *if p is such that $Z_s(p)$ is bidegree-wise surjective for some $s \leq r$, then the pushout $p'\colon A' \longrightarrow B'$ also satisfies $Z_s(p')$ is bidegree-wise surjective,*
2. *suppose p is such that $Z_r(p)$ is bidegree-wise surjective and is an r-weak equivalence, then the pushout $p'\colon A' \longrightarrow B'$ also satisfies $Z_{r+1}(p')$ is bidegree-wise surjective.*

Proof. Most cycle surjectivity conditions on p' follow directly from our description of the cycles of A' (and the equivalent descriptions for B') from Lemmas 3.7.1.1 and 3.7.1.2 along with $p(\alpha) = \beta$ and $p(\gamma) = \delta$. The missing cases are bidegree-wise surjectivity of $Z_{r+1}^{p-1,p-1+n}(p')$ and $Z_{r+1}^{p+r,p+r+n-1}(p')$.

For the former case of $Z_{r+1}^{p-1,p-1+n}(p')$ we consider an element of $(y, k\beta) \in Z_{r+1}^{p-1,p-1+n}(B')$ so that $dy + kd\beta = dy + kdb \in F_{p-r-2}B^{n+1}$ by Lemma 3.7.1.2 so then we have an element of $B_{r+1}^{p-r-1,p-r-1+n+1}(B)$ given by:

$$\left(R^n_{(p-1)}\{-y\} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} R^{n+1}_{(p-r-1)} \oplus R^{n+1}_{(p-r-2)}\{dy+kdb\} \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} R^{n+2}_{(p-2r-2)}\right) \longrightarrow B \tag{3.4}$$

which gives a commutative diagram of the form:

$$\begin{tikzcd} \mathcal{Z}_{r+1}(p-r-1, p-r-1+n+1) \arrow[r, "kda"] \arrow[d, "i"'] & A \arrow[d, two heads, "p", "\sim"'] \\ \mathcal{B}_{r+1}(p-r-1, p-r-1+n+1) \arrow[r] \arrow[ur, dotted, "h"'] & B \end{tikzcd}$$

where the morphism $\mathcal{B}_{r+1}(*,*) \to B$ is given by Equation (3.4). Since i is a generating r-cofibration and p an r-acyclic fibration there exists a lift h and we let $x = hy$. We then have $h(dy+kdb) = dx+kda$, so that $(x, k\alpha) \in F_{p-1}(A')^n$ with $dx+kd\alpha = dx+kda = h(dy+kdb) \in h\left(F_{p-r-2}B^{n+1}\right) \subseteq F_{p-r-2}A^{n+1}$ and so $(x,k\alpha) \in Z_{r+1}^{p-1,p-1+n}(A')$ is the required lift.

The latter case of $Z_{r+1}^{p+r,p+r+n-1}(p')$ is near identical, we consider an element $(y, k\delta) \in Z_{r+1}^{p+r,p+r+n-1}(B')$ so that $dy + kd\delta = dy + kb - k\beta \in F_{p-1}(B')^n$ by Lemma 3.7.1.2 or equivalently that $dy+kb \in F_{p-1}B^n$. Again then we have a boundary element of $B_{r+1}^{p,p+n}(B)$ given by:

$$\left(R^{n-1}_{(p+r)}\{-y\} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} R^{n}_{(p)} \oplus R^{n}_{(p-1)}\{dy+kb\} \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} R^{n+1}_{(p-r-1)}\right) \longrightarrow B \tag{3.5}$$

again giving a commutative diagram of the form:

$$\begin{tikzcd}
\mathcal{Z}_{r+1}(p,p+n) \arrow[r, "ka"] \arrow[d, "i"'] & A \arrow[d, "p", "\sim"'] \\
\mathcal{B}_{r+1}(p,p+n) \arrow[r] \arrow[ur, dotted, "h"] & B
\end{tikzcd}$$

whose morphism $\mathcal{B}_{r+1}(*,*) \to B$ is given by Equation (3.5) and for which a lift h exists. Define $x = hy$ so that $h(dy+kb) = dx+ka$. Now $(x, k\gamma) \in F_{p+r}(B')^{n-1}$ maps to $(y, k\delta)$ under p' and we have $d(x,k\gamma) = dx + kd\gamma = dx+ka-k\alpha = h(dy+kb)-k\alpha \in h\left(F_{p-1}B^n\right) + F_{p-1}(A')^n = F_{p-1}(A')^n$ so that $(x, k\gamma) \in Z_{r+1}^{p+r,p+r+n-1}(A')$ is the required lift.

These then show the remaining cases of part 2 of the lemma. ∎

Lemma 3.7.1.4. *The kernel of the pushout* $p'\colon A' \longrightarrow B'$ *is* $K = \ker(p\colon A \longrightarrow B)$.

Proof. This is clear from the description of A' and B' and that kernels are computed filtration degree-wise and cohomological degree-wise in $f\mathcal{C}$. ∎

Recall the map $w_{r+1}\colon B_{r+1}(-) \to Z_{r+1}(-)$ of Definition 1.6.0.2.

Proposition 3.7.1.5. *For* p *an* r*-weak equivalence which is* Z_r*-bidegree-wise surjective, the morphism* $E_{r+1}^{*,*}(p')\colon E_{r+1}^{*,*}(A') \longrightarrow E_{r+1}^{*,*}(B')$ *on the* $(r+1)$*-pages of the associated spectral sequences is injective.*

Proof. Consider a cycle z representing some class of $E_{r+1}^{*,*}(A')$ whose image under $E_{r+1}(p')$ is 0. We then have a boundary (c_0, c_1), where we write c_0 and c_1 for the two r-cycles, in $B_{r+1}^{*,*}(B')$ such that $p'(z) = w_{r+1}\left((c_0,c_1)\right)$. Since p is Z_r-surjective so too is p' by Proposition 3.7.1.3 so we can lift c_0 and c_1 to r-cycles e_0 and e_1 of A'. We then have that $z - w_{r+1}\left((e_0,e_1)\right)$ is an $(r+1)$-cycle of A' which is in the kernel of p'. By Lemma 3.7.1.4 the kernel of p' is $K = \ker(p)$ which is r-acyclic, since pullbacks of acyclic fibrations are acyclic fibrations, so the cycle $z - w_{r+1}\left((e_0,e_1)\right)$ is in fact an $(r+1)$-boundary say (k_0,k_1), hence the cycle z is an $(r+1)$-boundary $z = w_{r+1}\left((e_0+k_0, e_1+k_1)\right)$, proving injectivity of $E_{r+1}(p')$. ∎

Corollary 3.7.1.6. *Let* $p\colon A \longrightarrow B$ *be an* S*-trivial fibration for the model structure* $(f\mathcal{C})_S$ *(whose weak equivalences are the* r*-weak equivalences) of Theorem 3.1.0.2 and* $A \longrightarrow A'$ *the pushout of* A *along a generating cofibration. The pushout* p' *of* p *along* $A \longrightarrow A'$ *is an* S*-trivial fibration.*

Proof. If the generating cofibration is of the form $0 \longrightarrow \mathcal{Z}_s(*,*)$ then p' is of the form $A \oplus \mathcal{Z}_s(*,*) \longrightarrow B \oplus \mathcal{Z}_s(*,*)$ and the result is clear.

If the generating cofibration is of the form $\mathcal{Z}_{r+1}(*,*) \longrightarrow \mathcal{B}_{r+1}(*,*)$ and if $s \in S$ with $s \leq r$ so that p is Z_s-surjective then so too is p' by part 1 of Proposition 3.7.1.3. Since p is Z_r-surjective and also an r-weak equivalence the pushout is Z_{r+1}-surjective by part 2 of Proposition 3.7.1.3, and hence $E_{r+1}(p')$ is surjective by Lemma 1.6.0.11. Lastly by Proposition 3.7.1.5 since p is an r-weak equivalence and Z_r-bidegree-wise surjective $E_{r+1}(p')$ is bidegree-wise injective. Hence p' is an S-trivial fibration. ∎

Theorem 3.7.1.7. *The model categories* $(f\mathcal{C})_S$ *of Theorem 3.1.0.2 are left proper.*

Proof. By Proposition 3.7.0.1 it suffices to show that $(f\mathcal{C})_S$ is a finitely cofibrantly generated model structure and that in the double pushout

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "i"'] & A \arrow[r, two heads, "\sim", "p"'] \arrow[d, "f"] & B \arrow[d] \\
D \arrow[r] \arrow[ur, phantom, near end, "\ulcorner"] & C \arrow[r, "p'"'] \arrow[ur, phantom, near end, "\ulcorner"] & P
\end{tikzcd} \tag{3.6}$$

where i is a generating cofibration of $(f\mathcal{C})_S$ and p a trivial fibration of $(f\mathcal{C})_S$ that p' is also a trivial fibration of $(f\mathcal{C})_S$. The model structure is finitely cofibrantly generated by Lemma 1.2.1.16, and the double pushout condition follows from Corollary 3.7.1.6.

3.7.2 Left properness of $(b\mathcal{C})_S$

We follow a similar procedure here for bicomplexes, we show the double pushouts maintain the various surjectivity conditions required and then show injectivity of the $(r+1)$-page.

$$\begin{tikzcd}
\mathcal{ZW}_{r+1}^{p,p+n} \arrow[r] \arrow[d] & A \arrow[r, two heads, "\sim", "p"'] \arrow[d] & B \arrow[d] \\
\mathcal{BW}_{r+1}^{p,p+n} \arrow[r] \arrow[ur, phantom, near end, "\ulcorner"] & A' \arrow[r] \arrow[ur, phantom, near end, "\ulcorner"] & B'
\end{tikzcd} \tag{3.7}$$

We begin by describing the pushout A'. We denote by $(a_0, \ldots, a_r)$ the $(r+1)$-cycle of A determined by Equation (3.7) and its image under p by $p(a_0, \ldots, a_r) = (b_0, \ldots, b_r)$. The pushout can be computed bidegree-wise, we denote by $(\alpha_0, \ldots, \alpha_{r-1})$ the generators of the r-cycle $\mathcal{ZW}_r^{p+r,p+r+n}$ of $\mathcal{BW}_{r+1}^{p,p+n}$ and by γ the generator of $\mathcal{ZW}_0^{p,p+n-1}$ of $\mathcal{BW}_{r+1}^{p,p+n}$, so that the image a_0 is identified with $d_1\alpha_{r-1} + d_0\gamma$ in the pushout.

Lemma 3.7.2.1. *The pushout A' is given by the quotient of $A \oplus \mathcal{ZW}_r^{p+r,p+r+n} \oplus \mathcal{ZW}_0^{p,p+n-1}$, where the new generating elements of $\mathcal{ZW}_r^{p+r,p+r+n}$ are given by the α_i and $d_1\alpha_i$ and the generators of $\mathcal{ZW}_0^{p,p+n-1}$ by γ, $d_0\gamma$, $d_1\gamma$ and $d_0d_1\gamma$, by the relation $d_0\gamma = a_0 - d_1\alpha_{r-1}$.*

Next we classify the s-cycles of the pushout A' for $s \leq r+1$. Clearly in bidegrees $(p, p+m)$ with $m \neq n, n-1$ the s-cycles will be the same as those of A, we then need to understand them on the $n-1$ and n diagonals. We do not give as explicit a description as for filtered chain complexes but only enough so as to prove cycle surjectivity results. For the n diagonal we have new elements the $d_1\alpha_*$ and $d_1\gamma$ to consider. We also have $d_0\gamma$ but note that $a_0 = d_0\gamma + d_1\alpha_{r-1}$ so it does not introduce new elements not already introduced by the $d_1\alpha_*$. Note next that the $d_1\alpha_*$ all satisfy $d_1d_1\alpha_* = 0 = d_0d_1\alpha_*$ so we can ignore all $d_1\alpha_*$ when computing whether a sequence is an s-cycle or not. Consider now we have an s cycle in which $d_1\gamma$ appears in some component say:

$$(x_0, x_1, \ldots, x_{i-1}, x_i + \kappa d_1\gamma, x_{i+1}, \ldots, x_{s-1}) \tag{3.8}$$

in which the x_i are elements of A and we have removed any $d_1\alpha_*$. For this to be an s-cycle we need

1. $d_0x_0 = 0$,
2. $d_1x_j = d_0x_{j+1}$ when $j \neq i-1, s-1$, and
3. $d_1x_{i-1} = d_0(x_i + \kappa d_1\gamma)$, which since $d_0d_1\gamma = d_1a_0$ amounts to having $d_1x_{i-1} = d_0x_i + \kappa d_1a_0$.

So Equation (3.8) is an s-cycle of A' if and only if the following is an s-cycle of A:

$$(x_0, x_1, \ldots, x_{i-1} - \kappa a_0, x_i, x_{i+1}, \ldots, x_{s-1})$$

since $d_0a_0 = 0$. For those s-cycles that involve the bidegree of $d_1\gamma$ the s-cycles of A' without any of the $d_1\alpha_*$ are then in bijection with the s-cycles of A via the above translation. We summarise this in the following lemma which requires a certain degree of sensible interpretation with regard to in which degrees the $d_1\gamma$ and $d_1\alpha_*$ can appear. For example for $q \leq p-2$ all the coefficients κ and κ_* must be 0 in the following lemma.

Lemma 3.7.2.2. *The s-cycles of A' in bidegrees $(q, q+n)$ are of the form:*

$$(x_0, x_1, \ldots, x_{i-1} + \kappa a_0, x_i + \kappa d_1\gamma, x_{i+1}, \ldots, x_{s-1}) + (0, \ldots, 0, \kappa_0 d_1\alpha_0, \ldots, \kappa_{r-1}d_1\alpha_{r-1}, 0, \ldots, 0)$$

for some $\kappa, \kappa_* \in R$ and where the x_i are in appropriate bidegrees and the sequence $(x_i)_i$ is an s-cycle of A.

Consider now s-cycles along the $(n-1)$-diagonal. The new elements of the pushout A' are γ and the α_*. Suppose an s-cycle of A' has in some component a non-zero mutiple of an α_*, say $x_i + \kappa_j\alpha_j$. Then d_0 of this must equal d_1 of the $(i-1)$-component of the s-cycle and so the $(i-1)$-component must be of the form $x_{i-1} + \kappa_j\alpha_{j-1}$ where the coefficients of the α_* are equal. Iterating up our s-cycle must have a $\kappa\alpha_*$ in each component for $* \leq i$ and where $\kappa = \kappa_j$, this also shows $i \geq j$. Iterating down instead any subsequent components of the s-cycle must also be of the form $x_* + \kappa\alpha_{*+j-i}$ for $* \leq r-1$. If a component has a non-zero γ summand, $x_i + \lambda\gamma$ then the $(i-1)$-component must have non-zero α_{r-1} summand $x_{i-1} + \kappa\alpha_{r-1}$ since

$$d_0(x_i + \lambda\gamma) = d_1(x_{i-1} + \kappa\alpha_{r-1})$$

which gives $d_0x_i - d_1x_{i-1} = \kappa d_1\alpha_{r-1} - \lambda d_0\gamma$ and since the left hand side is in A so too is the right hand side but this can only happen if $\lambda = -\kappa$ so that $d_0x_i - d_1x_{i-1} = \kappa d_1\alpha_{r-1} + \kappa d_0\gamma = \kappa a_0$. So for an s-cycle of A' to have component with summand $-\kappa\gamma$ it must have all previous components having summands of the form $\kappa\alpha_*$ and the γ summand appears in a component of index r or higher so that the s-cycle is an $(r+1)$-cycle (or longer). These descriptions are summarised in the following lemma. We denote the datum of a j-cycle given by $(\kappa\alpha_0, \kappa\alpha_1 \dots, \kappa\alpha_j)$ by $R(\alpha_0, \alpha_1, \dots, \alpha_j)$ which is isomorphic to the R-module R which determines the coefficient κ.

Lemma 3.7.2.3. *For $s \leq r$ the s-cycles of A' along the $(n-1)$-diagonal are of the form $ZW_s(A) \oplus R(\alpha_0, \alpha_1, \dots, \alpha_j)$ where the module $R(\alpha_0, \alpha_1, \dots, \alpha_j)$ denotes a single copy of R determining the coefficient κ and where α_j shares a bidegree with the final component of the s-cycle. For $s = r+1$ the $r+1$ cycles are either of the above form $ZW_{r+1}(A) \oplus R(\alpha_0, \alpha_1, \dots, \alpha_j)$ or of the form $ZW_{r+1}(A) \oplus R(\alpha_0, \alpha_1, \dots, \alpha_{r-1}, \gamma)$ where again the module $R(\alpha_0, \alpha_1, \dots, \alpha_{r-1}, \gamma)$ determines the coefficient κ and $\lambda = -\kappa$, with γ sharing a bidegree with the final component of the $(r+1)$-cycle.*

Given these descriptions of the s-cycles we now show given ZW_s-surjectivity conditions on a map of bicomplexes $p\colon A \to B$ the induced map on pushouts $A' \to B'$ is also ZW_s-surjective where B' is the double pushout in

$$\begin{tikzcd}
\mathcal{ZW}_{r+1}(p,p+n) \arrow[r] \arrow[d] & A \arrow[r, "p", "\sim"'] \arrow[d] & B \arrow[d] \\
\mathcal{BW}_{r+1}(p,p+n) \arrow[r] & A' \arrow[r, "p'"'] \arrow[ul, phantom, "\ulcorner", very near start] & B' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and note that the pushout B' has a similar description as A does. We write $a_* = p(b_*)$ and let B have new elements β_* and δ where the induced map satisfies $p(\alpha_*) = \beta_*$ and $p(\gamma) = \delta$. The preceeding two lemmas give the following surjectivity results.

Lemma 3.7.2.4. *Suppose $p\colon A \longrightarrow B$ satisfies ZW_s-surjectivity for some $s \leq r+1$, then so too does p'.*

Proof. This is apparent from our descriptions of s-cycles of A'. We can lift s-cycles of B into s-cycles of A by assumption and replace β_* and δ with α_* and γ respectively. This finds an s-cycle preimage of any s-cycle of B' under p'.

Lemma 3.7.2.5. *The kernel of $p'\colon A \to B$ is $K = \ker(p\colon A \to B)$.*

Recall the map $w_{r+1}\colon BW_{r+1}(-) \longrightarrow ZW_{r+1}(-)$ of Definition 1.7.0.7.

Lemma 3.7.2.6. *Suppose $p\colon A \longrightarrow B$ is an r-weak equivalence which is also surjective on 0-cycles and r-cycles, i.e. an acyclic fibration in $(b\mathcal{C})_r$, then the induced morphism on the $(r+1)$-page of the spectral sequence $E_{r+1}^{*,*}(p')\colon E_{r+1}^{*,*}(A') \longrightarrow E_{r+1}^{*,*}(B')$ is injective.*

Proof. Consider a cycle $[z] \in E_{r+1}^{*,*}(A')$, represented by an $(r+1)$-cycle z, such that its image under $E_{r+1}(p')$ is 0, so that the $p'(z)$ is a boundary of B'. Hence $p(z') = w_{r+1}((c_0, c_1, c_2))$ where c_0 and c_2 are the r-cycles, and c_1 the 0-cycle making up an $(r+1)$-boundary. By assumption p is surjective on 0-cycles and r-cycles so by Lemma 3.7.2.4 so too is p'. Hence we can lift c_0 and c_2 to r-cycles e_0 and e_2 of A' and c_1 to a 0-cycle e_1 of A'. Consider now the $(r+1)$-cycle $z - w_{r+1}((e_0, e_1, e_2))$ of A', each component is in the kernel of p' which is $K = \ker(p)$ by Lemma 3.7.2.5 hence, since K is r-acyclic we have that the $(r+1)$-cycle $z - w_{r+1}((e_0, e_1, e_2))$ is an $(r+1)$-boundary, say $w_{r+1}((k_0, k_1, k_2))$. Hence z too is an $(r+1)$-boundary, $z = w_{r+1}((e_0 + k_0, e_1 + k_1, e_2 + k_2))$. This shows $E_{r+1}(p')$ is injective.

Corollary 3.7.2.7. *Let $p\colon A \longrightarrow B$ be an S-trivial fibration for the model structure $(b\mathcal{C})_S$ (whose weak equivalences are the r-weak equivalences) of Theorem 3.2.0.2 and $A \longrightarrow A'$ the pushout of A along a generating cofibration. The pushout p' of p along $A \longrightarrow A'$ is an S-trivial fibration.*

Proof. If the generating cofibration is of the form $0 \longrightarrow \mathcal{ZW}_s^{*,*}$ then p' is of the form $A \oplus \mathcal{ZW}_s^{*,*} \longrightarrow B \oplus \mathcal{ZW}_s^{*,*}$ and the result is clear.

If the generating cofibration is of the form $\mathcal{ZW}_{r+1}(*,*) \longrightarrow \mathcal{BW}_{r+1}(*,*)$ and if $s \in S$ so that p is surjective on s-cycles then so too is p' by Lemma 3.7.2.4. Since p is surjective on 0-cycles, r-cycles and also on the $(r+1)$-page of the spectral sequence it is surjective on $(r+1)$-cycles too by Lemma 1.7.0.12 and so again by Lemma 3.7.2.4 we have p' is also surjective on $(r+1)$-cycles and again by Lemma 1.7.0.12 p' is surjective on the $(r+1)$-page of the spectral sequence. Lastly, by Lemma 3.7.2.6, since p is an r-weak equivalence and ZW_0 and ZW_r-bidegree-wise surjective $E_{r+1}(p')$ is bidegree-wise injective. Hence p' is an S-trivial fibration.

Theorem 3.7.2.8. *The model categories $(b\mathcal{C})_S$ of Theorem 3.2.0.2 are left proper.*

Proof. By Proposition 3.7.0.1 it suffices to show that $(b\mathcal{C})_S$ is a finitely cofibrantly generated model structure and that in the double pushout

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "i"'] & A \arrow[r, two heads, "\sim", "p"'] \arrow[d, "f"] \arrow[dl, phantom, "\ulcorner", very near start] & B \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\
D \arrow[r] & C \arrow[r, "p'"'] & P
\end{tikzcd} \tag{3.9}$$

where i is a generating cofibration of $(b\mathcal{C})_S$ and p a trivial fibration of $(b\mathcal{C})_S$ that p' is also a trivial fibration of $(b\mathcal{C})_S$. The model structure is finitely cofibrantly generated by Lemma 1.7.0.16, and the double pushout condition follows from Corollary 3.7.2.7.

3.8 Cellularity

3.8.1 Cellularity of $(f\mathcal{C})_S$

The smallness conditions of Definition 1.4.10.1 have already been established so our main task here is to verify the effective monomorphism condition. We will do so by instead showing that cofibrations are regular monomorphisms and use Proposition 1.4.10.3 to obtain they are effective monomorphisms. To show a cofibration $f\colon A \longrightarrow B$ is a regular monomorphism we would like to show that f is the equaliser of some pair of morphisms, the obvious choice being the equaliser of the pair $0, q\colon B \longrightarrow B/A$ where q is the map from B to the cokernel B/A. This is in general not true for any morphism of filtered chain complexes however it will be true for the strict morphisms (Definition 1.2.1.7) so we will demonstrate that any cofibration in $(f\mathcal{C})_S$ is a strict morphism of filtered chain complexes.

Definition 3.8.1.1. For an R-module N we denote by $D_p^n(N)$ the filtered chain complex whose underlying chain complex is $D^n(N)$ and such that $F_{p-1}D_p^n(N) = 0$ and $F_pD_p^n(N) = D_p^n(N)$. We also write $D^n(N)$ for the filtered chain complex whose underlying complex is $D^n(N)$ and which is in all filtration degrees.

Lemma 3.8.1.2. *The filtered chain complexes $D^n(N)$ and $D_p^n(N)$ are acyclic and fibrant in the model structures $(f\mathcal{C})_S$.*

Lemma 3.8.1.3. *Cofibrations in $(f\mathcal{C})_S$ are inclusions.*

Proof. Let $i\colon A \longrightarrow B$ be a cofibration in $(f\mathcal{C})_S$ and consider the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"'] & D^{n-1}(A^n) \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & 0
\end{tikzcd}$$

where the disc object is in all filtration degrees and the map from A to the disc is the identity in degree n and the composite of the differential and identity in degree $n-1$. Since i is a cofibration and the disc an $\mathcal{S}$-acyclic fibrant object a lift exists. Since the map from A to the disc object is an inclusion in degree n so too is i.

In fact we have the stronger result that cofibrations are strict inclusions by taking appropriate disc objects.

Lemma 3.8.1.4. *Cofibrations in $(f\mathcal{C})_S$ are strict inclusions of filtered chain complexes.*

Proof. We let $f\colon A \longrightarrow B$ be a cofibration of $(f\mathcal{C})_S$, take $N = A^n/F_{p-1}A^n$, and consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "f"'] & D_p^{n-1}(N) \arrow[d, two heads, "\sim"] \\
B \arrow[r] \arrow[ur, dotted, "h"] & 0
\end{tikzcd}$$

where the morphism from A to the disc is given by the inclusion of $F_qA^n \longrightarrow A^n$ followed by the quotient $A^n \longrightarrow A^n/F_{p-1}A^n$ in degree n which fully determines the morphism. A lift h exists in the diagram since f is a cofibration and the disc object is acyclic and fibrant in $(f\mathcal{C})_S$. Consider then an element $a \in F_pA^n$. If $f(a) \in F_{p-1}B^n$ then $hf(a) = 0$ since the $(p-1)$-filtered part of the disc object is 0, hence the image of a in the disc object is 0 by commutativity of the diagram and so $a \in F_{p-1}A^n$. This shows the morphism f is strict.

Proposition 3.8.1.5. *The model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 are cellular.*

Proof. We need to show the three conditions of Definition 1.4.10.1 hold. By Lemma 1.2.1.16 the domains and codomains of the generating cofibrations are all small, as are the domains of the generating acyclic cofibrations hence they are small relative to the generating cofibrations. This shows the first two conditions. For condition 3 we need to demonstrate all cofibrations are effective monomorphisms. By Proposition 1.4.10.3 we need only check the cofibrations are regular monomorphisms. The cofibrations are certainly monomorphisms by Lemma 3.8.1.3 and by Lemma 3.8.1.4 they are strict morphisms of filtered chain complexes, hence by Lemmas 1.2.1.9 and 1.2.1.10 we can compute cokernels of cofibrations cohomological degree-wise and filtration degree-wise. This gives the cofibration $A \longrightarrow B$ is the kernel of its cokernel and so $A \longrightarrow B$ is the equaliser of the pair $0, q\colon B \longrightarrow B/A$ and hence a regular monomorphism. This shows any cofibration is a regular monomorphism and hence an effective monomorphism.

3.8.2 Cellularity of $(b\mathcal{C})_S$

Lemma 3.8.2.1. *Cofibrations in $(b\mathcal{C})_S$ are (degreewise split) monomorphisms.*

Proof. The proof is similar to that for chain complexes [Hov99, Proposition 2.3.9].

Proposition 3.8.2.2. *The model categories $(b\mathcal{C})_S$ of Theorem 3.2.0.2 are cellular.*

Proof. We need to show the three conditions of Definition 1.4.10.1 hold. By Lemma 1.7.0.16 the domains and codomains of the generating cofibrations are all finite, as are the domains of the generating acyclic cofibrations hence they are small relative to the generating cofibrations. This shows the first two conditions. For condition 3 we need to demonstrate all cofibrations are effective monomorphisms. By Proposition 1.4.10.3 we need only check the cofibrations are regular monomorphisms. The cofibrations are certainly monomorphisms by Lemma 3.8.2.1 and since $b\mathcal{C}$ is an abelian category any monomorphism $A \longrightarrow B$ is a kernel of some morphism $f\colon B \longrightarrow C$ and hence the equaliser of the pair $f, 0\colon B \longrightarrow C$, so $A \longrightarrow B$ is regular. This shows any cofibration is a regular monomorphism and hence an effective monomorphism.

3.9 Stability

3.9.1 Stability of $(f\mathcal{C})_S$

We show that the model structures of $(f\mathcal{C})_S$ on filtered chain complexes are stable model categories. Recall the definition of a stable model category, Definition 1.4.9.3. Recall too that in a pointed model category the loop functor on A can be computed by a homotopy pullback of the diagram

$$\begin{tikzcd}
\Omega A \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner h"] & * \arrow[d] \\
* \arrow[r] & A
\end{tikzcd}$$

and that such a homotopy pullback, in a right proper model category, can be computed by replacing the map $* \longrightarrow A$ by an acyclic fibration and taking a standard pullback as in Lemma 1.4.9.2.

For filtered chain complexes recall the notion of the r-suspension Σ^r, r-loops Ω^r and r-cone C_r functors of Definitions 1.2.1.18 and 1.6.0.12 and that there is a projection $\pi\colon C_r(A) \twoheadrightarrow \Sigma^r A$ which by Lemma 1.6.0.14 is surjective on all s-cycles with $s \leq r$ so in particular is a fibration for all model structure $(f\mathcal{C})_S$ with $\max S = r$ and such that $C_r(A)$ is r-acyclic. Applying the r-loops functor to this map we obtain a map $\Omega^r\pi\colon \Omega^r C_r(A) \to A$ which is surjective on all s-cycles for $s \leq r$. Note too that the kernel of π is simply $\Omega^r A$. By Lemma 1.4.9.2 we can then give a model for the loop functor on $\mathrm{Ho}\,(f\mathcal{C})_S$ on the level of the model category $(f\mathcal{C})_S$ by the pullback diagram:

$$\begin{tikzcd}
\Omega^r A \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^r C_r(A) \arrow[d] \\
* \arrow[r] & A
\end{tikzcd}$$

from which the following is immediate.

Lemma 3.9.1.1. *The loop functor on an object A in the model category $(f\mathcal{C})_S$ is given by $\Omega^r A$.*

Note that in any pointed model category the loop functor is always right adjoint to the suspension functor. Since our loop functor, Ω^r, is an automorphism of $f\mathcal{C}$ its adjoint must be its inverse which is given by Σ^r. We have then proved the following.

Proposition 3.9.1.2. *The model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 are stable model categories whose loops and suspension functors are given by Ω^r and Σ^r.*

3.9.2 Stability of $(b\mathcal{C})_S$

We now wish to show a similar result for $(b\mathcal{C})_S$ The proof of $(b\mathcal{C})_S$ being stable is similar to that of $(f\mathcal{C})_S$, we compute a pullback of a projection from a cone object, however this gives an object more unwieldy then a simple shift by an expected bidegree, however this pullback is weakly equivalent to the shift.

For bicomplexes recall the notion of the r-suspension and r-loops functors Σ^r and Ω^r given in Definition 1.2.2.8, and the r-cone functor C_r of Definition 1.7.0.14 and that the r-cone $C_r(A)$ on a bicomplex A is equivalent to tensoring the bicomplex by $\mathcal{ZW}_r(r, r-1)$. There is a projection of $\mathcal{ZW}_r(r, r-1)$ onto the bicomplex $R^{r,r-1}$ and this induces a projection, by tensoring by A, of $C_r(A)$ onto the r-suspension of A, i.e. we have a morphism $\pi\colon C_r(A) \to \Sigma^r A$ which we can apply the r-loop functor to to obtain $\Omega^r\colon \Omega^r C_r(A) \to A$. Note too that $\Omega^r C_r(A)$ is r-acyclic since $C_r(A)$ is r-acyclic by Lemma 1.7.0.15. By Lemma 1.4.9.2 we can then give a model for the loop functor on $\mathrm{Ho}\,(b\mathcal{C})_S$ on the level of the model category $(b\mathcal{C})_S$ by the pullback diagram:

$$\begin{tikzcd}
P \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^r C_r(A) \arrow[d] \\
* \arrow[r] & A
\end{tikzcd}$$

which is so far the same method we used for $(f\mathcal{C})_S$. Now however we must identify P. The bicomplex $\Omega^r C_r(A)$ is equivalent to tensoring by $\mathcal{ZW}_r(0, 0)$. We introduce the following notation.

Notation 3.9.2.1. The bicomplex $\mathcal{NW}_r(p, q)$ is given by the pullback of the projection $\mathcal{ZW}_r(p, q) \to R^{p,q}$ along the 0 map $0 \to R^{p,q}$.

The pullback P is then the bicomplex $\mathcal{NW}_r(0, 0) \otimes A$. We wish to show this is r-weakly equivalent to $\Omega^r A$. Note there is an inclusion $i\colon \Omega^r A \to \mathcal{NW}_r(0, 0) \otimes A$.

Lemma 3.9.2.2. *The inclusion $i\colon \Omega^r A \to \mathcal{NW}_r(0, 0) \otimes A$ is an r-weak equivalence.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_{r-1}$ denote the generators of $\mathcal{NW}_r(0, 0)$ in bidegrees $(-i, -i)$ and similarly $\beta_0, \beta_1, \ldots, \beta_i, \ldots, \beta_{r-1}$ denote the generators of $\mathcal{NW}_r(0, 0)$ in bidegrees $(-i, -i{+}1)$ so that $d^0\alpha_i = \beta_{i-1}$ for $0 \leq i \leq r-1$ and $d^1\alpha_i = \beta_{i-1}$. The inclusion $i\colon \Omega^r A \to \mathcal{NW}_r(0, 0) \otimes A$ is then just the map $a \mapsto \beta_{r-1} \otimes a$.

We consider the 0-page of the spectral sequence of $\mathcal{NW}_r(0, 0) \otimes A$. No element of the form $\Sigma_i \alpha_i \otimes a_i$ are 0-cycles since $d^0\alpha_i \neq 0$. We then need only consider elements of the form $\Sigma_i \beta_i \otimes a_i$ in the kernel of d_0 so that each $a_i \in \ker d_0^A$. Note however that $\Sigma_{i \leq r-2} \beta_i \otimes a_i = d^0 \Sigma_{i \leq r-1} \alpha_i \otimes a_i$ hence those elements are in the image of the differential and become 0 on the 0-page of the associated spectral sequence. Note too that no new elements are introduced with image under d^0 given by $\beta_{r-1} \otimes a_{r-1}$ (unless a_{r-1} were already in the image of d^0). Hence the inclusion $i\colon \Omega^r A \to \mathcal{NW}_r(0, 0) \otimes A$ induces an isomorphism on the 0-page of the spectral sequence and so on all subsequence pages of the spectral sequence.

We then have a model for the loops functors on the level of the model category $(b\mathcal{C})_S$ given by Ω^r and again since suspension is adjoint to the loops functor and we have modelled the loop functor by an auto-equivalence of $b\mathcal{C}$ the suspension functor can be modelled by Σ^r.

Lemma 3.9.2.3. *The loop functor on an object A in the model category $(b\mathcal{C})_S$ is given by $\Omega^r A$.*

And we have shown the following result.

Proposition 3.9.2.4. *The model categories $(b\mathcal{C})_S$ of Theorem 3.2.0.2 are stable model categories whose loops and suspension functors are given by Ω^r and Σ^r.*

3.10 Quillen equivalences between the posets

We consider the $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ adjunction of Proposition 2.1.0.2 which was shown to be a Quillen adunction between appropriate model structures

$$\begin{tikzcd} \mathcal{L}\colon (f\mathcal{C})_S \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (b\mathcal{C})_S \colon \mathrm{Tot}^{\Pi} \arrow[l, shift left] \end{tikzcd} .$$

Recall from Definitions 1.4.12.5 and 1.4.12.6 the notion of a right Bousfield localisation and from Theorem 1.4.12.7 the existence of right Bousfield localisations of right proper cellular model categories at the $\mathcal{K}$-cellular equivalences for some set of objects $\mathcal{K}$. These definitions and results can also be found in [GS13]. Since both model structures are right (and left) proper and cellular they admit right (and left) Bousfield localisations at any set of objects $\mathcal{K}$ (and morphisms $\mathcal{C}$) by Theorems 1.4.12.3 and 1.4.12.7. We also have the following result of Dugger.

Proposition 3.10.0.1 ([Dug01, Proposition A.5])**.** *For $\mathcal{M}$ a left proper, cofibrantly generated model category there exists a set W of cofibrant objects detecting weak equivalences, i.e. that $X \to Y$ is a weak equivalence if and only if the induced map on homotopy function complexes*

$$\mathrm{map}(A, X) \to \mathrm{map}(A, Y)$$

is a weak equivalence for all $A \in W$. Further the set W can be taken to be a set of cofibrant replacements for the domains and codomains of the generating cofibrations.

We will make use of these results along with the *cellularization principle* of Greenlees-Shipley to show that the $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ adjunction is in fact a Quillen equivalence. We likely do not have need for such a powerful result as we will show the particular cellularizations we take do not in fact alter the model categories. The following is Greenlees and Shipley's definition of smallness required to state the cellularization theorem which is a notion of smallness in the homotopy category. To avoid confusion we shall refer to it as *homotopically small.*

Definition 3.10.0.2. An object K is *homotopically small* if for any set of objects $\{Y_\alpha\}$ we have, in the homotopy category, the natural map $\bigoplus_\alpha [K, Y_\alpha] \to [K, \wedge_\alpha Y_\alpha]$ is an isomorphism.

Recall the notion of the $\mathcal{K}$ cellularization of $\mathcal{M}$ given in Definition 1.4.12.8.

Theorem 3.10.0.3 (The Cellularization Principle, [GS13, Theorem 2.7])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be right proper, stable, cellular model categories with a left Quillen functor $F\colon \mathcal{M} \to \mathcal{N}$ and adjoint U. Write Q for a cofibrant replacement functor in $\mathcal{N}$ and R for a fibrant replacement in $\mathcal{N}$.*

1. *Let $\mathcal{K} = \{A_\alpha\}$ be a set of objects in $\mathcal{M}$ with $FQ\mathcal{K} = \{FQA_\alpha\}$ the corresponding set in $\mathcal{N}$. Then F and U induce a Quillen adjunction:*

$$\begin{tikzcd} F\colon \mathcal{K}\text{-}\mathtt{cell}\text{-}\mathcal{M} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & FQ\mathcal{K}\text{-}\mathtt{cell}\text{-}\mathcal{N} \colon U \arrow[l, shift left] \end{tikzcd}$$

   between the $\mathcal{K}$-cellularization of $\mathcal{M}$ and the $FQ\mathcal{K}$-cellularization of $\mathcal{N}$.

2. *If $\mathcal{K} = \{A_\alpha\}$ is further a stable set of homotopically small objects in $\mathcal{M}$ such that for each $A \in \mathcal{K}$ we have FQA is small in $\mathcal{N}$ and the derived unit $QA \to URFQA$ is a weak equivalence in $\mathcal{M}$, then F and U induce a Quillen equivalence between the cellularizations:*

$$\mathcal{K}\text{-}\mathtt{cell}\text{-}\mathcal{M} \simeq_Q FQ\mathcal{K}\text{-}\mathtt{cell}\text{-}\mathcal{N} .$$

There is a third part to the theorem instead using a set of homotopically small objects of $\mathcal{N}$ we have no need of here. Our goal now is to use the celullarization principle to show that the adjunctions $\mathcal{L}\colon (f\mathcal{C})_S \underset{\perp}{\rightleftarrows} (b\mathcal{C})_S : \mathrm{Tot}^{\Pi}$ are Quillen equivalences. We will take $\mathcal{M} = (f\mathcal{C})_S, \mathcal{N} = (b\mathcal{C})_S$ and $\mathcal{K}$ will be the set obtained from Proposition 3.10.0.1 which consists of the domains and codomains of the generating cofibrations of $(f\mathcal{C})_S$ (they are already cofibrant objects). We take the functors to be $F = \mathcal{L}$ and $U = \mathrm{Tot}^{\Pi}$. Note that in this setting we already know $\mathcal{M}$ and $\mathcal{N}$ are right proper, stable and cellular. The first part of Theorem 3.10.0.3 is then immediate. To obtain the result of the second part we need to check the following:

1. $\mathcal{K}$ is a stable set of homotopically small objects,
2. for each $A \in \mathcal{K}$ that FQA is homotopically small (the A are already cofibrant so we needn't apply Q here),
3. the derived unit $QA \to URFQA = \mathrm{Tot}^{\Pi}\mathcal{L}A$ is a weak equivalence for all $A \in \mathcal{K}$ (A is already cofibrant and every object is fibrant).

This will establish the Quillen equivalence between the $\mathcal{K}$-cellulatization $\mathcal{K}$-`cell-` $(f\mathcal{C})_S$ and the $FQ\mathcal{K}$-cellularization $FQ\mathcal{K}$-`cell-` $(b\mathcal{C})_S$. The former is just $(f\mathcal{C})_S$ since the $\mathcal{K}$-cellular equivalences are simply r-quasi-isomorphisms and the fibrations are unchanged by right Bousfield localisations — the definition of $\mathcal{K}$-cellular equivalence was given in Definition 1.4.12.5 and that these are the r-quasi-isomorphisms follows from Proposition 3.10.0.1. It will then remain to show that:

4. right Bousfield localising $(b\mathcal{C})_S$ at $FQ\mathcal{K}$ doesn't change the model structure.

Theorem 3.10.0.4. *For S containing both 0 and r there is a Quillen equivalence between the S-model structure on filtered chain complexes and the S-model structure on bicomplexes given by the $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ adjunction:*

$$\mathcal{L}\colon (f\mathcal{C})_S \underset{\perp}{\rightleftarrows} (b\mathcal{C})_S : \mathrm{Tot}^{\Pi} \quad .$$

Proof. We will show the conditions 1–4 above hold. Note 1 is immediate since the generating cofibrations are closed under the loop and suspension functors. For condition 2 we already have that the A are cofibrant so we need only show that the $\mathcal{L}A$ are homotopically small for all $A \in \mathcal{K}$. Such an A is some t-cycle where $t \leq r+1$ and from Lemma 2.2.1.1 $\mathcal{L}\mathcal{Z}_t(*,*)$ is isomorphic to the direct sum of a witness t-cycle and an infinite number of witness 0-cycles. In the homotopy category this is isomorphic to just the constituent t-cycle and by [Hov99, Theorem 7.4.3] we have that this is homotopically small.

Condition 3 was shown in Proposition 2.2.1.2. For condition 4 we compute $\mathcal{L}\mathcal{K}$. The set $\mathcal{K}$ consists of some of the t-cycles for $t \leq r+1$. Applying the functor $\mathcal{L}$ to this set we obtain some bicomplexes, which by Lemma 2.2.1.1, are each isomorphic to a witness t-cycle and an infinite direct sum of witness 0-cycles. Applying the same result, Proposition 3.10.0.1, used to obtain $\mathcal{K}$ to the model category $(b\mathcal{C})_S$ we obtain a set $\mathcal{J}$ of objects detecting weak equivalences and such that right Bousfield localising $(b\mathcal{C})_S$ at $\mathcal{J}$ does not change the model structure. Note then the effects of localising at the sets $\mathcal{L}\mathcal{K}$ and $\mathcal{J}$ are the same since (up to direct sums) they both consist of the same witness t-cycle objects, noting too that the functor $\mathcal{L}$ introduces into $\mathcal{K}$ witness 0-cycles which are always present as generating acyclic cofibrations of the bicomplex model structures.

We have then shown that the conditions of the second part of the cellularization principle, Theorem 3.10.0.3, are satisfied and that the $\mathcal{K}$ and $FQ\mathcal{K}$-cellularisations do not change the model structure, therefore we have a Quillen equivalence:

$$\mathcal{L}\colon (f\mathcal{C})_S \underset{\perp}{\rightleftarrows} (b\mathcal{C})_S : \mathrm{Tot}^{\Pi} \quad .$$

∎

A priori one might not expect the model structures $(f\mathcal{C})_S$ and $(b\mathcal{C})_S$ to be Quillen equivalent. The reason being that the category of filtered chains 'contains much more information', e.g. the filtrations associated to the totalisation of a bicomplex is always Hausdorff and exhaustive which are not required of our filtered chain complexes. An explanation for this Quillen equivalence despite the descrepancy is that the S-model structure of filtered chain complexes 'only sees' the filtration within $(r+1)$ filtration degrees of any finite filtration stage.

3.11 Non-existence of certain Bousfield localisations

Consider in either filtered chain complexes or bicomplexes one of the model structures of the poset $\mathcal{M}_S$ whose weak equivalences are the r-weak equivalences. A natural question is: "Does this model structure admit a (left or right)

Bousfield localisation with weak equivalences the $(r+1)$-weak equivalences?". We will show here that no such left Bousfield localisations exist, the issue being that the S-generating cofibrations are contained in the $(r+1)$-weak equivalences.

Proposition 3.11.0.1. *Let $\mathcal{M}_S$ be one of the S-model structures of either poset, where $\mathcal{M}$ is either $f\mathcal{C}$ or $b\mathcal{C}$ whose weak equivalences are the r-weak equivalences. Then there is no left Bousfield localisation $\mathcal{M}_{new}$ of $\mathcal{M}_S$ whose weak equivalences are the $(r+1)$-weak equivalences.*

Proof. Suppose for contradiction such a left Bousfield localisation $\mathcal{M}_{new}$ exists, so that the cofibrations, fibrations and weak equivalences satisfy $\mathcal{C}_{new} = \mathcal{C}_S, \mathcal{F}_{new} \subset \mathcal{F}_S$ and $\mathcal{W}_{new} = \mathcal{W}_{r+1} \supset \mathcal{W}_r = \mathcal{W}_S$ respectively. Note that the $\mathcal{M}_{new}$ at least has a set of generating cofibrations $I_{new} = I_S$ and that $I_{new} \subset \mathcal{W}_{new}$. We will show as a consequence that all cofibrations $\mathcal{C}_{new} \subset \mathcal{W}_{new}$.

By assumption $\mathcal{M}_{new}$ is a model structure so that pushouts of acyclic cofibrations are acyclic cofibrations, hence pushouts along elements of I_{new} are acyclic cofibrations in $\mathcal{M}_{new}$. All I_S-`Cell` morphisms are now also acyclic cofibrations in $\mathcal{M}_{new}$. Indeed we can take such a transfinite composition $X\colon \lambda \to \mathcal{M}$ of morphisms that are pushouts along elements of I_S which is a composition of $(r+1)$-weak equivalences and consider it in the model category $\mathcal{M}_{r+1}$. This model structure is finitely generated by either Lemma 1.2.1.16 or [Hov99, Lemma 2.3.3] and the domains and codomains of I_r are finite relative to the whole category, so transfinite compositions of $(r+1)$-weak equivalences are $(r+1)$-weak equivalences by [Hov99, Corollary 7.4.2]. Hence in $\mathcal{M}_{r+1}$ the morphism $X_0 \to \operatorname{colim} X$ is an $(r+1)$-weak equivalence and so too then is a weak equivalence of the model structure $\mathcal{M}_{new}$ since $\mathcal{W}_{new} = \mathcal{W}_{r+1}$. By Proposition 1.4.2.10 all cofibrations $\mathcal{C}_{new}$ are retracts of elements of I_S-`Cell` and since the latter are all $(r+1)$-weak equivalences so too are all $\mathcal{C}_{new}$ since retracts preserve weak equivalences.

Now consider any morphism f in $\mathcal{M}_{new}$ and factor it as a cofibration followed by an acyclic fibration, by the above the cofibration is necessarily an $(r+1)$-weak equivalence and so too is the acyclic fibration since its acyclic. Hence so too is f which shows any morphism of $\mathcal{M}$ is an $(r+1)$-weak equivalence which gives the contradiction.

3.12 Bounded model structures on $(f\mathcal{C})_S$

Given a model category $\mathcal{M}$ and a category $\mathcal{C}$ one can often construct new model category structures by means of a transfer along an adjunction either of the form $F\colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{C} : U$ or $U\colon \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{M} : F$, e.g. Theorem 1.4.4.1 applies in the former case when suitable conditions are satisfied and more generally one has the results of [BHK$^+$15, GKR20] which give existence results for model structures transferred along adjunctions (the latter paper contains a corrected proof of a result of the former). One of the main results stated is the following theorem.

Theorem 3.12.0.1 ([BHK$^+$15, Theorem 2.23]). *For an adjunction $U\colon \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{M} : F$ of locally presentable categories with $\mathcal{M}$ cofibrantly generated by a pair of sets with cofibrations* Cof *and weak equivalences $\mathcal{W}$ such that $\left(U^{-1}\mathrm{Cof}\right)$-$\mathit{Inj} \subseteq U^{-1}\mathcal{W}$ then there is a left induced model structure on $\mathcal{C}$ cofibrantly generated by a pair of sets with cofibrations $U^{-1}\mathrm{Cof}$ and weak equivalences $U^{-1}\mathcal{W}$.*

This is shown by an application of [MR14, Theorem 3.2] which asserts that the 2-*category of combinatorial categories* have *pseudopullbacks* computed in the 2-*category of cellular categories* and as explained in [CR14, Remark A.3] one computes the transferred model structure by a pseudopullback of $\mathcal{C}_{\mathrm{triv}} \to \mathcal{M}_{\mathrm{triv}}$ along $\mathcal{M} \to \mathcal{M}_{\mathrm{triv}}$ where the triv subscripts denote a trivial *weak factorisation system* in which all morphisms are cofibrations.

Definition 3.12.0.2. Given an adjunction $U\colon \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{M} : F$ with $\mathcal{M}$ a model structure with weak equivalences $\mathcal{W}$ and cofibrations Cof if there exists a model structure on $\mathcal{C}$ whose weak equivalences are $U^{-1}\mathcal{W}$ and cofibrations $U^{-1}\mathrm{Cof}$ then this model structure on $\mathcal{C}$ is said to be *a left induced model structure.*

These then are some of the tools we have to transfer model structures along adjunctions and note all these more general results on transferred model structures involve checking some acyclicity condition.

As examples and to motivate what follows one has adjunctions involving the inclusion of non-negatively or non-positively graded cochain complexes into cochain complexes and one can induce, from the projective model structure on unbounded chain complexes, model structures on the bounded variants which for the non-positively graded cochain complexes is the usual bounded projective model structure. This bounded projective model structure can be found for instance in the discussion following [GJ09, Corollary 2.12]. In the following the functor i denotes the inclusion of bounded variants of a category into the full category.

First recall that from the unbounded projective model structure on chain complexes that there is a truncation functor $\tau\colon \mathcal{C}_R \to \mathcal{C}_R^{\geq}$ which is the identity on positive degrees and in degree 0 is the quotient by the the image of the differential $d\colon X^{-1} \to X_0$. This functor is left adjoint to the inclusion functor of non-negatively graded chain complexes into chain complexes:

$$\begin{tikzcd} \tau\colon \mathcal{C}_R \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}_R^{\geq} :i \arrow[l, shift left] \end{tikzcd} .$$

The Kan transfer theorem, Theorem 1.4.4.1, can then be used to obtain a cofibrantly generated model structure on non-negatively graded chain complexes.

To induce the projective model structure on non-positively graded cochain complexes note that one can't apply the Kan transfer theorem. The adjunction that one would try to use is of the form

$$\begin{tikzcd} n\colon \mathcal{C}_R \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}_R^{\leq} :i \arrow[l, shift left] \end{tikzcd}$$

where the functor n is the naïve truncation which simply forgets the portion of the cochain complex in positive degrees. Note then however that this adjunction does not satisfy the conditions of Theorem 1.4.4.1 since, for J the generating cofibrations of $\mathcal{C}_R$, i does not send relative nJ-`Cell` complexes to weak equivalences: the morphism from 0 to the disc with components in degrees 0 and 1 is in J, applying n to it gives the relative nJ-`Cell` morphism $0 \to R^0$ which is not a weak equivalence after applying i.

Instead then one can apply Theorem 3.12.0.1 to a different adjunction. The functor i is also a left adjoint with right adjoint given by κ which forgets the portion of the cochain complex in positive degrees and in degree 0 takes the kernel of the differential.

$$\kappa A^n := \begin{cases} \ker\left(d\colon A^0 \to A^1\right), & n = 0 \\ A^n, & n \leq -1 \end{cases}$$

The acyclicity condition is not hard to check here since bounded cofibrant objects are just the degreewise projective chain complexes. Furthermore we have that κI and κJ are generating sets for the bounded projective model structure.

We give existence of various bounded model structures obtained from those of $(f\mathcal{C})_S$ by a transfer theorem.

Note 3.12.0.3. In the following subsections imposing any of the boundedness conditions loses stability and shift-décalage functors so we don't necessarily have equivalences of the homotopy categories when we vary S.

3.12.1 S-Model structures on $f\mathcal{C}^{\geq}$

In [FGLW22, Proposition 5.11] the authors use the Kan transfer theorem in a similar way to obtain for each $r \geq 0$ cofibrantly generated model structures on their bounded n-truncated multicomplexes, bounded in the sense that $A^{p,q} = 0$ whenever $p \geq 0$. The case $n = 2$ gives bounded model structures on $b\mathcal{C}$ which easily generalises to bounded model structures on the $(b\mathcal{C})_S$, bounded in the same sense. We give here analogous bounded model structures on $(f\mathcal{C})_S$ again by application of the Kan transfer theorem. There is an adjunction we also denote $\tau \dashv i$ between the categories $f\mathcal{C}$ and $f\mathcal{C}^{\geq}$ where the latter denotes the category of filtered objects in $\mathcal{C}_R^{\geq}$. The right adjoint i is inclusions of a filtered bounded complex into filtered complexes and the left adjoint τ is given as

$$\tau A^n := \begin{cases} A^n, & n > 1 \\ A^0/\operatorname{im} d, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

where the filtration on τA^0 is $F_p \tau A^0 := (F_p A^0)/\operatorname{im} d$. We then verify this adjunction $\tau\colon f\mathcal{C} \underset{\perp}{\rightleftarrows} f\mathcal{C}^{\geq} :i$ satisfies the conditions of the Kan transfer theorem where we equip $(f\mathcal{C})_S$ with one of the S-model structures.

Lemma 3.12.1.1. *The adjunction* $\tau\colon (f\mathcal{C})_S \underset{\perp}{\rightleftarrows} f\mathcal{C}^{\geq} :i$ *satisfies the conditions of Theorem 1.4.4.1.*

Proof. All objects of $f\mathcal{C}^{\geq}$ are small by a similar proof to Lemma 1.2.1.16 which proves condition 1. For condition 2 note that the set τJ consists of s-cycles $\mathcal{Z}_s(p, n)$ for $n \geq 0$; the ones of the form $\mathcal{Z}_s(p, 0)$ become 0 under the functor τ. Hence applying i to a relative τJ_S-`Cell` is a weak equivalence.

Applying the Kan transfer theorem we then obtain the following corollary.

Corollary 3.12.1.2. *There is a cofibrantly generated model structure denoted* $\left(f\mathcal{C}^{\geq}\right)_S$ *on* $f\mathcal{C}^{\geq}$ *whose weak equivalences are the r-quasi isomorphisms and with generating cofibrantions* τI_S *and generating acyclic cofibrations* τJ_S.

3.12.2 S-Model structures on $f\mathcal{C}^{\leq}$

For the same reason as above with $n\colon \mathcal{C}_R \underset{\perp}{\rightleftarrows} \mathcal{C}_R^{\leq} \colon i$ we cannot apply the same method to equip $f\mathcal{C}^{\leq}$ with an S-model structure. Observe that the left adjoint to the inclusion functor $i\colon f\mathcal{C}^{\leq} \to f\mathcal{C}$ is given by naïve truncation:

$$n\colon f\mathcal{C} \underset{\perp}{\rightleftarrows} f\mathcal{C}^{\leq} \colon i$$

which simply forgets the portion of the filtered chain complex in positive cohomological degree. However then condition 2 of Theorem 1.4.4.1 is not satisfied: $n\mathcal{Z}_s(p, p+0)$ is an $nJ\text{-}\mathtt{Cell}$ complex but i does not send it to a weak equivalence in $(f\mathcal{C})_S$.

Instead then we would like to apply Theorem 3.12.0.1 to the adjunction

$$i\colon f\mathcal{C}^{\leq} \underset{\perp}{\rightleftarrows} f\mathcal{C} \colon \kappa \quad .$$

as before where here κ does the same kernel truncation ignoring filtration. The issue here however is that checking the acyclicity condition is not an option since our study of cofibrations of Chapter 4 is not complete enough. However recall that κI and κJ gave generating (acyclic) cofibrations for the bounded projective model structure on cochain complexes and we can ask do κI_S and κJ_S give generating (acyclic) cofibrations for a model structure on $f\mathcal{C}^{\leq}$.

Theorem 3.12.2.1. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r, the category $f\mathcal{C}^{\leq}$ admits a right proper cofibrantly generated model structure, which we denote $\left(f\mathcal{C}^{\leq}\right)_S$, where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $Z_s^{*,*+n}(f)$ is bidegree-wise surjective for $n \leq -1$ and $s \in S$, and*
3. *κI_S and κJ_S are the sets of generating cofibrations and generating acyclic cofibrations respectively.*

Furthermore $\left(f\mathcal{C}^{\leq}\right)_S$ is a finitely generated model category.

Proof. For $S = \{r\}$ one verifies that [CELW19, Propositions 3.12 & 3.13] still hold for I and J replaced by κI and κJ. This gives the model structures for $S = \{r\}$. One then applies the same method for the proof of Theorem 3.1.0.2 to obtain the remaining model structures.

3.12.3 Bounded model structures on $f_{\geq}\mathcal{C}$

We'd similarly like to bound the filtration, we consider then the subcategory of $f\mathcal{C}$ consisting of those objects A with $F_{-1}A = 0$. We denote this subcategory by $f_{\geq}\mathcal{C}$ and note that there is an adjunction $q\colon f\mathcal{C} \underset{\perp}{\rightleftarrows} f_{\geq}\mathcal{C} \colon i$ where again i is inclusion and the left adjoint q is given by $qA^n := A^n/F_{-1}A^n$ with filtration $F_p qA^n := F_pA^n/F_{-1}A^n$ for $p \geq 0$ and obvious induced differentials. This adjunction $q \dashv i$ has the same defects as that of $n \dashv i$ when trying to apply the Kan transfer theorem if $r > 1$; the relative qJ_S-cell complex given by $0 \to q\mathcal{Z}_r(0,0)$ is just $0 \to R^0_{(0)}$ which is not sent to a weak equivalence by i. However we can apply Kan transfer for the case $r = 0$.

Theorem 3.12.3.1. *For $r = 0$ the category $f_{\geq}\mathcal{C}$ admits a right proper cofibrantly generated model structure, which we denote $(f_{\geq}\mathcal{C})_0$, where:*

1. *weak equivalences are the E_r-weak-equivalences,*
2. *fibrations are morphisms of filtered chain complexes $f\colon A \to B$ such that $Z_0^{p,p+n}(f)$ is bidegree-wise surjective, and*
3. *qI_0 and qJ_0 are the sets of generating cofibrations and generating acyclic cofibrations respectively.*

Proof. All objects of $f_{\geq}\mathcal{C}$ are small by a similar proof to Lemma 1.2.1.16. A relative $qJ_0\text{-}\mathtt{Cell}$ complex is of the form $A \to A \oplus \bigoplus \mathcal{Z}_0(p, p+n)$ where the direct sum is over some number of copies of 0-cycles with $p \geq 0$. The functor i then sends this morphism to a 0-weak equivalence in $f\mathcal{C}$. This shows the 2 conditions of Theorem 1.4.4.1 are satisfied so we apply the Kan transfer theorem to $q\colon f\mathcal{C} \underset{\perp}{\rightleftarrows} f_{\geq}\mathcal{C} \colon i$.

There are other adjunctions between $f_{\geq}\mathcal{C}$ and $f\mathcal{C}$. We can exhibit i, the inclusion functor, as the left adjoint.

$$\begin{tikzcd} i\colon f_{\geq}\mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & f\mathcal{C} \arrow[l, shift left] \colon\zeta \end{tikzcd} \quad .$$

The functor ζ simply forgets the filtration information in filtration degrees $p < 0$ so that the composite functor $i\zeta$ on a filtered chain complex A is setting $F_{-1}i\zeta A = 0$ and $F_p i\zeta A = F_p A$ for higher p with the same underlying chain complex. The effect of ζ on a cycle $\mathcal{Z}_r(p, p+n) \in f\mathcal{C}$ is the identity if $p \geq r$ or a shortening (or concertina) of the cycle to $\mathcal{Z}_p(p, p+n)$ otherwise where we interpret these latter cycles in $f_{\geq}\mathcal{C}$.

The functor ζ is also a left adjoint to the functor η which given an $A \in f_{\geq}\mathcal{C}$ has $\eta A \in f\mathcal{C}$ as having the same underlying chain complex with the same filtration in non-negative degrees and where $F_p\eta A = F_0 A$ for all $p \leq 0$.

Lastly η is also left adjoint to a functor we denote $\nu\colon f\mathcal{C} \to f_{\geq}\mathcal{C}$. On an $A \in f\mathcal{C}$ the functor ν has the same filtration for positive filtration degrees however $F_0\nu A = \cap_{p<0} F_p A$.

We then have a chain of adjunctions filtered chain complexes and non-negatively bounded filtered chain complexes:

$$q \dashv i \dashv \zeta \dashv \eta \dashv \nu.$$

The author has been unable to show that the S-model structure on $f\mathcal{C}$ transfers, either by Theorem 1.4.4.1 or Theorem 3.12.0.1, along any of these adjunctions. Instead then to show existence of the remaining model structures $(f_{\geq}\mathcal{C})_S$ we follow the proof method of [CELW19] with an appropriate choice of generating cofibrations and acyclic cofibrations. We do so first for $S = \{r\}$ and then construct all S-model structures as was done for the $(f\mathcal{C})_S$.

Definition 3.12.3.2. We denote by $I_r^{\geq}$ and $J_r^{\geq}$ the sets of morphisms given by:

$$\begin{aligned} I_r^{\geq} &\coloneqq q\left\{\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)\right\} \\ J_r^{\geq} &\coloneqq \left\{0 \to \mathcal{Z}_r(p,n)\right\}_{p \geq r} \end{aligned}$$

We have then restricted our set of r-cycles by taking those which live in filtration degree 0 or greater and truncated the generating cofibrations. For those generating cofibration with $p \geq r+1$ this has no effect. Otherwise they are altered so as to introduce instead morphisms of the form:

$$\begin{tikzcd} & \left(R^{n}_{(p)}\right) \arrow[d, "\Delta"] \\ \left(R^{n-1}_{(p+r)} \arrow[r, "i_1"] & R^{n}_{(p)} \oplus R^{n}_{(p-1)}\right) \end{tikzcd}$$

when $r+1 > p \geq 1$. When we have $p = 0$ we obtain the new generating cofibration:

$$\begin{tikzcd} & \left(R^{n}_{(p)}\right) \arrow[d, "1"] \\ \left(R^{n-1}_{(p+r)} \arrow[r, "i_1"] & R^{n}_{(p)}\right) \end{tikzcd}$$

and for $0 > p \geq -r$ we obtain simply:

$$\begin{tikzcd} 0 \arrow[d] \\ \left(R^{n-1}_{(p+r)}\right) \end{tikzcd} \quad .$$

All smaller p result in the morphism $0\colon 0 \to 0$. As for the r-model structure these are easily seen to be the representing objects for the r-cycles and r-boundaries whose quotient gives the r-page of the associated spectral sequences. We now need to show a similar sequence of lemmas as in [CELW19] used to prove a model category structure hold. Recall the morphism $\varphi_r\colon \mathcal{Z}_r(p,n) \to \mathcal{B}_r(p,n)$ representing the inclusion of boundary elements into cycle elements.

Definition 3.12.3.3. We write $\mathcal{Z}_r^{\geq}(p,n) \coloneqq q\mathcal{Z}_r(p,n)$ and $\mathcal{B}_r^{\geq}(p,n) \coloneqq q\mathcal{B}_r(p,n)$. We also abuse notation and denote by φ_r the morphism $q\varphi_r$.

Lemma 3.12.3.4. *Pushouts of the generating cofibrations yield either* $0 \to \mathcal{Z}_{r+1}(p,n)$ *for* $p \geq r+1$ *or the morphism* $0 \to R^n_{(p)}$ *for* $0 \leq p \leq r$.

Definition 3.12.3.5. We say a morphism $f\colon A \to B$ of $f_{\geq}\mathcal{C}$ is *$K^{p,p+n}$-surjective* if there exist all lifts of diagrams of the form:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ R^n_{(p)} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} .$$

Corollary 3.12.3.6. *Any morphism* $f \in I_r^{\geq}\text{-}\mathtt{Inj}$ *satisfies* $Z_{r+1}^{p,p+*}$*-surjectivity for* $p \geq r+1$ *and* $K^{p,p+*}$*-surjectivity for* $r \geq p \geq 0$.

Definition 3.12.3.7. For $A \in f_{\geq}\mathcal{C}$ we define the R-modules $ZK_{r+1}^{p,p+n}(A) := \operatorname{Hom}(\mathcal{Z}_{r+1}^{\geq}(p,n), A)$ where we take the Hom objects as R-modules. Similarly we define the R-modules $BK_{r+1}^{p,p+n}(A) := \operatorname{Hom}(\mathcal{B}_{r+1}^{\geq}(p,n), A)$

These give the $(r+1)$-cycle and boundary objects in the bounded context of $f_{\geq}\mathcal{C}$ whose quotient gives the $(r+1)$-page of the associated spectral sequence. Explicitly then these are given by

$$ZK_{r+1}^{p,p+n}(A) = \begin{cases} \operatorname{Hom}(\mathcal{Z}_{r+1}(p,n), A), & \text{for } p \geq r+1 \\ \operatorname{Hom}(R^n_{(p)}, A), & \text{for } 0 \leq p \leq r \end{cases}$$

for the former and for the latter we have

$$BK_{r+1}^{p,p+n}(A) = \begin{cases} \operatorname{Hom}(\mathcal{B}_{r+1}(p,n), A), & \text{for } p \geq r+1 \\ \operatorname{Hom}(R^{n-1}_{(p+r)} \xrightarrow{i_1} R^n_{(p)} \oplus R^n_{(p-1)}, A), & \text{for } 1 \leq p \leq r \\ \operatorname{Hom}(R^{n-1}_{(p+r)} \xrightarrow{1} R^n_{(p)}, A), & \text{for } p = 0. \end{cases}$$

Lemma 3.12.3.8 ([CELW19, Lemma 2.8])**.** *For* $r \geq 0$ *and* $f\colon A \to B$ *a morphism of* $f_{\geq}\mathcal{C}$ *the following are equivalent:*

1. *the maps* $ZK_r(f)$ *and* $ZK_{r+1}(f)$ *are bidegree-wise surjective, and*

2. *the maps* $ZK_r(f)$ *and* $E_{r+1}(f)$ *are bidegree-wise surjective.*

Proof. This is shown in much the same way as in [CELW19].

Proposition 3.12.3.9 ([CELW19, Proposition 3.11])**.** *A morphism* f *of* $f_{\geq}\mathcal{C}$ *is* $J_r^{\geq}\text{-}\mathtt{Inj}$ *if and only if* $Z_r^{p,p+n}(f)$ *is surjective for all* $p \geq r$ *and all* n.

Proposition 3.12.3.10 ([CELW19, Proposition 3.12])**.** *We have* $I_r^{\geq}\text{-}\mathtt{Inj} = J_r^{\geq}\text{-}\mathtt{Inj} \cap \mathcal{E}_r$.

Proof. The proof is much the same as in [CELW19] with care taken for those cycle and boundary objects which have become truncated.

Take an $f\colon A \to B$ which is $I_r^{\geq}\text{-}\mathtt{Inj}$ and therefore satisfies $Z_{r+1}^{p,p+*}$-surjectivity for $p \geq r+1$ and $K^{p,p+*}$-surjectivity for $0 \leq p \leq r$ by Corollary 3.12.3.6. In the following diagram:

$$\begin{tikzcd} & & A \arrow[d, "f"] \\ \mathcal{Z}_{r+1}^{\geq}(p,n) \arrow[r] \arrow[urr, dashed, "\gamma"] & \mathcal{B}_{r+1}^{\geq}(p,n) \arrow[r, "g"'] \arrow[ur, dashed, "\psi"'] & B \end{tikzcd}$$

the lift γ exists by Corollary 3.12.3.6, and then so too does the lift ψ by $I_r^{\geq}$-injectivity. This shows that $f \in J_r^{\geq}\text{-}\mathtt{Inj}$. Since f is $Z_{r+1}^{p,p+*}$-surjectivity for $p \geq r+1$ and $K^{p,p+*}$-surjectivity for $0 \leq p \leq r$ by Corollary 3.12.3.6 it is then also E_{r+1}-bidegree-wise surjective. We check that $E_{r+1}(f)$ is also bidegree-wise injective. Let $[a]$ be a class in $E_{r+1}(A)$ represented by an $a \in ZK_{r+1}(A)$ such that $E_{r+1}(f)([a]) = 0$ i.e. that $[fa] = 0$ so there is a boundary element of $BK_{r+1}(B)$ whose image under φ_r is $f(a)$. Write b and c for the cycle components of this boundary image of

$BK_{r+1}(B)$ noting that for b we might have either $db = 0$ if $1 \leq p \leq r$ or $b = 0$ when $p = 0$ so that $f(a) = b + dc$ There is then a commuative diagram of the form

$$\begin{tikzcd}
\mathcal{Z}^{\geq}_{r+1}(p,n) \arrow[r, "a"] \arrow[d, "\varphi_{r+1}"'] & A \arrow[d, "f"] \\
\mathcal{B}^{\geq}_{r+1}(p,n) \arrow[r, "b+dc"] \arrow[ur, dashed, "b'+dc'"] & B
\end{tikzcd} \tag{3.10}$$

for which a lift exists since f is $I^{\geq}_r$-`Inj` with b' again perhaps having $db' = 0$ or $b' = 0$ for the same filtration indices. This shows injectivity of $E_{r+1}(f)$.

We now wish to show that if $f \in J^{\geq}_r$-`Inj` $\cap \mathcal{E}_r$ then $f \in I^{\geq}_r$-`Inj`. Given a solid diagram of the form of Equation (3.10) we demonstrate existence of a lift. Take a, b and c as above with similar restrictions according to filtration degree. Since $f \in \mathcal{E}_r$ we have by injectivity on the $(r+1)$-page that $[a] = 0$ so that $a \in BK^{p,p+n}(A)$ and we write $a = b' + dc'$ as before although note these are not necessarily lifts of b and c yet. We then have the equation $b - f(b') = d(f(c') - c)$ which shows the element $f(c') - c$ is an element of $ZK^{p+r,p+r+n-1}(B)$. By Lemma 3.12.3.8 we can lift this to an element u of $ZK^{p+r,p+r+n-1}_{r+1}(A)$ so that $f(u) = f(c') - c$ and $b - f(b') = df(u) = f(du)$. Setting $\beta = b' + du$ and $\gamma = c' - u$ gives a lift of the element of BK. ❂

Proposition 3.12.3.11 ([CELW19, Proposition 3.13]). *For all $r \geq 0$ and all $0 \leq k \leq r$ we have $J^{\geq}_r$-`Cof` $\subseteq \mathcal{E}_r$.*

Proof. The proof is identical to that of [CELW19] noting that their functor $\mathcal{M}_r$ restricts to an endo-functor on $f_{\geq}\mathcal{C}$ which enjoys the same properties as that on $f\mathcal{C}$. ❂

Theorem 3.12.3.12 (Theorem 3.14). *For every $r \geq 0$ the category $f_{\geq}\mathcal{C}$ admits a right proper cofibrantly generated model structures whose:*

- *weak equivalences are the E_r-quasi-isomorphisms,*
- *fibrations are morphisms that are $Z^{p,p+n}_r$-surjective for $p \geq r$ and all n, and*
- *generating cofibrations and generating acyclic cofibrations are given by $I^{\geq}_r$ and $J^{\geq}_r$ respectively.*

Proof. We verify conditions 1–5 of Theorem 1.4.2.9. Condition 1 is clear, condition 2 follows since all elements of $f_{\geq}\mathcal{C}$ are small via a similar proof as in Lemma 1.2.1.16, condition 3 follows from Proposition 3.12.3.11 and $J^{\geq}_r$-`Cell` $\subseteq$ $J^{\geq}_r$-`Cof` $\subseteq I^{\geq}_r$-`Cof` where the last inclusion follows since $I^{\geq}_r$-`Inj` $\subseteq J^{\geq}_r$-`Inj` by Proposition 3.12.3.10. Conditions 4 and 5 follows from Proposition 3.12.3.10. This gives a right proper (since every object is fibrant) cofibrantly generated model category as claimed. ❂

From these we can, as in the unbounded filtration setting, obtain S-model structures on $f_{\geq}\mathcal{C}$.

Definition 3.12.3.13. Let S be a subset of $\{0, 1, 2 \ldots, r\}$ containing r. We define the sets $I^{\geq}_S$ and $J^{\geq}_S$ of morphisms of $f_{\geq}\mathcal{C}$ as follows:

$$I^{\geq}_S := I^{\geq}_r \cup \bigcup_{s \in S} J^{\geq}_s \ ,$$
$$J^{\geq}_S := \bigcup_{s \in S} J^{\geq}_s \ .$$

Lemma 3.12.3.14. *We have $I^{\geq}_S$-`Inj` $= J^{\geq}_S$-`Inj` $\cap \mathcal{E}_r$.*

Proof. The third inequality in the following is the result $I^{\geq}_r$-`Inj` $= J^{\geq}_r$-`Inj` $\cap \mathcal{E}_r$ for the r-model structure on $f_{\geq}\mathcal{C}$.

$$\begin{aligned}
I^{\geq}_s &= \left(I^{\geq}_r \cup \bigcup_{s \in S} J^{\geq}_s \right) \text{-}\mathtt{Inj} \\
&= I^{\geq}_r\text{-}\mathtt{Inj} \cap \bigcap_{s \in S} J^{\geq}_s\text{-}\mathtt{Inj} \\
&= (J^{\geq}_r\text{-}\mathtt{Inj} \cap \mathcal{E}_r) \cap \bigcap_{s \in S} J^{\geq}_s\text{-}\mathtt{Inj} \\
&= J^{\geq}_S\text{-}\mathtt{Inj} \cap \mathcal{E}_r \ .
\end{aligned}$$

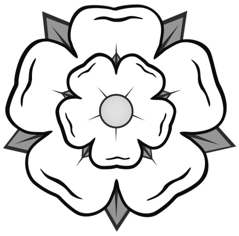
❂

The proof of the following is then much the same as for the S-model structure on $f\mathcal{C}$.

Theorem 3.12.3.15. *For every subset $S \subseteq \{0, 1, 2, \ldots, r\}$ containing r the category $f_{\geq}\mathcal{C}$ admits a right proper cofibrantly generated model structures, which we denote $(f_{\geq}\mathcal{C})_S$, whose:*

1. *weak equivalences are the E_r-quasi-isomorphisms,*
2. *fibrations are morphisms that for all $s \in S$ are $Z_s^{p,p+n}$-surjective for $p \geq s$ and all n, and*
3. *generating cofibrations and generating acyclic cofibrations are given by $I_S^{\geq}$ and $J_S^{\geq}$ respectively.*

Furthermore $(f_{\geq}\mathcal{C})_S$ is a finitely generated model category.



CHAPTER 4

Cofibrancy in $(f\mathcal{C})_r$

In this section we consider cofibrancy primarily in the model structure $(f\mathcal{C})_r$ with some interpretations for $(f\mathcal{C})_S$ more generally. Whilst we do not classify all cofibrant objects and cofibrations owing to difficulties involving the filtration we do classify those whose filtration is bounded in a suitable sense. We briefly recall how cofibrations in the projective model structure on unbounded chain complexes are classified. The full details of this can be found in [AFHnt] or [Hov99, §2.3] for a published account.

1. Firstly one considers for a cofibrant object lifts against the acyclic surjection of chain complexes from a disc object on N to a disc object on M coming from a surjection of R-modules $N \twoheadrightarrow M$. Necessary existence of a lift shows the cofibrant chain complex is degreewise projective.

2. Secondly one has, for any acyclic complex K, an acyclic surjection from the cone on K to K which the cofibrant object necessarily lifts against.

3. Thirdly these two restrictions on an object X, i.e. degreewise projective and has all lifts against $C(K) \to \Sigma K$ for any acyclic K can be shown to be sufficient for X to be cofibrant: for a lifting problem of X against an acylic surjection $E \overset{\sim}{\twoheadrightarrow} B$ one forms lifts degreewise irrespective of compatibility with the differential using the degreewise projective assumption and then by considering the difference of such a lift with the differentials $dh - hd$ which lands in the shift of the kernel K of $E \to B$ one can correct for the discrepancy of $dh - hd$ not being 0 using the second condition that X lifts against $C(K) \to \Sigma K$.

This then gives a classification of cofibrant objects. In the bounded setting one can completely remove the second condition involving all lifts against $C(K) \to \Sigma K$ for acyclic K since the homotopical correction can be acheived via an inductive argument starting in homological degree 0. We can then continue the classification to all cofibrations as follows.

4. By considering lifting problems of a cofibration, $A \to B$, against the surjection from a disc on N to 0 one can show that the cofibration is necessarily a degreewise split inclusion of R-modules.

5. Next note that the pushout of a cofibration is a cofibration hence the cokernel C of $A \to B$ is cofibrant and necessarily satisfies the classification conditions above.

6. By the degreewise split inclusion condition we can rewrite the cofibration as the inclusion of A into the twisted direct sum $A \oplus_\tau C \cong B$ for some twisted differential $\tau \colon C \to A$.

7. Given then a morphism of the form of the inclusion of a chain complex A into $A \oplus_\tau C$ where C is cofibrant one can constuct a lift against an acyclic fibration $Y \to X$; since C is cofibrant it can be lifted against the acyclic fibration irrespective of the twist differential τ to some morphism h, note one already has a lift of the A portion of $A \oplus_\tau C$ and there is no choice in the matter. Next consider the difference of this lift of C and the differential, i.e. $d^X h - (j\tau + hd^C)$ where j is the map from A to Y. Again this lands in the (shift of the) kernel K of $Y \to X$ and one uses existence of lifts of C against $C(K) \to \Sigma K$ to provide a homotopical correction for the incompatibility of the lift with the differential.

This then classifies all cofibrations in the projective model structure on unbounded chain complexes. In this section we follow this structure of proof to classify those cofibrations whose cokernel has a bounded assumption on the filtration. The author has been unable to remove this assumption in general however note there are cofibrant objects that don't satisfy this assumption. The reader should notice obvious replacements of disc objects to analogous *r-disc objects*, cones to r-cones and surjectivity conditions to Z_r-surjectivity conditions. The classification of such cofibrant objects is given as Proposition 4.1.0.16 where all but condition 5 are neccessary conditions to be cofibrant. Note the key differences when a filtration is involved for the $(f\mathcal{C})_r$ model structures where we now have that the graded pieces are projective and that the differential suppreses the filtration by r.

The main results (analogous to those in chain complexes) are listed below. We show conditions required of a cofibrant object in $(f\mathcal{C})_r$.

Lemma 4.1.0.1. *A cofibrant filtered chain complex A in the r-model structure on $f\mathcal{C}$ satisfies the following conditions:*

1. *$\frac{A^n}{F_pA^n}$ is a projective R-module for all $p, n \in \mathbb{Z}$,*
2. *A^n is a projective R-module for all $n \in \mathbb{Z}$,*
3. *the filtration on A is exhaustive, and*
4. *for a pure element $a \in F_pA^n$ we have $da \in F_{p-r}A^{n+1}$.*

Following this we show that with an added assumption on the boundedness of the filtration that this is sufficient to be cofibrant.

Proposition 4.1.0.16. *Given a filtered chain complex A such that the following conditions hold:*

1. *the graded pieces Gr_pA^n are projective for all $p, n \in \mathbb{Z}$,*
2. *for a pure element $a \in F_pA^n$ we have $da \in F_{p-r}A^{n+1}$ for all $p, n \in \mathbb{Z}$,*
3. *the filtration on A is exhaustive, and*
4. *whenever we have an r-acyclic filtered chain complex K and a morphism $A \to \Sigma^r K$ there is a lift in the following diagram:*

$$\begin{tikzcd} & C_r(K) \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & \Sigma^r K \end{tikzcd},$$

5. *and further such that for all n there is a $p(n) \in \mathbb{Z}$ such that $F_{p(n)}A^n = 0$ (i.e. the filtration is bounded below but not necessarily uniformly),*

then A is cofibrant in the r-model structure on $f\mathcal{C}$.

We also note that this added condition is not necessary of a cofibrant object of $(f\mathcal{C})_r$. In Chapter 5 we give a cofibrant replacement of the unit which does not satisfy this boundedness assumption. As in chain complexes we can show that cofibrations are degreewise split inclusions which are also strict giving an inclusion into a twisted direct sum interpretation of cofibrations.

Lemma 4.2.0.3. *An r-cofibration $i\colon A \to B$ is such that B is isomorphic to a twisted direct sum of A and the cokernel of i as filtered chain complexes.*

With an added condition of the twist map τ of the twisted direct sum above being an r-suppressive differential, called an r-supressive inclusion, we can give a subclass of the r-cofibrations.

Lemma 4.2.0.6. *An r-supressive inclusion $i\colon A \longrightarrow B$ whose cokernel C is cofibrant and such that for any n there is a $p(n)$ with $F_{p(n)}C^n = 0$ is an r-cofibration.*

We also give a result on the s-pages of a cofibrant $A \in (f\mathcal{C})_S$ for $s \leq r$ by considering the cellular objects and using the fact that all cofibrations are retracts of cellular cofibrations.

Proposition 4.1.0.14. *Let A be a cofibrant object of $(f\mathcal{C})_S$. Then for $k < r$ and $k \notin S$ the k-page differential d_k of A is 0.*

This result then says for a cofibrant object A of $(f\mathcal{C})_S$ and $k \notin S$ with $k < r$ that $E_kA \cong E_{k+1}A$. Lastly we show that the décalage functor preserves cofibrant objects for appropriate S-model structures. The shift functor automatically does so as it is a left Quillen functor.

Lemma 4.3.0.5. *Let B be a cofibrant object of $(f\mathcal{C})_{S+l}$, then $\mathrm{Dec}^l B$ is a cofibrant object in $(f\mathcal{C})_S$.*

4.1 Cofibrancy in filtered chain complexes

We give necessary conditions for an object to be cofibrant in the r-model structure on $f\mathcal{C}$ and show, with an added boundedness assumption on the filtration, that these conditions are sufficient to be cofibrant. Our list of necessary conditions is given in the following lemma. Unless stated otherwise all morphisms are morphisms of filtered chain complexes.

Lemma 4.1.0.1. *A cofibrant filtered chain complex A in the r-model structure on $f\mathcal{C}$ satisfies the following conditions:*

1. $\frac{A^n}{F_pA^n}$ *is a projective R-module for all $p, n \in \mathbb{Z}$,*
2. A^n *is a projective R-module for all $n \in \mathbb{Z}$,*
3. *the filtration on A is exhaustive, and*
4. *for a pure element $a \in F_pA^n$ we have $da \in F_{p-r}A^{n+1}$.*

We know of no such condition regarding the filtration being Hausdorff. Recall the notation of Notation 1.3.2.6. We will make use of the following fibrations.

Definition 4.1.0.2. Given a surjection of R-modules $\pi\colon N \longrightarrow M$ we define $\sigma_s^{p,p+n}$ to be the morphism given by

$$\sigma_s^{p,p+n}\colon \mathcal{Z}_{s+1}(p+s+1, p+s+1+n-1)(N) \longrightarrow \mathcal{Z}_s(p+s, p+s+n-1)(M)$$

which is given in homological degree n by π at and above filtration degree p and 0 otherwise, and in homological degree $n-1$ by 0 in filtration degree $p+s$ and π in all higher filtration degrees.

$$\begin{tikzcd}
\Big(N^{n-1}_{(p+s+1)} \arrow[r] \arrow[d, "\pi"] & N^{n}_{(p)}\Big) \arrow[d, "\pi"] \\
\Big(N^{n-1}_{(p+s)} \arrow[r] & N^{n}_{(p)}\Big)
\end{tikzcd}$$

Remark 4.1.0.3. These are similar to the morphisms $\alpha_s^{p,p+n}$ of Definition 3.3.0.1 with R-modules M and N instead of R.

Lemma 4.1.0.4. *For $0 \leq s < r$ the morphims $\sigma_s^{p,p+n}$ are r-acyclic fibrations.*

Proof. The domain and codomain are $s+1$ and s-cycles respectively on some R-module so are r-acyclic when $s < r$ hence the morphism is a weak equivalence. It is also an r-fibration since all elements in homological degree n are r-cycles and π is surjective, and in homological degree $n-1$ there are no r-cycles except 0 until the $(p+r)$-filtered part, but since $s+1 \leq r$ any r-cycle of the codomain is in the image of an r-cycle of the domain.

We also have the following result.

Lemma 4.1.0.5. *For $0 \leq s \leq r$ and surjection $\pi\colon N \longrightarrow M$ the morphism*

$$\rho_s^{p,p+n} := \mathcal{Z}_s(p+s, p+s+n-1)(\pi)\colon \mathcal{Z}_s(p+s, p+s+n-1)(N) \longrightarrow \mathcal{Z}_s(p+s, p+s+n-1)(M)$$

is an r-acyclic fibration.

Lemma 4.1.0.6. *A morphism $A \to \mathcal{Z}_s(p+s, p+s+n-1)(N)$ is equivalent to a morphism of R-modules*

$$\frac{A^n}{F_{p-1}A^n + dF_{p+s-1}A^{n-1}} \longrightarrow N \ .$$

In particular when $s = 0$ we have $A \to \mathcal{Z}_s(p+s, p+s+n-1)(N)$ is equivalent to a morphism of R-modules

$$\frac{A^n}{F_{p-1}A^n} \longrightarrow N \ .$$

Proof. Such a map is of the form $A \longrightarrow (N^{n-1}_{(p+s)} \to N^n_{(p)})$. In cohomological degree n this is then a map $A^n/F_{p-1}A^n \to N$. In cohomological degree $n-1$ we have the image of $F_{(p+s-1)}A^{n-1}$ is 0 in $N^{n-1}_{(p+s)}$ and so the image of $dF_{(p+s-1)}A^{n-1}$ is 0 in $N^n_{(p)}$. Together these imply the lemma.

Lemma 4.1.0.7. *If A is cofibrant in the r-model structure on $f\mathcal{C}$ then $\frac{A^n}{F_pA^n}$ is projective for all $p, n \in \mathbb{Z}$.*

Proof. Given a surjection $\pi \colon N \to M$ of R-modules we consider diagrams of the form:

$$\begin{tikzcd}
 & \mathcal{Z}_0(p, p+n-1)(N) \arrow[d, "\rho_0^{p,p+n}"] \\
A \arrow[r] \arrow[ur, dotted] & \mathcal{Z}_0(p, p+n-1)(M)
\end{tikzcd}$$

which necessarily have a lift since $\rho_0^{p,p+n}$ is an r-acyclic fibration by Lemma 4.1.0.5 and A is cofibrant. By Lemma 4.1.0.6 this diagram is then equivalent to one of the form

$$\begin{tikzcd}
 & N \arrow[d, "\pi"] \\
\frac{A^n}{F_{p-1}A^n} \arrow[r] \arrow[ur, dotted] & M
\end{tikzcd}$$

which given π was an arbitrary surjection shows that the $\frac{A^n}{F_{p-1}A^n}$ are projective.

Lemma 4.1.0.8. *If A is cofibrant in the r-model structure on $f\mathcal{C}$ then A^n is projective for all $n \in \mathbb{Z}$.*

Proof. The proof is the same as that of Lemma 4.1.0.7 except we take $p = -\infty$, so the 0-cycles live in all filtration degrees.

Lemma 4.1.0.9. *If A is cofibrant in the r-model structure on $f\mathcal{C}$ then the filtration on A is exhaustive.*

Proof. Given the filtered chain complex A we let $\bar{A} = \cup_p F_pA$, i.e. the union of all filtered pieces so that $\bar{A}$ is exhaustive. Note that the inclusion $\bar{A} \to A$ is an r-acyclic fibration, indeed the definiton of being an r-weak equivalence and r-fibration only rely on the filtered parts, never on any element of $A \backslash \bar{A}$. We then consider the lifting problem

$$\begin{tikzcd}
 & \bar{A} \arrow[d] \\
A \arrow[r, "id"'] \arrow[ur, dotted] & A
\end{tikzcd}$$

for which the lift must exist since A is cofibrant. This factorises the identity map through the union of its filtered parts and so $A = \bar{A}$.

The following sequence of lemmas show which differentials of a cofibrant object of $(f\mathcal{C})_S$ must be 0.

Lemma 4.1.0.10. *Given a filtered chain complex (or bicomplex) B whose k-page differential is 0 and a filtered chain complex (or bicomplex) A which is a retract of B then the k-page differential of A is also 0.*

Proof. Given A is a retract of B we have the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "id"] & B \arrow[r] & A
\end{tikzcd}$$

from which we obtain the diagram of k-pages

$$\begin{tikzcd}
E_kA \arrow[r] \arrow[d, "d_k^A"] & E_kB \arrow[r] \arrow[d, "d_k^B"] & E_kA \arrow[d, "d_k^A"] \\
E_kA \arrow[r] & E_kB \arrow[r] & A
\end{tikzcd}$$

whose horizontal composites are the identity and differential $d_k^B = 0$. A diagram chase shows that $d_k^A = 0$ too.

Lemma 4.1.0.11. *Suppose A has $d_k = 0$ with $k < r$, then so too does the pushout of A by a morphism of the form $\mathcal{Z}_{r+1}(p, p+n) \to \mathcal{B}_{r+1}(p, p+n)$.*

Proof. This can be seen by considering the s-cycle description of A' of Lemma 3.7.1.1. By commutativity of the diagram:

$$\begin{tikzcd}
\frac{Z_k^{p,p+n}(A)}{B_k^{p,p+n}(A)} \arrow[r, "d_k"] \arrow[d] & \frac{Z_k^{p-k,p-k+n+1}(A)}{B_k^{p-k,p-k+n+1}(A)} \arrow[d] \\
\frac{Z_k^{p,p+n}(A')}{B_k^{p,p+n}(A')} \arrow[r, "d_k"] & \frac{Z_k^{p-k,p-k+n+1}(A')}{B_k^{p-k,p-k+n+1}(A')}
\end{tikzcd}$$

δ_k remains 0 on those k-cycles of A in A' we need only consider new k-cycles. The pushout introduces α and γ, when considered as k-cycles representing a class of $E_k^{p-1,p-1+n}(A')$ and $E_k^{p+r,p+r+n-1}(A')$ respectively for $k < r$ one observes that under the differential d_k these classes become 0 since for example $d\alpha \in Z_{s-1}^{p-s,p-s+n+1}(A) \subseteq B_s^{p-s+1,p-s+1+n+1}(A')$ and similarly for $d\gamma$ with the appropriate indexing.

Lemma 4.1.0.12. *Suppose A has $d_k = 0$, then so too does the pushout of A by a morphism of the form $0 \to \mathcal{Z}_s(p, p+n)$ for $s \neq k$.*

Lemma 4.1.0.13. *Suppose A has $d_k = 0$ and we work in a model category $(f\mathcal{C})_S$ with $k \notin S$ and $k \leq r$, then so too does a relative I_S-cell whose domain is A.*

Proof. By Lemmas 4.1.0.11 and 4.1.0.12 we know that the pushout of a filtered chain complex with $d_k = 0$ by a generating cofibration in I_S also has trivial k-page differential. It now remains to show a transfinite composition of such pushouts preserves this property. We do so by transfinite induction [Sup60, §7.1], the base case is the statement that the filtered chain complex A has trivial d_k, the successor ordinal case is Lemmas 4.1.0.11 and 4.1.0.12 and it remains to show the limit ordinal case. Suppose then λ is a limit ordinal and we have a λ sequence of filtered chain complexes such that for each $\alpha < \lambda$ the filtered chain complex indexed by α has $d_k = 0$. Given an element $[x]$ of the k-page of the colimit of the λ sequence we can represent this by some k-cycle in the colimit of the λ sequence. There are indexing ordinals $\beta, \gamma < \lambda$ with elements y and z respectively such that the image of y in the colimit is x, and is in the same filtration degree, and that of z in the colimit is dx and of the same filtration degree. Taking the larger of these two ordinals we have that (the image of) y is a k-cycle and since the k-page differential on this stage of the λ-sequence is 0 so too then is that of $[x]$ in the colimit, this proves the statement for limit ordinals.

Proposition 4.1.0.14. *Let A be a cofibrant object of $(f\mathcal{C})_S$. Then for $k < r$ and $k \notin S$ the k-page differential d_k of A is 0.*

Proof. Any relative I_S-cell object has $d_k = 0$ for $k \notin S$ by Lemma 4.1.0.13. Since any cofibrant A in $(f\mathcal{C})_S$ is a retract of an I_S-cell object by Proposition 1.4.2.10 then the k-page differential of A is also 0 by Lemma 4.1.0.10.

Lemma 4.1.0.15. *If A is cofibrant in the r-model structure on $f\mathcal{C}$ then for a pure element $a \in F_pA^n$ we have $da \in F_{p-r}A^n$.*

Proof. By Proposition 4.1.0.14 the differentials $d_s = 0$ for $0 \leq s < r$ so that

$$Gr_pA^n = E_0^{p,p+n}A = E_1^{p,p+n}A = \ldots = E_{r-1}^{p,p+n}A = E_r^{p,p+n}A$$

and the first non-zero differential is on the r-page

$$\delta_r\colon E_r^{p,p+n}A = Gr_pA^n \longrightarrow Gr_{p-r}^{n+1} = E_r^{p-r,p-r+n+1}$$

and hence the differential d on A induces a morphism

$$d\colon \frac{F_pA^n}{F_{p-1}A^n} \longrightarrow \frac{F_{p-r}A^{n+1}}{F_{p-r-1}A^{n+1}}$$

showing that the differential d maps a pure element $a \in F_pA^n$ into $F_{p-r}A^{n+1}$. ∎

Proof of Lemma 4.1.0.1. The preceeding lemmas show the condition of Lemma 4.1.0.1. ∎

Similar results of have already been observed in the work of Cirici [Cir12, Lemma 4.3.15] but only in the restricted sense of their E_r-cofibrant dgas [Cir12, Definition 4.3.14] built as colimits of *KS-extensions* of [Cir12, Definition 4.3.13]. These E_r-cofibrant dgas are shown to have the left lifting property with respect to a class of r-acyclic fibrations, [Cir12, Proposition 4.3.17].

With these necessary conditions on cofibrant objects of $(f\mathcal{C})_r$ established we show now that with an extra assumption on the boundedness of the filtration, condition 5 in the following, that this is sufficient to be cofibrant. This is then not a full classification of the cofibrant objects of $f\mathcal{C}$. Indeed the cofibrant replacement of Definition 5.1.0.1 does not satisfy condition 5 but is still cofibrant.

Recall the r-cone C_r construction Definition 1.6.0.12 and the r-suspension Σ^r of Definition 1.2.1.18.

Proposition 4.1.0.16. *Given a filtered chain complex A such that the following conditions hold*

1. *the graded pieces Gr_pA^n are projective for all $p, n \in \mathbb{Z}$,*
2. *for a pure element $a \in F_pA^n$ we have $da \in F_{p-r}A^{n+1}$ for all $p, n \in \mathbb{Z}$,*
3. *the filtration on A is exhaustive,*
4. *whenever we have an r-acyclic filtered chain complex K and a morphism $A \to \Sigma^rK$ there is a lift in the following diagram:*

$$\begin{tikzcd} & C_r(K) \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & \Sigma^rK \end{tikzcd},$$

5. *and further such that for all n there is a $p(n) \in \mathbb{Z}$ such that $F_{p(n)}A^n = 0$ (i.e. the filtration is bounded below but not necessarily uniformly)*

then A is cofibrant in the r-model structure on $f\mathcal{C}$.

The proof of this is very similar to the classification of cofibrant objects in the (unbounded) projective model structure of chain complexes. We obtain a lift of graded (filtered) R-modules using assumptions 1–3 and 5 and then use the remaining assumption to correct this lift so it is compatible with the differentials.

We will use the following lemma implicitly for the proof.

Lemma 4.1.0.17. *Given a morphism $f\colon B \longrightarrow Y$ restricting to a map $f_A\colon A \longrightarrow X$ and short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f_A"] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "\tilde{f}"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

with C and Z projective then for any splittings the map f is isomorphic to one of the form

$$\begin{pmatrix} f_A & \tau \\ 0 & \tilde{f} \end{pmatrix} : A \oplus C \longrightarrow X \oplus Z$$

for some twist map $\tau\colon C \longrightarrow X$ and where f_C is the induced map between the quotients C and Z. ∎

Proof. We consider a lifting problem of the form

$$\begin{tikzcd}
 & Y \arrow[d, "\pi"] \\
A \arrow[r, "f"'] \arrow[ur, dotted] & X
\end{tikzcd} \tag{4.1}$$

where the morphism π is an r-acyclic fibration.

For each homological degree n we split A^n into its graded pieces as follows. There is a $p(n)$ such that $F_{p(n)}A^n = 0$ and we set $p = p(n) + 1$. There is then a short exact sequence of the form

$$0 \longrightarrow F_pA^n \longrightarrow F_{p+1}A^n \longrightarrow \frac{F_{p+1}A^n}{F_pA^n} \longrightarrow 0$$

in which the graded pieces are projective by assumption so we have a splitting

$$F_{p+1}A^n \cong F_pA^n \oplus \frac{F_{p+1}A^n}{F_pA^n}$$

and inducting up we obtain for each k

$$F_{p+k}A^n \cong F_pA^n \oplus \frac{F_{p+1}A^n}{F_pA^n} \oplus \frac{F_{p+2}A^n}{F_{p+1}A^n} \oplus \ldots \oplus \frac{F_{p+k-1}A^n}{F_{p+k-2}A^n} \oplus \frac{F_{p+k}A^n}{F_{p+k-1}A^n}$$

and further since the filtration is exhaustive we have

$$A^n \cong \bigoplus_{k=0}^{\infty} \frac{F_{p+k}A^n}{F_{p+k-1}A^n} \ .$$

Consider then the differential $d\colon A^n \longrightarrow A^{n+1}$ between this graded piece presentation of A^n and a corresponding one for A^{n+1}. By (repeated application of) Lemma 4.1.0.17 the differential restricted to the pieces representing $F_{p+k}A^n$ must have image in the graded pieces representing $F_{p+k}A^{n+1}$, and by the second assumption a pure element in the $F_{p+k}A^n/F_{p+k-1}A^n$ has image in $F_{p+k-r}A^{n+1}/F_{p+k-r-1}A^{n+1}$ so combining these two results we have that the image of a graded piece $F_{p+k}A^n/F_{p+k-1}A^n$ has image in the graded pieces $F_qA^{n+1}/F_{q-1}A^{n+1}$ for $q \leq p+k-r$.

Consider then f restricted to a graded piece $F_{p+k}A^n/F_{p+k-1}A^n$. Since elements of this graded piece thought of as elements of A are r-cycles, in $Z_r^{p+k,p+k+n}A$, so then too are their images under f. Since π is an r-fibration it is Z_r-bidegree-wise surjective so we can lift this restriction of the graded piece into Y using projectivity of the graded piece and Z_r-surjectivity. We do this for all graded pieces of the decomposition in all homological degrees irrespective of each other. This gives a lift of graded R-modules compatible with filtration but not necessarily with the differential. We denote this lift of filtered graded modules by G.

We now correct for the differential. Consider now the morphism

$$t \coloneqq d^YG - Gd^A \colon A \longrightarrow \Sigma Y$$

which is indeed a morphism of filtered chain complexes, not just of graded filtered modules, since

$$td^A = d^YGd^A - Gd^Ad^A = d^YGd^A = -d^{\Sigma Y}Gd^A = d^{\Sigma Y}d^YG - d^{\Sigma Y}Gd^A = d^{\Sigma Y}t \ .$$

Note too that since G is a lift (of graded filtered modules) we have $\pi t = 0$ and also since we have lifted using the surjectivity of r-cycles everywhere we in fact have

$$t\colon A \longrightarrow \Sigma^r K$$

where K is the kernel of π. Since π is an r-acyclic fibration its kernel is r-acyclic therefore so too is $\Sigma^r K$ and so, by the final assumption, we have a lift in the following diagram:

$$\begin{tikzcd}
 & C_r\left(K\right) \arrow[d] \\
A \arrow[r, "t"'] \arrow[ur, dotted, "T"] & \Sigma^r K
\end{tikzcd} \ .$$

Unwrapping the definition of the cone object, Definition 1.6.0.12, we have that T is a morphism given (on the p-filtered part) by

$$T = \begin{pmatrix} t \\ T_2 \end{pmatrix} : F_pA \to F_pC_r\left(K\right) = F_{p-r}K^{n+1} \oplus F_pK^n$$

and where the differential on the far right is given by

$$d^{C_r(K)} = \begin{pmatrix} d^{\Sigma^r K} & 0 \\ 1 & d^K \end{pmatrix} = \begin{pmatrix} -d^K & 0 \\ 1 & d^K \end{pmatrix}$$

and so, since T is a morphism of filtered chain complexes, we have that:

$$\begin{pmatrix} td^A \\ T_2d^A \end{pmatrix} = Td^A = d^{C_r(K)}T = \begin{pmatrix} -d^K & 0 \\ 1 & d^K \end{pmatrix} \begin{pmatrix} t \\ T_2 \end{pmatrix} = \begin{pmatrix} -d^Kt \\ t + d^KT_2 \end{pmatrix} \, . \tag{4.2}$$

We finally define our lift in Equation (4.1) to be $g := G + T_2$, where we interpret the image of T_2 in Y since $K \subset Y$, which is certainly a morphism of graded filtered modules and the second equation in Equation (4.2) shows it commutes with differentials. We have then shown that the object A with these assumptions lifts against any r-acyclic fibration and so is r-cofibrant. ✿

We include here an example demonstrating that a disc object in all filtration degrees is not cofibrant.

Example 4.1.0.18. By $\mathcal{Z}(-\infty, n)$ we mean the chain complex $R^n \longrightarrow R^{n+1}$ whose R-modules are in all filtration degrees. The filtered chain complexes $\mathcal{Z}(-\infty, n)$ are not cofibrant in any of the model structures $(f\mathcal{C})_S$. To see this consider the lifting problem:

$$\begin{tikzcd}
 & \bigoplus_{p\in\mathbb{Z}} \mathcal{Z}_0(p,n) \arrow[d, two heads, "\epsilon", "\sim"'] \\
\mathcal{Z}(-\infty,n) \arrow[r, "id"'] & \mathcal{Z}(-\infty,n)
\end{tikzcd}$$

in which the horizontal morphism is the identity and the morphism ϵ is the idenity whenever possible. The morphism ϵ is a 0-weak equivalence since its domain and codomain are both 0-acyclic and is also Z_k-bidegree-wise surjective for all $k \geq 0$ so is therefore an acylic fibration in all $(f\mathcal{C})_S$. The only morphism from $\mathcal{Z}(-\infty, n)$ to $\bigoplus_{p\in\mathbb{Z}} \mathcal{Z}_0(p, n)$ is the 0 morphism however so there is no lift and hence $\mathcal{Z}(-\infty, n)$ is not cofibrant in any of the $(f\mathcal{C})_S$.

4.2 Cofibrations in filtered chain complexes

We will show the class of maps of filtered chain complexes which are (cohomologically) degreewise split inclusions $A \to A \oplus_\tau C$ with cofibrant cokernel C with a bounded filtration and such that the differential τ supresses filtration by r are cofibrations. This is an imperfect characterisation due to the requirement that C have a bounded filtration.

Recall that we have already shown that cofibrations are strict inclusions, Lemmas 3.8.1.3 and 3.8.1.4. A consequence of this then is that for an element of A the pure filtration degree cannot be decreased by the map i but only by the differential of A.

Definition 4.2.0.1. The *twisted direct sum* of filtered chain complexes A and C, denoted $A \oplus_\tau C$, is a filtered chain complex whose underlying filtered graded modules is the direct sum $A \oplus C$ but whose differential is given by:

$$d^{A\oplus_\tau C} := \begin{pmatrix} d^A & \tau \\ 0 & d^C \end{pmatrix} : A \oplus C \longrightarrow A \oplus C$$

and we call $\tau \colon C \longrightarrow A$ the twist map.

Note that since $d^{A\oplus_\tau C}$ is a differential we must have $d^{A\oplus_\tau C} \circ d^{A\oplus_\tau C} = 0$ which holds if and only if $d^A\tau + \tau d^C = 0$. This is, up to sign and shift, the r-cone of a morphism of filtered chain complexes given as [CELW19, Definition 3.5]. There is an inclusion of filtered chain complexes $i \colon A \longrightarrow A \oplus_\tau C$ and projection $A \oplus_\tau C \to C$.

Lemma 4.2.0.2. *An r-cofibration, $i \colon A \longrightarrow B$, is a degree-wise split inclusion, after forgetting filtration, with cofibrant cokernel.*

Proof. Let C be the cokernel of i. The cokernel being cofibrant follows since the pushout of a cofibration is a cofibration. For the spliting we have that C^n is projective so the following short exact sequence splits:

$$0 \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow 0\ .$$

From this we have an isomorphism of filtered chain complexes between B and the twisted direct sum of A and C over some maps $\tau\colon C \longrightarrow A$ (which shift degree).

$$B \cong A \oplus_\tau C\ ,$$

where the filtration on the twisted direct sum is induced from that of B.

Lemma 4.2.0.3. *An r-cofibration $i\colon A \to B$ is such that B is isomorphic to a twisted direct sum of A and the cokernel of i as filtered chain complexes.*

Proof. Write C for the cokernel, as a filtered chain complex, of i. Since C is cofibrant C^n is a projective R-module for each n and so each short exact sequence of Lemma 4.2.0.2:

$$\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r, "i^n"'] & B \arrow[r, "q^n"'] & C^n \arrow[l, bend right, dashed, "s^n"'] \arrow[r] & 0
\end{tikzcd}$$

splits, as R-module morphisms, so that $B^n \cong A^n \oplus C^n$ as R-modules.

$$\begin{array}{rcl}
B^n & \xleftrightarrow{\quad\cong\quad} & A \oplus C \\[2em]
b & \longmapsto & (b - s^n q^n b, q^n b) \\[2em]
i^n a + s^n c & \longleftarrow\!\shortmid & (a, c)
\end{array}$$

We can then equip $A^n \oplus C^n$ with a filtration induced from B^n by this isomorphism making an isomorphism of filtered R-modules. This new filtration agrees with that on A^n: consider an $(a, 0) \in F_p(A^n \oplus C^n) \cong B^n$ which corresponds via the isomorphism to some $b \in F_p B^n$ via $b \mapsto (b - s^n q^n b, 0) = (b, 0)$ so that $i^n a = b$. Note that a cofibration cannot decrease the pure filtration degree of an element since they are strict inclusions by Lemma 3.8.1.4 so that $a \in F_p A^n$ and therefore the filtration on A^n agrees with that on $A^n \oplus C^n$.

We then have an induced differential $d\colon A^n \oplus C^n \to A^{n+1} \oplus C^{n+1}$ from that of B which preserves the filtration and squares to zero, since d^B does. Writing $\psi^n\colon B^n \to A^n \oplus C^n$ for this isomorphism and φ^n for its inverse we have:

$$\begin{aligned}
d(a, c) &= \psi^{n+1}(d^B(\varphi^n(a, c))) \\
&= \psi^{n+1}(d^B(i^n a + s^n c)) \\
&= \big(d^B(i^n a + s^n c) - s^{n+1} q^{n+1} d^B(i^n a + s^n c), q^{n+1} d^B(i^n a + s^n c)\big) \\
&= \big(i^{n+1} d^B a + d^B s^n c - s^{n+1} q^{n+1} i^{n+1} d^B a - s^{n+1} q^{n+1} d^B s^n c, q^{n+1} i^{n+1} d^B a + q^{n+1} d^B s^n c)\big) \\
&= \big(i^{n+1} d^B a + d^B s^n c - s^{n+1} q^{n+1} d^B s^n c, q^{n+1} d^B s^n c\big)\ .
\end{aligned}$$

However both i and q are morphisms of filtered chain complexes so in particular

$$d^C c = d^C q^n s^n c = q^{n+1} d^B s^n c$$

so that we have $d(a, c) = \big(d^A a + d^B s^n c - s^{n+1} q^{n+1} d^B s^n c, d^C c\big)$ and defining τ by

$$\begin{aligned}
\tau^n\colon C^n &\longrightarrow A^{n+1} \\
c &\longmapsto d^B s^n c - s^{n+1} q^{n+1} d^B s^n c
\end{aligned}$$

we have that as filtered chain complexes B is isomorphic to a twisted direct sum $A \oplus_\tau C$ of the filtered chain complexes A and C.

Definition 4.2.0.4. A (cohomologically) degree-wise split inclusion $i\colon A \longrightarrow B$ is an *r-supressive inclusion* if the maps τ suppress filtration by r.

Definition 4.2.0.5. An r-cofibration i is an *r-suppressive cofibration* if i is an r-supressive inclusion.

Note that since all cofibrations are strict inclusions this is really just requiring the splitting with the additional condition on τ. We will show the following which provides a subclass of the class of r-cofibrations.

Lemma 4.2.0.6. *An r-supressive inclusion $i\colon A \longrightarrow B$ whose cokernel C is r-cofibrant and such that for any n there is a $p(n)$ with $F_{p(n)}C^n = 0$ is an r-cofibration.*

We do not know whether all r-cofibrations are necessarily r-suppressive. The obvious choices of r-acyclic fibrations to use to prove such a statement, involving either path objects or cone objects, do not appear to impose the suppressive condition on the cofibration. We also do not have a counterexample to this statement. The author suspects that all r-cofibrations are necessarily suppressive.

Proof. Given such an i and an r-acyclic fibration $f\colon Y \longrightarrow X$ we consider a lifting problem of the form:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "i"'] & Y \arrow[d, "f"] \\
A \oplus_\tau C \arrow[r] & X
\end{tikzcd} .$$

As for the proof of Proposition 4.1.0.16 since C is cofibrant we can write each C^n as a direct sum of its graded pieces starting at some filtration level $p = p(n) + 1$:

$$C^n \cong \bigoplus_{k=0}^{\infty} \frac{F_{p+k}C^n}{F_{p+k-1}C^n} . \tag{4.3}$$

Since C is cofibrant d^C suppresses filtration by r and so too does the twist map τ by the suppressive inclusion assumption, so the differential $d^{A\oplus_\tau C}$ applied to an element $(0, c)$, where c is an element of one of the graded pieces of Equation (4.3), also suppresses filtration by r. So such a $c \in F_qC^n/F_{q-1}C^n$ has image in $Z_r^{q,q+n}(X)$. By r-cycle surjectivity of f we can lift the image of each graded piece (for each filtration and cohomological indexing) of C using these decompositions into r-cycles of Y. Denote this lift by G and note it is only a lift of filtered R-modules and is not necessarily compatible with any differentials. We next define $t^n\colon C^n \to Y^{n+1}$ by:

$$t := d^Y G - Gd^C - \varphi\tau .$$

As in the proof of Proposition 4.1.0.16 the image of t is in the kernel $K = \ker(f\colon Y \to X)$ and $d^K t + td^C = 0$ so that it is a map into the suspension of K, we need to check it is a map into the r-suspension of K however. This follows since every $c \in C^n$ is an r-cycle, G lifts r-cycles, and τ supresses filtration by r. Since d^C and τ both suppress filtration by p we have t is a map of filtered chain complexes from C into $\Sigma^r K$. By cofibrancy of C the lifting problem

$$\begin{tikzcd}
 & C_r(K) \arrow[d] \\
C \arrow[ur, dashed] \arrow[r, "t"'] & \Sigma^r K
\end{tikzcd}$$

has a solution $T = \begin{pmatrix} t \\ T_2 \end{pmatrix}$. The required lift of $A \oplus_\tau C$ is then given by $h := \begin{pmatrix} \varphi & G + T_2 \end{pmatrix}$. We have $G + T_2$ is a morphism of filtered R-modules which is a lift of $A \oplus_\tau C \to X$ since T_2 has image in the kernel, and using $T_2 d^C = t + d^K T_2$ we can show it commutes with differentials:

$$\begin{aligned}
d^Y h &= d^Y \begin{pmatrix} \varphi & G + T_2 \end{pmatrix} \\
&= \begin{pmatrix} d^Y\varphi & d^Y(G + T_2) \end{pmatrix} \\
&= \begin{pmatrix} \varphi d^A & d^Y G + d^K T_2 \end{pmatrix} \\
&= \begin{pmatrix} \varphi d^A & (t + Gd^C + \varphi\tau) + (T_2 d^C - t) \end{pmatrix} \\
&= \begin{pmatrix} \varphi d^A & Gd^C + \varphi\tau + T_2 d^C \end{pmatrix} \\
&= \begin{pmatrix} \varphi & G + T_2 \end{pmatrix} \begin{pmatrix} d^A & \tau \\ 0 & d^C \end{pmatrix} \\
&= \begin{pmatrix} \varphi & G + T_2 \end{pmatrix} d^{A\oplus_\tau C} ,
\end{aligned}$$

which shows this lift is indeed a lift of filtered chain complexes.

4.3 Shift-décalage adjunction on cofibrant objects

In [Cir12, Lemma 4.3.16] Cirici shows that the shift-décalage adjunction restricts to functors between their notions of E_r-cofibrant and E_{r+1}-cofibrant objects. In this section we show a similar results holds for the cofibrant objects of the model categories $(f\mathcal{C})_S$.

Consider the shift-décalage adjunction of Lemma 1.3.3.2 and recall the terminology of an object being l-supressive Definition 1.6.0.15. The shift functors S^l are inclusions of filtered chain complexes into the full subcategory of filtered chain complexes consisting of the l-supressive objects. The décalage functor conversely is not an inclusion on filtered chains, for example we have

$$\mathrm{Dec}\left(\mathcal{Z}_0(p, p+n)\right) \cong \mathrm{Dec}\left(\mathcal{Z}_1(p+1, p+1+n)\right) \ ,$$

whilst $\mathcal{Z}_0(p, p+n) \not\cong \mathcal{Z}_1(p+1, p+1+n)$ so that the adjunction is not an equivalence of categories. However restricting to the full subcategory of l-supressive objects the adjunction becomes an equivalence.

Definition 4.3.0.1. The full subcategory of $f\mathcal{C}$ consisting of the l-supressive objects is denoted by $\mathrm{Supp}_l\text{-}f\mathcal{C}$.

Slightly more generally we have the categories of l-supressive objects and $(l+k)$-supressive objects are equivalent.

Lemma 4.3.0.2. *There is an equivalence of categories:*

$$\begin{tikzcd} S^l \colon \mathrm{Supp}_k\text{-}f\mathcal{C} \arrow[r, shift left, ""{name=A}] & \mathrm{Supp}_{k+l}\text{-}f\mathcal{C} \colon \mathrm{Dec}^l \arrow[l, shift left, ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd} \ .$$

Proof. Given a k-supressive object A we have for any $a \in F_pA^n$ that $da \in F_{p-k}A^{n+1}$. Applying S^l to A we have that $a \in F_pA^n = F_{p-ln}S^lA^n$ and $da \in F_{p-k}A^{n+1} = F_{p-k-l(n+1)}S^lA^{n+1} = F_{p-ln-(k+l)}S^lA^{n+1}$. So that every element of S^lA^n is now a $(k+l)$-cycle, hence $(k+l)$-supressive.

Similarly suppose B is $(k+l)$-supressive so that for any $b \in F_pB^n$ we have that $db \in F_{p-(k+l)}B^{n+1}$.

$$\begin{aligned} b \in F_pB^n &= Z_l^{p,p+n}(B) \\ &= F_{p+ln}\mathrm{Dec}^l B^n \end{aligned}$$

and we have

$$\begin{aligned} db \in F_{p-(k+l)}B^n &= Z_l^{p-(k+l),p-(k+l)+n+1}(B) \\ &= F_{p+l(n+1)-(k+l)}\mathrm{Dec}^l B^n \\ &= F_{p+ln-k}\mathrm{Dec}^l B^n \end{aligned}$$

which shows that any element of $\mathrm{Dec}^l B^n$ is a k-cycle, hence $\mathrm{Dec}^l B$ is k-supressive.

We already know that $\mathrm{Dec}^l \circ S^l = id$ we now check that $S^l \circ \mathrm{Dec}^l = id$. Take B to be $(k+l)$-supressive, then:

$$\begin{aligned} F_pS^l \circ \mathrm{Dec}^l B^n &= F_{p+ln}\mathrm{Dec}^l B^n \\ &= Z_l^{p+ln-ln,p+ln-ln+n}B^n \\ &= Z_l^{p,p+n}B^n \\ &= B^n \end{aligned}$$

where the last equality follows since B is $(k+l)$-supressive and so l-supressive. So then S^l and Dec^l are inverse functors on these full subcategories and we have an equivalence of categories.

Corollary 4.3.0.3. *There is an equivalence of categories between filtered chain complexes and l-supressive objects.*

An l-supressive filtered chain complex is then precisely a filtered chain complex which is in the image of the functor S^l. Recall now that cofibrant objects in $(f\mathcal{C})_S$ are at least l-supressive where l is the smallest element of S. We'd now like to establish a similar result showing an equivalence of categories between the cofibrant objects of two of the model categories of Theorem 3.1.0.2. We know the shift functors already preserve cofibrants objects so we must show that so too does décalage to some extent.

Lemma 4.3.0.4. *Suppose a morphism ρ of filtered chain complexes is Z_s-surjective. Then $S^l\rho$ is Z_{s+l}-surjective.*

Proof. Consider $S^l\rho\colon S^lA \to S^lB$ and an element $b \in Z_{s+l}^{p,p+n}(S^lB)$, so that

$$\begin{aligned} b &\in F_pS^lB^n = F_{p+ln}B^n \\ db &\in F_{p-(s+l)}S^lB^{n+1} = F_{p-(s+l)+l(n+1)}B^{n+1} = F_{p+ln-s}B^{n+1} \end{aligned}$$

but since ρ is Z_s-surjective we can find a Z_s-lift of $b \in B$ and this gives a Z_{s+l}-lift of b in S^lB.

Recall the notation $S + l$ of Notation 3.5.0.1 for S a set and $l \in \mathbb{N}$.

Lemma 4.3.0.5. *Let B be a cofibrant object of $(f\mathcal{C})_{S+l}$, then Dec^lB is a cofibrant object in $(f\mathcal{C})_S$.*

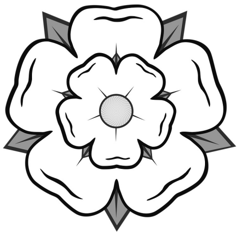
Proof. Consider the lifting problem

$$\begin{tikzcd} & Y \arrow[d, two heads, "\rho", "\sim"'] \\ \mathrm{Dec}^lB \arrow[r] & X \end{tikzcd} \tag{4.4}$$

where p is an acyclic fibration of $(f\mathcal{C})_S$. Applying S^l to the diagram gives

$$\begin{tikzcd} & S^lY \arrow[d, two heads, "S^l\rho", "\sim"'] \\ B \arrow[r] & S^lX \end{tikzcd} \tag{4.5}$$

where we have used Lemma 4.3.0.2 with $S^l\mathrm{Dec}^lB = B$ since B is l-supressive. Further the morphism $S^l\rho$ is an acyclic fibration now of $(f\mathcal{C})_{S+l}$; it is a fibration since S^l sends Z_k-surjections to Z_{k+l}-surjections by Lemma 4.3.0.4 and an $(r+l)$-weak equivalence by [CELW19, Lemma 3.27] which shows S^l sends r-weak equivalences to $(r+l)$-weak equivalences. Hence since B is cofibrant in $(f\mathcal{C})_{S+l}$ there is a lift in Equation (4.5). Applying Dec^l to this lift gives a lift in Equation (4.4).



CHAPTER 5

S-Model Structure of Filtered Chain Complexes is Monoidal

In this chapter we study the interaction of the S-model category structures $(f\mathcal{C})_S$ of Theorem 3.1.0.2 with the monoidal product on filtered chain complexes Definition 1.2.1.11. We will show that they satisfy the conditions, the pushout-product axiom and the unit axiom, of a monoidal model category structure of Definition 1.4.5.2 on $(f\mathcal{C})_S$. Hence we obtain an induced monoidal structure on the homotoy category $\mathrm{Ho}\,(f\mathcal{C})_S$ with unit given by a cofibrant replacement of the unit. The pertinent results for establishing this are listed below.

Example 5.1.0.7. *The monoidal unit in $(f\mathcal{C})_S$ is not cofibrant.*

Corollary 5.1.0.6. *A cofibrant replacement of the monoidal unit in $(f\mathcal{C})_S$, with $r = \max S$, is given by*

$$Q_r I := \left(\bigoplus_{i=0}^{\infty} R^0_{(-i)} \longrightarrow \bigoplus_{j=1}^{\infty} R^1_{(-r-j)} \right) .$$

where the differential is given by mapping each $R^0_{(-i)}$ diagonally onto the copies of R indexed as $R^1_{(-r-i)}$ and $R^1_{(-r-i-1)}$ for $i \geq 1$ and by the identity map from $R^0_{(0)}$ to $R^1_{(-r-1)}$.

Proposition 5.2.0.2. *The composite function $Q_r I \otimes A \longrightarrow I \otimes A \longrightarrow A$ is an r-weak equivalence for all (not necessarily cofibrant) A.*

Our characterisation of a subclass of cofibrations of Lemma 4.2.0.6 is sufficient to show that pushout-products of generating r-cofibrations are also r-cofibrations. This result constitutes most of this chapter and verifying pushout-products of a generating r-cofibration with a morphism of the form $0 \to \mathcal{Z}_s(*,*)$ are cofibrations is comparitively straightforward.

Lemma 5.3.2.1. *The pushout-product of $i\colon \mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ and $j\colon \mathcal{Z}_{r+1}(q,m) \to \mathcal{B}_{r+1}(q,m)$ in $(f\mathcal{C})_S$ where $\max S = r$ is a cofibration.*

Theorem 5.3.2.2. *Each of the model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 is a monoidal model category.*

Following work of Schwede and Shipley [SS00] we will then in Section 5.4 use the monoidal model structure in combination with an additional axiom, the monoid axiom Definition 1.4.6.1, to show that the category of filtered differential graded algebras can be equipped with an S-model structure. One can also infer S-model category structures on modules over a filtered differential graded algebra from the same monoid axiom.

Corollary 5.4.0.4. *The model categories $(f\mathcal{C})_S$ satisfy the monoid axiom.*

This implies existence of various model categories of module objects, Theorems 5.4.0.5 and 5.4.0.6. Furthermore we have the following theorem in which for A a filtered differential graded-commutative algebra we let T_A denote the free A-algebra functor.

Theorem 5.4.0.7. *For a fixed r, subset $S \subseteq \{0, 1, \ldots, r-1, r\}$ containing r and filtered differential graded-commutative algebra A there is a cofibrantly generated model category structure on A-algebras whose weak equivalences are the r-quasi-isomorphisms and fibrations those morphisms that are surjective on all s-cycles with $s \in S$. The generating cofibrations are given by $T_A I_S$ and generating acyclic cofibrations by $T_A J_S$.*

We conclude with a section using the work of Muro, [Mur15], to show that the r-model structures can be adapted, to Quillen equivalent ones, in which the unit is cofibrant Section 5.5.

Corollary 5.5.0.6. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r, the category $f\mathcal{C}$ admits a left and right proper cofibrantly generated monoidal model structure, which we denote $\left(\widetilde{f\mathcal{C}}\right)_S$, satisfying the monoid axiom where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *$\tilde{I}_S := I_S \cup \{0 \to R^0_{(0)}\}$ and $\tilde{J}_S := J_S \cup \{j \circ i_{Q_r I} \colon Q_r I \to D\}$ are the sets of generating cofibrations and generating trivial cofibrations respectively.*

One can also make a more symmetrical constrution via the same proof of Muro by forcing all $R^n_{(p)}$ to be cofibrant.

Corollary 5.5.0.9. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \ldots, r\}$ including r, the category $f\mathcal{C}$ admits a left and right proper cofibrantly generated monoidal model structure, which we denote $\left(\widehat{f\mathcal{C}}\right)_S$, satisfying the monoid axiom where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *$\hat{I}_S := I_S \cup \{0 \to R^n_{(p)}\}_{p,n\in\mathbb{Z}}$ and $\hat{J}_S := J_S \cup \{j \circ i_{Q_r R^n_{(p)}} \colon Q_r R^n_{(p)} \to D^n_{(p)}\}$ are the sets of generating cofibrations and generating trivial cofibrations respectively.*

5.1 A cofibrant replacement for the unit

In this section we show that the monoidal unit $R^0_{(0)}$ is not cofibrant and construct a cofibrant replacement for it. In the model categories of bicomplexes a cofibrant replacement for the unit was given in [FGLW22, Proposition 6.7] as an infinite staircase. The cofibrant replacement is similar to that for bicomplexes however depends on r unlike the case for bicomplexes.

Definition 5.1.0.1. We denote by $Q_r I$ the filtered chain complex given by:

$$Q_r I := \left(\bigoplus_{i=0}^{\infty} R^0_{(-i)} \longrightarrow \bigoplus_{j=1}^{\infty} R^1_{(-r-j)} \right) ,$$

where the differential is given by mapping each $R^0_{(-i)}$ diagonally onto the copies of R indexed as $R^1_{(-r-i)}$ and $R^1_{(-r-i-1)}$ for $i \geq 1$ and by the identity map from $R^0_{(0)}$ to $R^1_{(-r-1)}$.

We will first show that this is a cofibrant replacement for the unit in one of the model categories, $(f\mathcal{C})_S$, of Theorem 3.1.0.2 and then show $R^0_{(0)}$ is indeed not cofibrant. The object $Q_r I$ can be more easily pictured as follows where all arrows denote identity morphisms, the summands in cohomological degree 0 are displayed in the first column and those of degree 1 in the second column:

$$\begin{tikzcd}
R^0_{(0)} \arrow[r] & R^1_{(-r-1)} \\
R^0_{(-1)} \arrow[r] \arrow[ru] & R^1_{(-r-2)} \\
R^0_{(-2)} \arrow[r] \arrow[ru] & R^1_{(-r-3)} \\
\vdots \arrow[r] \arrow[ru] & \vdots
\end{tikzcd} .$$

Example 5.1.0.2. We provide a description of the pages of the associated spectral sequence of Q_rI. The E_0 page to the E_r page are isomorphic with a copy of R in each bidegree $(p, p+n)$ with $p \leq 0$ and $n = 0$ as well as a copy of R in each bidegree $(p, p+n)$ with $p \leq -r-1$ and $n = 1$. The differentials d_k are all 0 for $k \leq r-1$ and the r-page differentials d_r are the identities between bidegrees $(-p, -p)$ and $(-p-r, -p-r+1)$ for $p \geq 1$, Figure 5.1. We then have $E_\infty(Q_rI) = E_{r+1}(Q_rI)$ with a single copy of R in bidegree $(0,0)$ and 0 elsewhere.

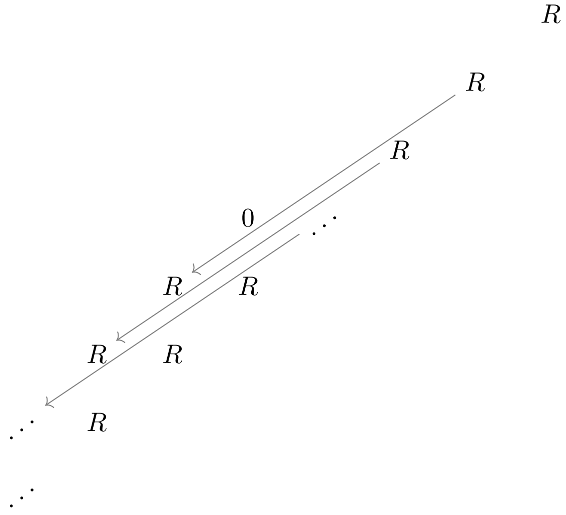


Figure 5.1: The r-page of the spectral sequence associated to Q_rI.

Notation 5.1.0.3. We will denote by $1^0_{(-k)}$ and $1^1_{(-r-1-k)}$ generators of the summands $R^0_{(-k)}$ and $R^1_{(-r-1-k)}$ of Q_rI.

Before we prove this is an r-cofibrant replacement of the unit note the following change of basis (displayed vertically) between two filtered chain complexes (displayed horizontally):

$$\begin{tikzcd}
\Big(R^n_{(p)} \arrow[r, "\Delta"] \arrow[d, "1"] & R^{n+1}_{(p)} \oplus R^{n+1}_{(p-1)}\Big) \arrow[d, "\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}"] \\
\Big(R^n_{(p)} \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & R^{n+1}_{(p)} \oplus R^{n+1}_{(p-1)}\Big)
\end{tikzcd} . \tag{5.1}$$

We prove that Q_rI is an r-cofibrant replacement for the unit for $r = 0$ and then appeal to the shift-décalage adjunction to obtain the result for all other S-model structures.

Proposition 5.1.0.4. *The filtered chain complex Q_0I of Definition 5.1.0.1 is a 0-cofibrant replacement for the unit.*

Proof. There is an obvious morphism of filtered chain complexes $Q_0I \longrightarrow R^0_{(0)}$ projecting onto the $R^0_{(0)}$ summand. This morphism is a 0-quasi-isomorphism; the copy of $R^0_{(0)}$ in Q_0I is a 1-cycle and not a 1-boundary. Any other 1-cycle in cohomological degree 0 must be a finite sum of the $1^0_{(-k)}$ whose coefficient of $1^0_{(0)}$ is non-zero, all other $1^0_{(-k)}$ for $k > 0$ become 0 on the 1-page. Similarly all finite sums of cohomological degree -1 generators can be seen to be 1-boundaries. Hence the morphism $Q_0I \longrightarrow R^0_{(0)}$ is a 0-quasi-isomorphism.

We now wish to show Q_0I is 0-cofibrant, i.e. given a 0-acylic fibration $f\colon E \longrightarrow B$ we can construct a lift in the diagram:

$$\begin{tikzcd}
 & E \arrow[d, two heads, "f", "\sim"'] \\
Q_0I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd} .$$

We proceed by *induction down the staircase*, note that by Lemma 1.6.0.7 the morphism $0 \longrightarrow \mathcal{Z}_1(0,0)$ is a 0-cofibration so that we can lift the *top of the staircase* $R^0_{(0)} \to R^1_{(-1)}$ against f.

The induction step lifts a *step of the staircase* against f subject to knowing where the *top of the step* is mapped to. I.e. we lift the following portion of $Q_0 I$

$$\begin{tikzcd}
 & R^1_{(-k)} \\
R^0_{(-k)} \arrow[ur] \arrow[r] & R^1_{(-k-1)}
\end{tikzcd},$$

subject to already knowing where the $R^1_{(-k)}$ summand is lifted to. This is equivalent to finding a lift in the diagram:

$$\begin{tikzcd}
\left(R^1_{(-k)}\right) \arrow[r] \arrow[d, "\binom{1}{0}"'] & E \arrow[d, two heads, "f", "\sim"'] \\
\left(R^0_{(-k)} \xrightarrow{\Delta} R^1_{(-k)} \oplus R^1_{(-k-1)}\right) \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd},$$

or equivalently, by the change of basis of Equation (5.1), a lift in the following diagram:

$$\begin{tikzcd}
\left(R^1_{(-k)}\right) \arrow[r] \arrow[d, "\Delta"'] & E \arrow[d, two heads, "f", "\sim"'] \\
\left(R^0_{(-k)} \xrightarrow{\binom{1}{0}} R^1_{(-k)} \oplus R^1_{(-k-1)}\right) \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}. \tag{5.2}$$

This lifting problem is then just a special case of a lift of $\mathcal{Z}_1(-k,1) \longrightarrow \mathcal{B}_1(-k,1)$ in which the images of two of the differentials are 0 in E and B respectively. Since f is a 0-acyclic fibration there is then a lift in Equation (5.2).

Lemma 5.1.0.5. *Applying the r-shift functor to $Q_0 I$ gives $Q_r I$, i.e. $S^r Q_0 I = Q_r I$.*

Corollary 5.1.0.6. *The filtered chain complex $Q_r I$ is an S-cofibrant replacement for the unit.*

Proof. By the shift-décalage adjunction a lift of $Q_r I = S^r Q_0 I$ against an S-acyclic fibration f is equivalently a lift of $Q_0 I$ against $\mathrm{Dec}^r(f)$. The lift in the latter exists since Dec^r takes S-acyclic fibrations to 0-acyclic fibrations by Proposition 3.5.0.2, and $Q_0 I$ is 0-cofibrant.

Example 5.1.0.7. The filtered chain complex $R^0_{(0)}$ is not cofibrant in any of the model structures $(f\mathcal{C})_S$ of Theorem 3.1.0.2. We have the projection $\pi \colon Q_r I \to R^0_{(0)}$ onto the unit which is the identity on the component given by $R^0_{(0)}$ and 0 otherwise. We've seen that this is a 0-weak equivalence for $r = 0$ and applying the shift functor S^r shows π is an r-weak equivalence. It is further a Z_s-bidegree-wise surjection for all $s \leq r$ since π is just projection onto one of it's components. We then consider the lifting problem:

$$\begin{tikzcd}
 & Q_r I \arrow[d, two heads, "\pi"', "\sim"] \\
R^0_{(0)} \arrow[r, "id"] & R^0_{(0)}
\end{tikzcd}.$$

If the unit were cofibrant a lift would exist in this diagram. The image of the generator 1 of $R^0_{(0)}$ under such a lift would firstly have to have differential 0 and secondly have image in only a finite number of the $\bigoplus_{i=0}^{\infty} R^0_{(-i)}$. Let q be greatest such that the image of the generator under the lift has non-zero image in the $R^0_{(-q)}$ component. We can then see that the differential of the lift of this generator is non-zero as there is a non-zero differential from the $R^0_{(-q)}$ component with image in $R^{-1}_{(-q-r-1)}$ which cannot be cancelled by a differential from $R^0_{(-q-1)}$ by maximality of q. This shows no such lift can exist and therefore that $R^0_{(0)}$ cannot be cofibrant.

5.2 Verification of the unit axiom

We would now like to verify the unit axiom, Definition 1.4.5.2 condition 2, with Q_rI as the choice of cofibrant replacement for the unit. We begin by identifying the $(r+1)$-cycles and $(r+1)$-boundaries in the tensor product $Q_rI\otimes A$ for some filtered chain complex A. Using the generators of Notation 5.1.0.3 we can write an element $q\in F_p(Q_rI\otimes A)^n$ as a finite sum of tensors:

$$q=\sum_{k\geq 0}1^0_{(-k)}\otimes a^n_{(p+k)}+\sum_{j\geq 0}1^1_{(-r-1-j)}\otimes a^{n-1}_{(p+r+1+j)}\ ,$$

with $a^m_{(q)}\in F_qA^m$. Note that there are no issues in this case arising from the definition of the filtered tensor product as we can view the underlying filtered graded module of $Q_rI\otimes A$ as being a direct sum of shifts of A (the shifts corresponding to the filtration and cohomological indexing of the R components of Q_rI). Further applying the differential of $Q_rI\otimes A$ to q we obtain:

$$dq=\sum_{k\geq 0}1^0_{(-k)}\otimes da^n_{(p+k)}+\sum_{j\geq 0}1^1_{(-r-1-j)}\otimes\left(-da^{n-1}_{(p+r+1+j)}+a^n_{(p+j)}+a^n_{(p+j+1)}\right)\ .$$

Lemma 5.2.0.1. *An element $q\in F_p(Q_rI\otimes A)^n$ is an $(r+1)$-cycle, i.e. $q\in Z^{p,p+n}_{r+1}(Q_rI\otimes A)$, if and only if the following hold:*

1. $a^n_{(p)}\in Z^{p,p+n}_{r+1}(A)$,
2. $a^n_{(p+k)}\in B^{p+k,p+k+n}_{r+1}(A)$, *for each* $k\geq 1$, *and*
3. $a^n_{(p+j)}-da^{n-1}_{(p+r+j)}\in F_{p-1+j}A^n$, *for each* $j\geq 1$.

Note that the third condition of the lemma is giving an explicit representation of $a^n_{(p+k)}$ as an $(r+1)$-boundary, and in fact $(3)\Rightarrow(2)$.

Proof. Assume q is an $(r+1)$-cycle, so then $dq\in F_{p-r-1}(Q_rI\otimes A)^{n+1}$ and we have:

1. $1^0_{(-k)}\otimes da^n_{(p+k)}\in F_{p-r-1}(Q_r\otimes A)^{n+1}$, for $k\geq 0$, and
2. $1^1_{(-r-1-j)}\otimes\left(-da^{n-1}_{(p+r+1+j)}+a^n_{(p+j)}+a^n_{(p+j+1)}\right)\in F_{p-r-1}(Q_rI\otimes A)^{n+1}$, for $j\geq 0$.

The first condition of the proof then says, for $k=0$, that $da^n_{(p)}\in F_{p-r-1}A^{n+1}$ giving the first condition of the lemma. For $k\geq 1$ it says that $da^n_{(p+k)}\in F_{p-r-1+k}A^{n+1}$. The second condition gives us that

$$-da^{n-1}_{(p+r+1+j)}+a^n_{(p+j)}+a^n_{(p+j+1)}\in F_{p+j}A^n$$

which can be rewritten as

$$-da^{n-1}_{(p+r+1+j)}+a^n_{(p+j+1)}\in F_{p+j}A^n\ .$$

Combined with the first condition we thus obtain a diagram of the form of Equation (5.3) which demonstrates the remaining two conditions.

$$\begin{tikzcd}
 & \Big(R^n_{(p+j+1)}\{a^n_{(p+j+1)}\} \arrow[r] \arrow[d, "\Delta"'] & R^{n+1}_{(p+j-r)}\Big) \arrow[d] \\
\Big(R^{n-1}_{(p+j+1+r)}\{a^{n-1}_{(p+r+1+j)}\} \arrow[r, "\binom{1}{0}"] & R^n_{(p+j+1)}\oplus R^n_{(p+j)} \arrow[r, "(0\ 1)"] & R^{n+1}_{(p+j-r)}\Big)
\end{tikzcd} \tag{5.3}$$

The reverse direction is similarly obtained.

Using this classification of $(r+1)$-cycles and $(r+1)$-boundaries we can obtain a result slightly stronger than the unit axiom for a monoidal model category. Since $Z^{*,*}_*(-)$ is functorial we have a morphism $Z^{p,p+n}_{r+1}(Q_rI\otimes A)\longrightarrow Z^{p,p+n}_{r+1}(A)$, and this is given by projection onto the $1^0_{(0)}\otimes -$ component followed by the isomorphism $R^0_{(0)}\otimes A\cong A$. This functoriality argument also shows condition 1 of the preceeding lemma. It also shows however that for the $(r+1)$-cycle q above, if the image is a boundary then so too is $a^n_{(p)}$ in A.

Proposition 5.2.0.2. *The composite function $Q_rI \otimes A \longrightarrow I \otimes A \longrightarrow A$ is an r-weak equivalence for all (not necessarily cofibrant) A.*

Proof. We will show the following map is both injective and surjective for all $p, n \in \mathbb{Z}$,

$$E^{p,p+n}_{r+1}\left(Q_rI \otimes A\right) = \frac{Z^{p,p+n}_{r+1}\left(Q_rI \otimes A\right)}{B^{p,p+n}_{r+1}\left(Q_rI \otimes A\right)} \longrightarrow \frac{Z^{p,p+n}_{r+1}\left(A\right)}{B^{p,p+n}_{r+1}\left(A\right)} = E^{p,p+n}_{r+1}\left(A\right)\ .$$

Surjectivity: Surjectivity of the map is clear since for an $a \in Z^{p,p+n}_{r+1}(A)$ we can take the element given by $1^0_{(0)} \otimes a \in Z^{p,p+n}_{r+1}(Q_rI \otimes A)$, which is an $(r+1)$-cycle by the classification of Lemma 5.2.0.1, whose image is a under the composite $Q_rI \otimes A \longrightarrow A$.

Injectivity: For injectivity we consider an $(r+1)$-cycle $q \in Z^{p,p+n}_{r+1}(Q_rI \otimes A)$ with image in $B^{p,p+n}_{r+1}(A)$. By Lemma 5.2.0.1 for $k \geq 1$ and the preceeding discussion for $k = 0$ this means that $a^n_{(p+k)} \in B^{p+k,p+k+n}_{r+1}(A)$. Lemma 5.2.0.1 also gives an explicit representation of $a^n_{(p+k)}$ as a boundary for $k \geq 1$ and we choose one for $k = 0$ and write these as

$$\begin{aligned} a^n_{(p+k)} &= da^{n-1}_{(p+r+k)} + b^n_{(p+k-1)} \\ B^{p+k,p+k+n}_{r+1}(A) &= dZ^{p+k,p+k+n-1}_{r}(A) + Z^{p+k+r-1,p+k+r-1+n}_{r}(A)\ . \end{aligned}$$

We can then rewrite the $1^0_{(-k)} \otimes a^n_{(p+k)}$ components as:

$$\begin{aligned} 1^0_{(-k)} \otimes a^n_{(p+k)} =& d\left(1^0_{(-k)} \otimes a^{n-1}_{(p+k+r)}\right) + 1^0_{(-k)} \otimes b^n_{(p+k-1)} \\ & - 1^1_{(-k-r)} \otimes a^{n-1}_{(p+k+r)} - 1^1_{(-k-r-1)} \otimes a^{n-1}_{(p+k+r)}\ , \end{aligned}$$

where the penultimate term is interpreted as 0 when $k = 0$. Rearranging the previous and summing over $k \geq 0$ gives:

$$\begin{aligned} q = \sum_{k\geq 0}\left(1^0_{(-k)} \otimes a^n_{(p+k)} + 1^1_{(-k-r)} \otimes a^{n-1}_{(p+k+r)}\right) = \sum_{k\geq 0}\Big(& d\left(1^0_{(-k)} \otimes a^{n-1}_{(p+k+r)}\right) + 1^0_{(-k)} \otimes b^n_{(p+k-1)} \\ & -1^1_{(-k-r-1)} \otimes a^{n-1}_{(p+k+r)}\Big)\ , \end{aligned}$$

and we now identify terms on the right as belonging to constituent r-cycles of $(r+1)$-boundaries. The term with differential applied to, $1^0_{(-k)} \otimes a^{n-1}_{(p+k+r)}$, is an element of $Z^{p+r,p+r+n-1}_{r}(Q_rI \otimes A)$ since $a^{n-1}_{(p+k+r)}$ is an r-cycle as are all $1^0_{(-k)}$, and hence its differential is in $B^{p,p+n}_{r+1}(Q_rI \otimes A)$. Similarly $b^n_{(p+k-1)}$ is an r-cycle and the final two terms of the right side sum are elements of $F_{p-1}(Q_rI \otimes A)^n$, hence are also elements of $B^{p,p+n}_{r+1}(Q_rI \otimes A)$. We have then written q as a sum of boundary elements proving injectivity.

The proof of Proposition 5.2.0.2 demonstrates the following result which is identical to that of Lemma 5.2.0.1 except we replace the remaining cycles with boundaries.

Corollary 5.2.0.3. *An element $q \in F_p(Q_rI \otimes A)$ is an $(r+1)$-boundary, i.e. $q \in B^{p,p+n}_{r+1}(Q_rI \otimes A)$, if and only if the following hold:*

1. $a^n_{(p)} \in B^{p,p+n}_{r+1}(A)$,
2. $a^n_{(p+k)} \in B^{p+k,p+k+n}_{r+1}(A)$, *for each* $k \geq 1$, *and*
3. $a^n_{(p+k)} - da^{n-1}_{(p+r+k)} \in F_{p-1+j}A^n$, *for each* $j \geq 1$.

In fact existence of a cofibrant replacement for the unit satisfying the unit axiom implies all cofibrant replacements of the unit satisfy the unit axiom [Mur15, Lemma 7].

5.3 Verification of the pushout-product axiom

With the unit axiom established it remains to show the pushout-product axiom holds for the (acyclic) cofibrations of $(f\mathcal{C})_S$. We will demonstrate this by verifying the pushout-product axiom of the generating (acyclic) cofibrations of $(f\mathcal{C})_S$ and bootstrapping the result to hold for all cofibrations with the aid of Lemma 1.4.5.5. We have sufficient knowledge regarding the cofibrations to show, using Lemma 4.2.0.6, that the pushout-products are cofibrations too.

5.3.1 Decompositions of certain tensor products

Recall the notations of Notation 1.2.1.13 and Section 1.8 and consider the tensor product $\mathcal{Z}_s(p,n) \otimes \mathcal{Z}_t(q,m)$, where w.l.o.g. $s \leq t$. This can be depicted as the filtered chain complex:

$$\begin{tikzcd}
R^{(n)+(m)}_{(p)+(q)}\{d\} \arrow[r] \arrow[d, "(-1)^n"] & R^{(n+1)+(m)}_{(p-s)+(q)}\{c\} \arrow[d, "(-1)^{n+1}"] \\
R^{(n)+(m+1)}_{(p)+(q-t)}\{b\} \arrow[r] & R^{(n+1)+(m+1)}_{(p-s)+(q-t)}\{a\}
\end{tikzcd}$$

where we have given names a, b, c and d to the generator 1 of each copy of R. There is a change of basis compatible with the filtration to the filtered chain complex with generators a, $(-1)^n b$, $(-1)^n b + c$ and d. Note that if $s < t$ there is not a change of basis to a, $(-1)^n b + c$, c and d as we can no longer refer to the element b in filtration degree $p + q - t$ unless we use both $(-1)^n b + c$ and b which both live in filtration degree $p - s + q$ which is greater than $p - t + q$.

Our change of basis to a, b,$(-1)^n b + c$ and d is such that the differentials of d and b are $(-1)^n b + c$ and a respectively and both d and b are therefore s-cycles. We have then shown the following lemma.

Lemma 5.3.1.1. *For $s \leq t$, there is an isomorphism of filtered chain complexes*

$$\mathcal{Z}_t(q,m) \otimes \mathcal{Z}_s(p,n) \cong \mathcal{Z}_s(p+q, n+m) \oplus \mathcal{Z}_s(p+q-t, n+m+1) \ .$$

Corollary 5.3.1.2. *In the model category $(f\mathcal{C})_S$ with $s, t \in S$ the pushout-product of $0 \to \mathcal{Z}_t(q,m)$ and $0 \to \mathcal{Z}_s(p,n)$ is an acyclic cofibration.*

Proof. By Lemma 5.3.1.1 the pushout-product is isomorphic to $0 \to \mathcal{Z}_s(p+q, n+m) \oplus \mathcal{Z}_s(p+q-t, n+m+1)$ which is the direct sum of two acyclic cofibrations, since $s \in S$, hence is an acyclic cofibration.

We now wish to show a similar decomposition for the tensor product of $\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ with $\mathcal{Z}_s(q,m)$, for $s \leq r$, which is the pushout-product of $\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ and $0 \to \mathcal{Z}_s(q,m)$. This morphism of filtered chain complexes can be depicted as in Figure 5.2 with generators as indicated. We now compute a change of basis, for the domain we use as a basis: $A, B, B + (-1)^n C$ and D. For the codomain we use $a, b, b + (-1)^n c, b + (-1)^n c + (-1)^n d$, e, $e + f$, $f + (-1)^{n-1} g$ and h. We have written each list of generators so that for those of the same cohomological degree they appear in ascending filtration order. This makes it clear that this change of basis is compatible with the filtration so we have an isomorphism of filtered chain complexes, not just of chain complexes. We now observe that under this new basis the morphism is the direct sum of the following four morphisms with the dashed arrows indicating the morphisms of filtered chain complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[r] \arrow[d, dashed] & 0 \arrow[d, dashed] \\
R\{h\} \arrow[r] & R\{f + (-1)^{n-1} g\} & R\{e\} \arrow[r] & R\{b + (-1)^n c\}
\end{tikzcd}$$

$$\begin{tikzcd}
R\{B\} \arrow[r] \arrow[d, dashed] & R\{(-1)^{n+1} A\} \arrow[d, dashed] & R\{D\} \arrow[r] \arrow[d, dashed] & R\{B + (-1)^n C\} \arrow[d, dashed] \\
R\{b\} \arrow[r] & R\{(-1)^{n+1} a\} & R\{e + f\} \arrow[r] & R\{b + (-1)^n c + (-1)^n d\}
\end{tikzcd}$$

which exhibits the pushout product of $\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ and $0 \to \mathcal{Z}_s(q,m)$ as the direct sum of four S-acyclic cofibrations, since $s \in S$, two of which are in fact isomorphisms. We have then proved the following lemma.

Lemma 5.3.1.3. *For $s \leq r$, there is a change of basis exhibiting the pushout-product of $\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ and $0 \to \mathcal{Z}_s(q,m)$ as the direct sum of morphisms:*

$$\begin{pmatrix} 0 \\ \downarrow \\ \mathcal{Z}_s(p+q+r, n+m-1) \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \downarrow \\ \mathcal{Z}_s(p+q-1, n+m) \end{pmatrix}$$

$$\oplus \begin{pmatrix} \mathcal{Z}_s(p+q-r-1, n+m+1) \\ \downarrow \\ \mathcal{Z}_s(p+q-r-1, n+m+1) \end{pmatrix} \oplus \begin{pmatrix} \mathcal{Z}_s(p+q, n+m) \\ \downarrow \\ \mathcal{Z}_s(p+q, n+m) \end{pmatrix} .$$

$$\begin{tikzcd}
 & & & R^{(n)+(m)}_{(p)+(q)}\{D\} \arrow[rr, "1"] \arrow[dl, "(-1)^n"] \arrow[dd, dashed, "{\begin{pmatrix}1\\1\end{pmatrix}}"] & & R^{(n+1)+(m)}_{(p-r-1)+(q)}\{B\} \arrow[dl, "(-1)^{n+1}"] \arrow[dd, dashed, "1"] \\
 & & R^{(n)+(m+1)}_{(p)+(q-s)}\{C\} \arrow[rr, "1"] \arrow[dd, dashed, "{\begin{pmatrix}1\\1\end{pmatrix}}"] & & R^{(n+1)+(m+1)}_{(p-r-1)+(q-s)}\{A\} \arrow[dd, dashed, "1"] & \\
 & R^{(n-1)+(m)}_{(p+r)+(q)}\{h\} \arrow[rr, "i_1"] \arrow[dl, "(-1)^{n-1}"] & & R^{(n)+(m)}_{(p)+(q)}\{f\} \oplus R^{(n)+(m)}_{(p-1)+(q)}\{e\} \arrow[rr, "\pi_2"] \arrow[dl, "(-1)^n id"] & & R^{(n+1)+(m)}_{(p-r-1)+(q)}\{b\} \arrow[dl, "(-1)^{n+1}"] \\
R^{(n-1)+(m+1)}_{(p+r)+(q-s)}\{g\} \arrow[rr, "i_1"] & & R^{(n)+(m+1)}_{(p)+(q-s)}\{d\} \oplus R^{(n)+(m+1)}_{(p-1)+(q+s)}\{c\} \arrow[rr, "\pi_2"] & & R^{(n+1)+(m+1)}_{(p-r-1)+(q-s)}\{a\} &
\end{tikzcd}$$

Figure 5.2: Pushout-product of $\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ and $0 \to \mathcal{Z}_s(q,m)$

As an immediate corollary we have the following:

Corollary 5.3.1.4. *In the model category $(f\mathcal{C})_S$ with $s \in S$ and $r = \max S$, the pushout-product of $\mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ and $0 \to \mathcal{Z}_s(q,m)$ is an acyclic cofibration.* ❂

5.3.2 Pushout-product of generating cofibrations of $(f\mathcal{C})_r$

Recall the construction of Lemma 1.2.1.9 which computed a colimit of a diagram $X \colon I \longrightarrow f\mathcal{C}$ as the composite $r \operatorname{colim}_I iX$. I.e. we interpret a filtered chain complex as an object of $\mathcal{C}_R^{\mathbb{Z}_+}$ via i, compute the colimit degreewise in $\mathcal{C}_R^{\mathbb{Z}_+}$ and then apply the reflector r. We will make use of this for computation of the pushouts in the pushout-products.

We now consider the case of the pushout-product of two generating cofibrations of $(f\mathcal{C})_r$. Let i be the morphism $\varphi_{r+1} \colon \mathcal{Z}_{r+1}(q,m) \longrightarrow \mathcal{B}_{r+1}(q,m)$ and j be the morphism $\varphi_{r+1} \colon \mathcal{Z}_{r+1}(p,n) \longrightarrow \mathcal{B}_{r+1}(p,n)$. Recall that i can be displayed as follows where the components of i are diagonal or identity maps:

$$\begin{tikzcd}
 & \Big(R^m_{(q)} \arrow[r] \arrow[d, "i^m"'] & R^{m+1}_{(q-r-1)}\Big) \arrow[d, "i^{m+1}"] \\
\Big(R^{m-1}_{(q+r)} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & R^m_{(q)} \oplus R^m_{(q-1)} \arrow[r, "(0\ 1)"'] & R^{m+1}_{(q-r-1)}\Big)
\end{tikzcd} ,$$

and similarly for j:

$$\begin{tikzcd}
 & \Big(R^n_{(p)} \arrow[r] \arrow[d, "j^n"'] & R^{n+1}_{(p-r-1)}\Big) \arrow[d, "j^{n+1}"] \\
\Big(R^{n-1}_{(p+r)} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & R^n_{(p)} \oplus R^n_{(p-1)} \arrow[r, "(0\ 1)"'] & R^{n+1}_{(p-r-1)}\Big)
\end{tikzcd} .$$

Recall too the notation of Notation 1.2.1.13. Our first goal is to compute the domain of the pushout-product $i \boxtimes j$ which is given by the pushout:

$$\begin{tikzcd}
\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n) \arrow[r] \arrow[d] & \mathcal{B}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n) \arrow[d] \\
\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{B}_{r+1}(p,n) \arrow[r] & \operatorname{dom}(i \boxtimes j) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} . \tag{5.4}$$

The three components of this pushout are depicted in Figure 5.3 as $\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{B}_{r+1}(p,n)$, $\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n)$ and $\mathcal{B}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n)$ respectively. The demarcated boxes illustrate the maps in the pushout Figure 5.3; a demarcated box of the centre $\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n)$ maps via the identity or diagonal as appropriate into the demarcated boxes of the same type in the other two subdiagrams of Figure 5.3.

Similarly the tensor product of the codomain $\mathcal{B}_{r+1}(q,m) \otimes \mathcal{B}_{r+1}(p,n)$ can be depicted as in Figure 5.4. We give names to each of the R-modules of these figures to simplify the notation. The R-modules of Figure 5.3 correspond to those of Figure 5.5 and similarly those of Figure 5.4 correspond with those of Figure 5.6. This change of notation will be used in the diagrams in $\mathcal{C}_R^{\mathbb{Z}_\infty}$. Converting the three filtered chain complexes of Figure 5.3 to $\mathbb{Z}_\infty$-diagrams of chain complexes via the reflector-inclusion adjunction of Lemma 1.2.1.9 gives Figures 5.7 to 5.9.

Under this new labelling of Figure 5.5 the maps of the pushout become:

$$\begin{array}{llll}
\Delta \colon I \longrightarrow B \oplus C & 1 \colon J \longrightarrow E & \Delta \colon K \longrightarrow F \oplus G & 1 \colon L \longrightarrow H \\
\Delta \colon I \longrightarrow N \oplus P & \Delta \colon J \longrightarrow Q \oplus T & 1 \colon K \longrightarrow S & 1 \colon L \longrightarrow U
\end{array} \tag{5.5}$$

and with reference to this labelling we compute the pushout in $\mathbb{Z}_\infty$-chains of Figures 5.8 and 5.9 over Figure 5.7. We also list the defining maps from the three components of the pushout in Figure 5.5 to Figure 5.6.

$$\begin{array}{llll}
\Delta \colon A \to \underline{C} \oplus \underline{E} & \Delta \colon B \to \underline{F} \oplus \underline{I} & \Delta \colon C \to \underline{H} \oplus \underline{K} & 1 \colon D \to \underline{J} \\
\Delta \colon E \to \underline{L} \oplus \underline{N} & 1 \colon F \to \underline{M} & 1 \colon G \to \underline{O} & 1 \colon H \to \underline{P} \\
\Delta \colon I \to \underline{F} \oplus \underline{I} \oplus \underline{H} \oplus \underline{K} & \Delta \colon J \to \underline{L} \oplus \underline{N} & \Delta \colon K \to \underline{M} \oplus \underline{O} & 1 \colon L \to \underline{P} \\
\Delta \colon M \to \underline{B} \oplus \underline{D} & \Delta \colon N \to \underline{F} \oplus \underline{H} & 1 \colon O \to \underline{G} & \Delta \colon P \to \underline{I} \oplus \underline{K} \\
1 \colon Q \to \underline{L} & \Delta \colon S \to \underline{M} \oplus \underline{O} & 1 \colon T \to \underline{N} & 1 \colon U \to \underline{P}
\end{array} \tag{5.6}$$

$$\begin{tikzcd}
R^{(n-1)+(m)}_{(p+r)+(q)} \arrow[r, "(-1)^{n-1}"] \arrow[d, "i_1"'] & R^{(n-1)+(m+1)}_{(p+r)+(q-r-1)} \arrow[d, "i_1"] \\
R^{(n)+(m)}_{(p)+(q)} \oplus R^{(n)+(m)}_{(p-1)+(q)} \arrow[r, "(-1)^n id"] \arrow[d, "\pi_2"'] & R^{(n)+(m+1)}_{(p)+(q-r-1)} \oplus R^{(n)+(m+1)}_{(p-1)+(q-r-1)} \arrow[d, "\pi_2"] \\
R^{(n+1)+(m)}_{(p-r-1)+(q)} \arrow[r, "(-1)^{n+1}"] & R^{(n+1)+(m+1)}_{(p-r-1)+(q-r-1)}
\end{tikzcd}$$

$$\begin{tikzcd}
R^{(n)+(m)}_{(p)+(q)} \arrow[r, "(-1)^{n}"] \arrow[d] & R^{(n)+(m+1)}_{(p)+(q-r-1)} \arrow[d] \\
R^{(n+1)+(m)}_{(p-r-1)+(q)} \arrow[r, "(-1)^{n+1}"] & R^{(n+1)+(m+1)}_{(p-r-1)+(q-r-1)}
\end{tikzcd}$$

$$\begin{tikzcd}
R^{(n)+(m-1)}_{(p)+(q+r)} \arrow[r, "(-1)^n i_1"] \arrow[d] & R^{(n)+(m)}_{(p)+(q)} \oplus R^{(n)+(m)}_{(p)+(q-1)} \arrow[r, "(-1)^n \pi_2"] \arrow[d, "id"] & R^{(n)+(m+1)}_{(p)+(q-r-1)} \arrow[d] \\
R^{(n+1)+(m-1)}_{(p-r-1)+(q+r)} \arrow[r, "(-1)^{n+1} i_1"] & R^{(n+1)+(m)}_{(p-r-1)+(q)} \oplus R^{(n+1)+(m)}_{(p-r-1)+(q-1)} \arrow[r, "(-1)^{n+1} \pi_2"] & R^{(n+1)+(m+1)}_{(p-r-1)+(q-r-1)}
\end{tikzcd}$$

Figure 5.3: Components of the pushout

Recall that in a diagram category of an abelian category colimits are computed index-wise, so we can compute colimits coordinate-wise in these figures. Despite the $\mathbb{Z}_\infty$ indexing not necessarily inducing inclusions from one stage to the next we shall still refer to it in the following as a filtration indexing for convenience.

In homological degree $n+1+m+1$: all filtration degrees, above $p-r-1+q-r-1$, have the same pushout of the form

$$\begin{tikzcd}
L \arrow[r, "1"] \arrow[d, "1"'] & H \arrow[d, dashed] \\
U \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & L
\end{tikzcd}$$

whose pushout can be taken to be L. Also note all induced maps from the filtration indexing are the identity between

$$\begin{tikzcd}
R^{(n-1)+(m-1)}_{(p+r)+(q+r)} \arrow[r, "(-1)^{n-1}i_1"] \arrow[d, "i_1"] & R^{(n-1)+(m)}_{(p+r)+(q)} & \oplus \arrow[d, "id"] & R^{(n-1)+(m)}_{(p+r)+(q-1)} \arrow[r, "(-1)^{n-1}\pi_2"] & R^{(n-1)+(m+1)}_{(p+r)+(q-r-1)} \arrow[d, "i_1"] \\
R^{(n)+(m-1)}_{(p)+(q+r)} & R^{(n)+(m)}_{(p)+(q)} & \oplus & R^{(n)+(m)}_{(p)+(q-1)} & R^{(n)+(m+1)}_{(p)+(q-r-1)} \\
\oplus \arrow[r, "(-1)^{n}id"] & \oplus & & \oplus \arrow[r, "(-1)^{n}id"] & \oplus \\
R^{(n)+(m-1)}_{(p-1)+(q+r)} \arrow[d, "\pi_2"] & R^{(n)+(m)}_{(p-1)+(q)} & \oplus \arrow[d, "id"] & R^{(n)+(m)}_{(p-1)+(q-1)} & R^{(n)+(m+1)}_{(p-1)+(q-r-1)} \arrow[d, "\pi_2"] \\
R^{(n+1)+(m-1)}_{(p-r-1)+(q+r)} \arrow[r, "(-1)^{n+1}i_1"] & R^{(n+1)+(m)}_{(p-r-1)+(q)} & \oplus & R^{(n+1)+(m)}_{(p-r-1)+(q-1)} \arrow[r, "(-1)^{n+1}\pi_2"] & R^{(n+1)+(m+1)}_{(p-r-1)+(q-r-1)}
\end{tikzcd}$$

Figure 5.4: Codomain of the pushout-product $i \boxtimes j$

$$\begin{tikzcd}
A \arrow[r, "(-1)^{n-1}"] \arrow[d] & D \arrow[d] \\
B & F \\
\oplus \arrow[r, "(-1)^{n}"] & \oplus \\
C \arrow[d] & G \arrow[d] \\
E \arrow[r, "(-1)^{n+1}"] & H
\end{tikzcd}$$

$$\begin{tikzcd}
I \arrow[r, "(-1)^{n}"] \arrow[d] & K \arrow[d] \\
J \arrow[r, "(-1)^{n+1}"] & L
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "(-1)^{n}i_1"] \arrow[d] & N \oplus P \arrow[r, "(-1)^{n}\pi_2"] \arrow[d] & S \arrow[d] \\
O \arrow[r, "(-1)^{n+1}i_1"'] & Q \oplus T \arrow[r, "(-1)^{n+1}\pi_2"'] & U
\end{tikzcd}$$

Figure 5.5: Named components of the pushout

these pushouts.

In homological degree $n + 1 + m$: we have in filtration degree $p - 1 + q - r - 1$ the pushout of T and G over 0 which is $G \oplus T$. In filtration degrees $p + q - r - 1$ and above we have the pushouts are of the form

$$\begin{tikzcd}
J \oplus K \arrow[r] \arrow[d] & E \oplus F \oplus G \arrow[d, dashed] \\
Q \oplus S \oplus T \arrow[r, dashed] & F \oplus G \oplus Q \oplus T \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{A} \arrow[r, "(-1)^{n-1}i_1"] \arrow[d, "i_1"'] & \underline{C} \oplus \underline{E} \arrow[r, "(-1)^{n-1}\pi_2"] \arrow[d] & \underline{J} \arrow[d, "i_1"] \\
\begin{array}{c}\underline{B}\\ \oplus\\ \underline{D}\end{array} \arrow[r] \arrow[d, "\pi_2"'] & \begin{array}{c}\underline{F} \oplus \underline{I}\\ \oplus \quad \oplus\\ \underline{H} \oplus \underline{K}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\underline{M}\\ \oplus\\ \underline{O}\end{array} \arrow[d, "\pi_2"] \\
\underline{G} \arrow[r, "(-1)^{n+1}i_1"'] & \underline{L} \oplus \underline{N} \arrow[r, "(-1)^{n+1}\pi_2"'] & \underline{P}
\end{tikzcd}$$

Figure 5.6: Named codomain of the pushout-product $i \boxtimes j$

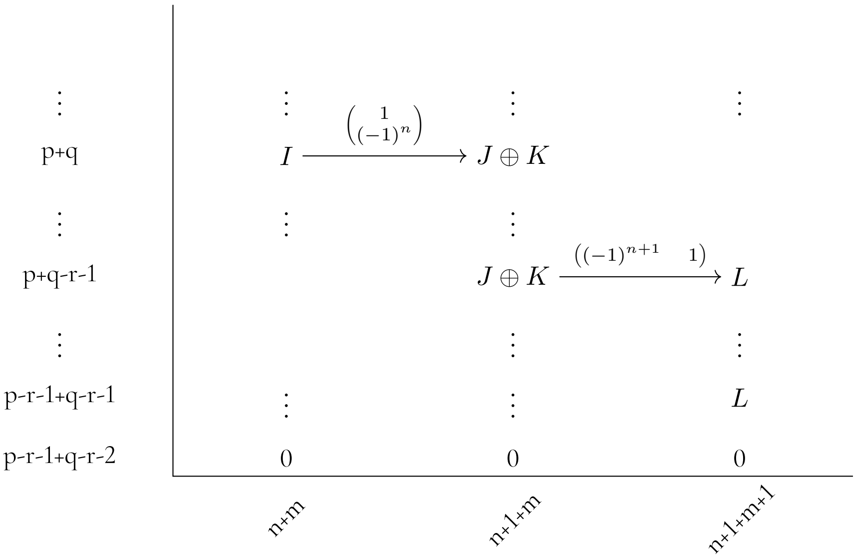


Figure 5.7: Representation in $\mathbb{Z}_\infty$ chains of $\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n)$

which, under the maps of Equation (5.5), identifies E with the diagonal of $Q \oplus T$ via J, and identifies S with the diagonal of $F \oplus G$. We can then take $F \oplus G \oplus Q \oplus T$ as the pushout. The induced maps from $\mathbb{Z}_\infty$ are the inclusion of $G \oplus T$ into $F \oplus G \oplus Q \oplus T$ from $p-1+q-r-1$ to $p+q-r-1$ and identity maps above this.

In homological degree $n+m$: we have in filtration degree $p+q-1$ the pushout of $C \oplus D$ and $O \oplus P$ over 0 giving $C \oplus D \oplus O \oplus P$. In filtration degrees $p+q$ and above the pushout is of the form

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & B \oplus C \oplus D \arrow[d, dashed] \\
N \oplus O \oplus P \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & B \oplus C \oplus D \oplus O \oplus P
\end{tikzcd}$$

where the maps of Equation (5.5) identifies the diagonal of $B \oplus C$ with the diagonal of $N \oplus P$ via I. The pushout is isomorphic to $B \oplus C \oplus D \oplus O \oplus P$, since we can express 1_N as $1_B + 1_C - 1_P$ under the identification from I. The induced maps from $\mathbb{Z}_\infty$, as before, are inclusions of submodules or the identity as appropriate.

In homological degree $n-1+m$: the pushout is simply $A \oplus M$ in filtration degree $p+r+q$ and above.

We can then depict this pushout in $\mathbb{Z}_\infty$-chains as Figure 5.10 where we have also calculated the induced differentials of the pushout. We now apply the reflector functor of the adjunction between $\mathbb{Z}_\infty$-indexed chains and filtered chain complexes which finishes the computation of the pushout in filtered chains.

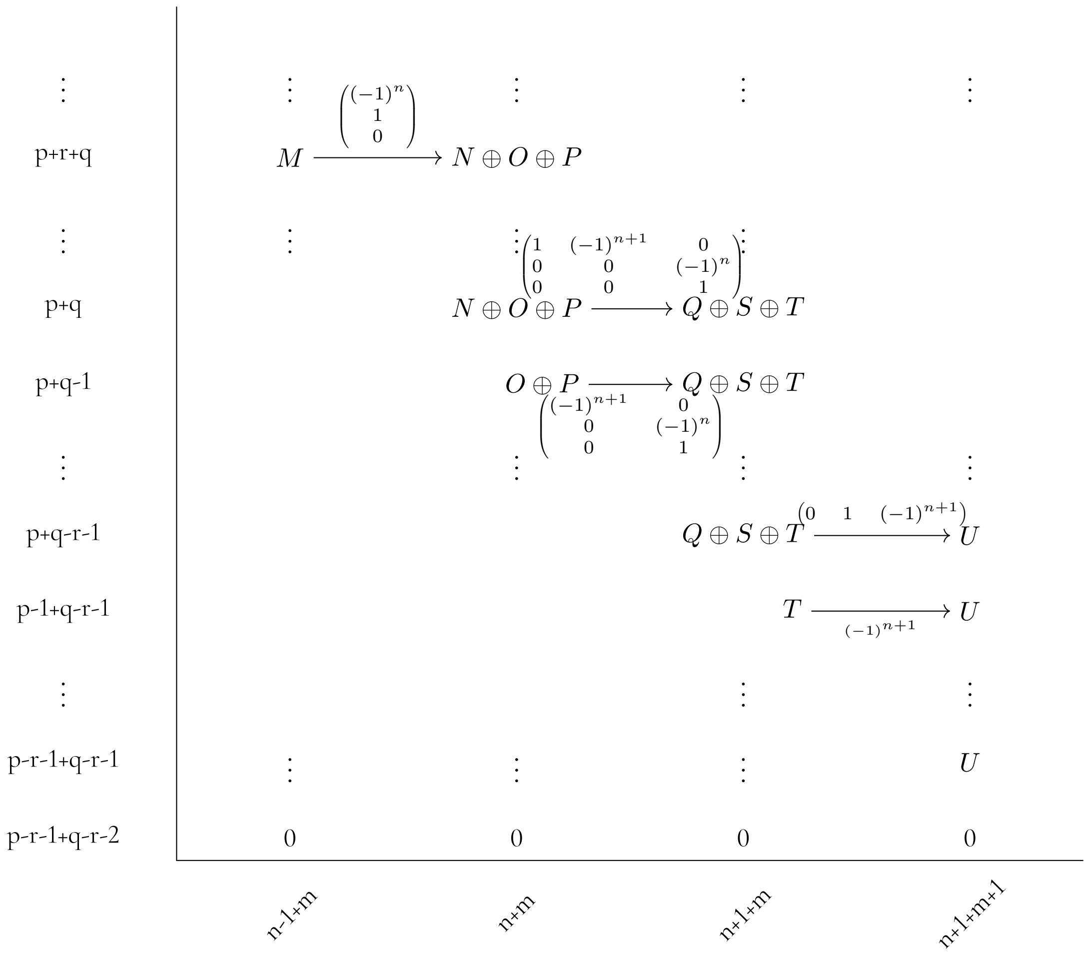


Figure 5.8: Representation in $\mathbb{Z}_\infty$ chains of $\mathcal{B}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n)$

In the notation of $A, B, \ldots$ the pushout is of the form:

$$\begin{tikzcd}
 & & A \arrow[rr, "(-1)^{n-1}"] \arrow[d, "\alpha"] & & D \arrow[d, "i_1"] \\
M \arrow[r, "\beta"] \arrow[d, "1"] & \begin{matrix}B\\ \oplus\\ C\end{matrix} & \oplus \arrow[d, "\gamma"] & P \arrow[r, "\epsilon"] & \begin{matrix}F\\ \oplus\\ G\end{matrix} \arrow[d, "\pi_2"] \\
O \arrow[r, "(-1)^{n+1}i_1"'] & Q & \oplus \arrow[dd, dashed, "i\boxtimes j"] & T \arrow[r, "(-1)^{n+1}\pi_2"'] & L \\
 & & & & \\
\underline{A} \arrow[r, "(-1)^{n-1}i_1"] \arrow[d, "i_1"'] & \underline{C} & \oplus \arrow[d] & \underline{E} \arrow[r, "(-1)^{n-1}\pi_2"] & \underline{J} \arrow[d, "i_1"] \\
\begin{matrix}\underline{B}\\ \oplus\\ \underline{D}\end{matrix} \arrow[r, "(-1)^n"] \arrow[d, "\pi_2"'] & \begin{matrix}\underline{F}\\ \oplus\\ \underline{H}\end{matrix} & \begin{matrix}\oplus\\ \\ \oplus\end{matrix} \arrow[d] & \begin{matrix}\underline{I}\\ \oplus\\ \underline{K}\end{matrix} \arrow[r, "(-1)^n"] & \begin{matrix}\underline{M}\\ \oplus\\ \underline{O}\end{matrix} \arrow[d, "\pi_2"] \\
\underline{G} \arrow[r, "(-1)^{n+1}i_1"'] & \underline{L} & \oplus & \underline{N} \arrow[r, "(-1)^{n+1}\pi_2"'] & \underline{P}
\end{tikzcd} \tag{5.7}$$

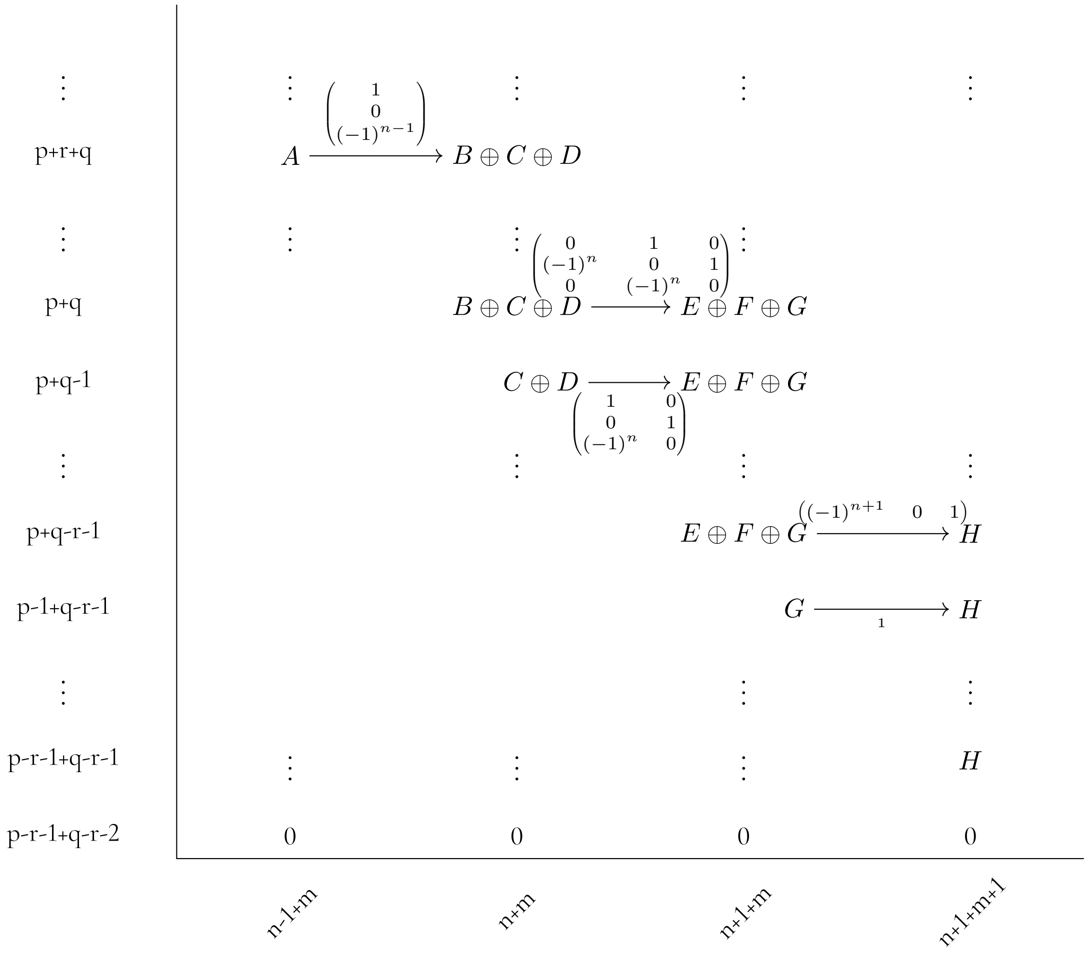


Figure 5.9: Representation in $\mathbb{Z}_\infty$ chains of $\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{B}_{r+1}(p,n)$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\text{p+r+q} & A \oplus M \arrow[r, "\begin{pmatrix} 1 & (-1)^n \\ 0 & (-1)^n \\ (-1)^{n-1} & 0 \\ 0 & 1 \\ 0 & (-1)^{n+1} \end{pmatrix}"] & B \oplus C \oplus D \oplus O \oplus P & & \\
\vdots & \vdots & & & \\
\text{p+q} & & B \oplus C \oplus D \oplus O \oplus P \arrow[r, "\begin{pmatrix} (-1)^n & 0 & 1 & 0 & (-1)^n \\ 0 & 1 & 0 & (-1)^{n+1} & 0 \\ 0 & (-1)^n & 0 & 0 & (-1)^n \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}"] & F \oplus Q \oplus G \oplus T & \\
\text{p+q-1} & & C \oplus D \oplus O \oplus P \arrow[r, "\begin{pmatrix} 0 & 1 & 0 & (-1)^n \\ 1 & 0 & (-1)^{n+1} & 0 \\ (-1)^n & 0 & 0 & (-1)^n \\ 1 & 0 & 0 & 1 \end{pmatrix}"'] & F \oplus Q \oplus G \oplus T & \\
\vdots & & \vdots & \vdots & \vdots \\
\text{p+q-r-1} & & & F \oplus Q \oplus G \oplus T \arrow[r, "\begin{pmatrix} 0 & 0 & 1 & (-1)^{n+1} \end{pmatrix}"] & L \\
\text{p-1+q-r-1} & & & G \oplus T \arrow[r, "\begin{pmatrix} 1 & (-1)^{n+1} \end{pmatrix}"'] & L \\
\vdots & & & \vdots & \vdots \\
\text{p-r-1+q-r-1} & \vdots & \vdots & \vdots & L \\
\text{p-r-1+q-r-2} & 0 & 0 & 0 & 0 \\
& \text{n-1+m} & \text{n+m} & \text{n+1+m} & \text{n+1+m+1}
\end{tikzcd}$$

Figure 5.10: Representation in $\mathbb{Z}_\infty$ chains of the pushout

The maps in Equation (5.7) α, β, γ and ϵ are given as follows:

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : A \longrightarrow B \oplus C \oplus P$$

$$\beta = \begin{pmatrix} (-1)^n \\ (-1)^n \\ (-1)^{n+1} \end{pmatrix} : M \longrightarrow B \oplus C \oplus P$$

$$\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} : B \oplus C \oplus P \longrightarrow Q \oplus T$$

$$\epsilon = \begin{pmatrix} (-1)^n & 0 & (-1)^n \\ 0 & (-1)^n & (-1)^n \end{pmatrix} : B \oplus C \oplus P \longrightarrow F \oplus G$$

We also describe the induced map

$$i \boxtimes j \colon \mathcal{Z}_{r+1}(q,m) \otimes \mathcal{B}_{r+1}(p,n) \coprod_{\mathcal{Z}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n)} \mathcal{B}_{r+1}(q,m) \otimes \mathcal{Z}_{r+1}(p,n) \longrightarrow \mathcal{B}_{r+1}(q,m) \otimes \mathcal{B}_{r+1}(p,n) \ .$$

With reference to Equation (5.6) we can describe this map by:

$$\begin{array}{llll} \Delta \colon A \longrightarrow \underline{C} \oplus \underline{E} & 1 \colon O \longrightarrow \underline{G} & 1 \colon G \longrightarrow \underline{O} & \Delta \colon B \longrightarrow \underline{F} \oplus \underline{I} \\ \Delta \colon M \longrightarrow \underline{B} \oplus \underline{D} & 1 \colon F \longrightarrow \underline{M} & 1 \colon T \longrightarrow \underline{N} & \Delta \colon C \longrightarrow \underline{H} \oplus \underline{K} \\ 1 \colon D \longrightarrow \underline{J} & 1 \colon Q \longrightarrow \underline{L} & 1 \colon L \longrightarrow \underline{P} & \Delta \colon P \longrightarrow \underline{I} \oplus \underline{K} \end{array}$$

One can easily verify that these maps of the R-modules commute with the differentials and so assemble to a map of filtered chain complexes. We also provide the translation back to the sub/superscript notation displaying the homological and filtration degrees in Figure 5.11.

It remains to show that the pushout product is indeed a generating cofibration for which we apply Lemma 4.2.0.6.

Lemma 5.3.2.1. *The pushout-product of* $i \colon \mathcal{Z}_{r+1}(p,n) \to \mathcal{B}_{r+1}(p,n)$ *and* $j \colon \mathcal{Z}_{r+1}(q,m) \to \mathcal{B}_{r+1}(q,m)$ *in* $(f\mathcal{C})_S$ *where* $\max S = r$ *is a cofibration.*

Proof. By Lemma 4.2.0.6 it is enough to show that $i \boxtimes j$ is an r-supressive inclusion whose cokernel C is cofibrant with for all n a $p(n)$ such that $F_{p(n)}C^n = 0$.

The cokernel is given, up to signs, by

$$\begin{tikzcd} R^{(n-1)+(m-1)}_{(p+r)+(q+r)} \arrow[r, "-1"] \arrow[d] & R^{(n-1)+(m)}_{(p+r)+(q-1)} \arrow[d] \\ R^{(n)+(m-1)}_{(p-1)+(q+r)} \arrow[r] & R^{(n)+(m)}_{(p-1)+(q-1)} \end{tikzcd}$$

which decomposes into the direct sum of two representing r-cycles by Lemma 5.3.1.1, hence is cofibrant. Further writing $\operatorname{dom}(i \boxtimes j)$ and $\operatorname{cod}(i \boxtimes j)$ for the domain and codomain of $i \boxtimes j$ we must have that, under the twisted direct sum decomposition of Lemma 4.2.0.3 of $i \boxtimes j$ into $i \boxtimes j \colon \operatorname{dom}(i \boxtimes j) \to \operatorname{dom}(i \boxtimes j) \oplus_\tau C$, that the twist τ surpresses filtration by r. Finally we have that the filtration on C is bounded below. Hence the pushout-product $i \boxtimes j$ is a cofibration in $(f\mathcal{C})_r$ and so too then in $(f\mathcal{C})_S$.

We have then shown the following theorem.

Theorem 5.3.2.2. *Each of the model categories* $(f\mathcal{C})_S$ *of Theorem 3.1.0.2 is a monoidal model category.*

Proof. The unit axiom was proved in Proposition 5.2.0.2 and the pushout-product axioms were proved in Lemma 5.3.2.1 and Corollaries 5.3.1.2 and 5.3.1.4 for the generating (acyclic) cofibrations, we can now apply Lemma 1.4.5.5 to obtain the result.

$$\begin{tikzcd}
& R^{n+m-1}_{(p+r)+(q)} \arrow[r, "(-1)^{n-1}"] \arrow[d, "\alpha"] & R^{n+m}_{(p+r)+(q-r-1)} \arrow[d, "i_1"] \\
R^{n+m-1}_{(p)+(q+r)} \arrow[r, "\beta"] \arrow[d, "1"] & \begin{matrix} R^{n+m}_{(p)+(q)} & \oplus & R^{n+m}_{(p)+(q-1)} \\ \oplus & & \\ R^{n+m}_{(p-1)+(q)} & & \end{matrix} \arrow[r, "\epsilon"] \arrow[d, "\gamma"] & \begin{matrix} R^{n+m+1}_{(p)+(q-r-1)} \\ \oplus \\ R^{n+m+1}_{(p-1)+(q-r-1)} \end{matrix} \arrow[d, "\pi_2"] \\
R^{n+m}_{(p-r-1)+(q+r)} \arrow[r, "(-1)^{n+1}i_1"] & R^{n+m+1}_{(p-r-1)+(q)} \oplus R^{n+m+1}_{(p-r-1)+(q-1)} \arrow[r, "(-1)^{n+1}\pi_2"] \arrow[d, dashed, "i\boxtimes j"] & R^{n+m+2}_{(p-r-1)+(q-r-1)} \\
R^{(n-1)+(m-1)}_{(p+r)+(q+r)} \arrow[r, "(-1)^{n-1}i_1"] \arrow[d, "i_1"] & R^{(n-1)+(m)}_{(p+r)+(q)} \oplus R^{(n-1)+(m)}_{(p+r)+(q-1)} \arrow[r, "(-1)^{n-1}\pi_2"] \arrow[d] & R^{(n-1)+(m+1)}_{(p+r)+(q-r-1)} \arrow[d, "1_1"] \\
\begin{matrix} R^{(n)+(m-1)}_{(p)+(q+r)} \\ \oplus \\ R^{(n)+(m-1)}_{(p-1)+(q+r)} \end{matrix} \arrow[r, "(-1)^n"] \arrow[d, "\pi_2"] & \begin{matrix} R^{(n)+(m)}_{(p)+(q)} & \oplus & R^{(n)+(m)}_{(p)+(q-1)} \\ \oplus & & \oplus \\ R^{(n)+(m)}_{(p-1)+(q)} & \oplus & R^{(n)+(m)}_{(p-1)+(q-1)} \end{matrix} \arrow[r, "(-1)^n"] \arrow[d] & \begin{matrix} R^{(n)+(m+1)}_{(p)+(q-r-1)} \\ \oplus \\ R^{(n)+(m+1)}_{(p-1)+(q-r-1)} \end{matrix} \arrow[d, "\pi_2"] \\
R^{(n+1)+(m-1)}_{(p-r-1)+(q+r)} \arrow[r, "(-1)^{n+1}i_1"] & R^{(n+1)+(m)}_{(p-r-1)+(q)} \oplus R^{(n+1)+(m)}_{(p-r-1)+(q-1)} \arrow[r, "(-1)^{n+1}\pi_2"] & R^{(n+1)+(m+1)}_{(p-r-1)+(q-r-1)}
\end{tikzcd}$$

Figure 5.11: The pushout-product of generating cofibrations

As a consequence we can infer the internal hom object preserves acyclic fibrations when its first component is cofibrant. Recall from Definition 1.2.1.14 that we have an internal Hom object right adjoint to the tensor product. Fixing the first object as a t-cycle we have a special case given by

$$F_q\underline{\mathrm{Hom}}\left(\mathcal{Z}_t(p,n),Y\right)^m = F_{p+q}Y^{n+m} \oplus F_{p-t+q}Y^{n+1+m}$$

with differential

$$F_{p+q}Y^{n+m} \oplus F_{p-t+q}Y^{n+1+m} \ni (x,y) \mapsto (dx-(-1)^m y, dy)\ .$$

Lemma 5.3.2.3. *Let $(f\mathcal{C})_S$ be one of the model structures of Theorem 3.1.0.2 with $s \in S$. Then the functor $\underline{\mathrm{Hom}}\left(\mathcal{Z}_s(p,n),-\right)$ preserves acyclic fibrations in $(f\mathcal{C})_S$.*

Proof. Let $\pi\colon Y \to X$ be an acyclic fibration in $(f\mathcal{C})_S$. We must show that

$$\underline{\mathrm{Hom}}\left(\mathcal{Z}_s(p,n),\pi\right) : \underline{\mathrm{Hom}}\left(\mathcal{Z}_s(p,n),Y\right) \longrightarrow \underline{\mathrm{Hom}}\left(\mathcal{Z}_s(p,n),X\right)$$

is an acyclic fibration, i.e. that it has the right lifting property with respect to all cofibrations $i\colon A \to B$ of $(f\mathcal{C})_S$ in

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & \underline{\mathrm{Hom}}\left(\mathcal{Z}_s(p,n),Y\right) \arrow[d, "\underline{\mathrm{Hom}}(\mathcal{Z}_s(p,n)\text{,}\pi)"] \\
B \arrow[r] & \underline{\mathrm{Hom}}\left(\mathcal{Z}_s(p,n),X\right)
\end{tikzcd}$$

or equivalently by the tensor hom adjunction that $\pi\colon Y \to X$ has the right lifting property in all diagrams of the form

$$\begin{tikzcd}
A \otimes \mathcal{Z}_s(p,n) \arrow[r] \arrow[d, "i\otimes\mathcal{Z}_s(p\text{,}n)"'] & Y \arrow[d, "\pi"] \\
B \otimes \mathcal{Z}_s(p,n) \arrow[r] & X
\end{tikzcd}\ .$$

All of the morphisms $(A \to B) \otimes \mathcal{Z}_s(p, n)$ are cofibrations in $(f\mathcal{C})_S$ by Theorem 5.3.2.2 hence a lift exists.

More generally one could replace $\mathcal{Z}_s(p, n)$ by any cofibrant object A and the functor $\underline{\mathrm{Hom}}(A, -)$ would preserve acyclic fibrations. And further if A were acyclic $\underline{\mathrm{Hom}}(A, -)$ would preserve all fibrations.

5.4 Model structures from the monoid axiom

With monoidal model structures established for filtered chain complexes by proving the unit and pushout-product axioms we now also demonstrate the monoid axiom which yields a model structure on algebra objects in the monoidal category of filtered chain complexes, i.e. model structures of filtered differential graded algebras. This then enhances previous work of Halperin and Tanré, [HT90], to equip filtered differential graded algebras with fully fledged model structures with r-weak equivalences.

Recall from Lemma 1.4.6.2 that in a cofibrantly generated monoidal model category $\mathcal{M}$ with generating acyclic cofibrations J that $\mathcal{M}$ satisfies the monoid axiom if every morphism of $(J \otimes \mathcal{M})\text{-}\mathtt{Cof}_{\mathtt{reg}}$ is a weak equivalence, where the notation $J\text{-}\mathtt{Cof}_{\mathtt{reg}}$ is of Notation 1.4.2.3.

Lemma 5.4.0.1. *Any map in $J_S \otimes f\mathcal{C}$ is an r-weak equivalence.*

Proof. Any map is of $J_S \otimes f\mathcal{C}$ is of the form $0 \to \mathcal{Z}_s(p, p+n) \otimes A$ for some $A \in f\mathcal{C}$ and $s \in S$, so is some shift of the s-cone, Definition 1.6.0.12, of A which is s-acyclic and hence r-acyclic.

Lemma 5.4.0.2. *A pushout of a filtered chain complex A by an element of $J \otimes f\mathcal{C}$ is a weak equivalence.*

Proof. The pushout is of the form $A \to A \oplus \mathcal{Z}_s \otimes B$ for some $B \in f\mathcal{C}$ which is a weak equivalence since $0 \to \mathcal{Z}_s \otimes B$ is a weak equivalence.

Proposition 5.4.0.3. *Every map of $(J \otimes f\mathcal{C})\text{-}\mathtt{Cof}_{\mathtt{reg}}$ is a weak equivalence.*

Proof. Since the model category $(f\mathcal{C})_S$ is a finitely cofibrantly generated model category by Lemma 1.2.1.16 then by [Hov99, Corollary 7.4.2] it suffices to show each pushout in the construction of a map of $(J \otimes \mathcal{M})\text{-}\mathtt{Cof}_{\mathtt{reg}}$ is a weak equivalence. This is the result of Lemma 5.4.0.2.

Corollary 5.4.0.4. *The model categories $(f\mathcal{C})_S$ satisfy the monoid axiom.*

Proof. By [SS00, Lemma 3.5 (2)] it suffices to show that maps in $(J \otimes f\mathcal{C})\text{-}\mathtt{Cof}_{\mathtt{reg}}$ are weak equivalences. This is the result of Proposition 5.4.0.3.

Each of the following results is immediate from Theorem 1.4.6.3, the fact that $(f\mathcal{C})_S$ is cofibrantly generated with all objects being small, and that $(f\mathcal{C})_S$ satisfies the monoid axiom. The statements about the sets of generating cofibrations and acyclic cofibrations follows from the proof of Theorem 1.4.6.3.

Theorem 5.4.0.5. *For a fixed r, subset $S \subseteq \{0, 1, \ldots, r-1, r\}$ containing r and filtered differential graded algebra A there is a model category structure on left A-modules whose weak equivalences are the r-quasi-isomorphisms and fibrations those morphisms that are surjective on all s-cycles with $s \in S$. The generating cofibrations are given by $A \otimes I_S$ and generating acyclic cofibrations by $A \otimes J_S$.*

Theorem 5.4.0.6. *For a fixed r, subset $S \subseteq \{0, 1, \ldots, r-1, r\}$ containing r and filtered differential graded-commutative algebra A there is a cofibrantly generated model category structure on A-modules whose weak equivalences are the r-quasi-isomorphisms and fibrations those morphisms that are surjective on all s-cycles with $s \in S$. The generating cofibrations are given by $A \otimes I_S$ and generating acyclic cofibrations by $A \otimes J_S$. Further this model category satisfies the monoid axiom*

In the following for A a filtered differential graded algebra we let T_A denote the free A-algebra functor.

Theorem 5.4.0.7. *For a fixed r, subset $S \subseteq \{0, 1, \ldots, r-1, r\}$ containing r and filtered differential graded-commutative algebra A there is a cofibrantly generated model category structure on A-algebras whose weak equivalences are the r-quasi-isomorphisms and fibrations those morphisms that are surjective on all s-cycles with $s \in S$. The generating cofibrations are given by $T_A I_S$ and generating acyclic cofibrations by $T_A J_S$.*

In particular taking $A = R^0_{(0)}$ the monoidal unit in this last result we obtain a model category of filtered differential graded algebras.

Corollary 5.4.0.8. *For a fixed r, subset $S \subseteq \{0, 1, \dots, r-1, r\}$ containing r there is a cofibrantly generated model category structure on filtered differential graded algebras whose weak equivalences are the r-quasi-isomorphisms and fibrations those morphisms that are surjective on all s-cycles with $s \in S$. The generating cofibrations are given by I_S and generating acyclic cofibrations by J_S.*

5.5 Forcing the unit to be cofibrant

In Example 5.1.0.7 we showed that the unit $R^0_{(0)}$ is not cofibrant in any of the model structures $(f\mathcal{C})_S$. Having a cofibrant unit does not seem an unreasonable requirement and is the case in many model structures of interest, most relevantly in the projective model structure on chain complexes the unit R^0 is cofibrant. Muro's paper [Mur15] provides a method of adapting a monoidal model category, by changing its fibrations and cofibrations but preserving weak equivalences, so that the unit becomes cofibrant. This method is in fact controlled in the sense that it is still a cofibrantly generated model category, with added generating cofibrations and generating acyclic cofibrations.

Definition 5.5.0.1. A model category with the structure of a closed symmetric monoidal category satisfies the *very strong unit axiom* if there exists a cofibrant replacement QI of the unit I of the tensor product such that for all X the morphism $QI \otimes X \to X$ is a weak equivalence.

This strengthing of the unit axiom removes the requirement that X be cofibrant. Recall that in Proposition 5.2.0.2 we showed that the model categories $(f\mathcal{C})_S$ satisfy not just the unit axiom but also the very strong unit axiom. Whilst existence of a cofibrant replacement of the unit satisfying the unit axiom implies all cofibrant replacements of the unit satisfy the unit axiom, Muro remarks the same does not necessarily hold for the very strong unit axiom if either the monoidal structure is not symmetric or it doesn't satisfy the monoid axiom. Since the model structures $(f\mathcal{C})_S$ are symmetric and satisfy the monoid axiom, Corollary 5.4.0.4, any cofibrant replacement of the unit satisfies the very strong unit axiom. One of Muro's theorems asserting existence of a modified model category with cofibrant unit states the following.

Theorem 5.5.0.2 ([Mur15, Theorem 3]). *Let $\mathcal{M}$ be a cofibrantly generated monoidal category with generating cofibrations I and generating acyclic cofibrations J satisfying the very strong unit axiom for a cofibrant replacement $\pi\colon Q\mathbb{I} \xrightarrow{\sim} \mathbb{I}$ of the unit $\mathbb{I}$. Let*

$$Q\mathbb{I} \coprod \mathbb{I} \underset{j}{\rightarrowtail} D \underset{q}{\xrightarrow{\sim}} \mathbb{I}$$

be a factorisation of $(\pi, id)\colon Q\mathbb{I} \coprod \mathbb{I} \to \mathbb{I}$ into a cofibration followed by a weak equivalence in $\mathcal{M}$. Let $i_{Q\mathbb{I}}\colon Q\mathbb{I} \to Q\mathbb{I} \coprod \mathbb{I}$ be the inclusion of $Q\mathbb{I}$. Assume further that the domains of I are small relative to $\tilde{I}$-`Cell` for $\tilde{I} := I \cup \{\emptyset \to \mathbb{I}\}$ and that $Q\mathbb{I}$ and the domains of J are small relative to $\tilde{J}$-`Cell` for $\tilde{J} := J \cup \{j \circ i_{Q\mathbb{I}}\colon Q\mathbb{I} \to D\}$. Then there is a cofibrantly generated monoidal model category $\tilde{M}$ with sets of generating cofibrations $\tilde{I}$ and generating acyclic cofibrations $\tilde{J}$ with the same underlying category and weak equivalences. Further if $\mathcal{M}$ is left (resp. right) proper so too is $\tilde{M}$ and if $\mathcal{M}$ is symmetric and satisfies the monoid axiom so too does $\tilde{M}$.

Muro also notes that there is then a monoidal Quillen equivalence $\mathcal{M} \underset{\perp}{\rightleftarrows} \tilde{\mathcal{M}}$, i.e. the left adjoint id is a monoidal functor and $id(Q\mathbb{I}) \to id(\mathbb{I})$ is a weak eqivalence. We wish to apply this theorem with $\mathcal{M} = (f\mathcal{C})_S$, $I = I_S$ and $J = J_S$ so for the unit $\mathbb{I} = R^0_{(0)}$ and cofibrant replacement $Q_r I$ of $R^0_{(0)}$ we must construct a factorisation of the morphism

$$(\pi, id)\colon Q_r I \coprod R^0_{(0)} \to R^0_{(0)} \ .$$

into a cofibration followed by a weak equivalence. We construct such a D and factorisation required of Theorem 5.5.0.2 as follows. We first introduce a shifted degree 1 projection map.

Definition 5.5.0.3. We denote by ρ_r the composite (degree 1) morphism

$$\rho_r := id^1_{-r} \circ \Sigma^r \pi \colon \Sigma^r Q_r I \xrightarrow{\sim} \Sigma^r R^0_{(0)} \cong R^{-1}_{(r)} \longrightarrow R^0_{(0)}$$

where the degree 1 morphism id^1_{-r} is the identity map that decreases filtration by r.

We denote by D the filtered chain complex given by:

$$D := \left(Q_r I \oplus R^0_{(0)}\right) \oplus_\tau \Sigma^r Q_r I$$

where the twist morphism τ is given in the first component as the degree 1 morphisms shifting filtraion by r that maps the components of $\Sigma^r Q_r I$ identically onto the corresponding components of $Q_r I$ and in the second component by the morphism $-\rho_r$. The first morphism j of the factorisation of the Theorem 5.5.0.2 is simply the inclusion of $Q_r I \oplus R^0_{(0)}$ into the first component of the twisted direct sum. We will show shortly that is indeed a cofibration. The second morphism q is given by, on the first component of the direct sum, the morphism $\pi\colon Q_r I \to R^0_{(0)}$ on the first component and the identity on the second, and on the second component of the twisted direct sum by 0. We will also show that this second map q is a weak equivalence.

Lemma 5.5.0.4. *The morphism* $j\colon Q_r I \oplus R^0_{(0)} \to D$ *is a cofibration in* $(f\mathcal{C})_S$*.*

Proof. The cokernel of the morphism is the filtered chain complex $\Sigma^r Q_r I$. Note that the proof of Lemma 4.2.0.6 can be used here depsite the condition on a bounded filtration of the cokernel not being satisfied since we have a good decomposition of $\Sigma^r Q_r I$ (the obvious one) into a direct sum in each cohomological degree. Using this decomposition one then applies the rest of the proof of Lemma 4.2.0.6 to show this morphism lifts against acyclic fibrations.

Lemma 5.5.0.5. *The morphism* $q\colon D \to R^0_{(0)}$ *is an* r*-weak equivalence.*

Recall the notion of an r-cone from Definition 1.6.0.12.

Proof. One can identify the chain complex D with the r-cone of the morphism $Q_r I \to Q_r I \oplus R^0_{(0)}$ which is the identity onto the first component and the map π onto the second. One can then identify the r-page of this filtered chain complex, by [CELW19, Remark 3.6], from which one deduces that the $(r+1)$-page is isomorphic to that of the associated spectral sequence of $R^0_{(0)}$. Hence the morphism $q\colon D \to R^0_{(0)}$ is an r-weak equivalence.

These two lemmas then provide a factorisation of $(\pi, id)\colon Q_r I \oplus R^0_{(0)} \to R^0_{(0)}$ into a cofibration followed by a weak equivalence as required by Theorem 5.5.0.2.

Corollary 5.5.0.6. *For every* $r \geq 0$ *and every subset* $S \subseteq \{0, 1, \ldots, r\}$ *including r, the category* $f\mathcal{C}$ *admits a left and right proper cofibrantly generated monoidal model structure, which we denote* $\left(\widetilde{f\mathcal{C}}\right)_S$*, satisfying the monoid axiom where:*

1. *weak equivalences are* E_r*-quasi-isomorphisms,*
2. $\tilde{I}_S \coloneqq I_S \cup \{0 \to R^0_{(0)}\}$ *and* $\tilde{J}_S \coloneqq J_S \cup \{j \circ i_{Q_r\mathbb{I}}\colon Q_r I \to D\}$ *are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Proof. The model categories $(f\mathcal{C})_S$ of Theorem 3.1.0.2 are right proper, since all objects are fibrant, left proper by Theorem 3.7.1.7 and monoidal by Theorem 5.3.2.2. Furthermore the morphism $\pi\colon Q_r I \to R^0_{(0)}$ is a cofibrant replacement of the unit which satisfies the very strong unit axiom by Proposition 5.2.0.2 and by Lemmas 5.5.0.4 and 5.5.0.5 has a factorisation into a cofibration followed by a weak equivalence. All objects of $f\mathcal{C}$ are small relative to the whole category by Lemma 1.2.1.17, hence all conditions of Theorem 5.5.0.2 are satisfied so the corrollary follows with generating cofibrations and acyclic cofibrations given by:

$$\begin{aligned}\tilde{I}_S &\coloneqq I_S \cup \left\{0 \to R^0_{(0)}\right\} \ , \\ \tilde{J}_S &\coloneqq J_S \cup \left\{j \circ i_{Q_r I}\colon Q_r I \to \left(Q_r I \oplus R^0_{(0)}\right) \oplus_\tau \Sigma^r Q_r I\right\} \ .\end{aligned}$$

Note 5.5.0.7. These model categories $\left(\widetilde{f\mathcal{C}}\right)_S$ are not finitely cofibrantly generated since the objects $Q_r I$ are not finite objects in $f\mathcal{C}$.

One apparent disadvantage to this construction in our setting is we lose our particularly nice description of the fibrations. They still must satisfy Z_s-bidegree-wise surjectivity but now with an added condition coming from the newly added morphisms of $\tilde{J}$.

Muro also provides a characterisation of cofibrant objects in $\widetilde{\mathcal{M}}$. Firstly an object X of $\mathcal{M}$ is said to be *cofibrant mod* $\mathbb{I}$ if it is a retract of an object Y which admits a cofibration $\coprod_T \mathbb{I} \to Y$ for some indexing set T. The characterisation is then given by the following result of Muro.

Proposition 5.5.0.8 ([Mur15, Corollary 11]). *For $\mathcal{M}$ satisfying the conditions of Theorem 5.5.0.2, an object X is cofibrant mod $\mathbb{I}$ in $\mathcal{M}$ if and only it is cofibrant in $\widetilde{\mathcal{M}}$.* ۞

The generating cofibrations $\tilde{I}_S$ and generating acyclic cofibrations $\tilde{J}_S$ given by Corollary 5.5.0.6 are not stable sets under the suspension Σ^r and loop Ω^r functors. We can make all $R^n_{(p)}$ cofibrant by adding appropriate generating cofibrations and acyclic cofibrations as for $\tilde{I}_S$ and $\tilde{J}_S$.

Corollary 5.5.0.9. *For every $r \geq 0$ and every subset $S \subseteq \{0, 1, \dots, r\}$ including r, the category $f\mathcal{C}$ admits a left and right proper cofibrantly generated monoidal model structure, which we denote $\left(\widehat{f\mathcal{C}}\right)_S$, satisfying the monoid axiom where:*

1. *weak equivalences are E_r-quasi-isomorphisms,*
2. *$\hat{I}_S := I_S \cup \{0 \to R^n_{(p)}\}_{p,n\in\mathbb{Z}}$ and $\hat{J}_S := J_S \cup \{j \circ i_{Q_r R^n_{(p)}} : Q_r R^n_{(p)} \to D^n_{(p)}\}$ are the sets of generating cofibrations and generating trivial cofibrations respectively.* ۞

Proof. The proof is identical to that of Corollary 5.5.0.6 noting that the proof of [Mur15, Theorem 3] does work by adding more of the $0 \to R^n_{(p)}$ to the generating cofibrations so long as we add the corresponding acyclic cofibrations. ۞

One of the consequence of [Mur15, Theorem 3] was that the monoid axiom still holds in the newly constructed model categories. We can then deduce as before model categories of modules and algebras where the unit is now cofibrant.

APPENDIX A

Cylinder Objects and Cosimplicial Frames

A.1 Cylinder objects in $(f\mathcal{C})_S$

Recall the definition of cylinder and path objects, Definition 1.4.1.11, and the notion of left and right homotopy, Definition 1.4.1.13, and that for cofibrant X and fibrant Y the notion of left and right homotopy is the same, [Hov99, Corollary 1.2.6], so that to compute the homotopy classes of maps from X to Y we can equivalently quotient $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ by either the left or right homotopy relation.

In the model categories $(f\mathcal{C})_S$ fibrations are generally easier to work with and understand, and constucting path objects is straightforward. Indeed the following gives a path object:

$$X \xrightarrow{\sim} \underline{\mathrm{Hom}}(I_r, X) \longrightarrow X \times X \ .$$

Construction of a cylinder object is not so simple. Consider the fold map on the unit of $f\mathcal{C}$, $R^0_{(0)} \coprod R^0_{(0)} \to R^0_{(0)}$. A first guess at a factorisation of this map to give a cylinder object might be a filtered generalisation of the interval object in chain complexes taking account of the $r = \max S$, i.e.

$$R^0_{(0)} \coprod R^0_{(0)} \longrightarrow I_r \longrightarrow R^0_{(0)} \ ,$$

however whilst the second map is indeed a weak equivalence the first map is not a cofibration; its cofibre is $R^{-1}_{(r)}$ which is not cofibrant. In Section 5.1 we constructed a cofibrant replacement of the unit so we consider now replacing the $R^{-1}_{(r)}$ portion of I_r by something weakly equivalent which is in addition cofibrant. Recall the cofibrant replacement of the unit denoted $Q_r I$ of Section 5.1 and projection map $\pi \colon Q_r I \xrightarrow{\sim} R^0_{(0)}$. Recall the shifted projection map ρ_r of Definition 5.5.0.3.

Definition A.1.0.1. We define the filtered chain complex $Q_r\mathrm{Cyl}_r$ to be:

$$Q_r\mathrm{Cyl}_r := \left(\left(R^0_{(0)} \oplus R^0_{(0)}\right) \oplus_\tau \Sigma^r Q_r I\right) \ ,$$

where the twist map τ is given by $\begin{pmatrix} \rho_r \\ -\rho_r \end{pmatrix}$.

Note we have an inclusion of the first component of the twisted direct sum and a fold map $Q_r\nabla$ which is the fold map on the first component of the twisted direct sum $R_{(0)} \oplus R_{(0)}$ and 0 on the second $\Sigma^r Q_r I$. We can now give a factorisation of the fold map $\nabla \colon R^0_{(0)} \oplus R^0_{(0)} \to R^0_{(0)}$ into a cofibration followed by a weak equivalence.

$$\nabla \colon R^0_{(0)} \oplus R^0_{(0)} \underset{i}{\rightarrowtail} Q_r\mathrm{Cyl}_r \underset{Q_r\nabla}{\overset{\sim}{\twoheadrightarrow}} R^0_{(0)} \ . \tag{A.1}$$

Lemma A.1.0.2. *The factorisation of Equation* (A.1) *factorises the fold map* $R^0_{(0)} \oplus R^0_{(0)} \to R^0_{(0)}$ *into a cofibration followed by an acylic fibration in* $(f\mathcal{C})_S$ *with* $\max S = r$.

Proof. The first map being a cofibration follows (almost) from Lemma 4.2.0.6 with a slight modification since the cokernel C is $\Sigma^r Q_r I$ does not satisfy the bounded filtration result; we do not have for each n a $p(n)$ such that $F_{p(n)}C^n = 0$. The issue is the decomposition of C into a direct sum of its filtered parts however note since $Q_r I$ is defined as a direct sum of shifts of the unit there is no issue with applying the method of the proof to find a lift, hence the first morphism is indeed a cofibration.

The second map is Z_k-surjective for all $0 \leq k \leq r$ hence is a fibration in $(f\mathcal{C})_S$. To see that it is an r-weak equivalence note that the twisted direct sum is isomorphic to

$$R^0_{(0)} \oplus \left(R^0_{(0)} \oplus_\tau \Sigma^r Q_r I \right) \tag{A.2}$$

where the twist map τ is now simply ρ_r; the change of basis here has the first $R^0_{(0)}$ component of Equation (A.2) as the first $R^0_{(0)}$ appearing in Definition A.1.0.1 and the $R^0_{(0)}$ component of the twisted direct sum is the anti-diagonal, generated by $(1, -1)$, of $R^0_{(0)} \oplus R^0_{(0)}$ in Definition A.1.0.1. Consequently Equation (A.1) becomes

$$\begin{tikzcd}
\nabla\colon R^0_{(0)} \oplus R^0_{(0)} \arrow[r, tail, "\begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}"'] & R^0_{(0)} \oplus \left(R^0_{(0)} \oplus_\tau \Sigma^r Q_r I \right) \arrow[r, two heads, "\sim", "\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}"'] & R^0_{(0)}
\end{tikzcd} .$$

The twisted direct sum component is isomorphic to the r-cone of Definition 1.6.0.12 on the morphism $\pi\colon Q_r I \to R^0_{(0)}$, and so by Lemma 1.6.0.13 is r-acyclic since π is an r-weak equivalence by Corollary 5.1.0.6. This completes the lemma.

We want to use this factorisation as foundation for a more general factorisation of the fold map $\nabla\colon K \oplus K \to K$, where K is a cofibrant object of $(f\mathcal{C})_S$. We have shown that $(f\mathcal{C})_S$ is a monoidal model category in Theorem 5.3.2.2 hence we can tensor the first map of Equation (A.1) by the cofibrant K and still have a cofibration. If the morphism $Q_r\nabla\colon Q_r\mathrm{Cyl}_r \to R^0_{(0)}$ remains an r-weak equivalence after tensoring by K then a cylinder object of K can be obtained by tensoring Equation (A.1) by K.

Recall that for a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

if B and C are flat so too is A; one sees this by considering the associated long exact sequence to the functor $- \otimes X$ for some R-module X.

Lemma A.1.0.3. *Let A be a filtered chain complex such that A^n and A^n/F_pA^n are flat for all $n, p \in \mathbb{Z}$, then $- \otimes A$ preserves kernels.*

Proof. Let Z be the kernel of a morphism $f\colon Y \to X$ of filtered chain complexes and recall the definition of the tensor product in filtered chain complexes Definition 1.2.1.11. For the tensor product of G and H we write φ for the morphism:

$$\varphi\colon \bigoplus_{q\in\mathbb{Z}} \bigoplus_{i\in\mathbb{Z}} F_{p-q}G^{n-i} \otimes F_qH^i \longrightarrow (G \otimes H)^n$$

or the restriction to some of its components. We now want to show the filtration on $Z \otimes A$ is the expected one, so we consider now an element α of $F_p(Y \otimes A)^n$ so that $\alpha = \varphi(\Sigma y_j \otimes a_j)$ for some $y_j \otimes a_J \in F_{p-q} \otimes F_qA$ for some q such that $(f \otimes A)(\alpha) = 0$. We restrict to consider the component $X^{n-i} \otimes A^i$ of $(X \otimes A)^n$ and similarly for Y in place of X. We are then supposing that the y_j are elements of Y^{n-i} for a fixed n and i and that a_j are elements of A^i for the same fixed i. Take m_1 to be the max filtration degree of the y_j and m_2 to be the max filtration degree of the a_j.

$$\begin{tikzcd}
F_{m_1}X^{n-i} \otimes F_{m_2}A^i \arrow[rr, "\varphi"] \arrow[dr] & & X^{n-i} \otimes A^i \subset (X \otimes A)^n \\
& X^{n-i} \otimes F_{m_2}A^i \arrow[ur] &
\end{tikzcd}$$

The morphism $F_{m_1}X^{n-i} \otimes F_{m_2}A^i \to X^{n-i} \otimes F_{m_2}A^i$ is an inclusion since it is the tensor product of the inclusion of a filtered piece by the flat module $F_{m_2}A^i$. The morphism $X^{n-i} \otimes F_{m_2}A^i \to X^{n-i} \otimes A^i$ is also an inclusion; to show this we consider the short exact sequence

$$0 \longrightarrow F_{m_2}A^i \longrightarrow A^i \longrightarrow \frac{A^i}{F_{m_2}A^i}$$

from which we obtain a long exact sequence from the left derived functors of $X^{n-i} \otimes -$:

$$\ldots \longrightarrow \mathrm{Tor}_1\left(X^{n-i}, \frac{A^i}{F_{m_2}A^i}\right) \longrightarrow X^{n-i} \otimes F_{m_2}A^i \longrightarrow X^{n-i} \otimes A^i \longrightarrow X^{n-i} \otimes \frac{A^i}{F_{m_2}A^i} \longrightarrow 0$$

of which the Tor_1 term is 0 since $A^i/F_{m_2}A^i$ is flat by assumption, hence $X^{n-i} \otimes F_{m-2}A^i \to X^{n-i} \otimes A^i$ is an inclusion.

We can then see that for $\varphi(\Sigma f y_j \otimes a_j)$ to be 0 we must have that $\Sigma f y_j \otimes a_j = 0$. Since we have an exact sequence of R-modules

$$0 \longrightarrow F_{m_1}Z^{n-i} \longrightarrow F_{m_1}Y^{n-i} \longrightarrow F_{m_1}X^{n-i}$$

we also have an exact sequence after tensoring by the flat $F_{m_2}A^i$

$$0 \longrightarrow F_{m_1}Z^{n-i} \otimes F_{m_2}A^i \longrightarrow F_{m_1}Y^{n-i} \otimes F_{m_2}A^i \longrightarrow F_{m_1}X^{n-i} \otimes F_{m_2}A^i$$

and so by exactness $\Sigma y_j \otimes a_j = \Sigma z_k \otimes a_k$.

Counterexample A.1.0.4. We list some counterexamples to show that the conditions A^n and A^n/F_pA^n are flat are necessary. For simplicity we work in filtered R-modules instead of chain complexes as the differentials play no role in flatness.

1. Take $R = \mathbb{Z}/4\mathbb{Z}$ and $I = 2\mathbb{Z}/4\mathbb{Z}$ as an R-module. Note too that $R/I \cong I$, $I \otimes I \cong I$ and that the natural map $I \otimes I \to I \otimes A \cong I$ is the zero map. We let A be the filtered module with $F_{-1}A = 0$, $F_0A = I$ and $F_1A = A = R$. Take $Z = I, Y = R$ and $X = I$ where the filtration on Z has $F_{-1}Z = 0$ and $F_0Z = Z$ and similarly for Y and X so that there is a pullback diagram in filtered R-modules:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} .$$

We now tensor this diagram $D \coloneqq (Z \to Y \to X)$ by A. The first two rows of the following diagram depicit the filtration of this tensor product in filtration degrees 1, 0 and the third the tensor product by F_0A:

$$\begin{array}{ll} D \otimes A: & I \otimes A \longrightarrow A \otimes A \longrightarrow I \otimes A \\ & \\ F_0(D \otimes A): & 0 \longrightarrow A \otimes I \longrightarrow 0 \\ & \\ F_0D \otimes F_0A: & I \otimes I \longrightarrow A \otimes I \longrightarrow I \otimes I \end{array}$$

where the first row is the tensor product of the underlying modules, and the second row is obtained by calculating the map φ which takes the image of the third row in the first. We can rewrite this as

$$\begin{array}{ll} F_1(D \otimes A): & I \longrightarrow A \longrightarrow I \\ & \\ F_0(D \otimes A): & 0 \longrightarrow A \otimes I \longrightarrow 0 \end{array}$$

and we can see that tensoring by A has not preserved the pullback diagram since in filtration degree F_0 we do not have that 0 is the kernel of the map $I \to 0$.

2. Similarly one can show that just the requirement that A^n and F_pA^n be flat are not sufficient either. Take Z, Y and X filtered $\mathbb{Z}$-modules whose -1^{st} filtration is 0, and 0^{th} filtration is the full module, respectively $\mathbb{Z}$, $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ so that there is a pullback diagram of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd},$$

   and take A to be the filtered $\mathbb{Z}$-module with $F_{-1}A = 0$, $F_0A = \mathbb{Z}$ and $F_1A = A = \mathbb{Z}$ however such that the inclusion $F_0A \to F_1A$ is multiplication by 2. One can check similarly to the previous example that tensoring the pullback diagram by A does not preserve the pullback property.

Lemma A.1.0.5. *Let A be a filtered chain complex such that either A^n is not flat for some n or A^n/F_pA^n is not flat for some n and p, then tensoring by A does not preserve kernels in general.*

Proof. The lemma is clear if A^n is not flat for some n. The differentials play no role so we show the lemma for filtered R-modules instead and suppose that A/F_pA is not flat for some p. Take a short exact sequence of R-modules $0 \to Z \to Y \to X \to 0$ such that $- \otimes A/F_pA$ does not preserve left exactness of this short exact sequence. We make Z, Y and X into filtered R-modules as before, by letting their -1^{st} filtration piece by 0 and the 0^{th} piece be the full R-module. We then compute $F_p\left(A \otimes Z\right)$ and similarly with Y and X in place of Z.

$$\begin{aligned} F_p\left(A \otimes Z\right) &= \operatorname{im}\left(F_pA \otimes F_0Z \longrightarrow A \otimes Z\right) \\ &= \operatorname{im}\left(F_pA \otimes Z \longrightarrow A \otimes Z\right) \end{aligned}$$

and we can compute the latter by the long exact sequence of the left derived functor of $- \otimes Z$ applied to the short exact sequence $0 \to F_pA \to A \to A/F_pA \to 0$. This gives

$$\begin{aligned} F_p\left(A \otimes Z\right) &= \operatorname{im}\left(F_pA \otimes Z \longrightarrow A \otimes Z\right) \\ &= \ker\left(A \otimes Z \longrightarrow A/F_pA \otimes Z\right) \ . \end{aligned}$$

We can assemble the following commutative diagram in which the rows are exact, any two composable vertical morphisms compose to 0 and the middle vertical complex is exact.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & ? \arrow[d] & \\ 0 \arrow[r] & F_p(A \otimes Z) \arrow[r, hook] \arrow[d] & A \otimes Z \arrow[r, two heads] \arrow[d, hook] & A/F_pA \otimes Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_p(A \otimes Y) \arrow[r, hook] \arrow[d] & A \otimes Y \arrow[r, two heads] \arrow[d, two heads] & A/F_pA \otimes Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_p(A \otimes X) \arrow[r, hook] & A \otimes X \arrow[r, two heads] \arrow[d, two heads] & A/F_pA \otimes X \arrow[r] & 0 \\ & & 0 & & \end{tikzcd}$$

where ? is the kernel of $A/F_pA \otimes Z \to A/F_pA \otimes Y$ and is non-zero by assumption. Computing the long exact sequence associated to this short exact sequence of chain complexes we get, using that the middle complex is exact, that the vertical homology at $F_p(A \otimes Z)$ is non-zero so that $F_p(A \otimes Z) \to F_p(A \otimes Y)$ is not an inclusion which shows that $A \otimes Z$ is not the kernel of $A \otimes Y \to A \otimes X$.

Corollary A.1.0.6. *For a filtered chain complex A the functor $- \otimes A$ preserves kernels if and only if A^n and A^n/F_pA^n are flat for each p and n.*

Corollary A.1.0.7. *Let K be cofibrant, then $- \otimes K$ preserves kernels.*

Proof. Recall from Lemma 4.1.0.1 that for a cofibrant K that K^n and K^n/F_pK^n are projective for all $p, n \in \mathbb{Z}$. Since projective R-modules are flat the corollary now follows from Lemma A.1.0.3.

Lemma A.1.0.8. *The tensor product of $Q_r\nabla\colon Q_r\mathrm{Cyl}_r \to R^0_{(0)}$ with a cofibrant K is an r-weak equivalence.*

Proof. We rewrite the morphism firstly using the change of basis of Equation (A.2) so it again becomes:

$$R^0_{(0)} \oplus \left(R^0_{(0)} \oplus_\tau \Sigma^r Q_r I\right) \xrightarrow[\begin{pmatrix}1 & 0 & 0\end{pmatrix}]{\sim} R^0_{(0)} \quad , \tag{A.3}$$

where the second component, $R^0_{(0)} \oplus_\tau \Sigma^r Q_r I$, is isomorphic to the r-cone on π, $C_r(\pi)$. Note that $C_r(\pi)$ is a kernel:

$$\begin{tikzcd}
C_r(\pi) \arrow[r] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q_r R^{-1}_{(r+1)} \arrow[d, two heads, "\sim"', "\pi"] \\
0 \arrow[r] & R^{-1}_{(r+1)}
\end{tikzcd} \quad ,$$

and so tensoring this diagram by a cofibrant K preserves the pullback by Corollary A.1.0.7. Further $\pi \otimes K$ is still a weak equivalence by Proposition 5.2.0.2 and also a fibration, so that the pullback $C_r(\pi) \otimes K \to 0$ is also an acyclic fibration, in particular an r-weak equivalence.

$$\begin{tikzcd}
C_r(\pi) \otimes K \arrow[r] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q_r R^{-1}_{(r+1)} \otimes K \arrow[d, two heads, "\sim"', "\pi\otimes K"] \\
0 \arrow[r] & R^{-1}_{(r+1)} \otimes K
\end{tikzcd}$$

This shows that tensoring Equation (A.3) by K is still a weak equivalence. ∎

We have then shown the following construction of a cylinder object on a cofibrant K in $(f\mathcal{C})_S$, (this cylinder object only depends on $r = \max S$).

Corollary A.1.0.9. *There is a factorisation of the fold map $\nabla\colon K \coprod K \to K$ in $(f\mathcal{C})_S$ into a cofibration followed by an acyclic fibration:*

$$Q_r\nabla\colon K \coprod K \longrightarrow Q_r I_r \otimes K \longrightarrow K \ .$$ ∎

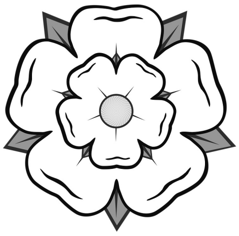
This then gives a way of computing homotopy classes of maps from a cofibrant X to a fibrant Y via left homotopies.

A.2 Higher factorisations and cosimplicial frames

In the previous section we gave a factorisation of the fold map $\nabla\colon R^0_{(0)} \coprod R^0_{(0)} \to R^0_{(0)}$ into a cofibration followed by a weak equivalence. This factorisation is the analogue in $(f\mathcal{C})_S$ of the factorisation of the fold map of simplicial sets $\Delta[0] \coprod \Delta[0] \to \Delta[1] \to \Delta[0]$ into a cofibration followed by weak equivalence. We could have written this instead as $\partial\Delta[1] \to \Delta[1] \to \Delta[0]$. In simplicial sets we can also factorise the collapse maps $\partial\Delta[n] \to \Delta[0]$ as $\partial\Delta[n] \to \Delta[n] \to \Delta[0]$ and we now sketch analogues to these 'higher dimensional factorisations' in $(f\mathcal{C})_S$. Recall to construct $\partial\Delta[n+1]$ from $\Delta[n]$ one takes $n+2$ copies of $\Delta[n]$ and identifies various boundaries (explicitly pushouts of coproducts of $\Delta[n]$ over $\Delta[n-1]$ subobjects). To then obtain $\Delta[n+1]$ from $\partial\Delta[n+1]$ one needs only add an $(n+1)$-dimensional cell whose boundary is $\partial\Delta[n+1]$. For the analogues in $(f\mathcal{C})_S$ we perform similar pushouts and to add an $(n+1)$-dimensional cell we use instead iterated suspensions of the cofibrant replacement of the unit. We introduce new notation to elicit the factorisations in simplcial sets. Denote by $R\Delta[0]$ the filtered chain complex $R^0_{(0)}$ and by $R\Delta[1]$ the filtered chain complex $Q_r\mathrm{Cyl}_r$. Consider the quotient of three copies of $R\Delta[1]$ where the end subobjects given by the $R\Delta[0]$ are identified as in the construction of $\partial\Delta[2]$ from three copies of $\Delta[1]$. We denote this filtered construction by $R\partial\Delta[2]$. Explicitly this filtered chain complex $R\partial\Delta[2]$ is given by:

$$\begin{tikzcd}
 & & R^0_{(0)} & & & & & & R^0_{(0)} & & \\
 & \Sigma^r Q_r I \arrow[ur, "\rho"] \arrow[ddl, "-\rho"'] & & \Sigma^r Q_r I \arrow[ul, "\rho"'] \arrow[ddr, "-\rho"] & {} \arrow[rr] & & {} & \Sigma^r Q_r I \arrow[ur, "\rho"] \arrow[ddl, "-\rho"'] & & \Sigma^r Q_r I \arrow[ul, "\rho"'] \arrow[ddr, "-\rho"] & \\
 & & & & & & & & \Sigma^{r2} Q_r I \arrow[ul, "-\rho"'] \arrow[ur, "\rho"] \arrow[d, "\rho"] & & \\
R^0_{(0)} & & \Sigma^r Q_r I \arrow[ll, "-\rho"] \arrow[rr, "\rho"'] & & R^0_{(0)} & & R^0_{(0)} & & \Sigma^r Q_r I \arrow[ll, "-\rho"] \arrow[rr, "\rho"'] & & R^0_{(0)}
\end{tikzcd}$$

and one can obtain higher dimensional versions in much a similar way.



APPENDIX B

Right Adjoint to Coproduct Totalisation

This appendix proves that the functor $\mathcal{R}$ of Definition 2.3.0.1 is right adjoint to the coproduct totalisation functor $\mathrm{Tot}^{\oplus}$. We need to describe natural maps between the hom sets $\mathrm{Hom}_{f\mathcal{C}}\left(\mathrm{Tot}^{\oplus}(K), C\right)$ and $\mathrm{Hom}_{b\mathcal{C}}\left(K, \mathcal{R}(C)\right)$ and show they are bijections.

We write π_2 for the projection onto the second component of any direct sum of two R-modules. Given a map $f\colon K \to \mathcal{R}(C)$ of bicomplexes we obtain a map of filtered chain complexes $\tilde{f}\colon \mathrm{Tot}^{\oplus}(K) \to C$ by the following diagram:

$$\begin{tikzcd}
\bigoplus_i K^{i,i+n} \arrow[r, dashed, "\tilde{f}^n"] \arrow[d, "\bigoplus f^{i,i+n}"'] & C \\
\bigoplus_i \mathcal{R}(C)^{i,i+n} \arrow[r, "\bigoplus \pi_2"'] & \bigoplus_i F_i C^n \arrow[u, "\Sigma"']
\end{tikzcd}$$

Note by construction $F_i\mathrm{Tot}^{\oplus}(K)^n$ is mapped by $\tilde{f}$ into F_iC^n, so that $\tilde{f}$ degreewise is a map of filtered R-modules. We check that $\tilde{f}$ commutes with differentials. First note the following equation, for an $(x, y) \in F_{i-1}C^{n+1} \oplus F_iC^n$:

$$d\pi_2(x, y) = dy = \pi_2 d_0(x, y) + (-1)^n \pi_2 d_1(x, y)\ . \tag{B.1}$$

Now in the following series of equalities we have the first follows from the definition of d on $\mathrm{Tot}^{\oplus}$, the second by definition of $\tilde{f}$, the third by commuting d_0 and d_1 past $f^{*,*}$ using f is a map of bicomplexes, the fourth reindexes the last part as we are summing over all i, the fifth by collecting terms, the sixth using equation B.1, and the last again by definition of $\tilde{f}$.

$$\begin{aligned}
\tilde{f}^{n+1} d(k_i)_i &= \tilde{f}^{n+1}(d_0 k_i + (-1)^n d_1 k_{i+1})_i \\
&= \sum_i \pi_2 f^{i,i+n+1}(d_0 k_i + (-1)^n d_1 k_{i+1}) \\
&= \sum_i \pi_2 d_0 f^{i,i+n} k_i + (-1)^n \pi_2 d_1 f^{i+1,i+1+n} k_{i+1} \\
&= \sum_i \pi_2 d_0 f^{i,i+n} k_i + (-1)^n \pi_2 d_1 f^{i,i+n} k_i \\
&= \sum_i \pi_2 (d_0 + (-1)^n d_1) f^{i,i+n} k_i \\
&= \sum_i d\pi_2 f^{i,i+n} k_i \\
&= d\tilde{f}^n (k_i)_i
\end{aligned}$$

This then shows $\tilde{f}$ is a map of filtered chain complexes.

Now suppose we are given a map, $g\colon \mathrm{Tot}^{\oplus}(K) \to C$ of filtered chain complexes. We define a map, $\hat{g}\colon K \to \mathcal{R}(C)$, of bicomplexes. We write $\eta^{i,i+n}\colon K^{i,i+n} \to \mathrm{Tot}^{\oplus}(K)^n$ for the inclusion of the i$^{\text{th}}$ summand.

$$\begin{aligned}\hat{g}^{i,i+n}\colon K^{i,i+n} &\to \mathcal{R}(C)^{i,i+n}\\ k &\mapsto \big((-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k, g\eta^{i,i+n} k\big)\end{aligned}$$

We check that the maps $\hat{g}^{i,i+n}$ commute with the differentials d_0 and d_1. The first equality in the following follows from the definition of $\hat{g}$, the second from the definition of d_1 on $\mathcal{R}(C)$, the third from $d_1 d_1 = 0$ and $(-1)^{n+1}(-1)^{n+1} = 1$, and the last again from the defintion of $\hat{g}$.

$$\begin{aligned}d_1\hat{g}^{i,i+n}(k) &= d_1\left((-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k, g\eta^{i,i+n} k\right)\\ &= \big(0, (-1)^{n+1}(-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k\big)\\ &= \big((-1)^{n} g\eta^{i-2,i-2+n+2} d_1 d_1 k, g\eta^{i-1,i-1+n+1} d_1 k\big)\\ &= \hat{g}^{i-1,i-1+n+1}(d_1 k)\end{aligned}$$

We will make use of the following equation relating differentials and the inclusion maps η:

$$d\eta^{i,i+n} k = \eta^{i,i+n+1} d_0 k + (-1)^n \eta^{i-1,i-1+n+1} d_1 k\ . \tag{B.2}$$

We show too that $\hat{g}^{i,i+n}$ commutes with d_0. In the following we then have the first equality follows from the definition of $\hat{g}$, the second from the definition of d_0 on $\mathcal{R}(C)$, the third commutes d and g using g is a map of filtered chain complexes, the fourth uses equation B.2 on the first component, the fifth uses equation B.2 on the second component, the sixth cancels the term involving $d_1 d_1$ in the first component and the terms of opposite sign in the second, the seventh commutes d_0 and d_1, and lastly we use the definition of $\hat{g}$.

$$\begin{aligned}d_0\hat{g}^{i,i+n}(k) &= d_0\left((-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k, g\eta^{i,i+n} k\right)\\ &= \big((-1)^{n} dg\eta^{i-1,i-1+n+1} d_1 k, dg\eta^{i,i+n} k + (-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k\big)\\ &= \big((-1)^{n} gd\eta^{i-1,i-1+n+1} d_1 k, gd\eta^{i,i+n} k + (-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k\big)\\ &= \big((-1)^{n} g\eta^{i-1,i-1+n+2} d_0 d_1 k + (-1)^{n+1}(-1)^{n}\eta^{i-2,i-2+n+2} d_1 d_1 k,\\ &\qquad gd\eta^{i,i+n} k + (-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k\big)\\ &= \big((-1)^{n} g\eta^{i-1,i-1+n+2} d_0 d_1 k + (-1)^{n+1}(-1)^{n}\eta^{i-2,i-2+n+2} d_1 d_1 k,\\ &\qquad g\eta^{i,i+n+1} d_0 k + (-1)^{n} g\eta^{i-1,i-1+n+1} d_1 k + (-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k\big)\\ &= \big((-1)^{n} g\eta^{i-1,i-1+n+2} d_0 d_1 k, g\eta^{i,i+n+1} d_0 k\big)\\ &= \big((-1)^{n} g\eta^{i-1,i-1+n+2} d_1 d_0 k, g\eta^{i,i+n+1} d_0 k\big)\\ &= \hat{g}^{i,i+n+1}(d_0 k)\end{aligned}$$

We now verify that $\hat{\tilde{f}} = f\colon K \to \mathcal{R}(C)$. The first equality in the following follows from the definition of $\hat{g}$, the second by definition of $\tilde{f}$, the third uses the equation $\pi_1(x,y) = (-1)^{n+1}\pi_2 d_1(x,y)$ when $(x,y) \in \mathcal{R}(C)^n$ and we take $(x,y) = f^{i,i+n}k$, the fourth equality is cancelling signs, and the last uses the definition of the projections π_*.

$$\begin{aligned}\hat{\tilde{f}}^{i,i+n}(k) &= \Big((-1)^{n+1}\tilde{f}\eta^{i-1,i-1+n+1} d_1 k, \tilde{f}\eta^{i,i+n} k\Big)\\ &= \big((-1)^{n+1}\pi_2 f^{i-1,i-1+n+1} d_1 k, \pi_2 f^{i,i+n} k\big)\\ &= \big((-1)^{n+1}(-1)^{n+1}\pi_1 f^{i,i+n} k, \pi_2 f^{i,i+n} k\big)\\ &= \big(\pi_1 f^{i,i+n} k, \pi_2 f^{i,i+n} k\big)\\ &= f^{i,i+n}(k)\end{aligned}$$

Finally we verify that $\tilde{\hat{g}} = g\colon \mathrm{Tot}^{\oplus}(K) \to C$. The first equality in the following follows from the definition of $\tilde{f}$, the second by definition of $\hat{g}$, the third from applying π_2, and the last by linearity of g.

$$\begin{aligned}
\tilde{\hat{g}}^n(k_i)_i &= \sum_i \pi_2 \hat{g}^{i,i+n} k_i \\
&= \sum_i \pi_2 \left((-1)^{n+1} g\eta^{i-1,i-1+n+1} d_1 k_i, g\eta^{i,i+n} k_i\right) \\
&= \sum_i g\eta^{i,i+n} k_i \\
&= g(k_i)_i
\end{aligned}$$

We have proven now the following proposition.

Proposition B.0.0.1. *There is an adjunction of categories* $\mathrm{Tot}^{\oplus}: b\mathcal{C} \underset{\perp}{\rightleftarrows} f\mathcal{C} : \mathcal{R}$. ❂

APPENDIX C

Questions

We discuss here some questions the author has regarding model structures related to spectral sequences. They are potential future directions of work.

Question C.1. The projective model structure on (bounded) chain complexes can be obtained via the Dold-Kan adjunction from transferring the Quillen model structure on simplicial sets along the free-forgetful adjuntion between simplicial sets and simplicial R-modules. Can the bounded model structures on filtered chain complexes be obtained by transfer along a filtered Dold-Kan adjunction in a similar way? What would such a model structure on filtered simplicial sets be describing?

Question C.2. Can we perform a series of left and right Bousfield localisations to pass from one of the r-model structures to an $(r+1)$-model structure? Or perhaps more simply are there non-trivial Bousfield localisations of any of the model structures in the posets?

Question C.3. Can we construct an model structure whose weak equivalences are those maps of filtered chain complexes that are isomorphisms on the ∞-page of the associated spectral sequence or those maps that are eventually isomorphisms on some page of the spectral sequences? We refer to such a tentative model structure as an ∞-*model structure*. The immediate problem is that the ∞-boundaries are not representable however they are still pro-representable. Perhaps then the question is better asked as "is there an ∞-model structure on the category of pro-filtered chain complexes"?

We first recall for a category $\mathcal{C}$ the category of pro-objects of $\mathcal{C}$ denoted $\texttt{pro-}\mathcal{C}$ and then give a brief overview of homotopy theories related to categories of pro-objects (or similar).

Definition C.0.0.1. A category I is said to be cofiltering if it is small, non-empty and satisfies:

1. for all $i, j \in I$ there exists a $k \in I$ with maps $k \to i$ and $k \to j$, and
2. for any pair of maps $f, g \colon i \rightrightarrows j$ there exists a $k \in I$ and arrow $h \colon k \to i$ such that $fh = gh$.

Definition C.0.0.2. The category of pro-objects in $\mathcal{C}$ denoted $\texttt{pro-}\mathcal{C}$ has objects all functors $X : I \to \mathcal{C}$ where I is a cofiltered category and has as morphisms between $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ the class

$$\mathrm{Hom}_{\texttt{pro-}\mathcal{C}}(X, Y) \coloneqq \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j) \ .$$

We have omitted the definition of composition which can be found explicitly in [EH76, Definition 2.1.1]. When $I = J$ natural transformations between X and Y are examples of morphisms of $\texttt{pro-}\mathcal{C}$ but there are many more morphisms than just the natural transformations (even when $I = J$). The category $\texttt{pro-}\mathcal{C}$ can be thought of as having objects the *formal cofiltered limits* of $\mathcal{C}$. This category is complete, essentially by definition, and further cocomplete whenever $\mathcal{C}$ is, see [Isa02] for constructions of limits and colimits in the category $\texttt{pro-}\mathcal{C}$. These constructions can be dualised to give a category of ind-objects of $\mathcal{C}$ using filtered diagrams. We relax the notion of *model category/structure* for this discussion to mean whatever it means in each of the following papers considered.

Artin and Mazur in [AM69] consider, without reference to any model structure, the pro-homotopy category i.e. pro objects in the usual homotopy category of spaces. They use this to define homotopy invariants of a locally noetherian scheme, using the étale topology, they refer to as the *étale homotopy type*. They further observe that this homotopy theory is "amenable to the techniques of classical algebraic topology" in that it admits Hurewicz and Whitehead theorems and one can work with Postnikov decompositions. Further one has good pro-finite and p-adic completions. The consequences to geometry this theory entails is considerable but we focus only on the good properties this homotopy theory has, see for instance [Fri82] as well.

Grossman, in [Gro75], considers a restriction of pro-objects and only considers the subcategory of *towers*, i.e. pro-objects of the form

$$\ldots \to X_{s+1} \to X_s \to \ldots \to X_0 \ .$$

Grossman then develops a model category structure on the category of towers of simplicial sets (although they remark that the definitions and proofs are not combinatorial).

Seperately homotopy theories of pro-objects were used in [EH76] where, under suitable restrictions on a model category $\mathcal{C}$, a model category structure on $\texttt{pro-}\mathcal{C}$ exists too and they study the homotopy theory comparing $\mathrm{Ho}(\texttt{pro-}\mathcal{C})$ with $\texttt{pro-}\mathrm{Ho}(\mathcal{C})$ and discuss the homotopy and homology groups thereof. They also apply this to generalized Steenrod homology in shape theory.

Isaksen develops the homotopy theory of pro-objects further and equips categories of pro-objects with model structures (under assumptions on $\mathcal{C}$). In [Isa01] the category of pro-simplicial sets, used in the étale homotopy theory in [AM69] and shape theory of [MS82], is given a model category structure (with non-functorial factorisations) which Isaksen notes is closely related to the "strict structure" of [EH76]. The weak equivalences in this model structure are appropriately the weak equivalences of [AM69]. This then generalises the model structure of [Gro75] on towers and puts the homotopy theory of [AM69] on a stronger footing; one can now work on the level of pro-simplicial sets so can work with strictly commutative diagrams rather than those commutative only up to homotopy for instance, although note that morphisms in the two different homotopy categories do not in general agree, see [Isa01, §8]. The construction of this model structure is somewhat involved in that firstly local systems are used to discuss weak equivalences in $\texttt{pro-}sSets$ instead of basepoints, since points of pro-simplicial sets may not exist, and secondly the model structure is not cofibrantly generated, [Isa01, Corollary 19.3], hence the non-functorial factorisations so the proof is not as simple as demonstrating a result like Theorem 1.4.2.9. Isaksen also remarks that the functoriality of the skeletal filtration on simplicial sets is vital to their proof so does not easily extend to pro-topological spaces for example.

In [Isa04] Isaksen extends the results of [EH76] to show existence of a *strict model structure* on $\texttt{pro-}\mathcal{C}$ whenever $\mathcal{C}$ is a proper model category. To describe this further we state some terminology found in [Isa04]. A *level representation* of a morphism of pro-objects $X \to Y$ is an isomorphic morphism, i.e.:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Y} \end{tikzcd}$$

in the $\texttt{pro-}\mathcal{C}$ such that $\hat{X}$ and $\hat{Y}$ are indexed by the same indexing set I and $\hat{f}$ is a natural transformation of the functors $X\colon I \to \mathcal{C}$ and $Y\colon I \to \mathcal{C}$. A morphism of pro-objects then satisfies a property *essentially levelwise* if there is a level representation for which all morphisms $\hat{f}_i$ satisfy the property. In the strict model structure on $\texttt{pro-}\mathcal{C}$ the *strict weak equivalences* are then the *essentially levelwise weak equivalences* and the *strict cofibrations* the *essentially levelwise cofibrations*. Again these model categories are not cofibrantly generated and despite every object of every pro-category being cosmall [CI04, Corollary 3.5] the strict model category of pro-simplicial sets is not fibrantly generated either, although adapting to pro-(κ-bounded simplicial sets) or relaxing to Chorny's notion of *class fibrantly generated*, [Cho06], one can obtain a form of fibrant generation. A similar result holds for the model structure of [Isa01].

In [CI04] Christensen and Isaksen consider localisations of the strict model structure of [Isa04]. Existence of these localisations is provided by [CI04, Theorem 4.4] and they use it to construct a model structure, [CI04, Theorem 6.5], on pro-spectra whose weak equivalences are isomorphisms on the colimits of the cohomotopy, [CI04, Definition 6.2]. Here the cohomotopy of a pro-spectrum X is given by:

$$\pi^n X \coloneqq \operatorname*{colim}_s \pi^n X_s \ .$$

They refer to this as the π^*-model structure. Their main result is that this π^*-model structure on pro-spectra is Quillen equivalent to the opposite of the usual stable model structure on spectra, [CI04, Corollary 8.6].

In [Isa05] Isaksen constructs two model categories on pro-spaces for each ring R. The cofibrations are those of the strict model structure and the weak equivalences are respectively the *R-cohomology weak equivalences* and the *R-homology weak equivalences* given as [Isa05, Definitions 5.2 & 5.4] which they remark, [Isa05, Remark 5.3], are distinct notions. Here the cohomology of a pro-spectrum X with M coefficients is given by:

$$H^n(X;M) \coloneqq \operatorname*{colim}_s H^n(X_s;M) \ ,$$

and the homology of a pro-spectrum X with M coefficients is the pro-group given by:

$$H_n(X;M) = \{H_n(X_s;M)\}_s \ .$$

They also shows that the R-homology weak equivalences are in fact those morphisms which are *M-cohomology weak equivalences* for all R-modules M, see [Isa05, Proposition 5.5].

In the paper [FI07] of Fausk and Isaksen the authors define the notion of a *filtered model category*, [FI07, Definition 4.1], on a category $\mathcal{C}$ which must satisfy a list of model structure like axioms, [FI07, Axioms 4.2–4.6], which is further a *proper filtered model category* if it in addition satisfies [FI07, Axioms 4.9 & and 4.10]. One of the conditions is that there are classes $\mathcal{W}_a$ for each $a \in A$ for some directed set A such that $\mathcal{W}_b \subseteq \mathcal{W}_a$ whenever $b \geq a$. Given a proper filtered model category on a category $\mathcal{C}$ there is then proper model structure on $\texttt{pro-}\mathcal{C}$, [FI07, Theorem 5.15], whose weak equivalences are those morphisms of pro-objects that are essentially levelwise $\mathcal{W}_a$ for all $a \in A$. This in fact recovers the model category of [Isa01], see [FI07, Example 7.2].

Some other model structures appearing on pro-categories are listed below.

- The model structures of Quick of pro-finite spaces, [Qui08, Theorem 2.12] and [Qui11, Theorem 2.3], pro-finite G-spaces, [Qui11, Theorem 2.20], and pro-finite G-spectra, [Qui11, Theorem 2.20].
- The $\mathbb{Z}/p$-model structure of Morel, [Mor96], on pro-finite spaces whose weak equivalences are those morphisms incuding isomorphisms on $\mathbb{Z}/p$-cohomology and where the Bousfield-Kan completion functor of [BK72] is a fibrant replacement functor.

With this overview of the literature regarding model structures on categories of pro-objects we provide now some commentary on the question of existence of an ∞-model structure on $\texttt{pro-}f\mathcal{C}$. We suggested an ∞-model structure on $\texttt{pro-}f\mathcal{C}$ in place of $f\mathcal{C}$ since the ∞-boundary functor is not representable but instead pro-representable. The motivation for having (pro-)representable objects is so that one can use them as domains and codomains for generating cofibrations and acyclic cofibrations for a model structure. However in each of the papers briefly reviewed above the model structures on $\texttt{pro-}\mathcal{C}$ have been seen to be not cofibrantly generated. This recurring shortcoming of model structures on $\texttt{pro-}\mathcal{C}$ suggests we should not expect a potential ∞-model structure on $\texttt{pro-}f\mathcal{C}$ to be cofibrantly generated either but perhaps constructed along lines similar to that of one of the preceeding model structures.

Note however that we can immediately deduce strict model structures on $\texttt{pro-}f\mathcal{C}$ (resp. $\texttt{pro-}b\mathcal{C}$) coming from the $(f\mathcal{C})_S$ (resp. $(b\mathcal{C})_S$) model structures which we record here.

Theorem C.0.0.3. *For $r \geq 0$ and $S \subseteq \{0,1,2,\dots,r\}$ containing r the category $\texttt{pro-}f\mathcal{C}$ can be equipped with a proper model structure (with non-functorial factorisations), which we denote $(\texttt{pro-}f\mathcal{C})_S$, whose weak equivalences are the essentially levelwise r-weak equivalences and cofibrations are the essentially levelwise S-cofibrations.*

Proof. Given such an S there is the $(f\mathcal{C})_S$ model structure of Theorem 3.1.0.2 which is right proper since every object is fibrant and left proper by Theorem 3.7.1.7. The result now follows by [Isa04, Theorem 4.15].

Theorem C.0.0.4. *For $r \geq 0$ and $S \subseteq \{0,1,2,\dots,r\}$ containing both 0 and r the category $\texttt{pro-}b\mathcal{C}$ can be equipped with a proper model structure (with non-functorial factorisations), which we denote $(\texttt{pro-}b\mathcal{C})_S$, whose weak equivalences are the essentially levelwise r-weak equivalences and cofibrations are the essentially levelwise S-cofibrations.*

Proof. Given such an S there is the $(b\mathcal{C})_S$ model structure of Theorem 3.2.0.2 which is right proper since every object is fibrant and left proper by Theorem 3.7.2.8. The result now follows by [Isa04, Theorem 4.15].

Note the last paper [FI07], whilst sounding ideal for our setup, does not yield a candidate ∞-model structure as the containment of r-weak equivalences is the wrong way:

$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \dots \ .$$

Question C.4. The $\mathcal{L} \dashv \mathrm{Tot}^{\Pi}$ adjunction can likely be generalised to an adjunction from filtered chain complexes to n-truncated multicomplexes but perhaps not from multicomplexes with all differentials. It is then feasible such adjunctions would form yet more Quillen equivalences between model structures on n-truncated multicomplexes and filtered chain complexes.

Question C.5. Perhaps related to Question C.1 is, given some category and a method of constructing a filtered chain complex from that category with the aim of running a spectral sequence, what conditions does one obtain on morphisms in that category to construct an r-(co)fibration or r-weak equivalence once one takes the associated filtered object. E.g. in an (subcategory of an) arrow category of spaces whose objects are fibrations of CW complexes/simplicial sets, to such an object one can associate a filtered chain complex whose spectral sequence is the Leray-Serre spectral sequence associated to the fibration. Given then a morphism between fibrations, what conditions on this morphism does one require to obtain an r-(co)fibration or r-weak equivalence on the filtered chain complexes?

And further can one transfer the S-model structures to these categories?

Question C.6. It would be informative to see to what extent one could formalise issues relating to convergence within the framework of these model categories (potentially the tentative ∞-model category). When spectral sequences are employed one wants to know the spectral sequence converges (and to what extent it converges, see [Boa99]). Frequently convergence is automatic from some boundedness conditions on the filtration, i.e. for each n there exists $p_1(n)$ and $p_2(n)$ such that $F_{p_1(n)}A^n = 0$ and $F_{p_2(n)}A^n = A^n$. Restricting to such a category of filtered chain complexes with bounded filtrations will be neither complete nor cocomplete so we cannot define a model structure on such a subcategory without relaxing what we mean by model category. However as seen in Section 3.12 if we impose a fixed filtration degree where boundedness must occur we can define model category structures.

Question C.7. The interaction between the various model structures, by which we mean the various inclusions of weak equivalences, fibrations or cofibrations across the model structures of $f\mathcal{C}$ and $b\mathcal{C}$, is worth further study (this also relates to the Question C.2). Of note in the literature where similar examples occur are Beke's model structures on simplicial sets. [Bek10, Propostion 2.1 & Theorem 2.2] gives a model structure on simplicial sets for each $n \geq 0$ whose weak equivalences are the usual weak equivalences of simplicial sets and satisfy the following proper inclusions:

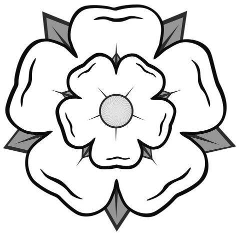
$$\mathrm{Fib}_0 \subset \mathrm{Fib}_1 \subset \ldots \subset \mathrm{Fib}_n \subset \ldots\ ,$$
$$\mathrm{Cof}_0 \supset \mathrm{Cof}_1 \supset \ldots \supset \mathrm{Cof}_n \supset \ldots\ .$$

Question C.8. All model structures considered in this thesis on filtered chain complexes or bicomplexes can be viewed as projective model structures in some sense. In analogy with chain complexes which have both a cofibrantly generated projective model structure and cofibrantly generated injective model structure (this is cofibrantly generated and not fibrantly generated, [Hov99, Theorem 2.3.13]) we ask are there analogous injective S-model structures on $f\mathcal{C}$ and $b\mathcal{C}$ and what is the analogous intepretation for the various S with fixed r of Proposition 4.1.0.14?

Such injective S-model structures are also likely to be Quillen equivalent to the ones considered within as there is a Quillen equivalence induced by the identity-identity adjunction; weak equivalences are the same on both sides and the projective cofibrations are a subclass of the injective monomorphisms which are simply the degreewise inclusions:

$$\begin{tikzcd} i\colon \mathcal{C}_R^{proj} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}_R^{inj} \colon i \arrow[l, shift left] \end{tikzcd} .$$

Question C.9. Does the adjunction $\mathrm{Tot}^{\oplus} \dashv \mathcal{R}$ of Proposition 2.3.0.2 left induce one model structures from $f\mathcal{C}$ to a model structure on $b\mathcal{C}$?



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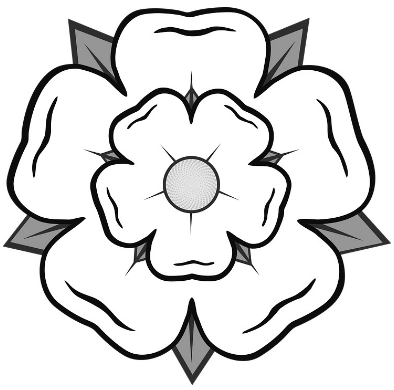

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