Aspects of representation theory: τ-exceptional sequences, modular Fuss-Catalan numbers and idempotent completion of extriangulated categories

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Abstract

This thesis is concerned with various aspects of the representation theory of finite dimensional algebras, with a focus on combinatorial and homological aspects. We explore the aspects of representation theory relating to tilting modules, cluster algebras, τ-exceptional sequences, and extriangulated categories.

The notion of a τ-exceptional sequence was introduced by Buan and Marsh in Buan & Marsh (2021) as a generalisation of an exceptional sequence for finite-dimensional algebras. We calculate the number of complete τ-exceptional sequences of certain classes of Nakayama algebras. In some cases, we obtain closed formulas which also count other well-known combinatorial sets and exceptional sequences of path algebras of Dynkin quivers.

The modular Catalan numbers $C_{k,n}$, introduced in Hein & Huang (2017) count equivalence classes of parenthesizations of $x_0 * \cdots * x_n$, where $*$ is a binary k-associative operation and k is a positive integer. The classical notion of associativity coincides with 1-associativity, in which case $C_{1,n} = 1$, and the single 1-equivalence class has size given by the Catalan number C_n. We introduce modular Fuss-Catalan numbers $C^m_{k,n}$ which count k-equivalence classes of parenthesizations of $x_0 * \cdots * x_n$ where $*$ is an m-ary k-associative operation for $m \geq 2$. Our main results are an explicit formula for $C^m_{k,n}$, and a characterisation of k-associativity.

Extriangulated categories were introduced by Nakaoka and Palu in Nakaoka & Palu (2019a) as a simultaneous generalisation of exact categories and triangulated categories. We show that the idempotent completion of an extriangulated category is also extriangulated. A

possible consequence of this is a methodology for constructing Krull-Remak-Schmidt extriangulated categories, since an additive category $\mathcal{A}$ has the Krull-Remak-Schmidt property if and only if $\mathcal{A}$ is idempotent complete and the endomorphism ring of every object is semi-perfect; see (Krause, 2015, Corollary 4.4).

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Chapter 1

Introduction

1.1 Introduction

This thesis is concerned with several aspects of the *representation theory of finite-dimensional algebras*. An *algebra* is any set where there are notions of addition, multiplication, and scalar multiplication which satisfy some natural properties. A motivating example of an algebra is the set of 3-dimensional vectors $\mathbb{R}^3$ with the usual notions of addition of vectors, the vector cross-product, and scalar multiplication. Whilst such examples of algebras are well understood, more exotic examples of algebras are less well understood and harder to study directly. Instead of studying these exotic examples directly, a common strategy is to see how these algebras act on other mathematical spaces called modules. For an algebra A, the set of its modules is called its *category of modules* and is donated by $\operatorname{mod} A$. The representation theory of finite-dimensional algebras is the study of the categories of modules of algebras. The categories of modules $\operatorname{mod} A$ are usually infinite, however, in the cases we are interested in, they have the *Krull-Remak-Schmidt* property; which states that every module M in $\operatorname{mod} A$ has a decomposition in terms of a subset of modules, which themselves cannot be decomposed called *indecomposable modules*. This is analogous to how any positive integer decomposes into a product of prime numbers. So to understand $\operatorname{mod} A$, and hence A, we have the smaller tasks of just needing to understand the indecomposable modules over A.

A first step in calculating the indecomposable modules of $\operatorname{mod} A$ is to calculate its *exceptional modules*. These are indecomposable modules M with the property of having no *self-extensions* of all degrees $i \geq 1$, which is written as $\operatorname{Ext}^i(M, M) = 0, i \geq 1$. However, calculating exceptional modules is itself a challenge. To overcome this challenge, one considers *exceptional sequences*. These are sequences of exceptional modules which satisfy certain properties. Exceptional sequences were first introduced in the context of algebraic geometry in the setting of *triangulated categories* by Bondal (1989); Gorodentsev (1989); Gorodentsev *et al.* (1987). They were then later introduced and studied in the context of the representation theory of finite-dimensional algebras by Crawley-Boevey (1993); Ringel (1994). Starting with an exceptional sequence, it is sometimes possible to generate the whole set of exceptional sequences by performing a process of *mutation* on the initial exceptional sequence, and hence obtain all the exceptional modules; see for example Bondal (1989); Ringel (1994).

Extriangulated categories were first introduced by Nakaoka and Palu Nakaoka & Palu (2019b), as a generalisation of both triangulated categories and *exact categories*. The category of modules $\operatorname{mod} A$ is a prototypical example of an *abelian category*, and abelian categories form one of the motivating examples for exact categories. Simply put, extriangulated categories represent the most abstract framework where the notion of *extensions* (therefore self-extensions) needed to define exceptional sequences, exists. Extriangulated categories with the Krull-Remak-Schmidt property are of interest in the literature; see for example Iyama *et al.* (2018); Zhu & Zhuang (2021).

Another strategy for understanding the category $\operatorname{mod} A$ comes from *tilting-theory*. Key ideas in tilting theory are those of *torsion pairs* (Happel, 1988, 4.1) and *tilting modules* (Assem *et al.*, 2006, Chapter VI, 2.1. Definition). A torsion pair is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of $\operatorname{mod} A$ which are maximal with respect to the property that there are no non-zero module homomorphisms from $\mathcal{T}$ to $\mathcal{F}$ i.e. $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$. Torsion pairs give us an alternative way of viewing the category $\operatorname{mod} A$ because, to each module M, there is a way of associating an extension δ_M in $\operatorname{Ext}^1(Z, X)$, for some $Z \in \mathcal{F}$ and $X \in \mathcal{T}$; see for example Proposition 2.3.16. A *split torsion pair* is a torsion pair that gives a complete description of $\operatorname{mod} A$ in the sense that every module M in $\operatorname{mod} A$ decomposes into the direct sum of modules $Z \in \mathcal{F}$ and $X \in \mathcal{T}$ above. Moreover, if $(\mathcal{T}, \mathcal{F})$ is a split-torsion pair, then each indecomposable module is in $\mathcal{T}$ or $\mathcal{F}$. So understanding split torsion pairs is sufficient for understanding $\operatorname{mod} A$. One source of torsion pairs is tilting modules; see for example Definition 2.3.22. For a tilting module M, we can construct a torsion pair $(\mathcal{T}(M), \mathcal{F}(M))$, where

$$\mathcal{T}(M) = \{X \in \operatorname{mod} A \mid \operatorname{Ext}^1(M, X) = 0\},$$

and

$$\mathcal{F}(M) = \{X \in \operatorname{mod} A \mid \operatorname{Hom}(M, X) = 0\};$$

see for example (Happel, 1988, 4.3 Lemma). Of particular interest to this thesis are combinatorics related to tilting modules in the category of modules $\operatorname{mod} \overrightarrow{\mathbb{A}_n}$; these are the combinatorics of the Catalan numbers C_n. Catalan numbers are a ubiquitous sequence of natural numbers in mathematics, which count many combinatorial sets including the set of triangulations of polygons; see Stanley (2015) for more on Catalan numbers. Also of particular interest to us is the appearance of Catalan numbers in the theory of cluster algebras of Dynkin type $\mathbb{A}_n$. Cluster algebras are a class of commutative algebras defined combinatorially by a process of iterated mutation which first appeared in Fomin & Zelevinsky (2002). In this thesis, our cluster algebras will be subrings in the field $\mathbb{F} = \mathbb{Q}(u_1, \dots, u_n)$ of rational functions in the indeterminates $u_1, \dots, u_n$. To define a cluster algebra, one starts the data of an initial *seed*. By applying all possible iterations of *mutations* to the initial seed, other seeds are obtained. A cluster algebra is then the subring of $\mathbb{F}$ generated from all the seeds obtained by this process of iterated mutation. For the cluster algebras of Dynkin type $\mathbb{A}_n$, the information in all the seeds admits a combinatorial description in terms of triangulations of $(n+3)$-sided polygons, which are counted by the Catalan number C_{n+1}; see (Fomin *et al.*, 2017, Corollary 5.3.6). Moreover, the mutation of the seeds of these cluster algebras can be described via the *flip* operation the triangulations of $(n+3)$-gons. This is mirrored in the case of tilting modules in $\operatorname{mod} \overrightarrow{\mathbb{A}_{n+1}}$, where *basic tilting modules* correspond to triangulations of $(n+3)$-gons, and a notion of *mutation* on tilting modules exists, which also corresponds to the *flip* operation the triangulations of $(n+3)$-gons; see (Buan & Krause, 2004, Theorem C). Motivated by the ubiquity of the Catalan numbers and their combinatorics, we investigate in this thesis a generalisation of these combinatorics, leading us to the notion of modular Fuss-Catalan numbers.

1.2 Outline

Chapter 2 is dedicated to recalling the prerequisite material for this thesis. An important notion to this thesis is that of a category, as categories form a majority of the framework and language for the work in this thesis. We start §2.1 by defining a category and functors. We then introduce the classes of categories that are of interest to us including; additive categories, abelian categories, triangulated categories, and extriangulated categories. In §2.2, we recall the theory of quiver representations and finite-dimensional algebras as pertaining to Chapter 3 of the thesis. In particular, we recall the construction of the Auslander-Reiten translations τ needed in the definition τ-exceptional sequences and recall the definition of the Nakayama algebras. In §2.3, we recall different notions of mutations from tilting theory, cluster algebras, exceptional sequences, and τ-exceptional sequences which motivate the work in Chapter 3 and Chapter 4.

Chapter 3 is about counting complete *τ-exceptional sequences* of Nakayama algebras. Complete τ-exceptional sequences, unlike compete exceptional sequences, always exist for finite-dimensional algebras. Hence a natural question is to count them. In this chapter we count the number of complete τ-exceptional sequences for certain classes of Nakayama algebras, and where possible establish connections to other well-known and understood combinatorial sets. The main results of this chapter are closed formulas and recurrence relations for the number of complete τ-exceptional sequences of some Nakayama algebras.

Chapter 4 introduces modular Fuss-Catalan numbers $C^m_{k,n}$. These numbers arise as a result of generalising the Catalan combinatorics in a natural way: by generalising *associativity* (flips) to the notion of *k-associativity*, and by replacing Catalan numbers with their "higher-dimensional" analogue, the Fuss-Catalan numbers. The main results of this chapter include an explicit formula for the modular Fuss-Catalan numbers $C^m_{k,n}$, and a characterisation of the notion of k-associativity.

Chapter 5 studies the idempotent completion of an extriangulated category. The main result of this chapter is that the idempotent completion of an extriangulated category is also an extriangulated category, with an inherited extriangulated structure. Applying these results to triangulated categories and exact categories, we recover the analogous results for triangulated categories and exact categories as in (Balmer & Schlichting, 2001, Theorem 1.5) (Bühler, 2010a, Proposition 6.13, Remark 7.8).

The results of Chapters 3,4, and 5 are new, except where otherwise stated. The results of Chapter 3 have been published in Msapato (2021a), the results of Chapter 4 have been published in Msapato (2022) (this paper was selected to be part of the Discrete Mathematics Editors' Choice 2022 list), and the results of Chapter 5 have been published in Msapato (2021b).

Chapter 2

Preliminaries

2.1 Category Theory

The notion of a Category and the language of Category Theory are essential to the work which is to be presented in this thesis. So we will begin by recalling the necessary and relevant concepts from category theory for this thesis. In particular, this section will cover: categories and functors, additive categories, abelian categories, triangulated categories, and extriangulated categories. Our main references for this section are, Leinster (2014), MacLane (2013), Happel (1988) and Nakaoka & Palu (2019b).

2.1.1 Categories, functors and natural transformations

Definition 2.1.1. (Leinster, 2014, Definition 1.1.1) A **category** $\mathcal{A}$ consists of the following data and axioms:

- a class of **objects**, which we will denote by $\mathrm{ob}(\mathcal{A})$;
- for any pair of objects A, B in $\mathrm{ob}(\mathcal{A})$, a class of **morphisms** or **arrows** from A to B, which we will denote by $\mathcal{A}(A, B)$;

- for any objects A, B, C in $\mathrm{ob}(\mathcal{A})$, a **composition** function

$$\circ\colon \mathcal{A}(B,C) \times \mathcal{A}(A,B) \to \mathcal{A}(A,C)$$

$$(g, f) \mapsto g \circ f,$$

  such that $\circ$ is **associative**, i.e. for each $f \in \mathcal{A}(A,B)$, $g \in \mathcal{A}(B,C)$ and $h \in \mathcal{A}(C,D)$, we have that $(h \circ g) \circ f = h \circ (g \circ f)$;

- for any object A, a morphism called the **identity** on A in $\mathcal{A}(A,A)$, which we denote by 1_A;

- for any $f \in \mathcal{A}(A,B)$, we have $f \circ 1_A = f = 1_B \circ f$.

There are many standard notation conventions in category theory. We will lay out the ones which we will be adopted in this thesis. Let $\mathcal{A}$ be a category and $A \in \mathrm{ob}(\mathcal{A})$. We will often write $A \in \mathcal{A}$ to also mean that A is an object of $\mathcal{A}$. For morphisms $f \in \mathcal{A}(A,B)$ and $g \in \mathcal{A}(B,C)$, it will be convenient to write gf for the composition $g \circ f$. In many diagrams to follow, we will write $A \xrightarrow{f} B$ to mean a morphism $f \in \mathcal{A}(A,B)$. We will also occasionally write $f\colon A \to B$ to mean a morphism $f \in \mathcal{A}(A,B)$. For particular categories such as the categories of modules, it is more standard to write $\mathrm{Hom}(A,B)$ or $\mathrm{Hom}_{\mathcal{A}}(A,B)$ for the class of morphisms from A to B in $\mathcal{A}$. In the special case of the class of morphisms from an object A in $\mathcal{A}$ to itself, we will write $\mathrm{End}(A)$ or $\mathrm{End}_{\mathcal{A}}(A)$ for $\mathcal{A}(A,A)$, and call the morphisms in $\mathrm{End}(A)$ endomorphisms of A .

When talking about a category, we are usually concerned with its morphisms and how the morphisms interact with each other through compositions. When discussing composition, it is often helpful to illustrate them using **commutative diagrams**. For example, the following diagram is said to be commutative (commutes)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

if $bf = ga$. In general, a diagram is said to be commutative if for all paths of arrows in the diagram starting and ending at the same objects, we have that the compositions of all the arrows in these paths are equal.

The morphisms between objects inform us how two objects are related to each other. A relationship between objects of a category that is often useful is the notion of an **isomorphism**. An isomorphism between objects of a category conveys that the objects are structurally the same, but not necessarily equal.

Definition 2.1.2. (Leinster, 2014, Definition 1.14) Let $\mathcal{A}$ be a category and A, B be objects in $\mathcal{A}$. A morphism $f\colon A \to B$ is an isomorphism if there exists a morphism $g\colon B \to A$ such that $gf = 1_A$ and $fg = 1_B$. The morphism g is called the **inverse** of f. We then say that A and B are isomorphic and we write $A \cong B$.

Definition 2.1.3. (Borceux, 1994, Definition 1.7.3) Let $\mathcal{A}$ be a category. A morphism $r\colon B \to C$ is called a **retraction** if there exists a morphism $q\colon C \to B$ such that $rq = 1_C$. Dually, morphism $s\colon A \to B$ is called a **section** if there exists a morphism $t\colon B \to A$ such that $ts = 1_A$.

For any given category $\mathcal{A}$, we can always define another category $\mathcal{A}^{\mathrm{op}}$ known as the **opposite** or **dual** category which is defined in the following way. The objects are given by $\mathrm{ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{A})$. For every pair of objects A, B in $\mathcal{A}$ and every morphism $f \in \mathcal{A}(A, B)$, there is a corresponding morphism f^{op} in $\mathcal{A}^{\mathrm{op}}(B, A)$. Formally, the arrow (morphism) f^{op} is the reverse of the arrow (morphism) f. Given morphisms f^{op} and g^{op} we define composition in $\mathcal{A}^{\mathrm{op}}$ as follows: $g^{\mathrm{op}} f^{\mathrm{op}} = (fg)^{\mathrm{op}}$ where fg is the composition of f and g in $\mathcal{A}$. The existence of the opposite category leads to the notion of the **principle of duality**. Simply put, for every categorical statement there is a dual statement which can be made by simply reversing all arrows i.e. passing to the opposite category. In this thesis we will often be using this principle, we will use the word, **dually** to indicate when we're invoking the principle of duality.

One of the strengths of category theory comes from the fact that it can be applied to many areas of mathematics by defining categories of the objects of interest. For example, in the area of group theory, one can define a category **Grp**, the category of all groups, where the objects are groups and the morphisms are group homomorphisms. The next notion is we recall is that of functors.

Definition 2.1.4. (Leinster, 2014, Definition 1.2.1) Let $\mathcal{A}$ and $\mathcal{B}$ be categories, a **covariant functor** $F\colon \mathcal{A} \to \mathcal{B}$ consists of the following data:

- a function from $\mathrm{ob}(\mathcal{A})$ to $\mathrm{ob}(\mathcal{B})$,

$$F\colon \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$$

$$A \mapsto F(A),$$

- for any pair of objects A and B in $\mathcal{A}$, a function

$$F\colon \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B))$$

$$f \mapsto F(f),$$

such that,

1. for all objects A in the category $\mathcal{A}$, we have that $F(1_A) = 1_{F(A)}$ and,
2. for all objects A, B, C and morphisms $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, we have that $F(g \circ f) = F(g) \circ F(f)$.

Contravariant functors are the dual covariant functors, in the sense that they reverse arrows. If $G\colon \mathcal{A} \to \mathcal{B}$ is a contravariant functor, then every morphism $f \in \mathcal{A}(A, B)$ maps to a morphism $G(f) \in \mathcal{B}(G(B), G(A))$. Formally, a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ is just a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$. Throughout the thesis, we will abuse terminology by referring to covariant functors simply as **functors**. When we are discussing functors, we are usually interested in functors with some special properties.

Definition 2.1.5. (Assem *et al.*, 2006, A.2. Functors, 2.1. Definition) Let $\mathcal{A}$, and $\mathcal{B}$ be categories. The **product category** $\mathcal{A} \times \mathcal{B}$ is the category defined as follows. The objects of $\mathcal{A} \times \mathcal{B}$ are pairs (A, B), where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A morphism $f\colon (A, B) \to (A', B')$ is a pair $f = (f_1\colon A \to A', f_2\colon B \to B')$ where $f_1 \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ and $f_2 \in \mathrm{Hom}_{\mathcal{B}}(B, B')$. A pair of morphisms (g_1, g_2) and (h_1, h_2) in $\mathcal{A} \times \mathcal{B}$ are composed as follows, $(g_1, g_2) \circ_{\mathcal{A}\times\mathcal{B}} (h_1, h_2) = (g_1 \circ_{\mathcal{A}} h_1, g_2 \circ_{\mathcal{B}} h_2)$ if and only g_1, h_1 are composable morphisms in $\mathcal{A}$ and g_2, h_2 are composable morphisms in $\mathcal{B}$. In other words, composition in $\mathcal{A}\times\mathcal{B}$ is given by componentwise composition in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Definition 2.1.6. (Assem *et al.*, 2006, A.2. Functors, 2.1. Definition) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories. A **bifunctor** is a functor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$.

Definition 2.1.7. (Leinster, 2014, Definitions 1.2.16,1.3.17) Let $\mathcal{A}, \mathcal{B}$ be categories and $F\colon \mathcal{A} \to \mathcal{B}$ a functor. If, for any pair of objects A, B in $\mathcal{A}$, the function

$$F\colon \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B))$$

$$f \mapsto F(f),$$

is injective, the functor F is said to be **faithful**. If instead the function is always surjective, the functor F is said to be **full**. In the case a functor is both faithful and full, we will say that it is **fully faithful**. If for any object $B \in \mathcal{B}$, there exists an object $A \in \mathcal{A}$ such that $F(A) \cong B$, it is said that F is **essentially surjective**.

Definition 2.1.8. (Leinster, 2014, Definition 1.2.18) Let $\mathcal{A}$ be a category. A **subcategory** $\mathcal{B}$ of $\mathcal{A}$ consists of a subclass of objects $\mathrm{ob}(\mathcal{B}) \subseteq \mathrm{ob}(\mathcal{A})$, such that for any pair of objects B, B' in $\mathrm{ob}(\mathcal{B})$, there is a corresponding subclass of morphisms $\mathcal{B}(B, B') \subseteq \mathcal{A}(B.B')$. Moreover it is required that $\mathcal{B}$ be closed under composition and identities.

Given a category $\mathcal{A}$ and a subcategory $\mathcal{B}$ of $\mathcal{A}$, there is a natural functor $i\colon \mathcal{B} \to \mathcal{A}$ called the inclusion functor defined as follows. On objects, $B \mapsto B$ for all $B \in \mathcal{B}$, and on morphisms $f \mapsto f$ for all morphisms f in $\mathcal{B}$. This functor is trivially faithful. In the case that the inclusion functor is also full, we call the subcategory $\mathcal{B}$ a **full subcategory**.

An isomorphism of objects in a category gives us a notion of objects being essentially the same. This notion is weaker than the notion of equality of objects. It is natural to ask for a notion of two categories being essentially the same. It turns out that in most practical situations, the notion of isomorphism between categories is too strong. A more appropriate notion turns out to be that of **equivalence**. But first, we need to recall **natural transformations**.

Definition 2.1.9. (Leinster, 2014, Definition 1.3.1) Let $\mathcal{A}$ and $\mathcal{B}$ be categories and F, G be two functors from $\mathcal{A}$ to $\mathcal{B}$. A natural transformation $\alpha\colon F \Rightarrow G$ from F to G consists of the following data:

A class of morphisms $F(A) \xrightarrow{\alpha_A} G(A)$ for every object $A \in \mathcal{A}$, such that that for every morphism $A \xrightarrow{f} B$ in $\mathcal{A}$, the following diagram commutes.

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"] & G(B)
\end{tikzcd}$$

If all the morphisms $F(A) \xrightarrow{\alpha_A} G(A)$ are isomorphisms, α is said to be a **natural isomorphism**.

Definition 2.1.10. (Leinster, 2014, Definition 1.3.15)(Assem *et al.*, 2006, A.2. Functors, 2.1 , 2.2. Definition) Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An **equivalence** between $\mathcal{A}$ and $\mathcal{B}$ consists of a pair of functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ together with a pair of natural isomorphisms,

$$\alpha\colon 1_{\mathcal{A}} \to G \circ F \text{ and } \beta\colon F \circ G \to 1_{\mathcal{B}}.$$

When there is an equivalence between two categories $\mathcal{A}$ and $\mathcal{B}$, the two categories are said to be **equivalent** and we write $\mathcal{A} \simeq \mathcal{B}$. The functors F and G are called **equivalences**, and G is called a **quasi-inverse** of F. A **duality** is the dual notion of equivalence between contravariant functors.

The following proposition is very useful in determining when a functor is an equivalence.

Proposition 2.1.11. (Leinster, 2014, Proposition 1.3.18) Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. Then F is an equivalence if and only if F is fully faithful and essentially surjective.

We will close this subsection by introducing the following key definitions.

Definition 2.1.12. Let $\mathcal{C}$ be any category, and let $f\colon X \to Z, g\colon Y \to Z$ be morphisms in $\mathcal{C}$. The **pullback** of f and g is an object P equipped with morphisms $p_X\colon P \to X$ and $p_Y\colon P \to Y$, such that the following diagram commutes (this diagram is said to be a pullback diagram), with the following universal property.

$$\begin{tikzcd}
P \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z,
\end{tikzcd}$$

For any object Q equipped with morphisms $q_X\colon Q \to X$, and $q_Y\colon Q \to Y$, satisfying $fq_X = gq_Y$; there is a unique morphism $u\colon Q \to P$ such that $p_X u = q_X$, and $p_Y u = q_Y$.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_X"] \arrow[ddr, bend right, "q_Y"'] & & \\
& P \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z,
\end{tikzcd}$$

Dually, given morphisms $f\colon X \to Y$ and $g\colon X \to Z$, the **pushout** of f and g is an object P equipped with morphisms $p_Y\colon Y \to P$, and $p_X\colon Z \to P$ such that the following diagram commutes (this diagram is said to be a pushout diagram), with the following universal property.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "p_X"] \\
Z \arrow[r, "p_Y"] & P
\end{tikzcd}$$

For any object Q equipped with morphisms $q_Z\colon Z \to Q$, and $q_Y\colon Y \to Q$, satisfying $q_Z g = q_Y f$; there is a unique morphism $v\colon P \to Q$ such that $vp_Z = q_Z$, and $vp_Y = q_y$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "p_X"] \arrow[ddr, bend left, "q_Y"] & \\
Z \arrow[r, "p_Y"] \arrow[drr, bend right, "q_Z"'] & P \arrow[dr, dashed, "v"] & \\
 & & Q
\end{tikzcd}$$

Proposition 2.1.13. (Freyd, 1964, Proposition 2.151, Theorem 2.15*) The pullback of a pair of morphisms, if it exists, is unique up to isomorphism. Dually, the pushout of a pair of morphisms, if it exists, is unique up to isomorphism.

2.1.2 Additive Categories

The categories we deal with in this work fall into one of the following types of categories; abelian categories, triangulated categories, and extriangulated categories. All of these types of categories fall into the class of categories known as additive categories. Hence, our next order of business is to define additive categories.

Definition 2.1.14. (MacLane, 2013, §I.5) Let $\mathcal{A}$ be a category. An object $T \in \mathcal{A}$ is called **terminal**, if for all objects $A \in \mathcal{A}$, there is exactly one morphism from A to T. Dually, an object $I \in \mathcal{A}$ is called **initial**, if for all objects $A \in \mathcal{A}$, there is exactly one morphism form I to A. If an object is both initial and terminal, we will call it a **zero object** and denote it by $\mathbf{0}$.

If a category $\mathcal{A}$ has a zero object $\mathbf{0}$, then the zero object is unique up to unique isomorphism. For any pair of objects A, B we have a morphism, called the **zero morphism** $0\colon A \to B$, which is the composition of the morphisms $A \to \mathbf{0}$, and $\mathbf{0} \to B$.

Definition 2.1.15. (Keller, 1997, §1.1) Let $\mathcal{A}$ be a category. The category $\mathcal{A}$ is **Ab-enriched** if for all objects $A, B \in \mathcal{A}$, the set of morphisms $\mathcal{A}(A, B)$ is an abelian group $(\mathcal{A}(A, B), +)$, in such a way that the composition function

$$\circ\colon \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$

is bilinear, i.e. for $f, g \in \mathcal{A}(A, B)$, and $e \in \mathcal{A}(B, C)$,

$$e \circ (f + g) = (e \circ f) + (e \circ g),$$

and for $f, g \in \mathcal{A}(B, C)$ and $e \in \mathcal{A}(A, B)$,

$$(f + g) \circ e = (f \circ e) + (g \circ e).$$

Definition 2.1.16. (Leinster, 2014, Definition 5.1.1) Let $\mathcal{A}$ be a category, and let A and B be objects in $\mathcal{A}$. The **product** of A and B is an object $A \Pi B$ that comes equipped with a pair of morphisms $\pi_A\colon A \Pi B \to A$, and $\pi_B\colon A \Pi B \to B$ called **projections** such that, for any object X and pair of morphisms $f\colon X \to A$, and $g\colon X \to B$, there exists a unique morphism $h\colon X \to A \Pi B$ such that the following diagram commutes.

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\
A & A\Pi B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B
\end{tikzcd}$$

We can define a dual notion of the coproduct by reversing the arrows in the above definition as follows.

Definition 2.1.17. (MacLane, 2013, §III.3) Let $\mathcal{A}$ be a category, and let A, B be objects in $\mathcal{A}$. The **coproduct** of A and B is an object $A \oplus B$ that comes equipped with a pair of morphisms $i_A \colon A \to A \oplus B$, and $i_B \colon B \to A \oplus B$ called **inclusions** such that, for any object X and pair of morphisms $f \colon A \to X$, and $g \colon B \to X$, there exists a unique morphism $h \colon A \oplus B \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
 & X & \\
A \arrow[ur, "f"] \arrow[r, "i_A"] & A \oplus B \arrow[u, dashed, "h"] & B \arrow[ul, "g"'] \arrow[l, "i_B"']
\end{tikzcd}$$

Proposition 2.1.18. (MacLane, 2013, Chapter VIII, §2, Theorem 2) Let $\mathcal{A}$ be an Ab-enriched category. The objects A and B in $\mathcal{A}$ have a product $A\Pi B$ if and only if they have a coproduct $A \oplus B$. Moreover, the coproduct and product of A and B are isomorphic.

Going forward, when dealing with additive categories we will write $A \oplus B$ for the product of two objects. We are now able to give the definition of an additive category.

Definition 2.1.19. (Keller, 1997, §1.1) An Ab-enriched category $\mathcal{A}$ is said to be **additive** if it has a zero object, and any pair of objects A and B in $\mathcal{A}$ admit a product $A\Pi B$.

A classic example of an additive category is the category **Ab** of abelian groups (note that **Ab** is a subcategory of **Grp**). The zero object in **Ab** is just the trivial group $\{0\}$. The product of two groups A and B in **Ab** is given by direct product of groups $A \times B$. Finally the sum of two morphisms (group homomorphisms) $f, g \colon A \to B$ is $f+g$ defined as $(f+g)(x) := f(x) + g(x)$ for all $x \in A$.

When considering a functor between two additive categories, it is desirable if the functor preserves the additive structure of these categories in the following sense.

Definition 2.1.20. (MacLane, 2013, Chapter VIII, §2) Let $\mathcal{A}$ and $\mathcal{B}$ be Ab-enriched categories. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is called **additive** if, for any pair of objects A and B in $\mathcal{A}$, the function

$$F \colon \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B))$$

is a group homomorphism. In particular, for a pair of morphisms $f, g \in \mathcal{A}(A, B)$, we have that $F(f+g) = F(f) + F(g)$.

Proposition 2.1.21. (MacLane, 2013, Chapter VIII, §2, Proposition 4) Let $\mathcal{A}$ and $\mathcal{B}$ be Ab-enriched categories. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is additive if and only if, for any objects A, B in $\mathcal{A}$,

$$F(A \oplus B) \cong F(A) \oplus F(B).$$

Classic examples of additive functors are the **Hom**-functors which are defined as follows for any additive $\mathcal{A}$ category. For any object $A \in \mathcal{A}$, we define the Hom-functors on A as follows,

$$\mathcal{A}(A, -) \colon \mathcal{A} \longrightarrow Ab$$

whereby

$$B \mapsto \mathcal{A}(A, B).$$

Dually, we can define the functor,

$$\mathcal{A}(-, A) \colon \mathcal{A}^{\mathrm{op}} \longrightarrow Ab$$

whereby

$$B \mapsto \mathcal{A}(B, A).$$

2.1.3 Abelian Categories

Abelian categories are additive categories with extra structure. The motivation and prototypical example of abelian categories is the category **Ab**. Abelian categories first appeared in Buchsbaum (1955) and a couple of years later in Grothendieck (1957). The abelian categories which will mostly be of interest to us here will be categories of modules over finite-dimensional algebras.

Definition 2.1.22. (Freyd, 1964, Chapter 1 §1.4) Let $\mathcal{A}$ be a category. A morphism $f\colon A \to B$ is called a **monomorphism** if, for any object $C \in \mathcal{A}$, and any pair of morphisms $g, h\colon C \to A$ such that $fg = fh$, we have $g = h$. If f is a monomorphism, we will say that f is **monic** for short. Dually, a morphism $f\colon A \to B$ is called an **epimorphism** if for any object $C \in \mathcal{A}$, and any pair of morphism $g, h\colon B \to C$ such that $gf = hf$, then $g = h$. If f is an epimorphism, we also say that f is **epic.**

Isomorphisms provide examples of morphisms which are both monomorphisms and epimorphisms; see for example (Freyd, 1964, Proposition 1.42).

Proposition 2.1.23. (Freyd, 1964, Proposition 1.41) Let $\mathcal{A}$ be a category and $f\colon A \to B$ and $g\colon B \to C$ be a pair of morphisms in $\mathcal{A}$. If gf is a monomorphism, then so is f. Moreover, if both f and g are monomorphisms, then so is gf.

By the principle of duality, we have the following dual statement.

Proposition 2.1.24. (Freyd, 1964, Proposition 1.42) Let $\mathcal{A}$ be a category and $f\colon A \to B$ and $g\colon B \to C$ be a pair of morphisms in $\mathcal{A}$. If gf is an epimorphism, then so is g. Moreover, if both f and g are epimorphisms, then so is gf.

In the category **Ab** or, more generally, the category **Grp**, monomorphisms are precisely the injective group homomorphisms, and epimorphisms are precisely the surjective group homomorphisms. Monomorphisms and epimorphisms in **Ab** can also be classified in the following way. Let $f\colon G \to H$ be a morphism in **Ab** (i.e. a group homomorphism). The kernel of f, denoted $\ker(f)$, is defined as follows;

$$\ker(f) := \{g \in G \mid f(g) = 0_H, \text{ where } 0_H \text{ is the identity element of } H\}.$$

It is well known that $\ker(f)$ is a subgroup of G. There is an inclusion map $i\colon \ker(f) \hookrightarrow G$ with the following properties; $fi = 0$ is the zero map, and for all morphisms $v\colon X \to G$ such that $fv = 0$, there exists a unique morphism $u\colon X \to \ker(f)$ such that $v = iu$.

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "u"'] \arrow[d, "v"] & \\
\ker(f) \arrow[r, hook, "i"] & G \arrow[r, "f"] & H
\end{tikzcd}$$

It is a well known result that the morphism f is injective (a monomorphism) if and only if $\ker(f) = \{0\}$, the trivial group i.e. the zero object of **Ab**. The inclusion map i is itself injective (a monomorphism). The dual notion of the kernel of f is known as the cokernel. The image of f, denoted $\mathrm{im}(f)$ is defined as follows;

$$\mathrm{im}(f) := \{h \in H \mid \text{ there exists } g \in G \text{ such that } f(g) = h\}.$$

The cokernel is the quotient group $\mathrm{coker}(f) = H/\mathrm{im}(f)$. There is a projection map $j\colon H \to \mathrm{coker}(f)$ with the following properties; $jf = 0$ is the zero map, and for all morphisms $v\colon H \to X$ such that $vf = 0$, there exists a unique morphism $u\colon \mathrm{coker}(f) \to X$ such that $uj = v$.

$$\begin{tikzcd}
G \arrow[r, "f"] & H \arrow[r, "j"] \arrow[dr, "v"'] & \mathrm{coker}(f) \arrow[d, dashed, "u"] \\
 & & X
\end{tikzcd}$$

It is also a well known result that the morphism f is surjective (an epimorphism) if and only if $\mathrm{coker}(f) = \{0\}$. The projective map j is itself surjective. The notions of kernel and cokernel generalise to the context of an arbitrary category in the following way.

Definition 2.1.25. Let $\mathcal{A}$ be a category and $f\colon A \to B$ a morphism in $\mathcal{A}$. The **kernel** of f is a morphism $k\colon K \to A$ such that $fk = 0$ and, for all morphisms $v\colon X \to A$ such that $fv = 0$, there exists a unique morphism $u\colon X \to K$ such that $ku = v$.

$$\begin{tikzcd} & X \arrow[dl, dashed, "u"'] \arrow[d, "v"] & \\ K \arrow[r, "k"] & A \arrow[r, "f"] & B \end{tikzcd}$$

When the kernel of f exists, we will write $\ker(f)\colon \mathrm{Ker}(f) \to A$ for the kernel of f.

Note that kernel of $f\colon A \to B$ in the category **Ab** defined prior is an example of a kernel according to the above definition. It is well known fact that the kernel of a morphism is monic.

Proposition 2.1.26. (Knapp, 2007, Proposition 4.34) Let $\mathcal{A}$ be a category, and let $k\colon K \to A$ be the kernel of a morphism $f\colon A \to B$ in $\mathcal{A}$. Then k is monic.

The notion of a cokernel in an arbitrary category is dual.

Definition 2.1.27. Let $\mathcal{A}$ be a category, and $f\colon A \to B$ a morphism in $\mathcal{A}$. The **cokernel** of f is a morphism $q\colon B \to Q$ such that; $qf = 0$, and for all morphisms $v\colon B \to X$ such that $vf = 0$, there exists a unique morphism $u\colon Q \to X$ such that $uq = v$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "q"] \arrow[dr, "v"'] & Q \arrow[d, dashed, "u"] \\ & & X \end{tikzcd}$$

When the cokernel of f exists, we will write $\mathrm{coker}(f)\colon B \to \mathrm{Coker}(f)$ for the cokernel of f.

Note that cokernel of $f\colon A \to B$ in the category $\mathbf{Ab}$ defined prior is an example of a cokernel according to the above definition. The dual statement that the cokernel of a morphism is epic is also well known and easy to prove. The proof of it is dual to the proof of the dual statement.

Proposition 2.1.28. Let $\mathcal{A}$ be a category, and let $q\colon B \to Q$ be the cokernel of a morphism $f\colon A \to B$ in $\mathcal{A}$. Then q is epic.

Proof. The proof is dual to that of Proposition 2.1.26. □

Definition 2.1.29. (MacLane, 2013, Chapter VIII, §3) Let $\mathcal{A}$ be a category, and $f\colon A \to B$ a morphism in $\mathcal{A}$. The image of f, denoted $\mathrm{im}(f)$, is the kernel of the cokernel of f. Dually, the coimage of f, denoted $\mathrm{coim}(f)$, is the cokernel of the kernel of f.

Unlike in the category $\mathbf{Ab}$, in a general category, a morphism need not have a kernel nor a cokernel. An abelian category is a category where it is true that every morphism has a kernel and cokernel as in the case with $\mathbf{Ab}$. In particular, abelian categories can be thought of as a generalisation of the category $\mathbf{Ab}$, they are defined as follows.

Definition 2.1.30. (MacLane, 2013, Chapter VIII, §3) An additive category $\mathcal{A}$ is said to be abelian if it has the following properties:

1. every morphism in $\mathcal{A}$ has a kernel and a cokernel,
2. every monomorphism in $\mathcal{A}$ is a kernel of a morphism in $\mathcal{A}$, and every epimorphism in $\mathcal{A}$ is a cokernel of a morphism in $\mathcal{A}$.

In the instance of the category $\mathbf{Ab}$, it is well known that a morphism $f\colon G \to H$ is monic if and only if its kernel is $\{0\} \to G$. Dually, f is epic if and only if its cokernel is $H \to \{0\}$. The same is true in a general abelian category.

Proposition 2.1.31. (Freyd, 1964, Theorem 2.17, Theorem 2.17*) Let $\mathcal{A}$ be an abelian category, and $f\colon A \to B$ a morphism in $\mathcal{A}$. Then f is monic if and only if $\ker(f) = 0\colon \mathbf{0} \to A$. Dually, f is epic if and only if $\mathrm{coker}(f) = 0\colon B \to \mathbf{0}$.

Proposition 2.1.32. (Freyd, 1964, Theorem 2.15) Let $\mathcal{A}$ be an abelian category. The pullback diagram of any pair of morphisms $f\colon X \to Z$, and $g\colon Y \to Z$ exists.

$$\begin{tikzcd}
P \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z,
\end{tikzcd}$$

Proposition 2.1.33. (Freyd, 1964, Theorem 2.15*) Let $\mathcal{A}$ be an abelian category. The pushout diagram of any pair of morphisms $f\colon X \to Y$, and $g\colon X \to Z$ exists.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "p_X"] \\
Z \arrow[r, "p_Y"] & P
\end{tikzcd}$$

Definition 2.1.34. (Gelfand & Manin, 2013, §6, 2 Definition) Let $\mathcal{A}$ be an abelian category. A **(cochain) complex** is a sequence of morphisms,

$$A^\bullet = \cdots \longrightarrow A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \xrightarrow{f_{i+2}} \cdots$$

such that $f_{i+1}f_i = 0$ for all integers i.

Since, for all integers i, the composition $f_{i+1}f_i = 0$, there is a unique morphism $a_i\colon A_i \to \mathrm{Ker}(f_{i+1})$ such that $f_i = \ker(f_{i+1})a_i$. Dually, there exists a unique morphism $b_{i+1}\colon \mathrm{Coker}(f_i) \to A_{i+2}$ such that $f_{i+1} = b_{i+1}\mathrm{coker}(f_i)$.

$$\begin{tikzcd}
 & \mathrm{Coker}(f_i) \arrow[dr, dashed, "b_{i+1}"] & \\
A_i \arrow[r, "f_i"] \arrow[dr, dashed, "a_i"'] & A_{i+1} \arrow[u, "\mathrm{coker}(f_i)"] \arrow[r, "f_{i+1}"] & A_{i+2} \\
 & \mathrm{Ker}(f_{i+1}) \arrow[u, "\ker(f_{i+1})"'] &
\end{tikzcd}$$

The $(i+1)^{\text{th}}$ **cohomology** of $A^\bullet$ is defined to be the object

$$H^{i+1}(A^\bullet) := \mathrm{Coker}(a_i) = \mathrm{Ker}(b_{i+1}).$$

Remark 2.1.35. We note that in the category **Ab** of abelian groups, the $(i+1)^{st}$ cohomology is given by the quotient group $H^{i+1}(A^\bullet)$ $=\ker(f_{i+1})/\mathrm{im}(f_i)$, see for example (Gelfand & Manin, 2013, §4, 4 Definition b).

Definition 2.1.36. (Popescu, 1973, Chapter 2 §2.3) Let $\mathcal{A}$ be an abelian category. A sequence of morphisms,

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots$$

in $\mathcal{A}$ is said to be **exact** at A_i if $\ker(f_i) = \mathrm{im}(f_{i-1})$ for all i. The sequence is an **exact sequence** if it is exact at every object A_i.

A **short exact** sequence is an exact sequence of the following form.

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

Short exact sequences are sometimes called **extensions**. We say that such a short exact sequence is an extension of C by A.

Proposition 2.1.37. (Kashiwara & Schapira, 2005, Corollary 7.4) Let

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

be a short exact sequence in an abelian category $\mathcal{A}$. The following statements are equivalent:

1. f is a section;
2. g is a retraction;
3. B is isomorphic to $A \oplus C$.

Such a sequence sequence is a called a **split exact** sequence or a **split extension**.

Given any morphism $f\colon A \to B$ in an abelian category $\mathcal{A}$, the following exact sequence can be constructed,

$$\mathbf{0} \longrightarrow \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow \mathbf{coker}(f) \longrightarrow \mathbf{0};$$

see for example (Freyd, 1964, Proposition 2.22). Short exact sequences are an important structural property of abelian categories, especially from the point of view of homological algebra. As such, when we consider functors between abelian categories, we often care about those functors which preserve exact sequences. Functors which preserve short exact sequences are known as **exact functors**, and they're defined as follows.

Definition 2.1.38. (Popescu, 1973, Chapter 3 §3.2) Let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant additive functor between two abelian categories $\mathcal{A}$ and $\mathcal{B}$. The functor F is said to be **left exact** if for any short exact sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

in $\mathcal{A}$, the sequence

$$\mathbf{0} \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact in $\mathcal{B}$. Dually, the functor F is said to be **right exact** if for any short exact sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

in $\mathcal{A}$, the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow \mathbf{0}$$

is exact in $\mathcal{B}$. Finally, the functor F is said to be **exact** if for any short exact sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

in $\mathcal{A}$, the sequence

$$\mathbf{0} \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow \mathbf{0}$$

is a short exact sequence in $\mathcal{B}$.

Example 2.1.39. (Freyd, 1964, Chapter 3 §3.1) Let $\mathcal{A}$ be an abelian category. For any object $A \in \mathcal{A}$, the Hom-functor $\mathcal{A}(A, -)\colon \mathcal{A} \to Ab$ is left exact. Dually, the functor $\mathcal{A}(-, A)\colon \mathcal{A} \to Ab$ is right exact.

Let $\mathcal{A}$ be an abelian category. The Hom-functors $\mathcal{A}(A, -)\colon \mathcal{A} \to Ab$ provide examples of left exact functors. Naturally, one may wonder, for what objects $P \in \mathcal{A}$ is the functor $\mathcal{A}(P, -)\colon \mathcal{A} \to Ab$ exact?

Definition 2.1.40. (Kashiwara & Schapira, 2005, Definition 8.4.1) Let $\mathcal{A}$ be an abelian category. An object P in $\mathcal{A}$ is called **projective** if the functor $\mathcal{A}(P, -)\colon \mathcal{A} \to Ab$ is exact. An object I in $\mathcal{A}$ is called **injective** if the functor $\mathcal{A}(-, I)\colon \mathcal{A} \to Ab$ is exact.

Proposition 2.1.41. (Freyd, 1964, Proposition 3.31) Let $\mathcal{A}$ be an abelian category. An object $P \in \mathcal{A}$ is projective if and only if for every epimorphism $f\colon X \to Y$ and every map $p\colon P \to Y$, there exists a map $q\colon P \to X$ such that $fq = p$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "q"'] \arrow[d, "p"] & \\ X \arrow[r, "f"] & Y \arrow[r] & \mathbf{0} \end{tikzcd}$$

Dually, an object $I \in \mathcal{A}$ is injective if and only if for every monomorphism $g\colon L \to M$, and every map $i\colon L \to I$, there exists a map $j\colon M \to I$ such that $jg = i$.

$$\begin{tikzcd} \mathbf{0} \arrow[r] & L \arrow[r, "g"] \arrow[d, "i"] & M \arrow[dl, dashed, "j"] \\ & I & \end{tikzcd}$$

Proposition 2.1.42. (Kashiwara & Schapira, 2005, Proposition 8.4.5) Let $\mathcal{A}$ be an abelian category, and let P_1 and P_2 be projective objects in $\mathcal{A}$. Then $P_1 \oplus P_2$ is a projective object if and only if P_1 and P_2 are projective objects. Dually, let I_1 and I_2 be injective objects in $\mathcal{A}$. Then $I_1 \oplus I_2$ is an injective object if and only if I_1 and I_2 are injective objects.

Definition 2.1.43. (Kashiwara & Schapira, 2005, Definition 8.4.1) Let $\mathcal{A}$ be an abelian category. It is said that $\mathcal{A}$ **has enough projectives** if for every A in $\mathcal{A}$, there exists a projective object P and an epimorphism $P \to A$. Dually, $\mathcal{A}$ **has enough injectives** if for every object A in $\mathcal{A}$, there exists an injective object I and a monomorphism $A \to I$.

Definition 2.1.44. (Popescu, 1973, Lemma 2.11) Let $\mathcal{A}$ be an abelian category and A an object in $\mathcal{A}$. A **projective resolution** of A is an exact sequence

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow \mathbf{0},$$

where each P_i is a projective object. Dually, an **injective resolution** of A is an exact sequence

$$\mathbf{0} \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots,$$

where each I_i is an injective object.

Projective resolutions and injective resolutions may be thought of as a measure of how close an object is to being projective and injective respectively in the following sense. The more terms there are in the projective (injective) resolution of an object, the further away it is from being projective (respectively, injective).

Proposition 2.1.45. (Popescu, 1973, Lemma 2.11) Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has enough projectives, then every object in $\mathcal{A}$ has a projective resolution. Dually, if $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution.

Let $\mathcal{A}$ be an abelian category with enough projectives and enough injectives. Let A be an arbitrary object in $\mathcal{A}$. Since $\mathcal{A}$ has enough projectives, we may consider the projective resolution of A:

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \longrightarrow \mathbf{0}.$$

Applying the Hom-functor $\mathcal{A}(-,B)$ to the above projective resolution we obtain

$$\mathbf{0} \longrightarrow \operatorname{Hom}(A,B) \xrightarrow{f_0^*} \operatorname{Hom}(P_0,B) \xrightarrow{f_1^*} \operatorname{Hom}(P_1,B) \xrightarrow{f_2^*} \operatorname{Hom}(P_2,B) \xrightarrow{f_3^*} \cdots.$$

The i^{th} **extension group** is defined to be i^{th} cohomology, denoted by

$$\operatorname{Ext}^i(A,B) = \ker(f_{i+1}^*)/\operatorname{im}(f_i^*).$$

This construction induces a functor $\operatorname{Ext}^i \colon \mathcal{A}^{\text{op}} \times \mathcal{A} \to Ab$ called the i^{th}**-extension functor**, see for example (Oppermann, 2016, §25, Example 25.4).

Remark 2.1.46. In the above construction of the extension functor $\operatorname{Ext}^i(-,B)$, after taking the projective resolution of $A \in \mathcal{A}$, we may instead apply any other right exact functor $F \colon \mathcal{A} \to \mathcal{B}$ to the projective resolution, before taking the i^{th} cohomology. This construction defines a functor $\mathbb{R}_i F \colon \mathcal{A} \to \mathcal{B}$ called the i^{th} **right derived functor of** F. See (Oppermann, 2016, Definition 25.1, §25) for more details. Analogous to the construction of $\operatorname{Ext}^i(-,B)$ we may also construct the extension functor $\operatorname{Ext}^i(A,-)$ dually.

Our main interest is the first extension functor $\operatorname{Ext}^1(-,-) \colon \mathcal{A}^{\text{op}} \times \mathcal{A} \to Ab$. The extension groups $\operatorname{Ext}^1(C,A)$ can be understood in terms of extensions of C by A (short exact sequences starting with A and ending with C), for any objects A, C in an abelian category $\mathcal{A}$. Let $\mathcal{A}$ be an abelian category, and let A, C be objects in $\mathcal{A}$. Let us denote by $\mathrm{E}(C,A)$ the set of equivalence classes of extensions of C by A,

$$\xi : \mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0},$$

with the following relation. Two extensions

$$\xi : \mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

and

$$\xi' : \mathbf{0} \longrightarrow A \xrightarrow{f'} B \xrightarrow{g'} C \longrightarrow \mathbf{0}$$

are said to be **equivalent** if there exists an isomorphism $b\colon B \to B'$, such that the following diagram commutes, see for example (Oppermann, 2016, §27).

$$\begin{tikzcd}
\xi : \mathbf{0} \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, equal] & \mathbf{0} \\
\xi' : \mathbf{0} \arrow[r] & A \arrow[r, "f'"] & B \arrow[r, "g'"] & C \arrow[r] & \mathbf{0}
\end{tikzcd}$$

Consider an extension of C by A,

$$\xi : \mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0},$$

and a morphism $a\colon A \to A'$. Since $\mathcal{A}$ is an abelian category, the pushout P of f and a exists by Proposition 2.1.33. By the universal property of the pushout applied to the morphism $g\colon B \to C$ and $0\colon A' \to C$, there exists a morphism $g'\colon P \to C$ such that the following is a short exact sequence,

$$a_*\xi : \mathbf{0} \longrightarrow A' \xrightarrow{f'} P \xrightarrow{g'} C \longrightarrow \mathbf{0},$$

and the following diagram commutes with the left square being a pushout square.

$$\begin{tikzcd}
\xi : \mathbf{0} \arrow[r] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, equal] & \mathbf{0} \\
a_*\xi : \mathbf{0} \arrow[r] & A' \arrow[r, "f'"] & P \arrow[r, "g'"] & C \arrow[r] & \mathbf{0}
\end{tikzcd}$$

Dually, given a morphism $c\colon C \to C'$, we can construct an short exact sequence $c^*\xi$. See (Oppermann, 2016, Construction 27.2) for more.

Definition 2.1.47. (Oppermann, 2016, Definition 27.3) Let $\xi \in \mathrm{E}(C, A)$,

$$\xi : \mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0},$$

and $\xi' \in \mathrm{E}'(C', A')$

$$\xi' : \mathbf{0} \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow \mathbf{0},$$

be extensions. The direct product of ξ and ξ' is defined to be,

$$\xi \oplus \xi' : \mathbf{0} \longrightarrow A \oplus A' \xrightarrow{f \oplus f'} B \oplus B' \xrightarrow{g \oplus g'} C \oplus C' \longrightarrow \mathbf{0}.$$

When $A = A'$ and $C = C'$, the **Baer sum** of ξ and ξ' is defined to be the extension

$$\xi + \xi' := \begin{pmatrix} 1 & 1 \end{pmatrix}_* \begin{pmatrix} 1 \\ 1 \end{pmatrix}^* (\xi \oplus \xi').$$

Theorem 2.1.48. (Oppermann, 2016, Theorem 27.4) Let $\mathcal{A}$ be an abelian category and A, C be objects in $\mathcal{A}$. The set $\mathrm{E}(C, A)$, together with the Baer sum, forms an abelian group, where the additive identity $0_{\mathrm{E}(C,A)}$ is given by the split exact sequence.

Theorem 2.1.49. (Oppermann, 2016, Theorem 27.5) Let $\mathcal{A}$ be an abelian category with enough projectives. Then for $A, B \in \mathcal{A}$, we have the following isomorphism,

$$\mathrm{E}(A, B) \cong \mathrm{Ext}^1(-, B)(A) = \mathrm{Ext}^1(A, B).$$

Dually, if the abelian category $\mathcal{A}$ has enough injectives, then

$$\mathrm{E}(A, B) \cong \mathrm{Ext}^1(A, -)(B) = \mathrm{Ext}^1(A, B).$$

Due to the above theorem, the first extension groups $\mathrm{Ext}^1(A, B)$ may be understood in terms of the group of extensions $\mathrm{E}(A, B)$. By Definition 2.1.40, projective objects P in an abelian category $\mathcal{A}$ are those objects for which the functor $\mathcal{A}(P, -)$ is an exact functor, and dually injective objects I are those for which the functor $\mathcal{A}(-, I)$ is an exact functor. The 1^{st} extension functor $\mathrm{Ext}^1(-, -)$ also provides a characterisation of projective objects and injective objects in a similar spirit.

Lemma 2.1.50. (Kashiwara & Schapira, 2005, Lemma 8.4.4) Let $\mathcal{A}$ be an abelian category and $P \in \mathcal{A}$. Then P is a projective objective if and only if $\mathrm{Ext}^1(P, A) = 0$ for all objects $A \in \mathcal{A}$. Dually, I is an injective object if and only if $\mathrm{Ext}^1(A, I) = 0$ for all objects $A \in \mathcal{A}$.

2.1.4 Triangulated Categories

In this subsection, we will recall additive categories known as **triangulated categories**. Naively speaking, triangulated categories are additive categories where the notion of short exact sequences is replaced by a notion of **distinguished triangles**, which satisfy certain axioms. Triangulated categories appear in the study in of abelian categories because given an abelian category $\mathcal{A}$, there exists an associated triangulated category known as the **derived category** $D^b(\mathcal{A})$ of bounded complexes. Our main reference for this subsection is Happel (1988).

Let $\mathcal{A}$ be an additive category and T an auto-equivalence of $\mathcal{A}$. A **sextuple** (X, Y, Z, u, v, w) in $\mathcal{A}$ is a sequence of morphisms of the following form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

A morphism of sextuples from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a triple (f, g, h) of morphisms in $\mathcal{A}$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'
\end{tikzcd}$$

If f, g and h are isomorphisms in $\mathcal{A}$, we say that the triple (f, g, h) is an isomorphism.

Definition 2.1.51. (Happel, 1988, §1) Let $\mathcal{A}$ be an additive category with auto-equivalence T. Let $\mathcal{T}$ be a set of sextuples of $\mathcal{A}$. The triple $(\mathcal{A}, T, \mathcal{T})$ is called a triangulated category if the following axioms hold. In this case, the elements of $\mathcal{T}$ are then called **distinguished triangles**.

(TR1) Every sextuple isomorphic to a distinguished triangle is again a distinguished triangle. Every morphism $u: X \to Y$ in $\mathcal{A}$ can be embedded into a distinguished triangle (X, Y, Z, u, v, w). The sextuple $(X, X, \mathbf{0}, 1_X, 0, 0)$ is a distinguished triangle.

(TR2) If (X, Y, Z, u, v, w) is a distinguished triangle, then $(Y, Z, TX, v, w, -Tu)$ is also a distinguished triangle.

(TR3) Given two distinguished triangles $(X, Y, Z, u, v, w), (X', Y', Z', u', v', w')$ and two morphisms $f\colon X \to X'$, $g\colon Y \to Y'$ such that $u'f = gu$, there exists a morphism $h\colon Z \to Z'$ such that (f, g, h) is a morphism from the first distinguished triangle to the second.

(TR4)(Octahedral Axiom) Given distinguished triangles

$$(X, Y, Z', u, i, i'), (Y, Z, X', v, j, j') \text{ and } (X, Z, Y', u \circ v, k, k'),$$

there exist morphisms $f\colon Z' \to Y'$, $g\colon Y' \to X'$ such that the following diagram commutes and the second column is a distinguished triangle. Moreover, we have that $T(u)k' = j'g$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "i'"] \arrow[d, "f"] & TX \arrow[d, equal] \\
X \arrow[r, "v \circ u"] & Z \arrow[r, "k"] \arrow[d, "j"] & Y' \arrow[r, "k'"] \arrow[d, "g"] & TX \\
 & X' \arrow[r, equal] \arrow[d, "j'"] & X' \arrow[d, "Ti \circ j'"] & \\
 & TY \arrow[r, "Ti"] & TZ &
\end{tikzcd}$$

In the sequel, we will often refer to distinguished triangles simply as **triangles** when $(\mathcal{A}, T, \mathcal{T})$ is a triangulated category.

2.1.5 Extriangulated categories.

In this section, we will recall mostly from Nakaoka & Palu (2019a) the basic theory of extriangulated categories needed for this thesis. Through out this subsection, $\mathcal{C}$ will be an additive category equipped with a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$, where **Ab** is the category of Abelian groups.

Definition 2.1.52. (Nakaoka & Palu, 2019a, Definition 2.1). Let A, C be objects of $\mathcal{C}$. Formally, an $\mathbb{E}$-extension is a triple (A, δ, C) with $\delta \in \mathbb{E}(C, A)$. We will write $\delta \in \mathbb{E}(C, A)$ to mean that δ is an $\mathbb{E}$-extension.

Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$, we have the following $\mathbb{E}$-extensions:

$$a_*\delta := \mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A'),$$

$$c^*\delta := \mathbb{E}(c^{\mathrm{op}}, A)(\delta) \in \mathbb{E}(C', A) \text{ and}$$

$$c^*a_*\delta = a_*c^*\delta := \mathbb{E}(c^{\mathrm{op}}, a)(\delta) \in \mathbb{E}(C', A').$$

We will abuse notation by writing $\mathbb{E}(c, -)$ instead of $\mathbb{E}(c^{\mathrm{op}}, -)$.

Definition 2.1.53. (Nakaoka & Palu, 2019a, Definition 2.3). Let (A, δ, C) and (A', δ', C') be any pair of $\mathbb{E}$-extensions. A morphism $(a, c)\colon \delta \to \delta'$ of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$ such that:

$$a_*\delta = c^*\delta'.$$

Lemma 2.1.54. (Nakaoka & Palu, 2019a, Remark 2.4). Let (A, δ, C) be an $\mathbb{E}$-extension. Then we have the following.

1. Any morphism $a \in \mathcal{C}(A, A')$ induces a morphism of $\mathbb{E}$-extensions,

$$(a, 1_C)\colon \delta \to a_*\delta.$$

2. Any morphism $c \in \mathcal{C}(C', C)$ induces a morphism of $\mathbb{E}$-extensions,

$$(1_A, c)\colon c^*\delta \to \delta.$$

Definition 2.1.55. (Nakaoka & Palu, 2019a, Definition 2.5). For any objects A, C in $\mathcal{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called a **split $\mathbb{E}$-extension**.

Definition 2.1.56. (Nakaoka & Palu, 2019a, Definition 2.6). Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions. Let $i_C\colon C \to C \oplus C'$ and $i_{C'}\colon C' \to C \oplus C'$ be the canonical inclusion maps. Let $p_A\colon A \oplus A' \to A$, and $p_{A'}\colon A \oplus A' \to A'$

be the canonical projection maps. By the biadditivity of $\mathbb{E}$ we have the following isomorphism.

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A')$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ via the above isomorphism. If $A = A'$ and $C = C'$, then the sum $\delta + \delta' \in \mathbb{E}(C, A)$ is obtained by

$$\delta + \delta' = \mathbb{E}(\Delta_C, \nabla_A)(\delta \oplus \delta'),$$

where $\Delta_C = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : C \to C \oplus C$, and $\nabla_A = \begin{pmatrix} 1, 1 \end{pmatrix} : A \oplus A \to A$.

Definition 2.1.57. (Nakaoka & Palu, 2019a, Definition 2.7). Let A, C be a pair of objects in $\mathcal{C}$. Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$, and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathcal{C}$ are said to be **equivalent** if there exists an isomorphism $b \in \mathcal{C}(B, B')$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, equal] \\ A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C \end{tikzcd}$$

We denote the equivalence class of a sequence $A \xrightarrow{x} B \xrightarrow{y} C$, by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

Definition 2.1.58. (Nakaoka & Palu, 2019a, Definition 2.8). Let A, B, C, A', B', C' be objects in the category $\mathcal{C}$.

1. We denote by 0 the equivalence class $[A \xrightarrow{\left[\begin{smallmatrix} 1_A \\ 0 \end{smallmatrix}\right]} A \oplus C \xrightarrow{[0\ 1_C]} C]$.
2. For any two equivalence classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we denote by $[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ the equivalence class $[A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$.

Definition 2.1.59. (Nakaoka & Palu, 2019a, Definition 2.9). Let $\mathfrak{s}$ be a correspondence associating an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. We say that $\mathfrak{s}$ is a **realisation** of $\mathbb{E}$ if the following condition $(\circ)$ holds.

$(\circ)$ Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be $\mathbb{E}$-extensions, with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then for any morphism $(a, c)\colon \delta \to \delta'$ of $\mathbb{E}$-extensions, there exists $b \in \mathcal{C}(B, B')$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, dashed, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C'
\end{tikzcd}$$

In this situation, we say that the triple of morphisms (a, b, c) realises (a, c). For $\delta \in \mathbb{E}(C, A)$, we say that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realises δ if $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

Definition 2.1.60. (Nakaoka & Palu, 2019a, Definition 2.10). A realisation $\mathfrak{s}$ is said to be an **additive realisation** if the following conditions are satisfied,

1. For any objects A, C in $\mathcal{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies
$$\mathfrak{s}(0) = 0.$$

2. For any pair of extensions δ and δ', we have that,
$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta').$$

We are now in a position to define an extriangulated category.

Definition 2.1.61. (Nakaoka & Palu, 2019a, Definition 2.12). Let $\mathcal{C}$ be an additive category. An **extriangulated category** is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ consisting of an additive category $\mathcal{C}$, a bifunctor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}}\mathcal{C} \times \to \mathbf{Ab}$, realisation $\mathfrak{s}$ satisfying the following axioms.

(ET1) The functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$ is a biadditive functor.

(ET2) The realisation $\mathfrak{s}$ is an additive realisation of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, realised as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ respectively. For any commutative diagram

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\
A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C'
\end{tikzcd}$$

there exists a morphism $c \in \mathcal{C}(C, C')$ such that $(a, c)\colon \delta \to \delta'$ is a morphism of $\mathbb{E}$-extensions and the triple (a, b, c) realises (a, c).

(ET3)$^{\mathrm{op}}$ The dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be any pair of $\mathbb{E}$-extensions, realised by the sequences, $A \xrightarrow{f} B \xrightarrow{f'} D$ and $B \xrightarrow{g} C \xrightarrow{g'} F$. Then there exists an object E in $\mathcal{C}$, a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[d, "d"] \\
A \arrow[r, "h"] & C \arrow[r, "h'"] \arrow[d, "g'"] & E \arrow[d, "e"] \\
& F \arrow[r, equal] & F
\end{tikzcd}$$

in $\mathcal{C}$ and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realised by the sequence $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities:

(i) $\mathfrak{s}((f')_*\delta') = [D \xrightarrow{d} E \xrightarrow{e} F]$.

(ii) $d^*\delta'' = \delta$.

(iii) $f_*\delta'' = e^*\delta'$.

(ET4)$^{\mathrm{op}}$ The dual of (ET4).

In this case, we call $\mathfrak{s}$ an $\mathbb{E}$-triangulation of $\mathcal{C}$.

Definition 2.1.62. (Bennett-Tennenhaus & Shah, 2021, Definition 2.31). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be extriangulated categories. A covariant additive functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called an **extriangulated functor** if there exists a natural transformation

$$\Gamma = \{\Gamma_{(C,A)}\}_{(C,A)\in\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}\colon \mathbb{E} \Rightarrow \mathbb{E}'(F^{\mathrm{op}}-, F-)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$, such that $\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$ implies that $\mathfrak{s}'(\Gamma_{(Z,X)})(\delta) = [F(A) \xrightarrow{F(x)} F(B) \xrightarrow{F(y)} F(C)]$. Here F^{op} is the **opposite functor** $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$ given by $F^{\mathrm{op}}(A) = F(A)$ and $F^{\mathrm{op}}(f^{\mathrm{op}}) = (F(f))^{\mathrm{op}}$. Furthermore, we say that F is an **extriangulated equivalence** if F is an equivalence of categories.

We will conclude this section by introducing some useful terminology from Nakaoka & Palu (2019a) and stating results about extriangulated categories which will be helpful for the rest of the thesis.

Definition 2.1.63. (Nakaoka & Palu, 2019a, Definition 2.5, Definition 3.9). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$.

1. A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a **conflation** if it realises some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$.

2. A morphism $f \in \mathcal{C}(A, B)$ is called an **inflation** if it admits some conflation $A \xrightarrow{f} B \longrightarrow C$.

3. A morphism $g \in \mathcal{C}(B, C)$ is called a **deflation** if it admits some conflation $A \longrightarrow B \xrightarrow{g} C$.

Definition 2.1.64. (Nakaoka & Palu, 2019a, Definition 2.19). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2).

1. If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realises $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$-**triangle** or **extriangle** and denote it by the following diagram.

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

2. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triple (a, b, c) realises $(a, c)\colon \delta \to \delta'$ we write it as in the following commutative diagram and call (a, b, c) a morphism of $\mathbb{E}$-triangles.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

Lemma 2.1.65. (Nakaoka & Palu, 2019a, Corollary 3.6). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Let (a, b, c) be a morphism of $\mathbb{E}$-triangles. If any two of $a, b, c,$ are isomorphisms, then so is the third.

Lemma 2.1.66. (Nakaoka & Palu, 2019a, Proposition 3.7). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Let

$$A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$$

be any $\mathbb{E}$-triangle in $\mathcal{C}$. If $f \in \mathcal{C}(A, X)$ and $h \in \mathcal{C}(C, Z)$ are isomorphisms, then

$$X \xrightarrow{a \circ f^{-1}} B \xrightarrow{h \circ b} Z \overset{f_* h^* \delta}{\dashrightarrow}$$

is again an $\mathbb{E}$-triangle.

Corollary 2.1.67. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Let

$$A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$$

be any $\mathbb{E}$-triangle in $\mathcal{C}$. Suppose we have the following commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[d, "h"] \\ X \arrow[r, "x"] & Y \arrow[r, "y"] & Z \end{tikzcd}$$

where the morphisms f, g, h are isomorphisms. Then it follows that

$$X \xrightarrow{x} Y \xrightarrow{y} Z \overset{f_* h^* \delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle.

Proof. By Proposition 2.1.66,

$$X \xrightarrow{a \circ f^{-1}} B \xrightarrow{h \circ b} Z \overset{f_* h^* \delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle. Now consider the following diagram.

$$\begin{tikzcd} X \arrow[r, "a \circ f^{-1}"] \arrow[d, equal] & B \arrow[r, "h \circ b"] \arrow[d, "g"] & Z \arrow[d, equal] \\ X \arrow[r, "x"] & Y \arrow[r, "y"] & Z \end{tikzcd}$$

Observe that it commutes, so it is an equivalence, which implies that

$$X \xrightarrow{x} Y \xrightarrow{y} Z \overset{f_* h^* \delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle. □

The following two propositions are special cases of propositions from Herschend *et al.* (2021) which are stated for general n-exangulated categories, but here we are restating them in the case of extriangulated categories which are in fact the same as 1-exangulated categories by (Herschend *et al.*, 2021, Proposition 4.3).

Proposition 2.1.68. (Herschend *et al.*, 2021, Proposition 3.2). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Let $(A \xrightarrow{a} B \xrightarrow{b} C, \delta)$ and $(X \xrightarrow{x} Y \xrightarrow{y} Z, \rho)$ be pairs consisting of a sequence of morphisms and an $\mathbb{E}$-extension. Then the following statements are equivalent.

1. $(A \oplus X \xrightarrow{a \oplus x} B \oplus Y \xrightarrow{b \oplus y} C \oplus Z, \delta \oplus \rho)$ is an $\mathbb{E}$-triangle.

2. Both of $(A \xrightarrow{a} B \xrightarrow{b} C, \delta)$ and $(X \xrightarrow{x} Y \xrightarrow{y} Z, \rho)$ are $\mathbb{E}$-triangles.

Proposition 2.1.69. (Herschend *et al.*, 2021, Corollary 3.3). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Suppose that

$$X \oplus A \xrightarrow{\begin{pmatrix} x & u \\ v & 1 \end{pmatrix}} Y \oplus A \xrightarrow{\begin{pmatrix} y & w \end{pmatrix}} Z \overset{\delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle. Then for $t = x - u \circ v$ and $p = [1, 0] \colon X \oplus A \to X$,

$$X \xrightarrow{t} Y \xrightarrow{y} Z \overset{p_*\delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle.

Proposition 2.1.70. (Nakaoka & Palu, 2019a, Corollary 3.12). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. For any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, the following sequences of natural transformations are exact.

$$\mathcal{C}(C, -) \xRightarrow{\mathcal{C}(y,-)} \mathcal{C}(B, -) \xRightarrow{\mathcal{C}(x,-)} \mathcal{C}(A, -) \xRightarrow{\delta^\#} \mathbb{E}(C, -) \xRightarrow{\mathbb{E}(y,-)} \mathbb{E}(B, -) \xRightarrow{\mathbb{E}(x,-)} \mathbb{E}(A, -)$$

$$\mathcal{C}(-, A) \xRightarrow{\mathcal{C}(-,x)} \mathcal{C}(-, B) \xRightarrow{\mathcal{C}(-,y)} \mathcal{C}(-, C) \xRightarrow{\delta_\#} \mathbb{E}(-, A) \xRightarrow{\mathbb{E}(-,x)} \mathbb{E}(-, B) \xRightarrow{\mathbb{E}(-,y)} \mathbb{E}(-, C)$$

The natural transformations $\delta^\#$ and $\delta_\#$ are defined as follows. Given any object X in $\mathcal{C}$, we have that

1. $(\delta^\#)_X \colon \mathcal{C}(A, X) \to \mathbb{E}(C, X)$; $g \mapsto f_*\delta$,

2. $(\delta_\#)_X \colon \mathcal{C}(X, C) \to \mathbb{E}(X, A)$; $f \mapsto f^*\delta$.

The exactness of the first sequence of natural transformations means that for any object X in $\mathcal{C}$, the sequence

$$\mathcal{C}(C,X) \xrightarrow{\mathcal{C}(y,X)} \mathcal{C}(B,X) \xrightarrow{\mathcal{C}(x,X)} \mathcal{C}(A,X) \xrightarrow{\delta^\#_X} \mathbb{E}(C,X) \xrightarrow{\mathbb{E}(y,X)} \mathbb{E}(B,X) \xrightarrow{\mathbb{E}(x,X)} \mathbb{E}(A,X)$$

is exact in **Ab** and likewise for the second sequence.

Proposition 2.1.71. (Nakaoka & Palu, 2019a, Proposition 3.3). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2). Then the following are equivalent.

1. $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET3) and (ET3)$^{\mathrm{op}}$.

2. For any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, the following sequences of natural transformations are exact.

$$\mathcal{C}(C,-) \xRightarrow{\mathcal{C}(y,-)} \mathcal{C}(B,-) \xRightarrow{\mathcal{C}(x,-)} \mathcal{C}(A,-) \xRightarrow{\delta^\#} \mathbb{E}(C,-) \xRightarrow{\mathbb{E}(y,-)} \mathbb{E}(B,-)$$

$$\mathcal{C}(-,A) \xRightarrow{\mathcal{C}(-,x)} \mathcal{C}(-,B) \xRightarrow{\mathcal{C}(-,y} \mathcal{C}(-,C) \xRightarrow{\delta_\#} \mathbb{E}(-,A) \xRightarrow{\mathbb{E}(-,x)} \mathbb{E}(-,B)$$

Lemma 2.1.72. (Nakaoka & Palu, 2019a, Lemma 3.2). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1),(ET2), (ET3), (ET3)$^{\mathrm{op}}$. Then for any $\mathbb{E}$-triangle,

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} ,$$

the following statements hold:

1. $y \circ x = 0$,

2. $x_* \delta = 0$,

3. $y^* \delta = 0$.

Proposition 2.1.73. (Liu & Nakaoka, 2019, Proposition 1.20). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

be an $\mathbb{E}$-triangle, let $f\colon A \to D$ be any morphism and let

$$D \xrightarrow{d} E \xrightarrow{e} F \overset{f_*\delta}{\dashrightarrow}$$

be an $\mathbb{E}$-triangle realising $f_*\delta$. Then there is a morphism g such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "f"] & B \arrow[r, "y"] \arrow[d, "g"] & C \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
D \arrow[r, "d"] & E \arrow[r, "e"] & C \arrow[r, dashed, "f_*\delta"] & {}
\end{tikzcd},$$

and $A \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g \end{pmatrix}} E \overset{e^*\delta}{\dashrightarrow}$ is an $\mathbb{E}$-triangle.

Dually, let

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

be an $\mathbb{E}$-triangle, let $h\colon E \to C$ be any morphism and let

$$A \xrightarrow{d} D \xrightarrow{e} E \overset{h^*\delta}{\dashrightarrow}$$

be an $\mathbb{E}$-triangle realising $h^*\delta$. Then there is a morphism $g\colon D \to B$ such that the following diagram commutes,

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, equal] & D \arrow[r, "e"] \arrow[d, "g"] & E \arrow[r, dashed, "h^*\delta"] \arrow[d, "h"] & {} \\
A \arrow[r, "x"] & B \arrow[r, "y"] & C \arrow[r, dashed, "\delta"] & {}
\end{tikzcd}$$

and $D \xrightarrow{\begin{pmatrix} -e \\ g \end{pmatrix}} E \oplus B \xrightarrow{\begin{pmatrix} h & y \end{pmatrix}} C \overset{d_*\delta}{\dashrightarrow}$ is an $\mathbb{E}$-triangle.

Corollary 2.1.74. (Herschend *et al.*, 2021, Proposition 3.5(2)). Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let

$$A \xrightarrow{a} B \xrightarrow{b} C \overset{\varepsilon}{\dashrightarrow}$$

and

$$X \xrightarrow{x} Y \xrightarrow{y} Z \overset{\delta}{\dashrightarrow}$$

be $\mathbb{E}$-triangles. Suppose we have the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal] & B \arrow[r, "b"] \arrow[d, "u"] & C \arrow[r, dashed, "\varepsilon"] & {} \\ A \arrow[r, "x"] & Y \arrow[r, "y"] & Z \arrow[r, dashed, "\delta"] & {} \end{tikzcd}$$

Then there exists a morphism $w\colon C \to Z$ such that $wb = yu, w^*\delta = \varepsilon$ and the following is an $\mathbb{E}$-triangle,

$$B \xrightarrow{\begin{pmatrix} -b \\ u \end{pmatrix}} C \oplus Y \xrightarrow{\begin{pmatrix} w & y \end{pmatrix}} Z \overset{a_*\delta}{\dashrightarrow} .$$

Lemma 2.1.75. (Herschend *et al.*, 2021, Lemma 4.1) Let $\mathcal{C}$ be an additive category with biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$. Let $X_\bullet = A \xrightarrow{x_1} X \xrightarrow{x_2} C$ and $Y_\bullet = A \xrightarrow{y_1} Y \xrightarrow{y_2} C$ be sequences of morphisms in $\mathcal{C}$. Suppose that the following sequences of functors are exact,

$$\mathcal{C}(C, -) \xRightarrow{\mathcal{C}(x_2,-)} \mathcal{C}(X, -) \xRightarrow{\mathcal{C}(x_1,-)} \mathcal{C}(A, -)$$

$$\mathcal{C}(-, A) \xRightarrow{\mathcal{C}(-,x_1)} \mathcal{C}(-, X) \xRightarrow{\mathcal{C}(-,x_2)} \mathcal{C}(-, C)$$

and likewise for $Y_\bullet$. Then for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "x_1"] \arrow[d, equal] & X \arrow[r, "x_2"] \arrow[d, "f"] & C \arrow[d, equal] \\ A \arrow[r, "y_1"] & Y \arrow[r, "y_2"] & C \end{tikzcd}$$

the following statements are equivalent.

1. $f_\bullet = (1_A, f, 1_C) \colon X_\bullet \to Y_\bullet$ is a homotopy equivalence.

2. $f_\bullet = (1_A, f, 1_C) \colon X_\bullet \to Y_\bullet$ is an equivalence of the sequences.

3. $f \colon X \to Y$ is an isomorphism.

Lemma 2.1.76. (Herschend *et al.*, 2021, Proposition 2.21) Let $\delta \in \mathbb{E}(C, A)$ be an extension, and let $X_\bullet = A \xrightarrow{x_1} X \xrightarrow{x_2} C$ and $Y_\bullet = A \xrightarrow{y_1} Y \xrightarrow{y_2} C$ be sequences of morphisms in $\mathcal{C}$. Suppose that the following sequences of functors are exact,

$$\mathcal{C}(C, -) \xRightarrow{\mathcal{C}(x_2, -)} \mathcal{C}(X, -) \xRightarrow{\mathcal{C}(x_1, -)} \mathcal{C}(A, -) \xRightarrow{\delta^\#} \mathbb{E}(C, -)$$

$$\mathcal{C}(-, A) \xRightarrow{\mathcal{C}(-, x_1)} \mathcal{C}(-, X) \xRightarrow{\mathcal{C}(-, x_2)} \mathcal{C}(-, C) \xRightarrow{\delta_\#} \mathbb{E}(-, A)$$

and likewise for $Y_\bullet$. Let $f_\bullet = (1_A, f, 1_C) \colon X_\bullet \to Y_\bullet$ be a commutative diagram.

$$\begin{tikzcd} A \arrow[r, "x_1"] \arrow[d, equal] & X \arrow[r, "x_2"] \arrow[d, "f"] & C \arrow[d, equal] \\ A \arrow[r, "y_1"] & Y \arrow[r, "y_2"] & C \end{tikzcd}$$

Suppose there exists a commutative diagram $g_\bullet = (1_A, g, 1_C) \colon Y_\bullet \to X_\bullet$;

$$\begin{tikzcd} A \arrow[r, "y_1"] \arrow[d, equal] & Y \arrow[r, "y_2"] \arrow[d, "g"] & C \arrow[d, equal] \\ A \arrow[r, "x_1"] & X \arrow[r, "x_2"] & C \end{tikzcd}$$

then $f_\bullet$ is a homotopic equivalence.

2.2 Quiver Representations & Finite-Dimensional Algebras

In this section, we will recall two classes of categories which we will use in this thesis, the categories of quiver representations, and the category of modules over a finite-dimensional algebra. Of particular interest to us is a class of finite-dimensional algebras known as, **Nakayama algebras**, which will be recall later in this section. Our main references for this section are Assem *et al.* (2006) and Schiffler (2014).

2.2.1 Quiver representations

Definition 2.2.1 (Definition 1.1)**.** Schiffler (2014) A **quiver** Q is a tuple (Q_0, Q_1, s, t), which consists of:

- a set Q_0, whose elements are called **vertices**,
- a set Q_1, whose elements are called **arrows**,
- a function $s\colon Q_1 \to Q_0$ called the **source map**,
- and a function $t\colon Q_1 \to Q_0$ called the **target map**.

A quiver $Q = (Q_0, Q_1, s, t)$ is called **finite** if both Q_0 and Q_1 are finite sets. We also say Q is **connected** if the underlying graph (Q_0, Q_1) is a connected graph.

All quivers from this point forward will be finite and connected. We will also assume that we have no oriented cycles or loops, see Definition 2.2.8. In practice, we will usually draw diagrams like the one below to represent quivers.

Example 2.2.2.
$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\ 2 \arrow[rr, "c"] & & 3 \end{tikzcd}$$

The above diagram should be taken to represent the quiver $Q = (Q_0, Q_1, s, t)$ where; $Q_0 = \{1, 2, 3\}, Q_1 = \{a, b, c\}$, and the maps s, t are defined by setting, $s(a) = 1, s(b) = 1, s(c) = 2, t(a) = 2, t(b) = 3, t(c) = 3$.

Definition 2.2.3. (Schiffler, 2014, Definition 1.2) Let $\mathbb{F}$ be a field, and Q a quiver. A **representation** of the quiver Q is a collection $V = (V_i, \phi_a)_{i\in Q_0, a\in Q_1}$, where

V_i is an $\mathbb{F}$-vector space for each vertex $i \in Q_0$, and where ϕ_a is a linear map $\phi_a\colon V_{s(a)} \to V_{t(a)}$ for each arrow $a \in Q_1$. A representation V is said to be **finite-dimensional** if each $\mathbb{F}$-vector space V_i is finite-dimensional, in which case the **dimension-vector** $\underline{\dim}(V) = (\dim(V_i))_{i\in Q_0}$. A representation is **non-zero** if there is one vertex $i \in Q_0$ such that $V_i \neq 0$.

Example 2.2.4. Consider the quiver in Example 2.2.2. The following is an example of a quiver representation.

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "1"] & \\ \mathbb{F}^2 \arrow[rr, "(1\ 1)"'] & & \mathbb{F} \end{tikzcd}$$

Definition 2.2.5. (Schiffler, 2014, Definition 1.3) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let $V = (V_i, \phi_a)$ and $V' = (V'_i, \phi'_a)$ be quiver representations over Q. A morphism $f\colon V \to V'$ of representations is a sequence $(f_i)_{i\in Q_0}$, of linear maps $f_i\colon V_i \to V'_i$, such that for every arrow $a \in Q_1$, the following diagram commutes.

$$\begin{tikzcd} V_{s(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{s(a)}"] & V_{t(a)} \arrow[d, "f_{t(a)}"] \\ V'_{s(a)} \arrow[r, "\phi'_a"] & V'_{t(a)} \end{tikzcd}$$

Let Q be a quiver. We denote by rep Q the category with objects given by finite-dimensional representation over Q, and morphisms as defined in Definition 2.2.5. The category rep Q is an abelian category with the property that for objects V, V' in rep Q, the group of morphisms $(\mathrm{rep})Q(V, V')$ is also an $\mathbb{F}$-vector space; see for example (Schiffler, 2014, Categories 3).

Definition 2.2.6. (Schiffler, 2014, Definition 1.5) Let Q be a quiver, and let $W \in$ rep Q be a non-zero representation. The representation W is called **indecomposable** if there do not exist any non-zero representations $U, V \in \mathrm{rep}Q$ such that $W \cong U \oplus V$.

Theorem 2.2.7. Krull-Remak-Schmidt Theorem (Schiffler, 2014, Theorem 1.2) Let Q be a quiver, and let $V \in \text{rep } Q$. Then there exists a positive integer n such that V can be decomposed as a direct sum,

$$V \cong V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

where each V_i is an indecomposable representation for $1 \leq i \leq n$. Moreover n is unique and this decomposition is unique up to permutation of the indecomposable modules V_i, and up to isomorphism of the indecomposable representations.

As a consequence of the above theorem, it is enough to understand the indecomposable representations of rep Q and the morphisms between them in order to understand the category rep Q. This is because every other representation of rep Q can be constructed as a direct sum of the indecomposable representations, and morphisms between general objects can be described in terms of indecomposable ones. We now recall some important classes of indecomposable representations.

Definition 2.2.8. (Schiffler, 2014, Definition 2.1) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let $i, j \in Q_0$. A **path** p from i to j of length k is a sequence

$$p = (i \mid a_1, a_2, \ldots, a_k \mid j)$$

such that $s(a_1) = i, t(a_k) = j$, and $s(a_h) = t(a_{h-1})$ for $2 \leq h \leq l$.

The **constant path** at vertex i is the path of length 0 which starts at i and ends at i, we will denote this path by $e_i = (i \mid\mid i)$. An arrow a with $s(a) = i$ and $t(a) = j$ can be considered as a path $p = (i \mid a \mid j)$ of length 1. If $i = j$, the path a is said to be a **loop**. A path $p = (i \mid a_1, a_2, \ldots, a_k \mid j)$ of length $l \geq 1$ is said to be an **oriented cycle** if $i = j$. A quiver with no oriented cycles is said to be **acyclic**.

Given paths $p = (i \mid a_1, a_2, \ldots, a_h \mid j)$ and $q = (j \mid b_1, b_2, \ldots, b_k \mid l)$ from j to l, we obtain a path $pq = (i \mid a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k \mid l)$ from i to l called, the **concatenation** of q and p.

Definition 2.2.9. (Schiffler, 2014, Definition 2.2) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let i be a vertex in Q_0. The **projective representation** at vertex i is the quiver representation

$$P_i = (P_{i_j}, \phi_a)_{j \in Q_0, a \in Q_1}$$

defined as follows. The vector space P_{i_j} has basis the set of all paths from i to j in the quiver Q. For $a \in Q_1$ with $s(a) = j$ and $t(a) = l$ considered as a path, the linear map $\phi_a \colon P_{i_j} \to P_{i_l}$ is defined on the basis of P_{i_j} as concatenation with a. That is to say, for a path p from i to j,

$$\phi_a(p) = pa.$$

Proposition 2.2.10. (Schiffler, 2014, Proposition 2.3, Proposition 2.8) Let Q be a quiver, and let i be a vertex of Q. The projective representation P_i is a projective object of the category rep Q. These are all the projective indecomposable representations of Q up to isomorphism.

The dual construction of a projective representation at a vertex i, is that of the injective representation at vertex i.

Definition 2.2.11. (Schiffler, 2014, Definition 2.2) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let i be a vertex in Q_0. The **injective representation** at vertex i is the quiver representation

$$I_i = (I_{i_j}, \phi_a)_{j \in Q_0, a \in Q_1}$$

defined as follows. The vector space I_{i_j} has basis the set of all paths from j to i in the quiver Q. For $a \in Q_1$ with $s(a) = j$ and $t(a) = l$ considered as a path, the linear map $\phi_a \colon I_{i_j} \to I_{i_j}$ is the linear map is defined on the basis of I_{i_j} by deleting the arrow a from all paths from j to i which start with a, and mapping to zero the paths from j to i which do not start with a. That is to say, for a path p from

i to j

$$\phi_a(p) = \begin{cases} p' & \text{if } p = ap'; \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Proposition 2.2.12. (Schiffler, 2014, Proposition 2.5, Proposition 2.8) Let Q be a quiver, and let i be a vertex of Q. The injective representation I_i is an injective object of the category rep Q. These are all the injective indecomposable representations up to isomorphism.

Definition 2.2.13. (Schiffler, 2014, Definition 2.2) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let i be a vertex in Q_0. The **simple representation** at vertex i is the quiver representation

$$S_i = (S_{i_j}, \phi_a)_{j \in Q_0, a \in Q_1}$$

defined as follows.

$$S_{i_j} = \begin{cases} \mathbb{F} & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

For any $a \in Q_1$, the linear map $\phi_a \colon S_{i_j} \to S_{i_l}$ is the zero map.

Proposition 2.2.14. (Schiffler, 2014, Proposition 2.8, Proposition 2.9) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let i be a vertex in Q_0. The representations S_i are simple indecomposable representations. These are all the simple indecomposable representations up to isomorphism.

2.2.2 Path Algebras

In this section, we recall a class of finite-dimensional algebras which are of interest to us. Recall that a finite-dimensional algebra is a finite-dimensional vector space which also has a compatible ring structure.

Definition 2.2.15. (Assem *et al.*, 2006, Chapter II, 1.2. Definition) Let Q be a quiver. The **path algebra** $\mathbb{F}Q$ of Q is the algebra whose underlying $\mathbb{F}$-vector space has basis all paths in Q. Multiplication on the basis elements of $\mathbb{F}Q$ is defined by concatenation of paths. In particular, for two paths $p = (i \mid a_1, a_2, \ldots, a_h \mid j)$ and $q = (j \mid b_1, b_2, \ldots, b_k \mid l)$, their product is defined to be

$$p \cdot q = \begin{cases} (i \mid a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k, \mid l) & \text{if } t(a_h) = s(b_1); \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The following is straightforward from the definition of a path algebra.

Lemma 2.2.16. (Schiffler, 2014, Lemma 4.3) Let Q be a quiver, and $\mathbb{F}Q$ its path algebra. The sum of the constant paths,

$$1_{\mathbb{F}Q} = \sum_{i \in Q_0} e_i,$$

is the identity element of $\mathbb{F}Q$.

It is also easy to observe that the path algebra of a quiver is finite-dimensional if and only the quiver is finite and acyclic. Moreover, from the definition of multiplication on the path algebra, it is easily observed that multiplication is associative. See for example (Assem *et al.*, 2006, Chapter II, 1.4. Lemma).

Let $l \geq 0$ be a positive integer, we shall denote by Q_l the set of all paths in Q of length greater than or equal to l. Furthermore, we denote by $\mathbb{F}Q_l$ the subspace of $\mathbb{F}Q$ generated by all paths of length greater than or equal to l. The path algebra $\mathbb{F}Q$ has the following natural decomposition as an $\mathbb{F}$-vector space,

$$\mathbb{F}Q = \bigoplus_{l \geq 0} \mathbb{F}Q_l.$$

In studying an algebra A over a field $\mathbb{F}$, the modules over A are of interest. So let us recall the definition of a module over an algebra, more generally, a module over a ring.

Definition 2.2.17. (Schiffler, 2014, Definition 4.8) Let R be a ring with an identity element $1_R \neq 0$. A **right R-module** M over R is an abelian group together with a binary operation $*$ called a right R-action,

$$M \times R \to M$$

$$(m, r) \mapsto m * r,$$

such that for any $m_1, m_2 \in M$ and $r_1, r_2 \in R$, we have that

1. $(m_1 + m_2) * (r_1 + r_2) = m_1 * r_1 + m_1 * r_2 + m_2 * r_1 + m_2 * r_2$,

2. $(m_1 * r_1) * r_2 = m_1 * (r_2 * r_2)$,

3. $m_1 * 1_R = m_1$.

Right R-modules over a ring R can be defined dually by having the ring act on the right. In the sequel, we will always consider right R-modules, hence we will just refer to them as R-modules or modules over R.

Definition 2.2.18. (Schiffler, 2014, Definition 4.10) Let M and N be R-modules. A **module homomorphism** from M to N is a map $f \colon M \to N$, such that for all $m, m' \in M$, and $r \in R$,

$$f(m + m') = f(m) + f(m'),$$

$$f(rm) = r * f(m).$$

Definition 2.2.19. (Schiffler, 2014, Definition 4.9) Let R be a finite-dimensional algebra and M a module over R. The module M is said to be **finitely generated** if there exists a finite set of elements $\{m_1, m_2, \ldots, m_s\} \subset M$, such that for every $m \in M$, there exists some $a_1, a_2, \ldots, a_s \in A$ such that,

$$m = a_1 * m_1 + a_2 * m_2 + \cdots + a_s * m_s.$$

Given a ring R, we define $\operatorname{mod} R$ to be the category with object all finitely generated R-modules and morphisms given by module homomorphisms. In particular, we will denote by $\operatorname{mod} \mathbb{F}Q$ the category of all finitely generated $\mathbb{F}Q$-modules.

2.2.3 Quiver representations of bound quivers

In the above exposition, we made the assumption that the quivers had no loops or oriented cycles. In the following, we will drop these assumptions.

Definition 2.2.20. (Schiffler, 2014, Definition 3.1) Let $Q = (Q_0, Q_1, s, t)$ be a quiver possibly with loops or oriented cycles. Two paths p, q in Q are said to be **parallel** if $s(p) = s(q)$ and $t(p) = t(q)$. A **relation** ρ is a linear combination $\rho = \sum_p \lambda_p p$ of parallel paths, each of which has length greater than or equal to 2, and the coefficients $\lambda_p \in \mathbb{F}$.

A **bound quiver** (Q, R) is a pair consisting of a quiver $Q = (Q_0, Q_1, s, t)$ and a set of relations R over Q. Bound quivers generalise quivers in the sense that a quiver is a bound quiver where the set of relations is empty. As with quivers, we have representations of bound quivers. Given a path $p = (i \mid a_1, a_2, \ldots, a_k \mid j)$, we set ϕ_p to be the linear map which is the following composition $\phi_p = \phi_{a_k} \ldots \phi_{a_2}\phi_{a_1}$.

Definition 2.2.21. (Schiffler, 2014, Definition 3.2) Let (Q, R) be a bound quiver. A representation of (Q, R) is a representation $V = (V_i, \phi_a)$ such that for each relation $\rho = \sum_p \lambda_p p \in R$, we have that $\phi_\rho = \sum_p \lambda_p \phi_p = 0$.

Given a bound quiver (Q, R), the category rep (Q, R) of representations of (Q, R) is similarly defined as the category rep Q of representations over Q; see for example (Schiffler, 2014, Definition 3.2). Like the category rep Q, the category rep (Q, R) is also an abelian category. The simple representations of (Q, R) are defined in the same way. The definitions of the projective and injective representations are similar. It is also true that as objects of rep (Q, R), the representations P_i, I_i, S_i are indecomposable. Moreover, the representation P_i is a projective object in rep (Q, R), and I_i is an injective object in rep (Q, R).

2.2.4 Path Algebras of bound quivers

Let Q be a quiver. If the quiver Q has oriented cycles, then the path algebra $\mathbb{F}Q$ is infinite-dimensional. However, for certain ideals of I of the path algebra $\mathbb{F}Q$, it is possible to obtain quotient algebras $\mathbb{F}Q/I$ which are finite-dimensional.

Definition 2.2.22. (Assem *et al.*, 2006, Chapter II, 1.9. Definition) Let Q be a quiver and $\mathbb{F}Q$ the associated path algebra. The **arrow ideal** R_Q of $\mathbb{F}Q$ is the two-sided ideal generated by all arrows in the quiver Q.

Considered as a vector space, the arrow ideal has the following vector space decomposition,

$$R_Q = \bigoplus_{l \geq 1} \mathbb{F}Q_l,$$

where $\mathbb{F}Q_l$ is the subspace of $\mathbb{F}Q$ with basis the set of all paths of length greater or equal to l. The n^{th} power of the arrow ideal R_Q^n can be decomposed as,

$$R_Q^n = \bigoplus_{l \geq n} \mathbb{F}Q_l.$$

Definition 2.2.23. (Schiffler, 2014, Definition 5.1) Let Q be a quiver. A two-sided ideal I of $\mathbb{F}Q$ is called an **admissible ideal** if there exists an integer $n \geq 2$ such that,

$$R_Q^n \subset I \subset R_Q^2.$$

Remark 2.2.24. (Schiffler, 2014, Remark 5.1,5.2 Definition 5.1) Every admissible ideal I is generated by a set of relations R, that is to say $I = \langle R \rangle$; see Definition 2.2.20. Hence given an admissible ideal I, we call the pair (Q, I) a **bound quiver**. Since $I = \langle R \rangle$, we can equivalently write (Q, R) for the bound quiver (Q, I).

The quotient algebra $\mathbb{F}Q/I$ is called a **bound quiver algebra**. The condition $R_Q^n \subset I$ guarantees that the algebra $\mathbb{F}Q/I$ is finite-dimensional.

Basic Connected Finite-Dimensional Algebras

Definition 2.2.25. (Assem *et al.*, 2006, I.4 Direct sum decompositions) Let A be an $\mathbb{F}$-algebra. An element $e \in A$ is **idempotent** if $e^2 = e$. Idempotent elements e_1, e_2 in A are **orthogonal** if $e_1 e_2 = e_2 e_1 = 0$. An idempotent element e is **primitive** if there does not exists two non-zero idempotent elements e_1, e_2 such that $e = e_1 + e_2$. We also call an idempotent element e **central** if $ea = ae$ for all $a \in A$.

For an $\mathbb{F}$-algebra A, the elements 0 and 1 are always idempotents. Moreover, for any idempotent element e, the element $1-e$ is also idempotent. Together the idempotents e and $1-e$, give an A-module decomposition of $A = Ae \oplus A(1-e)$.

Example 2.2.26. Let Q be a quiver and $\mathbb{F}Q$ its path algebra. For each $i \in Q_0$, the constant paths e_i are idempotent.

Proposition 2.2.27. (Assem *et al.*, 2006, I.4 Direct sum decompositions) Let A be a finite-dimensional algebra. There exists a set of primitive pairwise orthogonal idempotent elements $\{e_1, e_2, \ldots, e_n\}$ such that A admits the following A-module decomposition,

$$A = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_n.$$

Moreover, the modules e_iA are indecomposable modules for $1 \leq i \leq n$ and $1 = e_1 + e_2 + \cdots + e_n$. Conversely every set of idempotents $\{e_1, e_2, \ldots, e_n\}$ such that $1 = e_1 + e_2 + \cdots + e_n$ and the modules e_iA are indecomposable, induce an A-module decomposition of A. The set of of primitive pairwise orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$ is called a **complete set of primitive orthogonal idempotents** of A.

Definition 2.2.28. (Assem *et al.*, 2006, Chapter I, 6.1. Definition) Let A be an $\mathbb{F}$-algebra with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$. The algebra A is **basic** if $Ae_i \cong Ae_j$ if and only if $i = j$.

An A-module M is a **basic module** if each summand has no isomorphic copies in N and if each summand has multiplicity of one in the direct sum composition of M as an A-module.

Definition 2.2.29. (Assem *et al.*, 2006, I.4 Direct sum decompositions) An $\mathbb{F}$-algebra A is **connected** if A is not a direct product of two $\mathbb{F}$-algebras.

Theorem 2.2.30. (Assem *et al.*, 2006, Chapter II, 3.7. Theorem) Let A be a basic connected finite-dimensional $\mathbb{F}$-algebra. There exists a quiver Q and an admissible ideal I of $\mathbb{F}Q$ such that $A \cong \mathbb{F}Q/I$.

Example 2.2.31. (Assem *et al.*, 2006, Chapter V, 3.2. Theorem, 3.8. Proposition) **Nakayama Algebras** In this thesis, we will refer to the following class of algebras as Nakayama algebras. For a positive integer $n \geq 1$, let $\overrightarrow{\mathbb{A}_n}$ denote the linearly oriented type $\mathbb{A}$ Dynkin quiver with n vertices,

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n.$$

Figure 2.1: The linearly oriented type $\mathbb{A}$ quiver with n vertices.

Let $\overrightarrow{\mathbb{C}_n}$ be the linearly oriented n-cycle.

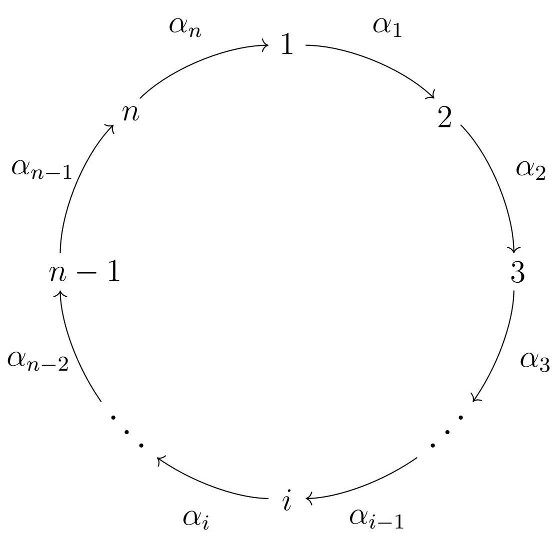


Figure 2.2: The linearly oriented n-cycle.

We denote by Γ_n^t the algebra $\mathbb{F}A_n/R_Q^t$ and by Λ_n^t the algebra $\mathbb{F}C_n/R_Q^t$ where $2 \leq t \leq n$. Throughout the text, we will write P_i for the indecomposable projective module at vertex i of the underlying quiver of the algebra A in question. Likewise we will write S_i for the simple A-module at vertex i.

Theorem 2.2.32. (Schiffler, 2014, Theorem 5.4) Let Q be a quiver, and let $I = \langle R \rangle$ be an admissible ideal generated by a set of relations R. Then there is an equivalence of categories,

$$\operatorname{mod} \mathbb{F}Q/I \simeq \operatorname{rep}(Q, I).$$

In particular, if Q is an acyclic quiver with no loops, we have that

$$\operatorname{mod} \mathbb{F}Q \simeq \operatorname{rep}Q.$$

By the above theorem, the representations of (Q, I) correspond to the modules of the bound algebra $\mathbb{F}Q/I$, so in the sequel, we will not make a distinction between representations and modules unless its necessary.

Theorem 2.2.33. Krull-Remark-Schmidt Theorem(Assem *et al.*, 2006, Chapter I, 4.10. Unique decomposition theorem) Let A be a finite-dimensional algebra over a field $\mathbb{F}$, and let M in $\operatorname{mod} A$ be a non-zero module. Then there exists a positive integer n such that M can be decomposed as a direct sum,

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n,$$

where each M_i is an indecomposable representation for $1 \leq i \leq n$. Moreover n is unique and this decomposition is unique up to permutation of the indecomposable modules M_i and up to isomorphism of the indecomposable modules M_i.

As it was with the category rep Q, we can observe from the above theorem that in order to understand the category $\operatorname{mod} A$, it is sufficient to understand the indecomposable modules and the homomorphisms between them.

2.2.5 Auslander-Reiten Theory

Let A be a basic connected finite-dimensional algebra, and $\operatorname{mod} A$ be its category of modules. By Theorem 2.2.33, it is enough to understand the indecomposable modules of $\operatorname{mod} A$ in order to understand all of the modules of $\operatorname{mod} A$. An analogous notion for the morphisms of $\operatorname{mod} A$ is that of **irreducible morphisms**. Irreducible morphisms may be thought of as the morphisms from which all morphisms of $\operatorname{mod} A$ may be constructed. An essential tool in the calculation of indecomposable modules and irreducible morphisms is **Auslander-Reiten theory**. The information of indecomposable modules and irreducible morphisms can capture by an **Auslander-Reiten** quiver. The Auslander-Reiten quiver of $\operatorname{mod} A$ has vertices given by the indecomposable modules in $\operatorname{mod} A$, and arrows given by the irreducible morphisms in $\operatorname{mod} A$. If $\operatorname{mod} A$ has finitely many indecomposable modules, the Auslander-Reiten quiver is a complete picture of $\operatorname{mod} A$; otherwise it is a good first approximation. The Auslander-Reiten quiver has the additional structure of **almost split sequence**; more will be said about this later. This subsection does not represent an extensive exposition on Auslander-Reiten theory, but simply what is required for the thesis. For a more complete exposition, the reader is referred to our main references for this subsection, Schiffler (2014) and Assem *et al.* (2006). Going forward, A will always be a finite-dimensional algebra.

Definition 2.2.34. (Schiffler, 2014, Definition 7.3) A morphism $f \colon M \to N$ in $\operatorname{mod} A$ is **irreducible** if the following is true:

- f is not a section,
- f is not a retraction,
- if $f = gh$, where $g \colon J \to N$ and $h \colon M \to J$ are composable morphisms, either g is a retraction or h is a section.

Definition 2.2.35. (Schiffler, 2014, Definition 7.1) A morphism $f \colon M \to N$ is **left minimal almost split** provided the following holds:

- if $h \colon N \to N$ is such that $hf = f$, then h an automorphism,
- f is not a section, and there doesn't a exist a morphism $g \colon N \to M$ such that $gf = 1_M$,
- and for each morphism $u \colon M \to U$ which is not section, there exists a morphism $u' \colon N \to U$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "u"] & N \arrow[dl, "u'"] \\ U & \end{tikzcd}$$

The dual notion to that of a left minimal almost split morphism is that of a **right minimal almost split morphism**.

Definition 2.2.36. (Schiffler, 2014, Definition 7.2) A short exact sequence in $\operatorname{mod} A$

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

is an **an almost split sequence** (or alternatively an **Auslander-Reiten sequence**) if f is a left minimal almost split morphism, and g is a right minimal almost split morphism.

Proposition 2.2.37. (Assem *et al.*, 2006, Chapter IV, 1.13. Theorem) A short exact sequence in $\operatorname{mod} A$

$$\mathbf{0} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathbf{0}$$

is an almost split sequence if and only if A and C are indecomposable modules, and f and g are irreducible morphisms.

Auslander-Reiten Translation

In the following, we recall a functor τ called the **Auslander-Reiten translation**. With it, we will show the existence of almost split sequences and construct Auslander-Reiten quivers. To construct τ, we first need to construct other functors starting with the **standard $\mathbb{F}$-duality**,

$$D = \operatorname{Hom}_{\mathbb{F}}(-, \mathbb{F})\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$$

from the category of right A-modules to the category of right A-modules. For a module $M \in \operatorname{mod} A$,

$$D(M) = \operatorname{Hom}_{\mathbb{F}}(M, \mathbb{F})$$

is the space of a linear maps from M to $\mathbb{F}$. For an A-module homomorphism $h\colon M \to N$,

$$D(h) = \operatorname{Hom}_{\mathbb{F}}(h, \mathbb{F})\colon D(N) \to D(M)$$

whereby

$$D(h)(g) = g \circ h.$$

$D(M)$ is a right A-module by the following A-action. For $a \in A$, $f \in \operatorname{Hom}_{\mathbb{F}}(M, \mathbb{F})$ and $m \in M$,

$$(f * a)(m) = f(am).$$

As the name suggest, the standard duality is a duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$. We abuse notation by also denoting the quasi-inverse of D by

$$D = \operatorname{Hom}_{\mathbb{F}}(-, \mathbb{F})\colon \operatorname{mod} A^{op} \to \operatorname{mod} A.$$

The quasi-inverse is defined dually from right A-modules to right A-modules; see (Assem *et al.*, 2006, Chapter I, 2.9. Standard dualities) for more details.

We will consider the A-dual functor

$$(-)^t = \operatorname{Hom}_A(-, A)\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$$

which admits a similar definition to the dual functor D. We will again abuse notation and denote by

$$(-)^t = \operatorname{Hom}_A(-, A)\colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} A.$$

The functor $(-)^t$ is generally not a duality, however it does induce a duality between the full subcategory of projective right A-modules proj A, and the subcategory of projective right A-modules proj A^{op}. The **Nakayama functor** $\nu = D(-)^t\colon \operatorname{mod} \to \operatorname{mod} A^{\mathrm{op}}$. Before we can define the Auslander-Reiten translate, we need to define the notion of a projective cover.

Definition 2.2.38. (Schiffler, 2014, Definition 2.4) Let M be an A-module. A **projective cover** of M is a projective A-module P together with a surjective A-module homomorphism $g\colon P \to M$, such that for any other surjective A-module homomorphism $g'\colon P' \to M$ with P' projective, then there exists a surjective A-module homomorphism such that $gh = g'$.

Injective covers are defined dually.

Definition 2.2.39. (Schiffler, 2014, Definition 2.5) A projective resolution in $\operatorname{mod} A$

$$\cdots \longrightarrow P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow \mathbf{0},$$

is **minimal** if f_0 is a projective cover, and each $f_i\colon P_i \to \ker(f_{i-1})$ is a projective cover for every $i > 0$.

Minimal projective resolutions are defined dually.

Proposition 2.2.40. (Assem *et al.*, 2006, Chapter I, 5.8. Theorem) Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal components of A. Any A-module M has a projective cover $g\colon P(M) \to M$ where

$$P(M) \cong (Ae_1)^{s_1} \oplus \cdots \oplus (Ae_n)^{s_n}$$

for some positive integers s_i for $1 \leq i \leq n$.

Definition 2.2.41. A **projective presentation** of a module $M \in \operatorname{mod} A$ is a projective resolution

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow \mathbf{0}.$$

If this is minimal, we call it a **minimal projective presentation**.

We are now ready to construct the Auslander-Reiten translation, we will follow the construction as in the section (Schiffler, 2014, 7.2 Auslander-Reiten Translation). Let M be an A-module and let

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow \mathbf{0}$$

be its projective presentation. Apply the Nakayama functor ν to the projective presentation, we may obtain the following exact sequence,

$$\mathbf{0} \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu f_1} \nu P_0 \xrightarrow{\nu f_0} \nu M \longrightarrow \mathbf{0},$$

where $\tau M := \ker(\nu f_1)$. Dually, let

$$\mathbf{0} \longrightarrow I_0 \xrightarrow{g_0} I_1 \xrightarrow{g_1} I_2$$

be an injective presentation of M. By applying the inverse Nakayama functor ν^{-1}, we may obtain the following exact sequence,

$$\mathbf{0} \longrightarrow \nu^{-1} I_0 \xrightarrow{\nu^{-1} g_0} \nu^{-1} I_1 \xrightarrow{\nu^{-1} g_1} \tau^{-1} M \longrightarrow \mathbf{0},$$

where $\tau^{-1}M := \operatorname{coker}(\nu^{-1} f_1)$. The module τM is called the **Auslander-Reiten translate** of M, and $\tau^{-1}M$ the **inverse Auslander-Reiten translate** of M. From this point forward, we will always donate by τ the **Auslander-Reiten translation** and τ^{-1} the **inverse Auslander-Reiten translation**. The Auslander-Reiten translation has the following properties when its applied to indecomposable modules.

Proposition 2.2.42. (Assem *et al.*, 2006, Chapter IV, 2.10 Proposition) Let M be an indecomposable module in $\operatorname{mod} A$.

- The module M is projective if and only if $\tau M = \mathbf{0}$.
- The module M is injective if and only if $\tau^{-1} M = \mathbf{0}$.
- If M is a non-projective module, then τM is an indecomposable non-injective module; moreover $\tau^{-1}\tau M = M$.
- If M is a non-injective module, then $\tau^{-1} M$ is an indecomposable non-projective module; moreover $\tau\tau^{-1} M = M$.

- An non-projective A-module N is isomorphic to a non-projective A-module M if and only if $\tau N \cong \tau M$.
- An non-injective A-module N is isomorphic to a non-injective A-module M if and only if $\tau^{-1} N \cong \tau^{-1} M$.

Definition 2.2.43. (Schiffler, 2014, Definition 7.9) Let M and N be A-modules, and let $P(M,N)$ to be the set of all A-module homomorphisms from M to N which factor through a projective A-module, and set

$$\underline{\mathrm{Hom}}(M,N) := \mathrm{Hom}(M,N)/P(M,N).$$

Dually, let $I(M,N)$ be the set of all A-module homomorphisms from M to N which factor through an injective A-module. Set

$$\overline{\mathrm{Hom}}(M,N) := \mathrm{Hom}(M,N)/I(M,N).$$

Theorem 2.2.44. (Assem *et al.*, 2006, Chapter IV, 2.13. Theorem)

The Auslander-Reiten formulas Let M and N be A-modules, there exists the following isomorphisms

$$\mathrm{Ext}^1(M,N) \cong D\underline{\mathrm{Hom}}(\tau^{-1}N, M) \cong D\overline{\mathrm{Hom}}(N, \tau M),$$

these isomorphisms are are functorial in both arguments, which is to say that

$$\mathrm{Ext}^1(-,N) \cong D\underline{\mathrm{Hom}}(\tau^{-1}N, -) \cong D\overline{\mathrm{Hom}}(N, \tau -)$$

and

$$\mathrm{Ext}^1(M,-) \cong D\underline{\mathrm{Hom}}(\tau^{-1}-, M) \cong D\overline{\mathrm{Hom}}(-, \tau M).$$

Theorem 2.2.45. (Assem *et al.*, 2006, Chapter IV, 3.1. Theorem) Let M be an indecomposable A-module

- If M is non-projective A-module, then there is the following almost split sequence in $\operatorname{mod} A$

$$\mathbf{0} \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow \mathbf{0}.$$

- If M is a non-injective A-module, then there is the following almost split sequence in $\operatorname{mod} A$

$$\mathbf{0} \longrightarrow M \longrightarrow F \longrightarrow \tau^{-1} M \longrightarrow \mathbf{0}.$$

Example 2.2.46. Let A be the algebra Γ_3^2 given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3,$$

subject to the relation $\alpha\beta = 0$. The Auslander-Reiten quiver of $\operatorname{mod}\Gamma_3^2$ is as follows,

$$\begin{tikzcd}
 & P_2 \arrow[dr] & & P_1 \arrow[dr] & \\
S_3 \arrow[ur] & & S_2 \arrow[ll, dashed, no head] \arrow[ur] & & S_1 \arrow[ll, dashed, no head]
\end{tikzcd}.$$

The Auslander-Reiten translates are indicated by the dashed lines, specifically $\tau S_1 = S_2$ and $\tau S_2 = S_3$. The other modules are projective, hence each of their Auslander-Reiten translates is the zero module.

2.3 Mutation: Cluster algebras, Tilting theory, τ-tilting theory, and exceptional sequences.

2.3.1 Cluster algebras

Cluster algebras are a class of commutative algebras defined combinatorially by a process of iterated mutation. They were first introduced in 2001 by S. Fomin and A. Zelevinsky in a series of seminal papers Fomin & Zelevinsky (2002), Fomin & Zelevinsky (2003), Berenstein *et al.* (2005), Fomin & Zelevinsky (2007) as an approach towards problems on total positivity Fomin (2010) and canonical bases in quantum groups. Since their inception, cluster algebras have become an object of study in their own right. They find uses in many other areas including representation theory Leclerc (2010), Poisson geometry Gekhtman *et al.* (2010) and integrable systems Williams (2014). Of particular interest to us in this thesis are the cluster algebras of type $\mathbb{A}$, and their combinatorics. Before we recall this class of cluster algebras, we first need to define cluster algebras of geometric type; in this text we will simply refer to them as cluster algebras.

Let n and m be natural numbers such that $n \leq m$. We set our ambient field to be $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_m)$, the field of rational functions over $\mathbb{Q}$ in m indeterminates. We shall also denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

Definition 2.3.1. (Fomin *et al.*, 2020, Definition 3.1.1) A **labelled seed** of geometric type in $\mathbb{F}$ is a pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ where:

- $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ is an m-tuple of elements forming a free generating set of $\mathbb{F}$ over $\mathbb{Q}$. We say that $\tilde{\mathbf{x}}$ is the extended cluster and $\mathbf{x} = (x_1, \ldots, x_n)$ is the cluster of the seed. The x_i where $i \in [n]$ are **cluster variables** and the x_i with $n + 1 \leq i \leq m$ are **frozen variables**.

- The **extended exchange matrix** $\tilde{\mathbf{B}}$ is an integer matrix $m \times n$ matrix where the top $n \times n$ matrix is a skew-symmetrizable matrix called the **exchange matrix**.

Definition 2.3.2. (Fomin *et al.*, 2020, Lemma 2.7.3, Definition 3.1.2) Let $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ be a labelled seed. Fix some $k \in [n]$. Then the **seed mutation** μ_k in direction k transforms $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ into a new labelled seed $\mu_k(\tilde{\mathbf{x}}, \tilde{\mathbf{B}}) = (\tilde{\mathbf{x}}', \tilde{\mathbf{B}}')$ which is defined in the following way.

- Given $\tilde{\mathbf{B}} = (b_{ij})$, then $\tilde{\mathbf{B}}' = (b'_{ij})$ where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0; \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0; \\ b_{ij} & \text{otherwise} \end{cases}$$

- The extended cluster $\tilde{\mathbf{x}}' = (x'_1, \ldots, x'_m)$ is defined as follows, $x'_j = x_j$ for $j \neq k$ and $x'_k \in \mathbb{F}$ satisfies the following relation called the **exchange relation**:

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

We say that two seeds $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ and $(\tilde{\mathbf{x}}', \tilde{\mathbf{B}}')$, are **mutationally equivalent** if we can obtain one seed from the other through a finite sequence of mutations.

Definition 2.3.3. (Fomin *et al.*, 2020, Definition 3.1.6) Let $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ be a labelled seed and S be the set of all the labelled seeds mutationally equivalent to $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$. The **cluster algebra** $A(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ with initial seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$, is the subring of $\mathbb{F}$ generated by all the cluster variables contained in the labelled seeds contained in S.

It is shown in (Fomin & Zelevinsky, 2003, Theorem 1.13) that the frozen variables do not play a significant role in the cluster algebras. So in the discussions to follow, we won't deal with frozen variable explicitly. In other words we shall assume $n = m$, unless stated otherwise.

It is often the case that two distinct labelled seeds $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ and $(\tilde{\mathbf{y}}, \tilde{\mathbf{C}})$ of a cluster algebra have the same cluster variables and exchange matrices up to relabelling of the cluster variables. Since such a pair of labelled seeds hold the same information, they are equivalent to each other (see Definition 2.3.4). So from the point of view of the cluster algebra it is enough to just consider the equivalence classes of the labelled seeds. The equivalence classes of labelled seeds will be called **unlabelled seeds**. Unlabelled seeds can naively be thought of as the result of replacing the

m-tuples $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ in Definition 2.3.1 with sets $\{x_1, \ldots, x_m\}$, in the sense that two seeds are no longer distinct if they have the same cluster variables but different labelling.

Definition 2.3.4. (Fomin & Zelevinsky, 2002, §7, Definition 7.1) Two labelled seeds $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ and $(\tilde{\mathbf{y}}, \tilde{\mathbf{C}})$ are equivalent if there exists a permutation σ of $[1, m]$ such that:

(i) $\sigma([1, n]) = [1, n]$,

(ii) $\sigma(i) = i$ for all $n + 1 \leq i \leq m$,

(iii) $y_i = x_{\sigma(i)}$,

(iv) $\tilde{\mathbf{C}}_{i,j} = \tilde{\mathbf{B}}_{\sigma(i),\sigma(j)}$.

An **unlabelled seed** $[(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})]$ is the equivalence class of the labelled seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$.

In the sequel, we only consider unlabelled seeds instead of labelled seeds. We will abuse notation and write $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ to denote the unlabelled seed of $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$.

Cluster Algebras via Quivers

If the exchange matrix $\tilde{\mathbf{B}}$ is skew-symmetric, the combinatorial data of a cluster algebras can be presented in the form of a quiver $Q(\tilde{\mathbf{B}})$ defined in the following way. Given $\tilde{\mathbf{B}}$, a skew-symmetric $n \times n$ integer matrix, we define the quiver $Q(\tilde{\mathbf{B}})$ to have n vertices labelled 1 to n corresponding to rows and columns of the matrix. If $b_{ij} > 0$, we draw b_{ij} arrows from i to j. If $b_{ij} < 0$, we draw b_{ij} arrows from j to i. This gives us a unlabelled seed in terms of the quiver $(\tilde{\mathbf{x}}, Q(\tilde{\mathbf{B}}))$ and we can then define mutation in terms of the quiver; see (Fomin *et al.*, 2020, 2.7 Matrix Mutation) for more details on this construction.

Definition 2.3.5. (Fomin *et al.*, 2020, Definition 2.1.2) Let $(\tilde{\mathbf{x}}, Q(\tilde{\mathbf{B}}))$ be a unlabelled seed. Fix some $k \in [n]$. Then the seed mutation μ_k in direction k transforms $(\tilde{\mathbf{x}}, Q(\tilde{\mathbf{B}}))$ into a new unlabelled seed $\mu_k(\tilde{\mathbf{x}}, Q(\tilde{\mathbf{B}})) = (\tilde{\mathbf{x}}', Q(\tilde{\mathbf{B}}'))$ which is defined in the following way.

- The mutation quiver $\mu_k(Q(\tilde{\mathbf{B}}))$ is obtained from the quiver $Q(\tilde{\mathbf{B}})$ as follows.

  a) For all paths $i \to k \to j$ add an arrow from i to j with multiplicity for each path.

  b) Cancel a maximal collection of 2-cycles.

  c) Reverse all arrows incident to k.

- The exchange relation is given by

$$x_k x'_k = \prod_{i \to k} x_i + \prod_{k \to j} x_j.$$

As above we can define a corresponding cluster algebra.

The following theorem by Fomin and Zelevinsky establishes a connection between combinatorics of Lie theory and cluster algebras.

Theorem 2.3.6. (Fomin & Zelevinsky, 2002, Theorem 3.1) (Fomin & Zelevinsky, 2003, Theorem 1.4) Let Q be a finite connected quiver on n vertices labelled by $[n]$, with no loops or 2-cycles. Then for the associated cluster algebra A_Q, we have the following:

1. All cluster variables are Laurent polynomials in the initial cluster variables $x_1, \ldots, x_n$.
2. A cluster algebra has finitely many cluster variables if and only if Q is mutationally equivalent to a quiver whose underlying graph is simply laced ADE Dynkin diagram (See Figure 2.3).

We have seen that there is a connection between cluster algebras with a seed where the exchange matrix is quiver. This connection in fact extends to quiver representations, where the simply-laced Dynkin diagrams play a key role.

Definition 2.3.7. (Schiffler, 2014, §3.2.1) A quiver Q is of **finite representation type** if the number of isomorphism classes of indecomposable representations in the category rep Q is finite.

Theorem 2.3.8. Gabriel's Theorem(Assem *et al.*, 2006, VII.5, 5.10. Theorem) Let Q be a finite, connected and acyclic quiver. The category rep Q is of finite type if and only if the underlying graph of Q is a simply laced Dynkin Diagram.

We have purposely omitted further details about this connection as they are not strictly relevant to the rest of this thesis. These details may be found in (Assem *et al.*, 2006, VII.5 Reflection functors and Gabriel's theorem) and Fomin & Zelevinsky (2003) for example.

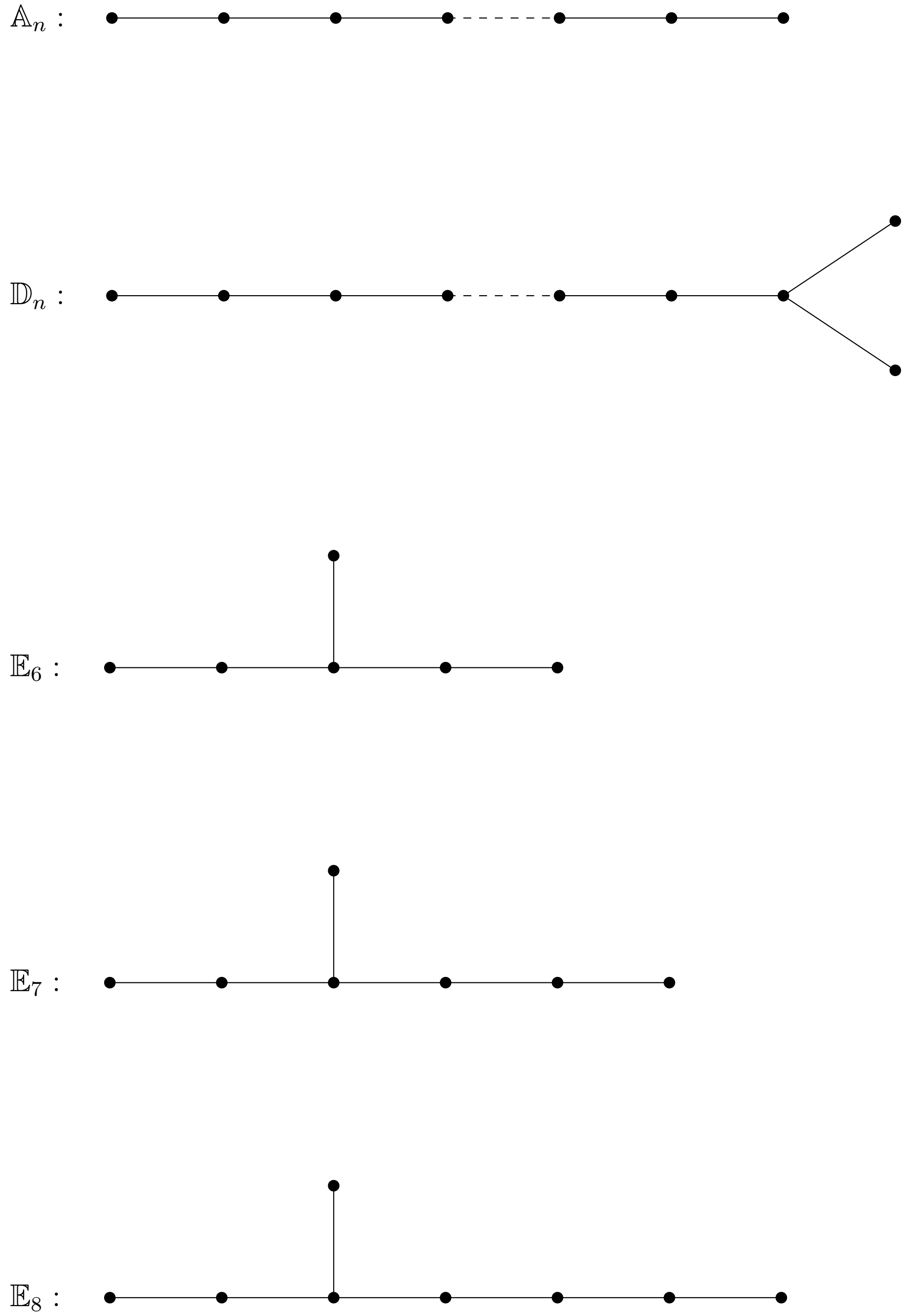


Figure 2.3: The simply laced Dynkin diagrams.

Cluster Algebras of Dynkin Type $\mathbb{A}$

The class of cluster algebras which we are interested in is the class of cluster algebras of type $\mathbb{A}$. This class of cluster algebras has an interesting combinatorial description in terms of triangulations on polygons.

Definition 2.3.9. (Fomin *et al.*, 2008, Definition 2.6) Let $\mathbf{P}_{n+3}$ be a convex $(n+3)$-gon with $n \geq 0$. A *diagonal* in $\mathbf{P}_{n+3}$ is any non-boundary edge connecting two vertices of $\mathbf{P}_{n+3}$. We then say that a triangulation T is a maximal collection of non-crossing diagonals in the polygon $\mathbf{P}_{n+3}$. We shall denote the set of triangulations of $\mathbf{P}_{n+3}$ by $\mathbf{\Delta}_{n+3}$.

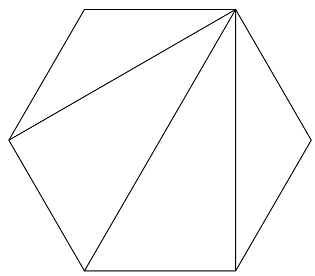


Figure 2.4: A triangulation of $\mathbf{P}_6$.

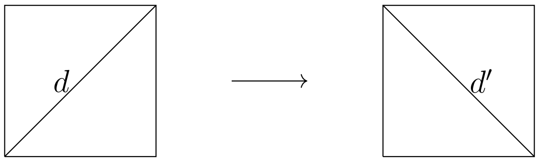
We now describe how to obtain a unlabelled seed given a triangulated polygon. Let T be a triangulation of $\mathbf{P}_{n+3}$. Label the diagonals of $\mathbf{P}_{n+3}$ by $1, 2, \dots, n$. We define the exchange matrix $\mathbf{B}(\tilde{\mathbf{T}}) = (b_{ij})$ obtained from T as follows:

$$b_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ label two sides of a triangle in } T, \text{ with } j \text{ following } i \text{ clockwise;} \\ -1 & \text{if } i \text{ and } j \text{ label two sides of a triangle in } T, \text{ with } i \text{ following } j \text{ clockwise;} \\ 0 & \text{if } i \text{ and } j \text{ do not belong to the same triangle in } T. \end{cases}$$

Let x_i be a variable corresponding to a diagonal i of $\mathbf{P}_{n+3}$. The resulting unlabelled seed is $(\tilde{\mathbf{x}} = \{x_1, x_2, \dots, x_n\}, \mathbf{B}(\tilde{\mathbf{T}}))$; see (Fomin *et al.*, 2008, Definition 4.1) for a more detailed exposition.

Definition 2.3.10. (Fomin *et al.*, 2008, Definition 3.5) Let T be a triangulation of $\mathbf{P}_{n+3}$, we can **flip** a diagonal d in T to obtain a new triangulation of $\mathbf{P}_{n+3}$

which we denote by T'. A flip in T at d is the move where we remove the diagonal labelled d in T and replace it with the unique diagonal d', the other diagonal of the quadrilateral housing d.



Remark 2.3.11. Note that the lengths of the diagonals d and d' satisfy Ptolemy's relation on the quadrilateral containing d and d'. Ptolemy's relation is precisely the exchange relation.

Theorem 2.3.12. (Fomin *et al.*, 2017, Corollary 5.3.6) Cluster variables the unlabelled seeds of a type $\mathbb{A}_n$ cluster algebra can be labelled by the diagonals of a convex $(n+3)$-gon $\mathbf{P}_{n+3}$ so that

- unlabelled seeds correspond to triangulations of the polygon $\mathbf{P}_{n+3}$
- mutations corresponds to flips, and
- exchange matrices are given by $\mathbf{B}(\tilde{\mathbf{T}})$ as in Definition 2.3.9.

Let n be a positive integer. The n^{th} Catalan number C_n is given by the closed formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is well known that number of triangulations of a convex $(n+3)$-gon is given by the Catalan number C_{n+1}, see for example Stanley (2015) for more on Catalan numbers. So it follows from the above theorem that the number of unlabelled seeds in a cluster algebra of type $\mathbb{A}_n$ is given by C_{n+1}.

2.3.2 Tilting theory

Tilting theory is a very important tool in the representation theory of finite-dimensional algebras, it originates from the study of **reflection functors** and coxeter functors; these functors play a crucial role in the proof of Gabriel's theorem (Theorem 2.3.8); see Bernstein *et al.* (1973) and Auslander *et al.* (1979). The fundamental objects of study in tilting theory are **tilting modules** and **torsion pairs**. Tilting modules were first axiomatised by Brenner and Butler in Brenner & Butler (1980). Later on Happel and Ringel provided an alternative and equivalent axiomatisation in Happel & Ringel (1982), which is the one that is more generally used in the literature. One of the central ideas in tilting theory is that the representation theory of certain finite-dimensional algebras A, is more easily studied by replacing the algebra A with a less complicated and related algebra B. In particular it turns out that for an algebra A and a tilting module T_A, the representation theory algebra of $B = \mathrm{End}(T_A)$ is closely related to that of A. However, in general the module category $\mathrm{mod}\, A$ is not equivalent to $\mathrm{mod}\, B$, but their derived categories are, in other words $D^b(\mathrm{mod}\, A) \simeq D^b(\mathrm{mod}\, B)$ (Happel, 1987, 1.7 Corollary). Tilting theory also provides us with a method for constructing equivalences between certain pairs of subcategories called torsion pairs of $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$, these torsion pairs provide an almost complete description of $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$; more on this later.

Through out this subsection, A will be a finite-dimensional algebra over a field $\mathbb{F}$.

Definition 2.3.13. (Happel, 1988, 4.1) A **torsion pair** (or a **torsion theory**) is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathrm{mod}\, A$ satisfying the following conditions:

(a) $\mathrm{Hom}_A(M, N) = \{0\}$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$.

(b) If $\mathrm{Hom}_A(M, N) = \{0\}$ for all $N \in \mathcal{F}$, then $M \in \mathcal{T}$.

(c) If $\mathrm{Hom}_A(M, N) = \{0\}$ for all $M \in \mathcal{T}$, then $N \in \mathcal{F}$.

In other words, a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathrm{mod}\, A$ are a torsion pair if there are no non-zero module homomorphisms from $\mathcal{T}$ to $\mathcal{F}$, and the two subcategories are maximal with respect to this property. The subcategory $\mathcal{T}$ is called the **torsion class** and the subcategory $\mathcal{F}$ is called the **torsion-free class**.

Definition 2.3.14. (Assem *et al.*, 2006, Chapter VI, 1.3. Definition) A **subfunctor** of the identify functor $\mathbf{1}\colon \operatorname{mod} A \to \operatorname{mod} A$ is a functor $t\colon \operatorname{mod} A \to \operatorname{mod} A$ such that for $M \in \operatorname{mod} A$, the A-module $t(M)$ is a submodule of M, and each A-module homomorphisms $f\colon M \to N$, restricts to an A-module homomorphism $t(f)\colon t(M) \to t(N)$. An **idempotent radical** is a subfunctor t of $mathbf1$ with the property that $t(t(M)) = t(M)$ and $t(M/t(M)) = \mathbf{0}$ for every A-module M.

Proposition 2.3.15. (Assem *et al.*, 2006, Chapter VI, 1.4. Proposition) Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod} A$. Then the following statements hold:

- The torsion class $\mathcal{T}$ is closed under images, direct sums, and extensions.
- The torsion-free class $\mathcal{F}$ is closed under submodules, direct sums, and extensions.
- There exists an idempotent radical functor t such that $\mathcal{T} = \{M \mid t(M) = M\}$ and $\mathcal{F} = \{N \mid t(N) = \mathbf{0}\}$.

The idempotent radical functor t associated to the torsion pair $(\mathcal{T}, \mathcal{F})$ is called the **torsion radical**. It induces short exact sequences of the following form in $\operatorname{mod} A$.

Proposition 2.3.16. (Assem *et al.*, 2006, Chapter VI, 1.5. Proposition) Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod} A$ and $t\colon \operatorname{mod} A \to \operatorname{mod} A$ the associated idempotent radical functor. For each A-module M, there exists a short exact sequence

$$\mathbf{0} \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow \mathbf{0}$$

with $t(M) \in \mathcal{T}$ and $M/t(M) \in \mathcal{F}$. This short exact sequence is unique up to the isomorphism of short exact sequences.

Remark 2.3.17. Consequently, torsion pairs provide us with an alternative way of viewing the category $\operatorname{mod} A$. In particular we can associate to the objects M of $\operatorname{mod} A$ the short exact sequences

$$\mathbf{0} \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow \mathbf{0}.$$

In instances where the objects of $\operatorname{mod} A$ can be completely described in terms of a torsion pair, in the sense that the above sequences split, (in other words each A-module $M \cong t(M) \oplus M/t(M)$), the torsion pair $(\mathcal{T}, \mathcal{F})$ is said to be **splitting**. If a torsion pair $(\mathcal{T}, \mathcal{F})$ is splitting, then each indecomposable module of $\operatorname{mod} A$ either lies in $\mathcal{T}$ or $\mathcal{F}$. So in this sense, a splitting torsion pair holds all information about the module category $\operatorname{mod} A$; see for example (Assem *et al.*, 2006, Chapter VI, 1.7. Proposition) and (Happel, 1988, §4.1) for more on splitting torsion pairs.

Definition 2.3.18. (Schiffler, 2014, Definition 5.5) Let M be an A-module. The **projective dimension** of M, denote $\operatorname{pd} M$, is the smallest positive integer d such that there exists a projective resolution of the following form.

$$\mathbf{0} \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow \mathbf{0}$$

If such a projective resolution does not exist, then we say that M has an infinite projective dimension and we write $\operatorname{pd} M = \infty$. The **injective dimension** of an A-module is defined dually. The **global dimension** $\operatorname{gldim} A$ of the algebra A is defined to be $\operatorname{gldim} A := \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} A\}$.

Definition 2.3.19. (Schiffler, 2014, Definition 5.2) An algebra A is **hereditary** if the submodule of a projective module is also projective.

Proposition 2.3.20. (Assem *et al.*, 2006, Chapter VII.1, 1.4. Theorem) An algebra A is hereditary if and only if $\operatorname{gldim} A \leq 1$.

Definition 2.3.21. (Geiß *et al.*, 2006, See for example §1 Introduction, 1.4) An A-module M is **rigid** if $\operatorname{Ext}^1(M, M) = 0$.

Definition 2.3.22. (Assem *et al.*, 2006, Chapter VI, 2.1. Definition) An A-module M is **partial tilting** if it satisfies the following:

(i) pd $M \leq 1$,

(ii) $\mathrm{Ext}^1_A(M, M) = \{0\}$.

Moreover, if there exists a short exact sequence in mod A

$$\mathbf{0} \longrightarrow A \longrightarrow M' \longrightarrow M'' \longrightarrow \mathbf{0}$$

with M' and M'' in add M, then M is called a **tilting module**. Where add M is the subcategory of mod A generated by the direct summands of M.

One reason why partial tilting modules are of interest is that they are a great source of torsion pairs.

Lemma 2.3.23. (Happel, 1988, See 4.2 Lemma) Let M be a partial tilting module and let

$$\mathcal{T}(M) := \{X \in \mathrm{mod}\, A \mid \text{there exists an integer } d \geq 0, \text{and an epimorphism } M^d \to X\}$$

and

$$\mathcal{F}(M) := \{X \in \mathrm{mod}\, A \mid \mathrm{Hom}(M, X) = 0\}.$$

The pair $(\mathcal{T}(M), \mathcal{F}(M))$ is a torsion pair in mod A.

The subcategory $\mathcal{T}(M)$ is the subcategory of mod A generated by M, and it sometimes denoted by Gen$M$; see for example (Assem *et al.*, 2006, Chapter VI, page 188). If M is a tilting module, then Gen$M$ has the following description.

Corollary 2.3.24. (Happel, 1988, 4.3 Lemma)(Assem *et al.*, 2006, Chapter VI, 2.5. Theorem) Let A be finite-dimensional algebra of finite global dimension.. Let M an A-module. Then M is a tilting module if and only if

$$\mathcal{T}(M) = \{X \in \mathrm{mod}\, A \mid \mathrm{Ext}^1(M, X) = 0\}.$$

The word *partial* in partial tilting has the connotation that partial tilting modules are *incomplete* in some sense. This is indeed the case in the sense that every partial tilting module can be completed to a tilting module in the following sense.

Lemma 2.3.25. (Assem *et al.*, 2006, Chapter VI, 2.4. Lemma) Let M be a partial tilting module in $\operatorname{mod} A$. There exists a module N in $\operatorname{mod} A$ such that $T = M \oplus N$ is a tilting module.

From this viewpoint, tilting modules are the *maximal* A-modules M satisfying pd $M \leq 1$ and $\operatorname{Ext}^1(M, M) = \{0\}$. In particular, tilting modules are maximal in the terms of the number of indecomposable direct summands they have up to isomorphism.

Corollary 2.3.26. (Happel, 1988, 6.2 Corollary) Let A be finite-dimensional algebra of finite global dimension. Let n be the number of isomorphism classes of simple A-modules. Let $M = \bigoplus_{i=1}^{r} = M_i^{r_i}$, with M_i indecomposable modules such that $M_i \ncong M_j$ when $i \neq j$, be a partial tilting module. Then M is a tilting module if and only if $r = n$. Furthermore, M is a **basic tilting module** if $r_i = 1$ for all $1 \leq i \leq n$.

Tilting modules also have the following characterisation.

Lemma 2.3.27. (Happel, 1988, 6.3 Lemma) Let A be a finite-dimensional hereditary algebra. Let M be a partial tilting A-module. Then M is a tilting module if and only if for any non-zero A-module N with $\operatorname{Ext}^1(N, N) = \{0\}$, it is the case that either $\operatorname{Ext}^1(N, M) \neq 0$ or $\operatorname{Hom}(N, M) \neq 0$.

Definition 2.3.28. (Happel, 1988, §5.1) Let A be a finite-dimensional hereditary algebra. A **tilted algebra** is an algebra of the form

$$B = \operatorname{End}_A(M),$$

where M is a tilting A-module.

Remark 2.3.29. The algebra A considered as an A module has the decomposition $A = \bigoplus_{i=1}^{n} P_i$, where the P_i are indecomposable projective modules such that $P_i \ncong P_j$ if $i \neq j$, and n is the number of isomorphism classes of simple A-modules; see (Assem *et al.*, 2006, I.4 Direct sum decompositions) and (Schiffler, 2014, Example 6.3) for more details. Due to Lemma 2.1.50, we have that $\mathrm{Ext}^1(A, A) = 0$, so A is a basic tilting A-module, and in fact $A \cong \mathrm{End}_A(A)^{\mathrm{op}}$. We will see later that after removing a projective module P_j (to then get an almost complete tilting module) from the direct sum decomposition of A as an A-module, we can replace P_j with another A-module to obtain another tilting module T, with tilted algebra $B = \mathrm{End}_A(T)$. In this sense T is a close approximation of A and hence its tilted algebra B is also a close approximation to the algebra A.

Definition 2.3.30. (Happel & Unger, 1989, §Introduction) Let A be a basic connected finite-dimensional hereditary algebra. Let n be the number of isomorphism classes of simple A-modules. Let $M = \bigoplus_{i=1}^{r} M_i^{r_i}$, with M_i indecomposable modules such that $M_i \ncong M_j$ when $i \neq j$ be a partial tilting module. The module M is said to be an **almost complete tilting module** if $r = n - 1$. A module N is a **complement** of an almost complete tilting module M if $M \oplus N$ is a tilting module.

By Lemma 2.3.25, it can be seen that every almost complete tilting module has a complement. In general an almost complete tilting module has at most two non-isomorphic complements, and under certain assumptions it has exactly two.

Definition 2.3.31. (Happel & Unger, 1989, §Introduction) Let A be a basic connected finite-dimensional hereditary algebra. An A-module M is **sincere** if for any indecomposable projective module P, it is the case that $\mathrm{Hom}(P, M) \neq 0$.

Theorem 2.3.32. (Happel & Unger, 1989, §Introduction, 2.3 Proposition) Let A be a basic connected finite-dimensional hereditary algebra. Let M be an almost complete tilting A-module. Then M has exactly two non-isomorphic complements if and only if M is a sincere module.

As a consequence of the above theorem, we have the following corollary.

Corollary 2.3.33. (Happel & Unger, 1989, 1.3 Corollary) Let A be a basic connected finite-dimensional hereditary algebra. Let M be an almost complete tilting A-module. If M is not a sincere module, then it has exactly one complement up to isomorphism.

When an almost complete module has exactly two non-isomorphic complements, they may be related in the following manner.

Theorem 2.3.34. (Happel & Unger, 1989, 1.1 Theorem) Let A be a basic finite-dimensional hereditary algebra. Let M be a sincere almost complete tilting A-module. Let X and Y be the two non-isomorphic complements of M, and suppose that $\mathrm{Ext}^1(Y, X) \neq 0$. Then there exists a short exact sequence

$$\mathbf{0} \longrightarrow X \xrightarrow{f} E \xrightarrow{g} Y \longrightarrow \mathbf{0}$$

with $E \in \mathrm{add}\ M$, where $\mathrm{add}\ M$ is the subcategory of $\mathrm{mod}\, A$ generated by the direct summands of M.

To conclude, we have seen that for a basic tilting module $M = \bigoplus_{i=1}^n M_i$ we may remove a summand M_j to get an almost complete tilting module M/M_j. If M/M_j is a sincere module, then there exists an A-module M_j^* which is not isomorphic to M_j such that $M' = M/M_j \oplus M_j^*$ is a basic tilting module. The module M' will be called a mutation of M in the direction j. Note that since not every almost complete tilting module is sincere, mutation is not always possible in any direction as in the situation of cluster algebras. To rectify this, τ-tilting theory was introduced to make mutation always possible. To conclude this subsection we will discuss the tilting theory of $\mathrm{mod}\, \overrightarrow{\mathbb{A}}_n$.

Tilting modules in $\operatorname{mod} \overrightarrow{\mathbb{A}_n}$

Of particular interest to this thesis will be the tilting theory of the module category $\operatorname{mod} \overrightarrow{\mathbb{A}_n}$ (the category of modules over the path algebra of the linearly oriented type $\mathbb{A}$ quiver with n vertices), especially in relation to the cluster combinatorics of type $\mathbb{A}$ cluster algebras. Before we can talk about this relationship, we first need to recall some combinatorial language. We start by defining the Tamari lattice.

Definition 2.3.35. A binary parenthesization of the binary product $x_0 * \cdots * x_n$ is a parenthesization where each product is binary. For example the parenthesization $(((x_0 * x_1) * x_2) * (x_3 * x_4))$ is binary whereas $((x_0 * x_1) * x_2 * x_3 * x_4)$ is not.

Definition 2.3.36. (Geyer, 1994, Definition 2.1) Let $*$ be a binary operation, and let $\mathbb{T}_n$ be the set of binary parenthesizations of the expression $x_0 * \cdots * x_n$ with $n+1$ symbols. The set $\mathbb{T}_n$ can be ordered by the associativity rule:

$$(x * y) * z \to x * (y * z).$$

So that for two parenthesizations t, t' in $\mathbb{T}_n$ we have that $t \leq t'$ if and only if we can obtain t' from t by finitely many applications of the associativity rule. This order is called the **Tamari lattice**.

There is a well-known bijection between binary parenthesizations of the expression $x_0 * \cdots * x_n$ and triangulations of the $n+2$ sided convex polygon $\mathbf{P}_{n+2}$, see for example (Gelfand *et al.*, 2008, Chapter 7, §3 ,page 240 and Figure 35). It is also well-known that under this bijection, the associativity rule corresponds to diagonal flips in triangulations; see for example (De Loera *et al.*, 2010, Chapter 1, §1.1, page 8). As a consequence, we may describe the Tamari Lattice in terms of triangulations and diagonal flips. Consider the graph whose vertices are triangulations of $(n+2)$-gons, and where there is an edge between two triangulations T and T' if and only if we can obtain T' from T by performing a flip on a diagonal of T or vice versa. This graph is the Hasse diagram of the Tamari lattice $\mathbb{T}_n$. The Hasse diagram of the Tamari lattice is given by 1-skeleton of the n-dimensional convex polytope known as the associahedron. The associahedron was first introduced by in Stasheff (1963) and was later given the description in terms of triangulations below; see (Lee, 1989, §2) or see for example (Ziegler, 2012, Chapter 0, page 18) for a description of the associahedron in terms of parenthesizations.

Recall from Theorem 2.3.12, that the clusters of a cluster algebra of type $\mathbb{A}_n$ admit a description by triangulations of a convex polygon $\mathbf{P}_{n+3}$ (counted by the Catalan number C_{n+1}), and the mutation of the clusters corresponds to diagonal flips. Hence the mutation graph of a cluster algebras of type $\mathbb{A}_n$ is given by the associahedron. A similar statement can be made with regard to the set of tilting modules in $\operatorname{mod}\overrightarrow{\mathbb{A}_n}$, thus relating the cluster combinatorics of type $\mathbb{A}_n$ cluster algebras with the tilting theory of the module category $\operatorname{mod}\overrightarrow{\mathbb{A}_n}$. It was shown by in Gabriel (1981) that the number of basic tilting modules in $\operatorname{mod}\overrightarrow{\mathbb{A}_n}$ equals the Catalan number C_n. Then Buan and Krause further showed the tilting modules in $\operatorname{mod}\overrightarrow{\mathbb{A}_n}$ form a Tamari lattice.

Definition 2.3.37. (Buan & Marsh, 2021, See for example §1)Let $\mathcal{C}$ be an additive category and $\mathcal{X} \subseteq \mathcal{C}$ be a collection. The **right perpendicular** and **left perpendicular** subcategories of $\mathcal{X}$ are defined as follows respectively.

$$\mathcal{X}^{\perp} := \{Y \in \mathcal{C} \mid \operatorname{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\},$$

$$^{\perp}\mathcal{X} := \{Y \in \mathcal{C} \mid \operatorname{Hom}(Y, X) = 0 \text{ for all } X \in \mathcal{X}\}.$$

Definition 2.3.38. (Buan & Krause, 2004, See for example §5) Let $\mathcal{C}$ be an additive category and $\mathcal{X} \subseteq \mathcal{C}$ be a collection of objects. The **right Ext-perpendicular** and **left Ext-perpendicular** subcategories of $\mathcal{X}$ are defined as follows respectively.

$$\mathcal{X}^{\perp_{\mathrm{Ext}}} := \{Y \in \mathcal{C} \mid \operatorname{Ext}^1(X, Y) = 0 \text{ for all } X \in \mathcal{X}\},$$

$$^{\perp_{\mathrm{Ext}}}\mathcal{X} := \{Y \in \mathcal{C} \mid \operatorname{Ext}^1(Y, X) = 0 \text{ for all } X \in \mathcal{X}\}.$$

The following defines a partial order on the set of isomorphism classes of basic tilting objects in $\operatorname{mod} A$.

Proposition 2.3.39. (Happel & Unger, 2005, See for example §1) Let T and U be basic tilting modules, we will write

$$T \leq U$$

if and only if

$$T^{\perp_{\mathrm{Ext}}} \subseteq U^{\perp_{\mathrm{Ext}}}.$$

This defines a partial order on the set of basic tilting modules up to isomorphism.

Proposition 2.3.40. (Buan & Krause, 2004, Proposition 5.4) Let T and U be basic tilting modules in $\operatorname{mod} A$, then T covers U or U covers T if and only if T and U have $n-1$ direct summands in common.

Therefore if T and U are basic tilting modules such that T covers U, and M is the direct sum of the $n-1$ direct summands they have in common, then T/M and U/M are the two complements of M. In particular U is a mutation of T.

Proposition 2.3.41. (Buan & Krause, 2004, Proposition 5.5) Let T and U be basic tilting modules. Then T covers U if and only if there are decompositions $T = X \oplus M$ and $U = Y \oplus M$ where X and Y are indecomposable modules with a monomorphism $X \to M$ and an epimorphism $M \to Y$.

Theorem 2.3.42. (Buan & Krause, 2004, Theorem C) The basic tilting modules of $\operatorname{mod} \overrightarrow{\mathbb{A}_n}$ are in correspondence with the triangulations of the convex $(n+2)$-gon $\mathbf{P}_{n+2}$. Moreover the set of basic tilting modules is a Tamari lattice with respect to partial order $T \leq U \Leftrightarrow T^{\perp_{\mathrm{Ext}}} \subseteq U^{\perp_{\mathrm{Ext}}}$. In other words, the mutation of basic tilting modules $\operatorname{mod} \overrightarrow{\mathbb{A}_n}$ gives rise to the Tamari lattice.

2.3.3 τ-tilting theory

Thus far, we have seen the notion of mutation in the context of cluster algebras and tilting theory. In the context of cluster algebras, clusters can always be mutated in any direction, whilst in the context of tilting theory, tilting modules can only be in the directions which give us since almost complete tilting modules. To rectify the fact that mutation is not always possible in tilting theory, a generalisation of tilting theory called τ-tilting theory was introduced, in particular **support τ-tilting pairs** were introduced as a generalisation of tilting modules where mutation is always possible. In this subsection, we will give a very brief introduction to τ-tilting theory and support τ-tilting pairs. Our primary reference for this subsection is Adachi *et al.* (2014). Throughout this subsection, A will be a basic finite-dimensional algebra over an algebraically closed field $\mathbb{F}$.

Definition 2.3.43. (Adachi *et al.*, 2014, Definition 0.1) Let n be the number of isomorphism classes of simple A-modules. Let M be an A-module and let m be the number of non-isomorphic indecomposable direct summands of M.

(i) M is **τ-rigid** if $\mathrm{Hom}_A(M, \tau M) = 0$

(ii) If the module M is τ-rigid and $m = n$, then M is said to be a **τ-tilting module**.

(iii) If the module M is τ-rigid and $m = n - 1$, then M is said to be an **almost complete τ-tilting module**

(iv) M is **support τ-tilting** if there is an idempotent element $e \in A$ such that M is a τ-tilting module in $\mathrm{mod}\, A/\langle e\rangle$.

Recall by the Auslander-Reiten formula in Theorem 2.2.44, we have that

$$\mathrm{Ext}^1(M, N) \cong D\overline{\mathrm{Hom}}(N, \tau M),$$

therefore a τ-rigid module is rigid in the sense of Definition 2.3.21. The converse statement is true when the projective dimension $\mathrm{pd}\, M \leq 1$. It can then be observed that a partial tilting module (Definition 2.3.22) is τ-rigid, and more specifically tilting modules are τ-tilting. So in this way, τ-tilting modules may be considered a generalisation of tilting modules. Like partial tilting modules, the number of non-isomorphic direct summands of τ-rigid modules cannot exceed the number of simple A-modules up to isomorphism, see (Adachi *et al.*, 2014, Proposition 1.3). According to Lemma 2.3.25, every partial tilting module can be completed to a tilting module. An analogous statement exists in this case.

Proposition 2.3.44. (Adachi *et al.*, 2014, Theorem 2.10) Let M be a τ-rigid module in $\operatorname{mod} A$. There exists a module N in $\operatorname{mod} A$ such that $T = M \oplus N$ is a τ-tilting module.

Although τ-tilting modules generalise tilting modules, from the point of view of mutation expressed above they are not the generalisation that is required. From the point of view of mutation, support τ-tilting pairs are the correct notion.

Definition 2.3.45. (Adachi *et al.*, 2014, Definition 0.3) Let (M, P) be a pair consisting of a $M \in \operatorname{mod} A$ and $P \in \operatorname{proj} A$. Let m and p be the number of non-isomorphic direct summands of M and P respectively.

(i) (M, P) is a τ-rigid pair if M is τ-rigid and $\operatorname{Hom}_A(P, M) = 0$.

(ii) (M, P) is a **support τ-tilting pair** if (M, P) is τ-rigid and $m + p = n$.

(iii) (M, P) is an **almost complete support τ-tilting pair** if (M, P) is τ-rigid and $m + p = n - 1$.

The pair (M, P) is basic if both M and P are basic modules. A pair (M, P) will be called a direct summand of a pair (M', P') if M is a direct summand of M' and P is a direct summand of P'.

The notion of a support τ-tilting pairs is related to that of τ-tilting modules by the following.

Proposition 2.3.46. (Adachi *et al.*, 2014, Proposition 2.3) Let (M, P) be a pair consisting of $M \in \operatorname{mod} A$ and $P \in \operatorname{proj} A$. Let $e \in A$ be an idempotent element such that $\operatorname{add} P = \operatorname{add} Ae$.

(i) Then (M, P) is a τ-rigid (respectively, support τ-tilting, almost complete support τ-tilting) pair if and only if M is a τ-rigid (τ-tilting, almost complete τ-tilting) in $\operatorname{mod} A/\langle e\rangle$.

(ii) In a support τ-tilting pair (M, P), the module P is completely determined by M. This is in the sense that if (M, P) and (M, Q) are support τ-tilting pairs in $\operatorname{mod} A$, then $\operatorname{add} P = \operatorname{add} Q$.

By the above proposition, support τ-tilting pairs give rise to τ-tilting modules. This statement is also true the other way. In particular any τ-tilting module is a **sincere support τ-tilting** module; see (Adachi *et al.*, 2014, Proposition 2.2) for more details. We conclude this subsection by stating the theorem which says that every basic almost complete support τ-tilting pair has exactly two complements.

Theorem 2.3.47. (Adachi *et al.*, 2014, Theorem 2.18) Let (M, P) be a basic almost complete support τ-tilting pair. Then (M, P) is a direct summand of precisely two non-isomorphic basic support τ-tilting pairs (T, Q) and (T', Q').

In the case of the above theorem, the basic support τ-tilting pairs (T, Q) and (T', Q') are said to be mutations of each other. Moreover, by the above theorem, mutation is always possible for a support τ-tilting pair; see (Adachi *et al.*, 2014, Definition 2.19) for more details.

2.3.4 Exceptional sequences

Motivated by an interest in mutation, we look to **exceptional sequences**. These are sequences of certain rigid modules with certain orthogonality conditions. Exceptional sequences were first introduced in the context of algebraic geometry by Bondal (1989), Gorodentsev (1989) and Gorodentsev *et al.* (1987), where they were defined in the setting of triangulated categories. They were later studied in the setting of categories of modules over a finite-dimensional algebra by Crawley-Boevey Crawley-Boevey (1993) and Ringel Ringel (1994), it is this setting which is of interest to us. For the remainder of this subsection, A will be a finite-dimensional hereditary algebra over an algebraically closed field $\mathbb{F}$.

Definition 2.3.48. (Crawley-Boevey, 1993, §1, Page 118) Let A be a finite-dimensional hereditary algebra over an algebraically closed field $\mathbb{F}$, and let n be the number of isomorphism classes of simple A-modules. An indecomposable module $M \in \operatorname{mod} A$ is **exceptional** if $\operatorname{Hom}_A(M, M) \cong \mathbb{F}$ and $\operatorname{Ext}^1(M, M) = 0$.

A sequence of exceptional modules $E = (M_1, \ldots, M_r)$ is an **exceptional sequence** of length r if for each pair (M_l, M_j) with $1 \leq l < j \leq r$, we have that $\operatorname{Hom}(M_j, M_l) = \operatorname{Ext}^1_A(M_j, M_l) = 0$. The sequence E is a **complete exceptional sequence** if $r = n$.

Exceptional sequences were first introduced in the setting of triangulated categories in. When defining exceptional sequences in triangulated categories, it is also required that all the extension groups of all degree Ext^i where $i \in \mathbb{Z}$ vanish; see for (Bondal, 1989, §2). For finite-dimensional algebras (including non-hereditary algebras), it is required in the definition of exceptional sequences that all the extension groups Ext^i where $i \geq 1$ vanish; see for example (Westin, 2020, 6.1 Exceptional sequences). The requirement that the higher extensions vanish for exceptional sequence is trivially satisfied for hereditary algebras, hence it is generally not stated when working in the context of hereditary algebras.

We have seen that partial tilting modules, and τ-rigid modules can be *completed* to tilting modules and τ-tilting modules respectively. An analogous statement exists for exceptional sequences.

Proposition 2.3.49. (Crawley-Boevey, 1993, Lemma 1) Let A be a finite-dimensional hereditary algebra over an algebraically closed field $\mathbb{F}$, and let

$$E = (M_1, \ldots, M_i, N_1, \ldots, N_j)$$

be an exceptional sequence in $\operatorname{mod} A$ where $i + j < n$. Then E can be extended to a complete exceptional sequence

$$E' = (M_1, \ldots, M_i, L_1, \ldots, L_k, N_1, \ldots, N_j).$$

So far we have presented mutation for tilting modules, clusters in cluster algebras, and support τ-tilting pairs. In all these settings, mutation has involved replacing one object with another. For example mutation of a basic tilting module involves replacing one summand of the tilting module with an appropriate choice of A-module. Likewise the mutation of a cluster in cluster algebras involves replacing one cluster variable with another. We will soon see that there is a notion of mutation for complete exceptional sequences. However mutation of complete exceptional sequences differs from the other notions of mutation we have seen so far in that it involves changing two entries of the sequence instead of one. In fact, two complete exceptional sequences cannot only differ in one position.

Proposition 2.3.50. (Crawley-Boevey, 1993, Lemma 2) Let $E = (M_1, \ldots, M_n)$ and $D = (N_1, \ldots, N_n)$ be two complete exceptional sequences in $\operatorname{mod} A$ that differ only in one position. Then $E = D$.

Let E be an exceptional sequence in $\operatorname{mod} A$. We will write $\mathcal{C}(E)$ for the smallest full subcategory of $\operatorname{mod} A$ containing the sequence E which is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms; see Crawley-Boevey (1993) for more.

Lemma 2.3.51. (Crawley-Boevey, 1993, Lemma 3) Let E be a complete exceptional sequence in $\operatorname{mod} A$. Then $\mathcal{C}(E) = \operatorname{mod} A$.

Lemma 2.3.52. (Crawley-Boevey, 1993, Lemma 6) Let (M, N) be an exceptional sequence in $\operatorname{mod} A$. Then there are unique A-modules $R_N M$ and $L_M N$ such that $(N, R_N M)$ and $(L_M N, M)$ are exceptional sequences in $\mathcal{C}(X, Y)$.

A consequence of the above lemmas is that we have a notion of mutation of exceptional sequence. This mutation is particularly interesting because it can be interpreted in terms of a group action by the Artin braid group. Let us recall the definition of the Artin braid group. The Artin braid group action on exceptional sequence was first observed Bondal (1989) and was later studied in the context of module categories by Ringel (1994) and Crawley-Boevey (1993).

Definition 2.3.53. (Kassel & Turaev, 2008, See for example, Definition 1.1) The **Artin braid group** B_r on r strands is the group defined by group presentation

$$\langle \sigma_1, \dots, \sigma_r \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } \mid i-j \mid \geq 2 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq r-2\rangle.$$

Theorem 2.3.54. (Crawley-Boevey, 1993, Lemma 8, Lemma 9, Theorem) Let $E = (M_1, \dots, M_r)$ be an exceptional sequence in $\operatorname{mod} A$. Then we have that:

- the sequence $E' = (M_1, \dots, M_{i-1}, M_{i+1}, N, M_{i+2}, \dots, M_r)$ is an exceptional sequence in $\mathcal{C}(E)$ if and only if $N \cong R_{M_{i+1}} M_i$,
- the sequence $E'' = (M_1, \dots, M_{i-1}, O, M_i, M_{i+2}, \dots, M_r)$ is an exceptional sequence in $\mathcal{C}(E)$ if and only if $O \cong L_{M_i} M_{i+1}$,
- the setting $\sigma_i E = E'$ and $\sigma_i^{-1} E = E''$ defines an action of the Artin braid group on r strands on to the set of exceptional sequences of length r in $\operatorname{mod} A$. Moreover, this group action is transitive if $r = n$.

To conclude, given an exceptional sequence in $\operatorname{mod} A$, it may be mutated to obtain two more exceptional sequences E' and E'' as above. The former is called the **right mutation** of E at i, whilst the latter is called the **left mutation** of E at i. It is the case that complete exceptional sequences always exist when A is a finite-dimensional hereditary algebra. This is not the case if A is non-hereditary, see for example 3.5.6. Buan and Marsh introduced τ-exceptional sequences Buan & Marsh (2021) as a generalisation of exceptional sequences that ensure that complete sequences always exists for finite-dimensional algebras.

Chapter 3

Counting τ-exceptional of Nakayama algebras

3.1 Introduction

Let A be a finite dimensional algebra over a field $\mathbb{F}$, where $\mathbb{F}$ is algebraically closed. Let $\operatorname{mod} A$ be the category of finitely generated A-modules. An A-module M is called **exceptional** if $\operatorname{Hom}(M, M) \cong \mathbb{F}$ and $\operatorname{Ext}^i_A(M, M) = 0$ for $i \geq 1$. A sequence of indecomposable modules $(M_1, M_2, \ldots, M_r)$ is called an **exceptional sequence** if for each pair (M_l, M_j) with $1 \leq l < j \leq r$, we have that $\operatorname{Hom}(M_j, M_l) = \operatorname{Ext}^i_A(M_j, M_l) = 0$ for $i \geq 1$, and each M_k is exceptional for $1 \leq k \leq r$. Exceptional sequences were first introduced in the context of algebraic geometry by Bondal (1989), Gorodentsev (1989) and Gorodentsev *et al.* (1987).

Exceptional sequences exhibit some interesting behaviours. It was shown in Crawley-Boevey (1993) and in Ringel (1994) that there is a transitive braid group action on the set of exceptional sequences. A characterisation of exceptional sequences for hereditary algebras was given in Igusa & Schiffler (2010) using the fact that the product of the corresponding reflections is the inverse Coxeter element of the Weyl group. The exceptional sequences for $\operatorname{mod} \mathbb{A}_r$, where $\mathbb{A}_r$ is the path algebra of a Dynkin type A quiver are classified in Garver *et al.* (2015) using combinatorial objects called strand diagrams. The exceptional sequences over path algebras of type A, were also characterised using non-crossing spanning trees in Araya (2013). A natural question for exceptional sequences is to ask how many there are. The number of them has been computed for all the hereditary Dynkin algebras in Seidel (2001) and Obaid *et al.* (2013).

Exceptional sequences have been subject to a number of generalisations. The **signed exceptional sequences** were introduced in Igusa & Todorov (2017). More recently, **weak exceptional sequences** were introduced and studied in Sen (2020b). Finally, the **signed τ-exceptional sequences** and **τ-exceptional sequences** were introduced in Buan & Marsh (2021). It is τ-exceptional sequences which are the subject of this chapter. An A-module M is called **τ-rigid** if $\mathrm{Hom}(M, \tau M) = 0$, see Definition 0.1 in Adachi *et al.* (2014). The **τ-perpendicular category** of M in $\mathrm{mod}\, A$ is the subcategory $J(M) = M^{\perp} \cap {}^{\perp}(\tau M)$, see Definition 3.3 in Jasso (2014). A sequence of indecomposable modules $(M_1, M_2, \dots, M_r)$ in $\mathrm{mod}\, A$ is called a τ-exceptional sequence if M_r is τ-rigid in $\mathrm{mod}\, A$ and $(M_1, M_2, \dots, M_{r-1})$ is a τ-exceptional sequence in $J(M_r)$.

Our main results are derivations of closed formulas for the number of complete τ-exceptional sequences in the module categories of certain Nakayama algebras. Most notably, we see that the complete τ-exceptional sequences over the linear radical square zero Nakayama algebras Γ_n^2 are counted by the **restricted Fubini numbers** $F_{n,\leqslant 2}$ Mezo (2014). The numbers $F_{n,\leqslant 2}$ count the number of ordered set partitions of the set $\{1, 2, \dots, n\}$ with blocks of size at most two. In the case for the cyclic Nakayama algebra Λ_n^n, we get that the complete τ-exceptional sequences are counted by the sequence n^n. We remark that this sequence also counts the number of complete exceptional sequences for the hereditary Dykin algebras of quivers of type B and C, as shown in Obaid *et al.* (2013), and full weak exceptional sequences over Λ_n^n, see (Sen, 2020a, Theorem 3.5). In fact, we show that the complete τ-exceptional sequences over Λ_n^n coincide with the full weak exceptional sequences over Λ_n^n, see Corollary 3.6.13.

We remark that Buan and Marsh showed in Buan & Marsh (2021) that there is a bijection between complete signed τ-exceptional sequences and basic ordered support τ-tilting modules over a finite-dimensional algebra. So, one way of counting signed τ-exceptional sequences would be to count ordered support τ-tilting modules, but this would not give the number of (unsigned) τ-exceptional sequences, which is what we consider here. In this direction, Asai (2018) gave a recurrence relation for the number of support τ-tilting modules over Nakayama algebras with a linearly oriented type A quiver. A recurrence relation for the number of τ-tilting modules over the same algebras as Asai was given in Adachi (2016) . More recently, the recurrence relations of Adachi and Asai were extended to τ-tilting modules and support τ-tilting modules over Nakayama algebras whose quiver is an oriented cycle in Gao *et al.* (2020). In Sen (2020a), the number of exceptional sequences over the linear radical square zero Nakayama algebras Γ_n^2 are counted. However, to date the numbers of exceptional sequences for other classes of Nakayama algebras have not been determined.

3.2 Definitions and Notation

Let A be a basic finite-dimensional algebra over a field $\mathbb{F}$ which is algebraically closed. Let $\operatorname{mod} A$ be the category of finite-dimensional A-modules. Denote by $\mathcal{P}(A)$ the full subcategory of projective objects in $\operatorname{mod} A$. If $\mathcal{T}$ is a subcategory of $\operatorname{mod} A$, we say an A-module M in $\mathcal{T}$ is Ext-**projective** in $\mathcal{T}$ if $\operatorname{Ext}^1_A(M, \mathcal{T}) = 0$; that is to say $\operatorname{Ext}^1_A(M, T) = 0$ for all $T \in \mathcal{T}$. We will then write $\mathcal{P}(\mathcal{T})$ to denote the direct sum of the indecomposable Ext-projective modules in $\mathcal{T}$. In everything that follows, we make the assumption that all subcategories are full, and closed under isomorphism. We will also take all objects to be basic where possible, and they will be considered up to isomorphism.

Recall that for an additive category $\mathcal{C}$, and an object X in $\mathcal{C}$, we denote by $\operatorname{add} X$ the additive subcategory of $\mathcal{C}$ generated by X. This is the subcategory of $\mathcal{C}$ with objects the direct summands of direct sums of copies of X. If $\mathcal{C}$ is skeletally small and Krull-Schmidt, we denote by $\operatorname{ind}(\mathcal{C})$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. For any basic object X in $\mathcal{C}$, let $\delta(X)$ denote the number of indecomposable direct summands of X. We fix $\delta(A)$ to be n, where $n \geq 1$ is a positive integer.

Definition 3.2.1. τ-perpendicular category (Jasso, 2014, Definition 3.3). Let M be a basic τ-rigid A-module. The **τ-perpendicular category** associated to

M is the subcategory of $\operatorname{mod} A$ given by $J_{\operatorname{mod} A}(M)_: = M^\perp \cap {}^\perp(\tau M)$. If there is no risk of ambiguity, we will write $J(M)$ for the subcategory $J_{\operatorname{mod} A}(M)$.

Definition 3.2.2. ***τ*-exceptional sequence** (Buan & Marsh, 2021, Definition 1.3). Let k be a positive integer. A sequence $(M_1, M_2, \ldots, M_k)$ in $\operatorname{ind}(\operatorname{mod} A)$ is called a **τ-exceptional sequence** in $\operatorname{mod} A$ if M_k is τ-rigid in $\operatorname{mod} A$ and $(M_1, M_2, \ldots, M_{k-1})$ is a τ-exceptional sequence in $J(M_k)$. If $k = n$ we say that the sequence is a **complete** τ-exceptional sequence.

Remark 3.2.3. By Theorem 3.3.5, there is an additive exact equivalence of categories between $J(M)$, the τ-perpendicular category of a module M and a category of modules $\operatorname{mod} D_M$. Therefore each τ-perpendicular category has an Auslander-Reiten translation through this equivalence. The recursiveness of the definition means that τ-rigidity is always with respect to the Auslander-Reiten translation of the equivalent category of modules in question, and likewise the τ-perpendicular categories are always taken with respect to the equivalent category of modules in question. In particular, the definition requires that M_{k-1} is τ-rigid in $J(M_k)$ with respect to the Auslander-Reiten translation of $\operatorname{mod} D_{M_k}$ (the category of modules equivalent to $J(M_k)$). Likewise, the definition also requires that $(M_1, M_2, \ldots, M_{k-2})$ is a τ-exceptional sequence in $J_{\operatorname{mod} D_{M_k}}(M_{k-1})$, the τ-perpendicular category of M_{k-1} in the category of modules equivalent to $J(M_k)$. The rest of the sequence $(M_1, M_2, \ldots, M_{k-2})$ and modules M_i, where $1 \leq i \leq k-2$ are treated similarly.

For a positive integer $n \geq 1$, we will write $(a)_n$ to stand for a modulo n. We will also write $[i, j]_n$ for the set $\{(i)_n, (i+1)_n, \ldots, (j-1)_n, (j)_n\}$.

3.3 Preliminary Results

In this section, we will state and prove results which will be used in later sections to calculate the number of τ-exceptional sequences over the algebras Γ_n^t and Λ_n^t. For this section, we fix an arbitrary finite-dimensional $\mathbb{F}$-algebra A.

Proposition 3.3.1. (Adachi *et al.*, 2014, Theorem 2.10). Let M be a τ-rigid A-module. Then the following holds:

1. The module M is Ext-projective in ${}^{\perp}(\tau M)$, which is to say that M is in $\mathrm{add}(\mathcal{P}({}^{\perp}(\tau M)))$.
2. The module $T_M := \mathcal{P}({}^{\perp}(\tau M))$ is a τ-tilting A-module.

The A-module T_M is called the **Bongartz completion** of M in $\mathrm{mod}\, A$.

Example 3.3.2. Let A be the algebra Γ_3^2 given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3,$$

subject to the relation $\alpha\beta = 0$. The Auslander-Reiten quiver of $\mathrm{mod}\, \Gamma_3^2$ is as follows,

$$\begin{tikzcd}
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
3 \arrow[ur] \arrow[rr, dashed, no head] & & 2 \arrow[ur] \arrow[rr, dashed, no head] & & 1
\end{tikzcd}.$$

For the Γ_3^2-module $M = 1$, $\mathrm{ind}({}^{\perp}(\tau 1)) = \mathrm{ind}({}^{\perp}2) = \{3, \begin{smallmatrix}1\\2\end{smallmatrix}, 1\}$. Therefore it is easy to see that $T_1 = \mathcal{P}({}^{\perp}(\tau 1)) = 3 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$. It is also easy to observe that T_1 is indeed a τ-tilting Γ_3^2-module.

Proposition 3.3.3. (Adachi *et al.*, 2014, Lemma 2.1). Let I be an ideal of A, and let M, N be A/I-modules. Then we have the following:

1. If $\operatorname{Hom}_A(M, \tau N) = 0$ then $\operatorname{Hom}_{A/I}(M, \tau_{A/I} N) = 0$.

2. If $I = \langle e \rangle$ for some idempotent $e \in A$, then it is the case that $\operatorname{Hom}_A(M, \tau N) = 0$ if and only if $\operatorname{Hom}_{A/I}(M, \tau_{A/I} N) = 0$.

The following lemma is well known and it will be important in the sequel.

Lemma 3.3.4. Let Q be a finite simple quiver with vertex set $\{1, 2, \ldots, n\}$. Let I be the ideal of $\mathbb{F}Q$ generated by relations on Q where each relation is a path in Q and take $A = \mathbb{F}Q/I$. For some $j \in \{1, 2, \ldots, n\}$ let $Q^{(j)}$ be the quiver obtained from Q by removing the vertex j and any arrows incident to j. Let $I^{(j)} \subset I$ be the ideal of $\mathbb{F}Q$ generated by the generating relations of I defined by paths of Q not containing the vertex j and take $B = \mathbb{F}Q^{(j)}/I^j$. Then $B \cong A/\langle e_j \rangle$ as an $\mathbb{F}$-algebra , where e_j is the idempotent at vertex j of $\mathbb{F}Q$.

Theorem 3.3.5. (Jasso, 2014, Theorem 3.8). Let A be a finite-dimensional algebra and M a basic τ-rigid A-module. Let T_M be the Bongartz completion of M in $\operatorname{mod} A$. Let $E_M = \operatorname{End}_A(T_M)$ and $D_M = E_M/\langle e_M \rangle$, where e_M is the idempotent corresponding to the projective E_M-module $\operatorname{Hom}_A(T_M, M)$. Then there is an additive exact equivalence of categories between the category $J(M)$, (the τ-perpendicular category of M in $\operatorname{mod} A$) and the category $\operatorname{mod} D_M$. Moreover, if M is indecomposable we have that $\delta(D_M) = \delta(A) - 1$.

We now prove some results which will be crucial in our strategy for calculating the number of τ-exceptional sequences in $\operatorname{mod} A$.

Definition 3.3.6. Interleaving. Let $X = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_t)$ be sequences. An interleaved sequence of X and Y is a sequence $Z = (Z_1, Z_2, \ldots, Z_{s+t})$ with $Z_i \in \{X_j : 1 \leq i \leq s\} \cup \{Y_j : 1 \leq j \leq t\}$ such that the subsequence of Z containing only elements X or Y is precisely X or Y respectively.

Example 3.3.7. Let $X = \left(5, \begin{matrix}4\\5\end{matrix}, 6\right)$ and $Y = \left(2, \begin{matrix}1\\2\end{matrix}\right)$ be sequences in $\operatorname{mod}\Gamma_6^2$. The sequence $Z = \left(2, 5, \begin{matrix}4\\5\end{matrix}, \begin{matrix}1\\2\end{matrix}, 6\right)$ is an interleaved sequence of X and Y. However $W = \left(\begin{matrix}4\\5\end{matrix}, 5, 2, 6, \begin{matrix}1\\2\end{matrix}\right)$ is not an interleaved sequence of X and Y because the subsequence containing only elements of X is not equal to X.

Let A and B be finite-dimensional $\mathbb{F}$-algebras and let $\operatorname{mod} A$ and $\operatorname{mod} B$ be the categories of finitely generated A-modules and B-modules respectively. We may consider the category $\operatorname{mod} A \oplus \operatorname{mod} B$, the direct product category of $\operatorname{mod} A$ and $\operatorname{mod} B$. The objects of $\operatorname{mod} A \oplus \operatorname{mod} B$ are pairs (M, N) with $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} B$. A morphism between a pair of objects, (M_1, N_1) and (M_2, N_2) in $\operatorname{mod} A \oplus \operatorname{mod} B$ is a pair of morphisms $(f : M_1 \to M_2, g : N_1 \to N_2)$ where $f \in \operatorname{mod} A$ and $g \in \operatorname{mod} B$. The indecomposable objects of $\operatorname{mod} A \oplus \operatorname{mod} B$ are pairs $(M, 0)$ and $(0, N)$ where M and N are indecomposable in their respective categories. The category $\operatorname{mod} A \oplus \operatorname{mod} B$ is an abelian category, in fact, there is an exact, additive equivalence to $\operatorname{mod}(A \times B)$. The category $\operatorname{mod} A \oplus \operatorname{mod} B$ also has an Auslander-Reiten translate $\tau_{A,B}$ which acts in the obvious way i.e. $\tau_{A,B}(M, 0) = (\tau_A M, 0)$ and $\tau_{A,B}(0, N) = (0, \tau_B N)$. It is easy to see that the above exact equivalence preserves the Auslander-Reiten translations, since irreducible morphisms, left minimal almost split and right minimal almost split morphisms are preserved under equivalence of categories. Let M be an A-module, we identify M with the object $(M, 0)$ in $\operatorname{mod} A \oplus \operatorname{mod} B$. We like wise identify the B-module N with the object $(0, N)$ in $\operatorname{mod} A \oplus \operatorname{mod} B$. It is easy to observe that $(M, 0)$ is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$ if and only if M is τ-rigid in $\operatorname{mod} A$. The similar statement for $(0, N)$ and N is also true.

Theorem 3.3.8. *Let A and B be finite-dimensional $\mathbb{F}$-algebras. Suppose $X = (X_1, X_2, \ldots, X_s)$ is a τ-exceptional sequence in $\operatorname{mod} A$ and $Y = (Y_1, Y_2, \ldots, Y_t)$ is a τ-exceptional sequence in $\operatorname{mod} B$. Suppose $Z = (Z_1, Z_2, \ldots, Z_{s+t})$ is an interleaved sequence of X and Y. Then Z is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$.*

Proof. We prove this by induction on $s+t$. For the base case, suppose $s+t=1$. Without loss of generality suppose $t=0$, so $Z=(X_1)$. By assumption, X_1 is τ-rigid in $\operatorname{mod} A$, so it is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$. This completes the base case.

Suppose the statement is true for $s+t=m$. We consider the $s+t=m+1$ case. Suppose the sequence $Z=(Z_1, Z_2, \ldots, Z_{m+1})$ is an interleaved sequence of $X=(X_1, X_2, \ldots, X_s)$ and $Y=(Y_1, Y_2, \ldots, Y_t)$, where X is a τ-exceptional sequence in $\operatorname{mod} A$ and Y is a τ-exceptional sequence in $\operatorname{mod} B$. Suppose without loss of generality that Z_{m+1} is in X i.e. $Z_{m+1}=X_s$. To show that Z is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$, we need to show that Z_{m+1} is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$ and that $(Z_1, Z_2, \ldots, Z_m)$ is a τ-exceptional sequence in $J_{(A,B)}(Z_{m+1})$, the τ-perpendicular category of Z_{m+1} in $\operatorname{mod} A \oplus \operatorname{mod} B$. By assumption, Z_{m+1} is τ-rigid in $\operatorname{mod} A$, so it is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$. Observe that $\operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(X_s, N) = \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(N, \tau X_s) = 0$ for all $N \in \operatorname{mod} B$, so it follows that

$$J_{(A,B)}(Z_{m+1}) =$$

$$\{U \in \operatorname{mod} A \oplus \operatorname{mod} B : \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(X_s, U) = \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(U, \tau_A X_s) = 0\}$$

$$= J_{\operatorname{mod} A}(X_s) \oplus \operatorname{mod} B,$$

where $J_{\operatorname{mod} A}(X_s)$ is the τ-perpendicular category of X_s in $\operatorname{mod} A$. By theorem 3.3.5, $J_{\operatorname{mod} A}(X_s)$ is equivalent to a category of modules over some finite-dimensional $\mathbb{F}$-algebra. By assumption, X is a τ-exceptional sequence in $\operatorname{mod} A$, thus $X'=(X_1, X_2, \ldots, X_{s-1})$ is a τ-exceptional sequence in $J_{\operatorname{mod} A}(X_s)$. Moreover, $Z'=(Z_1, Z_2, \ldots, Z_m)$ is an interleaved sequence of X' and Y, so it follows by the inductive hypothesis that Z' is a τ-exceptional sequence in $J_{\operatorname{mod} A}(X_s) \oplus \operatorname{mod} B = J_{(A,B)}(Z_{m+1})$, hence Z is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$. This completes the proof. □

We now prove the converse statement.

Theorem 3.3.9. Let A and B be finite-dimensional $\mathbb{F}$-algebras. Suppose $Z = (Z_1, Z_2, \ldots, Z_m)$ is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$. Then Z is an interleaved sequence of some $X = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_{m-s})$, where X is a τ-exceptional sequence in $\operatorname{mod} A$ and Y is a τ-exceptional sequence in $\operatorname{mod} B$.

Proof. We prove this by induction on m.

For the base case, suppose $m = 1$, so $Z = (Z_1)$ is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$. The module Z_1 either lies in $\operatorname{mod} A$ or $\operatorname{mod} B$. Suppose without loss of generality that $Z_1 \in \operatorname{mod} A$. So we define the sequence $X := (Z_1)$ and the sequence Y to be the empty sequence. The sequence Z is trivially an interleaved sequence of X and Y. As Z is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$, by definition Z_1 is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$, so Z_1 is τ-rigid in $\operatorname{mod} A$. This completes the base case.

Now suppose the statement is true for $m = k$. We consider the $m = k + 1$ case. The sequence $Z = (Z_1, Z_2, \ldots, Z_{k+1})$ is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$, so by definition, Z_{k+1} is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$ and the sequence $Z' = (Z_1, Z_2, \ldots, Z_k)$ is a τ-exceptional sequence in $J_{(A,B)}(Z_{k+1})$, the τ-perpendicular category of Z_{k+1} in $\operatorname{mod} A \oplus \operatorname{mod} B$. Suppose without loss of generality that $Z_{k+1} \in \operatorname{mod} A$. We then observe that $\operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(Z_{k+1}, N) = \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(N, \tau Z_{k+1}) = 0$ for all $N \in \operatorname{mod} B$, so it follows that

$$J_{(A,B)}(Z_{m+1}) =$$

$$\{U \in \operatorname{mod} A \oplus \operatorname{mod} B : \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(X_s, U) = \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(U, \tau_A X_s) = 0\}$$

$$= J_{\operatorname{mod} A}(Z_{k+1}) \oplus \operatorname{mod} B,$$

where $J_{\operatorname{mod} A}(Z_{k+1})$ is the τ-perpendicular category of Z_{k+1} in $\operatorname{mod} A$. By theorem 3.3.5 we have that $J_{\operatorname{mod} A}(Z_{k+1})$ is equivalent to a category of modules over some finite-dimensional $\mathbb{F}$-algebra. So we may apply the inductive hypothesis to Z', hence Z' is an interleaved sequence of some $X' = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_{k-s})$, where X' is a τ-exceptional sequence in $J_{\operatorname{mod} A}(Z_{k+1})$ and Y is a τ-exceptional sequence in $\operatorname{mod} B$. Since Z_{k+1} is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$, it is also τ-rigid $\operatorname{mod} A$, hence $X = (X_1, X_2, \ldots, X_s, Z_{k+1})$ is a τ-exceptional sequence in $\operatorname{mod} A$. Clearly Z is an interleaved sequence X and Y, so this completes the proof by induction.

□

We will now recall some standard definitions from Assem *et al.* (2006) which we require for the rest of this chapter. Recall that the radical of an A-module M, denoted by $\operatorname{rad}(M)$, is defined to be the intersection of all maximal submodules of M. The quotient $M/\operatorname{rad}(M)$ is known as the top of M and is denoted $\operatorname{top}(M)$. The socle of an A-module M denoted $\operatorname{soc}(M)$ is the sum of the simple submodules of M.

Definition 3.3.10. Radical Series (Assem *et al.*, 2006, V.1). Let M be an A-module. The radical series of M is defined to be the following sequence of submodules,

$$0 \subset \cdots \subset \operatorname{rad}^2(M) \subset \operatorname{rad}(M) \subset M.$$

Since the A-modules M are finite-dimensional as $\mathbb{F}$-vector spaces, there exists a least positive integer m such that $\operatorname{rad}^m(M) = 0$. The integer m is called the length of the radical series and we denote it by $l(M) = m$. We will also refer to $l(M)$ as the length of the module M.

Proposition 3.3.11. (Assem *et al.*, 2006, V.3.5, V.4.1, V.4.2). Let A be a basic connected Nakayama algebra and let M be an indecomposable A-module. Then there exists some $1 \leq i \leq n$ and $1 \leq j \leq l(P_i)$, such that $M \cong P_i/\operatorname{rad}^j(P_i)$ and $j = l(M)$. Moreover, if M is not projective, we have that $\tau M \cong \operatorname{rad}(P_i)/\operatorname{rad}^{j+1}(P_i)$ and $l(\tau M) = l(M)$.

So we see that modules M of Nakayama algebras are uniquely determined by their top, $\operatorname{top}(M)$ and their length $l(M)$.

Proposition 3.3.12. (Adachi, 2016, Lemma 2.4). Let $M = P_j/\operatorname{rad}^l(P_j)$ and $N = P_i/\operatorname{rad}^k(P_i)$ for $1 \leq i, j, k, l \leq n$. Then the following conditions are equivalent,

1. $\operatorname{Hom}(M, N) \neq 0$

2. $j \in [i, (i + k - 1)]_n$ and $(i + k - 1)_n \in [j, (j + l - 1)]_n$

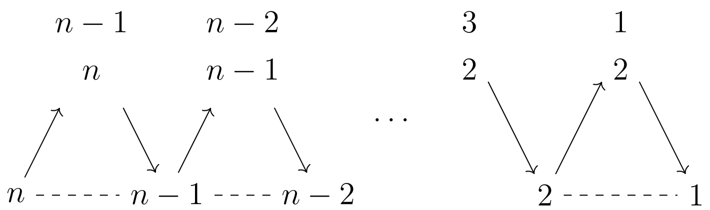
3.4 The Γ_n^2 case

Let $n \geq 1$ be a positive integer. In this section we will derive a closed formula for the number of complete τ-exceptional sequences in $\operatorname{mod}\Gamma_n^2$. Recall that we denote by A_n the linearly oriented quiver with n vertices,

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n.$$

The algebra Γ_n^2 is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}A_n/R_Q^2$. This is the path algebra of the quiver A_n modulo the relations $\alpha_i\alpha_{i+1} = 0$ for $1 \leq i \leq n-2$.

The category $\operatorname{mod}\Gamma_n^2$ has the following Auslander-Reiten quiver.



Our strategy for calculating the number of τ-exceptional sequences is straightforward. For each M in $\operatorname{ind}(\operatorname{mod}\Gamma_n^2)$, we will calculate the number of complete τ-exceptional sequences ending in M. If M is indecomposable, then either $M = P_i$, the projective at vertex i of A_n, or $M = S_i$, the simple at vertex i of A_n (notice that $S_n = P_n$). In the former case $\tau P_i = 0$ for $1 \leq i \leq n$ and in the latter case $\tau S_j = S_{j+1}$ for $1 \leq j \leq n-1$. In both cases we see that M is τ-rigid i.e. every indecomposable M in $\operatorname{mod}\Gamma_n^2$ is τ-rigid. We recall that a sequence of indecomposable modules $(M_1, M_2, \ldots, M_{n-1}, M)$ is a τ-exceptional sequence in $\operatorname{mod}\Gamma_n^2$ if M is τ-rigid, and $(M_1, M_2, \ldots, M_{n-1})$ is a τ-exceptional sequence in $J(M)$. Having seen that every indecomposable module M is τ-rigid, what is left to do is to calculate $J(M)$ for each indecomposable module. Theorem 3.3.5 and Lemma 3.3.4 are the main tools for these calculations.

Proposition 3.4.1. Let P_i be an indecomposable projective module in $\operatorname{mod}\Gamma_n^2$ for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in $\operatorname{mod}\Gamma_n^2$ is $J(P_i) \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i}^2$.

Proof. By definition $T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i))$. Since $\tau P_i = 0$, we have that ${}^{\perp}(\tau P_i) = \operatorname{mod}\Gamma_n^2$. As a result the Ext-projectives of ${}^{\perp}(\tau P_i)$ are just the projectives of $\operatorname{mod}\Gamma_n^2$, hence

$$T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i)) = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \operatorname{End}_{\Gamma_n^2}(T_{P_i})$ is precisely given by the path algebra of A_n^{op},

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \ldots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_i} i \xleftarrow{\alpha_{i+1}} i+1 \xleftarrow{\alpha_{i+2}} \ldots \xleftarrow{\alpha_n} n$$

modulo the relations $\alpha_j\alpha_{j-1} = 0$ for $3 \leq j \leq n$. Let $A_n^{\text{op}(i)}$ be the quiver obtained from A_n^{op} by removing the vertex i and any arrows incident to i,

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \ldots \xleftarrow{\alpha_{i-1}} i-1 \qquad\qquad i+1 \xleftarrow{\alpha_{i+2}} \ldots \xleftarrow{\alpha_n} n.$$

The quiver $A_n^{\text{op}(i)}$ has relations $\alpha_j\alpha_{j-1} = 0$ for $3 \leq j \leq i-1$ and $i+3 \leq j \leq n$. By Lemma 3.3.4, $D_{P_i} = E_{P_i}/\langle e_{P_i}\rangle$ is the path algebra of $A_n^{\text{op}(i)}$ modulo its relations. So it follows that $J(P_i) \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i}^2$ by Theorem 3.3.5. $\square$

Proposition 3.4.2. Let S_i be a simple non-projective module in $\operatorname{mod}\Gamma_n^2$ for some $1 \leq i \leq n-1$. Then the τ-perpendicular category of S_i in $\operatorname{mod}\Gamma_n^2$ is $J(S_i) \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i-1}^2 \oplus \operatorname{mod}\Gamma_1^2$.

Proof. By definition $T_{S_i} = \mathcal{P}({}^{\perp}(\tau S_i))$. Since S_i is a simple non-projective indecomposable module S_i, we have that $\tau S_i = S_{i+1}$. Note that the only indecomposable Γ_n^2-modules not in ${}^{\perp}(\tau S_i)$ are S_{i+1} and P_{i+1}. Observe also that $\mathrm{Ext}_{\Gamma_n^2}(P_j, {}^{\perp}(\tau S_i)) = 0$ if $j \neq i+1, 1 \leq j \leq n$. We also have that $\mathrm{Ext}_{\Gamma_n^2}(S_j, {}^{\perp}(\tau S_i)) \neq 0$ for $j \neq i$, and $1 \leq j \leq n$ because S_{j+1} is in ${}^{\perp}(\tau S_i)$ in these cases. By Proposition 3.3.1, S_i is Ext-projective in ${}^{\perp}(\tau S_i)$. Therefore

$$T_{S_i} = \mathcal{P}({}^{\perp}(\tau S_i)) = S_i \oplus \bigoplus_{j \neq i+1} P_j.$$

The $\mathbb{F}$-algebra $E_{S_i} = \mathrm{End}_{\Gamma_n^2}(T_{S_i})$ is the path algebra of the following quiver,

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \dots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_{v_{S_i}}} v_{S_i} \xleftarrow{\alpha_i} i \qquad i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_n} n$$

modulo the relations $\alpha_j \alpha_{j-1} = 0$ for $3 \leq j \leq i-1$ and $i+4 \leq j \leq n$. Here the vertex v_{S_i} is the one corresponding to the simple non-projective module S_i and the rest correspond to the projective modules P_j. Consider the following quiver obtained from the one above by removing the vertex v_{S_i} and any arrows incident to v_{S_i},

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \dots \xleftarrow{\alpha_{i-1}} i-1 \qquad i \qquad i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_n} n,$$

it has the relations $\alpha_j \alpha_{j-1} = 0$ for $3 \leq j \leq i-1$ and $i+4 \leq j \leq n$. By Lemma 3.3.4, $D_{S_i} = E_{S_i}/\langle e_{S_i} \rangle$ is the path algebra of this quiver modulo its relations. So it follows that $\operatorname{mod} D_{S_i} \cong \operatorname{mod} \Gamma_{i-1}^2 \oplus \operatorname{mod} \Gamma_{n-i-1}^2 \oplus \operatorname{mod} \Gamma_1^2$. By Theorem 3.3.5, the statement of this Proposition follows. □

Let us denote by G_n the number of complete τ-exceptional sequences of $\operatorname{mod} \Gamma_n^2$. When $n = 0, 1, 2$ the τ-exceptional sequences coincide with the "classical" exceptional sequences since the algebra Γ_n^2 is the hereditary Dynkin type A algebra $\mathbb{A}_n$ in this case. Hence, $G_0 = G_1 = 1$ and $G_2 = 3$.

Lemma 3.4.3. Let P_i be the indecomposable projective module in $\operatorname{mod}\Gamma_n^2$ at the vertex i of A_n for some $1 \leq i \leq n$. The number of complete τ-exceptional sequences in $\operatorname{mod}\Gamma_n^2$ ending in P_i is,

$$\binom{n-1}{n-i, i-1} G_{n-i} G_{i-1}.$$

Proof. Let $(X_1, X_2, \ldots, X_{n-1}, P_i)$ be a complete τ-exceptional sequence in $\operatorname{mod}\Gamma_n^2$ ending in P_i. Then by definition and the fact that $\delta(J(P_i)) = n-1$, the sequence $(X_1, X_2, \ldots, X_{n-1})$ is a τ-exceptional sequence in $J(P_i)$. So to count the number of complete τ-exceptional sequences in $\operatorname{mod}\Gamma_n^2$ ending in P_i, we just need to count the number of complete τ-exceptional sequences in $J(P_i)$. By Lemma 3.4.1, $J(P_i) \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i}^2$. By Theorem 3.3.8 and 3.3.9, the τ-exceptional sequences of $J(P_i)$ are interleavings of τ-exceptional sequences of $\operatorname{mod}\Gamma_{i-1}^2$ and $\operatorname{mod}\Gamma_{n-i}^2$. The number of interleaved sequences coming from a sequence of length $i-1$ and a sequence of length $n-i$ is precisely $\binom{n-1}{n-i,i-1}$. Thus the number of complete τ-exceptional sequences ending in P_i is $\binom{n-1}{n-i,i-1} G_{n-i} G_{i-1}$. $\square$

Lemma 3.4.4. Let S_i be the indecomposable simple non-projective module in $\operatorname{mod}\Gamma_n^2$ at the vertex i of A_n for some $1 \leq i \leq n-1$. The number of τ-exceptional sequences in $\operatorname{mod}\Gamma_n^2$ ending in S_i is,

$$\binom{n-1}{n-i-1, i-1} G_{n-i-1} G_{i-1}.$$

Proof. Let $(X_1, X_2, \ldots, X_{n-1}, S_i)$ be a complete τ-exceptional sequence in $\operatorname{mod}\Gamma_n^2$ ending in S_i. Then by definition and the fact that $\delta(J(S_i)) = n-1$, the sequence $(X_1, X_2, \ldots, X_{n-1})$ is a complete τ-exceptional sequence in $J(S_i)$. Hence to count the number of complete τ-exceptional sequences in $\operatorname{mod}\Gamma_n^2$ ending in S_i, we just need to count the number of complete τ-exceptional sequences in $J(S_i)$. By Lemma 3.4.2, $J(S_i) \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i-1}^2 \oplus \operatorname{mod}\Gamma_1^2$. The number of interleaved sequences coming from a sequence of length $i-1$, a sequence of length $n-i-1$ and a sequence of length 1 is precisely $\binom{n-1}{n-i-1,i-1,1} = \binom{n-1}{n-i-1,i-1}$. Thus the number of complete τ-exceptional sequences ending in S_i is $\binom{n-1}{n-i-1,i-1} G_{n-i-1}G_{i-1}$. $\square$

Theorem 3.4.5. Let G_n denote the number of complete τ-exceptional sequences in $\operatorname{mod}\Gamma_n^2$. Then G_n satisfies the recurrence relation,

$$G_n = \sum_{i=1}^{n} \binom{n-1}{n-i,i-1} G_{n-i}G_{i-1} + \sum_{i=1}^{n-1} \binom{n-1}{n-i-1,i-1} G_{n-i-1}G_{i-1},$$

with initial conditions $G_0 = G_1 = 1$.

Proof. Let M be an indecomposable in $\operatorname{mod}\Gamma_n^2$, then either M is projective or M simple non-projective. There are n projective indecomposable modules in $\operatorname{mod}\Gamma_n^2$ denoted by P_i for $1 \le i \le n$. There are $n-1$ simple non-projective indecomposable modules in $\operatorname{mod}\Gamma_n^2$ denoted by S_i for $1 \le i \le n-1$. Therefore by Lemma 3.4.3 and 3.4.4, $G_n = \sum_{i=1}^{n} \binom{n-1}{n-i,i-1} G_{n-i}G_{i-1} + \sum_{i=1}^{n-1} \binom{n-1}{n-i-1,i-1} G_{n-i-1}G_{i-1}$. $\square$

Theorem 3.4.5 allows us to calculate the first ten terms of the sequence $(G_n)_{n=0}^{\infty}$ as:

$$1, 1, 3, 12, 66, 450, 3690, 35280, 385560, 4740120, 6475140.$$

An ordered set partition of $\{1, 2, \ldots, n\}$ is a partition of the set $\{1, 2, \ldots, n\}$ together with a total order on the sets in the partition. We refer to the sets in an ordered partition as blocks. The **restricted Fubini** number $F_{n,\leqslant m}$ counts the number of ordered set partitions of $\{1, 2, \ldots, n\}$ with blocks of size at most m. The **restricted Stirling number of the second kind**, denoted by $\left\{ {n \atop k} \right\}_{\leqslant m}$, is the number of (unordered) partitions of $\{1, 2, \ldots, n\}$ into k subsets with the restriction that each block contains at most m elements. Therefore

$$F_{n,\leqslant m} = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\leqslant m}.$$

It is shown in (Mezo, 2014, Section 5.4) that the restricted Fubini numbers satisfy the recurrence:

$$F_{n,\leqslant m} = \sum_{l=1}^{m} \binom{n}{l} F_{n-l,\leqslant m}.$$

The sequence $(F_{n,\leqslant 2})$ is listed on the On-line Encyclopedia of Integer Sequences (OEIS) as the sequence A080599. The first terms of this sequence coincide with the first terms we calculated for (G_n) so we would like to prove that it is the case that $F_{n,\leqslant 2} = G_n$.

When $m = 2$ the recurrence for $F_{n,\leqslant m}$ is given as $F_{n,\leqslant 2} = nF_{n-1,\leqslant 2} + \binom{n}{2} F_{n-2,\leqslant 2}$. In (Gellert & Sanyal, 2017, Theorem 3.7), the authors derive the closed formula

$$F_{n,\leqslant 2} = \frac{n!}{\sqrt{3}} \left((\sqrt{3}-1)^{-n-1} - (-\sqrt{3}-1)^{-n-1} \right).$$

An exponential generating function for $F_{n,\leqslant m}$ is given in (Komatsu & Ramírez, 2018, Theorem 4):

$$\sum_{n=0}^{\infty} F_{n,\leqslant m} \frac{x^n}{n!} = \frac{1}{1 - x - \frac{x^2}{2!} - \dots \frac{x^m}{m!}}.$$

We will show that $G_n = F_{n,\leqslant 2}$ by showing that the exponential generating functions for G_n and $F_{n,\leqslant 2}$ coincide.

Theorem 3.4.6. Let G_n denote the number of complete τ-exceptional sequences in $\operatorname{mod} \Gamma_n^2$. The exponential generating function of G_n is as follows,

$$\sum_{n=0}^{\infty} G_n \frac{x^n}{n!} = \frac{1}{1 - x - \frac{x^2}{2!}}.$$

Therefore $G_n = F_{n,\leqslant 2}$ and

$$G_n = \frac{n!}{\sqrt{3}} \left((\sqrt{3}-1)^{-n-1} - (-\sqrt{3}-1)^{-n-1} \right).$$

Proof. First let us recall the recurrence relation for G_n.

$$G_n = \sum_{i=1}^{n} \binom{n-1}{n-i, i-1} G_{n-i} G_{i-1} + \sum_{i=1}^{n-1} \binom{n-1}{n-i-1, i-1} G_{n-i-1} G_{i-1}.$$

$$= \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!(i-1)!} G_{n-i} G_{i-1} + \sum_{i=1}^{n-1} \frac{(n-1)!}{(n-i-1)!(i-1)!} G_{n-i-1} G_{i-1}.$$

Therefore

$$G_{n+1} = \sum_{i=1}^{n+1} \frac{n!}{(n+1-i)!(i-1)!} G_{n+1-i} G_{i-1} + \sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} G_{n-i} G_{i-1}.$$

Let

$$g(x) = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!} \text{ with } g(0) = 1,$$

be the exponential generating function of G_n. We then have that the first derivative of $g(x)$ is $g'(x) = \sum_{n=0}^{\infty} G_{n+1} \frac{x^n}{n!}$. Expanding G_{n+1} in $g'(x)$ by the recurrence relation above we obtain the following.

$$\begin{aligned} g'(x) &= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n+1} \frac{n!}{(n+1-i)!(i-1)!} G_{n+1-i} G_{i-1} \right) \frac{x^n}{n!} + \\ &\quad \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} G_{n-i} G_{i-1} \right) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n+1} \frac{G_{n+1-i} G_{i-1}}{(n+1-i)!(i-1)!} \right) x^n + \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} \frac{G_{n-i} G_{i-1}}{(n-i)!(i-1)!} \right) x^n. \end{aligned}$$

Recall the Cauchy product of formal power series is as follows,

$$\left(\sum_{s=0}^{\infty} a_s x^s \right) \left(\sum_{t=0}^{\infty} b_t x^t \right) = \sum_{k=0}^{\infty} c_k x^k \text{ where } c_k = \sum_{l=0}^{k} a_l b_{k-l}.$$

By performing a change of variable in $g'(x)$ by setting $j = i - 1$ and factorising x from the right summand we write,

$$g'(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{G_{n-j} G_j}{(n-j)!j!} \right) x^n + x \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n-1} \frac{G_{n-j-1} Gj}{(n-j-1)!j!} \right) x^{n-1}.$$

Using the Cauchy product of formal power series, we obtain the following first order non-linear ordinary differential equation.

$$g'(x) = (g(x))^2 + x(g(x))^2 = (1+x)(g(x))^2 \text{ with initial conditions } g(0) = 1.$$

It is easy to check that the unique solution to this ODE is given by,

$$g(x) = \frac{-2}{-2 + x(x+2)} = \frac{1}{1 - x - \frac{x^2}{2}}.$$

This completes the proof. □

Remark 3.4.7. Here we focused on τ-exceptional sequences, but it's natural to ask what is known about the more classical exceptional sequences. It is shown in Sen (2020a) that the number of complete exceptional sequences of $\operatorname{mod}\Gamma_n^2$ are equal to the sum, $\sum_{j=1}^{n}\binom{n}{j}j^{n-j}$. The first ten terms of the sequence $(\sum_{j=1}^{n}\binom{n}{j}j^{n-j})_{n=1}^{\infty}$ are,

$$1, 3, 10, 41, 196, 1057, 6322, 41393, 293608, 2237921.$$

For comparison the number of complete τ-exceptional sequences of $\operatorname{mod}\Gamma_n^2$ are given by G_n, the first ten terms of the sequence $(G_n)_{n=1}^{\infty}$ are,

$$1, 3, 12, 66, 450, 3690, 35280, 385560, 4740120, 6475140.$$

3.5 The Λ_n^2 case

Let $n \geq 1$ be a positive integer. In this section we will derive a closed formula for the number of complete τ-exceptional sequences in $\operatorname{mod}\Lambda_n^2$. Recall that we denote by C_n the linearly oriented n-cycle.

$$\begin{tikzcd}
& 1 \arrow[dr, "\alpha_1"] & \\
n \arrow[ur, "\alpha_n"] & & 2 \arrow[d, "\alpha_2"] \\
n-1 \arrow[u, "\alpha_{n-1}"] & & 3 \arrow[dl, "\alpha_3"] \\
& \cdots \arrow[ul, "\alpha_{n-2}"] &
\end{tikzcd}$$

The algebra Λ_n^2 is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}C_n/R_Q^2$. This is the path algebra of the quiver C_n modulo the relations $\alpha_j\alpha_{(j+1)_n} = 0$ for $1 \leq j \leq n$.

The category $\operatorname{mod}\Lambda_n^2$ has the following Auslander-Reiten quiver.

$$\begin{tikzcd}[column sep=small]
\cdots & \begin{smallmatrix}n\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}n-1\\n\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}n-2\\n-1\end{smallmatrix} \arrow[dr] & & \cdots & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}n\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}n-1\\n\end{smallmatrix} & \cdots \\
& & n \arrow[ur] \arrow[rr, dashed, no head] & & n-1 \arrow[ur] \arrow[rr, dashed, no head] & & n-2 & & & 2 \arrow[ur] \arrow[rr, dashed, no head] & & 1 \arrow[ur] \arrow[rr, dashed, no head] & & n \arrow[ur] & &
\end{tikzcd}$$

We will use the same approach for calculating the number of complete τ-exceptional sequences for $\operatorname{mod}\Lambda_n^2$ as we did for $\operatorname{mod}\Gamma_n^2$. If M is indecomposable in $\operatorname{mod}\Lambda_n^2$, then $M = P_i$, the projective at vertex i of C_n, or $M = S_i$, the simple at vertex i of C_n. In the former case $\tau P_i = 0$ and in the latter case $\tau S_i = S_{(i+1)_n}$. In both cases M is τ-rigid i.e. every indecomposable M in $\operatorname{mod}\Lambda_n^2$ is τ-rigid. We recall that a sequence of indecomposable modules $(M_1, M_2, \ldots, M_{n-1}, M)$ is a τ-exceptional sequence in $\operatorname{mod}\Lambda_n^2$ if M is τ-rigid, and $(M_1, M_2, \ldots, M_{n-1})$ is a τ-exceptional sequence in $J(M)$. Having seen that every indecomposable module is τ-rigid, what is left to do is to calculate $J(M)$ for each indecomposable module. Theorem 3.3.5 and Lemma 3.3.4 are the main tools for these calculations.

Proposition 3.5.1. Let P_i be an indecomposable projective module in $\operatorname{mod}\Lambda_n^2$ for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in $\operatorname{mod}\Lambda_n^2$ is $J(P_i) \cong \operatorname{mod}\Gamma_{n-1}^2$.

Proof. By definition $T_{P_i} = \mathcal{P}({}^\perp(\tau P_i))$. Since P_i is projective, we have that $\tau P_i = 0$, therefore ${}^\perp(\tau P_i) = \operatorname{mod}\Lambda_n^2$. As a result the Ext-projectives of ${}^\perp(\tau P_i)$ are just the projectives of $\operatorname{mod}\Lambda_n^2$, hence

$$T_{P_i} = \mathcal{P}({}^\perp(\tau P_i)) = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \operatorname{End}_{\Lambda_n^2}(T_{P_i})$ is precisely given by the path algebra of the quiver C_n^{op},

$$\begin{tikzcd}
& 1 \arrow[dl, "\alpha_1"'] & \\
n \arrow[d, "\alpha_n"'] & & 2 \arrow[ul, "\alpha_2"'] \\
n-1 \arrow[dr, "\alpha_{n-1}"'] & & 3 \arrow[u, "\alpha_3"'] \\
& \cdots \arrow[ur, "\alpha_4"'] &
\end{tikzcd}$$

modulo the relations $\alpha_i\alpha_{(i-1)_n} = 0$ for $1 \leq i \leq n$.

Let $C_n^{\mathrm{op}(i)}$ be the quiver obtained from C_n^{op} by removing the vertex at i and any arrows incident to i.

$$i+1 \xleftarrow{\alpha_{i+2}} \cdots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{i-1}} i-1$$

It has the relations $\alpha_1\alpha_n = 0$ and $\alpha_j\alpha_{j-1} = 0$ for $i+2 \leq j \leq n$ and $2 \leq j \leq i-2$. By Lemma 3.3.4, $D_{P_i} = E_{P_i}/\langle e_{P_i}\rangle$ is the path algebra of the quiver $C_n^{\mathrm{op}(i)}$ modulo relations. It is easy to see that in fact D_{P_i} is isomorphic to Γ^2_{n-1}. Hence by Theorem 3.3.5, the τ-perpendicular category $J(M) \cong \operatorname{mod}\Gamma^2_{n-1}$. $\square$

Proposition 3.5.2. Let S_i be a simple module in $\operatorname{mod}\Lambda_n^2$ for some $1 \leq i \leq n$. Then the τ-perpendicular category of S_i in $\operatorname{mod}\Lambda_n^2$ is $J(S_i) \cong \operatorname{mod}\Gamma^2_{n-2} \oplus \operatorname{mod}\Gamma^2_1$.

Proof. By definition $T_{S_i} = \mathcal{P}({}^{\perp}(\tau S_i))$. Since S_i is a simple Λ_n^2-module, we have that $\tau S_i = S_{(i+1)_n}$. Note that the only Λ_n^2-modules not in ${}^{\perp}(\tau S_i)$ are $S_{(i+1)_n}$ and $P_{(i+1)_n}$. Observe also that $\operatorname{Ext}_{\Lambda_n^2}(P_j, {}^{\perp}(\tau S_i)) = 0$ for $j \neq (i+1)_n$ and $1 \leq j \leq n$. However for $j \neq i, i+1$, $\operatorname{Ext}_{\Lambda_n^2}(S_j, {}^{\perp}(\tau S_i)) \neq 0$ because $S_{(j+1)_n}$ is in ${}^{\perp}(\tau M)$. By Proposition 3.3.1, S_i is Ext-projective in ${}^{\perp}(\tau S_i)$. Hence

$$T_{S_i} = \mathcal{P}({}^{\perp}(\tau S_i)) = S_i \oplus \bigoplus_{j \neq (i+1)_n} P_j$$

is the Bongartz completion of S_i.

The $\mathbb{F}$-algebra $E_{S_i} = \operatorname{End}_{\Lambda_n^2}(T_{S_i})$ is given by the path algebra of the quiver,

$$i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_{v_{S_i}}} v_{S_i} \xleftarrow{\alpha i} i$$

modulo the relations $\alpha_{v_{S_i}}\alpha_{i-1} = 0 = \alpha_1\alpha_n$ and $\alpha_j\alpha_{j-1} = 0$ for $i+4 \le j \le n$ and $2 \le j \le i-1$. Here the vertex v_{S_i} is the one corresponding to the simple module S_i and the rest correspond to the projective modules P_j. By Lemma 3.3.4, $D_{S_i} = E_{S_i}/\langle e_{S_i}\rangle$ is the path algebra of the quiver obtained from the one above by removing the vertex v_{S_i},

$$i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{i-1}} i-1 \qquad\qquad i$$

modulo the relations $\alpha_1\alpha_n = 0$ and $\alpha_j\alpha_{j-1} = 0$ for $i+4 \le j \le n$ and $2 \le j \le i-1$. So it follows that $\operatorname{mod} D_{S_i} \cong \operatorname{mod} \Gamma^2_{n-2} \oplus \operatorname{mod} \Gamma^2_1$. By Theorem 3.3.5 the statement of this Proposition follows. □

Denote by L_n the number of complete τ-exceptional sequences in $\operatorname{mod} \Lambda_n^2$.

Theorem 3.5.3. Let L_n be the number of complete τ-exceptional sequences in $\operatorname{mod} \Lambda_n^2$. Then L_n satisfies the relation,

$$L_n = nG_{n-1} + n(n-1)G_{n-2},$$

with initial conditions $L_1 = 1$ and $L_2 = 4$, and where G_m denotes the number of complete τ-exceptional sequences in $\operatorname{mod} \Gamma^2_m$.

Proof. Suppose M is an indecomposable projective Λ_n^2-module, then by Lemma 3.5.1, the τ-perpendicular category $J(M) \cong \operatorname{mod} \Gamma^2_{n-1}$. Suppose that the sequence $(X_1, X_2, \dots, X_{n-1}, M)$ is a complete τ-exceptional sequence ending in M in $\operatorname{mod} \Lambda_n^2$. Then by the fact that $\delta(J(M)) = n-1$ and by definition, the sequence $(X_1, X_2, \dots, X_{n-1})$ is a complete τ-exceptional sequence in $J(M) \cong \operatorname{mod} \Gamma^2_{n-1}$. Hence the number of complete τ-exceptional sequences ending in M is G_{n-1}, which is the number of complete τ-exceptional sequences in $\operatorname{mod} \Gamma^2_{n-1}$.

Now suppose M is a simple Λ_n^2-module. By Lemma 3.5.2, the τ-perpendicular category $J(M) \cong \operatorname{mod}\Gamma_{n-2}^2 \oplus \operatorname{mod}\Gamma_1^2$. Arguing as above the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $J(M)$. Since $J(M) \cong \operatorname{mod}\Gamma_{n-2}^2 \oplus \operatorname{mod}\Gamma_1^2$, by Theorem 3.3.8 and 3.3.9, the τ-exceptional sequences of $J(M)$ are interleavings of τ-exceptional sequences of $\operatorname{mod}\Gamma_{n-2}^2$ and $\operatorname{mod}\Gamma_1^2$. The number of interleaved sequences coming from a sequence of length $n-2$ and a sequence of length 1 is precisely $\binom{n-1}{n-2,1} = (n-1)$. Thus the number of complete τ-exceptional sequences ending in M is $(n-1)G_{n-2}G_1 = (n-1)G_{n-2}$.

An arbitrary indecomposable Λ_n^2-module is either projective or simple. There are n projective modules and n simple modules up to isomorphism in $\operatorname{mod}\Lambda_n^2$, hence the number of complete τ-exceptional sequences in $\operatorname{mod}\Lambda_n^2$ is $L_n = nG_{n-1} + n(n-1)G_{n-2}$. It then follows easily that $L_1 = 1$ and $L_2 = 4$. $\square$

In the previous section we found the exponential generating function and closed formula for G_n. Using the above theorem, we can immediately do the same for L_n.

Theorem 3.5.4. Let L_n denote the number of complete τ-exceptional sequences in $\operatorname{mod}\Lambda_n^2$. The exponential generating function of L_n is as follows,

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = \frac{x + x^2}{1 - x - \frac{x^2}{2}} .$$

Proof. Let $h(x) = \sum_{n=0}^{\infty} L_n \frac{x^n}{n!}$ be the exponential generating function of L_n. Let

$$g(x) = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!}$$

be the exponential generating function of G_n. We then recall the recurrence relation of L_n,

$$L_n = nG_{n-1} + n(n-1)G_{n-2}.$$

Therefore the exponential generating function of L_n is,

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} nG_{n-1}\frac{x^n}{n!} + \sum_{n=0}^{\infty} n(n-1)G_{n-2}\frac{x^n}{n!}.$$

$$= \sum_{n=0}^{\infty} G_{n-1}\frac{x^n}{(n-1)!} + \sum_{n=0}^{\infty} G_{n-2}\frac{x^n}{(n-2)!}$$
$$= x\sum_{n=0}^{\infty} G_{n-1}\frac{x^{n-1}}{(n-1)!} + x^2\sum_{n=0}^{\infty} G_{n-2}\frac{x^{n-2}}{(n-2)!}.$$

Therefore

$$h(x) = xg(x) + x^2(g(x)) = (x + x^2)g(x).$$

By Theorem 3.4.6,

$$g(x) = \frac{1}{1 - x - \frac{x^2}{2}},$$

hence

$$h(x) = \frac{x + x^2}{1 - x - \frac{x^2}{2!}} .$$

□

Theorem 3.5.5. Let L_n denote the number of complete τ-exceptional sequences in $\operatorname{mod}\Lambda_n^2$. Then L_n is given by the closed formula,

$$L_n = \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-2} - (-\sqrt{3}-1)^{-n-2}\right) + \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-3} - (-\sqrt{3}-1)^{-n-3}\right).$$

Proof. It is immediate from the recurrence relation for L_n and Theorem 3.4.6 that,

$$L_n = n\frac{(n-1)!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-2} - (-\sqrt{3}-1)^{-n-2}\right)$$

$$+n(n-1)\frac{(n-2)!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-3} - (-\sqrt{3}-1)^{-n-3}\right)$$

$$= \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-2} - (-\sqrt{3}-1)^{-n-2}\right) + \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-3} - (-\sqrt{3}-1)^{-n-3}\right).$$

□

We calculate the first 10 terms of the sequence $(L_n)_{n=0}^{\infty}$ to be,

$$1, 4, 15, 84, 570, 4680, 44730, 488880, 6010200, 82101600.$$

In comparison to τ-exceptional sequences, there are no complete exceptional sequences in $\operatorname{mod}\Lambda_n^2$, as we will show. In general, not much is known about exceptional sequences over the Nakayama algebras Λ_n^2.

Proposition 3.5.6. There are no complete exceptional sequences in $\operatorname{mod}\Lambda_n^2$ when $n > 1$.

Proof. Suppose $M = (M_1, M_2, \ldots, M_n)$ is a complete exceptional sequence. Recall that an indecomposable module in $\operatorname{mod}\Lambda_n^2$ is either projective or simple. Since $\operatorname{Hom}(P_i, P_{(i+1)_n}) \neq 0$ for $1 \leq i \leq n$, the sequence M cannot consist entirely of just indecomposable projective modules, so M must contain at least one simple module.

Consider the simple module S_i for some $1 \leq i \leq n$. Then S_i has the following infinite exact sequence as its projective resolution.

$$\ldots \longrightarrow P_1 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

We observe that the projective resolution of S_i contains every projective indecomposable module of $\operatorname{mod}\Lambda_n^2$. We also observe that the only projective module P such that $\operatorname{Hom}(P, S_i) \neq 0$ is $P = P_i$. Hence applying the functor $\operatorname{Hom}(-, S_i)$ to the above projective resolution we get the following sequence.

$$0 \longrightarrow \operatorname{Hom}(S_i, S_i) \xrightarrow{f_0} \operatorname{Hom}(P_i, S_i) \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} 0 \xrightarrow{f_n} \operatorname{Hom}(P_i, S_i) \xrightarrow{f_{n+1}} 0 \xrightarrow{f_{n+2}} \ldots$$

We then observe that $\operatorname{Ext}^n(S_i, S_i) = \ker(f_{n+1})/\operatorname{im}(f_n) \neq 0$, so no simple module in $\operatorname{mod}\Lambda_n^2$ is exceptional. From this we conclude M cannot contain simple modules, a contradiction. $\square$

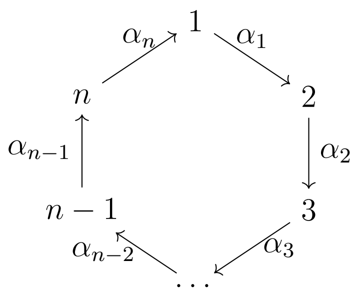
Unlike exceptional sequences, weak exceptional sequences number over Λ_n^2 have been studied. Weak exceptional sequences are defined in Sen (2020b) as follows.

Definition 3.5.7. (Sen, 2020b, Definition 1.1). Let A be a finite-dimensional algebra over a field $\mathbb{F}$, where $\mathbb{F}$ is algebraically closed. A left A-module M is called **weak exceptional** if $\mathrm{Hom}(M, M) \cong \mathbb{F}$ and $\mathrm{Ext}^1_A(M, M) = 0$. A sequence of indecomposable modules $(M_1, M_2, \ldots, M_r)$ is called a **weak exceptional sequence** if, for each pair (M_i, M_j) with $1 \leq i < j \leq r$, we have that $\mathrm{Hom}(M_j, M_i) = \mathrm{Ext}^1_A(M_j, M_i) = 0$ and each M_k is weak exceptional for $1 \leq k \leq r$.

It turns out that for weak exceptional sequences over Λ_n^2, the maximum length need not be n and in fact can exceed n. According to (Sen, 2020b, Theorem 1.6), if $n = 2m + 1$ is odd, the maximum length of a weak exceptional sequence over Λ_n^2 is equal to $3m + 1$. On the other hand, if $n = 2m$ is even, then the maximum length of a weak exceptional sequence over Λ_n^2 is $3m - 1$. A weak exceptional sequence with maximum length is called **full**. Again by (Sen, 2020b, Theorem 1.6), when $n = 2m$, the number of full weak exceptional sequences is given by $2m\left(\frac{8^m}{12} - \frac{(-1)^m}{3} + 1\right)$, and when $n = 2m + 1$, the number of full weak exceptional sequences is given by n.

3.6 The Λ_n^n case

Let $n \geq 1$ be a positive integer. In this section we will derive a closed formula for the number of complete τ-exceptional sequences in mod Λ_n^n. Recall that we denote by C_n the linearly oriented n-cycle.



The algebra Λ_n^n is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}C_n/R_Q^n$. This is the path algebra of the quiver C_n modulo the relations $\alpha_i\alpha_{(i+1)_n}\dots\alpha_{(i+(n-1))_n} = 0$ for $1 \leq j \leq n$.

Proposition 3.6.1. (Adachi, 2016, Proposition 2.5). Let A be a Nakayama algebra. Let M be an indecomposable non-projective module in $\operatorname{mod} A$. Then M is rigid if and only if $l(M) < n$ holds.

For our purposes, the following Proposition is a more convenient restatement of Proposition 3.3.12.

Proposition 3.6.2. Let M be an indecomposable Λ_n^n-module with length $1 \leq l(M) \leq n-1$. Then $\operatorname{Hom}(X, \tau M) \neq 0$ if and only if $\operatorname{top}(X) \cong \operatorname{top}(\operatorname{rad}^k(\tau M))$ for some $0 \leq k \leq l(M)-1$ and $l(\operatorname{rad}^k(\tau M)) \leq l(X)$.

Proof. All indecomposable modules in $\operatorname{mod}\Lambda_n^n$ have simple tops. By Proposition 3.3.11, for a Λ_n^n-module X, we have that $X = P_j/\operatorname{rad}^{l(X)}(P_j)$ hence $\operatorname{top}(X) = S_j$. Let $M = P_{i-1}/\operatorname{rad}^{l(M)}P_{i-1}$, then $\tau M = P_{(i)_n}/\operatorname{rad}^{l(M)}(P_{(i)_n})$ by Proposition 3.3.11 as well. Observe that for $0 \leq k \leq l(M)-1$, $\operatorname{rad}^k(\tau M) = P_{(i+k)_n}/\operatorname{rad}^{(l(M)-k)}(P_{(i+k)_n})$ thus $\operatorname{top}(\operatorname{rad}^k(\tau M)) = S_{(i+k)_n}$ and $l(\operatorname{rad}^k(\tau M)) = l(M)-k$. By Proposition 3.3.12 we have that,

$$\operatorname{Hom}(X, \tau M) \neq 0$$

if and only if

$$j \in [i, (i + l(M) - 1)]_n \text{ and } (i + l(M) - 1)_n \in [j, (j + l(X) - 1)]_n.$$

Suppose $\operatorname{Hom}(X, \tau M) \neq 0$, this implies that $j = (i + k)_n$ for some $0 \leq k \leq l(M) - 1$, and $(i + l(M) - 1)_n = (j + a)_n$ for some $0 \leq a \leq l(X) - 1$. It then immediately follows $\operatorname{top}(X) \cong \operatorname{top}(\operatorname{rad}^k(\tau M))$ and $l(\operatorname{rad}^k(\tau M)) \leq l(X)$.

For the converse, suppose that $\operatorname{top}(X) \cong \operatorname{top}(\operatorname{rad}^k(\tau M))$ and $l(\operatorname{rad}^k(\tau M)) \leq l(X)$. Then $j = (i + k)_n$ for some $0 \leq k \leq l(M) - 1$. Moreover, $l(\operatorname{rad}^k(\tau M)) = l(M) - k \leq l(X)$ which implies $i + l(M) - 1 \leq (i + k) + l(X) - 1$ therefore

$(i+l(M)-1)_n \in [j,(j+l(X)-1)]_n$. Hence by Proposition 3.3.12, $\mathrm{Hom}(X,\tau M) \neq 0$. □

By Proposition 3.6.1, every indecomposable module M of $\operatorname{mod}\Lambda_n^n$ is τ-rigid in $\operatorname{mod}\Lambda_n^n$ since it is either projective or has length $l(M) < n$. Hence, we once again adopt the same strategy for calculating the number of complete τ-exceptional sequences in $\operatorname{mod}\Lambda_n^n$ as we have done thus far. For each M in $\operatorname{ind}(\operatorname{mod}\Lambda_n^n)$, we will calculate the number of complete τ-exceptional sequences ending in M. By definition a sequence of indecomposable modules $(M_1, M_2, \ldots, M_{n-1}, M)$ is a τ-exceptional sequence in $\operatorname{mod}\Lambda_n^n$ if M is τ-rigid and $(M_1, M_2, \ldots, M_{n-1})$ is a τ-exceptional sequence in $J(M)$. Having seen that every indecomposable Λ_n^n-module M is τ-rigid, what is left to do is to calculate $J(M)$ for each indecomposable module. Theorem 3.3.5 and Lemma 3.3.4 are once again the main tools these calculations.

Proposition 3.6.3. Let M be an indecomposable Λ_n^n-module with length $1 \leq l(M) \leq n-1$ and $\mathrm{top}(M) = S_i$. Then for all $1 \leq k \leq l(M)-1$,

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{\substack{1\leq j\leq n \\ j\notin[i+1,i+l(M)]_n}} P_j.$$

Proof. Suppose the Λ_n^n-module M has top equal to $\mathrm{top}(M) = S_i$ and has length $1 \leq l(M) \leq n-1$ i.e. M is not projective. By Proposition 3.3.11, $M = P_i/\mathrm{rad}^{l(M)}(P_i)$ and $\tau M = \mathrm{rad}(P_i)/\mathrm{rad}^{l(M)+1}(P_i)$ with $l(M) = l(\tau M)$. It is easy to see that $\mathrm{top}(\tau M) = S_{(i+1)_n}$ hence $\tau M = P_{(i+1)_n}/\mathrm{rad}^{l(M)}(P_{(i+1)_n})$.

By Proposition 3.6.2, a Λ_n^n-module X is not in ${}^{\perp}(\tau M)$ if and only if $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^k(\tau M))$ for some $0 \leq k \leq l(M)-1$ and $l(\mathrm{rad}^k(\tau M)) \leq l(X)$. Let $X = P_j/\mathrm{rad}^{l(X)}(P_j)$ for some $1 \leq j \leq n$. The statement $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^k(\tau M))$ for some $0 \leq k \leq l(M)-1$ means that $j = (i+1+k)_n$ for some $0 \leq k \leq l(M)-1$. With this we are able to determine the Ext-projectives in ${}^{\perp}(\tau M)$.

Let $Y = P_l$ be the indecomposable project at the vertex l with $l \neq (i+1+k)_n$ for some $0 \leq k \leq l(M)-1$. Then P_l is in ${}^{\perp}(\tau M)$ by Proposition 3.6.2.

Moreover $\mathrm{Ext}_{\Lambda_n^n}(P_l, {}^{\perp}(\tau M)) = 0$ since P_l is a projective Λ_n^n-module. Hence P_l is Ext-projective in ${}^{\perp}(\tau M)$.

Let $Y = \mathrm{rad}^s(M)$ for some $1 \leq s \leq l(M) - 1$. Then observe that $Y = P_{(i+s)_n}/\mathrm{rad}^{(l(M)-s)}(P_{(i+s)_n})$ meaning $l(Y) = l(M) - s$. Recall a Λ_n^n-module X is not in ${}^{\perp}(\tau Y)$ if and only if $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^r(\tau Y))$ for some $0 \leq r \leq l(M) - s - 1$ and $l(\mathrm{rad}^r(\tau Y)) \leq l(X)$. Therefore if $X = P_j/\mathrm{rad}^{l(X)}(P_j)$, then $j = (i+1+s+r)_n$ for some $0 \leq r \leq l(M) - s - 1$. This implies that $\{X : \mathrm{Hom}(X, \tau Y) \neq 0\} \subset \{X : \mathrm{Hom}(X, \tau M) \neq 0\}$, which further implies that $\mathrm{Ext}_{\Lambda_n^n}(Y, N) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Lambda_n^n}(N, \tau Y) = 0$ for all N in ${}^{\perp}(\tau M)$ by the Auslander-Reiten formula. Hence $Y = \mathrm{rad}^s(M)$ is an Ext-projective in ${}^{\perp}(\tau M)$.

By Proposition 3.3.1, M is Ext-projective in ${}^{\perp}(\tau M)$. For every other indecomposable Λ_n^n-module Y, we have that τY is in ${}^{\perp}(\tau M)$, therefore $\mathrm{Ext}_{\Lambda_n^n}(Y, \tau Y) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Lambda_n^n}(\tau Y, \tau Y) \neq 0$ i.e. they are not Ext-projective in ${}^{\perp}(\tau M)$. By definition, $T_M = \mathcal{P}({}^{\perp}(\tau M))$, hence by the above arguments,

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{\substack{1 \leq j \leq n \\ j \notin [i+1, i+l(M)]_n}} P_j.$$

□

Proposition 3.6.4. Let P_i be an indecomposable projective module in $\mathrm{mod}\,\Lambda_n^n$ for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in $\mathrm{mod}\,\Lambda_n^n$ is $J(P_i) \cong \mathrm{mod}\,\mathbb{A}_{n-1}$, where $\mathbb{A}_{n-1}$ is the Dynkin type A hereditary algebra.

Proof. By definition $T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i))$. Since P_i is projective, we have that $\tau P_i = 0$ therefore ${}^{\perp}(\tau P_i) = \mathrm{mod}\,\Lambda_n^n$. As a result the Ext-projectives of ${}^{\perp}(\tau P_i)$ are just the projectives of $\mathrm{mod}\,\Lambda_n^n$ therefore

$$T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i)) = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \mathrm{End}_{\Lambda_n^2}(T_{P_i})$ is precisely given by the path algebra of the quiver C_n^{op},

$$\begin{tikzcd}
 & 1 \arrow[dl, "\alpha_1"'] & \\
n \arrow[d, "\alpha_n"'] & & 2 \arrow[ul, "\alpha_2"'] \\
n-1 \arrow[dr, "\alpha_{n-1}"'] & & 3 \arrow[u, "\alpha_3"'] \\
 & \cdots \arrow[ur, "\alpha_4"'] &
\end{tikzcd}$$

modulo the relations $\alpha_i\alpha_{(i+1)_n}\dots\alpha_{(i+(n-1))_n} = 0$ for $1 \leq j \leq n$.

By Lemma 3.3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $C_n^{\mathrm{op}(i)}$ which is the quiver obtained from C_n^{op} by removing the vertex i. More precisely, $C_n^{\mathrm{op}(i)}$ is the quiver,

$$i+1 \xleftarrow{\alpha_{i+2}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{i-1}} i-1$$

with no relations. It is easy to see that the path algebra $\mathbb{F}C_n^{\mathrm{op}(i)}$ is isomorphic to $\mathbb{A}_{n-1}$. Hence the Proposition follows by Theorem 3.3.5. $\square$

Proposition 3.6.5. Let M be an indecomposable Λ_n^n-module with length $1 \leq l(M) \leq n-1$ and $\mathrm{top}(M) = S_i$. Which is to say that $M = P_i/\mathrm{rad}^{l(M)}(P_i)$. Then the τ-perpendicular category of M in $\mathrm{mod}\,\Lambda_n^n$ is $J(M) \cong \mathrm{mod}\,\mathbb{A}_{l(M)-1} \oplus \mathrm{mod}\Lambda_{n-l(M)}^{n-l(M)}$, where $\mathbb{A}_m$ is the Dynkin type A hereditary algebra.

Proof. By Proposition 3.6.3 the Bongartz completion of M is,

$$T_M = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{\substack{1\leq j\leq n \\ j\notin[i+1,i+l(M)]_n}} P_j.$$

Hence the $\mathbb{F}$-algebra $E_M = \mathrm{End}_{\Lambda_n^n}(T_M)$ is the path algebra of the following quiver Q_n,

$$
\begin{tikzcd}
 & v_M & v_{\mathrm{rad}(M)} \arrow[l, "\alpha_{\mathrm{rad}}"'] & v_{\mathrm{rad}^2(M)} \arrow[l, "\alpha_{\mathrm{rad}^2}"'] & \cdots \arrow[l, "\alpha_{\mathrm{rad}^3}"'] & v_{\mathrm{rad}^{(l(M)-1)}(M)} \arrow[l, "\alpha_{\mathrm{rad}^{l(M)-1}}"'] \\
 & i \arrow[u, "\alpha"] \arrow[dr, "\alpha_i"] & & & & \\
(i-q)_n \arrow[ur, "\alpha_{(i-q)}"] & & (i-1)_n \arrow[d, "\alpha_{i-1}"] & & & \\
(i-q-1)_n \arrow[u, "\alpha_{(i-q-1)}"] & & (i-2)_n \arrow[dl, "\alpha_{i-2}"] & & & \\
 & \cdots \arrow[ul, "\alpha_{(i-q-2)}"] & & & &
\end{tikzcd}
$$

modulo the relations $\alpha_r \alpha_{r-1} \dots \alpha_{r-(n-l(M)-1)} = 0$ for $r \in [i, i-(n-l(M)-1)]_n$ and where $q = n - l(M) - 1$.

Let $Q_n^{(v_M)}$ be the quiver obtained from Q_n by removing the vertex v_M and any arrows incident to v_M. More precisely, $Q_n^{(v_M)}$ is the following quiver with two connected components,

$$
\begin{tikzcd}
 & & v_{\mathrm{rad}(M)} & v_{\mathrm{rad}^2(M)} \arrow[l, "\alpha_{\mathrm{rad}^2}"'] & \cdots \arrow[l, "\alpha_{\mathrm{rad}^3}"'] & v_{\mathrm{rad}^{(l(M)-1)}(M)} \arrow[l, "\alpha_{\mathrm{rad}^{l(M)-1}}"'] \\
 & i \arrow[dr, "\alpha_i"] & & & & \\
(i-q)_n \arrow[ur, "\alpha_{(i-q)}"] & & (i-1)_n \arrow[d, "\alpha_{i-1}"] & & & \\
(i-q-1)_n \arrow[u, "\alpha_{(i-q-1)}"] & & (i-2)_n \arrow[dl, "\alpha_{i-2}"] & & & \\
 & \cdots \arrow[ul, "\alpha_{(i-q-2)}"] & & & &
\end{tikzcd}
$$

and with relations $\alpha_r\alpha_{r-1}\dots\alpha_{r-(n-l(M)-1)} = 0$ for $r \in [i, i-(n-l(M)-1)]_n$. By Lemma 3.3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of $Q_n^{(v_M)}$ modulo relations. So it follows that $\operatorname{mod} D_M \cong \operatorname{mod} \mathbb{A}_{l(M)-1} \oplus \operatorname{mod}\Lambda_{n-l(M)}^{n-l(M)}$. So by Theorem 3.3.5, the statement of the Proposition follows. □

Theorem 3.6.6. Let H_n denote the number of complete τ-exceptional sequences in $\operatorname{mod} \Lambda_n^n$. Then H_n satisfies the recurrence relation,

$$H_n = n \sum_{i=1}^{n} \binom{n-1}{i-1} i^{i-2} H_{n-i},$$

with $H_0 = 1$.

Proof. Let M be an indecomposable Λ_n^n-module. Suppose $(M_1, M_2, \dots, M_{n-1}, M)$ is a complete τ-exceptional sequence in $\operatorname{mod} \Lambda_n^n$ ending in M. Then by definition and the fact that $\delta(J(M)) = n-1$, the sequence $(M_1, M_2, \dots, M_{n-1})$ is a complete τ-exceptional sequence in $J(M)$. It then follows that the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $J(M)$.

The length of M is $1 \leq l(M) \leq n$. For each possible value of $l(M)$, there are n indecomposable Λ_n^n-modules of that length. If $l(M) = n$ then M is projective and by Proposition 3.6.4, $J(M) \cong \operatorname{mod} \mathbb{A}_{n-1}$. The number of τ-exceptional sequences in $\operatorname{mod} \mathbb{A}_{n-1}$ was shown in [Seidel (2001) [Proposition 1.1]] to be $n^{n-2} = \binom{n-1}{n-1} n^{n-2} H_0$, where $H_0 = 1$.

If $1 \leq l(M) \leq n-1$, then by Proposition 3.6.5, the τ-perpendicular category of M is $J(M) \cong \operatorname{mod} \mathbb{A}_{l(M)-1} \oplus \operatorname{mod} \Lambda_{n-l(M)}^{n-l(M)}$. Arguing as above, the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $\operatorname{mod} \mathbb{A}_{l(M)-1} \oplus \operatorname{mod} \Lambda_{n-l(M)}^{n-l(M)}$. By Theorems 3.3.8 and 3.3.9, this is equal to

$$\binom{n-1}{n-l(M), l(M)-1} l(M)^{(l(M)-2)} H_{n-l(M)} = \binom{n-1}{l(M)-1} l(M)^{(l(M)-2)} H_{n-l(M)}.$$

So it follows that,

$$H_n = \sum_{l(M)=1}^{n} n\binom{n-1}{l(M)-1} l(M)^{l(M)-2} H_{n-l(M)} = n\sum_{i=1}^{n}\binom{n-1}{i-1} i^{i-2} H_{n-i}.$$

It is trivial to see that $H_1 = 1$. Using the recurrence we obtain $H_1 = \binom{0}{0}1^{-1}H_0 = 1$, therefore $H_0 = 1$. $\square$

We are now in a position to derive the exponential generating function of H_n. First we state the following results and definitions which will be useful in deriving the exponential generating function.

Lemma 3.6.7. (Wilf, 2005, Section 2.3 Rule 3′). Let $f = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $g = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$ be the generating functions of the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ respectively. Then the series fg is the exponential generating function of the sequence,

$$\left\{\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}\right\}_{n=0}^{\infty}.$$

The Lambert W function is defined to be the function $W(z)$ satisfying

$$W(z)e^{W(z)} = z.$$

The **tree function** $T(z)$ is defined by the equation $T(z) = -W(-z)$. The functions W and T have many applications in mathematics. For example, they appear in the enumeration of trees and the calculation of water-wave heights. The reader is referred to Corless *et al.* (1996) for more on Lambert's W function.

Lemma 3.6.8. (Corless *et al.*, 1996, Section 2, Equation 2.36). Let $a \geq 1$ and $n \geq 0$ be integers. Let $N(a,n) := a(a+n)^{n-1}$ be a function of two variables. For a fixed positive integer a, the exponential generating function of the sequence $N(a,n)$ is given by,

$$\sum_{n=0}^{\infty} a(a+n)^{n-1}\frac{x^n}{n!} = e^{-aW(-x)},$$

where W is Lambert's W function.

Theorem 3.6.9. The exponential generating function of H_n is,

$$\sum_{n=0}^{\infty} H_n \frac{x^n}{n!} = \frac{1}{1+W(-x)},$$

where W is Lambert's W function and H_n is given by the closed formula,

$$H_n = n^n.$$

Proof. Let a_n be the sequence $a_n = (n+1)^{n-1}$. Let $h(x) = \sum_{n=0}^{\infty} H_n \frac{x^n}{n!}$ and $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ be exponential generating functions of H_n and a_n respectively. Recall the recurrence relation of H_n is given by,

$$H_n = n \sum_{k=1}^{n} \binom{n-1}{k-1} k^{k-2} H_{n-k},$$

so,

$$\frac{H_n}{n} = \sum_{k=1}^{n} \binom{n-1}{k-1} k^{k-2} H_{n-k}.$$

We make the change of variable $j = k-1$ in $\frac{H_n}{n}$ to obtain the following.

$$\frac{H_n}{n} = \sum_{j=0}^{n-1} \binom{n-1}{j} (j+1)^{j-1} H_{n-(j+1)},$$

thus

$$\frac{H_{n+1}}{n+1} = \sum_{j=0}^{n} \binom{n}{j} (j+1)^{j-1} H_{n-j}.$$

We now study the exponential generating function of $\frac{H_{n+1}}{n+1}$,

$$\sum_{n=0}^{\infty} \frac{H_{n+1}}{n+1} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} (j+1)^{j-1} H_{n-j} \right) \frac{x^n}{n!}.$$

By Lemma 3.6.7, the right hand side is given the product $g(x)h(x)$. So we have,

$$\sum_{n=0}^{\infty} \frac{H_{n+1}}{n+1} \frac{x^n}{n!} = g(x)h(x).$$

We can manipulate the right hand side so that the exponent of x matches the factorial, hence

$$\frac{1}{x}\sum_{n=0}^{\infty} H_{n+1}\frac{x^{n+1}}{(n+1)!} = g(x)h(x),$$

so we can write the left hand side in terms of $h(x)$ as follows,

$$\frac{1}{x}(h(x) - H_0) = g(x)h(x).$$

Since $H_0 = 1$,

$$h(x) - 1 = xh(x)g(x).$$

By Lemma 3.6.8, $g(x) = e^{-W(-x)}$ therefore,

$$h(x) = \frac{1}{1 - xg(x)} = \frac{1}{1 - xe^{-W(-x)}}.$$

Recall that Lambert's W function is defined by the equation $x = W(x)e^{W(x)}$ (See Corless *et al.* (1996) for more on Lambert's W function), thus $-xe^{-W(-x)} = W(-x)$, giving us that,

$$h(x) = \frac{1}{1 + W(-x)} = \frac{1}{1 - T(x)},$$

where $T(x) = -W(-x)$ is called Euler's tree function, again see Corless *et al.* (1996). This exponential generating function is precisely the exponential generating function of the sequence n^n, see; Knuth & Pittel (1989) Section 2 equation 2.7 and Riordan (1968). $\square$

It is interesting to note that n^n is also the number of complete exceptional sequences over the hereditary algebras of type B and C; see section 5 of Obaid *et al.* (2013). On a more interesting note, n^n also counts the number of full weak exceptional sequences (see Definition 3.5.7) over Λ_n^n (Sen, 2020b, Theorem 1.4). The full weak exceptional sequences over Λ_n^n also have length n, so a natural question to ask is whether the complete τ-exceptional sequences over Λ_n^n coincide with the full weak exceptional sequences (see Definition 3.5.7) over Λ_n^n. We answer this question in the affirmative. First, we state the following well known result.

Lemma 3.6.10. (Bühler, 2010b, Lemma 10.20) Let $\mathcal{A}$ be an exact category and let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. Then if $\mathcal{B}$ is extension-closed in $\mathcal{A}$, the exact sequences $A \to B \to C$ in $\mathcal{A}$ with $A, B,$ and $C \in \mathcal{B}$ form an exact structure on $\mathcal{B}$. In particular for $X, Y \in \mathcal{B}$, we have that $\operatorname{Ext}^1_{\mathcal{A}}(X, Y) = \operatorname{Ext}^1_{\mathcal{B}}(X, Y)$.

Lemma 3.6.11. (Jasso, 2014, Proposition 3.6) Let A be a finite-dimensional $\mathbb{F}$ algebra and let M be a basic τ-rigid left A-module. Then the τ-perpendicular category $J(M)$ is extension-closed in $\operatorname{mod} A$.

Proposition 3.6.12. Let $M = (M_1, M_2, \ldots, M_n)$ be a complete τ-exceptional sequence in $\operatorname{mod} \Lambda_n^n$. Then M is also a full weak exceptional sequence in $\operatorname{mod} \Lambda_n^n$.

Proof. We will argue by induction on n. In the case of $n = 1$, there is only one indecomposable module which is both weak exceptional and τ-rigid, so the statement follows trivially. Suppose that the statement is true for all $1 \leq n \leq k$. Let us consider the $k+1$ case. Suppose $M = (M_1, M_2, \ldots, M_{k+1})$ is a τ-exceptional sequence in $\operatorname{mod} \Lambda_{k+1}^{k+1}$.

Let $l = l(M_{k+1})$ be the length of M_{k+1}, then $1 \leq l \leq k + 1$. By Propositions 3.6.4 and 3.6.5, the τ-perpendicular category

$$J(M_{k+1}) \cong \operatorname{mod} \mathbb{A}_{l-1} \oplus \Lambda_{k+1-l}^{k+1-l}.$$

By definition, the sequence $(M_1, M_2, \ldots, M_k)$ is τ-exceptional in $J(M_{k+1})$. By Theorems 3.3.8 and 3.3.9, we have that the sequence $(M_1, M_2, \ldots, M_k)$ is an interleaving of a complete exceptional sequence $X = (X_1, X_2, \ldots, X_{l-1})$ in $\operatorname{mod} \mathbb{A}_{l-1}$ with a complete τ-exceptional sequence $Y = (Y_1, Y_2, \ldots, Y_{k+1-l})$ in $\operatorname{mod} \Lambda_{k+1-l}^{k+1-l}$. By the inductive hypothesis, the sequence Y is a full weak exceptional sequence in $\operatorname{mod} \Lambda_{k+1-l}^{k+1-l}$. Moreover, we have that

$$\operatorname{Hom}_{J(M_{k+1})}(X_i, Y_j) = 0 = \operatorname{Hom}_{J(M_{k+1})}(Y_j, X_i),$$

where $1 \leq i \leq l-1$ and $1 \leq j \leq k+1-l$. Therefore, since the τ-perpendicular category $J(M_{k+1})$ is a full subcategory of $\operatorname{mod}\Lambda_{k+1}^{k+1}$, we also have that

$$\operatorname{Hom}_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(X_i, Y_j) = 0 = \operatorname{Hom}_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(Y_j, X_i),$$

where $1 \leq i \leq l-1$ and $1 \leq j \leq k+1-l$. By a similar argument, we also have that

$$\operatorname{Hom}_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(X_j, X_i) = 0$$

for $1 \leq i < j \leq l-1$ and

$$\operatorname{Hom}_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(Y_j, Y_i) = 0$$

for $1 \leq i < j \leq k+1-l$. By Lemma 3.6.10, since $J(M_{k+1})$ is an extension-closed subcategory of Λ_{k+1}^{k+1}, we can argue in a similar way that

$$\operatorname{Ext}^1_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(X_i, Y_j) = \operatorname{Ext}^1_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(Y_j, X_i) = 0,$$

where $1 \leq i \leq l-1$ and $1 \leq j \leq k+1-l$. By another similar argument,

$$\operatorname{Ext}^1_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(X_j, X_i) = 0$$

for $1 \leq i \leq j \leq l-1$ and

$$\operatorname{Ext}^1_{\operatorname{mod}\Lambda_{k+1}^{k+1}}(Y_j, Y_i) = 0$$

for $1 \leq i \leq j \leq k+1-l$. So we can conclude that the sequence $(M_1, M_2, \ldots, M_k)$ is weak exceptional in $\operatorname{mod}\Lambda_{k+1}^{k+1}$, hence $M = (M_1, M_2, \ldots M_{k+1})$ is a full weak exceptional sequence in $\operatorname{mod}\Lambda_{k+1}^{k+1}$. This completes the proof. $\square$

Corollary 3.6.13. The complete τ-exceptional sequences of $\operatorname{mod}\Lambda_n^n$ and the full weak exceptional sequences of $\operatorname{mod}\Lambda_n^n$ coincide.

Proof. Let us denote by T_n the set of complete τ-exceptional sequences in $\operatorname{mod}\Lambda_n^n$ and denote by W_n the set of full weak exceptional sequences in $\operatorname{mod}\Lambda_n^n$. Using Proposition 3.6.12, we can construct the following map, $f\colon T_n \to W_n$, where by $f(M) = M$. The map f is clearly injective and since $\mid T_n \mid = \mid W_n \mid = n^n$, the map f is bijective, but more precisely the complete τ-exceptional sequences of $\operatorname{mod}\Lambda_n^n$ and the full weak exceptional sequences of $\operatorname{mod}\Lambda_n^n$ coincide. □

Unlike τ-exceptional sequences, there are no complete exceptional sequences in $\operatorname{mod}\Lambda_n^n$, as we will show. In general, not much is known about exceptional sequences over the Nakayama algebras Λ_n^n.

Proposition 3.6.14. There are no exceptional sequences $(M_1, M_2, \ldots, M_l)$ in $\operatorname{mod}\Lambda_n^n$ of length $l > 1$. In particular, there are no complete exceptional sequences in $\operatorname{mod}\Lambda_n^n$ where $n > 1$.

Proof. Suppose $M = (M_1, M_2, \ldots, M_l)$ is an exceptional sequence of length $l > 1$. Every indecomposable projective module in $\operatorname{mod}\Lambda_n^n$ has length n, so by Proposition 3.3.12 we have that $\operatorname{Hom}(P_i, P_j) \neq 0$ for all $1 \leq i, j \leq n$. As a consequence of this M cannot contain more than one indecomposable projective module, in particular, if $l > 1$, then M must contain non-projective indecomposable modules.

Let N be a non-projective indecomposable module in $\operatorname{mod}\Lambda_n^n$. Observe that $N = \operatorname{rad}^k(P_i)$ for some $1 \leq i \leq n$ and $1 \leq k \leq n-1$. We can further observe that the length of N is $l(N) = n - k$ and that $\operatorname{top}(N) = S_{(i+k)_n}$. Further observe that N has the following infinite sequence as its projective resolution.

$$\ldots \longrightarrow P_i \longrightarrow P_{(i+k)_n} \longrightarrow P_i \longrightarrow P_{(i+k)_n} \longrightarrow P_i \longrightarrow P_{(i+k)_n} \longrightarrow N \longrightarrow 0$$

Since N has length $l(N) = n - k$ and $\operatorname{top}(N) = S_{(i+k)_n}$, we can write $N = P_{(i+k)_n}/\operatorname{rad}^{n-k}(P_{(i+k)_n})$. By Proposition 3.3.12, we can observe that $\operatorname{Hom}(P_i, N) = 0$ and $\operatorname{Hom}(P_{(i+k)_n}, N) \neq 0$. Therefore by applying the functor $\operatorname{Hom}(-, N)$ to the above projective resolution, we obtain the following sequence.

$$0 \longrightarrow \mathrm{Hom}(N,N) \xrightarrow{f_0} \mathrm{Hom}(P_{(i+k)_n},N) \xrightarrow{f_1} 0 \xrightarrow{f_2} \mathrm{Hom}(P_{(i+k)_n},N) \xrightarrow{f_3} 0 \xrightarrow{f_4} \dots$$

So we have that $\mathrm{Ext}^2(N,N) = \ker(f_3)/\mathrm{im}(f_2) \neq 0$. Which is to say any non-projective module in $\mathrm{mod}\,\Lambda_n^n$ is not exceptional. From this we conclude that M cannot contain non-projective modules. A contradiction. □

3.7 The Γ_n^{n-1} case

Let $n \geq 1$ be a positive integer. In this section we will study the combinatorics for the number of complete τ-exceptional sequence in $\mathrm{mod}\,\Gamma_n^{n-1}$. Recall that we denote by A_n the linearly oriented quiver with n vertices,

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n\ .$$

The algebra Γ_n^{n-1} is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}A_n/R_Q^{n-1}$. This is the path algebra of the quiver A_n modulo the relation $\alpha_1\alpha_2\dots\alpha_{n-1} = 0$.

Observe the following. Let M be an indecomposable module in $\mathrm{mod}\,\Gamma_n^{n-1}$, then M belongs to one of the following disjoint sets. The first set contains the indecomposable projective modules P_j for $1 \leq j \leq n$. The second set contains non-projective modules of the form $M = \mathrm{rad}^i(P_1)$ where $1 \leq i \leq n-2$ and P_1 is the indecomposable projective at vertex 1. The third set contains indecomposable modules which are neither projective or of the form $M = \mathrm{rad}^i(P_1)$ for $1 \leq i \leq n-2$. Any indecomposable module M in $\mathrm{mod}\,\Gamma_n^{n-1}$ has length $l(M) < n$, therefore by Proposition 3.6.1, every indecomposable module of $\mathrm{mod}\,\Gamma_n^{n-1}$ is τ-rigid.

Proposition 3.7.1. Let P_i be an indecomposable projective module in $\mathrm{mod}\,\Gamma_n^{n-1}$ for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in $\mathrm{mod}\,\Gamma_n^{n-1}$ is $J(P_i) \cong \mathrm{mod}\,\mathbb{A}_{n-i} \oplus \mathrm{mod}\,\mathbb{A}_{i-1}$, where $\mathbb{A}_j$ is the hereditary type A hereditary algebra.

Proof. Let P_i be an indecomposable projective with length $1 \leq l(P_i) \leq n-1$. By definition the Bongartz completion $T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i))$. Since P_i is projective,

$\tau P_i = 0$ therefore ${}^\perp(\tau P_i) = \operatorname{mod} \Gamma_n^{n-1}$, hence the Bongartz completion

$$T_{P_i} = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \operatorname{End}_{\Gamma_n^{n-1}}(T_{P_i})$ is precisely the algebra Γ_n^{n-1}, the path algebra of the quiver A_n^{op},

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \dots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_i} i \xleftarrow{\alpha_{i+1}} i+1 \xleftarrow{\alpha_{i+2}} \dots \xleftarrow{\alpha_n} n$$

modulo the relation $\alpha_n \alpha_{n-1} \dots \alpha_1 = 0$. Let $A_n^{\mathrm{op}(i)}$ be the quiver obtained from A_n^{op} by removing the vertex i and all arrows incident to i.

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \dots \xleftarrow{\alpha_{i-1}} i-1 \qquad\qquad i+1 \xleftarrow{\alpha_{i+2}} \dots \xleftarrow{\alpha_n} n$$

The quiver $A_n^{\mathrm{op}(i)}$ has no relations. By Lemma 3.3.4, $D_M = E_M / \langle e_M \rangle$ is the path algebra of the quiver $A_n^{\mathrm{op}(i)}$. Since $D_M = \mathbb{F}A_n^{\mathrm{op}(i)}$, it follows that $J(M) \cong \operatorname{mod} \mathbb{A}_{n-i} \oplus \operatorname{mod} \mathbb{A}_{i-1}$ by Theorem 3.3.5. $\square$

Proposition 3.7.2. Let M be an indecomposable module in $\operatorname{mod} \Gamma_n^{n-1}$ of the form $M = \operatorname{rad}^i(P_1)$ for some $1 \leq i \leq n-2$ with length $1 \leq l(M) \leq n-2$. Then the τ-perpendicular category of M in $\operatorname{mod} \Gamma_n^{n-1}$ is

$$J(M) \cong \begin{cases} \operatorname{mod} \mathbb{A}_{l(M)-1} \oplus \operatorname{mod} \mathbb{A}_1 \oplus \operatorname{mod} \mathbb{A}_1 & i = 1 \\ \operatorname{mod} \mathbb{A}_{n-l(M)} \oplus \operatorname{mod} \mathbb{A}_{l(M)-1} & i \neq 1 \end{cases},$$

where $\mathbb{A}_j$ is the hereditary type A hereditary algebra.

Proof. Consider P_j the indecomposable projective module at the vertex j in $\operatorname{mod} \Gamma_n^{n-1}$ with $j \neq 1$. Then it is easy to see that $P_j = \operatorname{rad}^{j-2}(P_2)$ and that P_j has length $l(P_j) = n - j + 1$. From this it follows that $\operatorname{rad}^q(P_j) = P_{j+q}$ where $0 \leq q \leq n-j$.

Let $M = \operatorname{rad}^i(P_1)$ for some $1 \leq i \leq n-1$. We observe that $l(M) = n-i-1$ and $\operatorname{top}(M) = S_{i+1}$. In accordance to Proposition 3.3.11, M may in fact be written as $M = P_{i+1}/\operatorname{rad}^{n-i-1}(P_{i+1})$. Using Proposition 3.3.11 again, we can see that

Auslander-Reiten translate of M is given by $\tau M = \mathrm{rad}(P_{i+1})/\mathrm{rad}^{n-i}(P_{i+1}) = P_{i+2}$ because $\mathrm{rad}(P_{i+1}) = P_{i+2}$ and $l(P_{i+1}) = n-(i+1)+1 = n-i$, hence $\mathrm{rad}^{n-i}(P_{i+1}) = 0$. So we see that the only indecomposable Γ_n^{n-1}-modules not in ${}^{\perp}(\tau M)$ are the projectives $P_j = \mathrm{rad}^s(P_{i+2})$ for $0 \leq s < n-i-1$, in other words $i+2 \leq j \leq n$ since $l(P_{i+2}) = n-i-1$.

We can now determine the Ext-projectives of ${}^{\perp}(\tau M)$ where $M = \mathrm{rad}^i(P_1)$. By the above calculation, we can say that for $1 \leq j \leq i+1$, the projective P_j is in ${}^{\perp}(\tau M)$ hence $\mathrm{Ext}_{\Gamma_n^{n-1}}(P_j, {}^{\perp}(\tau M)) = 0$.

Let $N = \mathrm{rad}^j(P_1)$ for some $j > i$. Arguing as above we can see that the only indecomposable Γ_n^{n-1}-modules not in ${}^{\perp}(\tau N)$ are the indecomposable projectives P_m where $j+2 \leq m \leq n$, so it follows that $\{X : \mathrm{Hom}(X, \tau N) \neq 0\} \subset \{X : \mathrm{Hom}(X, \tau M) \neq 0\}$. This implies that $\mathrm{Ext}_{\Gamma_n^{n-1}}(N, X) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Gamma_n^{n-1}}(X, \tau N) = 0$ for all X in ${}^{\perp}(\tau M)$ by the Auslander-Reiten formula. Hence $N = \mathrm{rad}^j(P_1)$ is an Ext-projective in ${}^{\perp}(\tau M)$.

For every other indecomposable Γ_n^{n-1} module Y, we have that τY is in ${}^{\perp}(\tau M)$, therefore since $\mathrm{Ext}_{\Gamma_n^{n-1}}(Y, \tau Y) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Gamma_n^{n-1}}(X, \tau N) \neq 0$. Therefore these modules are not Ext-projective in ${}^{\perp}(\tau M)$. By definition $T_M = \mathcal{P}({}^{\perp}(\tau M))$, so by the above arguments,

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=i+1}^{n-2} \mathrm{rad}^s(P_1) \oplus \bigoplus_{j=1}^{i+1} P_j.$$

In the case when $i \neq 1$ the $\mathbb{F}$-algebra $E_M = \mathrm{End}_{\Gamma_n^{n-1}}(T_M)$ is the path algebra of the quiver Q_n,

$$\begin{tikzcd}
 & & & i+1 \arrow[r, "\alpha_{i+1}"] \arrow[d, "\alpha"] & i \arrow[r, "\alpha_i"] & \dots \arrow[r, "\alpha_2"] & 1 \\
v_{\mathrm{rad}^{n-2}} \arrow[r, "\alpha_{\mathrm{rad}^{n-2}}"] & \dots \arrow[r, "\alpha_{\mathrm{rad}^{i+1}}"] & v_{\mathrm{rad}^{i+1}} \arrow[r, "\alpha_{\mathrm{rad}^{i+1}}"] & v_M \arrow[urrr, bend right, "\alpha_{v_M}"'] & & &
\end{tikzcd}.$$

By Lemma 3.3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n^{(v_M)}$ which is the quiver obtained from Q_n by removing the vertex v_M and all arrows incident to v_M. The quiver $Q_n^{(v_M)}$ has two connected components.

$$\begin{tikzcd} & & & i+1 \arrow[r, "\alpha_{i+1}"] & i \arrow[r, "\alpha_i"] & \dots \arrow[r, "\alpha_2"] & 1 \\ v_{\mathrm{rad}^{n-2}} \arrow[r, "\alpha_{\mathrm{rad}^{n-2}}"] & \dots \arrow[r, "\alpha_{\mathrm{rad}^{i+1}}"] & v_{\mathrm{rad}^{i+1}} \end{tikzcd}$$

Since $D_M = \mathbb{F}Q_n^{(v_M)}$, it follows that $J(M) \cong \operatorname{mod} \mathbb{A}_{i+1} \oplus \operatorname{mod} \mathbb{A}_{n-i-2}$ by Theorem 3.3.5. Recall that $l(M) = n - i - 1$, hence $J(M) \cong \operatorname{mod} \mathbb{A}_{n-l(M)} \oplus \operatorname{mod} \mathbb{A}_{l(M)-1}$.

When $i = 1$ however, the $\mathbb{F}$-algebra $E_M = \operatorname{End}_{\Gamma_n^{n-1}}(T_M)$ is the path algebra of the quiver Q'_n,

$$\begin{tikzcd} & & & & & 2 \arrow[d, "\alpha"] & \\ v_{\mathrm{rad}^{n-2}} \arrow[r, "\alpha_{\mathrm{rad}^{n-2}}"] & v_{\mathrm{rad}^{n-3}} \arrow[r, "\alpha_{\mathrm{rad}^{n-3}}"] & \dots \arrow[r, "\alpha_{\mathrm{rad}^{i+1}}"] & v_{\mathrm{rad}^{i+1}} \arrow[r, "\alpha_{\mathrm{rad}^{i+1}}"] & & v_M \arrow[r, "\alpha_{v_M}"] & 1 \end{tikzcd}$$

with no relations. By Lemma, 3.3.4 $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n'^{(v_M)}$ which is the quiver obtained from Q'_n by removing the vertex v_M and all arrows incident to v_M. This quiver has three connected components.

$$\begin{tikzcd} & & & & & 2 & \\ v_{\mathrm{rad}^{n-2}} \arrow[r, "\alpha_{\mathrm{rad}^{n-2}}"] & v_{\mathrm{rad}^{n-3}} \arrow[r, "\alpha_{\mathrm{rad}^{n-3}}"] & \dots \arrow[r, "\alpha_{\mathrm{rad}^{i+1}}"] & v_{\mathrm{rad}^{i+1}} & & & 1 \end{tikzcd}.$$

Since $D_M = \mathbb{F}Q_n'^{(v_M)}$, it follows that $J(M) \cong \operatorname{mod} \mathbb{A}_{n-3} \oplus \operatorname{mod} \mathbb{A}_1 \oplus \operatorname{mod} \mathbb{A}_1$ by Theorem 3.3.5. Since $l(\mathrm{rad}^1(P_1)) = n - 2$ then $J(M) \cong \operatorname{mod} \mathbb{A}_{l(M)-1} \oplus \operatorname{mod} \mathbb{A}_1 \oplus \operatorname{mod} \mathbb{A}_1$. $\square$

Proposition 3.7.3. Let M be an indecomposable Γ_n^{n-1}-module such that $M \neq \mathrm{rad}^k(P)$ for some indecomposable projective P and positive integer k. Suppose M

has length $1 \leq l(M) \leq n-2$, then the τ-perpendicular category of M in $\operatorname{mod}\Gamma_n^{n-1}$ is $J(M) \cong \operatorname{mod}\mathbb{A}_{l(M)-1} \oplus \operatorname{mod}\Gamma_{n-l(M)}^{n-l(M)-1}$.

Proof. By Proposition 3.3.11, we can write $M = P_i/\operatorname{rad}^{l(M)}(P_i)$ for some $1 \leq i \leq n-1$ and $\tau M = \operatorname{rad}(P_i)/\operatorname{rad}^{l(M)+1}(P_i)$. We will consider the case where $i \neq 1$ and $i = 1$ separately.

Suppose $i \neq 1$, then we have that $P_i = \operatorname{rad}^{i-2}(P_2)$ and $l(P_i) = n - i + 1$. Therefore, $\operatorname{rad}^q(P_i) = P_{i+q}$, in particular we have that $\operatorname{rad}(P_i) = P_{i+1}$, hence $\tau M = P_{i+1}/\operatorname{rad}^{l(M)}(P_{i+1})$. Now suppose that $i = 1$, hence $M = P_1/\operatorname{rad}^{l(M)}(P_1)$, then $\tau M = \operatorname{rad}(P_1)/\operatorname{rad}^{l(M)+1}(P_1)$. Observe that $\operatorname{top}(\tau M) = S_2$ and $l(\tau M) = l(M)$, hence $\tau M = P_2/\operatorname{rad}^{l(M)}(P_2)$. In either case of i, we have that $\tau M = P_{i+1}/\operatorname{rad}^{l(M)}(P_{i+1})$.

Now let $X = P_j/\operatorname{rad}^l(P_j)$ be an arbitrary indecomposable Γ_n^{n-1} module. By Proposition 3.3.12, $\operatorname{Hom}(X, \tau M) \neq 0$ if and only $j \in [i+1, i+l(M)]_n$ and $i+l(M) \in [j, j+l-1]_n$. From this it follows that P_j is not in ${}^\perp(\tau M)$ if $i+1 \leq j \leq i+l(M)$. Hence $\operatorname{Ext}_{\Gamma_n^{n-1}}(P_j, {}^\perp(\tau M)) = 0$ if $j \notin [i+1, i+l(M)]$.

Consider the module $\operatorname{rad}^s(M)$ for $1 \leq s \leq l(M) - 1$. The length of $\operatorname{rad}^s(M)$ is given by $l(\operatorname{rad}^s(M)) = l(M) - s$. Moreover, $\operatorname{rad}^s(M) = P_{i+s}/\operatorname{rad}^{l(M)-s}(P_{i+s})$, from which it follows that $\tau\operatorname{rad}^s(M) = P_{i+s+1}/\operatorname{rad}^{l(M)-s}(P_{i+s+1})$. Again let $X = P_j/\operatorname{rad}^l(P_j)$ be an arbitrary indecomposable Γ_n^{n-1} module. By Proposition 3.3.12, $\operatorname{Hom}(X, \tau\ \operatorname{rad}^s(M)) \neq 0$ if and only $j \in [i+s+1, i+l(M)]_n$ and $i+l(M) \in [j, j+l-1]_n$. Therefore $\{X : \operatorname{Hom}(X, \tau\operatorname{rad}^s(M)) \neq 0\} \subset \{X : \operatorname{Hom}(X, \tau M) \neq 0\}$, which implies that $\operatorname{Ext}_{\Gamma_n^{n-1}}(\operatorname{rad}^s(M), Y) \cong \mathrm{D}\overline{\operatorname{Hom}}_{\Gamma_n^{n-1}}(Y, \tau\operatorname{rad}^s(M)) = 0$ for all Y in ${}^\perp(\tau M)$. In other words, $\operatorname{rad}^s(M)$ is Ext-projective in ${}^\perp(\tau M)$.

By Proposition 3.3.1, M is Ext-projective in ${}^\perp(\tau M)$, so

$$\mathcal{P}({}^\perp(\tau M)) = M \oplus \bigoplus_{s=1}^{l(M)-1} \operatorname{rad}^s(M) \oplus \bigoplus_{j \notin [i+1, i+l(M)]} P_j.$$

By definition, the Bongartz completion $T_M = \mathcal{P}(^{\perp}(\tau M))$, so the $\mathbb{F}$-algebra $E_M = \mathrm{End}_{\Gamma_n^{n-1}}(T_M)$ is the path algebra of the quiver Q_n modulo relations (set $l(M) := m$),

$$\begin{tikzcd} n \arrow[r, "\alpha_n"] & \dots \arrow[r, "\alpha_{i+m+2}"] & i+m+1 \arrow[r, "\alpha_{i+m+1}"] & i \arrow[r, "\alpha_i"] \arrow[d, "\alpha"] & i-1 \arrow[r, "\alpha_{i-1}"] & \dots \arrow[r, "\alpha_2"] & 1 \\ v_{\mathrm{rad}^{m-1}} \arrow[r, "\alpha_{\mathrm{rad}^{m-1}}"] & \dots \arrow[r, "\alpha_{\mathrm{rad}^2}"] & v_{\mathrm{rad}^1} \arrow[r, "\alpha_{\mathrm{rad}^1}"] & v_M \end{tikzcd} .$$

Since the vertices of the top row of the quiver correspond to the indecomposable projectives of $\mathrm{mod}\,\Gamma_n^{n-1}$ and the arrows reflect the relations the corresponding maps between the projectives, we see that we have the relation

$$\alpha_n \alpha_{n-1} \dots \alpha_{i+m+1} \alpha_i \alpha_{i-1} \dots \alpha_1 = 0.$$

Let $Q_n^{(v_M)}$ be the quiver obtained from Q_n by removing the vertex v_M and all the arrows incident to v_M,

$$\begin{tikzcd} n \arrow[r, "\alpha_n"] & \dots \arrow[r, "\alpha_{i+m+2}"] & i+m+1 \arrow[r, "\alpha_{i+m+1}"] & i \arrow[r, "\alpha_i"] & i-1 \arrow[r, "\alpha_{i-1}"] & \dots \arrow[r, "\alpha_2"] & 1 \\ v_{\mathrm{rad}^{m-1}} \arrow[r, "\alpha_{\mathrm{rad}^{m-1}}"] & \dots \arrow[r, "\alpha_{\mathrm{rad}^2}"] & v_{\mathrm{rad}^1} \end{tikzcd}$$

with the relation $\alpha_n \alpha_{n-1} \dots \alpha_{i+m+1} \alpha_i \alpha_{i-1} \dots \alpha_1 = 0$. By Lemma 3.3.4, $D_M = E_M / \langle e_M \rangle$ is the path algebra of the quiver $Q_n^{(v_M)}$ modulo the relation

$$\alpha_n \alpha_{n-1} \dots \alpha_{i+m+1} \alpha_i \alpha_{i-1} \dots \alpha_1 = 0.$$

It follows that $J(M) \cong \mathrm{mod}\,\mathbb{A}_{l(M)-1} \oplus \mathrm{mod}\,\Gamma_{n-l(M)}^{n-l(M)-1}$ by Theorem 3.3.5. $\square$

Theorem 3.7.4. Let K_n denote the number of complete τ-exceptional sequences in $\mathrm{mod}\,\Gamma_n^{n-1}$. Then K_n satisfies the recurrence relation;

$$K_n = (n-1)(n-2)^{(n-3)} + \sum_{i=1}^{n} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} i^{i-2} +$$

$$\sum_{i=1}^{n-3}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)}i^{i-2}+\sum_{i=1}^{n-2}\binom{n-1}{i-1}(n-i-1)i^{i-2}K_{n-i}$$

with $K_1 = 1$.

Proof. Let M be an indecomposable module in $\operatorname{mod}\Gamma_n^{n-1}$. Suppose that the sequence $(X_1, X_2, \ldots, X_{n-1}, M)$ is a τ-exceptional sequence in $\operatorname{mod}\Gamma_n^{n-1}$. Then by definition and the fact that $\delta(J(M)) = n-1$, the sequence $(X_1, X_2, \ldots, X_{n-1})$ is a complete τ-exceptional sequence in $J(M)$. Hence the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $J(M)$.

Suppose M is projective, hence $M = P_i$ for some $1 \leq i \leq n$, then by Proposition 3.7.1 the τ-perpendicular category $J(M) \cong \operatorname{mod}\mathbb{A}_{n-i} \oplus \operatorname{mod}\mathbb{A}_{i-1}$. The number of complete τ-exceptional sequences in $\operatorname{mod}\mathbb{A}_l$ is precisely the number of complete exceptional sequence in $\operatorname{mod}\mathbb{A}_l$ which is shown in [Seidel (2001) [Proposition 1.1]] to be $(l+1)^{(l-1)}$. Therefore by Theorems 3.3.8 and 3.3.9 the number of complete τ-exceptional sequence ending in $M = P_i$ is $\binom{n-1}{i-1}(n-i+1)^{n-i-1}i^{i-2}$.

Suppose $M = \operatorname{rad}^i(P_1)$ for some $1 \leq i \leq n-2$. If $i = 1$ then we saw in Proposition 3.7.2 that $J(M) \cong \operatorname{mod}\mathbb{A}_{n-3} \oplus \operatorname{mod}\mathbb{A}_1 \oplus \operatorname{mod}\mathbb{A}_1$. Arguing as above it follows that the number of complete τ-exceptional sequences ending in $\operatorname{rad}^1(P_1)$ is $\binom{n-1}{n-3,1,1}(n-2)^{(n-4)}2^0 2^0 = (n-1)(n-2)^{(n-3)}$. If it is the case that $2 \leq i \leq n-2$, then $J(M) \cong \operatorname{mod}\mathbb{A}_{n-l(M)} \oplus \operatorname{mod}\mathbb{A}_{l(M)-1}$. Therefore the number of complete τ-exceptional sequences ending in $M = \operatorname{rad}^i(P_1)$ for some $2 \leq i \leq n-2$ is $\binom{n-1}{l(M)-1}(n-l(M)+1)^{n-l(M)-1}l(M)^{l(M)-2}$, where $l(M)$ is the length of M.

Finally suppose that M is not of the form $\operatorname{rad}^i(P)$ for some indecomposable projective module P. By Proposition 3.7.3, $J(M) \cong \operatorname{mod}\mathbb{A}_{l(M)-1} \oplus \operatorname{mod}\Gamma_{n-l(M)}^{n-l(M)-1}$ where $l(M)$ is the length of M. Therefore the number of complete τ-exceptional sequences ending in M is $\binom{n-1}{l(M)-1}K_{n-l(M)}l(M)^{l(M)-2}$. Observe that in this case the length of M is $1 \leq l(M) \leq n-2$ and for each fixed value of $l(M)$ there are

$n - l(M) - 1$ indecomposable modules M such that $M \neq \mathrm{rad}^i(P)$.

By counting the number of complete τ-exceptional sequences ending in each indecomposable Γ_n^{n-1}-module M, the recurrence relation of K_n follows. It is also trivial to see that $K_1 = 1$. □

Theorem 3.7.5. Let $h(x) = \sum_{n=0}^{\infty} K_n \frac{x^n}{n!}$ be the exponential generating function of K_n. Then $h(x)$ satisfies the first order linear ODE,

$$h'(x)(1 - xe^{-W(-x)}) + h(x)e^{-W(-x)} = 2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x).$$

Proof. Let $h(x) = \sum_{n=0}^{\infty} K_n \frac{x^n}{n!}$ be the exponential generating function of K_n. Let $a(n) = (n+1)^{n-1}$. Let $g(x) = \sum_{n=0}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}$. Then $g(x) = e^{-W(-x)}$ by Lemma 3.6.8, where $W(x)$ is Lambert's W function. By the only Proposition in Section 6 of Obaid *et al.* (2013),

$$2(n+2)^{n-1} = \sum_{i=0}^{n} \binom{n}{i} a(i)a(n-i).$$

So it follows from Lemma 3.6.7 that

$$(g(x))^2 = \sum_{n=0}^{\infty} 2(n+2)^{n-1}\frac{x^n}{n!}. \tag{3.1}$$

We make the following observations about

$$\sum_{i=1}^{n} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} \cdot i^{i-2}.$$

With the change of variable $j = i - 1$,

$$\sum_{i=1}^{n} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} \cdot i^{i-2} = \sum_{j=0}^{n-1} \binom{n-1}{j} (n-j)^{(n-j-2)}(j+1)^{j-1} = 2(n+1)^{n-2},$$

as shown in the proof of the only Proposition in Section 6 of Obaid *et al.* (2013). We also observe that

$$\sum_{i=1}^{n} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} \cdot i^{i-2}$$

$$= \sum_{i=1}^{n-3} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} \cdot i^{i-2} + n^{n-2} + (n-1)^{n-2} + \frac{3}{2}(n-1)(n-2)^{n-3},$$

so,

$$\sum_{i=1}^{n-3} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} \cdot i^{i-2} = 2(n+1)^{n-2} - n^{n-2} - (n-1)^{n-2} - \frac{3}{2}(n-1)(n-2)^{n-3}. \tag{3.2}$$

As a result we can write the recurrence for K_{n+1} (from Theorem 3.7.4) in the following way ,

$$K_{n+1} = 2 \cdot 2(n+2)^{n-1} - (n+1)^{n-1} - n^{n-1} - \frac{1}{2}(n)(n-1)^{n-2} + \sum_{i=1}^{n-1} \binom{n}{i-1} (n-i) K_{n+1-i} \cdot i^{i-2}. \tag{3.3}$$

Making the change of variable $j = i - 1$ we get,

$$K_{n+1} = 2 \cdot 2(n+2)^{n-1} - (n+1)^{n-1} - n^{n-1} - \frac{1}{2}(n)(n-1)^{n-2} + \sum_{j=0}^{n-2} \binom{n}{j} (n-j-1) K_{n-j} \cdot (j+1)^{j-1}.$$

We will now study the exponential generating function of K_{n+1}. To do this we look at the exponential generating function of each of the summands on the right hand side. We have already seen from (3.1) that

$$\sum_{n=0}^{\infty} 2(n+2)^{n-1} \frac{x^n}{n!} = (g(x))^2 \tag{3.4}$$

To deal with the rest of the summands of K_{n+1} in (3.3) but the last one, we first re-organise them in the following way using equation (3.2). Let

$$\phi(n) = \sum_{i=1}^{n-2} \binom{n-1}{i-1} (n-i+1)^{(n-i-1)} \cdot i^{i-2} = 2(n+2)^{n-1} - (n+1)^{n-1} - n^{n-1} - \frac{3}{2} n(n-1)^{n-2}.$$

The change of variable $j = i - 1$ gives us

$$\phi(n) = \sum_{j=0}^{n-3} \binom{n}{j} (n-j)^{n-j-2} (j+1)^{j-1}.$$

We have $\phi(n) = 0$ for $n = 0, 1, 2$ since the sum is empty for these values of n. This further implies that,

$$\sum_{n=0}^{\infty} \phi(n)\frac{x^n}{n!} = \sum_{n=2}^{\infty} \phi(n)\frac{x^n}{n!}$$

$$= \sum_{n=2}^{\infty} 2(n+2)^{n-1}\frac{x^n}{n!} - \sum_{n=2}^{\infty}(n+1)^{n-1}\frac{x^n}{n!} - \sum_{n=2}^{\infty} n^{n-1}\frac{x^n}{n!} - \frac{3}{2}\sum_{n=2}^{\infty} n(n-1)^{n-2}\frac{x^n}{n!}$$

$$= \left((g(x))^2 - 1 - 2x\right) - \sum_{n=2}^{\infty}(n+1)^{n-1}\frac{x^n}{n!} - \sum_{n=2}^{\infty} n^{n-1}\frac{x^n}{n!} - \frac{3}{2}\sum_{n=2}^{\infty} n(n-1)^{n-2}\frac{x^n}{n!},$$

by (3.1). Lemma 3.6.8 resolves the second summand. The third summand is resolved by Corless *et al.* (1996) in Section 2, page 4. This was previously was done in Pólya *et al.* (1937). This has been translated into English; see(Polya & Read, 2012, Page 48)). To resolve the fourth summand we use the fact the exponential generating function is a right index shift and multiplication by n of the 3rd summand. Right index shifting is equivalent to formal integration and by Rule 2′ in Section 2.3 page 41 of Wilf (2005) multiplication by n is equivalent to differentiating and then multiplying the exponential generating function by x (This is also given on the OEIS A055541). Therefore.

$$\sum_{n=0}^{\infty} \phi(n)\frac{x^n}{n!} = [e^{-2W(-x)} - 1 - 2x] - [e^{-W(-x)} - 1 - x] - [-W(-x) - x] - \frac{3}{2}[-xW(-x)]$$

$$= e^{-2W(-x)} - 1 - 2x - e^{-W(-x)} + 1 + x + W(-x) + x + \frac{3}{2}xW(-x)$$

$$= e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{3}{2}xW(-x). \tag{3.5}$$

Now let us study the final summand of (3.3)

$$\sum_{j=0}^{n-2} \binom{n}{j}(n-j-1)K_{n-j} \cdot (j+1)^{j-1}.$$

Notice that the term $\binom{n}{n-1}(n-(n-1)-1)K_1 n^{n-2} = 0$ and $\binom{n}{n}(n-n-1)K_0 n + 1^{n-1} = 0$ since $K_0 = 0$. Therefore,

$$\sum_{j=0}^{n-2}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1} = \sum_{j=0}^{n}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}.$$

By Lemma 3.6.7,

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}\right)\frac{x^n}{n!} =$$

$$\left(\sum_{n=0}^{\infty}(n-1)K_n\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}\right).$$

By Rule 2′ in Section 2.3 page 41 of Wilf (2005)

$$\left(\sum_{n=0}^{\infty}(n-1)K_n\frac{x^n}{n!}\right) = x\frac{d}{dx}\left(\sum_{n=0}^{\infty}K_n\frac{x^n}{n!}\right) - \left(\sum_{n=0}^{\infty}K_n\frac{x^n}{n!}\right) = xh'(x) - h(x).$$

By Lemma 3.6.8,

$$\left(\sum_{n=0}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}\right) = e^{-W(-x)}.$$

Therefore

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}\right)\frac{x^n}{n!} = (xh'(x)-h(x))e^{-W(-x)}. \quad (3.6)$$

By Rule 1′ in Section 2.3 page 41 of Wilf (2005),

$$\sum_{n=0}^{\infty}K_{n+1}\frac{x^n}{n!} = h'(x).$$

We now write the exponential generating function of K_{n+1}, using the expression of K_{n+1} in (3.3) and the exponential generating functions of the summands of K_{n+1} obtained as in (3.4), (3.5) and (3.6).

$$\sum_{n=0}^{\infty}K_{n+1}\frac{x^n}{n!} = 2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x) + (xh'(x)-h(x))e^{-W(-x)},$$

so

$$h'(x) = 2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x) + (xh'(x) - h(x))e^{-W(-x)},$$

Therefore we have the following first order linear ODE,

$$h'(x) + \frac{h(x)e^{-W(-x)}}{(1 - xe^{-W(-x)})} = \frac{2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x)}{(1 - xe^{-W(-x)})}.$$

This ODE is of the form,

$$h'(x) + Q(x)h(x) = F(x),$$

so we may apply the integrating factor method and give a general solution for $h(x)$,

$$h(x) = e^{-V(x)} \int V(x)F(x)dx + C,$$

where $V(X)$ is the integrating factor,

$$V(X) = \int Q(x)dx = \int \frac{e^{-W(-x)}}{1 - xe^{-W(-x)}}dx.$$

Unfortunately, we are unable to evaluate $V(X)$ so we leave $h(x)$ as it is.

□

3.8 Justification

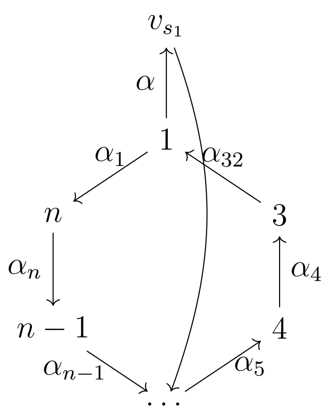
In this section we would like to justify why we only look at the four cases above. Our approach to counting the number of complete τ-exceptional sequences in the above module categories relied upon Theorems 3.3.8 and 3.3.9. We also took advantage of the fact that the τ-perpendicular categories of indecomposable modules M were of the form $J(M) \cong \mathcal{C} \oplus \mathcal{D}$ with $\mathcal{C}$ and $\mathcal{D}$ being module categories in the the two families Γ_n^t or Λ_n^t. It is our claim that these four cases, $\Gamma_n^2, \Gamma_n^{n-1}, \Lambda_n^2, \Lambda_n^n$ are the only ones were all the τ-perpendicular categories $J(M)$ are of this form. In other words, our approach only works on these four cases.

Proposition 3.8.1. Fix a positive integers $t \geq 3$. For $n \geq t+1$, let $A = \Lambda_n^t$. Then there exists an A-module M such that the τ-perpendicular category $J(M)$ is not a direct sum of module categories over algebras of the form $\Lambda_{n'}^{t'}$ or $\Gamma_{n'}^{t'}$ for $2 \leq t' \leq n' < n$.

Proof. We prove this by counter-example. Set $M = S_1$, the simple module at vertex 1 of the quiver C_n of A. Note that other simple modules also work, but for simplicity we choose S_1. The Auslander-Reiten translate of S_1 is $\tau S_1 = S_2$. Using Proposition 3.3.11 and 3.3.12, we can say that $\mathrm{Hom}(X, S_2) \neq 0$ if and only if $X = P_2/\mathrm{rad}^{l(X)}(P_2)$ where $l(X)$ is the length of X. It also follows that P_2 is the only projective with non-zero maps to S_2. Therefore all other indecomposable projective modules P_j with $j \neq 2$ are in ${}^{\perp}(\tau S_1)$, hence they are Ext-projectives in ${}^{\perp}(\tau S_1)$. By Proposition 3.3.1, the module S_1 is Ext-project in ${}^{\perp}(\tau S_1)$. We can thus conclude that,

$$\mathcal{P}({}^{\perp}(\tau S_1)) = \bigoplus_{j \neq 2} P_j \oplus S_1.$$

By definition the Bongartz completion of M in $\mathrm{mod}\, A$ is $T_M = \mathcal{P}(\tau S_1)$. Let Q_n be the following quiver,



,

where the vertices labelled j correspond to the projective P_j and the vertex v_{s_1} corresponds to the simple S_1 and the arrows correspond to the irreducible maps between their respective modules. The $\mathbb{F}$-algebra $E_M = \mathrm{End}_A(T_M)$ is the path algebra of the quiver modulo relations. Let $Q_n^{v_{s_1}}$ be the quiver obtained from Q_n by removing the vertex v_{s_1} and any arrows incident to v_{s_1},

$$\begin{tikzcd}
 & 1 \arrow[dl, "\alpha_1"'] & \\
n \arrow[d, "\alpha_n"'] & & 3 \arrow[ul, "\alpha_{32}"'] \\
n-1 \arrow[dr, "\alpha_{n-1}"'] & & 4 \arrow[u, "\alpha_4"'] \\
 & \cdots \arrow[ur, "\alpha_5"'] &
\end{tikzcd}$$

by Lemma 3.3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n^{v_{s_1}}$ modulo relations. We have the relation $\alpha_{t+1}\alpha_t \ldots \alpha_4\alpha_{32} = 0$ involving $t-1$ arrows because it corresponds to $\mathrm{Hom}_A(P_{t+1}, P_2) = 0$ since in $\mathrm{mod}\,\Lambda_n^t$ the composition of t maps between projectives is 0. However, at the same time we have that the composition of the $t'-1$ arrows $\alpha_1\alpha_n \ldots \alpha_{n-(t'-3)} \neq 0$ for $2 \leq t' \leq t$. Therefore as a module category $J(M)$ cannot be a direct sum of module categories of the form $\mathrm{mod}\,\Lambda_{n'}^{t'}$ or $\mathrm{mod}\,\Gamma_{n'}^{t'}$ as required. $\square$

Proposition 3.8.2. Fix a positive integers $t \geq 3$. For $n \geq t+2$, let $A = \Gamma_n^t$. Then there exists an A-module M such that the τ-perpendicular category $J(M)$ is not a direct sum of module categories over algebras of the form $\Gamma_{n'}^{t'}$ or $\Lambda_{n'}^{t'}$ for $2 \leq t' \leq n' < n$.

Proof. The argument is similar to that for the previous proposition. We prove this by counter-example. Set $M = S_1$, the simple module at vertex 1 of the quiver A_n of A. The Auslander-Reiten translate of S_1 is $\tau S_1 = S_2$. By Proposition 3.3.11 and 3.3.12, $\mathrm{Hom}(X, S_2) \neq 0$ if and only if $X = P_2/\mathrm{rad}^{l(X)}(P_2)$ where $l(X)$ is the length of X. It also follows that P_2 is the only projective with non-zero maps to S_2. Therefore all other indecomposable projective modules P_j with $j \neq 2$ are in ${}^{\perp}(\tau S_1)$, hence they are Ext-projectives in ${}^{\perp}(\tau S_1)$. By Proposition 3.3.1, the module S_1 is Ext-project in ${}^{\perp}(\tau S_1)$ We can thus conclude that,

$$\mathcal{P}(\tau S_1) = \bigoplus_{j \neq 2} P_j \oplus S_1.$$

By definition the Bongartz completion of M in $\operatorname{mod} A$ is $T_M = \mathcal{P}(\tau S_1)$. Let Q_n be the following quiver,

$$v_{s_1} \xleftarrow{\alpha} 1 \xleftarrow{\alpha_{32}} 3 \xleftarrow{\alpha_4} 4 \xleftarrow{\alpha_5} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \ ,$$

where the vertices labelled j correspond to the projective P_j and the vertex v_{s_1} corresponds to the simple S_1 and the arrows correspond to the irreducible maps between their respective modules. The $\mathbb{F}$-algebra $E_M = \operatorname{End}_A(T_M)$ is the path algebra of the quiver Q_n modulo relations. Let $Q_n^{v_{s_1}}$ be the quiver obtained from Q_n by removing the vertex v_{s_1} and any arrows incident to v_{s_1},

$$1 \xleftarrow{\alpha_{32}} 3 \xleftarrow{\alpha_4} 4 \xleftarrow{\alpha_5} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \ .$$

by Lemma 3.3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n^{v_{s_1}}$ modulo relations. We have the relation $\alpha_{t+1}\alpha_t \dots \alpha_4\alpha_{32} = 0$ involving $t-1$ arrows because it corresponds to $\operatorname{Hom}_A(P_{t+1}, P_2) = 0$ since in $\operatorname{mod} \Gamma_n^t$ the composition of t maps between projectives is 0. However, at the same time we have that the composition of the $t'-1$ arrows $\alpha_n\alpha_{n-1} \dots \alpha_{n-(t'-2)} \neq 0$. Therefore as a module category $J(M)$ cannot be a direct sum of module categories of the form $\operatorname{mod} \Lambda_{n'}^{t'}$ or $\operatorname{mod} \Gamma_{n'}^{t'}$ as required. □

So we have shown that our strategy for deriving recurrences for the number of complete τ-exceptional sequences over Nakayama algebras only works in the four cases we've studied. However, the statements of Theorems 3.3.8 and 3.3.9 are general enough that a similar strategy may be applied to other algebras, and may prove as effective for counting the τ-exceptional sequences for the module categories of those algebras.

Chapter 4

Modular Fuss-Catalan numbers

4.1 Introduction

The Catalan numbers are a ubiquitous sequence of natural numbers with a rich mathematical history. They appear in mathematics in widely different contexts and count an ever growing list of sequences of combinatorial sets, see Stanley (2015) for more on Catalan numbers. The n^{th} Catalan number C_n, where n is a non-negative integer is given by the closed formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Suppose $*$ is an m-ary operation for $m \geq 1$. An m-ary parenthesization of the m-ary product $x_0 * \cdots * x_n$ is a parenthesization where each product is m-ary. For example, when $m = 3$ the parenthesization $((x_0 * x_1 * x_2) * x_3 * x_4)$ is 3-ary whereas $((x_0 * x_1) * x_2 * x_3 * x_4)$ is not. The Fuss-Catalan numbers are a natural generalisation of Catalan numbers introduced by Fuss in Fuss (1791). They can be thought of as "higher-dimensional" Catalan numbers. For example, the Catalan number C_n

counts the number of binary parenthesizations of the expression $x_0 * \cdots * x_n$, whereas the Fuss-Catalan number C_n^m counts the number of m-ary parenthesizations of the expression $x_0 * \cdots * x_n$, where m and n are non-negative integers, such that $m \geq 1$. The n^{th} Fuss-Catalan number with parameter m is given by the closed formula:

$$C_n^m = \frac{1}{(m-1)n+1}\binom{mn}{n}.$$

When $m = 2$, we recover the Catalan numbers from the Fuss-Catalan numbers, that is to say $C_n^2 = C_n$. The modular Catalan numbers, introduced in Hein & Huang (2017), count equivalence classes of parenthesizations of $x_0 * \cdots * x_n$, where $*$ is a binary operation satisfying the k-associative law, which generalises the usual notion of associativity. In this chapter we introduce and study a higher-dimensional version of the modular Catalan numbers, which we call modular Fuss-Catalan numbers.

Let X be a non-empty set with a binary operation $\star : X^2 \to X$, and let n be a positive integer. If $\star$ is associative, then the general associativity law states that the expression $x_1 \star \cdots \star x_n$ is unambiguous for all $x_1, \ldots, x_n \in X$. Which is to say, all possible parenthesizations of the expression result in the same evaluation. The order of operation of a binary operation $\star$ is *left-justified* if the order of operation is understood to be from left to right, in which case we write $x_1 \star \cdots \star x_n$ to mean $((\ldots((x_1 \star x_2) \star x_3) \cdots \star x_{n-1}) \star x_n)$. From this point onwards, it will be our convention to treat all binary operation as left-justified. Let $k \geq 1$ be a positive integer. There is a notion of k-associativity for binary operations which generalises the usual notion of associativity. A binary operation $\star$ is *k-associative* if

$$(x_1 \star x_2, \star \cdots \star x_{k+1}) \star x_{k+2} = x_1 \star (x_2 \star \cdots \star x_{k+1} \star x_{k+2}) \text{ for all } x_1, \ldots, x_{k+2} \in X.$$

By setting $k = 1$, we recover the classical notion of associativity for binary operations. In the case where $k > 1$, the general associativity law no longer holds, which is to say in general the evaluation of the expression $x_1 \star x_2 \star \cdots \star x_n$ depends on its parenthesization. The k-associative binary operations are studied in Hein & Huang (2017).

Fix a positive integer $m \geq 2$. An *m-ary operation* on X is a map $* : X^m \to X$. Another way to generalise associativity of binary operations is to consider associative m-ary operations. An m-ary operation $*$ is *associative* if for $1 \leq j \leq m-1$,

$$\begin{aligned} & x_1 * \cdots * x_{j-1} * (x_j * x_{j+1} * \cdots * x_{j+(m-1)}) * x_{j+(m-1)+1} * x_{j+(m-1)+2} * \cdots * x_{m+(m-1)} \\ = & x_1 * \cdots * x_{j-1} * x_j * (x_{j+1} * \cdots * x_{j+(m-1)} * x_{j+(m-1)+1}) * x_{j+(m-1)+2} * \cdots * x_{m+(m-1)} \end{aligned} \tag{4.1}$$

for all $x_1, \ldots, x_{m+(m-1)} \in X$, see for example (Post, 1940, §1).

As in the case for associative binary operations, there is a general associativity law stating that the expression $x_1 * \cdots * x_n$ is independent of m-ary parenthesization (see for example (Andres, 2009, Theorem 2.1)). Which is to say that all possible parenthesizations of the expression result in the same evaluation. We note that n is not arbitrary in this case, but is of the form $n = m + g(m-1)$ for some integer $g \geq 1$. Associative m-ary operations are important for the study of m-semigroups and polyadic groups. These are generalisations of semigroups and groups where we consider associative m-ary operations instead of associative binary operations. The m-semigroups were introduced in Dörnte (1929) and polyadic groups were introduced in Post (1940) and Sankappanavar & Burris (1981).

In this chapter we will study m-ary k-associative operations, which are a further generalisation of associative binary operations that combines the two generalisations above. The order of operation of an m-ary operation $*$ is *left-justified* if the order of operation is understood to be from left to right, hence for an integer $g \geq 1$, we write $x_1 * \cdots * x_{m+g(m-1)}$ to mean

$$((\ldots((x_1 * \cdots * x_m) * x_{m+1} * \cdots * x_{m+(m-1)}) \cdots * x_{m+(g-1)(m-1)}) x_{m+(g-1)(m-1)+1} \cdots x_{m+g(m-1)}).$$

From this point onwards, it will be our convention to treat all m-ary operations as left-justified. An m-ary operation $*$ is k-*associative* if for $1 \leq j \leq m-1$, the following equality holds:

$$\begin{aligned} & x_1 * \cdots * x_{j-1} * (x_j * x_{j+1} * \cdots * x_{j+k(m-1)}) * x_{j+k(m-1)+1} * x_{j+k(m-1)+2} * \cdots * x_{m+k(m-1)} \\ = & x_1 * \cdots * x_{j-1} * x_j * (x_{j+1} * \cdots * x_{j+k(m-1)} * x_{j+k(m-1)+1}) * x_{j+k(m-1)+2} * \cdots * x_{m+k(m-1)}. \end{aligned} \tag{4.2}$$

We note that the terminology "k-associativity" is used by Wardlaw in Wardlaw (2001) to mean associativity of k-ary operations. This is not to be confused with the notion of k-associativity we consider here, which is a generalisation of associativity for m-ary operations (and in the case of binary operations coincides with the notion of k-associativity as introduced in Hein & Huang (2017)).

Let $*$ be a k-associative m-ary operation, and $g \geq 1$ be a positive integer. For $n = m + g(m-1)$ and $k > 1$, the expression $x_1 * \cdots * x_n$ is ambiguous without a parenthesization to clarify the order of operation, which is to say the general associativity law no longer holds. Let p and p' be two parenthesizations of $x_1 * \cdots * x_n$. If we can obtain p' from p via a sequence of finitely many left side to right side applications of the k-associative property (4.2), then we write $p \preceq_k p'$. The k-*associative order* is the induced partial order on the set of parenthesizations of $x_1 * \cdots * x_n$. The k-components are the connected components of the k-associative order. Two parenthesizations of $x_1 * \cdots * x_n$ are k-*equivalent* if they lie in the same k-component. When $k = 1$ and $m = 2$, we recover the well-known *Tamari lattice* (see for example Geyer (1994)). In general, determining whether two parenthesizations are k-equivalent is a non-trivial problem.

Cluster algebras are a class of commutative algebras defined combinatorially by a process of iterated mutation. They were first introduced in Fomin & Zelevinsky (2002), Fomin & Zelevinsky (2003), Berenstein *et al.* (2005), Fomin & Zelevinsky (2007) as an approach towards problems on total positivity Fomin (2010) and canonical bases in quantum groups. Since their inception, cluster algebras have become an object of study in their own right. They find uses in many other areas including representation theory Leclerc (2010), Poisson geometry Gekhtman *et al.* (2010) and integrable systems Williams (2014).

To define a cluster algebra over the field $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_n)$ of rational functions in the indeterminates $u_1, \ldots, u_n$, one starts with a *seed*. A seed is a pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ which consists of a set of variables $\tilde{\mathbf{x}} = \{v_1, \ldots, v_n\}$, which freely generates the field $\mathbb{F}$, and an integer matrix $\tilde{\mathbf{B}}$ called the *exchange matrix*. By applying a certain mutation rule μ_k in a direction k, where $1 \leq k \leq n$, to the seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$, we obtain another seed $\mu_k(\tilde{\mathbf{x}}, \tilde{\mathbf{B}}) = (\tilde{\mathbf{x}}', \tilde{\mathbf{B}}')$ consisting of a free generating set of variables $\tilde{\mathbf{x}}'$ and exchange matrix $\tilde{\mathbf{B}}'$. Let $\mathcal{S}$ be the set of variables obtained from performing all possible finite sequences of mutations to the seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$. The cluster algebra with initial seed $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$, which we denote by $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ is the subring of $\mathbb{F}$ generated by all the variables in $\mathcal{S}$.

Of particular interest to us are the cluster algebras of type A_n (where n is a positive integer), and their combinatorics. The seeds of a cluster algebra of type A_n can be encoded as triangulations of an $(n+3)$-gon; see (Fomin *et al.*, 2017, Lemma 5.3.1). Mutating a seed in a cluster algebra of type A_n turns out to be equivalent to performing a *flip* on a diagonal of the corresponding triangulation; see (Fomin *et al.*, 2017, Corollary 5.3.6). The number of seeds of a cluster algebra of type A_n is equal to the number of triangulations of an $(n+3)$-gon, which is the Catalan number C_{n+1}; see (Fomin *et al.*, 2017, Corollary 5.3.6). Consider the graph whose vertices are triangulations of $(n+3)$-gons, and where there is an edge between two triangulations T and T' if and only if we can obtain T' from T by performing a flip on a diagonal of T or vice versa. This graph is a regular graph where each vertex has degree n. In fact this graph is the 1-skeleton of the n-dimensional convex polytope known as the *associahedron*, see for example (Fomin *et al.*, 2020, §1.2). This 1-skeleton is usually realised as follows: the vertices are binary parenthesizations of $x_1 \star \cdots \star x_{n+2}$, and the edges represent applications of the associativity rule. Because of this correspondence, k-associativity is of interest in the study of cluster algebras. In particular, viewing k-associativity as a mutation rule, it might be possible to generalise the definition of cluster algebras to a wider class of objects. If this is possible, then m-ary k-associativity could extend this generalisation to a higher-dimensional setting.

4.2 m-ary Trees

In studying k-equivalence, it is often more convenient to do so by appealing to other sequences of combinatorial sets counted by the Fuss-Catalan numbers. In this section we will study k-equivalence via m-ary trees. In order to do this, we use a known bijection between parenthesizations of m-ary expressions and m-ary trees outlined in (Hilton & Pedersen, 1991, §0). For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$, and $n = m + g(m-1)$.

Definition 4.2.1. m-ary Tree (Stanley, 2015, §4, A14(b)). An *m-ary tree* is a rooted tree with the property that each node either has 0 or m linearly ordered children. A *leaf* is a node with no children and the unique node without a parent is the *root* of the tree. For a node with m-children, the l^{th} child refers to the l^{th} node below when counting from left to right, and likewise the l^{th} subtree refers to the l^{th} subtree below when counting from left to right.

The objects we are calling m-ary trees in this thesis are commonly referred to as *full m-ary trees* in the wider literature. There are multiple ways in which we can traverse (systematically examine the nodes of the tree so that each node is visited only once) the nodes of an m-ary tree. In this thesis, it will be our convention to traverse m-ary trees by the pre-order traversal method. Recall that the *pre-order traverse* is defined recursively as follows.

Definition 4.2.2. Pre-order traverse(Knuth, 1997, §2.3.1, page 319,336) If an m-ary tree is empty, then do nothing. Otherwise,

- Visit the root
- Traverse the 1^{st} subtree of the root
- Traverse the 2^{nd} subtree of the root
- $\ldots$
- Traverse the m^{th} subtree of the root.

It will be our convention in this thesis to draw m-ary trees with the root at the top and leaves below the root. We shall denote the set of m-ary trees with n leaves by $\mathbf{B}_n^m$. We will enumerate the leaves by the order in which the leaves are visited in the pre-order traverse. Hence enumerating by 1 the first leaf to be visited in the pre-order traverse, by 2 the second leaf to be visited in the pre-order traverse, and so on up to n for the last leaf to be visited in the pre-order traverse. We will endow the m-ary trees with an additional edge labelling with labels from the set $\{l_1, \ldots, l_m\}$. An edge will be given the label l_i if it links a node with its i^{th} child. See the figure below for an example.

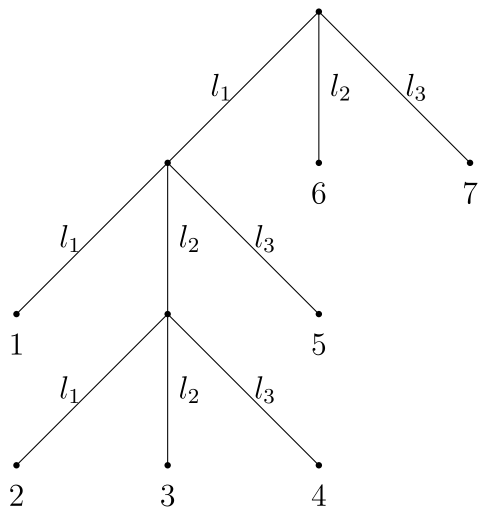


Figure 4.1: A labelled 3-ary tree.

Definition 4.2.3. Tag. Let $t_1, \ldots, t_m$ be m-ary trees. We define the *tag* of $t_1, \ldots, t_m$ to be the m-ary tree $t_1 \wedge \cdots \wedge t_m$, which has the tree t_i as the subtree rooted at the i^{th} child of the root for $1 \leq i \leq n$.

The following bijection is well-known, see for example (Hilton & Pedersen, 1991, §0).

Proposition 4.2.4. (Hilton & Pedersen, 1991, §0) Let X be a non-empty set and let $* : X^m \to X$ be an m-ary operation. Take $x_1, \ldots, x_n$ in X. There is a bijection between the set of m-ary trees on n leaves and the set of m-ary parenthesizations of the expression $x_1 * \cdots * x_n$ which is defined in the following way. Let t be an m-ary tree with n leaves where the i^{th} leaf of t is labelled ε_i. Consider the tree t expressed as a bracketed tag of its leaves ε_i, where the ε_i are thought of as trees consisting of just a root. The bijection maps t to the parenthesization obtained by replacing $\wedge$ with $*$ and replacing ε_i with x_i. The inverse map from the set of m-ary parenthesizations of the expression $x_1 * \cdots * x_n$ to m-ary trees with n leaves acts in the naturally opposite way.

Example 4.2.5. Let t be the tree in Figure 4.1. Thinking of the leaves of t as 3-ary trees consisting of just a root, assign to each leaf i the label ε_i. We can write t as a tag of the leaves ε_i so $t = (\varepsilon_1 \wedge (\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4) \wedge \varepsilon_5) \wedge \varepsilon_6 \wedge \varepsilon_7$. Under the bijection in Proposition 4.2.4 the tree t is assigned to the parenthesization of the x_i given by $(x_1 * (x_2 * x_3 * x_4) * x_5) * x_6 * x_7$.

Definition 4.2.6. Right k-rotation. Let $k \geq 1$ be a positive integer. Let $t_1, t_2, \ldots, t_{(m-1)+k(m-1)}$ be m-ary trees. Let $1 \leq j \leq m-1$. Suppose that $t \in \mathbf{B}_n^m$ has a subtree,

$$s = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$$

rooted at some node v in t. The *right k-rotation* of t at v is the operation of replacing s with the subtree

$$s' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_j \wedge (t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1}) \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

Remark 4.2.7. It should be clear that under the bijection in Proposition 4.2.4, right k-rotation of m-ary trees corresponds to a left side to right side application of the k-associative rule in (4.2). We can also define a left k-rotation dually by switching the roles of s and s' in the definition above. In this case, a left k-rotation corresponds to a right side to left side application of the k-associative rule in (4.2).

Definition 4.2.8. Let t and t' be m-ary trees with n leaves. If we can obtain t' from t by applying finitely many right k-rotations to t, then we write $t \preceq_k t'$. The *k-associative order* is the induced partial order on $\mathbf{B}_n^m$. The *k-components* are the connected components (connected components of the Hasse diagram) of $\mathbf{B}_n^m$ under the k-associative order. Two m-ary trees with n leaves are *k-equivalent* if they belong to the same k-component of $\mathbf{B}_n^m$.

Example 4.2.9. The example below shows a right 2-rotation on a 3-ary tree. We apply the right 2-rotation at v.

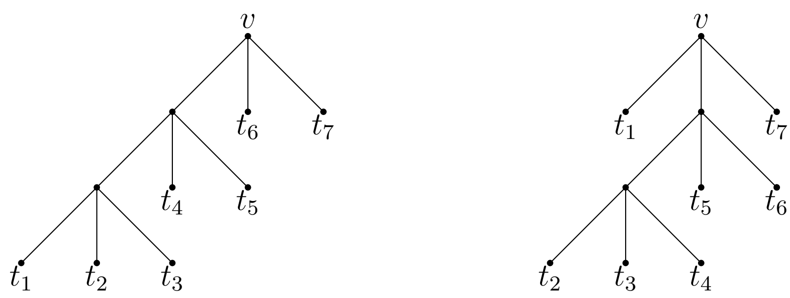


Figure 4.2: The tree on the right is a result of a right 2-rotation of the tree on the left at v.

The subtree rooted at v is $s = (t_1 \wedge t_2 \wedge t_3 \wedge t_4 \wedge t_5) \wedge t_6 \wedge t_7 = ((t_1 \wedge t_2 \wedge t_3) \wedge t_4 \wedge t_5) \wedge t_6 \wedge t_7$. The subtree s is then replaced by the subtree $s' = t_1 \wedge (t_2 \wedge t_3 \wedge t_4 \wedge t_5 \wedge t_6) \wedge t_7 = t_1 \wedge ((t_2 \wedge t_3 \wedge t_4) \wedge t_5 \wedge t_6) \wedge t_7$ at v.

The following proposition is a generalisation of (Hein & Huang, 2017, Proposition 2.5).

Proposition 4.2.10. Let t be an m-ary tree such that we can perform a right k-rotation of t at some node v. If $k = pk'$ for some positive integers p and k', then the right k-rotation at v can be decomposed into a sequence of p right k'-rotations of t. The same holds for left k-rotations

Proof. We argue by induction on p. The case for $p = 1$ is trivial. Suppose for induction that the statement is true for some $p \geq 1$. Suppose that $k = (p+1)k'$ for some positive integer k'.

Suppose we have a tree t which we can right k-rotate at some node v. Denote by s the subtree of t rooted at v. For some $1 \leq j \leq m-1$,

$s = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$.

The right k-rotation replaces the subtree s with the subtree s' where $s' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_j \wedge (t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1} \wedge t_{j+pk'(m-1)+2} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1}) \wedge \cdots \wedge t_{(m-1)+k(m-1)}$.

We will show that the result of this right k-rotation can also be obtained by performing $(p+1)$ right k'-rotations.

Let r be the following subtree of s, which is rooted at the j^{th} child of the root of s,

$$r = (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1} \wedge \cdots \wedge t_{j+k(m-1)}).$$

We can write

$$r = ((t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)}) \wedge t_{j+pk'(m-1)+1} \wedge \cdots \wedge t_{j+k(m-1)})$$

since the tag operation is left-justified. Performing a right pk'-rotation of t at the j^{th} child of the root of s, we replace r with

$$r' = (t_j \wedge (t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1}) \wedge \cdots \wedge t_{j+k(m-1)}).$$

By the inductive hypothesis, this right pk'-rotation is the result of p right k'-rotations.

Set

$$u = (t_{j+1} \wedge \cdots \wedge t_{j+m-1} \wedge t_{j+m} \wedge \cdots \wedge t_{j+pk'(m-1)} \wedge t_{j+pk'(m-1)+1}),$$

then

$$r' = (t_j \wedge u \wedge \cdots \wedge t_{j+k(m-1)}).$$

Thus the right pk'-rotation of t at the j^{th} child of the root of s, replaces s with q at the node v in t where,

$$q = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge u \wedge t_{j+pk'(m-1)+2} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

We then perform a right k'-rotation of t at v. This replaces q with the subtree,

$$q' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_j \wedge (u \wedge t_{j+pk'(m-1)+2} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1}) \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

It is easy to see that $s' = q'$, therefore the result of performing the right $k = (p+1)k'$-rotation at v is precisely the result of performing $(p+1)$ right-k' rotations. The proof for left k-rotations is similar. □

Definition 4.2.11. Path. Let t be an m-ary tree and n a positive integer. A *path* p in t of length n from a node v to a node w is a sequence $p = (v_0, \ldots, v_n)$ of nodes such that $v_0 = v, v_n = w$ and (v_i, v_{i+1}) is an edge in t for $1 \leq i \leq n-1$.

Definition 4.2.12. Depth. Let t be an m-ary tree with n leaves and edges labelled by labels from the set $\{l_1, \ldots, l_m\}$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\delta_j^{l_i}(t)$ be the number of edges labelled l_i in the unique path from the root to the j^{th} leaf. Let $\delta^{l_i}(t) = (\delta_1^{l_i}(t), \ldots, \delta_n^{l_i}(t))$ and set $\delta(t) = (\delta^{l_1}(t), \ldots, \delta^{l_n}(t))$. The *depth* of t is the n-tuple $\delta(t)$.

Example 4.2.13. Let t be the tree in Figure 4.1. The depth of tree t is given by

$$\delta(t) = ((2,2,1,1,1,0,0),(0,1,2,1,0,1,0),(0,0,0,1,1,0,1)).$$

The following lemmas are easy to verify.

Lemma 4.2.14. Suppose that t is an m-ary tree with n leaves and depth

$$\delta(t) = (\delta^{l_1}(t), \ldots, \delta^{l_m}(t)).$$

It then follows that $\delta_n^{l_m}(t) \neq 0$ and $\delta_n^{l_i}(t) = 0$ for $i \neq m$. Dually, $\delta_1^{l_1}(t) \neq 0$ and $\delta_1^{l_i}(t) = 0$ for $i \neq 1$.

This is because the unique path from the root to the n^{th} leaf involves choosing the m^{th} child at each stage. Similarly for the dual statement.

Lemma 4.2.15. Suppose that t is an m-ary tree with n leaves and depth

$$\delta(t) = (\delta^{l_1}(t), \ldots, \delta^{l_m}(t)).$$

It then follows that $\delta^{l_{m-1}}_{n-1}(t) = 1$, moreover for $1 \leq i \leq m-2$, we have that $\delta^{l_i}_{n-1}(t) = 0$.

This is because the unique path from the root to the $(n-1)^{\text{th}}$ leaf involves choosing the m^{th} child at every stage but one, in which case, we choose the $(m-1)^{\text{th}}$ child.

We shall prove the following result in the next section.

Theorem 4.2.16. Suppose that t and t' are a pair of m-ary trees with n leaves with depths $\delta(t) = (\delta^{l_1}(t), \ldots, \delta^{l_m}(t))$ and $\delta(t') = (\delta^{l_1}(t'), \ldots, \delta^{l_m}(t'))$ respectively. It then follows that t and t' are k-equivalent if and only if

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1)$$

where the addition on the n-tuples is componentwise.

The strength of the theorem is that it allows us to determine the k-equivalence of m-ary trees from simply reading their depths. The case $m = 2$ is known, see (Hein & Huang, 2017, Proposition 2.11)). We shall prove the case for general $m \geq 2$. To do this, we appeal to another sequence of combinatorial sets counted by the Fuss-Catalan numbers, the m-Dyck paths. The setting of m-Dyck path turns out to be a more natural setting for studying k-equivalence.

4.3 m-Dyck Paths

In this section we prove Theorem 4.2.16. In order to do so, we appeal to a generalisation of Dyck paths known as m-Dyck paths to further study k-equivalence. We prove the theorem by first proving an m-Dyck path version of it. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m-1)$.

Definition 4.3.1. m-Dyck Path. An *m-Dyck path* is a lattice path in $\mathbb{Z}^2$ starting at $(0,0)$ consisting of up-steps (m,m) and down-steps $(1,-1)$, which remains above the x-axis and ends on the x-axis. The length of a Dyck path is defined to be the number of down-steps it has.

Definition 4.3.2. Translated m-Dyck Path. Let a,b be non-negative integers both not equal to 0. A *translated m-Dyck path* is a lattice path in $\mathbb{Z}^2$ starting at the point (a,b) consisting of up-steps (m,m) and down-steps $(1,-1)$, which remains above the line $y=b$ and ends on the line $y=b$.

We denote the set of m-Dyck paths of length n by $\mathbf{D}_n^m$. Where it is convenient, we refer to these paths as Dyck paths instead of m-Dyck paths. When referring to a translated m-Dyck path that is a sub path of a larger m-Dyck path, we will call it a *sub m-Dyck path* or just *sub-Dyck path*. The following lemma is straight forward, so we state it without proof.

Lemma 4.3.3. For every m-Dyck path D of length n, we can write

$$D = N^{d_1}S\ldots SN^{d_n}S,$$

where N denotes the up-step $(1,1)$ and S denotes the down-step (1,-1). Note that when $m \neq 1$ the up-steps (1,1) are not steps on the path D since by definition up-steps of D are of the form (m,m). Here N^{d_i} is taken to mean a sequence of d_i consecutive up-steps N. The d_i are non-negative integer multiples of m such that $d_1+\cdots+d_n=n$, and $d_1+\cdots+d_j \geq j$ for $1 \leq j < n$. The latter conditions on the d_i are because the m-Dyck paths start at $(0,0)$ and end on the x-axis whilst remaining above the x-axis. Moreover the n-tuple $d(D)=(d_1,\ldots,d_n)$ is unique to each m-Dyck path D.

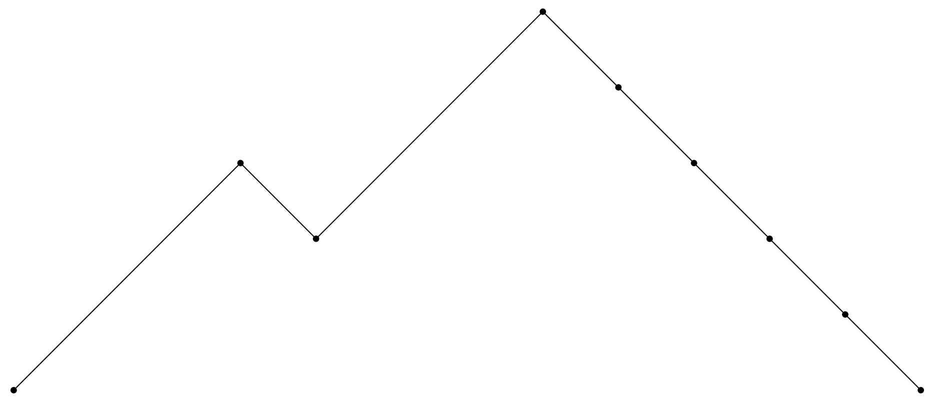


Figure 4.3: A 3-Dyck path with 2 (3,3) up-steps. $D = N^3SN^3SN^0SN^0SN^0SN^0S$.

Let $D = N^{d_1}S \dots SN^{d_n}S$ be an m-Dyck path of length n. When expressing D in this way, if $d_i = 0$, we will omit N^{d_i} from the expression. In this form we will also write S^l to mean a sequence of l consecutive S steps. For example, $D = N^3SN^3SN^0SN^0SN^0SN^0S = N^3SN^3S^5$.

We can express any m-ary tree as the tag of the m-ary sub-trees rooted at the children of the root. Therefore, for t an m-ary tree with n leaves, we write

$$t = t_1 \wedge \cdots \wedge t_m,$$

where for $1 \leq i \leq m$ each t_i is an m-ary tree with n_i leaves and $n_1 + \cdots + n_m = n$.

Let ε be the element of the singleton set $\mathbf{B}_0^m$, so ε is the m-ary tree which consists of just a root. Let $\mathbf{B}^m$ be the set m-ary trees with any appropriate number of leaves, and likewise let $\mathbf{D}^{m-1}$ be the set of $(m-1)$-Dyck paths of any appropriate length. We construct a map $\sigma_m \colon \mathbf{B}^m \to \mathbf{D}^{m-1}$ from the set of m-ary trees to the set of $(m-1)$-Dyck paths. We define σ_m inductively in the following way,

$$\sigma_m(t) = \begin{cases} N^0S^0 & \text{if } t = \varepsilon; \\ N^{m-1}\sigma_m(t_1)S\sigma_m(t_2)S \dots S\sigma_m(t_m) & \text{otherwise.} \end{cases}$$

This construction generalises a well known map between binary trees (2-ary trees) and Dyck paths (1-Dyck paths); see for example (Bernardi & Bonichon, 2009, Page 58, Tamari Lattice, Paragraph 2).

Example 4.3.4. Consider the following 3-ary tree $t = \varepsilon \wedge \varepsilon \wedge (\varepsilon \wedge \varepsilon \wedge \varepsilon)$. We calculate $\sigma_3(t)$,

$$\begin{aligned}\sigma_3(t) &= N^2\sigma_3(\varepsilon)S\sigma_3(\varepsilon)S\sigma_3(\varepsilon \wedge \varepsilon \wedge \varepsilon)\\ &= N^2N^0S^0SN^0S^0SN^2\sigma(\varepsilon)S\sigma(\varepsilon)S\sigma(\varepsilon)\\ &= N^2S^2N^2S^2.\end{aligned}$$

See Figure 4.4 below.

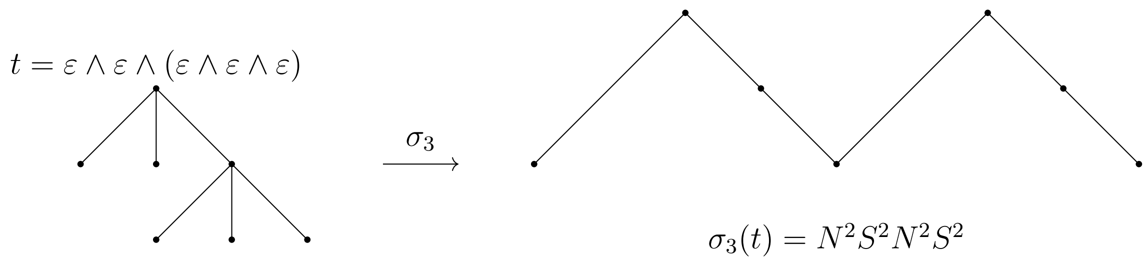
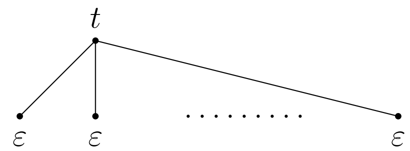


Figure 4.4: The image under σ_3 of the 3-ary tree $t = \varepsilon \wedge \varepsilon(\varepsilon \wedge \varepsilon \wedge \varepsilon)$.

Lemma 4.3.5. The map $\sigma_m \colon \mathbf{B}^m \to \mathbf{D}^{m-1}$ sends m-ary trees with n leaves to $(m-1)$-Dyck paths of length $n-1$.

Proof. We argue by induction. Recall that $n = m + g(m-1)$ for some integer $g \geq 0$. We prove the result by induction on g. When $g = 0$ there is only one tree to consider, namely $t = \varepsilon \wedge \cdots \wedge \varepsilon$.



It is easy to see that $\sigma_m(t) = N^{m-1}S^{m-1}$ which is an $(m-1)$-Dyck path of length $m-1$.

Now suppose that the result holds for $n = m + g'(m-1)$ with $0 \leq g' \leq g$. We consider the $g+1$ case. If t is an m-ary tree with $m + (g+1)(m-1)$ leaves, then we may write $t = t_1 \wedge \cdots \wedge t_m$ with the $t_i \in B^m_{n_i}$ and $n_1 + \cdots + n_m = m + (g+1)(m-1)$. By definition $\sigma_m(t) = N^{m-1}\sigma_m(t_1)S\sigma_m(t_2)S \ldots S\sigma_m(t_m)$ and by the inductive hypothesis, each $\sigma(t_i)$ is an $(m-1)$-Dyck paths of length $n_i - 1$. In the expression for $\sigma_m(t)$ we have $m-1$ down-steps S following the N^{m-1} inbetween the $\sigma_m(t_i)$. Therefore, the length of $\sigma_m(t)$ is $(n_1 - 1) + \cdots + (n_m - 1) + (m-1)$ which is equal to $m + (g+1)(m-1) - 1$ as required.

What is left is to show that $\sigma_m(t)$ is weakly above the x-axis. By Lemma 4.3.3, each $\sigma_m(t_i)$ can be written in the form $\sigma_m(t_i) = N^{d^i_1}S \ldots SN^{d^i_{n_i}}S$, where $d^i_1 + \cdots + d^i_{n_i} = n_i$, and $d^i_1 + \cdots + d^i_r \geq r$ for $1 \leq r < n_i$. We can likewise write that $\sigma_m(t) = N^{d_1}S \ldots SN^{d_n}S$ where $d_1 = (m-1) + d^1_1$ and $d_n = 0$, and for $2 \leq h < n$ either $d_h = 0$ or $d_h = d^i_u$ for some appropriate i and u. Since there are $m-1$ down-steps S following the N^{m-1} inbetween the $\sigma_m(t_i)$, and the $\sigma_m(t_i)$ are all weakly above the x-axis. It follows that $d^1_1 + \cdots + d_j \geq j - (m-1)$, therefore $d_1 + \cdots + d_j \geq j$, for $1 \leq j \leq n$, where there is equality if $j = n$. Thus $\sigma_m(t)$ is weakly above the x-axis, hence the map σ_m is indeed from $\mathbf{B}^m_n$ to $\mathbf{D}^{m-1}_{n-1}$. □

By the lemma above, σ_m induces a map $\sigma_{m,n} \colon \mathbf{B}^m_n \to \mathbf{D}^{m-1}_{n-1}$. This map is in fact a bijection between $\mathbf{B}^m_n$ and $\mathbf{D}^m_{n-1}$. For any tree $t = t_1 \wedge \cdots \wedge t_m$, where for $1 \leq i \leq m$ each t_i is an m-ary tree with n_i leaves and $n_1 + \cdots + n_m = n$. The map $\sigma_{m,n}$ is defined as follows:

$$\sigma_{m,n}(t) = N^{m-1}\sigma_{m,n_1}(t_1)S\sigma_{m,n_2}(t_2)S \ldots S\sigma_{m,n_m}(t_m).$$

Proposition 4.3.6. The map $\sigma_{m,n} \colon \mathbf{B}^m_n \to \mathbf{D}^{m-1}_{n-1}$ is a bijection.

Proof. It is well known that both the finite sets $\mathbf{B}^m_n$ and $\mathbf{D}^{m-1}_{n-1}$ have cardinality $\frac{1}{(m-1)n+1}\binom{mn}{n}$, see for example (Heubach *et al.*, 2008, §3). Therefore, in order to show that $\sigma_{m,n}$ is a bijection, it suffices to show that it is a surjection. We argue by induction on n. When $n = 0$, it is trivial.

If $D \in \mathbf{D}_{n-1}^{m-1}$, then the first step of D is an up-step $N^{k_1(m-1)}$ where $k_1 \geq 1$ is an integer, so we can write $D = N^{m-1}N^{k_1(m-1)-(m-1)}\dots S$ as in Lemma 4.3.3. Let $(x_1', m-2)$ be the first point on D with y-coordinate $m-2$ after the point $(m-1, m-1)$. The step in D ending at $(x_1', m-2)$ must be a down-step S starting at $(x_1, y_1) = (x_1'+1, m-1)$. The part of D from $(m-1, m-1)$ to (x_1, y_1) is a translated $(m-1)$-Dyck path D_1, so we see that the path D starts as $N^{m-1}D_1S$. Let $(x_2', m-3)$ be the first point on D with y-coordinate $m-3$ after the point $(x_1', m-2)$. As above, the step in D ending at $(x_2', m-3)$ must be a down-step S starting at $(x_2, y_2) = (x_2'-1, m-2)$. The part of D from $(x_1', m-2)$ to (x_2, y_2) is a translated $(m-1)$-Dyck path D_2. Therefore, $D = N^{m-1}D_1SD_2S\dots S$, and continuing this argument we see that can be write $D = N^{m-1}D_1SD_2S\dots D_mS$, where the D_i are translated $(m-1)$-Dyck paths for $1 \leq i \leq m$.

Regarding the translated $(m-1)$-Dyck paths D_i as $(m-1)$-Dyck paths, they each have length $n_i < n$ for $1 \leq i \leq m$. Hence by the inductive hypothesis, for each D_i there exists an m-ary tree t_i such that $D_i = \sigma_{m,n_i}(t_i)$. It then follows that $D = \sigma_{m,n}(t_1 \wedge t_2 \wedge \dots \wedge t_m)$.

□

Going forward, we drop the subscripts on $\sigma_{m,n}$ and just write σ when it is clear what is meant from the context.

Proposition 4.3.7. Suppose t is an m-ary tree with n leaves and depth

$$\delta(t) = (\delta^{l_1}(t), \dots, \delta^{l_m}(t)).$$

It follows that

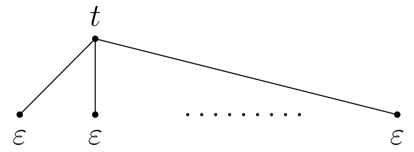
$$\sigma(t) = N^{d_1}S\dots SN^{d_{n-1}}SN^{d_n}$$

where the d_i are given by

$$d_1 = (m-1)\delta_1^{l_1}(t),$$

$$d_j = \left(\sum_{i=1}^{m} (m-i)(\delta_j^{l_i}(t) - \delta_{j-1}^{l_i}(t)) \right) + 1, \text{ for } 2 \leq j \leq n.$$

Proof. Recall that n satisfies the equation $n = m + g(m-1)$ for some integer $g \geq 0$. We prove the result by induction on g. When $g = 0$ there is only one tree to consider, namely $t = \varepsilon \wedge \cdots \wedge \varepsilon$.



For this tree, $\delta_j^{l_i} = \delta_{ij}$, where the right side is the usual Kronecker delta function. We also have that $\sigma(t) = N^{m-1}SN^0SN^0S \ldots N^0SN^0 = N^{m-1}S^{m-1}$. We now need to verify that the exponents of the Ns satisfy the relations above. Indeed $d_1 = m-1 = (m-1)\delta_1^{l_1}$. Moreover $\sum_{i=1}^{m}(m-i)(\delta_j^{l_i} - \delta_{j-1}^{l_i}) + 1 = (m-j) - (m-(j-1)) + 1 = (m-m) + ((j-1)-j) + 1 = 0 = d_j$ for $2 \leq j \leq n$.

Now suppose that the result holds for $n = m + g'(m-1)$ for all $g' \leq g$. We consider the $g+1$ case. Let t be an m-ary tree with $n = m + (g+1)(m-1)$ leaves. We may then write $t = t_1 \wedge \cdots \wedge t_m$ where each t_i is the subtree rooted at the i^{th} child of the root of t. Each subtree t_i has $n_i < n$ leaves and $n_1 + \cdots + n_m = n$. In writing t as the tag of its sub-trees at the root, we partition the leaves of t. We identify each leaf of t with a pair (h, j) if it lies in the subtree t_h and it is the j^{th} leaf in the pre-order traverse of t_h where $1 \leq j \leq n_h$. Therefore, for the leaf identified by (h, j),

$$\delta_{h,j}^{l_i}(t) = \begin{cases} \delta_{h,j}^{l_i}(t_h) + 1 & \text{if } i = h; \\ \delta_{h,j}^{l_i}(t_h) & \text{otherwise.} \end{cases} \tag{4.3}$$

By the inductive hypothesis $\sigma(t_h) = N^{d_{h,1}}SN^{d_{h,2}} \ldots SN^{h,n_h}$, where

$$d_{h,1}(t_h) = (m-1)\delta_{h,1}^{l_1}(t_h)$$

and

$$d_{h,j}(t) = \sum_{i=1}^{m}(m-i)(\delta_{h,j}^{l_i}(t_h) - \delta_{h,j-1}^{l_i}(t_h)) + 1, \text{ for } 2 \leq j \leq n_h.$$

By definition, $\sigma(t) = N^{m-1}\sigma(t_1)S\sigma(t_2)\ldots S\sigma(t_m)$, so

$$\begin{aligned}\sigma(t) &= N^{m-1}N^{d_{1,1}}SN^{d_{1,2}}S\ldots SN^{d_{1,n_1}}SN^{d_{2,1}}SN^{d_{2,2}}S\ldots N^{d_{2,n_2}}SN^{d_{3,1}}S\ldots SN^{d_{m,n_m}} \\ &= N^{(m-1)+d_{1,1}}SN^{d_{1,2}}S\ldots SN^{d_{1,n_1}}SN^{d_{2,1}}SN^{d_{2,2}}S\ldots N^{d_{2,n_2}}SN^{d_{3,1}}S\ldots SN^{d_{m,n_m}}\end{aligned}$$

Now we verify that the exponents of the Ns satisfy the required relations. We see that

$$(m-1) + d_{1,1} = (m-1) + (m-1)\delta_{1,1}^{l_1}(t_1) = (m-1)(\delta_{1,1}^{l_1}(t_1) + 1) = (m-1)\delta_{1,1}^{l_1}(t),$$

so the first exponent satisfies the required relation. We also see that

$$d_{h,j} = \sum_{i=1}^{m}(m-i)(\delta_{h,j}^{l_i}(t_h) - \delta_{h,(j-1)}^{l_i}(t_h)) + 1 = \sum_{i=1}^{m}(m-i)(\delta_{h,j}^{l_i}(t) - \delta_{h,(j-1)}^{l_i}(t)) + 1, \text{ for } 2 \leq j \leq n_h,$$

by (4.3), therefore the $d_{j,h}$ also satisfy the required relation for t.

The only exponents left to verify are the $d_{h,1}$ for $2 \leq h \leq m$. In this case, the leaf $(h-1, n_{h-1})$ is the rightmost leaf in the subtree t_{h-1}, so by Lemma 4.2.14, $\delta_{h-1,n_{h-1}}^{l_i}(t_{h-1}) = 0$ when $i \neq m$. Therefore, by (4.3), $\delta_{h-1,n_{h-1}}^{l_i}(t) = 0$ when $i \neq m, h-1$, so $\delta_{h-1,n_{h-1}}^{l_{h-1}}(t) = 1$. The leaf $(h,1)$ is the leftmost leaf in the subtree t_h, so by a dual statement of Lemma 4.2.14, $\delta_{h,1}^{l_i}(t_h) = 0$ when $i \neq 1$. Therefore, by (4.3), $\delta_{h,1}^{l_i}(t) = 0$ when $i \neq 1, h$ and $\delta_{h,1}^{l_h}(t) = 1$. It follows that,

$$\begin{aligned}\sum_{i=1}^{m}(m-i)(\delta_{h,1}^{l_i}(t) - \delta_{h-1,n_{h-1}}^{l_i}(t)) + 1 &= (m-1)\delta_{h,1}^{l_1}(t_h) + (m-h) - (m-(h-1)) \\ &\quad - (m-m)\delta_{h-1,n_{h-1}}^{l_m}(t_{h-1}) + 1 \\ &= (m-1)\delta_{h,1}^{l_1}(t_h) \\ &= d_{h,1}\end{aligned}$$

therefore the $d_{h,1}$ also satisfy the required relations for $2 \leq h \leq m$. This completes the proof. $\square$

Remark 4.3.8. It is important to note that $d_n = 0$ in Proposition 4.3.7 since otherwise D is not a Dyck path. We can observe that $d_n = 0$ by referencing Lemma 4.2.14 and Lemma 4.2.15. Hence in the proposition above, $\sigma(t)$ is indeed a Dyck path. Also note that since $d_n = 0$, this form of $\sigma(t)$ is the same as that given in Lemma 4.3.3.

The bijection between m-ary trees with n leaves and $(m-1)$-Dyck paths of length $n-1$ induces an operation corresponding to k-rotation on Dyck paths, which we shall call a *k-compression*. Recall in the definition of a right k-rotation we replace a sub-tree of the form,

$$s = t_1 \wedge t_2 \wedge t_3 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$$

by a subtree of the form,

$$s' = t_1 \wedge t_2 \wedge \cdots \wedge t_j \wedge (t_{j+1} \wedge t_{j+2} \cdots \wedge t_{j+k(m-1)+1}) \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}.$$

It is easy to see that

$$\sigma(s) = N^{m-1} D_1 S D_2 S \ldots D_{j-1} S N^{k(m-1)} D_j S D_{j+1} S \ldots S D_{m+k(m-1)},$$

and

$$\sigma(s') = N^{m-1} D_1 S D_2 S \ldots D_{j-1} S D_j S N^{k(m-1)} D_{j+1} S \ldots S D_{m+k(m-1)},$$

where $D_i = \sigma(t_i)$.

Definition 4.3.9. Right k-Compression. Let $k \geq 1$ and $1 \leq j \leq m-1$ be positive integers. Let D be an $(m-1)$-Dyck path of length $n-1$. Suppose D contains a sub-Dyck path of the form

$$X = N^{m-1} D_1 S D_2 S \ldots D_{j-1} S N^{k(m-1)} D_j S D_{j+1} S \ldots S D_{m+k(m-1)},$$

where the D_i are (possibly translated) Dyck paths which may be empty. A *right k-compression* at X is the operation of replacing X with the sub-Dyck path

$$X' = N^{m-1}D_1SD_2S\dots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S\dots SD_{m+k(m-1)}.$$

A *left k-compression* is the inverse operation of replacing X' with X. Let D, D' be $(m-1)$-Dyck paths of length $n-1$. Write $D \preceq_k D'$ to mean that D' can be obtained from D by applying finitely many right k-compressions. The *k-associative order* is the induced partial order on $\mathbf{D}^{m-1}_{n-1}$. The *k-components* are the connected components of $\mathbf{D}^{m-1}_{n-1}$ under the k-associative order. Two $(m-1)$-Dyck paths of length $n-1$ are *k-equivalent* if they belong to the same k-component.

Let $\mathbb{M} \subset \mathbb{N}^n$ be the set of n-tuples of non-negative integers $(e_1, \dots, e_n)$ satisfying the following relations,

$$e_1 + \dots + e_n = n-1,$$

$$(m-1)|e_i \text{ for } 1 \leq i \leq n,$$

$$e_1 + \dots + e_{j-1} \geq j-1 \text{ for all } 2 \leq j \leq n.$$

Notice that it follows from the first and last relation that $e_n = 0$.

Proposition 4.3.10. The map $d\colon \mathbf{D}^{m-1}_{n-1} \to \mathbb{M}$ maps an $(m-1)$-Dyck path of length $(n-1)$ $D = N^{d_1}S\dots SN^{d_n}$ to the n-tuple $d(D) = (d_1, \dots, d_n)$. This map is a bijection.

Proof. Let $D = N^{d_1}S\dots SN^{d_n}$ be an $(m-1)$-Dyck path. Let $d(D) = (d_1, \dots, d_n)$. Note that $d_n = 0$ by Remark 4.3.8, so the form of D is precisely as in Lemma 4.3.3. By Lemma 4.3.3 the tuple $(d_1, \dots, d_{n-1})$ is unique, so the map d is well-defined. Since D is an $(m-1)$-Dyck path, by definition $(m-1)|d_i$. All Dyck paths start and end on the x-axis, therefore they must go up the same number of times as they go down. Hence if a path has length $n-1$, which is the number of down-steps S, then $d_1 + \dots + d_n = n-1$. By definition, Dyck paths cannot go below the x-axis, this is to say that $d_1 + \dots + d_{j-1} \geq j-1$ for all $j \geq 2$.

Let $f\colon \mathbb{M} \to \mathbf{D}_{n-1}^{m-1}$ be the map given by $f(e_1, \ldots, e_n) = N^{e_1}S \ldots SN^{e_n}$. This is a $(m-1)$-dyck path by the arguments similar to those above. It is easy to see that $f(d(D)) = D$, and $d(f((e_1, \ldots, e_n))) = (e_1, \ldots, e_n)$. Therefore, d is indeed a bijection. $\square$

Proposition 4.3.11. Let D, D' be $(m-1)$-Dyck paths of length $n-1$ with $d(D) = (d_1, \ldots, d_n)$ and $d(D') = (d'_1, \ldots, d'_n)$. Suppose that we can obtain D' from D by applying a right k-compression to D. There then exist $1 \leq j < i \leq n$ such that $d'_i = d_i + k(m-1)$, $d'_j = d_j - k(m-1)$ and $d'_h = d_h$ for $h \neq i, j$.

Proof. Recall from the definition of right k-compression, there exists a sub-dyck path

$$X = N^{m-1}D_1SD_2S \ldots D_{a-1}SN^{k(m-1)}D_aSD_{a+1}S \ldots SD_{m+k(m-1)}$$

in D which we then replace with the sub-dyck path

$$X' = N^{m-1}D_1SD_2S \ldots D_{a-1}SD_aSN^{k(m-1)}D_{a+1}S \ldots SD_{m+k(m-1)}$$

to get D'. In replacing X with X' we are simply moving the substring $N^{k(m-1)}$ from the immediate left of the (possibly translated) Dyck path D_a to the immediate left of (possibly translated) Dyck path D_{a+1}. Write $D = N^{d_1}S \ldots SN^{d_n}$. Since we have the sub-dyck path X in D, we have the sub-strings $N^{k(m-1)}D_a = N^{d_j}S \ldots S$ and $D_{a+1} = N^{d_i} \ldots S$ in D for some $1 \leq j < i \leq n$. Therefore, in replacing X with X' to get D' (moving the $N^{k(m-1)}$ up-steps) we can observe that we have the sub-strings $D_a = N^{d_j - k(m-1)}S \ldots S$ and $N^{k(m-1)}D_{a+1} = N^{d_i + k(m-1)}S \ldots S$ in the Dyck path D'. This proves the statement of the proposition. $\square$

Remark that if we replace right k-compression with left k-compression in the proposition above, we get that $j > i$ instead.

Corollary 4.3.12. Let D, D' be $(m-1)$-Dyck paths of length $n-1$. If D and D' are k-equivalent, then $d(D) \equiv d(D') \bmod k(m-1)$.

Proof. This is an immediate consequence of Proposition 4.3.11. □

Let D be an m-Dyck path of length n. A dyck path D is *k-minimal* if it is minimal in its k-equivalence class. That is to say there does not exist a Dyck path $D' \in \mathbf{D}_n^m$ such that $D' \preceq_k D$. Let $p = (x, y)$ in $\mathbb{Z}^2$ be a point on the m-Dyck path D. The *level* of the point p is the integer y, and we say that p is on the y^{th} level.

Proposition 4.3.13. An $(m-1)$-Dyck path D is minimal if and only if for $d(D) = (d_1, \ldots, d_n)$, we have that $d_i < k(m-1)$ for all $i \neq 1$.

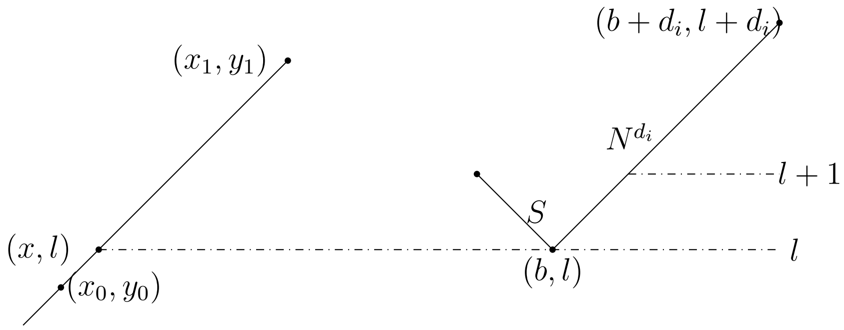
Proof. Suppose that $d_i < k(m-1)$ for all $i \neq 1$ and D is not minimal. We can then left k-compress D to obtain another dyck path D'. By Proposition 4.3.11 there is some $j > 1$ such that the j-th entry of $d(D')$ is $d'_j = d_j - k(m-1)$. By the assumption that $d_i < k(m-1)$ for $i \neq 1$, we must have that $d'_j < 0$, a contradiction. Thus D must be minimal.

Recall D is of the form $D = N^{d_1}S \ldots SN^{d_i}S \ldots SN^{d_n}$. Suppose that D is minimal and there exists some $i \neq 0$ such that $d_i \geq k(m-1)$. We will show that D is not minimal by demonstrating that we can left k-compress D. That is to say we will show that there is a sub-Dyck path X' in D which required to perform a left k-compression, where

$$X' = N^{m-1}D_1SD_2S \ldots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S \ldots SD_{(m-1)+k(m-1)}$$

for $1 \leq j \leq (m-1)$.

Suppose the up-step N^{d_i} starts at some point (b, l) and ends at $(b + d_i, l + d_i)$. The immediately preceding down-step S starts at $(b-1, l+1)$ and ends at (b, l). Let $0 \leq x \leq b-1$ be maximal such that the point (x, l) is on the Dyck path D. By the maximality, the point (x, l) is part of an up-step. Let U to be the up-step in D beginning at (x, l) if (x, l) is at the start of an up-step; otherwise let U to be the up-step containing (x, l). Let (x_1, y_1) be the end point of the up-step U. Let $(x_0, y_0) = (x_1 - (m-1), y_1 - (m-1))$, this is the start point of the up-step U. See the figure below.



Let (x_2, y_2) be the point at which it is the first time the Dyck path goes below the level y_1 after the point (x_1, y_1). That is $y_2 = y_1 - 1$. We can then observe that subpath D_1 starting from (x_1, y_1) and ending at $(x_2 - 1, y_2 + 1)$ is a translated $(m-1)$-Dyck path. Note that it could happen that $(x_1, y_1) = (x_2 - 1, y_2 + 1)$, in this case D_1 is just the empty $(m-1)$-Dyck path.

Let (x_3, y_3) be the point at which it is the first time the Dyck path goes below the level $y_1 - 1$ after the point (x_2, y_2), that is $y_3 = y_1 - 2$. We define D_2 to be the path starting from (x_2, y_2) to $(x_3 - 1, y_3 + 1)$. As before D_2 is a translated m-Dyck path which starts and ends on the $(y_1 - 1)^{\text{th}}$ level.

Let $j = y_1 - l$. We can repeat this procedure to define translated m-Dyck paths $D_3, D_4, \ldots, D_j$. Here each m-Dyck path D_r starts at the point (x_r, y_r) and ends at the point $(x_{r+1} - 1, y_{r+1} + 1)$, where the start and end points are defined as above and $y_{r+1} = y_r - 1 = y_1 - r$. Note that the translated m-Dyck path D_j begins on the level $y_j = y_1 - (j-1) = l + 1$, so the last point of D_j is $(x_{j+1} - 1, l + 1)$ for some $x_{j+1} - 1 \leq b$. We claim that $(x_{j+1} - 1, l + 1) = (b - 1, l + 1)$. The point $(b - 1, l + 1)$ is the last time we are on the $(l+1)^{\text{th}}$ level before the N^{d_i} up-step. By construction, there is a down-step from $(x_{j+1} - 1, l + 1)$ to (x_{j+1}, l). By the maximality of x we have that $x_{j+1} = b$ or $x_{j+1} = x$. Note that $x_{j+1} \geq x_1 > x$, so we have that $x_{j+1} = b$.

So far we have constructed a subpath from (x_0, y_0) to $(b, l+1)$ given by

$$X'' = N^{m-1} D_1 S D_2 S \dots S D_j S,$$

where the S down-steps are the down steps from $(x_i - 1, y_i + 1)$ to (x_i, y_i). Note that $y_i = y_1 - (i-1)$ for $2 \le i < j$ and the S after D_j is the one from $(b-1, l+1)$ to (b, l). The N^{m-1} is the up-step U from (x_0, y_0) to (x_1, y_1).

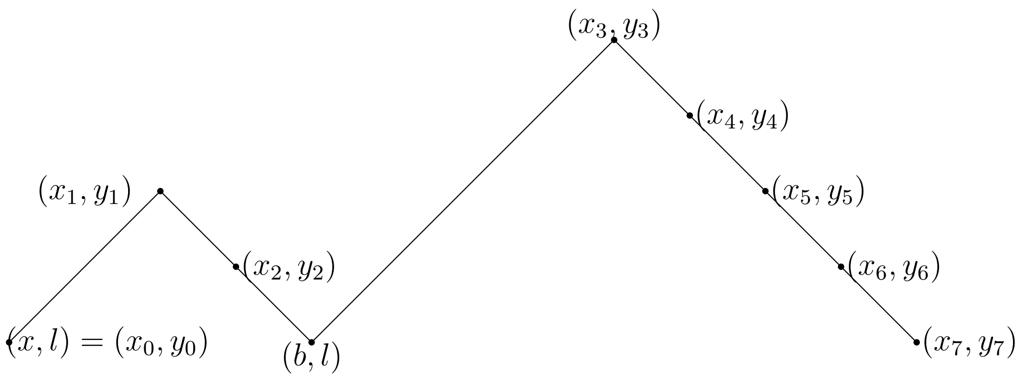
Since $d_i \ge k(m-1)$, there is an up-step $N^{k(m-1)}$ from (b, l) to $(x_{j+1}, y_{j+1}) = (b + k(m-1), l + k(m-1))$. We define D_{j+1} to be the path from (x_{j+1}, y_{j+1}) to $(x_{j+2} - 1, y_{j+2} + 1)$ where (x_{j+2}, y_{j+2}) is the point at which the Dyck path first sits on level $l + k(m-1) - 1$ after (x_{j+1}, y_{j+1}). In the same fashion we define the $(m-1) - j + k(m-1)$ sub paths $D_{j+2}, D_{j+3} \dots D_{(m-1)+k(m-1)}$. These are all translated m-Dyck paths by the same arguments as above. By how we construct the Dyck paths, we see that the path $D_{(m-1)+k(m-1)}$ ends on level $y_0 = y_1 - (m-1)$.

We have thus successfully constructed the sub-Dyck path of D,

$$X' = N^{m-1} D_1 S D_2 S \dots D_{j-1} S D_j S N^{k(m-1)} D_{j+1} S \dots S D_{(m-1)+k(m-1)}.$$

As before the S are the intermediate down steps between the D_i and the D_i may also be empty. $\square$

We illustrate the constructive proof above with an example for the case where $k = 2$ and $m = 3$.



The $N^{m-1} = N^2$ up-step is the one from (x_0, y_0) to (x_1, y_1). The translated 2-Dyck paths D_1 and D_2 are the empty paths N^0S^0 at (x_1, y_1) and (x_2, y_2) respectively. The $N^{k(m-1)} = N^{2(2)}$ up-step is the one from (b, l) to (x_3, y_3). The rest of the translated 2-Dyck paths $D_3, \ldots, D_6$ are the empty paths at $(x_4, y_4), \ldots, (x_7, y_7)$ respectively. Therefore, X' in this case is the whole path above.

We now show that minimal m-Dyck paths do exist and that they are unique in each k-equivalence class.

Proposition 4.3.14. Every k-equivalence class contains a unique minimal Dyck path.

Proof. To show existence, we consider the bijection $d \colon \mathbf{D}_{n-1}^{m-1} \to \mathbb{M}$ from Proposition 4.3.10 where $d(D) = (d_1, \ldots, d_n)$. Endow $\mathbb{N}^n$ with the standard lexicographic order. By Proposition 4.3.11, d is order reversing. That is $D \prec_k D'$ implies that $d(D') <_{\text{lex}} d(D)$. Recall that the lexicographic order is a partial order therefore it has no cycles because of anti-symmetry. Hence suppose that there is a k-equivalence class with no minimal Dyck path. Take D belonging to such a class and repeatedly left k-compress it. Since there is no minimal element in this class, we can do this indefinitely. As a result we obtain the descending chain.

$$\cdots \prec_k D^{(a)} \prec_k \cdots \prec_k D^1 \prec_k D.$$

Applying d to the descending chain, we get the ascending chain.

$$d(D) <_{\text{lex}} d(D^1) <_{\text{lex}} \cdots <_{\text{lex}} d(D^a) <_{\text{lex}} \ldots$$

Since $\mathbf{D}_{n-1}^{m-1}$ is finite, this ascending chain must be a cycle. Since the lexicographic order is anti-symmetric, this cycle must contain only one element. Therefore, if a k-equivalence class does not have a minimal element, it only contains one Dyck path in which case that Dyck path is trivially minimal. This is a contradiction to our assumption. Thus every k-equivalence class has a minimal Dyck path.

Suppose we have two minimal Dyck paths D and D' in an equivalence class. By Proposition 4.3.13 all but the first entries of $d(D)$ and $d(D')$ are strictly less than $k(m-1)$. But since D and D' are k-equivalent, $d(D) \equiv d(D') \bmod k(m-1)$. This means all but the first entries of $d(D)$ and $d(D')$ are equal. The equality of these entries forces the first entries to also be equal since clearly it cannot be the case otherwise. Therefore, $d(D) = d(D')$ which implies $D = D'$. Therefore, the minimal Dyck paths are unique in their equivalence classes. □

Theorem 4.3.15. Suppose that D and D' are m-Dyck paths of length n. The m-Dyck paths D and D' are k-equivalent if and only if $d(D) \equiv d(D') \bmod k(m-1)$.

Proof. Suppose that D and D' are k-equivalent. Furthermore, suppose without loss of generality that we obtain D' from D by application of a finite sequence of k-compressions. From Proposition 4.3.11, we see that a k-compression maps $d(D)$ to an n-tuple which is congruent to $d(D)$ modulo $k(m-1)$. Therefore, since $d(D')$ an n-tuple which is a result of a finite sequence of k-compressesions on D, then $d(D) \equiv d(D') \bmod k(m-1)$.

Suppose now that $d(D) \equiv d(D') \bmod k(m-1)$. Consider their respective minimal representatives in their k-equivalence classes D_{min} and D'_{min} respectively. It then follows that

$$d(D_{min}) \equiv d(D) \equiv d(D') \equiv d(D'_{min}) \bmod k(m-1).$$

Therefore, $d(D_{min}) \equiv d(D'_{min}) \bmod k(m-1)$, hence by Proposition 4.3.13 we obtain that $d(D_{min}) = d(D'_{min})$ which means $D_{min} = D'_{min}$. Therefore, D and D' belong to the same k-equivalence class. Therefore, we conclude D and D' are k-equivalent. □

Theorem 4.3.16. Suppose that t and t' are a pair of m-ary trees with n leaves and depth $\delta(t) = (\delta^{l_1}(t), \dots, \delta^{l_m}(t))$ and $\delta(t') = \delta^{l_1}(t'), \dots, \delta^{l_m}(t'))$ respectively.

The trees t and t' are k-equivalent if and only if

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1),$$

where the addition on the n-tuples is componentwise.

Proof. Suppose that t and t' are k-equivalent, then their corresponding Dyck paths $D = \sigma(t)$ and $D' = \sigma(t')$ respectively are also k-equivalent. Therefore, by Theorem 4.3.15, $d(D) \equiv d(D') \bmod k(m-1)$. By Proposition 4.3.7,

$$d_1 = (m-1)\delta_1^{l_1}(t),$$

$$d_j = \sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t) - \delta_{j-1}^{l_i}(t)) + 1, \text{ for } j > 1.$$

Since $d_1 \equiv d_1' \bmod k(m-1)$, we have that $(m-1)\delta_1^{l_1}(t) \equiv (m-1)\delta_1^{l_1}(t') \bmod k(m-1)$. Furthermore, we observe that from the structure of the of m-ary trees that $\delta_1^{l_i}(t) = 0$ and $\delta_1^{l_i}(t') = 0$ for $i \neq 1$. Thus

$$(m-1)\delta_1^{l_i}(t) \equiv (m-1)\delta_1^{l_i}(t') \bmod k(m-1) \text{ for } 1 \leq i \leq m,$$

hence

$$\sum_{i=1}^{m}(m-1)\delta_1^{l_i}(t) \equiv \sum_{i=1}^{m}(m-1)\delta_1^{l_i}(t') \bmod k(m-1).$$

From the fact that,
$d_2 = \sum_{i=1}^{m}(m-i)(\delta_2^{l_i}(t) - \delta_1^{l_i}(t)) + 1 \equiv d_2' = \sum_{i=1}^{m}(m-i)(\delta_2^{l_i}(t') - \delta_1^{l_i}(t')) + 1 \bmod k(m-1)$,

we conclude that,

$$\sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t') \bmod k(m-1).$$

From this congruence and the congruence

$$d_3 = \sum_{i=1}^{m}(m-i)(\delta_3^{l_i}(t) - \delta_2^{l_i}(t)) + 1 \equiv d_3' = d_3 = \sum_{i=1}^{m}(m-i)(\delta_3^{l_i}(t') - \delta_2^{l_i}(t')) + 1,$$

we conclude that,

$$\sum_{i=1}^{m}(m-i)\delta_3^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_3^{l_i}(t') \bmod k(m-1).$$

Continuing in this manner we obtain the following,

$$\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t') \bmod k(m-1), \text{ for } 1 \leq j \leq n.$$

This is the same as saying,

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

Now for the converse, suppose that

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

This implies that,

$$\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t) \equiv \sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t') \bmod k(m-1), \text{ for } 1 \leq j \leq n.$$

This further implies that

$$d_1 = (m-1)\delta_1^{l_1}(t) \equiv d_1' = (m-1)\delta_1^{l_1}(t) \bmod k(m-1)$$

$$d_j = \sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t)-\delta_{j-1}^{l_i}(t))+1 \equiv d_j' = \sum_{i=1}^{m}(m-i)(\delta_j^{l_i}(t')-\delta_{j-1}^{l_i}(t'))+1 \bmod k(m-1).$$

Therefore, by Theorem 4.3.15, $D = \sigma(t)$ and $D' = \sigma(t')$ are k-equivalent which implies that t and t' are k-equivalent. □

4.4 An Application to m-ary operations

The main aim of this section is to prove Theorem 4.4.4, which gives a characterisation of k-equivalence. To do so, we introduce a particular k-associative m-ary operation which will be denoted by $\circ$. This operation will be used to evaluate m-ary parenthesizations and we will show that this operation characterises k-equivalence. This is to say that two parenthesizations will be k-equivalent (k-associative) if and only if their evaluations under this operation are equal. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m-1)$.

Let $A = \mathbb{C}\langle u_1, \ldots, u_n \rangle$ be the free unital associative algebra over $\mathbb{C}$ in n indeterminates $u_1, u_2, \ldots, u_n$. We define a binary operation $\circ$ on A as follows. Let ω be an element of A of order $k(m-1)$, for example $\omega = e^{\frac{2\pi i}{k(m-1)}}$. For a, b in A, we define $a \circ b = \omega \cdot a + b$, where $\cdot$ and $+$ are the multiplication and addition operations in A respectively. This is taken to be a left-justified operation. Sometimes we will omit the $\cdot$ for convenience. The binary operation $\circ$ on A induces an m-ary operation on A^m defined in the following way,

$$a_1 \circ a_2 \cdots \circ a_m = \omega^{m-1} \cdot a_1 + \omega^{m-2} \cdot a_2 + \cdots + \omega \cdot a_{m-1} + a_m. \tag{4.4}$$

It is easy to see by direct calculation that the following two lemmas are true.

Lemma 4.4.1. The binary operation $\circ$ on A is $k(m-1)$-associative.

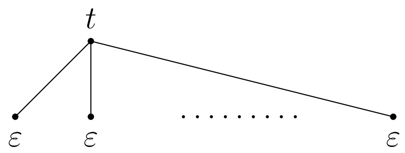
Lemma 4.4.2. The m-ary operation on A^m induced by the binary operation $\circ$ on A is k-associative.

Let X be a non-empty set and let $* : X^m \to X$ be an m-ary operation. Take $x_1, \ldots, x_n$ in X. Recall that there is a bijection between the set of m-ary trees on n leaves and the set of m-ary parenthesizations of the expression $x_1 * \cdots * x_n$, see Proposition 4.2.4. We will write $p_t = p(x_1 * \cdots * x_n)_t$ to be the m-ary parenthesization of the expression $x_1 * \cdots * x_n$ corresponding to the m-ary tree t. We denote the evaluation of p_t with respect to $\circ$ by $p(u_1 \circ \cdots \circ u_n)_t$. When there is no risk of confusion, we omit the subscript t.

Lemma 4.4.3. Suppose that $p(x_1 * \cdots * x_n)_t$ is an m-ary parenthesization of $x_1 * \cdots * x_n$ corresponding to the m-ary tree on n leaves t, which has depth $\delta(t) = (\delta^{l_1}(t), \ldots, \delta^{l_m}(t))$ be the depth of t. It follows that

$$p(u_1 \circ \cdots \circ u_n)_t = \omega^{\sum\limits_{i=1}^{m}(m-i)\delta_1^{l_i}(t)} \cdot u_1 + \omega^{\sum\limits_{i=1}^{m}(m-i)\delta_2^{l_i}(t)} \cdot u_2 + \cdots + \omega^{\sum\limits_{i=1}^{m}(m-i)\delta_n^{l_i}(t)} \cdot u_n.$$

Proof. Recall that n satisfies the equation $n = m + g(m-1)$ for some integer $g \geq 0$. We prove the result by induction on g. When $g = 0$ there is only on tree to consider, namely $t = \varepsilon \wedge \cdots \wedge \varepsilon$.



For this tree, $\delta_j^{l_i} = \delta_{ij}$, where the right side is the usual Kronecker delta function. it is easy to see that the statement holds in this case by the definition of $u_1 \circ \cdots \circ u_m$ in (4.4).

Now suppose that the result holds for $n = m + g'(m-1)$ for all $g' \leq g$. We consider the $g+1$ case. Let t be an m-ary tree with $n = m+(g+1)(m-1)$ leaves. We may write $t = t_1 \wedge \cdots \wedge t_m$ where each t_i is the subtree rooted at the i^{th} child of the root of t. Each subtree t_i has $n_i < n$ leaves and $n_1 + \cdots + n_m = n$. In writing t as the tag of its sub-trees at the root, we partition the leaves of t. We identify each leaf of t with a tuple (h, j) if it lies in the subtree t_h and it is the j^{th} leaf in the pre-order traverse of t_h, where $1 \leq j \leq n_h$. Therefore, for the leaf identified with (h, j),

$$\delta_{h,j}^{l_i}(t) = \begin{cases} \delta_{h,j}^{l_i}(t_h) + 1 & \text{if } i = h; \\ \delta_{h,j}^{l_i}(t_h) & \text{otherwise.} \end{cases} \tag{4.5}$$

From the equation above, it follows that,

$$(m-i)\delta_{h,j}^{l_i}(t) = \begin{cases} (m-i)\delta_{h,j}^{l_i}(t_h) + (m-i) & \text{if } i = h; \\ (m-i)\delta_{h,j}^{l_i}(t_h) & \text{otherwise.} \end{cases} \tag{4.6}$$

The identification of the leaves with the tuples (h, j) gives another labelling of the variables u_s, where $1 \leq s \leq n$. Since the variable u_s corresponds to the s^{th} leaf of t, and the s^{th} leaf is identified with (h, j), we write $u_{(h,j)}$ for u_s. Hence

$$p(u_1 \circ u_2 \circ \cdots \circ u_n)_t = p(u_{(1,1)} \circ u_{(1,2)} \ldots u_{(m,n_m)})_t.$$

It is then easy to see that,

$$p(u_{(1,1)} \circ u_{(1,2)} \circ \cdots \circ u_{(m,n_m)})_t = p(u_{(1,1)} \circ \cdots \circ u_{(1,n_1)})_{t_1} \circ p(u_{(2,1)} \circ \cdots \circ \cdots \circ u_{(2,n_2)})_{t_2} \circ \cdots \circ p(u_{(m,1)} \circ \cdots \circ u_{(m,n_m)})_{t_m}$$

$$= \omega^{m-1} p(u_{(1,1)} \circ \cdots \circ u_{(1,n_1)})_{t_1} + \omega^{m-2} p(u_{(2,1)} \circ \cdots \circ \cdots \circ u_{(2,n_2)})_{t_2} + \cdots + p(u_{(m,1)} \circ \cdots \circ u_{(m,n_m)})_{t_m}.$$

By in the inductive assumption,

$$p(u_{(h,1)} \circ u_{(h,2)} \circ \cdots \circ u_{(h,n_h)})_{t_h} = \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,1)}(t_h)} \cdot u_{(h,1)} + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,2)}(t_h)} \cdot u_{(h,2)} + \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,n_h)}(t_h)} \cdot u_{(h,n_h)}$$

from which it follows that,

$$\omega^{m-h} p(u_{(h,1)} \circ u_{(h,2)} \circ \cdots \circ u_{(h,n_h)})_{t_h} = \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,1)}(t_h)+(m-h)} \cdot u_{(h,1)} + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,2)}(t_h)+(m-h)} \cdot u_{(h,2)} + \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,n_h)}(t_h)+(m-h)} \cdot u_{(h,n_h)}.$$

By equation (4.6),

$$\begin{aligned} \left(\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,j)}(t_h)\right) + (m-h) &= \left(\sum_{\substack{i=1\\ i\neq h}}^{m}(m-i)\delta^{l_i}_{(h,j)}(t_h)\right) + (m-h)\delta^{l_h}_{(h,j)}(t_h) + (m-h) \\ &= \left(\sum_{\substack{i=1\\ i\neq h}}^{m}(m-i)\delta^{l_i}_{(h,j)}(t)\right) + (m-h)\delta^{l_h}_{(h,j)}(t) \\ &= \sum_{i=1}^{m}(m-i)\delta^{l_i}_{(h,j)}(t) \end{aligned}$$

Therefore,

$$p(u_{(1,1)} \circ u_{(1,2)} \circ \cdots \circ u_{(m,n_m)})_t = \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(1,1)}(t)} \cdot u_{(1,1)} + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(1,2)}(t)} \cdot u_{(1,2)} + \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta^{l_i}_{(m,n_m)}(t)} \cdot u_{(m,n_m)},$$

as required. This completes the proof.

$\square$

We are now able to prove our first main results.

Theorem 4.4.4. Suppose that $p = p(x_1 * \cdots * x_n)_t$ and $p' = p'(x_1 * \cdots * x_n)_{t'}$ are two m-ary parenthesizations of $x_1 * \cdots * x_n$ corresponding to the m-ary trees on n leaves t and t' respectively. It then follows that p and p' are k-equivalent with respect to k-associativity if and only if,

$$p(u_1 \circ \cdots \circ u_n)_t = p'(u_1 \circ \cdots \circ u_n)_{t'}.$$

Proof. Suppose the parenthesizations p and p' are k-equivalent. It follows that the trees t and t' are also k-equivalent. By Theorem 4.3.16,

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

Therefore,

$$p(u_1 \circ u_2 \cdots \circ u_n)_t = p'(u_1 \circ u_2 \cdots \circ u_n)_{t'}$$

by Lemma 4.4.3.

Suppose that

$$p(u_1 \circ u_2 \cdots \circ u_n)_t = p'(u_1 \circ u_2 \cdots \circ u_n)_{t'},$$

then

$$\omega^{\sum_{i=1}^{m}(m-i)\delta_1^{l_i}(t)} \cdot u_1 + \omega^{\sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t)} \cdot u_2 + \cdots + \omega^{\sum_{i=1}^{m}(m-i)\delta_n^{l_i}(t)} \cdot u_n$$

$=$

$$\omega^{\sum_{i=1}^{m}(m-i)\delta_1^{l_i}(t')} \cdot u_1 + \omega^{\sum_{i=1}^{m}(m-i)\delta_2^{l_i}(t')} \cdot u_2 + \cdots + \omega^{\sum_{r=1}^{m}(m-i)\delta_n^{l_i}(t')} \cdot u_n.$$

Since $u_1, \ldots, u_n$ are algebraically independent and hence linearly independent in A, the coefficients of the u_i on each side of the equation must be equal.

Hence

$$\omega^{\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t)} = \omega^{\sum_{i=1}^{m}(m-i)\delta_j^{l_i}(t')} \text{ for } 1 \leq j \leq n.$$

Since ω has order $k(m-1)$ this implies,

$$\sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1}(m-i)\delta^{l_i}(t') \bmod k(m-1).$$

Hence t and t' are k-equivalent by Theorem 4.3.16 which implies that p and p' are also k-equivalent by Remark 4.2.7. □

Example 4.4.5. In example 4.2.9 we saw that the 3-ary parenthesization

$$((x_1x_2x_3)x_4x_5)x_6x_7$$

is 2-equivalent to

$$x_1((x_2x_3x_4)x_5x_6)x_7.$$

Let us check the theorem above for this example.
The depth of the first tree is

$$(\delta^{l_1} = (3,2,2,1,1,0,0), \delta^{l_2} = (0,1,0,1,0,1,0), \delta^{l_3} = (0,0,1,0,1,0,1)).$$

Therefore, the valuation of $((x_1x_2x_3)x_4x_5)x_6x_7$ with respect to $\circ$ is

$$\omega^6x_1 + \omega^5x_2 + \omega^4x_3 + \omega^3x_4 + \omega^2x_5 + \omega x_6 + x_7.$$

The depth of $x_1((x_2x_3x_4)x_5x_6)x_7$ is

$$(\delta^{l_1} = (1,2,1,1,0,0,0), \delta^{l_2} = (0,1,2,1,2,1,0), \delta^{l_3} = (0,0,0,1,0,1,1)),$$

hence the valuation of $x_1((x_2x_3x_4)x_5x_6)x_7$ with respect to $\circ$ is

$$\omega^2x_1 + \omega^5x_2 + \omega^4x_3 + \omega^3x_4 + \omega^2x_5 + \omega x_6 + x_7.$$

Since ω has order 4 the valuations are equal.

4.5 Modular Fuss-Catalan Number

Recall that we define the modular Fuss-Catalan number $C_{k,n}^m$ to be the number of k-equivalence classes of parenthesizations of $x_0 * \cdots * x_n$. In the previous sections we saw that k-associativity corresponds to k-rotation and k-compression. Therefore, $C_{k,n}^m$ also counts the k-equivalence classes of $(m-1)$-Dyck paths of length n. In this section we follow the strategy of (Hein & Huang, 2015, §5) to derive an explicit formula for $C_{k,n}^m$. By Proposition 4.3.14, each k-equivalence class has a unique minimal element. Therefore, to count the number of k-equivalence classes, we just need to count the number of minimal elements. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m-1)$.

Assume that l is a positive integer in $\{1, \ldots, n\}$ such that $(m-1)$ divides l. Let N denote the up-step $(1,1)$ and S denote the down-step $(1,-1)$ in $\mathbb{Z}^2$. Denote by $\mathbf{L}_{k,n,l}'^m$ the set of all strings (lattice paths) of the form $N^l S N^{i_1} S N^{i_2} \ldots S N^{i_n}$ such that $i_1 + i_2 + \cdots + i_n = n - l$ where $(m-1)|i_p$ for all $1 \leq p \leq n$ and $0 \leq i_1, \ldots, i_n < k(m-1)$. Thus $\mathbf{L}_{k,n,l}'^m$ is a set of lattice paths of length n where the first up-step is of size l. For integers $1 \leq j \leq k$, denote by m_j the number of $(j-1)(m-1)$s appearing among the $i_1, \ldots, i_n \in \{0, m-1, 2(m-1), \ldots, (k-1)(m-1)\}$.

It is easy to see that

$$|\mathbf{L}_{k,n,l}'^m| = \sum_{\substack{m_1 + \cdots + m_k = n \\ m_2 + 2m_3 + \cdots + (k-1)m_k = \frac{n-l}{m-1}}} \binom{n}{m_1, m_2, \ldots, m_k}.$$

For a string $w = N^l S N^{i_1} S N^{i_2} \ldots S N^{i_n}$ in $\mathbf{L}_{k,n,l}'^m$ and j in $\{0, 1, \ldots, n-1\}$ we define

$$w^{\bullet j} = N^l S N^{i_{j+1}} S \ldots S N^{i_n} S N^{i_1} S \ldots S N^{i_j}.$$

Let $\mathbf{L}_{k,n,l}^m$ be the subset of strings in $\mathbf{L}_{k,n,l}'^m$ which are $(m-1)$-Dyck paths of length n. Since $(m-1)$-Dyck paths can be thought of as 1-Dyck paths where up-steps come in multiples of $m-1$, the following lemmas follow by similar arguments to Lemma 5.5 and Lemma 5.6 from Hein & Huang (2015), which can be thought of as the $m = 2$ case. Thus we will state the followings lemmas without proof.

Lemma 4.5.1. For a string w in $\mathbf{L}_{k,n,l}'^m$ the set $\{0 \leq j \leq n-1 : w^{\bullet j} \in \mathbf{L}_{k,l,n}^m\}$ has cardinality l.

Let ϕ be the following map,

$$\phi : \mathbf{L}^m_{k,n,l} \times \{0, 1, \ldots, n-1\} \to \mathbf{L}'^m_{k,n,l},$$

$$(w, j) \mapsto w^{\bullet j}.$$

Lemma 4.5.2. For a string w in $\mathbf{L}'^m_{k,n,l}$, the fibre $\phi^{-1}(w)$ of ϕ over w has cardinality $|\phi^{-1}(w)| = l$.

By Proposition 4.3.14, the modular Fuss-Catalan number counts the number of minimal $(m-1)$-Dyck paths. Moreover by Proposition 4.3.13, minimal Dyck paths D satisfy $d(D) = (d_1, \ldots, d_n)$ where $d_i < k(m-1)$ for $i \neq 1$. Combining the results of Proposition 4.3.13, Proposition 4.3.14 and Lemma 4.5.2,

$$|\mathbf{L}^m_{k,n,l}| = \frac{l}{n}|\mathbf{L}'^m_{k,n,l}|.$$

Let $\mathbf{L}^m_{k,n}$ be the set of minimal $(m-1)-$Dyck paths, then

$$|\mathbf{L}^m_{k,n}| = \sum_{\substack{1 \leq l \leq n \\ (m-1)|l}} |\mathbf{L}^m_{k,n,l}|.$$

Therefore,

$$C^m_{k,n} = |\mathbf{L}^m_{k,n}| = \sum_{\substack{1 \leq l \leq n \\ m-1|l}} \frac{l}{n} \sum_{\substack{m_1+\cdots+m_k=n \\ m_2+2m_3+\cdots+(k-1)m_k=\frac{n-l}{m-1}}} \binom{n}{m_1, \ldots m_k}$$

is the number of minimal $(m-1)$-Dyck paths of length n, so by Proposition 4.3.6, it is the number of minimal m-ary trees of length $n+1$. This completes the proof.

Example 4.5.3. In this example, we will count the number of 2-equivalence classes of 3-ary trees with 7

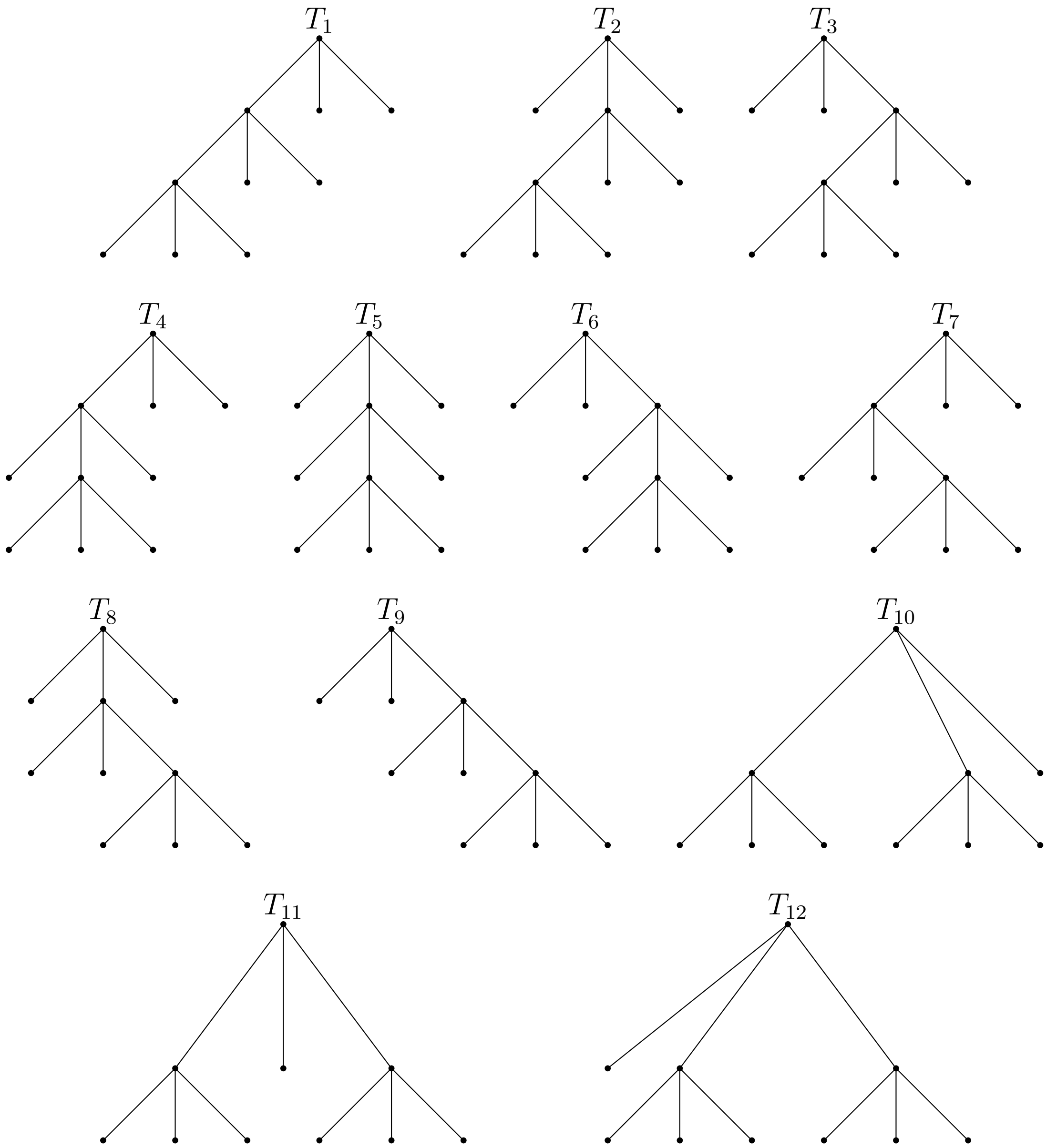


Figure 4.5: The complete list of the 3-ary trees with 7 leaves.

Observe that the trees T_1, T_2 and T_3 in the top row correspond to the following parenthesizations

$$((x_1 * x_2 * x_3) * x_4 * x_5) * x_6 * x_7,$$

$$x_1 * ((x_2 * x_3 * x_4) * x_5 * x_6) * x_7,$$

$$x_1 * x_2 * ((x_3 * x_4 * x_5) * x_6 * x_7)$$

respectively. We can see that we get the tree T_2 from the tree T_1 by a 2-rotation at the root of T_1, and likewise we get the tree T_3 from the tree T_2 by a 2-rotation at the root of T_2. Therefore T_1, T_2 and T_3 belong to the same 2-equivalence class. Further observe that the other trees cannot be 2-rotated because they do not contain a subtree of form required to perform a 2-rotation. We conclude that $C^3_{2,6} = 10$. Let us check this against the explicit formula we have derived.

$$\begin{aligned} C^3_{2,6} &= \sum_{\substack{1 \le l \le 6 \\ 2 | l}} \frac{l}{6} \sum_{\substack{m_1+m_2=6 \\ m_2=\frac{6-l}{2}}} \binom{6}{m_1, m_2} \\ &= \frac{2}{6}\binom{6}{4,2} + \frac{4}{6}\binom{6}{5,1} + \frac{6}{6}\binom{6}{6,0} \\ &= \frac{1}{3}(15) + \frac{2}{3}(6) + 1(1) \\ &= 10. \end{aligned}$$

Chapter 5

Idempotent completion of extriangulated categories

5.1 Introduction.

Extriangulated categories were introduced in Nakaoka & Palu (2019a) as a simultaneous generalisation of exact categories and triangulated categories in the context of the study of cotorsion pairs. The known classes of examples of extriangulated categories include exact categories, triangulated categories and extension-closed subcategories of triangulated categories; see (Nakaoka & Palu, 2019a, Example 2.13, Remark 2.18, Proposition 3.22(1)). There are many extriangulated categories which a neither exact nor triangulated. For example, it is shown in (Jin, 2020, Theorem 2.4) that the category of Cohen-Macaulay differential graded modules over certain Gorenstein differential graded algebras is extriangulated. Another is the subcategory $K^{[-1,0]}(\text{proj }\Lambda)$, which is the subcategory of complexes concetrated in degree -1 and degree 0 in $K^{\text{b}}(\text{proj }\Lambda)$, where Λ is an Artin algebra; see (Padrol *et al.*, 2019, Proposition 4.39). For other constructions of extriangulated categories which are neither exact nor triangulated; see for example, (Nakaoka & Palu, 2019a, Proposition 3.30), (Zhou & Zhu, 2018, Example 4.14 and Corollary 4.12).

Let $\mathcal{A}$ be an additive category. A morphism $p\colon A \to A$ in $\mathcal{A}$ is said to be *idempotent* if $p^2 = p$. The category $\mathcal{A}$ is said to be *idempotent complete* (or *have split idempotents*) if every idempotent morphism in $\mathcal{A}$ admits a kernel. Every additive category $\mathcal{A}$ can be embedded fully faithfully into an idempotent complete category $\tilde{\mathcal{A}}$ called the *idempotent completion* (also called the Karoubi envelope); see for example (Bühler, 2010a, Remark 6.3). The property of being idempotent complete is often desirable and has interesting consequences. For example, in defining n-Abelian categories for $n \geq 2$, one of the required axioms is that the underlying additive category is idempotent complete, see (Jasso, 2016, Definition 3.1). Abelian categories (note that 1-Abelian categories are precisely Abelian categories) are idempotent complete, however for $n \geq 2$, idempotent completeness is independent of the other axioms of n-Abelian categories. One important consequence of idempotent completeness in the definition of n-Abelian categories is that of the existence of n-pushouts, see (Jasso, 2016, Theorem 3.8). Furthermore, an additive category $\mathcal{A}$ is Krull-Schmidt if and only if it is idempotent complete and the endormorphism ring of every object is semi-perfect, see (Krause, 2015, Corollary 4.4). So by taking the idempotent completion of extriangulated categories, one could possibly obtain Krull-Schmidt extriangulated categories.

When $\mathcal{A}$ is a triangulated category, it has been shown that the idempotent completion $\tilde{\mathcal{A}}$ is also triangulated; see (Balmer & Schlichting, 2001, Theorem 1.5). It has also been shown (Liu & Sun, 2014, Theorem 2.16) that the idempotent completion of a left triangulated category is again left triangulated, and likewise for right triangulated categories. When $\mathcal{A}$ is an exact category, it has also been shown that the idempotent completion is exact; see (Bühler, 2010a, Proposition 6.13). We will unify these results by showing that when $\mathcal{A}$ is extriangulated then the idempotent completion is also extriangulated. In doing so, we also add to the family of examples of extriangulated categories.

Independent work by Wang *et al.* (2022) has also shown that the idempotent completion of an extriangulated category is extriangulated. Although the result is the same, our work offers a different perspective. For example, the description of Ext^1 functor of the idempotent completion which give is different to the description of Wang *et al.* (2022). Our description of the biadditive functor has the advantage of allowing us to easily observe that the Ext^1-groups of an idempotent completion

$\tilde{\mathcal{A}}$ are subgroups of the Ext^1-groups of $\mathcal{A}$. In particular, the Ext^1 bifunctor on $\tilde{\mathcal{A}}$ behaves like a subbifunctor of the Ext^1 bifunctor on $\mathcal{A}$, in the sense of (Herschend *et al.*, 2021, Definition 3.7). Our alternative perspective also leads us to a proof of the main theorem which is quite different to the proof presented in Balmer & Schlichting (2001) for the triangulated case and Wang *et al.* (2022) for extriangulated case. In our work, the role of the idempotent morphisms is clarified and the extriangles of the idempotent completion have an explicit description which isn't available in the treatment of Balmer & Schlichting (2001) and Wang *et al.* (2022).

5.2 Idempotent completeness.

In this section, we recall the basic theory of idempotent completions of additive categories.

Let us set the common notation for this section. Let $\mathcal{A}$ be an additive category. Given objects X, Y in $\mathcal{A}$ we will write $\mathcal{A}(X,Y)$ for the group of morphisms $X \to Y$. For an object X in $\mathcal{A}$ we denote the identity morphism of X by 1_X.

Definition 5.2.1. (Karoubi, 1968, Definition 1.2.1,1.2.2). Let $\mathcal{A}$ be an additive category. We say that $\mathcal{A}$ is idempotent complete if for every idempotent morphism $p\colon A \to A$ (i.e. $p^2 = p$) in $\mathcal{A}$, there is a decomposition $A \cong K \oplus I$ of A such that $p \cong \begin{pmatrix} 0 & 0 \\ 0 & 1_I \end{pmatrix}$ with respect to this decomposition.

Suppose $r\colon A \to B$ is a retraction with section $s\colon B \to A$, that is to say $rs = 1_B$. Then it can be observed that the morphism sr is an idempotent morphism. Such an idempotent gives a decomposition of A in the sense of Definition 5.2.1 if the morphism r admits a kernel k. See (Bühler, 2010a, Remark 7.4) for more details.

Definition 5.2.2. (Borceux, 1994, Definitions 6.5.3 ,6.5.1) An idempotent morphism $p\colon A \to A$ in $\mathcal{A}$ is said to *split* if there is a retraction $r\colon A \to B$ and section $s\colon B \to A$ such that $sr = p$ and $rs = 1_B$.

Proposition 5.2.3. (Bühler, 2010a, Remark 6.2)(Borceux, 1994, Proposition 6.5.4). Let $\mathcal{A}$ be an additive category. The following statements are equivalent:

1. $\mathcal{A}$ is idempotent complete;
2. Every idempotent morphism in $\mathcal{A}$ admits a kernel;
3. Every idempotent morphism in $\mathcal{A}$ admits a cokernel;
4. Every idempotent morphism in $\mathcal{A}$ splits.

For this reason, idempotent complete categories are often referred to as *categories with split idempotents.* Every additive category $\mathcal{A}$ embeds fully faithfully into an idempotent complete category $\tilde{\mathcal{A}}$. The category $\tilde{\mathcal{A}}$ is commonly referred to as the *idempotent completion* or as the *Karoubi envelope* of $\mathcal{A}$.

Definition 5.2.4. (Balmer & Schlichting, 2001, 1.2 Definition). Let $\mathcal{A}$ be an additive category. The *idempotent completion* of $\mathcal{A}$ is denoted by $\tilde{\mathcal{A}}$ and is defined as follows. The objects of $\tilde{\mathcal{A}}$ are the pairs (A, p) where A is an object of $\mathcal{A}$ and $p\colon A \to A$ is an idempotent morphism. A morphism in $\tilde{\mathcal{A}}$ from (A, p) to (B, q) is a morphism $\sigma\colon A \to B \in \mathcal{A}$ such that $\sigma p = q\sigma = \sigma$. For any object (A, p) in $\tilde{\mathcal{A}}$, the identity morphism $1_{(A,p)} = p$.

Proposition 5.2.5. (Bühler, 2010a, See e.g. Remark 6.3). Let $\mathcal{A}$ be an additive category. The Karoubi envelope $\tilde{\mathcal{A}}$ is an idempotent complete category. The biproduct in $\tilde{\mathcal{A}}$ is defined as $(A, p) \oplus (B, q) = (A \oplus B, p \oplus q)$. There is a fully faithful additive functor $i_{\mathcal{A}}\colon \mathcal{A} \to \tilde{\mathcal{A}}$ defined as follows. For an object A in $\mathcal{A}$, we have that $i_{\mathcal{A}}(A) = (A, 1_A)$ and for a morphism f in $\mathcal{A}$, we have that $i_{\mathcal{A}}(f) = f$.

The Karoubi envelope is unique with respect to the following universal property.

Proposition 5.2.6. (Bühler, 2010a, Proposition 6.10). Let $\mathcal{A}$ be an additive category and let $\mathcal{B}$ be an idempotent complete category. For every additive functor $F\colon \mathcal{A} \to \mathcal{B}$, there exists a functor $\tilde{F}\colon \tilde{\mathcal{A}} \to \mathcal{B}$ and a natural isomorphism $\alpha\colon F \Rightarrow \tilde{F} i_{\mathcal{A}}$.

5.3 Idempotent completion of extriangulated categories.

For the rest of this section, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $\tilde{\mathcal{C}}$ be the idempotent completion of $\mathcal{C}$.

Theorem 5.3.1. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let $\tilde{\mathcal{C}}$ be the idempotent completion of $\mathcal{C}$. Then $\tilde{\mathcal{C}}$ is extriangulated. Moreover, in this case the embedding $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$ is an extriangulated functor.

Our first step in proving the above theorem is the construction of a bifunctor $\mathbb{F} \colon \tilde{\mathcal{C}}^{\mathrm{op}} \times \tilde{\mathcal{C}} \to Ab$ for the extriangulated structure. Given a pair of objects (X, p) and (Y, q) in $\tilde{\mathcal{C}}$, we define $\mathbb{F}$ on objects by setting,

$$\mathbb{F}((X,p),(Y,q)) := p^* q_* \mathbb{E}(X,Y) = \{p^* q_* \delta \mid \delta \in \mathbb{E}(X,Y)\}.$$

Lemma 5.3.2. Let $p \colon X \to X$ and $q \colon Y \to Y$ be morphisms in $\mathcal{C}$. Then $p^* q_* \mathbb{E}(X,Y) = \{p^* q_* \delta \mid \delta \in \mathbb{E}(X,Y)\}$ is a subgroup of $\mathbb{E}(X,Y)$.

Proof. Observe that $p^* q_* \mathbb{E}(X,Y)$ is the image of $\mathbb{E}(X,Y)$ under the group homomorphism $\mathbb{E}(p,q)$, therefore $p^* q_* \mathbb{E}(X,Y)$ is a subgroup of $\mathbb{E}(X,Y)$. □

By Lemma 5.3.2, $p^* q_* \mathbb{E}(X,Y)$ is an Abelian group. We now need to define $\mathbb{F}$ on morphisms.

Let $\tilde{\alpha} \colon (X,p) \to (Y,q)$ and $\tilde{\beta} \colon (U,e) \to (V,f)$ be any pair of morphisms in $\tilde{\mathcal{C}}$. By definition these are morphisms $\alpha \colon X \to Y$ and $\beta \colon U \to V$ in $\mathcal{C}$ such that $\alpha p = q\alpha = \alpha$ and $\beta e = f\beta = \beta$. Take $\varepsilon \in \mathbb{F}((Y,q),(U,e))$, we have that $\varepsilon = q^* e_* \delta_\varepsilon$ for some $\delta_\varepsilon \in \mathbb{E}(Y,U)$, hence we observe that

$$\beta_* \alpha^* \varepsilon = \beta_* \alpha^* q^* e_* \delta_\varepsilon = \beta_* e_* \alpha^* q^* \delta_\varepsilon = (\beta e)_* (q\alpha)^* \delta_\varepsilon$$

$$= (f\beta)_* (\alpha p)^* \delta_\varepsilon = f_* \beta_* p^* \alpha^* \delta_\varepsilon = p^* f_* (\alpha^* \beta_* \delta_\varepsilon).$$

Since $\alpha^* \beta_* \delta_\varepsilon$ is in $\mathbb{E}(X,V)$ we have that $\alpha^* \beta_* \varepsilon$ is an element of $\mathbb{F}((X,p),(V,f))$.

For the pair $(\tilde{\alpha}, \tilde{\beta})$ we define $\mathbb{F}(\tilde{\alpha}^{\mathrm{op}}, \tilde{\beta})\colon \mathbb{F}((Y,q),(U,e)) \to \mathbb{F}((X,p),(V,f))$ as follows. For $\varepsilon \in \mathbb{F}((Y,q),(U,e))$ we set $\mathbb{F}(\tilde{\alpha}^{\mathrm{op}}, \tilde{\beta})(\varepsilon) := \beta_*\alpha^*\varepsilon$. It is easy to observe that $\mathbb{F}$ preserves identity morphisms from the above definition. Let $(\tilde{\alpha_1}, \tilde{\beta_1})$ and $(\tilde{\alpha_2}, \tilde{\beta_2})$ be a pair of composable morphisms in $\tilde{\mathcal{C}}^{\mathrm{op}} \times \tilde{\mathcal{C}}$ and $(\tilde{\alpha_1}\tilde{\alpha_2}, \tilde{\beta_1}\tilde{\beta_2})$ be their composition. Then,

$$\mathbb{F}((\tilde{\alpha_1}\tilde{\alpha_2})^{\mathrm{op}}, \tilde{\beta_1}\tilde{\beta_2})(\varepsilon) = \mathbb{F}(\tilde{\alpha_2}^{\mathrm{op}}\tilde{\alpha_1}^{\mathrm{op}}, \tilde{\beta_1}\tilde{\beta_2})(\varepsilon)$$
$$= \beta_{1*}\beta_{2*}(\alpha_2\alpha_1)^*\varepsilon = \beta_{1*}\beta_{2*}\alpha_1^*\alpha_2^*\varepsilon = \beta_{1*}\alpha_1^*\beta_{2*}\alpha_2^*\varepsilon,$$

so $\mathbb{F}$ preserves composition. This completes the definition of the bifunctor $\mathbb{F}\colon \tilde{\mathcal{C}}^{\mathrm{op}} \times \tilde{\mathcal{C}} \to Ab$.

Our next step will be to verify that $\mathbb{F}\colon \tilde{\mathcal{C}}^{\mathrm{op}} \times \tilde{\mathcal{C}} \to Ab$ is a biadditive functor. We will only show that $\mathbb{F}$ is additive in the second argument since the proof for additivity in the first argument is dual.

Proposition 5.3.3. Fix (X,p) in $\tilde{\mathcal{C}}$. Then the functor $\mathbb{F}((X,p),-)\colon \tilde{\mathcal{C}} \to Ab$ is an additive functor.

Proof. For the zero object $(0, 1_0)$ in $\tilde{\mathcal{C}}$, we have that $\mathbb{F}((X,p),(0,1_0)) = p^*(1_0)_*\mathbb{E}(X,0) = \{0\}$.

Now let (U,e) and (V,f) be any pair of objects in $\tilde{\mathcal{C}}$. Denote by $\mathbb{F}_X^{U\oplus V}$ the Abelian group $\mathbb{F}((X,p),(U\oplus V, e\oplus f)) = \{p^*(e\oplus f)_*\delta \mid \delta \in \mathbb{E}(X, U\oplus V)\}$. We also denote by $\mathbb{F}_X^U$ the Abelian group $\mathbb{F}((X,p),(U,e)) = \{p^*e_*\varepsilon \mid \varepsilon \in \mathbb{E}(X,U)\}$. We likewise denote by $\mathbb{F}_X^V$ the Abelian group $\mathbb{F}((X,p),(V,f)) = \{p^*f_*\tau \mid \tau \in \mathbb{E}(X,V)\}$.

Since $\mathbb{E}$ is a biadditive functor, there is a group isomorphism $\varphi\colon \mathbb{E}(X, U\oplus V) \to \mathbb{E}(X,U) \oplus \mathbb{E}(X,V)$, where an $\mathbb{E}$-extension $\delta \in \mathbb{E}(X, U\oplus V)$ corresponds to $\varphi(\delta) = (\delta_U, \delta_V)$ for some $\delta_U \in \mathbb{E}(X,U)$ and $\delta_V \in \mathbb{E}(X,V)$.

Define the map $G\colon \mathbb{F}_X^{U\oplus V} \to \mathbb{F}_X^U \oplus \mathbb{F}_X^V$ by setting $G(p^*(e\oplus f)_*\delta) = (p^*e_*\delta_U, p^*f_*\delta_V)$ where $\varphi(\delta) = (\delta_U, \delta_V)$ for some $\delta_U \in \mathbb{E}(X,U)$ and $\delta_V \in \mathbb{E}(X,V)$. Observe that for any pair $p^*(e\oplus f)_*\delta$ and $p^*(e\oplus f)_*\varepsilon$ where $\varphi(\delta) = (\delta_U, \delta_V)$ and $\varphi(\varepsilon) = (\varepsilon_U, \varepsilon_V)$ we have that $\varphi(\delta + \varepsilon) = \varphi(\delta) + \varphi(\varepsilon)$. So

$$G(p^*(e\oplus f)_*\delta + p^*(e\oplus f)_*\varepsilon) = G(p^*(e\oplus f)_*(\delta+\varepsilon)) = (p^*e_*(\delta_U + \varepsilon_U), p^*e_*(\delta_V + \varepsilon_V))$$

$$= (p^*e_*\delta_U, p^*f_*\delta_V) + (p^*e_*\varepsilon_U, p^*f_*\varepsilon_V) = G(p^*(e\oplus f)_*\delta) + G(p^*(e\oplus f)_*\varepsilon).$$

Hence G is a group homomorphism.

Define the map $H\colon \mathbb{F}_X^U \oplus \mathbb{F}_X^V \to \mathbb{F}_X^{U\oplus V}$ by setting $H(p^*e_*\delta_U, p^*f_*\delta_V) = p^*(e\oplus f)_*\delta$ where $\varphi^{-1}(\delta_U, \delta_V) = \delta$ for some $\delta \in \mathbb{E}(X, U\oplus V)$, and where $\delta_U \in \mathbb{E}(X,U)$ and $\delta_V \in \mathbb{E}(X,V)$. Take any pair $(p^*e_*\delta_U, p^*f_*\delta_V)$ and $(p^*e_*\varepsilon, p^*f_*\varepsilon_V)$ in $\mathbb{F}_X^U \oplus \mathbb{F}_X^V$. Then if $\varphi^{-1}(\delta_U, \delta_V) = \delta$ and $\varphi^{-1}(\varepsilon_U, \varepsilon_V) = \varepsilon$ then $\varphi^{-1}((\delta_U+\varepsilon_U, \delta_V+\varepsilon_V)) = \varphi^{-1}(\delta_U, \delta_V) + \varphi^{-1}(\varepsilon_U, \varepsilon_V)$. So

$$H((p^*e_*\delta_U, p^*f_*\delta_V) + (p^*e_*\varepsilon_U, p^*f_*\varepsilon_V)) = H((p^*e_*(\delta_U+\varepsilon_U), p^*f_*(\delta_V+\varepsilon_V))$$

$$= p^*(e\oplus f)_*(\delta+\varepsilon) = p^*(e\oplus f)_*\delta + p^*(e\oplus f)_*\varepsilon = H(p^*e_*\delta_U, p^*f_*\delta_V) + H(p^*e_*\varepsilon_U, p^*f_*\varepsilon_V).$$

Hence H is a group homomorphism.

We claim that $G\circ H = 1_{\mathbb{F}_X^U\oplus\mathbb{F}_X^V}$. Take $(p^*e_*\delta_U, p^*f_*\delta_V) \in \mathbb{F}_X^U \oplus \mathbb{F}_X^V$ and suppose $\varphi^{-1}(\delta_U, \delta_V) = \delta$. Then $H((p^*e_*\delta_U, p^*f_*\delta_V)) = p^*(e\oplus f)_*\delta$. Since $\phi(\delta) = (\delta_U, \delta_V)$, we have that

$$GH((p^*e_*\delta_U, p^*f_*\delta_V)) = G(p^*(e\oplus f)_*\delta) = (p^*e_*\delta_U, p^*f_*\delta_V).$$

We also claim that $H\circ G = 1_{\mathbb{F}_X^{U\oplus V}}$. Take $p^*(e\oplus f)_*\delta \in \mathbb{F}_X^{U\oplus V}$ and suppose $\varphi(\delta) = (\delta_U, \delta_V)$. Then $G(p^*(e\oplus f)_*\delta) = (p^*e_*\delta_U, p^*f_*\delta_V)$. Since $\varphi^{-1}(\delta_U, \delta_V) = \delta$, we have that

$$H(G(p^*(e\oplus f)_*\delta)) = H((p^*e_*\delta_U, p^*f_*\delta_V)) = p^*(e\oplus f)_*\delta.$$

This shows that $\mathbb{F}((X,p),(U\oplus V, e\oplus f)) \cong \mathbb{F}((X,p),(U,e)) \oplus \mathbb{F}((X,p),(V,f))$. Therefore the functor $\mathbb{F}((X,p),-)\colon \tilde{\mathcal{C}} \to Ab$ is additive. $\square$

Proposition 5.3.4. Fix (X,p) in $\tilde{\mathcal{C}}$. Then the functor $\mathbb{F}(-,(X,p))\colon \tilde{\mathcal{C}}^{\mathrm{op}} \to Ab$ is an additive functor.

Proof. The proof is dual to the proof of the previous proposition. $\square$

Having verified that the functor $\mathbb{F}$ is biadditive, the next thing we need to do is to define a correspondence which will be a realisation. In order to define the correspondence, we need the following lemmas. This lemma is a generalisation of (Balmer & Schlichting, 2001, Lemma 1.13) in the setting of extriangulated categories. The proof is also a straightforward adaptation.

Lemma 5.3.5. Let $(\mathcal{A}, \mathbb{G}, \mathfrak{t})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Let A, B, C be objects of $\mathcal{A}$. Let δ be an extension in $\mathbb{G}(C, A)$ with $\mathfrak{t}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$. Let $(e, f) \colon \delta \to \delta$ be a morphism of $\mathbb{G}$-extensions where $e \colon A \to A$ and $f \colon C \to C$ are idempotent morphisms. Then there exists an idempotent morphism $g \colon B \to B$ such that the triple (e, g, f) realises the morphism of $\mathbb{G}$-extensions $(e, f) \colon \delta \to \delta$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "e"] & B \arrow[r, "b"] \arrow[d, dashed, "g"] & C \arrow[d, "f"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \end{tikzcd}$$

Proof. Since $(e, f) \colon \delta \to \delta$ is a morphism of $\mathbb{G}$-extensions and $\mathfrak{t}$ is a realisation, there exists a morphism $i \colon B \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "e"] & B \arrow[r, "b"] \arrow[d, dashed, "i"] & C \arrow[d, "f"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \end{tikzcd}$$

Let $h := i^2 - i$. Then we have that $ha = (i^2 - i)a = 0$ and $bh = b(i^2 - i) = 0$ from the commutativity of the above diagram and the fact that e and f are idempotent. By the exact sequences in Proposition 2.1.71, b is a weak cokernel of a so there exists $\bar{h} \colon C \to B$ such that $h = \bar{h}b$. So we observe that $h^2 = \bar{h}bh = 0$.

Let $g = i + h - 2ih$. Since the morphisms i and h commute and $h^2 = 0$ we have that $g^2 = i^2 + 2ih - 4i^2h$. Since $i^2 = i + h$ we have that $g^2 = i + h + 2ih - 4ih = g$. We have that $ga = ia + ha - 2iha = ia = ae$, since $ha = 0$. We likewise have that $bg = bi + bh - 2bih = bi + bh - 2bhi = bi = fb$. Therefore, the above diagram commutes if we replace i with g. This completes the proof. □

Lemma 5.3.6. Let $(\mathcal{A}, \mathbb{G}, \mathfrak{t})$ be an extriangulated category. Let A, B, C be objects of $\mathcal{A}$. Let δ be an extension in $\mathbb{G}(C, A)$ with $\mathfrak{t}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$. Let $(e, f)\colon \delta \to \delta$ be a morphism of $\mathbb{G}$-extensions realised by (e, i, f) where $e\colon A \to A$ and $i\colon B \to B$ are idempotent morphisms. Then there exists an idempotent morphism $g\colon C \to C$ such that $(e, g)\colon \delta \to \delta$ is a morphism of $\mathbb{G}$-extensions realised by (e, i, g). Dually if we instead assume that $i\colon B \to B$ and $f\colon C \to C$ are idempotent. Then there exists an idempotent morphism $k\colon A \to A$ such that $(k, f)\colon \delta \to \delta$ is a morphism of $\mathbb{G}$-extensions realised by (k, i, f).

Proof. Since $(e, f)\colon \delta \to \delta$ is realised by (e, i, f), we have the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "e"] & B \arrow[r, "b"] \arrow[d, "i"] & C \arrow[d, "f"] \\
A \arrow[r, "a"] & B \arrow[r, "b"] & C
\end{tikzcd}$$

Set $h := f^2 - f$. Then we have that

$$hb = f^2b - fb = f(bi) - bi = (fb)i - bi = bi^2 - bi = bi - bi = 0.$$

We also have that

$$h^*\delta = (f^2 - f)^*\delta = f^*f^*\delta - f^*\delta = f^*e_*\delta - e_*\delta$$

$$= e_*(f^*\delta) - e_*\delta = (e^2)_*\delta - e_*\delta = e_*\delta - e_*\delta = 0.$$

By Proposition 2.1.70 we have the following exact sequence in Ab.

$$\mathcal{C}(C, A) \xrightarrow{\mathcal{C}(C,a)} \mathcal{C}(C, B) \xrightarrow{\mathcal{C}(C,b)} \mathcal{C}(C, C) \xrightarrow{(\delta_\#)_C} \mathbb{E}(C, A) \xrightarrow{\mathbb{E}(C,a)} \mathbb{E}(C, B) \xrightarrow{\mathbb{E}(C,b)} \mathbb{E}(C, C)$$

Since $h^*\delta = (\delta_\#)_C(h) = 0$, it follows from the exactness of the above sequence that there exists a morphism $\bar{h}\colon C \to B$ such that $h = b\bar{h}$. From this we can observe that $h^2 = b\bar{h}b\bar{h} = hb\bar{h} = 0$. Now set $g := f + h - 2fh$, as f and h commute and $h^2 = 0$, we then have that

$$g^2 = (f + h - 2fh)^2 = f^2 + 2fh - 4f^2h.$$

By noting that $f^2 = f + h$ we obtain

$$g^2 = f + h - 2fh = g.$$

It is then easy to check that,

$$g^*\delta = f^*\delta + h^*\delta - 2h^*f^*\delta = e_*\delta + 0 - 2h^*e_*\delta = e_*\delta - 2e_*h^*\delta = e_*\delta - 2e_*0 = e_*\delta,$$

and

$$gb = (f + h - 2fh)b = fb + hb - 2fhb = bi + 0 - 2f0 = bi.$$

We have shown that $g\colon C \to C$ is an idempotent morphism such that $(e, g)\colon \delta \to \delta$ is a morphism of $\mathbb{G}$-extensions realised by (e, i, g). The proof of the other statement is dual. $\square$

An analogue of Lemma 5.3.5 where we replace the idempotent morphisms with split idempotent morphisms can also be obtained.

Lemma 5.3.7. Let $(\mathcal{A}, \mathbb{G}, \mathfrak{t})$ be a triple satisfying (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. Let A, B and C be objects of $\mathcal{A}$. Let δ be an extension in $\mathbb{G}(C, A)$ with $\mathfrak{t}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$. Let $(e, f)\colon \delta \to \delta$ be a morphism of $\mathbb{G}$-extensions where $e\colon A \to A$ and $f\colon C \to C$ are idempotent morphisms that split. Then there exists an idempotent morphism $g\colon B \to B$ that splits such that the triple (e, g, f) realises the morphism of $\mathbb{G}$-extensions (e, f).

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "e"] & B \arrow[r, "b"] \arrow[d, dashed, "g"] & C \arrow[d, "f"] \\
A \arrow[r, "a"] & B \arrow[r, "b"] & C
\end{tikzcd}$$

Proof. Since e splits, there is an object $X \in \mathcal{A}$ and morphisms $e_2 \colon A \to X$ and $e_1 \colon X \to A$ such that $e = e_1 e_2$ and $e_2 e_1 = 1_X$. Likewise, for f there is an object $Z \in \mathcal{A}$ and morphisms $f_2 \colon C \to Z$ and $f_1 \colon Z \to C$ such that $f = f_1 f_2$ and $f_2 f_1 = 1_Z$.

Suppose that $\mathfrak{t}(e_{2*} f_1^* \delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$. Consider the following diagram of $\mathbb{G}$-triangles.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "e_1"] & Y \arrow[r, "y"] \arrow[d, dashed, "r_1"] & Z \arrow[r, dashed, "e_{2*}f_1^*\delta"] \arrow[d, "f_1"] & {} \\
A \arrow[r, "a"] \arrow[d, "e_2"] & B \arrow[r, "b"] \arrow[d, dashed, "r_2"] & C \arrow[r, dashed, "\delta"] \arrow[d, "f_2"] & {} \\
X \arrow[r, "x"] & Y \arrow[r, "y"] & Z \arrow[r, dashed, "e_{2*}f_1^*\delta"] & {}
\end{tikzcd} \tag{5.1}$$

Observe that

$$e_{1*}(e_{2*} f_1^* \delta) = e_* f_1^* \delta = f_1^*(e_* \delta) = f_1^*(f^* \delta) = f_1^*(f_1 f_2)^* \delta = ((f_2 f_1) f_1)^* \delta = f_1^* \delta.$$

Therefore $(e_1, f_1) \colon e_{2*} f_1^* \delta \to \delta$ is a morphism of $\mathbb{G}$-extensions. So, by the axiom (ET2), there exists a morphism $r_1 \colon Y \to B$ such that the top row of the above diagram commutes. Also observe that

$$f_2^*(e_{2*} f_1^* \delta) = e_{2*} f_2^* f_1^* \delta = e_{2*}(f^* \delta) = e_{*2}(e_* \delta) = ((e_2 e_1) e_2)_* \delta = e_{2*} \delta.$$

Therefore $(e_2, f_2) \colon \delta \to e_{2*} f_1^* \delta$ is a morphism of $\mathbb{G}$-extensions. So by axiom (ET2), there exists a morphism $r_2 \colon B \to Y$ such that the bottom row of the above diagram commutes. Collapsing the above diagram into the diagram below, we obtain the following morphism of $\mathbb{G}$-triangles and commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, equal, "e_2 e_1 = 1_X"] & Y \arrow[r, "y"] \arrow[d, "r_2 r_1"] & Z \arrow[r, dashed, "e_{2*}f_1^*\delta"] \arrow[d, equal, "f_2 f_1 = 1_Z"] & {} \\
X \arrow[r, "x"] & Y \arrow[r, "y"] & Z \arrow[r, dashed, "e_{2*}f_1^*\delta"] & {}
\end{tikzcd} \tag{5.2}$$

By Lemma 2.1.65, the morphism r_2r_1 is an automorphism of Y. That is to say, there exists $h\colon Y \to Y$ such that $r_2r_1h = 1_Y$ and $hr_2r_1 = 1_Y$. Set $g := (r_1h)r_2\colon B \to B$. Observe that

$$g^2 = r_1h(r_2r_1h)r_2 = r_1h(1_Y)r_2 = r_1hr_2 = g,$$

so g is an idempotent morphism. Moreover,

$$r_2(r_1h) = 1_Y.$$

So g is in fact a split idempotent. Now consider the following diagram.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "e"] & B \arrow[r, "b"] \arrow[d, dashed, "g"] & C \arrow[d, "f"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \end{tikzcd} \tag{5.3}$$

Note that, by the commutativity of diagram (5.2),

$$r_2r_1x = x$$

so

$$x = (hr_2r_1)x = hx,$$

and similarly

$$yr_2r_1 = y,$$

so

$$y = y(r_2r_1h) = yh.$$

Using the fact that diagram (5.1) commutes, we further observe that

$$ga = r_1h(r_2a) = r_1(hx)e_2 = (r_1x)e_2 = a(e_1e_2) = ae,$$

and

$$bg = (br_1)hr_2 = f_1(yh)r_2 = (f_1y)r_2 = (f_1f_2)b = fb,$$

so diagram (5.3) commutes. This completes the proof. □

Definition 5.3.8. Let $\mathfrak{r}$ be the correspondence between $\mathbb{F}$-extensions and equivalence classes of sequences of morphisms in $\tilde{\mathcal{C}}$ defined as follows. For any objects Z, X in $\mathcal{C}$ and idempotent morphisms $p\colon Z \to Z, q\colon X \to X$ in $\mathcal{C}$, let $\delta = p^*q_*\varepsilon$ be an extension in $\mathbb{F}((Z,p),(X,q))$ such that

$$\mathfrak{s}(p^*q_*\varepsilon) = [X \xrightarrow{x} Y \xrightarrow{y} Z].$$

We set

$$\mathfrak{r}(\delta) := [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)],$$

where $r\colon Y \to Y$ is an idempotent morphism such that $rx = xq$ and $yr = py$ obtained by application of Lemma 5.3.5.

Remark 5.3.9. Before we can proceed any further, we need to show that the above definition of $\mathfrak{r}$ is well-defined in the following sense. Given an $\mathbb{F}$-extension δ, $\mathfrak{r}(\delta)$ is defined in terms of a choice of the representative $\mathfrak{s}(\delta)$. We will show that it is independent of this choice. Moreover, in the above definition, the idempotent morphism $r\colon Y \to Y$ such that $rx = xq$ and $yr = py$ need not be unique. We will show that all choices of such an idempotent give equivalent sequences.

Lemma 5.3.10. Let $\delta = p^*q_*\varepsilon$ be an extension in $\mathbb{F}((Z,p),(X,q))$ such that

$$\mathfrak{s}(p^*q_*\varepsilon) = [X \xrightarrow{x} Y \xrightarrow{y} Z].$$

The following sequences of functors are exact;

$$\tilde{\mathcal{C}}((Z,p),-) \xRightarrow{\tilde{\mathcal{C}}(py,-)} \tilde{\mathcal{C}}((Y,r),-) \xRightarrow{\tilde{\mathcal{C}}(xq,-)} \tilde{\mathcal{C}}((X,q),-) \xRightarrow{\delta^\#_-} \mathbb{F}((Z,p),-)$$

$$\tilde{\mathcal{C}}(-,(X,q)) \xRightarrow{\tilde{\mathcal{C}}(-,xq)} \tilde{\mathcal{C}}(-,(Y,r)) \xRightarrow{\tilde{\mathcal{C}}(-,py)} \tilde{\mathcal{C}}(-,(Z,p)) \xRightarrow{\delta^-_\#} \mathbb{F}(-,(X,q))$$

where $r\colon Y \to Y$ is an idempotent morphism such that $rx = xq$ and $yr = py$, obtained by application of Lemma 5.3.5. In particular, py is a weak cokernel of xq and xq is a weak kernel of py.

Proof. We will only show the exactness of the first sequence; the proof of the exactness of the second sequence is dual. Since $(\mathcal{C}, \mathbb{E}, \mathfrak{r})$ is an extriangulated category, the following sequence

$$\mathcal{C}(Z,-) \xRightarrow{\mathcal{C}(y,-)} \mathcal{C}(Y,-) \xRightarrow{\mathcal{C}(x,-)} \mathcal{C}(X,-) \xRightarrow{\delta\#_-} \mathbb{E}(Z,-) \tag{5.4}$$

is exact.

We will start by showing exactness at $\tilde{\mathcal{C}}((Y,r),-)$. Since (5.4) is exact we have that $y \circ x = 0$. So it follows that for $f \in \tilde{\mathcal{C}}((Z,p),(A,e))$,

$$(\tilde{\mathcal{C}}(xq,(A,e))\tilde{\mathcal{C}}(py,(A,e)))(f) = (f \circ py) \circ (xq) = fp(y \circ x)q = 0.$$

That is to say $\mathrm{im}(\tilde{\mathcal{C}}(py,(A,e))) \subseteq \ker(\tilde{\mathcal{C}}(xq,(A,e)))$.

Let $g \in \tilde{\mathcal{C}}((Y,r),(A,e))$ be such that

$$\tilde{\mathcal{C}}(xq,(A,e))(g) = g \circ xq = 0,$$

we have that $gxq = grx = 0$. By the exactness of the sequence 5.4, y is a weak cokernel of x, so we have that there exists a morphism $h\colon Z \to A$ such that $gr = hy$. Since $gr = hy$ we have that

$$gr = gr^2 = hyr = hpy.$$

Moreover for the morphism $ehp\colon (Z,p) \to (A,e)$ we have that

$$gr = egy = ehpy = ehppy = (ehp) \circ py.$$

This is to say $gr \in \mathrm{im}(\tilde{\mathcal{C}}(py,(A,e)))$, in particular $\ker(\tilde{\mathcal{C}}(xq,(A,e))) \subseteq \mathrm{im}\tilde{\mathcal{C}}(py,(A,e)))$.

What is left is to prove exactness at $\tilde{\mathcal{C}}((X,q),-)$. Take a morphism

$$f\colon (Y,r)\to (A,e) \text{ in } \tilde{\mathcal{C}}((Y,r),(A,e)).$$

Then

$$(\delta^{\#}_{(A,e)}\circ \tilde{\mathcal{C}}(xq,(A,e)))(f) = (fxq)_*\delta = (f(xq))_*\delta = (f(rx))_*\delta = ((fr)x)_*\delta) = 0$$

by the exactness of (5.4). We conclude that $\operatorname{im}(\tilde{\mathcal{C}}(xq,(A,e)) \subseteq \ker(\delta^{\#}_{(A,e)})$.

Take a morphism $g\colon (X,q)\to (A,e)\in \tilde{\mathcal{C}}((X,q),(A,e))$. Suppose

$$\delta^{\#}_{(A,e)}(g) = g_*\delta = 0.$$

Since g is also a morphism in $\mathcal{C}$ and δ is an $\mathbb{E}$-extension, we have by the exactness of (5.4) that there exists $h\colon Y\to A$ such that $g = hx$. Now consider the morphism

$$h' = ehr\colon (Y,r)\to (A,e).$$

We have that

$$h'xq = (ehr)xq = eh(rx)q = eh(xq)q = e(hx)q = e(g)q = g.$$

We conclude that $\ker(\delta^{\#}_{(A,e)}) \subseteq \operatorname{im}(\tilde{\mathcal{C}}(xq,(A,e))$. Therefore we have exactness at $\tilde{\mathcal{C}}((X,q),-)$ as required. $\square$

Proposition 5.3.11. Let δ be an extension in $\mathbb{F}((C,p),(A,q))$ realised under $\mathfrak{s}$ by the following sequences,

$$A \xrightarrow{a} B \xrightarrow{b} C,$$

$$A \xrightarrow{x} Y \xrightarrow{y} C.$$

Then given idempotents $r\colon B\to B$ and $w\colon Y\to Y$ such that

$$aq = ra,\ pb = br \text{ and } xq = wx,\ py = yw \tag{5.5}$$

the following sequences are equivalent,

$$(A,q) \xrightarrow{aq} (B,r) \xrightarrow{pb} (C,p),$$

$$(A,q) \xrightarrow{xq} (Y,w) \xrightarrow{py} (C,p).$$

That is to say, $\mathfrak{r}$ is well-defined.

Proof. Since the sequences $A \xrightarrow{a} B \xrightarrow{b} C$, and $A \xrightarrow{x} Y \xrightarrow{y} C$ both realise δ, they are by definition equivalent in $\mathcal{C}$. That is to say we have the following commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, equal] & B \arrow[r, "b"] \arrow[d, "f"] & C \arrow[d, equal] \\ A \arrow[r, "x"] & Y \arrow[r, "y"] & C \end{tikzcd} \tag{5.6}$$

where the morphism $f\colon B \to Y$ is an isomorphism. Now consider the following diagram.

$$\begin{tikzcd} (A,q) \arrow[r, "aq"] \arrow[d, equal] & (B,r) \arrow[r, "pb"] \arrow[d, "wfr"] & (C,p) \arrow[d, equal] \\ (A,q) \arrow[r, "xq"] & (Y,w) \arrow[r, "py"] & (C,p) \end{tikzcd} \tag{5.7}$$

From the relations in (5.5) and those arising from the commutative diagram (5.6), we can observe the following,

$$wf(ra)q = wf(aq)q = wf(aq) = w(fa)q = w(x)q = (wx)q = (xq)q = xq, \tag{5.8}$$

$$p(yw)fr = p(py)fr = (py)fr = p(yf)r = p(b)r = p(br) = p(pb) = pb. \tag{5.9}$$

That is to say, diagram (5.7) is a commutative diagram. Now consider the following diagram.

$$\begin{tikzcd} (A,q) \arrow[r, "xq"] \arrow[d, equal] & (Y,w) \arrow[r, "py"] \arrow[d, "rf^{-1}w"] & (C,p) \arrow[d, equal] \\ (A,q) \arrow[r, "aq"] & (B,r) \arrow[r, "pb"] & (C,p) \end{tikzcd} \tag{5.10}$$

From the relations in (5.5) and those arising from the commutative diagram (5.6), we can observe the following,

$$rf^{-1}(wx)q = rf^{-1}(xq)q = r(f^{-1}x)q = r(a)q = (ra)q = (aq)q = aq, \tag{5.11}$$

$$p(br)f^{-1}w = p(pb)f^{-1}w = p(bf^{-1})w = p(y)w = p(yw) = p(py) = py. \tag{5.12}$$

That is to say, diagram (5.10) is a commutative diagram, and by Lemma 2.1.76 we have that $wfr_\bullet = (1_{(A,q)}, wfr, 1_{(C,p)})$ is a homotopy equivalence, and hence by Lemma 2.1.75, the morphism wfr is an isomorphism. We conclude that (5.7) is an equivalence, that is to say $\mathfrak{r}$ is well-defined. □

From Proposition 5.3.11 we conclude that $\mathfrak{r}$ is well-defined in the sense of Remark 5.3.9.

Lemma 5.3.12. Let δ be an extension in $\mathbb{F}((Z,p),(X,q))$ with $\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$ and $\mathfrak{r}(\delta) = [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)]$. Suppose that $(X,q) \xrightarrow{u} (W,s) \xrightarrow{v} (Z,p)$ is another sequence realising δ as an $\mathbb{F}$-extension. Then $u = u_1q$ and $v = pv_1$ for some $u_1 \colon X \to W$ and $v_1 \colon W \to Z$.

Proof. Since $(X,q) \xrightarrow{u} (W,s) \xrightarrow{v} (Z,p)$ realises δ, there is an equivalence,

$$\begin{tikzcd}
(X,q) \arrow[r, "xq"] \arrow[d, equal] & (Y,r) \arrow[r, "py"] \arrow[d, "f"] & (Z,p) \arrow[d, equal] \\
(X,q) \arrow[r, "u"] & (W,s) \arrow[r, "v"] & (Z,p)
\end{tikzcd}$$

where f is an isomorphism. Since the above diagram commutes $fxq = u$ and $vf = py$, so set $u_1 = fx$ and $v_1 = yf^{-1}$. □

Proposition 5.3.13. Let $\mathfrak{r}$ be the correspondence defined above. Then $\mathfrak{r}$ is an additive realisation of $\mathbb{F}$.

Proof. Let $\delta = p^*q_*\varepsilon \in \mathbb{F}((Z,p),(X,q))$ and $\delta' = p'^*q'_*\varepsilon' \in \mathbb{F}((Z',p'),(X',q'))$ be $\mathbb{F}$-extensions with

$$\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z],$$

$$\mathfrak{r}(\delta) = [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)]$$

and

$$\mathfrak{s}(\delta') = [X' \xrightarrow{x'} Y' \xrightarrow{y'} Z'],$$

$$\mathfrak{r}(\delta') = [(X',q') \xrightarrow{x'q'} (Y',r') \xrightarrow{p'y'} (Z',p')].$$

Suppose that we have a morphism of $\mathbb{F}$-extensions $(a,c)\colon \delta \to \delta'$ for some $a \in \tilde{\mathcal{C}}((X,q),(X',q')$ and $c \in \tilde{\mathcal{C}}((Z,p),(Z',p')$, that is to say

$$\mathbb{F}((Z,p),a)(\delta) = \mathbb{F}(c,(X,q))(\delta').$$

In other words, we have the following diagram in $\tilde{\mathcal{C}}$.

$$\begin{tikzcd}
(X,q) \arrow[r, "xq"] \arrow[d, "a"] & (Y,r) \arrow[r, "py"] & (Z,p) \arrow[d, "c"] \\
(X',q') \arrow[r, "x'q'"] & (Y',r') \arrow[r, "p'y'"] & (Z',p')
\end{tikzcd} \tag{5.13}$$

By definition we have that $\delta = p^*q_*\varepsilon$ and $\delta' = p'^*q'_*\varepsilon'$ for some $\varepsilon \in \mathbb{E}(Z,X)$ and some $\varepsilon' \in \mathbb{E}(Z',X')$. Moreover the morphism $a \in \mathcal{C}(X,X')$ is such that $aq = q'a = a$, likewise for $c \in \mathcal{C}(Z,Z')$ we have that $cp = p'c = c$. We also have by definition that

$$\mathbb{F}((Z,p),a)(\delta) = a_*(p^*q_*\varepsilon) = \mathbb{F}(c,(X,q))(\delta') = c^*(p'^*q'_*\varepsilon').$$

Therefore we have a morphism of $\mathbb{E}$-extensions $(a,c)\colon p^*q_*\varepsilon \to p'^*q'_*\varepsilon'$. In other words we have the following solid diagram in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "a"] & Y \arrow[r, "y"] \arrow[d, dashed, "b"] & Z \arrow[r, dashed, "p^*q_*\varepsilon"] \arrow[d, "c"] & {} \\
X \arrow[r, "x'"] & Y \arrow[r, "y'"] & Z \arrow[r, dashed, "p'^*q'_*\varepsilon'"] & {}
\end{tikzcd}$$

Since $\mathfrak{s}$ is a realisation, there exists a morphism $b\colon Y \to Y'$ making the above diagram commute. Recall that by Lemma 5.3.5, we have that $rx = xq$, $yr = py$, $r'x' = x'q'$ and $y'r' = p'y'$. It then follows that $r'br\colon (Y,r) \to (Y',r')$ makes diagram (5.13) commute since,

$$r'brxq = r'b(rxq) = r'b(xq^2) = r'(bx)q = r'x'(aq) = (r'x')a = x'q'a$$

and

$$p'y'r'br = p'(y'r')br = (p'p')y'br = p'(y'b)r = p'c(yr) = (p'c)py = cpy.$$

So we conclude that $\mathfrak{r}$ is a realisation of $\mathbb{F}$.

Now we verify additivity of $\mathfrak{r}$. For any pair $(Z,p),(X,q)$, we have that $0 = p^*q_*0$ and

$$\mathfrak{s}(0) = [X \xrightarrow{\begin{bmatrix} 1_X \\ 0 \end{bmatrix}} X \oplus Z \xrightarrow{[0\ 1_Z]} Z] = 0.$$

By definition we have that

$$\mathfrak{r}(0) = [(X,q) \xrightarrow{\begin{bmatrix} q \\ 0 \end{bmatrix}} (X,q) \oplus (Z,p) \xrightarrow{[0\ p]} (Z,p)],$$

since $q = 1_{(X,q)}$ and $p = 1_{(Z,p)}$, we have that,

$$\mathfrak{r}(0) = 0.$$

Now take a pair of $\mathbb{F}$-extensions $\delta = p^*q_*\varepsilon \in \mathbb{F}((Z,p),(X,q))$ and $\delta' = p'^*q'_*\varepsilon' \in \mathbb{F}((Z',p'),(X',q'))$. Since $\mathfrak{s}$ is an additive realisation we have that

$$\mathfrak{s}(p^*q_*\varepsilon \oplus p'^*q'_*\varepsilon') = \mathfrak{s}(p^*q_*\varepsilon) \oplus \mathfrak{s}(p'^*q'_*\varepsilon').$$

As $\mathfrak{s}(p^*q_*\varepsilon) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$ and $\mathfrak{s}(p'^*q'_*\varepsilon') = [X' \xrightarrow{x'} Y' \xrightarrow{y'} Z']$, we have that

$$\mathfrak{s}(p^*q_*\varepsilon \oplus p'^*q'_*\varepsilon') = [X \oplus X' \xrightarrow{x \oplus x'} Y \oplus Y' \xrightarrow{y \oplus y'} Z \oplus Z']$$

By the definition of $\mathfrak{r}$ we have that,

$$\mathfrak{r}(\delta) = [(X, q) \xrightarrow{xq} (Y, r) \xrightarrow{py} (Z, p)],$$

$$\mathfrak{r}(\delta') = [(X', q') \xrightarrow{x'q'} (Y', r') \xrightarrow{p'y'} (Z', p')] \text{ and}$$

$$\mathfrak{r}(\delta \oplus \delta') = \ [(X \oplus X', q \oplus q') \xrightarrow{(x \oplus x')(q \oplus q')} (Y \oplus Y', r \oplus r') \xrightarrow{(p \oplus p')(y \oplus y')} (Z \oplus Z', p \oplus p')] \ .$$

We have that

$$(x \oplus x')(q \oplus q') = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & q' \end{pmatrix} = \begin{pmatrix} xq & 0 \\ 0 & x'q' \end{pmatrix} = (xq) \oplus (x'q'),$$

likewise $(p \oplus p')(y \oplus y') = py \oplus p'y'$. It is also easy to check that $r \oplus r'$ is idempotent and satisfies the required equations arising from Lemma 5.3.5. So it follows that

$$\mathfrak{r}(\delta \oplus \delta') = \mathfrak{r}(\delta) \oplus \mathfrak{r}(\delta').$$

This completes the proof. □

So far we have constructed the triple $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$. Since $\tilde{\mathcal{C}}$ is the idempotent completion of $\mathcal{C}$, it is an additive category. Propositions 5.3.3, 5.3.4 and 5.3.13 show that the triple $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies axioms (ET1) and (ET2) of the definition of an extriangulated category, see Definition 2.1.61. So what is left is to verify axioms (ET3), (ET3)$^{\mathrm{op}}$, (ET4) and (ET4)$^{\mathrm{op}}$.

Proposition 5.3.14. The triple $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies the axioms (ET3) and (ET3)$^{\mathrm{op}}$.

Proof. Let $\delta = p^*q_*\varepsilon \in \mathbb{F}((Z,p),(X,q))$ and $\delta' = (p')^*(q')_*\varepsilon' \in \mathbb{F}((Z',p'),(X',q'))$ be $\mathbb{F}$-extensions with

$$\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z],$$

$$\mathfrak{r}(\delta) = [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)],$$

whereby $rx = xq$ and $py = yr$ by Lemma 5.3.5 and

$$\mathfrak{s}(\delta') = [X' \xrightarrow{x'} Y' \xrightarrow{y'} Z'],$$

$$\mathfrak{r}(\delta') = [(X',q') \xrightarrow{x'q'} (Y',r') \xrightarrow{p'y'} (Z',p')],$$

whereby $r'x' = x'q'$ and $p'y' = y'r'$ by Lemma 5.3.5. Suppose we have the following commutative diagram in $\tilde{\mathcal{C}}$. Note that we have that $q'a = aq = a$ and $r'b = br = b$ by the definition of morphisms in $\tilde{\mathcal{C}}$.

$$\begin{tikzcd}
(X,q) \arrow[r, "xq"] \arrow[d, "a"] & (Y,r) \arrow[r, "py"] \arrow[d, "b"] & (Z,p) \\
(X',q') \arrow[r, "x'q'"] & (Y',r') \arrow[r, "p'y'"] & (Z',p')
\end{tikzcd} \tag{5.14}$$

We then have the following diagram in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "a"] & Y \arrow[r, "y"] \arrow[d, "r'br"] & Z \arrow[r, dashed, "p^*q_*\varepsilon"] \arrow[d, dashed, "c"] & {} \\
X' \arrow[r, "x'"] & Y' \arrow[r, "y'"] & Z' \arrow[r, dashed, "p'^*q'_*\varepsilon'"] & {}
\end{tikzcd} \tag{5.15}$$

Using the above relations, we have that

$$r'brx = r'bxq = r'x'q'a = r'r'x'a = r'x'a = x'q'a = x'a$$

hence the left square of diagram (5.15) commutes. Since $\mathcal{C}$ is an extriangulated category, there exists $c\colon Z \to Z'$ such that the diagram commutes and $a_*(p^*q_*\varepsilon) = c^*(p'^*q'_*\varepsilon')$.

Consider the morphism $p'cp\colon (Z,p) \to (Z,p')$, we have that

$$p'c(pp)y = p'c(py) = p'(cy)r = p'y'r'b(rr) = p'y'(r'br) = p'y'b,$$

so $p'cp$ makes diagram (5.14) commute.

We also have that,

$$(p'cp)^*(p'^*q'_*\varepsilon') = p^*c^*p'^*p'^*q'_*\varepsilon' = p^*c^*p'^*q'_*\varepsilon' = p^*a_*p^*q_*\varepsilon = a_*p^*p^*q_*\varepsilon = a_*p^*q_*\varepsilon$$

therefore we have a morphism of $\mathbb{F}$-extensions $(a, p'cp)\colon \delta \to \delta'$, as required. This verifies (ET3). The proof for (ET3)$^{\mathrm{op}}$ is dual. $\square$

Before we can prove that $\tilde{\mathcal{C}}$ satisfies (ET4) and (ET4)$^{\mathrm{op}}$. We first need to prove the upcoming statements, which will play an important part in our proof of (ET4) and (ET4)$^{\mathrm{op}}$.

Lemma 5.3.15. Let δ be an extension in $\mathbb{F}((Z,p),(X,q))$ where $\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$ and $\mathfrak{r}(\delta) = [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)]$. Then the following sequences of natural transformations are exact.

$$\mathbb{F}(-,(X,q)) \xrightarrow{\mathbb{F}(-,xq)} \mathbb{F}(-,(Y,r)) \xrightarrow{\mathbb{F}(-,py)} \mathbb{F}(-,(Z,p))$$

$$\mathbb{F}((Z,p),-) \xrightarrow{\mathbb{F}(py,-)} \mathbb{F}((Y,r),-) \xrightarrow{\mathbb{F}(xq,-)} \mathbb{F}((X,q),-)$$

Proof. Let (A,e) be any object in $\tilde{\mathcal{C}}$. We need to show that the following sequence in *Ab* is exact.

$$\mathbb{F}((A,e),(X,q)) \xrightarrow{\mathbb{F}((A,e),xq)} \mathbb{F}((A,e),(Y,r)) \xrightarrow{\mathbb{F}((A,e),py)} \mathbb{F}((A,e),(Z,p)).$$

Take $\theta \in \mathbb{F}((A,e),(Y,r))$. Recall that, by definition, we have that $e^*\theta = r_*\theta = \theta$. Suppose that $\mathbb{F}((A,e),py)(\theta) = (py)_*\theta = 0$. Recall that, by construction $py = yr$, hence we have that $0 = (py)_*\theta = (yr)_*\theta = y_*(r_*\theta)$. In particular we have that $r_*\theta = \theta \in \ker(\mathbb{E}(A,y))$. By Proposition 2.1.70, the following sequence is exact in *Ab*.

$$\mathbb{E}(A,X) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(A,Y) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(A,Z)$$

Therefore we have that $r_*\theta = \theta = x_*(\sigma)$ for some $\sigma \in \mathbb{E}(A,X)$. Observe the following,

$$\theta = e^*\theta = e^*x_*\sigma = x_*(e^*\sigma),$$

thus

$$(xq)_*(e^*\sigma) = (rx)_*(e^*\sigma) = r_*(x_*e^*\sigma) = r_*(\theta) = \theta.$$

Therefore we have that $(xq)_*(q_*e^*\sigma) = \theta$. In other words, $\theta \in \operatorname{im}(\mathbb{F}((A,e),xq))$ and in particular, $\ker(\mathbb{F}((A,e),py)) \subseteq \operatorname{im}(\mathbb{F}((A,e),xq))$.

Now take $\sigma \in \mathbb{F}((A,e),(X,q))$, then

$$(py)_*((xq)_*\sigma) = (py)_*(x_*q_*\sigma) = p_*(yx)_*(q_*\sigma) = 0$$

since $y \circ x = 0$ by Lemma 2.1.72. Hence $\operatorname{im}(\mathbb{F}((A,e),xq)) \subseteq \ker(\mathbb{F}((A,e),py))$.

The proof of the dual statement is dual. This completes the proof. □

Remark 5.3.16. Since $\tilde{\mathcal{C}}$ satisfies (ET3) and (ET3)$^{\mathrm{op}}$ we have that $\tilde{\mathcal{C}}$ satisfies Proposition 2.1.71. By Lemma 5.3.15, we see that $\tilde{\mathcal{C}}$ induces long exact sequences as in Proposition 2.1.70 without requiring that $\tilde{\mathcal{C}}$ is an extriangulated category as a priori.

Corollary 5.3.17. Let δ be an extension in $\mathbb{F}((Z,p),(X,q))$ where $\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$ and $\mathfrak{r}(\delta) = [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)]$. Suppose that $(X,q) \xrightarrow{u} (W,s) \xrightarrow{v} (Z,p)$ is another sequence realising δ as an $\mathbb{F}$-extension. Then the following sequences of natural transformations are exact.

$$\mathbb{F}(-,(X,q)) \xRightarrow{\mathbb{F}(-,u)} \mathbb{F}(-,(W,s)) \xRightarrow{\mathbb{F}(-,v)} \mathbb{F}(-,(Z,p))$$

$$\mathbb{F}((Z,p),-) \xRightarrow{\mathbb{F}(v,-)} \mathbb{F}((W,s),-) \xRightarrow{\mathbb{F}(u,-)} \mathbb{F}((X,q),-)$$

Proof. Since $(X,q) \xrightarrow{u} (W,s) \xrightarrow{v} (Z,p)$ realises δ, there is an equivalence,

$$\begin{tikzcd} (X,q) \arrow[r, "xq"] \arrow[d, equal] & (Y,r) \arrow[r, "py"] \arrow[d, "f"] & (Z,p) \arrow[d, equal] \\ (X,q) \arrow[r, "u"] & (W,s) \arrow[r, "v"] & (Z,p) \end{tikzcd}$$

Then, for any object $(A,e) \in \tilde{\mathcal{C}}$ we have the following commutative diagram, where by Lemma 5.3.15, the top row is exact.

$$\begin{tikzcd} \mathbb{F}((A,e),(X,q)) \arrow[r, "(xq)_*"] \arrow[d, equal] & \mathbb{F}((A,e),(Y,r)) \arrow[r, "(py)_*"] \arrow[d, "f_*"] & \mathbb{F}((A,e),(Z,p)) \arrow[d, equal] \\ \mathbb{F}((A,e),(X,q)) \arrow[r, "u_*"] & \mathbb{F}((A,e),(W,s)) \arrow[r, "v_*"] & \mathbb{F}((A,e),(Z,p)) \end{tikzcd}$$

From the above commutative diagram, it is easy to see that the bottom row is also exact. $\square$

The following proposition is an analogue of Proposition 2.1.73 in $\tilde{\mathcal{C}}$. Remarkably, we are able to prove the statement of the following proposition without requiring $\tilde{\mathcal{C}}$ to be extriangulated unlike in the statement of Proposition 2.1.73. We only require that $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies axioms (ET1) and (ET2).

Proposition 5.3.18. Let $\delta = p^*q_*\varepsilon \in \mathbb{F}((C,p),(A,q))$ be an $\mathbb{F}$-extension where

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \text{ and } \mathfrak{r}(\delta) = [(A,q) \xrightarrow{xq} (B,r) \xrightarrow{py} (C,p)].$$

Let $h\colon (E,w) \longrightarrow (C,p)$ be any morphism and suppose

$$\mathfrak{s}(h^*\delta) = [A \xrightarrow{d} D \xrightarrow{e} E] \text{ and } \mathfrak{r}(h^*\delta) = [(A,q) \xrightarrow{dq} (D,s) \xrightarrow{we} (E,w)].$$

Then there exists a morphism $g\colon (D,s) \longrightarrow (B,r)$ such that $(1_{(A,q)}, h)\colon h^*\delta \to \delta$ is realised by $(1_{(A,q)}, g, h)$. Moreover

$$\mathfrak{r}((dq)_*\delta) = [(D,s) \xrightarrow{\begin{pmatrix} -es \\ gs \end{pmatrix}} (E,w) \oplus (B,r) \xrightarrow{\begin{pmatrix} h & py \end{pmatrix}} (C,p)].$$

Proof. We apply Proposition 2.1.73 to $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, the morphism $h\colon E \to C$ and $A \xrightarrow{d} D \xrightarrow{e} E \overset{h^*\delta}{\dashrightarrow}$. Then there is a morphism $\bar{g}\colon D \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, equal] & D \arrow[r, "e"] \arrow[d, "\bar{g}"] & E \arrow[r, dashed, "h^*\delta"] \arrow[d, "h"] & {} \\ A \arrow[r, "x"] & B \arrow[r, "y"] & C \arrow[r, dashed, "\delta"] & {} \end{tikzcd} \tag{5.16}$$

and that

$$D \xrightarrow{\begin{pmatrix} -e \\ \bar{g} \end{pmatrix}} E \oplus B \xrightarrow{\begin{pmatrix} h & y \end{pmatrix}} C \overset{d_*\delta}{\dashrightarrow}.$$

Since $h\colon (E, w) \longrightarrow (C, p)$ is a morphism in $\tilde{\mathcal{C}}$ we have that $h = hw$, therefore

$$h^*\delta = w^*h^*\delta = w^*h^*p^*q_*\varepsilon = w^*q_*(h^*p^*\varepsilon).$$

In other words $h^*\delta \in \mathbb{F}((E, w), (A, q))$, so we have that

$$\mathfrak{r}(h^*\delta) = [(A, q) \xrightarrow{dq} (D, s) \xrightarrow{we} (E, w)],$$

where $s\colon D \to D$ is an idempotent morphism such that $dq = sd$ and $we = es$. Consider the following diagram.

$$\begin{tikzcd} (A,q) \arrow[r, "dq"] \arrow[d, equal] & (D,s) \arrow[r, "we"] \arrow[d, "g=r\bar{g}s"] & (E,w) \arrow[r, dashed, "h^*\delta"] \arrow[d, "h"] & {} \\ (A,q) \arrow[r, "xq"] & (B,r) \arrow[r, "py"] & (C,p) \arrow[r, dashed, "\delta"] & {} \end{tikzcd} \tag{5.17}$$

By diagram (5.16) and the relations $h = hw = ph$, we can observe that

$$(r\bar{g}s)dq = r\bar{g}(dq) = r(\bar{g}d)q = rxq = xq \text{ and}$$

$$py(r\bar{g}s) = (pyr)\bar{g}s = (py)\bar{g}s = p(y\bar{g})s = p(he)s = (ph)es = h(es) = hwe.$$

Therefore diagram (5.17) commutes and $(1_{(A,q)}, h)$ is realised by $(1_{(A,q)}, g, h)$.

Since $dq = sd$, we have that

$$(dq)_*\delta = (sd)_*p^*q_*\varepsilon = s_*d_*p^*q_*\varepsilon = s_*p^*(d_*q_*\varepsilon).$$

That is to say, $(dq)_*\delta \in \mathbb{F}((C,p),(D,s))$. By definition $(E,w)\oplus(B,r) = (E \oplus B, w \oplus r)$, where $w \oplus r = \begin{pmatrix} w & 0 \\ 0 & r \end{pmatrix}$, which we observe is an idempotent morphism. Also observe that,

$$\begin{pmatrix} w & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} -e \\ g \end{pmatrix} = \begin{pmatrix} -we \\ rg \end{pmatrix} = \begin{pmatrix} -es \\ gs \end{pmatrix} = \begin{pmatrix} -e \\ g \end{pmatrix} s \text{ and}$$

$$\begin{pmatrix} h & y \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} hw & yr \end{pmatrix} = \begin{pmatrix} h & py \end{pmatrix}.$$

Therefore

$$\mathfrak{r}((dq)_*\delta) = [(D,s) \xrightarrow{\begin{pmatrix} -es \\ gs \end{pmatrix}} (E,w)\oplus(B,r) \xrightarrow{\begin{pmatrix} h & py \end{pmatrix}} (C,p)],$$

as required. $\square$

Corollary 5.3.19. Let $\varepsilon = p^*q_*\sigma \in \mathbb{F}((C,p),(A,q))$ and $\delta = t^*q_*\theta \in \mathbb{F}((Z,t),(A,q))$ be $\mathbb{F}$-extensions where

$$\mathfrak{s}(\varepsilon) = [A \xrightarrow{a} B \xrightarrow{b} C] \text{ and } \mathfrak{r}(\varepsilon) = [(A,q) \xrightarrow{aq} (B,r) \xrightarrow{pb} (C,p)]$$

and

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} Y \xrightarrow{y} Z] \text{ and } \mathfrak{r}(\delta) = [(A,q) \xrightarrow{xq} (Y,s) \xrightarrow{ty} (Z,t)].$$

Suppose we have the following diagram, where the left square commutes i.e. $uaq = xq$.

$$\begin{tikzcd}
(A,q) \arrow[r, "aq"] \arrow[d, equal] & (B,r) \arrow[r, "pb"] \arrow[d, "u"] & (C,p) \arrow[r, dashed, "\varepsilon"] & {} \\
(A,q) \arrow[r, "xq"] & (Y,s) \arrow[r, "ty"] & (Z,t) \arrow[r, dashed, "\delta"] & {}
\end{tikzcd}$$

Then there exists a morphism $w\colon (C,p) \to (Z,t)$ in $\tilde{\mathcal{C}}$ such $wpb = tyu, w^*\delta = \varepsilon$ and that the following is an $\mathbb{F}$-triangle,

$$(B,r) \xrightarrow{\begin{pmatrix} -pb \\ u \end{pmatrix}} (C,p) \oplus (Y,s) \xrightarrow{\begin{pmatrix} w & ty \end{pmatrix}} (Z,t) \overset{(aq)_*\delta}{\dashrightarrow} .$$

Proof. This statement is the analogue of the statement of (Herschend *et al.*, 2021, Proposition 3.6) (Corollary 2.1.74). The proof of (Herschend *et al.*, 2021, Proposition 3.6) is a consequence of (ET2) and Proposition 2.1.73, or axioms (R0) and (EA2) respectively in the language of Herschend *et al.* (2021). We have shown that $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies (ET2) in Proposition 5.3.13 and Proposition 5.3.18 shows that $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies Proposition 2.1.73. Hence the statement of the corollary follows, by using an argument as in Herschend *et al.* (2021). $\square$

So far we have shown that the triple $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies the axioms (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$. We are now in a position to prove axioms (ET4) and (ET4)$^{\mathrm{op}}$.

Proposition 5.3.20. The triple $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ satisfies the axioms (ET4) and (ET4)$^{\mathrm{op}}$.

Proof. Let $(D,p), (A,q), (F,t)$ and (B,r) be objects in $\tilde{\mathcal{C}}$ and let $\delta \in \mathbb{F}((D,p),(A,q))$ and $\delta' \in \mathbb{F}((F,t),(B,r))$ be $\mathbb{F}$-extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{f'} D],$$

and

$$\mathfrak{s}(\delta') = [B \xrightarrow{g} C \xrightarrow{g'} F]$$

in the extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Then by definition

$$\mathfrak{r}(\delta) = [(A,q) \xrightarrow{fq} (B,r) \xrightarrow{pf'} (D,p)],$$

for some idempotent $r\colon B \to B$ where

$$fq = rf \text{ and } pf' = f'r \tag{5.18}$$

and

$$\mathfrak{r}(\delta') = [(B,r) \xrightarrow{gr} (C,s) \xrightarrow{tg'} (F,t)],$$

for some idempotent $s\colon C \to C$ where

$$gr = sg \text{ and } tg' = g's. \tag{5.19}$$

We must show that there exists an object $(E,w) \in \tilde{\mathcal{C}}$, an $\mathbb{F}$-extension $\delta'' \in \mathbb{F}((E,w),(A,q))$ such that the following diagram commutes,

$$\begin{tikzcd}
(A,q) \arrow[r, "fq"] \arrow[d, equal] & (B,r) \arrow[r, "pf'"] \arrow[d, "gr"] & (D,p) \arrow[d, "\bar{d}"] \\
(A,q) \arrow[r, "hq"] & (C,s) \arrow[r, "wh'"] \arrow[d, "tg'"] & (E,w) \arrow[d, "\bar{e}"] \\
& (F,t) \arrow[r, equal] & (F,t)
\end{tikzcd}$$

and that the following compatibilities hold,

(1) $\mathfrak{r}((pf')_*\delta') = [(D,p) \xrightarrow{\bar{d}} (E,w) \xrightarrow{\bar{e}} (F,t)]$.

(2) $(\bar{d})^*\delta'' = \delta$.

(3) $(fq)_*\delta'' = (\bar{e})^*\delta'$.

Since $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is extriangulated we can apply (ET4) to the above $\mathbb{E}$-triangles to get an object E in $\mathcal{C}$, a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[d, "d"] \\
A \arrow[r, "h"] & C \arrow[r, "h'"] \arrow[d, "g'"] & E \arrow[d, "e"] \\
& F \arrow[r, equal] & F
\end{tikzcd} \tag{5.20}$$

in $\mathcal{C}$ and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E,A)$ where

$$\mathfrak{s}(\delta'') = [A \xrightarrow{h} C \xrightarrow{h'} E]$$

such that the following compatibilities are satisfied:

(i) $\mathfrak{s}((f')_*\delta') = [D \xrightarrow{d} E \xrightarrow{e} F]$.

(ii) $d^*\delta'' = \delta$.

(iii) $f_*\delta'' = e^*\delta'$.

Recall that since $\delta' \in \mathbb{F}((F,t),(B,r))$, then by definition $\delta' = t^*r_*\varepsilon'$ for some $\varepsilon' \in \mathbb{E}(F,B)$. Also recall that $pf' = f'r$ by (5.18), so we have that

$$f'_*\delta' = f'_*t^*r_*\varepsilon' = t^*f'_*r_*\varepsilon' = t^*(f'r)_*\varepsilon' = t^*(pf')_*\varepsilon' = t^*p_*(f'_*\varepsilon').$$

In other words $f'_*\delta' \in \mathbb{F}((F,t),(D,p))$ and so we have by definition that

$$\mathfrak{r}(f'_*\delta') = [(D,p) \xrightarrow{dp} (E,v) \xrightarrow{te} (F,t)], \tag{5.21}$$

where $v\colon E \to E$ is an idempotent such that

$$dp = vd \text{ and } te = ev. \tag{5.22}$$

Now consider the element $\delta'' \in \mathbb{E}(E,A)$. We are going to show that $\delta'' \in \mathbb{F}((E,v),(A,q))$. Note that by the compatibility (iii), we have that $f_*\delta'' = e^*\delta'$. Recall the relations $fq = rf, te = ev$ and $t^*\delta' = \delta'$ by (5.18) and (5.22). We can then observe that

$$(fq)_*\delta'' = (rf)_*\delta'' = r_*f_*\delta'' = r_*e^*t^*\delta'$$

$$= r_*(te)^*\delta' = r_*(ev)^*\delta' = r_*v^*(e^*\delta') \in \mathbb{F}((E,v),(B,r)).$$

Consider the $\mathbb{F}$-triangle,

$$(A,q) \xrightarrow{fq} (B,r) \xrightarrow{pf'} (D,p) \overset{\delta}{\dashrightarrow} ,$$

and the following diagram arising from it.

$$\begin{tikzcd}
 & \mathbb{F}((E,v),(A,q)) \arrow[r, "(fq)_*"] \arrow[d, hook] & \mathbb{F}((E,v),(B,r)) \arrow[r, "(pf')_*"] \arrow[d, hook] & \mathbb{F}((E,v),(D,p)) \arrow[d, hook] \\
\tilde{\mathcal{C}}((E,v),(D,p)) \arrow[ur, bend left, "\delta_\#"] \arrow[r, "\delta_\#"] & \mathbb{E}(E,A) \arrow[r, "(fq)_*"] & \mathbb{E}(E,B) \arrow[r, "(pf')_*"] & \mathbb{E}(E,D)
\end{tikzcd}$$

Note that the vertical inclusion maps are due to the fact that the $\mathbb{F}$-extension groups are subgroups of the respective $\mathbb{E}$-extension groups and the diagram commutes. By Lemma 5.3.15, the top row is exact in Ab. Moreover, the sequence obtained by appending the morphism $\delta_\#\colon \tilde{\mathcal{C}}((E,v),(D,p)) \to \mathbb{F}((E,v),(A,q))$ to the top row is exact by Proposition 2.1.71 and Lemma 5.3.15.

Observe that $(fq)_*\delta'' \in \mathbb{F}((E,v),(B,r))$ and

$$(pf')_*((fq)_*\delta'') = p_*(f'f)_*q_*\delta'' = p_*0_*q_*\delta'' = 0.$$

That is to say, $(fq)_*\delta''$ is in the kernel of $(pf')_*$ in the top row. So there exists an $\mathbb{F}$-extension $\sigma \in \mathbb{F}((E,v),(A,q))$ such that $(fq)_*\sigma = (fq)_*\delta''$, so $(fq)_*(\sigma - \delta'') = 0$. That is to say $\sigma - \delta''$ is in the kernel of $(fq)_*$ in the top row, therefore there exists $k \in \tilde{\mathcal{C}}((E,v),(D,p))$ such that $\delta_\#(k) = \sigma - \delta''$. In other words $\delta'' = \sigma - \delta_\#(k) \in \mathbb{F}((E,v),(A,q))$ as required.

Consider the solid part of the following diagram.

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "q"] & C \arrow[r, "h'"] \arrow[d, "s"] & E \arrow[r, dashed, "\delta''"] \arrow[d, dashed, "w"] & {} \\
A \arrow[r, "h"] & C \arrow[r, "h'"] & E \arrow[r, dashed, "\delta''"] & {}
\end{tikzcd} \tag{5.23}$$

By (5.19) we have that,

$$sh = s(gf) = (sg)f = g(rf) = (gf)q = hq,$$

therefore the solid square commutes, so by axiom (ET3) for $\mathcal{C}$, there exists a morphism $u\colon E \to E$ such that $(q,u)\colon \delta'' \to \delta''$ is a morphism of $\mathbb{E}$-extensions realised by (q,s,u). Since q and s are idempotent, it follows from Lemma 5.3.6

that there exists an idempotent morphism $w\colon E \to E$ such that $(q,w)\colon \delta'' \to \delta''$ is a morphism of $\mathbb{E}$-extensions and diagram (5.23) commutes. As

$$\delta'' = q_*\delta'' = w^*\delta'' = w^*q_*\delta'',$$

we have that $\delta'' \in \mathbb{F}((E,w),(A,q))$, and since $s\colon C \to C$ is an idempotent such that

$$sh = hq \text{ and } wh' = h's,$$

and furthermore

$$\mathfrak{s}(\delta'') = [A \xrightarrow{h} C \xrightarrow{h'} E],$$

we have that

$$\mathfrak{r}(\delta'') = [(A,q) \xrightarrow{hq} (C,s) \xrightarrow{wh'} (E,w)].$$

Applying Corollary 5.3.19 to the following solid commutative diagram,

$$\begin{tikzcd}
(A,q) \arrow[r, "fq"] \arrow[d, equal] & (B,r) \arrow[r, "pf'"] \arrow[d, "gr"] & (D,p) \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\bar{d}"] & {} \\
(A,q) \arrow[r, "hq"] & (C,s) \arrow[r, "wh'"] & (E,w) \arrow[r, dashed, "\delta''"] & {}
\end{tikzcd} \tag{5.24}$$

we get a morphism $\bar{d}\colon (D,p) \to (E,w)$ such that $\bar{d} \circ pf' = wh' \circ gr, \bar{d}^*\delta'' = \delta'$ and that

$$(B,r) \xrightarrow{\begin{pmatrix} -pf' \\ gr \end{pmatrix}} (D,p) \oplus (C,s) \xrightarrow{\begin{pmatrix} \bar{d} & wh' \end{pmatrix}} (E,w) \overset{(fq)_*\delta''}{\dashrightarrow},$$

is an $\mathbb{F}$-triangle.

Applying Corollary 5.3.19 to the following solid commutative diagram,

$$\begin{tikzcd}
(B,r) \arrow[r, "\begin{pmatrix} -pf' \\ gr \end{pmatrix}"] \arrow[d, equal] & (D,p)\oplus(C,s) \arrow[r, "\begin{pmatrix} \bar{d} & wh' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] & (E,w) \arrow[r, dashed, "(fq)_*\delta''"] \arrow[d, dashed, "\bar{e}"] & {} \\
(B,r) \arrow[r, "gr"] & (C,s) \arrow[r, "tg'"] & (F,t) \arrow[r, dashed, "\delta'"] & {}
\end{tikzcd} \tag{5.25}$$

we get a morphism $\bar{e}\colon (E,w) \to (F,t)$ such that $\bar{e}\circ\begin{pmatrix} \bar{d} & wh' \end{pmatrix} = tg'\circ\begin{pmatrix} 0 & 1 \end{pmatrix}$, $\bar{e}^*\delta' = (fq)_*\delta''$ and that,

$$(D,p)\oplus(C,s) \xrightarrow{\begin{pmatrix} -\bar{d} & -wh' \\ 0 & 1 \end{pmatrix}} (E,w)\oplus(C,s) \xrightarrow{\begin{pmatrix} \bar{e} & tg' \end{pmatrix}} (F,t) \overset{\begin{pmatrix} -pf' \\ gr \end{pmatrix}^{*}\delta'}{\dashrightarrow},$$

is an $\mathbb{F}$-triangle.

By Proposition 2.1.69 applied to the above $\mathbb{F}$-triangle, the following is an $\mathbb{F}$-triangle.

$$(D,p) \xrightarrow{-\bar{d}} (E,w) \xrightarrow{\bar{e}} (F,t) \overset{(-pf')_*\delta'}{\dashrightarrow}$$

This $\mathbb{F}$-triangle is isomorphic using the triple $(-1_{(D,p)}, 1_{(E,w)}, 1_{(F,t)})$.

$$(D,p) \xrightarrow{\bar{d}} (E,w) \xrightarrow{\bar{e}} (F,t) \overset{(pf')_*\delta'}{\dashrightarrow}, \tag{5.26}$$

Hence, by Corollary 2.1.67, we have that (5.26) is an $\mathbb{F}$-triangle, so

$$\mathfrak{r}((pf')_*\delta') = [(D,p) \xrightarrow{\bar{d}} (E,w) \xrightarrow{\bar{e}} (F,t)].$$

Now consider the following diagram.

$$\begin{tikzcd}
(A,q) \arrow[r, "fq"] \arrow[d, equal] & (B,r) \arrow[r, "pf'"] \arrow[d, "gr"] & (D,p) \arrow[d, "\bar{d}"] \\
(A,q) \arrow[r, "hq"] & (C,s) \arrow[r, "wh'"] \arrow[d, "tg'"] & (E,w) \arrow[d, "\bar{e}"] \\
 & (F,t) \arrow[r, equal] & (F,t)
\end{tikzcd} \tag{5.27}$$

Using the relations arising from the commutative diagram (5.24), we have that the top squares of the above diagram commute and that $\bar{d}^*\delta'' = \delta$. From the relations arising from the commutative diagram (5.25) we have that the bottom right square of the above diagram commutes and that $(fq)_*\delta'' = \bar{e}^*\delta'$.

To conclude we have shown that there exists an object $(E, w) \in \tilde{\mathcal{C}}$, an $\mathbb{F}$-extension $\delta'' \in \mathbb{F}((E, w), (A, q))$ such that the following diagram commutes,

$$\begin{tikzcd}
(A,q) \arrow[r, "fq"] \arrow[d, equal] & (B,r) \arrow[r, "pf'"] \arrow[d, "gr"] & (D,p) \arrow[d, "\bar{d}"] \\
(A,q) \arrow[r, "hq"] & (C,s) \arrow[r, "wh'"] \arrow[d, "tg'"] & (E,w) \arrow[d, "\bar{e}"] \\
& (F,t) \arrow[r, equal] & (F,t)
\end{tikzcd}$$

and that the following compatibilities hold.

(1) $\mathfrak{r}((pf')_*\delta') = [(D, p) \xrightarrow{\bar{d}} (E, w) \xrightarrow{\bar{e}} (F, t)]$.

(2) $(\bar{d})^*\delta'' = \delta$.

(3) $(fq)_*\delta'' = (\bar{e})^*\delta'$.

This completes the proof of (ET4). The proof of (ET4)$^{\text{op}}$ is dual. $\square$

Having shown that $(\tilde{\mathcal{C}}, \mathbb{E}, \mathfrak{r})$ satisfies (ET4) and (ET4)$^{\text{op}}$, we can now conclude that $(\tilde{\mathcal{C}}, \mathbb{E}, \mathfrak{r})$ is an extriangulated category. Recall that there is a fully faithful additive functor $i_{\mathcal{C}} : \mathcal{C} \to \tilde{\mathcal{C}}$ defined as follows. For an object A of $\mathcal{C}$, we have that $i_{\mathcal{C}}(A) = (A, 1_A)$ and for a morphism f in $\mathcal{C}$, we have that $i_{\mathcal{C}}(f) = f$. We will show that this functor is an extriangulated functor in the sense of (Bennett-Tennenhaus & Shah, 2021, Definition 2.31). In particular, the functor $i_{\mathcal{C}}$ preserves the extriangulated structure of $\mathcal{C}$.

Proposition 5.3.21. Let $\mathcal{C}$ be an extriangulated category and $\tilde{\mathcal{C}}$ be its idempotent completion. Then the functor $i_{\mathcal{C}} : \mathcal{C} \to \tilde{\mathcal{C}}$ is an extriangulated functor.

Proof. It is easy to see that the functor $i_{\mathcal{C}}$ is a covariant additive functor. So all that is left is to define a natural transformation

$$\Gamma = \{\Gamma_{(C,A)}\}_{(C,A)\in\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \colon \mathbb{E} \Rightarrow \mathbb{F}(i_{\mathcal{C}}^{\mathrm{op}}-, i_{\mathcal{C}}-)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$, such that for any $\mathbb{E}$-extension δ if

$$\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$$

then

$$\mathfrak{r}(\Gamma_{(Z,X)}(\delta)) = [i_{\mathcal{C}}(A) \xrightarrow{i_{\mathcal{C}}(x)} i_{\mathcal{C}}(B) \xrightarrow{i_{\mathcal{C}}(y)} i_{\mathcal{C}}(C)].$$

First note that by definition, $\mathbb{F}((C,1),(A,1)) = \mathbb{E}(C,A)$. So given a pair of objects A, C in $\mathcal{C}$ we define $\Gamma_{(C,A)} \colon \mathbb{E}(C,A) \to \mathbb{F}((C,1),(A,1))$ by setting $\Gamma_{(C,A)}(\delta) = \delta$ for all $\delta \in \mathbb{E}(C,A)$. Given a morphism $(f,g)\colon (C,A) \to (Z,X)$ in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{E}(C,A) \arrow[r, "\Gamma_{(C,A)}"] \arrow[d, "\mathbb{E}(f,g)"'] & \mathbb{F}((C,1),(A,1)) \arrow[d, "\mathbb{F}(f,g)"] \\
\mathbb{E}(X,Z) \arrow[r, "\Gamma_{(Z,X)}"] & \mathbb{F}((X,1),(Z,1)).
\end{tikzcd}$$

For $\delta \in \mathbb{E}(C,A)$, we have that $\mathbb{E}(f,g)(\delta) = f^*g_*\delta$. Therefore $\Gamma_{(Z,X)}(\mathbb{E}(f,g)(\delta)) = f^*g_*\delta$. On the other hand, $\Gamma_{(C,A)}(\delta) = \delta$ and $\mathbb{F}(f,g)(\delta) = f^*g_*\delta$. So the diagram commutes.

Let A, C be an objects in $\mathcal{C}$ and δ be any extension in $\mathbb{E}(C,A)$. Then by definition, we have that $\Gamma_{(C,A)}(\delta) = \delta \in \mathbb{F}((C,1),(A,1))$. Suppose

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C].$$

Then

$$\begin{aligned}
\mathfrak{r}(\Gamma_{(C,A)}(\delta)) &= [(A,1) \xrightarrow{x} (B,1) \xrightarrow{y} (C,1)] \\
&= [i_{\mathcal{C}}(A) \xrightarrow{i_{\mathcal{C}}(x)} i_{\mathcal{C}}(B) \xrightarrow{i_{\mathcal{C}}(y)} i_{\mathcal{C}}(C)].
\end{aligned}$$

So we conclude that $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$ is in fact an extriangulated functor as required. $\square$

Theorem 5.3.22. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let $\tilde{\mathcal{C}}$ be the idempotent completion of $\mathcal{C}$. Then $\tilde{\mathcal{C}}$ is extriangulated. Moreover, the embedding $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$ is an extriangulated functor.

Proof. This follows from the following Propositions 5.3.3, 5.3.4, 5.3.13, 5.3.14, 5.3.20 and 5.3.21. $\square$

In the introduction we made the claim that Theorem 5.3.22 unifies the analogous results for exact categories and triangulated categories. We will clarify this claim starting with the triangulated case.

For the rest of this subsection, let $(\mathcal{C}, \Sigma, \Delta)$ be a triangulated category with an additive category $\mathcal{C}$, shift functor Σ and a collection of distinguished triangles Δ. Balmer and Schlichting showed in (Balmer & Schlichting, 2001, Theorem 1.12) that the idempotent completion of $(\mathcal{C}, \Sigma, \Delta)$ is again a triangulated category $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \tilde{\Delta})$. By (Nakaoka & Palu, 2019a, Proposition 3.22), the triangulated category $(\mathcal{C}, \Sigma, \Delta)$ may be viewed as an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, therefore from this viewpoint the idempotent completion is also an extriangulated category $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ by Theorem 5.3.22. We will show that $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ has the structure of a triangulated category $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \Theta)$, and that this triangulated structure coincides with the triangulated structure $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \tilde{\Delta})$.

We start by recalling how the triangulated structure $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \tilde{\Delta})$ is defined in (Balmer & Schlichting, 2001, Definition 1.10). The shift functor $\Sigma \colon \mathcal{C} \to \mathcal{C}$ induces the functor $\tilde{\Sigma} \colon \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ as follows. For an object (A, e), we have that $\tilde{\Sigma}(A, e) := (\Sigma A, \Sigma e)$, and for a morphism $f \colon (X, q) \to (Y, p)$, we have that $\tilde{\Sigma}(f) := \Sigma(f) \colon (\Sigma X, \Sigma q) \to (\Sigma Y, \Sigma p)$. Elements of $\tilde{\Delta}$ are the following sequences of morphisms.

Definition 5.3.23. (Balmer & Schlichting, 2001, Definition 1.10) A sequence of morphisms

$$t \colon (A, q) \xrightarrow{x} (B, r) \xrightarrow{y} (C, p) \xrightarrow{\delta} (\Sigma A, \Sigma q)$$

is a distinguished triangle in $\tilde{\Delta}$ if there exists a sequence of morphisms

$$t'\colon (A',q') \xrightarrow{x'} (B',r') \xrightarrow{y'} (C',p') \xrightarrow{\delta'} (\Sigma A', \Sigma q')$$

such that $t \oplus t'$ is isomorphic to the image of a distinguished triangle in Δ under the embedding $i_{\mathcal{C}}\colon \mathcal{C} \to \tilde{\mathcal{C}}$.

By (Balmer & Schlichting, 2001, Theorem 1.12), the triple $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \tilde{\Delta})$ is a triangulated category, in particular $\tilde{\Sigma}$ is an auto-equivalence on $\tilde{\mathcal{C}}$.

Now let us recall how the extriangulated structure $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is defined in (Nakaoka & Palu, 2019a, Proposition 3.22).

- For any objects $A, C \in \mathcal{C}$, the biadditive functor $\mathbb{E}$ is such that $\mathbb{E}(C,A) := \mathcal{C}(C, \Sigma A)$.

  For $\delta \in \mathbb{E}(C,A)$, $a \in \mathcal{C}(A,A')$, and $c \in \mathcal{C}(C',C)$, we have that

  $$a_*\delta := \Sigma a \circ \delta \text{ and } c^*\delta := \delta \circ c.$$

- For an extension $\delta \in \mathbb{E}(C,A) = \mathcal{C}(C, \Sigma A)$ with a distinguished triangle

  $$t\colon A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \Sigma A$$

  δ is realised as

  $$\mathfrak{s}(\delta) := [A \xrightarrow{x} B \xrightarrow{y} C].$$

Now let us consider the extriangulated category $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$.

Lemma 5.3.24. $\mathbb{F}(-,-) = \tilde{\mathcal{C}}(-, \tilde{\Sigma}-)$.

Proof. Let (C,p) and (A,q) be any pair of objects in $\tilde{\mathcal{C}}$. By definition

$$\mathbb{F}((C,p),(A,q)) := p^*q_*\mathbb{E}(C,A) = \{p^*q_*\delta \mid \delta \in \mathbb{E}(C,A)\}.$$

By the definition of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ above, we have that

$$\mathbb{F}((C,p),(A,q)) = p^*q_*\mathcal{C}(C, \Sigma A) = \{\Sigma q \circ \delta \circ p \mid \delta \in \mathcal{C}(C, \Sigma A\}.$$

From this it can be observed that that $\mathbb{F}((C,p),(A,q)) \subseteq \tilde{\mathcal{C}}((C,p),\tilde{\Sigma}(A,q))$.

We will now show the opposite inclusion. Take a morphism f in $\tilde{\mathcal{C}}((C,p),\tilde{\Sigma}(A,q))$, then by definition f is a morphism in $\mathcal{C}(C,\Sigma A)$ such that $\Sigma q \circ f = f = f \circ p$, from this it may be observed that $f = \Sigma q \circ f \circ p$, therefore $f \in \mathbb{F}((C,p),(A,q)$, hence $\tilde{\mathcal{C}}((C,p),\tilde{\Sigma}(A,q)) \subseteq \mathbb{F}((C,p),(A,q))$. We conclude that

$$\mathbb{F}((C,p),(A,q)) = \tilde{\mathcal{C}}((C,p),\tilde{\Sigma}(A,q)).$$

For any morphisms $f\colon (X,q) \to (Y,p)$, $g\colon (U,e) \to (V,i)$, and $\mathbb{F}$-extension $\delta\colon (Y,p) \to (\Sigma U, \Sigma e)$,

$$\mathbb{F}(f,g)(\delta) = f^* g_* \delta = \Sigma g \circ \delta \circ f,$$

we also have that

$$\tilde{\mathcal{C}}(f,\Sigma g)(\delta) = \Sigma g \circ \delta \circ f.$$

Therefore $\mathbb{F}(-,-) = \tilde{\mathcal{C}}(-,\tilde{\Sigma}-)$ as required. $\square$

The category $(\tilde{\mathcal{C}},\mathbb{F},\mathfrak{r})$ is extriangulated with auto-equivalence $\tilde{\Sigma}$, such that $\mathbb{F}(-,-) = \tilde{\mathcal{C}}(-,\Sigma-)$, so by (Nakaoka & Palu, 2019a, Proposition 3.22) it has has the structure of a triangulated category $(\tilde{\mathcal{C}},\tilde{\Sigma},\Theta)$, where Θ is defined as follows. The collection Θ is defined to be collection of the sequences

$$(A,q) \xrightarrow{xq} (B,r) \xrightarrow{py} (C,p) \xrightarrow{\delta} (\Sigma A, \Sigma q).$$

for which

$$\mathfrak{r}(\delta) = [(A,q) \xrightarrow{xq} (B,r) \xrightarrow{py} (C,p)].$$

We will show that the collections Θ and $\tilde{\Delta}$ are equal, thus confirming that the triangulated structure of $(\tilde{\mathcal{C}},\mathbb{F},\mathfrak{r})$ which is given as $(\tilde{\mathcal{C}},\tilde{\Sigma},\Theta)$ coincides with the triangulated structure of $(\tilde{\mathcal{C}},\tilde{\Sigma},\tilde{\Delta})$.

Lemma 5.3.25. For any $\mathbb{F}$-triangle $((A,q) \xrightarrow{xq} (B,r) \xrightarrow{py} (C,p),\delta)$ in $(\tilde{\mathcal{C}},\mathbb{F},\mathfrak{r})$ there exists an $\mathbb{F}$-triangle $((A',q') \xrightarrow{x'q'} (B',r') \xrightarrow{p'y'} (C',p'),\delta')$ such that their direct sum is isomorphic to the image of an $\mathbb{E}$-triangle in $(\mathcal{C},\mathbb{E},\mathfrak{s})$ under the embedding $i_{\mathcal{C}}\colon \mathcal{C} \to \tilde{\mathcal{C}}$.

Proof. Recall that for any object $(X, w) \in \tilde{\mathcal{C}}$, we have that

$$(X, w) \oplus (X, 1-w) := (X \oplus X, w \oplus (1-w)) \cong (X, 1);$$

see for example (Thomason & Trobaugh, 1990, A.9.1). Consider an $\mathbb{F}$-extension $\delta = p^* q_* \varepsilon = \Sigma q \circ \varepsilon \circ p$ where $\varepsilon \in \mathbb{E}(C, A)$, and $p \colon C \to C, q \colon A \to A$ are idempotent morphisms. Suppose that δ has an $\mathbb{F}$-triangle

$$(A, q) \xrightarrow{xq} (B, r) \xrightarrow{py} (C, p) \overset{\delta}{\dashrightarrow} .$$

Recall from Definition 5.3.8, that we have the $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} .$$

Now consider the following diagram.

$$\begin{tikzcd}
(A \oplus A, q \oplus (1-q)) \arrow[r, "xq \oplus x(1-q)"] \arrow[d, "\begin{pmatrix} q & (1-q) \end{pmatrix}"] & (B \oplus B, r \oplus (1-r)) \arrow[r, "py \oplus (1-p)y"] \arrow[d, "\begin{pmatrix} r & (1-r) \end{pmatrix}"] & (C \oplus C, p \oplus (1-p)) \arrow[d, "\begin{pmatrix} p & (1-p) \end{pmatrix}"] \\
(A, 1) \arrow[r, "x"] & (B, 1) \arrow[r, "y"] & (C, 1)
\end{tikzcd}$$

The vertical morphisms are isomorphisms, for example, by calculation we can observe that the morphisms

$$\begin{pmatrix} q & (1-q) \end{pmatrix} : (A \oplus A, q \oplus (1-q)) \longrightarrow (A, 1)$$

and

$$\begin{pmatrix} q \\ 1-q \end{pmatrix} : (A, 1) \longrightarrow (A \oplus A, q \oplus (1-q))$$

are mutual inverses. Making use of the fact that $rx = xq$ and $yr = py$ (see Definition 5.3.8), it can also be observed that the squares of the diagram commute. It can be further observed that the bottom row of morphisms give an $\mathbb{F}$-triangle

$$(A, 1) \xrightarrow{x} (B, 1) \xrightarrow{y} (C, 1) \overset{\delta}{\dashrightarrow} .$$

Since the diagram commutes, and the vertical morphisms are isomorphisms, by Corollary 2.1.67 the top row of morphisms realises the $\mathbb{F}$-extension,

$$\begin{pmatrix} q \\ 1-q \end{pmatrix}_* \begin{pmatrix} p & (1-p) \end{pmatrix}^* \delta = \begin{pmatrix} \Sigma q \\ \Sigma(1-q) \end{pmatrix} \circ \delta \circ \begin{pmatrix} p & (1-p) \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} = \delta \oplus 0,$$

hence we have the $\mathbb{F}$-triangle

$$(A \oplus A, q \oplus (1-q)) \xrightarrow{xq \oplus x(1-q)} (B \oplus B, r \oplus (1-r)) \xrightarrow{py \oplus (1-p)y} (C \oplus C, p \oplus (1-p)) \overset{\delta \oplus 0}{\dashrightarrow} . \tag{5.28}$$

Since (5.28) is an $\mathbb{F}$-triangle, by Proposition 2.1.68, its direct summands are also $\mathbb{F}$-triangles, namely,

$$(A, 1-q) \xrightarrow{x(1-q)} (B, 1-r) \xrightarrow{(1-p)y} (C, 1-p) \overset{0}{\dashrightarrow}$$

is an $\mathbb{F}$-triangle. This concludes the proof.

□

Proposition 5.3.26. The triangulated structure of $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \Theta)$ coincides with that of $(\tilde{\mathcal{C}}, \tilde{\Sigma}, \tilde{\Delta})$, that is to say $\Theta = \tilde{\Delta}$.

Proof. Recall that the distinguished triangles of $\tilde{\Delta}$ are direct summands of images of distinguished triangles in Δ under the embedding $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$ (see Definition 5.3.23). It follows from Lemma 5.3.25 and the definition of Θ that the distinguished triangles of Θ are direct summands of images of distinguished triangles in Δ under the embedding $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$, hence $\Theta \subseteq \tilde{\Delta}$.

We will now show the opposite inclusion. Consider a triangle in $\tilde{\Delta}$

$$t \colon (A, q) \xrightarrow{a} (B, r) \xrightarrow{b} (C, p) \xrightarrow{\delta} (\Sigma A, \Sigma q).$$

By the definition of $\tilde{\Delta}$, there exists $t' \in \tilde{\Delta}$ such that $t \oplus t' \cong i_{\mathcal{C}}(\tau)$ where τ is a distinguished triangle in $(\mathcal{C}, \Sigma, \Delta)$. Suppose that

$$t' \colon (A', q') \xrightarrow{a'} (B', r') \xrightarrow{b'} (C', p') \xrightarrow{\delta'} (\Sigma A', \Sigma q'),$$

and

$$i_{\mathcal{C}}(\tau)\colon (X,1) \xrightarrow{x} (Y,1) \xrightarrow{y} (Z,1) \xrightarrow{\varepsilon} (\Sigma X,1),$$

we then have the following isomorphism of triangles in $(\mathcal{C},\Sigma,\tilde{\Delta})$

$$\begin{tikzcd}
(A\oplus A', q\oplus q') \arrow[r, "a\oplus a'"] \arrow[d, "\varphi_1"] & (B\oplus B', r\oplus r') \arrow[r, "b\oplus b'"] \arrow[d, "\varphi_2"] & (C\oplus C', p\oplus p') \arrow[r, "\delta\oplus\delta'"] \arrow[d, "\varphi_3"] & (\Sigma A\oplus \Sigma A', \Sigma q\oplus \Sigma q') \arrow[d, "\Sigma\varphi_1"] \\
(X,1) \arrow[r, "x"] & (Y,1) \arrow[r, "y"] & (Z,1) \arrow[r, "\varepsilon"] & (\Sigma X,1).
\end{tikzcd}$$

Assign to t, t' and $i_{\mathcal{C}}(\tau)$ the pairs

$$((A,q) \xrightarrow{a} (B,r) \xrightarrow{b} (C,p),\delta), \tag{5.29}$$

$$((A',q') \xrightarrow{a'} (B',r') \xrightarrow{b'} (C',p'),\delta'), \tag{5.30}$$

and

$$((X,1) \xrightarrow{x} (Y,1) \xrightarrow{y} (Z,1),\varepsilon) \tag{5.31}$$

respectively.

The isomorphism $(\varphi_1,\varphi_2,\varphi_3)$ induces an isomorphism of pairs (5.29)$\oplus$(5.30) $\cong$ (5.31). It can be observed that (5.31) is an $\mathbb{F}$-triangle because it is the image of a distinguished triangle in Δ. Since (5.29) $\oplus$ (5.30) $\cong$ (5.31), it follows that (5.29) $\oplus$ (5.30) is also an $\mathbb{F}$-triangle by Corollary 2.1.67. By Proposition 2.1.68, it follows that (5.29) is an $\mathbb{F}$-triangle, therefore by the definition of Θ we have that $t \in \Theta$. So we conclude that $\tilde{\Delta} \subseteq \Theta$, therefore $\Theta = \tilde{\Delta}$ as required. $\square$

Let $(\mathcal{A},\mathcal{E})$ be an exact category, with certain smallness conditions on $\mathcal{A}$, the exact category $(\mathcal{A},\mathcal{E})$ can be viewed as an extriangulated category $(\mathcal{A},\mathbf{E}^1,\mathbf{s})$ in an analogous way to triangulated categories; see (Nakaoka & Palu, 2019a, Example 2.13). Conversely, an extriangulated category in which every inflation is a monomorphism, and every deflation is an epimorphism, has the structure of an exact category; see (Nakaoka & Palu, 2019a, Corollary 3.18). Bühler showed in (Bühler, 2010a, Proposition 6.13) that the idempotent completion of $(\mathcal{A}, \mathcal{E})$ is an exact category $(\tilde{\mathcal{A}}, \tilde{\mathcal{E}})$, whereby a sequence of morphisms

$$t\colon (A, q) \xrightarrow{x} (B, r) \xrightarrow{y} (C, p)$$

is an *exact sequence* in $\tilde{\mathcal{E}}$ if there exists a sequence of morphisms

$$t'\colon (A', q') \xrightarrow{x'} (B', r') \xrightarrow{y'} (C', p')$$

such that $t \oplus t'$ is isomorphic to the image of an exact sequence in $\mathcal{E}$ under the embedding $i_{\mathcal{A}}\colon \mathcal{A} \to \tilde{\mathcal{A}}$; see (Bühler, 2010a, Proposition 6.13). By Theorem 5.3.22 the idempotent completion of $(\mathcal{A}, \mathbf{E}^1, \mathbf{s})$ is an extriangulated category $(\tilde{\mathcal{A}}, \mathbf{F}^1, \mathbf{r})$. As in the triangulated case above, we can show that the category $(\tilde{\mathcal{A}}, \mathbf{F}^1, \mathbf{r})$ has an exact structure $(\tilde{\mathcal{A}}, \mathcal{F})$. It is then true that $(\tilde{\mathcal{A}}, \tilde{\mathcal{E}}) = (\tilde{\mathcal{A}}, \mathcal{F})$ by arguments analogous to the ones in the triangulated case shown above.

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