Presentations of strict monoidal categories and strict monoidal categories of welded tangle-oids

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The candidate confirms that the work submitted is her own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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Abstract

In this thesis, we address combinatorial descriptions of *welded knotoids* from the point of view of strict monoidal categories.

To this end, we address combinatorial presentations of strict monoidal categories, by generators and relations. We do this by addressing presentations of a closely related type of categorical object, which we call $\frac{1}{2}$-*monoidal categories* (essentially sesqui-categories on a single object). A key part of the construction relies on the construction of the free $\frac{1}{2}$-monoidal category over what we call a *monoidal graph* (a graph with monoidal structure on the set of vertices). We prove that the category of what we called *slideable* $\frac{1}{2}$-*monoidal categories* is equivalent to the category of strict monoidal categories. We prove that there exists a *slidealisation functor*, sending a $\frac{1}{2}$-monoidal category to a slideable $\frac{1}{2}$-monoidal category. We use this to obtain combinatorial presentations of strict monoidal categories from combinatorial presentations of $\frac{1}{2}$-monoidal categories.

We use this formalism to define presentations of strict monoidal categories of welded tangle-oids, generalising work of Lambropoulou, Turaev, Kaufmann and others on *knotoids*.

Given a finite group G, more generally a finite group acting on a finite abelian group, we construct a functor from the monoidal category of welded tangle-oids to a strictified version of the monoidal category of vector spaces.

Contents

List of abbreviations

$\mathbb{N}$: the set of natural numbers, that contains 0.

$\mathbb{Z}$: the set of integers.

$\mathbb{Z}^+$: the set of positive integers.

$\mathbb{C}(G)$: G is a group, the set of map $f\colon G \to \mathbb{R}$, we write for all $g \in G$, $f(g) = \sum_{g\in G} f_g g$.

$\mathcal{SETS}$: the category of sets. 3.1.10

$\mathcal{SET}$: the class of all sets.

$\hom_{\mathcal{SETS}}(A, B)$: for A, B sets, this denotes the set of functions from A to B.

Mat: the category of matrices. 3.1.2

Top: the category of topological spaces with continuous maps as morphisms. 3.1.11

$\mathcal{M}$: the category of monoids. 4.1.6

$\mathcal{M}_{\text{ab}}$: the category of abelian monoids. 4.1.7

SmC: the category of small categories. 3.2.4

$\mathbf{Graphs}$: the category of graphs. 3.3.3

$\mathbf{\frac{1}{2}-MC}$: the category of $\frac{1}{2}$-monoidal categories. 5.1.13

$\mathbf{s\frac{1}{2}-MC}$: the category of slideable $\frac{1}{2}$-monoidal categories. 5.1.14

$\mathbf{StriMC}$: the category of strict monoidal categories. 5.3.6

$\mathbf{MGraphs}$: the category of monoidal graphs. 6.1.4

$UWTC$: the unoriented welded tangle-oids category. 7.2.2

$OWTC$: the oriented welded tangle-oids category. 7.3.2

Chapter 1

Introduction

Here we briefly put our work into a broader research context.

We will argue that our work fits into a couple of distinct research contexts. 'Presentation' of algebraic structures is a useful tool in areas such as *representation theory* and in *construction* of new examples from old. In *category theory*: Our approach is in the spirit of sesquicategories (see e.g. [Haz96] and references therein) in 'categorification' (making an algebraic structure more categorical - for whatever reason).
In low-dimensional topology: Our approach is in the spirit of the Siefert–Van Kampen Theorem, in the sense of taking infinite geometric-topological systems and combinatorialising by (finite) presentation.

In this thesis, I will address welded knotoids from the point of view of presentations of strict monoidal categories (reviewed in Chapter 5). To this end, I will

- define what it means to present a strict monoidal category (by generators and relations), from the point of view of the more general 1/2-monoidal categories,
- define a strict monoidal category of welded tangle-oids using a presentation. Welded knotoids are morphisms from the unit object to itself,
- construct functorial invariants of welded tangle-oids.

1.0.1 Structure of the thesis.

- We provide the motivation for this work in section 2.
- In section 3, we explain our notation and conventions for categories, particularly free categories, and congruence relations. In section 3.5, we define presentations of categories, and in doing so explain how a relation in the set of morphisms of a category (a *congruence template* Definition 3.5.1), gives rise to a congruence relation called the *closure of the congruence template*, Theorem 3.5.5. We then use this to say what we mean by a *presentation of a category*, Definition 3.5.8. We apply this framework to the construction of the *combinatorial braid category.*
- In section 4, we review monoids and the monoid abelianisation functor. This is in order to prepare for a similar argument when defining the *slidealisation functor* for $\frac{1}{2}$*-monoidal categories*, defined in the following section.
- In section 5, we discuss strict monoidal categories from the point of view of $\frac{1}{2}$-monoidal categories. In Definition 5.1.2, we define what a $\frac{1}{2}$-monoidal category is, and in Definition 5.1.5 we define slideable $\frac{1}{2}$-monoidal categories. In section 5.2.2 we construct our crucial slidealisation functor, from the category of $\frac{1}{2}$-monoidal categories to the category of slideable $\frac{1}{2}$-monoidal categories, and prove that it is a left adjoint to the inclusion functor from the category of slideable $\frac{1}{2}$-monoidal categories to the category of $\frac{1}{2}$-monoidal categories. In section 5.3 we give the definition of a strict monoidal category. In section 5.4 we address the equivalence between slideable $\frac{1}{2}$-monoidal categories and strict monoidal categories, Theorem 5.4.6.
- In section 6 we define free strict monoidal categories over monoidal graphs. We define monoidal graphs R in Definition 6.1.1, these are graphs with a monoid structure on the set of vertices. To each monoidal graph R we associate a graph R^*, which we call the *extent* of the monoidal graph R, in 6.1.2. In Definition 6.1.7, we define *free* $\frac{1}{2}$*-monoidal category-triples*. In proposition 6.1.9, we prove that the free category $P(R^*)$ over the graph R^* naturally becomes a $\frac{1}{2}$-monoidal category denoted $\Omega(R)$. In Proposition 6.1.10, we prove that $\Omega(R)$ is a free $\frac{1}{2}$-monoidal category.

- Then in section 6.2, we address presentations of strict monoidal categories. We first explain how a congruence template in the underling category of a $\frac{1}{2}$-monoidal category gives rise to a $\frac{1}{2}$-*monoidal congruence*, 6.2.1 called the $\frac{1}{2}$-*monoidal closure* of the congruence template: Theorem 6.2.7, Definition 6.2.8. In Definition 6.2.9, we define presentations of $\frac{1}{2}$-monoidal categories. In Definition 6.2.10, we define presentation of strict monoidal categories. In section 6.2.2 we apply this to the construction of the monoidal combinatorial braid category.

- In section 6.3 we sketch the definition of the monoidal category of tangles, because our categories welded tangle-oids can be seen as a generalisation of it.

- In section 7.2 we define the strict monoidal category of *unoriented welded tangle-oids* $UWTC$, Definition 7.2.2. In section 7.3 we define the strict monoidal category of *oriented welded tangle-oids* $OWTC$, Definition 7.3.2. In section 7.4.1 we construct functorial invariants from finite group for the $UWTC$, Theorem 7.4.1. In section 7.4.2 we construct functorial invariants from group acting on abelian group for $UWTC$, Theorem 7.4.3.

Chapter 2

Motivating Ideas

2.1 Presentations of groups.

A presentation of a group is a way to define the group (by set of ‘generators’ and ‘relations’) that is useful when studying group homomorphisms from this group (i.e. when studying representations). The method was introduced by Walther von Dyck [Rob96]. Generalising this idea is a key paradigm for us, so we will review it next.

In this section we assume familiarity with the category of sets — see 3.1.10 for details; and the category of groups (which we also take to imply familiarity with the language of basic group theory and so on).

Here the underlying-set functor from the category of groups, see for example [ML13, page 14], to the category of sets, is denoted U. It is such that if $(G, \circ, e)$ is a group then we have that $U(G, \circ, e) = G$. If $f\colon (G, \circ, e) \to (G', \circ', e')$ is a homomorphism then $U(f)\colon G \to G'$ is given by f, regarded as a function only.

2.1.1 Free group

Definition 2.1.1. *(See for example [Rob96, Rot12, MRR88]). A group* $(G, \circ, e)$ *is a* free group*, on a set* X*, if there is a function* $\delta\colon X \to G$*, that satisfies the following universal* *property.*

Given any group $(A, \bullet, e')$, and any function $f\colon X \to A$, there is a unique group homomorphism $F\colon (G, \circ, e) \to (A, \bullet, e')$ that makes the diagram below, in the category of sets, commute:

$$\begin{array}{ccc} X & \xrightarrow{\delta} & U(G, \circ, e) \\ & \searrow^{f} & \big\downarrow{\scriptstyle U(F)} \\ & & U(A, \bullet, e') \end{array}$$

An important point about such a G (if it exists — see below) is that a representation of it in A (i.e. a homomorphism to A) is determined by the image of X. We do not need to give the image of every element of G.

Given a set X, it is not directly clear that a free group on X exists (or if multiple non-isomorphic free groups exist). And when we generalise later this will be even less clear. So let us have one example.

Example 2.1.2. *(See for example [Rot12]). We claim that $(\mathbb{Z}, +, 0)$ is a free group on the set $X = \{1\}$.*
To see this we may proceed as follows. Given a group $(A, \bullet, e')$, and

$$f\colon \{1\} \to U(A, \bullet, e'),$$

the next diagram commutes:

$$\begin{array}{ccc} \{1\} & \xrightarrow{\delta} & U(\mathbb{Z}, +, 0) \\ & \searrow^{f} & \big\downarrow{\scriptstyle U(F)} \\ & & U(A, \bullet, e') \end{array} \tag{2.1}$$

where:

$$\begin{aligned}\delta\colon \{1\} &\to U(\mathbb{Z},+,0)\\ 1 &\mapsto 1\end{aligned}$$

and F is given as follows.

Explanation of the group homomorphism F:
First let us give F as a set map.

$$\begin{aligned}F\colon \mathbb{Z} &\to A\\ a &\mapsto f(1)^a.\end{aligned}$$

- *Here if $a > 0$, $f(1)^a = \underbrace{f(1) \bullet \ldots \bullet f(1)}_{a \text{ times}}$.*
- *If $a < 0$, $f(1)^a = \underbrace{f(1)^{-1} \bullet \ldots \bullet f(1)^{-1}}_{a \text{ times}}$.*
- *If $a = 0$, $f(1)^a = e'$.*

Finally observe that this map is indeed a group homomorphism.

Clearly the diagram (2.1) *commutes.*

Also F is unique, because for any such group homomorphism $F'\colon (\mathbb{Z},+) \to (A,\bullet)$, then $F'(1) = f(1)$, because the diagram (2.1) *commutes. So the homomorphism gives, for $a > 0$*

$$\begin{aligned}F'(a) &= F'(\underbrace{1 + \cdots + 1}_{a \text{ times}})\\ &= \underbrace{F'(1) \bullet \ldots \bullet F'(1)}_{a \text{ times}}\\ &= \underbrace{f(1) \bullet \ldots \bullet f(1)}_{a \text{ times}}.\end{aligned}$$

If $a < 0$*,*

$$\begin{aligned} F'(a) &= F'(\underbrace{(-1) + \cdots + (-1)}_{a \text{ times}}) \\ &= \underbrace{F'(1)^{-1} \bullet \ldots \bullet F'(1)^{-1}}_{a \text{ times}} \\ &= \underbrace{f(1)^{-1} \bullet \ldots \bullet f(1)^{-1}}_{a \text{ times}}. \end{aligned}$$

If $a = 0$*,*

$$F'(a) = F'(0) = f(1)^0 = e'.$$

Therefore:

$$F = F'.$$

Theorem 2.1.3. *(See for example [Rot12], Theorem 11.1). Given any set* X*, there is a free group* M_X *on X.*

Later on this thesis we will address closely related free monoids, and also free $\frac{1}{2}$-monoidal categories.

Proposition 2.1.4. *(See for example [Rot12], Corollary 11.2.). Every group* G *is a quotient of a free group.*

2.1.2 Presentation of groups

Definition 2.1.5. *(See for example [Rob96, Rot12]). Let* S *be a set, and* F *a free group on* S*. We write*

$$\langle S|R\rangle,$$

for a pair consisting of S *and a subset* $R \subset F$*. We will call the elements of* R *'relations'. This pair is a 'presentation' of a group* G *if*

$$G \cong F/\sim,$$

where $\sim$ the normal closure of $R \subset F$, i.e. the intersection of all normal subgroups of F that contains the relation set R.

Example 2.1.6. *The cyclic group of order n has the presentation:*

$$\langle x \mid x^n = e \rangle.$$

Here and after we abused notation and instead of putting $\langle \{x\} \mid \{x^n\} \rangle$ we write $\langle x \mid x^n = e \rangle$.

2.1.3 Presentation of braid groups

In this section we define braid groups in two different ways one by using geometric braids, and another by using Artin's presentation. We assume familiarity with the notion of isotopy, as described for example in [Kas12, JM19].

Definition 2.1.7. *(See for example [Oht02, Chapter 2], and [KT08, Lie11]). A* geometric braid *on $n \in \mathbb{N}$ strings is a set $B \subset \mathbb{R}^2 \times [0,1]$ that is composed of n disjoint topological intervals (where a topological interval is the image of injective continuous map from the unit interval into $\mathbb{R}^3$) such that the following are satisfied:*

1. $B \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), \ldots, (n,0,0)\}$,
2. $B \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), (2,0,1), \ldots, (n,0,1)\}$,
3. *If $t \in [0,1]$, then $B \cap (\mathbb{R}^2 \times \{t\})$ has cardinality n.*

Definition 2.1.8. *(See for example [KT08]). Two geometric braids $B_1, B_2 \subset \mathbb{R}^2 \times [0,1]$ are called* ambient isotopic *if there is a continuous map:*

$$h\colon (\mathbb{R}^2 \times [0,1]) \times [0,1] \to \mathbb{R}^2 \times [0,1],$$

such that

1. *for all $(x,y,z) \in \mathbb{R}^2 \times [0,1]$, $h(x,y,z,0) = (x,y,z)$;*

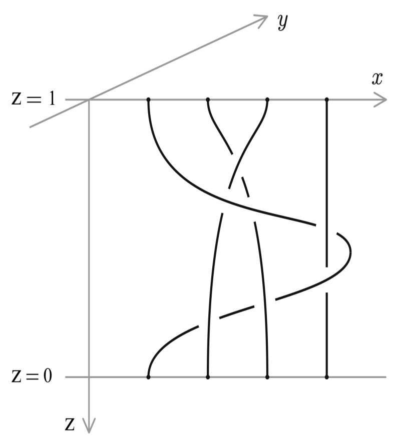
2. *given any* $t \in [0,1]$, *the map*

$$\begin{aligned} \mathbb{R}^2 \times [0,1] &\to \mathbb{R}^2 \times [0,1] \\ (x,y,z) &\mapsto h(x,y,z,t) \end{aligned}$$

is a homeomorphism;

3. *for all* $t \in [0,1]$ *and* $(x,y) \in \mathbb{R}^2$, *we have*
   - $h(x,y,0,t) = (x,y,0)$,
   - $h(x,y,1,t) = (x,y,1)$;
4. $h(B_1, 1) = B_2$;
5. *for all* $t \in [0,1]$, $h(B_1, t)$ *is a geometric braid.*

Example 2.1.9. *([KT08], page 5). A rough sketch of a geometric braid in four strings.*



Proposition 2.1.10 (Geometric braid groups)**.** *(See for example [FN62]).*

Let $n \in \mathbb{N}$, *the geometric braid group* $\mathcal{B}_n = (\mathcal{B}_n, \circ)$ *is such that*

1. $\mathcal{B}_n$ *is the set of equivalence classes under ambient isotopy of geometric braids on* n *strings.*
2. *If* $[\alpha], [\beta] \in \mathcal{B}_n$, *their composition is such that*

$$[\alpha] \circ [\beta] = \left[\left\{(x,y,\frac{z}{2}) \mid (x,y,z) \in \alpha\right\}\right] \cup \left[\left\{(x,y,\frac{z}{2}+\frac{1}{2}) \mid (x,y,z) \in \beta\right\}\right].$$

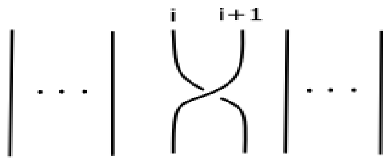
3. *The identity element is the equivalence class of the geometric braid* Id_n *with* n *parallel strings, namely*

$$Id_n = \{(1, 2, \ldots, n)\} \times \{0\} \times [0, 1] \subset \mathbb{R}^2 \times [0, 1].$$

Note that the composition $[\alpha] \circ [\beta]$ does not depend on the choice of representatives, see [Kas12, Section X.6.1]. The proof that $\mathcal{B}_n$ is a group is also in [Kas12, Section X.6.1].

2.1.4 Artin's presentation:

Note that any geometric braid in $\mathcal{B}_n$ can be obtained by multiplying a finite number of $\sigma_1, \ldots, \sigma_{n-1}$, and their inverses. Here σ_i is the equivalence class of a geometric braid as sketched in the figure below.



Definition 2.1.11. *(See for example [KT08, Lie11, Cas10]). Let* $n \geq 2$ *be an integer. The* Artin braid group B_n *is the group formally generated by:*

$$\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\},$$

and relations

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{if } |i - j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & \text{for all } i \in \{1, \ldots, n-1\}. \end{cases}$$

Lemma 2.1.12. *([KT08],Lemma1.2.). Let* G *be a group with elements* $g_1, g_2, \ldots, g_{n-1}$

that satisfy the braid relations, this means

$$\begin{cases} g_i g_j = g_j g_i, & \textit{if } |i-j| \geq 2, \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, & \textit{for all } i \in \{1, \ldots, n-1\}. \end{cases}$$

Then there is a unique group homomorphism $f \colon B_n \to G$*, such that*

$$g_i = f(\sigma_i), \textit{for all } i = 1, 2, \ldots, n-1.$$

Proof. See [KT08]. □

Example 2.1.13. *Consider the cyclic group in example 2.1.6, there is a unique group homomorphism*

$$f \colon B_n \to \langle x \mid x^n = e \rangle,$$

such that

$$f(\sigma_i) = x, \textit{for each } i \in \{1, \ldots, n-1\}.$$

This assignment clearly satisfies the braid relations.

Several other examples arise as particular cases of the invariant of welded tangle-oids in section 7.

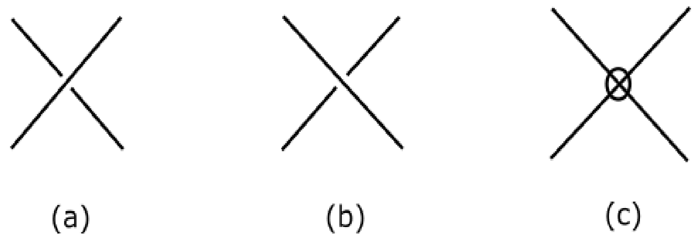
2.2 Presentation of welded braid groups

Definition 2.2.1 (Knot)**.** *(See for example [Car12]). A* knot *is an embedding of the circle into* $\mathbb{R}^3$.

A knot in $\mathbb{R}^3$ can be projected 'regularly' onto $\mathbb{R}^2$. Projection here is onto one of the many possible (oriented) $\mathbb{R}^2$ subspaces of $\mathbb{R}^3$. 'Regular' means that the projection is injective everywhere except at a finite number of crossing points, which are the projections of only two points of the knot (where both points have tangents, and the projected tangents are not colinear — see for example [KMY19]). Keeping track of the positive normal to the subspace $\mathbb{R}^2$, this give the over under crossing information. This data in the projection is called 'classical crossing'.

The algebraic-combinatorial aspects of knot theory can be tracked purely at the level of such projections. The projected formalism is then also amenable to direct generalisation (i.e. without reference to knots in 3d).

In this thesis, we define presentations of categories of *welded tangle-oids* that can be seen as generalizations of welded braid groups. The theory of virtual knots was introduced by L. Kauffman see [Kau21, Kau00]. In the virtual knots there is a new crossing that is not a classical crossing which is virtual crossing. You cannot switch over and under in a virtual crossing. However the idea is not that a virtual crossing is just an ordinary graphical vertex. Rather, the idea is that the virtual crossing is not really there. In the next diagrams: (a) and (b) are classical crossings and (c) a virtual crossing:



The virtual braid group arises naturally in virtual knot theory, see [Kau00, Kau21, KL06]. This group is closely related to the welded braid group that was introduced by R. Fenn, R. Rimányi and C. Rourke [FRR97].

Definition 2.2.2. *[Kam07, Kau00, KL04, KL06, Kau21]. The* virtual braid group *of degree $n \in \mathbb{N}$, VB_n, is the group formally presented in the following way: generators:*

$$\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau_1, \tau_2, \ldots, \tau_{n-1}\},$$

and relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & \textit{if } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \sigma_i\sigma_i^{-1} = 1, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \tau_i^2 = 1, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \tau_i\tau_j = \tau_j\tau_i, & \textit{if } |i-j| \geq 2, \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \sigma_i\tau_j = \tau_j\sigma_i, & \textit{if } |i-j| \geq 2, \\ \sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1}, & \textit{for all } i \in \{1,\ldots,n-1\}. \end{cases}$$

Definition 2.2.3. *[FRR97, Kam07]. The* welded braid group *is the group formally presented in the same was as as the virtual braid group, however adding one more relation. Generators:*

$$\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau_1, \tau_2, \ldots, \tau_{n-1}\},$$

and relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & \textit{if } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \sigma_i\sigma_i^{-1} = 1, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \tau_i^2 = 1, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \tau_i\tau_j = \tau_j\tau_i, & \textit{if } |i-j| \geq 2, \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \sigma_i\tau_j = \tau_j\sigma_i, & \textit{if } |i-j| \geq 2, \\ \sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1}, & \textit{for all } i \in \{1,\ldots,n-1\}, \\ \tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}, & \textit{for all } i = 1,2,\ldots,n-1. \end{cases}$$

Chapter 3

Categories

In this section I will explain the notion of a category. This is a type of algebraic structure that I use heavily later.

3.1 Collections, sets and classes

Sets:
Sets can be thought of as the usual sets of intuitive set theory. One construction that can be performed with sets is the set of all subsets $\mathcal{P}(X)$ of the set X (called the power set of X).

But with this intuition of the sets we can not treat the collection of all sets as a set. As in "Russell's paradox" if we consider the set that contains elements which are not elements of themselves, the collection of all these sets can not be a set. So this lead to define another concept "*Classes*" [AHS04].
Classes:
The concept of "*class*" has been created to deal with large collections of sets. The members of each class are sets, so the member of the class are sets.

A class that is not a set is called a *proper class*, and a class that is a set is sometimes called a *small class* [AHS04].

3.1.1 Small categories

Definition 3.1.1. *A* small precategory $\mathcal{C}$ *is an ordered quadruple consisting of:*

1. *A set* $ob(\mathcal{C})$.
2. *For each pair* $A, B \in ob(\mathcal{C})$ *a set* $\hom_{\mathcal{C}}(A, B)$.
3. *For each triple* $(A, B, C) \in ob(\mathcal{C}) \times ob(\mathcal{C}) \times ob(\mathcal{C})$ *a function called composition*

$$\hom_{\mathcal{C}}(A, B) \times \hom_{\mathcal{C}}(B, C) \to \hom_{\mathcal{C}}(A, C),$$
$$(f, g) \mapsto f \star g.$$

4. *For each* $A \in ob(\mathcal{C})$ *an element* id_A *in* $\hom_{\mathcal{C}}(A, A)$.

Notation. We might write the data giving a small precategory in the form

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_, \_), \star, \mathrm{id}_{\_}).$$

This means that we given a set $ob(\mathcal{C})$; and a function from $ob(\mathcal{C}) \times ob(\mathcal{C})$ to a class of sets; and for every $f \in \hom_{\mathcal{C}}(A, B)$ and $g \in \hom_{\mathcal{C}}(B, C)$ a way to construct an element $f \star g \in \hom_{\mathcal{C}}(A, C)$; and a suitable function id from $ob(\mathcal{C})$.

Example 3.1.2. *In general giving the 'composition' function might be very hard. To get started we can make this a bit easier by using a construction we already have.*

Let $\hom_{Mat}(m, n)$ *denote the set of* $m \times n$ *complex matrices (that is, with* m *rows and* n *columns). Let* $\cdot$ *denote matrix multiplication. Let* id_n *denote the* $n \times n$ *unit matrix. Then consider the quadruple*

$$Mat = (\mathbb{N}, \hom_{Mat}(\_, \_), \cdot, \mathrm{id}_{\_}).$$

If n *or* m *is equal to zero then the only matrix we have is the matrix with no elements.*

Claim. *We claim this gives the data for a small precategory.*

Idea of proof. *It will be clear that the first, second and fourth components are as required, so it remains to check the third component. This is non-trivial, even in this familiar case, since matrix multiplication requires that matrices conform in the given order.*

An example to show the order is correct.

Let $f = \begin{bmatrix} x & x' \\ z & z' \\ w & w' \end{bmatrix} \in hom_{Mat}(3,2)$, *and* $g = \begin{bmatrix} m & n & k & t \\ m' & n' & k' & t' \end{bmatrix} \in hom_{Mat}(2,4)$.

Therefore: $f.g \in hom_{Mat}(3,4)$.

Definition 3.1.3. *(See for example [AHS04, Par70, Lei14, ML13]). A* small category *is a small precategory* $(ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id}_{\_})$ *that satisfies the following axioms*

- *(A1: associativity axiom): for every quadruple* $(A, B, C, D) \in ob(\mathcal{C}) \times ob(\mathcal{C}) \times ob(\mathcal{C}) \times ob(\mathcal{C})$ *and every* $f \in \hom_{\mathcal{C}}(A, B)$*,* $g \in \hom_{\mathcal{C}}(B, C)$ *and* $h \in \hom_{\mathcal{C}}(C, D)$ *we have:*
$$(f \star g) \star h = f \star (g \star h).$$

- *(A2: unit axiom): for every pair* $(A, B) \in ob(\mathcal{C}) \times ob(\mathcal{C})$ *and any* $f \in \hom_{\mathcal{C}}(A, B)$ *we have* $\mathrm{id}_A \star f = f$*, and* $f \star \mathrm{id}_B = f$*.*

Proposition 3.1.4. *The small precategory* Mat *is a small category.*

Proof. To show that precategory Mat is a small category we must show it satisfies the axioms in 3.1.3:

(A1: associativity axiom): for every quadruple $(m, n, s, r) \in \mathbb{N}^4$ and every $f \in \hom_{Mat}(m, n), g \in hom_{Mat}(n, s)$ and $h \in \hom_{Mat}(s, r)$, we have:

$$(f \cdot g) \cdot h,$$

and

$$f \cdot (g \cdot h),$$

and we require to show that these are equal. In particular that $((f.g).h)_{ij}$ and $(f.(g.h))_{ij}$ are equal. These are given by

$$((f.g).h)_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{s}(f_{ik}g_{kl})h_{lj} = \sum_{k=1}^{n}\sum_{l=1}^{s} f_{ik}(g_{kl}h_{lj}),$$

where we used associativity of complex multiplication, and

$$(f.(g.h))_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{s} f_{ik}(g_{kl}h_{lj}).$$

The right hand sides of these are equal.

(A2: unit axiom): for every pair $(m, n) \in \mathbb{N}^2$, and every $f \in \hom_{Mat}(m, n)$ we have:

$$f.\mathrm{id}_m,$$

and

$$\mathrm{id}_n.f,$$

and we requie to show that these are equal to f. In particular that $(\mathrm{id}_m.f)_{ij}$ and $(f.\mathrm{id}_n)_{ij}$ are equal to f_{ij}. These are given by

$$(f.\mathrm{id}_m)_{ij} = \sum_{k=1}^{m} f_{ik}\mathrm{id}_{kj} = f_{ij}\mathrm{id}_{jj} = f_{ij}.$$

(Note: $\mathrm{id}_{kj} = 1$ if $k = j$ and $\mathrm{id}_{kj} = 0$ if $k \neq j$)

$$(\mathrm{id}_n.f)_{ij} = \sum_{k=1}^{n} \mathrm{id}_{ik} f_{kj} = \mathrm{id}_{ii} f_{ij} = f_{ij}.$$

□

Definition 3.1.5 (Precategory)**.** *The definition of* precategory *is the same as the definition of a small precategory but the collection of objects is a class instead of set.*

Definition 3.1.6. *[Categories] A* category *is like a small category, but we relax the requirement that the collection of objects is a set.*

3.1.2 Subcategories

Definition 3.1.7. *(See for example [ML13, Lei14, Awo10]). Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star_{\mathcal{C}}, \mathrm{id}^{\mathcal{C}}_{\_})$$

be a category. A subcategory of $\mathcal{C}$ *is a pair*

$$\mathcal{S} = (ob(\mathcal{S}), \hom_{\mathcal{S}}(\_,\_))$$

consisting of

1. *A subclass* $ob(\mathcal{S})$ *of the class of objects of* $\mathcal{C}$*.*
2. *Given* $(A, B) \in ob(\mathcal{S}) \times ob(\mathcal{S})$ *a subset* $\hom_{\mathcal{S}}(A, B) \subset \hom_{\mathcal{C}}(A, B)$*.*

These are required to satisfy

1. *Given* $f \in \hom_{\mathcal{S}}(A, B)$ *and* $g \in \hom_{\mathcal{S}}(B, C)$*, then* $f \star_C g \in \hom_{\mathcal{S}}(A, C)$ *(here* $\star_C$ *denotes the composition in* $\mathcal{C}$ *).*
2. *For all* $A \in ob(\mathcal{S})$*, it holds that* $\mathrm{id}^{\mathcal{C}}_A \in \hom_{\mathcal{S}}(A, A)$*. (Here* $\mathrm{id}^{\mathcal{C}}_A$ *denotes the identity of* A *in* $ob(\mathcal{C})$*).*

Proposition 3.1.8. *(See for example [ML13, AHS04]). Let* $\mathcal{S} = (ob(S), \hom_{\mathcal{S}}(\_,\_))$ *be a subcategory. Then we have a category*

$$\mathcal{S} = (ob(S), \hom_{\mathcal{S}}(\_,\_), \star_S, \mathrm{id}^{\mathcal{S}}_{\_}),$$

such that

1. *the class of objects is* $ob(\mathcal{S})$*;*
2. *given* $(A, B) \in ob(\mathcal{S}) \times ob(\mathcal{S})$*, we have a set* $\hom_{\mathcal{S}}(A, B)$*;*

3. *given* $f \in \hom_{\mathcal{S}}(A, B)$ *and* $g \in \hom_{\mathcal{S}}(B, C)$*, the composition in* $\mathcal{S}$ *is*

$$f \star_{\mathcal{S}} g = f \star_{\mathcal{C}} g;$$

4. *for all* $A \in ob(\mathcal{S})$*, we have* $\mathrm{id}_A^{\mathcal{C}} = \mathrm{id}_A^{\mathcal{S}}$.

Proof. Let $\mathcal{S}$ be a subcategory of the category $\mathcal{C}$, we want to prove the category axioms:

1. Associativity: for every quadruple

$$(A, B, C, D) \in ob(\mathcal{S}) \times ob(\mathcal{S}) \times ob(\mathcal{S}) \times ob(\mathcal{S}),$$

and morphisms:

$$f \in \hom_{\mathcal{S}}(A, B) \subset \hom_{\mathcal{C}}(A, B),$$

$$g \in \hom_{\mathcal{S}}(B, C) \subset \hom_{\mathcal{C}}(B, C),$$

$$h \in \hom_{\mathcal{S}}(C, D) \subset \hom_{\mathcal{C}}(C, D),$$

we have:

$$f \star_{\mathcal{S}} g = f \star_{\mathcal{C}} g.$$

Therefore:

$$f \star_{\mathcal{S}} (g \star_{\mathcal{S}} h) = (f \star_{\mathcal{S}} g) \star_{\mathcal{S}} h.$$

2. Unit: let

$$(A, B) \in ob(\mathcal{S}) \times ob(\mathcal{S}),$$

$$f \in hom_{\mathcal{S}}(A, B),$$

$$\mathrm{id}_A^{\mathcal{S}} = \mathrm{id}_A^{\mathcal{C}} \text{ and } \mathrm{id}_B^{\mathcal{S}} = \mathrm{id}_B^{\mathcal{C}}.$$

Therefore:

$$f \star_{\mathcal{S}} \mathrm{id}_B^{\mathcal{S}} = f \star_{\mathcal{C}} \mathrm{id}_B^{\mathcal{S}} = f,$$

and

$$\mathrm{id}_A^{\mathcal{S}} \star_{\mathcal{S}} f = \mathrm{id}_A^{\mathcal{S}} \star_{\mathcal{C}} f = f.$$

□

Definition 3.1.9. *(See for example [AHS04]). A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is* full *if for each $X, Y \in \mathcal{B}$, we have:*

$$\hom_{\mathcal{B}}(X, Y) = \hom_{\mathcal{C}}(X, Y).$$

3.1.3 Examples of categories

Example 3.1.10. *(See for example [AHS04]). The precategory*

$$\mathcal{SETS} = (\mathcal{SET}, \hom_{\mathcal{SETS}}(\_,\_), \star, 1_{\_})$$

is a category where

1. *the class $\mathcal{SET}$ is the class of all sets,*
2. *given a pair of objects $(A, B) \in \mathcal{SET} \times \mathcal{SET}$, the set $\hom_{\mathcal{SETS}}(A, B)$ is the set of all functions from A to B,*
3. *for every triple of objects $(A, B, C) \in \mathcal{SET} \times \mathcal{SET} \times \mathcal{SET}$ and every $f \in \hom_{\mathcal{SETS}}(A, B),\ g \in \hom_{\mathcal{SETS}}(B, C)$, we have*

$$f \star g = g \circ f,$$

4. *for all $A \in \mathcal{SET}$, an arrow $1_A \in \hom_{\mathcal{SETS}}(A, A)$ where for all $x \in A$, $1_A(x) = x$.*

Proof. Note that the above quadruple is formally a precategory (with the usual caveat about collections as above). We will show that $\mathcal{SETS}$ satisfies the category axioms.

1. Associative, for all $f : A \to B, g : B \to C, h : C \to D$, $x \in A$, we have:

$$\begin{aligned} h \circ (g \circ f)(x) &= h(g \circ f(x)) \\ &= h(g(f(x))) \\ &= (h \circ g)(f(x)) \\ &= (h \circ g) \circ f(x). \end{aligned}$$

   Therefore

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. Unit, for every pair of objects $(A, B) \in \mathcal{SET} \times \mathcal{SET}$ and any $f : A \to B, x \in A$, we have

$$\begin{aligned} 1_B \circ f(x) = 1_B(f(x)) = f(x), \\ f \circ 1_A(x) = f(1_A(x)) = f(x). \end{aligned}$$

   Therefore

$$1_B \circ f = f = f \circ 1_A.$$

□

Note we will also allow categories also to be formulated 'backwards', so we use notation for composition from the SET convention. So for example we may write that the quadruple $(\mathcal{SET}, hom_{\mathcal{SETS}}(-,-), \circ, 1_{-})$ is a precategory and a category (even though composition is backwards).

Example 3.1.11. *(See for example [AHS04]). The precategory*

$$Top = (ob(Top), \hom_{Top}(\_,\_), \circ, \mathrm{id}_{\_})$$

is a category for which

1. *the class $ob(Top)$ is the class of all topological spaces,*

2. *given a pair of objects* $(A,B) \in ob(Top) \times ob(Top)$, *the set* $\hom_{Top}(A,B)$ *is the set of all continuous maps from* A *to* B,

3. *for every triple of objects*$(A,B,C) \in ob(Top) \times ob(Top) \times ob(Top)$ *we define the composition as*

$$\hom_{Top}(A,B) \times \hom_{Top}(B,C) \to \hom_{Top}(A,C)$$
$$(f,g) \mapsto g \circ f,$$

   in other words we put $f \star g = g \circ f$,

4. *for all* $A \in ob(Top)$, *the arrow* $\mathrm{id}_A\colon A \to A$ *is given by the identity map* $A \to A$. *(Note that* $\mathrm{id}_A \in \hom_{Top}(A,A)$ *as the identity map is always continuous).*

Proof. Let us first prove that the composition map is well-defined i.e, if

$$f \in \hom_{Top}(A,B) \text{ and } g \in \hom_{Top}(B,C),$$

Hence

$$g \circ f \in \hom_{Top}(A,C),$$

i.e, $g \circ f\colon A \to C$ is a continuous map; this follow from the fact that the composition map of continuous maps is continuous. Let U an open set in $C \Rightarrow g^{-1}(U)$ an open set in $B \Rightarrow f^{-1}(g^{-1}(U))$ is an open set in A, but

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U).$$

Therefore $g \circ f$ is a continuous map.
Second: Top satisfies the category axioms as follow:

1. Associativity: for all $f : A \to B, g : B \to C, h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. Unit: for all $f : A \mapsto B$, we have:

$$\mathrm{id}_B \circ f = f \text{ and } f \circ \mathrm{id}_A = f.$$

$\square$

Definition 3.1.12. *We have a precategory*

$$Vec = (ob(Vec), \hom_{Vec}(\_,\_), \circ, \mathrm{id}_{\_})$$

of vector spaces where

1. *$ob(Vec)$ is the class of finite dimensional of vector spaces;*
2. *for each pair of objects $V, U \in ob(Vec)$*

$$\hom_{Vec}(V, U)$$

   is the set of all linear map from V to U;
3. *for each $V, U, W \in ob(Vec)$, and $f \in \hom_{Vec}(V, U)$, $g \in \hom_{Vec}(U, W)$, we have*

$$f \star g = g \circ f;$$

4. *for each object $V \in ob(Vec)$, the identity map $\mathrm{id}_V\colon V \to V$ is a linear map.*

Proposition 3.1.13. *The precategory of vector spaces*

$$Vec = (ob(Vec), \hom_{Vec}(\_,\_), \circ, \mathrm{id}_{\_})$$

is a category.

Proof. We want to prove the category axioms,

1. Associativity, for each $U, V, W, Q \in ob(Vec)$ and linear map $f\colon U \to V, g\colon V \to$

W and $h\colon W \to Q$, and x an element of V we have

$$\begin{aligned} h \circ (g \circ f)(x) &= h(g \circ f(x)) \\ &= h(g(f(x))) \\ &= (h \circ g)(f(x)) \\ &= (h \circ g) \circ f(x). \end{aligned}$$

2. Unit, for each $V, U \in ob(Vec)$, and a linear map $f\colon V \to U$. We have

$$\begin{aligned} f \circ \mathrm{id}_V(x) &= f(\mathrm{id}_V(x)) \\ &= f(x). \\ \mathrm{id}_U \circ f(x) &= \mathrm{id}_U(f(x)) \\ &= f(x). \end{aligned}$$

□

Example 3.1.14 (Example of subcategory)**.** *Consider the category of topological spaces Top, there is a subcategory*

$$DTop = (ob(DTop), \hom_{DTop}(\_, \_)),$$

such that:

1. *$ob(DTop)$ are discrete topological spaces, so $ob(DTop) \subset ob(Top)$,*

2. *for all $A, B \in ob(DTop)$, $\hom_{DTop}(A, B)$ is the set of homeomorphism map from A to B, so*

$$\hom_{DTop}(A, B) \subset \hom_{Top}(A, B).$$

3.2 Functors

In this section we define functors between categories.

Definition 3.2.1. *(See for example [ML13, Lei14, Awo10]). Given categories* $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star_{\mathcal{C}}, \mathrm{id}^{\mathcal{C}}_{\_})$ *and* $\mathcal{D} = (ob(\mathcal{D}), \hom_{\mathcal{D}}(\_,\_), \star_{\mathcal{D}}, \mathrm{id}^{\mathcal{D}}_{\_})$*, a functor*

$$F = (F_0, F_1) : \mathcal{C} \to \mathcal{D},$$

consists of

1. *for each* $A \in ob(\mathcal{C})$*, an object* $F_0(A) \in ob(\mathcal{D})$*;*
2. *for each* $f \in \hom_{\mathcal{C}}(A, B)$ *an arrow* $F_1(f) \in \hom_{\mathcal{D}}(F_0A, F_0B)$*;*

such that

1. *for all* $f : A \to B,\ g : B \to C$*, we have:*

$$F_1(g \star_{\mathcal{C}} f) = (F_1 g) \star_{\mathcal{D}} (F_1 f);$$

2. *for all* $A \in ob(\mathcal{C})$*, we have:* $F_1(\mathrm{id}^{\mathcal{C}}_A) = \mathrm{id}^{\mathcal{D}}_{F_0A}$.

Example 3.2.2. *1. For any category*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id}_{\_}),$$

there is the 'identity' functor

$$\mathrm{Id} \colon \mathcal{C} \to \mathcal{C};$$

such that

(a) *for each* $A \in ob(\mathcal{C})$*, an object* $\mathrm{Id}_0\ A\ = A$*;*

(b) *for each* $f : A \to B \in \mathcal{C}$*, an arrow* $\mathrm{Id}_1\ f\ = f$.

To check that the axioms are satisfied we observe the following,

(a) *for all* $f : A \to B, g : B \to C$ *in* $\mathcal{C}$*, we have*

$$\mathrm{Id}(g \star f) = g \star f = (\mathrm{Id} g) \star (\mathrm{Id} f);$$

(b) *for all* $A \in ob(\mathcal{C})$*, we have* $\mathrm{Id}(\mathrm{id}_A) = \mathrm{id}_A = \mathrm{id}_{\mathrm{Id}A}$.

2. *For any categories:*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star_{\mathcal{C}}, \mathrm{id}_{\_}^{\mathcal{C}}),$$

and

$$\mathcal{D} = (ob(\mathcal{D}), \hom_{\mathcal{D}}(\_,\_), \star_{\mathcal{D}}, \mathrm{id}_{\_}^{\mathcal{D}}),$$

and $B \in ob(\mathcal{D})$*, there is a constant functor*

$$C_B\colon \mathcal{C} \to \mathcal{D},$$

such that

(a) *for each* $A \in ob(\mathcal{C})$*, an object* $C_B A = B$*;*

(b) *for each* $f\colon A \to A' \in \hom_{\mathcal{C}}(A, A')$*, an arrow* $C_B(f\colon A \to A') = \mathrm{id}_B\colon B \to B$.

That satisfies the axioms

(a) *for all morphisms* $f : A \to A', g : A' \to A''$*, we have:*

$$C_B(g \star_{\mathcal{C}} f) = \mathrm{id}_B = \mathrm{id}_B \star_D \mathrm{id}_B = (C_B g) \star_{\mathcal{D}} (C_B f);$$

(b) *for all* $A \in ob(\mathcal{C})$*, we have:* $C_B(\mathrm{id}_A) = \mathrm{id}_B = \mathrm{id}_{(C_B A)}$.

Note: in the next example we use the same convention for the composition for the category Mat and category $\mathcal{SETS}$.

Example 3.2.3. *(See for example [Per19]). Consider the category of* Mat *in 3.1.4*

$$Mat = (\mathbb{N}, \hom_{Mat}(\_,\_), \cdot, \mathrm{id}_{\_}),$$

and the category of $\mathcal{SETS}$ *in 3.1.10*

$$\mathcal{SETS} = (\mathcal{SET}, \hom_{\mathcal{SETS}}(\_,\_), \star, 1_{\_}).$$

We have a functor

$$F = (F_0, F_1)\colon Mat \to \mathcal{SETS},$$

such that

1. *given a natural number* $n \in ob(Mat)$, $F_0(n) = \mathbb{R}^n$;
2. *given* $f\colon n \to m$ *then* $F_1(f)\colon F_0(n) \to F_0(m)$ *is the map given by*

$$v \mapsto f^{tr}.v$$

where each $v \in \mathbb{R}^n$ *is understood as a column vector (and thus both as an element inside an object in* $\mathcal{SETS}$ *and as a morphism in* Mat*!). (Here* $(-)^{tr}$ *denotes the usual matrix transpose.)*

For example if $f\colon 2 \to 3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$, *then* $F_1(f)\colon \mathbb{R}^2 \to \mathbb{R}^3$. *Let* $v = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$, *then* $F_1(f)(v) = \begin{pmatrix} a_1 & a_4 \\ a_2 & a_5 \\ a_3 & a_6 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_4b_2 \\ a_2b_1 + a_5b_2 \\ a_3b_1 + a_6b_2 \end{pmatrix} \in \mathbb{R}^3.$

Proof. To prove F is a functor we have to prove the functor axioms 3.2.1.

- Let $f \in \hom_{Mat}(n, m)$, and $g \in \hom_{Mat}(m, k)$and let $v \in \mathbb{R}^n$, we have

$$\begin{aligned} F(f.g)(v) &= (f.g)^{tr} v \in \mathbb{R}^k \\ &= (g^{tr}.f^{tr}) = g^{tr}(f^{tr} v) \\ &= F(g).(F(f).V) = F(f) \star F(g)(v). \end{aligned}$$

  Therefore

$$F_1(f.g) = F_1(f) \star F_1(g).$$

- Let $m \in \mathbb{N}$,

$$F_1(\mathrm{id}_m)\colon F_0(m) \to F_0(m)$$
$$\mathbb{R}^m \mapsto \mathbb{R}^m.$$

Then

$$F_1(\mathrm{id}_m) = 1_{F_0(m)}.$$

□

Note this could have been formulated as a functor from Mat to Vec.

3.2.1 Properties of functors

First let us think about composition of functors.

Proposition 3.2.4. *Consider the formal quadruple*

$$\mathsf{SmC} = (ob(\mathsf{SmC}), \hom_{\mathsf{SmC}}(\_, \_), \star, \mathrm{id}_{\_})$$

where

1. *the $ob(\mathsf{SmC})$ is the class of all small categories,*
2. *for each pair $(A, B) \in ob(\mathsf{SmC}) \times ob(\mathsf{SmC})$, the set $\hom_{\mathsf{SmC}}(A, B)$ is the set of all functors from A to B,*
3. *for every triple of objects*

$$(A, B, C) \in ob(\mathsf{SmC}) \times ob(\mathsf{SmC}) \times ob(\mathsf{SmC}),$$

and every

$$F \in \hom_{\mathsf{SmC}}(A, B),\ G \in \hom_{\mathsf{SmC}}(B, C),$$

we have

$$\begin{aligned} \hom_{\mathsf{SmC}}(A,B) \times \hom_{\mathsf{SmC}}(B,C) &\to \hom_{\mathsf{SmC}}(A,C) \\ (F,G) &\mapsto \ G \star F, \end{aligned}$$

where for all $X \in ob(A)$, and morphism f in A, we have

$$(G \star F)_0(X) = G_0(F_0(X)),$$
$$(G \star F)_1(f) = G_1(F_1(f)),$$

(so far the construction is only formal. We need to show that $G \star F$ is a functor,)

4. *for all $A \in ob(\mathsf{SmC})$, there is a functor $ID_A \in \hom_{\mathsf{SmC}}(A,A)$.*

This is a category.

Proof. To show that SmC is a precategory we have only to show that $G \star F$ is a functor. For this let $A \in ob(\mathsf{SmC})$, $f \in \hom_A(X,Y)$ and $g \in \hom_A(Y,Z)$, we have

$$\begin{aligned} (G \star F)_1(g \star_A f) &= G_1(F_1(g \star_A f)) \\ &= G_1(F_1(g) \star_B F_1(f)) \\ &= G_1(F_1(g)) \star_C G_1(F_1(f)) \\ &= G_1 \star F_1(g) \star_C G_1 \star F_1(f), \end{aligned}$$

which verifies functor-axiom 1. For axiom 2

$$\begin{aligned} (G \star F)_1(\mathrm{id}_A) &= G_1(F_1(\mathrm{id}_A)) \\ &= G_1(\mathrm{id}_{F_0(A)}) \\ &= \mathrm{id}_{G_0(F_0(A))} \\ &= \mathrm{id}_{G_0 \star F_0(A)}. \end{aligned}$$

as required. Thus our quadruple is a precategory.

Next we want to show that the precategory SmC satisfies the category axioms:

1. Associativity, for all $F\colon A \to B,\ \ G\colon B \to C$ and $H\colon C \to D$, and $X \in ob(A)$ we have

$$\begin{aligned}(H \star (G \star F))_0(X) &= H_0(G_0 \star F_0(X)) \\ &= H_0(G_0(F_0(X)) \\ &= (H_0 \star G_0)(F_0(X)) \\ &= (H_0 \star G_0) \star F_0(X),\end{aligned}$$

and for all morphism f in A, we have

$$\begin{aligned}H_1 \star (G_1 \star F_1)(f) &= H_1(G_1 \star F_1(f)) \\ &= H_1(G_1(F_1(f)) \\ &= (H_1 \star G_1)(F_1(f)) \\ &= (H_1 \star G_1) \star F_1(f).\end{aligned}$$

Then

$$H \star (G \star F) = (H \star G) \star F.$$

2. Unit, for all $F\colon A \to B$, and $X \in ob(A)$, f morphism in A, we have

$$\begin{aligned}(\mathrm{Id}_B \star F)_0(X) &= \mathrm{Id}_0(F_0(X)) = F_0(X), \\ (\mathrm{Id}_B \star F)_1(f) &= \mathrm{Id}_1(F_1(f)) = F_1(f), \\ (F \star \mathrm{Id}_A)_0(X) &= F_0(\mathrm{Id}_0(X)) = F_0(X), \\ (F \star \mathrm{Id}_A)_1(f) &= F_1(\mathrm{Id}_1(f)) = F_1(f).\end{aligned}$$

Therefore

$$\mathrm{Id}_B \star F = F = F \star \mathrm{Id}_A.$$

□

Definition 3.2.5 (Faithful functor)**.** *(See for example [AHS04]). A functor*

$$F\colon \mathcal{C} \to D$$

is called a faithful functor *if it is injective on set of morphisms between objects, i.e all the maps*

$$F_1\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(F_0A, F_0B)$$

are injective.

Example 3.2.6. *(See for example [ML13]). Let Grp be a category of groups such that the objects are groups and the morphisms are the group homomorphisms. There is a faithful functor*

$$U\colon Grp \to \mathcal{SETS},$$

where U is the forgetful functor that forgets the structure of groups.

3.2.2 Product categories

Definition 3.2.7. *(See for example [ML13]). Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_\,,\_), \star, \mathrm{id}_{\_})$$

be a category. We define a precategory

$$\mathcal{C} \times \mathcal{C} = (ob(\mathcal{C} \times \mathcal{C}), \hom_{\mathcal{C}\times\mathcal{C}}(\_\,,\_), \star, \mathrm{id}_{\_})$$

as follows

1. $ob(\mathcal{C} \times \mathcal{C}) = ob(\mathcal{C}) \times ob(\mathcal{C}) = \{(x, y) \mid x, y \in ob(\mathcal{C})\}$,

2. *given a pair of objects* $((x, x'), (y, y'))$ *in* $ob(\mathcal{C} \times \mathcal{C}) \times ob(\mathcal{C} \times \mathcal{C})$*,*

$$\begin{aligned}\hom_{\mathcal{C}\times\mathcal{C}}((x, x'), (y, y')) &:= \hom_{\mathcal{C}}(x, x') \times \hom_{\mathcal{C}}(y, y') \\ &= \left\{ (f, g) \middle| \begin{array}{c} f : x \to y, \\ g : x' \to y' \end{array} \right\}\end{aligned}$$

3. *for every triple of objects* $((x, x'), (y, y'), (z, z')) \in ob(\mathcal{C} \times \mathcal{C}) \times ob(\mathcal{C} \times \mathcal{C}) \times ob(\mathcal{C} \times \mathcal{C})$*, and every* $f = (f_1, f_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((x, x'), (y, y'))$*, and* $g = (g_1, g_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((y, y'), (z, z'))$*, we define*

$$g \star f = (g_1, g_2) \star (f_1, f_2) = (g_1 \star f_1, g_2 \star f_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((x, x'), (z, z')),$$

4. *for all* $(x, x') \in ob(\mathcal{C} \times \mathcal{C})$*,*

   $\mathrm{id}_{(x,x')} = (\mathrm{id}_x, \mathrm{id}_{x'}) \in \hom_{\mathcal{C}\times\mathcal{C}}((x, x'), (x, x'))$.

Proposition 3.2.8. *(See for example [ML13]). The precategory*

$$\mathcal{C} \times \mathcal{C} = (ob(\mathcal{C} \times \mathcal{C}), \hom_{\mathcal{C}\times\mathcal{C}}(\_, \_), \star, \mathrm{id}_{\_})$$

in definition 3.2.7 is a category.

Proof. We want to prove the precategory $\mathcal{C} \times \mathcal{C}$ satisfies the category axioms

1. Associativity, for all $f = (f_1, f_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((x, x'), (y, y'))$, $g = (g_1, g_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((y, y'), (z, z'))$ and $h = (h_1, h_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((z, z'), (w, w'))$, we have:

$$\begin{aligned}h \star (g \star f) &= (h_1, h_2) \star ((g_1, g_2) \star (f_1, f_2)) \\ &= (h_1, h_2) \star (g_1 \star f_1, g_2 \star f_2) \\ &= (h_1 \star g_1 \star f_1, h_2 \star g_2 \star f_2) \\ &= (h_1 \star g_1, h_2 \star g_2) \star (f_1, f_2) \\ &= ((h_1, h_2) \star (g_1, g_2)) \star (f_1, f_2) \\ &= (h \star g) \star f.\end{aligned}$$

2. Unit, for all $(x, x'), (y, y') \in ob(\mathcal{C} \times \mathcal{C})$ and

$$f = (f_1, f_2) \in \hom_{\mathcal{C}\times\mathcal{C}}((x, x'), (y, y')),$$

we have

$$(f_1, f_2) \star (\mathrm{id}_x, \mathrm{id}_{x'}) = (f_1 \star \mathrm{id}_x, f_2 \star \mathrm{id}_{x'}) = (f_1, f_2) = (\mathrm{id}_y, \mathrm{id}_{y'}) \star (f_1, f_2).$$

□

3.2.3 Natural transformations

Definition 3.2.9. *(See for example [AHS04, AT10]). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A* natural transformation

$$t\colon F \to G$$

is a family of morphisms in $\mathcal{D}$ indexed by objects A of $\mathcal{C}$,

$$\Big(t_A\colon FA \to GA\Big)_{A\in ob(\mathcal{C})},$$

such that, for each $f\colon A \to B$, a morphism in $\mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle t_A}\downarrow & & \downarrow{\scriptstyle t_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

Example 3.2.10. *[AT10] Let $\mathcal{SETS}$ be the category of set and*

$$\mathrm{Id}\colon \mathcal{SETS} \to \mathcal{SETS}$$

be the identity functor from Example 3.2.2, and let

$$\delta\colon \mathcal{SETS} \to \mathcal{SETS} \times \mathcal{SETS}$$

[recall that product categories are defined in Section 3.2.2] be the functor such that for each $A \in ob(\mathcal{SETS})$ *we have*

$$A \mapsto A \times A,$$

and for all $f \in \hom_{\mathcal{SETS}}(A, B)$,

$$f \mapsto f \times f.$$

Then there is a natural transformation

$$t\colon F \to G$$

given a set A*,*

$$t_A\colon A \to A \times A,$$

such that for all $x \in A$*, we have*

$$t_A\colon x \in A \mapsto (x, x) \in A \times A.$$

This makes the next diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle t_A}\big\downarrow & & \big\downarrow{\scriptstyle t_B} \\ A \times A & \xrightarrow[f\times f]{} & B \times B \end{array}$$

3.2.4 Adjoint functors

In this section we define adjoint functors in order to use it later where slidealisation functor from $\frac{1}{2}$-monoidal categories to slideable $\frac{1}{2}$-monoidal categories is a left adjoint to the inclusion functor from slideable $\frac{1}{2}$-monoidal categories to $\frac{1}{2}$-monoidal categories.

One of the definitions of adjunction can be found in [BW90, Section 13.2], also you can see for example [ML13, Par70, AHS04].

Definition 3.2.11. *(See for example [BW90, Section 13.2]). Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories and let* $F\colon \mathcal{C} \to \mathcal{D}$ *and* $G\colon \mathcal{D} \to \mathcal{C}$ *be functors. An adjunction is a triple* (F, G, δ)*, where* δ *is a natural transformation* $\delta\colon \mathrm{id} \to G \circ F$*, that for any objects* $A \in \mathcal{C}$ *and* $B \in \mathcal{D}$

and any arrow $f\colon A \to GB$, there is a unique arrow $g\colon FA \to B$ that makes the next diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F(A)) \\ & \searrow^{f} & \downarrow{\scriptstyle G(g)} \\ & & G(B). \end{array}$$

We say that F is left adjoint *to G and G is right adjoint to F and δ_A is a universal arrow from A to G.*

The next definition is the definition of a right adjoint functor $G\colon \mathcal{D} \to \mathcal{C}$.

Definition 3.2.12. *(See for examples [AHS04, Chapter V]). A functor*

$$G\colon \mathcal{D} \to \mathcal{C}$$

is a right adjoint *if for each $A \in ob(\mathcal{C})$ there are $F_A \in ob(\mathcal{D})$ and a morphism $\delta_A\colon A \to G(F_A)$ such that*

> *given $B \in ob(\mathcal{D})$ and morphism $f\colon A \to G(B)$, there is a unique morphism $g\colon F_A \to B$ that makes the next diagram commute.*

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ & \searrow^{f} & \downarrow{\scriptstyle G(g)} \\ & & G(B) \end{array}$$

The morphism δ_A is called a universal arrow from A to G.

Lemma 3.2.13. *(See for example [AHS04, Chapter V]). Let $G\colon \mathcal{D} \to \mathcal{C}$ be a right adjoint functor. Choose for each object A of $\mathcal{C}$ a universal arrow $\delta_A\colon A \to G(F_A)$, from A to G. Then there is a unique functor $F\colon \mathcal{C} \to \mathcal{D}$, where for all $A \in ob(\mathcal{C})$, $F(A) = F_A$, such that $\delta = (\delta_A\colon A \to G(F_A))_{A \in ob(\mathcal{C})}$ is a natural transformation from $\mathrm{id}_{\mathcal{C}}$ to $G \circ F$, i.e, such that for all $A, B \in ob(\mathcal{C})$ and morphism $g\colon A \to B$, the next diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G(F(g))} \\ B & \xrightarrow[\delta_B]{} & G(F_B) \end{array}$$

Moreover that (F, G, δ) *is an adjunction.*

Proof. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a right adjoint functor, then from the Def.3.2.12, for each $A \in ob(\mathcal{C})$ there are $F_A \in ob(\mathcal{D})$ and a given universal arrow $\delta_A\colon A \to G(F_A)$, such that

> Given $B \in ob(\mathcal{D})$ and morphism $f\colon A \to G(B)$, there is a unique morphism $g\colon F_A \to B$ that makes the next diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ & \searrow^{f} & \Big\downarrow{\scriptstyle G(g)} \\ & & G(B) \end{array}$$

We want to construct F. The value of F on objects is given. On morphisms, let $g\colon A \to B$, then we have universal arrows $\delta_A\colon A \to G(F_A)$ and $\delta_B\colon B \to G(F_B)$. Then the value of $F(g\colon A \to B)$ is the unique map from F_A to F_B in $\mathcal{D}$ that makes the next diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle G(F(g))} \\ B & \xrightarrow[\delta_B]{} & G(F_B) \end{array}$$

Now we want to prove F is a functor. For all morphisms $g\colon A \to B$ and $f\colon B \to C$ in $\mathcal{C}$, we have that the next diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle G(F(g))} \\ B & \xrightarrow{\delta_B} & G(F_B) \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle G(F(f))} \\ C & \xrightarrow[\delta_C]{} & G(F_C) \end{array}$$

Also we have the next diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ {\scriptstyle f\star g}\Big\downarrow & & \Big\downarrow{\scriptstyle G(F(f\star g))} \\ C & \xrightarrow[\delta_C]{} & G(F_C) \end{array}$$

Hence from the universal property

$$F(f \star g) = F(f) \star F(g).$$

Also the next two diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ {\scriptstyle \mathrm{id}_A}\big\downarrow & & \big\downarrow{\scriptstyle G(F(\mathrm{id}_A))} \\ A & \xrightarrow[\delta_A]{} & G(F_A) \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & G(F_A) \\ {\scriptstyle \mathrm{id}_A}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{GF_A} = G(\mathrm{id}_{F_A})} \\ A & \xrightarrow[\delta_A]{} & G(F_A) \end{array}$$

Then

$$F(\mathrm{id}_A) = \mathrm{id}_{FA}.$$

Therefore F is a functor and $\delta\colon \mathrm{id} \to G \circ F$ is a natural transformation and (F, G, δ) is an adjunction.

□

3.3 Graphs and categories

In this section, we define the free categories over graphs.

Definition 3.3.1 (See for example [Hig71])**.** *A* directed graph*, or simply* “graph”,

$$X = (V(X), E(X), \delta_1, \delta_2)$$

consists of

1. *A set* $V = V(X)$. *The elements of* V *are called* vertices.

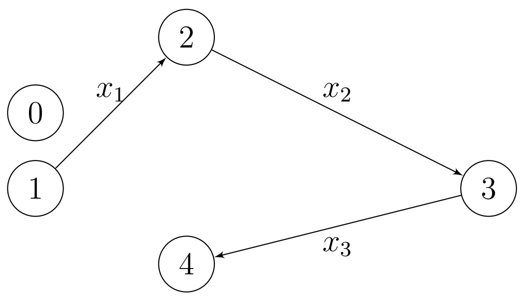
2. *A set* $E = E(X)$. *The elements of* E *are called* edges.

3. *A map, called* "incidence map"

$$\delta\colon E \to V \times V$$
$$x \mapsto (\delta_1 x, \delta_2 x).$$

We call $\delta_1 x$ *the source of* x *and* $\delta_2 x$ *the target of* x.

Example 3.3.2. *Consider the next graph* $(\mathbb{Z}_5, \{x_1, x_2, x_3\}, \delta_1, \delta_2)$ *such that the functions* δ_i *are determined by the following picture*



Definition 3.3.3. *(See for example [Hig71]). A* graph-map $\theta\colon A \to B$ *is a pair of maps* (V_θ, E_θ), *where* $V_\theta\colon V(A) \to V(B)$ *and* $E_\theta\colon E(A) \to E(B)$ *which preserves incidences, i.e, for all edge* x *of* A,

$$\delta_i(E_\theta(x)) := V_\theta(\delta_i x); (i = 1, 2).$$

Note the class of graphs and maps between them, with the evident identities and composition, can be arranged into a category **Graphs**.

3.3.1 On freeness and forgetful functors

Lemma 3.3.4 (The forgetful functor from the category of small categories to the category of graphs)**.** *(See for example [ML13]). We have a* forgetful functor

$$U\colon \mathsf{SmC} \to \mathbf{Graphs}.$$

It is defined in the following way. Let $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id}_{\_})$ *be a category, then*

1. $$U(\mathcal{C}) = (Mor(\mathcal{C}) \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} ob(\mathcal{C})),$$

   where

   $$Mor(\mathcal{C}) := \bigcup_{(x,y)\in ob(\mathcal{C})\times ob(\mathcal{C})} (\{x\} \times \hom_{\mathcal{C}}(x,y) \times \{y\}),$$

   where for every $f \in \hom_{\mathcal{C}}(x,y)$

   $$\delta_1(x,f,y) = x, \;\; \delta_2(x,f,y) = y.$$

2. *Given small categories* $\mathcal{C}_1 = (ob(\mathcal{C}_1), \hom_{\mathcal{C}_1}(\_,\_), \star, \mathrm{id}_{\_})$*, and* $\mathcal{C}_2 = (ob(\mathcal{C}_2), \hom_{\mathcal{C}_2}(\_,\_), \star, \mathrm{id}_{\_})$*. Given* $F = (F_0, F_1) \in \hom_{\mathsf{SmC}}(\mathcal{C}_1, \mathcal{C}_2)$*.*

   $$U(\mathcal{C}_1) = \left(ob(\mathcal{C}_1), \bigcup_{(x,y)\in ob(\mathcal{C}_1)\times ob(\mathcal{C}_1)} \{x\} \times \hom_{\mathcal{C}_1}(x,y) \times \{y\}, \delta_1, \delta_2 \right),$$

   $$U(\mathcal{C}_2) = \left(ob(\mathcal{C}_2), \bigcup_{(x',y')\in ob(\mathcal{C}_2)\times ob(\mathcal{C}_2)} \{x'\} \times \hom_{\mathcal{C}_2}(x',y') \times \{y'\}, \delta_1', \delta_2' \right).$$

   $$U(F) = U(F_0, F_1) = (U(F)_0, U(F)_1),$$

such that

$$U(F)_0 \colon ob(\mathcal{C}_1) \to ob(\mathcal{C}_2),$$
$$U(F)_1(x, f \colon x \to y, y) = (F_0(x), F_1(x \xrightarrow{f} y), F_0(y)).$$

Definition 3.3.5. *(See for example [Hig71]).* A free-category-triple *on a graph X is a triple $(X, \mathcal{C}, \delta_X)$ where $\mathcal{C}$ is a category and $\delta_X \colon X \to U(\mathcal{C})$ is a graph-map such that the following universal property is satisfied.*

> *Given a category A and a graph-map $\theta \colon X \to U(A)$, there is a unique functor $\theta^* \colon \mathcal{C} \to A$ such that $\theta = U(\theta^*) \circ \delta$ in the category of* $\mathbf{Graphs}$.

$$\begin{array}{ccc} X & \xrightarrow{\delta} & U(\mathcal{C}) \\ & \searrow^{\theta} & \downarrow{\scriptstyle U(\theta^*)} \\ & & U(A) \end{array}$$

Here U is the forgetful functor from the category of small categories SmC *to the category of* $\mathbf{Graphs}$.

Therefore $\delta_X \colon X \to U(\mathcal{C})$ is a universal arrow from X to U.

Following from definition 3.2.12, the forgetful functor U is a right adjoint if for all objects X in category of $\mathbf{Graphs}$ there is a universal arrow $\delta_X \colon X \to U$.

Theorem 3.3.6. *A free-category-triple over a graph is unique, up to a unique isomorphism. Given any two free-categories-triples over a graph X, $(X, \mathcal{C}, \delta)$ and $(X, \mathcal{D}, \theta)$, there is a unique functor $f \colon \mathcal{C} \to \mathcal{D}$ making the next diagram commute.*

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & U(\mathcal{C}) \\ & \searrow^{\theta} & \downarrow{\scriptstyle U(f)} \\ & & U(\mathcal{D}) \end{array}$$

Moreover f is an isomorphism.

Proof. First, we want to prove f exists. Since $(X, \mathcal{C}, \delta_X)$ is a free-category-triple, given a category A and a graph-map $\theta: X \to U(A)$, there is a functor $f: \mathcal{C} \to A$ such that $\theta = U(f) \circ \delta$. Assume $A = \mathcal{D}$, so the functor f exists.

Second, we want to prove that the functor f is unique. Assume there is $h: \mathcal{C} \to \mathcal{D}$ another functor such that

$$U(h) \circ \delta = \theta,$$

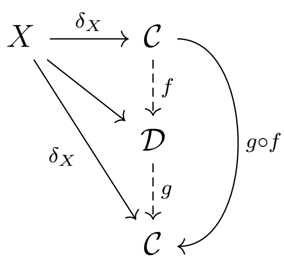
but we have

$$U(f) \circ \delta = \theta.$$

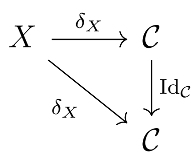
Therefore the functor f is a unique functor i.e,

$$f = h.$$

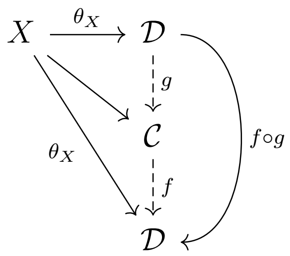
Third, we want to prove that the functor f is an isomorphism. Let $g: \mathcal{D} \to \mathcal{C}$ be the functor that given by the universal property where $(X, \mathcal{D}, \theta_X)$ is a free-category-triple. So we have to prove $f \circ g = \mathrm{Id} = g \circ f$. Since $(X, \mathcal{C}, \delta_X)$ and $(X, \mathcal{D}, \theta_X)$ are free-category-triples, so we have the commutative diagram



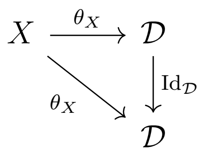
But the map $\mathrm{Id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ is the only map makes the next diagram commute.



Then, $\mathrm{Id}_{\mathcal{C}} = g \circ f$. By a same argument, the next diagram commutes.



The map $\mathrm{Id}_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$ is the only map makes the next diagram commute.



Then, $\mathrm{Id}_D = f \circ g$. □

3.3.2 Paths on a graph

Definition 3.3.7 (Path)**.** *Let*

$$(V(X), E(X), \delta_1, \delta_2)$$

be a graph. A sequence of edges $(x_1, x_2, \ldots, x_n)$, $x_i \in E(X)$, $n \in \mathbb{N}$ *is a* path *if* $\delta_2 x_i = \delta_1 x_{i+1}$ *for all* $i = 1, 2, ..., n-1$.

A path for which $\delta_1 x_1 = \delta_2 x_n$ *is called a closed path.*

Example 3.3.8. *From our Example 3.3.2 we have the path* (x_1, x_2), *where* $\delta_2 x_1 = \delta_1 x_2 = 2$.

Definition 3.3.9. *(See for example [Hig71]). Let* $X = (V(X), E(X), \delta_1, \delta_2)$ *be a graph. For each pair* $i \neq j \in V(X)$, *we define* $Mor_X(i, j)$ *to be the set of 'paths' from* i *to* j, *i.e.,*

$$Mor_X(i,j) = \{(x_1, x_2, ..., x_n) \mid \textit{for some } n \in \mathbb{Z}^+, x_1, \ldots, x_n \in E(X), \delta_1 x_1 = i, \delta_2 x_n = j,$$
$$\delta_2(x_k) = \delta_1(x_{k+1}) \forall k \in \{1, \ldots, n-1\}\}.$$

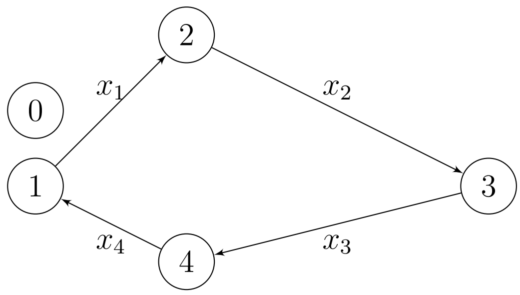
$$Mor_X(i,i) = \{(x_1, x_2, ..., x_n) \mid \textit{for some } n \in \mathbb{Z}^+, x_1, \ldots, x_n \in E(X), \delta_1 x_1 = i = \delta_2 x_n, \delta_2(x_k) = \delta_1(x_{k+1}) \forall k \in \{1, \ldots, n-1\}\} \cup \phi_i.$$

Remark 3.3.10. *ϕ_i is called the empty path from i, or the path of length 0 from i.*

Example 3.3.11. *Consider the graph*

$$X = (\mathbb{Z}_5, \{x_1, x_2, x_3, x_4\}, \delta_1, \delta_2),$$

such that the functions δ_i are determined by the following picture



$$Mor_X(2,4) = \{(x_2, x_3), (x_2, x_3, x_4, x_1, x_2, x_3), (x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3), \ldots, (x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, \ldots, x_4, x_1, x_2, x_3), \ldots\}.$$

$$Mor_X(1,1) = \{\phi_1\} \cup \{(x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4), \ldots, (x_1, x_2, x_3, x_4, \ldots, x_1, x_2, x_3, x_4), \ldots\}.$$

$$Mor_X(0,0) = \{\phi_0\}.$$

$$Mor_X(0,1) = \{\ \}.$$

Note that $\{\ \}$ is the empty set.

Definition 3.3.12. *(See for example [Hig71].*

Let X be a graph. Then the 4-tuple

$$P(X) = (V(X), Mor_X(i,j), \bullet, \phi_{\_})$$

is a precategory where

1. *$ob(P(X)) = V(X)$.*

2. *For each pair $i, j \in V(X), Mor_X(i, j)$ is as defined in 3.3.9.*

3. *For each triple $i, j, k \in V(X)$ the multiplication of paths is*

$$Mor(i, j) \times Mor(j, k) \to Mor(i, k)$$
$$(p, q) \mapsto p \bullet q$$

   given as follows

   Let $p = (x_1, \ldots, x_n) \in Mor_X(i, j)$ and $q = (y_1, \ldots, y_m) \in Mor_X(j, k)$. Then

$$p \bullet q = (x_1, x_2 \ldots, x_n, y_1, y_2, \ldots, y_m) \in Mor_X(i, k)$$

   this multiplication is defined whenever $\delta_2(p) = \delta_1(q)$.

   If $p = \phi_j$,

$$\phi_j \bullet q = q.$$

   If $q = \phi_j$

$$p \bullet \phi_j = p.$$

4. *For each $i \in V(X)$, ϕ_i is the identity on i.*

Proposition 3.3.13. *(See for example [Hig71]). The precategory*

$$P(X) = (V(X), Mor_X(i, j), \bullet, \phi_{\_})$$

is a category.

Proof. We want to check the category axioms in 3.1.3

1. Associative, let $p_1 = (x_1, \ldots, x_n) \in Mor_X(i, j)$, $p_2 = (y_1, \ldots, y_m) \in Mor(j, k)$ and $p_3 = (z_1, \ldots, z_r) \in Mor(k, s)$, we have

   $p_1 \bullet (p_2 \bullet p_3) = p_1 \bullet (y_1, \ldots, y_m, z_1, \ldots, z_r) = (x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r),$

$$(p_1 \bullet p_2) \bullet p_3 = (x_1, \ldots, x_n, y_1, \ldots, y_m) \bullet p_3 = (x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r).$$

Then

$$p_1 \bullet (p_2 \bullet p_3) = (p_1 \bullet p_2) \bullet p_3.$$

2. Unit, let $p = (x_1, \ldots, x_n) \in Mor_X(i, j)$, then

$$\phi_i \bullet p = p = p \bullet \phi_j.$$

□

Theorem 3.3.14. *(See for example [Hig71]). Let X be a graph. Then*

$$(X, P(X), \delta_X)$$

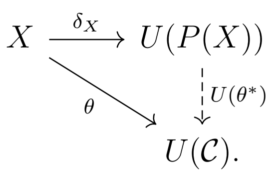
is a free-category-triple on graph X (in the sense of Def. 3.3.5) where

$$P(X) = (V(X), Mor_X(i, j), \bullet, \phi_{\_}),$$

and for all $a \in V(X)$, $x \in E(X)$ the graph map δ_X is the map

$$\delta_X: X \to U(P(X))$$
$$x \mapsto x$$

Proof. We want to prove that given a category $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_, \_), \star, \mathrm{id}_{\_})$ and a graph map $\theta: X \to U(\mathcal{C})$, there is a unique functor $\theta^*: P(X) \to \mathcal{C}$, that makes the diagram commute



1. Existences, define a functor θ^* by

$$\theta^* = (\theta^*_0, \theta^*_1): P(X) \to \mathcal{C},$$

that on objects

$$a \mapsto V_\theta(a),$$

and on morphism

$$(x_1, x_2, \ldots, x_n) \mapsto E_\theta(x_1) \star E_\theta(x_2) \star \ldots \star E_\theta(x_n).$$

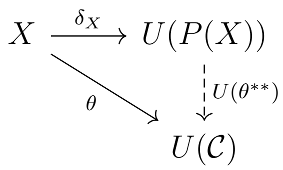
The map $\theta^*\colon P(X) \to \mathcal{C}$ is a functor since for all $p = (x_1, x_2, \ldots, x_n) \in Mor_X(i, j)$ and $q = (y_1, y_2, \ldots, y_m) \in Mor_X(j, k)$, we have

$$\begin{aligned}
\theta_1^*(p \bullet q) &= \theta_1^*(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \\
&= E_\theta(x_1) \star E_\theta(x_2) \star \ldots \star E_\theta(x_n) \star E_\theta(y_1) \star E_\theta(y_2) \star \ldots \star E_\theta(y_m) \\
&= \Big(E_\theta(x_1) \star E_\theta(x_2) \star \ldots \star E_\theta(x_n)\Big) \star \Big(E_\theta(y_1) \star E_\theta(y_2) \star \ldots \star E_\theta(y_m)\Big) \\
&= \theta_1^*(x_1, x_2, \ldots, x_n) \star \theta_1^*(y_1, y_2, \ldots, y_m) \\
&= \theta_1^*(p) \star \theta_1^*(q). \\
\theta_1^*(\phi_a) &= E_\theta(\phi_a) \\
&= \phi_{V_\theta(a)}.
\end{aligned}$$

2. Uniqueness, suppose there is a functor

$$\theta^{**} = (\theta_0^{**}, \theta_1^{**})\colon P(X) \to \mathcal{C}$$

that makes the next diagram commute



Then on objects

$$\theta_0^{**}(a) = V_\theta(a),$$

and on morphisms

$$\begin{aligned}\theta^{**}(p) &= \theta^{**}(x_1, \ldots, x_n)\\ &= \theta^{**}(x_1) \star \ldots \star \theta^{**}(x_n)\\ &= E_\theta(x_1) \star \ldots \star E_\theta(x_n).\end{aligned}$$

Therefore there is a unique functor i.e.,

$$\theta^* = \theta^{**}.$$

□

Example 3.3.15. *1. Let X be a graph with one vertex and one edge x from the vertex to itself. Then the category*

$$P(X) = (\{i\}, Mor_X(i, i), \bullet, \phi_{\_}).$$

Here

$$Mor_X(i, i) = \{\phi_i, p, p \bullet p, p \bullet p \bullet p, \ldots\},$$

where $p = (x)$ (path of x).

Then the free-category-triple on X is

$$(X, P(X), \delta_X).$$

2. *Let X be a graph with two vertices a, b and an edge x from a to b, with*

$$P(X) = (\{a, b\}, Mor_X(\_, \_), \bullet, \phi_{\_}).$$

where for all $i, j \in V(X)$ we have,

$$\begin{aligned}Mor_X(a, b) =&\{(x)\},\\ Mor_X(a, a) =&\{\phi_a\},\\ Mor_X(b, b) =&\{\phi_b\}.\end{aligned}$$

Then the free-category-triple on X is

$$(X, P(X), \delta_X).$$

3.4 Quotient categories

In this section we review congruence relation and quotient categories.

Definition 3.4.1 (Precongruence)**.** *Let $\mathcal{C}$ be a category, A precongruence relation R on $\mathcal{C}$ is given by for each pair $X, Y \in ob(\mathcal{C})$, an equivalence relation $R_{X,Y}$ on $\hom_{\mathcal{C}}(X, Y)$.*

Example 3.4.2. *Let $\mathcal{C}$ be the category Mat see 3.1.4. Let $X, Y \in ob(\mathcal{C})$, and $f, g \colon X \to Y$. We define a relation R such that $f \sim_{R_{X,Y}} g$, if they are projectively equivalent, i.e. there is $a \in \mathbb{C}, a \neq 0$, such that*

$$f = ag.$$

That is an equivalence relation, since

1. *$f \sim_{R_{X,Y}} f$, where $f = (1)f$.*
2. *Let $f \sim_{R_{X,Y}} g$, this means there is $a \in \mathbb{C} \setminus \{0\}$ such that $f = ag$, so, $g = a^{-1}f$, then $g \sim_{R_{X,Y}} f$.*
3. *Let $f \sim_{R_{X,Y}} g$ and $g \sim_{R_{X,Y}} h$, there are $a, b \in \mathbb{C} \setminus \{0\}$, such that, $f = ag$ and $g = bh$ then $f = a(bh) = (ab)h$, this mean $f \sim_{R_{X,Y}} h$.*

Definition 3.4.3 (Congruence)**.** *(See for example [ML13, page 51-52 ;VO95]). A precongruence relation R on a category $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id}_{\_})$ is called a congruence if it satisfies the following,*

for each $f_1 \sim_{R_{X,Y}} f_2$ and $g_1 \sim_{R_{Y,Z}} g_2$, we have

$$g_1 \star f_1 \sim_{R_{X,Z}} g_2 \star f_2.$$

Example 3.4.4. *The precongruence in Example 3.4.2 is a congruence .*
Let $f_1, f_2 \in \hom_{Mat}(X,Y), f_1 \sim_{R_{X,Y}} f_2$ *and* $g_1, g_2 \in \hom_{Mat}(Y,Z), g_1 \sim_{R_{Y,Z}} g_2$*, we have*

$f_1 \cdot g_1 \in \hom_{Mat}(X,Z)$ *and* $f_2 \cdot g_2 \in \hom_{Mat}(X,Z)$*. Moreover*

$$f_1 = af_2,$$
$$g_1 = bg_2,$$

for some $a, b \in \mathbb{C} \setminus \{0\}$*. Hence*

$$\begin{aligned} f_1 \cdot g_1 &= (af_2) \cdot (bg_2) \\ &= (ab)(f_2 \cdot g_2). \end{aligned}$$

Therefore

$$f_1 \cdot g_1 \sim_{R_{X,Z}} f_2 \cdot g_2.$$

Proposition 3.4.5 (Quotient precategory)**.** *(See for example [ML13, page 51]). Let* R *be a congruence relation on a category* $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star_{\mathcal{C}}, \mathrm{id}_{\_})$*, then the quotient*

$$\mathcal{C}/R := (ob(\mathcal{C}), \hom_{\mathcal{C}/R}(\_,\_), \star_{\mathcal{C}/R}, \mathrm{id}'_{\_})$$

is a precategory, where

1. $ob(\mathcal{C}/R) = ob(\mathcal{C})$.

2. *For* $X, Y \in ob(\mathcal{C})$*,*

$$\hom_{\mathcal{C}/R}(X,Y) := \hom_{\mathcal{C}}(X,Y)/R_{X,Y}.$$

(Given $f \in \hom_{\mathcal{C}}(X,Y)$*, the equivalence class to which it belong is denoted* $[f]_{R_{X,Y}}$*).*

3. *Given* $f \in \hom_{\mathcal{C}/R}(X,Y), g \in \hom_{\mathcal{C}/R}(Y,Z)$*, then*

$$[g]_{R_{Y,Z}} \star_{\mathcal{C}/R} [f]_{R_{X,Y}} := [g \star_{\mathcal{C}} f]_{R_{X,Z}}.$$

4. *For all* $X \in ob(\mathcal{C}/R)$, *we have* $\mathrm{id}'_X := [\mathrm{id}_X]$.

Proof. Given $f \in \hom_{\mathcal{C}/R}(X,Y)$, $g \in \hom_{\mathcal{C}/R}(Y,Z)$, we want to prove the composition

$$[g]_{R_{Y,Z}} \star_{\mathcal{C}/R} [f]_{R_{X,Y}} = [g \star_{\mathcal{C}} f]_{R_{X,Z}}$$

is well defined, namely that it is independent of the choice of representatives. We want to prove if $f \sim f'$ and $g \sim g'$, then

$$g \star_{\mathcal{C}} f \sim g' \star_{\mathcal{C}} f'.$$

This comes directly from the definition of congruence relation 3.4.3. □

To $\mathcal{C}/R$ we call the quotient precategory.

Proposition 3.4.6. *(See for example [ML13, page 51-52 ;VO95]). The quotient precategory*

$$\mathcal{C}/R = (ob(\mathcal{C}), \hom_{\mathcal{C}/R}(\_,\_), \star_{\mathcal{C}/R}, \mathrm{id}'_{\_})$$

of a category $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star_{\mathcal{C}}, \mathrm{id}_{\_})$ *is a category.*

Proof. We need to check associtiviy and identity axioms in 3.1.3 .

1. For all $[f]_{R_{X,Y}}$, $[g]_{R_{Y,Z}}$ and $[h]_{R_{Z,W}}$ in $\mathcal{C}/R$,

$$\begin{aligned}
[h]_{R_{Z,W}} \star_{\mathcal{C}/R} ([g]_{R_{Y,Z}} \star_{\mathcal{C}/R} [f]_{R_{X,Y}}) =& [h]_{R_{Z,W}} \star_{\mathcal{C}/R} ([g \star_{\mathcal{C}} f]_{R_{X,Z}}) \\
=& [h \star_{\mathcal{C}} (g \star_{\mathcal{C}} f)]_{R_{X,W}} \\
=& [(h \star_{\mathcal{C}} g) \star_{\mathcal{C}} f]_{R_{X,W}} \\
=& [(h \star_{\mathcal{C}} g)]_{R_{Y,W}} \star_{\mathcal{C}/R} [f]_{R_{X,Y}} \\
=& ([h]_{R_{Z,W}} \star_{\mathcal{C}/R} [g]_{R_{Y,Z}}) \star_{\mathcal{C}/R} [f]_{R_{X,Y}}.
\end{aligned}$$

2. For all $[f]_{R_{X,Y}}$ in $\mathcal{C}/R$,

$$\begin{aligned}[f]_{R_{X,Y}} \star_{\mathcal{C}/R} \mathrm{id}'_X &= [f]_{R_{X,Y}} \star_{\mathcal{C}/R} [\mathrm{id}_X] \\ &= [f \star_{\mathcal{C}} \mathrm{id}_X]_{R_{X,Y}} \\ &= [f]_{R_{X,Y}} \\ &= [\mathrm{id}_Y \star_{\mathcal{C}} f]_{R_{X,Y}} \\ &= [\mathrm{id}_Y] \star_{\mathcal{C}/R} [f]_{R_{X,Y}} \\ &= \mathrm{id}'_Y \star_{\mathcal{C}/R} [f]_{R_{X,Y}}.\end{aligned}$$

□

Example 3.4.7. *Consider the category of Mat and the congruence relation R in 3.4.4, so we have a quotient category Mat/R, where*

1. *ob*$(Mat/R) =$ *ob*(Mat).
2. $\hom_{Mat/R}(X, Y) = \hom_{Mat}(X, Y)/R$.

Let $f, g \in \hom_{Mat}(2, 2)$,

$$f = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}, g = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

So, $f = 2g$*, then* $[f] = [g] \in \hom_{Mat/R}(2, 2)$

Proposition 3.4.8. *(See for example [ML13, page 51-52]). Let* $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_, \_), \star_{\mathcal{C}}, \mathrm{id}_{\_})$ *be a category and R a congruence. There is a functor F from the category $\mathcal{C}$ to the quotient category* $\mathcal{C}/R = (ob(\mathcal{C}), \hom_{\mathcal{C}/R}(\_, \_), \star_{\mathcal{C}/R}, \mathrm{id}'_{\_})$*, where*

1. $F(X) = X, \forall X \in ob(\mathcal{C})$.
2. $F(f) = [f]_{R_{X,Y}} \, \forall f \in \hom_{\mathcal{C}}(X, Y)$.

Proof. We want to prove F is a functor, let $f\colon X \to Y$ and $g\colon Y \to Z$ morphisms in $\mathcal{C}$,

$$\begin{aligned}
F(g \star_{\mathcal{C}} f) &= [g \star_{\mathcal{C}} f]_{R_{X,Z}} \\
&= [g]_{R_{Y,Z}} \star_{\mathcal{C}/R} [f]_{R_{X,Y}} \\
&= F(g) \star_{\mathcal{C}/R} F(f). \\
F(\mathrm{id}_X) &= [\mathrm{id}_X] \\
&= [\mathrm{id}_{F(X)}] \\
&= \mathrm{id}'_{F(X)}.
\end{aligned}$$

□

3.5 Presentations of small categories

3.5.1 Congruence closure

In this section we recall the definition of congruence and that it yields a quotient category and a functor. We will give a way to pass from a collection of relations W to the 'smallest' congruence containing W. This is implicitly used in the literature however I could not find a place where it is rigorously treated.

Definition 3.5.1 (Congruence Template)**.** *Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_\,,\_), \star, \mathrm{id}_{\_})$$

be a category. A congruence template *is given by a relation – not necessarily an equivalence relation –* $W_{X,Y}$ *for each objects* $X, Y \in ob(\mathcal{C})$*. We write* $f \sim_{W_{X,Y}} g$ *to say that* $f, g\colon X \to Y$ *are related by* $W_{X,Y}$*.*

Definition 3.5.2. *Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_\,,\_), \star, \mathrm{id}_{\_})$$

be a category that has a congruence template $W_{X,Y}$ *for all* $X, Y \in ob(\mathcal{C})$*. We say*

$f, g\colon X \to Y$ are related in $\overline{W}_{X,Y}$, and we put

$$f \sim_{\overline{W}_{X,Y}} g$$

if there are $A, B \in ob(\mathcal{C})$, $f', g'\colon A \to B$ and $\alpha\colon X \to A$, $\beta\colon B \to Y$, hence fitting into the diagram below

$$X \xrightarrow{\alpha} A \underset{g'}{\overset{f'}{\rightrightarrows}} B \xrightarrow{\beta} Y$$

such that $f = \beta \star f' \star \alpha$ and $g = \beta \star g' \star \alpha$, and $f' \sim_{W_{A,B}} g'$ or $g' \sim_{W_{A,B}} f'$.

Lemma 3.5.3. *Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_, \_), \star, \mathrm{id}_{\_})$$

be a category that has a congruence template W. Let $X, Y, X', Y' \in ob(\mathcal{C})$. Let also $f, g\colon X \to Y$, $m\colon X' \to X$ and $n\colon Y \to Y'$ be morphisms such that

$$f \sim_{\overline{W}_{X,Y}} g.$$

Then

$$n \star f \star m \sim_{\overline{W}_{X',Y'}} n \star g \star m.$$

Proof. Let $f, g\colon X \to Y$ be morphisms such that

$$f \sim_{\overline{W}_{X,Y}} g.$$

Then there are $A, B \in ob(\mathcal{C})$, $f', g'\colon A \to B$ and $\alpha\colon X \to A$, $\beta\colon B \to Y$, such that $f = \beta \star f' \star \alpha$ and $g = \beta \star g' \star \alpha$, and $f' \sim_{W_{A,B}} g'$ or $g' \sim_{W_{A,B}} f'$. Hence

$$\begin{aligned} n \star f \star m &= n \star \beta \star f' \star \alpha \star m \\ &= (n \star \beta) \star f' \star (\alpha \star m), \\ n \star g \star m &= n \star \beta \star g' \star \alpha \star m \\ &= (n \star \beta) \star g' \star (\alpha \star m). \end{aligned}$$

Therefore

$$n \star f \star m \sim_{\overline{W}_{X',Y'}} n \star g \star m.$$

□

Definition 3.5.4. *Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_, \_), \star, \mathrm{id}_{\_})$$

be a category, with a congruence template $\{W_{X,Y}\}_{(X,Y)\in ob(\mathcal{C})\times ob(\mathcal{C})}$*. Consider the relation* $\overline{W}_{X,Y}$ *in 3.5.2, defined in* $\hom_{\mathcal{C}}(X,Y)$*, for all* $X, Y \in ob(\mathcal{C})$*. Let*

$$\overline{\overline{W}}_{X,Y}$$

be the transitive, reflexive closure of $\overline{W}_{X,Y}$*. (So* $\overline{\overline{W}}_{X,Y}$ *is an equivalence relation in* $\hom_{\mathcal{C}}(X,Y)$*).*
Then $f, g\colon X \to Y$ *are related in* $\overline{\overline{W}}_{X,Y}$ *we write*

$$f \sim_{\overline{\overline{W}}_{X,Y}} g,$$

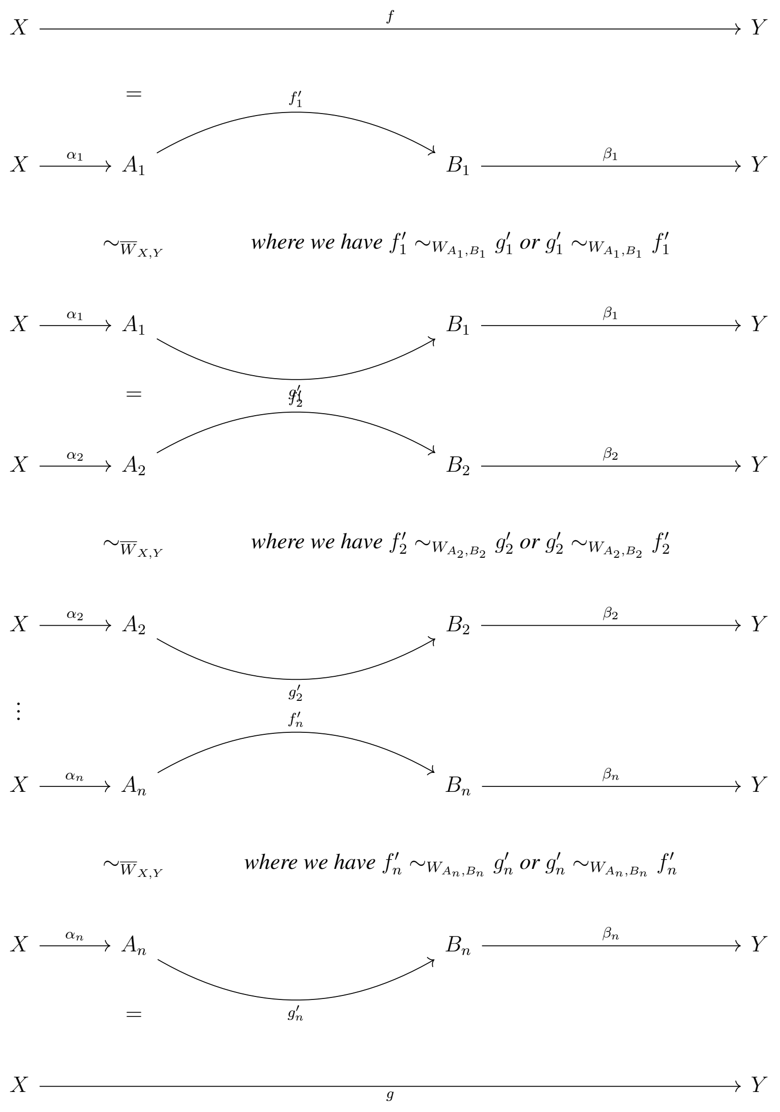
if $f = g$ *or there exists an* $n \in \mathbb{N}$*,* $(A_1, A_2, \ldots, A_n) \in ob(\mathcal{C})^n$*, and* $(B_1, B_2, \ldots, B_n) \in ob(\mathcal{C})^n$*, and for all* $i \in \{1, 2, \ldots, n\}$*, morphisms*

$$\alpha_i\colon X \to A_i,$$
$$f_i'\colon A_i \to B_i,$$
$$g_i'\colon A_i \to B_i,$$
$$\beta_i\colon B_i \to Y,$$

such that

$$
\begin{aligned}
&f = f_1 = \beta_1 \star f_1' \star \alpha_1 \\
&\quad \sim_{\overline{W}_{X,Y}} g_1 = \beta_1 \star g_1' \star \alpha_1, \text{ where we have } f_1' \sim_{W_{A_1,B_1}} g_1' \text{ or } g_1' \sim_{W_{A_1,B_1}} f_1' \\
&\quad = f_2 = \beta_2 \star f_2' \star \alpha_2 \\
&\quad \sim_{\overline{W}_{X,Y}} = g_2 = \beta_2 \star g_2' \star \alpha_2, \text{ where we have } f_2' \sim_{W_{A_2,B_2}} g_2' \text{ or } g_2' \sim_{W_{A_2,B_2}} f_2' \\
&\vdots \\
&\quad = f_n = \beta_n \star f_n' \star \alpha_n \\
&\quad \sim_{\overline{W}_{X,Y}} g_n = \beta_n \star g_n' \star \alpha_n = g, \text{ where we have } f_n' \sim_{W_{A_n,B_n}} g_n' \text{ or } g_n' \sim_{W_{A_n,B_n}} f_n'.
\end{aligned}
$$

I.e,



Theorem 3.5.5. *The equivalence relations* $\overline{\overline{\overline{W}}}_{X,Y}$ *on* $\hom_{\mathcal{C}}(X,Y)$*, for all* $(X,Y) \in$

$ob(\mathcal{C}) \times ob(\mathcal{C})$, are a congruence in $\mathcal{C}$, see 3.4.3. I.e, if

$$f \sim_{\overline{\overline{W}}_{X,Y}} f' \text{ and } g \sim_{\overline{\overline{W}}_{Y,Z}} g',$$

then

$$g \star f \sim_{\overline{\overline{W}}_{X,Z}} g' \star f'.$$

Proof. Suppose $f \sim_{\overline{\overline{W}}_{X,Y}} f'$, so $f = f'$ or there exists $n \in \mathbb{N}$ and there are $f_1, f_2, \ldots, f_n \colon X \to Y$, such that

$$f = f_1 \sim_{\overline{W}_{X,Y}} f_2, f_2 \sim_{\overline{W}_{X,Y}} f_3, \ldots, f_{n-1} \sim_{\overline{W}_{X,Y}} f_n = f'.$$

Suppose $g \sim_{\overline{\overline{W}}_{Y,Z}} g'$, so $g = g'$ or there exists $m \in \mathbb{N}$ and there are $g_1, g_2, \ldots, g_m \colon Y \to Z$, such that

$$g = g_1 \sim_{\overline{W}_{Y,Z}} g_2, g_2 \sim_{\overline{W}_{Y,Z}} g_3, \ldots, g_{m-1} \sim_{\overline{W}_{Y,Z}} g_m = g'.$$

Then, by Lemma 3.5.3,

$$\begin{aligned} g \star f = g \star f_1 &\sim_{\overline{W}_{X,Z}} g \star f_2, \\ g \star f_2 &\sim_{\overline{W}_{X,Z}} g \star f_3, \\ &\vdots \\ g \star f_{n-1} &\sim_{\overline{W}_{X,Z}} g \star f_n = g \star f'. \end{aligned}$$

Therefore

$$g \star f \sim_{\overline{\overline{W}}_{X,Z}} g \star f'.$$

By similarity

$$g \star f' = g_1 \star f' \sim_{\overline{W}_{X,Z}} g_2 \star f',$$
$$g_2 \star f' \sim_{\overline{W}_{X,Z}} g_3 \star f'$$
$$\vdots$$
$$g_{m-1} \star f' \sim_{\overline{W}_{X,Z}} g_m \star f' = g' \star f'.$$

Therefore

$$g \star f' \sim_{\overline{\overline{W}}_{X,Z}} g' \star f'.$$

Hence

$$g \star f \sim_{\overline{\overline{W}}_{X,Z}} g' \star f'.$$

□

Definition 3.5.6. *The congruence $\overline{\overline{W}}_{X,Y}$ is called the closure of the congruence template $W_{X,Y}$.*

Example 3.5.7. *Consider the category Mat in 3.1.2 and let it have the following congruence template $W_{m,n}$.*
If $m, n \in \mathbb{N}$ and $A, B \in \hom_{Mat}(m, n)$ then

$$A \sim_{W_{m,n}} B, \text{ if } \begin{cases} A = B & , \text{ where } m \neq n, \\ B = \lambda A & , \text{ for some } \lambda \neq 0, \text{ if } m = n. \end{cases}$$

Let $m, n \in N$. Let $U, V \in \hom_{Mat}(m, n)$. Suppose $V = \lambda U$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ then $U \sim_{\overline{\overline{W}}_{m,n}} V$.

Proof.

$$U = \mathrm{id}_m.\mathrm{id}_m.U,$$
$$V = \mathrm{id}_m.\lambda\mathrm{id}_m.U.$$

By using Def. 3.5.2,

$$U \sim_{\overline{W}_{m,n}} V.$$

Therefore

$$U \sim_{\overline{\overline{W}}_{m,n}} V.$$

□

Definition 3.5.8 (Category presented by a graph and a congruence template)**.** *Let $\mathcal{C}$ be the free category on a graph G, see 3.3.14, and let $W_{X,Y}$, $X, Y \in ob(\mathcal{C})$ be a congruence template in $\mathcal{C}$. Let $\overline{\overline{W}}_{X,Y}$ be its closure. To the quotient category*

$$\mathcal{C}/\overline{\overline{W}}$$

we call the category presented by G and W.

3.5.2 Example: the combinatorial braid category

We now use free category and closures of congruence templates, and the ensuing notion of a presentation of a category, to give an example.

Example 3.5.9. *(See [Ver13, section 4]). Consider the graph*

$$\beta = (\mathbb{N}, E, \delta_1, \delta_2),$$

where the set of edges is

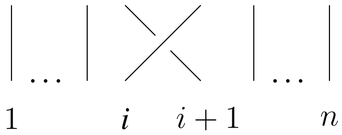
$$E = \left\{ X^+_{(i,n)} \mid n \geq 2,\ i \in \{1, \dots, n-1\} \right\} \cup \left\{ X^-_{(i,n)} \mid n \geq 2,\ i \in \{1, \dots, n-1\} \right\},$$

and the incidence function is

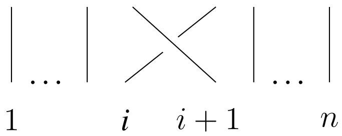
$$\delta_1 X^+_{(i,n)} = \delta_1 X^-_{(i,n)} = n \qquad \delta_2 X^+_{(i,n)} = \delta_2 X^-_{(i,n)} = n. \tag{3.1}$$

We will represent the formal symbols $X^{\pm}_{(i,n)}$ as

$X^{+}_{(i,n)} =$



$X^{-}_{(i,n)} =$



Consider the category (see 3.3.13)

$$P(\beta) = (\mathbb{N}, Mor_{P(\beta)}(n,m), \bullet, \phi_{\_}).$$

So, we have the free-category-triple in sense of Def. 3.3.5

$$(\beta, P(\beta), \delta).$$

We define the formal braid category to be the following quotient

$$P(\beta) \Big/ \overline{\overline{W}}_{n,m}.$$

Where, given $m, n \in \mathbb{N}$, then $W_{n,m}$ is the relation in $Mor_{P(\beta)}(n,m)$ defined as below

- *If $m \neq n$ it is clear, from incidence map on* (3.1) *that $Mor_{P(\beta)}(n,m) = \emptyset$. So $W_{n,m}$ is the unique equivalence relation on the empty set.*
- *$m, n = 0$. Then $Mor_{P(\beta)}(0,0) = \{\phi_0\}$. The relation $W_{(0,0)}$ is the unique equivalence relation such that $\phi_0 \sim_{W_{0,0}} \phi_0$.*
- *$m, n = 1$. Then $Mor_{P(\beta)}(1,1) = \{\phi_1\}$. The relation $W_{(1,1)}$ is the unique equivalence relation such that $\phi_1 \sim_{W_{1,1}} \phi_1$.*
- *$m, n = 2$. $Mor_{P(\beta)}(2,2)$ is the set of words in $X^{+}_{1,2}$ and $X^{-}_{1,2}$. The two pairs of*

related elements are : (note that we omitted • from the notation.)

$$X^+_{(1,2)}X^-_{(1,2)} \sim_{W_{2,2}} \phi_2,$$
$$X^-_{(1,2)}X^+_{(1,2)} \sim_{W_{2,2}} \phi_2.$$

- $m, n = 3$. $Mor_{P(\beta)}(3,3)$ *is the set of words in* $X^+_{1,3}, X^+_{2,3}$ *and* $X^-_{1,3}, X^-_{2,3}$. *The related elements are*

$$X^+_{(1,3)}X^-_{(1,3)} \sim_{W_{3,3}} \phi_3,$$
$$X^-_{(1,3)}X^+_{(1,3)} \sim_{W_{3,3}} \phi_3,$$
$$X^+_{(2,3)}X^-_{(2,3)} \sim_{W_{3,3}} \phi_3,$$
$$X^-_{(2,3)}X^+_{(2,3)} \sim_{W_{3,3}} \phi_3,$$
$$X^+_{(1,3)}X^+_{(2,3)}X^+_{(1,3)} \sim_{W_{3,3}} X^+_{(2,3)}X^+_{(1,3)}X^+_{(2,3)},$$
$$X^-_{(1,3)}X^-_{(2,3)}X^-_{(1,3)} \sim_{W_{3,3}} X^-_{(2,3)}X^-_{(1,3)}X^-_{(2,3)}.$$

- $m, n = 4$. $Mor_{P(\beta)}(4,4)$ *is the set of words in* $X^+_{1,4}, X^+_{2,4}, X^+_{3,4}$ *and* $X^-_{1,4}, X^-_{2,4}, X^-_{3,4}$. *The related elements are*

$$X^+_{(1,4)}X^-_{(1,4)} \sim_{W_{4,4}} \phi_4,$$
$$X^-_{(1,4)}X^+_{(1,4)} \sim_{W_{4,4}} \phi_4,$$
$$X^+_{(2,4)}X^-_{(2,4)} \sim_{W_{4,4}} \phi_4,$$
$$X^-_{(2,4)}X^+_{(2,4)} \sim_{W_{4,4}} \phi_4,$$
$$X^+_{(3,4)}X^-_{(3,4)} \sim_{W_{4,4}} \phi_4,$$
$$X^-_{(3,4)}X^+_{(3,4)} \sim_{W_{4,4}} \phi_4,$$
$$X^+_{(1,4)}X^+_{(3,4)} \sim_{W_{4,4}} X^+_{(3,4)}X^+_{(1,4)},$$
$$X^+_{(1,4)}X^+_{(2,4)}X^+_{(1,4)} \sim_{W_{4,4}} X^+_{(2,4)}X^+_{(1,4)}X^+_{(2,4)},$$
$$X^+_{(2,4)}X^+_{(3,4)}X^+_{(2,4)} \sim_{W_{4,4}} X^+_{(3,4)}X^+_{(2,4)}X^+_{(3,4)},$$
$$X^-_{(1,4)}X^-_{(2,4)}X^-_{(1,4)} \sim_{W_{4,4}} X^-_{(2,4)}X^-_{(1,4)}X^-_{(2,4)},$$
$$X^-_{(2,4)}X^-_{(3,4)}X^-_{(2,4)} \sim_{W_{4,4}} X^-_{(3,4)}X^-_{(2,4)}X^-_{(3,4)}.$$

- *General* $n \geq 5$ *case. As in* $n = 4$.

Chapter 4

Monoids and their abelianisation

The aim of this chapter is understanding the abelianisation functor. This is because it is very similar to an operation later sending a $\frac{1}{2}$-monoidal category into a slideable $\frac{1}{2}$-monoidal category by imposing slidealisation conditions. A slideable $\frac{1}{2}$-monoidal category is the same thing as a strict monoidal category, as we discuss below.

4.1 Monoid definition and examples

Definition 4.1.1 (Monoid)**.** *(See for example [Ber15, Fac21]). A* monoid $(G, \bullet, e)$ *is given by a triple consisting of*

1. *a set G,*
2. *a map $G \times G \to G$, denoted by $(x, y) \in G \times G \mapsto x \bullet y$,*
3. *$e \in G$ (e is called the identity),*

such that the following axioms are satisfied

1. *associativity, for all $x, y, z \in G$, the equation*

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

holds,

2. *unit, for every element $x \in G$, the equations*

$$e \bullet x = x \bullet e = x$$

hold.

Definition 4.1.2 (Abelian monoid)**.** *(See for example [Red14]). A monoid $(A, \bullet, e)$ is called* abelian *if every $x, y \in A$, we have*

$$x \bullet y = y \bullet x.$$

Example 4.1.3. 1. *The set of natural numbers under addition $(\mathbb{N}, +, 0)$ is an abelian monoid.*

2. *The set of natural numbers $\mathbb{N}$ under multiplication with 1 as an identity element $(\mathbb{N}, \times, 1)$ is an abelian monoid.*

3. *Every group is a monoid.*

Definition 4.1.4 (Monoid map)**.** *(See for example [Fac21, HZ97]). Let $(M, \bullet, e)$ and $(N, \bullet', e')$ be monoids. A monoid map*

$$f\colon (M, \bullet, e) \to (N, \bullet', e')$$

is a set map $f\colon M \to N$ such that

$$f(x \bullet y) = f(x) \bullet' f(y),$$

and

$$f(e) = e'.$$

Proposition 4.1.5. *The precategory of monoids*

$$\mathcal{M} = (ob(\mathcal{M}), \hom_{\mathcal{M}}(\_, \_), \circ, \mathrm{id}_{\_})$$

is given by

1. $ob(\mathcal{M})$ *is the class of all monoids.*

2. *Given a pair of objects* $(A, B) \in ob(\mathcal{M}) \times ob(\mathcal{M})$*, the set* $\hom_{\mathcal{M}}(A, B)$ *is the set of all monoids maps from* A *to* B*.*

3. *For every triple of objects* $(A, B, C) \in ob(\mathcal{M}) \times ob(\mathcal{M}) \times ob(\mathcal{M})$ *and every* $f \in \hom_{\mathcal{M}}(A, B)$ *and* $g \in \hom_{\mathcal{M}}(B, C)$ *we have*

$$g \circ f \in \hom_{\mathcal{M}}(A, C).$$

4. *For all* $A \in ob(\mathcal{M})$*, an arrow* $\mathrm{id}_A \in \hom_{\mathcal{M}}(A, A)$ *(the usual identity on the underlying set of* A*).*

Proof. It is clear the first, second and fourth components are as required, so it remains to check the third component.
Let $(A, B, C) \in ob(\mathcal{M}) \times ob(\mathcal{M}) \times ob(\mathcal{M})$ and $f \in \hom_{\mathcal{M}}(A, B)$ and $g \in \hom_{\mathcal{M}}(B, C)$.
So, $f \in \hom_{\mathcal{SETS}}(A, B)$ and $g \in \hom_{\mathcal{SETS}}(B, C)$, then

$$g \circ f \in \hom_{\mathcal{SETS}}(A, C).$$

Also,

$$\begin{aligned} g \circ f(X \bullet_A Y) &= g(f(X) \bullet_B f(Y)) \\ &= g(f(X)) \bullet_C g(f(Y)) \\ &= g \circ f(X) \bullet_C g \circ f(Y), \\ g \circ f(e_A) &= g(e_B) \\ &= e_C. \end{aligned}$$

Therefore

$$g \circ f \in \hom_{\mathcal{M}}(A, C).$$

□

Proposition 4.1.6. *The precategory in the definition 4.1.5 is a category .*

Proof. We can see easily that $\mathcal{M}$ satisfy the category axioms. □

Definition 4.1.7. *We have a full subcategory* $\mathcal{M}_{\text{ab}} = (ob(\mathcal{M}_{\text{ab}}), \hom_{\mathcal{M}}(\_, \_))$*, where*

1. $ob(\mathcal{M}_{\text{ab}})$ *is the class of abelian monoids, so* $ob(\mathcal{M}_{\text{ab}}) \subset ob(\mathcal{M})$*.*
2. *Given a pair of objects* $(A, B) \in ob(\mathcal{M}_{\text{ab}}) \times ob(\mathcal{M}_{\text{ab}})$*, the set* $\hom_{\mathcal{M}_{\text{ab}}}(A, B) = \hom_{\mathcal{M}}(A, B)$*.*

We now address the abelianisation of a monoid. This is in preparation for the definition of the slidealisation of a $\frac{1}{2}$-monoidal category.

Lemma 4.1.8. *Let* $A = (A, \bullet, e)$ *be a monoid. There is another monoid* $A_{\text{ab}} = (A_{ab}, \bullet_{ab}, e_{ab})$*. Here*

1. $A_{ab} = A/\sim$*. Here* $\sim$ *is the transitive closure of* $\sim_0$*, we say* $x \sim_0 y$ *if there exist* $a, b, s, t \in A$ *such that* $x = t \bullet a \bullet b \bullet s$ *and* $y = t \bullet b \bullet a \bullet s$*. Note that* $\sim_0$ *is reflexive and symmetric.*
2. *The product* $\bullet_{ab}$ *in* A_{ab} *is*
$$[x] \bullet_{ab} [y] = [x \bullet y].$$
3. *The identity is* $e_{ab} = [e]$*.*

Moreover, the map $\rho_A \colon A \to A_{ab}$ *such that*

$$\rho_A(x) = [x]$$

is a monoid morphism.

Definition 4.1.9 (Abelianisation of a monoid)**.** *Let* $A = (A, \bullet, e)$ *be a monoid, the abelianisation of* A *is* $A_{\text{ab}} = (A_{ab}, \bullet_{ab}, e_{ab})$ *that is defined as in Lem.4.1.8.*

4.1.1 Proof of Lemma 4.1.8

First of all let us prove that $[x] \bullet_{ab} [y] = [x \bullet y]$ is well-defined, namely that it is independent of the choice of representatives. We want to prove if $x \sim x'$ and $y \sim y'$, then

$$x \bullet y \sim x' \bullet y'$$

Suppose $x \sim x'$. So there exists $n \in \mathbb{N}$, $(t_1, t_2, \ldots, t_n) \in A^n$, $(a_1, a_2, \ldots, a_n) \in A^n$, $(b_1, b_2, \ldots, b_n) \in A^n$, $(s_1, s_2, \ldots, s_n) \in A^n$ and $(x_1, x_2, \ldots, x_n) \in A^n$, such that

$$\begin{aligned} x = x_1 =& t_1 \bullet a_1 \bullet b_1 \bullet s_1, \\ x_2 =& t_1 \bullet b_1 \bullet a_1 \bullet s_1 \\ =& t_2 \bullet a_2 \bullet b_2 \bullet s_2, \\ x_3 =& t_2 \bullet b_2 \bullet a_2 \bullet s_2 \\ =& t_3 \bullet a_3 \bullet b_3 \bullet s_3, \\ \vdots& \\ x' = x_n =& t_{n-1} \bullet b_{n-1} \bullet a_{n-1} \bullet s_{n-1} \\ =& t_n \bullet a_n \bullet b_n \bullet s_n. \end{aligned}$$

So,

$$x = x_1 \sim_0 x_2, x_2 \sim_0 x_3, \ldots, x_{n-1} \sim_0 x_n = x'.$$

Similarly, suppose $y \sim y'$, so there are $m \in \mathbb{N}$, $(h_1, h_2, \ldots, h_m) \in A^m$, $(c_1, c_2, \ldots, c_m) \in$

A^m, $(d_1, d_2, \ldots, d_m) \in A^m$, $(k_1, k_2, \ldots, k_m) \in A^m$ and $(y_1, y_2, \ldots, y_m) \in A^m$, such that

$$\begin{aligned} y = y_1 =& h_1 \bullet c_1 \bullet d_1 \bullet k_1, \\ y_2 =& h_1 \bullet d_1 \bullet c_1 \bullet k_1 \\ =& h_2 \bullet c_2 \bullet d_2 \bullet k_2, \\ y_3 =& h_2 \bullet d_2 \bullet c_2 \bullet k_2 \\ =& h_3 \bullet c_3 \bullet d_3 \bullet k_3, \\ &\vdots \\ y' = y_m =& h_{m-1} \bullet d_{m-1} \bullet c_{m-1} \bullet k_{m-1} \\ =& h_m \bullet c_m \bullet d_m \bullet k_m. \end{aligned}$$

So,

$$y = y_1 \sim_0 y_2, y_2 \sim_0 y_3, \ldots, y_{m-1} \sim_0 y_m = y'.$$

Now, we prove that $x \bullet y \sim x' \bullet y'$. We first prove $x \bullet y \sim x' \bullet y$. Then we prove $x' \bullet y \sim x' \bullet y'$.

$$\begin{aligned} x \bullet y = x_1 \bullet y =& t_1 \bullet a_1 \bullet b_1 \bullet (s_1 \bullet y), \\ x_2 \bullet y =& t_1 \bullet b_1 \bullet a_1 \bullet (s_1 \bullet y) \\ =& t_2 \bullet a_2 \bullet b_2 \bullet (s_2 \bullet y), \\ x_3 \bullet y =& t_2 \bullet b_2 \bullet a_2 \bullet (s_2 \bullet y) \\ =& t_3 \bullet a_3 \bullet b_3 \bullet (s_3 \bullet y), \\ &\vdots \\ x' \bullet y = x_n \bullet y =& t_{n-1} \bullet b_{n-1} \bullet a_{n-1} \bullet (s_{n-1} \bullet y) \\ =& t_n \bullet a_n \bullet b_n \bullet (s_n \bullet y). \end{aligned}$$

So,

$$x \bullet y = x_1 \bullet y \sim_0 x_2 \bullet y, x_2 \bullet y \sim_0 x_3 \bullet y, \ldots, x_{n-1} \bullet y \sim_0 x_n \bullet y = x' \bullet y.$$

We have just proven

$$x \bullet y \sim x' \bullet y.$$

Analogously

$$\begin{aligned}
x' \bullet y = x' \bullet y_1 =& (x' \bullet h_1) \bullet c_1 \bullet d_1 \bullet k_1, \\
x' \bullet y_2 =& (x' \bullet h_1) \bullet d_1 \bullet c_1 \bullet k_1 \\
=& (x' \bullet h_2) \bullet c_2 \bullet d_2 \bullet k_2, \\
x' \bullet y_3 =& (x' \bullet h_2) \bullet d_2 \bullet c_2 \bullet k_2 \\
=& (x' \bullet h_3) \bullet d_3 \bullet c_3 \bullet k_3, \\
&\vdots \\
x' \bullet y' = x' \bullet y_m =& (x' \bullet h_{m-1}) \bullet d_{m-1} \bullet c_{m-1} \bullet k_{m-1} \\
=& (x' \bullet h_m) \bullet c_m \bullet d_m \bullet k_m.
\end{aligned}$$

So,

$$x' \bullet y = x' \bullet y_1 \sim_0 x' \bullet y_2, x' \bullet y_2 \sim_0 x' \bullet y_3, \dots, x' \bullet y_{m-1} \sim_0 x' \bullet y_m = x' \bullet y'.$$

This mean

$$x' \bullet y \sim x' \bullet y'.$$

Therefore

$$x \bullet y \sim x' \bullet y'.$$

Secondly, we want to prove associativity. Let $x, y, z \in A$

$$\begin{aligned}
([x] \bullet_{ab} [y]) \bullet_{ab} [z] =& [x \bullet y] \bullet_{ab} [z] \\
=& [(x \bullet y) \bullet z] \\
=& [x \bullet (y \bullet z)] \\
=& [x] \bullet_{ab} ([y] \bullet_{ab} [z]).
\end{aligned}$$

Now, we want to prove the unit axiom

$$\begin{aligned}[e_{ab}] \bullet_{ab} [x] =& [e \bullet x] \\ =& [x], \\ [x] \bullet_{ab} [e_{ab}] =& [x \bullet e] \\ =& [x].\end{aligned}$$

The map $\rho_A \colon A \to A_{ab}$ is a monoid morphism. Let $x, y \in A$, we have

$$\begin{aligned}\rho_A(x \bullet y) =& [x \bullet y] \\ =& [x] \bullet_{ab} [y] \\ =& \rho_A(x) \bullet_{ab} \rho_A(y), \\ \rho_A(e) =& [e] \\ =& e_{ab}.\end{aligned}$$

□

4.1.2 Examples of abelianisation of monoids

Example 4.1.10. *Given a set X, $(X^*, \star, \emptyset)$ is a monoid where*

1. $\forall i \in \{1, 2, ..., m\}$

$$X^* = \{[x_1][x_2]...[x_m] \mid m \in \mathbb{Z},\, x_i \in X\} \cup \{\emptyset\}$$

is a set;

2. *The monoid operation is such that*

$$\star \colon X^* \times X^* \to X^*$$
$$[x_1][x_2] \dots [x_n] \star [y_1][y_2] \dots [y_m] = [x_1] \dots [x_n][y_1] \dots [y_m].$$

If $[x_1][x_2]\dots[x_n]=\emptyset$,

$$\emptyset\star[y_1][y_2]\dots[y_m]\mapsto[y_1][y_2]\dots[y_m].$$

If $[y_1][y_2]\dots[y_m]=\emptyset$,

$$[x_1][x_2]\dots[x_n]\star\emptyset\mapsto[x_1][x_2]\dots[x_n].$$

3. $\emptyset\in X^*$ *is the identity.*

Proof. $(X^*,\star,\emptyset)$ are satisfied the monoid axioms

1. Associativity, for all $u,v,w\in X^*$, such that $u=[x_1]\dots[x_n]$, $v=[y_1]\dots[y_m]$ and $w=[z_1]\dots[z_r]$.

$$\begin{aligned}(u\star v)\star w=&([x_1]\dots[x_n][y_1]\dots[y_m])\star[z_1]\dots[z_r]\\=&[x_1]\dots[x_n][y_1]\dots[y_m][z_1]\dots[z_r],\\u\star(v\star w)=&[x_1]\dots[x_n]\star([y_1]\dots[y_m][z_1]\dots[z_r])\\=&[x_1]\dots[x_n][y_1]\dots[y_m][z_1]\dots[z_r].\end{aligned}$$

Therefore
$$(u\star v)\star w=u\star(v\star w).$$

2. Unit, for every $w\in X^*$,
$$\emptyset\star w=w\star\emptyset=w.$$

□

Example 4.1.11. *Consider the monoid* $A=(\{x,y\}^*,\star,\phi\})$*, define*

$$\{x,y\}^+=\{x^ny^m\mid n,m\in\mathbb{N}\}.$$

So, $(\{x,y\}^+, *, x^0y^0)$ *where*

$$x^n y^m * x^{n'} y^{m'} = x^{n+n'} y^{m+m'}$$

is clearly an abelian monoid. We have an isomorphism

$$F_{\mathrm{ab}}(\{x,y\}^*, \star, \phi\}) \xrightarrow{f} (\{x,y\}^+, *, x^0y^0),$$

where

$$f([w]) = x^{\chi_x(w)} y^{\chi_y(w)}.$$

Here $\chi_x(w)$ *is the number of* x*'s in the word* w*,* $\chi_y(w)$ *is the number of* y*'s in the word* w*.*

Proof. 1. We want to prove that f is well defined i.e,

$$\text{if } w \sim_0 w', \text{ then } \chi_x(w) = \chi_x(w') \text{ and } \chi_y(w) = \chi_y(w')$$

Suppose $w \sim_0 w'$, then there are $a, b, s, t \in \{x,y\}^*$ such that $w = t \star a \star b \star s$, $w' = t \star b \star a \star s$ clearly this operation preserves $\chi_x(w)$ and $\chi_y(w)$. Now we want to show that

$$\text{if } w \sim w', \text{ then } \chi_x(w) = \chi_x(w') \text{ and } \chi_y(w) = \chi_y(w').$$

Suppose $w \sim w'$, then there are $w_1, w_2, \ldots, w_n$, such that

$$w = w_1 \sim_0 w_2, w_2 \sim_0 w_3, \ldots, w_{n-1} \sim_0 w_n = w'.$$

Then from this $\chi_x(w) = \chi_x(w')$ and $\chi_y(w) = \chi_y(w')$.

2. f is a monoid map.

$$\begin{aligned} f([w]\star[w']) &= x^{\chi_x(w\star w')}y^{\chi_y(w\star w')} \\ &= x^{\chi_x(w)+\chi_x(w')}y^{\chi_y(w)+\chi_y(w')} \\ &= x^{\chi_x(w)}y^{\chi_y(w)}x^{\chi_x(w')}y^{\chi_y(w')} \\ &= f([w]) * f([w']), \\ f(\phi) &= x^{\chi_x(\phi)}y^{\chi_y(\phi)} \\ &= x^0y^0. \end{aligned}$$

3. The map f is surjective. For each x^ny^m, there is a word $w \in \{x,y\}^*$ such that $\chi_x(w) = n$, $\chi_y(w) = m$, e.g.

$$w = \underbrace{x\ldots x}_{n \text{ times}}\underbrace{y\ldots y}_{m \text{ times}}.$$

4. The map f is injective. Let $w, w' \in A$ and suppose that

$$f([w]) = f([w']).$$

So,

$$\chi_x(w) = \chi_x(w'),$$

and

$$\chi_y(w) = \chi_y(w').$$

Let us argue that $[w] \sim [w']$. Because w and w' contain the same number of x's and y's, we can go from the word w to the word w' by a finite number of operations flipping the order of consecutive symbols x and y. If w and w' are connected by a single flip of consecutive symbols, then, there exist words α and α' and $w = \alpha\star x\star y\star\alpha'$ and $w' = \alpha\star y\star x\star\alpha'$, or the other way around. If we have $n \in \mathbb{N}$ flips, we do one flip every time to have series of equivalent words i.e,

$$w = w_1 \sim w_2, w_2 \sim w_3, \ldots, w_{n-1} \sim w_n = w'.$$

Then f is an isomorphism and $(\{x,y\}^+, *, x^0y^0)$ is isomorphic to the abelian monoid of the monoid $A = (\{x,y\}^*, \star, \emptyset)$. $\square$

Remark 4.1.12. *Instead of this section we can apply the previous section of quotient categories to define abelianisation of a monoid. We can see a monoid* $(A, \bullet, e)$ *as a category* $\mathcal{C}$ *with one object* x *such that* $\hom_{\mathcal{C}}(x,x) = A$*. Consider the relation* W *let* $y, z \in A$*, we say* $y \sim_W z$*, if there exist* $a, b \in A$*, such that* $y = a \bullet b$ *and* $z = b \bullet a$*. Then we can define the transitive closure* $\overline{\overline{W}}$ *of the relation* $\overline{W}$*, then we apply Theorem 3.5.5. Hence we can have the quotient category* $\mathbb{C}/\overline{\overline{W}}$ *which is the abelianisation of the monoid* $(A, \bullet, e)$*.*

4.1.3 Monoid abelianisation as a functor

In this section we define abelianisation functor from the category of monoids $\mathcal{M}$ to the category of abelian monoids $\mathcal{M}_{\text{ab}}$, which is left adjoint to the inclusion functor G from $\mathcal{M}_{\text{ab}}$ to $\mathcal{M}$.

Lemma 4.1.13. *Let* $g\colon (A, \bullet, e) \to (B, \bullet, e)$ *be a monoid map and let* $x, x' \in A$ *and* $x \sim_0 x'$*, then*

$$g(x) \sim_0 g(x').$$

Proof. Suppose $x \sim_0 x'$, so there exist $a, b, s, t \in A$ such that

$$x = t \bullet a \bullet b \bullet s,$$

and

$$x' = t \bullet b \bullet a \bullet s.$$

So, we have

$$\begin{aligned} g(x) &= g(t \bullet a \bullet b \bullet s) \\ &= g(t) \bullet g(a) \bullet g(b) \bullet g(s), \\ g(x') &= g(t \bullet b \bullet a \bullet s) \\ &= g(t) \bullet g(b) \bullet g(a) \bullet g(s). \end{aligned}$$

Hence

$$g(x) \sim_0 g(x').$$

□

Corollary 4.1.14. *Let* $g\colon (A, \bullet, e) \to (B, \bullet, e)$ *be a monoid map and let* $x, x' \in A$ *and* $x \sim x'$*, then*

$$g(x) \sim g(x').$$

Proof. Suppose $x \sim x'$. So there exists $n \in \mathbb{N}$ and $(x_1, x_2, \ldots, x_n) \in A^n$, such that

$$x = x_1 \sim_0 x_2, x_2 \sim_0 x_3, \ldots, x_{n-1} \sim_0 x_n = x'.$$

By Lemma 4.1.13 we have

$$g(x) = g(x_1) \sim_0 g(x_2), g(x_2) \sim_0 g(x_3), \ldots, g(x_{n-1}) \sim_0 g(x_n) = g(x').$$

Hence

$$g(x) \sim g(x').$$

□

Proposition 4.1.15. *Let* $(A, \bullet, e)$ *and* $(B, \bullet, e)$ *be monoids and let* $g\colon A \to B$ *be a monoid map. Then there is a monoid map* $g_{\mathrm{ab}}\colon A_{\mathrm{ab}} \to B_{\mathrm{ab}}$*, such that, given* $x \in A$

$$g_{\mathrm{ab}}([x]) := [g(x)].$$

Furthermore the next diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle\rho_A}\downarrow & & \downarrow{\scriptstyle\rho_B} \\ A_{\mathrm{ab}} & \xrightarrow[g_{\mathrm{ab}}]{} & B_{\mathrm{ab}} \end{array}$$

So the crucial fact that if $G\colon \mathcal{M}_{\mathrm{ab}} \to \mathcal{M}$ is the inclusion functor, then the family of maps $\rho_A\colon A \to G(F(A)) = A_{ab}$, where A is a monoid, is a natural transformation $\mathrm{id} \to G \circ F$.

Proof. Note that

$$g_{\mathrm{ab}}([x]) = [g(x)]$$

is well defined, because if $x \sim x'$, then $g(x) \sim g(x')$, see 4.1.14. Also, the diagram commutes

$$\begin{array}{ccc} x & \xmapsto{g} & g(x) \\ {\scriptstyle\rho_A}\downarrow & & \downarrow{\scriptstyle\rho_B} \\ [x] & \xmapsto[g_{\mathrm{ab}}]{} & [g(x)] \end{array}$$

□

Proposition 4.1.16. *Let $\mathcal{M}$ be the category of monoids, as defined in 4.1.6 and $\mathcal{M}_{\mathrm{ab}}$ the category of abelian monoids as defined in 4.1.7. There is a functor*

$$F\colon \mathcal{M} \to \mathcal{M}_{\mathrm{ab}},$$

where

1. *For all $A \in ob(\mathcal{M})$, $F(A) = A_{ab}$.*
2. *For all $f \in \hom_{\mathcal{M}}(A, B), F(f) = f_{ab}$.*

Proof. We want to prove F is a functor.

1. Let A, B and C be monoids. If $f \in \hom_{\mathcal{M}}(A, B), g \in \hom_{\mathcal{M}}(B, C)$, we have,

for all $x \in A$

$$F(g \circ f)([x]) = (g \circ f)_{\mathrm{ab}}([x]) = [(g \circ f)(x)] \text{ by 4.1.15}$$
$$= [g(f(x)] = g_{\mathrm{ab}}([f(x)]) \text{ by 4.1.15}$$
$$= g_{\mathrm{ab}}(f_{\mathrm{ab}}([x]) = (g_{\mathrm{ab}} \circ f_{\mathrm{ab}})([x])$$
$$= (F(g) \circ F(f))([x]).$$

2. For all $A \in ob(\mathcal{M})$, given $x \in A$

$$F(\mathrm{id}_A([x])) = (\mathrm{id}_A)_{\mathrm{ab}}([x]) = [\mathrm{id}_A(x)] = \mathrm{id}_{A_{\mathrm{ab}}}([x]) = \mathrm{id}_{F(A)}([x]).$$

□

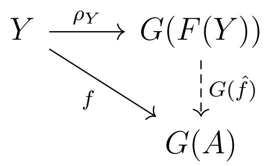
In the proposition below, G denotes the inclusion functor $\mathcal{M}_{ab} \to \mathcal{M}$. This gives that ρ_Y is a universal arrow.

Proposition 4.1.17. *Given any monoid $(Y, \bullet_Y, e_Y)$ there exists an abelian monoid $F(Y)$ and a monoid map $\rho_Y: Y \to G(F(Y))$ satisfying the following universal property*

Given any abelian monoid $(A, \bullet_A, e_A)$, and a monoid map

$$f: Y \to G(A),$$

there exists a unique monoid map $\hat{f}: F(Y) \to A$ that makes the diagram commute.



Moreover the family of all $\rho_Y: Y \to G \circ F(Y)$ is a natural transformation . I.e, we have an adjunction (F, G, ρ).

Proof. Let Y be a monoid. So there exists a monoid map

$$\rho_Y : Y \to F(Y)$$
$$y \mapsto [y]$$

Let A be an abelian monoid. Consider a monoid map $f: Y \to G(A)$. First, we want to prove a map $\hat{f}: F(Y) \to A$ is exists. We define $\hat{f}$ by

$$\hat{f}: F(Y) \to A$$
$$[y] \mapsto f(y),$$

that is independence of representatives. If $x \sim_0 y$, so there exist $a, b, s, t \in Y$ such that

$$x = t \bullet a \bullet b \bullet s$$

and

$$y = t \bullet b \bullet a \bullet s.$$

$$\begin{aligned}\hat{f}([x]) = f(x) &= f(t \bullet a \bullet b \bullet s)\\ &= f(t) \bullet f(a) \bullet f(b) \bullet f(s).\\ \hat{f}([y]) = f(y) &= f(t \bullet b \bullet a \bullet s)\\ &= f(t) \bullet f(b) \bullet f(a) \bullet f(s).\end{aligned}$$

Therefore,

$$f(x) = f(y).$$

The map $\hat{f}$ is a monoid map because

1. For all $x, y \in Y$,

$$\begin{aligned}\hat{f}([x] \bullet_{ab} [y]) &= \hat{f}([x \bullet_Y y]) = f(x \bullet_Y y)\\ &= f(x) \bullet_A f(y) = \hat{f}([x]) \bullet_A \hat{f}([y]).\end{aligned}$$

2. $\hat{f}([e_Y]) = f(e_Y) = e_A$.

Second, we want to prove the uniqueness. Suppose there is a monoid map

$$\hat{f}_1 \colon F(Y) \to A$$

which makes the diagram commute

$$\begin{array}{ccc} Y & \xrightarrow{\rho_Y} & G(F(Y)) \\ & \searrow^{f} & \downarrow{\scriptstyle G(\hat{f}_1)} \\ & & G(A) \end{array}$$

So,

$$\hat{f}_1([y]) = f(y).$$

Therefore there is a unique monoid map i.e,

$$\hat{f} = \hat{f}_1.$$

From 4.1.15, for all $A, B \in ob(\mathcal{M})$, the next diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \rho_B} \\ A_{\text{ab}} & \xrightarrow[g_{\text{ab}}]{} & B_{\text{ab}} \end{array}$$

where $G \circ F(A) = A_{\text{ab}}$. Then $\rho \colon \text{id} \to G \circ F$ is a natural transformation, i.e, we have an adjunction (F, G, ρ). □

4.2 Free Monoid

Lemma 4.2.1. *(See for example [BW90]).We have a forgetful functor*

$$U \colon \mathcal{M} \to \mathcal{SETS},$$

from the category of monoids to the category of sets such that:

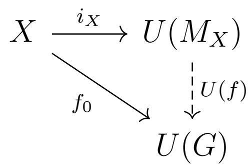
$$U(G, \bullet, e) = G,$$

and sends a monoid-map to underlying set-map as shown below:

$$U((G, \bullet, e) \xrightarrow{f} (G', \bullet', e')) = (G \xrightarrow{f} G').$$

Definition 4.2.2. *(See for example [BW90]). A* free-monoid-triple *on a set X is a triple (X, M_X, i_X) where M_X is a monoid, and $i_X : X \to U(M_X)$ is a map of sets that satisfies the following universal property*

> Given any monoid G, and any set map $f_0 : X \to U(G)$, there is a unique monoid map $f\colon M_X \to G$ that makes the diagram below, in the category of sets, *commute:*



U is the forgetful functor $U\colon \mathcal{M} \to \mathcal{SETS}$ and $i_X\colon X \to U(M_X)$ is a universal arrow. The forgetful functor U is a right adjoint if for all objects X of the category $\mathcal{SETS}$ there is a universal arrow $i_X\colon X \to U(M_X)$.

Let us prove that free monoid triples on a set X do exist. In order to prve that U is a right adjoint and in oder to construct a corresponding left adjoint.

Theorem 4.2.3. *(See for example [BW90]). Given a set X, (X, X^*, i) is a free-monoid-triple, where*

$$i_X\colon X \to U(X^*, \star, \emptyset)$$
$$x \mapsto [x].$$

Proof. We want to prove that given a monoid $(G, \bullet, e_G)$ and a set-map $f\colon X \to U(G)$, there is a unique monoid map $\bar{f}\colon (X^*, \star, \emptyset) \to (G, \bullet, e_G)$, that makes the diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & U(X^*) \\ & \searrow^{f} & \downarrow{\scriptstyle U(\bar{f})} \\ & & U(G) \end{array}$$

1. Existence; define a map $\bar{f}$ by

$$\bar{f}\colon (X^*, \star, \emptyset) \to (G, \bullet, e_G)$$
$$[x_1] \ldots [x_m] \mapsto f(x_1) \bullet f(x_2) \bullet \ldots \bullet f(x_m).$$

$\bar{f}$ is a monoid map since:

i. for every word $w, w' \in X$, we have

$$\bar{f}(w \star w') = \bar{f}(w) \bullet \bar{f}(w').$$

That is because: if $w \neq \emptyset$ and $w' \neq \emptyset$, then

$$\begin{aligned} &\bar{f}([x_1] \ldots [x_n] \star [y_1] \ldots [y_m]) \\ &\qquad = \bar{f}([x_1] \ldots [x_n][y_1] \ldots [y_m]) \\ &\qquad = f(x_1) \bullet f(x_2) \bullet \ldots \bullet f(x_n) \bullet f(y_1) \bullet \ldots \bullet f(y_m) \\ &\qquad = (f(x_1) \bullet f(x_2) \bullet \ldots \bullet f(x_n)) \bullet (f(y_1) \bullet \ldots \bullet f(y_m)) \\ &\qquad = \bar{f}([x_1] \ldots [x_n]) \bullet \bar{f}([y_1] \ldots [y_m]). \end{aligned}$$

if $w' = \emptyset$, then

$$\bar{f}(w \star \emptyset) = \bar{f}(w),$$
$$\bar{f}(w) \bullet \bar{f}(\emptyset) = \bar{f}(w) \bullet e_G = \bar{f}(w).$$

if $w = \emptyset$, then

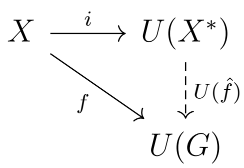
$$\bar{f}(\emptyset \star w') = \bar{f}(w'),$$
$$\bar{f}(\emptyset) \bullet \bar{f}(w') = e_G \bullet \bar{f}(w') = \bar{f}(w').$$

ii. $\bar{f}(\emptyset) = f(\emptyset) = e_G$.

2. Uniqueness; suppose there is a monoid map

$$\hat{f}\colon (X^*, \star, \emptyset) \to (G, \bullet, e_G)$$

which makes the next diagram commute:



So, as $\hat{f}$ is a monoid map

$$\begin{aligned}\hat{f}([x_1][x_2]\ldots[x_n]) &= \hat{f}([x_1]) \bullet \hat{f}([x_2]) \bullet \ldots \bullet \hat{f}([x_m]) \\ &= f(x_1) \bullet f(x_2) \bullet \ldots \bullet f(x_m).\end{aligned}$$

Therefore there is a unique monoid map i.e,

$$\bar{f} = \hat{f}.$$

$\square$

The main aim of previous chapters was as preparation for the more involved free monoidal categories.

Chapter 5

$\frac{1}{2}$-monoidal categories and monoidal categories

In this chapter we discuss strict monoidal categories from the point of view of $\frac{1}{2}$-monoidal categories.

5.1 $\frac{1}{2}$-monoidal categories

In this section we set up some machinery for what we call $\frac{1}{2}$-monoidal categories . This is useful for technical aspects of (free)-monoidal categories.

5.1.1 The cat's whiskers: $\frac{1}{2}$-monoidal categories

The idea of $\frac{1}{2}$-monoidal category comes for example from Street (Edited by M. Hazewinkel) [Haz96, chapter 15]. In this paper Street define a sesquicategory[1] as an analogous object to a 2-category, however skipping the interchange law. Also see [PR97, Cra99].

We prove in section 5.4.1 that a slideable $\frac{1}{2}$-monoidal category gives a strict monoidal category. After that in section 6 we define the free $\frac{1}{2}$-monoidal category over monoidal

[1] $(1 + \frac{1}{2})$ is the meaning of "sesqui"

graph (monoidal graphs are defined in 6.1.1).

Definition 5.1.1 (Pre-$\frac{1}{2}$-monoidal structure)**.** *(See [PR97]). Let*

$$\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id}_{\_})$$

be a category. A pre-$\frac{1}{2}$-monoidal structure

$$(\mathcal{C}, I, \otimes_0, \Theta_{(-)}, {}_{(-)}\Theta)$$

in $\mathcal{C}$ is given by

1. *For each pair of objects x and y in $ob(\mathcal{C})$, another object $x \otimes_0 y$ in $ob(\mathcal{C})$.*
2. *An object $I \in ob(\mathcal{C})$.*
3. *For each morphism $f\colon x \to y$ and object z, a morphism $x \otimes_0 z \xrightarrow{\Theta z(f)} y \otimes_0 z$. We will use the notation*

$$\left(x \otimes_0 z \xrightarrow{\Theta z(f)} y \otimes_0 z\right) = \left(x \otimes_0 z \xrightarrow{f\Theta z} y \otimes_0 z\right).$$

4. *For each morphism $f\colon x \to y$ and object z, a morphism $z \otimes_0 x \xrightarrow{z\Theta(f)} z \otimes_0 y$. We will use the notation*

$$\left(z \otimes_0 x \xrightarrow{z\Theta(f)} z \otimes_0 y\right) = \left(z \otimes_0 x \xrightarrow{z\Theta f} z \otimes_0 y\right).$$

Definition 5.1.2 ($\frac{1}{2}$-monoidal category)**.** *(See [PR97]). Let $\mathcal{C}$ be a category exactly as above. A pre-$\frac{1}{2}$-monoidal structure on $\mathcal{C}$*

$$(\mathcal{C}, I, \otimes_0, \Theta_{(-)}, {}_{(-)}\Theta)$$

gives a $\frac{1}{2}$-monoidal category

$$(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

if the following is satisfied

1. $(ob(\mathcal{C}), \otimes_0, I)$ *is a monoid.*

2. *Let* A *be an object of* $\mathcal{C}$*. Then the pair of assignments*

   - $ob(\mathcal{C}) \to ob(\mathcal{C})$ *such that* $x \mapsto x \otimes_0 A$*,*
   - *given objects* x, y*, consider*

$$f \in \hom_{\mathcal{C}}(x, y) \mapsto \Theta_A(f)\colon x \otimes_0 A \to y \otimes_0 A$$

   is a functor $\mathcal{C} \to \mathcal{C}$*. (We will denote this functor by* $\#_A\colon \mathcal{C} \to \mathcal{C}$*.)*

3. *Let* A *be an object of* $\mathcal{C}$*. Then the pair of assignments*

   - $ob(\mathcal{C}) \to ob(\mathcal{C})$ *such that* $x \mapsto A \otimes_0 x$*.*
   - *Given objects* x, y*, consider*

$$f \in \hom_{\mathcal{C}}(x, y) \mapsto {}_A\Theta(f)\colon A \otimes_0 x \to A \otimes_0 y.$$

   is a functor $\mathcal{C} \to \mathcal{C}$*. (We will denote this functor by* ${}_A\#\colon \mathcal{C} \to \mathcal{C}$*.)*

4. *For each* $A, B \in ob(\mathcal{C})$

$${}_A\# \circ {}_B\# = {}_{A \otimes_0 B}\#. \tag{5.1}$$

$$\#_A \circ \#_B = \#_{B \otimes_0 A}. \tag{5.2}$$

$$\#_A \circ {}_B\# = {}_B\# \circ \#_A. \tag{5.3}$$

$$\#_I = \mathrm{id}_{\mathcal{C}}. \tag{5.4}$$

$${}_I\# = \mathrm{id}_{\mathcal{C}}. \tag{5.5}$$

   Here $\mathrm{id}_{\mathcal{C}}$ *is the identity functor* $\mathcal{C} \to \mathcal{C}$*.*

Example 5.1.3. *Fix an abelian group* $(G, \bullet)$*. We have a category*

$$\mathcal{C} = (\mathbb{Z}, \hom_{\mathcal{C}}(\_,\_), \bullet, 1).$$

1. *Set of objects is* $\mathbb{Z}$*.*

2. *For all* $m, n \in \mathbb{Z}$, $\hom_{\mathcal{C}}(m,n) = \begin{cases} \phi, & \textit{if } m \neq n \\ G, & \textit{if } m = n. \end{cases}$

3. *The composition in this category is given by the product in the group* G. *This means that given any* $n \in \mathbb{Z}$

$$\hom_{\mathcal{C}}(n,n) \times \hom_{\mathcal{C}}(n,n) \to \hom_{\mathcal{C}}(n,n)$$
$$(n \xrightarrow{g} n, n \xrightarrow{h} n) \mapsto n \xrightarrow{g \bullet h} n.$$

4. *For each* $n \in \mathbb{Z}$ *the identity morphisms* $\mathrm{id}_n = 1_G \in \hom_{\mathcal{C}}(n,n)$.

We have $\frac{1}{2}$*-monoidal category such that (we denote the product of two integers* a *and* b *as* $a.b$*)*

1. $1 \in \mathbb{Z}$ *is the identity object.*

2. *For all* $n, m \in \mathbb{Z}$*;*

$$m \otimes_0 n = m.n$$

3. *For all* $A \in \mathbb{Z}$ *and* $n \xrightarrow{g} n$*;*

$$\#_A(n \xrightarrow{g} n) = n.A \xrightarrow{g^A} n.A$$

$$_A\#(n \xrightarrow{g} n) = A.n \xrightarrow{g^A} A.n$$

This satisfies the $\frac{1}{2}$*-monoidal category axioms*

1. *We want to prove* $\#_A$ *and* $_A\#$ *are functors for all* $A, n \in \mathbb{Z}$ *and for all* $g, h \in \hom_{\mathcal{C}}(n,n)$ *we have*

$$\begin{aligned} \#_A(g \bullet h) &= n.A \xrightarrow{(g \bullet h)^A} n.A \\ &= n.A \xrightarrow{g^A \bullet h^A} n.A \\ &= (n.A \xrightarrow{g^A} n.A) \bullet (n.A \xrightarrow{h^A} n.A) \\ &= \#_A(g) \bullet \#_A(h) \end{aligned}$$

Therefore $\#_A$ is a functor, by similarity $_A\#$ is a functor.

2. *To verify the axiom* (5.1)*, for each* $A, B, n \in ob(\mathcal{C})$ *and* $g \in \hom_{\mathcal{C}}(n, n)$.

$$\begin{aligned} {}_A\# \circ {}_B\#(n \xrightarrow{g} n) &= {}_A\#(B.n \xrightarrow{g^B} B.n) \\ &= n.B.A \xrightarrow{(g^B)^A} n.B.A \\ &= A.B.n \xrightarrow{g^{A.B}} A.B.n \\ &= {}_{A.B}\#(n \xrightarrow{g} n) \\ &= {}_{A\otimes_0 B}\#(n \xrightarrow{g} n). \end{aligned}$$

3. *To verify the axiom* (5.2)*, for each* $A, B, n \in ob(\mathcal{C})$ *and* $g \in \hom_{\mathcal{C}}(n, n)$*;*

$$\begin{aligned} \#_A \circ \#_B(n \xrightarrow{g} n) &= \#_A(n.B \xrightarrow{g^B} n.B) \\ &= n.B.A \xrightarrow{(g^B)^A} n.B.A \\ &= n.B.A \xrightarrow{g^{A.B}} n.B.A \\ &= \#_{B.A}(n \xrightarrow{g} n) \\ &= \#_{B\otimes_0 A}(n \xrightarrow{g} n). \end{aligned}$$

4. *To verify the axiom* (5.3)*, for each* $A, B, n \in ob(\mathcal{C})$ *and* $g \in \hom_{\mathcal{C}}(n, n)$*;*

$$\begin{aligned} \#_A \circ {}_B\#(n \xrightarrow{g} n) &= \#_A(B.n \xrightarrow{g^B} B.n) \\ &= n.B.A \xrightarrow{(g^B)^A} n.B.A \\ &= n.B.A \xrightarrow{(g^A)^B} n.B.A \\ &= {}_B\# \circ \#_A(n \xrightarrow{g} n). \end{aligned}$$

5. *To verify the axiom* (5.4) *and* (5.5)*,*

$$\begin{aligned} \#_1(n \xrightarrow{g} n) &= n \xrightarrow{g} n, \\ {}_1\#(n \xrightarrow{g} n) &= n \xrightarrow{g} n. \end{aligned}$$

Example 5.1.4. *Let* $(A, \bullet, 1)$ *be a monoid. We have a category* $\mathcal{C}$*, such that*

1. *the set of objects is* $\{1\}$*,*

2. $\hom_{\mathcal{C}}(1,1) = A$*,*

3. *the composition in the category* $\mathcal{C}$ *is given by the product* $\bullet$ *in the monoid* $(A, \bullet, 1)$*,*

4. *the identity morphism is* $1 \in \hom_{\mathcal{C}}(1,1)$*.*

We have $\frac{1}{2}$*-monoidal category such that*

1. 1 *is the identity object,*

2. $1 \otimes_0 1 = 1$*,*

3. *for all* $g\colon 1 \to 1$

$$\#_1(1 \xrightarrow{g} 1) = 1 \xrightarrow{g} 1.$$
$$_1\#(1 \xrightarrow{g} 1) = 1 \xrightarrow{g} 1.$$

This satisfies the $\frac{1}{2}$*-monoidal category axioms.*

Definition 5.1.5. *Let* $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id})$ *be a category. A* $\frac{1}{2}$*-monoidal category* $(\mathcal{C}, \otimes_0, I, \#_{(-)}, {}_{(-)}\#)$ *is called* slideable *if given objects* $x, y.z, w$ *and a pair of morphisms* $f\colon x \to y$ *and* $g\colon z \to w$ *we have*

$$(f\Theta w) \star (x\Theta g) = (y\Theta g) \star (f\Theta z).$$

Where $f\Theta w\colon x \otimes_0 w \to y \otimes_0 w$*.*

This mean that the next diagram commutes

$$\begin{array}{ccc} x \otimes_0 z & \xrightarrow{x\Theta g} & x \otimes_0 w \\ {\scriptstyle f\Theta z}\downarrow & & \downarrow{\scriptstyle f\Theta w} \\ y \otimes_0 z & \xrightarrow[y\Theta g]{} & y \otimes_0 w \end{array}$$

Example 5.1.6. *The* $\frac{1}{2}$*-monoidal category in example 5.1.3 is slideable .*

Proof. Let $m, n \in \mathbb{Z}$ and $g\colon m \to m$, $h\colon n \to n$. We have

$$\begin{aligned}(g\Theta n) \bullet (m\Theta h) &= (m.n \xrightarrow{g^n} m.n) \bullet (m.n \xrightarrow{h^m} m.n)\\ &= (m.n \xrightarrow{h^m} m.n) \bullet (m.n \xrightarrow{g^n} m.n)\\ &= (m\Theta h) \bullet (g\Theta n).\end{aligned}$$

□

Example 5.1.7. *The $\frac{1}{2}$-monoidal category in example 5.1.4 is slideable if the monoid A is abelian.*

Proof. Let $g, h \in \hom_{\mathcal{C}}(1,1)$ we have

$$\begin{aligned}(g\Theta 1) \bullet (1\Theta h) &= (1 \xrightarrow{g} 1) \bullet (1 \xrightarrow{h} 1)\\ &= (1 \xrightarrow{h} 1) \bullet (1 \xrightarrow{g} 1)\\ &= (1\Theta h) \bullet (g\Theta 1).\end{aligned}$$

□

5.1.2 Matrix elements and $\frac{1}{2}$-monoidal categories

Definition 5.1.8 (Matrix elements)**.** *Let V and W be finite dimensional vector spaces with bases X and Y respectively. Let $f\colon V \to W$ be a linear map. The* matrix elements *of f with respect to basis X and Y, denoted (using Dirac notation)*

$$\langle x \mid f \mid y\rangle \in \mathbb{C}, \text{ where } x \in X, y \in Y$$

are defined by

$$f(x) = \sum_{y \in Y} \langle x \mid f \mid y\rangle\, y,$$

where $x \in X$.

We only consider pointed spaces, i,e V and W will be the free vector spaces $\mathbb{C}(X)$

and $\mathbb{C}(Y)$ over the finite sets X and Y. A linear map is uniquely specified by matrix elements. This mean if we have a function

$$\begin{aligned} X \times Y &\to \mathbb{C} \\ (x, y) &\mapsto \langle x \mid f \mid y \rangle, \end{aligned}$$

then we have a unique linear map

$$\mathbb{C}(X) \xrightarrow{f} \mathbb{C}(Y),$$

such that

$$\sum_{x \in X} a_x x \mapsto \sum_{x \in X} \sum_{y \in Y} a_x \langle x \mid f \mid y \rangle \, y.$$

Now, we discuss the matrix elements of a linear map

$$f \colon \mathbb{C}(X^m) \to \mathbb{C}(X^n).$$

Here X a nonempty finite set and $m, n \in \mathbb{N}$. In case $m, n \neq 0$, then

$$X^m = \{(x_1, x_2, \ldots, x_m) \mid x_1, \ldots, x_m \in X\}$$

is a basis of $\mathbb{C}(X^m)$ and

$$X^n = \{(y_1, y_2, \ldots, y_n) \mid y_1, \ldots, y_n \in X\}$$

is a basis of $\mathbb{C}(X^n)$. The matrix elements of f are denoted

$$\langle (x_1, x_2, \ldots, x_m) \mid f \mid (y_1, y_2, \ldots, y_n) \rangle \in \mathbb{C},$$

and the linear map f is such that on the basis X^m of $\mathbb{C}(X^m)$ we have

$$f(x_1, x_2, \ldots, x_m) = \sum_{(y_1,\ldots,y_n)\in X^n} \langle (x_1, x_2, \ldots, x_m) \mid f \mid (y_1, y_2, \ldots, y_n)\rangle \, (y_1, y_2, \ldots, y_n).$$

If $m = 0$ and $n > 0$, then $\mathbb{C}(X^0) \cong \mathbb{C}$. We will take $\{1\}$, to be the basis of $\mathbb{C}$. So given a linear map

$$f\colon \mathbb{C} \to \mathbb{C}(X^n),$$

the matrix elements of f are $\langle 1 \mid f \mid (y_1, y_2, \ldots, y_n)\rangle$, and the linear map f is such that on the basis $\{1\}$ of $\mathbb{C}$

$$f(1) = \sum_{(y_1,\ldots,y_n)\in X^n} \langle 1 \mid f \mid (y_1, y_2, \ldots, y_n)\rangle \, (y_1, y_2, \ldots, y_n).$$

If $n = 0$, so

$$f\colon \mathbb{C}(X^m) \to \mathbb{C},$$

the matrix elements of f are $\langle (x_1, x_2, \ldots, x_m) \mid f \mid 1\rangle$, and the linear map f is such that on the basis X^m of $\mathbb{C}(X^m)$

$$f(x_1, x_2, \ldots, x_m) = \langle (x_1, x_2, \ldots, x_m) \mid f \mid 1\rangle.$$

If $m = n = 0$, so

$$f\colon \mathbb{C} \to \mathbb{C}$$

the matrix elements of f are $\langle 1 \mid f \mid 1\rangle$, and the linear map is such that on the basis $\{1\}$ of $\mathbb{C}$ we have

$$f(1) = \langle 1 \mid f \mid 1\rangle.$$

Example 5.1.9. *Let* $X = \{a, b\}$. *Consider the linear map*

$$f\colon \mathbb{C}(X^2) \to \mathbb{C}(X^3)$$

such that on the bases X^2 *and* X^3 *we have.*

$$(x_1, x_2) \mapsto (x_1, x_2, x_2).$$

The basis of $\mathbb{C}(X^2)$ *is* $\{(a, a), (a, b), (b, a), (b, b)\}$*, and the basis of* $\mathbb{C}(X^3)$ *is*
$\{(a, a, a), (a, a, b), (a, b, b), (b, b, b), (b, b, a), (b, a, a), (a, b, a), (b, a, b)\}$*, so explicitly*

$$f(A_1(a, a) + A_2(a, b) + A_3(b, a) + A_4(b, b)) \\ = A_1(a, a, a) + A_2(a, b, b) + A_3(b, a, a) + A_4(b, b, b).$$

So explicitly

$$\langle (x, y) \mid f \mid (w, z, t) \rangle = \begin{cases} 1, & \textit{if } (x, y, y) = (w, z, t), \\ 0, & \textit{otherwise}. \end{cases} \tag{5.6}$$

Also, consider the linear map,

$$g\colon \mathbb{C} \to \mathbb{C}(X^2),$$

such that on the bases $\{1\}$ *of* $\mathbb{C}$*, we have*

$$1 \mapsto (a, a),$$

$$g(1) = 1(a, a) + 0(a, b) + 0(b, a) + 0(b, b).$$

So,

$$\langle 1 \mid g \mid (x,y)\rangle = \begin{cases} 1, & \textit{if } (x,y) = (a,a), \\ 0, & \textit{otherwise}. \end{cases}$$

Proposition 5.1.10. *Let X be a nonempty finite set, we have a category*

$$Vec_X = (\mathbb{N}, \hom_{Vec_X}(\_,\_), \circ, \mathrm{id}_\_),$$

where

1. *$ob(Vec_X) = \mathbb{N}$,*
2. *for each objects $m, n \in \mathbb{N}$,*

   $$\hom_{Vec_X}(m,n) = \hom_{Vec}(\mathbb{C}(X^m), \mathbb{C}(X^n)),$$

   where we put $\mathbb{C}(X^0) := \mathbb{C}$. Here $\hom_{Vec}$ is defined in 3.1.13
3. *for each linear maps $f \in \hom_{Vec_X}(m,n) = \hom_{Vec}(\mathbb{C}(X^m), \mathbb{C}(X^n))$ and $g \in \hom_{Vec_X}(n,s) = \hom_{Vec}(\mathbb{C}(X^n), \mathbb{C}(X^s))$, then*

   $$g \circ f \in \hom_{Vec_X}(m,s) = \hom_{Vec}(\mathbb{C}(X^m), \mathbb{C}(X^s)).$$

4. *for each object $m \in \mathbb{N}$ the identity of m in Vec_X is the identity map*

   $$\mathrm{id}_m \colon \mathbb{C}(X^m) \to \mathbb{C}(X^m),$$

   which is a linear map.

Proof. We have proved the category axioms in 3.1.13, where we prove that Vec is a category, since Vec_X is essentially a subcategory.

□

Remark It is a very interesting point that $\hom_{Vec}(V,W)$ does not 'remember' a basis of V, even if V was constructed using a basis. So if we want to give a concrete element explicitly then we probably first have to give a basis. On the other hand $\hom_{Vec_X}(m,n)$ comes with a basis for each underlying space. Thus although we gave $\hom_{Vec_X}(m,n)$ above by identifying it with a set of morphisms in Vec, this only works if we know X, and works *per* X.

Proposition 5.1.11. *Let X be a non-empty finite set . Consider the category*

$$Vec_X = (\mathbb{N}, \hom_{Vec_X}(\_,\_), \circ, \mathrm{id}_{\_}).$$

We have a slideable $\frac{1}{2}$-monoidal category

$$(Vec_X, +, 0, \#_{(-)}, {}_{(-)}\#),$$

where

1. *for each $m, n \in \mathbb{N}$, $m \otimes_0 n = m + n$;*
2. *$I = 0$;*
3. *consider objects $m, n \neq 0$. Suppose we have a linear map $f\colon \mathbb{C}(X^m) \to \mathbb{C}(X^n)$. We know that $X^m = \{(x_1, x_2, \dots, x_m) \mid x_1, \dots, x_m \in X\}$ is a basis of $\mathbb{C}(X^m)$ and $X^n = \{(y_1, y_2, \dots, y_n) \mid y_1, \dots, y_n \in X\}$ is a basis of $\mathbb{C}(X^n)$.*

   Let $A > 0$ be an object, then $\#_A(f)$ is the linear map

$$\#_A(f)\colon \mathbb{C}(X^{m+A}) \to \mathbb{C}(X^{n+A})$$

   such that on the basis X^{m+A} of $\mathbb{C}(X^{m+A})$ has the form

$$(x_1, x_2, \dots, x_m, a_1, a_2, \dots, a_A) \mapsto \sum_{(y_1,\dots,y_n)\in X^n} \langle (x_1, x_2, \dots, x_m) \mid f \mid (y_1, y_2, \dots, y_n)\rangle \, (y_1, y_2, \dots, y_n, a_1, a_2, \dots, a_A).$$

Therefore the matrix elements is

$$\langle (x_1, x_2, \dots, x_m, a_1, a_2, \dots, a_A) \mid \#_A(f) \mid (y_1, y_2, \dots, y_n, a'_1, a'_2, \dots, a'_A) \rangle$$
$$= \begin{cases} \langle (x_1, x_2, \dots, x_m) \mid f \mid (y_1, y_2, \dots, y_n) \rangle, & \textit{if } (a_1, \dots, a_A) = (a'_1, \dots, a'_A), \\ 0, & \textit{otherwise}. \end{cases}$$

If A=0, then

$$\#_A(f) = f.$$

If $m = 0$, and $A, n > 0$ the linear map

$$\#_A(f) \colon \mathbb{C}(X^A) \to \mathbb{C}(X^{n+A})$$

is such that on the basis X^A of $\mathbb{C}(X^A)$ we have

$$(a_1, a_2, \dots, a_A) \mapsto \sum_{(y_1, \dots, y_n) \in X^n} \langle 1 \mid f \mid (y_1, y_2, \dots, y_n) \rangle$$
$$(y_1, y_2, \dots, y_n, a_1, \dots, a_A).$$

The matrix elements is

$$\langle (a_1, a_2, \dots, a_A) \mid \#_A(f) \mid (y_1, y_2, \dots, y_n, a'_1, a'_2, \dots, a'_A) \rangle$$
$$= \begin{cases} \langle 1 \mid f \mid (y_1, y_2, \dots, y_n) \rangle, & \textit{if } (a_1, \dots, a_A) = (a'_1, \dots, a'_A), \\ 0, & \textit{otherwise}. \end{cases}$$

If $n = 0$,

$$\#_A(f) \colon \mathbb{C}(X^{m+A}) \to \mathbb{C}(X^A)$$

is such that on the basis X^{m+A} of $\mathbb{C}(X^{m+A})$,

$$(x_1, x_2 \dots, x_m, a_1, \dots, a_A) \mapsto \langle (x_1, x_2, \dots, x_m) \mid f \mid 1 \rangle \, (a_1, \dots, a_A).$$

The matrix elements is

$$\langle (x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_A) \mid \#_A(f) \mid (a'_1, a'_2, \ldots, a'_A) \rangle$$
$$= \begin{cases} \langle (x_1, x_2, \ldots, x_m) \mid f \mid 1 \rangle, & \textit{if } (a_1, \ldots, a_A) = (a'_1, \ldots, a'_A) \\ 0, & \textit{otherwise}. \end{cases}$$

4. *For each objects m, n, suppose we have a linear map $f\colon \mathbb{C}(X^m) \to \mathbb{C}(X^n)$. We know that $X^m = \{(x_1, x_2, \ldots, x_m) \mid x_1, \ldots, x_m \in X\}$ is a basis of $\mathbb{C}(X^m)$ and $X^n = \{(y_1, y_2, \ldots, y_n) \mid y_1, \ldots, y_n \in X\}$ is a basis of $\mathbb{C}(X^n)$. Let A be an object, then ${}_A\#(f)$ is the linear map*

$${}_A\#(f)\colon \mathbb{C}(X^{A+m}) \to \mathbb{C}(X^{A+n})$$

such that on the basis X^{A+m} of $\mathbb{C}(X^{A+m})$ we have

$$(a_1, a_2, \ldots, a_A, x_1, x_2, \ldots, x_m) \mapsto \sum_{(y_1, y_2, \ldots, y_n) \in X^n} \langle (x_1, x_2, \ldots, x_m) \mid f \mid (y_1, y_2, \ldots, y_n) \rangle$$
$$(a_1, a_2, \ldots, a_A, y_1, y_2, \ldots, y_n).$$

The matrix elements is

$$\langle (a_1, a_2, \ldots, a_A, x_1, x_2, \ldots, x_m) \mid {}_A\#(f) \mid (a'_1, a'_2, \ldots, a'_A, y_1, y_2, \ldots, y_n) \rangle$$
$$= \begin{cases} \langle (x_1, x_2, \ldots, x_m) \mid f \mid (y_1, y_2, \ldots, y_n) \rangle, & \textit{if } (a_1, \ldots, a_A) = (a'_1, \ldots, a'_A), \\ 0, & \textit{otherwise}. \end{cases}$$

If $m = 0$ or $n = 0$ or $A = 0$, ${}_A\#(f)$ is similar as before in $\#_A(f)$.

Proof. We require to show first that the $\frac{1}{2}$-monoidal category conditions 1-4 from 5.1.2 are satisfied, and then slideability.

- 1. $(\mathbb{N}, +, 0)$ is a monoid.

- 2. For each $A \in \mathbb{N}$, ${}_A\#$ is a functor, let $f \in \hom_{Vec_X}(m,n)$

  and $g \in \hom_{Vec_X}(n,k)$, we have

$$f(x_1,\dots,x_m) = \sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle\, (y_1,\dots,y_n).$$

$$g(y_1,\dots,y_n) = \sum_{(z_1,\dots,z_k)} \langle (y_1,\dots,y_n) \mid g \mid (z_1,\dots,z_k)\rangle\, (z_1,\dots,z_k).$$

$$g\circ f(x_1,\dots,x_m) = \sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle \sum_{(z_1,\dots,z_k)} \langle (y_1,\dots,y_n) \mid g \mid (z_1,\dots,z_k)\rangle\, (z_1,\dots,z_k).$$

$${}_A\#(g\circ f)(a_1,\dots,a_A,x_1,\dots,x_m) = \sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle \sum_{(z_1,\dots,z_k)} \langle (y_1,\dots,y_n) \mid g \mid (z_1,\dots,z_k)\rangle\, (a_1,\dots,a_A,z_1,\dots,z_k).$$

$$\begin{aligned}{}_A\#(g)\circ {}_A\#(f)(a_1,\dots,a_A,x_1,\dots,x_m) &= {}_A\#(g)\Big({}_A\#(f)(a_1,\dots,a_A,x_1,\dots,x_m)\Big)\\ &= {}_A\#(g)\Big(\sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle (a_1,\dots,a_A,y_1,\dots,y_n)\Big)\\ &= \sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle \sum_{(z_1,\dots,z_k)} \langle (y_1,\dots,y_n) \mid g \mid (z_1,\dots,z_k)\rangle\\ &\qquad (a_1,\dots,a_A,z_1,\dots,z_k).\end{aligned}$$

Therefore

$${}_A\#(g\circ f) = {}_A\#(g)\circ {}_A\#(f).$$

In the following calculations, we will identify a linear map $f\colon \mathbb{C}(X^m) \to \mathbb{C}(X^n)$ with its restriction $X^m \to \mathbb{C}(X^n)$.

$$
\begin{aligned}
&{}_A\#(\mathrm{id}_{\mathbb{C}(X^m)})\\
&\quad = {}_A\#\Big((x_1,\dots,x_m)\mapsto \sum_{(x'_1,\dots,x'_m)} \langle (x_1,\dots,x_m) \mid \mathrm{id} \mid (x'_1,\dots,x'_m)\rangle \\
&\qquad\qquad (x'_1,\dots,x'_m)\Big)\\
&=(a_1,\dots,a_A,x_1,\dots,x_m)\mapsto \sum_{(x'_1,\dots,x'_m)} \langle (x_1,\dots,x_m) \mid \mathrm{id} \mid (x'_1,\dots,x'_m)\rangle \\
&\qquad\qquad (a_1,\dots,a_A,x'_1,\dots,x'_m)\\
&=\mathrm{id}_{{}_A\#(\mathbb{C}(X^m))}.
\end{aligned}
$$

Hence ${}_A\#$ is a functor.

3. By similar argument $\#_A$ is a functor.

- 4. For each $A, B \in \mathbb{N}$ and a morphism $f \in \hom_{Vec_X}(m,n)$, we have

1. To verify axiom in (5.1),

$$
\begin{aligned}
&{}_A\# \circ {}_B\#\Big((x_1,\dots,x_m)\mapsto f(x_1,\dots,x_m)\Big)\\
&\quad = {}_A\#\Big({}_B\#((x_1,\dots,x_m)\mapsto f(x_1,\dots,x_m))\Big)\\
&\quad = {}_A\#\Big((b_1,\dots,b_B,x_1,\dots,x_m)\mapsto \\
&\qquad\qquad \sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle\, (b_1,\dots,b_B,y_1,\dots,y_n)\Big)\\
&\quad = (a_1,\dots,a_A,b_1,\dots,b_B,x_1,\dots,x_m)\mapsto \\
&\qquad\qquad \sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle\, (a_1,\dots,a_A,b_1,\dots,b_B,y_1,\dots,y_n)\\
&\quad = {}_{(A+B)}\#\Big((x_1,\dots,x_m)\mapsto f(x_1,\dots,x_m)\Big).
\end{aligned}
$$

2. By an analogous calculation the axiom in (5.2) is satisfied.

3. To verify axiom in (5.3),

$$
\begin{aligned}
&\#_A \circ {}_B\#\Big((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m)\Big) \\
&\quad = \#_A\Big({}_B\#((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m))\Big) \\
&\quad = \#_A\Big((b_1, \ldots, b_B, x_1, \ldots, x_m) \mapsto \\
&\qquad\qquad \sum_{(y_1, \ldots, y_n)} \langle (x_1, \ldots, x_m) \mid f \mid (y_1, \ldots, y_n) \rangle \, (b_1, \ldots, b_B, y_1, \ldots, y_m)\Big) \\
&\quad = (b_1, \ldots, b_B, x_1, \ldots, x_m, a_1, \ldots, a_A) \mapsto \\
&\qquad\qquad \sum_{(y_1, \ldots, y_n)} \langle (x_1, \ldots, x_m) \mid f \mid (y_1, \ldots, y_n) \rangle \, (b_1, \ldots, b_B, y_1, \ldots, y_n, a_1, \ldots, a_A) \\
&\quad = {}_B\#\Big((x_1, \ldots, x_m, a_1, \ldots, a_A) \mapsto \\
&\qquad\qquad\quad \sum_{(y_1, \ldots, y_n)} \langle (x_1, \ldots, x_m) \mid f \mid (y_1, \ldots, y_n) \rangle \, (y_1, \ldots, y_n, a_1, \ldots, a_A)\Big) \\
&\quad = {}_B\# \circ \#_A\Big((x_1, \ldots, x_m) \mapsto \\
&\qquad\qquad\qquad \sum_{(y_1, \ldots, y_n)} \langle (x_1, \ldots, x_m) \mid f \mid (y_1, \ldots, y_n) \rangle \, (y_1, \ldots y_n)\Big) \\
&\quad = {}_B\# \circ \#_A\Big((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m)\Big).
\end{aligned}
$$

4. To verify axioms in (5.4) and (5.5),

$$
\#_0(f) = f.
$$

$$
{}_0\#(f) = f.
$$

In case $m = 0$.

5. To verify axiom in (5.1),

$$
\begin{aligned}
{}_A\# \circ {}_B\#\Big(1 \mapsto f(1)\Big) &= {}_A\#\Big({}_B\#(1 \mapsto f(1))\Big) \\
&= {}_A\#\Big((b_1, \ldots, b_B) \mapsto \sum_{(y_1,\ldots,y_n)} \langle 1 \mid f \mid (y_1, \ldots, y_n)\rangle\, (b_1, \ldots, b_B, y_1, \ldots, y_n)\Big) \\
&= (a_1, \ldots, a_A, b_1, \ldots, b_B) \mapsto \\
&\qquad \sum_{(y_1,\ldots,y_n)} \langle 1 \mid f \mid (y_1, \ldots, y_n)\rangle\, (a_1, \ldots, a_A, b_1, \ldots, b_B, y_1, \ldots y_n) \\
&= {}_{(A+B)}\#(1 \mapsto f(1)).
\end{aligned}
$$

6. By an analogous calculation the axiom in (5.2) is satisfied.

7. To verify axiom in (5.3),

$$
\begin{aligned}
\#_A \circ {}_B\#(1 \mapsto f(1)) &= \#_A\Big({}_B\#(1 \mapsto f(1))\Big) \\
&= \#_A\Big((b_1, \ldots, b_B) \mapsto \sum_{(y_1,\ldots,y_n)} \langle 1 \mid f \mid (y_1, \ldots, y_n)\rangle\, (b_1, \ldots, b_B, y_1, \ldots, y_n)\Big) \\
&= (b_1, \ldots, b_B, a_1, \ldots, a_A) \mapsto \\
&\qquad \sum_{(y_1,\ldots,y_n)} \langle 1 \mid f \mid (y_1, \ldots, y_n)\rangle\, (b_1, \ldots, b_B, y_1, \ldots, y_n, a_1, \ldots, a_A) \\
&= {}_B\#\Big((a_1, \ldots, a_A) \mapsto \sum_{(y_1,\ldots,y_n)} \langle 1 \mid f \mid (y_1, \ldots, y_n)\rangle\, (y_1, \ldots y_n, a_1, \ldots, a_A)\Big) \\
&= {}_B\# \circ \#_A\Big(1 \mapsto \sum_{(y_1,\ldots,y_n)} \langle 1 \mid f \mid (y_1, \ldots, y_n)\rangle\, (y_1, \ldots, y_n)\Big) \\
&= {}_B\# \circ \#_A(1 \mapsto f(1)).
\end{aligned}
$$

8. To verify axiom in (5.4) and (5.5),

$$
\begin{aligned}
\#_0(f) &= f. \\
{}_0\#(f) &= f.
\end{aligned}
$$

In case $n = 0$.

9. To verify axiom in (5.1),

$$
\begin{aligned}
&{}_A\# \circ {}_B\#\Big((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m)\Big)\\
&\quad = {}_A\#\Big({}_B\#((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m))\Big)\\
&\quad = {}_A\#\Big((b_1, \ldots, b_B, x_1, \ldots, x_m) \mapsto \langle (x_1, \ldots, x_m) \mid f \mid 1\rangle\, (b_1, \ldots, b_B)\Big)\\
&\quad = (a_1, \ldots, a_A, b_1, \ldots, b_B, x_1, \ldots, x_m) \mapsto\\
&\qquad\qquad \langle (x_1, \ldots, x_m) \mid f \mid 1\rangle\, (a_1, \ldots, a_A, b_1, \ldots, b_B)\\
&\quad = {}_{(A+B)}\#\Big((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m)\Big).
\end{aligned}
$$

10. By an analogous calculation the axiom in (5.2) is satisfied.

11. To verify axiom in (5.3),

$$
\begin{aligned}
&\#_A \circ {}_B\#\Big((x_1, \ldots, x_m) \mapsto (f(x_1, \ldots, x_m)\Big)\\
&\quad = \#_A({}_B\#\Big((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m)\Big)\\
&\quad = \#_A\Big((b_1, \ldots, b_B, x_1, \ldots, x_m) \mapsto \langle (x_1, \ldots, x_m) \mid f \mid 1\rangle\, (b_1, \ldots, b_B)\Big)\\
&\quad = (b_1, \ldots, b_B, x_1, \ldots, x_m, a_1, \ldots, a_A) \mapsto\\
&\qquad\qquad \langle (x_1, \ldots, x_m) \mid f \mid 1\rangle\, (b_1, \ldots, b_B, a_1, \ldots, a_A)\\
&\quad = {}_B\#\Big((x_1, \ldots, x_m, a_1, \ldots, a_A) \mapsto \langle (x_1, \ldots, x_m) \mid f \mid 1\rangle\, (a_1, \ldots, a_A)\Big)\\
&\quad = {}_B\# \circ \#_A\Big((x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m)\Big).
\end{aligned}
$$

12. To verify axiom in (5.4) and (5.5),

$$
\begin{aligned}
\#_0(f) &= f.\\
{}_0\#(f) &= f.
\end{aligned}
$$

Slideable 5.1.5 because for all morphisms $f \in \hom_{Vec_X}(m,n)$ and $g \in \hom_{Vec_X}(k,s)$,

$$(f\Theta s)\circ(m\Theta g) = f\Theta s\Big((x_1,\dots,x_m,w_1,\dots,w_k)\mapsto \sum_{(z_1,\dots,z_s)} \langle (w_1,\dots,w_k) \mid g \mid (z_1,\dots,z_s)\rangle\,(x_1,\dots,x_m,z_1,\dots,z_s)\Big)$$

$$=(x_1,\dots,x_m,w_1,\dots,w_k) \mapsto \sum_{(y_1,\dots,y_n,z_1,\dots,z_s)} \langle (w_1,\dots,w_k) \mid g \mid (z_1,\dots,z_s)\rangle\,\langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle\,(y_1,\dots,y_n,z_1,\dots,z_s).$$

$$(n\Theta g)\circ(f\Theta k) = n\Theta g\Big((x_1,\dots,x_m,w_1,\dots,w_k)\mapsto \Big(\sum_{(y_1,\dots,y_n)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle\,(y_1,\dots,y_n,w_1,\dots,w_k)\Big)$$

$$= (x_1,\dots,x_m,w_1,\dots,w_k)\mapsto \sum_{(y_1,\dots,y_n,z_1,\dots,z_s)} \langle (x_1,\dots,x_m) \mid f \mid (y_1,\dots,y_n)\rangle\langle (w_1,\dots,w_k) \mid g \mid (z_1,\dots,z_s)\rangle\,(y_1,\dots,y_n,z_1,\dots,z_s).$$

Therefore

$$(f\Theta s)\circ(m\Theta g) = (n\Theta g)\circ(f\Theta k).$$

□

5.1.3 Category of $\frac{1}{2}$-monoidal categories.

Definition 5.1.12 ($\frac{1}{2}$-monoidal functor)**.** *A* $\frac{1}{2}$-monoidal functor

$$F\colon(\mathcal{C}, I_{\mathcal{C}}, \otimes_0, \#_{(-)}, {}_{(-)}\#) \to (\mathcal{D}, I_{\mathcal{D}}, \otimes_0', \#_{(-)}', {}_{(-)}\#')$$

between $\frac{1}{2}$*-monoidal categories* $(\mathcal{C}, I_{\mathcal{C}}, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ *and* $(\mathcal{D}, I_{\mathcal{D}}\otimes_0', \#_{(-)}', {}_{(-)}\#')$ *is a functor 3.2.1*

$$F = (F_0, F_1)\colon \mathcal{C}\to\mathcal{D},$$

such that the following holds

1. $$F_0\colon (ob(\mathcal{C}), \otimes_0, I_{\mathcal{C}}) \to (ob(\mathcal{D}), \otimes_0', I_{\mathcal{D}})$$

   is a monoid map (hence in particular for all objects $x, y \in ob(\mathcal{C})$*,*
   $F_0(I_{\mathcal{C}}) = I_{\mathcal{D}}$ *and* $F_0(x \otimes_0 y) = F_0(x) \otimes_0' F_0(y)$*;*

2. *for all* $A, x, y \in ob(\mathcal{C})$ *and* $f \in \hom_{\mathcal{C}}(x, y)$ *we have*

$$F_1(\#_A(f)) = \#'_{F_0(A)}(F_1(f)),$$
$$F_1({}_A\#(f)) = {}_{F_0(A)}\#'(F_1(f)).$$

Proposition 5.1.13 (Category of $\frac{1}{2}$-monoidal categories)**.** *Consider the precategory of all* $\frac{1}{2}$*-monoidal categories*

$$\mathbf{\frac{1}{2}{-}MC} \;=\; \left(ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big), \hom_{\mathbf{\frac{1}{2}-MC}}(\_,\_), \star, \mathrm{id}_{\_}\right),$$

where

1. *the class* $ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big)$ *is the class of all* $\frac{1}{2}$*-monoidal categories,*

2. *for each pair* $(A, B) \in ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big) \times ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big)$*, the set* $\hom_{\mathbf{\frac{1}{2}-MC}}(A, B)$ *is the set of all* $\frac{1}{2}$*-monoidal functors* $A \to B$*,*

3. *for each triple of objects*

$$(A, B, C) \in ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big) \times ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big) \times ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big),$$

   and for every

$$F \in \hom_{\mathbf{\frac{1}{2}-MC}}(A, B) \text{ and } G \in \hom_{\mathbf{\frac{1}{2}-MC}}(B, C),$$

we have

$$\hom_{\frac{1}{2}\mathbf{MC}}(A,B) \times \hom_{\frac{1}{2}\mathbf{MC}}(B,C) \to \hom_{\frac{1}{2}\mathbf{MC}}(A,C)$$
$$(F,G) \mapsto G \star F,$$

where $\star$ *is defined in 3.2.4,*

4. *for all* $\mathcal{C} \in ob\Big(\frac{1}{2}\text{–}\mathbf{MC}\Big)$*, there is an identity functor* $\mathrm{id}_{\mathcal{C}} \in \hom_{\frac{1}{2}\mathbf{MC}}(\mathcal{C},\mathcal{C})$*, that is a* $\frac{1}{2}$*-monoidal functor.*

This precategory is a category.

Proof. First, we want to prove $G \star F$ is a $\frac{1}{2}$-monoidal functor. Let $F{:}\,\mathcal{C} \to \mathcal{D}$ and $G{:}\,\mathcal{D} \to \mathcal{E}$, for all $A, x \in ob(\mathcal{C})$ and all morphism $f \in \hom_{\mathcal{C}}(y,z)$, we have

$$\begin{aligned} G \star F(\#_A(x)) &= G(\#_{F(A)}(F(x))) \\ &= \#_{G(F(A))}(G(F(x))) \\ &= \#_{G\star F(A)}(G \star F(x)). \\ G \star F(\#_A(f) &= G(\#_{F(A)}(F(f))) \\ &= \#_{G(F(A))}(G(F(f))) \\ &= \#_{G\star F(A)}(G \star F(f)). \end{aligned}$$

Similarly,

$$\begin{aligned} G \star F({}_A\#(x) &= {}_{G\star F(A)}\#(G \star F(x)), \\ G \star F({}_A\#(f)) &= {}_{G\star F(A)}\#(G \star F(f)). \end{aligned}$$

Second, the associativity and unit are proved in 3.2.4. □

Lemma 5.1.14. *We have a full subcategory*

$$\mathbf{s}\frac{\mathbf{1}}{\mathbf{2}}\text{–}\mathbf{MC} = (ob\Big(\mathbf{s}\frac{\mathbf{1}}{\mathbf{2}}\text{–}\mathbf{MC}\Big), \hom_{\mathbf{s}\frac{1}{2}\mathbf{MC}}(\_\,,\_)),$$

where

1. *$ob\Big(\mathbf{s\frac{1}{2}{-}MC}\Big)$ is the class of slideable $\frac{1}{2}$-monoidal categories so*

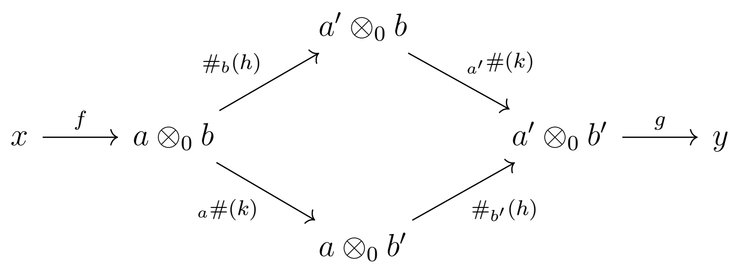
$$ob\Big(\mathbf{s\frac{1}{2}{-}MC}\Big) \subset ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big).$$

2. *Given a pair of objects $(A, B) \in ob(\mathbf{s\frac{1}{2}{-}MC}) \times ob\Big(\mathbf{s\frac{1}{2}{-}MC}\Big)$, the set $\hom_{\mathbf{s\frac{1}{2}{-}MC}}(A, B) = \hom_{\mathbf{\frac{1}{2}{-}MC}}(A, B)$.*

Proof. We have $ob\Big(\mathbf{s\frac{1}{2}{-}MC}\Big) \subset ob\Big(\mathbf{\frac{1}{2}{-}MC}\Big)$, so we can consider the full subcategory $\mathbf{s\frac{1}{2}{-}MC}$. □

5.2 Towards the slidealisation functor: preliminaries

Lemma 5.2.1. *A $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ as in Def. 5.1.2 is* slideable *if, and only if, the following diagram is commutative*



(5.7)

for all $x, y, a, b, a', b' \in ob(\mathcal{C})$, for all choices of morphisms $h\colon a \to a'$, $k\colon b \to b'$, and for all choices of morphism $f\colon x \to a \otimes_0 b$, and $g\colon a' \otimes_0 b' \to y$. The commutativity of the diagram is equivalent to the equation below

$$g \circ {}_{a'}\#(k) \circ \#_b(h) \circ f = g \circ \#_{b'}(h) \circ {}_a\#(k) \circ f.$$

Proof. Suppose a $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ is slideable. Choose objects $a, b, a', b' \in ob(\mathcal{C})$, choose morphisms $h\colon a \to a'$, $k\colon b \to b'$, and also morphism $f\colon x \to a \otimes_0 b$, and $g\colon a' \otimes_0 b' \to y$.

So from the definition 5.1.5 we have

$$ {}_{a'}\#(k) \circ \#_b(h) = \#_{b'}(h) \circ {}_a\#(k). $$

By composing the morphisms f, g from the right and lift we have

$$ g \circ {}_{a'}\#(k) \circ \#_b(h) \circ f = g \circ \#_{b'}(h) \circ {}_a\#(k) \circ f. $$

Now, on the other hand suppose the diagram (5.7) commutes.

Let

$$ x = a \otimes_0 b,\ y = a' \otimes_0 b',\ f = \mathrm{id}_x \text{ and } g = \mathrm{id}_y, $$

hence

$$ {}_{a'}\#(k) \circ \#_b(h) = \#_{b'}(h) \circ {}_a\#(k). $$

□

Definition 5.2.2. *Consider a $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$. Let $x, y \in ob(\mathcal{C})$, we define a relation $\sim_0$ on $\hom_{\mathcal{C}}(x, y)$. Given $L, R \in \hom_{\mathcal{C}}(x, y)$, $L \sim_0 R$ if*

> *there exist objects $a, b, a', b' \in ob(\mathcal{C})$, morphisms $h\colon a \to a'$, $k\colon b \to b'$ and morphisms $f\colon x \to a \otimes_0 b$, and $g\colon a' \otimes_0 b' \to y$, such that*
>
> $$ L = g \circ {}_{a'}\#(k) \circ \#_b(h) \circ f, $$
>
> $$ R = g \circ \#_{b'}(h) \circ {}_a\#k \circ f. $$
>
> *I.e, (where L and R are obtained by the chosen compositions)*

$$ L = \left(x \xrightarrow{f} a \otimes_0 b \xrightarrow{\#_b(h)} a' \otimes_0 b \xrightarrow{{}_{a'}\#(k)} a' \otimes_0 b' \xrightarrow{g} y \right) $$

$$ \sim_0 $$

$$R = \left(x \xrightarrow{f} a \otimes_0 b \xrightarrow{{}_a\#(k)} a \otimes_0 b' \xrightarrow{\#_{b'}(h)} a' \otimes_0 b' \xrightarrow{g} y\right)$$

Lemma 5.2.3. *Let* $\mathcal{C} = (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ *be a* $\frac{1}{2}$*-monoidal category and* $x, y, z, w, A \in ob(\mathcal{C})$*. Let* $L, L' \colon x \to y$*,* $R \colon y \to z$ *and* $R' \colon w \to x$*. Suppose that*

$$L \sim_0 L',$$

then

1. $R \circ L \sim_0 R \circ L'$,
2. $L \circ R' \sim_0 L' \circ R'$,
3. ${}_A\#L \sim_0 {}_A\#L'$,
4. $\#_A L \sim_0 \#_A L'$.

Proof. Suppose $L \sim_0 L'$. So, there exist objects $a, b, a', b' \in ob(\mathcal{C})$, morphisms $h \colon a \to a'$, $k \colon b \to b'$, and morphisms $f \colon x \to a \otimes_0 b$, and $g \colon a' \otimes_0 b' \to y$ such that

$$L = g \circ {}_{a'}\#(k) \circ \#_b(h) \circ f,$$

$$L' = g \circ \#_{b'}(h) \circ {}_a\#(k) \circ f.$$

We have

1.

$$R \circ L = (R \circ g) \circ {}_{a'}\#(k) \circ \#_b(h) \circ f,$$
$$R \circ L' = (R \circ g) \circ \#_{b'}(h) \circ {}_a\#(k) \circ f.$$

Hence by definition

$$R \circ L \sim_0 R \circ L'.$$

2. Analogously,

$$L \circ R' = g \circ {}_{a'}\#(k) \circ \#_b(h) \circ (f \circ R'),$$
$$L' \circ R' = g \circ \#_{b'}(h) \circ {}_{a}\#(k) \circ (f \circ R').$$

Hence,

$$L \circ R' \sim_0 L' \circ R'.$$

3. We want to prove ${}_A\#(L) \sim_0 {}_A\#(L')$. L and L' are given by the compositions below.

$$L = \left(x \xrightarrow{f} a \otimes_0 b \xrightarrow{\#_b(h)} a' \otimes_0 b \xrightarrow{{}_{a'}\#(k)} a' \otimes_0 b' \xrightarrow{g} y\right)$$

$$L' = \left(x \xrightarrow{f} a \otimes_0 b \xrightarrow{{}_{a}\#(k)} a \otimes_0 b' \xrightarrow{\#_{b'}(h)} a' \otimes_0 b' \xrightarrow{g} y\right)$$

We now apply the functor ${}_A\#\colon \mathcal{C} \to \mathcal{C}$ to both diagrams. And conclude that ${}_A\#L$ and ${}_A\#L'$ are given by the compositions below

$${}_A\#(L) = A \otimes_0 x \xrightarrow{{}_A\#(f)} A \otimes_0 a \otimes_0 b \xrightarrow{{}_A\#(\#_b(h))} A \otimes_0 a' \otimes_0 b \xrightarrow{{}_A\#({}_{a'}\#(k))} A \otimes_0 a' \otimes_0 b' \xrightarrow{{}_A\#(g)} A \otimes_0 y$$

(We have used the fact that ${}_A\#\mathcal{C} \to \mathcal{C}$ is functor, and therefore preserves compositions.) By using $\frac{1}{2}$-monoidal category axioms we have

$$
{}_A\#(L) = A \otimes_0 x \xrightarrow{{}_A\#(f)} (A \otimes_0 a) \otimes_0 b \xrightarrow{\#_b({}_A\#(h))} (A \otimes_0 a') \otimes_0 b \xrightarrow{{}_{(A\otimes_0 a')}\#(k)} (A \otimes_0 a') \otimes_0 b' \xrightarrow{{}_A\#(g)} A \otimes_0 y
$$

Analogously

$$
{}_A\#(L') = A \otimes_0 x \xrightarrow{{}_A\#(f)} A \otimes_0 a \otimes_0 b \xrightarrow{{}_A\#({}_a\#(k))} A \otimes_0 a \otimes_0 b' \xrightarrow{{}_A\#(\#_{b'}(h))} A \otimes_0 a' \otimes_0 b' \xrightarrow{{}_A\#(g)} A \otimes_0 y
$$

By using $\frac{1}{2}$-monoidal category axioms we have

$$
{}_A\#(L') = A \otimes_0 x \xrightarrow{{}_A\#(f)} (A \otimes_0 a) \otimes_0 b \xrightarrow{{}_{(A\otimes_0 a)}\#(k)} (A \otimes_0 a) \otimes_0 b', \xrightarrow{\#_{b'}({}_A\#(h))} (A \otimes_0 a') \otimes_0 b' \xrightarrow{{}_A\#(g)} A \otimes_0 y
$$

so,

$$
{}_A\#(L) = ({}_A\#(g)) \circ ({}_{(A\otimes_0 a')}\#(k)) \circ (\#_b({}_A\#(h))) \circ ({}_A\#(f)),
$$

and

$$
{}_A\#(L') = ({}_A\#(g)) \circ (\#_{b'}({}_A\#(h))) \circ ({}_{(A\otimes_0 a)}\#(k)) \circ ({}_A\#(f)).
$$

Therefore

$$
{}_A\#(L) \sim_0 {}_A\#(L').
$$

4. By the similar way

$$
\#_A(L) \sim_0 \#_A(L').
$$

□

Lemma 5.2.4. *Let $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ be a $\frac{1}{2}$-monoidal category. Then*
(I) There exists another $\frac{1}{2}$-monoidal category $F(\mathcal{C}) = (\mathcal{C}', I', \otimes_0', \#'_{(-)}, {}_{(-)}\#')$, where

1. $ob(\mathcal{C}') = ob(\mathcal{C})$.

2. *Given* $x, y \in ob(\mathcal{C})$*,* $\hom_{\mathcal{C}'}(x, y) = \hom_{\mathcal{C}}(x, y)/\sim$ *, where*

$$\sim = closure(\sim_0),$$

where closure means the transitive and symmetric closure, note that $\sim_0$ *is reflexive.*

3. *The composition* $\circ$ *in* $\mathcal{C}'$*, let* $L: y \to z$ *and* $R: x \to y$

$$[L] \circ [R] = [L \circ R].$$

4. *Identity morphisms in* $\mathcal{C}'$*. Given* $A \in ob(\mathcal{C}')$*,* $\mathrm{id}'_A = [\mathrm{id}_A]$*.*
5. *Given* $[f]: x \to y$ *and an object* A*,* $\#'_A([f]) = [\#_A(f)]$*.*
6. *Given* $[f]: x \to y$ *and an object* A*,* ${}_A\#'([f]) = [{}_A\#(f)]$*.*
7. *The unit object,* $I' = I$*.*

(II) $F(\mathcal{C})$ *is slideable.*

5.2.1 Proof of Lemma 5.2.4

Explanation of the equivalence relation

Suppose that $L, L': x \to y$. Then $L \sim L'$ if there are morphisms $(L_1, L_2, \ldots, L_n)$, such that

$$L = L_1 \sim_0 L_2, L_2 \sim_0 L_3, \ldots, L_{n-1} \sim_0 L_n = L',$$

whenever $L_i \sim_0 L_{i+1}$ or $L_{i+1} \sim_0 L_i$, for all $i = 1, 2, \ldots, n-1$.

Proof that the composition is well defined

We want to prove $[R] \circ' [L] = [R \circ L]$ is well defined. So, we want to prove that if $L \sim L'$ and $R \sim R'$, then

$$R \circ L \sim R' \circ L'.$$

Suppose $L, L': x \to y$, and $L \sim L'$. Then there are morphisms $(L_1, L_2, \ldots, L_n)$, such that $L = L_1$, $L' = L_n$, and for all $i = 1, 2, \ldots, n-1$,

$$L_i \sim_0 L_{i+1} \text{ or } L_{i+1} \sim_0 L_i.$$

Similarly, suppose $R, R': y \to z$ and $R \sim R'$. Then there are non-negative integer m and morphisms $(R_1, R_2, \ldots, R_m)$, such that $R = R_1$, $R_m = R'$, and for all $i = 1, 2, \ldots, m-1$,

$$R_i \sim_0 R_{i+1} \text{ or } R_{i+1} \sim_0 R_i.$$

Now, we prove $R \circ L \sim R' \circ L'$. So, first prove $R \circ L \sim R' \circ L$, then prove $R' \circ L \sim R' \circ L'$. We have $R = R_1$, $R_m = R'$, and for all $i = 1, 2, \ldots, m-1$,

$$R_i \sim_0 R_{i+1} \text{ or } R_{i+1} \sim_0 R_i.$$

by Lemma 5.2.3 we have

$$R_i \circ L \sim_0 R_{i+1} \circ L \text{ or } R_{i+1} \circ L \sim_0 R_i \circ L.$$

Hence,

$$R \circ L \sim R' \circ L.$$

Now we want to prove $R' \circ L \sim R' \circ L'$. We have $L = L_1$, $L' = L_n$, and for all $i = 1, 2, \ldots, n-1$,

$$L_i \sim_0 L_{i+1} \text{ or } L_{i+1} \sim_0 L_i.$$

by Lemma 5.2.3 we have $L = L_1$, $L' = L_n$, and for all $i = 1, 2, \ldots, n-1$,

$$R' \circ L_i \sim_0 R' \circ L_{i+1} \text{ or } R' \circ L_{i+1} \sim_0 R' \circ L_i.$$

Hence

$$R' \circ L \sim R' \circ L'.$$

Therefore

$$R \circ L \sim R' \circ L'.$$

Proof that $F(\mathcal{C})$ is a category

Now, we want to prove category axioms

1. Associativity, for all $f \in \hom_{\mathcal{C}}(x, y), g \in \hom_{\mathcal{C}}(y, z)$ and $h \in \hom_{\mathcal{C}}(z, w)$, we have (note that we already proved that composition descends to the quotient):

$$\begin{aligned}[h] \circ ([g] \circ [f]) =& [h] \circ ([g \circ f]) = [h \circ (g \circ f)] \\ =& [(h \circ g) \circ f] = [h \circ g] \circ [f] = ([h] \circ [g]) \circ [f].\end{aligned}$$

2. Unit, for all $f \in \hom_{\mathcal{C}}(x, y)$;

$$\begin{aligned}[\mathrm{id}_y] \circ [f] =& [\mathrm{id}_y \circ f] = [f], \\ [f] \circ [\mathrm{id}_x] =& [f \circ \mathrm{id}_x] = [f].\end{aligned}$$

Proof that the $\#'_{(-)}, {}_{(-)}\#'$ is well defined

We want to prove

$$\#'_A[L] = [\#_A L]$$

is well defined for given $[L]: x \to y$ and an object A. So, we want to prove that

$$\text{if } L \sim R, \text{ then } \#_A(L) \sim \#_A(R).$$

Suppose $L, R: x \to y$, and $L \sim R$. So there is a non-negative integer n and n-tuples $(L_1, L_2, \ldots, L_n) \in \mathcal{C}^n$, such that

$$L = L_1 \sim_0 L_2, L_2 \sim_0 L_3, \ldots, L_{n-1} \sim_0 L_n = R,$$

by Lemma 5.2.3, we have

$$\#_A(L) = \#_A(L_1) \sim_0 \#_A(L_2), \#_A(L_2) \sim_0 \#_A(L_3), \ldots, \#_A(L_{n-1}) \sim_0 \#_A(L_n) = \#_A(R).$$

Therefore

$$\#_A(L) \sim \#_A(R).$$

By a similar argument

$$_A\#(L) \sim {}_A\#(R).$$

Proof of the axioms of $\frac{1}{2}$-monoidal category.

For each $A, B \in ob(\mathcal{C})$ (noting that we already proved that left-whiskering and right-whiskering, $_{(-)}\#$ and $\#_{(-)}$ descend to quotient).

1. To verify the axiom in (5.1)

$$\begin{aligned} {}_A\#' \circ {}_B\#'([L]) =& [{}_A\# \circ {}_B\#(L)] \\ =& [{}_{A\otimes_0 B}\#(L)] \\ =& {}_{A\otimes_0 B}\#'([L]). \end{aligned}$$

2. By a similar argument the axiom in (5.2) is satisfied

$$\#'_A \ \circ \#'_B([L]) = \#'_{B\otimes_0 A}([L]).$$

3. To verify the axiom in (5.3)

$$\begin{aligned} \#'_A \ \circ {}_B\#'([L]) =& [\#_A \circ {}_B\#(L)] \\ =& [{}_B\# \circ \#_A(L)] \\ =& {}_B\#' \circ \#'_A([L]). \end{aligned}$$

4. To verify the axiom in (5.4)

$$\#'_I([L]) = [\#_I(L)] = [L].$$

5. To verify the axiom in (5.5)

$$_I\#'([L]) = [_I\#(L)] = [L].$$

(II) We want to prove $F(\mathcal{C})$ is slideable. From the definition of $F(\mathcal{C})$, for all $a, b, a', b' \in ob(\mathcal{C})$, for all choices of morphisms $h: a \to a'$, $k: b \to b'$, and for all choices of morphism $f: x \to a \otimes_0 b$, and $g: a' \otimes_0 b' \to y$, we have

$$g \circ {_{a'}\#}(k) \circ \#_b(h) \circ f {\sim} g \circ \#_{b'}(h) \circ {_a\#}(k) \circ f.$$

Hence

$$[g \circ {_{a'}\#}(k) \circ \#_b(h) \circ f] = [g \circ \#_{b'}(h) \circ {_a\#}(k) \circ f]$$
$$[g] \circ {_{a'}\#}([k]) \circ \#_b([h]) \circ [f] = [g] \circ \#_{b'}([h]) \circ {_a\#}([k]) \circ [f].$$

Hence $F(\mathcal{C})$ is slideable.

□

5.2.2 The slidealisation functor: conclusion

Lemma 5.2.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\frac{1}{2}$-monoidal categories and let $G: \mathcal{C} \to \mathcal{D}$ be a $\frac{1}{2}$-monoidalfunctor and $x, y \in ob(\mathcal{C})$, $f, f' \in \hom_{\mathcal{C}}(x, y)$, such that $f \sim_0 f'$, then $G(f) \sim_0 G(f')$.*

Proof. Assume $f \sim_0 f'$, then there exist objects $a, b, a', b' \in ob(\mathcal{C})$, morphisms $h: a \to a'$, $k: b \to b'$, and morphisms $w: x \to a \otimes_0 b$, and $g: a' \otimes_0 b' \to y$ such that

$$f = g \circ {_{a'}\#}k \circ \#_b h \circ w,$$

$$f' = g \circ \#_{b'} h \circ {_a\#}k \circ w.$$

So,

$$\begin{aligned}G(f) &= G(g)\circ G({}_{a'}\#k)\circ G(\#_b h)\circ G(w)\\ &= G(g)\circ {}_{G(a')}\#G(k)\circ \#_{G(b)}G(h)\circ G(w),\\ G(f') &= G(g)\circ G(\#_{b'}h)\circ G({}_a\#k)\circ G(w)\\ &= G(g)\circ \#_{G(b')}G(h)\circ {}_{G(a)}\#G(k)\circ G(w).\end{aligned}$$

Therefore

$$G(f)\sim_0 G(f').$$

□

Lemma 5.2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\frac{1}{2}$-monoidal categories and let $G\colon\mathcal{C}\to\mathcal{D}$ be a $\frac{1}{2}$-monoidal functor and $x, y\in ob(\mathcal{C})$, $f, f'\in \hom_{\mathcal{C}}(x, y)$, such $f\sim f'$, then $G(f)\sim G(f')$.*

Proof. Assume $f\sim f'$, then there exist objects $n\in\mathbb{N}$ and an n-tuple $(f_1, f_2, \ldots, f_n)$ such that

$$f = f_1\sim_0 f_2, f_2\sim_0 f_3, \ldots, f_{n-1}\sim_0 f_n = f'.$$

So,

$$G(f) = G(f_1)\sim_0 G(f_2), G(f_2)\sim_0 G(f_3), \ldots, G(f_{n-1})\sim_0 G(f_n) = G(f').$$

Therefore

$$G(f)\sim G(f').$$

□

Lemma 5.2.7. *Let $\mathcal{C}$ be a $\frac{1}{2}$-monoidal category and $F(\mathcal{C})$ be the slideable $\frac{1}{2}$-monoidal category as defined in Lem.5.2.4. So there is a $\frac{1}{2}$-monoidal functor $\rho_{\mathcal{C}}\colon\mathcal{C}\to F(\mathcal{C})$ such* *that for all* $A, B \in ob(\mathcal{C})$ *and* $f \in \hom_{\mathcal{C}}(A, B)$ *we have*

$$\rho_{\mathcal{C}}(A) := A,$$
$$\rho_{\mathcal{C}}(f) := [f].$$

Proof. Let $A, B, D \in ob(\mathcal{C})$, $f \in \hom_{\mathcal{C}}(A, B)$ and $g \in \hom_{\mathcal{C}}(B, D)$ we have $\rho_{\mathcal{C}}$ is satisfied the axioms of a functor

$$\rho_{\mathcal{C}}(g \circ f) = [g] \circ [f] = \rho_{\mathcal{C}}(g) \circ \rho_{\mathcal{C}}(f).$$

$$\rho_{\mathcal{C}}(\mathrm{id}_A) = [\mathrm{id}_A] = \mathrm{id}'_A = \mathrm{id}'_{\rho_{\mathcal{C}}(A)}.$$

Also, $\rho_{\mathcal{C}}$ is satisfied the axioms of a $\frac{1}{2}$-monoidal functor

$$\rho_{\mathcal{C}}({}_A\#(f)) = [{}_A\#(f)] = {}_A\#'([f]) = {}_{\rho_{\mathcal{C}}(A)}\#'\rho_{\mathcal{C}}(f).$$

By a similar argument

$$\rho_{\mathcal{C}}(\#_A(f)) = \#'_{\rho_{\mathcal{C}}(A)}\rho_{\mathcal{C}}(f).$$

□

Proposition 5.2.8. *Let* $\delta\colon \mathcal{C} \to \mathcal{D}$ *be a* $\frac{1}{2}$*-monoidal functor between* $\frac{1}{2}$*-monoidal categories. Then there is a* $\frac{1}{2}$*-monoidal functor* $\delta'\colon F(\mathcal{C}) \to F(\mathcal{D})$ *between slideable* $\frac{1}{2}$*-monoidal categories, such that for all* $A, B \in ob(\mathcal{C})$ *and* $f \in \hom_{\mathcal{C}}(A, B)$
on objects

$$\delta'(A) := \delta(A),$$

on morphisms

$$\delta'([f]) := [\delta(f)].$$

Furthermore the next diagram commutes, $\rho_{\mathcal{C}}$ *the functor from* $\mathcal{C}$ *to* $F(\mathcal{C})$*.*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \delta\ } & \mathcal{D} \\ {\scriptstyle \rho_{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{\mathcal{D}}} \\ F(\mathcal{C}) & \xrightarrow[\delta']{} & F(\mathcal{D}) \end{array}$$

Proof. Note that

$$\delta'([f]) = [\delta(f)]$$

is well defined because if $f \sim f'$, then $\delta(f) \sim \delta(f')$, see 5.2.6. $\delta'\colon F(\mathcal{C}) \to F(\mathcal{D})$ preserves compositions, identities and whiskerings . For all $A, B, D \in ob(\mathcal{C})$, $f \in \hom_{\mathcal{C}}(A, B)$ and $g \in \hom_{\mathcal{C}}(B, D)$, we have $\rho_{\mathcal{C}}$ satisfies the axioms of a functor

$$\begin{aligned} \delta'([g] \circ [f]) &= \delta'([g \circ f]) \\ &= [\delta(g \circ f)] \\ &= [\delta(g) \circ \delta(f)] \\ &= [\delta(g)] \circ [\delta(f)] \\ &= \delta'([g]) \circ \delta'([f]). \end{aligned}$$

Also,

$$\delta'(\mathrm{id}'_A) = \delta'([\mathrm{id}_A]) = [\delta(\mathrm{id}_A)] = [\mathrm{id}_{\delta(A)}] = [\mathrm{id}_{\delta'(A)}] = \mathrm{id}'_{\delta'(A)}.$$

Also, $\rho_{\mathcal{C}}$ is satisfied the axioms of a $\frac{1}{2}$-monoidal functor

$$\begin{aligned} \delta'({}_A\#'[f]) &= \delta'([{}_A\#(f)]) \\ &= [\delta({}_A\#(f))] \\ &= [{}_{\delta(A)}\#(\delta(f))] \\ &= {}_{\delta(A)}\#'[\delta(f)] \\ &=_{\delta'(A)} \#'\delta'([f]). \end{aligned}$$

By a similar argument

$$\delta'(\#'_A[f]) = \#'_{\delta'(A)}\delta'([f]).$$

Also, the diagram commutes

$$\begin{array}{ccc} f & \xmapsto{\delta} & \delta(f) \\ {\scriptstyle\rho_{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle\rho_{\mathcal{D}}} \\ [f] & \xmapsto[\delta']{} & [\delta(f)] \end{array}$$

$\square$

Proposition 5.2.9. *Let* $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}$ *be the category of* $\frac{1}{2}$*-monoidal categories as defined in 5.1.13, and* $\mathbf{s}\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}$ *be the category of slideable* $\frac{1}{2}$*-monoidal categories as defined in 5.1.14. Then there is a functor*

$$\mathfrak{F} = (\mathfrak{F}_0, \mathfrak{F}_1) \colon \frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC} \to \mathbf{s}\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC},$$

such that

- *for each* $\mathcal{C} \in ob(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC})$*,* $\mathfrak{F}_0(\mathcal{C}) = F(\mathcal{C})$*,*
- *for each* $\delta \in \hom_{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}}(\mathcal{C}, \mathcal{D})$*,* $\mathfrak{F}_1(\delta) = \delta'$*, that defined in 5.2.8.*

Proof. 1. For all $\delta \in \hom_{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}}(\mathcal{C}, \mathcal{D})$ and $\lambda \in \hom_{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}}(\mathcal{D}, \mathcal{E})$, given a morphism f in $\mathcal{C}$

$$\begin{aligned} \mathfrak{F}_1(\lambda \circ \delta)([f]) &= (\lambda \circ \delta)'([f]) \\ &= [(\lambda \circ \delta)(f)] \\ &= [\lambda(\delta(f))] \\ &= \lambda'[\delta(f)] \\ &= (\lambda' \circ \delta')([f]) \\ &= (\mathfrak{F}_1(\lambda) \circ \mathfrak{F}_1(\delta))([f]). \end{aligned}$$

2. For all $\mathcal{C} \in ob(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC})$, given $x \in ob(\mathcal{C})$

$$\mathfrak{F}_1(\mathrm{id}_{\mathcal{C}}(x)) = \mathrm{id}'_{F(\mathcal{C})}(x) = \mathrm{id}'_{\mathfrak{F}_0(\mathcal{C})}(x),$$

given $f \in \hom_{\mathcal{C}}(x, y)$

$$\mathfrak{F}_1(\mathrm{id}_{\mathcal{C}}([f])) = \mathrm{id}'_{F(\mathcal{C})}([f]) = \mathrm{id}'_{\mathfrak{F}_0(\mathcal{C})}([f]).$$

Therefore

$$\mathfrak{F}_1(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}'_{\mathfrak{F}_0(\mathcal{C})}.$$

□

In the next theorem, the functor

$$G\colon \mathbf{s}\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC} \to \frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}$$

is the inclusion functor .

Theorem 5.2.10. *Given any $\frac{1}{2}$-monoidal category*

$$\mathcal{C} = (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#),$$

there is a slideable $\frac{1}{2}$-monoidal category $\mathfrak{F}_0(\mathcal{C}) = F(\mathcal{C})$ and a $\frac{1}{2}$-monoidal functor $\rho_{\mathcal{C}}\colon \mathcal{C} \to G(F(\mathcal{C}))$ satisfying the following universal property.

Given any slideable $\frac{1}{2}$-monoidal category $\mathcal{D}$, and a $\frac{1}{2}$-monoidal functor

$$\delta\colon \mathcal{C} \to G(\mathcal{D}).$$

Then, there exist a unique $\frac{1}{2}$-monoidal functor $\hat{\delta}\colon F(\mathcal{C}) \to \mathcal{D}$ that makes the next diagram commute.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\rho_{\mathcal{C}}} & G(F(\mathcal{C})) \\ & \searrow^{\delta} & \downarrow{\hat{\delta}} \\ & & G(\mathcal{D}) \end{array} \tag{5.8}$$

Note that G is missing in the vertical arrow since it is inclusion functor. Moreover the *family maps* $\rho_{\mathcal{C}}: \mathcal{C} \to G(F(\mathcal{C}))$ *is a natural transformation. I,e the triple* $(\mathfrak{F}, G, \rho)$ *is an adjunction.*

Proof. There exists a $\frac{1}{2}$-monoidal functor

$$\rho_{\mathcal{C}}: \mathcal{C} \to G(F(\mathcal{C}))$$

that on objects, given $x \in ob(\mathcal{C})$,

$$x \mapsto x,$$

on morphisms, given $L \in \hom_{\mathcal{C}}(x, y)$,

$$L \mapsto [L].$$

First, we want to prove a functor $\hat{\delta}: F(\mathcal{C}) \to \mathcal{D}$ is exists. We define $\hat{\delta}$ by

$$\hat{\delta}: F(\mathcal{C}) \to \mathcal{D}$$

that on objects, given $x \in ob(\mathcal{C})$,

$$x \mapsto \delta(x),$$

on morphisms, given $L \in \hom_{\mathcal{C}}(x, y)$,

$$[L] \mapsto \delta(L).$$

Independent of representatives. Suppose $L \sim_0 L'$. So, there exist objects $a, b, a', b' \in ob(\mathcal{C})$, morphisms $h: a \to a'$, $k: b \to b'$, and morphisms $f: x \to a \otimes_0 b$, and $g: a' \otimes_0 b' \to y$ such that

$$L = g \circ {}_{a'}\#k \circ \#_b h \circ f,$$

$$L' = g \circ \#_{b'} h \circ {}_a\#k \circ f.$$

We have

$$\begin{aligned}\delta(L) &= \delta(g \circ {}_{a'}\#k \circ \#_b h \circ f) \\ &= \delta(g) \circ \delta({}_{a'}\#k) \circ \delta(\#_b h) \circ \delta(f) \\ &= \delta(g) \circ {}_{\delta(a')}\#\delta(k) \circ \#_{\delta(b)}\delta(h) \circ \delta(f), \\ \delta(L') &= \delta(g \circ \#_{b'} h \circ {}_a\#k \circ f) \\ &= \delta(g) \circ \delta(\#_{b'} h) \circ \delta({}_a\#k) \circ \delta(f) \\ &= \delta(g) \circ \#_{\delta(b')}\delta(h) \circ ({}_{\delta(a)}\#\delta(k)) \circ \delta(f).\end{aligned}$$

Therefore since $\mathcal{D}$ is slideable

$$\delta(L) = \delta(L').$$

The $\frac{1}{2}$-monoidal functor $\hat{\delta}$ is unique that making diagram (5.8) commute. Suppose we have $\delta_1 \colon F(\mathcal{C}) \to \mathcal{D}$ another $\frac{1}{2}$-monoidal functor, making diagram (5.8) commute. So on objects

$$[x] \mapsto \delta(x),$$

and on morphisms

$$[L] \mapsto \delta(L).$$

Then

$$\hat{\delta} = \delta_1.$$

Note that $G(F(\mathcal{C})) = F(\mathcal{C})$, from Pro.5.2.8, the next diagram commutes.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\lambda} & \mathcal{D} \\ {\scriptstyle \rho_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathcal{D}}} \\ F(\mathcal{C}) & \xrightarrow[\lambda']{} & F(\mathcal{D}) \end{array}$$

Therefore $\rho \colon \mathrm{id} \to G \circ \mathfrak{F}$ is a natural transformation, i.e, $(\mathfrak{F}, G, \rho)$ is an adjunction. $\square$

5.3 Strict monoidal categories

There are two main ways of thinking about strict monoidal catgories. One uses 'whiskers' as in §5.1.1 — composition of objects and morphisms to morphisms. The other gives a full 'tensor' product on morphisms. There are (at least) two ways of giving the axioms for this full tensor product. Here we give the one way.

Definition 5.3.1 (Pre-monoidal structure)**.** *(See [Kas12]). Let $\mathcal{C}$ be a category. A* pre-monoidal structure *in $\mathcal{C}$ is a triple $(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$ where I is an object in $\mathcal{C}$ and $\otimes$ is given by*

1. *Given pair of objects x and y another object $x \otimes_0 y$.*
2. *Given a pair of morphisms $f\colon x \to y$ and $g\colon z \to w$ another morphism $(f \otimes_1 g)\colon x \otimes_0 z \to y \otimes_0 w$.*

Definition 5.3.2 (Strict monoidal category)**.** *(See [Kas12]). A pre-monoidal structure $(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$, as defined in Def. 5.3.1, gives a* strict monoidal category*, written $(\mathcal{C}, I, \otimes)$, if*

1. *$(ob(\mathcal{C}), \otimes_0, I)$ is a monoid. I.e.,*
   (a) *for all $x, y, z \in ob(\mathcal{C})$,*
   $$(x \otimes_0 y) \otimes_0 z = x \otimes_0 (y \otimes_0 z),$$
   (b) *for all $x \in ob(\mathcal{C})$,*
   $$x \otimes_0 I = I \otimes_0 x = x.$$
2. *$(\hom(C), \otimes_1, \mathrm{id}_I)$ is a monoid. I.e.,*
   (a) *for all morphisms $f, g, h \in \mathcal{C}$,*
   $$(f \otimes_1 g) \otimes_1 h = f \otimes_1 (g \otimes_1 h),$$

(b) *for all morphism* $f \in \mathcal{C}$,

$$\mathrm{id}_I \otimes_1 f = f \otimes_1 \mathrm{id}_I = f,$$

3. *for all* $A, B \in ob(\mathcal{C})$, *we have* $\mathrm{id}_A \otimes_1 \mathrm{id}_B = \mathrm{id}_{A \otimes_0 B}$,

4. *for all* $f\colon x \to y$, $f'\colon y \to z$, $g\colon u \to v$ *and* $g'\colon v \to w$, *we have*

$$(f' \otimes_1 g') \circ (f \otimes_1 g) = (f' \circ f) \otimes_1 (g' \circ g).$$

Note(3 and 4) mean that $\otimes := (\otimes_0, \otimes_1)$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Example 5.3.3. *(See for example [Kas12] P.286) Consider the category*

$$Mat = (\mathbb{N}, \hom_{Mat}(\_, \_), \cdot, \mathrm{id}_{\_})$$

in 3.1.4. We have a strict monoidal category

$$(Mat, 0, \otimes = (+, \otimes_1))$$

where: for all $n, m \in \mathbb{N}$, *we have*

$$n \otimes_0 m = n + m.$$

For all $f \in \hom_{Mat}(n, m)$ *and* $g \in \hom_{Mat}(r, s)$ *we have*

$$f \otimes_1 g = \begin{pmatrix} f & 0_{n,s} \\ 0_{r,m} & g \end{pmatrix} \in \hom_{Mat}(n + r, m + s)$$

where $0_{n,s}$ *is the zero matrix* $\in \hom_{Mat}(n, s)$, *e.g, if* $f = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ *and* $g = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix}$, *then*

$$f \otimes_1 g = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 & b_6 \end{pmatrix}$$

Proof. We want to prove $(Mat, 0, \otimes = (+, \otimes_1))$ is a strict monoidal category. It is manifestly a pre-monoidal structure, so we want to satisfy the axioms in 5.3.2

1. the object structure is a monoid because $(\mathbb{N}, +, 0)$ is a monoid,

2. we require to show that $(\hom(Mat), \otimes_1, \mathrm{id}_0)$ is a monoid.

   For all $f \in \hom_{Mat}(n, m), g \in \hom_{Mat}(r, s)$ and $h \in \hom_{Mat}(t, k)$ we have

$$(f \otimes_1 g) \otimes_1 h = \begin{pmatrix} f & 0_{n,s} \\ 0_{r,m} & g \end{pmatrix} \otimes_1 h = \begin{pmatrix} f & 0_{n,s} & 0_{n,k} \\ 0_{r,m} & g & 0_{r,k} \\ 0_{t,m} & 0_{t,s} & h \end{pmatrix}$$

$$f \otimes_1 (g \otimes_1 h) = f \otimes_1 \begin{pmatrix} g & 0_{r,k} \\ 0_{t,s} & h \end{pmatrix} = \begin{pmatrix} f & 0_{n,s} & 0_{n,k} \\ 0_{r,m} & g & 0_{r,k} \\ 0_{t,m} & 0_{t,s} & h \end{pmatrix},$$

   so,

$$(f \otimes_1 g) \otimes_1 h = f \otimes_1 (g \otimes_1 h).$$

   For all matrix f, the next axioms hold

$$\mathrm{id}_0 \otimes_1 f = f = f \otimes_1 \mathrm{id}_0,$$

   because id_0 is the matrix with no elements.

3. For all $n, m \in \mathbb{N}$

$$\mathrm{id}_n \otimes_1 \mathrm{id}_m \in \hom_{Mat}(n + m, n + m),$$
$$\mathrm{id}_{n+m} \in \hom_{Mat}(n + m, n + m),$$

such that

$$\mathrm{id}_n = \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{n}, \qquad \mathrm{id}_m = \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{m}$$

$$\mathrm{id}_n \otimes_1 \mathrm{id}_m = \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 & 0 & \dots & 1}_{n} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \underbrace{0 & 0 & \dots & 1}_{m} \end{pmatrix}}_{n+m}.$$

Hence

$$\mathrm{id}_n \otimes_1 \mathrm{id}_m = \mathrm{id}_{n+m},$$

4. for all $f \in \hom_{Mat}(n, m)$, $f' \in \hom_{Mat}(m, r)$, $g \in \hom_{Mat}(t, k)$ and $g' \in \hom_{Mat}(k, s)$ we have

$$(f \otimes_1 g).(f' \otimes_1 g') = \begin{pmatrix} f & 0_{n,k} \\ 0_{t,m} & g \end{pmatrix} . \begin{pmatrix} f' & 0_{m,s} \\ 0_{k,r} & g' \end{pmatrix} = \begin{pmatrix} ff' & 0_{n,s} \\ 0_{t,r} & gg' \end{pmatrix} = (f.f') \otimes_1 (g.g').$$

□

5.3.1 Strict monoidal functors and the category $\mathbf{StriMC}$

Some references for strict monoidal functors are [Bén67, TV17a, AHS04, ML13, Kas12, TV17b].

Definition 5.3.4 (Strict monoidal functor)**.** *A* strict monoidal functor F *between the strict monoidal categories* $(\mathcal{C}, I_{\mathcal{C}}, \otimes_{\mathcal{C}} = (\otimes_0, \otimes_1))$ *and* $(\mathcal{D}, I_{\mathcal{D}}, \otimes_{\mathcal{D}} = (\otimes_0, \otimes_1))$ *is a functor*

$$F = (F_0, F_1)\colon \mathcal{C} \to \mathcal{D},$$

such that for all $X, Y \in ob(\mathcal{C})$ *and all morphisms* $f, g \in \mathcal{C}$

$$\begin{aligned} F_0(I_{\mathcal{C}}) &= I_{\mathcal{D}}, \\ F_0(X \otimes_{\mathcal{C}} Y) &= F_0(X) \otimes_{\mathcal{D}} F_0(Y), \\ F_1(f \otimes_{\mathcal{C}} g) &= F_1(f) \otimes_{\mathcal{D}} F_1(g). \end{aligned}$$

There is an example of strict monoidal functor in §7.4.

Proposition 5.3.5. *We have a precategory*

$$\mathbf{StriMC} = (ob(\mathbf{StriMC}), \hom_{\mathbf{StriMC}}(\_, \_), \star, \mathrm{id}_{\_}),$$

where

1. *the* $ob(\mathbf{StriMC})$ *is the class of all strict monoidal categories,*

2. *for each pair* $(A, B) \in ob(\mathbf{StriMC}) \times ob(\mathbf{StriMC})$*, the set* $\hom_{\mathbf{StriMC}}(A, B)$ *is the set of strict monoidal functors,*

3. *for each triple of objects* $(A, B, C) \in ob(\mathbf{StriMC}) \times ob(\mathbf{StriMC}) \times ob(\mathbf{StriMC})$*, and for every* $F \in \hom_{\mathbf{StriMC}}(A, B)$*,* $G \in \hom_{\mathbf{StriMC}}(B, C)$*, we have*

$$\begin{aligned} \hom_{\mathbf{StriMC}}(A, B) \times \hom_{\mathbf{StriMC}}(B, C) &\to \hom_{\mathbf{StriMC}}(A, C) \\ (G, F) &\mapsto G \star F, \end{aligned}$$

where $\star$ is defined in 3.2.4,

4. *for all $A \in ob(\mathbf{StriMC})$, there is an identity functor*

$$\mathrm{id}_A \in \hom_{\mathbf{StriMC}}(A, A).$$

Proof. Item 1 does not require proof.
Item 2 requires only to note that the class of strict monoidal functors here is a set because for all strct monoidal category $(\mathcal{C}, I, \otimes)$, $\mathcal{C}$ is small category.
Item 3 we want to prove $G \star F$ is a strict monoidal functor. Let x, y objects and f, g morphisms in category A

$$G \star F(I_A) = G(F(I_A)) = G(I_B) = I_C.$$

$$\begin{aligned} G \star F(x \otimes_0 y) &= G(F(x \otimes_0 y)) \\ &= G(F(x) \otimes_0 F(y)) \\ &= G(F(x)) \otimes_0 G(F(y)) \\ &= G \star F(x) \otimes_0 G \star F(y). \end{aligned}$$

$$\begin{aligned} G \star F(f \otimes_1 g) &= G(F(f \otimes_1 g)) \\ &= G(F(f) \otimes_1 F(g)) \\ &= G(F(f)) \otimes_1 G(F(g)) \\ &= G \star F(f) \otimes_1 G \star F(g). \end{aligned}$$

Item 4: We require to show that for all $A \in ob(\mathbf{StriMC})$, the identity functor id_A is a strict monoidal functor.

Given objects x, y, z in A and morphisms f, g in A, we have

$$\begin{aligned}
\mathrm{id}_A(I_A) &= I_A, \\
\mathrm{id}_A(x \otimes_0 y) &= x \otimes_0 y = \mathrm{id}_A(x) \otimes_0 \mathrm{id}_A(y), \\
\mathrm{id}_A(f \otimes_1 g) &= f \otimes_1 g = \mathrm{id}_A(f) \otimes_1 \mathrm{id}_A(g).
\end{aligned}$$

□

Proposition 5.3.6 (Category of strict monoidal categories)**.** *The precategory of strict monoidal categories*

$$\mathbf{StriMC} = (ob(\mathbf{StriMC}), \hom_{\mathbf{StriMC}}(\_, \_), \star, \mathrm{id}_{\_}),$$

that define in 5.3.5 is a category.

Proof. Associativity and unit are inherited from the category of categories as in 3.2.4.

□

5.4 Equivalence Theorems

5.4.1 Slideable $\frac{1}{2}$-monoidal categories and strict monoidal categories

In this subsection, we prove that the category of strict monoidal categories is isomorphic to the category of slideable 1/2-monoidal categories, seen as a subcategory of the category of 1/2 monoidal categories.

Lemma 5.4.1. *Consider a strict monoidal category $(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$ with underlying category $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \star, \mathrm{id}_{\_})$. The strict monoidal category $(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$ gives rise to a $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$, where $\mathcal{C}$, I and $\otimes_0$ do not change and the $\#$ functors are as follows.*

Given $A \in ob(\mathcal{C})$, the functors $\#_A, {}_A\#$ are defined as given $B, X, Y \in ob(\mathcal{C})$ and morphism $f \in \hom_{\mathcal{C}}(X, Y)$

$$\#_A B = B \otimes_0 A, \tag{5.9}$$

$$\#_A(X \xrightarrow{f} Y) = (X \xrightarrow{f} Y) \otimes_1 (A \xrightarrow{\mathrm{id}} A), \tag{5.10}$$

$${}_A\# B = A \otimes_0 B, \tag{5.11}$$

$${}_A\#(X \xrightarrow{f} Y) = (A \xrightarrow{\mathrm{id}} A) \otimes_1 (X \xrightarrow{f} Y). \tag{5.12}$$

Proof. We have to prove the axioms of $\frac{1}{2}$-monoidal category 5.1.2.

1. First of all we must prove that if A is an object then $\#_A, {}_A\#$ are functors $\mathcal{C} \to \mathcal{C}$. Given $X, Y, Z \in ob(\mathcal{C})$, $f \in \hom_{\mathcal{C}}(X, Y)$ and $g \in \hom_{\mathcal{C}}(Y, Z)$.

$$\begin{aligned} {}_A\#(f \star g) =& \mathrm{id}_A \otimes_1 (f \star g), \text{ by (5.12)} \\ =& (\mathrm{id}_A \star \mathrm{id}_A) \otimes_1 (f \star g) \\ =& (\mathrm{id}_A \otimes_1 f) \star (\mathrm{id}_A \otimes_1 g), \text{ by def. 5.3.2} \\ =& {}_A\#(f) \star {}_A\#(g), \text{ by (5.12).} \end{aligned}$$

$$
\begin{aligned}
{}_A\#(\mathrm{id}_B) &= \mathrm{id}_A \otimes_1 \mathrm{id}_B, \text{ by (5.12)}\\
&= \mathrm{id}_{A\otimes_0 B}, \text{ by def. 5.3.2}\\
&= \mathrm{id}_{{}_A\# B}, \text{ by (5.11).}
\end{aligned}
$$

By similar way $\#_A: \mathcal{C} \to \mathcal{C}$ is a functor.

2. Let $A, B \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y)$,

$$
\begin{aligned}
{}_A\# \circ {}_B\#(f) &= {}_A\#({}_B\#(f))\\
&= {}_A\#(\mathrm{id}_B \otimes_1 f), \text{ by (5.12)}\\
&= \mathrm{id}_A \otimes_1 (\mathrm{id}_B \otimes_1 f), \text{ by (5.12)}\\
&= (\mathrm{id}_A \otimes_1 \mathrm{id}_B) \otimes_1 f, \text{ by def. 5.3.2}\\
&= \mathrm{id}_{A\otimes_0 B} \otimes_1 (f), \text{ by def. 5.3.2}\\
&= {}_{A\otimes_0 B}\#(f), \text{ by (5.12).}
\end{aligned}
$$

3. We can prove by similar way, $\#_A \circ \#_B = \#_{B\otimes_0 A}$.

4. Let $A, B \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y)$,

$$
\begin{aligned}
\#_A \circ {}_B\#(f) &= \#_A({}_B\#(f))\\
&= \#_A(\mathrm{id}_B \otimes_1 f), \text{ by (5.12)}\\
&= (\mathrm{id}_B \otimes_1 f) \otimes_1 \mathrm{id}_A\ , \text{ by (5.10)}\\
&= \mathrm{id}_B \otimes_1 (f \otimes_1 \mathrm{id}_A)\ , \text{ by def. 5.3.2}\\
&= \mathrm{id}_B \otimes_1 (\#_A(f))\ , \text{ by (5.10)}\\
&= {}_B\#(\#_A(f))\ , \text{ by (5.12)}\\
&= {}_B\# \circ \#_A(f).
\end{aligned}
$$

5. Let $A \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y)$,

$$\begin{aligned} \#_I(A) =& A \otimes_0 I \text{ , by (5.9)} \\ =& A \text{ , by def. 5.3.2} \\ \#_I(f) =& f \otimes_1 \mathrm{id}_I \text{ , by (5.10)} \\ =& f\text{, by def. 5.3.2.} \end{aligned}$$

Therefore, $\#_I = \mathrm{id}_{\mathcal{C}}$.

6. We can prove 5 by similarity, ${}_I\# = \mathrm{id}_{\mathcal{C}}$.

□

Theorem 5.4.2. *The strict monoidal category $\mathcal{C} = (\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$ in 5.3.2 gives rise to a slideable $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$, where $\mathcal{C}$, I and $\otimes_0$ do not change. Given $A \in ob(\mathcal{C})$, the functor $\#_A$, ${}_A\#$ define by given $B, X, Y \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y)$*

$$\begin{aligned} \#_A B &= B \otimes_0 A \\ \#_A(X \xrightarrow{f} Y) &= (X \xrightarrow{f} Y) \otimes_1 (A \xrightarrow{\mathrm{id}} A) \\ {}_A\# B &= A \otimes_0 B \\ {}_A\#(X \xrightarrow{f} Y) &= (A \xrightarrow{\mathrm{id}} A) \otimes_1 (X \xrightarrow{f} Y). \end{aligned}$$

Proof. We proved the axioms of $\frac{1}{2}$-monoidal category in 5.4.1. We want to prove now the slideable that mean we want to prove for all $f: X \to Y$ and $g: Z \to W$, the following diagram commutes

$$\begin{array}{ccc} X \otimes_0 Z & \xrightarrow{X\Theta g} & X \otimes_0 W \\ {\scriptstyle f\Theta Z}\downarrow & & \downarrow{\scriptstyle f\Theta W} \\ Y \otimes_0 Z & \xrightarrow[Y\Theta g]{} & Y \otimes_0 W \end{array}$$

i.e. we have

$$(f\Theta W) \star (X\Theta g) = (Y\Theta g) \star (f\Theta Z).$$

$$\begin{aligned}(f\Theta W)\star(X\Theta g) =&(f\otimes_1 \mathrm{id}_W)\star(\mathrm{id}_X\otimes_1 g), \text{ by (5.10)(5.12)}\\ =&(f\star\mathrm{id}_X)\otimes_1(\mathrm{id}_W\star g), \text{ by def. 5.3.2}\\ =&(\mathrm{id}_Y\star f)\otimes_1(g\star\mathrm{id}_Z), \text{ by category axioms.}\\ =&(\mathrm{id}_Y\otimes_1 g)\star(f\otimes_1\mathrm{id}_Z), \text{ by def. 5.3.2}\\ =&(Y\Theta g)\star(f\Theta Z), \text{ by (5.10)(5.12).}\end{aligned}$$

□

Lemma 5.4.3. *There is a faithful functor $\mathfrak{f}$ from the category* $\mathbf{StriMC}$ *of strict monoidal categories (as in 5.3.6) to the category* $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{-MC}$ *of $\frac{1}{2}$-monoidal categories (as in Prop. 5.1.13), such that*

1. *on objects, given strict monoidal category $\mathcal{C} = (\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$,*

$$\mathfrak{f}(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1)) = (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#),$$

   from Lemma 5.4.1,

2. *on morphisms, given a strict monoidal functor*

$$F\colon(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1)) \to (\mathcal{D}, I, \otimes' = (\otimes'_0, \otimes'_1)),$$

$$\mathfrak{f}(F) = F.$$

Proof. We want to prove that given a strict monoidal functor

$$F\colon(\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1)) \to (\mathcal{D}, I, \otimes' = (\otimes'_0, \otimes'_1))$$

satisfies the axioms of $\frac{1}{2}$ monoidal functor in 5.1.12. For all $A, B \in ob(\mathcal{C})$ and f, g

morphisms in $\mathcal{C}$, we have

$$\begin{aligned}F({}_A\#(B)) &= F(A \otimes_0 B), \text{ by (5.11)}\\ &= F(A) \otimes_0 F(B), \text{ because } F \text{ is strict monoidal functor}\\ &= {}_{F(A)}\#(F(B)), \text{ by (5.11)},\\ F({}_A\#(g)) &= F(\mathrm{id}_A \otimes_1 g), \text{ by (5.12)}\\ &= F(\mathrm{id}_A) \otimes_1 F(g) \text{ , because } F \text{ is strict monoidal functor}\\ &= {}_{F(A)}\#(F(g)), \text{ by (5.12).}\end{aligned}$$

By a similar argument

$$\begin{aligned}F(\#_A(B)) &= \#_{F(A)}(F(B)),\\ F(\#_A g) &= \#_{F(A)}(F(g)).\end{aligned}$$

$F(I) = I$ for strict monoidal categories gives the same equation of $\frac{1}{2}$-monoidal categories. □

Theorem 5.4.4. *Consider a category $\mathcal{C} = (ob(\mathcal{C}), \hom_{\mathcal{C}}(\_,\_), \circ, \mathrm{id}_\_)$. A slideable $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ with underlying category $\mathcal{C}$ gives rise to a strict monoidal category $\mathcal{C} = (\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1))$ where ($\mathcal{C}$, I and $\otimes_0$ do not change; and) the functor $\otimes = (\otimes_0, \otimes_1)$ is defined as*
Given $A, B, X, Y, Z, W \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y), g \in \hom_{\mathcal{C}}(Z, W)$,

$$B \otimes_0 A = \#_A B \tag{5.13}$$

$$f \otimes_1 g = (\#_W f) \circ ({}_X\# g) \tag{5.14}$$

$$= ({}_Y\# g) \circ (\#_Z f) \tag{5.15}$$

$$\mathrm{id}_A = \#_I A = {}_I\# A \tag{5.16}$$

Proof. We have to prove the axioms of strict monoidal category in 5.3.2

1. for all $X, Y, Z \in ob(\mathcal{C})$, from the definition

$$(X \otimes_0 Y) \otimes_0 Z = X \otimes_0 (Y \otimes_0 Z),$$

2. for all morphisms $f \in \hom_{\mathcal{C}}(X, Y), g \in \hom_{\mathcal{C}}(Z, W), h \in \hom_{\mathcal{C}}(R, S)$, we have

$$\begin{aligned}(f \otimes_1 g) \otimes_1 h =& ((\#_W f) \circ ({}_X\#g)) \otimes_1 h, \text{ by (5.14)}\\ =& \#_S((\#_W f) \circ ({}_X\#g)) \circ {}_{X\otimes_0 Z}\#h, \text{ by (5.14)}\\ =& (\#_S(\#_W f) \circ \#_S({}_X\#g)) \circ {}_{(X\otimes_0 Z)}\#h, \text{ because } \#_S \text{ is a functor}\\ =& \#_S(\#_W f) \circ (\#_S({}_X\#g) \circ {}_{(X\otimes_0 Z)}\#h), \text{ by the associtivity of category.}\\ =& \#_{(W\otimes_0 S)}f \circ (({}_X\#\#_S g) \circ ({}_X\#_Z\#h)), \text{ by (5.2),(5.3),(5.1)}\\ =& \#_{(W\otimes_0 S)}f \circ ({}_X\#(\#_S g \circ {}_Z\#h)), \text{ because } {}_X\# \text{ is a functor}\\ =& f \otimes_1 (g \otimes_1 h), \text{ by (5.14),}\end{aligned}$$

3. for all $f \colon X \to Y, f' \colon Y \to Z, g \colon U \to V$ and $g' \colon V \to W$,

$$\begin{aligned}(f' \otimes_1 g') \circ (f \otimes_1 g) =& ((\#_W f') \circ ({}_Y\#g')) \circ ((\#_V f) \circ ({}_X\#g)), \text{ by (5.14)}\\ =& (\#_W f') \circ (({}_Y\#g') \circ (\#_V f)) \circ ({}_X\#g)\text{*}\\ =& (\#_W f') \circ ((\#_W f) \circ ({}_X\#g')) \circ ({}_X\#g)), \text{ by (5.14),(5.15)}\\ =& ((\#_W f') \circ (\#_W f)) \circ (({}_X\#g') \circ ({}_X\#g))\text{*}\\ =& (\#_W(f' \circ f)) \circ ({}_X\#(g' \circ g)), \text{ because } \#_W \text{ and } {}_X\# \text{ are functors}\\ =& (f' \circ f) \otimes_1 (g' \circ g), \text{ by (5.14).}\end{aligned}$$

*By the associativity of the category. □

Lemma 5.4.5. *There is a faithful functor $\mathfrak{g}$ from the category $\mathbf{s\frac{1}{2}{-}MC}$ of slideable $\frac{1}{2}$-monoidal categories to the category $\mathbf{StriMC}$ of strict monoidal categories (as in 5.3.6) such that*

1. on objects, given $\frac{1}{2}$-monoidal category $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$, we have

$$\mathfrak{g}(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#) = (\mathcal{C}, I, \otimes = (\otimes_0, \otimes_1)),$$

from Theorem 5.4.4,

2. *on morphisms, given a $\frac{1}{2}$-monoidal functor*

$$G\colon (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#) \to (\mathcal{D}, I, \otimes_0', \#'_{(-)}, {}_{(-)}\#'),$$

$$\mathfrak{g}(G) = G.$$

Proof. We want to prove that given a $\frac{1}{2}$-monoidal functor

$$G\colon (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#) \to (\mathcal{D}, I, \otimes_0', \#'_{(-)}, {}_{(-)}\#'),$$

G satisfies the axioms of strict monoidal functor in 5.3.4.

1. $G(I_{\mathcal{C}}) = I_{\mathcal{D}}$,

2. for all $A, B \in ob(\mathcal{C})$, we have

$$\begin{aligned} G(A \otimes_0 B) &= G({}_A\#B), \text{ by (5.13)} \\ &= {}_{G(A)}\#G(B), G \text{ is } \frac{1}{2}\text{-monoidal functor} \\ &= G(A) \otimes_0 G(B), \text{ by (5.13)}, \end{aligned}$$

3. for all $X, Y, Z, W \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y), \hom_{\mathcal{C}}(Z, W)$, we have

$$\begin{aligned} G(f \otimes_1 g) &= G(\#_W(f) \circ {}_X\#(g)), \text{ by (5.14)} \\ &= G(\#_W(f)) \circ G({}_X\#(g)), G \text{ is a functor} \\ &= \#_{G(W)}G(f) \circ {}_{G(X)}\#G(g), G \text{ is } \frac{1}{2}\text{-monoidal functor.} \\ &= G(f) \otimes_1 G(g), \text{ by (5.14)}. \end{aligned}$$

□

Theorem 5.4.6. *The category of slideable $\frac{1}{2}$-monoidal categories and the category of strict monoidal categories are isomorphic. Let $\mathcal{C} = (\mathcal{C}, I_{\mathcal{C}}, \otimes = (\otimes_0, \otimes_1))$ be a strict* *monoidal category and functor $\mathfrak{f}$ that was defined in Lem. 5.4.3 and for all slideable $\frac{1}{2}$-monoidal category $\mathcal{D} = (\mathcal{D}, I_{\mathcal{D}}, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ and functor $\mathfrak{g}$ that defined in Lem. 5.4.5, we have*

$$\mathfrak{f}(\mathfrak{g}(\mathcal{D})) = \mathcal{D} \quad \textit{and} \quad \mathfrak{g}(\mathfrak{f}(\mathcal{C})) = \mathcal{C}.$$

Proof.

$$\mathfrak{g}(\mathcal{D}) = (\mathcal{D}, I_{\mathcal{D}}, \otimes = (\otimes_0, \otimes_1)).$$

$I_{\mathcal{D}}$ and $\otimes_0$ are the same in $\mathcal{D}$ and $\mathfrak{g}(\mathcal{D})$ by definition. Given $X, Y, Z, W \in ob(\mathcal{D})$ and $f \in \hom_{\mathcal{D}}(X, Y), g \in \hom_{\mathcal{D}}(Z, W)$,

$$f \otimes_1 g = (\#_W f) \circ ({}_X\# g) \text{ , by (5.14)}$$

$$\mathfrak{f}(\mathfrak{g}(\mathcal{D})) = (\mathcal{D}, I_{\mathcal{D}}, \otimes_0, \#'_{(-)}, {}_{(-)}\#').$$

$I_{\mathcal{D}}$ and $\otimes_0$ are the same in $\mathfrak{g}(\mathcal{D})$ and $\mathfrak{f}(\mathfrak{g}(\mathcal{D}))$ by definition. We want to prove $\#$ and $\#'$ are equal. Given $A, X, Y \in ob(\mathcal{D})$ and $f \in \hom_{\mathcal{D}}(X, Y)$,

$$\begin{aligned}
\#'_A(f) &= f \otimes_1 \mathrm{id}_A, \text{ by (5.10)} \\
&= (\#_A(f)) \circ ({}_X\#(\mathrm{id}_A)), \text{ by (5.14)} \\
&= (f \otimes_1 \mathrm{id}_A) \circ (\mathrm{id}_X \otimes_1 \mathrm{id}_A), \text{ by (5.10)(5.12)} \\
&= \#_A(f), \\
{}_A\#'(f) &= \mathrm{id}_A \otimes_1 f \\
&= (\#_Y(\mathrm{id}_A)) \circ (({}_A\#(f)), \text{ by (5.14)} \\
&= (\mathrm{id}_A \otimes_1 Y) \circ (\mathrm{id}_A \otimes_1 f), \text{ by (5.10)(5.12)} \\
&= {}_A\#(f).
\end{aligned}$$

Therefore

$$\mathfrak{f}(\mathfrak{g}(\mathcal{D})) = \mathcal{D}.$$

Also,

$$\mathfrak{f}(\mathcal{C}) = (\mathcal{C}, I_{\mathcal{C}}, \otimes_0, \#_{(-)}, {}_{(-)}\#).$$

$I_{\mathcal{C}}$ and $\otimes_0$ are the same in $\mathcal{C}$ and $\mathfrak{f}(\mathcal{C})$ by definition. Given $A, X, Y \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y)$,

$$\#_A(f) = f \otimes_1 \mathrm{id}_A, \text{ by (5.10)},$$
$$_A\#(f) = \mathrm{id}_A \otimes_1 f, \text{ by (5.12)}.$$

$$\mathfrak{g}(\mathfrak{f}(\mathcal{C})) = (\mathcal{C}, I_{\mathcal{C}}, \otimes' = (\otimes_0, \otimes_1'))$$

$I_{\mathcal{C}}$ and $\otimes_0$ are the same in $\mathfrak{f}(\mathcal{C})$ and $\mathfrak{g}(\mathfrak{f}(\mathcal{C}))$ by definition. We want to prove $\otimes_1$ and $\otimes_1'$ are equal. Given $X, Y, Z, W \in ob(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(X, Y), g \in \hom_{\mathcal{C}}(Z, W)$,

$$\begin{aligned} f \otimes_1' g &= (\#_W f) \circ ({}_X\# g), \text{ by (5.14)} \\ &= (f \otimes_1 \mathrm{id}_W) \circ (\mathrm{id}_X \otimes_1 g), \text{ by (5.10)(5.12)} \\ &= f \otimes_1 g. \end{aligned}$$

Therefore

$$\mathfrak{g}(\mathfrak{f}(\mathcal{C})) = \mathcal{C}.$$

So the category of slideable $\frac{1}{2}$-monoidal categories and the category of strict monoidal categories are isomorphic. □

Chapter 6

Free strict monoidal categories

In §3.3 we defined a left adjoint to the forgetful functor from categories to graphs. We now want to define a free strict monoidal category. We do this by defining a left adjoint to the forgetful functor from $\frac{1}{2}$-monoidal category to a monoidal graph that is a graph together with a monoid structure in the set of vertices.

6.1 The category of monoidal graphs

In this section we define a free strict monoidal category over a monoidal graph — that is a graph 3.3.1 that its vertices have the monoidal structure (in a sense that we explain).

Definition 6.1.1. *A monoidal graph is a 6-tuple* $(V(R), \otimes, e, E(R), \delta_1, \delta_2)$ *consisting of*

1. *a set* $V = V(R)$*;*
2. *a binary operation* $\otimes$ *in* $V(R)$*;*
3. *an element* $e \in V(R)$*;*
4. *a set* $E = E(R)$*. The elements of* E *are called edges;*

5. *a map, called* incidence map

$$\delta\colon E \to V \times V$$
$$x \mapsto (\delta_1(x), \delta_2(x)).$$

We call $\delta_1 x$ the source of x and $\delta_2 x$ the target of x.

6. $(V, \otimes, e)$ *is a monoid.*

Definition 6.1.2. *Let*

$$R = (V(R), \otimes, e, E(R), \delta_1, \delta_2)$$

be a monoidal graph. The extent *of the monoidal graph R is the graph*

$$R^* = (V(R), T(R), \delta_1^*, \delta_2^*),$$

where

$$T(R) = V(R) \times E(R) \times V(R)$$

is the set of edges for the graph R^*. The incidence maps are

$$\delta_1^*(i, x, j) = i \otimes \delta_1(x) \otimes j,$$

$$\delta_2^*(i, x, j) = i \otimes \delta_2(x) \otimes j,$$

where $(i, x, j) \in T(R)$.

Example 6.1.3. *Consider the monoidal graph*

$$R = (\mathbb{N}, +, 0, \{x_1, x_2, x_3, x_4\}, \delta_1, \delta_2),$$

where

$$\delta_1(x_1) = 2, \qquad \delta_2(x_1) = 6,$$
$$\delta_1(x_2) = 6, \qquad \delta_2(x_2) = 8,$$
$$\delta_1(x_3) = 8, \qquad \delta_2(x_3) = 10,$$
$$\delta_1(x_4) = 1, \qquad \delta_2(x_4) = 5.$$

Then there is the extent of the monoidal graph,

$$R^* = (\mathbb{N}, \{y_{i1j}, y_{i2j}, y_{i3j}, y_{i4j} \mid \forall i, j \in \mathbb{N}\}, \delta_1^*, \delta_2^*),$$

such that

$$\delta_1^*(y_{i1j}) = \delta_1(i, x_1, j) = i + 2 + j,$$
$$\delta_2^*(y_{i1j}) = \delta_2(i, x_1, j) = i + 6 + j,$$
$$\delta_1^*(y_{i2j}) = \delta_1(i, x_2, j) = i + 6 + j,$$
$$\delta_2^*(y_{i2j}) = \delta_2(i, x_2, j) = i + 8 + j,$$
$$\delta_1^*(y_{i3j}) = \delta_1(i, x_3, j) = i + 8 + j,$$
$$\delta_2^*(y_{i3j}) = \delta_2(i, x_3, j) = i + 10 + j,$$
$$\delta_1^*(y_{i4j}) = \delta_1(i, x_4, j) = i + 1 + j,$$
$$\delta_2^*(y_{i4j}) = \delta_2(i, x_4, j) = i + 5 + j.$$

Definition 6.1.4 (Monoidal graph map)**.** *Let A, B monoidal graphs such that*

$$A = (V(A), \otimes_A, e_A, E(A), \delta_1, \delta_2),$$

and

$$B = (V(B), \otimes_B, e_B, E(B), \delta_1, \delta_2).$$

A monoidal graph map $\theta\colon A \to B$ *is a pair of maps* (V_θ, E_θ)*, where* $V_\theta\colon V(A) \to V(B)$

and $E_\theta \colon E(A) \to E(B)$*, which preserves incidences, i.e, for all edge* x *of* A*,*

$$\delta_i(E_\theta(x)) = V_\theta(\delta_i x); (i = 1, 2),$$

and moreover preserves the monoid structures in the the sets of vertices, i.e. for all $i, j \in A$*,*

$$V_\theta(i \otimes_A j) = V_\theta(i) \otimes_B V_\theta(j),$$
$$V_\theta(e_A) = e_B.$$

Proposition 6.1.5. *We have a category*

$$(ob(\mathbf{MGraphs}), \hom_{\mathbf{MGraphs}}(\_, \_), \star, \mathrm{id}_\_),$$

where

1. $ob(\mathbf{MGraphs})$ *is the class of all monoidal graphs,*
2. *for all monoidal graph* A *and* B*,* $\hom_{\mathbf{MGraphs}}(A, B)$*, is the set of all monoidal graph maps from* A *to* B*,*
3. *for all monoidal graphs* A, B *and* C *and for all monoidal graph maps* $f = (V_f, E_f) \colon A \to B$*,* $g = (V_g, E_g) \colon B \to C$*, and*

   $$f \star g \colon A \to C$$

   is the monoidal graph map from A *to* C*, such that for all* $a \in V(A)$*, we have*

   $$(V_f \star V_g)(a) = V_g(V_f(a)),$$

   and for all $x \in E(A)$*, we have*

   $$(E_f \star E_g)(x) = E_g(E_f(x)),$$

4. *for all monoidal graph A, the identity morphism $\mathrm{id}_A\colon A \to A$ is a monoidal graph map, such that for all $a \in V(A)$, we have*

$$(V_{\mathrm{id}_A})(a) = a,$$

and for all $x \in E(A)$, we have

$$(E_{\mathrm{id}_A})(x) = x.$$

Proof. Note that the monoidal graph map is a graph map preserving monoidal structure in the sets of vertices. First we want to prove for all monoidal graphs A, B and C and for all monoidal graph maps $f\colon A \to B$, $g\colon B \to C$

$$f \star g\colon A \to C$$

is a monoidal graph map from A to C, for all $i, j \in V(A)$, we have

$$\begin{aligned}(V_f \star V_g)(i \otimes_A j) &= V_g(V_f(i \otimes_A j)) \\ &= V_g(V_f(i) \otimes_B V_f(j)) \\ &= V_g(V_f(i)) \otimes_C V_g(V_f(j)) \\ &= V_f \star V_g(i) \otimes_C V_f \star V_g(j).\end{aligned}$$

$$\begin{aligned}(V_f \star V_g)(e_A) &= V_g(V_f(e_A)) \\ &= V_g(e_B) = e_C.\end{aligned}$$

Second we want to prove the category axioms for all monoidal graphs A, B, C and D and for all monoidal graph maps $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, and for all $i \in V(A)$, $x \in E(A)$ we have

$$\begin{aligned}(V_f \star (V_g \star V_h))(i) &= V_g \star V_h(V_f(i))\\ &= V_h(V_g(V_f(i)))\\ &= V_g(V_f(i)) \star V_h\\ &= ((V_f \star V_g) \star V_h)(i).\end{aligned}$$

$$\begin{aligned}(E_f \star (E_g \star E_h))(x) &= E_g \star E_h(E_f(x))\\ &= E_h(E_g(E_f(x)))\\ &= E_g(E_f(x)) \star E_h\\ &= ((E_f \star E_g) \star E_h)(x).\end{aligned}$$

□

Definition 6.1.6. *(The forgetful functor from the category of $\frac{1}{2}$-monoidal categories to the category of monoidal graphs). We have a* forgetful functor

$$U\colon \mathbf{\frac{1}{2}{-}MC} \to \mathbf{MGraphs},$$

such that for the $\frac{1}{2}$-monoidal category $\mathcal{C} = (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ we have,

1.

$$U(\mathcal{C}) = (ob(\mathcal{C}), I, \otimes_0, Mor(\mathcal{C}) \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} ob(\mathcal{C})),$$

where

$$Mor(\mathcal{C}) = \bigcup_{(x,y)\in ob(\mathcal{C})\times ob(\mathcal{C})} (\{x\} \times \hom_{\mathcal{C}}(x,y) \times \{y\}),$$

where for every $f\colon x \to y$,

$$\delta_1(x,f,y) = x, \;\; \delta_2(x,f,y) = y.$$

2. *Let* $\mathcal{C} = (\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ *and* $\mathcal{D} = (\mathcal{D}, I', \otimes_0', \#'_{(-)}, {}_{(-)}\#')$ *be* $\frac{1}{2}$*-monoidal categories and let*

$$F = (F_0, F_1)\colon \mathcal{C} \to \mathcal{D}$$

be a $\frac{1}{2}$*-monoidal functor, we have*

$$U(\mathcal{C}) = \left(ob(\mathcal{C}), I, \otimes_0, \bigcup_{(x,y)\in ob(\mathcal{C})\times ob(\mathcal{C})} (\{x\} \times \hom_{\mathcal{C}}(x,y) \times \{y\}) \right),$$

$$U(\mathcal{D}) = \left(ob(\mathcal{D}), I', \otimes_0' \bigcup_{(x',y')\in ob(\mathcal{D})\times ob(\mathcal{D})} (\{x'\} \times \hom_{\mathcal{D}}(x',y') \times \{y'\}) \right)$$

and

$$U(F) = U(F_0, F_1) = (U(F)_0, U(F)_1),$$

where

$$U(F)_0 = F_0,$$

and

$$U(F)_1(x, f\colon x \to y, y) = (F_0(x), F_1(x \xrightarrow{f} y), F_0(y)).$$

Definition 6.1.7. *A* free-$\frac{1}{2}$-monoidal category-triple *on a monoidal graph* R *is a triple* $(R, \mathcal{C}, \delta_R)$*, where* $\mathcal{C}$ *is a* $\frac{1}{2}$*-monoidal category and* $\delta_R\colon R \to U(\mathcal{C})$ *is a monoidal graph-map such that the following universal property is satisfied.*

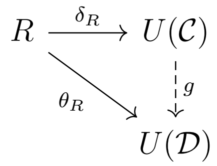
> *Given a* $\frac{1}{2}$*-monoidal category* A *and a monoidal graph map* $\theta\colon R \to U(A)$*, there is a unique* $\frac{1}{2}$*-monoidal functor* $\theta^*\colon \mathcal{C} \to A$ *such that* $\theta = \delta_R \star U(\theta^*)$*, i,e the next diagram commutes*

$$\begin{array}{ccc} R & \xrightarrow{\delta_R} & U(\mathcal{C}) \\ & {\scriptstyle\theta}\searrow & \downarrow{\scriptstyle U(\theta^*)} \\ & & U(A) \end{array}$$

Where U *is the forgetful functor from the category of* $\frac{1}{2}$*-monoidal categories to the category of monoidal graphs. This is to say that the arrow* δ_R *is a universal arrow from* R *to* U*.*

We want to prove that U is a right adjoint by proving that a universal arrow $\delta_R\colon R \to U(\mathcal{C})$ – what we called a free 1/2-monoidal category triple – exists for all R.

Proposition 6.1.8. *Given any free-$\frac{1}{2}$-monoidal category-triple $(R, \mathcal{C}, \delta)$, a universal arrows from object R to $U(C)$, if it exists, is unique up to ismorphism. I.e, if there is another free-$\frac{1}{2}$-monoidal category-triple $(R, \mathcal{D}, \theta)$, then there exists a unique functor $g\colon \mathcal{C} \to \mathcal{D}$ that makes the diagram commute. Moreover, that unique functor is an isomorphism.*



Proof. The proof is similar to the proof in 3.3.6. □

We want to construct the free $\frac{1}{2}$-monoidal category $\Omega(R)$ over a monoidal graph R. As a first step, we first consider the extent R^* of the monoidal graph R, namely

$$R^* = (V(R), e, T(R), \delta_1^*, \delta_2^*),$$

and then consider the free category over R^* 3.3.12, that is

$$P(R^*) = (V(R), \hom_{P(R^*)}(\_,\_), \bullet, \phi).$$

Here, for all $a, b \in V(R)$, we have

$$\begin{aligned}\hom_{P(R^*)}(a,b) := \{(a_1, r_1, b_1)(a_2, r_2, b_2)...(a_n, r_n, b_n) \mid \text{ for some } n \in \mathbb{N}, a = \delta_1^*(a_1, r_1, b_1),\\ b = \delta_2^*(a_n, r_n, b_n),\ \delta_2^*(a_i, r_i, b_i) = \delta_1^*(a_{i+1}, r_{i+1}, b_{i+1}),\\ , \qquad r_i \in E(R), i \in \{1, ..., n-1\}\}.\end{aligned}$$

Proposition 6.1.9. *Let*

$$P(R^*) = (V(R), \hom_{P(R^*)}(a,b), \bullet, \phi),$$

where R^* is the extent of monoidal graph R, then

$$\Omega(R) := (P(R^*), e, \otimes_0, ({}_a\#)_{a\in V(R)}, (\#_b)_{b\in V(R)})$$

is a $\frac{1}{2}$-monoidal category. Here $a, b \in V(R)$,

$$\begin{aligned}
&{}_a\#((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)) \\
&\qquad := (a \otimes_0 a_1, x_1, b_1), (a \otimes a_2, x_2, b_2), \ldots, (a \otimes_0 a_n, x_n, b_n). \\
&\#_b((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)) \\
&\qquad := (a_1, x_1, b_1 \otimes_0 b), (a_2, x_2, b_2 \otimes_0 b), \ldots, (a_n, x_n, b_n \otimes_0 b).
\end{aligned}$$

Proof. First we want to prove that for all $a \in V(R)$, $\#_a$ and ${}_a\#$ are functors, for all morphisms $f, g \in P(R^*)$ where

$$f = (a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n),$$

and

$$g = (h_1, y_1, l_1), (h_2, y_2, l_2), \ldots, (h_m, y_m, l_m),$$

such that

$$\delta_2(a_n, x_n, b_n) = \delta_1(h_1, y_1, l_1),$$

we have

$$\begin{aligned}
\#_a(f \bullet g) = \#_a\Big(&(a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n), (h_1, y_1, l_1), \\
&(h_2, y_2, l_2), \ldots, (h_m, y_m, l_m)\Big). \\
= &(a_1, x_1, b_1 \otimes_0 a), (a_2, x_2, b_2 \otimes_0 a), \ldots, (a_n, x_n, b_n \otimes_0 a), (h_1, y_1, l_1 \otimes_0 a), \\
&(h_2, y_2, l_2 \otimes_0 a), \ldots, (h_m, y_m, l_m \otimes_0 a). \\
= &\Big((a_1, x_1, b_1 \otimes_0 a), (a_2, x_2, b_2 \otimes_0 a), \ldots, (a_n, x_n, b_n \otimes_0 a)\Big) \\
&\Big((h_1, y_1, l_1 \otimes_0 a), (h_2, y_2, l_2 \otimes_0 a), \ldots, (h_m, y_m, l_m \otimes_0 a)\Big) \\
= &\#_a(f) \bullet \#_a(g).
\end{aligned}$$

Let $i \in V(R)$,

$$\#_a(\phi_i) = \#_a(e, \phi_i, e) = (e, \phi_i, a) = \phi_{\#_a(i)},$$

where ϕ_i is the identity morphism i.e, the empty path in the path category over the extent of monoidal graph R^*, where every edge write as $V \times E \times V$.

By a similar argument ${}_a\#$ is a functor. We want to verify the remaining axioms of $\frac{1}{2}$-monoidal category in 5.1.2.

1. To verify the axiom in (5.1)

$$\begin{aligned}
&{}_a\# \circ {}_b\#((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)) \\
&\qquad = {}_a\#((b \otimes_0 a_1, x_1, b_1), (b \otimes_0 a_2, x_2, b_2), \ldots, (b \otimes_0 a_n, x_n, b_n)) \\
&\qquad = (a \otimes_0 b \otimes_0 a_1, x_1, b_1), (a \otimes_0 b \otimes_0 a_2, x_2, b_2), \ldots, (a \otimes_0 b \otimes_0 a_n, x_n, b_n) \\
&\qquad = {}_{a\otimes_0 b}\#((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)).
\end{aligned}$$

2. To verify the axiom in (5.2)

$$\begin{aligned}
&\#_a \circ \#_b((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)) \\
&\qquad = \#_a((a_1, x_1, b_1 \otimes_0 b), (a_2, x_2, b_2 \otimes_0 b), \ldots, (a_n, x_n, b_n \otimes_0 b)) \\
&\qquad = (a_1, x_1, b_1 \otimes_0 b \otimes_0 a), (a_2, x_2, b_2 \otimes_0 b \otimes_0 a), \ldots, (a_n, x_n, b_n \otimes_0 b \otimes_0 a) \\
&\qquad = \#_{b\otimes_0 a}((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)).
\end{aligned}$$

3. To verify the axiom in (5.3)

$$\begin{aligned}
&\#_a \circ {}_b\#((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)) \\
&\qquad = \#_a((b \otimes_0 a_1, x_1, b_1), (b \otimes_0 a_2, x_2, b_2), \ldots, (b \otimes_0 a_n, x_n, b_n)) \\
&\qquad = ((b \otimes_0 a_1, x_1, b_1 \otimes_0 a), (b \otimes_0 a_2, x_2, b_2 \otimes_0 a), \ldots, (b \otimes_0 a_n, x_n, b_n \otimes_0 a)) \\
&\qquad = {}_b\#((a_1, x_1, b_1 \otimes_0 a), (a_2, x_2, b_2 \otimes_0 a), \ldots, (a_n, x_n, b_n \otimes_0 a)) \\
&\qquad = {}_b\# \circ \#_a((a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)).
\end{aligned}$$

4. To verify the axiom in (5.4)

$$\begin{aligned}&\#_e((a_1,x_1,b_1),(a_2,x_2,b_2),\ldots,(a_n,x_n,b_n))\\&\quad=(a_1,x_1,b_1\otimes_0 e),(a_2,x_2,b_2\otimes_0 e),\ldots,(a_n,x_n,b_n\otimes_0 e)\\&\quad=(a_1,x_1,b_1),(a_2,x_2,b_2),\ldots,(a_n,x_n,b_n).\end{aligned}$$

5. To verify the axiom in (5.5)

$$\begin{aligned}&{}_e\#((a_1,x_1,b_1),(a_2,x_2,b_2),\ldots,(a_n,x_n,b_n))\\&\quad=(e\otimes_0 a_1,x_1,b_1),(e\otimes_0 a_2,x_2,b_2),\ldots,(e\otimes_0 a_n,x_n,b_n)\\&\quad=(a_1,x_1,b_1),(a_2,x_2,b_2),\ldots,(a_n,x_n,b_n).\end{aligned}$$

So,

$${}_e\# = \#_e = \mathrm{id}_{P(R^*)}.$$

□

Proposition 6.1.10. *Let R be a monoidal graph. Then $(R, \Omega(R), \delta)$ is a free $\frac{1}{2}$- monoidal category triple, where*

$$\delta\colon R \to U(\Omega(R))$$

that is on vertices

$$a \mapsto a,$$

and on edges

$$x \mapsto (e, x, e).$$

Proof. We want to prove that given a $\frac{1}{2}$-monoidal category

$$\mathcal{C} = (\mathcal{C}, I_{\mathcal{C}}, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

and a monoidal graph map $F\colon R \to U(\mathcal{C})$, there is a unique $\frac{1}{2}$-monoidal functor 5.1.12

$$F'\colon \Omega(R) \to \mathcal{C}$$

that makes the next diagram commute

$$\begin{array}{ccc} R & \xrightarrow{\delta} & U(\Omega(R)) \\ & {}_{F}\searrow & \downarrow{\scriptstyle U(F')} \\ & & U(\mathcal{C}). \end{array} \tag{6.1}$$

First, we want to prove the existence; define the map

$$F'\colon \Omega(R) \to \mathcal{C}$$

that is on objects

$$a \mapsto F_0(a),$$

and on morphisms

$$(a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n) \mapsto$$
$$\Big(F_0(a_1)\Theta F_1(x_1)\Theta F_0(b_1)\Big)\star\Big(F_0(a_2)\Theta F_1(x_2)\Theta F_0(b_2)\Big)\star\ldots\star\Big(F_0(a_n)\Theta F_1(x_n)\Theta F_0(b_n)\Big).$$

Note Θ is whiskering where ${}_B\# \circ \#_A(x \xrightarrow{f} y) = (B \otimes x \otimes_0 A \xrightarrow{B\Theta f\Theta A} B \otimes y \otimes_0 A)$.

Now, we want to prove F' is a functor, for all morphism $f, g \in \Omega(R)$

$$f = (a_1, x_1, b_1), (a_2, x_2, b_2), \ldots, (a_n, x_n, b_n)$$

and

$$g = (h_1, y_1, l_1), (h_2, y_2, l_2), \ldots, (h_m, y_m, l_m),$$

such that

$$\delta_2(a_n, x_n, b_n) = \delta_1(h_1, y_1, l_1),$$

we have

$$\begin{aligned}
F_1'(f \bullet g) &= F_1'((a_1, x_1, b_1), (a_2, x_2, b_2), \dots, (a_n, x_n, b_n), (h_1, y_1, l_1), \\
&\qquad (h_2, y_2, l_2), \dots, (h_m, y_m, l_m)) \\
&= \Big(F_0(a_1)\Theta F_1(x_1)\Theta F_0(b_1)\Big) \star \Big(F_0(a_2)\Theta F_1(x_2)\Theta F_0(b_2)\Big) \star \dots \star \\
&\Big(F_0(a_n)\Theta F_1(x_n)\Theta F_0(b_n)\Big) \star \Big(F_0(h_1)\Theta F_1(y_1)\Theta F_0(l_1)\Big) \star \\
&\Big(F_0(h_2)\Theta F_1(y_2)\Theta F_0(l_2)\Big) \star \dots \star \Big(F_0(h_m)\Theta F_1(y_m)\Theta F_0(l_m)\Big) \\
&= \Bigg(\Big(F_0(a_1)\Theta F_1(x_1)\Theta F_0(b_1)\Big) \star \Big(F_0(a_2)\Theta F_1(x_2)\Theta F_0(b_2)\Big) \star \dots \star \\
&\Big(F_0(a_n)\Theta F_1(x_n)\Theta F_0(b_n)\Big)\Bigg) \star \Bigg(\Big(F_0(h_1)\Theta F_1(y_1)\Theta F_0(l_1)\Big) \star \\
&\Big(F_0(h_2)\Theta F_1(y_2)\Theta F_0(l_2)\Big) \star \dots \star \Big(F_0(h_m)\Theta F_1(y_m)\Theta F_0(l_m)\Big)\Bigg) \\
&= F'(f) \star_{\mathcal{C}} F'(g).
\end{aligned}$$

For each $a \in ob(\Omega(R))$

$$F_1'(\phi_a) = F_1(\phi_a) = \phi_{F_0(a)} = \phi_{F_0'(a)}.$$

Hence F' is a functor. Now we want to prove F' is a $\frac{1}{2}$-monoidal functor in 5.1.12

1. Let e be the identity element,
$$F'(e) = I_{\mathcal{C}}.$$
2. For all $a, b \in ob(\Omega(R))$ we have
$$\begin{aligned}
F_0'(a \otimes_0 b) &= F_0(a \otimes_0 b) \\
&= F_0(a) \otimes_0 F_0(b) \\
&= F_0'(a) \otimes_0 F_0'(b).
\end{aligned}$$

3. For all $A \in ob(\Omega(R)$ and $f \in \hom_{P(R^*)}(i,j)$ we have

$$F_1'({}_A\#(f)) = F_1'\Big({}_A\#\Big((a_1,x_1,b_1),(a_2,x_2,b_2),\ldots,(a_n,x_n,b_n)\Big)\Big)$$

$$= F_1'\Big((A\otimes_0 a_1,x_1,b_1),(A\otimes_0 a_2,x_2,b_2),\ldots,(A\otimes_0 a_n,x_n,b_n)\Big)$$

$$= \Big(F_0(A\otimes_0 a_1)\Theta F_1(x_1)\Theta F_0(b_1)\Big)\star\Big(F_0(A\otimes_0 a_2)\Theta F_1(x_2)\Theta F_0(b_2)\Big)$$

$$\star\ldots\star\Big(F_0(A\otimes_0 a_n)\Theta F_1(x_n)\Theta F_0(b_n)\Big)$$

$$= \Big((F_0(A)\otimes F_0(a_1))\Theta F_1(x_1)\Theta(F_0(b_1))\Big)\star$$

$$\Big((F_0(A)\otimes_0 F_0(a_2))\Theta F_1(x_2)\Theta(F_0(b_2))\Big)$$

$$\star\ldots\star\Big((F_0(A)\otimes_0 F_0(a_n))\Theta F_1(x_n)\Theta F_0(b_n)\Big)$$

$$\stackrel{*}{=}\Big(F_0(A)\Theta(F_0(a_1)\Theta F_1(x_1)\Theta F_0(b_1))\Big)\star$$

$$\Big(F_0(A)\Theta(F_0(a_2)\Theta F_1(x_2)\Theta F_0(b_2))\Big)\star\ldots\star$$

$$\Big(F_0(A)\Theta(F_0(a_n)\Theta F_1(x_n)\Theta F_0(b_n))\Big)$$

$$\stackrel{**}{=}_{F_0(A)}\#\Big(\Big(F_0(a_1)\Theta F_1(x_1)\Theta F_0(b_1)\Big)\star\Big(F_0(a_2)\Theta F_1(x_2)\Theta F_0(b_2)\Big)$$

$$\star\ldots\star\Big(F_0(a_n)\Theta F_1(x_n)\Theta F_0(b_n)\Big)\Big)$$

$$=_{F_0(A)}\#F_1'\Big((a_1,x_1,b_1),(a_2,x_2,b_2),\ldots,(a_n,x_n,b_n)\Big)$$

$$=_{F_0(A)}\#F_1'(f)$$

$$=_{F_0'(A)}\#F_1'(f).$$

Therefore

$$F'({}_A\#(f)) =_{F'(A)}\#(F'(f)).$$

* follow from (5.1).

** Since $\#_{F_0(A)}$ is a functor.

4. By a similar argument

$$F'(\#_A(f)) = \#_{F'(A)}(F'(f)).$$

Second uniqueness; suppose we have another $\frac{1}{2}$-monoidal functor

$$F'' \colon \Omega(R) \to \mathcal{C}$$

that makes the diagram (6.1) commute. Then

$$\begin{aligned} F''(f) =& F''\Big((a_1, x_1, b_1), (a_2, x_2, b_2), \dots, (a_n, x_n, b_n)\Big) \\ =& F''(a_1, x_x, b_1) \star F''(a_2, x_2, b_2) \star \dots \star F''(a_n, x_n, b_n) \\ =& \Big(F_0(a_1)\Theta F_1(x_1)\Theta F_0(b_1)\Big) \star \Big(F_0(a_2)\Theta F_1(x_2)\Theta F_0(b_2)\Big) \\ & \star \dots \star \Big(F_0(a_n)\Theta F_1(x_n)\Theta F_0(b_n)\Big) \\ =& F'(f). \end{aligned}$$

□

6.2 Presentations of Strict Monoidal Categories

In this section we define presentation of strict monoidal categories by defining presentations of $\frac{1}{2}$-monoidal categories . To this end we define $\frac{1}{2}$-monoidal congruence , then we define $\frac{1}{2}$-monoidal closure $\overline{\underline{W}}$ of a congruence template W. Then we quotient the free $\frac{1}{2}$-monoidal category by $\overline{\underline{W}}$. Finally we have the presentation of strict monoidal category by applying the slidealisation functor to the quotient.

Definition 6.2.1 ($\frac{1}{2}$-monoidal congruence)**.** *Let $(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$ be a $\frac{1}{2}$-monoidal category. A $\frac{1}{2}$-*monoidal congruence *R is a congruence relation 3.4.3 in the category $\mathcal{C}$ such that, for all object A, and morphisms $f, g \colon X \to Y$ if $f \sim_{R_{X,Y}} g$. then*

$$\#_A(f) \sim_{R_{X\otimes_0 A, Y\otimes_0 A}} \#_A(g),$$

and

$${}_A\#(f) \sim_{R_{A\otimes_0 X, A\otimes_0 Y}} {}_A\#(g).$$

Proposition 6.2.2. *Let R be a $\frac{1}{2}$-monoidal congruence on a $\frac{1}{2}$-monoidal category*

$$(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#).$$

Consider the quotient category

$$\mathcal{C}/R = (ob(\mathcal{C}), \hom_{\mathcal{C}/R}(\_,\_), \star_{\mathcal{C}/R}, \mathrm{id}'_{\_}),$$

as defined in 3.4.6. (Note that R is in particular a congruence in $\mathcal{C}$, so it makes sense to take the quotient category). Then

$$(\mathcal{C}/R, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

is a $\frac{1}{2}$-monoidal category, where for all $A \in ob(\mathcal{C})$, and $f \in \hom_{\mathcal{C}/R}(X, Y)$

$$
\begin{aligned}
{}_A\#([f]_{R_{X,Y}}) &= [{}_A\#(f)]_{R_{A\otimes_0 X, A\otimes_0 Y}}, \\
\#_A([f]_{R_{X,Y}}) &= [\#_A(f)]_{R_{X\otimes_0 A, Y\otimes_0 A}}.
\end{aligned}
$$

Proof. First we want to prove that

$$
\begin{aligned}
{}_A\#([f]_{R_{X,Y}}) &= [{}_A\#(f)]_{R_{A\otimes_0 X, A\otimes_0 Y}}, \\
\#_A([f]_{R_{X,Y}}) &= [\#_A(f)]_{R_{X\otimes_0 A, Y\otimes_0 A}}.
\end{aligned}
$$

are well defined.

Let $f \sim_{R_{X,Y}} g$, we have

$$
\begin{aligned}
{}_A\#(f) &\sim_{R_{A\otimes_0 X, A\otimes_0 Y}} {}_A\#(g), \\
\#_A(f) &\sim_{R_{X\otimes_0 A, Y\otimes_0 A}} \#_A(g).
\end{aligned}
$$

This follows from the definition of $\frac{1}{2}$-monoidal congruence in 6.2.1.
Secondly, we want to prove the $\frac{1}{2}$-monoidal category axioms.
We want to prove for all $A \in ob(\mathcal{C}/R)$, ${}_A\#$ and $\#_A$ are functors. Given $X, Y, Z \in$

$ob(\mathcal{C}/R)$, $[f] \in \hom_{\mathcal{C}/R}(X,Y)$ and $[g] \in \hom_{\mathcal{C}/R}(Y,Z)$.

$$\begin{aligned}
{}_A\#([g]_{R_{Y,Z}} \star [f]_{R_{X,Y}}) &= {}_A\#([g \star f]_{R_{X,Z}}) \\
&= [{}_A\#(g \star f)]_{R_{A\otimes_0 X, A\otimes_0 Z}} \\
&= [{}_A\#(g) \star {}_A\#(f)]_{R_{A\otimes_0 X, A\otimes_0 Z}} \\
&= [{}_A\#(g)]_{R_{A\otimes_0 Y, A\otimes_0 Z}} \star [{}_A\#(f)]_{R_{A\otimes_0 X, A\otimes_0 Y}} \\
&= {}_A\#([g]_{R_{Y,Z}}) \star {}_A\#([f]_{R_{X,Y}}).
\end{aligned}$$

Also,

$${}_A\#(\mathrm{id}'_X) = {}_A\#([\mathrm{id}_X]) = [{}_A\#(\mathrm{id}_X)] = [\mathrm{id}_{{}_A\#(X)}] = \mathrm{id}'_{{}_A\#(X)}.$$

By a similar argument

$$\#_A([g]_{R_{Y,Z}} \star [f]_{R_{X,Y}}) = \#_A([g]_{R_{Y,Z}}) \star \#_A([f]_{R_{X,Y}}).$$

$$\#_A(\mathrm{id}'_X) = \mathrm{id}'_{\#_A(X)}.$$

For all $A, B \in ob(\mathcal{C})$ and morphisms $[f] \in \hom_{\mathcal{C}/R}(X,Y)$, we have

$$\begin{aligned}
{}_A\#({}_B\#([f]_{R_{X,Y}})) &= {}_A\#([{}_B\#(f)]_{R_{B\otimes_0 X, \otimes_0 Y}}) \\
&= [{}_A\#({}_B\#(f))]_{R_{A\otimes_0 B\otimes_0 X, A\otimes_0 B\otimes_0 Y}} \\
&\overset{by (5.1)}{=} [{}_{A\otimes_0 B}\#(f)]_{R_{A\otimes_0 B\otimes_0 X, A\otimes_0 B\otimes_0 Y}} \\
&= {}_{A\otimes_0 B}\#([f]_{R_{X,Y}}).
\end{aligned}$$

$$\begin{aligned}
\#_A(\#_B([f]_{R_{X,Y}})) &= \#_A([\#_B(f)]_{R_{X\otimes_0 B, Y\otimes_0 B}}) \\
&= [\#_A(\#_B(f))]_{R_{X\otimes_0 B\otimes_0 A, X\otimes_0 B\otimes_0 A}} \\
&\overset{by (5.2)}{=} [\#_{B\otimes_0 A}(f)]_{R_{X\otimes_0 B\otimes_0 A, Y\otimes_0 B\otimes_0 A}} \\
&= \#_{B\otimes_0 A}([f]_{R_{X,Y}}).
\end{aligned}$$

$$\begin{aligned}\#_A({}_B\#([f]_{R_{X,Y}})) &= \#_A([{}_B\# f]_{R_{B\otimes_0 X, B\otimes_0 Y}})\\ &= [\#_A({}_B\#(f))]_{R_{B\otimes_0 X\otimes_0 A, B\otimes_0 Y\otimes_0 A}}\\ &\overset{by(5.3)}{=} [{}_B\#(\#_A(f))]_{R_{B\otimes_0 X\otimes_0 A, B\otimes_0 Y\otimes_0 A}}\\ &= {}_B\#(\#_A([f]_{R_{X,Y}})).\end{aligned}$$

$$\begin{aligned}\#_I([f]_{R_{X,Y}}) &= [\#_I(f)]_{R_{X,Y}}\\ &\overset{by(5.4)}{=} [f]_{R_{X,Y}}.\\ {}_I\#([f]_{R_{X,Y}}) &\overset{by(5.5)}{=} [{}_I\#(f)]_{R_{X,Y}}\\ &= [f]_{R_{X,Y}}.\end{aligned}$$

□

Proposition 6.2.3. *In the conditions of the previous proposition. If the 1/2-monoidal category is slideable, then so is the quotient.*

Proof. Assume

$$(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

is a slideable $\frac{1}{2}$-monoidal category. Then if given a pair of morphisms $f\colon x \to y$ and $g\colon z \to w$ we have

$$(f\Theta w) \star (x\Theta g) = (y\Theta g) \star (f\Theta z).$$

Assume

$$(\mathcal{C}/R, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

is a quotient of the slideable $\frac{1}{2}$-monoidal category. Let $[f] \in \hom_{\mathcal{C}/R}(x, y)$ and $[g] \in \hom_{\mathcal{C}/R}(z, w)$, we have

$$\begin{aligned}([f]\Theta w) \star (x\Theta[g]) =& ([f\Theta w]) \star ([x\Theta g])\\ =& [(f\Theta w) \star (x\Theta g)]\\ =& [(y\Theta g) \star (f\Theta z)]\\ =& (y\Theta[g]) \star ([f]\Theta z).\end{aligned}$$

Hence the quotient $\frac{1}{2}$-monoidal category is slideable. □

6.2.1 The 1/2-monoidal closure of a congruence template

We defined in 3.5.6 the closure of a congruence template, defined in 3.5.1, in order to define presentations of categories. We now define the 1/2-monoidal closure of a congruence template, in order to define presentations of 1/2 monoidal and monoidal categories.

Note: We will use the notation ${}_x\#_y(f)$ instead of ${}_x\# \circ \#_y(f) = \#_y \circ {}_x\#(f)$.

Definition 6.2.4. *Let*

$$(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

be a $\frac{1}{2}$-monoidal category such that $\mathcal{C}$ has a congruence template $W_{X,Y}$, for all $X, Y \in ob(\mathcal{C})$. We say $f, g\colon X \to Y$ are related in $\underline{W}_{X,Y}$,and we put

$$f \sim_{\underline{W}_{X,Y}} g$$

if there are $A, B, x, y \in ob(\mathcal{C})$, $f', g'\colon A \to B$ and

$$\alpha\colon X \to x \otimes_0 A \otimes_0 y, \text{ and } \beta\colon x \otimes_0 B \otimes_0 y \to Y,$$

hence fitting into the diagram below

$$X \xrightarrow{\alpha} x \otimes_0 A \otimes_0 y \underset{x\Theta g'\Theta y}{\overset{x\Theta f'\Theta y}{\rightrightarrows}} x \otimes_0 B \otimes_0 y \xrightarrow{\beta} Y$$

such that $f = \beta \star ({}_x\#_y f') \star \alpha$ and $g = \beta \star ({}_x\#_y g') \star \alpha$, and $f' \sim_{W_{A,B}} g'$ or $g' \sim_{W_{A,B}} f'$.

Lemma 6.2.5. *Let*

$$(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)$$

be a $\frac{1}{2}$-monoidal category such that $\mathcal{C}$ has a congruence template $W_{X,Y}$ for all $X, Y \in ob(\mathcal{C})$. Let also $U \in ob(\mathcal{C})$, $f, g\colon X \to Y$, $m\colon X' \to X$ and $n\colon Y \to Y'$ be morphisms

such that

$$f \sim_{\underline{W}_{X,Y}} g.$$

Then

$$
\begin{aligned}
{}_U\#(n \star f \star m) &\sim_{\underline{W}_{U\otimes_0 X', U\otimes_0 Y'}} {}_U\#(n \star g \star m),\\
\#_U(n \star f \star m) &\sim_{\underline{W}_{X'\otimes_0 U, Y'\otimes_0 U}} \#_U(n \star g \star m).
\end{aligned}
$$

In particular (putting $U = I$) given then

$$n \star f \star m \sim_{\underline{W}_{X',Y'}} n \star g \star m$$

as maps $X' \to Y'$.

Proof. We have

$$f \sim_{\underline{W}_{X,Y}} g.$$

So, there are $A, B, x, y \in ob(\mathcal{C})$, $f', g' \colon A \to B$ and

$$\alpha \colon X \to x \otimes_0 A \otimes_0 y \text{ and } \beta \colon x \otimes_0 B \otimes_0 y \to Y$$

such that $f = \beta \star ({}_x\#_y f') \star \alpha$ and $g = \beta \star ({}_x\#_y g') \star \alpha$ and $f' \sim_{W_{A,B}} g'$ or $g' \sim_{W_{A,B}} f'$. Thus

$$
\begin{aligned}
f &= \beta \star ({}_x\#_y f') \star \alpha\\
n \star f \star m &= n \star \beta \star ({}_x\#_y f') \star \alpha \star m\\
&= (n \star \beta) \star ({}_x\#_y f') \star (\alpha \star m),
\end{aligned}
$$

and

$$
\begin{aligned}
{}_U\#(n \star f \star m) &= {}_U\#((n \star \beta) \star ({}_x\#_y f') \star (\alpha \star m))\\
&= {}_U\#(n \star \beta) \star {}_U\#({}_x\#_y f') \star {}_U\#(\alpha \star m)\\
&\overset{by (5.1)}{=} {}_U\#(n \star \beta) \star ({}_{U\otimes_0 x}\#_y f') \star {}_U\#(\alpha \star m).
\end{aligned}
$$

Also,

$$
\begin{aligned}
g &= \beta \star ({}_x\#_y g') \star \alpha \\
n \star g \star m &= n \star \beta \star ({}_x\#_y g') \star \alpha \star m \\
&= (n \star \beta) \star ({}_x\#_y g') \star (\alpha \star m),
\end{aligned}
$$

and

$$
\begin{aligned}
{}_U\#(n \star g \star m) &= {}_U\#((n \star \beta) \star ({}_x\#_y g') \star (\alpha \star m)) \\
&= {}_U\#(n \star \beta) \star {}_U\#({}_x\#_y g') \star {}_U\#(\alpha \star m) \\
&\overset{by(5.1)}{=} {}_U\#(n \star \beta) \star ({}_{U\otimes_0 x}\#_y g') \star {}_U\#(\alpha \star m).
\end{aligned}
$$

Therefore

$$
{}_U\#(n \star f \star m) \sim_{\underline{W}_{U\otimes_0 X', U\otimes_0 Y'}} {}_U\#(n \star g \star m).
$$

By a similar calculation.

$$
\#_U(n \star f \star m) \sim_{\underline{W}_{X'\otimes_0 U, Y'\otimes_0 U}} \#_U(n \star g \star m).
$$

□

Definition 6.2.6. *Let*

$$
(\mathcal{C}, I, \otimes_0, \#_{(-)}, {}_{(-)}\#)
$$

be a $\frac{1}{2}$*-monoidal category with a congruence template* $\{W_{X,Y}\}_{(X,Y)\in ob(\mathcal{C})\times ob(\mathcal{C})}$*. Consider the relation* $\underline{W}_{X,Y}$ *in 6.2.4, defined in* $\hom_{\mathcal{C}}(X,Y)$*, for all* $X, Y \in ob(\mathcal{C})$*. For each* $(X,Y) \in ob(\mathcal{C}) \times ob(\mathcal{C})$*, let*

$$
\overline{\underline{W}}_{X,Y}
$$

be the transitive, reflexive closure of $\underline{W}_{X,Y}$*. (So* $\overline{\underline{W}}_{X,Y}$ *is a relation in* $\hom_{\mathcal{C}}(X,Y)$*). If* $f, g\colon X \to Y$ *are related in* $\overline{\underline{W}}_{X,Y}$ *we write*

$$
f \sim_{\overline{\underline{W}}_{X,Y}} g,
$$

$f = g$ *or there exists an* $n \in \mathbb{N}$*,* $(A_1, A_2, \ldots, A_n) \in ob(\mathcal{C})^n$*,* $(B_1, B_2, \ldots, B_n) \in ob(\mathcal{C})^n$*,*$(x_1, x_2, \ldots, x_n) \in ob(\mathcal{C})^n$ *and* $(y_1, y_2, \ldots, y_n) \in ob(\mathcal{C})^n$ *for all* $i \in \{1, 2, \ldots, n\}$*,* *morphisms*

$$\alpha_i : X \to x_i \otimes_0 A_i \otimes_0 y_i,$$
$$f'_i : A_i \to B_i,$$
$$g'_i : A_i \to B_i,$$
$$\beta_i : x_i \otimes_0 y_i \otimes_0 B_i \to Y,$$

such that

$$f = f_1 = \beta_1 \star ({}_{x_1}\#_{y_1} f'_1) \star \alpha_1$$
$$\sim_{\underline{W}_{X,Y}} g_1 = \beta_1 \star ({}_{x_1}\#_{y_1} g'_1) \star \alpha_1, \textit{ where we have } f'_1 \sim_{W_{A_1,B_1}} g'_1 \textit{ or } g'_1 \sim_{W_{A_1,B_1}} f'_1$$
$$= f_2 = \beta_2 \star ({}_{x_2}\#_{y_2} f'_2) \star \alpha_2$$
$$\sim_{\underline{W}_{X,Y}} = g_2 = \beta_2 \star ({}_{x_2}\#_{y_2} g'_2) \star \alpha_2, \textit{ where we have } f'_2 \sim_{W_{A_2,B_2}} g'_2 \textit{ or } g'_2 \sim_{W_{A_2,B_2}} f'_2$$
$$\vdots$$
$$= f_n = \beta_n \star ({}_{x_n}\#_{y_n} f'_n) \star \alpha_n$$
$$\sim_{\underline{W}_{X,Y}} g_n = \beta_n \star ({}_{x_n}\#_{y_n} g'_n) \star \alpha_n = g, \textit{ where we have } f'_n \sim_{W_{A_n,B_n}} g'_n \textit{ or } g'_n \sim_{W_{A_n,B_n}} f'_n.$$

Theorem 6.2.7. *The equivalence relations $\overline{\underline{W}}_{X,Y}$ on $\hom_{\mathcal{C}}(X,Y)$, for all $(X,Y) \in ob(\mathcal{C}) \times ob(\mathcal{C})$, are a $\frac{1}{2}$-monoidal congruence in $\mathcal{C}$ 6.2.1. I.e, if*

$$f \sim_{\overline{\underline{W}}_{X,Y}} f' \textit{ and } g \sim_{\overline{\underline{W}}_{Y,Z}} g'.$$

Then

$$g \star f \sim_{\overline{\underline{W}}_{X,Z}} g' \star f',$$
$$\#_A(f) \sim_{\overline{\underline{W}}_{X \otimes_0 A, Y \otimes_0 A}} \#_A(f'),$$
$${}_A\#(f) \sim_{\overline{\underline{W}}_{A \otimes_0 X, A \otimes_0 A}} {}_A\#(f').$$

Proof. Suppose

$$f \sim_{\overline{\underline{W}}_{X,Y}} f'.$$

Then $f = f'$ or there exists $n \in \mathbb{N}$ and there are $f_1, f_2, \ldots, f_n : X \to Y$, such that

$$f = f_1 \sim_{\underline{W}_{X,Y}} f_2, f_2 \sim_{\underline{W}_{X,Y}} f_3, \ldots, f_{n-1} \sim_{\underline{W}_{X,Y}} f_n = f'.$$

Suppose

$$g \sim_{\overline{\underline{W}}_{Y,Z}} g',$$

then $g = g'$ or there exists $m \in \mathbb{N}$ and there are $g_1, g_2, \ldots, g_m : Y \to Z$, such that

$$g = g_1 \sim_{\underline{W}_{Y,Z}} g_2, g_2 \sim_{\underline{W}_{Y,Z}} g_3, \ldots, g_{m-1} \sim_{\underline{W}_{Y,Z}} g_m = g'.$$

Then by Lemma 6.2.5

$$\begin{aligned} g \star f = g \star f_1 &\sim_{\underline{W}_{X,Z}} g \star f_2, \\ g \star f_2 &\sim_{\underline{W}_{X,Z}} g \star f_3, \\ &\vdots \\ g \star f_{n-1} &\sim_{\underline{W}_{X,Z}} g \star f_n = g \star f'. \end{aligned}$$

Therefore

$$g \star f \sim_{\overline{\underline{W}}_{X,Z}} g \star f'.$$

Similarly

$$\begin{aligned} g \star f' = g_1 \star f' &\sim_{\underline{W}_{X,Z}} g_2 \star f', \\ g_2 \star f' &\sim_{\underline{W}_{X,Z}} g_3 \star f', \\ &\vdots \\ g_{m-1} \star f' &\sim_{\underline{W}_{X,Z}} g_m \star f' = g' \star f'. \end{aligned}$$

Therefore,

$$g \star f' \sim_{\overline{\underline{W}}_{X,Z}} g' \star f'.$$

Hence,

$$g \star f \sim_{\overline{\underline{W}}_{X,Z}} g' \star f'.$$

Also,

$$\#_A(f) = \#_A(f_1) \sim_{\underline{W}_{X\otimes_0 A, Y\otimes_0 A}} \#_A(f_2),$$
$$\#_A(f_2) \sim_{\underline{W}_{X\otimes_0 A, Y\otimes_0 A}} \#_A(f_3),$$
$$\vdots$$
$$\#_A(f_{n-1}) \sim_{\underline{W}_{X\otimes_0 A, Y\otimes_0 A}} \#_A(f_n) = \#_A(f').$$

Therefore,

$$\#_A(f) \sim_{\overline{\underline{W}}_{X\otimes_0 A, Y\otimes_0 A}} \#_A(f').$$

$$_A\#(f) = {}_A\# f_1 \sim_{\underline{W}_{A\otimes_0 X, A\otimes_0 Y}} {}_A\#(f_2),$$
$$_A\#(f_2) \sim_{\underline{W}_{A\otimes_0 X, A\otimes_0 Y}} {}_A\#(f_3),$$
$$\vdots$$
$$_A\#(f_{n-1}) \sim_{\underline{W}_{A\otimes_0 X, A\otimes_0 Y}} {}_A\#(f_n) = {}_A\#(f').$$

Therefore,

$$_A\#(f) \sim_{\overline{\underline{W}}_{A\otimes_0 X, A\otimes_0 Y}} {}_A\#(f').$$

□

Definition 6.2.8 (1/2-monoidal closure of a congruence template)**.** *The equivalence relations $\overline{\underline{W}}_{X,Y}$ is called the $\frac{1}{2}$-monoidal closure of congruence template $W_{X,Y}$.*

Definition 6.2.9 (Presentation of $\frac{1}{2}$-monoidal category)**.** *Let $\mathcal{C}$ be a free $\frac{1}{2}$-monoidal category over the monoidal graph R. Let $W_{X,Y}$ be a congruence template on the $\mathcal{C}$. We say that*

$$\mathcal{C}/\overline{\underline{W}}_{X,Y}$$

is the $\frac{1}{2}$-monoidal category that is presented by R and relations $W_{X,Y}$.

Definition 6.2.10 (Presentation of strict monoidal category)**.** *Let $\mathcal{C}$ be a free $\frac{1}{2}$-monoidal category over the monoidal graph R. Let $W_{X,Y}$ be a congruence template on $\mathcal{C}$. We say that*

$$\mathfrak{F}(\mathcal{C}/\overline{\underline{W}}_{X,Y})$$

is the slideable $\frac{1}{2}$-monoidal category that is presented by R and relations $W_{X,Y}$. Here we applied the slidealisation functor $\mathfrak{F}$ in Proposition 5.2.9.

By using Theorem 5.4.6, then

$$\mathfrak{F}(\mathcal{C}/\underline{\overline{W}}_{X,Y})$$

has equivalent information to a strict monoidal category.

6.2.2 Example: the monoidal combinatorial braid category

Example 6.2.11. *Consider the monoidal graph*

$$\beta = (\mathbb{N}, E, \otimes_0, 0, \delta_1, \delta_2),$$

where for all $n, m \in \mathbb{N}$, we have

$$m \otimes_0 n = m + n,$$

and $E = \{X^+, X^-\}$,

$$\delta_1 X^+ = \delta_2 X^+ = \delta_1 X^- = \delta_2 X^- = 2.$$

Consider the path category over the extent β^* of the monoidal graph β (see 3.3.13).

$$P(\beta^*) = (\mathbb{N}, \hom_{P(\beta^*)}(n, m), \bullet, \phi_{\text{-}}),$$

we have a $\frac{1}{2}$-monoidal category

$$\Omega(\beta) = (P(\beta^*), \otimes_0, 0, {}_n\#, \#_m)$$

where

$$\delta_1({}_n\#_m X^+) = \delta_2({}_n\#_m X^+) = \delta_1({}_n\#_m X^-) = \delta_2({}_n\#_m X^-) = n + 2 + m \tag{6.2}$$

So, we have the free-$\frac{1}{2}$-monoidal category-triple in sense of Def. 6.1.7

$$(\beta, \Omega(\beta), \delta).$$

We define the monoidal combinatorial braid category to the following quotient

$$\mathfrak{F}\left(\Omega(\beta) \Big/ \overline{W}_{n,m}\right).$$

Where, given $m, n \in \mathbb{N}$*, then* $W_{n,m}$ *is the relation in* $\hom_{P(\beta^*)}(n, m)$ *defined as below (note that we omitted* $\bullet$ *from the notation)*

- *If* $m \neq n$ *it is clear from incidences maps* (6.2) *that* $\hom_{P(\beta^*)}(n, m) = \emptyset$*. So* $W_{n,m}$ *is the unique equivalence relation on the empty set.*
- $m, n = 0$*. Then* $\hom_{P(\beta^*)}(0, 0) = \{\phi_0\}$*. The relation* $W_{0,0}$ *is the unique equivalence relation such that* $\phi_0 \sim_{W_{0,0}} \phi_0$*.*
- $m, n = 1$*. Then* $\hom_{P(\beta^*)}(1, 1) = \{\phi_1\}$*. The relation* $W_{1,1}$ *is the unique equivalence relation such that* $\phi_1 \sim_{W_{1,1}} \phi_1$*.*
- $m, n = 2$*.* $\hom_{P(\beta^*)}(2, 2)$ *is the set of words in* X^+ *and* X^-*. The two pairs of related elements are*

$$X^+X^- \sim_{W_{2,2}} \phi_2,$$
$$X^-X^+ \sim_{W_{2,2}} \phi_2.$$

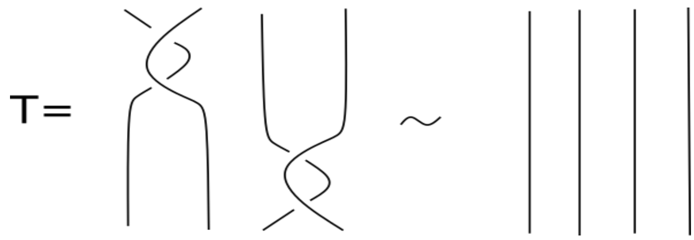
- $m, n = 3$*.* $\hom_{P(\beta^*)}(3, 3)$ *is the set of words in* ${}_0\#_1X^+, {}_1\#_0X^+$ *and* ${}_0\#_1X^-, {}_1\#_0X^-$*. The related elements are*

$${}_1\#_0X^+{}_0\#_1X^+{}_1\#_0X^+ \sim_{W_{3,3}} {}_0\#_1X^+{}_1\#_0X^+{}_0\#_1X^+,$$
$${}_1\#_0X^-{}_0\#_1X^-{}_1\#_0X^- \sim_{W_{3,3}} {}_0\#_1X^-{}_1\#_0X^-{}_0\#_1X^-.$$

In the next example we apply the congruence template $W_{2,2}$,

$$T = (\#_2(X^+X^-))({}_2\#(X^+X^-)) \sim_{W_{2,2}} ({}_2\#\phi_2)(\#_2\phi_2) = \phi_4\phi_4 = \phi_4.$$

I.e,



6.3 Tangles

In this section, we first define tangles from the geometric point of view. Then we give, a presentation of the tangle category by generators and relations. These constructions are discussed for example by Kassel [Kas12] and Turaev [Tur90].

Kassel constructed a presentation for the tangle category [Kas12, Chapter XII] in similar way to group presentation. In group presentation, let G be a group and F be a subset of G and R be a set of pairs of words in the alphabet F. Then (F, R) is a presentation of the group G if the two following conditions are satisfied:

- the subset F generates G,
- two words a and b in the alphabet F represent the same element in G if and only if one may pass from a to b by operations replacing any subword of the form c by a subword of the form d where (c, d) belongs to R.

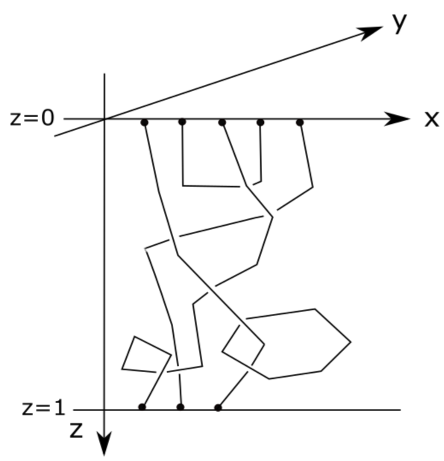
So Kassel took a collection of morphisms from the strict monoidal category $(\mathcal{C}, \otimes, I)$, of tangles as generators. Then he define the equivalence between words in order to have strict monoidal category. This presentation of the tangle category is not completely combinatorial because it uses the category of tangles to explain how the generators and relations for the monoidal category of tangles are to be interpreted.

We provided in section §6.2 a combinatorial presentation of strict monoidal categories, by using presentations of $\frac{1}{2}$-monoidal categories. So we can use this to define formally category of tangles by generators and relations.

Definition 6.3.1. *(See for example [Kas12, Tur90, BS19, CR89]). Let n, m two integers, so a* geometric tangle *T from n to m is an embedded union of polygonal circles and intervals in $\mathbb{R}^2 \times I$, such that:*

$$T \cap (\mathbb{R}^2 \times \{0\}) = \{1, 2, \ldots, n\} \times \{0\} \times \{0\},$$
$$T \cap (\mathbb{R}^2 \times \{1\}) = \{1, 2, \ldots, m\} \times \{0\} \times \{1\}.$$

Example 6.3.2. *The next picture represents a geometric tangle T from 5 to 3.*



Definition 6.3.3. *(See for example [Kas12, Tur90, BS19, CR89]). Two geometric tangles $T_1, T_2 \subset \mathbb{R}^2 \times [0,1]$ are called ambient isotopic if there is a continuous map:*

$$h\colon (\mathbb{R}^2 \times [0,1]) \times [0,1] \to \mathbb{R}^2 \times [0,1],$$

such that

1. *for all $(x, y, z, 0) \in (\mathbb{R}^2 \times [0,1]) \times [0,1]$, $h(x, y, z, 0) = (x, y, z)$;*

2. *given any* $t \in [0, 1]$*, the map*

$$\begin{aligned} \mathbb{R}^2 \times [0,1] &\to \mathbb{R}^2 \times [0,1] \\ (x, y, z) &\mapsto h(x, y, z, t) \end{aligned}$$

is a homeomorphism;

3. *for all* $t \in [0, 1]$ *and* $(x, y) \in \mathbb{R}^2$
   - $h(x, y, 0, t) = (x, y, 0)$,
   - $h(x, y, 1, t) = (x, y, 1)$.
4. $h(T_1, 1) = T_2$,
5. $h(T_1, t)$ *is a tangle for all* $t \in [0, 1]$.

Proposition 6.3.4. *(See for example[BL98]). We have a strict monoidal category of tangles where*

1. *the set of objects is the set* $\mathbb{N}$ *of natural number,*
2. *for all* $n, m \in \mathbb{N}$*, the set of morphism from* m *to* n *is the set of equivalence classes of ambient isotopy of* (m, n)*-tangles,*
3. *composition and identities are defined as in the case of braids,*
4. *the tensor product* $T_1 \otimes T_2$ *is the isotopy class of the tangle obtained by placing* L_2 *to the right of* T_1*. (For the precise way to do this see [Kas12, Section XII.2]).*

Definition 6.3.5 (Category of combinatorial tangles)**.** *(See for example [CDM12]). The* strict monoidal category $\mathcal{T}$ *is the strict monoidal category (in the sense of Definition 5.3.2), whose set of objects is the set* $\mathbb{N}$*, where the object monoid is* $(\mathbb{N}, +, 0)$.

The monoidal graph

$$\beta = (\mathbb{N}, E(\beta), \otimes_0, 0, \delta_1, \delta_2),$$

where for all $m, n \in \mathbb{N}$*,* $m \otimes_0 n = m + n$*, and*

$$E(\beta) = \{X_+, X_-, \cup, \cap\},$$

where

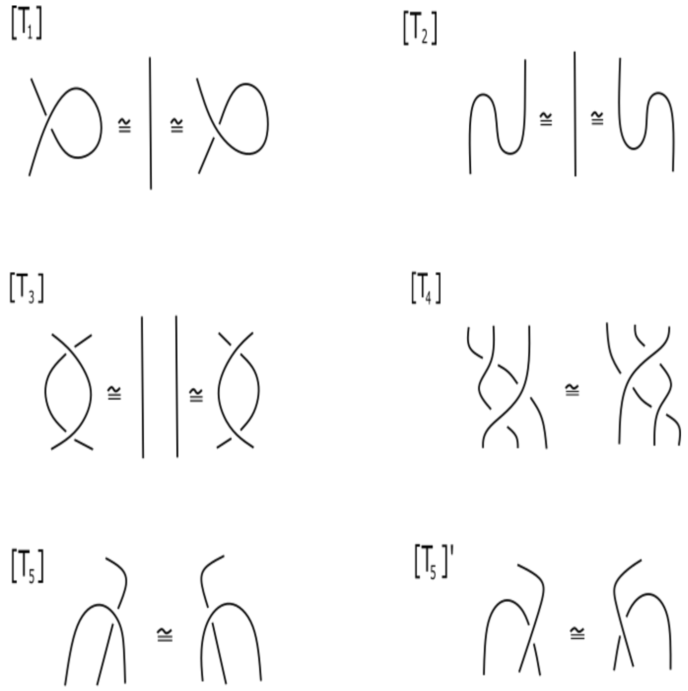
$$\delta_1 X_+ = 2, \qquad \delta_2 X_+ = 2,$$
$$\delta_1 X_- = 2, \qquad \delta_2 X_- = 2,$$
$$\delta_1 \cup = 0, \qquad \delta_2 \cup = 2,$$
$$\delta_1 \cap = 2, \qquad \delta_2 \cap = 0,$$

that satisfy the following relations

- $[T_1] : (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \cong \mathrm{id}_1 \cong (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[T_2] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \cong \mathrm{id}_1 \cong (\mathrm{id}_1 \otimes \cap)(\cup \otimes \mathrm{id}_1)$.
- $[T_3] : X_- X_+ \cong \mathrm{id}_2 \cong X_+ X_-$.
- $[T_4] : (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \cong (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[T_5] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-) \cong (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)$.
- $[T_5]' : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+) \cong (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)$.

Note here similarly to the monoidal combinatorial braid category we have a $1/2$*-monoidal closure to the congruence template* W *defined in the* $\frac{1}{2}$*-monoidal category* $\Omega(\beta)$ *and the pairs above are the only pairs related by* $W_{1,1}, W_{2,2}, W_{3,3}, W_{3,1}, W_{1,3}$.

These relations can be presented geometrically as:

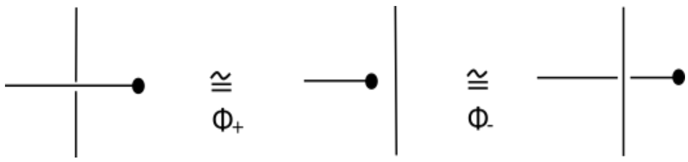


Chapter 7

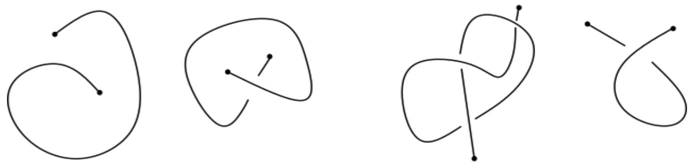
Welded Tangle-oid Categories

7.1 Introduction

The theory of knotoids was introduced by Turaev and Lambropoulou [Tur12, GKL16]. " Knotoids are represented by diagrams in a surface which differ from the usual knot diagrams in that the underlying curve is a segment rather than a circle. Knotoid diagrams are considered up to Reidemeister moves applied away from the endpoints of the underlying segment" [Tur12]. "The two endpoints may lie in different regions of the diagram. They may move within their regions by planar isotopy, but they are not allowed to cross over or under any arcs of the diagram. These are the 'forbidden moves' of the theory" [GKL16].



Below we can see a geometric representation of four different knotoids.



(1) (2) (3) (4)

My idea to consider welded tangle-oids categories arose from a talk by Sofia Lambropoulou [GKL16] in the conference Loops in Leeds" 1-4 July 2019. Welded tangleoids generalise the welded virtual arcs defined in [KM08] and also considered in [Sat00], the definition of welded virtual arcs is similar to the definition of welded virtual knots with addition move $[WT_1 3]'$ and $[WT_1 4]'$ in the definition of welded tangle-oids categories below. Also, we can see welded tangle-oids categories as a generalisation of tangle categories by added welded virtual arcs.

We here consider a formulation of welded knotoids within a strict monoidal category of welded tangle-oids, which we define by using generators and relations. We will take the presentation of the category of tangles as our starting point.

7.2 Unoriented Welded Tangle-oids

We will define a monoidal category of unoriented welded tangle-oids by giving a presentation, as defined in Definition 6.2.10, by using presentation of slideable $\frac{1}{2}$-monoidal categories . We proved in 5.4.6 slideable $\frac{1}{2}$-monoidal categories are isomorphic to strict monoidal categories.

Definition 7.2.1. *Consider the monoidal graph*

$$\beta = (\mathbb{N}, E(\beta), \otimes_0, 0, \delta_1, \delta_2),$$

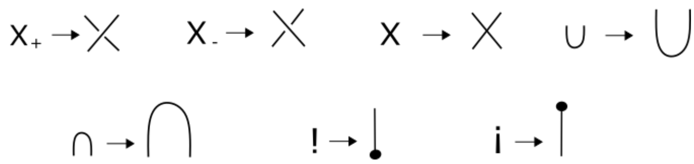
where for all $m, n \in \mathbb{N}$*,* $m \otimes_0 n = m + n$*,and*

$$E(\beta) = \{X_+, X_-, X, \cup, \cap, ¡, !\},$$

the incidence maps

$$\begin{array}{cccc} \delta_1 X_+ = 2, & \delta_2 X_+ = 2, & \delta_1 X_- = 2, & \delta_2 X_- = 2, \\ \delta_1 X = 2, & \delta_2 X = 2, & \delta_1 \cup = 0, & \delta_2 \cup = 2, \\ \delta_1 \cap = 2, & \delta_2 \cap = 0, & \delta_1 ¡ = 1, & \delta_2 ¡ = 0, \\ \delta_1 ! = 0, & \delta_2 ! = 1. & & \end{array}$$

These generators can be presented geometrically as



Consider the path category, see 3.3.13 over β^*, the extent of the monoidal graph β.

$$P(\beta^*) = (\mathbb{N}, \hom_{P(\beta^*)}(n, m), \bullet, \phi_{\text{-}}).$$

Therefore

$$\Omega(\beta) = (P(\beta^*), \otimes_0, 0, {}_n\#, \#_m)$$

is a $\frac{1}{2}$-monoidal category, whose set of objects is the set of natural numbers, where for all $n, m, k \in \mathbb{N}$;

$${}_n\#_m(k) = n \otimes_0 k \otimes_0 m = n + k + m,$$

and for all generating morphism $(f \colon k \to k') \in E(\beta)$, we have

$${}_n\#_m(f) = n + k + m \xrightarrow{n \Theta f \Theta m} n + k' + m.$$

Then we have the free-$\frac{1}{2}$-monoidal category-triple in sense of Definition 6.1.7

$$(\beta, \Omega(\beta), \delta).$$

Definition 7.2.2 (Unoriented welded tangloids category)**.** *The* unoriented welded tangloids category $UWTC$ *is the strict monoidal category formally presented by*

$$\mathfrak{F}\Big(\Omega(\beta)\Big/\underline{\overline{W}}\Big),$$

where $\Omega(\beta)$ *defined in Definition 7.2.1 and* $\underline{\overline{W}}$ *is the* $\frac{1}{2}$*-monoidal closure of the congruence template* W *that is defined as follows.*

Given $m, n \in \mathbb{N}$*, then* $W_{m,n}$ *is the relation in* $\hom_{P(\beta^*)}(m,n)$*, defined as (the picture will follow)*
In $\hom_{P(\beta^*)}(1,1)$*, we have the only relations*

- $[WT_1]$ *:* $(\mathrm{id}_1 \otimes \cap)(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)(\cup \otimes \mathrm{id}_1)$.
- $[WT_2] : (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_3] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(\cup \otimes \mathrm{id}_1)$.
- $[WT_4] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\mathrm{id}_1 \otimes \cap)(\cup \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(2,2)$*, we have the only relation*

- $[WT_5] : X_- X_+ \sim_{W_{2,2}} \mathrm{id}_2 \sim_{W_{2,2}} X_+ X_-$.

In $\hom_{P(\beta^*)}(3,3)$*, we have the only relations*

- $[WT_6] : (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[WT_7] : (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)(X \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X)(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[WT_8] : (X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)$.

In $\hom_{P(\beta^*)}(3,1)$*, we have the only relations*

- $[WT_9] : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)$.
- $[WT_9]' : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)$.
- $[WT_9]'' : (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(1,3)$*, we have the only relations*

- $[WT_{10}] : (\mathrm{id}_1 \otimes X_+)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_{10}]' : (\mathrm{id}_1 \otimes X_-)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_{10}]'' : (\mathrm{id}_1 \otimes X)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.

In $\hom_{P(\beta^*)}(1,0)$*, we have the only relation*

- $[WT_{11}] : \cap(\mathrm{id}_1 \otimes !) \sim_{W_{1,0}}$ ¡ $\sim_{W_{1,0}} \cap(! \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(0,1)$*, we have the only relation:*

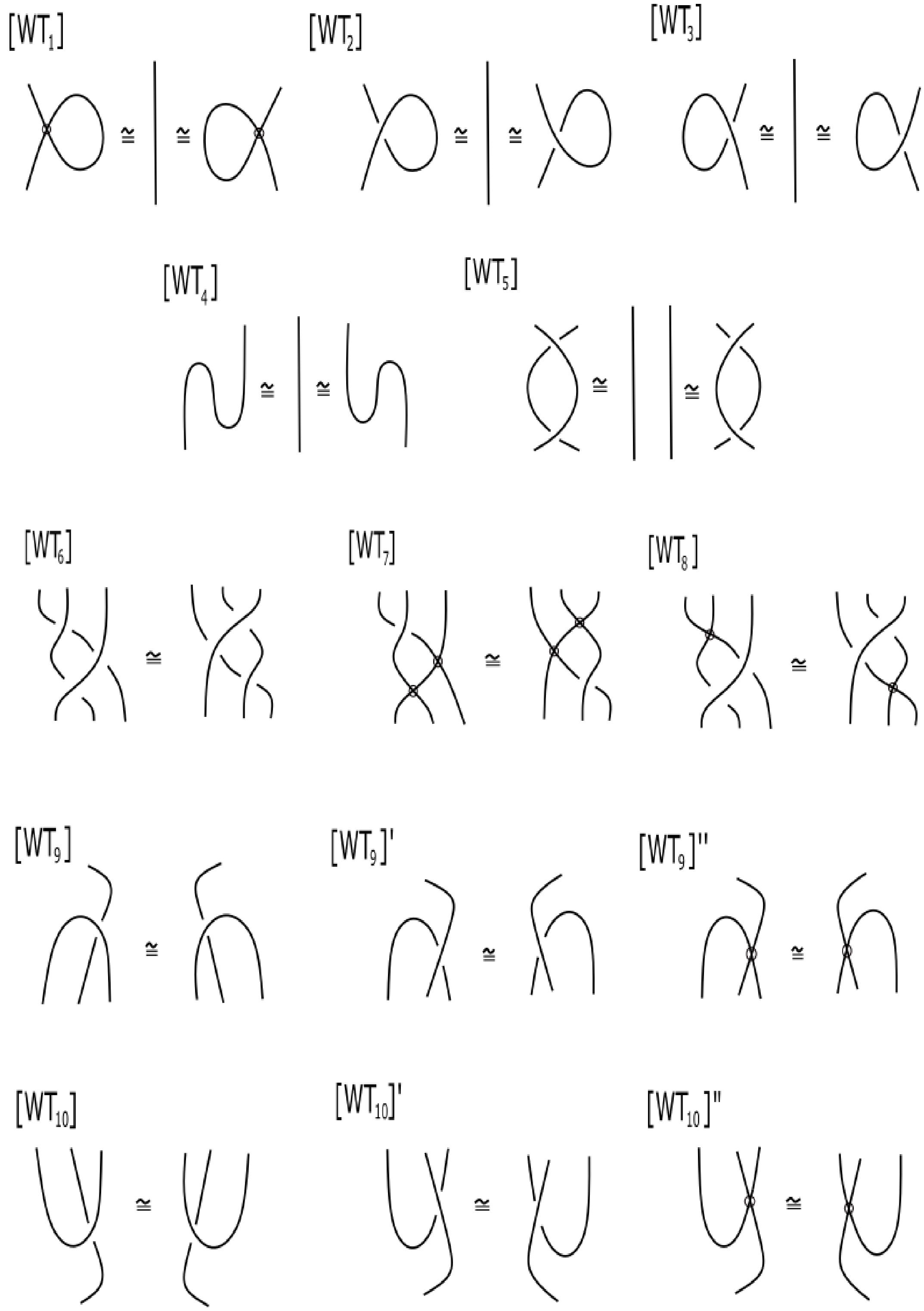
- $[WT_{12}] : (\mathrm{id}_1 \otimes$ ¡$)\cup \sim_{W_{0,1}} ! \sim_{W_{0,1}}$ (¡ $\otimes \mathrm{id}_1)\cup$.

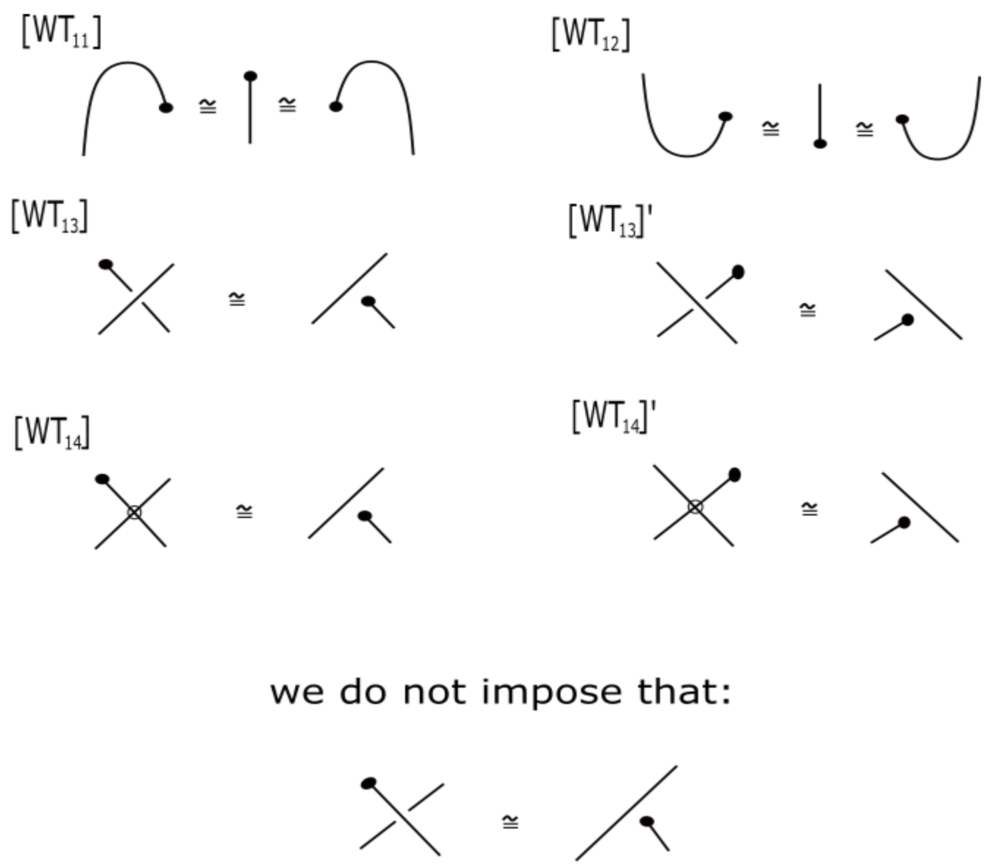
In $\hom_{P(\beta^*)}(2,1)$*, we have the only relations*

- $[WT_{13}] :$ (¡ $\otimes \mathrm{id}_1)X_+ \sim_{W_{2,1}} \mathrm{id}_1 \otimes$ ¡.
- $[WT_{13}]' : (\mathrm{id}_1 \otimes$ ¡$)X_- \sim_{W_{2,1}}$ ¡ $\otimes \mathrm{id}_1$.
- $[WT_{14}] :$ (¡ $\otimes \mathrm{id}_1)X \sim_{W_{2,1}} \mathrm{id}_1 \otimes$ ¡.
- $[WT_{14}]' : (\mathrm{id}_1 \otimes$ ¡$)X \sim_{W_{2,1}}$ ¡ $\otimes \mathrm{id}_1$.

Note that we do not impose that in $\hom_{P(\beta^*)}(2,1)$***:***
(¡ $\otimes \mathrm{id}_1)X_- \nsim_{W_{2,1}} \mathrm{id}_1 \otimes$ ¡.

These relations can be present geometrically as (note we read the diagram from bottom to top)





7.3 Oriented Welded Tangle-oids

Now we will define the oriented case of welded tangle-oids categories by using presentation of slideable $\frac{1}{2}$-monoidal categories.

Definition 7.3.1. *Consider the monoidal graph*

$$\beta = (\{+,-\}^*, E(\beta), \otimes_0, \emptyset, \delta_1, \delta_2),$$

where for all words $u, v \in \{+,-\}^*$*, we have*

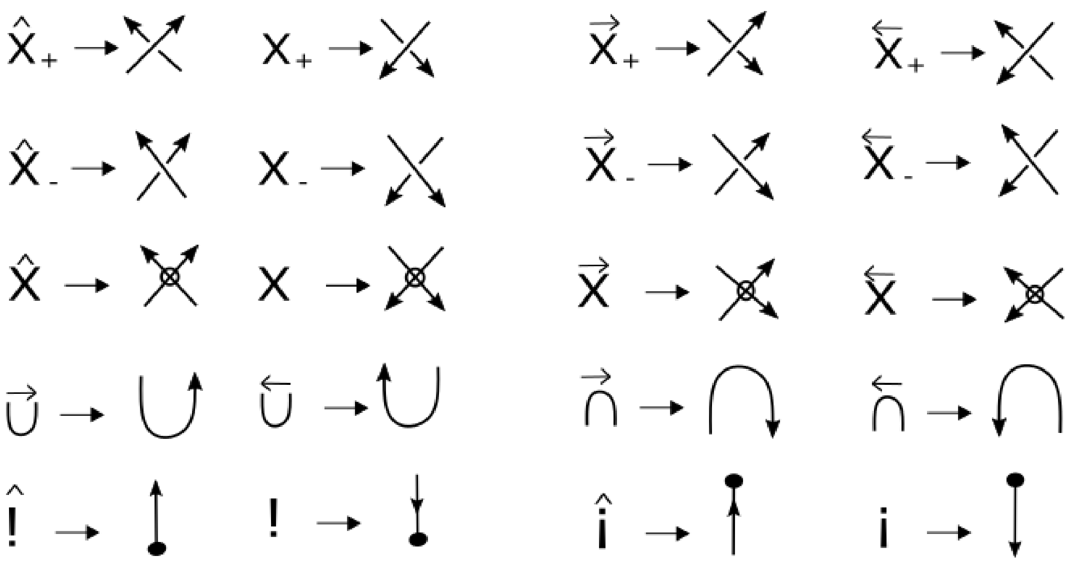
$$u \otimes_0 v = uv.$$

Note $\{+,-\}^*$ *means the monoid of words on set* $\{+,-\}$.

$$E(\beta)=\{\overleftarrow{\cup},\overrightarrow{\cup},\overleftarrow{\cap},\overrightarrow{\cap},\hat{X}_+,X_+,\overleftarrow{X_+},\overrightarrow{X_+},\hat{X}_-,X_-,\overleftarrow{X_-},\overrightarrow{X_-},\hat{X},X,\overleftarrow{X},\overrightarrow{X},\hat{\text{¡}},\text{¡},\hat{!},!\}.$$

$$\begin{array}{cccc}
\delta_1\hat{X}_+=++, & \delta_2\hat{X}_+=++, & \delta_1X_+=--, & \delta_2X_+=--,\\
\delta_1\overleftarrow{X_+}=-+, & \delta_2\overleftarrow{X_+}=+-, & \delta_1\overrightarrow{X_+}=+-, & \delta_2\overrightarrow{X_+}=-+,\\
\delta_1\hat{X}_-=++, & \delta_2\hat{X}_-=++, & \delta_1X_-=--, & \delta_2X_-=--,\\
\delta_1\overleftarrow{X_-}=-+, & \delta_2\overleftarrow{X_-}=+-, & \delta_1\overrightarrow{X_-}=+-, & \delta_2\overrightarrow{X_-}=-+,\\
\delta_1\hat{X}=++, & \delta_2X=++, & \delta_1X=--, & \delta_2X=--,\\
\delta_1\overleftarrow{X}=-+, & \delta_2\overleftarrow{X}=+-, & \delta_1\overrightarrow{X}=+-, & \delta_2\overrightarrow{X}=-+,\\
\delta_1\overleftarrow{\cap}=-+, & \delta_2\overleftarrow{\cap}=\emptyset, & \delta_1\overrightarrow{\cap}=+-, & \delta_2\overrightarrow{\cap}=\emptyset,\\
\delta_1\overleftarrow{\cup}=\emptyset, & \delta_2\overleftarrow{\cup}=+-, & \delta_1\overrightarrow{\cup}=\emptyset, & \delta_2\overrightarrow{\cup}=-+,\\
\delta_1\hat{\text{¡}}=+, & \delta_2\hat{\text{¡}}=\emptyset, & \delta_1\text{¡}=-, & \delta_2\text{¡}=\emptyset,\\
\delta_1\hat{!}=\emptyset, & \delta_2\hat{!}=+, & \delta_1!=\emptyset, & \delta_2!=-.
\end{array}$$

These generators can be present geometrically as



Consider the path category, see 3.3.13 over β^*, the extent of monoidal graph β (see 3.3.13).

$$P(\beta^*) = (\{+,-\}^*, \hom_{P(\beta^*)}(\_,\_), \bullet, \phi_\_).$$

Therefore

$$\Omega(\beta) = (P(\beta^*), \otimes_0, \emptyset, {}_u\#, \#_v)$$

is a $\frac{1}{2}$-monoidal category where for all sequences u, v, k of $\{+,-\}^*$,

$${}_u\#_v(k) = n \otimes_0 k \otimes_0 m = ukv,$$

and for all generators morphism $(f\colon k \to k') \in E(\beta)$, we have

$${}_u\#_v(f) = u \otimes_0 k \otimes_0 v \xrightarrow{u\Theta f \Theta v} u \otimes_0 k' \otimes_0 v.$$

Then we have the free-$\frac{1}{2}$-monoidal category-triple in sense of Def. 6.1.7

$$(\beta, \Omega(\beta), \delta).$$

Definition 7.3.2 (Oriented welded tangle-oids category)**.** *The* oriented welded tangle-oids category $OWTC$ *is the slideable $\frac{1}{2}$-monoidal category formally presented as*

$$\mathfrak{F}\Big(\Omega(\beta) \Big/ \overline{W}\Big).$$

Where $\Omega(\beta)$ defined in Def.7.2.1 and $\overline{W}$ is the $\frac{1}{2}$-monoidal closure of congruence template W that define as follows. The pairs of morphisms below are the only ones that W relates.

Given $u, v \in \{+,-\}^*$, then $W_{u,v}$ is the relation in $\hom_{P(\beta^*)}(u, v)$, defined as below

In $\hom_{P(\beta^*)}(+,+)$, we have the relations

- $[WT_1]$: $(\mathrm{id}_+ \otimes \overrightarrow{\cap})(\hat{X} \otimes \mathrm{id}_-)(\mathrm{id}_+ \otimes \overleftarrow{\cup}) \sim_{W_{+,+}} \mathrm{id}_+ \sim_{W_{+,+}} (\overleftarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_- \otimes \hat{X})(\overrightarrow{\cup} \otimes \mathrm{id}_+)$.

- $[WT_2]$: $(\mathrm{id}_+ \otimes \overrightarrow{\cap})(\hat{X}_+ \otimes \mathrm{id}_-)(\mathrm{id}_+ \otimes \overleftarrow{\cup}) \sim_{W_{+,+}} \mathrm{id}_+ \sim_{W_{+,+}} (\mathrm{id}_+ \otimes \overrightarrow{\cap})(\hat{X}_- \otimes \mathrm{id}_-)(\mathrm{id}_+ \otimes \overleftarrow{\cup})$.

- $[WT_3]$: $(\overleftarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_- \otimes \hat{X}_-)(\overrightarrow{\cup} \otimes \mathrm{id}_+) \sim_{W_{+,+}} \mathrm{id}_+ \sim_{W_{+,+}} (\overleftarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_- \otimes \hat{X}_+)(\overrightarrow{\cup} \otimes \mathrm{id}_+)$

- $[WT_4]$: $(\overrightarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \overrightarrow{\cup}) \sim_{W_{+,+}} \mathrm{id}_+ \sim_{W_{+,+}} (\mathrm{id}_+ \otimes \overleftarrow{\cap})(\overleftarrow{\cup} \otimes \mathrm{id}_+)$.

In $\hom_{P(\beta^*)}(++,++)$*, we have the only relation:*

- $[WT_5]$: $\hat{X}_-\hat{X}_+ \sim_{W_{++,++}} \mathrm{id}_+ \otimes \mathrm{id}_+ \sim_{W_{++,++}} \hat{X}_+\hat{X}_-$.

In $\hom_{P(\beta^*)}(+++,+++)$*, we have the only relations:*

- $[WT_6]$: $(\hat{X}_+ \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \hat{X}_+)(\hat{X}_+ \otimes \mathrm{id}_+) \sim_{W_{+++,+++}} (\mathrm{id}_+ \otimes \hat{X}_+)(\hat{X}_+ \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \hat{X}_+)$.

- $[WT_7]$: $(\hat{X}_+ \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \hat{X})(\hat{X} \otimes \mathrm{id}_+) \sim_{W_{+++,+++}} (\mathrm{id}_+ \otimes \hat{X})(\hat{X} \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \hat{X}_+)$.

- $[WT_8]$: $(\hat{X}\otimes\mathrm{id}_+)(\mathrm{id}_+\otimes\hat{X}_+)(\hat{X}_+\otimes\mathrm{id}_+) \sim_{W_{+++,+++}} (\mathrm{id}_+\otimes\hat{X}_+)(\hat{X}_+\otimes\mathrm{id}_+)(\mathrm{id}_+\otimes\hat{X})$

In $\hom_{P(\beta^*)}(++-,+)$*, we have the only relations:*

- $[WT_9]$: $(\overrightarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \overrightarrow{X}_-) \sim_{W_{++-,+}} (\mathrm{id}_+ \otimes \overrightarrow{\cap})(\hat{X}_+ \otimes \mathrm{id}_-)$.

In $\hom_{P(\beta^*)}(-++,+)$*, we have the only relations:*

- $[WT_9]'$: $(\overleftarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_- \otimes \hat{X}_+) \sim_{W_{-++,+}} (\mathrm{id}_+ \otimes \overrightarrow{\cap})(\overleftarrow{X}_- \otimes \mathrm{id}_-)$.

- $[WT_9]''$: $(\overleftarrow{\cap} \otimes \mathrm{id}_+)(\mathrm{id}_- \otimes \hat{X}) \sim_{W_{-++,+}} (\mathrm{id}_+ \otimes \overrightarrow{\cap})(\overleftarrow{X} \otimes \mathrm{id}_-)$.

In $\hom_{P(\beta^*)}(+,-++)$*, we have the relations:*

- $[WT_{10}] : (\mathrm{id}_- \otimes \hat{X}_+)(\overrightarrow{\cup} \otimes \mathrm{id}_+) \sim_{W_{+,-++}} (\overrightarrow{X}_- \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes \overrightarrow{\cup})$.

In $\hom_{P(\beta^*)}(+, ++-)$, *we have the relations:*

- $[WT_{10}]' : (\mathrm{id}_+ \otimes \overleftarrow{X}_-)(\overleftarrow{\cup} \otimes \mathrm{id}_+) \sim_{W_{+,++-}} (\hat{X}_+ \otimes \mathrm{id}_-)(\mathrm{id}_+ \otimes \overleftarrow{\cup})$.
- $[WT_{10}]'' : (\mathrm{id}_+ \otimes \overleftarrow{X})(\overleftarrow{\cup} \otimes \mathrm{id}_+) \sim_{W_{+,++-}} (\hat{X} \otimes \mathrm{id}_-)(\mathrm{id}_+ \otimes \overleftarrow{\cup})$.

In $\hom_{P(\beta^*)}(+, \emptyset)$, *we have the relation:*

- $[WT_{11}] : \overrightarrow{\cap}(\mathrm{id}_+ \otimes !) \sim_{W_{+,\emptyset}} \hat{¡} \sim_{W_{+,\emptyset}} \overleftarrow{\cap}(! \otimes \mathrm{id}_+)$.

In $\hom_{P(\beta^*)}(\emptyset, +)$, *we have the relation:*

- $[WT_{12}] : (\mathrm{id}_+ \otimes ¡)\overleftarrow{\cup} \sim_{W_{\emptyset,+}} \hat{!} \sim_{W_{\emptyset,+}} (¡ \otimes \mathrm{id}_+)\overrightarrow{\cup}$.

In $\hom_{P(\beta^*)}(++, +)$, *we have the relations:*

- $[WT_{13}] : (\hat{¡} \otimes \mathrm{id}_+)\hat{X}_+ \sim_{W_{++,+}} \mathrm{id}_+ \otimes \hat{¡}$.
- $[WT_{13}]' : (\mathrm{id}_+ \otimes \hat{¡})\hat{X}_- \sim_{W_{++,+}} \hat{¡} \otimes \mathrm{id}_+$.
- $[WT_{14}] : (\hat{¡} \otimes \mathrm{id}_+)\hat{X} \sim_{W_{++,+}} \mathrm{id}_+ \otimes \hat{¡}$.
- $[WT_{14}]' : (\mathrm{id}_+ \otimes \hat{¡})\hat{X} \sim_{W_{++,+}} \hat{¡} \otimes \mathrm{id}_+$.

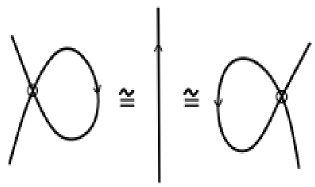
Note that we do not impose that in $\hom_{P(\beta^*)}(++, +)$***:***

$(\hat{¡} \otimes \mathrm{id}_+)\hat{X}_- \nsim_{W_{++,+}} \mathrm{id}_+ \otimes \hat{¡}$.

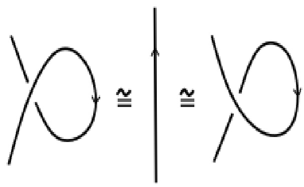
These relations can be presented geometrically as

Note: *We read the diagram from bottom to top.*

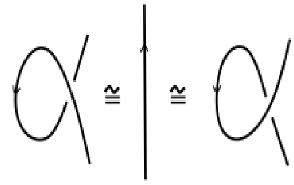
[WT_1]



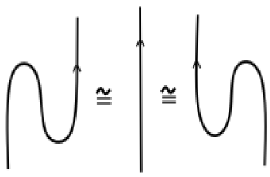
[WT_2]



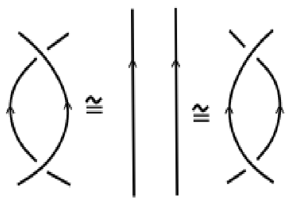
[WT_3]



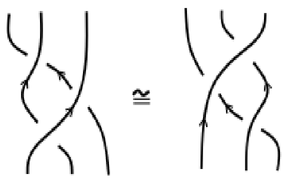
[WT_4]



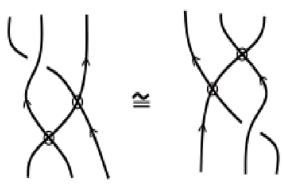
[WT_5]



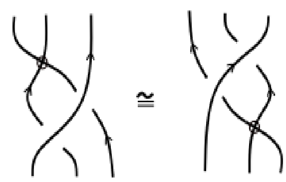
[WT_6]



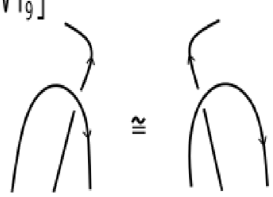
[WT_7]



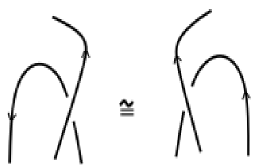
[WT_8]



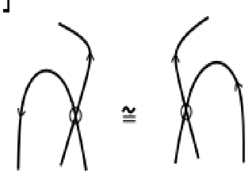
[WT_9]



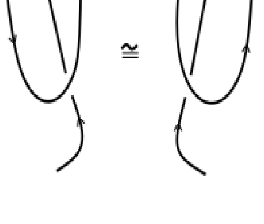
[WT_9]'



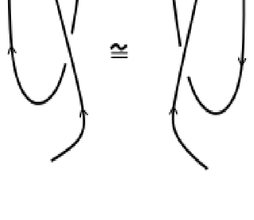
[WT_9]''



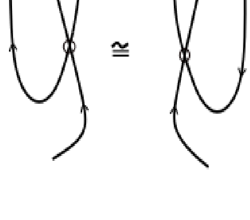
[WT_{10}]

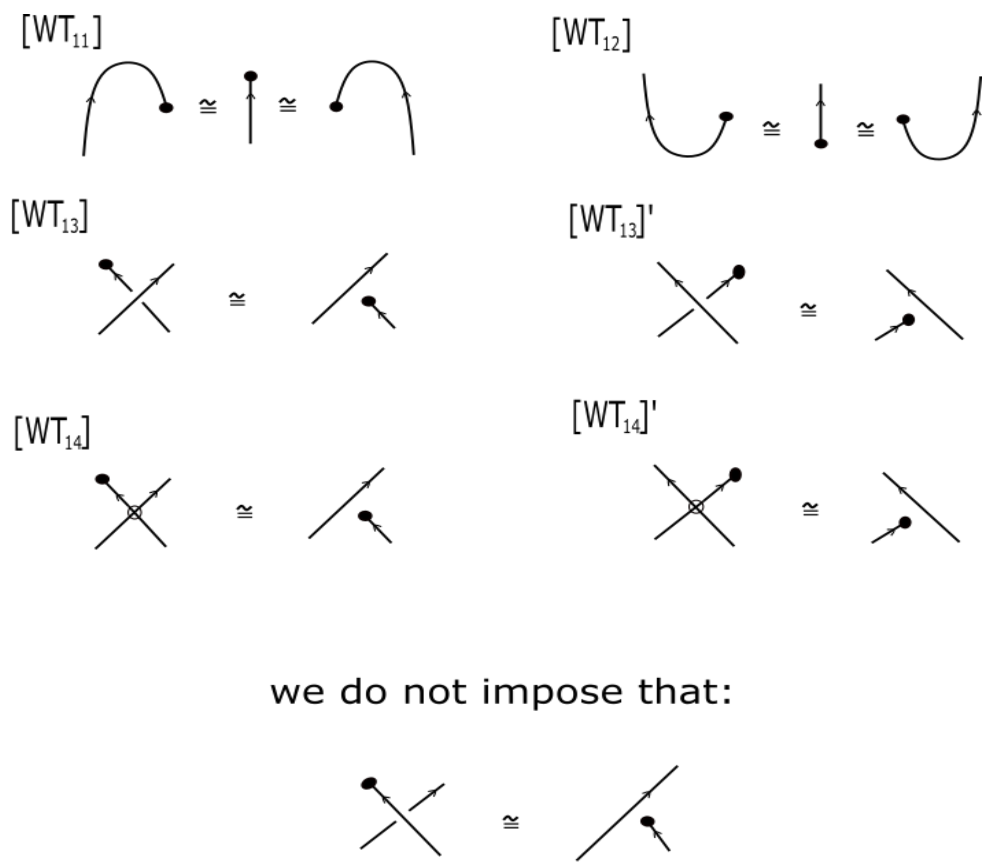


[WT_{10}]'



[WT_{10}]''





Note: In the oriented case of welded tangleois, the level of complexity of the relations is a lot higher, in comparison with the unoriented case, as there are several other cases depending on orientations. To the relations above we still need to add more relations with different orientation conventions.

7.4 Functorial invariants for Welded Tangle-oids categories

In this section we define functorial invariants for welded tangle-ids categories from a finite group and from a group acting on an abelian group. Useful references are [KM08, BMM18, DMM21]. is inspired by the invariants of tangles and welded virtual arcs in [KM08].

7.4.1 Functorial invariants from finite groups for the UWTC

Theorem 7.4.1. *Let G be a finite group. There is a unique strict monoidal functor F from the category of welded tangle-oids to the strict monoidal category Vec_G defined in 5.1.11, such that on objects for all $n \in \mathbb{N}$*

$$F(n) = n.$$

On morphisms

- $$F(\cup)\colon F(0) \to F(2)$$

  is the map in Vec

  $$\begin{aligned} &\mathbb{C} \to \mathbb{C}(G \times G) \\ &t \mapsto t \sum_{g \in G} (g, g^{-1}) \end{aligned}$$

  So the matrix elements of $F(\cup)$ are

  $$\langle 1 \mid F(\cup) \mid (g,h) \rangle = \begin{cases} 1, & \textit{if}\, h = g^{-1}, \\ 0, & otherwise. \end{cases}$$

- $$F(\cap): F(2) \to F(0)$$

  is the map in Vec
  $$\mathbb{C}(G \times G) \to \mathbb{C}$$
  such that on the basis $G \times G$ *of* $\mathbb{C}(G \times G)$*, we have*
  $$(g, h) \mapsto \delta(g, h^{-1}).$$
  So the matrix elements of $F(\cap)$ *are*
  $$\langle (g, h) \mid F(\cap) \mid 1 \rangle = \begin{cases} 1, & \textit{if } h = g^{-1}, \\ 0, & otherwise. \end{cases}.$$

- $$F(X_+): F(2) \to F(2)$$

  is the map in Vec
  $$\mathbb{C}(G \times G) \to \mathbb{C}(G \times G)$$
  such that, on the basis $G \times G$ *of* $\mathbb{C}(G \times G)$*, we have*
  $$(g, h) \mapsto (ghg^{-1}, g).$$
  So the matrix elements of $F(X_+)$ *are*
  $$\langle (g, h) \mid F(X_+) \mid (g', h') \rangle = \begin{cases} 1, & \textit{if } g' = ghg^{-1}, h' = g \\ 0, & otherwise \end{cases}$$

- $$F(X_-): F(2) \to F(2)$$

is the map in Vec

$$\mathbb{C}(G \times G) \to \mathbb{C}(G \times G)$$

such that, on the basis $G \times G$ of $\mathbb{C}(G \times G)$, we have

$$(g, h) \mapsto (h, h^{-1}gh).$$

So the matrix elements of $F(X_-)$ are

$$\langle (g,h) \mid F(X_-) \mid (g',h') \rangle = \begin{cases} 1, & \text{if } g' = h, h' = h^{-1}gh \\ 0, & otherwise. \end{cases}$$

- $$F(X) \colon F(2) \to F(2)$$

  is the map in Vec

  $$\mathbb{C}(G \times G) \to \mathbb{C}(G \times G)$$

  such that, on the basis $G \times G$ of $\mathbb{C}(G \times G)$, we have

  $$(g, h) \mapsto (h, g).$$

  So the matrix elements of $F(X)$ are

  $$\langle (g,h) \mid F(X) \mid (g',h') \rangle = \begin{cases} 1, & \text{if } g' = h, h' = g \\ 0, & otherwise. \end{cases}$$

- $$F(!) \colon F(0) \to F(1)$$

  is the map in Vec

  $$\mathbb{C} \to \mathbb{C}(G)$$
  $$t \mapsto t \sum_{g \in G} g$$

So the matrix elements of $F(!)$ are

$$\langle 1 \mid F(!) \mid g \rangle = 1.$$

- $$F(¡) \colon F(1) \to F(0)$$

  is the map in Vec

  $$\mathbb{C}(G) \to \mathbb{C}$$

  such that, on the basis G of $\mathbb{C}(G)$, we have

  $$g \mapsto 1.$$

  So the matrix elements of $F(¡)$ are

  $$\langle g \mid F(¡) \mid 1 \rangle = 1.$$

Proof. **Step 1:** $\Omega(\beta)$ is a free -$\frac{1}{2}$-monoidal category-triple over the monoidal graph β and $\delta \colon \beta \to U(\Omega(\beta))$ is a monoidal graph map 6.1.4, such that, on objects, $n \mapsto n$, and on morphisms $x \mapsto (0, x, 0)$. Let $A = Vec_G$, there is a monoidal graph map $\theta \colon \beta \to U(Vec_G)$ that on objects $n \in \mathbb{N}$.

$$n \mapsto n,$$

and on morphisms $E(\beta) = \{X_+, X_-, X, \cup, \cap, ¡, !\}$, θ define as F in the statement of this theorem.

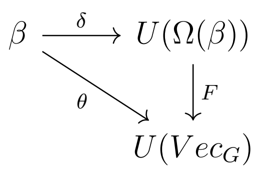
To prove θ is a monoid map need to show for all $x \in E(\beta)$, $\delta_i(E_\theta(x)) = V_\theta(\delta_i x)$.

$$\delta_1(E_\theta(X_+)) = 2 = V_\theta(\delta_1 X_+),$$
$$\delta_2(E_\theta(X_+)) = 2 = V_\theta(\delta_2 X_+),$$
$$\delta_1(E_\theta(X_-)) = 2 = V_\theta(\delta_1 X_-),$$
$$\delta_2(E_\theta(X_-)) = 2 = V_\theta(\delta_2 X_-),$$
$$\delta_1(E_\theta(X)) = 2 = V_\theta(\delta_1 X),$$
$$\delta_2(E_\theta(X)) = 2 = V_\theta(\delta_2 X),$$
$$\delta_1(E_\theta(\cup)) = 0 = V_\theta(\delta_1 \cup),$$
$$\delta_2(E_\theta(\cup)) = 2 = V_\theta(\delta_2 \cup),$$
$$\delta_1(E_\theta(\cap)) = 2 = V_\theta(\delta_1 \cap),$$
$$\delta_2(E_\theta(\cap)) = 0 = V_\theta(\delta_2 \cap),$$
$$\delta_1(E_\theta(¡)) = 1 = V_\theta(\delta_1 ¡),$$
$$\delta_2(E_\theta(¡)) = 0 = V_\theta(\delta_2 ¡),$$
$$\delta_1(E_\theta(!)) = 0 = V_\theta(\delta_1 !),$$
$$\delta_2(E_\theta(!)) = 1 = V_\theta(\delta_2 !).$$

Also for all $n, m \in N$,

$$V_\theta(n+m) = n+m = V_\theta(n) + V_\theta(m).$$

By applying the universal property, we have a unique $\frac{1}{2}$-monoidal functor $F\colon \Omega(\beta) \to Vec_G$, that makes the next diagram commute.



Step 2: Consider the $\frac{1}{2}$-monoidal functor $F\colon \Omega(\beta) \to Vec_G$. Let W be the congruence template that was defined in 7.2.2. We have for all objects $n, m \in ob(\Omega(\beta))$, and

morphisms $f, g\colon m \to n$.

- If $f \sim_W g$ then $F(f) = F(g)$, by a series of explicit calculations that we will do in step 4.

- If $f \sim_{\underline{W}} g$, then there are objects x, y, A, B, and morphisms $f', g'\colon A \to B$, $\alpha\colon m \to x + A + y$ and $\lambda\colon x + B + y \to n$, all in the category $\Omega(\beta)$, such that $f = \lambda \star ({}_x\#_y f') \star \alpha$ and $g = \lambda \star ({}_x\#_y g') \star \alpha$, and $f' \sim_{W_{A,B}} g'$.
  We know $F(f') = F(g')$. Therefore, since F is a 1/2-monoidal functor

  $$F(f) = F(\lambda) \star ({}_{F(x)}\#_{F(y)} F(f')) \star F(\alpha),$$

  $$F(g) = F(\lambda) \star ({}_{F(x)}\#_{F(y)} F(g')) \star F(\alpha).$$

  Hence
  $$F(f) = F(g).$$

- If $f \cong_{\underline{W}} g$, then there is $n \in \mathbb{N}$, n-morphisms $f_1, f_2, \dots, f_n$, such that $f = f_1, f_n = g$ and

  $$f = f_1 \sim_{\underline{W}} f_2 \sim_{\underline{W}} f_3 \sim_{\underline{W}} \dots \sim_{\underline{W}} f_n = g.$$

  Then
  $$F(f) = F(f_1) = F(f_2) = \dots = F(f_n) = F(g).$$

- The $\frac{1}{2}$-monoidal functor $F\colon \Omega(\beta) \to Vec_G$ "descends" to $\Omega(\beta)/\overline{\underline{W}}$. This means that there exists a unique $\frac{1}{2}$-monoidal functor $F'\colon \Omega(\beta)/\overline{\underline{W}} \to Vec_G$, that on objects
  $$F'(n) = n,$$

  and on morphisms
  $$F'([f]) = F(f),$$

F' is $\frac{1}{2}$-monoidal functor because

$$\begin{aligned} F'({}_A\#_B([f])) &= F'([{}_A\#_B(f)]) \\ &= F({}_A\#_B(f)) \\ &= {}_{F(A)}\#_{F(B)}(F(f)) \\ &= {}_{F'(A)}\#_{F'(B)}(F'([f])). \end{aligned}$$

Note that the diagram commutes

$$\begin{array}{ccc} \Omega(\beta) & \xrightarrow{P} & \Omega(\beta)/\underline{\overline{W}} \\ & \underset{F}{\searrow} & \downarrow F' \\ & & Vec_G \end{array}$$

Step 3: Recall that Vec_G is slideable. From the universal property of slidification 5.2.10, we have a unique functor $F''\colon \mathfrak{F}(\Omega(\beta)/\underline{\overline{W}}) \to Vec_G$ that makes the next diagram commute.

$$\begin{array}{ccc} \Omega(\beta)/\underline{\overline{W}} & \longrightarrow & \mathfrak{F}(\Omega(\beta)/\underline{\overline{W}}) \\ & \underset{F'}{\searrow} & \downarrow F'' \\ & & Vec_G \end{array}$$

Step 4: Let us now show the explicit calculations. Here $[WT_i]$, $i = 1, \ldots, 14$, is the relation of welded tangle-oids categories as in the definition.

- Let $g \in G$. We have

$$\begin{aligned} [WT_1]\, g &\xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h \in G} (g, h, h^{-1}) \xrightarrow{F(X \otimes \mathrm{id}_1)} \sum_{h \in G} (h, g, h^{-1}) \\ &\xrightarrow{F(\mathrm{id}_1 \otimes \cap)} \sum_{h \in G} h\, \delta(g, h) = g. \\ g &\xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h \in G} (h, h^{-1}, g) \xrightarrow{F(\mathrm{id}_1 \otimes X)} \sum_{h \in G} (h, g, h^{-1}) \\ &\xrightarrow{F(\cap \otimes \mathrm{id}_1)} \sum_{h \in G} \delta(h, g^{-1}) h^{-1} = g. \end{aligned}$$

- Let $g \in G$. We have

$$[WT_2]\, g \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h\in G}(g,h,h^{-1}) \xrightarrow{F(X_+ \otimes \mathrm{id}_1)} \sum_{h\in G}(ghg^{-1},g,h^{-1})$$
$$\xrightarrow{F(\mathrm{id}_1 \otimes \cap)} \sum_{h\in G} ghg^{-1}\delta(g,h) = g.$$
$$g \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h\in G}(g,h,h^{-1}) \xrightarrow{F(X_- \otimes \mathrm{id}_1)} \sum_{h\in G}(h,h^{-1}gh,h^{-1})$$
$$\xrightarrow{F(\mathrm{id}_1 \otimes \cap)} \sum_{h\in G} h\,\delta(h^{-1}gh,h) = g.$$

Since $h^{-1}gh = h \Leftrightarrow h^{-1}g = 1 \Leftrightarrow h = g$.

- Let $g \in G$. We have

$$[WT_3]\, g \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h\in G}(h,h^{-1},g) \xrightarrow{F(\mathrm{id}_1 \otimes X_-)} \sum_{h\in G}(h,g,g^{-1}h^{-1}g)$$
$$\xrightarrow{F(\cap \otimes \mathrm{id}_1)} \sum_{h\in G} \delta(h,g^{-1})g^{-1}h^{-1}g = g.$$
$$g \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h\in G}(h,h^{-1},g) \xrightarrow{F(\mathrm{id}_1 \otimes X_+)} \sum_{h\in G}(h,h^{-1}gh,h^{-1})$$
$$\xrightarrow{F(\cap \otimes \mathrm{id}_1)} \sum_{h\in G} \delta(h,h^{-1}g^{-1}h)h^{-1} = g.$$

Since $h = h^{-1}g^{-1}h \Leftrightarrow h^{-1} = g$.

- Let $g \in G$. We have

$$[WT_4]\, g \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h\in G}(g,h,h^{-1}) \xrightarrow{F(\cap \otimes \mathrm{id}_1)} \sum_{h\in G} \delta(g,h^{-1})h^{-1} = g.$$
$$g \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h\in G}(h,h^{-1},g) \xrightarrow{F(\mathrm{id}_1 \otimes \cap)} \sum_{h\in G} h\,\delta(h^{-1},g^{-1}) = g.$$

- Let $(g,h) \in G \times G$. We have

$$[WT_5]\, (g,h) \xrightarrow{F(X_+)} (ghg^{-1},g) \xrightarrow{F(X_-)} (g,g^{-1}ghg^{-1}g) = (g,h).$$
$$(g,h) \xrightarrow{F(X_-)} (h,h^{-1}gh) \xrightarrow{F(X_+)} (hh^{-1}ghh^{-1},h) = (g,h).$$

- Let $(g,h,k) \in G \times G \times G$. We have

$$[WT_6]\,(g,h,k) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} (ghg^{-1},g,k) \xrightarrow{F(\mathrm{id}_1\otimes X_+)} (ghg^{-1},gkg^{-1},g) \xrightarrow{F(X_+\otimes \mathrm{id}_1)}$$
$$(ghg^{-1}gkg^{-1}gh^{-1}g^{-1},ghg^{-1},g) = (ghkh^{-1}g^{-1},ghg^{-1},g).$$
$$(g,h,k) \xrightarrow{F(\mathrm{id}_1\otimes X_+)} (g,hkh^{-1},h) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} (ghkh^{-1}g^{-1},g,h) \xrightarrow{F(\mathrm{id}_1\otimes X_+)}$$
$$(ghkh^{-1}g^{-1},ghg^{-1},g).$$

- Let $(g,h,k) \in G \times G \times G$. We have

$$[WT_7]\,(g,h,k) \xrightarrow{F(X\otimes \mathrm{id}_1)} (h,g,k) \xrightarrow{F(\mathrm{id}_1\otimes X)} (h,k,g) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} (hkh^{-1},h,g).$$
$$(g,h,k) \xrightarrow{F(\mathrm{id}_1\otimes X_+)} (g,hkh^{-1},h) \xrightarrow{F(X\otimes \mathrm{id}_1)} (hkh^{-1},g,h)$$
$$\xrightarrow{F(\mathrm{id}_1\otimes X)} (hkh^{-1},h,g).$$

- Let $(g,h,k) \in G \times G \times G$. We have

$$[WT_8]\,(g,h,k) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} (ghg^{-1},g,k) \xrightarrow{F(\mathrm{id}_1\otimes X_+)} (ghg^{-1},gkg^{-1},g)$$
$$\xrightarrow{F(X\otimes \mathrm{id}_1)} (gkg^{-1},ghg^{-1},g).$$
$$(g,h,k) \xrightarrow{F(\mathrm{id}_1\otimes X)} (g,k,h) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} (gkg^{-1},g,h)$$
$$\xrightarrow{F(\mathrm{id}_1\otimes X_+)} (gkg^{-1},ghg^{-1},g).$$

- Let $(g,h,k) \in G \times G \times G$. We have

$$[WT_9]\,(g,h,k) \xrightarrow{F(\mathrm{id}_1\otimes X_-)} (g,k,k^{-1}hk) \xrightarrow{F(\cap\otimes \mathrm{id}_1)} \delta(g,k^{-1})k^{-1}hk$$
$$= \begin{cases} ghg^{-1}, & \text{if } g = k^{-1} \\ 0, & \text{if } g \neq k^{-1}. \end{cases}$$
$$(g,h,k) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} (ghg^{-1},g,k) \xrightarrow{F(\mathrm{id}_1\otimes \cap)} ghg^{-1}\delta(g,k^{-1})$$
$$= \begin{cases} ghg^{-1}, & \text{if } g = k^{-1} \\ 0, & \text{if } g \neq k^{-1}. \end{cases}$$

- Let $(g, h, k) \in G \times G \times G$. We have

$$[WT_9]'\,(g,h,k) \xrightarrow{F(\mathrm{id}_1 \otimes X_+)} (g, hkh^{-1}, h) \xrightarrow{F(\cap \otimes \mathrm{id}_1)} \delta(g, hk^{-1}h^{-1})h$$
$$= \begin{cases} h, & \text{if } g = hk^{-1}h^{-1} \\ 0, & \text{if } g \neq hk^{-1}h^{-1}. \end{cases}$$
$$(g,h,k) \xrightarrow{F(X_- \otimes \mathrm{id}_1)} (h, h^{-1}gh, k) \xrightarrow{F(\mathrm{id}_1 \otimes \cap)} h\,\delta(h^{-1}gh, k^{-1})$$
$$= \begin{cases} h, & \text{if } h^{-1}gh = k^{-1} \\ 0, & \text{if } h^{-1}gh \neq k^{-1}. \end{cases}$$

Note: $g = hk^{-1}h^{-1} \Leftrightarrow h^{-1}gh = k^{-1}$.

- Let $(g, h, k) \in G \times G \times G$. We have

$$[WT_9]''\,(g,h,k) \xrightarrow{F(\mathrm{id}_1 \otimes X)} (g, k, h) \xrightarrow{F(\cap \otimes \mathrm{id}_1)} \delta(g, k^{-1})h$$
$$= \begin{cases} h, & \text{if } g = k^{-1} \\ 0, & \text{if } g \neq k^{-1}. \end{cases}$$
$$(g,h,k) \xrightarrow{F(X \otimes \mathrm{id}_1)} (h, g, k) \xrightarrow{F(\mathrm{id}_1 \otimes \cap)} h\,\delta(g, k^{-1})$$
$$= \begin{cases} h, & \text{if } g = k^{-1} \\ 0, & \text{if } g \neq k^{-1}. \end{cases}$$

- Let $g \in G$. We have

$$[WT_{10}]\,g \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h \in G} (h, h^{-1}, g) \xrightarrow{F(\mathrm{id}_1 \otimes X_+)} \sum_{h \in G} (h, h^{-1}gh, h^{-1}).$$
$$g \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h \in G} (g, h, h^{-1}) \xrightarrow{F(X_- \otimes \mathrm{id}_1)} \sum_{h \in G} (h, h^{-1}gh, h^{-1}).$$

- Let $g \in G$. We have

$$[WT_{10}]'\,g \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h \in G} (h, h^{-1}, g) \xrightarrow{F(\mathrm{id}_1 \otimes X_-)} \sum_{h \in G} (h, g, g^{-1}h^{-1}g).$$
$$g \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h \in G} (g, h, h^{-1}) \xrightarrow{F(X_+ \otimes \mathrm{id}_1)} \sum_{h \in G} (ghg^{-1}, g, h^{-1}).$$

- Let $g \in G$. We have

$$[WT_{10}]''\, g \xrightarrow{F(\cup\otimes \mathrm{id}_1)} \sum_{h\in G}(h,h^{-1},g) \xrightarrow{F(\mathrm{id}_1\otimes X)} \sum_{h\in G}(h,g,h^{-1}).$$

$$g \xrightarrow{F(\mathrm{id}_1\otimes\cup)} \sum_{h\in G}(g,h,h^{-1}) \xrightarrow{F(X\otimes \mathrm{id}_1)} \sum_{h\in G}(h,g,h^{-1}).$$

- Let $g \in G$. We have

$$[WT_{11}]\, g \xrightarrow{F(\mathrm{id}\otimes !)} \sum_{h\in G}(g,h) \xrightarrow{F(\cap)} \sum_{h\in G}\delta(g,h^{-1}) = 1.$$

$$g \xrightarrow{F(¡)} 1.$$

$$g \xrightarrow{F(!\otimes \mathrm{id}_1)} \sum_{h\in G}(h,g) \xrightarrow{F(\cap)} \sum_{h\in G}\delta(h,g^{-1}) = 1.$$

-

$$[WT_{12}]\, 1 \xrightarrow{F(\cup)} \sum_{g\in G}(g,g^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes ¡)} \sum_{g\in G} g.$$

$$1 \xrightarrow{F(!)} \sum_{g\in G} g.$$

$$1 \xrightarrow{F(\cup)} \sum_{g\in G}(g,g^{-1}) \xrightarrow{F(¡\otimes \mathrm{id}_1)} \sum_{g\in G} g^{-1}.$$

- Let $(g,h) \in G\times G$. We have

$$[WT_{13}]\, (g,h) \xrightarrow{F(X_+)} (ghg^{-1},g) \xrightarrow{F(¡\otimes \mathrm{id}_1)} g.$$

$$(g,h) \xrightarrow{F(\mathrm{id}_1\otimes ¡)} g.$$

- Let $(g,h) \in G\times G$. We have

$$[WT_{13}]'\, (g,h) \xrightarrow{F(X_-)} (h,h^{-1}gh) \xrightarrow{F(\mathrm{id}_1\otimes ¡)} h.$$

$$(g,h) \xrightarrow{F(¡\otimes \mathrm{id}_1)} h.$$

Let $(g,h) \in G \times G$. We have

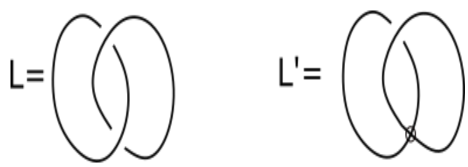
$$[WT_{14}]\,(g,h) \xrightarrow{F(X)} (h,g) \xrightarrow{F(\text{¡}\otimes \mathrm{id}_1)} g.$$
$$(g,h) \xrightarrow{F(\mathrm{id}_1\otimes\text{¡})} g.$$

$$[WT_{14}]'\,(g,h) \xrightarrow{F(X)} (h,g) \xrightarrow{F(\mathrm{id}_1\otimes\text{¡})} h.$$
$$(g,h) \xrightarrow{F(\text{¡}\otimes \mathrm{id}_1)} h.$$

□

The following example is in [KM08].

Example 7.4.2. *Consider the following morphisms of the category of $UWTC$.*



We want to calculate the functorial invariant from finite group.

$$F(L) = 1 \xrightarrow{F(\cup\otimes\cup)} \sum_{g,h\in G} (g, g^{-1}, h, h^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes X_+\otimes \mathrm{id}_1)} \sum_{g,h\in G} (g, g^{-1}hg, g^{-1}, h^{-1})$$
$$\xrightarrow{F(\mathrm{id}_1\otimes X_+\otimes \mathrm{id}_1)} \sum_{g,h\in G} (g, g^{-1}hg^{-1}h^{-1}g, g^{-1}hg, h^{-1}) \xrightarrow{F(\cap\otimes\cap)}$$
$$\sum_{g,h\in G} \delta(g, g^{-1}hgh^{-1}g)\delta(g^{-1}hg, h) = \mid \{(g,h) \mid gh = hg\} \mid .$$

$$F(L') = 1 \xrightarrow{F(\cup\otimes\cup)} \sum_{g,h\in G}(g,g^{-1},h,h^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes X\otimes \mathrm{id}_1)} \sum_{g,h\in G}(g,h,g^{-1},h^{-1})$$
$$\xrightarrow{F(\mathrm{id}_1\otimes X_+\otimes \mathrm{id}_1)} \sum_{g,h\in G}(g,hg^{-1}h^{-1},h,h^{-1}) \xrightarrow{F(\cap\otimes\cap)}$$
$$\sum_{g,h\in G}\delta(g,hgh^{-1})\delta(h,h) = \mid \{(g,h) \mid gh = hg\} \mid .$$

Then

$$F(L) = F(L').$$

We can see in this example although the two knots is different but the invariants are the same, so this invariants do not separate all knots. Now we will define another invariants and we will see after that if this invariants will separate this knots in the example.

7.4.2 Functorial invariants from group acting on abelian group for the UWTC

This is construction is inspired by the invariants of welded virtual arcs in [KM08].

Theorem 7.4.3. *Let G be a finite group act on an abelian group A . There is a strict monoidal functor F from the category of welded tangl-oids to the strict monoidal category of $Vec_{G\times A}$, such that for all $n \in \mathbb{N}$,*

$$F(n) = n.$$

Moreover

- $$F(\cup)\colon F(0) \to F(2)$$

is the map in Vec

$$\begin{aligned} \mathbb{C} &\to \mathbb{C}((G\times A)\times(G\times A)) \\ t &\mapsto t\sum_{g\in G, a\in A}(g,a,g^{-1},a^{-1}). \end{aligned}$$

So the matrix elements of $F(\cup)$ *are*

$$\langle 1 \mid F(\cup) \mid (g,a,h,b)\rangle = \begin{cases} 1, & \textit{if } h=g^{-1}, b=a^{-1}, \\ 0, & otherwise. \end{cases}$$

- $$F(\cap)\colon F(2)\to F(0)$$

  is the map in Vec

  $$\mathbb{C}((G\times A)\times(G\times A))\to\mathbb{C}$$

  such that, on the basis $(G\times A)\times(G\times A)$*, we have*

  $$(g,a,h,b)\mapsto \delta(g,h^{-1})\delta(a,b^{-1}).$$

  So the matrix elements of $F(\cap)$ *are*

  $$\langle (g,a,h,b) \mid F \mid 1\rangle = 1.$$

- $$F(X_+)\colon F(2)\to F(2)$$

  is the map in Vec

  $$\mathbb{C}((G\times A)\times(G\times A))\to\mathbb{C}((G\times A)\times(G\times A))$$

  such that, on the basis $(G\times A)\times(G\times A)$*, we have*

  $$(g,a,h,b)\mapsto \Big(ghg^{-1},\ (g\triangleright b),\ g,\ (g\triangleright b^{-1})ab\Big).$$

So the matrix elements of $F(X_+)$ *are*

$$\langle (g,a,h,b) \,| F(X_+) \,|\, (g',a',h',b') \rangle = \begin{cases} 1, & \text{if } g' = ghg^{-1},\ a' = g \triangleright b,\ h' = g,\ b' = g \triangleright b^{-1}ab, \\ 0, & otherwise. \end{cases}$$

- $$F(X_-)\colon F(2) \to F(2)$$

  is the map in Vec

  $$\mathbb{C}((G \times A) \times (G \times A)) \to \mathbb{C}((G \times A) \times (G \times A))$$

  such that, on the basis $(G \times A) \times (G \times A)$*, we have*

  $$(g,a,h,b) \mapsto \Big(h,\ (h^{-1} \triangleright a^{-1})ab,\ h^{-1}gh,\ (h^{-1} \triangleright a)\Big).$$

  So the matrix elements of $F(X_-)$ *are:*

  $$\langle (g,a,h,b) \,| F(X_-) \,|\, (g',a',h',b') \rangle = \begin{cases} 1, & \text{if } g' = h,\ a' = h^{-1} \triangleright a^{-1}ab,\ h' = h^{-1}gh,\ b' = h^{-1} \triangleright a, \\ 0, & otherwise. \end{cases}$$

- $$F(X)\colon F(2) \to F(2)$$

  is the map in Vec

  $$\mathbb{C}((G \times A) \times (G \times A)) \to \mathbb{C}((G \times A) \times (G \times A))$$

  such that, on the basis $(G \times A) \times (G \times A)$*, we have*

  $$(g,a,h,b) \mapsto (h,b,g,a).$$

So the matrix elements of $F(X)$ are

$$\langle (g,a,h,b) \mid F(X) \mid (g',a',h',b')\rangle = \begin{cases} 1, & \text{if } g'=h,\ a'=b,\ h'=g,\ b'=a, \\ 0, & otherwise. \end{cases}$$

- $$F(!)\colon F(0) \to F(1)$$

  is the map in Vec

  $$\begin{aligned} \mathbb{C} &\to \mathbb{C}(G\times A) \\ t &\mapsto t\sum_{g\in G}(g,0_A). \end{aligned}$$

  So the matrix elements of $F(!)$ are

  $$\langle 1 \mid F(!) \mid (g,a)\rangle = \begin{cases} 1, & \text{if } a = 0_A, \\ 0, & otherwise. \end{cases}$$

- $$F(¡)\colon F(1) \to F(0)$$

  is the map in Vec

  $$\mathbb{C}(G\times A) \to \mathbb{C}$$

  such that, on the basis $G \times A$, we have

  $$\begin{aligned} \mathbb{C}(G\times A) &\to \mathbb{C} \\ (g,a) &\mapsto \begin{cases} 1 & \text{if } a = 0_A, \\ 0 & otherwise. \end{cases} \end{aligned}$$

  So the matrix elements of $F(¡)$ are

  $$\langle (g,a) \mid F(¡) \mid 1\rangle = 1.$$

Proof. • Let $(g,a) \in G \times A$, we have

$$[WT_1]\,(g,a) \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h\in G, b\in A} (g,a,h,b,h^{-1},b^{-1}) \xrightarrow{F(X\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} (h,b,g,a,h^{-1},b^{-1}) \xrightarrow{F(\mathrm{id}_1 \otimes \cap)} \sum_{h\in G, b\in A} (h,b)\delta(g,h)\delta(a,b) = (g,a).$$

$$(g,a) \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h\in G, b\in A} (h,b,h^{-1},b^{-1},g,a) \xrightarrow{F(\mathrm{id}_1 \otimes X)}$$

$$\sum_{h\in G, b\in A} (h,b,g,a,h^{-1},b^{-1}) \xrightarrow{F(\cap \otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} \delta(h,g^{-1})\delta(b,a^{-1})(h^{-1},b^{-1}) = (g,a).$$

• Let $(g,a) \in G \times A$, we have

$$[WT_2]\,(g,a) \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h\in G, b\in A} (g,a,h,b,h^{-1},b^{-1}) \xrightarrow{F(X_+\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} \Big(ghg^{-1},\ (g \triangleright b),\ g,\ (g \triangleright b^{-1})ab,\ h^{-1},\ b^{-1}\Big) \xrightarrow{F(\mathrm{id}_1 \otimes \cap)}$$

$$\sum_{h\in G, b\in A} \Big(ghg^{-1}, g \triangleright b\Big)\Big(\delta(g,h)\delta((g \triangleright b^{-1})ab, b)\Big) = (g,a).$$

Note: $\delta((g \triangleright b^{-1})ab, b) = 1 \Leftrightarrow (g \triangleright b^{-1})ab = b \Leftrightarrow g \triangleright b^{-1} = a^{-1} \Leftrightarrow g \triangleright b = a.$

$$(g,a) \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h\in G, b\in A} (g,a,h,b,h^{-1},b^{-1}) \xrightarrow{F(X_-\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} \Big(h,\ (h^{-1} \triangleright a^{-1})ab,\ h^{-1}gh,\ (h^{-1} \triangleright a),\ h^{-1},\ b^{-1}\Big) \xrightarrow{F(\mathrm{id}_1 \otimes \cap)}$$

$$\sum_{h\in G, b\in A} \Big(h,\ (h^{-1} \triangleright a^{-1})ab\Big)\Big(\delta(h^{-1}gh, h)\delta(h^{-1} \triangleright a, b)\Big) = (g,a).$$

Note: $\delta(h^{-1}gh, h) = 1 \Leftrightarrow h^{-1}gh = h \Leftrightarrow g = h.$

$\delta(h^{-1} \triangleright a, b) = 1 \Leftrightarrow h^{-1} \triangleright a = b \Leftrightarrow h^{-1} \triangleright a^{-1} = b^{-1}.$

- Let $(g, a) \in G \times A$, we have

$$[WT_3]\,(g,a) \xrightarrow{F(\cup\otimes \mathrm{id}_1)} \sum_{h\in G, b\in A} (h, b, h^{-1}, b^{-1}, g, a) \xrightarrow{F(\mathrm{id}_1\otimes X_-)}$$

$$\sum_{h\in G, b\in A} \Big(h,\ b,\ g,\ (g^{-1} \triangleright b)b^{-1}a,\ g^{-1}h^{-1}g,\ (g^{-1} \triangleright b^{-1})\Big) \xrightarrow{F(\cap\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} \Big(\delta(h, g^{-1})\delta(b, ((g^{-1} \triangleright b)b^{-1}a)^{-1}\Big)\Big(g^{-1}h^{-1}g,\ g^{-1} \triangleright b^{-1}\Big) = (g, a).$$

Note: $\delta(b, ((g^{-1} \triangleright b)b^{-1}a)^{-1}) = 1 \Leftrightarrow b = (g^{-1} \triangleright b^{-1})a^{-1}b \Leftrightarrow a = g^{-1} \triangleright b^{-1}$.

$$(g,a) \xrightarrow{F(\cup\otimes \mathrm{id}_1)}= \sum_{h\in G, b\in A} (h, b, h^{-1}, b^{-1}, g, a) \xrightarrow{F(\mathrm{id}_1\otimes X_+)}$$

$$\sum_{h\in G, b\in A} \Big(h,\ b,\ h^{-1}gh,\ (h^{-1} \triangleright a), h^{-1},\ (h^{-1} \triangleright a^{-1})b^{-1}a\Big) \xrightarrow{F(\cap\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} \Big(\delta(h, h^{-1}g^{-1}h)\delta(b, h^{-1} \triangleright a^{-1})\Big)\Big(h^{-1},\ (h^{-1} \triangleright a^{-1})b^{-1}a\Big) = (g, a).$$

- Let $(g, a) \in G \times A$, we have

$$[WT_4]\,(g,a) \xrightarrow{F(\mathrm{id}_1\otimes\cup)} \sum_{h\in G, b\in A} (g, a, h, b, h^{-1}, b^{-1}) \xrightarrow{F(\cap\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} \delta(g, h^{-1})\delta(a, b^{-1})(h^{-1}, b^{-1}) = (g, a).$$

$$(g,a) \xrightarrow{F(\cup\otimes \mathrm{id}_1)} \sum_{h\in G, b\in A} (h, b, h^{-1}, b^{-1}, g, a) \xrightarrow{F(\mathrm{id}_1\otimes\cap)}$$

$$\sum_{h\in G, b\in A} (h, b)\delta(h^{-1}, g^{-1})\delta(b^{-1}, a^{-1}) = (g, a).$$

- Let $(g,a,h,b) \in (G \times A) \times (G \times A)$, we have

$$[WT_5]\,(g,a,h,b) \xrightarrow{F(X_+)} \Big(ghg^{-1},\,(g \triangleright b),\,g,\,(g \triangleright b^{-1})ab\Big) \xrightarrow{F(X_-)}$$

$$\Big(g,\,(g^{-1} \triangleright (g \triangleright b)^{-1})(g \triangleright b)(g \triangleright b^{-1})ab),\,g^{-1}ghg^{-1}g,\,g^{-1} \triangleright g \triangleright b\Big)$$

$$= (g,a,h,b) * .$$

$$(g,a,h,b) \xrightarrow{F(X_-)} \Big(h,\,(h^{-1} \triangleright a^{-1})ab,\,h^{-1}gh,\,(h^{-1} \triangleright a)\Big) \xrightarrow{F(X_+)}$$

$$\Big(hh^{-1}ghh^{-1},\,(h \triangleright h^{-1} \triangleright a),\,h,\,h \triangleright (h^{-1} \triangleright a)^{-1}(h^{-1} \triangleright a^{-1})ab(h^{-1} \triangleright a)\Big)$$

$$= (g,a,h,b) **.$$

* $g^{-1} \triangleright (g \triangleright b)^{-1}(g \triangleright b)(g \triangleright b^{-1})ab = g^{-1} \triangleright (g \triangleright b)^{-1}(g \triangleright bb^{-1})ab) = g^{-1} \triangleright (g \triangleright b)^{-1}(ab) = g^{-1} \triangleright (g \triangleright b^{-1})(ab) = b^{-1}ab = a.$
** $h \triangleright (h^{-1} \triangleright a)^{-1}(h^{-1} \triangleright a^{-1})ab(h^{-1} \triangleright a) = h \triangleright h^{-1} \triangleright a^{-1}(h^{-1} \triangleright a^{-1})(h^{-1} \triangleright a)(ab) = a^{-1}ab = b.$

- Let $(g,a,h,b) \in (G \times A) \times (G \times A)$, we have

$$[WT_6]\,\Big(g,a,h,b,k,c\Big) \xrightarrow{F(X_+ \otimes \mathrm{id}_1)} \Big(ghg^{-1},\,(g \triangleright b),\,g,\,(g \triangleright b^{-1})ab,\,k,\,c\Big) \xrightarrow{F(\mathrm{id}_1 \otimes X_+)}$$

$$\Big(ghg^{-1},\,(g \triangleright b),\,gkg^{-1},\,(g \triangleright c),\,g,\,(g \triangleright c^{-1})((g \triangleright b^{-1})ab)c\Big) \xrightarrow{F(X_+ \otimes \mathrm{id}_1)}$$

$$\Big(ghg^{-1}gkg^{-1}gh^{-1}g^{-1},\,ghg^{-1} \triangleright (g \triangleright c),\,ghg^{-1},$$

$$ghg^{-1} \triangleright (g \triangleright c)^{-1}(g \triangleright b)(g \triangleright c),\,g,(g \triangleright c^{-1})((g \triangleright b^{-1})abc)\Big)$$

$$= \Big(ghkh^{-1}g^{-1},\,(gh \triangleright g^{-1} \triangleright g \triangleright c),\,ghg^{-1},\,(gh \triangleright g^{-1} \triangleright g \triangleright c^{-1})(g \triangleright bc),$$

$$g,\,(g \triangleright (bc)^{-1})abc\Big)$$

$$= \Big(ghkh^{-1}g^{-1},\,(gh \triangleright c),\,ghg^{-1},(gh \triangleright c^{-1})(g \triangleright bc),\,g,\,(g \triangleright (bc)^{-1})abc\Big).$$

$$\Big(g,a,h,b,k,c\Big) \xrightarrow{F(\mathrm{id}_1\otimes X_+)} \Big(g,\ a,\ hkh^{-1},\ (h\triangleright c),\ h,\ (h\triangleright c^{-1})bc\Big) \xrightarrow{F(X_+\otimes \mathrm{id}_1)}$$
$$\Big(ghkh^{-1}g^{-1},\ (g\triangleright h\triangleright c),\ g,\ (g\triangleright (h\triangleright c)^{-1})a(h\triangleright c),\ h,\ (h\triangleright c^{-1})bc\Big) \xrightarrow{F(\mathrm{id}_1\otimes X_+)}$$
$$\Big(ghkh^{-1}g^{-1},\ (g\triangleright h\triangleright c),\ ghg^{-1},\ g\triangleright ((h\triangleright c^{-1})bc),$$
$$g,\ g\triangleright ((h\triangleright c^{-1})(bc))^{-1}(g\triangleright (h\triangleright c)^{-1}a(h\triangleright c))((h\triangleright c^{-1})bc)\Big)$$
$$= \Big(ghkh^{-1}g^{-1},\ (gh\triangleright c),\ ghg^{-1},\ (gh\triangleright c^{-1})(g\triangleright bc),\ g,$$
$$(g\triangleright h\triangleright c)(g\triangleright (bc)^{-1})(g\triangleright h\triangleright c^{-1})a(h\triangleright c)(h\triangleright c^{-1})bc\Big)$$
$$= \Big(ghkh^{-1}g^{-1},\ (gh\triangleright c),\ ghg^{-1},\ (gh\triangleright c^{-1})(g\triangleright bc),\ g,\ (g\triangleright (bc)^{-1})abc\Big).$$

- Let $(g,a,h,b) \in (G\times A)\times(G\times A)$, we have

$$[WT_7]\ \Big(g,a,h,b,k,c\Big) \xrightarrow{F(X\otimes \mathrm{id}_1)} \Big(h,b,g,a,k,c\Big) \xrightarrow{F(\mathrm{id}_1\otimes X)}$$
$$\Big(h,b,k,c,g,a\Big) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} \Big(hkh^{-1},\ (h\triangleright c),\ h,\ (h\triangleright c^{-1})bc,\ g,\ a\Big).$$

$$\Big(g,a,h,b,k,c\Big) \xrightarrow{F(\mathrm{id}_1\otimes X_+)} \Big(g,\ a,\ hkh^{-1},\ (h\triangleright c),\ h,\ (h\triangleright c^{-1})bc\Big)$$
$$\xrightarrow{F(X\otimes \mathrm{id}_1)} \Big(hkh^{-1},\ (h\triangleright c),\ g,\ a,\ h,\ (h\triangleright c^{-1})bc\Big)$$
$$\xrightarrow{F(\mathrm{id}_1\otimes X)} \Big(hkh^{-1},\ (h\triangleright c),\ h,\ (h\triangleright c^{-1})bc,\ g,\ a\Big).$$

- Let $(g,a,h,b) \in (G\times A)\times(G\times A)$, we have

$$[WT_8]\ \Big(g,a,h,b,k,c\Big) \xrightarrow{F(X_+\otimes \mathrm{id}_1)} \Big(ghg^{-1},\ (g\triangleright b),\ g,\ (g\triangleright b^{-1})ab,\ k,\ c\Big) \xrightarrow{F(\mathrm{id}_1\otimes X_+)}$$
$$\Big(ghg^{-1},\ (g\triangleright b),\ gkg^{-1},\ (g\triangleright c),\ g,\ (g\triangleright c^{-1})(g\triangleright b^{-1})abc\Big) \xrightarrow{F(X\otimes \mathrm{id}_1)}$$
$$\Big(gkg^{-1},\ (g\triangleright c),\ ghg^{-1},\ (g\triangleright b),\ g,\ (g\triangleright c^{-1})(g\triangleright b^{-1})abc\Big).$$

$$\Big(g,a,h,b,k,c\Big) \xrightarrow{F(\mathrm{id}_1\otimes X)} \Big(g,a,k,c,h,b\Big) \xrightarrow{F(X_+\otimes \mathrm{id}_1)}$$
$$\Big(gkg^{-1},\ (g\triangleright c),\ g,\ (g\triangleright c^{-1})ac,\ h,\ b\Big) \xrightarrow{F(\mathrm{id}_1\otimes X_+)}$$
$$\Big(gkg^{-1},\ (g\triangleright c),\ ghg^{-1},\ (g\triangleright b),\ g,\ (g\triangleright b^{-1})(g\triangleright c^{-1})acb\Big).$$

- Let $(g,a,h,b)\in(G\times A)\times(G\times A)$, we have

$$[WT_9]\Big(g,a,h,b,k,c\Big)\xrightarrow{F(\mathrm{id}_1\otimes X_-)}\Big(g,\ a,\ k,\ (k^{-1}\triangleright b^{-1})bc,\ k^{-1}hk,\ (k^{-1}\triangleright b)\Big)$$
$$\xrightarrow{F(\cap\otimes\mathrm{id}_1)}\Big(\delta(g,k^{-1})\delta(a,\ (k^{-1}\triangleright b)b^{-1}c^{-1})(k^{-1}hk,\ k^{-1}\triangleright b)\Big)$$
$$=\begin{cases}(ghg^{-1},g\triangleright b), & \text{if } g=k^{-1}, a=(k^{-1}\triangleright b)b^{-1}c^{-1},\\ 0 & otherwise.\end{cases}$$

$$\Big(g,a,h,b,k,c\Big)\xrightarrow{F(X_+\otimes\mathrm{id}_1)}\Big(ghg^{-1},(g\triangleright b),\ g,\ (g\triangleright b^{-1})ab,\ k,\ c\Big)\xrightarrow{F(\mathrm{id}_1\otimes\cap)}$$
$$\Big((ghg^{-1},\ g\triangleright b)\delta(g,k^{-1})\delta((g\triangleright b^{-1})ab,c^{-1})\Big)$$
$$=\begin{cases}(ghg^{-1},g\triangleright b), & \text{if } g=k^{-1}, c^{-1}=(g\triangleright b^{-1})ab\\ 0 & otherwise.\end{cases}$$

Note: $\delta(g,k^{-1})=1\Leftrightarrow g=k^{-1}$ and,
$\delta(a,(k^{-1}\triangleright b)b^{-1}c^{-1})=1\Leftrightarrow\delta((g\triangleright b^{-1})ab,c^{-1})=1.$

- Let $(g,a,h,b)\in(G\times A)\times(G\times A)$, we have

$$[WT_9]'\Big(g,a,h,b,k,c\Big)\xrightarrow{F(\mathrm{id}_1\otimes X_+)}\Big(g,\ a,\ hkh^{-1},\ (h\triangleright c),\ h,\ (h\triangleright c^{-1})bc\Big)$$
$$\xrightarrow{F(\cap\otimes\mathrm{id}_1)}\Big(\delta(g,hk^{-1}h^{-1})\delta(a,h\triangleright c^{-1})(h,(h\triangleright c^{-1})bc)\Big)$$
$$=\begin{cases}(h,abc) & \text{if } g=hk^{-1}h^{-1}, a=h\triangleright c^{-1},\\ 0 & otherwise.\end{cases}$$

$$\Big(g,a,h,b,k,c\Big)\xrightarrow{F(X_-\otimes\mathrm{id}_1)}\Big(h,(h^{-1}\triangleright a^{-1})ab,\ h^{-1}gh,\ (h^{-1}\triangleright a),\ k,\ c\Big)$$
$$\xrightarrow{F(\mathrm{id}_1\otimes\cap)}\Big((h,(h^{-1}\triangleright a^{-1})ab)\delta(h^{-1}gh,k^{-1})\delta(h^{-1}\triangleright a,c^{-1})\Big)$$
$$=\begin{cases}(h,abc) & \text{if } k^{-1}=h^{-1}gh, c^{-1}=h^{-1}\triangleright a,\\ 0 & otherwise.\end{cases}$$

- Let $(g, a, h, b) \in (G \times A) \times (G \times A)$, we have

$$[WT_9]'' \Big(g, a, h, b, k, c\Big) \xrightarrow{F(\mathrm{id}_1 \otimes X)} \Big(g,\ a,\ k,\ c,\ h,\ b\Big)$$
$$\xrightarrow{F(\cap \otimes \mathrm{id}_1)} \Big(\delta(g, k^{-1})\delta(a, c^{-1})(h, b)\Big)$$
$$= \begin{cases} (h, b) & \text{if } g = k^{-1}, a = c^{-1}, \\ 0 & otherwise. \end{cases}$$

$$\Big(g, a, h, b, k, c\Big) \xrightarrow{F(X \otimes \mathrm{id}_1)} \Big(h, b,\ g,\ a,\ k,\ c\Big)$$
$$\xrightarrow{F(\mathrm{id}_1 \otimes \cap)} \Big((h, b)\delta(g, k^{-1})\delta(a, c^{-1})\Big)$$
$$= \begin{cases} (h, b) & \text{if } g = k^{-1}, a = c^{-1}, \\ 0 & otherwise. \end{cases}$$

- Let $(g, a) \in (G \times A)$, we have

$$[WT_{10}]\,(g, a) \xrightarrow{F(\cup \otimes \mathrm{id}_1)} \sum_{h \in G, b \in A} (h, b, h^{-1}, b^{-1}, g, a) \xrightarrow{F(\mathrm{id}_1 \otimes X_+)}$$
$$\sum_{h \in G, b \in A} (h,\ b,\ h^{-1}gh,\ (h^{-1} \triangleright a),\ h^{-1},\ (h^{-1} \triangleright a^{-1})b^{-1}a).$$

$$(g, a) \xrightarrow{F(\mathrm{id}_1 \otimes \cup)} \sum_{h \in G, b \in A} (g, a, h, b, h^{-1}, b^{-1}) \xrightarrow{F(X_- \otimes \mathrm{id}_1)}$$
$$\sum_{h \in G, b \in A} (h,\ (h^{-1} \triangleright a^{-1})ab,\ h^{-1}gh,\ (h^{-1} \triangleright a),\ h^{-1},\ b^{-1}).$$

- Let $(g,a) \in (G \times A)$, we have

$$[WT_{10}]'\,(g,a) \xrightarrow{F(\cup\otimes \mathrm{id}_1)} \sum_{h\in G, b\in A} (h,b,h^{-1},b^{-1},g,a) \xrightarrow{F(\mathrm{id}_1\otimes X_-)}$$

$$\sum_{h\in G, b\in A} (h,\, b,\, g,\, (g^{-1}\triangleright b)b^{-1}a,\, g^{-1}h^{-1}g,\, (g^{-1}\triangleright b^{-1}))\, * \,.$$

$$(g,a) \xrightarrow{F(\mathrm{id}_1\otimes\cup)} \sum_{h\in G, b\in A} (g,a,h,b,h^{-1},b^{-1}) \xrightarrow{F(X_+\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} (ghg^{-1},\, (g\triangleright b),\, g,\, (g\triangleright b^{-1})ab,\, h^{-1},\, b^{-1})\, **.$$

* and ** are equal because the $\sum$ it is over all $h \in G$ and $b \in A$.

- Let $(g,a) \in (G \times A)$, we have

$$[WT_{10}]''\,(g,a) \xrightarrow{F(\cup\otimes \mathrm{id}_1)} \sum_{h\in G, b\in A} (h,b,h^{-1},b^{-1},g,a) \xrightarrow{F(\mathrm{id}_1\otimes X)}$$

$$\sum_{h\in G, b\in A} (h,\, b,\, g,\, a,\, h^{-1},\, b^{-1}).$$

$$(g,a) \xrightarrow{F(\mathrm{id}_1\otimes\cup)} \sum_{h\in G, b\in A} (g,a,h,b,h^{-1},b^{-1}) \xrightarrow{F(X\otimes \mathrm{id}_1)}$$

$$\sum_{h\in G, b\in A} (h,\, b,\, g,\, a,\, h^{-1},\, b^{-1}).$$

- Let $(g, a) \in (G \times A)$, we have

$$[WT_{11}]\ (g,a) \xrightarrow{F(\mathrm{id}\otimes !)} \sum_{h\in G}(g,a,h,0_A) \xrightarrow{F(\cap)} \sum_{h\in G}\delta(g,h^{-1})\delta(a,0_A)$$

$$= \begin{cases} 1 & \text{if } a = 0_A, \\ 0 & otherwise. \end{cases}$$

$$(g,a) \xrightarrow{F(\text{¡})} \begin{cases} 1 & \text{if } a = 0_A, \\ 0 & otherwise. \end{cases}$$

$$(g,a) \xrightarrow{F(!\otimes \mathrm{id}_1)} \sum_{h\in G}(h,0_A,g,a) \xrightarrow{F(\cap)} \sum_{h\in G}\delta(h,g^{-1})\delta(0_A,a^{-1})$$

$$= \begin{cases} 1 & \text{if } a = 0_A, \\ 0 & otherwise. \end{cases}$$

-

$$[WT_{12}]\ 1 \xrightarrow{F(\cup)} \sum_{g\in G, a\in A}(g,a,g^{-1},a^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes \text{¡})} \sum_{g\in G}(g,0_A).$$

$$1 \xrightarrow{F(!)} \sum_{g\in G}(g,0_A).$$

$$1 \xrightarrow{F(\cup)} \sum_{g\in G, a\in A}(g,a,g^{-1},a^{-1}) \xrightarrow{F(\text{¡}\otimes \mathrm{id}_1)} \sum_{g\in G}(g^{-1},0_A).$$

- Let $(g,a,h,b) \in (G\times A)\times(G\times A)$, we have

$$[WT_{13}]\,(g,a,h,b) \xrightarrow{F(X_+)} (ghg^{-1}, (g\triangleright b),\ g,\ (g\triangleright b^{-1})ab)$$
$$\xrightarrow{F(\mathsf{j}\otimes \mathrm{id}_1)} \begin{cases}(g,a) & \text{if } g\triangleright b = 0_A,\\ 0 & otherwise.\end{cases}$$

$$(g,a,h,b) \xrightarrow{F(\mathrm{id}_1\otimes \mathsf{j})} \begin{cases}(g,a) & \text{if } b = 0_A,\\ 0 & otherwise.\end{cases}$$

- Let $(g,a,h,b) \in (G\times A)\times(G\times A)$, we have

$$[WT_{13}]'(g,a,h,b) \xrightarrow{F(X_-)} (h,\ (h^{-1}\triangleright a^{-1})ab,\ h^{-1}gh,\ (h^{-1}\triangleright a))$$
$$\xrightarrow{F(\mathrm{id}_1\otimes \mathsf{j})} \begin{cases}(h,b) & \text{if } h^{-1}\triangleright a = 0_A,\\ 0 & otherwise.\end{cases}$$

$$(g,a,h,b) \xrightarrow{F(\mathsf{j}\otimes \mathrm{id}_1)} \begin{cases}(h,b) & \text{if } a = 0_A,\\ 0 & otherwise.\end{cases}$$

- Let $(g,a,h,b) \in (G\times A)\times(G\times A)$, we have

$$[WT_{14}]\,(g,a,h,b) \xrightarrow{F(X)} (h,b,g,a) \xrightarrow{F(\mathsf{j}\otimes \mathrm{id}_1)} \begin{cases}(g,a) & \text{if } b = 0_A,\\ 0 & otherwise.\end{cases}$$

$$(g,a,h,b) \xrightarrow{F(\mathrm{id}_1\otimes \mathsf{j})} \begin{cases}(g,a) & \text{if } b = 0_A,\\ 0 & otherwise.\end{cases}$$

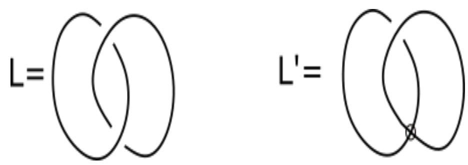
- Let $(g,a,h,b) \in (G\times A)\times(G\times A)$, we have

$$[WT_{14}]'(g,a,h,b) \xrightarrow{F(X)} (h,b,g,a) \xrightarrow{F(\mathrm{id}_1\otimes \mathsf{i})} \begin{cases}(h,b) & \text{if } a=0_A,\\ 0 & otherwise.\end{cases}$$

$$(g,a,h,b) \xrightarrow{F(\mathsf{i}\otimes \mathrm{id}_1)} \begin{cases}(h,b) & \text{if } a=0_A,\\ 0 & otherwise.\end{cases}$$

□

Example 7.4.4. *Consider the following morphisms of the category of $UWTC$.*



We want to calculate the functorial invariant from finite group.

$$\begin{aligned}F(L) =& 1 \xrightarrow{F(\cup\otimes\cup)} \sum_{g,h\in G,a,b\in A}(g,a,g^{-1},a^{-1},h,b,h^{-1},b^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes X_+\otimes \mathrm{id}_1)}\\ &\sum_{g,h\in G,a,b\in A}\Big(g,\ a,\ g^{-1}hg,\ (g^{-1}\triangleright b), g^{-1},\ (g^{-1}\triangleright b^{-1})a^{-1}b,\ h^{-1},\ b^{-1}\Big) \xrightarrow{F(\mathrm{id}_1\otimes X_+\otimes \mathrm{id}_1)}\\ &\sum_{g,h\in G,a,b\in A}\Big(g,\ a,\ (g^{-1}hgg^{-1}g^{-1}h^{-1}g),\ \Big(g^{-1}hg\triangleright((g^{-1}\triangleright b^{-1})a^{-1}b)\Big),\ g^{-1}hg,\\ &g^{-1}hg\triangleright((g^{-1}\triangleright b^{-1})a^{-1}b)^{-1}(g^{-1}\triangleright b)((g^{-1}\triangleright b^{-1})a^{-1}b), h^{-1}, b^{-1}\Big) =\\ &\sum_{g,h\in G,a,b\in A}(g,a,\ (g^{-1}hg^{-1}h^{-1}g),\ (g^{-1}hg\triangleright a^{-1})(g\triangleright b), g^{-1}hg,\\ &((g\triangleright b^{-1})(g^{-1}hg\triangleright a)a^{-1}b),\ h^{-1}, b^{-1}) \xrightarrow{F(\cap\otimes\cap)}\\ &\sum_{g,h\in G,a,b\in A}\delta\Big(g, g^{-1}hgh^{-1}g\Big)\delta\Big(a, (g^{-1}hg\triangleright a)(g\triangleright b^{-1})\Big)\\ &\delta\Big(g^{-1}hg, h\Big)\delta\Big(((g\triangleright b^{-1})(g^{-1}hg\triangleright a)a^{-1}b), b\Big)\\ &=\mid \{(g,a,h,b)\mid gh=hg, a=(g^{-1}hg\triangleright a)(g\triangleright b^{-1}), b=(g\triangleright b^{-1})(g^{-1}hg\triangleright a)a^{-1}b)\}\mid .\end{aligned}$$

$$
\begin{aligned}
F(L') =& 1 \xrightarrow{F(\cup\otimes\cup)} \sum_{g,h\in G,a,b\in A} (g,a,g^{-1},a^{-1},h,b,h^{-1},b^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes X\otimes \mathrm{id}_1)} \\
& \sum_{g,h\in G,a,b\in A} (g,a,h,b,g^{-1},a^{-1},h^{-1},b^{-1}) \xrightarrow{F(\mathrm{id}_1\otimes X_+\otimes \mathrm{id}_1)} \\
& \sum_{g,h\in G,a,b\in A} (g,a,hg^{-1}h^{-1},(h\triangleright a^{-1}),h,(h\triangleright a)ba^{-1},h^{-1},b^{-1}) \xrightarrow{F(\cap\otimes\cap)} \\
& \sum_{g,h\in G,a,b\in A} \delta(g,hgh^{-1})\delta(a,h\triangleright a)\delta(h,h)\delta((h\triangleright a)ba^{-1},b) \\
& =\mid \{(g,a,h,b) \mid gh=hg, a=h\triangleright a\} \mid
\end{aligned}
$$

Therefore

$$F(L) \neq F(L'),$$

then this invariants separate these two knots.

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