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Assembly maps for finite asymptotic dimension via coarse geometry

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Abstract

Coarse geometry is the study of the large-scale structure of geometric spaces, in contrast to the better known field of topology, which is the study of small-scale structures. Coarse geometry has been around for about twenty-five years and has significant applications in geometric topology and in the study of curvature.

The Novikov conjecture concerns homotopy invariance of higher signatures and has generated a huge quantity of research in fields ranging from algebraic K-theory to geometric functional analysis. One way of obtaining a positive solution to this conjecture is to prove the coarse Baum–Connes conjecture using coarse geometric methods and then to apply the notion of descent.

The coarse Baum–Connes conjecture is true for a wide variety of spaces, and in particular, spaces of finite asymptotic dimension. Mitchener formulated a version of the coarse Baum–Connes conjecture for a general class of coarse invariants, not just the K-theory of the Roe C^*-algebra. This generalisation has applications to geometry and algebra beyond those of the original conjecture and these generalisations are explored in this thesis.

The underlying idea is that Wright's proof of the coarse Baum–Connes conjecture for finite asymptotic dimension does not rely on the precise definition of the Roe C^*-algebra, and depends only on the coarse geometry.

In this thesis, we construct a generalised coarse assembly map from a coarsely excisive functor. To show that this map is an isomorphism for spaces of finite asymptotic dimension, we begin by constructing a sequence of coarsening spaces which approximate the original space of finite asymptotic dimension. We then relate each side of the assembly map to the direct limit of homology groups of these coarsening spaces, equipped with the C_0 coarse structure for the domain and the bounded coarse structure for the codomain. It is then shown that these direct limits agree under certain conditions, allowing us to conclude the main result. This result is then applied to C^*-category and algebraic K-theory, as well as equivariant versions of these theories.

Contents

Introduction

The concern of traditional topology is with the small-scale structure of spaces. The significance of a metric for a topologist lies in the collection of open sets that it generates. But a great deal of information is lost in the passage from the metric to its topology and only the very small-scale structure of the metric is reflected in the topology. Coarse geometry (introduced in [Roe96]) is the study of spaces from a large-scale point of view and can be seen as the dual to the idea of topology. This approach to geometry is fruitful as the coarse properties of a metric space are of interest in various contexts. The idea of coarse geometry is to view a space through successively blurred lenses and studying the geometry that remains at the end of this coarsening process. Large-scale properties of a space, such as boundedness, do not depend on the small-scale features of the space.

The coarse geometry of a space can be visualised by viewing the space from a further and further distance. The fundamental example of this are the spaces $\mathbb{R}$ and $\mathbb{Z}$. If you zoom further and further away from $\mathbb{Z}$, all of the points look as if they are getting closer and closer together until eventually in the limit blur to look like $\mathbb{R}$. These are considered to be equivalent from a coarse point of view as they "behave the same way at infinity" and share the same similarities when viewed from a great distance. Similarly, every space which is finite in size is equivalent to a single point as far as coarse geometry is concerned. All that matters is the large-scale geometry of spaces that are infinitely large.

In topology the importance of the metric is not in its numerical values but in the open sets that it defines, so we can abstract from metric spaces to topological spaces by defining the concept of continuity using open sets rather than using a metric. The passage from metric spaces to coarse spaces is a large-scale analogue of this process but instead we focus on abstraction with respect to the large-scale structure and form a collection of controlled sets satisfying certain large-scale conditions. Any topological property of a space can be defined entirely in terms of open sets, and analogously any coarse property can be defined entirely in terms of controlled sets.

For any metric space there is a naturally associated coarse structure called the metric coarse structure. In [Wri05], Wright introduces a refinement of this structure called the C_0 coarse structure, which is more delicate at infinity but is easier to study. For the metric coarse structure a coarse equivalence is a coarse map which is invertible up to some bounded error, but for the C_0 coarse structure a coarse equivalence is required to be invertible up to an error which tends to zero at infinity.

The coarse geometry of a space X is often studied via its Roe C^*-algebra $C^*(X)$ as introduced by Roe in [Roe96] to study index theory on open manifolds. There is a

C^*-algebra $D^*(X)$ containing $C^*(X)$ as an ideal and a short exact sequence

$$0 \longrightarrow C^*(X) \longrightarrow D^*(X) \longrightarrow D^*(X)/C^*(X) \longrightarrow 0$$

By Bott periodicity and exactness of K-theory there is a six-term exact sequence of K-theory groups

$$\begin{array}{ccccc} K_0(C^*(X)) & \longrightarrow & K_0(D^*(X)) & \longrightarrow & K_0(D^*(X)/C^*(X)) \\ {\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle A} \\ K_1(D^*(X)/C^*(X)) & \longleftarrow & K_1(D^*(X)) & \longleftarrow & K_1(C^*(X)) \end{array}$$

Kasparov showed in [Kas75] that $K_*(X)$ (the locally finite K-homology groups of X) and $K_{*+1}(D^*(X)/C^*(X))$ are isomorphic. The boundary maps

$$A : K_*(X) \to K_*(C^*(X))$$

are called the assembly maps and connect the locally finite K-homology of X to the K-theory of the Roe C^*-algebra of X. We can ask for which metric spaces the assembly map is an isomorphism.

The functor $K_*(C^*(X))$ is a coarse geometric object in the sense that it is functorial for coarse maps and invariant under coarse homotopy, and the functor $K_*(X)$ is a topological object in the sense that it is functorial for proper maps and invariant under proper homotopy. For these reasons, we can not expect this assembly map to be an isomorphism for cases other than for spaces whose large-scale geometry and small-scale geometry are the same (such spaces are known as uniformly contractible spaces). The assembly map A can be modified via a process of coarsening of the left hand side to obtain coarse K-homology, and another assembly map

$$A_\infty : K_*^{\text{coarse}}(X) \to K_*(C^*(X))$$

It is now a much more reasonable question to ask under what conditions the assembly map A_∞ is an isomorphism. This question is known as the coarse Baum–Connes conjecture (see [Roe93]). There are multiple reasons for wanting to prove such a conjecture. Firstly, the K-theory of the Roe C^*-algebra contains coarse geometric information but is very difficult to compute and analyse. The locally finite K-homology of a space is much easier to compute, and the coarse K-homology is also significantly easier to compute than the K-theory of the Roe C^*-algebra. Therefore an isomorphism serves as an explanation of the right hand side of A_∞. The second (and perhaps most important) reason is that the coarse Baum–Connes conjecture has significant applications in areas of geometric topology, such as surgery theory, allowing us to make many geometric deductions.

The notion of descent in coarse geometry (see Chapter 8 of [Roe96]) asserts that if G is a group which is classified by a finite complex such that the coarse Baum–Connes assembly map is an isomorphism for the underlying metric space $|G|$ (the group G equipped with the word length metric) then the Baum–Connes assembly map

$$K_*^G(\underline{E}G) \to K_*(C_r^*(G))$$

is injective. The injectivity of an equivariant assembly map usually gives implications in topology and the surjectivity gives implications in analysis. The coarse Baum–Connes conjecture also directly implies the analytic Novikov conjecture. Explicitly, if G is a group which is classified by a finite complex such that the coarse Baum–Connes assembly map is an isomorphism for the metric space $|G|$ then the map

$$K_*(BG) \to K_*(C_r^*(G))$$

is injective. In the case where G is a torsion free group, injectivity of the Baum–Connes conjecture is equivalent to the analytic Novikov conjecture.

It is well known that the analytic Novikov conjecture implies the topological Novikov conjecture. The topological Novikov conjecture concerns homotopy invariance of higher signatures and has stimulated a vast amount of research in many fields ranging from algebraic K-theory to geometric functional analysis. The validity of the topological Novikov conjecture has been established, by a variety of techniques, for many groups.

In [Yu98], Yu proves that the coarse Baum–Connes conjecture holds for proper metric spaces of finite asymptotic dimension using methods that are analytic in flavour and therefore uses coarse geometric techniques to provide a proof of the topological Novikov conjecture for groups with finite asymptotic dimension admitting a finite classifying space. In [Wri05], Wright shows that the coarse Baum–Connes conjecture can be interpreted as the relation between the K-theory for the C_0 and the metric coarse structures and gives a new proof of Yu's result using a more geometric method.

The concept of an assembly map has been generalised beyond the K-theory of the Roe C^*-algebra and there are many other important assembly maps arising from the fields of algebraic K-theory, algebraic L-theory and topological K-theory. For example, it is well known that the topological Novikov conjecture is equivalent to rational injectivity of the assembly map

$$H_*(BG; \mathbf{L}(\mathbb{Z})) \to L_*(\mathbb{Z}G)$$

in algebraic L-theory. In analogy to the coarse Baum–Connes conjecture we can ask under which assumptions are these assembly maps isomorphisms. In [Bar03], Bartels gives a proof that the assembly map in algebraic K-theory is split injective for groups of finite asymptotic dimension admitting a finite classifying space.

In [Mit10], Mitchener introduces the concept of a coarsely excisive functor $\mathbf{E}$ from the coarse category to the category of spectra (such a functor has the properties so that its homotopy groups form a coarse homology theory) and defines a coarse assembly map

$$\Omega\mathbf{E}(\mathcal{O}X) \to \mathbf{E}(X)$$

for any coarsely excisive functor $\mathbf{E}$. The continuously controlled coarse structure on the open cone $\mathcal{O}X$ is the unique coarse structure such that if $f : X \to Y$ is continuous then the induced map $f_* : \mathcal{O}X \to \mathcal{O}Y$ is coarse, allowing us to form a pathway between topology and coarse geometry. The functor $X \mapsto \mathbf{E}(\mathcal{O}X)$ is properly excisive functor. By taking homotopy groups, we can form an assembly map

$$h_*(\mathcal{O}X) \to h_{*-1}(X)$$

where the left hand side is a locally finite homology theory and the right hand side is a coarse homology theory.

By the same process of coarsening as with going between locally finite K-homology and coarse K-homology, we have a coarsened assembly map

$$k_*^{\text{coarse}}(X) \to h_{*-1}(X)$$

where $k_*(X) = h_*(\mathcal{O}X)$.

As with the case of the coarse Baum–Connes conjecture there is a notion of descent for any coarse assembly map, implying results about an equivariant version of the coarse assembly map such as a generalised version of the analytic Novikov conjecture.

In this thesis, we tackle the assembly maps in algebraic and topological K-theory in a universal way by giving a proof that the coarsened assembly map is an isomorphism for spaces of finite asymptotic dimension for any coarsely excisive functor $\mathbf{E}$. To do this, we prove that this result holds for $\mathbb{N}_0$ and then build on this by using the fact that all infinitely uniformly discrete proper metric spaces are C_0 coarsely equivalent to $\mathbb{N}_0$. We prove analogues of results from [Wri05] using geometric methods. Specifically, using the coarsening space, we show that the left and right hand sides of the coarsened assembly map are the direct limits of the coarse homology of partial coarsening spaces equipped with the C_0 coarse structure and fusion coarse structure respectively. A category theoretic argument is given to allow us to determine that these direct limits are isomorphic. We then show that the functors representing algebraic and C^*-category K-theory are coarsely excisive, which shows that the Baum–Connes assembly map and the Farrell–Jones assembly map fit into this generalised theory.

Thesis outline

In Chapter 1 we outline the basic material used in this thesis, looking in particular at the notions of coarse spaces and coarse maps needed to describe large-scale equivalence, firstly for metric spaces and then more generally for coarse structures. We introduce two coarse structures of interest to us; the C_0 coarse structure and the hybrid coarse structure, establishing some basic results about the relationship between them.

Chapter 2 details the theory of a coarse invariant playing a crucial role in this thesis known as asymptotic dimension, an analogue of Lebesgue covering dimension in topology. We define asymptotic dimension in three different ways and show them to be equivalent. We compute some examples of asymptotic dimension and discuss finitely generated groups coming from geometric group theory, looking in particular at hyperbolic groups, a large class of groups whose asymptotic dimension is finite.

In Chapter 3 we explain the ideas of coarse homotopy and coarse homology. These require the concept of a generalised ray, the space $[0, \infty)$ equipped with a coarse structure with desirable properties. We state a theorem relating the homotopy groups of a space with the coarse homotopy groups of its open cone. We present an axiomatic definition of a locally finite homology theory and we show that it is possible to use a process of coarsening to define a coarse homology theory for every locally finite homology theory. This chapter also features some background material on the coarse Baum-Connes conjecture.

The focus of Chapter 4 is on almost flasque spaces, a generalisation of flasque spaces. We give many examples of almost flasque spaces and show that being almost flasque is a coarse invariant. The idea of the number of ends of a space is discussed and we present a new result at the end of this chapter by showing that an almost flasque geodesic metric space must have one end.

Chapter 5 is a survey of the theory of assembly as developed by Weiss and Williams, Davis and Lück, and Mitchener. To do this, we introduce the idea of spectra and define and give background theory on properly excisive and coarsely excisive functors. Both the assembly map and the coarse assembly map are defined and isomorphism conjecture of these maps are discussed. We define the open cone of a topological space and show that the continuously controlled coarse structure is the coarse structure such that a proper continuous map between spaces induces a coarse map between their open cones, allowing us to move from a coarsely excisive functor to a properly excisive functor by taking open cones. This theory is then developed in an equivariant setting and the notion of descent is applied.

Chapter 6 contains a proof that the coarse assembly map is an isomorphism for spaces of finite asymptotic dimension. The proof follows the same structure as Wright's geometric proof of the coarse Baum–Connes conjecture for finite asymptotic dimension. Namely; we firstly prove the conjecture for uniformly discrete metric spaces with the C_0 coarse structure, by using an argument of building the C_0 coarse structure on $\mathbb{N}$ via an increasing limit of smaller coarse structures. The coarsening space is introduced and it is shown that it is almost flasque when equipped with both the C_0 and the hybrid coarse structures. We also show that both sides of the assembly map are direct limits of the coarse homology of the partial coarsening spaces with each of these coarse structures. Using a technical category theoretic result, we show that the partial coarsening spaces with the C_0 and fusion coarse structures eventually agree in the case of discrete spaces. This allows us to conclude with our main result, by using an induction argument on the decomposition of the coarsening space.

The aim of Chapter 7 is to apply the main result in the previous chapter to functors that are of interest in algebraic K-theory and the K-theory of C^*-categories. An introduction to Waldhausen K-theory is given, and in particular, we explicitly construct the spectrum $\mathbf{K}\mathcal{A}$ of a Waldhausen category $\mathcal{A}$. We show that any additive category can be considered as a Waldhausen category and we define an additive category $\mathcal{A}(X)$ capturing coarse properties of X using ideas from controlled topology. Similarly we give some background material on C^*-categories and their K-theory, and define an additive C^*-category $\mathcal{A}^*(X)$. The functors $X \mapsto \mathbf{K}\mathcal{A}(X)$ and $X \mapsto \mathbf{K}\mathcal{A}^*(X)$ are shown to be coarsely excisive. We then discuss the implications of this using descent and end by showing how the original coarse Baum–Connes and Baum–Connes conjectures for finite asymptotic dimension drop out as a corollary of the main result.

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Chapter 1

Coarse geometry

Coarse geometry is the study of large-scale properties of spaces, and is considered in many ways as the "opposite" to topology, where the idea is to study the small-scale properties of spaces. To see this, note that the metric $d'(x,x') = \min\{d(x,x'),1\}$ defines the same topology as d but that d' erases all information about distances greater than 1, and so loses the large-scale information of the space and focuses on the small-scale properties. Coarse geometry concentrates on the dual procedure, the metric d' defined by $d'(x,x') = \lceil d(x,x') \rceil$ defines the same coarse structure as d but d' loses the small-scale information of the space. The metric d' does not worry about the fine details of the space.

Coarse geometry has a geometrical interpretation; the coarse geometry of a space can be visualised by viewing the space from a further and further distance. The fundamental example of this are the spaces $\mathbb{R}$ and $\mathbb{Z}$. If you zoom further and further away from $\mathbb{Z}$, all of the points look as if they are getting closer and closer together until eventually in the limit blur to look like $\mathbb{R}$. These are considered to be equivalent from a coarse point of view as they "behave the same way at infinity" and share the same similarity when viewed from a great distance. This approach is fruitful as the coarse geometry of a metric space usually determines its relevant geometric properties.

1.1 Coarse geometry for metric spaces

Definition 1.1 (Coarse map). A map $f\colon X \to Y$ between metric spaces is said to be:

- *coarsely proper* if the inverse image $f^{-1}(B) \subseteq X$ is bounded whenever $B \subseteq Y$ is bounded;
- *bornologous* if for each $R > 0$ there exists an $S_R > 0$ such that $d(x,x') < R$ implies that $d(f(x), f(x')) < S_R$;

- *coarse* if it is both coarsely proper and bornologous.

Remark 1.2. If $f\colon X \to Y$ is a coarse map and $A \subseteq X$ is unbounded then $f(A) \subseteq Y$ is also unbounded (since if $f(A)$ was bounded, $A \subseteq f^{-1}(f(A))$ would also be bounded). In other words, for all sequences (a_n) in A with $a_n \to \infty$, it follows that $f(a_n) \to \infty$ also.

Examples 1.3. The following maps are coarse maps:

- $j\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by $j(x) = \sqrt{x}$;
- $k\colon \mathbb{R} \to \mathbb{Z}$ defined by $k(x) = \lfloor x \rfloor$, the map in which each $x \in \mathbb{R}$ is mapped to the greatest integer less than or equal to x;
- $m\colon \mathbb{R} \to \mathbb{R}_+$ defined by $m(x) = |x|$;
- $n\colon \mathbb{R} \to \mathbb{R}$ defined by $n(x) = ax + b$ for any $a \neq 0$.

Non-Examples 1.4.

- The map $g\colon \mathbb{N} \to \mathbb{N}$ defined by $g(x) = n$ for any natural number n is not a coarse map, since the inverse image of the bounded set $\{n\}$ is unbounded.
- The map $h\colon \mathbb{N} \to \mathbb{N}$ defined by $h(x) = x^2$ is not a coarse map, since we have

$$d(x, x+1) < 2$$

for each x, but

$$d(h(x), h(x+1)) = 2x + 1,$$

which cannot be bounded for all x.

Definition 1.5 (Close maps). Let X and Y be metric spaces. Two coarse maps $f, g\colon X \to Y$ are said to be *close* if there exists a $C > 0$ such that

$$d(f(x), g(x)) \leq C$$

for all $x \in X$.

Definition 1.6 (Coarsely equivalent). Let X and Y be metric spaces, and let $f\colon X \to Y$ be a coarse map. The spaces X and Y are said to be *coarsely equivalent* (and f is said to be a *coarse equivalence*) if there exists a coarse map $g\colon Y \to X$ such that fg is close to id_Y and gf is close to id_X.

Remark 1.7. The composition of two coarse maps is again a coarse map, and metric spaces being coarsely equivalent is an equivalence relation.

Examples 1.8.

- The spaces $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent. We can see this geometrically by drawing $\mathbb{Z}$ on a number line and then zooming back further and further to see that in the limit the points of $\mathbb{Z}$ have blurred together and now look like $\mathbb{R}$. To satisfy the definition, define $f\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ by the inclusion, and $g\colon \mathbb{R} \to \mathbb{Z}$ by $g(x) = \lfloor x \rfloor$.

- If B is a bounded set then B is coarsely equivalent to a point. We can see this geometrically by drawing B, zooming back further and further away, and in the limit it shall look like a point. The coarse maps satisfying the definition are the inclusion $\{\mathrm{pt}\} \hookrightarrow B$ and constant map $B \to \{\mathrm{pt}\}$.

The following proposition shows that for coarsely equivalent metric spaces, the coarsely proper condition comes for free.

Proposition 1.9. *If X and Y are metric spaces then X and Y are coarsely equivalent if and only if there exist bornologous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that fg is close to id_Y and gf is close to id_X.*

Proof. The 'if' direction is clear since all coarse maps are bornologous maps. If $B \subseteq Y$ is bounded and $x, x' \in f^{-1}(B)$ then there exists a constant $D > 0$ such that $d(f(x), f(x')) \leq D$ for all $x, x' \in f^{-1}(B)$. As g is bornologous there exists a constant $D' > 0$ such that $d(gf(x), gf(x')) \leq D'$. Now it follows that

$$\begin{aligned} d(x, x') &\leq d(x, gf(x)) + d(gf(x), gf(x')) + d(gf(x'), x') \\ &\leq C + D' + C, \end{aligned}$$

as gf is close to the identity. It follows that $f^{-1}(B)$ is bounded. The case for g is identical so it follows that X and Y are coarsely equivalent. □

1.2 Coarse structures

In topology the importance of the metric is not in its numerical values but in the open sets that it defines, so we can abstract from metric spaces to topological spaces by defining continuity by using open sets rather than using a metric. The passage from metric spaces to coarse spaces is a coarse analogue of this process in topology, but instead we focus on abstraction with respect to the large-scale structure, with the behaviour being modelled on the theory for metric spaces.

Definition 1.10 (Coarse structure). A *coarse structure* $\mathcal{E}$ on a set X is a collection of subsets of $X \times X$ which satisfy the following axioms:

- if $M \in \mathcal{E}$ and $N \in \mathcal{E}$ then $M \cup N \in \mathcal{E}$;
- if $M \in \mathcal{E}$ and $N \subseteq M$ then $N \in \mathcal{E}$;
- if $M \in \mathcal{E}$ then $M^T \in \mathcal{E}$, where

$$M^T = \{(x', x)\colon (x, x') \in M\}$$

  is the *transpose* of M;

- if $M \in \mathcal{E}$ and $N \in \mathcal{E}$ then $M \circ N \in \mathcal{E}$, where

$$M \circ N = \{(x, x'') \in X \times X : (x, x') \in M, (x', x'') \in N \text{ for some } x' \in X\}$$

  is the *product* of M and N.

Elements of $\mathcal{E}$ are called *controlled sets.*

Definition 1.11 (Unital weakly connected coarse structure). A coarse structure $\mathcal{E}$ on X is said to be *unital* if $\Delta_X \in \mathcal{E}$, where $\Delta_X = \{(x, x) : x \in X\}$ is the *diagonal* of $X \times X$, and *weakly connected* if $\{(x, x')\} \in \mathcal{E}$ for all $x, x' \in X$.

All coarse structures mentioned in this thesis will be assumed to be unital and weakly connected unless stated otherwise. A coarse structure which is not weakly connected is called a *disconnected* coarse structure.

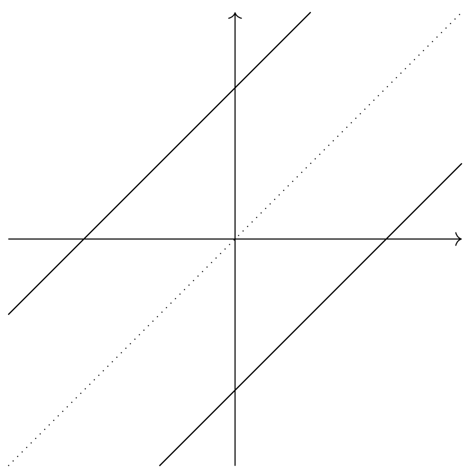
Definition 1.12 (Coarse space). A *coarse space* $(X, \mathcal{E})$ is a set X equipped with a coarse structure $\mathcal{E}$.

The coarse space $(X, \mathcal{E})$ will be abbreviated to X when it is clear from the context what the coarse structure $\mathcal{E}$ is.

Examples 1.13.
- The *metric coarse structure* $\mathcal{E}_d$ on a metric space X is the collection of all subsets $M \subseteq X \times X$ such that $\sup\{d(x, x') : (x, x') \in M\}$ is finite. That is, each controlled set is a subset of an *R-neighbourhood of the diagonal*

$$N_R = \{(x, x') \in X \times X : d(x, x') \leq R\}$$

  for some R.



An example of a set for the metric coarse structure on $\mathbb{R}$.

- The *discrete coarse structure* $\mathcal{E}_{\text{discrete}}$ on X is the collection of all subsets $M \subseteq X \times X$ such that $x = x'$ for all but finitely many $(x, x') \in M$. In other words, the controlled sets are those which contain only finitely many points off the diagonal.

- The *minimal coarse structure* (or *trivial coarse structure*) $\mathcal{E}_{\min}$ on X is the collection of all subsets of the diagonal on X.
- The *maximal coarse structure* $\mathcal{E}_{\max}$ on X is the collection of all subsets of $X \times X$.

Two other important examples of a coarse structure, the C_0 coarse structure and the hybrid coarse structure, are defined in Section 1.2.1 and 1.2.2 respectively. The continuously controlled coarse structure will be introduced in Chapter 3.

Definition 1.14. If $\mathcal{E}$ and $\mathcal{E}'$ are coarse structures with $\mathcal{E} \subseteq \mathcal{E}'$ then it is said that $\mathcal{E}'$ *coarsens* $\mathcal{E}$.

Examples 1.15. The metric coarse structure $\mathcal{E}_d$ coarsens the discrete coarse structure $\mathcal{E}_{\text{discrete}}$. The coarse structure $\mathcal{E}_{\min}$ is least coarse of all coarse structure, and $\mathcal{E}_{\max}$ is the most coarse of all coarse structures.

Remark 1.16. If $(X, \mathcal{E})$ is a coarse space and $Y \subseteq X$ then $(Y, \mathcal{E}|_Y)$ is also a coarse space, where $\mathcal{E}|_Y = \{M \subseteq Y \times Y : M \in \mathcal{E}\}$.

Remark 1.17. If $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ are coarse spaces, then $(X \times Y, \mathcal{E}_{X\times Y})$ is also a coarse space, where $\mathcal{E}_{X\times Y}$ is the coarse structure generated by the collection $\{M_X \times M_Y : M_X \in \mathcal{E}_X, M_Y \in \mathcal{E}_Y\}$, known as the *product coarse structure.*

Let $(X, \mathcal{E})$ be a coarse space, $M \in \mathcal{E}$ and $S \subseteq X$. Define

$$M(S) = \{x \in X : (x, x') \in M \text{ for some } x' \in S\}.$$

If S has only one element, say $S = \{x\}$ then we write M_x or $M(x)$ instead of $M(\{x\})$.

It is easy to show that for the metric coarse structure, sets of the form $M(x)$ are subsets of open balls, and so the following definition of a bounded set is a reasonable one.

Definition 1.18 (Bounded set). If X is a coarse space then a subset $B \subseteq X$ is *bounded* if it is a subset of $M(x)$ for some controlled set M and for some $x \in X$.

It can be easily shown that a subset of a bounded set is bounded and (for a connected coarse structure) the union of two bounded sets is also bounded, and that in the metric coarse structure case the bounded sets are precisely those which are metrically bounded.

Proposition 1.19 (Proposition 2.16 of [Roe03]). *If X is a coarse space and $B \subseteq X$ then the following conditions are equivalent:*

- *B is bounded;*
- *$B \times B$ is controlled;*
- *$B \times \{p\}$ is controlled for some $p \in X$;*
- *the inclusion map $B \hookrightarrow X$ is close to a constant map.* □

Examples 1.20. It is easy to check the following:

- the bounded sets for $(X, \mathcal{E}_d)$ are the metrically bounded subsets of X;
- the bounded sets for $(X, \mathcal{E}_{\text{discrete}})$ are the finite subsets of X;
- for $(X, \mathcal{E}_{\text{min}})$, the only sets which are bounded are the singletons of X;
- for $(X, \mathcal{E}_{\text{max}})$, every subset of X is bounded.

Definition 1.21 (Coarse map). A map $f\colon X \to Y$ between coarse spaces is said to be:

- *coarsely proper* if $f^{-1}(B) \subseteq X$ is bounded whenever $B \subseteq Y$ is bounded;
- *controlled* if $(f \times f)(M)$ is controlled whenever M is controlled;
- *coarse* if it is both coarsely proper and controlled.

Remark 1.22. If X is a coarse space and $B \subseteq X$ is bounded then $B \times B$ is controlled by Proposition 1.19. If $f\colon X \to Y$ is a controlled map then the set $(f \times f)(B \times B) = f(B) \times f(B)$ is controlled, so again by Proposition 1.19, $f(B)$ is bounded. Thus a controlled map sends bounded sets to bounded sets.

Example 1.23. The identity map $\mathrm{id}\colon (\mathbb{R}, \mathcal{E}_{\text{min}}) \to (\mathbb{R}, \mathcal{E}_d)$ is a controlled map (but is not coarse) as $\mathcal{E}_{\text{min}}$ coarsens $\mathcal{E}_d$, but the identity map $\mathrm{id}'\colon (\mathbb{R}, \mathcal{E}_d) \to (\mathbb{R}, \mathcal{E}_{\text{min}})$ is not a controlled map since $(\mathrm{id}' \times \mathrm{id}')(M) \notin \mathcal{E}_{\text{min}}$ for most $M \in \mathcal{E}_d$.

Example 1.24. It can be easily seen that a map $h\colon (X, \mathcal{E}_{\text{discrete}}) \to (Y, \mathcal{E}_{\text{discrete}})$ is coarse if and only if it is coarsely proper. In other words, a map $h\colon (X, \mathcal{E}_{\text{discrete}}) \to (Y, \mathcal{E}_{\text{discrete}})$ is always controlled.

Non-Example 1.25. If $(X, \mathcal{E})$ is a coarse space where X is unbounded then the projection map $\pi\colon X \times X \to X$ (where $X \times X$ is equipped with the product coarse structure) defined by $\pi(x, x') = x$ is not a coarse map as for each bounded $B \subseteq X$, the inverse image $\pi^{-1}(B) = B \times X$ is unbounded.

Definition 1.26 (Close maps). Let X be a coarse space, and let S be a set. The maps $f, g\colon S \to X$ are *close* if the set

$$\{(f(s), g(s))\colon s \in S\}$$

is a controlled set.

Proposition 1.27. *If $f\colon X \to Y$ is a coarse map and g is close to f then g is also a coarse map.*

Proof. The assumption of closeness means that the set $C = \{(f(x), g(x))\colon x \in X\}$ is controlled. The set $(f \times f)(M)$ is controlled whenever M is controlled as f is a coarse map. It can be easily checked that $(g \times g)(M) \subseteq C^T \circ (f \times f)(M) \circ C$, and also that $g^{-1}(B) \subseteq f^{-1}(C(B))$ and so it follows that g is coarse.

□

Definition 1.28 (Coarse equivalence). Let X and Y be coarse spaces, and let $f\colon X \to Y$ be a coarse map. The spaces X and Y are said to be *coarsely equivalent* (and f is said to be a *coarse equivalence*) if there exists a map $g\colon Y \to X$ such that fg is close to id_Y and gf is close to id_X.

Example 1.29. Let X be a coarse space, $B \subseteq X$ be bounded and $\{\mathrm{pt}\} \subseteq B$. Define a map $f\colon B \to \{\mathrm{pt}\}$ and a map $g\colon \{\mathrm{pt}\} \hookrightarrow B$ by the inclusion map. The set $\{((f \circ g)(\mathrm{pt}), \mathrm{pt})\} = \{(\mathrm{pt}, \mathrm{pt})\}$ is controlled and the set $\{((g \circ f)(b), b)\colon b \in B\} = \{(\mathrm{pt}, b)\colon b \in B\}$ is controlled by Proposition 1.19. It follows that for any coarse structure, any bounded set is coarsely equivalent to a point.

Example 1.30. Let $(X, \mathcal{E}_{\mathrm{min}})$ and $(Y, \mathcal{E}_{\mathrm{min}})$ be coarse spaces. It is easy to check that X and Y are coarsely equivalent if and only if there is a bijection $f\colon X \to Y$. For example, $\mathbb{R}$ and $\mathbb{Z}$ are not coarsely equivalent for this coarse structure. The spaces $\mathbb{R}$ and $(-\pi/2, \pi/2)$ are coarsely equivalent because there exists a bijection given by $x \mapsto \arctan x$.

Sometimes it is useful to consider a slightly different type of map between coarse spaces, known as a coarse embedding.

Definition 1.31 (Coarse embedding). A map $f\colon X \to Y$ between coarse spaces is said to be a *coarse embedding* (or *rough map*) if it is controlled and if $(f \times f)^{-1}(M)$ is controlled whenever M is controlled.

It can be shown that every coarse equivalence is a coarse embedding, and that any surjective coarse embedding is a coarse equivalence. For a proof, see Proposition 1.4.4 of [Moh13]. It is also easy to see that any coarse embedding is also a coarse map.

It is useful to be able to define a coarse structure on a disjoint union of coarse spaces.

Definition 1.32 (Disjoint union I). If $\{(X_i, \mathcal{E}_i)\colon i \in I\}$ is a collection of coarse spaces then the *disjoint union* $(\bigsqcup_{i \in I} X_i, \mathcal{E})$ can be considered a coarse space when equipped with the coarse structure $\mathcal{E}$ defined by $\mathcal{E} = \{\bigsqcup_{i \in I} M_i : M_i \in \mathcal{E}_i\}$.

Definition 1.33 (Disjoint union II). If $\{(X_i, \mathcal{E}_i)\colon i \in I\}$ is a collection of coarse spaces and $\mathcal{B}_i$ is the collection of bounded subsets of $(X_i, \mathcal{E}_i)$ then the *disjoint union* $(\bigsqcup_{i \in I} X_i, \mathcal{E})$ can be considered a coarse space when equipped with the coarse structure $\mathcal{E}$ defined by $\mathcal{E} = \bigcup_{i \in I} \mathcal{E}_i \cup \bigcup_{i, i' \in I} \{B \times B'\colon B \in \mathcal{B}_i, B' \in \mathcal{B}_{i'}\}$.

The first definition is simpler than the second definition, but has the disadvantage that the disjoint union of spaces X_i is not a weakly connected coarse structure. The second definition fixes this issue. The first one will be denoted by $\bigsqcup_\infty X_i$ and the second one simply by $\bigsqcup X_i$.

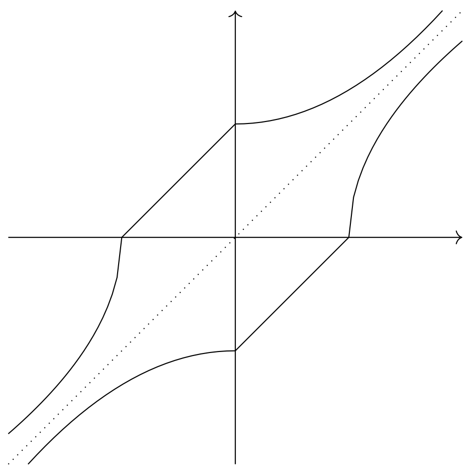
1.2.1 The C_0 coarse structure

The C_0 coarse structure was first introduced in [Wri02]. It is a generalisation of the discrete coarse structure and is a refinement of the metric coarse structure in the sense that the metric coarse structure coarsens the C_0 coarse structure. The C_0 coarse structure provides easier methods of computing certain coarse invariants, and hence makes it easier to study the large-scale properties of the space.

Definition 1.34. Let X be a metric space. A sequence $(x_n) \in X$ *tends to infinity*, written $(x_n) \to \infty$, if for every metrically bounded set $B \subseteq X$ there exists an N such that $x_n \notin B$ if $n \geq N$.

Definition 1.35 (C_0 coarse structure). The C_0 *coarse structure* $\mathcal{E}_0$ on a metric space X is the collection of all subsets $M \subseteq X \times X$ (equipped with the product metric) such that if whenever we have a sequence $(x_n, x'_n) \in M$ with $(x_n, x'_n) \to \infty$ then we also have $d(x_n, x'_n) \to 0$.

Elements of $\mathcal{E}_0$ are called C_0 *controlled sets*. The coarse space $(X, \mathcal{E}_0)$ will be abbreviated to X_0.



An example of a set for the C_0 coarse structure on $\mathbb{R}$.

The picture above shows that the ends at infinity pinch down to zero. It is clear from the picture that each C_0 controlled set is also controlled for the metric coarse structure. In other words, the identity map $X_0 \to X$ is a coarse map.

The following result is well known by coarse geometry experts but the proof cannot be found in the current literature.

Proposition 1.36. *Let X be a metric space. A set $M \subseteq X \times X$ is C_0 controlled if and only if for all $\epsilon > 0$, we can write $M = B \cup A_\epsilon$ where $B \subseteq X \times X$ is metrically bounded and $d(x, x') < \epsilon$ for all $(x, x') \in A_\epsilon$.*

Proof. Suppose M is C_0 controlled, and choose $\epsilon > 0$. Define the sets $B = \{(x, x') \in M : d(x, x') \geq \epsilon\}$ and $A_\epsilon = \{(x, x') \in M : d(x, x') < \epsilon\}$. Suppose B is not metrically bounded and choose a sequence B_n of bounded increasing subsets of B such that $B = \bigcup_{n=1}^{\infty} B_n$. If $C \subseteq B$ is bounded then $C \subseteq B_m$ for some m. Define a sequence $(x_n, x'_n) \in B$ where $(x_i, x'_i) \in B_i$ for each i. For $N \geq m$, $(x_N, x'_N) \notin C$, and hence $(x_n, x'_n) \to \infty$. Thus as B is a C_0 controlled set, $d(x_n, x'_n) \to 0$, which is a contradiction as $(x_n, x'_n) \in B$.

For the reverse argument, let (x_n, x'_n) be a sequence in M such that $(x_n, x'_n) \to \infty$. Let $\epsilon > 0$, and write $M = B \cup A_\epsilon$ where B is metrically bounded and $d(x, x') < \epsilon$ for all $(x, x') \in A_\epsilon$. Since B is metrically bounded there exists an N such that $(x_n, x'_n) \notin B$ for $n \geq N$. Therefore $(x_n, x'_n) \in A_\epsilon$ for $n \geq N$, and so $d(x_n, x'_n) < \epsilon$ for $n \geq N$, and since ϵ is arbitrary, it follows that $d(x_n, x'_n) \to 0$ and thus that M is C_0 controlled. □

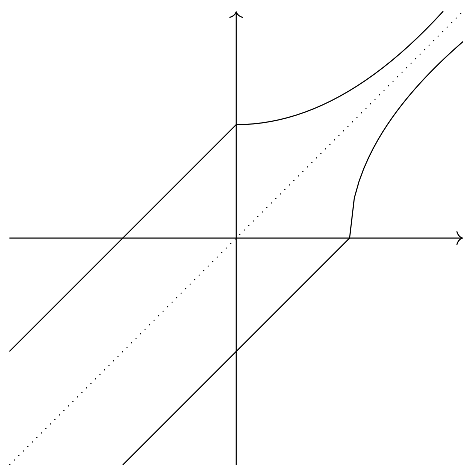
1.2.2 The hybrid coarse structure

The hybrid coarse structure was also first introduced in [Wri02]. It is a also a refinement of the metric coarse structure as the metric coarse structure coarsens the hybrid coarse structure, but not as refined, as the hybrid structure coarsens the C_0 coarse structure.

Definition 1.37 (Hybrid coarse structure). Let X be a metric space equipped with a function $\pi : X \to \mathbb{R}_+$ and define $X_i = \pi^{-1}([0, i])$. The *hybrid coarse structure for* π on X is the collection $\mathcal{E}_{h,\pi}$ of all subsets $M \subseteq X \times X$ satisfying the following:

- M is controlled for the metric coarse structure;
- $\sup\{d(x, x') : (x, x') \in M \backslash (X_i \times X_i)\} \to 0$ as $i \to \infty$.

Elements of $\mathcal{E}_{h,\pi}$ are called *π-hybrid controlled sets* (or simply *hybrid controlled sets* if the map π is clear from context). The coarse space $(X, \mathcal{E}_{h,\pi})$ will be abbreviated to $X_{h,\pi}$ (or X_h if the map π is clear from context).



An example of a set for the hybrid coarse structure on $\mathbb{R}$.

Proposition 1.38. *Let X be a metric space equipped with a map $\pi\colon X \to \mathbb{R}_+$. A set $M \subseteq X \times X$ is hybrid controlled if and only if for all $\epsilon > 0$, we can write $M = B \cup A_\epsilon$ where $B \subseteq X_i \times X_i$ for some i and $B \subseteq N_R$ for sufficiently large R and $d(x, x') < \epsilon$ for all $(x, x') \in A_\epsilon$.*

Proof. Suppose that M is hybrid controlled, and choose $\epsilon > 0$. Define $B = \{(x, x') \in M \colon d(x, x') \geq \epsilon\}$ and $A_\epsilon = \{(x, x') \in M \colon d(x, x') < \epsilon\}$. By assumption of metric control, $M \subseteq N_R$ for sufficiently large R, and therefore $B \subseteq N_R$. Suppose that for all i, $B \nsubseteq X_i \times X_i$, and choose a sequence $(x_n, x'_n) \in B$ with $(x_i, x'_i) \in A \backslash (X_i \times X_i)$. Since $\sup\{d(x, x') \colon (x, x') \in A \backslash (X_i \times X_i)\} \to 0$ as $i \to \infty$, we must have $d(x_i, x'_i) \to 0$ and so $(x_i, x'_i) \notin B$, which is a contradiction. Thus $B \subseteq X_i \times X_i$ for some i.

Conversely, suppose that for all $\epsilon > 0$, we can write $M = B \cup A_\epsilon$ where $B \subseteq X_i \times X_i$ for some i and $B \subseteq N_R$ for sufficiently large R. As A_ϵ is an ϵ-neighbourhood of the diagonal, $M = B \cup A_\epsilon \subseteq N_R$ and so M is controlled for the metric coarse structure. As $B \subseteq X_i \times X_i$, it follows that $M \backslash (X_i \times X_i) \subseteq M \backslash B = A_\epsilon$. And hence if $(x, x') \in M \backslash (X_i \times X_i)$ then $d(x, x') < \epsilon$. It follows that $\sup\{d(x, x') \colon (x, x') \in A \backslash (X_i \times X_i)\} \to 0$ as $i \to \infty$, and so M is hybrid controlled. □

Proposition 1.39. *Let X be a metric space equipped with a controlled map $\pi\colon X \to \mathbb{R}_+$. If $M \subseteq X \times X$ is C_0 controlled then M is hybrid controlled.*

Proof. If M is C_0 controlled then by Proposition 1.36, for all $\epsilon > 0$, we can write $M = B \cup A_\epsilon$ with B metrically bounded and $d(x, x') < \epsilon$ for all $(x, x') \in A_\epsilon$.

As B is metrically bounded, there exists Q and a basepoint $(y_0, y'_0) \in B$ such that $d((y, y'), (y_0, y'_0)) \leq Q$ for all $(y, y') \in B$. As π is controlled, there exists an S such that $d((\pi(y), \pi(y')), (\pi(y_0), \pi(y'_0))) \leq S$. Thus $d(\pi(y), \pi(y_0)) \leq S$ and $d(\pi(y'), \pi(y'_0)) \leq S$. Thus there exists an i such that $\pi(y), \pi(y') \leq i$ for all $(y, y') \in B$ and it follows that $B \subseteq X_i \times X_i$ for some i.

Suppose that B is metrically bounded, but that for all $R > 0$ there is a sequence $(x_n, x'_n) \in B$ with $(x_R, x'_R) \notin N_R$ for each R. Then $(x_n, x'_n) \in B$ with $(x_n, x'_n) \to \infty$. This contradicts the fact that B is metrically bounded, and hence $B \subseteq N_R$ for sufficiently large R. It follows from Proposition 1.38 that M is hybrid controlled. □

It is easy to see why the converse to Proposition 1.39 is not true. The following is a concrete example of this.

Example 1.40. Let $X = \mathbb{R}$ with the usual metric and $\pi\colon \mathbb{R} \to \mathbb{R}_+$ be the controlled map

$$\pi(x) = \begin{cases} \ln(x) & \text{if } x > 1 \\ 0 & \text{if } x \leq 1. \end{cases}$$

It is easy to see that π is controlled, using the facts that $\ln(R+1) \leq R$ for $R \geq 0$ and $|\ln(x) - \ln(y)| \leq |x - y|$ for $x, y \in \mathbb{R}$.

Define $X_i = \pi^{-1}([0,i]) = (-\infty, e^i]$. Fix k and let

$$M_R = \{(x,x') \colon d(x,x') \leq R \text{ and } x, x' \leq e^k\} \cup \{(x,x) \colon x > e^k\}.$$

It follows that $\sup\{d(x,x') \colon (x,x') \in M_R \backslash (X_i \times X_i)\} \to 0$ as $i \to \infty$ since $M_R \backslash (X_i \times X_i) = \{(x,x) \colon x > e^i\}$ for $i > k$, and hence M_R is hybrid controlled. But $(x, x+R) \in M_R$ for all $x \leq 0$ and $d(x, x+R) = R \not\to 0$ as $x \to -\infty$. It follows that M_R is not C_0 controlled.

Note that if $M \subseteq \mathbb{R} \times \mathbb{R}$ is hybrid controlled with π defined instead by $\pi(x) = |x|$ then M is also C_0 controlled. The reason for this comes out in the following result.

Proposition 1.41. *Let X be a metric space equipped with a coarse map $\pi \colon X \to \mathbb{R}_+$. Then $M \subseteq X \times X$ is C_0 controlled if and only if it is hybrid controlled.*

Proof. By Proposition 1.39, any C_0 controlled set M is hybrid controlled. Since π is coarse, each X_i is bounded (and so $X_i \times X_i$ is bounded). If $(x_n, x_n') \in M \to \infty$ then for each i there exists an N such that $(x_n, x_n') \in M \backslash (X_i \times X_i)$ for $n \geq N$. As M is hybrid controlled, $\sup\{d(x_n, x_n') \colon (x_n, x_n') \in A \backslash (X_i \times X_i)\} \to 0$ as $i \to \infty$, and so $d(x_n, x_n') \to 0$. It follows that M is C_0 controlled. □

1.3 Uniformly discrete and bounded geometry

Definition 1.42 (Uniformly discrete). A metric space X is said to be *uniformly discrete* (or *δ-discrete*) if there exists a $\delta > 0$ such that $d(x,x') \geq \delta$ whenever $x \neq x'$.

For example, $\mathbb{Z}$ is uniformly discrete but $\mathbb{R}$ is not. Every group equipped with the word length metric (see Chapter 2) is uniformly discrete.

The following definition comes from page 13 of [Roe96].

Definition 1.43 (Bounded geometry). A uniformly discrete metric space X is said to have *bounded geometry* if for every $R > 0$ there exists a k such that $|B(x,R)| \leq k$ for all $x \in X$. A metric (not neccessarily uniformly discrete) space is said to have *bounded geometry* if every uniformly discrete subset has bounded geometry.

Examples 1.44.
- Every subset of a metric space with bounded geometry also has bounded geometry.
- The space $\mathbb{R}^n$ has bounded geometry.
- The free group on 2 generators $|F_2|$ (see Chapter 2) with generating set $S = \{a, b\}$ has bounded geometry.

To see this, let $M \subseteq |F_2| \times |F_2|$ be a controlled set. Choose $x \in |F_2|$ and recall that $M(x) = \{x' \in F_2 \colon |x'x^{-1}|_S < R\}$. If $|x'x^{-1}|_S \leq R$ then $x' = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} x$ where $x_j = a$ or b, $e_j = \pm 1$ for each j and $n \leq R$. There are 4 possible choices for $x_1^{e_1}$. There are then 3 choices for each subsequent $x_j^{e_j}$ (so that there are no subwords of the form aa^{-1}). Thus for each $x \in |F_2|$, $|M(x)| \leq \sum_{n=1}^{R} 4(3)^{n-1}$ and $|F_2|$ has bounded geometry.

Non-Example 1.45. If H is an infinite dimensional Hilbert space then H has an orthonormal basis $S = \{e_n \colon n \in \mathbb{N}\}$. A metric can be given on H, induced by the inner product. It follows that $d(e_n, e_m) = \sqrt{2}$ for $n \neq m$, and thus that S is uniformly discrete. However, S does not have bounded geometry as the ball $\{e_m \colon d(e_n, e_m) \leq R\}$ is infinite for $R \geq \sqrt{2}$ and therefore H also does not have bounded geometry.

Chapter 2

Asymptotic dimension

Asymptotic dimension was first introduced by Gromov in [Gro93]. This is the coarse analogue of Lebesgue covering dimension, which plays a crucial role in the theory of topological spaces. Three equivalent definitions of asymptotic dimension will be given here, along with some basic examples. We shall show that asymptotic dimension is invariant under coarse equivalence, and state some useful results about asymptotic dimension.

Some basic definitions of geometric group theory are given here so that asymptotic dimension for finitely generated groups can be discussed. In particular, hyperbolic spaces and groups are introduced as these are a class of examples which have finite asymptotic dimension.

Finite asymptotic dimension is an important geometric property, and is a condition used to show that certain isomorphism conjectures in K-theory hold.

For more on the theory of asymptotic dimension, see [BD07], [Gra05] and [Gro93].

2.1 Geometric group theory

Geometric group theory is the study of groups by regarding them as metric spaces. This powerful way of looking at groups enables us to deduce results about groups which satisfy certain geometric conditions.

The word length metric is a way of measuring the distance between two elements of a group, and thus when equipped with this metric we are able to think about these group as metric spaces.

Definition 2.1 (Lipschitz). A map $f\colon X \to Y$ between metric spaces is *Lipschitz* (or *A-Lipschitz*) if there is an $A > 0$ such that $d(f(x), f(x')) \leq Ad(x, x')$ for all $x, x' \in X$.

It is easy to see that any Lipschitz map is controlled, and that if X and Y are proper and $f\colon X \to Y$ is proper and Lipschitz then f is coarse.

Definition 2.2 (Quasi-isometry). A map $f\colon X \to Y$ between metric spaces is a *quasi-isometry* if there exist constants $A \geq 1$ and $B, C \geq 0$ such that

$$\frac{1}{A}d(x,x') - B \leq d(f(x), f(x')) \leq Ad(x,x') + B$$

for all $x, x' \in X$ and for every $y \in Y$, there is an $x \in X$ with $d(f(x), y) \leq C$.

A quasi-isometry is the geometric group theory version of a coarse equivalence. The following theorem is well known, the proof is straightforward but is technical (see [BD07]).

Theorem 2.3. *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a map. If f is a quasi-isometry then it is a coarse equivalence. Moreover, if X and Y are geodesic, then f is a quasi-isometry if and only if it is a coarse equivalence.* □

Definition 2.4 (Words). Let S be a set. A *word* (of length n) with *alphabet* S is a finite sequence $a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_n^{\epsilon_n}$ where $a_1, a_2, \ldots, a_n \in S$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n = \pm 1$.

The *concatenation* of two words $w_1 = a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_m^{\epsilon_m}$ and $w_2 = b_1^{\epsilon'_1} b_2^{\epsilon'_2} \ldots b_n^{\epsilon'_n}$ is given by

$$w_1 w_2 = a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_m^{\epsilon_m} b_1^{\epsilon'_1} b_2^{\epsilon'_2} \ldots b_n^{\epsilon'_n}.$$

Two words are said to be *equivalent* if one can be obtained from the other by adding and deleting subwords of the form $a_j a_j^{-1}$. A word containing no strings of the form $a_j a_j^{-1}$ is called a *reduced word*. Each equivalence class contains a unique reduced word.

Definition 2.5 (Finitely generated group). A set S is said to be a *generating set* for a group G if every element of G can be written as a word with alphabet S. A group is said to be *finitely generated* if it has a finite generating set.

Definition 2.6 (Word length). The *word length* $|g|_S$ of $g \in G$ with alphabet S is defined to be the smallest n such that g is a word of length n with alphabet S. The word length of the identity element (the empty word) is defined to be 0.

The following proposition is straightforward to prove.

Proposition 2.7. *A group G with generating set S can be equipped with a metric (called the* word length metric*) defined by the formula*

$$d(g, g') = |g^{-1} g'|_S.$$

□

A group G with generating set S equipped with the word length metric shall be denoted by $|G|_S$.

If $S = G$, the word length metric in this case is simply the discrete metric. This metric contains no interesting geometric information about G so our attention will be restricted to finite generating sets, and all groups mentioned from now on will be assumed to be finitely generated.

It is clear that the word length metric depends on the choice of S. However, the following result tells us that from a coarse point of view, the geometric properties of a group are independent of the choice of generating set.

Proposition 2.8. *If G is a finitely generated group and S and S' are two choices of generating set then the identity map $|G|_S \to |G|_{S'}$ is a quasi-isometry.*

Proof. Define $S_L = \{|s|_{S'} \colon s \in S\}$ and $S'_L = \{|s'|_S \colon s' \in S'\}$. That is, S_L is the set of lengths of words in S written using alphabet S' and S'_L is the set of lengths of words in S' written using alphabet S. If $\lambda = \max(S_L \cup S'_L)$ then it follows that $\lambda^{-1} d_S(g, g') \leq d_{S'}(g, g') \leq \lambda d_S(g, g')$, and so the identity map is a quasi-isometry. □

Definition 2.9 (Cayley graph). Let G be a finitely generated group with generating set S. The *Cayley graph* $\mathrm{Cay}(G, S)$ is constructed combinatorially as follows:

- for each element $g \in G$, assign a vertex;
- for any $g \in G$ and $s \in S$, connect the vertices corresponding to the elements g and gs by an edge.

Observe that $g, g' \in G$ are adjacent in $\mathrm{Cay}(G, S)$ (connected by an edge) if and only if $d(g, g') = 1$. Thus the metric distance between any two vertices is the length of the shortest geodesic path between them. This can be extended further to all elements of $\mathrm{Cay}(G, S)$ by defining a metric d_C on $\mathrm{Cay}(G, S)$ by

$$d_C(x, y) = \inf\{\mathrm{Length}(\alpha) \colon \alpha \text{ is a path from } x \text{ to } y\}.$$

Proposition 2.10. *If G is a finitely generated group with generating set S then the inclusion map $|G|_S \hookrightarrow \mathrm{Cay}(G, S)$ is a quasi-isometry.* □

Proposition 2.11. *If G is a finitely generated group and S and S' are two choices of generating set then there is a quasi-isometry $\mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S')$.*

Proof. Choose a map $\varphi \colon \mathrm{Cay}(G, S) \to |G|_S$ defined by $\varphi(x) = g$ where g is such that $d_C(x, g) \leq 1/2$. This can always be done since any vertex is of distance 1 away from some other vertex. It is clear that $d(\varphi(x), \varphi(y)) \leq d_C(\varphi(x), x) + d(x, y) + d_C(y, \varphi(y)) \leq d(x, y) + 1$. Similarly, one can check that $d(\varphi(x), \varphi(y)) \geq d_C(x, y) - 1$.

The identity map $|G|_S \to |G|_{S'}$ is also a quasi-isometry, as is the inclusion map $|G|_{S'} \hookrightarrow \mathrm{Cay}(G, S')$. Composition of these with the map φ gives us a map $\mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S')$, which is a quasi-isometry as it is a composition of quasi-isometries. □

As our interest is in coarse properties, we can therefore write $|G|$ instead of $|G|_S$ and $\mathrm{Cay}(G)$ instead of $\mathrm{Cay}(G, S)$ as the generating set does not affect the coarse structure.

The metric space $|G|$ is not a geodesic space since the values of the word length metric are natural numbers. Being geodesic is a nice property to have, and the result above tells us that $|G|$ and $\mathrm{Cay}(G)$ can be considered the same geometrically. Therefore, as a geodesic space, $\mathrm{Cay}(G)$ is a natural (and useful) way of viewing $|G|$.

2.2 Asymptotic dimension for metric spaces

Definition 2.12. Let X be a metric space, and let $\mathcal{U}$ be a cover of X.

- A *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$.
- The *multiplicity* $\mu(\mathcal{U})$ of $\mathcal{U}$ is defined to be the maximum number of sets in $\mathcal{U}$ with a non-empty intersection.
- The *mesh* of $\mathcal{U}$ is defined to be the supremum of the diameter of sets from $\mathcal{U}$. The cover $\mathcal{U}$ is *uniformly bounded* if it has finite mesh.
- The *Lebesgue number* $L(\mathcal{U})$ of $\mathcal{U}$ is the largest positive L such that for each $x \in X$, there exists a $U \in \mathcal{U}$ such that $B(x, L) \subseteq U$.

Definition 2.13 (Asymptotic dimension I). A metric space X has *asymptotic dimension* less than or equal to n if for every uniformly bounded open cover $\mathcal{V}$, there is a uniformly bounded open cover $\mathcal{U}$ with $\mu(\mathcal{U}) \leq n+1$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Definition 2.14 (Asymptotic dimension II). A metric space X has *asymptotic dimension* less than or equal to n if for all $L > 0$ there exists a uniformly bounded open cover $\mathcal{U}$ of X such that $\mu(\mathcal{U}) \leq n + 1$ and $L(\mathcal{U}) \geq L$.

In each of the definitions of asymptotic dimension, $\mathrm{asdim}(X) = n$ if $\mathrm{asdim}(X) \leq n$ and $\mathrm{asdim}(X) \nleq n-1$, and $\mathrm{asdim}(X) = \infty$ if there exists no n such that $\mathrm{asdim}(X) \leq n$.

Both definitions of asymptotic dimension are equivalent (see Section 3 of [BD07]). As the arguments are straightforward, we give them here. Suppose that $\mathrm{asdim}(X) \leq n$ by Definition 2.13. Let $L > 0$ and set $\mathcal{V} = \{B(x, L) \colon x \in X\}$. It is easy to see that $\mathcal{V}$ is a uniformly bounded open cover of X. Thus there exists a uniformly bounded open cover $\mathcal{U}$ with $\mu(\mathcal{U}) \leq n+1$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Thus for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$. That is, for every $x \in X$ there is a $U \in \mathcal{U}$ such that $B(x, L) \subseteq U$, so $L(\mathcal{U}) \geq L$.

For the reverse claim, suppose $\mathrm{asdim}(X) \leq n$ by Definition 2.14. Let $\mathcal{V}$ be a uniformly bounded open cover, with $\mathrm{mesh}(\mathcal{V}) \leq C$. There exists a uniformly bounded open cover $\mathcal{U}$ such that $\mu(\mathcal{U}) \leq n+1$ and $L(\mathcal{U}) \geq C$. Since $\mathrm{Diam}(V) \leq C$ for all $V \in \mathcal{V}$, we have $V \subseteq B(x, C)$ for some $x \in X$. Since $L(\mathcal{U}) \geq C$, this means $B(x, C) \subseteq U$ for some $U \in \mathcal{U}$. It follows that $\mathcal{V}$ is a refinement of $\mathcal{U}$.

The following is another equivalent definition of asymptotic dimension (Section 3 of [BD07]). It is often the easiest definition to use for checking whether or not a space has finite asymptotic dimension.

Definition 2.15 (Asymptotic dimension III). A metric space X has *asymptotic dimension* less than or equal to n if for all $L > 0$ there exists a uniformly bounded open cover $\mathcal{U}$ of X such that the cover $\mathcal{U}$ consists of $n + 1$ families $\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}$ and each family is L-disjoint (the distance between two sets in the family is always bigger than L).

Some properties of asymptotic dimension are given below.

Proposition 2.16 (Section 9.2 of [Roe03]). *If X and Y are metric spaces and $A, B \subseteq X$ with $X = A \cup B$ then:*

- $\mathrm{asdim}(A) \leq \mathrm{asdim}(X)$;
- $\mathrm{asdim}(X) = \max\{\mathrm{asdim}(A), \mathrm{asdim}(B)\}$;
- $\mathrm{asdim}(X \times Y) \leq \mathrm{asdim}(X) + \mathrm{asdim}(Y)$. □

Proposition 2.17. *If X and Y are metric spaces and $f \colon X \to Y$ is a coarse embedding then*

$$\mathrm{asdim}(X) \leq \mathrm{asdim}(Y).$$

Proof. It is shown in Theorem 1 of [BD05] that the definition of asymptotic dimension is equivalent if the requirement for the uniformly bounded cover to be open is dropped. Suppose that $\mathrm{asdim}(Y) \leq n$. For each $L > 0$, let S_L be the constant such that if $d(x, y) < L$ then $d(f(x), f(y)) < S_L$. By assumption there is a uniformly bounded cover $\mathcal{U}_{S_L}$ of Y with $\mu(\mathcal{U}_{S_L}) \leq n + 1$ and $L(\mathcal{U}_{S_L}) \geq S_L$. The assumption that f is a coarse embedding means that for all $R > 0$ there exists an $S_R > 0$ such that $d(y, y') < R$ implies $d(f^{-1}(y), f^{-1}(y')) < S_R$.

There exists an R such that for each $U \in \mathcal{U}_{S_L}$ there is a y_0 with $U \subseteq B(y_0, R)$ as $\mathcal{U}_{S_L}$ is uniformly bounded. Therefore there is an S such that $f^{-1}(U) \subseteq B(f^{-1}(y_0), S)$ for each $f^{-1}(U) \in f^{-1}(\mathcal{U}_{S_L})$ and it follows that $f^{-1}(\mathcal{U}_{S_L})$ is a uniformly bounded cover of X.

It is also easy to verify that $\mu(f^{-1}(\mathcal{U}_{S_L})) \leq n+1$. Let $x \in X$. Then there exists a $U \in \mathcal{U}_{S_L}$ such that $B(f(x), S_L) \subseteq U$. Thus

$$\begin{aligned} B(x, L) &\subseteq f^{-1}(B(f(x), S_L)) \\ &\subseteq f^{-1}(U), \end{aligned}$$

and so it follows that $L(f^{-1}(\mathcal{U}_{S_L})) \geq L$, and hence that $\operatorname{asdim}(X) \leq \operatorname{asdim}(Y)$. □

Remark 2.18. It follows from Proposition 2.17 that asymptotic dimension is invariant under coarse equivalence. Even more generally it implies that if $f\colon X \to Y$ is a coarse embedding and there exists a coarse embedding $g\colon Y \to X$ then $\operatorname{asdim}(X) = \operatorname{asdim}(Y)$.

Example 2.19. $\operatorname{asdim}(\mathbb{R}) = 1$. To see that $\operatorname{asdim}(\mathbb{R}) \leq 1$, choose $L > 0$ and define

$$\mathcal{U}_0 = \{(4Lz - 2L, 4Lz + 2L)\colon z \in \mathbb{Z}\}$$

and

$$\mathcal{U}_1 = \{(4Lz, 4Lz + 4L)\colon z \in \mathbb{Z}\}.$$

Then the set $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$ is a cover of $\mathbb{R}$ which satisfies $\mu(\mathcal{U}) = 2$. Choose $x \in \mathbb{R}$, and use I to denote an open interval containing x. The point x is no more than $2L$ away from an end of this interval. If x is more than L away from this end, then $(x-L, x+L)$ is a subset of I, so also of $\mathcal{U}$. If x is less than L away then $(x-L, x+L)$ is a subset of J, where J is the 'next interval' in $\mathcal{U}$. This shows that $L(\mathcal{U}) = L$ and proves the claim that $\operatorname{asdim}(\mathbb{R}) \leq 1$.

Now we need to show that $\operatorname{asdim}(\mathbb{R}) \nleq 0$. If $\operatorname{asdim}(\mathbb{R}) \leq 0$ then for all $L > 0$ there exists a uniformly bounded open cover $\mathcal{U}$ of X satisfying $\mu(\mathcal{U}) \leq 1$. This is impossible as $\mathbb{R}$ is connected. Hence $\operatorname{asdim}(\mathbb{R}) = 1$.

Examples 2.20.
- $\operatorname{asdim}(B) = 0$ for every bounded set B. This follows as given any cover $\mathcal{V}$ of B, there is a one element cover $\mathcal{U}$ of B such that $\mathcal{V}$ refines $\mathcal{U}$.
- $\operatorname{asdim}(\{n^2\colon n \in \mathbb{N}\}) = 0$ as given any cover $\mathcal{V}$ of $\{n^2\colon n \in \mathbb{N}\}$, we can construct a disjoint cover $\mathcal{U}$ of $\{n^2\colon n \in \mathbb{N}\}$ such that $\mathcal{V}$ refines $\mathcal{U}$ by blending sets together and getting rid of any unnecessary overlaps.
- $\operatorname{asdim}(\mathbb{Z}) = 1$ by Example 2.19 and Proposition 2.17.
- $\operatorname{asdim}(\mathbb{R}^n) = n$ by a higher dimensional version of Example 2.19 (see Corollary 3.6 of [Gra05]).

Asymptotic dimension and topological dimension have some striking differences. For example, the asymptotic dimension of a unit square and a unit cube is 0 (as they are bounded) but they have topological dimensions 2 and 3 respectively.

Example 2.21 (Proposition 9.8 of [Roe03]). If T is a tree then $\operatorname{asdim}(T) \leq 1$.

Example 2.22. Let G be a finitely generated free group. By Proposition 2.10, $|G|$ and $\mathrm{Cay}(G)$ are coarsely equivalent, so $\mathrm{asdim}(|G|) = \mathrm{asdim}(\mathrm{Cay}(G))$. Since the Cayley graph of a free group is a tree, it follows from Example 2.21 that $\mathrm{asdim}(|G|) \leq 1$.

Example 2.23 (Proposition 8.1 of [Yu98]). Let $X = \bigsqcup_{n=1}^{\infty} S^{2n}$ where S^{2n} is the $2n$-sphere of radius 1. Equip each S^{2n} with the standard Riemannian metric d_{2n} and then equip X with a metric d such that $d|_{S^{2n}} = nd_{2n}$ (so that the radii of the spheres grow infinitely large) and that the Hausdorff distance $d(S^{2n}, S^{2n'}) > \max\{n, n'\}$ if $n \neq n'$ (so that the distance between the spheres grow infinitely large). Then X has infinite asymptotic dimension.

The following proposition is useful because it implies finite asymptotic dimension for groups with certain properties.

Proposition 2.24 (Lemma 9.16 of [Roe03]). *If X and Y are metric spaces where Y has finite asymptotic dimension and $f\colon X \to Y$ is a Lipschitz map where for each $R > 0$, the inverse images $f^{-1}(B(y, R))$ have finite asymptotic dimension uniformly in y then X also has finite asymptotic dimension.* □

Corollary 2.25 (Corollary 9.19 of [Roe03]). *Let*

$$0 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of finitely generated groups. If K and H have finite asymptotic dimension, then so does G. □

2.2.1 Hyperbolic spaces and groups

Hyperbolic spaces, such as the Poincaré half-plane model $\mathbb{H} = \{(x, y) : y > 0, x, y \in \mathbb{R}\}$ equipped with the Poincaré metric, are spaces with constant negative curvature, which is the main difference from Euclidean spaces, which have zero curvature. Hyperbolic groups are groups which share many geometric properties of hyperbolic spaces, and these are well studied objects in geometric group theory. Hyperbolic groups are of interest to us as they are a large class of examples which have finite asymptotic dimension. To be able to define a hyperbolic group, firstly we need to define a hyperbolic space. Hyperbolic spaces are straightforward to define by a simple and easy to visualise property of triangles.

Definition 2.26 (Geodesic triangle). A *geodesic triangle* $[x, y] \cup [x, z] \cup [z, y]$ in a metric space X is the union of three geodesic segments $[x, y], [x, z], [z, y]$ joined together to form a triangle in X.

Definition 2.27 (δ-thin). A geodesic triangle is said to be *δ-thin* if each side is contained in the δ-neighbourhoods of each of the other two sides.

Definition 2.28 (δ-hyperbolic space). A geodesic space X is said to be a *δ-hyperbolic space* if every geodesic triangle is δ-thin.

Definition 2.29 (Hyperbolic space). A geodesic space is said to be a *hyperbolic space* if it is δ-hyperbolic for some $\delta \geq 0$.

Large-scale invariance of hyperbolic space is well known.

Proposition 2.30 (Theorem 1.9, Chapter III of [BH99]). *If X and Y are coarsely equivalent geodesic spaces then X is hyperbolic if and only if Y is hyperbolic.* □

Examples 2.31.

- Bounded geodesic spaces are hyperbolic. If the distance between any two points is at most B then any side of a triangle is in a B-neighbourhood of the union of the other two sides.
- A tree is a 0-hyperbolic metric space. Any two points are connected by a shortest path, so the tree is geodesic. Any side of a triangle is automatically contained inside the union of the other two sides.

Non-Example 2.32. The Euclidean plane $\mathbb{R}^2$ is not a hyperbolic space, as equilateral triangles can grow as large as they wish, meaning that no such δ exists so that all geodesic triangles are δ-thin (so Euclidean space does not satisfy the thin triangles property).

Definition 2.33 (Hyperbolic group). A finitely generated group G is said to be a *hyperbolic group* if $\mathrm{Cay}(G)$ is a hyperbolic space.

Examples 2.34.

- Finite groups are hyperbolic as their Cayley graphs are bounded.
- Finitely generated free groups are hyperbolic as their Cayley graphs are trees.

Non-Example 2.35. The group $\mathbb{Z}^2$ is not a hyperbolic group. This follows from Proposition 2.30 as $\mathrm{Cay}(\mathbb{Z}^2)$ is coarsely equivalent to $\mathbb{R}^2$, which is not a hyperbolic space by Non-Example 2.32.

Proposition 2.36 (Theorem 9.25 of [Roe03]). *If G is a hyperbolic group then G has finite asymptotic dimension.* □

2.3 Asymptotic dimension for coarse spaces

Asymptotic dimension can be easy generalised from metric spaces to coarse spaces. Metric balls can simply be replaced with controlled balls. Asymptotic dimension for coarse spaces is reviewed here, but is not used in this thesis as the focus is always on metric spaces.

The following definition comes from Chapter 9 of [Roe03].

Definition 2.37. Let X be a coarse space, $L \subseteq X \times X$ be a controlled set and $\mathcal{U}$ be a cover of X.

- A cover $\mathcal{U}$ is said to have *appetite* L if for all $x \in X$, there exists a $U \in \mathcal{U}$ such that $L(x) \subseteq U$.
- A cover $\mathcal{U}$ is said to be *uniformly bounded* if $\bigcup_{U \in \mathcal{U}} U \times U$ is a controlled set.

Definition 2.38 (Asymptotic dimension for coarse spaces). A coarse space has *asymptotic dimension* less than or equal to n if for every controlled set $L \subseteq X \times X$, there is a uniformly bounded cover $\mathcal{U}$ of X such that $\mu(\mathcal{U}) \leq n+1$ and $\mathcal{U}$ has appetite L.

This agrees with our previous definitions if the coarse structure used is the metric coarse structure. There are various other equivalent definitions of asymptotic dimension for coarse structures, see [Gra05]. All of the properties stated for the metric case (such as coarse invariance) also hold in the general case.

Chapter 3

Coarse homotopy and coarse homology

The main definitions and results of coarse homotopy theory are contained within this chapter. Coarse homotopy is the coarse analogue of homotopy in topology. A homotopy between continuous maps is required to take place over the interval $[0, 1]$ (so in a finite time). A coarse homotopy requires that the restriction of coarse maps to bounded sets also takes places over a finite time, but coarse homotopy for the coarse maps themselves are allowed to take an infinite time. We describe a coarse version of the cylinder $X \times [0, 1]$ used in ordinary homotopy theory which is best suited for our setup. References for more information about coarse homotopy theory include [Nor09], [Moh13] and [MNS18].

The concept of a coarse homology theory is also introduced here, and it will be shown how to define a coarse homology theory on the coarse category for each locally finite homology theory, using a process of coarsening via a coarsening family or (more intuitively for the metric coarse structure) the Rips complex. The coarse Baum–Connes conjecture is of great interest in coarse geometry as there are interesting geometrical and topological consequences for many areas of mathematics. This conjecture concerns a map (known as an assembly map) which is defined between the coarse K-homology of a space and the K-theory of the Roe C^*-algebra of a space. The conjecture asks whether or not this map is an isomorphism under certain conditions. Examples of spaces for which this conjecture holds and does not hold will be given.

An axiomatic definition of a locally finite homology theory (and a relative version) will be introduced in this chapter, allowing us to expand on the process of coarsening, creating a pathway between topological homology and coarse homology.

3.1 Coarse homotopy theory

Definition 3.1 (Coarse topological space). A *coarse topological space* X is a coarse space X equipped with a Hausdorff topology such that every controlled set is contained in an open set, and the closure of every bounded set is compact. The topology and coarse structure of a coarse space X equipped with a topology are said to be *compatible* if this condition holds.

Remark 3.2. Coarse topological spaces have to be locally compact since for each $x \in X$ the singleton set (x, x) is controlled, so is contained in an open set U. It follows that $U(x)$ is an open neighbourhood of x, but is bounded, so x is contained in the compact neighbourhood closure $\overline{U(x)}$.

Remark 3.3. If $X \subseteq \mathbb{R}^n$ then every compact subset of X is closed and metrically bounded, so it follows that if B is bounded with respect to the coarse structure of a coarse topological space then B is metrically bounded because the closure of every bounded set B is compact, so B itself is also metrically bounded.

Definition 3.4 (Generalised ray). Consider the coarse topological space $[0, \infty)$, which we will denote by R_+. The space R_+ is called a *generalised ray* if the following conditions are satisfied:

- the set
$$M + M' = \{(u + u', v + v') \colon (u, v) \in M, (u', v') \in M'\}$$
is controlled if $M, M' \subseteq R_+ \times R_+$ are controlled;
- the set
$$M^s = \{(u, v) \in R_+ \times R_+ : x \leq u \leq y \text{ and } x \leq v \leq y, (x, y) \in M\}$$
is controlled if $M \subseteq R_+ \times R_+$ is controlled;
- for each $a \in R_+$, the set
$$a + M = \{(a + u, a + v) \colon (u, v) \in M\}$$
is controlled if $M \subseteq R_+ \times R_+$ is controlled.

Example 3.5. The coarse space $(R_+, \mathcal{E}_d)$ is a generalised ray. The notation $\mathbb{R}_+$ will be reserved only for this specific ray.

Definition 3.6 (Continuously controlled coarse structure). Let $(X, \mathcal{E})$ be a coarse topological space, and suppose that X is a topologically dense subset of a Hausdorff space Y (so that Y is a compactification of X). Define the *boundary* ∂X of X by $Y \backslash X$. An open subset $M \subseteq X \times X$ is said to be *strongly controlled* if $M \in \mathcal{E}$ and if the closure $\overline{M}$ of M in $Y \times Y$ satisfies

$$\overline{M} \cap ((Y \times \partial X) \cup (\partial X \times Y)) \subseteq \Delta_{\partial X}.$$

The *continuously controlled coarse structure* $\mathcal{E}_{cc}$ with respect to Y is the collection of subsets $M \subseteq X \times X$ such that M is a composite of subsets of strongly controlled open sets.

It can be shown that $(X, \mathcal{E}_{cc})$ is also a coarse topological space (see Proposition 2.4 of [Mit10]), and that $M \subseteq X \times X$ is continuously controlled if and only if for all sequences $(x_n, x'_n) \in M$ then $x_n \to x \in \partial X$ if and only if $x'_n \to x$ and $M \in \mathcal{E}$.

Proposition 3.7. *The coarse space* $(\mathrm{R}_+, \mathcal{E}_{cc})$ *is a generalised ray.*

Proof. The space R_+ is a topologically dense subset of the one-point compactification $[0, \infty] = R_+ \cup \{\infty\}$. Since $\partial R_+ = \{\infty\}$, the condition for $M \subseteq R_+ \times R_+$ is be continuously controlled is that

$$\overline{M} \cap ([0, \infty] \times \{\infty\} \cup \{\infty\} \times [0, \infty]) \subseteq \{(\infty, \infty)\}$$

Let $M, N \subseteq R_+ \times R_+$ be continuously controlled. If $(x, y) \in \overline{M + N} \cap ([0, \infty] \times \{\infty\} \cup \{\infty\} \times [0, \infty])$ then $x = u + u'$ and $y = v + v'$ for $(u, v) \in \overline{M}$ and $(u', v') \in \overline{N}$. By assumption, $x = \infty$ or $y = \infty$. Without loss of generality, suppose that $x = \infty$. Then $u = \infty$ or $u' = \infty$, and if $u = \infty$ then $v = \infty$ as $(u, v) \in M$ and M is continuously controlled, so $x = y = \infty$. The other cases are the same. It follows that $M + N$ is continuously controlled.

If $(u, v) \in \overline{M^s} \cap ([0, \infty] \times \{\infty\} \cup \{\infty\} \times [0, \infty])$ then $x \leq u \leq y$ and $x \leq v \leq y$ for $(x, y) \in \overline{M}$. As $u = \infty$ or $v = \infty$, it follows that $y = \infty$ so $x = \infty$ as M is continuously controlled. Thus $u = v = \infty$.

If $a \in R_+$ and $(x, y) \in \overline{a + M} \cap ([0, \infty] \times \{\infty\} \cup \{\infty\} \times [0, \infty])$ then $x = a + u$ and $y = a + v$ where $(u, v) \in \overline{M}$. By assumption, $x = \infty$ or $y = \infty$. Then $u = \infty$ or $v = \infty$. Then $u = v = \infty$ as M is continuously controlled, so $x = y = \infty$ and $a + M$ is continuously controlled.

It follows that $(R_+, \mathcal{E}_{cc})$ is a generalised ray. □

Non-Example 3.8. The coarse space R_+ with the C_0 coarse structure is not a generalised ray. To see this, note that the set $\Delta_{R_+} \cup \{(0, R), (R, 0)\}$ is controlled for any $R > 0$ as each singleton set is controlled. The first and second conditions for R_+ with the C_0 coarse structure to be a generalised ray would imply that the set

$$\Delta_R = \{(x, y) \in R_+ \times R_+ \colon d(x, y) \leq R\}$$

is also controlled, which is not true for the C_0 coarse structure.

Definition 3.9 (p-cylinder). Let X be a coarse space equipped with a controlled map $p\colon X \to R_+$. The *p-cylinder of X* is defined to be the space

$$I_pX = \{(x,t) \in X \times R_+ \colon t \leq p(x)+1\}.$$

Note that a different choice of p will produce a different p-cylinder.

Definition 3.10 (Coarse homotopy). Two coarse maps $f, g\colon X \to Y$ are *coarsely homotopic* if there exists a controlled map $p : X \to R_+$ and a coarse map $H\colon I_pX \to Y$ such that $f(x) = H(x,0)$ and $g(x) = H(x, p(x)+1)$. The coarse map H is called a *coarse homotopy.*

Proposition 3.11 (Theorem 2.4 of [MNS18]). *The notion of maps being coarsely homotopic is an equivalence relation.* □

An outline of the proof is given here. It is easy to verify that a coarse map $f\colon X \to Y$ is coarsely homotopic to itself. Choose any controlled map $p\colon X \to R_+$ and define $H(x,t) = f(x)$ for all t.

Let $g\colon X \to Y$ be a coarse map and suppose that f is coarsely homotopic to $g\colon X \to Y$ by $F\colon I_pX \to Y$. Define $G\colon I_pX \to Y$ by $G(x,t) = F(x, p(x)+1-t)$. It follows that g is coarsely homotopic to to f.

Let $h\colon X \to Y$ be a coarse map and suppose f is coarsely homotopic to g (by $H\colon I_pX \to Y$) and g is coarsely homotopic to h (by $H'\colon I_qX \to Y$). Note that the map $p+q$ is controlled. Define $J\colon I_{p+q+1}X \to Y$ by

$$J(x,t) = \begin{cases} H(x,t) & 0 \leq t \leq p(x)+1 \\ H'(x, t-(p(x)+1)) & p(x)+1 \leq t \leq p(x)+q(x)+2. \end{cases}$$

It follows that f is coarsely homotopic to h.

Definition 3.12 (Coarse homotopy equivalence). Let $f\colon X \to Y$ be a coarse map. Then f is said to be a *coarse homotopy equivalence* if there is a coarse map $g\colon Y \to X$ such that $g \circ f$ is coarsely homotopic to id_X and $f \circ g$ is coarsely homotopic to id_Y.

Example 3.13. Let $f, g\colon X \to Y$ be close coarse maps, and choose a controlled map $p\colon X \to R_+$. Then f and g are coarsely homotopic via the map $H\colon I_pX \to Y$ defined by

$$H(x,t) = \begin{cases} f(x) & t < 1 \\ g(x) & t \geq 1. \end{cases}$$

In particular, this shows that every coarse equivalence is a coarse homotopy equivalence.

Example 3.14. The map $i\colon X \hookrightarrow I_pX$ defined by $i(x) = (x,0)$ is a coarse homotopy equivalence, with coarse homotopy inverse given by the projection map $\pi\colon I_pX \to X$

defined by $\pi(x,t) = x$. Define a coarse map $H\colon I_{p\circ\pi}(I_pX) \to I_pX$ by

$$H((x,t),s) = \begin{cases} (x, s+t) & s \leq (p\circ\pi)(x,t) \\ (x,0) & s > (p\circ\pi)(x,t). \end{cases}$$

Note that $H((x,t),0) = \mathrm{id}_{I_pX}(x,t)$ and $H((x,t),(p\circ\pi)(x,t)+1) = (i\circ\pi)(x,t)$. Thus we have a coarse homotopy between $i\circ\pi$ and id_{I_pX}. The other direction is clear because $\pi\circ i = \mathrm{id}_X$.

3.1.1 Coarse homotopy groups

Definition 3.15 (Pointed coarse space). Let X be a coarse space and let R_+ be a generalised ray. A *basepoint* for X is a coarse map $i_X\colon R_+ \to X$ such that $p_X \circ i_X$ is close to id_{R_+} where $p_X\colon X \to R_+$ is a controlled map. A *pointed coarse space* is a coarse space with a basepoint.

Definition 3.16 (Base-point preserving). A coarse map $f\colon X \to Y$ between pointed coarse spaces is said to be *base-point preserving* if $f\circ i_X = i_Y$.

Definition 3.17 (Coarse π_0). Let X be a pointed coarse space. The *coarse homotopy set of* 0^{th} *degree* is defined by

$$\pi_0^{\text{coarse}}(X;R_+) = [R_+;X]^{\text{coarse}}$$

where $[R_+;X]^{\text{coarse}}$ is the set of coarse homotopy classes of coarse maps $R_+ \to X$.

The coarse homotopy class of a map f will be denoted by $[f]$.

Example 3.18. If B is a bounded coarse space then there are no coarse maps from R_+ to B, so it follows that $\pi_0^{\text{coarse}}(B;R_+) = \emptyset$.

The following example shows us that a coarse space does not have to be bounded to have empty coarse homotopy.

Proposition 3.19. *If $X = \{n^2\colon n\in\mathbb{N}\}$ then $\pi_0^{coarse}(X;\mathbb{R}_+) = \emptyset$.*

Proof. There are no coarse maps $f\colon \mathbb{R}_+ \to \{n^2\colon n\in\mathbb{N}\}$. To see this, note that there are infinitely many pairs $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ with $d(x,y)\leq 1$. If f was coarse, there must be some $S>0$ such that for all (x,y) with $d(x,y)\leq 1$, we have $d(f(x),f(y)) < S$. But only finitely many pairs $f(x), f(y)$ are of distance less than S apart. Thus f cannot be bornologous, so cannot be coarse. □

Proposition 3.20. *If $A_i = \{n^i\colon n\in\mathbb{N}\}$ then the spaces A_i and A_j are coarsely equivalent for $i,j\geq 2$.*

Proof. There is a bijection $\phi_{i,j}\colon A_i \to A_j$ defined by $\phi_{i,j}(n^i) = n^j$. It is clear that $\phi_{i,j}^{-1} = \phi_{j,i}$. To show that $\phi_{i,j}$ is a coarse equivalence, it suffices to show that the map $\phi_{i,j}$ is coarse. Fix $R > 0$ and note that $A_i = A_{i,R} \sqcup \{n^i \colon n \geq n_R\}$ where $A_{i,R} = \{n^i \colon n \leq n_R\}$ and $n_R = \min\{n \colon d(n^i, (n+1)^i) > R\}$. As $\phi_{i,j}(A_{i,R})$ is finite and A_j is totally ordered, there is a largest element, say m^j. Now let $S = d(m^j, (m-1)^j)$. Then $d(m^j, (m+1)^j) > S$ and $\phi_{i,j}(A_{i,R}) = A_{j,S}$. It follows that $\phi_{i,j}$ is bornologous, so is coarse. The result follows. □

It will follow from Proposition 3.19 and invariance under coarse equivalence (see Proposition 3.25) that $\pi_0^{\text{coarse}}(\{n^i \colon n \in \mathbb{N}\}; \mathbb{R}_+) = \emptyset$ for $i \geq 2$.

The following examples can be found in [Moh13]. A sketch of the proofs are given here.

Example 3.21 (Proposition 2.3.3 of [Moh13]). $\pi_0^{\text{coarse}}(\mathbb{R}_+; \mathbb{R}_+) = \{0\}$.

To prove this, it can be shown that any coarse map $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is coarsely homotopic to the identity map on $\mathbb{R}_+$. The easiest way to do this is to show that f and $f + \text{id}_{\mathbb{R}_+}$ are coarsely homotopic, and that $f + \text{id}_{\mathbb{R}_+}$ and $\text{id}_{\mathbb{R}_+}$ are also coarsely homotopic. The result then follows by the transitivity property of being coarsely homotopic.

Example 3.22 (Proposition 2.3.5 of [Moh13]). Let $f_+, f_- \colon \mathbb{R}_+ \to \mathbb{R}$ be defined by $f_+(x) = x$ and $f_-(x) = -x$. Then $\pi_0^{\text{coarse}}(\mathbb{R}; \mathbb{R}_+) = \{[f_+], [f_-]\}$.

To prove this, it can be shown that any coarse map $f\colon \mathbb{R}_+ \to \mathbb{R}$ is close to a Lipschitz coarse map $g\colon \mathbb{R}_+ \to \mathbb{R}$, which allows the intermediate value theorem to be applied to show that g is either eventually always positive or always negative. It is then shown that g is coarsely homotopic to either f_+ or f_- respectively.

Example 3.23 (Proposition 2.3.8 of [Moh13]). $\pi_0^{\text{coarse}}(\mathbb{R}^2; \mathbb{R}_+) = \{0\}$.

To prove this, it can shown that any coarse map $f\colon \mathbb{R}_+ \to \mathbb{R}^2$ is coarsely homotopic to the map $i\colon \mathbb{R}_+ \to \mathbb{R}^2$ defined by $i(s) = (s, 0)$, using the fact that f can be written in polar coordinates as $f(x) = (r(x), \theta(x))$ with θ bounded.

Definition 3.24 (Coarse homotopy groups). Let X be a pointed coarse space and let R_+ be a generalised ray. For $n \geq 1$, the *coarse homotopy groups of n^{th} degree* are defined by

$$\pi_n^{\text{coarse}}(X; R_+) = [(R_+ \sqcup R_+)^{n+1}; X]_+^{\text{coarse}}$$

where $[(R_+ \sqcup R_+)^{n+1}; X]_+^{\text{coarse}}$ is the set of coarse homotopy classes of base-point preserving maps $(R_+ \sqcup R_+)^{n+1} \to X$.

It can be shown that $\pi_n^{\text{coarse}}(X; R_+)$ is a group for $n \geq 1$, and is an abelian group for $n \geq 2$. For a proof, see Proposition 3.8 of [MNS18].

Note that $R_+ \sqcup R_+$ is coarsely equivalent to the set $\mathbb{R}$ where both $[0, \infty)$ and $(-\infty, 0]$ have the same coarse structure as R_+. In $R_+ \sqcup R_+$ we write t with $t \geq 0$ for a coordinate in the first copy of R_+ and $-t$ with $t \geq 0$ for a coordinate in the second copy of R_+. This is used when defining the group operation below.

The group operation is given by is $[f].[g] = [f \star g]$ where $f \star g\colon (R_+ \sqcup R_+)^{n+1} \to X$ defined by

$$(f \star g)(x_0, x_1, \ldots, x_n) = \begin{cases} f(x_0 - \frac{x_1}{2}, \frac{x_1}{2}, x_2, \ldots, x_n) & x_1 \leq x_0 \\ g(\frac{x_0}{2}, x_1 - \frac{x_0}{2}, x_2, \ldots, x_n) & x_0 \leq x_1 \end{cases}$$

Given a coarse map $f : X \to Y$, there is an induced map $f_*\colon \pi_n^{\text{coarse}}(X; R_+) \to \pi_n^{\text{coarse}}(Y; R_+)$ defined by $f_*([h]) = [f \circ h]$. It is easy to see that this map is a group homomorphism as $f \circ (h \star h') = (f \circ h) \star (f \circ h')$ for $h, h'\colon (R_+ \sqcup R_+)^{n+1} \to X$.

Proposition 3.25. *If $f, g\colon X \to Y$ are coarsely homotopic then the induced maps $f_*, g_*\colon \pi_n^{coarse}(X; \mathrm{R}_+) \to \pi_n^{coarse}(Y; \mathrm{R}_+)$ are equal.*

Proof. As f and g are coarsely homotopic, there is a coarse map $H\colon I_p R_+ \to Y$ for some controlled map $p\colon R_+ \to Y$ such that $H(x, 0) = f(x)$ and $H(x, p(x) + 1) = g(x)$. Define a map $H'\colon I_{p\circ h} R_+ \to Y$ by $H'(x, t) = H(h(x), t)$. Then H' is coarse and $H'(x, 0) = (f \circ h)(x)$ and $H'(x, p(x) + 1) = (g \circ h)(x)$. It follows that $f \circ h$ and $g \circ h$ are coarsely homotopic, so $[f \circ h] = [g \circ h]$. Thus $f_* = g_*$. □

3.1.2 Coarse homotopy of the open cone

Let X be a subset of the unit sphere of some Hilbert space H and define the *metric cone* of X by

$$\mathcal{O}_H X = \{tx\colon t \geq 0, x \in X\}.$$

The following result from [MNS18] provides us with a pathway between the topological homotopy of a space and the coarse homotopy of its open cone.

Theorem 3.26 (Theorem 5.6 of [MNS18]). *If X is a finite dimensional simplicial complex realised as a subset of the unit sphere of a Hilbert space H then there is an isomorphism between the groups $\pi_n(X)$ and $\pi_n^{coarse}(\mathcal{O}_H X; \mathbb{R}_+)$ for each n.* □

As the metric cone of the k-sphere S^k is $\mathbb{R}^{k+1}$ (where H is also $\mathbb{R}^{k+1}$) we obtain the following:

Corollary 3.27. *For each n, there is an isomorphism*

$$\pi_n(S^k) \cong \pi_n^{coarse}(\mathbb{R}^{k+1}; \mathbb{R}_+).$$

A sketch of the proof of Theorem 3.26 is given here. Let X be a subset of the unit sphere of some real Hilbert space H. For a continuous map $f\colon X \to Y$, the authors in [MNS18] define a map $f_{\mathcal{O}}\colon \mathcal{O}_H X \to \mathcal{O}_H Y$ by $f_{\mathcal{O}}(tx) = tf(x)$, known as a radial map. This map is not a coarse map in general, but it is if f is Lipschitz. They then define a map $\Psi\colon \pi_n(X) \to \pi_n^{\text{coarse}}(\mathcal{O}_H X; \mathbb{R}_+)$ by $\Psi([f]) = [f_{\mathcal{O}}]$, and show that this map is an isomorphism. They show that if $f\colon \mathbb{R}^k \to \mathcal{O}_H X$ is a coarse map, then f is coarsely homotopic to a radial proper Lipshitz map (of the form $f_{\mathcal{O}}$). Using these facts, an inverse map to Ψ is constructed.

Examples 3.28. By results in ordinary homotopy theory, we can quickly compute some coarse homotopy groups:

$$\pi_j^{\text{coarse}}(\mathbb{R}^{j+1}; \mathbb{R}_+) \cong \pi_j(S^j) = \mathbb{Z}$$

for $j \geq 1$,

$$\pi_j^{\text{coarse}}(\mathbb{R}^{2}; \mathbb{R}_+) \cong \pi_j(S^1) = \{0\}$$

for $j \geq 2$ and

$$\pi_1^{\text{coarse}}(\mathbb{R}^{j+1}; \mathbb{R}_+) \cong \pi_1(S^j) = \{0\}$$

for $j \geq 2$.

3.2 Coarse homology and locally finite homology theories

3.2.1 Coarse homology theories

Definition 3.29 (Coarsely excisive decomposition). A decomposition $X = A \cup B$ of a coarse space X is called *coarsely excisive* if for all controlled sets $m \subseteq X \times X$ there exists a controlled set $M \subseteq X \times X$ such that

$$m(A) \cap m(B) \subseteq M(A \cap B).$$

Roughly speaking, a decomposition being coarsely excisive means that intersections of neighbourhoods are neighbourhoods of intersections.

Example 3.30. The decomposition $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$ is coarsely excisive (with the metric coarse structure). To see this, choose $R > 0$ and let $N_R(S) = \{x \in \mathbb{R} \colon d(x, S) \leq R\}$. It is easy to see that

$$N_R((-\infty, 0]) \cap N_R([0, \infty)) = [-R, R] = N_R(\{0\})$$

and so it follows that $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$ is a coarsely excisive decomposition.

In the same way, it can be shown that $\mathbb{R} = (-\infty, c_1) \cup (c_2, \infty)$ for any $c_1 > c_2$ is coarsely excisive.

Non-Example 3.31. The decomposition $\mathbb{R} = (-\infty, 0] \cup (0, \infty)$ is not a coarsely excisive decomposition. The intersection of $(-\infty, 0]$ and $(0, \infty)$ is empty, so to be coarsely excisive, for all $R > 0$, the intersection $N_R((-\infty, 0]) \cap N_R((0, \infty))$ would have to be empty. This clearly does not hold for any $R > 0$.

Non-Example 3.32 (Section 1, Example 2 of [HRY93]). If

$$M = \{(x, y) \in \mathbb{R}^2 \colon x > 0, y \in \{0, 1\} \text{ or } x = 0 \text{ and } 0 \leq y \leq 1\}$$

then $M = A \cup B$ where $A = \{(x, y) \in M \colon y \leq 1/2\}$ and $B = \{(x, y) \in M \colon y \geq 1/2\}$. Since $A \cap B = \{(0, 1/2)\}$ is a single point, but $N_1(A)$ and $N_1(B)$ are equal to M, it follows that $M = A \cup B$ is not a coarsely excisive decomposition.

Definition 3.33 (Path metric). A *path* in a metric space X is a continuous map $\gamma : [a, b] \to X$. The *length* of a path $\gamma : [a, b] \to X$ is defined by writing

$$\text{Length}(\gamma) = \sup \left\{ \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)) \,:\, a = t_0 < t_1 < \ldots < t_N = b \right\},$$

where N can vary depending on the partitioning of $[a, b]$. A metric d on X is called a *path metric* if for any two points $x, x' \in X$ we have

$$d(x, x') = \inf\{\text{Length}(\gamma) : \gamma \text{ is a path from } x \text{ to } x'\}.$$

Lemma 3.34. *If X is a path metric space and $X = A \cup B$ where A and B are both closed then $X = A \cup B$ is a coarsely excisive decomposition.*

Proof. Fix $R > 0$ and note that as X is a path metric space, the intersection $A \cap B$ is non-empty. The R-neighbourhood of A in the case of a path metric space is the set

$$N_R(A) = \{x \in X \colon \text{there is a path } \gamma_A \colon x \to a \in A \text{ such that} \, \text{Length}(\gamma_a) < R\}.$$

If $x \in N_R(A) \cap N_R(B)$ then there is a path $\gamma_A \colon x \to a$ for some $a \in A$ with $\text{Length}(\gamma_A) < R$ and a path $\gamma_B \colon x \to b$ for some $b \in B$ with $\text{Length}(\gamma_B) < R$.

Suppose $x \in A$ and let $\gamma : x \to b$ be a path from x to $b \in B$ with $\text{Length}(\gamma) < R$, that is, a continuous map $\gamma \colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = b$. It can be assumed that $x \notin B$ as otherwise $x \in A \cap B$ and the result follows.

Define $S = \sup\{T \in [0, 1] \colon \gamma(t) \notin B \text{ for all } 0 \leq t \leq T\}$ so that for $0 \leq t < S$, $\gamma(t) \in A$. The sequence $\gamma(S - 1/n)$ converges to $\gamma(S)$, so as A is closed, $\gamma(S) \in A$. There is a sequence ϵ_n such that $0 < \epsilon_n \leq 1/n$ and $\gamma(S + \epsilon_n) \in B$ since if $\gamma(S + \epsilon) \notin B$ for all $0 < \epsilon \leq 1/n$ then $\sup\{T \in [0, 1] \colon \gamma(t) \notin B \text{ for all } 0 \leq t \leq T\} \geq S + 1/n$. The sequence $\gamma(S + \epsilon_n)$ converges to $\gamma(S)$, and as B is closed, $\gamma(S) \in B$.

So $\gamma(S) \in A \cap B$ and $d(\gamma(S), x) \leq \text{Length}(\gamma) < R$. So $x \in N_R(A \cap B)$ and hence $N_R(A) \cap N_R(B) \subseteq N_R(A \cap B)$ and thus $X = A \cup B$ is a coarsely excisive decomposition. □

Definition 3.35 (Coarse homology theory). A *coarse homology theory* is a sequence of covariant functors $h_*^{\text{coarse}} : \text{Coarse} \to \text{Groups}$ such that:

- if $f, g \colon X \to Y$ are coarsely homotopic maps then the induced maps $f_*, g_* \colon h_*^{\text{coarse}}(X) \to h_*^{\text{coarse}}(Y)$ are equal;
- if $X = A \cup B$ is a coarsely excisive decomposition where $i \colon A \cap B \hookrightarrow A$, $j \colon A \cap B \hookrightarrow B$, $k \colon A \hookrightarrow X$ and $l \colon B \hookrightarrow X$ are the inclusion maps then there is a natural map

$$d \colon h_*^{\text{coarse}}(X) \to h_{*-1}^{\text{coarse}}(A \cap B)$$

and a long exact sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow h_*^{\text{coarse}}(A \cap B) & \xrightarrow{\alpha} & h_*^{\text{coarse}}(A) \oplus h_*^{\text{coarse}}(B) & \xrightarrow{\beta} & h_*^{\text{coarse}}(X) \\ & & & & \downarrow d \\ & & \cdots \longleftarrow & & h_{*-1}^{\text{coarse}}(A \cap B) \end{array}$$

called the *coarse Mayer–Vietoris sequence.*
The maps α and β are defined by $\alpha = (i_*, -j_*)$ and $\beta = k_* + l_*$.

The notation $h_*^c(X)$ may sometimes be used instead of $h_*^{\text{coarse}}(X)$.

Just as in ordinary homology theory where the study is often of pairs of topological spaces using the Eilenberg-Steenrod axioms, it is also possible to form a version of coarse homology for pairs of coarse spaces.

Definition 3.36 (Category of coarse pairs). A *pair of coarse spaces* (X, A) is a coarse space X and a subspace $A \subseteq X$. A *coarse map of pairs* $f : (X, A) \to (X', A')$ is a coarse map $f : X \to X'$ such that $f(A) \subseteq A'$. The *category of coarse pairs* is the category where the objects are pairs of coarse spaces and the morphisms are the coarse maps of pairs.

Definition 3.37 (Relative coarse homotopy). A *relative coarse homotopy* $F : I_p(X, A) \to (X', A')$ is a coarse homotopy $F : I_pX \to X'$ such that $F(a, t) \in A'$ for $a \in A$ and $t \in R$.

Definition 3.38 (Relative coarse homology theory). A *relative coarse homology theory* is a sequence of covariant functors $h_*^{\text{coarse}} : \text{CoarsePairs} \to \text{Groups}$ such that:

- if $f, g \colon (X, A) \to (Y, B)$ are relatively coarsely homotopic maps then the induced maps $f_*, g_* \colon h_*^{\text{coarse}}(X, A) \to h_*^{\text{coarse}}(Y, B)$ are equal;

- the inclusions $i : (A, \emptyset) \hookrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ induce a long exact sequence

$$\ldots \to h_*^{\text{coarse}}(A, \emptyset) \to h_*^{\text{coarse}}(X, \emptyset) \to h_*^{\text{coarse}}(X, A) \to h_{*-1}^{\text{coarse}}(A, \emptyset) \to \ldots$$

- if $X = A \cup B$ is coarsely excisive then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $h_*^{\text{coarse}}(A, A \cap B) \to h_*^{\text{coarse}}(X, B)$.

It can be shown that the assignment $h_*^{\text{coarse}}(X) := h_*^{\text{coarse}}(X, \emptyset)$ is a (non-relative) coarse homology theory. This follows from the long exact sequences from the pairs $(A, A \cap B)$ and (X, B) and excision.

Definition 3.39 (X-module). Let X be a coarse topological space. A Hilbert space H is an *X-module* if there exists a $*$-homomorphism $\rho\colon C_0(X) \to B(H)$.

Example 3.40. The Hilbert space $l^2(\mathbb{N})$ is an $\mathbb{N}$-module. To see this, note that $f \in C_0(\mathbb{N})$ is a sequence (a_n) in $\mathbb{N}$ such that there exists an N with $a_n = 0$ for $n \geq N$. Define $\rho((a_n))(b_n) = (a_n b_n)$ for $(b_n) \in l^2(\mathbb{N})$. We know that $|a_n| \leq M$ for all $n \in \mathbb{N}$, so that $\sum_{n=1}^{\infty} |a_n b_n|^2 \leq M^2 \sum_{n=1}^{\infty} |b_n|^2$ (so that $(a_n b_n) \in l^2(\mathbb{N})$) and $||(a_n b_n)||_{l^2} \leq M||(b_n)||_{l^2}$ and therefore the map $\rho : C_0(\mathbb{N}) \to B(l^2(\mathbb{N}))$ is the required $*$-homomorphism.

Definition 3.41 (Compact and locally compact). An operator T on an X-module H is *compact* if the image $\overline{T[B]}$ is compact whenever B is bounded and *locally compact* if $\rho(\varphi)T$ and $T\rho(\varphi)$ are compact for all $\varphi \in C_0(X)$.

We usually suppress mention of the $*$-homomorphism ρ, and write $\rho(\varphi)T$ and $T\rho(\varphi)$ simply as φT and $T\varphi$.

Definition 3.42 (Finite propagation). The *support* of an operator T on an X-module H is defined to be the set of unions of open subsets $U \times V$ such that there exists $\varphi \in C_0(U)$ and $\psi \in C_0(V)$ so that $\varphi T \psi = 0$.

Definition 3.43 (Roe C^*-algebra). If X is a coarse topological space and H is an X-module then the *Roe C^*-algebra* $C^*(X; H)$ is defined to be the norm closure of the locally compact, finite propagation operators.

Example 3.44. The K-theory of the Roe C^*-algebra is the fundamental example of a coarse homology theory. See Lemma 3.5 of [Roe96] for proof of invariance under coarse homotopy and Section 4 of [HRY93] for proof of the existence of the coarse Mayer–Vietoris sequence.

3.2.2 Locally finite homology theories

The aim of this section is to give a concrete definition of a locally finite homology theory. Although the concept of a locally finite homology theory is used and referred to in the area of coarse geometry, a formal definition has not been presented in the current literature.

Definition 3.45 (Proper map). Let X, Y be topological spaces. A map $f\colon X \to Y$ is said to be *proper* if $f^{-1}(K) \subseteq X$ is compact whenever $K \subseteq Y$ is compact.

Definition 3.46 (Proper homotopy). Let $f, g\colon X \to Y$ be proper continuous maps. Then f and g are said to be *proper homotopic* if there exists a proper continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

The category where the objects are topological spaces and the morphisms are proper continuous maps is denoted ProperTop.

Definition 3.47 (Locally finite homology theory). A *locally finite homology theory* is a sequence of covariant functors h_*^{lf} : ProperTop $\to$ Groups such that:

- if $f, g : X \to Y$ are properly homotopic then the induced maps $f_*, g_* : h_*^{\mathrm{lf}}(X) \to h_*^{\mathrm{lf}}(Y)$ are equal;
- for an open inclusion $i : U \hookrightarrow X$ with $U \subseteq X$ open, there is an induced map $i^* : h_*^{\mathrm{lf}}(X) \to h_*^{\mathrm{lf}}(U)$ such that if $j : U' \hookrightarrow U$ is an open inclusion then $(i \circ j)^* = j^* \circ i^*$ and $(\mathrm{id})^* = \mathrm{id}$. Furthermore, if $f : X \to Y$ is proper, $V \subseteq Y$ open and $U = f^{-1}(V)$ then the diagram

$$\begin{array}{ccc} h_*^{\mathrm{lf}}(X) & \longrightarrow & h_*^{\mathrm{lf}}(U) \\ \downarrow & & \downarrow \\ h_*^{\mathrm{lf}}(Y) & \longrightarrow & h_*^{\mathrm{lf}}(V) \end{array}$$

  commutes;
- the sequence

$$\ldots \to h_*^{\mathrm{lf}}(\{\mathrm{pt}\}) \to h_*^{\mathrm{lf}}(V^+) \to h_*^{\mathrm{lf}}(V) \to h_{*-1}^{\mathrm{lf}}(\{\mathrm{pt}\}) \to \ldots$$

  is a long exact sequence for $V \subset V^+$ open;
- if there exist U, V such that X is covered by $U \cup V$ then there is a long exact sequence

$$\begin{array}{ccccccc} \ldots \longrightarrow & h_*^{\mathrm{lf}}\left(\overline{U} \cap \overline{V}\right) & \longrightarrow & h_*^{\mathrm{lf}}\left(\overline{U}\right) \oplus h_*^{\mathrm{lf}}\left(\overline{V}\right) & \longrightarrow & h_*^{\mathrm{lf}}(X) \\ & & & & & \downarrow \\ & & & \ldots & \longleftarrow & h_{*-1}^{\mathrm{lf}}\left(\overline{U} \cap \overline{V}\right) \end{array}$$

induced by the proper inclusions $\overline{U}\cap\overline{V}\hookrightarrow\overline{U}, \overline{U}\cap\overline{V}\hookrightarrow\overline{V}, \overline{U}\hookrightarrow X$ and $\overline{V}\hookrightarrow X$.

Remark 3.48. Applying the second axiom of a locally finite homology theory to the proper map $\{\mathrm{pt}\}\to X^+$ and $X\subset X^+$ open gives us a commutative diagram

$$\begin{array}{ccc} h_*^{\mathrm{lf}}(\{\mathrm{pt}\}) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ h_*^{\mathrm{lf}}(X^+) & \longrightarrow & h_*^{\mathrm{lf}}(X) \end{array}$$

It follows that the composition $h_*^{\mathrm{lf}}(\{\mathrm{pt}\})\to h_*^{\mathrm{lf}}(X^+)\to h_*^{\mathrm{lf}}(X)$ is zero and the map $h_*^{\mathrm{lf}}(\{\mathrm{pt}\})\to h_*^{\mathrm{lf}}(X^+)$ is split injective. In the case where $X=R_+$, as $\{\mathrm{pt}\}$ is proper homotopy equivalent to the one-point compactification of the ray R_+, the long exact sequence condition of a locally finite homology theory implies that $h_*^{\mathrm{lf}}(R_+)=0$.

Remark 3.49. Consider the commutative ladder

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & h_*^{\mathrm{lf}}(\{\mathrm{pt}\}) & \longrightarrow & h_*^{\mathrm{lf}}(X^+) & \underset{\beta_X}{\overset{\alpha_X}{\rightleftarrows}} & h_*^{\mathrm{lf}}(X) & \longrightarrow & h_{*-1}^{\mathrm{lf}}(\{\mathrm{pt}\}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \gamma_{X^+}} & & \downarrow{\scriptstyle \gamma_X} & & \downarrow & & \\ \cdots & \longrightarrow & h_*^{\mathrm{lf}}(\{\mathrm{pt}\}) & \longrightarrow & h_*^{\mathrm{lf}}(U^+) & \xrightarrow{\alpha_U} & h_*^{\mathrm{lf}}(U) & \longrightarrow & h_{*-1}^{\mathrm{lf}}(\{\mathrm{pt}\}) & \longrightarrow & \cdots \end{array}$$

coming from the long exact sequence axiom for a locally finite homology theory.

If $i:U\hookrightarrow X$ is an open inclusion, one can define a wrong way map $i^c:X^+\to U^+$ between the one-point compactifications of X and U by

$$i^c(x)=\begin{cases} i^{-1}(x) & \text{if } x\in U \\ \{\infty\} & \text{if } x\in X^+\backslash U. \end{cases}$$

This map is a proper continuous map, which allows us to form a map $(i^c)_*:h_*^{\mathrm{lf}}(X^+)\to h_*^{\mathrm{lf}}(U^+)$.

Since $\gamma_X\circ\alpha_X=\alpha_U\circ\gamma_{X^+}$ we can define $\gamma_X:h_*^{\mathrm{lf}}(X)\to h_*^{\mathrm{lf}}(U)$ by the composition $\alpha_U\circ\gamma_{X^+}\circ\beta_X$ where β_X is a right inverse to α_X, which exists because of surjectivity coming from the short exact sequence $h_*^{\mathrm{lf}}(\{\mathrm{pt}\})\to h_*^{\mathrm{lf}}(X^+)\to h_*^{\mathrm{lf}}(X)$ from Remark 3.48. Thus it is sufficient to require the wrong way maps for an open inclusion $X\hookrightarrow X^+$.

Example 3.50. Locally finite K-homology (also known as analytic K-homology or Kasparov's K-homology) is the fundamental example of a locally finite homology theory. This is introduced in Section 5.2 of [HR00]. Locally finite K-homology of a space X can be defined via K-theory of C^*-algebras by defining

$$K_*(X)=K_{*+1}(D^*(X)/C^*(X))$$

as shown in Corollary 5.9 of [Roe96].

Definition 3.51 (Uniformly contractible). A metric space X is called *uniformly contractible* if for every $R > 0$ there is $S > 0$ such that for every $x \in X$, the inclusion $B(x, R) \hookrightarrow B(x, S)$ is homotopic to a constant map.

Definition 3.52 (Metric simplicial complex). A metric space X is called a *metric simplicial complex* if it is a simplicial complex equipped with a path metric which coincides on each simplex with the standard metric.

Definition 3.53 (Coarsening). A *coarsening* of a bounded geometry metric space X is a uniformly contractible metric simplicial complex EX equipped with a coarse equivalence $X \to EX$.

Example 3.54. $\mathbb{R}^n$ is a coarsening of $\mathbb{Z}^n$. This follows as $\mathbb{Z}^n$ has bounded geometry and is coarsely equivalent to $\mathbb{R}^n$ which is uniformly contractible and is a metric simplicial complex, when uniformly tesselated by triangles.

If the functors $X \mapsto h_*^{\mathrm{lf}}(X)$ define a locally finite homology theory and a metric space X has a coarsening EX then the coarse homology of X can be defined by

$$h_*^{\mathrm{coarse}}(X) = h_*^{\mathrm{lf}}(EX).$$

As coarsenings do not always exist (for example, $\{n^2 \colon n \in \mathbb{N}\}$ does not have a coarsening), a more general way to define the coarse homology of a space is needed.

Recall from Definition 2.12 that the Lebesgue number $L(\mathcal{U})$ of a cover $\mathcal{U}$ is the largest positive L such that for each $x \in X$, there exists a $U \in \mathcal{U}$ such that $B(x, L) \subseteq U$.

Definition 3.55 (Anti-Čech sequence). An *anti-Čech sequence* $\mathcal{U}_*$ for a metric space X is a family $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X with the properties that the Lebesgue number $L(\mathcal{U}_n) \to \infty$ as $n \to \infty$ and $\mathrm{diam}(\mathcal{U}_n) \leq L(\mathcal{U}_{n+1})$.

Definition 3.56 (Nerve of a cover). The *nerve* of a cover $\mathcal{U}$, denoted by $|\mathcal{U}|$ or $N_{\mathcal{U}}$, is the simplical complex defined abstractly to have the elements of $\mathcal{U}$ as vertices, and the sets $[U_1, U_2, \ldots, U_k]$ span a simplex if and only if $U_1 \cap U_2 \cap \ldots \cap U_k \neq \emptyset$.

If $\mathcal{U}_*$ is an anti-$\breve{C}$ech sequence then for each $V \in \mathcal{U}_i$ there exists a $U \in \mathcal{U}_{i+1}$ with $V \subseteq U$. A *simplicial connecting map* is a map $\phi_i \colon N_{\mathcal{U}_i} \to N_{\mathcal{U}_{i+1}}$ which sends a vertex $[V]$ of $N_{\mathcal{U}_i}$ to a vertex $[U]$ of $N_{\mathcal{U}_{i+1}}$ where $V \subseteq U$. We make the convention that an anti-$\breve{C}$ech sequence comes equipped with a particular choice of connecting maps for each i.

The following definitions come from [Mit01].

Definition 3.57 (Good cover). A *good cover* of a coarse space X is a cover $\{B_i \colon i \in I\}$ such that each B_i is bounded and every $x \in X$ lies in at most finitely many of the sets B_i.

Definition 3.58 (Coarsening). A *coarsening* of a good cover $\mathcal{U}$ is a good cover $\mathcal{V}$ such that there is a map $\phi\colon \mathcal{U} \to \mathcal{V}$ (known as the *coarsening map*) with $U \subseteq \phi[U]$ for all $U \in \mathcal{U}$.

Definition 3.59 (Coarsening family). A *coarsening family* $(\mathcal{U}_i, \phi_{ij})$ is a directed family of good covers of X such that there is a family of controlled sets (M_i) satisfying the following conditions:

- for all $U \in \mathcal{U}_i$ there is a point $x \in X$ such that $U \subseteq M_i(x)$;
- if $x \in X$ and $i < j$ then there is a set $U \in \mathcal{U}_j$ such that $M_i(x) \subseteq U$;
- if $M \subseteq X \times X$ is a controlled set then $M \subseteq M_i$ for some $i \in I$.

It is shown in Proposition 3.4 of [Mit01] that every unital coarse topological space has a coarsening family.

It can be checked that an anti-Čech sequence for a metric space X is a coarsening family for X.

Theorem 3.60 (Theorem 3.10 of [Mit01]). *If the functors $X \mapsto h_*^{lf}(X)$ define a locally finite homology theory on the category of CW-complexes then the functors $X \mapsto h_*^{coarse}(X)$ defined by*

$$h_*^{coarse}(X) = \varinjlim_{n\to\infty} h_*^{lf}(|\mathcal{U}_n|)$$

form a coarse homology theory on the coarse category, where $\mathcal{U}_*$ is a coarsening family for X. □

The Rips complex provides us with another way to coarsen and obtain a coarse homology theory.

Definition 3.61 (Rips complex). If X is a uniformly discrete metric space then the *Rips complex* $R_d(X)$ is a simplicial complex where each element of X is a vertex of $R_d(X)$ and a set of points $\{x_1, \ldots, x_n\}$ spans an n-simplex if and only if $d(x_i, x_j) \leq d$ for every i, j.

The proof of the following proposition is the same as that of Theorem 3.60.

Proposition 3.62. *If the functors $X \mapsto h_*^{lf}(X)$ define a locally finite homology theory on the category of CW-complexes then the functors $X \mapsto h_*^{coarse}(X)$ defined by*

$$h_*^{coarse}(X) = \varinjlim_{d\to\infty} h_*^{lf}(R_d(X))$$

form a coarse homology theory on the category of uniformly discrete metric spaces and coarse maps. □

Let X be a coarse paracompact topological space equipped with an open coarsening family $\mathcal{U}_i$. Let $\{\varphi_U : U \in \mathcal{U}_i\}$ be a partition of unity with $\operatorname{Supp}\varphi_U \subseteq U$ for each $U \in \mathcal{U}_i$. For each $x \in X$, there are only finitely many sets $U \in \mathcal{U}_i$ such that $x \in U$. The sum $\sum_{U\in\mathcal{U}_i} \varphi_U(x)U$ represents a point in the simplex spanning the vertices represented by these sets. Therefore it is possible to define a proper continuous map $\kappa_i : X \to |\mathcal{U}_i|$ by $\kappa_i(x) = \sum_{U\in\mathcal{U}_i} \varphi_U(x)U$. A map $c\colon h_*^{\mathrm{lf}}(X) \to h_*^{\mathrm{coarse}}(X)$ is obtained by applying locally finite homology and taking direct limits. The map c is called the *coarsening map.*

The following proposition tells us that the small-scale and large-scale topology of uniformly contractible spaces are the same.

Proposition 3.63 (Proposition 3.8 of [HR95]). *If functors $X \mapsto h_*^{lf}(X)$ and $X \mapsto h_*^{coarse}(X)$ define a locally finite homology theory and the associated coarse homology theory respectively then the coarsening map $c\colon h_*^{lf}(X) \to h_*^{coarse}(X)$ is an isomorphism if X is a uniformly contractible bounded geometry metric simplicial complex.* □

Definition 3.64 (Relative locally finite homology theory). A *relative locally finite homology theory* is a sequence of covariant functors h_*^{lf} : PPairs → Groups where PPairs is the category of pairs of topological spaces (X, A) where $A \subseteq X$ and the inclusion map $A \hookrightarrow X$ is proper, and the morphisms are proper maps $f : (X, A) \to (Y, B)$ such that:

- if $f, g : (X, A) \to (Y, B)$ are properly homotopic then the induced maps $f_*, g_* : h_*^{\mathrm{lf}}(X, A) \to h_*^{\mathrm{lf}}(Y, B)$ are equal;
- for an open inclusion $i : (U, V) \to (X, A)$ with $U \subseteq X, V \subseteq A$ open, there is an induced map $i^* : h_*^{\mathrm{lf}}(X, A) \to h_*^{\mathrm{lf}}(U, V)$ such that if $j : (U', V') \hookrightarrow (U, V)$ is an open inclusion then $(i \circ j)^* = j^* \circ i^*$ and $(\mathrm{id})^* = \mathrm{id}$. Furthermore, if $f : (X, A) \to (Y, B)$ is proper, C is an open subset of Y, D is an open subset of B and $U = f^{-1}(C), V = f^{-1}(D)$ then there is a commutative diagram
$$\begin{array}{ccc} h_*^{\mathrm{lf}}(X, A) & \longrightarrow & h_*^{\mathrm{lf}}(U, V) \\ \downarrow & & \downarrow \\ h_*^{\mathrm{lf}}(Y, B) & \longrightarrow & h_*^{\mathrm{lf}}(C, D) \end{array}$$
- there is an isomorphism
$$h_*^{\mathrm{lf}}(V^+, \{\mathrm{pt}\}) \to h_*^{\mathrm{lf}}(V, \emptyset);$$
- if (X, A) is a pair then there is a natural long exact sequence
$$\ldots \to h_*^{\mathrm{lf}}(A, \emptyset) \to h_*^{\mathrm{lf}}(X, \emptyset) \to h_*^{\mathrm{lf}}(X, A) \to h_{*-1}^{\mathrm{lf}}(A, \emptyset) \to \ldots$$
- (Excision) if $Z \subseteq A \subseteq X$ where Z is open and $\overline{Z} \subseteq A^\circ$ then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism $i^* : h_*^{\mathrm{lf}}(X, A) \to h_*^{\mathrm{lf}}(X \setminus Z, A \setminus Z)$.

Remark 3.65. Just as in ordinary homology theory, by taking $B = X\backslash Z$ so that $A \cap B = A\backslash Z$, the excision axiom is equivalent to the condition that if there exist A, B such that X is covered by $A^\circ \cup B^\circ$ then the inclusion $i : (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $i_* : h_*^{\mathrm{lf}}(B, A \cap B) \to h_*^{\mathrm{lf}}(X, A)$.

Proposition 3.66. *The assignment $X \mapsto h_*^{lf}(X) := h_*^{lf}(X, \emptyset)$ forms a (non-relative) locally finite homology theory.*

Proof. Suppose there exist U, V such that X is covered by $U^\circ \cup V^\circ$. As $U \subseteq \overline{U}$ and therefore $U^\circ \subseteq \left(\overline{U}\right)^\circ$ it follows that X is also covered by $\left(\overline{U}\right)^\circ \cup \left(\overline{V}\right)^\circ$. Excision (in Remark 3.65) implies that there is an isomorphism $c : h_*^{\mathrm{lf}}\left(\overline{U}, \overline{U} \cap \overline{V}\right) \to h_*^{\mathrm{lf}}\left(X, \overline{V}\right)$. The diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & h_*^{\mathrm{lf}}\left(\overline{U}\cap\overline{V}\right) & \xrightarrow{i_*} & h_*^{\mathrm{lf}}\left(\overline{U}\right) & \longrightarrow & h_*^{\mathrm{lf}}\left(\overline{U},\overline{U}\cap\overline{V}\right) & \xrightarrow{i_2} & h_{*-1}^{\mathrm{lf}}\left(\overline{U}\cap\overline{V}\right) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle j*} & & \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle c} & & \downarrow & \\
\cdots \longrightarrow & h_*^{\mathrm{lf}}\left(\overline{V}\right) & \xrightarrow{k_*} & h_*^{\mathrm{lf}}(X) & \xrightarrow{i_1} & h_*^{\mathrm{lf}}\left(X,\overline{V}\right) & \longrightarrow & h_{*-1}^{\mathrm{lf}}\left(\overline{V}\right) & \longrightarrow \cdots
\end{array}$$

implies that there is a long exact sequence

$$\cdots \longrightarrow h_*^{\mathrm{lf}}\left(\overline{U}\cap\overline{V}\right) \xrightarrow{\alpha} h_*^{\mathrm{lf}}\left(\overline{U}\right) \oplus h_*^{\mathrm{lf}}\left(\overline{V}\right) \xrightarrow{\beta} h_*^{\mathrm{lf}}(X) \xrightarrow{\gamma} h_{*-1}^{\mathrm{lf}}\left(\overline{U}\cap\overline{V}\right) \longrightarrow \cdots$$

where $\alpha = (i_*, -j_*)$, $\beta = k_* - l_*$ and $\gamma = i_2 \circ c^{-1} \circ i_1$. □

It is also possible to coarsen a relative locally finite homology theory.

Lemma 3.67. *If the functors $(X, A) \mapsto h_*^{lf}(X, A)$ define a relative locally finite homology theory then the functors $X \mapsto h_*^{coarse}(X)$ defined by*

$$h_*^{coarse}(X, A) := \varinjlim_i h_*^{lf}\left(|\mathcal{U}_i^X|, |\mathcal{U}_i^A|\right)$$

form a relative coarse homology theory where $\mathcal{U}_i^X$ is a coarsening family for X and $\mathcal{U}_i^A$ is the coarsening family for A defined by $\mathcal{U}_i^A = \{U \cap A : U \in \mathcal{U}_i^X\}$.

Proof. If X and X' are coarse spaces with coarsening families $\mathcal{U}_*^X$ and $\mathcal{U}_*^{X'}$ respectively, then a coarse map $f\colon (X, A) \to (X', A')$ induces a map $\widetilde{f}\colon (|\mathcal{U}_i^X|, |\mathcal{U}_i^A|) \to (|\mathcal{U}_j^{X'}|, |\mathcal{U}_j^{A'}|)$ of simplicial sets. It can be shown that the map $\widetilde{f}$ is proper as the map f is coarse. By functoriality of h_*^{lf} under proper continuous maps, the map $f_*\colon h_*^{\mathrm{lf}}(|\mathcal{U}_i^X|, |\mathcal{U}_i^A|) \to h_*^{\mathrm{lf}}(|\mathcal{U}_j^{X'}|, |\mathcal{U}_j^{A'}|)$ is a homomorphism of abelian groups, and this is also preserved by taking direct limits, so h_*^{coarse} is functorial under coarse maps.

If $f, g\colon (X, A) \to (X', A')$ are relatively coarsely homotopic, then the coarse homotopy $F\colon I_p(X, A) \to X'$ induces a proper continuous map F_* between the nerves of $I_p(X, A)$

and X'. It can be checked that F_* is a proper homotopy between f_* and g_*. Thus $h_*^{\mathrm{lf}}(f_*) = h_*^{\mathrm{lf}}(g_*)$ and by taking the direct limit it follows that $h_*^{\mathrm{coarse}}(f) = h_*^{\mathrm{coarse}}(g)$.

Existence of a long exact sequence

$$\ldots \to h_*^{\mathrm{coarse}}(A, \emptyset) \to h_*^{\mathrm{coarse}}(X, \emptyset) \to h_*^{\mathrm{coarse}}(X, A) \to h_{*-1}^{\mathrm{coarse}}(A, \emptyset) \to \ldots$$

follows immediately from the existence of a long exact sequence

$$\ldots \to h_*^{\mathrm{lf}}(|\mathcal{U}_i^A|, \emptyset) \to h_*^{\mathrm{lf}}(|\mathcal{U}_i^X|, \emptyset) \to h_*^{\mathrm{lf}}(|\mathcal{U}_i^X|, |\mathcal{U}_i^A|) \to h_{*-1}^{\mathrm{lf}}(|\mathcal{U}_i^A|, \emptyset) \to \ldots$$

for each i and the fact that a direct limit of a long exact sequence is again a long exact sequence.

If $X = A \cup B$ is coarsely excisive then as for each i the nerve $|\mathcal{U}_i^X|$ is covered by $|\mathcal{U}_i^A|^\circ \cup |\mathcal{U}_i^B|^\circ$ it follows from excision that we have isomorphisms $h_*^{\mathrm{lf}}(|\mathcal{U}_i^B|, |\mathcal{U}_i^{A\cap B}|) \to h_*^{\mathrm{lf}}(|\mathcal{U}_i^X|, |\mathcal{U}_i^A|)$ as $|\mathcal{U}_i^A| \cap |\mathcal{U}_i^B| = |\mathcal{U}_i^{A\cap B}|$. It follows by taking the direct limit that the map $h_*^{\mathrm{coarse}}(B, A \cap B) \to h_*^{\mathrm{coarse}}(X, A)$ is an isomorphism.

□

3.3 The coarse Baum–Connes conjecture

The coarse Baum–Connes conjecture is of interest for many reasons. This conjecture asks whether or not the assembly map $A_\infty \colon K_*^{\mathrm{coarse}}(X) \to K_*(C^*(X))$ (defined below) is an isomorphism, and this has many implications in other areas of mathematics (usually of a topological nature from the injectivity side, and of a analytic nature from the surjectivity side). The descent principle states that if G is a group which is classified by a finite complex and the coarse Baum–Connes conjecture is true for $|G|$, then the analytic Novikov conjecture is true for G. There are also certain conditions on the group G for which satisfying the coarse Baum–Connes conjecture implies the injectivity of the Baum–Connes assembly map, leading us to results on positive scalar curvature, for example. The coarse K-homology of a space is usually easier to compute than the K-theory of the Roe C^*-algebra, so this conjecture also serves as being an explanation of the right hand side of A_∞.

For any coarse topological space X, there is a short exact sequence

$$0 \longrightarrow C^*(X) \longrightarrow D^*(X) \longrightarrow D^*(X)/C^*(X) \longrightarrow 0.$$

By Example 3.50 (and Bott periodicity), the diagram

$$\begin{array}{ccccc} K_1(C^*(X)) & \longrightarrow & K_1(D^*(X)) & \longrightarrow & K_0(X) \\ {\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle A} \\ K_1(X) & \longleftarrow & K_0(D^*(X)) & \longleftarrow & K_0(C^*(X)) \end{array}$$

is an exact cyclic sequence.

The maps labelled as A are the called *assembly maps.*

Conjecture 3.68 (Coarse Baum–Connes conjecture). *If X is a metric space of bounded geometry then the assembly map*

$$A\colon K_*(X) \to K_*(C^*(X))$$

is an isomorphism.

The conjecture has been shown to be false in general, see [HLS02]. The right hand side of A is functorial for coarse maps, but the left hand side is functorial for proper continuous maps. Thus the only spaces that the conjecture could be expected to hold for are those which are uniformly contractible. The bounded geometry condition is required here as shown in [DFW03]. The authors construct a uniformly contractible metric on $\mathbb{R}^8$ which does not have bounded geometry and such that the assembly map fails to be injective.

One way to overcome this obstacle is to coarsen the K-homology on the left hand side of the assembly map. This gives us functoriality for coarse maps on both sides, and thus it can be expected that this conjecture might hold for a much larger class.

By taking the direct limit of the assembly maps for each $|\mathcal{U}_i|$, we obtain a map $A_\infty\colon K_*^{\text{coarse}}(X) \to K_*(C^*(X))$ such that the diagram

$$\begin{array}{ccc} K_*(X) & & \\ {\scriptstyle c}\downarrow & \searrow^{A} & \\ K_*^{\text{coarse}}(X) & \xrightarrow[A_\infty]{} & K_*(C^*(X)) \end{array}$$

commutes.

Conjecture 3.69 (Coarse Baum–Connes conjecture II). *If X is a metric space of bounded geometry then the assembly map*

$$A_\infty\colon K_*^{coarse}(X) \to K_*(C^*(X))$$

is an isomorphism.

Theorem 3.70 (Coarse Baum–Connes for finite asymptotic dimension). *Let X be a proper metric space of bounded geometry and finite asymptotic dimension. The assembly map*

$$A_\infty \colon K_*^{coarse}(X) \to K_*(C^*(X))$$

is an isomorphism. □

This theorem was first proved by Yu in [Yu98] using methods of an analytic flavour. Another proof was provided by Wright in [Wri02] using methods of a more geometric nature (to be more specific, using the C_0 and hybrid coarse structures). The space of infinite asymptotic dimension given in Example 2.23 does not satisfy the coarse Baum–Connes conjecture.

Chapter 4

Almost flasque spaces

In this chapter, the concept of an almost flasque space is introduced. These are a generalisation of flasque spaces (introduced by Roe in [Roe96]). We will give some concrete examples of both flasque and almost flasque spaces. The conditions for a space to be flasque are conditions which cause the K-theory of the Roe C^*-algebra to be trivial, a useful result for computation when combined with coarsely excisive decompositions and the coarse Mayer–Vietoris sequence. Flasque spaces can be thought of as a coarse version of the notion of "trivial spaces" such as contractible sets in topology. For example, if X is a contractible space then one can show that the open cone of X is coarsely homotopic to the space $\mathbb{R}_+$, our fundamental example of a flasque space.

Many of the results and examples in this chapter are well known and the proofs have been given for completeness. The proofs of Propositions 4.4, 4.14 and 4.16 are straightforward but are not in the current literature. Proposition 4.18 and Theorem 4.27 are new results.

4.1 Flasque spaces

Definition 4.1 (Flasque space). A coarse space X is said to be *flasque* if there exists a map $\alpha\colon X \to X$ such that the following conditions are satisfied:

- for all bounded sets $B \subseteq X$, $B \cap \alpha^k(X) = \emptyset$ for sufficiently large k;
- if $M \subseteq X \times X$ is controlled, then $\bigcup_{k=1}^{\infty} (\alpha \times \alpha)^k(M)$ is controlled;
- the map α is close to the identity map.

Remark 4.2. By Proposition 1.27, the map α is a coarse map since it is close to the identity map (which is coarse).

Example 4.3. The spaces $\mathbb{N}$ and $\mathbb{R}_+$ (with the metric coarse structures) are flasque. The map α which shifts everything to the right by 1 satisfies the above properties for both cases.

It is possible to generalise the flasqueness of $\mathbb{R}_+$ slightly further.

Proposition 4.4. *If* R_+ *is a generalised ray then* R_+ *is flasque.*

Proof. Let $\alpha\colon R_+ \to R_+$ be the map defined by $\alpha(x) = x + 1$.

If $B \subseteq R_+$ is bounded then the closure of B is compact, and therefore is metrically bounded, so B is metrically bounded. Thus $B \cap \alpha^k(R_+) = \emptyset$ for sufficiently large k.

If $M \subseteq R_+ \times R_+$ is controlled then the set

$$\begin{aligned}\bigcup_{k=1}^{\infty} (\alpha \times \alpha)^k(M) &= \left\{(\alpha^k(x), \alpha^k(y))\colon (x,y) \in M, k \in \mathbb{N}\right\} \\ &= \{(x+k, y+k)\colon (x,y) \in M, k \in \mathbb{N}\} \\ &= M + \Delta_{\mathbb{N}}\end{aligned}$$

is also controlled.

By the weakly connected axiom of a coarse structure, the singleton set $\{(0,1)\}$ is controlled. For each $x \in R_+$, the set

$$\begin{aligned}\{(x, \alpha(x))\colon x \in R_+\} &= \{(x, x+1)\colon x \in R_+\} \\ &= \Delta_{R_+} + \{(0,1)\}\end{aligned}$$

is therefore also controlled. It follows that R_+ is flasque. □

Example 4.5. The space $\mathbb{R}_+$ with the C_0 coarse structure is not a generalised ray (see Non-Example 3.8), but is flasque. In [Wri03], it is shown that the map $\alpha\colon \mathbb{R}_+ \to [1, \infty)$ defined by $\alpha(t) = t + \frac{1}{t+1}$ satisfies the required properties of flasqueness. It is shown that if A is C_0 controlled then $(\alpha \times \alpha)(A)$ is contained in A, so is C_0 controlled too. It follows that $\bigcup_{k=1}^{\infty} (\alpha \times \alpha)^k(A)$ is C_0 controlled. It is easy to see that it is C_0 close to the identity.

Non-Example 4.6. The space $\mathbb{R}$ is not flasque. Intuitively, shifting points off to the infinity direction will introduce new points coming from the negative infinity direction. To prove this rigourously, suppose for a contradiction that $\mathbb{R}$ is flasque. Then there exists a map $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that if $d(x, x') < R$ then there exists an $S > 0$ such that $d(\alpha^k(x), \alpha^k(x')) < S$ for all k and such that there is a $C > 0$ such that $d(\alpha(x), x) \leq C$ for all $x \in \mathbb{R}$.

Choose $x \geq 0$ and let B be the interval $(-r, r)$ for some $r > \max\{S, C, x\}$ (i.e. so that $x \in B$). By assumption, there exists an N such that $(-r, r) \cap \alpha^k(\mathbb{R}) = \emptyset$ for $k \geq N$. i.e. for all $x' \in \mathbb{R}$, either $\alpha^k(x') \geq r$ or $\alpha^k(x') \leq -r$ for all $k \geq N$.

Choose a fixed $M \geq N$ and let B' be the interval $(-T, T)$ for $T > MC$. Now choose $y \geq T$ (i.e. outside B'). Since $d(\alpha^M(y), y) \leq MC$, it follows that $\alpha^M(y) \geq r$.

Choose an R-path $x, x_1, \ldots, x_j, y$. Then $d(\alpha^M(x_i), \alpha^M(x_{i+1})) < S$. Suppose without loss of generality that $\alpha^M(x_i) \geq r$ and $\alpha^M(x_{i+1}) \leq -r$. Then

$$d(\alpha^M(x_i), \alpha^M(x_{i+1})) \geq 2r > S,$$

which is a contradiction. Hence if $\alpha^M(x_i) \geq r$ then $\alpha^M(x_{i+1}) \geq r$. As $\alpha^M(y) \geq r$ then it follows that $\alpha^M(x) \geq r$. The same argument shows that if $x \leq 0$ then $\alpha^M(x) \leq -r$.

Recall $d(\alpha^k(-1), \alpha^k(1)) < S$ for all k. Since $\alpha^k(-1) \leq -r$ and $\alpha^k(1) \geq r$ for $k \geq M$, they lie within a bounded set A. It follows that $A \cap \alpha^k(\mathbb{R}) \neq \emptyset$ for $k \geq M$, which is a contradiction.

Non-Example 4.7. The space $X = \{n^2 : n \in \mathbb{N}\}$ (with the metric coarse structure) is not flasque. To see this, suppose that there exists a C such that $d(\alpha(n^2), n^2) \leq C$. Observe there exists an m such that $d((n-1)^2, n^2) > C$ and $d(n^2, (n+1)^2) > C$ for all $n \geq m$. Thus we must have $\alpha(m^2) = m^2$. But then $\{m^2\} \cap \alpha^k(X) \neq \emptyset$ for all k.

Proposition 4.8. *If X is a coarse space and Y is a flasque space then the space $X \times Y$ is flasque.*

Proof. Let $\beta\colon Y \to Y$ be the flasque map for Y. Define $\alpha\colon X \times Y \to X \times Y$ by $\alpha(x, y) = (x, \beta(y))$.

If $B \subseteq X \times Y$ is bounded then $B \subseteq \{(x', y') : (x_0, x') \in M_X, (y_0, y') \in M_Y\}$ for some $(x_0, y_0) \in X \times Y$ and some controlled sets $M_X \subseteq X \times X$ and $M_Y \subseteq Y \times Y$. Hence $B = B_X \times B_Y$, where B_X is bounded for the coarse structure on X and B_Y is bounded for the coarse structure on Y. Therefore

$$\begin{aligned} B \cap \alpha^k(X \times Y) &= (B_X \times B_Y) \cap (X \times \beta^k(Y)) \\ &= (B_X \cap X) \times (B_Y \cap \beta^k(Y)) \\ &= \emptyset \end{aligned}$$

for sufficiently large k as β is the flasque map for Y.

If $M \subseteq (X \times Y) \times (X \times Y)$ is a controlled set then $M \subseteq \{(u, v, x, y) : (u, x) \in$

$M_X, (v, y) \in M_Y\}$ for some controlled $M_X \subseteq X \times X$ and $M_Y \subseteq Y \times Y$. The set

$$\bigcup_{k=1}^{\infty} (\alpha \times \alpha)^k (M) = \{(u, \beta^k(v), x, \beta^k(y)) \colon (u, v, x, y) \in M, k \in \mathbb{N}\}$$

is controlled as $(u, x) \in M_X$ and $(\beta^k(v), \beta^k(y))$ is contained in some controlled set for each k by flasqueness of Y.

Similarly, the set $\{((x, t), \alpha(x, t)) \colon (x, t) \in X \times Y\}$ is controlled as Δ_X is controlled and $\{(t, \beta(t)) \colon t \in Y\}$ is controlled as Y is flasque. It follows that $X \times Y$ is flasque. □

Remark 4.9. Observe that in Proposition 4.8, X itself does not have to be flasque. For example, the space $\{n^2 \colon n \in \mathbb{N}\} \times \mathbb{R}_+$ is flasque.

We have the following result for flasque spaces in K-theory, by using an Eilenberg swindle.

Proposition 4.10 (Proposition 9.4 of [Roe96]). *If X is a flasque space then $K_*(C^*(X)) = 0$.* □

It immediately follows that $K_*(C^*(R_+)) = 0$ for any generalised ray R_+.

Example 4.11. Flasque spaces can be useful in computations, and as an example we compute $K_*(C^*(\mathbb{R}))$. The decomposition $\mathbb{R} = \mathbb{R}_- \cup \mathbb{R}_+$ is coarsely excisive (Example 3.30). It follows from the coarse Mayer–Vietoris sequence that we have a long exact sequence

$$\begin{array}{ccccc} \dots \longrightarrow K_*(C^*(\{0\})) & \longrightarrow & K_*(C^*(\mathbb{R}_-)) \oplus K_*(C^*(\mathbb{R}_+)) & \longrightarrow & K_*(C^*(\mathbb{R})) \\ & & & & \downarrow \\ & \dots \longleftarrow & K_{*-1}(C^*(\mathbb{R}_-)) \oplus K_{*-}(C^*(\mathbb{R}_+)) & \longleftarrow & K_{*-1}(C^*(\{0\})) \end{array}$$

By Proposition 4.10, both $\mathbb{R}_+$ and $\mathbb{R}_-$ are flasque and so there is an isomorphism $K_*(C^*(\mathbb{R})) \to K_{*-1}(C^*(\{0\}))$. It can be shown that $C^*(\{0\})$ is the algebra of compact operators, and it is well known that this has K-theory groups $\mathbb{Z}$ for even degree and trivial for odd degree, and hence that $K_*(C^*(\mathbb{R}))$ is $\mathbb{Z}$ for odd degree, and trivial for even degree (and so it follows that $\mathbb{R}$ is not flasque (in confirmation with Non-Example 4.6)).

Remark 4.12. It can also be shown that if X is flasque then $K_*(D^*(X)) = 0$. It follows from six-term exact sequence in K-theory that $K_*^{\mathrm{coarse}}(X) = 0$, and so also follows that the coarse Baum–Connes conjecture is true for X.

4.2 Almost flasque spaces

The following definition of an almost flasque space comes from Definition 3.1 of [Wri02].

Definition 4.13 (Almost flasque space). A coarse space X is said to be *almost flasque* if there exists a sequence of maps $\alpha_k \colon X \to X$ (with $\alpha_0 = \mathrm{id}_X$) such that the following conditions are satisfied:

- (Properly supported) for all bounded sets $B \subseteq X$, $B \cap \alpha_k(X) = \emptyset$ for all but finitely many k;
- (Uniformly supported) if $M \subseteq X \times X$ is controlled, then there exists a controlled set $B_M \subseteq X \times X$ such that

$$(x, x') \in M \text{ implies } (\alpha_k(x), \alpha_k(x')) \in B_M$$

  for all k;
- (Uniformly close steps) there exists a controlled set $C \subseteq X \times X$ such that

$$(\alpha_k(x), \alpha_{k+1}(x)) \in C,$$

  for all k and for all $x \in X$.

Proposition 4.14. *If X is a flasque space then X is almost flasque.*

Proof. Let $\alpha \colon X \to X$ be the flasque map for X and define a family of maps α_k by $\alpha_k = \alpha^k$. It is easy to see that the first conditions of flasque and almost flasque are equivalent.

Let $M \subseteq X \times X$ be controlled. If $(x, x') \in M$ then for each m, $(\alpha^m(x), \alpha^m(x')) \in \bigcup_{k=1}^{\infty} (\alpha \times \alpha)^k(M)$, and this set is controlled by flasqueness of X.

It can easily be seen that

$$\{(\alpha_k(x), \alpha_{k+1}(x)) \colon x \in X\} \subseteq \{(\alpha_0(x), \alpha_1(x)) \colon x \in X\}$$

for all k, which is controlled as X is flasque. □

Remark 4.15. It is unknown if the converse to this result is true. To date, there are no known examples of almost flasque spaces which are not flasque.

Proposition 4.16. *If X is a coarse space and Y is an almost flasque space then the space $X \times Y$ is almost flasque.*

Proof. Let $\beta_k \colon Y \to Y$ be the family of maps for the almost flasqueness. Define $\alpha_k \colon X \times Y \to X \times Y$ by $\alpha_k(x, y) = (x, \beta_k(y))$.

If $B \subseteq X \times Y$ is bounded then $B \subseteq \{(x', y') \colon (x_0, x') \in M_X, (y_0, y') \in M_Y\}$ for some controlled sets $M_X \subseteq X \times X$ and $M_Y \subseteq Y \times Y$.

It can be seen that $B = B_X \times B_Y$ where B_X and B_Y are bounded since $\{x_0\} \times B_X$ and $\{y_0\} \times B_Y$ are controlled.

Then

$$\begin{aligned} B \cap \alpha_k(X \times Y) &= (B_X \times B_Y) \cap (X \times \beta_k(Y)) \\ &= (B_X \cap X) \times (B_Y \cap \beta_k(Y)) \\ &= \emptyset \end{aligned}$$

for all but finitely many k.

Let $M \subseteq (X \times Y) \times (X \times Y)$, and suppose $(x, y, x', y') \in M$, so that $(x, x') \in M_X$ and $(y, y') \in M_Y$ for some controlled $M_X \subseteq X \times X$ and $M_Y \subseteq Y \times Y$. Then

$$(\alpha_k(x, y), \alpha_k(x', y')) = (x, \beta_k(y), x', \beta_k(y')).$$

Reordering the factors, we get $(x, x', \beta_k(y), \beta_k(y'))$ which is contained inside a controlled set for all k as Y is almost flasque.

To verify the final condition, note that

$$(\alpha_k(x, y), \alpha_{k+1}(x, y)) = (x, \beta_k(y), x, \beta_{k+1}(y)),$$

Reordering the factors again, we get $(x, x, \beta_k(y), \beta_{k+1}(y)) \in \Delta_X \times C_Y$ for all k and $(x, y) \in X \times Y$, which is controlled as Y is almost flasque.

It follows that $X \times Y$ is almost flasque. □

Proposition 4.17 (Lemma 3.11 of [Wri05]). *If X is an almost flasque space then $K_*(C^*(X)) = 0$.* □

Proposition 4.18. *If X and Y are coarsely equivalent spaces then X is almost flasque if and only if Y is almost flasque.*

Proof. Let X be an almost flasque space with family of maps $\alpha_k \colon X \to X$ and suppose that Y is coarsely equivalent to X, where the coarse equivalences are given by $f \colon X \to Y$ and $g \colon Y \to X$. Define $\beta_k \colon Y \to Y$ by $\beta_k = f \circ \alpha_k \circ g$. To show that Y is almost flasque, assume for a contradiction that the first condition for almost flasqueness fails. Then there exists some $B \subseteq Y$ bounded such that

$$\beta_k(Y) \cap B \neq \emptyset$$

for infinitely many k. Then for each k,

$$\begin{aligned}\alpha_k(X) \cap f^{-1}(B) &\supseteq \alpha_k(g(Y)) \cap f^{-1}(B)\\ &= f^{-1}(\beta_k(Y)) \cap f^{-1}(B)\\ &= f^{-1}(\beta_k(Y) \cap B)\\ &\neq \emptyset.\end{aligned}$$

as $\beta_k(Y) \cap B \neq \emptyset$ and the inverse image of a non empty set is again non empty. Hence $\alpha_k(X) \cap f^{-1}(B) \neq \emptyset$ for infinitely many k, a contradiction that X is almost flasque. For the second condition of almost flasqueness, let $M \subseteq Y \times Y$ be a controlled set. There exists a controlled set $B_{g(M)} \subseteq X \times X$ such that $(\alpha_k \times \alpha_k)(g(M)) \subseteq B_{g(M)}$ for all k. By applying f,

$$(\beta_k \times \beta_k)(M) = (f \circ \alpha_k \times f \circ \alpha_k)(g(M)) \subseteq f(B_{g(M)})$$

for all k.

For the final condition of almost flasqueness, we know that there exists a controlled $C \in X \times X$ such that for all k and $x \in X$, $(\alpha_k \times \alpha_{k+1})(\Delta_X) \subseteq C$. And hence that $(\alpha_k \times \alpha_{k+1})(g(\Delta_Y)) \subseteq C$ as $g(\Delta_Y) \subseteq \Delta_X$. By applying f, we have that

$$(\beta_k \times \beta_{k+1})(\Delta_Y) = f(\alpha_k \times \alpha_{k+1})(g(\Delta_Y)) \subseteq f(C),$$

for all k and $y \in Y$. It follows that Y is almost flasque, and that being almost flasque is a coarse invariant. The result also shows the other direction, and hence if X and Y are coarsely equivalent, then either both are almost flasque or neither are. □

4.3 Ends of space

Definition 4.19 (Ray). Let X be a proper geodesic space. A *ray* in X is a proper continuous map $r\colon [0,\infty) \to X$. We say that two rays $r_1, r_2\colon [0,\infty) \to X$ *converge to the same end* if for every compact subset $K \subseteq X$ there exists $C \geq 0$ such that the images $r_1[C,\infty)$ and $r_2[C,\infty)$ lie in the same path-component of the space $X \backslash K$.

We can see that the notion of two rays converging to the same end is an equivalence relation. We define the *set of ends* of X by

$$\text{Ends}(X) = \{\text{end}(r) \,|\, r \text{ is a ray}\},$$

where $\text{end}(r)$ (or sometimes $[r]$) denotes the equivalence class of a ray r. We say that $|\,\text{Ends}(X)|$ is the *number of ends* of X.

Example 4.20. We shall show that $\mathbb{R}$ has two ends. We have two rays r_1, r_2 in $\mathbb{R}$

defined by $r_1(t) = t$ and $r_2(t) = -t$. We can see that $\mathrm{end}(r_1) \neq \mathrm{end}(r_2)$ by definition, since there is no $C \geq 1$ such that $[C, \infty)$ and $(-\infty, -C]$ lie in the same path component of $\mathbb{R}\backslash[-1, 1]$. It follows that there are atleast two equivalence classes in $\mathrm{Ends}(X)$.

Let r_3 be a ray in $\mathbb{R}$, we need to show that $\mathrm{end}(r_3) = \mathrm{end}(r_1)$ or $\mathrm{end}(r_3) = \mathrm{end}(r_2)$. Since r_3 is proper, we know that $r_3^{-1}[\{0\}]$ is compact. Thus $r_3^{-1}[\{0\}]$ is bounded and so there is a $b \geq 0$ such that if $r_3(t) = 0$ then $t < b$.

Hence there exists an $R > 0$ such that $0 \notin r_3[R, \infty)$. By the intermediate value theorem, $r_3[R, \infty) \subseteq (0, \infty)$ or $r_3[R, \infty) \subseteq (-\infty, 0)$.

Suppose without loss of generality that $r_3[R, \infty) \subseteq (0, \infty)$. Then $\mathrm{end}(r_3) = \mathrm{end}(r_1)$, since for every $[c, d] \subseteq \mathbb{R}$, the images $[C, \infty)$ and $r_3[R, \infty)$ lie in the same path component of

$$\mathbb{R}\backslash[c, d] = (-\infty, c) \cup (d, \infty),$$

providing that $C = \min\{d + 1, \inf r_3[R, \infty)\}$.

It follows that $\mathrm{end}(r_3) = \mathrm{end}(r_1)$, so that $\mathbb{R}$ has two ends.

Examples 4.21. It is also an easy exercise to verify that a geodesic space X has 0 ends if and only if it is bounded, and that the space $\mathbb{R}_+$ has one end. For a proof, see Theorem 8.32 of [BH99].

The following definition and lemmas are technical results needed for the proof of our main theorem.

Definition 4.22 (r-path). Let X be a metric space, and let $r > 0$. An *r-path* from x to y is a sequence of points $x = x_0, x_1, x_2, \ldots, x_n = y$ such that $d(x_i, x_{i-1}) \leq r$ for all i.

Lemma 4.23 (Lemma 8.28(1) of [BH99]). *Let X be a proper geodesic space, and let $r_1, r_2 \colon [0, \infty) \to X$ be rays. Let $r > 0$, and $x_0 \in X$. Then $\mathrm{end}(r_1) = \mathrm{end}(r_2)$ if and only if for every $R > 0$ there exists $S > 0$ such that there is an r-path from $r_1(t)$ to $r_2(t)$ in $X\backslash B(x_0, R)$ whenever $t > S$.* □

Lemma 4.24 (Lemma 8.28(2) of [BH99]). *If X is a proper geodesic space and $r \colon [0, \infty) \to X$ is a ray then there exists a geodesic ray $\gamma \colon [0, \infty) \to X$ with $\mathrm{end}(r) = \mathrm{end}(\gamma)$.* □

Theorem 4.25 (Proposition 8.29 of [BH99]). *If X and Y are proper geodesic spaces and $f \colon X \to Y$ is a coarse map then there exists a functorially induced map $f_* \colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ defined by*

$$f_*(\mathrm{end}(r)) = \mathrm{end}(f(r)).$$

If we have coarse maps $f, g \colon X \to Y$ which are close, then $f_* = g_*$. □

Corollary 4.26. *If $f\colon X \to Y$ is a coarse equivalence then the induced map*

$$f_*\colon \text{Ends}(X) \to \text{Ends}(Y)$$

is a bijection. Therefore, the number of ends of a space is a coarse invariant. □

Theorem 4.27. *If X is an almost flasque geodesic metric space then X has one end.*

Proof. Suppose that X is a geodesic metric space with more than one end. We will show that X is not almost flasque. By assumption, there are proper continuous maps $r_1, r_2\colon [0,\infty) \to X$ such that there exists a compact $K \subseteq X$ with the property that for sufficiently large C, the images $r_1[C,\infty)$ and $r_2[C,\infty)$ lie in different path components of $X\backslash K$.

We assume that X is almost flasque and derive a contradiction. Suppose that $\alpha_k\colon X \to X$ are the almost flasque maps, so that if $d(x,x') < R$ then $d(\alpha_k(x), \alpha_k(x')) < S$ for all k. We will show that there exists an M such that for each $c \geq C$, $\alpha_k(r_1(c))$ and $r_1(c)$ lie in the same path component for $k \geq M$.

Choose $x \in r_1[C,\infty)$ and $x' \in r_2[C,\infty)$ and choose an R with $d(x,x') < R$, so that for all k, $d(\alpha_k(x), \alpha_k(x')) < S$. Since $\alpha_k(x)$ and $\alpha_k(x')$ lie in different path components for $k \geq M$, the set $A = \{(\alpha_k(x), \alpha_k(x'))\colon k \geq M\}$ is bounded. It follows that $A \cap \alpha_k(X) \neq \emptyset$ for $k \geq M$, which is a contradiction.

To prove the claim, fix $R > 0$. Then there exists an $S > 0$ such that $d(x,x') < R$ implies

$$d(\alpha_k(x), \alpha_k(x')) < S$$

for all k.

Let B be a ball containing K, of radius greater than $\max(D,S)$, where D is the constant such that $d(\alpha_k(x), \alpha_{k+1}(x)) \leq D$ for each k and all x. Since B is bounded and r_1, r_2 are proper, then there exists a $\gamma \in [0,\infty)$ such that $r_1(\gamma')$ and $r_2(\gamma')$ are outside B for $\gamma' \geq \gamma$.

If B' is a ball with the same centre of larger radius, containing $r_1(\gamma)$, then we have an N such that $\alpha_k(r_1(\gamma)) \cap B' = \emptyset$ for $k \geq N$.

If $\alpha_M(r_1(\gamma))$ and $r_1(\gamma)$ are in different path components, for a fixed $M \geq N$ choose a ball B'' of radius greater than MD. Choose $d \in [0,\infty)$ such that $r_1(d)$ lies outside B''. It follows that $\alpha_M(r_1(d))$ and $r_1(d)$ must lie in the same path component as

$$d(\alpha_M(r_1(d)), r_1(d)) \leq MD$$

by the triangle inequality.

Choose an R-path between $r_1(\gamma)$ and $r_1(d)$ (choosing points along the geodesic segment if necessary). Call this R-path $r_1(c_0), r_1(c_1), \ldots, r_1(c_{j-1}), r_1(c_j)$ with $c_0 = \gamma$ and $c_j = d$.

So

$$d(\alpha_M(r_1(c_i)), \alpha_M(r_1(c_{i+1}))) < S.$$

If $\alpha_M(r_1(c_i))$ and $\alpha_M(r_1(c_{i+1}))$ were in different path components, then we would have

$$d(\alpha_M(r_1(c_i)), \alpha_M(r_1(c_{i+1}))) > 2r(B) > S$$

(by assumption and the first property of almost flasqueness). Hence they do lie in the same path component, so $\alpha_M(r_1(\gamma))$ and $r_1(\gamma)$ do too. □

Chapter 5

Assembly maps and descent

Assembly maps were first introduced by Quinn in [Qui95] and play a central role in the area of surgery theory. An *excisive* functor is a functor from the category of spaces to the category of spectra which is homotopy invariant, preserves homotopy pushout squares and preserves coproducts up to homotopy equivalence. These properties imply that if $\mathbf{F}$ is an excisive functor then $\pi_*(\mathbf{F})$ is a generalised homology theory.

The goal of assembly is to approximate homotopy invariant functors from spaces to spectra by excisive functors from spaces to spectra. Weiss and Williams prove in [WW95] that there exists a best approximation, characterized by a universal property. Specifically, if $\mathbf{F}\colon \text{Spaces} \to \text{Spectra}$ is a homotopy invariant functor then there exists an excisive functor $\mathbf{F}^{\%}\colon \text{Spaces} \to \text{Spectra}$ and a natural transformation $\alpha_{\mathbf{F}}\colon \mathbf{F}^{\%} \to \mathbf{F}$ depending functorially on $\mathbf{F}$, such that $\alpha_{\mathbf{F}}\colon \mathbf{F}^{\%}(\{\text{pt}\}) \to \mathbf{F}(\{\text{pt}\})$ is a homotopy equivalence. The map $\alpha_{\mathbf{F}}$ is called the *assembly map*. Thus there is an induced map

$$(\alpha_{\mathbf{F}})_*\colon \pi_*(\mathbf{F}^{\%}(X)) \to \pi_*(\mathbf{F}(X))$$

where the functors $X \mapsto \pi_*(\mathbf{F}^{\%}(X))$ form a generalised homology theory. Weiss and Williams show that this map is the unique in the sense that if there is another such map $\beta\colon h_* \to \pi_*(\mathbf{F})$ for some generalised homology theory h_* then $\beta = \alpha_{\mathbf{F}} \circ T$ where $T\colon h_* \to \pi_*(\mathbf{F}^{\%})$ is an isomorphism of homology theories. Homotopy invariant functors of interest in algebraic K-theory and L-theory have been well studied via their assembly maps.

In [DL98], Davis and Lück introduce a new viewpoint on assembly and generalise the construction of Weiss and Williams to an equivariant setting to be able to study the Baum–Connes conjecture in topological K-theory and the Farrell–Jones conjectures in algebraic K- and L-theory.

In [Mit10], Mitchener introduces the concept of a *coarse assembly map* via a coarse decomposition of the open cone for any *coarsely excisive* functor, a coarse analogue of an excisive functor. By taking open cones, a coarsely excisive functor produces an excisive functor and the coarse assembly map links this excisive functor to the coarsely excisive functor. An equivariant version is also developed and it is shown that the Novikov conjecture, the Baum–Connes conjecture and the Farrell–Jones conjecture also fit into this picture.

The aim of this chapter is to explore the known results of isomorphism conjectures for assembly maps and coarse assembly maps. An important result is the notion of descent, which says that a coarsely assembly map being an isomorphism implies that the corresponding equivariant coarse assembly map is injective. Some of the theory in this chapter, particularly in the construction of the generalised assembly map and the decomposition of the open cone, differ from that in [Mit10]. This modified theory removes some of the limitations that were present in [Mit10], allowing us to create a more robust framework for assembly maps.

5.1 Spectra

A *spectrum* is a construction in algebraic topology which generalises the idea of the homotopy groups of spaces. An important concept for spectra is that they are designed so that suspension of spectrum is invertible up to homotopy. Spectra are important for stable homotopy theory, and in particular the study of stable homotopy groups of spheres, which are well known for being very difficult to calculate. The Freudenthal suspension theorem makes it easier to calculate these homotopy groups. Specifically this states that the groups $\pi_{n+k}(S^n)$ stabilize for $n \geq k+2$. Our interest in spectra comes from Brown's representability theorem, which states that every generalised homology theory can be represented by a spectrum. Specifically, if h_* is a generalised homology theory then there is a spectrum $\mathbf{H}$ such that $h_*(X) = \pi_*(\mathbf{H}(X))$ where $\mathbf{H}(X) = \mathbf{H} \wedge X$.

Definition 5.1 (Suspension). The *suspension* of a topological space X is the quotient space $\Sigma X = (X \times [0,1])/\sim$ where $\sim$ is the equivalence relation $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$.

If X is a pointed space, it can be shown that ΣX is homeomorphic to the smash product $S^1 \wedge X$ of the unit circle and X.

Definition 5.2 (Spectra). A *spectrum* $\mathbf{E} = \{E_n\}_{n=0}^{\infty}$ is a sequence of pointed spaces equipped with pointed continuous maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$ for $n \geq 0$ (or equivalently, maps $\sigma_n \colon E_n \to \Omega E_{n+1}$).

Definition 5.3 (Function between spectra). A *function* $f \colon \mathbf{E} \to \mathbf{F}$ between spectra is a sequence $\{f_n\colon E_n \to F_n\}$ of maps such that the square

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma f_n} & \Sigma F_n \\ {\scriptstyle\sigma_n}\downarrow & & \downarrow{\scriptstyle\sigma'_n} \\ E_{n+1} & \xrightarrow[f_{n+1}]{} & F_{n+1} \end{array}$$

commutes.

Definition 5.4 (Homotopy groups). The *homotopy groups* of a spectrum $\mathbf{E}$ are defined by the direct limit

$$\pi_k(\mathbf{E}) = \varinjlim_{n} \pi_{n+k}(E_n)$$

for each k via the induced maps $(\sigma_n)_k\colon \pi_k(\Sigma E_n) \to \pi_k(E_{n+1})$.

Using the suspension functor it is possible to produce a spectrum from any pointed space.

Examples 5.5. The *suspension spectrum* $\mathbf{\Sigma^\infty X}$ of a pointed space X is the spectrum given by $(\Sigma^\infty X)_n = \Sigma^n X$. The homotopy groups in this case are the stable homotopy groups $\pi_n^S(X)$.

The *sphere spectrum* $\mathbf{S} = \{S^n\}$ is defined to be the suspension spectrum of the zero dimensional sphere S^0.

It is possible to define the suspension $\mathbf{\Sigma E}$ of a spectrum $\mathbf{E}$ by setting $(\Sigma E)_n = E_{n+1}$. This process of suspension has a one-sided inverse $\mathbf{\Sigma^{-1}E}$ defined by $(\Sigma^{-1}E)_n = E_{n-1}$ for $n \geq 1$ and $(\Sigma^{-1}E)_0 = \{\mathrm{pt}\}$.

Definition 5.6 (Weak homotopy equivalence). A map $f\colon \mathbf{E} \to \mathbf{F}$ between spectra is said to be a *weak homotopy equivalence* if the functorially induced maps

$$f_*\colon \pi_n(\mathbf{E}) \to \pi_n(\mathbf{F})$$

are isomorphisms.

Definition 5.7 (Homotopy lifting property). A map $\pi\colon A \to B$ between topological spaces is said to have the *homotopy lifting property* for X if for any homotopy $f\colon X \times [0,1] \to B$ and for any map $\widetilde{f_0}\colon X \to A$ lifting $f_0 = f|_{X\times\{0\}}$ (so that $f_0 = \pi(\widetilde{f_0})$), there exists a homotopy $\widetilde{f}\colon X \times [0,1] \to A$ lifting f (so that $f = \pi(\widetilde{f})$) with $\widetilde{f_0} = \widetilde{f}|_{X\times\{0\}}$.

Definition 5.8 (Weak fibration). A map $p : E \to X$ between pointed topological spaces is called a *Serre fibration* if p has the homotopy lifting property for all cubes. Pick basepoints $e_0 \in E, b_0 \in B$ such that $b_0 = p(e_0)$. The inverse image $F = p^{-1}(\{b_0\})$ is called the *fibre of* p. The sequence $F \hookrightarrow E \xrightarrow{p} X$ is called a *weak fibration of spaces*. A *weak fibration of spectra* is a sequence $\mathbf{F} \hookrightarrow \mathbf{E} \longrightarrow \mathbf{X}$ of spectra such that each sequence of maps of spaces $F_n \hookrightarrow E_n \longrightarrow X_n$ is a weak fibration.

5.2 Coarsely excisive and properly excisive functors

The following definition is based on the definition of a coarsely excisive functor in [Mit10].

Definition 5.9 (Coarsely excisive functor). A functor $\mathbf{E}$ from the category of coarse spaces to the category of spectra is *coarsely excisive* if the following conditions are satisfied:

- the spectrum $\mathbf{E}(X)$ is weakly contractible whenever the coarse space X is almost flasque;
- the functor $\mathbf{E}$ takes coarse homotopy equivalences to weak homotopy equivalences;
- if $X = A \cup B$ is a coarsely excisive decomposition then there is a homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{E}(A \cap B) & \longrightarrow & \mathbf{E}(A) \\ \downarrow & & \downarrow \\ \mathbf{E}(B) & \longrightarrow & \mathbf{E}(X); \end{array}$$

- if $(X, \mathcal{E}_n)$ is a coarse space for each n and $(X, \mathcal{E}) = (X, \bigcup_{n=1}^{\infty} \mathcal{E}_n)$ then $\pi_*(\mathbf{E}(X, \mathcal{E}))$ and $\varinjlim \pi_*(\mathbf{E}(X, \mathcal{E}_n))$ are isomorphic.

Examples of coarsely excisive functors are given in Chapter 7.

Remark 5.10. The definition of coarsely excisive functor given here differs slightly from the one in [Mit10]. In the definition here, it is required that $\mathbf{E}(X)$ is weakly contractible for almost flasque spaces, not just for flasque spaces. Additionally, the last property is new here. The reasons for including this will be explained in Chapter 6.

Definition 5.11 (Relative coarsely excisive functor). A functor $\mathbf{E}$ from the category of pairs of coarse spaces to the category of spectra is *relatively coarsely excisive* if the following conditions are satisfied:

- the spectrum $\mathbf{E}(X, \emptyset)$ is weakly contractible whenever the coarse space X is almost flasque;
- the functor $\mathbf{E}$ takes relative coarse homotopy equivalences to weak homotopy equivalences;
- for the pair (X, A) there exists a weak fibration

$$\mathbf{E}(A, \emptyset) \to \mathbf{E}(X, \emptyset) \to \mathbf{E}(X, A);$$

- for every coarsely excisive decomposition $X = A \cup B$, the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces a weak homotopy equivalence

$$\mathbf{E}(A, A \cap B) \to \mathbf{E}(X, B);$$

- if $(X, \mathcal{E}_n)$ is a coarse space for each n and $A \subseteq X$ with $(X, \mathcal{E}) = (X, \bigcup_{n=1}^{\infty} \mathcal{E}_n)$ and $(A, \mathcal{E}|_A) = (A, \bigcup_{n=1}^{\infty} \mathcal{E}_n|_A)$ then there is an isomorphism between the groups $\pi_*(\mathbf{E}((X, \mathcal{E}), (A, \mathcal{E}|_A)))$ and $\varinjlim \pi_*(\mathbf{E}((X, \mathcal{E}_n), (A, \mathcal{E}_n|_A)))$.

These axioms are such that if $\mathbf{E}$ is a coarsely excisive functor then the functors $X \mapsto \pi_*(\mathbf{E}(X))$ form a coarse homology theory, and if $\mathbf{E}$ is a relative coarsely excisive functor then the functors $X \mapsto \pi_*(\mathbf{E}(X, A))$ form a relative coarse homology theory.

Remark 5.12. If $\mathbf{E}$ is a relative coarsely excisive functor, the assigment $X \mapsto \mathbf{E}(X) := \mathbf{E}(X, \emptyset)$ is a coarsely excisive functor because if $X = A \cup B$ is coarsely excisive then there exists weak fibrations

$$\mathbf{E}(A \cap B, \emptyset) \to \mathbf{E}(A, \emptyset) \to \mathbf{E}(A, A \cap B)$$

and

$$\mathbf{E}(B, \emptyset) \to \mathbf{E}(X, \emptyset) \to \mathbf{E}(X, B)$$

By definition, the map $\mathbf{E}(A, A \cap B) \to \mathbf{E}(X, B)$ is a weak homotopy equivalence. The homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{E}(A \cap B) & \longrightarrow & \mathbf{E}(A) \\ \downarrow & & \downarrow \\ \mathbf{E}(B) & \longrightarrow & \mathbf{E}(X) \end{array}$$

immediately follows.

The following two definitions are new and are based on the axioms for a locally finite homology theory and a relative locally finite homology theory respectively.

Definition 5.13 (Properly excisive functor). A functor $\mathbf{E}$ from the category ProperTop to the category of spectra is *properly excisive* if the following conditions are satisfied:

- the functor $\mathbf{E}$ takes proper homotopy equivalences to weak homotopy equivalences;
- for an open inclusion $i : U \hookrightarrow X$ with $U \subseteq X$ open, there is an induced map $i^* : \mathbf{E}(X) \to \mathbf{E}(U)$ such that if $j : U' \hookrightarrow U$ is an open inclusion then $(i \circ j)^* = j^* \circ i^*$ and $(\mathrm{id})^* = \mathrm{id}$. Furthermore, if $f : X \to Y$ is proper, V is an open subset of Y and $U = f^{-1}(V)$ then the diagram

$$\begin{array}{ccc} \mathbf{E}(X) & \longrightarrow & \mathbf{E}(U) \\ \downarrow & & \downarrow \\ \mathbf{E}(Y) & \longrightarrow & \mathbf{E}(V) \end{array}$$

  commutes;

- the sequence $\mathbf{E}(\{\mathrm{pt}\}) \to \mathbf{E}(V^+) \to \mathbf{E}(V)$ is a weak fibration for $V \subset V^+$ open;
- if there exist U, V such that X is covered by $U \cup V$ then there is a homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{E}\left(\overline{U} \cap \overline{V}\right) & \longrightarrow & \mathbf{E}\left(\overline{U}\right) \\ \downarrow & & \downarrow \\ \mathbf{E}\left(\overline{V}\right) & \longrightarrow & \mathbf{E}(X). \end{array}$$

Definition 5.14 (Relative properly excisive functor). A functor $\mathbf{E}$ from the category PPairs to the category of spectra is *relative properly excisive* if the following conditions are satisfied:

- the functor $\mathbf{E}$ takes proper homotopy equivalences to weak homotopy equivalences;
- for an open inclusion $i : (U, V) \to (X, A)$ with $U \subseteq X, V \subseteq A$ open, there is an induced map $i^* : \mathbf{E}(X, A) \to \mathbf{E}(U, V)$ such that if $j : (U', V') \hookrightarrow (U, V)$ is an open inclusion then $(i \circ j)^* = j^* \circ i^*$ and $(\mathrm{id})^* = \mathrm{id}$. Furthermore, if $f : (X, A) \to (Y, B)$ is proper, C is an open subset of Y, D is an open subset of B and $U = f^{-1}(C), V = f^{-1}(D)$ then there is a commutative diagram

$$\begin{array}{ccc} \mathbf{E}(X, A) & \longrightarrow & \mathbf{E}(U, V) \\ \downarrow & & \downarrow \\ \mathbf{E}(Y, B) & \longrightarrow & \mathbf{E}(C, D); \end{array}$$

- there is a weak homotopy equivalence

$$\mathbf{E}(V^+, \{\mathrm{pt}\}) \to \mathbf{E}(V, \emptyset);$$

- if (X, A) is a pair then there is a weak fibration

$$\mathbf{E}(A, \emptyset) \to \mathbf{E}(X, \emptyset) \to \mathbf{E}(X, A);$$

- (Excision) if $Z \subseteq A \subseteq X$ where Z is open and $\overline{Z} \subseteq A^\circ$ then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces a weak homotopy equivalence $i^* : \mathbf{E}(X, A) \to \mathbf{E}(X \setminus Z, A \setminus Z)$.

These axioms are such that if $\mathbf{E}$ is a properly excisive functor then the functors $X \mapsto \pi_*(\mathbf{E}(X))$ is a locally finite homology theory, and if $\mathbf{E}$ is a relative properly excisive functor then the functors $X \mapsto \pi_*(\mathbf{E}(X, A))$ is a relative locally finite homology theory.

5.3 Coarse cones

In this section, let X be a coarse topological space.

Definition 5.15 (Pointed cone). The *pointed cone on* X is defined to be the pointed cone on the one-point compactification X^+ with basepoint ∞. That is,

$$\mathbf{C}X = X^+ \times [0,1]/\sim$$

where $(x,0) \sim (\infty,0) \sim (\infty,t)$ for each $x \in X$ and $0 \leq t \leq 1$.

Definition 5.16 (Open cone). The *open cone on* X is the coarse space

$$\mathcal{O}X = X \times (0,1)$$

with continuously controlled coarse structure arising from the compactification $\mathbf{C}X$. Note that this depends only on the topology of X, and not on the coarse structure.

Definition 5.17 (Coarse annulus). The *coarse annulus on* X is the space

$$\mathcal{A}X = X \times [0,1)$$

equipped with the continuously controlled coarse structure arising from the given coarse structure on X, the maximal proper coarse structure on $[0,1)$ and the compactification $X^+ \times [0,1]/\sim$ where $(\infty,0) \sim (\infty,t)$ for all $0 \leq t \leq 1$.

Definition 5.18 (Coarse cusp). The *coarse cusp on* X is the space

$$\mathcal{C}X = X \times (0,1]$$

equipped with the continuously controlled structure arising from the given coarse structure on X, the maximal coarse structure on $(0,1]$ and the one-point compactification $(X \times (0,1])^+$.

Remark 5.19. If X has a proper coarse structure then the continuously controlled coarse structure on the one-point compactification X^+ recovers the original coarse structure.

In the situation of the continuously controlled coarse structure on $\mathcal{O}X$ with respect to $\mathbf{C}X$ observe that $\partial\mathcal{O}X = \mathbf{C}X \backslash \mathcal{O}X = X \times \{0,1\}$.

Recall that a subset $M \subseteq \mathcal{O}X \times \mathcal{O}X$ is strongly controlled if M is controlled with respect to the ambient coarse structure on $\mathcal{O}X$ and if $\overline{M}$ is the closure of the set M in the space $\mathbf{C}X \times \mathbf{C}X$ then

$$\overline{M} \cap ((\mathbf{C}X \times X \times \{0,1\}) \cup (X \times \{0,1\} \times \mathbf{C}X)) \subseteq \Delta_{X \times \{0,1\}}.$$

The following is a technical result providing an equivalent definition of the continuously controlled coarse structure on $\mathcal{O}X$. The proof cannot be found in any current literature.

Lemma 5.20. *A subset $M \subseteq \mathcal{O}X \times \mathcal{O}X$ is strongly controlled if and only if for every convergent sequence $([x_n, s_n], [y_n, t_n])$ in M with either of s_n or t_n converging to 0 or 1, it follows that s_n and t_n both converge to 0 or 1, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.*

Proof. Let $([x_n, s_n], [y_n, t_n])$ be a sequence in $M \subseteq \mathcal{O}X \times \mathcal{O}X$ converging to $([x, s], [y, t])$ in $\overline{M} \subseteq \mathbf{C}X \times \mathbf{C}X$ with either $s = 0$ or 1 or $t = 0$ or 1, and suppose that

$$\overline{M} \cap ((\mathbf{C}X \times X \times \{0, 1\}) \cup (X \times \{0, 1\} \times \mathbf{C}X)) \subseteq \Delta_{X\times\{0,1\}}$$

Since either $s = 0$ or 1 or $t = 0$ or 1, it follows that $([x, s], [y, t]) \in (\mathbf{C}X \times X \times \{0, 1\}) \cup (X \times \{0, 1\} \times \mathbf{C}X)$. By the above condition, $([x, s], [y, t]) \in \Delta_{X\times\{0,1\}}$, so $x = y$ and $s = t = 0$ or 1.

Conversely suppose that for every sequence $([x_n, s_n], [y_n, t_n])$ in $M \subseteq \mathcal{O}X \times \mathcal{O}X$ converging to $([x, s], [y, t])$ in $\overline{M} \subseteq \mathbf{C}X \times \mathbf{C}X$ with either $s = 0$ or 1 or $t = 0$ or 1, we have $x = y$ and $s = t = 0$ or 1.

Let $([x, s], [y, t])$ be the limit of a convergent sequence in $\overline{M}$ with $s = 0$ or 1 or $t = 0$ or 1. It follows that $([x, s], [y, t])$ is also in $(\mathbf{C}X \times X \times \{0, 1\}) \cup (X \times \{0, 1\} \times \mathbf{C}X)$. By assumption, $s = t = 0$ or 1 and $x = y$, so

$$\overline{M} \cap ((\mathbf{C}X \times X \times \{0, 1\}) \cup (X \times \{0, 1\} \times \mathbf{C}X)) \subseteq \Delta_{X\times\{0,1\}}$$

as required. □

Suppose that X is a subset of the unit sphere of some Hilbert space H and let $\varphi\colon [0, 1) \to [0, \infty)$ be a homeomorphism. There is an induced map $\varphi_*\colon \mathcal{O}X \to H$ defined by $\varphi_*([x, t]) = \varphi(t)x$. The coarse space $\mathcal{O}_\varphi X$ is defined to be $\mathrm{Im}(\varphi_*)$ equipped with the metric coarse structure coming from the metric of the Hilbert space H.

Proposition 5.21 (Proposition 6.2.1 of [HR00]). *Let X be a compact subset of the unit sphere of a Hilbert space H. If $\varphi\colon [0, 1) \to [0, \infty)$ is a homeomorphism then any controlled set for $\mathcal{O}_\varphi X$ is also controlled for $\mathcal{O}X$. If M is a controlled set for $\mathcal{O}X$ then there is a homeomorphism $\varphi\colon [0, 1) \to [0, \infty)$ such that the set M is controlled for the space $\mathcal{O}_\varphi X$.* □

The following result is known, but the proof cannot be found in the current literature.

Proposition 5.22. *If $f\colon X \to Y$ is a proper continuous map then the induced map $f_*\colon \mathcal{O}X \to \mathcal{O}Y$ defined by $f_*([x, t]) = [f(x), t]$ is coarse.*

Proof. Let $M \subseteq \mathcal{O}X \times \mathcal{O}X$ be a strongly controlled set, and let $([x_n, s_n], [y_n, t_n])$ be a sequence in M converging to $([x, s], [y, t])$ with either $s = 0$ or 1 or $t = 0$ or 1. By Lemma 5.20, we know that $x = y$ and $s = t = 0$ or 1. By definition $(f_* \times f_*)([x_n, s_n], [y_n, t_n]) = ([f(x_n), s_n], [f(y_n), t_n])$ and as f is continuous, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$. By Lemma 5.20, $(f_* \times f_*)(M)$ is strongly controlled.

If $K \subseteq \mathcal{O}Y$ is compact then $K \subseteq K_Y \times K_{(0,1)}$ with K_Y and $K_{(0,1)}$ compact. $f_*^{-1}(K) \subseteq f_*^{-1}(K_Y \times K_{(0,1)}) = f^{-1}(K_Y) \times K_{(0,1)}$ is compact as f is proper and $f_*^{-1}(x, t) = (f^{-1}(x), t)$. As $\mathcal{O}X$ is a coarse topological space, $B \subseteq \mathcal{O}Y$ is bounded if and only if $\overline{B} \subseteq \mathbf{C}Y$ is compact. As f_* is proper, $f_*^{-1}(\overline{B})$ is compact, so $\overline{f_*^{-1}(B)} \subseteq \mathbf{C}X$ is compact (as a closed subset), which is if and only if $f_*^{-1}(B) \subseteq \mathcal{O}X$ is bounded. It follows that if B is bounded then $f_*^{-1}(B)$ is bounded, and so f_* is a coarse map. □

Proposition 5.23. *Every inclusion $X \hookrightarrow X^+$ induces a map $h_*(\mathcal{O}X^+) \to h_*(\mathcal{O}X)$.*

Proof. Using reparametrisation, write $\mathcal{O}X = X \times (-\infty, \infty)$ and $\mathcal{A}X = X \times [0, \infty)$. Let X^+ be the one-point compactification of X with the point at infinity denoted by $\{\infty\}$. It is clear that $\mathcal{O}X^+$ is coarsely equivalent to $\mathcal{A}X^+ \cup \{\infty\} \times (-\infty, \infty)$. It is also true that $\mathcal{O}X$ and $\mathcal{A}X^+$ are coarsely equivalent. To see this, choose a basepoint $x_0 \in X$ and define $f : \mathcal{O}X \to \mathcal{A}X^+$ by

$$f(x,t) = \begin{cases} (x,t) & \text{if } t \geq d(x, x_0) \\ (\infty, |t| + d(x, x_0)) & \text{if } t < d(x, x_0) \end{cases}$$

Note that $x \to \infty$ if and only if $t \to \infty$. The map f is a coarse equivalence.

The decomposition $\mathcal{O}X^+ = \mathcal{A}X^+ \cup \{\infty\} \times (-\infty, \infty)$ is coarsely excisive so there is a coarse Mayer-Vietoris sequence

$$\ldots \longrightarrow h_*(\{\infty\} \times [0,\infty)) \longrightarrow h_*(\mathcal{A}X^+) \oplus h_*(\{\infty\} \times (-\infty,\infty)) \longrightarrow h_*(\mathcal{O}X^+) \longrightarrow \ldots$$

which reduces to

$$\ldots \longrightarrow \{0\} \longrightarrow h_*(\mathcal{O}X) \oplus h_*(\mathcal{O}\{\infty\}) \xrightarrow{\cong} h_*(\mathcal{O}X^+) \longrightarrow \{0\} \longrightarrow \ldots$$

as $\{\infty\} \times [0, \infty)$ is flasque and the map $f : \mathcal{O}X \to \mathcal{A}X^+$ is a coarse equivalence. This gives us an isomorphism $h_*(\mathcal{O}X) \oplus h_*(\mathcal{O}\{\infty\}) \to h_*(\mathcal{O}X^+)$ and hence the inverse isomorphism gives us a map $h_*(\mathcal{O}X^+) \to h_*(\mathcal{O}X)$. □

The following proof is based on Theorem 4.9 in [Mit10].

Theorem 5.24. *If $\boldsymbol{E}$ is a relative coarsely excisive functor then the mapping $(X, A) \mapsto \boldsymbol{E}(\mathcal{O}X, \mathcal{O}A)$ is a relative properly excisive functor.*

Proof. The spaces $[0, 1)$ and $[0, \infty)$ are homeomorphic via the homeomorphism $\varphi\colon [0, 1) \to [0, \infty)$ defined by $\varphi(t) = t/(1-t)$. The space $R_+ = [0, \infty)$ is a generalised ray equipped with the continuously controlled structure with respect to the compactification $\overline{R_+} = [0, \infty) \cup \{\infty\}$. The map $p\colon \mathcal{O}X \to R_+$ defined by $p([x, t]) = \varphi(t)$ is a controlled map.

If $f\colon X \to Y$ is a proper homotopy equivalence then there exists a proper continuous map $g\colon Y \to X$ and a proper continuous map $H\colon X \times [0, 1] \to Y$ such that $H(x, 0) = gf(x)$ and $H(x, 1) = \mathrm{id}_X(x)$. By Lemma 5.22, the induced map $H_*\colon \mathcal{O}(X\times[0, 1]) \to \mathcal{O}Y$ defined by $H_*([(x, s), t]) = [H(x, s), t]$ is coarse. The map $i\colon I_p\mathcal{O}X \to \mathcal{O}(X \times [0, 1])$ defined by $i([x, t], s) = [(x, (1-t)s), t]$ is a coarse equivalence, with inverse given by j : $\mathcal{O}(X \times [0, 1]) \to\colon I_p\mathcal{O}X$ defined by $j([x, t], s) = [(x, s), t/(1-s)]$. By composition there is a coarse map $H'\colon I_p\mathcal{O}X \to \mathcal{O}Y$ defined by $H' = H_* \circ i$ such that that $H'([x, t], 0) = [gf(x), t] = (gf)_*([x, t])$ and $H'([x, t], p([x, t]) + 1) = [x, t] = (\mathrm{id}_X)_*([x, t])$. It follows that f_* is a coarse homotopy equivalence.

Proposition 5.23 provides us with a map $i^* : \mathbf{E}(X^+) \to \mathbf{E}(X)$ for any open inclusion $i : X \hookrightarrow X^+$.

To show that the map $\mathbf{E}(\mathcal{O}V^+, \mathcal{O}\{\mathrm{pt}\}) \to \mathbf{E}(\mathcal{O}V, \emptyset)$ is a weak homotopy equivalence, note that $\mathcal{O}V^+ = \mathcal{O}V \cup \mathcal{O}\{\mathrm{pt}\}$ is a coarsely excisive decomposition and $\mathcal{O}V \cap \mathcal{O}\{\mathrm{pt}\}$ is empty. The result now follows from coarse excision.

Suppose $Z \subseteq A \subseteq X$ with Z open and $\overline{Z} \subseteq A^\circ$. Decompose $\mathcal{O}X$ as $\mathcal{O}(X\backslash Z) \cup \mathcal{O}A$. Observe that $\mathcal{O}(X\backslash Z) \cap \mathcal{O}A = \mathcal{O}(A\backslash Z)$ as $\mathcal{O}(A\backslash Z) = \mathcal{O}A\backslash\mathcal{O}Z$. To prove this is a coarsely excisive decomposition, note that the condition $\overline{Z} \subseteq A^\circ$ implies that $X = X\backslash\overline{Z} \cup A^\circ$. Observe that $X\backslash\overline{Z}$ and A° are both open, so $\mathcal{O}X = \mathcal{O}(X\backslash\overline{Z}) \cup \mathcal{O}A^\circ$ is a coarsely excisive decompostion. It follows that the map $\mathbf{E}(\mathcal{O}X, \mathcal{O}A) \to \mathbf{E}(\mathcal{O}(X\backslash Z), \mathcal{O}(A\backslash Z))$ is a weak homotopy equivalence, as $\mathcal{O}(X\backslash\overline{Z})$ is coarsely equivalent to $\mathcal{O}(X\backslash Z)$ and $\mathcal{O}A^\circ$ is coarsely equivalent to $\mathcal{O}A$.

□

It therefore follows that if $\mathbf{E}$ is a coarsely excisive functor then the functors $X \mapsto \pi_*(\mathbf{E}(\mathcal{O}X))$ form a locally finite homology theory.

5.4 Generalised assembly

From now on, the notation $h_*(X)$ will be used to denote $\pi_*(\mathbf{E}(X))$ for a coarsely excisive functor $\mathbf{E}$. It follows from Proposition 3.60 that there is a coarse homology theory defined by

$$k_*^{\mathrm{coarse}}(X) = \varinjlim_i k_*(|\mathcal{U}_i|)$$

where $k_*(X) = h_*(\mathcal{O}X)$.

We now show how the properly excisive functor $X \mapsto \mathbf{E}(\mathcal{O}X)$ for a coarsely excisive functor $X \mapsto \mathbf{E}(X)$ fits into the picture of generalised assembly.

Note that $\mathcal{C}X$ and $\mathcal{A}X$ are reparametrisations of the subspaces $X \times (0, 1/2]$ in $\mathcal{O}X$ and $X \times [1/2, 1)$ in $\mathcal{O}X$ respectively. The decomposition $\mathcal{O}X = \mathcal{C}X \cup \mathcal{A}X$ is therefore coarsely excisive.

Lemma 5.25. *The space $\mathcal{C}X$ is flasque.*

Proof. By Remark 5.19, the coarse space $\mathcal{C}X$ is $X \times (0, 1]$ with its given coarse structure on X. As $(0, 1]$ is flasque, it follows that $\mathcal{C}X$ is flasque. □

Remark 5.26. As the decomposition $\mathcal{O}X = \mathcal{C}X \cup \mathcal{A}X$ is coarsely excisive, there is a homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{E}(X) & \longrightarrow & \mathbf{E}(\mathcal{A}X) \\ \downarrow & & \downarrow \\ \mathbf{E}(\mathcal{C}X) & \longrightarrow & \mathbf{E}(\mathcal{O}X) \end{array}$$

As the space $\mathcal{C}X$ is flasque, it follows that there exists a weak fibration

$$\mathbf{E}(X) \to \mathbf{E}(\mathcal{A}X) \to \mathbf{E}(\mathcal{O}X)$$

and therefore a long exact sequence

$$\ldots \to h_*(X) \to h_*(\mathcal{A}X) \to h_*(\mathcal{O}X) \to h_{*-1}(X) \to \ldots$$

where $h_*(X) := \pi_*(\mathbf{E}(X))$.

The *coarse assembly map* associated to the functor $\mathbf{E}$ is the boundary map

$$\delta \colon \Omega\mathbf{E}(\mathcal{O}X) \to \mathbf{E}(X)$$

of this long exact sequence.

It is not known whether or not the coarse assembly map is unique.

The *coarse isomorphism conjecture* associated to the functor $\mathbf{E}$ asserts that the coarse assembly map is a weak homotopy equivalence for X where X has some desired properties.

Remark 5.27. It is easy to check that the coarse isomorphism conjecture holds for X if and only if $\mathcal{A}X$ is weakly contractible.

It is straightforward to prove the following (using the map $\alpha : \mathcal{A}\{\mathrm{pt}\} \to \mathcal{A}\{\mathrm{pt}\}$ defined by $\alpha([\mathrm{pt}, t]) = [\mathrm{pt}, 1/(2 - t)]$).

Proposition 5.28. *The space $\mathcal{A}\{\mathrm{pt}\}$ is almost flasque.*

It follows that the coarse isomorphism conjecture holds for a point, or equivalently, any bounded set. This theory has an advantage over the theory described in [Mit10], where it not clear whether or not the coarse isomorphism conjecture holds for a point in general.

The following theorem shows that the coarse isomorphism conjecture is true for a large number of spaces.

Theorem 5.29 (Theorem 5.9 of [Mit10]). *If $\boldsymbol{E}$ is a coarsely excisive functor then the coarse isomorphism conjecture holds for finite coarse CW-complexes.* □

5.5 Equivariant assembly

The Baum–Connes and Farrell–Jones conjectures are the main reason for studying equivariant assembly. The Baum–Connes conjecture states that if G is a discrete group then the map $K_*^G(\underline{E}G) \to K_*(C_r^*(G))$ is an isomorphism. In the case where G is torsion free (there are no non-trivial elements of finite order) then this reduces to asking if the map $K_*(BG) \to K_*(C_r^*(G))$ is an isomorphism. The Farrell–Jones conjecture states that the map $H_*^G(\underline{E}G; \mathbf{K}(R)) \to K_*(RG)$ is an isomorphism, where $H_*^G(-; \mathbf{K}(R))$ is an appropriate G-homology theory. In the case where G is torsion free and R is a regular ring then this reduces to asking if the map $H_*(BG; \mathbf{K}(R)) \to K_*(RG)$ is an isomorphism where $H_*(-; \mathbf{K}(R))$ is the homology theory associated to the algebraic K-theory spectrum of R satisfying $H_*(\mathrm{pt}; \mathbf{K}(R)) = \pi_*(\mathbf{K}(R)) = K_*(R)$ and RG is the group ring with R associative with unit.

There are no known groups for which either of these conjectures is false. The Farrell–Jones conjecture was originally of interest by topologists for the case where $R = \mathbb{Z}$ in the L-theory variant as this implies several famous conjectures such as the Novikov conjecture and the Borel conjecture regarding topological rigidity in surgery theory.

5.5.1 The Davis–Lück assembly map

In [DL98], Davis and Lück give an equivariant analogue of the theorem of Weiss and Williams and show that that there exists a best approximation of G-homotopy invariant functors by G-excisive functors, characterized by a universal property. Specifically, if $\mathbf{F}\colon G\text{-Spaces} \to \text{Spectra}$ is a G-homotopy invariant functor then there exists an G-excisive functor $\mathbf{F}^{\%}\colon G\text{-Spaces} \to \text{Spectra}$ and a natural transformation $\alpha_{\mathbf{F}}\colon \mathbf{F}^{\%} \to \mathbf{F}$ such that $\alpha_{\mathbf{F}}\colon \mathbf{F}^{\%}(G/H) \to \mathbf{F}(G/H)$ is a stable equivalence for every finite subgroup H of G. It is also shown that the pair $(\mathbf{F}^{\%}, \alpha_{\mathbf{F}})$ is unique up to weak equivalence.

Definition 5.30 (Topological group). A *topological group* G is a group that is also a topological space with the condition that the group operations

$$G \times G \to G\colon (g, g') \mapsto gg'$$

and

$$G \to G\colon g \mapsto g^{-1}$$

are continuous.

Definition 5.31 (Discrete group). A *discrete group* is a topological group G equipped with the discrete topology, that is, every subset of G is open.

Observe that any group can be considered as a discrete group. All groups in this section will be assumed to be discrete.

Definition 5.32 (Group action). If G is a group and X is a set then a *group action* of G on X (also known as a G*-action*) is a map $\varphi\colon G \times X \to X$ defined by $\varphi(g, x) = g.x$ such that the conditions $e.x = x$ for all $x \in X$ and $(gg').x = g.(g'.x)$ for all $g, g' \in G$ and all $x \in X$ are satisfied. The set X in this situation is called a G*-set*.

For a G-set, the *orbit* of $x \in X$ is the set $G.x = \{g.x\colon g \in G\}$. There is an equivalence relation on X by saying $x \sim y$ if and only if there exists a $g \in G$ with $g.x = y$ (equivalent to $x \sim y$ if and only if $G.x = G.y$). The orbits are therefore the equivalence classes under this relation and the set of all orbits of X under the action of G is known as the *quotient* of the action and is denoted by X/G.

Definition 5.33 (Equivariant map). An *equivariant map* $f\colon X \to Y$ between G-sets is a map such that $f(g.x) = g.f(x)$ for all $g \in G$ and all $x \in X$.

Definition 5.34 (G-CW-complex). A G*-CW-complex* X is the union of G-spaces X^n such that X^0 is a disjoint union of orbits of the form G/H for some subgroup H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along attaching G-maps $G/H \times S^n \to X^n$. The attaching map is determined by its restriction $S^n \to (X^n)^H$.

Definition 5.35 (G-properly excisive functor). A functor $\mathbf{E}_G$ from the category of topological spaces with a G-action that are homotopy equivalent to G-CW-complexes to the category of spectra is called G*-properly excisive* if the following conditions are satisfied:

- the functor $\mathbf{E}_G$ takes G-proper homotopy equivalences of spaces to weak homotopy equivalences of spectra;
- for an open inclusion $i : U \hookrightarrow X$ with open G-invariant $U \subseteq X$ (that is, $G.U = U$), there is an induced map $i^* : \mathbf{E}_G(X) \to \mathbf{E}_G(U)$ such that if $j : U' \hookrightarrow U$ is an open inclusion then $(i \circ j)^* = j^* \circ i^*$ and $(\mathrm{id})^* = \mathrm{id}$. Furthermore, if $f : X \to Y$

is G-proper, V is an open G-invariant subset of Y and $U = f^{-1}(V)$ then the diagram

$$\begin{array}{ccc} \mathbf{E}_G(X) & \longrightarrow & \mathbf{E}_G(U) \\ \downarrow & & \downarrow \\ \mathbf{E}_G(Y) & \longrightarrow & \mathbf{E}_G(V) \end{array}$$

commutes;

- the sequence $\mathbf{E}_G(\{\mathrm{pt}\}) \to \mathbf{E}_G(V^+) \to \mathbf{E}_G(V)$ is a weak fibration for open G-invariant $V \subset V^+$;
- if there exist G-invariant U, V such that X is covered by $U \cup V$ then there is a homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{E}_G\left(\overline{U} \cap \overline{V}\right) & \longrightarrow & \mathbf{E}_G\left(\overline{U}\right) \\ \downarrow & & \downarrow \\ \mathbf{E}_G\left(\overline{V}\right) & \longrightarrow & \mathbf{E}_G(X). \end{array}$$

Definition 5.36 (Davis-Lück assembly map). If G is a discrete group and $\mathbf{E}_G$ is a G-excisive functor then the *Davis-Lück assembly map* is the map

$$\alpha_X \colon \mathbf{E}_G(X) \to \mathbf{E}_G(\{\mathrm{pt}\})$$

induced by the projection map $X \to \{\mathrm{pt}\}$.

The ***$\mathbf{E}_G$-assembly map*** is the induced map $(\alpha_X)_* \colon \pi_*(\mathbf{E}_G(X)) \to \pi_*(\mathbf{E}_G(\{\mathrm{pt}\}))$.

A G-space X is *free* if for each $x \in X$, $g.x = x$ implies that $g = e$. The space EG is defined to be a weakly contractible free G-CW-complex. This space is unique up to G-homotopy equivalence (see [DL98]) and the quotient space EG/G is the classifying space BG (i.e. the fundamental group of BG is isomorphic to G and all higher homotopy groups are trivial). A choice of BG is called a *model*, e.g. S^1 is a model for $B\mathbb{Z}$.

A family $\mathcal{F}$ of subgroups of a group G is a collection which is closed under conjugation and finite intersections. Important examples of this are $\mathcal{F} = \mathcal{FIN}$, the family of finite subgroups of G, and $\mathcal{F} = \mathcal{VA}$, the family of all virtually cyclic subgroups (that is, has a cyclic subgroup of finite index).

Definition 5.37 (Classifying space for $\mathcal{F}$). A *classifying space for $\mathcal{F}$* is a space $\mathcal{E}_{\mathcal{F}}(G)$ with the property that the fixed point set $\mathcal{E}_{\mathcal{F}}(G)^H = \{x \in \mathcal{E}_{\mathcal{F}}(G) \colon h.x = x \text{ for all } h \in H\}$ is G-contractible if $H \in \mathcal{F}$ and empty if $H \notin \mathcal{F}$.

As a special case, if $\mathcal{F}$ is the family consisting of only the trivial subgroup of G then $\mathcal{E}_{\mathcal{F}}(G) = EG$, and if $\mathcal{F}$ consists of all finite subgroups then this is the classifying space $\underline{E}G$. It can be shown that $\mathcal{E}_{\mathcal{F}}(G)$ is unique up to G-homotopy and that there is always a choice of a uniformly contractible $\mathcal{E}_{\mathcal{F}}(G)$.

It is shown in [DL98] that both of the Baum–Connes and the Farrell–Jones assembly maps fit into this framework. Specifically, the Baum–Connes assembly map is the Davis–Lück assembly map

$$K_*^G(\mathcal{E}_{\mathcal{FIN}}(G)) \to K_*(C_r^*(G))$$

where $\mathbf{E}_G = \mathbf{K}^{\text{top}}$ and $X = \mathcal{E}_{\mathcal{FIN}}(G)$.

The Farrell–Jones assembly map in algebraic K-theory is the Davis–Lück assembly map

$$H_*^G(\mathcal{E}_{\mathcal{VA}}(G); \mathbf{K}(R)) \to K_*(RG)$$

where $\mathbf{E}_G = \mathbf{K}^{\text{alg}}$ and $X = \mathcal{E}_{\mathcal{VA}}(G)$.

Lemma 2.4 of [DL98] shows that $\pi_*(\mathbf{K}^{\text{top}}(\{\text{pt}\})) = K_*(C_r^*(G))$ and $\pi_*(\mathbf{K}^{\text{alg}}(\{\text{pt}\})) = K_*(RG)$.

5.5.2 Equivariant coarse assembly

We now give the required definitions and results in order to discuss the coarse version of equivariant assembly.

Definition 5.38 (Coarse G-space). A *coarse G-space* is a coarse space X equipped with a G-action $\varphi : G \times X \to X$ such that for each $g \in G$, the map $x \mapsto \varphi(g, x) = g.x$ is coarse.

Definition 5.39 (Cobounded). A subset A of a coarse G-space X is said to be *cobounded* if there is a bounded subset B of X such that $A \subseteq G.B$.

Definition 5.40 (Coarse G-category). The *coarse G-category* is the category of coarse G-spaces and controlled equivariant maps with the condition that the inverse image of each cobounded set is cobounded (known as *coarse G-maps*).

If X is a coarse G-space and $p\colon X \to R_+$ is coarse G-map, the group G acts on the cylinder I_pX by writing $g.(x, t) = (g.x, t)$. The inclusion maps $i_0, i_1\colon X \to I_pX$ defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, p(x) + 1)$ can easily be verified to be coarse G-maps.

Definition 5.41 (Coarse G-homotopy). Let $f, g\colon X \to Y$ be coarse G-maps. A *coarse G-homotopy* between f and g is a coarse G-map $H\colon I_pX \to Y$ for some controlled G-map $p\colon X \to R$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Definition 5.42 (Coarse G-homotopy equivalence). A coarse G-map $f\colon X \to Y$ is a *coarse G-homotopy equivalence* if there exists a coarse G-map $g\colon Y \to X$ such that $g \circ f$ is coarsely G-homotopic to id_X and $f \circ g$ is coarsely G-homotopic to id_Y.

Definition 5.43 (G-almost flasque space). A G-coarse space X is called *G-almost flasque* if there exists a sequence of equivariant maps $\alpha_k\colon X \to X$ (with $\alpha_0 = \mathrm{id}_X$) satisfying the following conditions:

- for all bounded sets $B \subseteq X$, $B \cap \alpha_k(X) = \emptyset$ for all but finitely many k;
- if $M \subseteq X \times X$ is controlled, then there exists a controlled set $B_M \subseteq X \times X$ such that
$$(x, x') \in M \text{ implies } (\alpha_k(x), \alpha_k(x')) \in B_M$$
for all k;
- there exists a controlled set $C \subseteq X \times X$ such that
$$(\alpha_k(x), \alpha_{k+1}(x)) \in C,$$
for all k and for all $x \in X$.

Definition 5.44 (Coarsely G-excisive decomposition). A *coarsely G-excisive decomposition* $X = A \cup B$ of a coarse G-space X is a coarsely excisive decomposition where A and B are both coarse G-spaces.

Definition 5.45 (Coarsely G-excisive functor). A functor $\mathbf{E}_G$ from the coarse G-category to the category of spectra is called *coarsely G-excisive* if the following conditions are satisfied:

- the spectrum $\mathbf{E}_G(X)$ is weakly contractible whenever X is G-almost flasque;
- the functor $\mathbf{E}_G$ takes all coarse G-homotopy equivalences to weak homotopy equivalences;
- if $X = A \cup B$ is a coarsely G-excisive decomposition then there is a homotopy pushout diagram
$$\begin{array}{ccc} \mathbf{E}_G(A \cap B) & \longrightarrow & \mathbf{E}_G(A) \\ \downarrow & & \downarrow \\ \mathbf{E}_G(B) & \longrightarrow & \mathbf{E}_G(X); \end{array}$$
- if X is a cobounded coarse G-space then the constant map $c\colon X \to \{\mathrm{pt}\}$ induces a stable equivalence $c_*\colon \mathbf{E}_G(X) \to \mathbf{E}_G(\{\mathrm{pt}\})$.

The last condition of a coarsely G-excisive functor is an equivariant version of requiring a bounded set to be coarsely equivalent to a point.

Definition 5.46 (Cocompact). A G-space X is *cocompact* if there is a compact subset $K \subseteq X$ such that $X = G.K$.

The following is proved similarly to Theorem 5.24.

Theorem 5.47. *If $\mathbf{E}_G$ is a coarsely G-excisive functor then the mapping $X \mapsto \mathbf{E}_G(\mathcal{O}X)$ defines a G-properly excisive functor on the category of cocompact Hausdorff G-spaces.*

□

It is immediate that if $\mathbf{E}_G$ is a coarsely G-excisive functor then the functors $X \mapsto \pi_*(\mathbf{E}_G(X))$ form a G-locally finite homology theory.

The following is proved similarly to Remark 5.26.

Lemma 5.48. *If X is a coarse Hausdorff G-space and $\boldsymbol{E}_G$ is a coarsely G-excisive functor then the sequence of spectra*

$$\boldsymbol{E}_G(X) \to \boldsymbol{E}_G(\mathcal{A}X) \to \boldsymbol{E}_G(\mathcal{O}X)$$

is a weak fibration. □

As in the non-equivariant case, there is an associated boundary map

$$\delta_G \colon \Omega\mathbf{E}_G(\mathcal{O}X) \to \mathbf{E}_G(X)$$

called the *equivariant coarse assembly map.*

Definition 5.49. The *Novikov conjecture* associated to the functor $\mathbf{E}_G$ asserts that the equivariant coarse assembly map $\delta_G \colon \Omega\mathbf{E}_G(\mathcal{O}EG) \to \mathbf{E}_G(EG)$ is injective at the level of stable homotopy groups for some coarse structure on EG compatible with the topology.

It is possible to significantly generalise the Novikov conjecture.

Definition 5.50. The $(\boldsymbol{E}_G, \mathcal{F})$*-isomorphism conjecture* asserts that the equivariant coarse assembly map $\delta_G \colon \Omega\mathbf{E}_G(\mathcal{O}\mathcal{E}_{\mathcal{F}}(G)) \to \mathbf{E}_G(\mathcal{E}_{\mathcal{F}}(G))$ is a stable equivalence for some coarse structure on $\mathcal{E}_{\mathcal{F}}(G)$ compatible with the topology.

The following result is known as the notion of descent.

Theorem 5.51 (Theorem 9.5 of [Mit10]). *Let $\boldsymbol{E}_G$ be a coarsely G-excisive functor. Let X be a free coarse G-space, that is, as a topological space, G-homotopy equivalent to a finite G-CW-complex. If the coarse isomorphism conjecture holds for $\boldsymbol{E}$ and the space X then the map $\delta_G \colon \Omega\boldsymbol{E}_G(\mathcal{O}X) \to \boldsymbol{E}_G(X)$ is injective at the level of stable homotopy groups.* □

Recall that Proposition 3.63 states that the coarsening map

$$c \colon k_*^{\mathrm{coarse}}(X) \to h_*(\mathcal{O}X)$$

is an isomorphism for X uniformly contractible, where $k_*(X) = h_*(\mathcal{O}X)$.

Corollary 5.52. *Let $\boldsymbol{E}_G$ be a coarsely G-excisive functor. Let X be a free coarse G-space, that is, as a topological space, G-homotopy equivalent to a finite G-CW-complex and additionally suppose that X is uniformly contractible. If the map $k_*^{\mathrm{coarse}}(X) \to h_{*-1}(X)$ is an isomorphism then the map $(\delta_G)_* \colon k_*^G(X) \to h_{*-1}^G(X)$ is injective.* □

The Baum–Connes conjecture and Farrell–Jones conjecture also fit into this picture in the following sense:

Theorem 6.17 of [Mit04] says that if G acts cocompactly on a space X then the Baum–Connes assembly map is stably equivalent to the equivariant coarse assembly map $\delta_G \colon \Omega\mathbf{K}\mathcal{V}^*_G(\mathcal{O}X) \to \mathbf{K}\mathcal{V}^*_G(X)$ where the coarsely excisive functor $X \mapsto \mathbf{K}\mathcal{V}^*_G(X)$ will also be defined in Chapter 7.

Theorem 8.7 of [Mit10] says that if G acts cocompactly on a space X then the Farrell–Jones assembly map is stably equivalent to the equivariant coarse assembly map $\delta_G \colon \Omega\mathbf{K}\mathcal{A}_G(\mathcal{O}X) \to \mathbf{K}\mathcal{A}_G(X)$ where the coarsely excisive functor $X \mapsto \mathbf{K}\mathcal{A}_G(X)$ will be defined in Chapter 7.

Chapter 6

An isomorphism for finite asymptotic dimension

The following notation will be used throughout this chapter. For a coarsely excisive functor $\mathbf{E}$ the notation $h_*(X)$ will be used to denote the coarse homology theory $\pi_*(\mathbf{E}(X))$, the notation $k_*(X)$ to denote the locally finite homology theory $h_*(\mathcal{O}X)$ and thus $k_*^{\text{coarse}}(X)$ to denote the coarse homology theory obtained by coarsening.

The aim of this chapter is to show that the coarsened assembly map associated to a coarsely excisive functor is an isomorphism under the assumption of finite asymptotic dimension, that is, if W is a proper metric space with bounded geometry and finite asymptotic dimension then the map

$$\lambda\colon k_*^{\text{coarse}}(W) \to h_{*-1}(W)$$

is an isomorphism.

The coarsening space $X(W,\mathcal{U}_*)$ of a space W and an anti-Čech sequence $\mathcal{U}_*$ is a simplicial complex equipped with the spherical metric which gets rid of the small-scale topology of W. Spherical simplices are a modification of the standard simplices which allow for more curvature and provide the Lipschitz constants necessary to show that the coarsening space with the hybrid coarse structure is almost flasque. The assumption of finite asymptotic dimension implies that the coarsening space is finite dimensional, a key requirement for our proof. The coarse isomorphism conjecture will be verified for infinite uniformly discrete proper metric spaces with the C_0 coarse structure, and this will be used to form an inductive proof for any finite dimensional simplicial complex with the C_0 coarse structure. It will be shown that the left hand side of the coarse assembly map is the direct limit of partial coarsening spaces with the C_0 coarse structure and that the right hand side of the assembly map is the direct limit of partial coarsening spaces with the hybrid coarse structure, and we shall use these results to show that this map is an isomorphism. Using a new category theoretic result, we show that the partial coarsening spaces with the C_0 and fusion coarse structures eventually agree in the case of discrete spaces, and then use an induction argument on the decomposition of the coarsening space to prove the main result more generally.

Many proofs in this chapter are based on those in [Wri02] and [Wri05] and their geometric arguments are included here for completeness, showing that the theory can be generalised.

6.1 Spherical metrics for simplicial complexes

Definition 6.1 (Spherical simplex). The *spherical m-simplex* is the simplex Δ_m defined by

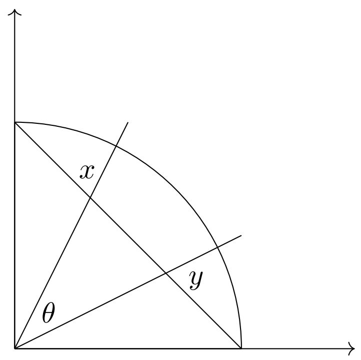
$$\Delta_m = \{(t_0, \dots, t_m) \in \mathbb{R}^{m+1} \colon \sum_{i=0}^{m} t_i^2 = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

Thus it is the intersection of the m-sphere in $\mathbb{R}^{m+1}$ and the positive quadrant of $\mathbb{R}^{m+1}$. A spherical 0-simplex is therefore a point, a spherical 1-simplex is the arc of a quarter circle of radius 1 and a spherical 2-simplex is the surface of the 2-sphere where all coordinates are positive, and so on.

Lemma 6.2 (Lemma A.3. of [Wri05]). *If σ_f is the standard m-simplex in $\mathbb{R}^{m+1}$ equipped with the Euclidean metric inherited from $\mathbb{R}^{m+1}$ and σ_s is the spherical m-simplex then the radial projection from σ_s to σ_f is a $(m+1)^{1/2}$-Lipschitz map with a $(m+1)^{-1/2}$-Lipschitz inverse.* □

We outline the proof for the case where $m = 1$. The argument for the higher dimensions is very similar.

Consider the following diagram



Let a, b be the distances between the origin and the points x, y, respectively and let s

be the distance between x and y. The angle between these two lines is denoted by θ.

It is easy to see that $1/\sqrt{2} \leq a, b \leq 1$, and that the area of the triangle with corners x, y and the origin is $\frac{1}{2}ab\sin\theta$ and is also equal to $s/2\sqrt{2}$. Here $1/4\sin\theta \leq \frac{1}{2}ab\sin\theta$ so $1/2\sin\theta \leq s/\sqrt{2} \leq \sin\theta$. As $0 \leq \theta \leq \pi/2$ and $\sin\theta \leq \theta$ then $s \leq \sqrt{2}\theta$ and so $\sin\theta \leq \sqrt{2}s$, thus $\theta \leq \sqrt{2}$.

A simplicial complex is said to be *locally finite* if each vertex belongs to only finitely many simplices.

Definition 6.3 (Uniform spherical metric). A *uniform spherical metric* on a locally finite simplicial complex is a metric such that the following conditions hold:

- each m-simplex is isometric to the spherical m-simplex;
- the restriction of the metric to each component is a path metric;
- for all $R > 0$ there exists a finite subcomplex K such that if x and y lie in different components and $x \notin K$ or $y \notin K$ then $d(x, y) > R$.

A uniform spherical metric always exists and is unique up to coarse equivalence for both the metric and the C_0 coarse structures, see [Wri05].

Lemma 6.4 (Lemma A.5 of [Wri05]). *If X is a locally finite simplicial complex with a uniform spherical metric then for any vertex v_0 and any simplex σ in the same component of X there is a sequence of adjacent vertices $v_0, v_1, \ldots, v_k$ with $v_k \in \sigma$ and $d(v_0, \sigma) = k\pi/2$.* □

The anti-Čech property (Definition 3.55) of $\mathcal{U}_*$ implies that for each $V \in \mathcal{U}_i$ there exists a $U \in \mathcal{U}_{i+1}$ with $V \subseteq U$. The simplicial connecting maps are the maps $\phi_i \colon |\mathcal{U}_i| \to |\mathcal{U}_{i+1}|$ which map a vertex $[V]$ of $|\mathcal{U}_i|$ to a vertex $[U]$ of $|\mathcal{U}_{i+1}|$ where $V \subseteq U$.

Lemma 6.5 (Lemma 5.13 of [Wri02]). *If $\mathcal{U}_*$ is an anti-Čech sequence of covers of a countable discrete metric space then there exists a subsequence $\mathcal{U}_{i_*}$ and a sequence of connecting maps $\phi_{i_k} \colon |\mathcal{U}_{i_k}| \to |\mathcal{U}_{i_{k+1}}|$ such that for each k and for each finite subcomplex K of $|\mathcal{U}_{i_k}|$ there is a j such that $\phi_{i_{k+j}} \circ \ldots \circ \phi_{i_k}(K)$ is a vertex.* □

Anti-Čech sequences will always be assumed to have this property since we can always pass to this subsequence if necessary.

6.2 Infinite uniformly discrete proper metric spaces

To show that the coarse isomorphism conjecture holds for finite dimensional simplicial complexes with the C_0 coarse structure, it suffices to show that it holds for uniformly discrete proper metric spaces. It will be shown that up to C_0 coarse equivalence, these spaces are either a point or $\mathbb{N}$. In [Wri05], the coarse Baum–Connes conjecture for uniformly discrete proper metric spaces is achieved by using C^*-algebra and K-theory techniques that are not available to us here. For the general case, we'll need to approach this more geometrically. It will be shown that an infinite uniformly discrete proper metric space W with the C_0 coarse structure is the direct limit of W with coarse structures which "build up" the C_0 coarse structure.

Lemma 6.6. *If W is an infinite uniformly discrete proper metric space then W_0 is coarsely equivalent to $\mathbb{N}_0$.*

Proof. For each $w \in W$, the set W can be written as the union $\bigcup_{n=1}^{\infty} \overline{B}(w,n)$ of closed balls. Since W is proper each $\overline{B}(w,n)$ is compact, and by uniform discreteness, each $\overline{B}(w,n)$ is finite. It follows that W is countable so there is a bijection $\varphi\colon W \to \mathbb{N}$. We will show that φ is a C_0 coarse equivalence.

By uniform discreteness if $M \subseteq W \times W$ is C_0 controlled then M can be written as $K \sqcup A$ where K is a metrically bounded set and A is a subset of the diagonal of W. Thus

$$(\varphi \times \varphi)(M) = (\varphi \times \varphi)(K \sqcup A) = (\varphi \times \varphi)(K) \sqcup N$$

where $N \subseteq \Delta_{\mathbb{N}}$ and $(\varphi \times \varphi)(K)$ is metrically bounded. It follows $(\varphi \times \varphi)(M)$ is also C_0 controlled. The inverse image of a bounded set is again bounded since cardinality is preserved by a bijection. The same argument applies to the inverse of φ, showing that φ^{-1} is coarse, and thus that φ is a coarse equivalence. □

Definition 6.7. For each $n \in \mathbb{N}$, define $\mathcal{E}_n$ to be the coarse structure on $\mathbb{N}$ where the controlled sets are all the subsets of the diagonal and all the subsets of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$.

Remark 6.8. Each of the coarse structures $\mathcal{E}_n$ is a disconnected coarse structure as the singleton sets $\{(m, m')\}$ are not controlled in $\mathcal{E}_n$ if $m, m' > n$ and $m \neq m'$. Recall that this also means that the union of two bounded sets is not necessarily again bounded set. The disconnectedness does not cause any problems, and allows us to coarsely decompose the coarse space $(\mathbb{N}, \mathcal{E}_{\min})$ into singleton sets.

Proposition 6.9. *For each n, the coarse spaces $(\mathbb{N}, \mathcal{E}_n)$ and $(\mathbb{N}, \mathcal{E}_{n+1})$ are coarsely equivalent.*

Proof. Define $\alpha\colon (\mathbb{N}, \mathcal{E}_n) \to (\mathbb{N}, \mathcal{E}_{n+1})$ by $\alpha(k) = k + 1$ and $\beta\colon (\mathbb{N}, \mathcal{E}_{n+1}) \to (\mathbb{N}, \mathcal{E}_n)$ by

$$\beta(k) = \begin{cases} k - 1 & \text{for } k > 1 \\ 1 & \text{for } k = 1 \end{cases}$$

It is easy to see that α and β send controlled sets to controlled sets. A bounded set for $\mathcal{E}_n$ is either a subset of $\{1, 2, \ldots, n\}$ or a singleton set $\{k\}$ for any $k \in \mathbb{N}$. The inverse image of a singleton set under α is again a singleton set and $\alpha^{-1}(\{1,\ldots,n\}) = \{1,\ldots,n-1\}$ which is bounded in $\mathcal{E}_{n-1}$. The inverse image of a singleton set under β is again a singleton set except for $\beta^{-1}(\{1\}) = \{1,2\}$, and $\beta^{-1}(\{1,\ldots,n\}) = \{1,\ldots,n+1\}$. It is easy to see that $\{(\alpha\circ\beta(k),k)\colon k\in\mathbb{N}\} = \{(k,k)\colon k>1\}\cup\{(2,1)\}$ which is controlled in $\mathcal{E}_{n+1}$ and that $\beta\circ\alpha$ is the identity on $(\mathbb{N},\mathcal{E}_n)$. □

Proposition 6.10. *If W is an infinite uniformly discrete proper metric space then*

$$k_*^{coarse}(W_0) \cong \varinjlim_n k_*^{coarse}(W,\mathcal{E}_n).$$

Proof. Define $\mathcal{U}_i$ to be the cover of $\mathbb{N}$ consisting of all subsets of $\{1,\ldots,i\}$ and all singletons. Let $\mathcal{U}_* = \mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3,\ldots$ be the sequence of covers of $\mathbb{N}$. It will be shown that $\mathcal{U}_*$ is a coarsening family for $\mathbb{N}_0$ with family of controlled sets defined by $M_i = \{1,\ldots,i\}\times\{1,\ldots,i\}\cup\Delta_{\mathbb{N}}$.

Each $\mathcal{U}_i$ is a good cover as all subsets of $\{1,\ldots,i\}$ and all singletons of $\mathbb{N}$ are bounded with respect to the C_0 coarse structure and each natural number is only in finitely many sets of each $\mathcal{U}_i$. If $U\in\mathcal{U}_i$ then U is either a subset of $\{1,\ldots,i\}$ or U is a singleton set. For the former, $U\subseteq M_i(n)$ for any $n\le i$ and for the latter, $U\subseteq M_i(n)$ when $U=\{n\}$. Choose $n\in\mathbb{N}$ and suppose that $i<j$. If $n\le i$ then $M_i(n)\subseteq\{1,\ldots,i\}$ and if $n>i$ then $M_i(n)=\{n\}$ and both of these sets are in $\mathcal{U}_k$ for any $k\ge i$. It is clear that if M is C_0 controlled then $M\subseteq M_i$ for some i and it therefore follows that $\mathcal{U}_*$ is a coarsening family for $\mathbb{N}_0$.

It is also easy to check that the collection $\mathcal{U}_*^k = \mathcal{U}_k,\mathcal{U}_k,\mathcal{U}_k,\ldots$ is a coarsening family for $(\mathbb{N},\mathcal{E}_k)$ where $M_i^k = M_k$ for each i. If $U\in\mathcal{U}_k$ then $U\subseteq M_k(x)$ as before, for all $n\in\mathbb{N}$ we have $M_k(n)\subseteq\{1,\ldots,k\}$ if $n\le k$ or $\{n\}$ if $n>k$, and if M is $\mathcal{E}_k$ controlled then $M\subseteq M_k$.

By definition $k_*^{\text{coarse}}(\mathbb{N}_0) = \varinjlim_i k_*(|\mathcal{U}_i|)$ and $k_*^{\text{coarse}}(\mathbb{N},\mathcal{E}_k) = \varinjlim_i k_*(|\mathcal{U}_i^k|)\cong k_*(|\mathcal{U}_k|)$. It immediately follows that $k_*^{\text{coarse}}(\mathbb{N}_0)\cong\varinjlim_k k_*^{\text{coarse}}(\mathbb{N},\mathcal{E}_k)$.

□

Remark 6.11. It follows from the coarse Mayer–Vietoris sequence that $k_*^{\text{coarse}}(X\sqcup Y)$ and $k_*^{\text{coarse}}(X)\oplus k_*^{\text{coarse}}(Y)$ are isomorphic if $X\sqcup Y$ is a coarsely excisive decomposition (that is, if X and Y are “infinitely” far apart, so that the every M-ball around X does not touch the M-ball around Y). Every decomposition $X\sqcup Y$ with the minimal coarse structure is a coarsely excisive decomposition as with this structure $m(X)=X$ for each controlled set m and therefore $m(X)\cap m(Y)=\emptyset$.

Recall that a coarsely excisive functor has the property that if $(X,\mathcal{E}) = (X,\bigcup_{n=1}^{\infty}\mathcal{E}_n)$ then

$$h_*(X,\mathcal{E})\cong\varinjlim_n h_*(X,\mathcal{E}_n).$$

Lemma 6.12. *If W is an infinite uniformly discrete proper metric space then the coarse isomorphism conjecture holds for W_0, that is, the coarse assembly map induces an isomorphism $k_*^{coarse}(W_0) \to h_*(W_0)$.*

Proof. We can write

$$\mathbb{N} = B \cup \{n+1\} \cup \{n+2\} \cup \ldots$$

where each set is bounded with respect to the $\mathcal{E}_n$ coarse structure, and the union is coarsely disconnected. Thus

$$h_*(\mathbb{N}, \mathcal{E}_n) = h_*(B) \oplus h_*(\{n+1\}) \oplus \ldots$$

and therefore $\varinjlim_n h_*(\mathbb{N}, \mathcal{E}_n) = \varinjlim_n (h_*(B) \oplus h_*(\{n+1\}) \oplus \ldots)$.

By the property of a coarsely excisive functor,

$$h_*(\mathbb{N}_0) \cong \varinjlim_n h_*(\mathbb{N}, \mathcal{E}_n) \cong \prod_{k=1}^{\infty} h_*(\{k\}) / \bigoplus_{k=1}^{\infty} h_*(\{k\})$$

and by Proposition 6.10, $k_*^{\text{coarse}}(\mathbb{N}_0) \cong \varinjlim_n k_*^{\text{coarse}}(\mathbb{N}, \mathcal{E}_n)$ and

$$\varinjlim_n k_*^{\text{coarse}}(\mathbb{N}, \mathcal{E}_n) \cong \prod_{k=1}^{\infty} k_*^{\text{coarse}}(\{k\}) / \bigoplus_{k=1}^{\infty} k_*^{\text{coarse}}(\{k\}).$$

By Proposition 5.28, the coarse assembly isomorphism holds for a singleton set, and the result follows. □

The following proposition is based on Theorem 3.17 of [Wri05].

Proposition 6.13. *If W is an infinite uniformly discrete proper metric space then*

$$k_*^{coarse}(W_0) \cong \varinjlim_{\substack{K_i \subseteq W \\ compact}} k_*(W/K_i).$$

Proof. Let $\mathcal{U}$ be a cover of $\mathbb{N}$ where the only sets which include elements of $\mathbb{N}$ greater than i are singletons, i.e. $\mathcal{U} = K_i \cup S$ where K_i is the finite union of non-singletons and S is a union of singleton sets. Clearly this cover is a good cover for $\mathbb{N}_0$.

Then the cover $\mathcal{U}$ can be coarsened to $\mathcal{U}_i = \mathcal{P}(K_i) \cup W \backslash K_i$ where $\mathcal{P}(K_i)$ is the power set of K_i. It has already been shown in Proposition 6.10 that this is a coarsening family for $\mathbb{N}_0$.

The nerve $|\mathcal{U}_i|$ is the union of a simplicial star (corresponding to the set K_i) and the discrete space $W \backslash K_i$. The simplicial star of a vertex is contractible so $|\mathcal{U}_i|$ and W/K_i

are homotopy equivalent. It follows that

$$k_*^{\text{coarse}}(W_0) = \varinjlim_i k_*(|\mathcal{U}_i|) \cong \varinjlim_{\substack{K_i \subseteq W \\ \text{compact}}} k_*(W/K_i).$$

□

Definition 6.14 (Relatively connected). A subspace Y of a topological space X is said to be *relatively connected* if each connected component of X contains at most one connected component of Y.

Lemma 6.15 (Lemma A.8 of [Wri05]). *Let X be a finite-dimensional simplicial complex equipped with the uniform spherical metric. Let Y be a subcomplex of $X^{(n)}$ (the n^{th} barycentric subdivision of X) and let $Y_\sigma = Y \cap \sigma$ for each simplex σ of X. If for each simplex σ of X, the subcomplex Y_σ of $X^{(n)}$ is connected and each edge of X has non-empty intersection with Y then if Y is relatively connected in X then the inherited metric on Y is coarsely equivalent and C_0 coarsely equivalent to any uniform spherical metric on Y, and this coarse equivalence is bi-Lipschitz on components.* □

Proposition 6.16 (Theorem 3.18 of [Wri05]). *If W is a finite dimensional simplicial complex equipped with a uniform spherical metric then the map $k_*^{coarse}(W_0) \to h_{*-1}(W_0)$ is an isomorphism.*

Proof. If $W^{(2)}$ is the second barycentric subdivision of W and Y_k is the union of simplicial stars in $W^{(2)}$ about the barycentres of the k-simplices of W then

$$W = Y_0 \cup \ldots \cup Y_m$$

where m is the dimension of W and each space Y_k is a disjoint union of uniformly seperated stars.

Define $\widetilde{Y_k}$ to be the union $Y_k \cup \text{skel}(W^{(2)})$ (the 1-skeleton of $W^{(2)}$). These sets are now relatively connected, allowing us to apply Lemma 6.15.

Let G_k be the graph consisting of edges of $W^{(2)}$ which are not contained in Y_k. It can then be seen that $\widetilde{Y_k} = Y_k \cup G_k$, and by subdividing G_k again, that $G_k = V_k \cup E_k$ where V_k is the union of uniformly seperated stars about the vertices of G_k and E_k is the union of uniformly seperated stars about the edges of G_k.

Let $Z_0 = \widetilde{Y_0}$ and define $Z_k = Z_{k-1} \cup \widetilde{Y_k}$ inductively so that $Z_m = W$. Observe that each of $Z_k, \widetilde{Y_k}$ and G_k are relatively connected in W. Lemma 6.15 tells us that the metrics they inherit as subsets of W are C_0 coarsely equivalent to uniformly spherical metrics.

Each of the spaces Y_k, V_k and E_k are either compact (if they consists of finitely many stars) or are coarsely homotopy equivalent to the infinite uniformly discrete set consisting of the k-barycentres of W. The homotopy is continuous and is contractive on each star so is a C_0 coarse homotopy as the stars are uniformly seperated. By Lemma 6.12 it follows that the maps

$$k_*^{\text{coarse}}((Y_k)_0) \to h_{*-1}((Y_k)_0)$$

$$k_*^{\text{coarse}}((V_k)_0) \to h_{*-1}((V_k)_0)$$

and

$$k_*^{\text{coarse}}((E_k)_0) \to h_{*-1}((E_k)_0)$$

are isomorphisms.

Observe also that each $Y_k \cap G_k$ and $V_k \cap E_k$ are uniformly discrete so the maps

$$k_*^{\text{coarse}}((Y_k \cap G_k)_0) \to h_{*-1}((Y_k \cap G_k)_0)$$

and

$$k_*^{\text{coarse}}((V_k \cap E_k)_0) \to h_{*-1}((V_k \cap E_k)_0)$$

are also isomorphisms by Lemma 6.12.

Applying the 5-Lemma to the diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & k_*^{\text{coarse}}((V_k \cap E_k)_0) & \longrightarrow & k_*^{\text{coarse}}((V_k)_0) \oplus k_*^{\text{coarse}}((E_k)_0) & \longrightarrow & k_*^{\text{coarse}}((G_k)_0) & \longrightarrow \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
\ldots \longrightarrow & h_{*-1}((V_k \cap E_k)_0) & \longrightarrow & h_{*-1}((V_k)_0) \oplus h_{*-1}((E_k)_0) & \longrightarrow & h_{*-1}((G_k)_0) & \longrightarrow \ldots
\end{array}$$

tells us that the map $k_*^{\text{coarse}}((G_k)_0) \to h_{*-1}((G_k)_0)$ is an isomorphism and therefore also applying the 5-Lemma to the diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & k_*^{\text{coarse}}((Y_k \cap G_k)_0) & \longrightarrow & k_*^{\text{coarse}}((Y_k)_0) \oplus k_*^{\text{coarse}}((G_k)_0) & \longrightarrow & k_*^{\text{coarse}}((\widetilde{Y}_k)_0) & \longrightarrow \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
\ldots \longrightarrow & h_{*-1}((Y_k \cap G_k)_0) & \longrightarrow & h_{*-1}((Y_k)_0) \oplus h_{*-1}((G_k)_0) & \longrightarrow & h_{*-1}((\widetilde{Y}_k)_0) & \longrightarrow \ldots
\end{array}$$

tells us that the map $k_*^{\text{coarse}}((\tilde{Y}_k)_0) \to h_{*-1}((\tilde{Y}_k)_0)$ is an isomorphism.

The isomorphism holds for $Z_0 = \tilde{Y}_k$ and by repeating the above argument m times we get an isomorphism for Z_1, Z_2 and so on until $Z_m = W$. □

6.3 The coarsening space

Definition 6.17. Let W be a proper metric space, and let $\mathcal{U}_*$ be an anti-Čech sequence for W. The *coarsening space* of $(W,\mathcal{U}_*)$ is the telescope

$$X = X(W,\mathcal{U}_*) = |\mathcal{U}_1| \times [1,2] \cup_{\phi_1} |\mathcal{U}_2| \times [2,3] \cup_{\phi_2} \dots$$

Recall that

$$|\mathcal{U}_{i-1}| \times [i-1,i] \cup_{\phi_i} |\mathcal{U}_i| \times [i,i+1] = \frac{(|\mathcal{U}_{i-1}| \times [i-1,i]) \sqcup (|\mathcal{U}_i| \times [i,i+1])}{\sim}$$

where $(x,i) \sim (\phi_i(x),i)$ for each $i \in \mathbb{N}$.

Each $|\mathcal{U}_i| \times [i,i+1]$ is equipped with the product metric, where each simplex σ of $|\mathcal{U}_i|$ is given the spherical metric and $[i,i+1]$ is given the metric it inherits from being a subset of Euclidean space. The coarsening space X is then equipped with the largest metric bounded above by the product metric on each $\sigma \times [i,i+1]$.

Denote the projection map $|\mathcal{U}_i| \times [i,i+1] \to [i,i+1]$ by π.

Definition 6.18. The *partial coarsening spaces* of $(W,\mathcal{U}_*)$ are the spaces

$$X_i = \pi^{-1}([1,i]) = |\mathcal{U}_1| \times [1,2] \cup_{\phi_1} |\mathcal{U}_2| \times [2,3] \cup_{\phi_2} \dots \cup_{\phi_{i-1}} |\mathcal{U}_i| \times [i,i+1]$$

equipped with the metric inherited from the coarsening space.

Definition 6.19. The *collapsing map* from X to $\pi^{-1}([t,\infty))$ is the map (defined for $t > 1$) by

$$\Phi_t(x,s) = \begin{cases} (\phi_{i'-1} \circ \dots \circ \phi_i(x), t) & \text{if } s \le t, \\ & (x,s) \in |\mathcal{U}_i| \times [i,i+1) \text{ and } t \in [i',i'+1) \\ (x,s) & \text{if } s \ge t. \end{cases}$$

The idea of the collapsing map is to collapse part of the coarsening space down to a point.

In the following proof, we shall use the fact that Φ_t is 1-Lipschitz and has the properties $\Phi_{t'} \circ \Phi_t = \Phi_{t'}$ for $t' \ge t$ and $d(\Phi_t(x,s), \Phi_{t'}(x,s)) \le |t-t'|$.

Lemma 6.20 (Theorem 4.5 of [Wri05]). *If W is a uniformly discrete bounded geometry metric space and $\mathcal{U}_*$ is an anti-Čech sequence for W then the coarsening space $X(W,\mathcal{U}_*)$ equipped with the C_0 coarse structure is almost flasque.*

Proof. Firstly, choose a basepoint (x_0,s_0) in X and then for $k \ge 1$, define maps

$\alpha_k \colon X \to X$ by $\alpha_k(x,s) = \Phi_{r_k(x,s)}(x,s)$, where

$$r_k(x,s) = \begin{cases} \log k - d((x,s),(x_0,s_0)) & \text{if } d((x,s),(x_0,s_0)) \leq \log k \\ 0 & \text{otherwise} \end{cases}$$

If K is a bounded subset of X then K must lie within some X_i. It then follows that $\Phi_i(K)$ is also bounded. As Φ_i is a coarse map, it then follows that $K' = \Phi_i^{-1}\Phi_i(K)$ is also bounded.

It can be shown that K' has the property that if $\Phi_t(x,s) \in K'$ for some t, then $(x,s) \in K'$, using the fact that $\Phi_t \circ \Phi_t = \Phi_t$; to see this note that if $\Phi_t(x,s) \in K'$ then $\Phi_t(x,s) \in \Phi_t^{-1}\Phi_t(K)$. So $\Phi_t(x,s) = \Phi_t \circ \Phi_t(x,s) \in \Phi_t\Phi_t^{-1}\Phi_t(K) \subseteq \Phi_t(K)$ and hence $(x,s) \in \Phi_t^{-1}\Phi_t(K) = K'$.

As $K \subseteq K'$, to show that $\alpha_k(X) \cap K = \emptyset$ for sufficiently large k, it suffices to show that $\alpha_k(K') \cap K = \emptyset$ for sufficiently large k. For see this; suppose that $\alpha_k(X) \cap K \neq \emptyset$ and take $(x,s) \in \alpha_k(X) \cap K$. Then $(x,s) = \alpha_k(y,s')$ for some (y,s'). Since $(x,s) \in K \subseteq K'$ then $\alpha_k(y,s') \in K'$ then $\alpha_k(y,s') \in \alpha_k^{-1}\alpha_k(K)$. So $\alpha_k(y,s') = \alpha_k \circ \alpha_k(y,s') \in \alpha_k\alpha_k^{-1}\alpha_k(K) \subseteq \alpha_k(K)$. Hence $(y,s') \in \alpha_k^{-1}\alpha_k(K) = K'$. So $(x,s) \in \alpha_k(K')$ also, and so $\alpha_k(K') \cap K \neq \emptyset$.

To show that $\alpha_k(K') \cap K = \emptyset$ for sufficiently large k, observe that the set $K' \subseteq B((x_0,s_0),R)$ in X for some R. So for $(x,s) \in K'$, we have $d((x_0,s_0),(x,s)) \leq R$ so that $r_k(x,s) \geq \log k - R$. Thus if $\log k > R + i$ then the set $\alpha_k(K')$ does not meet X_i and therefore does not meet K.

If M is a C_0 controlled subset of $X \times X$ then for each $\epsilon > 0$ there exists a metrically bounded K_ϵ and a set A_ϵ with $d((x,s),(x',s')) < \epsilon/2$ for all $((x,s),(x',s')) \in A_\epsilon$ such that $M = K_\epsilon \cup A_\epsilon$. The inequality

$$\begin{aligned} d(\alpha_k(x,s),\alpha_k(x',s')) &\leq d(\Phi_{r_k(x,s)}(x,s),\Phi_{r_k(x',s')}(x,s)) + \\ &\qquad d(\Phi_{r_k(x',s')}(x,s),\Phi_{r_k(x',s')}(x',s')) \\ &\leq d(r_k(x,s),r_k(x',s')) + d((x,s),(x',s')) \\ &\leq 2d((x,s),(x',s')). \end{aligned}$$

shows that α_k expand distances by at most a factor of 2 and therefore

$$d(\alpha_k(x,s),\alpha_k(x',s')) < \epsilon \text{ for } ((x,s),(x',s')) \in A_\epsilon$$

for each k. Also, K_ϵ is metrically bounded so the coarsening map eventually collapses K_ϵ down to a point so that for k sufficiently large we have $\alpha_k(x,s) = \alpha_k(x',s')$ for all $((x,s),(x',s')) \in K_\epsilon$. Then $(\alpha_k \times \alpha_k)(K_\epsilon)$ lies inside the union of a bounded set B and the diagonal.

The set $(\alpha_k \times \alpha_k)(M)$ is a subset of the union of B and a set on which the distance between any two coordinates is less than ϵ for each k, and is therefore C_0 controlled.

It is easy to see that α_1 is close to the identity; α_1 is the identity outside of a bounded set.

Observe that

$$\begin{aligned} d(\alpha_k(x,s), \alpha_{k+1}(x,s)) &\leq r_{k+1}(x,s) - r_k(x,s) \\ &\leq \log(k+1) - \log k \\ &< 1/k. \end{aligned}$$

For a fixed k' there exists a bounded set outside of which α_k is the identity for $k \leq k'$. Then for $k < k'$, the set $C = \{(\alpha_k(x,s), \alpha_{k+1}(x,s)) \colon k \in \mathbb{N}\}$ lies in the union of a bounded set and the diagonal. If $k \geq k'$ then $d(\alpha_k(x,s), \alpha_{k+1}(x,s)) < 1/k'$ and hence C is the union of a bounded set and the diagonal, and a set which the distance is bounded by $1/k'$. By choosing k' to be sufficiently large, C is a controlled set so the proof is complete. □

Recall that π is the map on X to $[1, \infty)$ arising from the projection maps $|\mathcal{U}_i| \times [i, i+1] \to [i, i+1]$. In the proof, X_0 denotes the space X equipped with the C_0 coarse structure, and not to be confused with X_i which denotes $\pi^{-1}([1,i])$ for $i > 1$.

Theorem 6.21 (Theorem 4.7 of [Wri05]). *If W is a uniformly discrete bounded geometry metric space then*

$$k_*^{coarse}(W) \cong \varinjlim_i h_{*-1}(X_i(W, \mathcal{U}_*)_0).$$

Proof. Observe that

$$k_*^{\text{coarse}}(W) = \varinjlim_i k_*(|\mathcal{U}_i|) \cong \varinjlim_i \varinjlim_{\substack{K \subseteq |\mathcal{U}_i| \\ \text{compact}}} k_*(|\mathcal{U}_i|/K)$$

as for any compact K of $|\mathcal{U}_i|$ and for j sufficiently large, $\phi_{i+j} \circ \ldots \circ \phi_i(K)$ is a contractible subset of $|\mathcal{U}_{i+j+1}|$ and therefore

$$k_*^{\text{coarse}}(W) \cong \varinjlim_i k_*^{\text{coarse}}(|\mathcal{U}_i|_0) \cong \varinjlim_i h_{*-1}(|\mathcal{U}_i|_0)$$

where the first isomorphism is by Proposition 6.13 and the second isomorphism is by Proposition 6.16.

It will be shown that $h_*(X_i(W, \mathcal{U}_*)_0)$ and $h_*(|\mathcal{U}_i|_0)$ are isomorphic. Observe that $\pi^{-1}\{i\} = |\mathcal{U}_i| \times \{i\}$ so that the spaces $\pi^{-1}\{i\}$ and $|\mathcal{U}_i|$ are C_0 coarsely equivalence as they are identical as simplicial complexes and carry C_0 coarsely equivalent metrics. To see this, firstly note that $|\mathcal{U}_i|$ is equipped with the uniform spherical metric d_S, and $\pi^{-1}\{i\}$ is equipped with the metric d_X inherited from $X(W,\mathcal{U}_*)$. If $M \subseteq \pi^{-1}\{i\} \times \pi^{-1}\{i\}$ is C_0 controlled then we can write $M = B \cup A_1$ where B is metrically bounded and $d_X(x,x') \leq 1$ for $(x,x') \in A_1$. If $(x,x') \in A_1$ then $d_X(x,x') = d_S(x,x')$ so A_1 is C_0 controlled for d_S. The set B is also bounded for d_S, so it follows that A is C_0 controlled for d_S. For the converse, if $|\mathcal{U}_i|$ is connected, we can simply reverse the above argument. If $|\mathcal{U}_i|$ is not connected, write $M = B \cup A_1$ with $d_S(x,x') \leq \min\{1,\epsilon\}$ for $(x,x') \in A_1$ where ϵ is smaller than the least distance between any two components of $|\mathcal{U}_i|$, and again apply the above argument.

The decomposition $X_0 = (X_i)_0 \cup \pi^{-1}[i,\infty)_0$ is coarsely excisive, so there is a coarse Mayer–Vietoris sequence

$$\ldots \to h_{*+1}(X_0) \to h_*(\pi^{-1}\{i\}_0) \to h_*((X_i)_0) \oplus h_*(\pi^{-1}[i,\infty)_0) \to h_*(X_0) \to \ldots$$

Lemma 6.20 implies that that $h_*(X_0) = 0$ and also implies that $h_*(\pi^{-1}[i,\infty)_0) = 0$ by choosing the anti-Čech sequence starting at $\mathcal{U}_i$. It follows that the map $h_*(\pi^{-1}\{i\}_0) \to h_*((X_i)_0)$ is an isomorphism and therefore that

$$k_*^{\text{coarse}}(W) \cong \varinjlim_i h_{*-1}(X_i(W,\mathcal{U}_*)_0).$$

□

Definition 6.22 (Degree of a cover). The *degree* of a cover $\mathcal{U}$ of X is defined to be

$$\sup_{x\in X} |\{U \in \mathcal{U} \colon x \in U\}|.$$

Lemma 6.23. *If X is a uniformly discrete space with bounded geometry and $\mathcal{U}$ is a cover of X with finite mesh then $\mathcal{U}$ has finite degree and $|\mathcal{U}|$ is finite dimensional with*

$$\text{degree}(\mathcal{U}) = \dim(|\mathcal{U}|) + 1.$$

Additionally, $|\mathcal{U}|$ is locally finite with a uniform bound on the number of simplices meeting at a point.

Proof. If $\mathcal{U}$ has finite mesh then every $U \subseteq \mathcal{U}$ is finite by uniform discreteness of X. If X is finite then it is clear that $\mathcal{U}$ has finite degree. If X is infinite and some $x \in X$ belongs to an infinite number of sets of $\mathcal{U}$ then this would contradict the finite mesh of the cover, and so $\mathcal{U}$ has finite degree. If degree$(\mathcal{U}) = m$ then each $x \in X$ belongs to at most m sets of the cover, and there is an x which belongs to exactly m sets. These m sets form an $m-1$ simplex so $|\mathcal{U}|$ is finite dimensional with dimension $m-1$ and therefore degree$(\mathcal{U}) = \dim(|\mathcal{U}|) + 1$.

□

Proposition 6.24 (Theorem 9.9(c) of [Roe03]). *If W is a uniformly discrete metric space with asymptotic dimension at most m then there exists an anti-Čech sequence $\mathcal{U}_*$ for W with* $\mathrm{degree}(\mathcal{U}_i) \leq m+1$ *for all i. Correspondingly there is an anti-Čech sequence such that* $\dim(|\mathcal{U}_i|) \leq m$ *for all i.*

The following constructions show us the existence of maps required to show that $X(W,\mathcal{U}_*)$ with the hybrid coarse structure is almost flasque.

Lemma 6.25 (Lemma 5.7 of [Wri05]). *Let W be a uniformly discrete bounded geometry metric space of asymptotic dimension at most m and let $\mathcal{U}_*$ be an anti-Čech sequence for W with degrees bounded by $m+1$. For each i and each $\epsilon > 0$, there is an $i' > i$, and a partition of unity $\{h_U\}$ of $|\mathcal{U}_i|$ indexed by sets $U \in \mathcal{U}_i'$ such that:*

- *all the maps h_U are ϵ-Lipschitz;*
- *for x in the interior of a simplex σ of $|\mathcal{U}_i|$, if $[V_1], \ldots, [V_j]$ are the vertices of σ and $U \in \mathcal{U}_{i'}$ with $h_U(x) \neq 0$ then U contains the intersection $V_1 \cap \ldots \cap V_j$.*

□

Proposition 6.26 (Proposition 5.8 of [Wri05]). *Let W be a uniformly discrete bounded geometry metric space of asymptotic dimension at most m, and let $\mathcal{U}_*$ be an anti-Čech sequence for W with finite degrees.*

Then there is a sequence i_j with $i_1 = 1$ such that there exists maps $\beta_j \colon \pi^{-1}([i_j, \infty)) \to \pi^{-1}([i_{j+1}, \infty))$ such that:

- *$d(\beta_j(x,s), \beta_j(x',s')) \leq \frac{1}{j} d((x,s),(x',s'))$ for $x, x' \in |\mathcal{U}_{i_j}|$ with $d(x,x') < j$;*
- *β_j is 4-Lipschitz;*
- *$\beta_j(x,s) = (x,s)$ for $x \in X$ with $s \geq i_{j+1}$;*
- *If $x \in X$ with $i_j \leq s \leq i_{j+1}$ then $\beta_j(x,s) \in |\mathcal{U}_{i_{j+1}}|$ and there is a simplex σ of $|\mathcal{U}_{i_{j+1}}|$ containing both $\Phi_{i_{j+1}}(x,s)$ and $\beta_j(x,s)$, hence $\Phi_{i_{j+1}}$ is linearly homotopic to β_j as a map from $\pi^{-1}([i_j,\infty)) \to \pi^{-1}([i_{j+1},\infty))$.*

□

Remark 6.27. Observe that for $(x,s) \in X$ we have $\beta_j \circ \beta_{j-1} \circ \ldots \circ \beta_1(x,s) \in |\mathcal{U}_{i_{j+1}}|$.

Lemma 6.28 (Lemma A.9 of [Wri05]). *Let X be a path metric space, and let Y be a simplicial complex with uniform spherical metric. Let η_0, $\eta_1 \colon X \to Y$ be Lipschitz maps with Lipschitz constant $\lambda \geq 1/2$, and suppose that for all x the images $\eta_0(x)$ and $\eta_1(x)$ lie in a common simplex. Let η_t denote the linear homotopy from η_0 to η_1. Then for each $x \in X$ the map $t \mapsto \eta_t(x)$ is Lipschitz with constant at most 2, and the map $\eta \colon X \times [0,1] \to Y$ is Lipschitz with constant at most 4λ.* □

Theorem 6.29 (Theorem 5.9 of [Wri05]). *If W is a uniformly discrete bounded geometry metric space with finite asymptotic dimension and $\mathcal{U}_*$ is an anti-Čech sequence for W then the coarse space $X(W,\mathcal{U}_*)_h$ is almost flasque.*

Proof. By Proposition 6.26 there is a sequence i_j with $i_1 = 1$ and a sequence of maps

$$\beta_j \colon \pi^{-1}([i_j,\infty)) \to \pi^{-1}([i_{j+1},\infty))$$

for each j with the property that

$$d(\beta_j(x,s),\beta_j(x',s')) \leq \frac{1}{j} d((x,s),(x',s')) \tag{6.1}$$

for $(x,s),(x',s') \in |\mathcal{U}_{i_j}|$ with $d((x,s),(x',s')) < j$.

Now we can define a map $\alpha_{i_j} \colon X \to \pi^{-1}([i_j,\infty))$ (for each i_j) by

$$\alpha_{i_j} = \Phi_{i_j} \circ \beta_{j-2} \circ \ldots \circ \beta_1.$$

By Proposition 6.26, Φ_{i_j} is linearly homotopic to β_{j-1} via a homotopy $\gamma_{j,t}$ for $i_j \leq t \leq i_{j+1}$ and such that $\gamma_{j,i_j} = \Phi_{i_j}$ and $\gamma_{j,i_{j+1}} = \beta_{j-1}$.

Now we can define a map $\alpha_t \colon X \to \pi^{-1}([t,\infty))$ for $t \in [1,\infty)$ by

$$\alpha_t = \Phi_t \circ \gamma_{j,t} \circ \beta_{j-2} \circ \ldots \circ \beta_1.$$

For ease of notation, we now write ρ_j for the map $\beta_j \circ \beta_{j-1} \circ \ldots \circ \beta_1$.

Let t_k be an increasing sequence tending to infinity with $t_0 = 1$ and $t_{k+1} - t_k \to 0$ as $k \to \infty$, and with the integers i_j a subsequence of t_k. If $B \subseteq X$ is bounded then $B \subseteq X_i$ for some i. If $t_k > i$, then $\alpha_{t_k}(X) \cap B = \emptyset$, so it follows that the maps α_{t_k} are properly supported.

We know from Proposition 6.26 that β_j is 4-Lipschitz for each j. We claim that ρ_j is λ-Lipschitz with λ independent of j. Let j be the largest integer such that $i_j \leq s, s'$. Then by Proposition 6.26 we have

$$(x,s) = \rho_1(x,s) = \rho_2(x,s) = \ldots = \rho_{j-1}(x,s)$$

and

$$(x',s') = \rho_1(x',s') = \rho_2(x',s') = \ldots = \rho_{j-1}(x',s').$$

If $d((x,s),(x',s')) < 1/4$ then

$$\begin{aligned} d(\rho_j(x,s),\rho_j(x',s')) &\leq 4d(\rho_{j-1}(x,s),\rho_{j-1}(x',s')) \\ &= 4d((x,s),(x',s')) \\ &< 1. \end{aligned}$$

Both $\rho_j(x,s)$ and $\rho_j(x',s')$ lie in $|\mathcal{U}_{j+1}|$ and are of distance at most 1 apart. Now inductively it follows that for $j' \geq j+2$

$$\begin{aligned} d(\rho_{j'}(x,s),\rho_{j'}(x',s')) &\leq d(\rho_{j'-1}(x,s),\rho_{j'-1}(x',s')) \\ &< 4d((x,s),(x',s')), \end{aligned}$$

as $\beta_{j'}$ is contractive on scales less than 1 in $|\mathcal{U}_{j'}|$.

If $d((x,s),(x',s')) \geq 1/4$ then as $\rho_j(x,s)$ must lie in a simplex containing $\Phi_{i_{j+1}}(x,s)$ and $\rho_j(x',s')$ must lie in a simplex containing $\Phi_{i_{j+1}}(x',s')$, it follows that

$$\begin{aligned} d(\rho_j(x,s),\rho_j(x',s')) &\leq d(\Phi_{i_{j+1}}(x,s),\Phi_{i_{j+1}}(x',s')) + \pi \\ &\leq (1+4\pi)d((x,s),(x',s')) \end{aligned}$$

for all j.

Hence

$$d(\rho_j(x,s),\rho_j(x',s')) \leq (1+4\pi)d((x,s),(x',s'))$$

for all j, and so ρ_j is $(1+4\pi)$-Lipschitz.

As Φ_{i_j} and β_{j-1} are 1-Lipschitz and 4-Lipschitz respectively, it follows from Lemma 6.28 that $\gamma_{j,t}$ is 16-Lipschitz. Hence

$$\begin{aligned} d(\alpha_{t_k}(x,s),\alpha_{t_k}(x',s')) &\leq d(\gamma_{j,i_j}\circ\rho_{j-2}(x,s),\gamma_{j,i_j}\circ\rho_{j-2}(x',s')) \\ &\leq 16d(\rho_{j-2}(x,s),\rho_{j-2}(x',s')) \\ &\leq 16(1+4\pi)d((x,s),(x',s')). \end{aligned}$$

so α_{t_k} is $16(1+4\pi)$-Lipschitz.

If M is a hybrid controlled set then we can write $M = \Delta_\epsilon \cup A_\epsilon$ where Δ_ϵ is an ϵ-neighbourhood of the diagonal and $A_\epsilon \subseteq X_i \times X_i$ lying within R of the diagonal for some i and R sufficiently large. It is clear that $d(\alpha_{t_k}(x,s),\alpha_{t_k}(x',s')) \leq 16(1+4\pi)\epsilon$ for all k and for all $((x,s),(x',s)) \in \Delta_\epsilon$.

On A_ϵ,

$$d(\alpha_{t_k}(x,s),\alpha_{t_k}(x',s')) \leq (1+4\pi)d((x,s),(x',s')) < (1+4\pi)R.$$

Choose j large enough so that $(1+4\pi)R < j$, so that in particular we have $d(\rho_{j-1}(x,s), \rho_{j-1}(x',s')) < j$. Since the images $\rho_{j-1}(x,s)$ and $\rho_{j-1}(x',s')$ lie in $|U_{i_j}|$, we have

$$\begin{aligned} d(\rho_j(x,s), \rho_j(x',s')) &\leq 1/jd(\rho_{j-1}(x,s), \rho_{j-1}(x',s')) \\ &\leq ((1+4\pi)/j)\, d((x,s),(x',s')). \end{aligned}$$

For any $t \geq i_{j+2}$, the map α_t is a composition of $\Phi_t \circ \gamma_{l,t} \circ \beta_{l-2} \circ \ldots \circ \beta_{j+1}$ with $\beta_j \circ \ldots \circ \beta_1$ for some $l \geq j+2$. The former is $16(1+4\pi)$-Lipschitz. Hence using the above inequality, for $t \geq i_{j+2}$ and $((x,s),(x',s')) \in A_\epsilon$ we have

$$d(\alpha_t(x,s), \alpha_t(x',s')) \leq 16(1+4\pi)^2/jd((x,s),(x',s')) \leq 16(1+4\pi)^2 R/j.$$

If j is sufficiently large that $i_j \geq i$ and $(1+4\pi)R/j < \epsilon$ then for $((x,s),(x',s')) \in \Delta_\epsilon$ the pairs $(\alpha_{t_k}(x,s), \alpha_{t_k}(x',s'))$ lie in a $16(1+4\pi)\epsilon$-neighbourhood of the diagonal for all k. This is also true for $((x,s),(x',s')) \in A_\epsilon$ if k is such that $t_k \geq i_{j+2}$. Note that the set of pairs with $t_k < i_{j+2}$ lies within $X_{i_{j+2}} \times X_{i_{j+2}}$ for $((x,s),(x',s')) \in A_\epsilon$ and hence $(\alpha_k \times \alpha_k)(M)$ lies in the union of $X_{i_{j+2}} \times X_{i_{j+2}}$ with a $16(1+4\pi)\epsilon$-neighbourhood of the diagonal, and so as ϵ is arbitrary, the set of all $(\alpha_k \times \alpha_k)(M)$ is hybrid controlled.

For all k, there is some j such that $t_k, t_{k+1} \in [i_j, i_{j+1}]$. Then we have

$$\begin{aligned} d(\Phi_{t_k} \circ \gamma_{j,t_k} \circ \rho_{j-2}(x,s), \Phi_{t_k} \circ \gamma_{j,t_{k+1}} \circ \rho_{j-2}(x,s)) \\ &\leq d(\gamma_{j,t_k} \circ \rho_{j-2}(x,s), \gamma_{j,t_{k+1}} \circ \rho_{j-2}(x,s)) \\ &\leq 2|t_{k+1} - t_k| \end{aligned}$$

and

$$d(\Phi_{t_k} \circ \gamma_{j,t_{k+1}} \circ \rho_{j-2}(x,s), \Phi_{t_{k+1}} \circ \gamma_{j,t_{k+1}} \circ \rho_{j-2}(x,s)) \leq |t_{k+1} - t_k|$$

and therefore that $d(\alpha_{t_k}(x,s), \alpha_{t_{k+1}}(x,s)) \leq 3|t_{k+1} - t_k| \to 0$ as $k \to \infty$.

For any $\epsilon > 0$, there is a fixed k' such that for $k \geq k'$ we have

$$d(\alpha_{t_k}(x,s), \alpha_{t_{k+1}}(x,s)) < \epsilon.$$

For $k < k'$, if $s \geq t_{k'}$ then $s \geq i_j$ for some $i_j \geq t_{k'} \geq t_k$. In this case, $\Phi_{t_k}(x,s) = (x,s)$ and since $k+1 \leq k'$, $\Phi_{t_{k+1}}(x,s) = (x,s)$ and hence

$$(x,s) = \alpha_{t_k}(x,s) = \alpha_{t_{k+1}}(x,s).$$

If $s \leq t_{k'}$ then $(x,s) \in \pi^{-1}([1,t_{k'}])$ and so $\Phi_{t_k}(x,s) = (x,s)$ as $t_k \leq s \leq t_{k'}$, so $\alpha_{t_k}(x,s) \in \pi^{-1}([1,t_{k'}])$, and similarly $\alpha_{t_{k+1}}(x,s) \in \pi^{-1}([1,t_{k'}])$.

It follows that the set $\{(\alpha_{t_k}(x,s), \alpha_{t_{k+1}}(x,s)) : (x,s) \in X\}$ is hybrid controlled as either pairs lie within ϵ of the diagonal or they lie within some X_i for some i. □

Proposition 6.30 (Proposition 2.16 of [Wri05]). *Let X be a proper, seperable coarse space, and $\mathcal{U}$ be a locally finite uniformly bounded open cover of X. If $|\mathcal{U}|$ has the uniform spherical metric (and the corresponding bounded coarse structure), and $\eta_{\mathcal{U}} \colon |\mathcal{U}| \to X$ is any map with the property that if $y \in Star([V])$ of $|\mathcal{U}|$ then $\eta_{\mathcal{U}}(y) \in V$, then $\eta_{\mathcal{U}}$ is coarse and any two such maps are close.* □

Definition 6.31 (Fusion coarse structure). Let X be a proper metric space equipped with a map $\pi \colon X \to \mathbb{R}_+$ and let $X_i = \pi^{-1}[0,i]$. The *fusion coarse structure* on X is the coarse structure whose controlled sets are metrically controlled subsets M of $X \times X$ for which there exists an i such that the distance restricted to $M \backslash (X_i \times X_i)$ is a C_0 function.

Theorem 6.32 (Theorem 4.12 of [Wri05]). *If W is a uniformly discrete bounded geometry metric space then*

$$h_*(W) \cong \varinjlim_i h_*(X_i(W,\mathcal{U}_*)_f).$$

Proof. It shall be shown that for each i, the spaces $X_i(W,\mathcal{U}_*)_f$ and W are coarsely equivalent.

It will follow that

$$h_*(W) \cong \varinjlim_i h_*(W) \cong \varinjlim_i h_*(X_i(W,\mathcal{U}_*)_f).$$

Let $\Psi \colon W \to \pi^{-1}\{i\} \hookrightarrow X_i(W,\mathcal{U}_*)$ be any map taking $w \in W$ to a vertex $[V]$ of $\pi^{-1}\{i\}$ with $w \in V$.

Let $\eta \colon |\mathcal{U}_i| \to W$ be any map such that if (x,s) lies in the star about a vertex $[V]$ then $\eta(x,s) \in V$ (which exist by Proposition 6.30). Note that $\Psi \circ \eta$ is close to the inclusion of $|\mathcal{U}_i|$ in $X_i(W,\mathcal{U}_*)$ as required. Define $\zeta \colon X_i(W,\mathcal{U}_*) \to W$ to be the composition $\zeta = \eta \circ \Phi_i$. If $(x,s),(x',s') \in X_i(W,\mathcal{U}_*)$ with $d((x,s),(x',s')) < 2j$ then as

$$d(\Phi_{i+j}(x,s), \Phi_{i+j}(x',s')) \leq d((x,s),(x',s')) < 2j,$$

there is a path in $|\mathcal{U}_{i+j}|$ from $\Phi_{i+j}(x,s)$ to $\Phi_{i+j}(x',s')$ of length at most $2j$. Hence by Lemma 6.4, there exists a sequence of open sets $V_0,\dots,V_k$ in $\mathcal{U}_{i+j}$ with the intersection of consecutive pairs non-empty, such that $\zeta(x,s) \in V_0$ and $\zeta(x',s') \in V_k$.

Then

$$\begin{aligned} d([V_0],[V_k]) &\leq d([V_0],(x,s)) + d((x,s),(x',s')) + d((x',s'),[V_k]) \\ &\leq 2j + \pi \end{aligned}$$

so $k\pi/2 \leq (2j+\pi)$ and k is no greater than $4j/\pi + 2$. Thus if $d((x,s),(x',s')) < 2j$ then

$$d(\zeta(x,s),\zeta(x',s')) \leq (4j/\pi + 2)\,\mathrm{Diam}(\mathcal{U}_{i+j})$$

and ζ is also proper, so it follows that ζ is coarse.

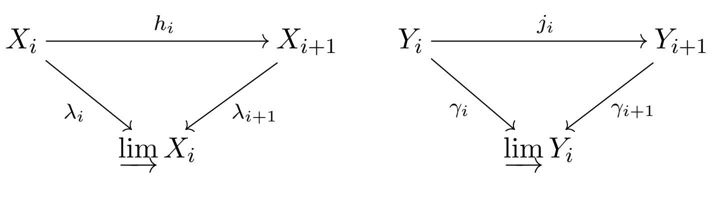
If $w, w' \in W$ with $d(w,w') < R$ then let $j \geq 0$ be such that $\mathcal{U}_{i+j}$ has Lebesgue number at least R. It follows that there exists $[V] \in |\mathcal{U}_{i+j}|$ with $w, w' \in V$ and hence such that $\Phi_{i+j}(\Psi(w))$ and $\Phi_{i+j}(\Psi(w'))$ are vertices of $|\mathcal{U}_{i+j}|$ which are adjacent to $[V]$. Thus if $d(w,w') < R$ then $d(\Psi(w),\Psi(w')) < 2j+\pi$ and hence Ψ is also coarse.

Let $w \in W$, so that $\Psi(w) = ([V],i)$, where $w \in V$. Observe that $(\zeta \circ \Psi)(w) = \zeta([V],i) = (\eta \circ \Phi_i)([V],i) = \eta([V]) \in V$ as $[V]$ is clearly in $\mathrm{Star}([V])$. It follows that $d((\zeta\circ\Psi)(w),w) \leq \mathrm{Diam}(\mathcal{U}_i)$. It is easy to see that $d((\Psi\circ\zeta)(x,s),(x,s)) \leq i + \pi$, so Ψ is a coarse equivalence as required. □

The following result is new. The proof is straightforward but cannot be found in the current literature.

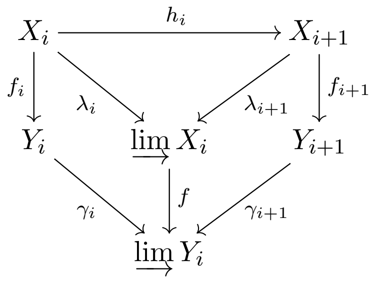
Lemma 6.33. *If $f_i : X_i \to Y_i$ and $g_i : Y_i \to X_{i+1}$ are morphisms for each i such that $g_i \circ f_i : X_i \to X_{i+1}$ and $f_{i+1} \circ g_i : Y_i \to Y_{i+1}$ are direct systems then the direct limits $\varinjlim X_i$ and $\varinjlim Y_i$ are isomorphic.*

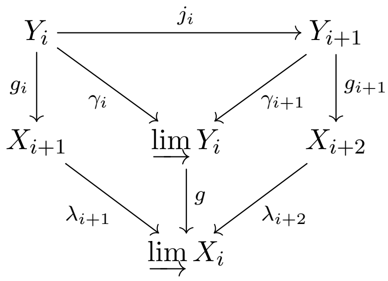
Proof. Write $h_i = g_i \circ f_i : X_i \to X_{i+1}$ and $j_i = f_{i+1} \circ g_i : Y_i \to Y_{i+1}$. By assumption, we have commuting diagrams



By the universal property there exists unique maps $f : \varinjlim X_i \to \varinjlim Y_i$ and $g : \varinjlim Y_i \to$

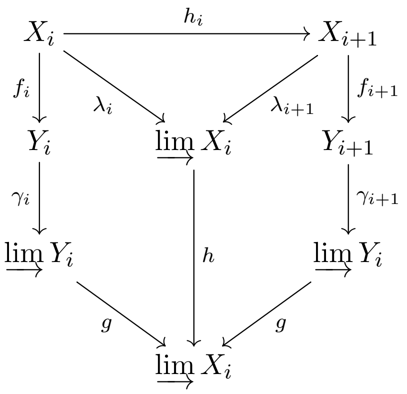
$\varinjlim X_i$ such that the diagrams





commute.

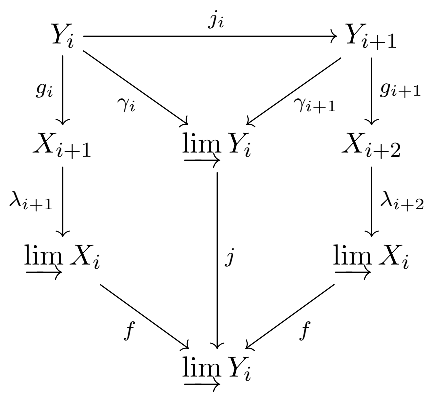
By the universal property, we also have a unique $h : \varinjlim X_i \to \varinjlim X_i$ such that the diagram



commutes.

Note that $(g \circ \gamma_i) \circ f_i = \lambda_{i+1} \circ g_i \circ f_i = \lambda_{i+1} \circ h_i = \lambda_i$. Thus the identity map fits into this diagram so $h = \mathrm{id}$. Also note that $g \circ (f \circ \lambda_i) = g \circ (\gamma_i \circ f_i)$ and thus $g \circ f$ fits into the diagram also and so $g \circ f = \mathrm{id}$.

Likewise, there is a unique $j : \varinjlim Y_i \to \varinjlim Y_i$ such that the diagram



commutes.

Note that $(f \circ \lambda_{i+1}) \circ g_i = (\gamma_{i+1} \circ f_{i+1}) \circ g_i = \gamma_{i+1} \circ j_i = \gamma_i$ and thus the identity map fits into this diagram so $j = \mathrm{id}$. Also note that $(f \circ \lambda_{i+1}) \circ g_i = (f \circ \lambda_{i+1}) \circ g_i = f \circ g \circ \gamma_i$

and thus $f \circ g$ fits into the diagram also and so $f \circ g = \text{id}$. It follows that f is an isomorphism so $\varinjlim X_i$ and $\varinjlim Y_i$ are isomorphic. □

Proposition 6.34. *For $X = X(W, \mathcal{U}_*)$, there is an isomorphism*

$$\varinjlim_i h_*(X_0, (X_i)_0) \to \varinjlim_i h_*(X_f, (X_i)_f).$$

Proof. We can rewrite X as the union of $\pi^{-1}\left(\bigcup[2i-1, 2i]\right)$ and $\pi^{-1}\left(\bigcup[2i, 2i+1]\right)$. Using the collapsing map, these spaces coarse homotopy retract onto $\pi^{-1}(2\mathbb{N})$ and $\pi^{-1}(2\mathbb{N}+1)$ respectively. The intersection of this decomposition is $\pi^{-1}(\mathbb{N})$. By the coarse Mayer–Vietoris sequence, it is enough to prove that there is an isomorphism

$$\varinjlim_i h_*(\pi^{-1}(I)_0, (\pi^{-1}(I) \cap X_i)_0) \to \varinjlim_i h_*(\pi^{-1}(I)_f, (\pi^{-1}(I) \cap X_i)_f)$$

for every $I \subseteq \mathbb{N}$.

We can decompose $\pi^{-1}(I)$ as the union $Y_0 \cup \ldots \cup Y_n$ where each Y_m is the union of stars around the barycentres of m-simplices in the second barycentric subdivision of $\pi^{-1}(I)$. We can prove the result inductively for Z_k where $Z_k = Y_0 \cup \ldots \cup Y_k$. As $\pi^{-1}(I)$ is a uniform metric simplicial complex, the induction process here is similar to that of Proposition 6.16. The final step is to prove that the result holds for the base space, Y_0. This is coarsely homotopy equivalent to a discrete space D consisting of the 0-simplices in the second barycentric subdivision of $\pi^{-1}(I)$. The claim has been reduced to showing that the map

$$\varinjlim_i h_*(D_0, (D \cap X_i)_0) \to \varinjlim_i h_*(D_f, (D \cap X_i)_f)$$

is an isomorphism.

Any fusion controlled set of D is the union of a fusion controlled set of $D \cap X_j$ for some j and a C_0 controlled set of $\pi^{-1}[j, \infty)$. It can also be shown that any fusion controlled set of $D \cap X_j$ is also C_0 controlled. It follows that for sufficiently large i, there exist coarse maps $(D_0, (D \cap X_i)_0) \to (D_f, (D \cap X_i)_f)$ and $(D_f, (D \cap X_i)_f) \to (D_0, (D \cap X_{i+1})_0)$. Therefore by Lemma 6.33, the result holds.

□

Remark 6.35. Since X has a path metric, the decomposition $X = X_i \cup \pi^{-1}[i, \infty)$ is coarsely excisive. This follows from Lemma 3.34 since if (x_n, s_n) is a sequence in X_i converging to (x, s) in X then as $s_n \leq i$ for all n it follows that $s \leq i$ so X_i is closed. Similarly if (x_n, s_n) is a sequence in $\pi^{-1}[i, \infty)$ converging to (x, s) in X then as $s_n \geq i$ for all n it follows that $s \geq i$ so $\pi^{-1}[i, \infty)$ is closed. It can be shown that these decompositions for the C_0 and fusion coarse structures are also coarsely excisive.

Therefore there are coarse Mayer–Vietoris sequences and a commutative ladder

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & h_*(\pi^{-1}\{i\}_0) & \longrightarrow & h_*((X_i)_0) \oplus h_*(\pi^{-1}[i,\infty)_0) & \longrightarrow & h_*(X_0) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & h_*(\pi^{-1}\{i\}_f) & \longrightarrow & h_*((X_i)_f) \oplus h_*(\pi^{-1}[i,\infty)_f) & \longrightarrow & h_*(X_f) & \longrightarrow & \ldots
\end{array}$$

It is well known that the direct limit of a short exact sequence is again a short exact sequence, so we have a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \varinjlim_i h_*(\pi^{-1}\{i\}_0) & \longrightarrow & \varinjlim_i h_*((X_i)_0) & \longrightarrow & \varinjlim_i h_*(X_0) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \varinjlim_i h_*(\pi^{-1}\{i\}_f)) & \longrightarrow & \varinjlim_i h_*((X_i)_f) & \longrightarrow & \varinjlim_i h_*(X_f) & \longrightarrow & \ldots
\end{array}$$

as the spaces $\pi^{-1}[i,\infty)_0$ and $\pi^{-1}[i,\infty)_f$ are almost flasque.

As X_0 is also almost flasque, the top row of the above ladder implies that the map $\varinjlim_i h_*(\pi^{-1}\{i\}_0) \to \varinjlim_i h_*((X_i)_0)$ is an isomorphism. It follows from the relative exact sequence

$$\ldots \to h_*(X_0) \to h_*(X_0,(X_i)_0) \to h_{*-1}((X_i)_0) \to h_{*-1}(X_0) \to \ldots$$

for the pair $(X_0,(X_i)_0)$ that the map $h_*(X_0,(X_i)_0) \to h_{*-1}((X_i)_0)$ is an isomorphism, and therefore that the map $\varinjlim_i h_*(X_0,(X_i)_0) \to \varinjlim_i h_{*-1}((X_i)_0)$ is also an isomorphism. The same exact sequence exists for the fusion case, and an argument similar to Theorem 5.10 of [Wri05] shows that $h_*(X_h)$ and $h_*(X_f)$ are isomorphic, and so $h_*(X_f)$ vanishes by Theorem 6.29.

Applying the above arguments to the fusion case tells us that asking for an isomorphism $\varinjlim_i h_{*-1}(\pi^{-1}\{i\}_0) \to \varinjlim_i h_{*-1}(\pi^{-1}\{i\}_f)$ is equivalent to asking for an isomorphism $\varinjlim h_*(X_0,(X_i)_0) \to \varinjlim h_*(X_f,(X_i)_f)$.

Consider the commuting diagram

$$\begin{array}{ccc}
\varinjlim_i h_{*-1}(\pi^{-1}\{i\}_0) & \longrightarrow & \varinjlim_i h_{*-1}(\pi^{-1}\{i\}_f) \\
\downarrow & & \downarrow \\
\varinjlim_i h_{*-1}((X_i)_0) & & \varinjlim_i h_{*-1}((X_i)_f) \\
\downarrow & & \downarrow \\
k_*^{\text{coarse}}(W) & \xrightarrow{\lambda} & h_{*-1}(W)
\end{array}$$

The map $\varinjlim_i h_{*-1}(\pi^{-1}\{i\}_0) \to \varinjlim_i h_{*-1}(\pi^{-1}\{i\}_f)$ is an isomorphism by Proposition 6.34 and the above remarks. The map $\varinjlim_i h_{*-1}((X_i)_0) \to k_*^{\text{coarse}}(W)$ is an isomorphism by Theorem 6.21. The map $\varinjlim_i h_{*-1}((X_i)_f) \to h_{*-1}(W)$ is an isomorphism by Theorem 6.32. Combining all of the above results together, we have proved the following result:

Theorem 6.36. *If W is a proper metric space of bounded geometry and finite asymptotic dimension then the coarse assembly map*

$$\lambda\colon k_*^{coarse}(W) \to h_{*-1}(W)$$

is an isomorphism.

Chapter 7

Applications

It was shown in the previous chapter that the coarse assembly map is an isomorphism for spaces of finite asymptotic dimension. As well as the coarse Baum–Connes assembly map there are other assembly maps of interest, in the areas of algebraic K-theory and C^*-category K-theory. These include the Loday assembly map, the Farrell–Jones assembly map and the Baum–Connes assembly map in topological K-theory. We construct coarsely excisive functors for these assembly maps and give isomorphism and descent results for these under the assumption of finite asymptotic dimension. The new results in this chapter including showing that these functors satisfy the definition of coarsely excisive as introduced in Chapter 5.

It will be shown that the original example of the coarse Baum–Connes conjecture for finite asymptotic dimension is a special case of the version for additive C^*-categories.

7.1 Waldhausen K-theory

The aim is to be able to define the algebraic K-theory of an additive category $\mathcal{A}$. It is well known that algebraic K-theory groups are complicated to define in higher dimensions, and it was observed by Quillen that the correct way to define the higher algebraic K-theory groups would be as the homotopy groups of a certain loop space. Furthermore, it is possible to construct a spectrum whose homotopy groups are the algebraic K-theory groups, and there are many benefits of doing this over the loop space construction. Waldhausen's $S_\bullet$-construction provides a recipe for constructing the loop space $\Omega|wS_\bullet\mathcal{C}|$ and the spectrum $\mathbf{K}(\mathcal{C})$ for a Waldhausen category $\mathcal{C}$ (a category with a notion of cofibrations and weak equivalences), and this is more general than the notions of an additive category and of exact categories defined by Quillen. For more on the Waldhausen $S_\bullet$-construction, see [Rog10], [Mit02] and [Car05].

Definition 7.1 (Pointed category). A *pointed category* $\mathcal{C}$ is a category with a zero object $0_\mathcal{C}$. That is, for every object $A \in \mathcal{C}$, there exists unique morphisms $0_\mathcal{C} \to A$ and $A \to 0_\mathcal{C}$ in $\mathcal{C}$.

Definition 7.2 (Category with cofibrations). A small pointed category $\mathcal{C}$ is called a *category with cofibrations* if there is a subcategory $\mathrm{co}(\mathcal{C})$ of $\mathcal{C}$ (called the *category of cofibrations*), closed under composition, whose morphisms are called *cofibrations* (denoted by $A \rightarrowtail B$) such that the following conditions are satisfied:

- every isomorphism $A \rightarrowtail B$ in $\mathcal{C}$ is a cofibration;
- the unique morphism $0 \rightarrowtail A$ is a cofibration for all objects $A \in \mathcal{C}$;
- if $A \rightarrowtail B$ is a cofibration in $\mathcal{C}$ and $A \to C$ is a morphism in $\mathcal{C}$ then the pushout diagram

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & C \cup_A B \end{array}$$

  exists in $\mathcal{C}$ and the canonical morphism $C \rightarrowtail C \cup_A B$ is a cofibration.

Observe that the first and second conditions each imply that $\mathrm{co}(\mathcal{C})$ has the same objects as $\mathcal{C}$. A category with cofibrations will be denoted by $(\mathcal{C}, \mathrm{co}(\mathcal{C}))$ or $\mathcal{C}$ when the subcategory is clear from context.

Example 7.3. If $A \rightarrowtail B$ is a cofibration, the unique map $A \to 0$ can be used to form the pushout diagram

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \twoheaddownarrow \\ 0 & \rightarrowtail & B/A \end{array}$$

in $\mathcal{C}$, where the pushout $B \cup_A 0$ is written as B/A. The induced map $B \twoheadrightarrow B/A$ is known as a *quotient map.*

Definition 7.4 (Cofiber sequence). A *cofiber sequence* is a diagram of the form

$$A \rightarrowtail B \twoheadrightarrow B/A$$

where the first map is a cofibration and the second map is the associated quotient map.

Definition 7.5 (Sequence of cofibrations). A *sequence of cofibrations* is a diagram of the form

$$A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_q$$

where each map $A_i \rightarrowtail A_{i+1}$ is a cofibration.

Example 7.6. The unique cofibrations $0 \rightarrowtail A$ and $0 \rightarrowtail B$ can be used to form the pushout diagram

$$\begin{array}{ccc} 0 & \rightarrowtail & A \\ \downarrow & & \downarrow \\ B & \rightarrowtail & A \cup_0 B \end{array}$$

for the coproduct $A \vee B = A \cup_0 B$.

Example 7.7. Let Set_+ be the pointed category where the objects are sets with a distinguished base-point and the morphisms are base-point preserving functions, and let $\mathrm{co}(\mathrm{Set}_+)$ be the subcategory of injective functions.

Clearly any bijection is injective, and for any $A \in \mathrm{Set}_+$ the map $0 \to A$ is also injective.

Consider the pushout diagram

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h_B} \\ C & \xrightarrow{h_C} & C \cup_A B \end{array}$$

where $f\colon A \to B$ is injective and $C \cup_A B$ is the space $(C \sqcup B)/\sim$ generated by the equivalence relation $g(a) \sim f(a)$ for all $a \in A$. The map $h_C\colon C \to C \cup_A B$ is the map defined by the composition of the inclusion map $i\colon C \hookrightarrow C \sqcup B$ and the quotient map $\pi\colon C \sqcup B \to C \cup_A B$, and the map h_B is defined similarly. If $h_C(c) = h_C(c')$ and $c, c' \notin \operatorname{Im} g$ then $\pi(c) = \pi(c')$ implies that $c = c'$. If $c, c' \in \operatorname{Im} g$ then $c = g(a)$ and $c' = g(a')$ for some $a, a' \in A$ and so if $\pi(c) = \pi(c')$ then $(\pi \circ f)(a) = (\pi \circ g)(a) = (\pi \circ g)(a') = (\pi \circ f)(a')$ by commutativity. It follows that $a = a'$ and so $c = c'$ by injectivity of f. Thus h_C is injective, and $(\mathrm{Set}_+, \mathrm{co}(\mathrm{Set}_+))$ is a category with cofibrations.

Definition 7.8 (Exact functor for cofibration categories). A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories with cofibrations is an *exact functor* if $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$, $F(\mathrm{co}(\mathcal{C})) = \mathrm{co}(\mathcal{D})$ and if for each cofibration $A \rightarrowtail B$ in $\mathcal{C}$ and morphism $A \to C$ in $\mathcal{C}$, the image

$$\begin{array}{ccc} F(A) & \rightarrowtail & F(B) \\ \downarrow & & \downarrow \\ F(C) & \rightarrowtail & F(C \cup_A B) \end{array}$$

of the pushout square

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & C \cup_A B \end{array}$$

in $\mathcal{C}$ is a pushout square in $\mathcal{D}$.

It can be shown that the composition of two exact functors is again an exact functor, and so the category of small categories with cofibrations and exact functors can be formed.

Definition 7.9 (Waldhausen category). A category with cofibrations $\mathcal{C}$ is called a *Waldhausen category* if there is a subcategory $\mathrm{w}(\mathcal{C})$ of $\mathcal{C}$ (called the *category of weak equivalences*), closed under composition, whose morphisms are called *weak equivalences* (denoted by $A \xrightarrow{\sim} B$) such that:

- every isomorphism $A \to B$ in $\mathcal{C}$ is a weak equivalence;
- if $B \rightarrowtail A$ and $\overline{B} \rightarrowtail \overline{A}$ are cofibrations and each of $A \xrightarrow{\sim} \overline{A}$, $B \xrightarrow{\sim} \overline{B}$ and $C \xrightarrow{\sim} \overline{C}$ are weak equivalences then the diagram

$$\begin{array}{ccccc} A & \longleftarrow\!\!\!< & B & \longrightarrow & C \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ \overline{A} & \longleftarrow\!\!\!< & \overline{B} & \longrightarrow & \overline{C} \end{array}$$

commutes and the pushout morphism

$$A \cup_B C \rightarrowtail \overline{A} \cup_{\overline{B}} \overline{C}$$

is also a weak equivalence.

Observe that the first condition implies that $\mathrm{w}(\mathcal{C})$ has the same objects as $\mathcal{C}$. A Waldhausen category will be denoted by $(\mathcal{C}, \mathrm{co}(\mathcal{C}), \mathrm{w}(\mathcal{C}))$ or $\mathcal{C}$ when the subcategories are clear from context.

The following example is important for defining the algebraic K-theory of rings.

Example 7.10. For a ring R, define $\mathcal{P}(R)$ to be the category where the objects are finitely generated projective left R-modules and the morphisms are the R-module homomorphisms. It is shown in Section 8.1 of [Rog10] that $\mathcal{P}(R)$ is a Waldhausen category where the cofibrations are taken to be the injective R-module homomorphisms $f \colon P \to Q$ such that the cokernel Q/P is finitely generated projective and the weak equivalences are taken to be the isomorphisms.

Definition 7.11 (Exact functor for Waldhausen categories). A functor $F \colon \mathcal{C} \to \mathcal{D}$ between Waldhausen categories is an *exact functor* if $F(\mathrm{w}(\mathcal{C})) = \mathrm{w}(\mathcal{D})$ and if it is an exact functor as a functor between categories with cofibrations.

As in the case of categories with cofibrations, the composition of two exact functors is again an exact functor, so the category of small Waldhausen categories and exact functors can be formed.

Definition 7.12 (Category $[q]$). For $q \geq 0$, the ordered set

$$[q] = \{0 < 1 < \ldots < q\}$$

of integers from 0 to q can be viewed as a small category where the objects are the integers from 0 to q and there is a unique morphism $i \to j$ for $i \leq j$ and no morphism $i \to j$ if $i > j$. Observe that by uniqueness, the morphism $i \to j$ factors as the composite $i \to i+1 \to \ldots \to j-1 \to j$ for $i \leq j$.

Definition 7.13 (Arrow category). The *arrow category* $\mathrm{Ar}(\mathcal{C})$ of a category $\mathcal{C}$ is the category where the objects are the morphisms $f\colon A \to B$ in $\mathcal{C}$ and the morphisms $F\colon f \to f'$ from $f\colon A \to B$ to $f'\colon A' \to B'$ are the pairs $F = (F_A, F_B)$ of morphisms $F_A\colon A \to A'$ and $F_B\colon B \to B'$ in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle F_A}\downarrow & & \downarrow{\scriptstyle F_B} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

commutes.

Definition 7.14 (Functor category). The *functor category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for small categories $\mathcal{C}$ and $\mathcal{D}$ is the category where the objects are functors $F\colon \mathcal{C} \to \mathcal{D}$ (known as $\mathcal{C}$-shaped diagrams in $\mathcal{D}$) and the morphisms are natural transformations between them.

A functor $A\colon [q] \to \mathcal{C}$ is a diagram

$$A(0) \to A(1) \to \ldots \to A(q)$$

and so it can be easily seen that $\mathrm{Fun}([0], \mathcal{C}) = \mathcal{C}$ and $\mathrm{Fun}([1], \mathcal{C}) = \mathrm{Ar}(\mathcal{C})$.

Definition 7.15 (Category $S_q\mathcal{C}$). For a Waldhausen category $\mathcal{C}$, define a category $S_q\mathcal{C}$ as follows. Define the objects to be functors $A\colon \mathrm{Ar}[q] \to \mathcal{C}$ such that $A(i \to j) = A_{ij}$ with $A_{ii} = 0$ for each i and for all triples $i \leq j \leq k$ there is a cofiber sequence $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$, or equivalently, let the objects be sequences of cofibrations

$$\{A_{ij}\} = \; A_{00} \rightarrowtail A_{01} \rightarrowtail \ldots \rightarrowtail A_{0q}$$

together with a choice of quotients $A_{ij} = A_{0j}/A_{0i}$ defined (by the pushout property) for $i \leq j$. Define a morphism $f\colon \{A_{ij}\} \to \{B_{ij}\}$ in $S_q\mathcal{C}$ to be the collection of morphisms $f_{ij}\colon A_{ij} \to B_{ij}$ in $\mathcal{C}$ for all $i \leq j$ such that the square

$$\begin{array}{ccc} A_{ij} & \xrightarrow{f_{ij}} & B_{ij} \\ \downarrow & & \downarrow \\ A_{i'j'} & \xrightarrow{f_{i'j'}} & B_{i'j'} \end{array}$$

commutes for each morphism $(i, j) \to (i', j')$ in $\mathrm{Ar}[q]$, that is, for all $i \leq i'$ and

$j \leq j'$ such that the square

$$\begin{array}{ccc} i & \longrightarrow & j \\ \downarrow & & \downarrow \\ i' & \longrightarrow & j' \end{array}$$

commutes.

By the universal property of quotients, objects of $S_q\mathcal{C}$ can be pictured as the upper triangular commutative diagrams with zeros along the diagonal

$$\begin{array}{ccccccccc}
0 = A_{00} & \rightarrowtail & A_{01} & \rightarrowtail & A_{02} & \rightarrowtail & \dots & \rightarrowtail & A_{0q} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 = A_{11} & \rightarrowtail & A_{12} & \rightarrowtail & \dots & \rightarrowtail & A_{1q} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & 0 = A_{22} & \rightarrowtail & \dots & \rightarrowtail & A_{2q} \\
 & & & & & & & & \downarrow \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & 0 = A_{qq}.
\end{array}$$

Examples 7.16 (Category $S_n\mathcal{C}$ for $n = 0, 1, 2$). An object in the category $S_0\mathcal{C}$ is simply the diagram

$$0$$

in $\mathcal{C}$ with quotient defined by $A_{00} = 0$. Thus $S_0\mathcal{C}$ is the category with single object 0 and single morphism $0 \to 0$.

An object in the category $S_1\mathcal{C}$ is a diagram

$$\begin{array}{ccc} 0 & \rightarrowtail & A_{01} \\ & & \downarrow \\ & & 0 \end{array}$$

in $\mathcal{C}$ with quotients defined by $A_{00} = A_{11} = 0$. The diagram can be viewed as the object A_{01} in $\mathcal{C}$ with no quotient, and hence $S_1\mathcal{C}$ is naturally isomorphic to the category $\mathcal{C}$.

An object in $S_2\mathcal{C}$ is a commutative diagram

$$\begin{array}{ccccc} 0 & \rightarrowtail & A_{01} & \rightarrowtail & A_{02} \\ & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ & & 0 & \rightarrowtail & A_{12} \\ & & & & \downarrow\!\!\!\downarrow \\ & & & & 0 \end{array}$$

in $\mathcal{C}$ where each horizontal morphism is a cofibration and the square is a pushout square defining the quotient A_{12}. This category is naturally isomorphic to the category where the objects are the cofibration sequences $A \rightarrowtail B \twoheadrightarrow B/A$ in $\mathcal{C}$ and morphisms from $A \rightarrowtail B \twoheadrightarrow B/A$ to $\overline{A} \rightarrowtail \overline{B} \twoheadrightarrow \overline{B}/\overline{A}$ to be the commutative diagrams

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow & & \downarrow & & \downarrow \\ \overline{A} & \rightarrowtail & \overline{B} & \twoheadrightarrow & \overline{B}/\overline{A} \end{array}$$

in $\mathcal{C}$.

Definition 7.17 (Category $\operatorname{co} S_q\mathcal{C}$). Define $\operatorname{co} S_q\mathcal{C}$ to be the subcategory of $S_q\mathcal{C}$ where the morphisms $f\colon \{A_{ij}\} \to \{B_{ij}\}$ are defined to be the collection of maps $f_{ij}\colon A_{ij} \to B_{ij}$ in $\mathcal{C}$ such that $f_{ij}\colon A_{ij} \rightarrowtail B_{ij}$ is a cofibration in $\mathcal{C}$ for each $1 \le i \le j \le q$.

Definition 7.18 (Category $wS_q\mathcal{C}$). Define $wS_q\mathcal{C}$ to be the subcategory of $S_q\mathcal{C}$ where the morphisms $f\colon \{A_{ij}\} \to \{B_{ij}\}$ are defined to be the collection of maps $f_{ij}\colon A_{ij} \to B_{ij}$ in $\mathcal{C}$ such that $f_{ij}\colon A_{ij} \xrightarrow{\sim} B_{ij}$ is a weak equivalence in $\mathcal{C}$ for each $1 \le i \le j \le q$.

It can be shown that $(S_q\mathcal{C}, \operatorname{co} S_q\mathcal{C}, wS_q\mathcal{C})$ is again a Waldhausen category, see Lemma 8.3.16 of [Rog10].

We can consider the collection $S_\bullet\mathcal{C} = \{S_q\mathcal{C}\colon q \in \mathbb{N}\}$ of all these categories, and it can again be shown that $S_\bullet\mathcal{C}$ is a Waldhausen category (with the cofibrations of $S_\bullet\mathcal{C}$ being the collection of cofibrations of each $S_q\mathcal{C}$ and the weak equivalences of $S_\bullet\mathcal{C}$ being the collection of weak equivalences of each $S_q\mathcal{C}$), see Proposition 8.3.20 of [Rog10].

Definition 7.19 (Waldhausen K-theory space). The *Waldhausen K-theory space* is the space $K(\mathcal{C}, w) = \Omega|wS_\bullet\mathcal{C}|$. The *Waldhausen K-theory groups* are defined to be the homotopy groups

$$K_n(\mathcal{C}) = \pi_n(K(\mathcal{C}, w))$$

for $n \ge 1$.

Definition 7.20 (Algebraic K-theory of rings). Let R be a ring. The *algebraic K-theory space of* R is the space $K(R) = \Omega|wS_\bullet\mathcal{P}(R)|$, that is, the algebraic K-theory space of the Waldhausen category $\mathcal{P}(R)$. The *algebraic K-theory groups* of R are defined to be the homotopy groups

$$K_n(R) = \pi_n(K(R))$$

for $n \geq 1$.

Since $S_\bullet\mathcal{C}$ of a Waldhausen category $\mathcal{C}$ is again a Waldhausen category, this process can be repeated to obtain a sequence of Waldhausen categories $S_\bullet^{(n)}\mathcal{C} = S_\bullet S_\bullet \dots S_\bullet\mathcal{C}$ for each n, and therefore a sequence of topological spaces $|wS_\bullet^{(n)}\mathcal{C}|$ for each n. It is therefore possible to form a Waldhausen K-theory spectrum as well as a Waldhausen K-theory space.

Definition 7.21 (Waldhausen K-theory spectrum and groups). The *Waldhausen K-theory spectrum* $\mathbf{K}(\mathcal{C})$ is the spectrum with spaces $\mathbf{K}(\mathcal{C})_n = |wS_\bullet^{(n)}\mathcal{C}|$ and structure maps $|wS_\bullet^{(n)}\mathcal{C}| \to \Omega|wS_\bullet^{(n+1)}\mathcal{C}|$. The *Waldhausen K-theory groups* are defined to be the stable homotopy groups

$$K_n(\mathcal{C}) = \pi_n(\mathbf{K}(\mathcal{C}))$$

for $n \in \mathbb{Z}$.

There is an action of the symmetric group Σ_n on the Waldhausen category $wS_\bullet^{(n)}\mathcal{C}$ defined by permuting the order in which the $S_\bullet$ constructions are made. There is also an induced Σ-action on the geometric realisation $|wS_\bullet^{(n)}\mathcal{C}|$, and therefore the spectrum $\mathbf{K}(\mathcal{C})$ is a symmetric spectrum.

7.2 Controlled algebraic K-theory groups

Assembly maps are often studied using techniques coming from controlled topology. An additive category $\mathcal{A}(X)$ with dependance on the coarse structure of X will be introduced and we show that this is a Waldhausen category. It will be shown that the functor $X \mapsto \mathbf{K}\mathcal{A}(X)$ representing the algebraic K-theory of X is coarsely excisive and we will apply descent to obtain results about the Farrell–Jones and Loday assembly maps in algebraic K-theory.

Definition 7.22 (R-linear category). Let R be a ring. An *R-linear* category is a category $\mathcal{A}$ in which each morphism set $\mathrm{Mor}_{\mathcal{A}}(A, B)$ is a left R-module and the composition of morphisms is bilinear.

Definition 7.23 (Biproduct). Let $\mathcal{A}$ be an R-linear category. An object $A \oplus B$ is called the *biproduct* of objects A and B if it comes equipped with morphisms $i_A \colon A \to A \oplus B$, $i_B \colon B \to A \oplus B$, $p_A \colon A \oplus B \to A$ and $p_B \colon A \oplus B \to B$ satisfying $p_A \circ i_A = \mathrm{id}_A$, $p_B \circ i_B = \mathrm{id}_B$, $p_A \circ i_B = p_B \circ i_A = 0$ and $i_A \circ p_A + i_B \circ p_B = \mathrm{id}_{A \oplus B}$.

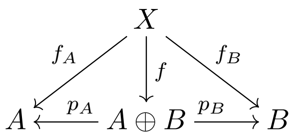
To avoid ambiguity, the morphism $i_A\colon A \to A \oplus B$ will sometimes be denoted by $i_A^{A\oplus B}$ and the morphism $p_A\colon A \oplus B \to A$ by $p^A_{A\oplus B}$, etc.

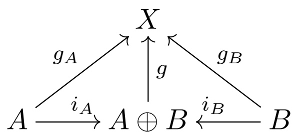
Definition 7.24 (Additive category). An R-linear category is said to be *additive* if it is equipped a zero object and each pair of objects has a biproduct.

The following result is straightforward to prove, but the proof cannot be found in the current literature.

Proposition 7.25. *If $\mathcal{A}$ is an R-linear category, then an object $A \oplus B$ is a biproduct of A and B if and only if it is simultaneously a product and a coproduct in $\mathcal{A}$.*

Proof. If $A \oplus B$ is a biproduct then the diagrams





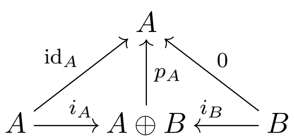
where $f\colon X \to A \oplus B$ and $g\colon A \oplus B \to X$ are defined by $f = i_A \circ f_A + i_B \circ f_B$ and $g = g_A \circ p_A + g_B \circ p_B$ are product and coproduct diagrams respectively.

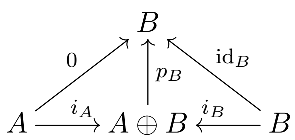
To see this, observe that $p_A \circ f = p_A \circ i_A \circ f_A + p_A \circ i_B \circ f_B = f_A$ and that $p_B \circ f = p_B \circ i_A \circ f_A + p_B \circ i_B \circ f_B = f_B$ by the biproduct identities. To show that f is unique, if there exists an $f'\colon X \to A \oplus B$ such that $p_A \circ f' = f_A$ and $p_B \circ f' = f_B$ then

$$\begin{aligned} f - f' &= \mathrm{id}_{A\oplus B} \circ (f - f') \\ &= (i_A \circ p_A + i_B \circ p_B) \circ (f - f') \\ &= i_A \circ p_A \circ f + i_B \circ p_B \circ f - i_A \circ p_A \circ f' - i_B \circ p_B \circ f' \\ &= 0 \end{aligned}$$

and so f is unique. The case for g is identical.

Conversely, if the product and coproduct diagrams both exist then there are maps $i_A\colon A \to A \oplus B$ and $i_B\colon B \to A \oplus B$ such that for any X and morphisms $f_B\colon B \to X$ and $f_A\colon A \to X$ then there exists a unique map $f\colon A \oplus B \to X$ such that $f \circ i_A = f_A$ and $f \circ i_B = f_B$. In particular, the coproduct diagrams





exist, where the unique maps have been labelled as p_A and p_B respectively. Thus there are maps such that $p_A \circ i_A = \mathrm{id}_A$ and $p_A \circ i_B = 0$, $p_B \circ i_B = \mathrm{id}_B$ and $p_B \circ i_A = 0$. The coproduct diagram

$$\begin{tikzcd} & A \oplus B & \\ A \arrow[ur, "i_A"] \arrow[r, "i_A"'] & A \oplus B \arrow[u, "f"] & B \arrow[ul, "i_B"'] \arrow[l, "i_B"] \end{tikzcd}$$

also exists. The identity morphism $\mathrm{id}_{A\oplus B}$ clearly satisfies this diagram. It is easy to check that $i_A \circ p_A + i_B \circ p_B$ also satifies this diagram and thus $i_A \circ p_A + i_B \circ p_B = \mathrm{id}_{A\oplus B}$ by uniqueness, and therefore $A \oplus B$ is a biproduct of A and B. □

Remark 7.26. It is easy to check that an object A in an R-linear category is a coproduct if and only if it is a product, and that in any additive category, A is the biproduct of A and 0 and also the biproduct of 0 and A.

The following is stated in Proposition 5.1 of [Mit02] without proof. For completeness, we give a proof here.

Proposition 7.27. *If $\mathcal{A}$ is an additive R-linear category, then $\mathcal{A}$ is a Waldhausen category where the cofibrations are defined to be morphisms isomorphic to morphisms of the form $i_A \colon A \to A \oplus B$ and weak equivalences are isomorphisms.*

Proof. If $f \colon C \to D$ is an isomorphism then f is isomorphic to $i_C \colon C \to C \oplus 0$ via the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, equal] & D \arrow[d, "i_C \circ f^{-1}"] \\ C \arrow[r, "i_C"] & C \oplus 0 \end{tikzcd}$$

so is therefore also a cofibration. The unique morphism $0 \to A$ is obviously of the form $0 \to 0 \oplus A$, so is a cofibration.

Suppose that $A \to A \oplus B$ is a cofibration and $A \to C$ is a morphism and that the diagram

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d] & A \oplus B \arrow[d] \\ C \arrow[r, "i_C"'] & C \oplus B \end{tikzcd}$$

commutes. To see that this is a pushout diagram, suppose that X is an object with maps $\gamma_{A\oplus B} \colon A \oplus B \to X$ and $\gamma_C \colon C \to X$ such the diagram

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "g"] & A \oplus B \arrow[d, "f"'] \arrow[ddr, "\gamma_{A\oplus B}"] & \\ C \arrow[r, "i_C"] \arrow[drr, "\gamma_C"'] & C \oplus B \arrow[dr, "\theta"] & \\ & & X \end{tikzcd}$$

commutes, where $f\colon A\oplus B\to C\oplus B$ is defined by

$$f = i_C^{C\oplus B}\circ g\circ p_{A\oplus B}^A + i_B^{C\oplus B}\circ p_{A\oplus B}^B.$$

Observe that $p_{C\oplus B}^B\circ f = p_{A\oplus B}^B$ and $p_{C\oplus B}^C\circ f = g\circ p_{A\oplus B}^A$.

Define a map $\theta\colon C\oplus B\to X$ by

$$\theta = \gamma_C\circ p_{C\oplus B}^C + \gamma_{A\oplus B}\circ i_B^{A\oplus B}\circ p_{C\oplus B}^B.$$

This map makes the diagram commute as $\theta\circ i_C = \gamma_C$ and

$$\begin{aligned}\theta\circ f &= (\gamma_C\circ p_{C\oplus B}^C + \gamma_{A\oplus B}\circ i_B^{A\oplus B}\circ p_{C\oplus B}^B)\circ f\\ &= \gamma_C\circ g\circ p_{A\oplus B}^A + \gamma_{A\oplus B}\circ i_B^{A\oplus B}\circ p_{A\oplus B}^B\\ &= \gamma_{A\oplus B}\circ i_A^{A\oplus B}\circ p_{A\oplus B}^A + \gamma_{A\oplus B}\circ i_B^{A\oplus B}\circ p_{A\oplus B}^B\\ &= \gamma_{A\oplus B}\end{aligned}$$

To see that θ is unique, suppose there exists $\phi\colon C\oplus B\to X$ such that $\phi\circ f = \gamma_{A\oplus B}$ and $\phi\circ i_C^{C\oplus B} = \gamma_C$. Observe then that $\phi\circ i_B^{C\oplus B} = \gamma_{A\oplus B}\circ i_B^{A\oplus B}$ and therefore

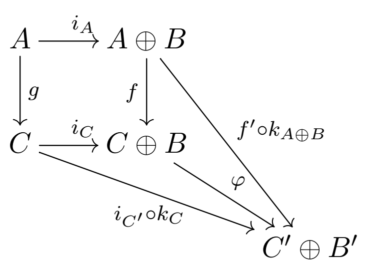
$$\begin{aligned}\phi &= \phi\circ i_C^{C\oplus B}\circ p_{C\oplus B}^C + \phi\circ i_B^{C\oplus B}\circ p_{C\oplus B}^B\\ &= \gamma_C\circ p_{C\oplus B}^C + \gamma_{A\oplus B}\circ i_B^{A\oplus B}\circ p_{C\oplus B}^B\\ &= \theta.\end{aligned}$$

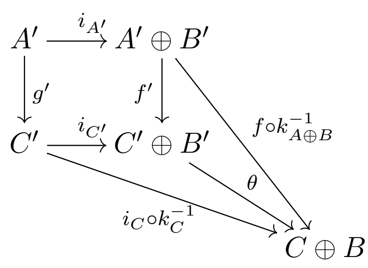
To verify the gluing property, consider the commutative diagram

$$\begin{array}{ccccc} A & \rightarrowtail & A\oplus B & \longrightarrow & C\\ \downarrow{\scriptstyle k_A} & & \downarrow{\scriptstyle k_{A\oplus B}} & & \downarrow{\scriptstyle k_C}\\ A' & \rightarrowtail & A'\oplus B' & \longrightarrow & C' \end{array}$$

where each of the maps $k_A, k_{A\oplus B}$ and k_C are isomorphisms.

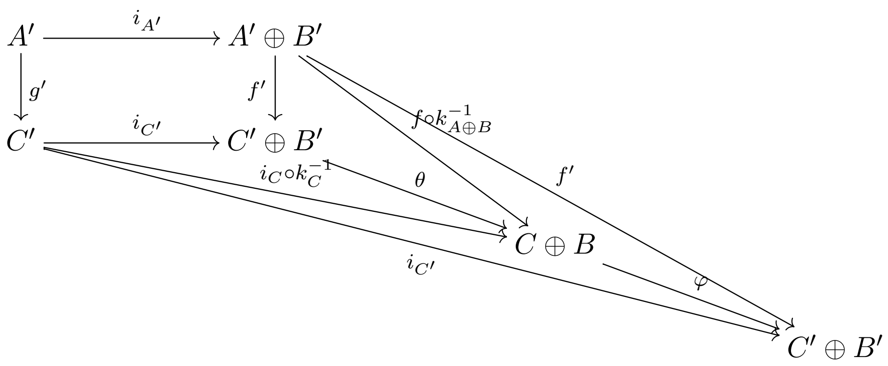
From the pushout diagram for $C\oplus B$, there are unique maps φ and θ such that the diagrams





commute.

Putting these together, the diagram



commutes as

$$\varphi \circ \theta \circ f' = \varphi \circ f \circ k_{A\oplus B}^{-1} = f' \circ k_{A\oplus B} \circ k_{A\oplus B}^{-1} = f'$$

and

$$\varphi \circ \theta \circ i_{C'} = \varphi \circ i_C \circ k_C^{-1} = i_{C'} \circ k_C \circ k_C^{-1} = i_{C'}.$$

Therefore the map $\varphi \circ \theta$ satisfies the diagram and is unique. It is easy to see that $\mathrm{id}_{C'\oplus B'}$ also satisfies the diagram and therefore $\varphi \circ \theta = \mathrm{id}_{C'\oplus B'}$. Similarly doing this the other way round gives us $\theta \circ \varphi = \mathrm{id}_{C\oplus B}$, so that the pushout morphism $\varphi\colon C \oplus B \to C' \oplus B'$ is an isomorphism as required. □

The results in this subsection therefore give us a description of the spectrum whose fundamental groups are the algebraic K-theory groups of an additive R-linear category.

Definition 7.28 (Geometric $\mathcal{A}$-module). Let X be a metric space, and $\mathcal{A}$ be an additive R-linear category. A functor M from the category of metrically bounded subsets of X to the category $\mathcal{A}$ is said to be a *geometric $\mathcal{A}$-module* over X if the following conditions are satisfied:

- for each metrically bounded set B, the natural map

$$\bigoplus_{x\in B} M_x \to M(B)$$

  induced by the inclusion maps is an isomorphism;
- the *support* of M

$$\operatorname{Supp} M = \{x \in X \colon M_x \neq 0\}$$

  is a locally finite subset of X.

Definition 7.29. A *morphism* $\phi\colon M \to N$ between geometric $\mathcal{A}$-modules over X is a collection of morphisms $\phi_{x,x'}\colon M_{x'} \to N_x$ in the category $\mathcal{A}$ such that:

- for each $x \in X$, the morphism $\phi_{x,x'} \neq 0$ for only finitely many points $x' \in X$;
- for each $x' \in X$, the morphism $\phi_{x,x'} \neq 0$ for only finitely many points $x \in X$.

The composition of morphisms $\phi\colon M \to N$ and $\psi\colon N \to P$ is defined by the formula

$$(\psi \circ \phi)_{x,y}(\lambda) = \sum_{z \in X} \psi_{x,z} \circ \phi_{z,y}(\lambda).$$

The *support* of a morphism $\phi\colon M \to N$ is defined to be the set

$$\operatorname{Supp}\phi = \{(x,y) \in X \times X \colon \phi_{x,y} \neq 0\}.$$

Definition 7.30. Let $(X, \mathcal{E})$ be a coarse space and $\mathcal{A}$ be an additive R-linear category. The category $\mathcal{A}(X)$ is defined to be the category consisting of all geometric $\mathcal{A}$-modules over X and morphisms ϕ between them such that $\operatorname{Supp}\phi \in \mathcal{E}$, that is, controlled with respect to the coarse structure.

It can be shown that $\mathcal{A}(X)$ is again an additive R-linear category. Thus we can form the algebraic K-theory spectrum $\mathbf{K}\mathcal{A}(X)$ of $\mathcal{A}(X)$ and hence the algebraic K-theory groups $K_*(\mathcal{A}(X))$.

For any coarse map $f\colon X \to Y$, there is an induced additive functor $f_*\colon \mathcal{A}(X) \to \mathcal{A}(Y)$ defined as follows:

- for a geometric $\mathcal{A}$-module M over X, the geometric $\mathcal{A}$-module $f_*[M]$ over Y is defined by writing
$$f_*[M](S) = M(f^{-1}(S));$$
- for a morphism $\phi\colon M \to N$ in $\mathcal{A}(X)$, the morphism $f_*[\phi]\colon f_*[M] \to f_*[N]$ in $\mathcal{A}(Y)$ is defined by writing
$$f_*[\phi]_{h_1,h_2} = \sum_{\substack{g_1 \in f^{-1}(h_1) \\ g_2 \in f^{-1}(h_2)}} \phi_{g_1,g_2}.$$

Using these induced maps, there is a functor $X \mapsto \mathcal{A}(X)$, and so there is a functor $X \mapsto \mathbf{K}\mathcal{A}(X)$.

The proofs of the following two results are based on that of Theorem 6.3 in [Mit10].

Proposition 7.31. *If X is an almost flasque space then the spectrum $\boldsymbol{K}\mathcal{A}(X)$ is weakly contractible (so that $K_*(\mathcal{A}(X)) = \{0\}$).*

Proof. Let $\alpha_k\colon X \to X$ be the collection of maps which satisfy the almost flasqueness condition, and let M be a geometric $\mathcal{A}$-module over X. It will be shown that the direct sum

$$\bigoplus_{k=0}^{\infty}(\alpha_k)_*[M]$$

is also a geometric $\mathcal{A}$-module over X.

Since the map $\bigoplus_{x\in B} M_x \to M(B)$ is an isomorphism for all B, it follows that by replacing B with $(\alpha_k)^{-1}(B)$, the map

$$\bigoplus_{x\in B}(\alpha_k)_*[M]_x \to (\alpha_k)_*[M](B)$$

is an isomorphism for each k.

By the first condition of almost flasqueness, for each bounded set B, the intersection $B \cap \alpha_k(X)$ is empty for all but finitely many k. Thus $(\alpha_k)^{-1}(B)$ is non-empty for only finitely many k. Hence we can write $\bigoplus_{k=0}^{\infty}(\alpha_k)_*[M](B) = \bigoplus_{k\in J_B}(\alpha_k)_*[M](B)$, where $J_B \subseteq \mathbb{N}$ is a finite subset which depends on B. There is also a finite subset $I_B \subseteq \mathbb{N}$ such that $\oplus_{k=0}^{\infty} \oplus_{x\in B} (\alpha_k)_*[M]_x = \oplus_{k\in I_B} \oplus_{x\in B} (\alpha_k)_*[M]_x$. Observe that $I_B \subseteq J_B$ since $x \in B$.

The map

$$\bigoplus_{k=0}^{\infty}\bigoplus_{x\in B}(\alpha_k)_*[M]_x \to \bigoplus_{k=0}^{\infty}(\alpha_k)_*[M](B)$$

is the map

$$\bigoplus_{k\in I_B}\bigoplus_{x\in B}(\alpha_k)_*[M]_x \to \bigoplus_{k\in J_B}(\alpha_k)_*[M](B),$$

so is an isomorphism.

Hence the map

$$\bigoplus_{x\in B}\bigoplus_{k=0}^{\infty}(\alpha_k)_*[M]_x \to \bigoplus_{k=0}^{\infty}(\alpha_k)_*[M](B)$$

is also an isomorphism, where the terms in the finite summations have been reordered.

To show that $\mathrm{Supp}(\bigoplus_{k=0}^{\infty}(\alpha_k)_*[M])$ is locally finite, observe that for each bounded $B \subseteq X$, $\{x \in B\colon \oplus_{k=0}^{\infty} (\alpha_k)_*[M]_x \neq 0\} = \{x \in B\colon \oplus_{k\in K_B} (\alpha_k)_*[M]_x \neq 0\}$ for some finite $K_B \subseteq \mathbb{N}$. It is easy to see that the sum of finitely many geometric $\mathcal{A}$-modules is locally finite.

For each morphism $\phi\colon M \to N$ in the category $\mathcal{A}(X)$, because each map $\alpha_k\colon X \to X$ is a coarse map, there is an induced morphism

$$(\alpha_k)_*[\phi]\colon (\alpha_k)_*[M] \to (\alpha_k)_*[N]$$

and hence an induced morphism

$$\bigoplus_{k=0}^{\infty}(\alpha_k)_*[\phi]\colon \bigoplus_{k=0}^{\infty}(\alpha_k)_*[M] \to \bigoplus_{k=0}^{\infty}(\alpha_k)_*[N]$$

in the category $\mathcal{A}(X)$ defined by

$$\left(\bigoplus_{k=0}^{\infty}(\alpha_k)_*[\phi]\right)_{x,y}(\lambda_0 \oplus \lambda_1 \oplus \dots) = \bigoplus_{k=0}^{\infty}[(\alpha_k)_*[\phi]_{x,y}(\lambda_k)] .$$

To verify that $\operatorname{Supp}(\bigoplus_{k=0}^{\infty}(\alpha_k)_*[\phi])$ is controlled, observe that

$$\operatorname{Supp}\left(\bigoplus_{k=0}^{\infty}(\alpha_k)_*[\phi]\right) \subseteq \{(\alpha_k(x), \alpha_k(y))\colon (x,y) \in \operatorname{Supp}\phi \text{ and } k \in \mathbb{N}\}$$

which is controlled by the second property of almost flasqueness.

It is stated in Remark 3.13 of [Bar03] that close maps induce the same maps on algebraic K-theory. It then follows from the third property of almost flasqueness that $(\alpha_k)_* = \mathrm{id}_*$ for each k. Therefore

$$\begin{aligned}\bigoplus_{k=0}^{\infty}(\alpha_k)_*[\lambda] &= [\lambda] \oplus (\alpha_1)_*[\lambda] \oplus (\alpha_2)_*[\lambda] \oplus \dots \\ &= [\lambda] \oplus [\lambda] \oplus \dots \\ &= [\lambda] \oplus ([\lambda] \oplus [\lambda] \oplus \dots).\end{aligned}$$

By an Eilenberg swindle, $[\lambda] = 0$, so that $K_*(\mathcal{A}(X)) = 0$. □

Proposition 7.32. *If $X = A \cup B$ is a coarsely excisive decomposition then there is a long exact sequence*

$$\begin{array}{ccccccc} \dots \longrightarrow & K_*(\mathcal{A}(A\cap B)) & \longrightarrow & K_*(\mathcal{A}(A)) \oplus K_*(\mathcal{A}(B)) & \longrightarrow & K_*(\mathcal{A}(X)) \\ & & & & & \big\downarrow \delta_* \\ & & & \dots \longleftarrow & & K_{*-1}(\mathcal{A}(A\cap B)) \end{array}$$

Proof. Consider the sequence

$$0 \longrightarrow \mathcal{A}(A\cap B) \xrightarrow{(i_*,j_*)} \mathcal{A}(A) \oplus \mathcal{A}(B) \xrightarrow{k_*-l_*} \mathcal{A}(X) \longrightarrow 0 \tag{7.1}$$

defined using the inclusion maps $i\colon A\cap B \hookrightarrow A, j\colon A\cap B \hookrightarrow B, k\colon A \hookrightarrow X$ and $l\colon B \hookrightarrow X$.

It will be shown that this sequence is exact. It is easy to see that the functor (i_*, j_*)

is injective, and that the functor $k_* - l_*$ is surjective on each morphism set as for each $\phi \in \mathcal{A}(X)$ we have $(k_* - l_*)(\phi|_A, -\phi|_{B\setminus A}) = \phi$. It is also clear that

$$(k_* - l_*) \circ (i_*, j_*) = 0,$$

and so $\mathrm{Im}(i_*, j_*) \subseteq \mathrm{Ker}(k_* - l_*)$.

Let $\phi_A\colon M_A \to N_A$ and $\phi_B\colon M_B \to N_B$ be morphisms between geometric $\mathcal{A}$-modules over A and B respectively such that $(\phi_A, \phi_B) \in \mathrm{Ker}(k_* - l_*)$. Then $k_*(\phi_A) = l_*(\phi_B)$, and since k_* and l_* are inclusion maps, this means that $\phi_A = \phi_B$. That is to say, $(\phi_A)_{x,y} = (\phi_B)_{x,y}$ for all $x, y \in A \cap B$, and $\phi_A = \phi_B = 0$ otherwise.

Define the restriction $\phi|_{A\cap B}\colon M_A|_{A\cap B} \to N_A|_{A\cap B}$ by $\phi|_{A\cap B} = \phi_A|_{A\cap B} = \phi_B|_{A\cap B}$. The support of $\phi|_{A\cap B}$ is controlled since it is a subset of the support of ϕ_A. It follows that $(\phi_A, \phi_B) = (i_*, j_*)(\phi|_{A\cap B})$, so $(\phi_A, \phi_B) \in \mathrm{Im}(i_*, j_*)$. Hence $\mathrm{Ker}(k_* - l_*) \subseteq \mathrm{Im}(i_*, j_*)$, and (7.1) is a short exact sequence.

By the fibration theorem in algebraic K-theory (see [PW85]), there is a fibration

$$\mathbf{K}\mathcal{A}(A \cap B) \longrightarrow \mathbf{K}\mathcal{A}(A) \vee \mathbf{K}\mathcal{A}(B) \longrightarrow \mathbf{K}\mathcal{A}(X)$$

It follows that we have a long exact sequence

$$\begin{array}{ccccccc} \dots \longrightarrow & K_*(\mathcal{A}(A\cap B)) & \longrightarrow & K_*(\mathcal{A}(A)) \oplus K_*(\mathcal{A}(B)) & \longrightarrow & K_*(\mathcal{A}(X)) \\ & & & & & \downarrow \delta_* \\ & & & \dots & \longleftarrow & K_{*-1}(\mathcal{A}(A\cap B)) \end{array}$$

□

The proof of the following result is based on that of Theorem 11.2 in [HPR97].

Proposition 7.33. *If $f\colon X \to Y$ is a coarse homotopy equivalence then the induced map $\boldsymbol{K}f\colon \boldsymbol{K}\mathcal{A}(X) \to \boldsymbol{K}\mathcal{A}(Y)$ is a weak homotopy equivalence.*

Proof. Let $f, g\colon X \to Y$ be coarsely homotopic. Then there exists a coarse map $H\colon I_pX \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$, where $i_0, i_1\colon X \to I_pX$ are defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, p(x) + 1)$ respectively. Therefore there are induced maps

$$H_* \circ (i_0)_* = f_*\colon K_*(\mathcal{A}(X)) \to K_*(\mathcal{A}(Y))$$

$$H_* \circ (i_1)_* = g_*\colon K_*(\mathcal{A}(X)) \to K_*(\mathcal{A}(Y))$$

By considering two different coarsely excisive decompositions of $X \times \mathbb{R}$, it will be shown that $f_* = g_*$. Define A, A', B, B' by $A = \{(x,t)\colon t \leq 0\}$, $A' = \{(x,t)\colon t \leq p(x)\}$, and $B = B' = \{(x,t) : t \geq 0\}$. Then $X \times \mathbb{R} = A \cup B$ and $X \times \mathbb{R} = A' \cup B'$ are both coarsely excisive decompositions. Note that $A \cap B = X$ and $A' \cap B' = I_pX$.

By Proposition 7.32, we have the following diagram

$$\begin{array}{ccccccccc} \dots \longrightarrow & K_*(\mathcal{A}(X)) & \longrightarrow & K_*(\mathcal{A}(A)) \oplus K_*(\mathcal{A}(B)) & \longrightarrow & K_*(\mathcal{A}(X \times \mathbb{R})) & \longrightarrow \dots \\ & \downarrow {\scriptstyle (i_0)_*} & & \downarrow {\scriptstyle \beta_*} & & \| & \\ \dots \longrightarrow & K_*(\mathcal{A}(I_pX)) & \longrightarrow & K_*(\mathcal{A}(A')) \oplus K_*(\mathcal{A}(B')) & \longrightarrow & K_*(\mathcal{A}(X \times \mathbb{R})) & \longrightarrow \dots \end{array}$$

Each of A, A' and B can be easily seen to be flasque. The map β_* is then an isomorphism since it follows from Proposition 7.31 that each of $K_*(\mathcal{A}(A))$, $K_*(\mathcal{A}(A'))$, $K_*(\mathcal{A}(B))$, and $K_*(\mathcal{A}(B'))$ are trivial.

By the 5-Lemma, the map $(i_0)_* \colon K_*(\mathcal{A}(X)) \to K_*(\mathcal{A}(I_pX))$ is an isomorphism.

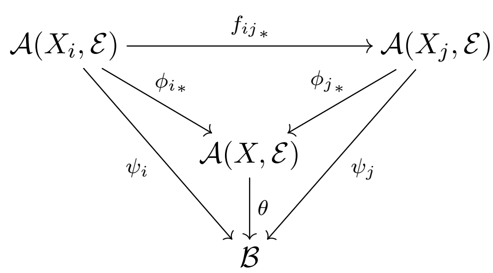
By definition $q_* \circ (i_0)_* = \text{id}$ (where $q \colon I_pX \to X$ is defined by $q(x,t) = x$), and since $(i_0)_*$ is an isomorphism, $q_* = (i_0)_*^{-1}$ and so $(i_0)_* \circ q_* = \text{id}$. The same argument applies to the map $(i_1)_*$. Inverses are unique, so it follows that $(i_0)_* = (i_1)_*$ and therefore that $f_* = g_*$. □

It follows that if $f \colon X \to Y$ is a coarse homotopy equivalence then the map $f_* \colon K_*(\mathcal{A}(X)) \to K_*(\mathcal{A}(Y))$ is an isomorphism.

Proposition 7.34. *If* $(X, \mathcal{E}) = \varinjlim_i (X_i, \mathcal{E})$ *with* $X_i \subseteq X_{i+1}$ *for each* i*, then* $\mathcal{A}(X, \mathcal{E}) = \varinjlim_i \mathcal{A}(X_i, \mathcal{E})$.

Proof. Since $(X, \mathcal{E}) = \varinjlim_i (X_i, \mathcal{E})$ in the coarse category, there exist coarse maps $f_{ij} \colon (X_i, \mathcal{E}) \to (X_j, \mathcal{E})$ and $\varphi_i \colon (X_i, \mathcal{E}) \to (X, \mathcal{E})$ for all i, j. By the partial ordering $X_i \subseteq X_{i+1}$, each of these maps can be taken to be the inclusion maps.

By applying the functor $\mathcal{A}$, there is an induced diagram



in the category of additive categories and additive functors such that ${\phi_j}_* \circ {f_{ij}}_* = \phi_{i*}$. Suppose there exists $\mathcal{B}$ and morphisms $\psi_i \colon \mathcal{A}(X_i, \mathcal{E}) \to \mathcal{B}$ and $\psi_j \colon \mathcal{A}(X_j, \mathcal{E}) \to \mathcal{B}$ such that $\psi_j \circ {f_{ij}}_* = \psi_i$. We must show that $\theta \colon \mathcal{A}(X, \mathcal{E}) \to \mathcal{B}$ exists and is unique.

For $M \in \mathcal{A}(X, \mathcal{E})$, define $M^x \in \mathcal{A}(X_i, \mathcal{E})$ by

$$M^x(B) = \begin{cases} M_x & x \in B \\ 0 & x \notin B \end{cases}$$

for $B \subseteq X_i$. Observe that $M = \oplus_{x \in X} M^x$ so it follows that

$$\theta(M) = \theta(\oplus_{x \in X} M^x) = \oplus_{x \in X} \psi_i(M^x).$$

For $\phi \colon M \to N \in \mathcal{A}(X, \mathcal{E})$, define $\phi_i \colon M|_{X_i} \to N|_{X_i} \in \mathcal{A}(X_i, \mathcal{E})$ by

$$(\phi_i)_{x,y} = \begin{cases} \phi_{x,y} & \text{for } x \in X_i \backslash X_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that $\phi = \oplus_{i=1}^{\infty} \phi_i$ and so follows that

$$\theta(\phi) = \theta(\oplus_{i=1}^{\infty} \phi_i) = \oplus_{i=1}^{\infty} \theta(\phi_i) = \oplus_{i=1}^{\infty} \psi_i(\phi_i).$$

Thus θ exists and is unique, and so $\varinjlim \mathcal{A}(X_i, \mathcal{E}) = \mathcal{A}(X, \mathcal{E})$.

□

Recall from Definition 6.7 that $\mathcal{E}_i$ denotes the coarse structure on $\mathbb{N}$ where the controlled sets are all the subsets of the diagonal and all the subsets of $\{1, 2, \ldots, i\} \times \{1, 2, \ldots, i\}$.

The proof of the following is similar to Theorem 3.98 of [Mit00]

Proposition 7.35. *If* $\mathcal{A}(X, \mathcal{E}) = \bigcup_{n=1}^{\infty} \mathcal{A}(X, \mathcal{E}_n)$ *then* $\mathcal{A}(X, \mathcal{E}) = \varinjlim_n \mathcal{A}(X, \mathcal{E}_n)$.

It is straightforward to verify that $\mathcal{A}(\mathbb{N}_0) = \bigcup_{n=1}^{\infty} \mathcal{A}(\mathbb{N}, \mathcal{E}_n)$ so it follows that $\mathcal{A}(\mathbb{N}_0) = \varinjlim \mathcal{A}(\mathbb{N}, \mathcal{E}_n)$.

The combination of the five results above lead us to the following result.

Theorem 7.36. *The functor $X \mapsto \boldsymbol{K}\mathcal{A}(X)$ is a coarsely excisive functor.*

By Theorem 6.36 we have the following.

Theorem 7.37. *If X is a proper metric space of finite asymptotic dimension and bounded geometry then the assembly map*

$$\lambda \colon K_*^{coarse}(\mathcal{A}(X)) \to H_{*-1}(\mathcal{A}(X))$$

is an isomorphism.

Applying the notion of descent (Theorem 5.51) gives us the following additional result.

Theorem 7.38. *If X is a free coarse G-space that is G-homotopy equivalent to a finite G-CW-complex and additionally X has bounded geometry, finite asymptotic dimension and is uniformly contractible then the Farrell–Jones assembly map*

$$\lambda\colon K_*(\mathcal{A}_G(X)) \to H_{*-1}(\mathcal{A}_G(X))$$

is injective for the space X and the group G.

In particular, if G has finite asymptotic dimension then the space EG also has finite asymptotic dimension, has bounded geometry and is uniformly contractible so this implies the Novikov conjecture in algebraic K-theory for proper metric spaces of finite asymptotic dimension. Additionally, if $X = E_{\mathcal{VC}}(G)$ then this implies the injectivity side of the Farrell–Jones conjecture for G.

7.3 C^*-categories

The notion of a C^*-category was introduced in [GLR85], and K-theory of a C^*-category in [Mit00]. A C^*-category is a closed subcategory of the category of all Hilbert spaces and bounded linear operators. This generalised the concept of a C^*-algebra. One main advantage of C^*-categories is that they are more natural than C^*-algebras. For example, the Roe algebra $C^*(X)$ of a coarse space X and corresponding C^*-algebra $D^*(X)$ as more naturally viewed as C^*-categories as all representations can be consided at once, rather than by selecting a sufficiently large representation as in the C^*-algebra case. The construction of the K-theory functors for C^*-categories and the K-theory spectrum of a C^*-categories as constructed in [Mit00]. An overview of this is given here.

Definition 7.39 (*-category). An *involution* on an $\mathbb{F}$-linear category $\mathcal{A}$ is a collection of maps

$$^*\colon \operatorname{Mor}(A, B) \to \operatorname{Mor}(B, A)$$

for each pair $A, B \in \mathcal{A}$ such that:

- $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$ for all $x, y \in \operatorname{Mor}(A, B)$ and $\alpha, \beta \in \mathbb{F}$;
- $(x^*)^* = x$ for all $x \in \operatorname{Mor}(A, B)$;
- $(xy)^* = y^* x^*$ if $x \in \operatorname{Mor}(B, C)$ and $y \in \operatorname{Mor}(A, B)$.

A *-*category* is an $\mathbb{F}$-linear category equipped with an involution.

Definition 7.40 (Banach category). A *Banach category* is an $\mathbb{F}$-linear category such that each set $\mathrm{Mor}(A, B)$ is a Banach space and $||xy|| \leq ||x||||y||$ for all $x \in \mathrm{Mor}(B, C)$ and $y \in \mathrm{Mor}(A, B)$.

Definition 7.41 (C^*-category). A *C^*-category* is a Banach *-category such that the C^*-identity

$$||x||^2 = ||x^*x||$$

holds for all $x \in \mathrm{Mor}(A, B)$ and for every $x \in \mathrm{Mor}(A, B)$ there exists $y \in \mathrm{Mor}(A, A)$ such that $x^*x = y^*y$.

Observe that each set $\mathrm{Mor}(A, A)$ is unital C^*-algebra, and that a unital C^*-algebra is a C^*-category with one object.

Example 7.42. The category of all Hilbert spaces and bounded linear maps is a C^*-category.

Definition 7.43 (C^*-functor). A functor $F\colon \mathcal{A} \to \mathcal{B}$ between C^*-categories is said to be a *C^*-functor* if it is linear on $\mathrm{Mor}(A, B)$ for all $A, B \in \mathcal{A}$ and such that $F(x^*) = F(x)^*$ for all $x \in \mathrm{Mor}(A, B)$.

The following definition is slightly weaker than the definition of an C^*-isomorphism, but is still useful.

Definition 7.44 (Unitary transformation). A *unitary transformation* $U\colon F \to G$ between C^*-functors $F, G\colon \mathcal{A} \to \mathcal{B}$ is a collection of unitary elements $U_A\colon F(A) \to G(A)$ for each $A \in \mathcal{A}$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(x)} & F(B) \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle U_B} \\ G(A) & \xrightarrow{G(x)} & G(B) \end{array}$$

commutes for all $x \in \mathrm{Mor}(A, B)$.

Definition 7.45 (Equivalence of C^*-categories). An *equivalence of C^*-categories* is a C^*-functor $F\colon \mathcal{A} \to \mathcal{B}$ such that there exists a C^*-functor $G\colon \mathcal{B} \to \mathcal{A}$ and unitary transformations $GF \to 1_{\mathcal{A}}$ and $FG \to 1_{\mathcal{B}}$.

Similarly to the case with C^*-algebras and $*$-homomorphisms, it can be shown that a C^*-functor $F\colon \mathcal{A} \to \mathcal{B}$ satisfies $||F(x)|| \leq ||x||$ for all $x \in \mathrm{Mor}(A, B)$ (Proposition 3.14 of [Mit00]) and that the norm on a C^*-category is unique (Corollary 3.16 of [Mit00]). It is also possible to define a non-unital C^*-category to generalise C^*-algebras, and the concept of unitisation also exists for C^*-categories, see Section 3.2 of [Mit00].

Example 7.46. There are C^*-categories $C^*_{\max}(\mathcal{G})$ and $C^*_r(\mathcal{G})$ for a groupoid $\mathcal{G}$ (a category where all morphisms are invertible) which generalise the C^*-algebra versions which occur as the right hand side of the assembly maps in the Baum–Connes and Novikov conjectures.

Definition 7.47 (Grading). A *grading* on a C^*-category $\mathcal{A}$ is a C^*-functor $\alpha\colon \mathcal{A} \to \mathcal{A}$ such that $\alpha(A) = A$ for all $A \in \mathcal{A}$ and $\alpha^2 = \mathrm{id}_{\mathcal{A}}$.

A C^*-algebra equipped with a grading is called a *graded C^*-algebra* (or sometimes a *C^*-superalgebra*). The identity functor is a grading so any C^*-category can be considered a graded C^*-category, and so the notion of graded C^*-categories can be considered a generalisation of C^*-categories.

For a graded C^*-category with grading $\alpha\colon \mathcal{A} \to \mathcal{A}$, define

$$\mathrm{Mor}(A,B)_{\mathrm{even}} = \{x \in \mathrm{Mor}(A,B)\colon \alpha(x) = x\}$$

and

$$\mathrm{Mor}(A,B)_{\mathrm{odd}} = \{x \in \mathrm{Mor}(A,B)\colon \alpha(x) = -x\}.$$

It can be shown that for a graded C^*-category, $\mathrm{Mor}(A,B)_{\mathrm{even}}$ and $\mathrm{Mor}(A,B)_{\mathrm{odd}}$ are closed subspaces of $\mathrm{Mor}(A,B)$ such that $\mathrm{Mor}(A,B) = \mathrm{Mor}(A,B)_{\mathrm{even}} \oplus \mathrm{Mor}(A,B)_{\mathrm{odd}}$ and that the composition of two even morphisms or two odd morphisms is even, and the composition of an odd and an even morphism is even, see Proposition 3.72 of [Mit00].

Conversely, if a C^*-category $\mathcal{A}$ has closed subspaces $\mathrm{Mor}(A,B)_{\mathrm{even}}$ and $\mathrm{Mor}(A,B)_{\mathrm{odd}}$ such that $\mathrm{Mor}(A,B) = \mathrm{Mor}(A,B)_{\mathrm{even}} \oplus \mathrm{Mor}(A,B)_{\mathrm{odd}}$ and that the composition of two even morphisms or two odd morphisms is even, and the composition of an odd and an even morphism is even, then there is a unique grading $\alpha\colon \mathcal{A} \to \mathcal{A}$ defined by $\alpha(x+y) = x - y$ for all $x \in \mathrm{Mor}(A,B)_{\mathrm{even}}$ and $y \in \mathrm{Mor}(A,B)_{\mathrm{odd}}$ and $\mathrm{Mor}(A,B)_{\mathrm{even}} = \{x \in \mathrm{Mor}(A,B)\colon \alpha(x) = x\}$ and $\mathrm{Mor}(A,B)_{\mathrm{odd}} = \{x \in \mathrm{Mor}(A,B)\colon \alpha(x) = -x\}$ for all objects $A, B \in \mathcal{A}$.

Definition 7.48 (Graded C^*-functor). A *graded C^*-functor* is a C^*-functor between graded C^*-categories that takes odd morphisms to odd morphisms and even morphisms to even morphisms.

Definition 7.49 (Supersymmetries). For every object A of a graded C^*-category, the *supersymmetries of A* are the odd self-adjoint involutions, that is, the set

$$SS(A) = \{x \in \mathrm{Mor}(A,A)_{\mathrm{odd}}\colon x^* = x \text{ and } x^2 = 1\}.$$

The set of supersymmetries can be seen to be a topological space (as a subset of $\mathrm{Mor}(A,A)$). For $x, y \in SS(A)$ we can define an equivalence relation $x \sim y$ if x and y lie in the same path component.

Definition 7.50 (Reference supersymmetry). A *reference supersymmetry* E in a unital graded C^*-category $\mathcal{A}$ is a collection of supersymmetries $E_A \in SS(A)$ such that $E_A \sim -E_A$ for each object $A \in \mathcal{A}$. If additionally the C^*-category has a direct sum, then it is also required that $E_A \oplus E_B = E_{A\oplus B}$ for all objects $A, B \in \mathcal{A}$.

For $x \in SS(A)$ and $y \in SS(B)$ define the equivalence relation $x \sim_E y$ if $x \oplus E_B \oplus E_C \sim E_A \oplus y \oplus E_C$ for some $C \in \mathcal{A}$.

Definition 7.51 (K_1 group). The set $K_1^{(E)}(\mathcal{A})$ is defined to be the collection of equivalence classes $\langle x\rangle_E$ of supersymmetries $x \in SS(A)$ for $A \in \mathcal{A}$.

It can be shown that $K_1^{(E)}(\mathcal{A})$ is an abelian group equipped with operation defined by $\langle x\rangle_E + \langle y\rangle_E = \langle x \oplus y\rangle_E$. The identity element is $\langle E\rangle_E$ and $\langle x\rangle_E^{-1} = \langle -ExE\rangle_E$ for all x. It can also be shown that $K_1^{(E)}(\mathcal{A})$ is independent of choice of reference supersymmetry, and will henceforth be shortened to $K_1(\mathcal{A})$.

It can be shown that K_1 is a covariant functor from the category of small graded unital C^*-categories (with direct sum and reference supersymmetry) to the category of abelian groups. There is also a definition for the non-unital case. See Section 4.5 of [Mit00].

Definition 7.52 (Suspension). The *suspension* of a C^*-category $\mathcal{A}$ is the C^*-category

$$S\mathcal{A} = \{f \in C([0,1], \mathcal{A})\colon f(0) = f(1) = 0\}.$$

Definition 7.53 (K-theory groups). The *K-theory groups of* $\mathcal{A}$ are defined by

$$K_{n+1}(\mathcal{A}) = K_n(S\mathcal{A})$$

for $n \geq 1$.

Thus by repeated application of the suspension functor, $K_n(\mathcal{A}) = K_1(S^{n-1}\mathcal{A})$ for $n \geq 2$.

It can be shown that the K-theory functors are continuous, that is, $K_n(\varinjlim \mathcal{A}_i) = \varinjlim K_n(\mathcal{A}_i)$ (Section 4.9 of [Mit00]) and that Bott periodicity holds, i.e., if $\mathcal{A}$ is a complex graded C^*-category then $K_n(\mathcal{A}) \cong K_{n+2}(\mathcal{A})$ and that if $\mathcal{A}$ is a real graded C^*-category then $K_n(\mathcal{A}) \cong K_{n+8}(\mathcal{A})$ (Section 6.2 of [Mit00]).

Theorem 7.54 (Section 6.4 of [Mit00]). *There exists a spectrum $\boldsymbol{K}$ such that the following conditions are satisfied:*

- *if $f, g\colon \mathcal{A} \to \mathcal{B}$ are homotopic graded C^*-functors then the induced maps of spectra $f_*, g_*\colon \boldsymbol{K}(\mathcal{A}) \to \boldsymbol{K}(\mathcal{B})$ are also homotopic;*

- *if*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a short exact sequence of C^**-categories then the induced sequence*

$$\boldsymbol{K}\mathcal{A} \to \boldsymbol{K}\mathcal{B} \to \boldsymbol{K}\mathcal{C}$$

of spectra is a fibration up to weak homotopy equivalence;

- *for any small graded* C^**-category* $\mathcal{A}$*, there are natural isomorphisms*

$$K_n(\mathcal{A}) \cong \pi_n(\boldsymbol{K}(\mathcal{A}))$$

for all $n \in \mathbb{Z}$*.*

□

Definition 7.55. Let $(X, \mathcal{E})$ be a coarse space and $\mathcal{A}$ be an additive C^*-category. The category $\mathcal{A}^b(X)$ is defined to be the category consisting of geometric $\mathcal{A}$-modules over X and morphisms ϕ between them such that $\operatorname{Supp}\phi \in \mathcal{E}$ (that is, controlled with respect to the coarse structure), and such that the map

$$T_\phi \colon \bigoplus_{x\in X} M_x \to \bigoplus_{x\in X} N_x$$

defined by the formula

$$T_\phi(v) = \sum_{x\in X} \phi_{x,y}(v)$$

for $v \in M_y$ is a bounded linear map.

Observe that $\mathcal{A}^b(X)$ has all of the properties of a C^*-category apart from the morphism sets being complete. The category $\mathcal{A}^*(X)$ is defined to be the completion of the category $\mathcal{A}^b(X)$.

The following is proved similarly to Theorem 7.36.

Theorem 7.56. *The functor* $X \mapsto \boldsymbol{K}\mathcal{A}^*(X)$ *is a coarsely excisive functor.* □

By Theorem 6.36 we have the following.

Theorem 7.57. *If* W *is a proper metric space of finite asymptotic dimension and bounded geometry then the assembly map*

$$\mu \colon K_*^{coarse}(\mathcal{A}^*(W)) \to H_{*-1}(\mathcal{A}^*(W))$$

is an isomorphism.

Applying the notion of descent we have the following.

Theorem 7.58. *If W is a free coarse G-space that is G-homotopy equivalent to a finite G-CW-complex and additionally has finite asymptotic dimension and bounded geometry then the assembly map*

$$\mu\colon K_*(\mathcal{A}_G^*(W)) \to H_{*-1}(\mathcal{A}_G^*(W))$$

is injective for W and G.

The following theorem shows us how the original C^*-algebra version fits into the C^*-category version.

Theorem 7.59 (Section 2.2 of [HP04]). *Let $\mathcal{V}$ be the C^*-category where the objects are the Hilbert spaces $\mathbb{C}^n$ and the morphisms are bounded linear maps $\mathbb{C}^n \to \mathbb{C}^{n'}$. If W is a seperable coarse topological space with bounded geometry then there is an isomorphism*

$$K_*(\mathcal{V}^*(W)) \cong K_*(C^*(W)).$$

□

Combining Theorem 7.57 and Theorem 7.59 gives us the following result, the coarse Baum-Connes conjecture:

Corollary 7.60 (Coarse Baum-Connes conjecture). *If W is a metric space of finite asymptotic dimension and bounded geometry then the assembly map*

$$\mu\colon K_*^{coarse}(W) \to K_*(C^*(W))$$

is an isomorphism.

Combining Theorem 7.58 and Theorem 7.59 gives us the following result, the injectivity of the Baum-Connes conjecture:

Corollary 7.61 (Injectivity of Baum-Connes conjecture). *If G is a discrete group of finite asymptotic dimension and bounded geometry then the assembly map*

$$\mu\colon K_*^G(\underline{E}G) \to K_*(C_r^*(G))$$

is injective.

The assembly map in Corollary 7.60 is not necessarily the same map as in the original coarse Baum–Connes conjecture (Conjecture 3.69), but this map has the same range and domain, all of the same features of the original map and has the same consequences. In particular, the Baum–Connes assembly map coming from descent is unique up to stable equivalence.

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