ABUNDANT SEMIGROUPS AND CONSTRUCTIONS

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Abstract

This thesis studies two classes of semigroups, given by presentations, with regard to weak regularity properties. Since its introduction by Fountain in the late 1970s, the study of abundant and related semigroups has given upward thrust to this fruitful and deep research area. The class of abundant semigroups extends that of regular semigroups in a natural way and is itself contained in the class of weakly abundant semigroups. We are interested in the properties of abundance and weak abundance as not only do they arise from a number of different directions and there are many natural examples, but also (weakly) abundant semigroups have enough structure to allow for the development of a coherent theory.

The study of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$ over a biordered set $\mathcal{E}$ began with the seminal work of Nambooripad in the 1970s. Given the universal nature of such semigroups, it is natural to investigate their structure. In 2016 Gould and Yang [16] showed that $\mathrm{IG}(\mathcal{B})$, where B is a band, is always a weakly abundant semigroup, but is not necessarily abundant. Moreover, they constructed a 10-element normal band B for which $\mathrm{IG}(\mathcal{B})$ is not abundant. Following these discoveries another interesting question comes out very naturally: what kind of normal bands are such that $\mathrm{IG}(\mathcal{B})$ is abundant? Our main result shows that if B is an iso-normal band, then $\mathrm{IG}(\mathcal{B})$ is an abundant semigroup.

The above considerations of the structure of $\mathrm{IG}(\mathcal{B})$ led us to introduce the notion of graph product of semigroups. We first consider the special case of free product and show that the free product of (weakly) abundant semigroups is (weakly) abundant. To answer the questions of whether the graph product of (weakly) abundant semigroups is (weakly) abundant we introduce a special form for the elements of graph products, and use this to answer the foregoing questions in the positive.

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Preface

Motivated by Fountain's definition of *abundant semigroup* in [40], we study in this thesis a (weak) abundancy of two different constructions of semigroups. The concept of (weakly) abundant is a natural generalization of the concept of regular. We say a semigroup S is *regular* if each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains an idempotent, whereas a semigroup S is *abundant* if each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class contains an idempotent. *Weak abundancy* is weaker property than abundancy. A semigroup is weakly abundant if each $\widetilde{\mathcal{L}}$-class and $\widetilde{\mathcal{R}}$-class of S contain an idempotent. A binary relation $\mathcal{R}^*$ on S is defined by the rule that $a\,\mathcal{R}^*\,b$ if the elements a and b are related by Green's relation $\mathcal{R}$ in some oversemigroup of S. The relation $\mathcal{L}^*$ is defined dually. A third set of relations, were introduced in[68], extending the stander version of Green's relations and used to define the weakly abundant semigroup. From the definition of the relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$, it easily deduced that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$ and $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$. The first main theorem in this thesis prove that if B is an *iso-normal band*, then the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ is abundant.

Now we explain what we mean by the notation $\mathrm{IG}(\mathcal{B})$ over a band B. Let S be a semigroup with a set of idempotents $E = E(S)$. In 1979, Nambooripad [72] described the structure of E as a biordered set $\mathcal{E}$, a notion arising as a generalization of the semilattice of idempotents in inverse semigroups. Conversely, Easdown [31], in 1984, proved the significant result that every biordered set $\mathcal{E}$ occurs as $E(S)$ for some semigroup S. Hence we lose nothing by assuming that a biordered set $\mathcal{E}$ is of the form $E(S)$ for a semigroup S.

The subsemigroup of S generated by the set of idempotents E of S is denoted by $\langle E \rangle$. If $S = \langle E \rangle$, then we say that S is *idempotent generated.* The importance of idempotent generated semigroups was evident in 1966, Howie [62] showed that every semigroup may be embedded into one that is idempotent generated.

One of the central construction of this thesis is the *free idempotent generated semigroup* $\mathrm{IG}(\mathcal{E})$, where $\mathcal{E}$ is biordered set. This is built from $\mathcal{E}$ via a presentation. Specifically,

$$\mathrm{IG}(\mathcal{E}) = \langle E \colon e \circ f = ef,\ (e, f) \text{ basic pair},\ e, f \in E \rangle$$

where $e \circ f$ is the word of length 2 with letters e and f. Note that if (e, f) is a *basic pair*, then $ef, fe \in E$. It is important to understand $\mathrm{IG}(\mathcal{E})$ if one is interested in understanding an arbitrary idempotent generated semigroup with a biordered set

$\mathcal{E}$.

One popular approach to investigating the structure of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$ was studying the behavior of its maximal subgroups. From the 1970s, it was conjectured that the maximal subgroups of $\mathrm{IG}(\mathcal{E})$ are always free [70]. In 2009, Brittenham, Margolis, and Meakin provided a counter example [6]. However, in 2011 McElwee [71] declared a non-free example of maximal subgroup of $\mathrm{IG}(\mathcal{E})$. Prompted by this significant result, in 2012, Gray and Ruškuc [54] showed that any group occurs as the maximal subgroup of some $\mathrm{IG}(\mathcal{E})$. In 2014, Gould and Yang [48] found direct proof of the same result, arising from a natural biordered set.

However, little was known about the overall structure of $\mathrm{IG}(\mathcal{E})$, other than that it not always regular, even where $\mathcal{E}$ is a semilattice. In 2016, Gould and Yang [16] investigated the free idempotent generated semigroups $\mathrm{IG}(\mathcal{E})$, where $\mathcal{E}$ is a biordered set with trivial products. They proved that for such an $\mathcal{E}$ the semigroup $\mathrm{IG}(\mathcal{E})$ is abundant. Moreover, if $\mathcal{E}$ is a finite biordered set with trivial products, then $\mathrm{IG}(\mathcal{E})$ has a solvable word problem. Further, they proved that if B is a band with a biordered set $\mathcal{B}$, then $\mathrm{IG}(\mathcal{B})$ is always endowed with significant property, namely, weakly abundant with the congruence condition. This led them to the conjecture that if B is a normal band, then $\mathrm{IG}(\mathcal{B})$ is abundant. However, they disproved this conjecture by a counter-example [16]. They gave an example of a 10-element normal band B for which $\mathrm{IG}(\mathcal{B})$ is not abundant. Motivated by this counter example we would be interested to determine, which special bands B have biordered set $\mathcal{B}$, such that the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ is abundant?

The above question led us to the concept of an *iso-normal band*. Simply put an iso-normal band is isomorphic to a direct product of semilattice and a rectangular band. This is equivlent to $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$, where each $\phi_{\alpha,\beta}$ is an isomorphism. We suppose that $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ is an iso-normal band and we start by looking at two special cases: one of them is the case where Y is a *diamond* and the other is the case where Y is a *fan*. A semilattice Y is called a *diamond* if $Y = \{\alpha, \beta, \gamma, \delta\}$, where α and δ are the upper and the lower bounds of Y and β is incomparable with γ. A semilattice Y is called a *fan* if Y has a lower bound δ and for any α, $\beta \in Y$, where $\alpha \neq \delta \neq \beta \neq \alpha$, α and β are incomparable. With two different strategies, we prove that $\mathrm{IG}(\mathcal{B})$ is abundant, where B is an iso-normal band over a diamond semilattice and a fan semilattice. Then we put these together, with additional techniques, to show that $\mathrm{IG}(\mathcal{B})$ is always abundant for an iso-normal band B.

During our work to prove the abundancy of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over an iso-normal band B, we show that the free product of abundant semigroups is an abundant semigroup. It is known that the free product and the (restricted) external direct product of semigroups are special cases of graph products. Following these results, another interesting question comes out very naturally: is the graph product of abundant semigroups always abundant?

In 1990, graph products of groups were introduced by Green in her thesis [55]. This concept was studied by many authors, such as Hermiller and Meier [60]. The graph product of monoids is defined in the same way as the graph product of groups and has been studied specifically by Veloso da Costa, Fohry and Kuske [13], [12], [38]. In 2008, Fountain and Kambites [42] were able to show that the graph product of cancellative monoids is cancellative. All these previous works have only focused on the graph product of monoids. However, the graph product of semigroups is a somewhat different construction. Therefore, it is valuable to introduce this notation and to consider its structure. In 2021, Gould and Yang [17] showed that the graph product of semigroups can always be embedded into a graph product of monoids. The second main theorem of this thesis shows that the graph product of abundant semigroups is always abundant. In addition, it has proved that the graph product of weakly abundant semigroups is weakly abundant.

The rest of this thesis is devoted to working on the description of the relations $\mathcal{R}^*$, $\mathcal{L}^*$ and $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ on the graph product of semigroups $\mathscr{GP}$. We succeed in giving a complete characterization of these relations on the graph product of semigroups.

Now let me explain the main content of each chapter of this thesis:

Chapter 1: We will present some basic definitions and results of semigroup theory. We end this chapter by providing a brief introduction to classes of bands such as semilattices, rectangular bands, and normal bands, which will be frequently used in the whole thesis.

Chapter 2: We give some preliminaries of semigroup constructions. Emphasis is made on exploring the structure of external direct products, free products, and graph products of semigroups.

Chapter 3: In this chapter, we briefly recall the definitions of the binary relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ and corresponding to these binary relations, the concepts of abundant semigroups, weakly abundant semigroups and their one-sided versions. These are introduced in a very natural way, as a generalization of the notion of a regular semigroup.

Chapter 4: In Section 4.1, we prove that the external direct product of (left-right) abundant semigroups is always (left-right) abundant. Further, we prove that the external direct product of (left-right) weakly abundant semigroups is (left-right) weakly abundant. In Section 4.2, we show that the classes of left abundant semigroups and left weakly abundant semigroups are closed under graph products of semigroups.

Chapter 5: In Section 3.1, we give the abstract definition of the concept of a *biordered set* $\mathcal{E}$ and show that $\mathcal{E}$ is the generating set of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$. Moreover, we present the significant results obtained by Nambooripad [72] and Easdown [32]. By these results, we lose nothing by assuming that a biordered set $\mathcal{E}$ is of the form $E(S)$ for some semigroup S. In Section 3.2, we recall the idempotent generated semigroups and present the importance of these semigroups. Section 3.3 is divided into four subsections. In Subsection 5.3.1, we give an overview of free idempotent generated semigroups $\mathrm{IG}(\mathcal{E})$, basic definitions, preliminary results, and several pleasant properties, particularly with respect to Green's relations. In Subsection 5.3.2, we define a special form of the elements of $\mathrm{IG}(\mathcal{B})$, where B is a band, called *normal form*. Note that this form is unique for the elements of $\mathrm{IG}(\mathcal{E})$ if $\mathcal{E}$ is a biordered set with trivial basic products [16]. In Subsection 5.3.3, we recall further results that have been obtained so far in the current research direction of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over a normal band B. In Subsection 5.3.4, we present the results of the word problem of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$, where $\mathcal{B}$ is finite and has trivial basic products.

Chapter 6: One of the main results in this thesis is proven in this chapter. In 2014, Gould and Yang [80] gave an example of a 10-element normal band for which $\mathrm{IG}(\mathcal{B})$ is not abundant, so our goal in this chapter is to find some special classes of band B for which $\mathrm{IG}(\mathcal{B})$ is abundant. This chapter is organised as follows. In Section 6.1, we give an alternative proofs of some known results about the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$, where B is a normal band. In Section 6.2, we introduce a special kind of normal band, called an *iso-normal* *band*. Then we prove that an iso-normal band $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is isomorphic to the direct product $B_\tau \times Y$ for any chosen τ in Y. Moreover, any direct product $R \times Z$, where R is a rectangular band and Z is a semilattice is isomorphic to an iso-normal band. In Section 6.3, we investigate the general structure of IG($\mathcal{B}$) over an iso-normal band where Y is a diamond or a fan semilattice. We show that in each case the free idempotent generated semigroup IG($\mathcal{B}$) is abundant. Unlike the case of semilattices and rectangular bands, we may lose the uniqueness of normal forms of the elements in IG($\mathcal{B}$), where B is an iso-normal band. To overcome this problem, the concepts *complete form* and *double normal form* are introduced. These forms are used in our whole work in this chapter. The main result is obtained in Section 6.5, that the free idempotent generated semigroup IG($\mathcal{B}$) over an iso-normal band is always abundant. Further, the word problem of IG($\mathcal{B}$) is solvable if B is a finite iso-normal band.

Chapter 7: In this chapter, our main concern is the graph product of semigroups. In Section 7.1, we recall the notation of the graph product of semigroups and describe the universal nature of this construction. In Section 7.2, we show that every element in the graph product of semigroups may be represented by reduced form. Further, we introduce important forms of the elements of the graph products of semigroups, called *left complete reduced forms* and *complete reduced forms*. To prove the abundancy of the graph product $\mathscr{G}\mathscr{P}$ of abundant semigroups, we give a characterization of idempotents in $\mathscr{G}\mathscr{P}$ in Section 7.3. At the beginning of Section 7.4, we obtain three maps of the graph product of semigroups $\mathscr{G}\mathscr{P}$, which we need to prove the main result in this chapter, together with the special forms for the elements of $\mathscr{G}\mathscr{P}$ that mentioned earlier. The main result is that the graph product of left abundant semigroups is always left abundant. Dually result holds of right abundant. Hence we get that the graph product of abundant semigroups is abundant. In Section 7.5, we prove that the graph product of weakly abundant semigroups is weakly abundant. We end this chapter by giving a complete characterization of the relation $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ on a graph product of semigroups.

Chapter 8: Our results in this thesis throw up many questions in need of further investigation. We will give a brief proposal for our further work.

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Dedication

My mother and my father, for their endless love, support and encouragement.

To my husband, for always being there for me, and kept me going strong.

To my super heroes, Haya, Dody (Abdullah) and Noura, I love you with every part of me.

Author's Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapters 1 and 2 mainly present the basics of semigroup theory which are frequently used in the whole thesis. Chapter 3 is devoted to recalling the definition of (weakly) abundant semigroups and their properties. Chapter 4 is the first chapter of new work in this thesis. The results are my own, making some small use of existing approaches. Chapter 5 is a summary of the recent results obtained in the area of free idempotent generated semigroups. The main results contained in Chapters 6 and 7 are my own, unless referenced otherwise. Chapter 6 form my joint paper [49] with Victoria Gould and Dandan Yang.

Chapter 7 is devoted to work that will form a joint paper with Victoria Gould and Dandan Yang [50].

Chapter 8 is about some questions we would like to work with in the near future.

Chapter 1

Preliminaries I: Semigroup fundamentals

In this chapter, we recall some basic definitions and results of semigroup theory which will be frequently used in the whole thesis. The results and definitions are taken from a number of introductory text books including [64] and [72].

Throughout this thesis, mappings are written on the right of their arguments. Hence the composition of mappings is from the left to the right.

1.1 Basic definitions and results

A *semigroup* $(S, \cdot)$ is a non-empty set S together with an associative binary operation $\cdot$ defined on S, that is for all $x, y, z \in S$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

We follow the usual convention of denoting the product $x \cdot y$ by juxtaposition xy, and we call the binary operation $\cdot$ a multiplication on S. We will write a semigroup $(S, \cdot)$ often more simply by S. Throughout this thesis, we will denote an arbitrary semigroup by S.

An element $f \in S$ is called a *left identity* for S if $fs = s$ for all s in S, and it is called a *right identity* if $sf = s$ for all s in S. An element e in S is called an *identity* (2-sided identity) if it is both a right and a left identity. Moreover, if S has a right identity e and a left identity f, then $e = f$ is an identity of S.

A *monoid* M is a semigroup with an identity, usually denoted by 1 or 1_M when M is not clear. The identity will be unique if it exists.

A *group* G is defined to be a monoid in which every element is a unit, that means for every a in G, there is an element b in G with $ab = 1 = ba$, where 1 is the identity of G.

By the above, it is clear that every group is a monoid, and every monoid is a semigroup.

Example 1.1.1. The natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$, are all semigroups with respect to both addition and multiplication. The sets $\mathbb{Z}$ and $\mathbb{Q}$ are monoids with respect to both addition and multiplication, whereas $\mathbb{N}$ is a monoid with respect to multiplication but not addition. The integers form a group with respect to addition.

Definition 1.1.2. A *subsemigroup* of a semigroup S is a non-empty subset T of S which is closed under the multiplication of S.

For any semigroup S, the set consisting of all units of S is a subsemigroup of S. This subsemigroup is a group, called the *group of units*, or the *unit group* of S.

Let A and B be subsets of a semigroup S. We define the *product* of A and B by the rule

$$AB = \{ab\colon a \in A, b \in B\}.$$

If $A = \{a\}$ we can write AB as aB and we follow the same convention in other situations. If $T \subseteq S$, we write $T^2 = TT$. Therefore, T is a subsemigroup if and only if $T \neq \emptyset$ and $T^2 \subseteq T$. If T also forms a group under the restriction of the operation of S to T, then T is called a *subgroup* of S. The set of all subsets of S, $\mathcal{P}(S) = \{A | A \subseteq S\}$, equipped by the above-defined operation, is a semigroup, called the *power semigroup of* S.

A *submonoid* of a monoid M is a subsemigroup T of M such that $1 \in T$, where 1 is the identity of M.

Example 1.1.3. The set of positive rational numbers $\{x \in \mathbb{Q} : x > 0\}$ is a subsemigroup of the semigroup of real numbers under addition. $(\mathbb{R}, +)$.

The intersection of two (or more) subgroups is always subgroup and the intersection of two (or more) submonoids is always submonoid. However, the intersection of two (or more) subsemigroups of a given semigroup S may be empty, as illustrated by the following easy example.

Example 1.1.4. For the semigroup $(\mathbb{R}, +)$. The sets

$$P = \{x \in \mathbb{R} : x > 0\}$$

and

$$Q = \{x \in \mathbb{R} : x \leq 0\}$$

are subsemigroups of the semigroup $(\mathbb{R}, +)$, but $P \cap Q = \emptyset$.

Notice that if the intersection of finitely many subsemigroups of S is non-empty, then it forms a subsemigroup of S.

A semigroup, monoid or group is said to be *trivial* if it has exactly one element. Note that a trivial semigroup or monoid is, in fact, a group. If S is a semigroup we can add an element 1 to S, and extend the multiplication to $S \cup \{1\}$ by defining

$$s \cdot 1 = s = 1 \cdot s,$$

for all s in S, and

$$1 \cdot 1 = 1.$$

Hence $S \cup \{1\}$ becomes monoid with identity 1, as the operation $\cdot$ is associative in $S \cup \{1\}$ [64]. The monoid S^1 is defined by

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid} \\ S \cup \{1\} & \text{if } S \text{ is not a monoid} \end{cases}$$

with multiplication as above. We called S^1 the *semigroup with adjoined identity.* By the above we can see that every semigroup S can be embedded into a monoid S^1.

Example 1.1.5. Let I, J be non-empty sets and let $T = I \times J$, with binary operation defined on T by

$$(i, j)(k, l) = (i, l)$$

where $(i, j), (k, l) \in T$. Then T is a semigroup, called the *rectangular band* on $I \times J$.

An element $z \in S$ is called a *left zero* of S if $zs = z$ for all s in S, and it is called a *right zero* of S if $sz = z$ for all s in S. An element $z \in S$ is called a *zero* of S if it is both a right and a left zero. It is clear that a zero element is unique if it exists.
Further, if S has no zero, it is easy to adjoin an element 0 to S and extend the multiplication to $S \cup \{0\}$ by defining

$$s0 = 0s = 00 = 0,$$

for all s in S. It is clear that the operation is associative in $S \cup \{0\}$. We write the semigroup $S \cup \{0\}$ as

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup \{0\} & \text{if } S \text{ is no zero element.} \end{cases}$$

We called S^0 the *semigroup obtained from* S *by adjoining a zero* if necessary.

In any set X if we define a multiplication by the rule

$$ab = a,$$

for any $a, b \in X$, then X is a semigroup. This kind of semigroup is called a *left zero semigroup*. Dually, we may define *right zero semigroups*. Note that if S is a left zero semigroup (or a right zero semigroup), then any non-empty subset of S is a subsemigroup of S.

In the following definition we collect some properties of semigroups.

Definition 1.1.6. (i) A semigroup S is *commutative* if $ab = ba$ for all a, b in S;

(ii) S is *left cancellative* if for all $a, b, c \in S$, $ab = ac$ implies $b = c$;

(iii) S is *right cancellative* if for all $a, b, c \in S$, $ba = ca$ implies $b = c$;

(iv) S is *cancellative* if it is both left and right cancellative.

It is clear that groups are cancellative.

Example 1.1.7. Let X be a non-empty set. The full transformation semigroup on X, $\mathcal{T}_X$, is the set of all mappings from X into X. Note that $(\mathcal{T}_X, \circ)$ is non-commutative semigroup if $|X| > 2$, where here $\circ$ is the composition of functions. The symmetric group $(\mathcal{S}_X, \circ)$, the set of all bijections from X onto X, is a subgroup of $\mathcal{T}_X$. The group $\mathcal{S}_X$ is the unit group of $\mathcal{T}_X$. If $|X| = 1$ or 2, then $\mathcal{S}_X$ is commutative.

The associative law enables us to drop brackets from any product. So we can unambiguously define powers of elements: if $x \in S$ and $n \in \mathbb{N}$, then

$$x^n = \underbrace{x \cdots x}_{n}.$$

We are now going to define the special elements in semigroups called *idempotents*. The study of idempotents pervades this thesis. We will see in Chapter 5 that the set of idempotents of a semigroup possesses an inherent structure, called a *biordered set*.

Definition 1.1.8. An element e in S is called an *idempotent* if $e^2 = e$.

The set of all idempotents in a semigroup S is denoted by $E(S)$. In any semigroup zeroes and identities are always idempotents. Moreover, if $e \in S$ is an idempotent, then $\{e\}$ is a subsemigroup (indeed, subgroup) of S. In a group the identity is the only idempotent. However, semigroups can consist of entirely of idempotents. If $E(S) = S$, then we say S is a *band*. A commutative band is called a *semilattice*. Note that any rectangular band is indeed a band since any $(i, j) \in T$ is an idempotent, as $(i, j)^2 = (i, j)(i, j) = (i, j)$. However, a rectangular band is not commutative if it has more than one member. We will see shortly an alternative order theoretic approach to semilattices.

A *binary relation* ρ between two sets A and B is a subset of $A \times B$. If $(a, b) \in \rho$, we say a and b are ρ-related, and write $a \mathrel{\rho} b$. If $A = B$, we say $\rho \subseteq A \times A$ is a *binary relation on A*.

The relation $A \times A$ and $\emptyset$ are called the *universal relation* and *empty relation* on A, respectively. The set $\{(a, a) : a \in A\}$ is called the *equality* or *diagonal* relation. We denote this relation by I_A or simply I where A is understood. For any relation ρ on A we define the relation ρ^{-1}, the converse of ρ, as the set

$$\rho^{-1} = \{(a, b) \in A \times A \colon (b, a) \in \rho\}.$$

The set of all binary relations on A is denoted by $\mathcal{B}_A$ and forms a monoid with identity I and a zero $\emptyset$ under the multiplication given by the rule

$$\rho \circ \lambda = \{(a,b) \in A \times A \colon (\exists c \in A)(a,c) \in \rho \text{ and } (c,b) \in \lambda\}.$$

In the following we will review the basic properties of relations. For a given relation ρ on a set A we say that:

(i) ρ is *reflexive*, if for all $a \in A$, $(a,a) \in \rho$;

(ii) ρ is *symmetric*, if for all $a,b \in A$, $(a,b) \in \rho$ implies $(b,a) \in \rho$;

(iii) ρ is *anti-symmetric*, if for all $a,b \in A$, (a,b), $(b,a) \in \rho$ implies $a = b$;

(iv) ρ is *transitive*, if (a,b), $(b,c) \in \rho$ implies $(a,c) \in \rho$.

Next we define some different kinds of relations:

(i) A relation ρ is called *pre-order (quasi-order)* on a set A if it is reflexive and transitive. We often denote ρ by $\preceq$.

(ii) A relation ρ on a set A is a *partial order* if it is reflexive, antisymmetric and transitive. We often denote a partial order ρ by $\leq$. If $\leq$ is a partial order, the pair $(A, \leq)$ is called a *partially ordered set.*

(iii) A relation ρ is an *equivalence* relation on a set A if it is reflexive, symmetric and transitive. The *equivalence class* of the element a is defined by

$$a\rho = \{b \in A \colon a \ \rho \ b\}.$$

We often denote the *equivalence class* of the element a by $[a]$.

Note that if $\preceq$ is a pre-order on a set A, then we can define a relation ρ by $a \, \rho \, b$ if $a \preceq b$ and $b \preceq a$, for any $a, b \in A$. Clearly that ρ is an equivalence relation. Moreover, we define a relation $\leq$ on a set A/ρ by $[a] \leq [b]$ if $a \preceq b$, for any $[a], [b] \in A/\rho$. It is clear that $\leq$ is a partial order on A/ρ.

A family $\pi = \{A_i : i \in I\}$ of subsets of a set A is said to form a *partition* of A if :

(P_1) each A_i is non-empty;

(P_2) for all $i, j \in I$, either $A_i = A_j$ or $A_i \cap A_j = \emptyset$;

(P_3) $\cup\{A_i : i \in I\} = A$.

It is easy to see that an equivalence relation ρ on a set A partitions A into equivalence classes. Conversely, corresponding to any partition of A, there exists an equivalence relation ρ on A having the elements of the partitions as its equivalence classes.

In the following we define the kernel relation.

Definition 1.1.9. Let $\alpha : X \longrightarrow Y$ be a map. Define a relation ρ on the set X by

$$a \ \rho \ b \Longleftrightarrow a\alpha = b\alpha,$$

where $a, b \in X$. We call ρ the *kernel* of α, denoted by $\ker \alpha$.

It is clear that $\ker \alpha$ is an equivalence relation.

Lemma 1.1.10. *The intersection of equivalence relations on a set A is an equivalence relation on A.*

An *equivalence relation generated by* some relation σ on A is the smallest equivalence relation containing the relation σ. It is the intersection of all equivalence relations containing σ on A, usually denoted by σ^e. Note that σ^e always exists as the set of all equivalence relations containing σ on A is not empty, since $A \times A$ is one of them.

The next result gives us a useful way to find ρ^e for any relation ρ on a set A. Let ρ be an arbitrary reflexive relation on a set A. Then we say that

$$\rho^\infty = \bigcup \{\rho^n : n \geq 1\}$$

is the *transitive closure* of the relation ρ. It is known that ρ^∞ is the smallest transitive relation on A containing ρ.

Lemma 1.1.11. *Let ρ be any binary relation on A. Then the smallest equivalence relation on A containing ρ is given by*

$$\rho^e = (\rho \cup \rho^{-1} \cup 1_A)^\infty.$$

For two binary relations ρ and σ on a set A we denote $(\rho \cup \sigma)^e$ by $\rho \vee \sigma$.

Lemma 1.1.12. *Let ρ and σ be two equivalence relations on a set A such that $\rho \circ \sigma = \sigma \circ \rho$. Then*

$$\rho \vee \sigma = \rho \circ \sigma = \sigma \circ \rho.$$

In the following we will define some special kinds of binary relations on semigroups.

Let ρ be a binary relation on a semigroup S. We say that ρ is *left compatible* if for any a, b and c in S, we have that

$$a \ \rho \ b \Longrightarrow ca \ \rho \ cb.$$

We say that ρ is *right compatible* if for any a, b and c in S we have that

$$a \ \rho \ b \Longrightarrow ac \ \rho \ bc.$$

A relation ρ said to be *compatible* if it is both left and right compatible.

Definition 1.1.13. We say that a left (right) compatible equivalence relation is a *left (right) congruence* on S, and a compatible equivalence relation is called a *congruence* on S.

Notice that an equivalence relation ρ on S is a congruence if and only if for any $a, b, c, d \in S$, if $a \ \rho \ b$ and $c \ \rho \ d$, then $ac \ \rho \ bd$.

Lemma 1.1.14. *Let ρ be a congruence on a semigroup S. Then $(S/\rho, \cdot)$ forms a semigroup where the set S/ρ is given as*

$$S/\rho = \{a\rho \colon a \in S\},$$

and the multiplication $\cdot$ is defined by the rule that

$$(a\rho) \cdot (b\rho) = (ab)\rho.$$

The semigroup in the above lemma is called a *quotient* semigroup. It is clear that if S is a monoid, then so is S/ρ, with identity [1].

Lemma 1.1.15. *The intersection of a non-empty family of congruences on a semigroup S is a congruence on S.*

For any relation ρ on S there is a unique smallest congruence $\rho^\natural$ on S containing ρ, which is the intersection of all the congruences on S containing ρ.

The following result gives us a characterization of the smallest congruence $\rho^\natural$ on a semigroup S containing ρ.

Lemma 1.1.16. *For any fixed binary relation ρ on a semigroup S, the smallest congruence $\rho^\natural$ containing ρ is defined by $\rho^\natural = (\rho^c)^e$, where*

$$\rho^c = \{(xay, xby) \colon x, y \in S^1,\ a\ \rho\ b\}.$$

Now it is time for us to give the alternative order theoretic approach to semilattices. If Y is a non-empty subset of a partially ordered set X, we say that an element c of X is a *lower bound* of Y if $c \leq y$ for every $y \in Y$. If the set of lower bounds of Y is non-empty and has a maximum element d, we say that d is the *greatest lower bound* or *meet* of Y. The element d is unique if it exists, and we write

$$d = \bigwedge\{y : y \in Y\}.$$

If $Y = \{a, b\}$, we write $d = a \wedge b$. We say that $(X, \leq)$ is a *lower semilattice* if $a \wedge b$ exists for all $a, b \in X$. In any lower semilattice $(X, \leq)$, for any $a, b \in X$, we have

$$a \leq b \quad \text{if and only if} \quad a \wedge b = a.$$

Proposition 1.1.17. *Let $(E, \leq)$ be a lower semilattice. Then $(E, \wedge)$ is a commutative semigroup consisting entirely of idempotents, and for any $e, f \in E$*

$$e \leq f \quad \textit{if and only if} \quad e \wedge f = e.$$

Conversely, suppose that $(E, \cdot)$ is a commutative semigroup of idempotents. Then the relation $\leq$ on E defined by

$$e \leq f \quad \textit{if and only if} \quad ef = e.$$

is a partial order on E, with respect to which $(E, \leq)$ is a lower semilattice.
In $(E, \leq)$, the meet $e \wedge f$ of e and f is their product ef.

In the following we define a binary relation $\leq$ on $E(S)$, in Section 1.2 we will return to define this binary relation on arbitrary semigroup S. The binary relation $\leq$ on $E(S)$ is defined by the rule

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

We show that the binary relation $\leq$ is a partial order on $E(S)$. For any $e, f, g \in E$ we have $e \leq e$ and if $e \leq f$ and $f \leq e$, then $ef = fe = e = f$. Moreover, if $e \leq f$ and $f \leq g$, then

$$ef = fe = e \ \text{ and } \ fg = gf = f,$$

so that

$$eg = efg = ef = e \ \text{ and } \ ge = gfe = fe = e$$

and hence $e \leq g$. Therefore, $(E, \leq)$ is a partially ordered set. In fact, $E(S)$ has much richer structure that we will explain in the following sections.

We say that an element a in S is *regular* if there exists an element b in S such that $aba = a$. It is clear that an idempotent is regular. A semigroup is called *regular* if every element of S is regular. It is obvious that every band is regular.

For an element a of a semigroup S we say that a' is an *inverse* of a if

$$a = aa'a \quad \text{and} \quad a' = a'aa'.$$

It is clear that an inverse element is regular. Conversely, any regular element has an inverse, since if a is a regular element, then there exists $b \in S$ such that $a = aba$. Then $a' = bab$ is an inverse of a, as

$$aa'a = a(bab)a = (aba)ba = aba = a$$

and

$$a'aa' = (bab)a(bab) = (bab)(aba)b = (bab)(a)b = b(aba)b = b(a)b = a'.$$

For each a in S we denote the set of all inverses of a by $V(a)$. A semigroup S is *inverse* if every element has a unique inverse. Every group is inverse as for each a in a group G, $V(a) = \{a^{-1}\}$, where a^{-1} is the usual group inverse of a in G.

For any non-empty subset X of a semigroup S there is at least one subsemigroup of S containing X, namely S itself. Let $\{P_i : i \in I\}$ be the collection of all the subsemigroups of S containing the set X. Then

$$P = \bigcap_{i \in 1} P_i$$

is called the *subsemigroup generated by* X, denoted by $\langle X \rangle$. Note that $\langle X \rangle$ is the smallest subsemigroup of S containing X. We say that X generates $\langle X \rangle$. If $S = \langle X \rangle$, then we say X a *generating set* for S. A semigroup is *finitely generated* if there exists a finite set $X \subseteq S$ such that $S = \langle X \rangle$. It is easy to see that $\langle X \rangle = \{x_1 x_2 \ldots x_n : \ n \in \mathbb{N},\ x_i \in X \text{ for all } 1 \leq i \leq n\}$. For example, any semigroup is generated by itself. If $X = \{x\}$ is a singleton set, then

$$\langle x \rangle = \{x, x^2, x^3, \cdots\}.$$

We say that $\langle x \rangle$ is a *monogenic subsemigroup* of S generated by x.

Example 1.1.18. The semigroup of natural numbers $\mathbb{N}$ under addition is generated by the element 1.

Let S be a semigroup and let A be a subset of S. Then

$$S^1A = \{sa : s \in S^1, a \in A\} = SA \cup A.$$

If $A = \{a\}$ is a singleton subset of S, then we simply write $S^1A = S^1a$, with similar conventions for other products. We have three key subsets of S which are given by:

(i) $S^1a = Sa \cup \{a\}$;

(ii) $aS^1 = aS \cup \{a\}$;

(iii) $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$.

Notice that S satisfies $S^1a = Sa$ if and only if $a \in Sa$ if and only if $a = ua$ for some $u \in S$. We give three examples of cases where this occurs:

(i) if S is a monoid, so that $a = 1a$;

(ii) if $a \in E(S)$, since $aa = a$;

(iii) if a is a regular, then there exists $x \in S$ such that $axa = a$ and $a = (xa)a$.

Let T be a non-empty subset of a semigroup S. Then T is a *left ideal* of S if for all $s \in S$ and $t \in T$, we have $st \in T$. We call T a *right ideal* of S if for all $s \in S$ and $t \in T$, we have $ts \in T$. We call T a *(two-sided) ideal* of S if it is both a left ideal and a right ideal of S.

Lemma 1.1.19. *Let S be a semigroup and $a \in S$. Then aS^1 (S^1a, S^1aS^1) is the smallest right (left, two-sided) ideal containing a.*

Definition 1.1.20. Let S be a semigroup, $a \in S$. We call aS^1 the *principal right ideal* generated by a.

Similarly, we call S^1a the *principal left ideal* generated by a. Moreover, S^1aS^1 is called the *principal ideal generated by a*.

Example 1.1.21. Let $I \times J$ be a rectangular band and let $i \in I$. For any $(m, n) \in I \times J$ and $(i, j) \in \{i\} \times J$ we have that $(i, j)(m, n) = (i, n) \in \{i\} \times J$. Then $\{i\} \times J$ is a principal right ideal of $I \times J$.

Lemma 1.1.22. *The following statements are equivalent:*

(i) $S^1a \subseteq S^1b$*;*

(ii) $a \in S^1b$*;*

(iii) $a = tb$*, where* $t \in S^1$*;*

(iv) $a = b$ *or* $a = tb$ *for some* $t \in S$*.*

We remark here that a dual result holds for principal right ideals.

Notice that S is an ideal of itself and if S is semigroup with 0, then $\{0\}$ is an ideal of S. An ideal A of S is *proper* if $A \neq S$ and if S has 0, then an ideal A is proper if $A \neq S$ and $A \neq \{0\}$. A semigroup S without 0 is called *simple* if S is the only ideal of S. A semigroup with 0 is called 0-*simple* if S and $\{0\}$ are the only ideals of S and $S^2 \neq \{0\}$.

We know that homomorphisms of algebraic structures preserve all the basic operations, which we now explicitly define. The same is true for semigroup and monoid homomorphisms.

Definition 1.1.23. Let S and T be semigroups. A map $\varphi : S \to T$ is called a *(semigroup) morphism* if for all $x, y \in S$, we have that

$$(xy)\varphi = (x\varphi)(y\varphi).$$

Note that a morphism φ from S into itself is called *an endomorphism.* The set of all endomorphisms is a monoid under composition, denoted by $\operatorname{End} S$.

If S and T are monoids, $\varphi : S \to T$ is a semigroup morphism and if in addition,

$$1_S\varphi = 1_T,$$

then we say φ is a *(monoid) morphism.*

If a morphism φ is onto (surjective), we call it an *epimorphism.* An injective morphism is called a *monomorphism.* If φ is bijective morphism, then φ is called an *isomorphism* and we say that S and T are *isomorphic*, denoted by $S \cong T$. Notice that two isomorphic semigroups (or monoids) have exactly the same algebraic properties.

Before giving some examples of semigroup morphisms, we fix the following standard notation for maps.

(i) Let $\theta : S \longrightarrow T$ be a map from S to T. Let S' be a subset of S. We denote the restriction of θ to S' as $\theta_{|_{S'}}$.

(ii) Let S and T be sets and let $\{S_i : i \in I\}$ be a partition of S. Let $\theta_i : S_i \longrightarrow T$ be a map from S_i to T, for each $i \in I$. Let $\theta = \bigcup_{i\in I} \theta_i$ be the map $\theta : S \longrightarrow T$ which is given by

$$s\theta = s\theta_i, \quad \text{if } s \in S_i.$$

(iii) Let S be a semigroup. The identity map $I_S : S \to S$ defined by $sI_S = s$, for all s of S.

Example 1.1.24. Let S be a semigroup and let $e \in E(S)$. Define the constant map

$$C_e : S \to S$$

by $s \longmapsto e$ for all s of S. Then the map C_e is a morphism.

Let S and T be semigroups and $\alpha : S \to T$ be a morphism. Then the set

$$\text{Im } \alpha = \{s\alpha : s \in S\}$$

is a subsemigroup of T. If α is a monoid morphism, then Im α is a submonoid of T. We say S is *embedded* in T if α is an injective morphism. Let $\varphi : S \to T$ be an epimorphism. We say that T is a *homomorphic image* of S.

The following result is an essential lemma of the morphism for semigroups [64].

Lemma 1.1.25. *Let ρ be a congruence on a semigroup S. The mapping*

$$\rho^\natural : S \to S/\rho,$$

defined by

$$a \longmapsto a\rho$$

is an epimorphism.

The next result is the Fundamental Theorem of Morphisms for semigroups.

Theorem 1.1.26. *Let $\theta : S \longrightarrow T$ be a semigroup morphism. Then* $\ker\theta$ *is a congruence on S. Moreover, $S/\ker\theta$ isomorphic to Im θ.*

1.2 Green's relations

We introduce an important tool for analyzing the ideals of a semigroup S and related notions of structure, introduced by J.A. Green and subsequently called Green's relations. These relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are equivalences, that characterize the elements of S in terms of the principal ideals they generate.

We first define the binary relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ on a semigroup S by the rule, for any $a, b \in S$

$$\begin{aligned} a \ \leq_{\mathcal{L}} \ b &\Longleftrightarrow S^1a \subseteq S^1b, \\ a \ \leq_{\mathcal{R}} \ b &\Longleftrightarrow aS^1 \subseteq bS^1, \\ a \ \leq_{\mathcal{J}} \ b &\Longleftrightarrow S^1aS^1 \subseteq S^1bS^1. \end{aligned}$$

We called these relations *Green's left, right and two sided quasi-orders*, respectively. All of these relations are quasi-orders. Also, it is easy to see that $\leq_{\mathcal{L}}$ is right compatible and similarly the relation $\leq_{\mathcal{R}}$ is left compatible.

It is easy to see from Lemma 1.1.17 that if e and f are idempotents, then

$$e \ \leq_{\mathcal{L}} \ f \Longleftrightarrow ef = e \text{ and } e \ \leq_{\mathcal{R}} \ f \Longleftrightarrow fe = e.$$

Now we are in a position to define *Green's relations* as follows:

(i) $a \,\mathcal{L}\, b$ if and only if $S^1a = S^1b$;

(ii) $a \,\mathcal{R}\, b$ if and only if $aS^1 = bS^1$;

(iii) $a \,\mathcal{H}\, b$ if and only if $a \,\mathcal{L}\, b$ and $a \ \mathcal{R} \ b$;

(iv) $a \,\mathcal{D}\, b$ if there exists $c \in S$ with $a \,\mathcal{R}\, c \,\mathcal{L}\, b$;

(v) $a \,\mathcal{J}\, b$ if and only if $S^1aS^1 = S^1bS^1$.

The relations $\mathcal{L}$ and $\mathcal{R}$ and $\mathcal{J}$ are the equivalence relations associated with the quasi-orders $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$, respectively. It is easy to check that $\mathcal{L}$ is a right congruence on S and $\mathcal{R}$ is a left congruence on S. Moreover, it is clear by the definition of $\mathcal{H}$ that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Lemma 1.2.1. *Let a, b be elements of a semigroup S. Then $a \,\mathcal{L}\, b$ if and only if there exist $x, y \in S^1$ such that*

$$xa = b, \ yb = a.$$

Dually, $a\,\mathcal{R}\,b$ if and only if there exist $x', y' \in S^1$ such that

$$ax' = b, \;\; by' = a.$$

Example 1.2.2. (i) In the commutative semigroup ($\mathbb{N}$,+) we have that

$$\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J} = I = \{(a, a) : a \in \mathbb{N}\}.$$

(ii) In any group G, we have $\mathcal{L} = \mathcal{R} = \mathcal{H}$. Since for any $a, b \in G$ we have $aG^1 = G = bG^1$ and $G^1a = G = G^1b$. Then $a\,\mathcal{R}\,b$, $a\,\mathcal{L}\,b$ and $\mathcal{R} = \mathcal{L} = \omega = G \times G$. As $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, then $\mathcal{H} = G \times G$. Moreover, we have $\omega = \mathcal{D} = \mathcal{J}$.

The following result showing that $\mathcal{L}$ and $\mathcal{R}$ commute [64].

Lemma 1.2.3. *For any semigroup S we have $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.*

From the above lemma and from the definition of the relation $\mathcal{D}$, it is clear that $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ and $\mathcal{D}$ is an equivalence relation. Moreover, the relation $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$.

For an element $a \in S$, we denote the $\mathcal{L}$-class, the $\mathcal{R}$-class, the $\mathcal{D}$-class, the $\mathcal{H}$-class and the $\mathcal{J}$ -class of a by L_a, R_a, D_a, H_a and J_a, respectively. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we have $H_a = L_a \cap R_a$. Also, it is clear that every $\mathcal{D}$-class of S is a union of the $\mathcal{L}$-classes, and also a union of $\mathcal{R}$-classes. By Lemma 1.2.3 and the definition of $\mathcal{D}$ relation we have

$$a \;\mathcal{D}\; b \iff L_a \cap R_b \neq \emptyset \iff R_a \cap L_b \neq \emptyset.$$

It is useful to visualize a $\mathcal{D}$-class D of a semigroup S as an eggbox diagram, [10]. An eggbox is a grid whose rows represent $\mathcal{R}$-classes of D, columns represent $\mathcal{L}$-classes of D, and cells of the grid represent $\mathcal{H}$-classes of D, as depicted by the figure below.

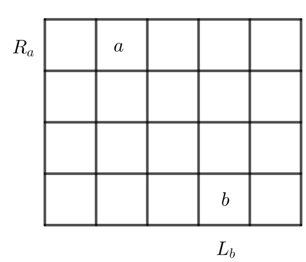


Figure 1.1: The egg-box of a typical $\mathcal{D}$-class.

Since $\mathcal{L}$, $\mathcal{R} \subseteq \mathcal{J}$ and $\mathcal{D}$ is the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$, we have $\mathcal{D} \subseteq \mathcal{J}$. We may depict the relations between Green's relations by the following *Hasse diagram.*

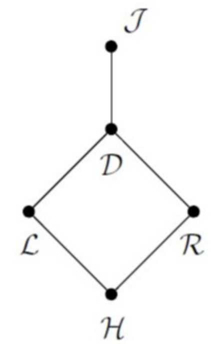


Figure 1.2: Hasse diagram of Green's relations

Lemma 1.2.4. *A semigroup S is simple if and only if it has a single $\mathcal{J}$-class, $\mathcal{J} = S \times S$.*

Example 1.2.5. For any rectangular band T we always have that $\mathcal{D} = \mathcal{J} = T \times T$ and for all $a, b \in T$ we have that

$$a \,\mathcal{R}\, ab \,\mathcal{L}\, b.$$

We end this section with some useful results of semigroups.

Lemma 1.2.6. *For any idempotent e in S. The element e is the left identity of its $\mathcal{R}$-class R_e and a right identity of its $\mathcal{L}$-class L_e.*

Lemma 1.2.7. *For each idempotent e in S, H_e is the maximal subgroup of S containing identity e.*

The following is one of the important theorems of semigroups, usually called Green's Theorem.

Theorem 1.2.8. *For each $\mathcal{H}$-class H of a $\mathcal{D}$-class D in S, we either have H is a subgroup or $H^2 \cap H = \emptyset$. Moreover, H is a subgroup if and only if H contains an idempotent of S, so that no $\mathcal{H}$-class can contain more than one idempotent.*

Lemma 1.2.9. *Let a and b be elements of S such that $a \;\mathcal{D}\; b$. Then $|H_a| = |H_b|$. Moreover, any two group $\mathcal{H}$-classes within the same $\mathcal{D}$-class are isomorphic.*

1.3 Semilattices and strong semilattices of semigroups

In the following we introduce an important semigroup construction used in this thesis. Let Y be a semilattice and S be a semigroup. We say that S is a *semilattice Y of subsemigroups* S_α, $\alpha \in Y$, if the following hold:

(i) S is a disjoint union of subsemigroups S_α, where $\alpha \in Y$

(ii) for any $\alpha, \beta \in Y$, $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

The semilattice Y is called the *structure semilattice of S*.

For an element a of S, if $a \in S_\alpha$, we may denote this by writing a as a_α. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice of semigroups. The map

$$\sigma\colon S \longrightarrow Y,$$

defined by the rule

$$a_\alpha \sigma = \alpha,$$

is an epimorphism.

Let S be a semilattice Y of subsemigroups S_α, $\alpha \in Y$. Then for any $a \in S_\alpha$, $b \in S_\beta$, by definition, $ab \in S_{\alpha\beta}$, but there is no clear location of ab within $S_{\alpha\beta}$. To determine this we need the notion of *strong semilattice of semigroups.*

Suppose that we have a semilattice Y and a family of disjoint semigroups S_α, where $\alpha \in Y$, and suppose that, for all $\alpha \geq \beta$ in Y, there exists a morphism $\varphi_{\alpha,\beta}\colon S_\alpha \longrightarrow S_\beta$ such that:

(i) for all $\alpha \in Y$, $\varphi_{\alpha,\alpha} = 1_{S_\alpha}$;

(ii) for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$, we have $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

We define a multiplication on the set $S = \bigcup_{\alpha \in Y} S_\alpha$ by the rule that for each $x \in S_\alpha$ and each $y \in S_\beta$,

$$xy = (x\varphi_{\alpha,\alpha\beta})(y\varphi_{\beta,\alpha\beta}),$$

where the multiplication on the right hand side is $S_{\alpha\beta}$.

It is clear that the operation extends the multiplication in each S_α, $\alpha \in Y$. The set $S = \bigcup_{\alpha \in Y} S_\alpha$, with the multiplication defined above, forms a semigroup, called *a strong semilattice Y of semigroups* $S_\alpha, \alpha \in Y$. We denote this semigroup by $S = \mathscr{S}(Y; S_\alpha, \varphi_{\alpha,\beta})$. Note that the multiplication extends that in each S_α.

It is important to note that if $S = \mathscr{S}(Y; S_\alpha, \varphi_{\alpha,\beta})$ is a strong semilattice of Y of semigroups S_α, $\alpha \in Y$, then it is certainly a semilattice Y of semigroups S_α, $\alpha \in Y$. However, the converse is not always true, as we see in the next example.

Example 1.3.1. Any non normal band B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, but it can not be a strong semilattice of any semigroups, as we will see in Section 1.5.

1.4 Regular semigroups and completely simple semigroups

Recall that $a \in S$ is *regular* if there exists $x \in S$ such that $a = axa$ and a semigroup S is called a *regular semigroup* if all its elements are regular. Groups and rectangular bands are examples of regular semigroups.

Recall that if a is regular element of S, then we have that $S^1 a = Sa$. Hence, if S is regular, Green's relations can be expressed in terms of S rather than S^1. That is we can define the Green's relations on S by the rules

$$a \,\mathcal{L}\, b \Longleftrightarrow Sa = Sb;$$
$$a \,\mathcal{R}\, b \Longleftrightarrow aS = bS;$$
$$a \,\mathcal{J}\, b \Longleftrightarrow SaS = SbS.$$

Lemma 1.4.1. *If a is a regular element of a semigroup S, then every element of D_a is regular.*

Thus for every $\mathcal{D}$-class D in S, either all elements of D are regular or none of them are regular. We say that a $\mathcal{D}$-class is $\mathcal{D}$*-regular* if it contains one regular element (consists entirely of regular elements). Note that this is not true for $\mathcal{J}$ [64, Exercise 26 Chapter 5]. Any idempotent $e \in S$ is a regular, we have that any $\mathcal{D}$-class containing an idempotent is regular. Moreover, every regular $\mathcal{D}$-class contains at least one idempotent, since if a in a regular $\mathcal{D}$-class with $a = axa$, we know that $ax, xa \in E(S)$.

It is easy to see that if a is a regular element with $axa = a$, then $ax \,\mathcal{R}\, a \,\mathcal{L}\, xa$. Also both ax and xa are idempotents. Hence we have the following result.

Lemma 1.4.2. *In a regular $\mathcal{D}$-class, each $\mathcal{R}$-class and each $\mathcal{L}$-class contains an idempotent.*

Let S be a semigroup and let $(E, \leq)$ be a partially ordered set, where $E = E(S)$. We say an idempotent $e \in S$ is a *primitive* if it is a minimal non-zero element of the set $E(S)$. If e is primitive, and $0 \neq f \in E(S)$ with $f \leq e$, then $f = e$.

Definition 1.4.3. A semigroup S is called a *completely simple (completely* 0-*simple)* if S is simple (0-simple) and contains a primitive idempotent.

Lemma 1.4.4. *Every completely* 0*-simple semigroup S is a regular semigroup with exactly two $\mathcal{D}$-classes, namely $\{0\}$ and $D = S \backslash \{0\}$.*

Let G be a group, let I and Λ be non empty index sets and let $P = (p_{\lambda,i})$ be an $\Lambda \times I$ matrix with entries in $G \cup \{0\}$. Suppose that P is regular, which means there is no row or column of P consists entirely of zeros,

$$(\forall i \in I)(\exists \lambda \in \Lambda) p_{\lambda,j} \neq 0 \text{ and } (\forall \lambda \in \Lambda)(\exists i \in I) p_{\lambda,j} \neq 0.$$

Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define multiplication on S by

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda,j}, h, \mu) & \text{if } p_{\lambda,j} \neq 0 \\ 0 & \text{else} \end{cases} \tag{1.1}$$

and $(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0$. We denoted S under this multiplication by $M^0[G, I, \Lambda, P]$, called the $I \times \Lambda$ *Rees matrix semigroup over G with regular sandwich matrix P.*

We end this section by these results which taken from [78], in 1940 by Rees.

Theorem 1.4.5. *The semigroup $M^0[G, I, \Lambda, P]$ constructed in the above manner is a completely* 0*-simple semigroup; conversely, every completely* 0*-simple semigroup is isomorphic to one constructed in this way.*

Corresponding to completely simple semigroups, we have the following simplified version of the Rees Theorem.

Theorem 1.4.6. *Let G be a group, let I and Λ be non-empty sets and let $P = (p_{\lambda,i})$ be a $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$, and define a multiplication on S by*

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda,j}b, \mu).$$

Then S is a completely simple semigroup. Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

The completely simple semigroup $S = I \times G \times \Lambda$, with the multiplication in the above result, denoted by

$$\mathcal{M}(G; I, \Lambda, P).$$

For further details, we refer readers to [78].

1.5 Bands

The main goal of this section is to introduce some special kinds of bands that are frequently mentioned in this thesis such as *left normal bands, right normal bands*, and *normal bands.* Recall that a semigroup S is a *band* if it consists entirely of idempotents. Usually, we denote a band by B.

In the previous section of this chapter, we have already defined a partial order $\leq$ on the set of idempotents $E(S)$ of a semigroup S by the rule that, for any idempotents $e, f \in S$

$$e \leq f \Longleftrightarrow ef = fe = e.$$

Certainly this relation can be used to partially order any band B.

Bands have the following useful property.

Theorem 1.5.1. *For any band B we have $\mathcal{D} = \mathcal{J}$.*

The following figure contained some special kinds of bands which are required for our work. In fact, all of these classes are *varieties* of bands; for further details see [44].

$\mathcal{R}e\mathcal{B}$	regular band : $zxyz = zxzyz$
$\mathcal{LRB}$	left regular band : $xy = xyx$
$\mathcal{RRB}$	right regular band : $xy = yxy$
$\mathcal{NB}$	normal bands : $zxyz = zyxz$
$\mathcal{LNB}$	left normal bands : $zxy = zyx$
$\mathcal{RNB}$	right normal bands : $xyz = yxz$
$\mathcal{RB}$	rectangular bands : $xyx = x$
$\mathcal{LZ}$	left zero semigroups : $xy = x$
$\mathcal{RZ}$	right zero semigroups : $xy = y$
$\mathcal{B}$	bands (idempotent semigroups) : $x^2 = x$
$\mathcal{Z}$	null semigroup : $xy = z$
$\mathcal{C}$	commutative semigroup : $xy = yx$
$\mathcal{SL}$	semilattices : $x^2 = x, xy = yx$
$\mathcal{T}$	trivial semigroups : $x = y$

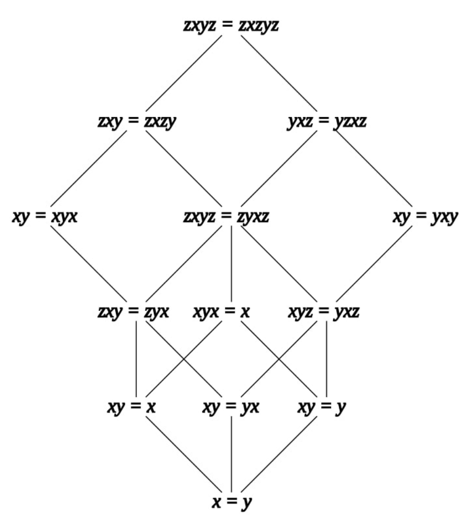


Figure 1.3: Lattice of varieties of regular bands

The following useful result from [66], illustrates a necessary and sufficient condition for any band to be a rectangular band.

Lemma 1.5.2. *A band B is rectangular if and only if for any $e, f, g \in B$, we have $efg = eg$.*

The above lemma means in bands the identities (or equations) $efe = e$ and $efg = eg$ are equivalent.

Now we are in the position to give the well known decomposition theorem of bands in terms of semilattices of semigroups.

Lemma 1.5.3. *Let B be a band. Then B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Further, $Y = B/\mathcal{D}$ is a semilattice and each B_α is a $\mathcal{D}$-class of B.*

Note that a band B is a semilattice if and only if each B_α is trivial.

In this thesis we focus on a special kind of band called a *normal band*. It is clear from the figure above that a normal band B is a *left normal band* and *right normal band*.

We end this section with the important result of normal bands.

Lemma 1.5.4. *A band B is normal band if and only if it is a strong semilattice $\mathscr{B}(Y; B_\alpha, \varphi_{\alpha,\beta})$, where $Y = B/\mathcal{D}$ and each B_α is a $\mathcal{D}$-class of B.*

Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band. We say B is *pliant (trivial normal band)* if for every $\alpha \in Y$, there exist an $a_\alpha \in B_\alpha$ such that for all $\beta > \alpha$ and each $u \in B_\beta$, we have $u\varphi_{\beta,\alpha} = a_\alpha$.

A band $B = \bigcup_{\alpha \in Y} B_\alpha$ is Y*-basic (simple band)* if it is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, where B_α is either a left zero band or a right zero band. For example any left or right regular band (that is, where every B_α is left zero, or every B_α is right zero) is Y-basic.

Chapter 2

Preliminaries II: Semigroup constructions

In this chapter, we outline some basic semigroup constructions required in this thesis such as direct product, free product, and graph product. The definitions and results are taken from the standard books on introductory semigroup theory [55], [64], [65] and [81].

2.1 Semigroup and monoid presentations

Let A be a non-empty set. Let A^+ be the set of non-empty finite sequences (strings) $a_1 \circ a_2 \circ \ldots \circ a_n$ formed from A. We call the sequences of A^+ *words*. Denote a binary operation on A^+ by $\circ$, that is, for all $a_1 \circ a_2 \circ \ldots \circ a_n$, $b_1 \circ b_2 \circ \ldots \circ b_m \in A^+$

$$(a_1 \circ a_2 \circ \ldots \circ a_n) \circ (b_1 \circ b_2 \circ \ldots \circ b_m) = a_1 \circ a_2 \circ \ldots \circ a_n \circ b_1 \circ b_2 \circ \ldots \circ b_m.$$

The set A^+ with respect to the above operation forms a semigroup, call the *free semigroup* on A. An element a in A is called a *generator* and A is the *generating set* of A^+. We say that two words $a_1 \circ a_2 \circ \ldots \circ a_n$ and $b_1 \circ b_2 \circ \ldots \circ b_m$ are *equal* if $m = n$ and $a_i = b_i$ for every $1 \leq i \leq n$. By adjoining an empty word (containing no letters) denoted by 1, into A^+, we obtain the *free monoid* $A^* = A^+ \cup \{1\}$ on A.

An abstract way to define a *free semigroup* on A can be given as follows. A semigroup F is called a *free semigroup* on A if we have the following:

(F1) there is a map $\alpha\colon A \to F$;

(F2) for every semigroup S and every map $\phi\colon A \to S$ there exists a unique morphism $\psi\colon F \to S$ such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & F \arrow[dl, "\psi"] \\
S &
\end{tikzcd}
$$

Figure 2.1: The commutative diagram for a free semigroup

It is easy to prove that A^+ is a free semigroup by using the above abstract definition of free semigroups. Let $\alpha\colon A \to A^+$ be a map defined by $a\alpha = a$, for any a in A. Then for any given semigroup S and an arbitrary map $\phi\colon A \to S$, we define $\psi\colon A^+ \to S$ by

$$(a_1 \circ a_2 \circ \ldots \circ a_n)\psi = (a_1\phi)(a_2\phi)\cdots(a_n\phi).$$

It is easy to check that ψ is the unique morphism from A^+ to S such that $\alpha\psi = \phi$. Then we have the following commuting diagram:

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & A^+ \arrow[dl, "\psi"] \\
S &
\end{tikzcd}
$$

Figure 2.2: The commutative diagram for the free semigroup A^+

By taking $A = S$ it is easy to see that the following result is true.

Lemma 2.1.1. *[64] Every semigroup may be expressed up to an isomorphism as a quotient of a free semigroup.*

In the following we define a *semigroup presentation.*

Definition 2.1.2. A *semigroup presentation* is an ordered pair $\langle A|R\rangle$, where R is a binary relation on A^+. The pair $(u, v) \in R$ is called a *defining relation.* The semigroup defined by a presentation $\langle A|R\rangle$ is A^+/ρ, where ρ is the smallest congruence generated by R.

2.2 External direct products of semigroups

There are numerous ways of building new structures from the already existing ones. One of the most well known ways is *direct product* of algebraic structures such as groups and semigroups.

Definition 2.2.1. [64] Let S_1, $S_2, \ldots$, S_n be semigroups and let

$$S_1 \times \cdots \times S_n = \{(s_1, \ldots, s_n) : s_i \in S_i, \text{ for all } 1 \leq i \leq n\}.$$

Define a binary operation on $S \times \cdots \times S_n$ by

$$(s_1, \ldots, s_n)(t_1, \ldots, t_n) = (s_1t_1, \ldots, s_nt_n),$$

where $(s_1, \ldots, s_n), (t_1, \ldots, t_n) \in S_1 \times \cdots \times S_n$. The set $S_1 \times \cdots \times S_n$ forms a semigroup with respect to the above operation, called the *(external) direct product* of the semigroups S_1, S_2, $\ldots$, S_{n-1} and S_n, denoted by

$$\prod_{i=1}^{n} S_i = \{(s_1, s_2, \ldots, s_n) \colon s_i \in S_i\}.$$

If S_1, $S_2, \ldots$, S_n are monoids with identities 1_1, 1_2, $\ldots$, 1_n, respectively, then the *external direct product* $S_1 \times \cdots \times S_n$ has identity $(1_1, \ldots, 1_n)$.

Let $M = S \times T$ and $M_1 = S_1 \times T_1$ be two external direct products of semigroups, and let

$$\psi_1 \colon S \longrightarrow S_1$$

and

$$\psi_2 \colon T \longrightarrow T_1$$

be morphisms. Define a map $\varphi \colon M \longrightarrow M_1$ by

$$(s, t)\varphi = (s\psi_1, t\psi_2).$$

This map is a morphism. Moreover, if ψ_1 and ψ_2 are injective, then φ is injective. Similarly, if ψ_1 and ψ_2 are surjective, then φ is surjective .

2.3 Free products of semigroups

In this section we study a different way to building a new semigroup from existing ones, called the *free product* of semigroups.

Let $\{S_i\colon i \in I\}$ be a set of pairwise disjoint semigroups. If

$$s \in \mathcal{S} = \bigcup\{S_i\colon i \in I\},$$

then there is a unique k in I such that $s \in S_k$ and we write $\sigma(s) = k$.

Let $\mathscr{F}\mathscr{P}$ be the set of all finite strings $s = (s_1, s_2, \ldots, s_n)$, where $n \in \mathbb{N}$, and $s_r \in \mathcal{S}$ for all $1 \le r \le n$, such that $\sigma(s_r) \neq \sigma(s_{r+1})$, for all $1 \le r \le n-1$. Let $s = (s_1, s_2, \ldots, s_n)$ and $t = (t_1, t_2, \ldots, t_m)$ be elements of $\mathscr{F}\mathscr{P}$. We define the product of s and t in $\mathscr{F}\mathscr{P}$ by

$$s \star t = \begin{cases} (s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_m) & \text{if } \sigma(s_n) \neq \sigma(t_1) \\ (s_1, s_2, \ldots, s_n t_1, t_2, \ldots, t_m) & \text{if } \sigma(s_n) = \sigma(t_1). \end{cases} \tag{2.1}$$

It is easy to check that the above operation is associative. The set $\mathscr{F}\mathscr{P}$ with respect to this operation forms a semigroup, called the *(semigroup) free product of the family* $\{S_i\colon i \in I\}$, denoted by

$$\mathscr{F}\mathscr{P} = \Pi^\star\{S_i\colon i \in I\}.$$

We may write $\mathscr{F}\mathscr{P}$ more simply as $\mathscr{F}\mathscr{P} = \Pi^\star S_i$. If $I = \{1, 2, \ldots, n\}$, we can write $\mathscr{F}\mathscr{P} = S_1 \star S_2 \star \cdots \star S_n$. We say an element $s = (s_1, s_2, \ldots, s_m)$ of $\mathscr{F}\mathscr{P}$ has *length* m and this is denoted by $|s| = m$.

We end this section by giving an alternative approach to define free products of semigroups, using a universal property.

Proposition 2.3.1. *Let $\mathscr{F}\mathscr{P} = \Pi^\star\{S_i\colon i \in I\}$ be the free product of a set $\{S_i\colon i \in I\}$ of disjoint semigroups. Then for each i in I there exists a monomorphism $\theta_i\colon S_i \longrightarrow \mathscr{F}\mathscr{P}$ given by*

$$s_i\theta_i = (s_i), \quad s_i \in S_i.$$

Further, if T is a semigroup and there is a morphism $\psi_i\colon S_i \longrightarrow T$ for each i, then there is a unique morphism $\gamma\colon \mathscr{F}\mathscr{P} \longrightarrow T$ such that the diagram

$$\begin{tikzcd} S_i \arrow[r, "\theta_i"] \arrow[d, "\psi_i"'] & \mathscr{F}\mathscr{P} \arrow[dl, "\gamma"] \\ T & \end{tikzcd}$$

Figure 2.3

commutes for every i in I.

It is clear that for any $s_i \in S_i$, where $i \in I$, the element (s_i) of $\mathscr{F}\mathscr{P}$ has length equal 1.

The property that given in Proposition 2.3.1 is uniquely defines free product as we see in the following result.

Proposition 2.3.2. *Let $\{S_i \colon i \in I\}$ be a family of semigroups, and let H be a semigroup such that*

(i) *there exists a monomorphism $\alpha_i \colon S_i \longrightarrow H$ for each $i \in I$,*

(ii) *if T is a semigroup and if there exists a morphism $\beta_i \colon S_i \longrightarrow T$ for every i in I, then there exists a unique morphism $\delta \colon H \longrightarrow T$ such that the diagram*

$$\begin{tikzcd} S_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_i"'] & H \arrow[dl, "\delta"] \\ T & \end{tikzcd}$$

Figure 2.4

commutes for every i in I.

Then H is isomorphic to $\mathscr{F}\mathscr{P} = \Pi^{\star}\{S_i \colon i \in I\}$.

As we will see later, the free product of the semigroups $\{S_i : i \in I\}$ can defined as a quotient

$$\mathscr{F}\mathscr{P} = X^+/\rho_1,$$

where $X = \bigcup\{S_i : i \in I\}$, and

$$\rho_1 = \langle\{(x \circ y, xy) : x, y \in S_i, i \in I\}\rangle.$$

2.4 Graph products

In this section, we outline the basic definitions and results concerning graphs and graph products of semigroups. These results and definitions are required for this thesis.

2.4.1 Graphs

In this subsection, we define graphs and recall some basic properties associated with graphs which are all essential in this thesis. The results and definitions of graphs are taken from some references, including [65] and [55].

Definition 2.4.1. A *graph* $\Gamma = (V, E)$ is a non-empty set V of *vertices* together with a set E, of 2-element subsets of V, whose elements are called *edges*.

Every element of E contains exactly two elements. For convenience we may write $(\alpha, \beta) \in E$ rather that $\{\alpha, \beta\} \in E$. Notice that in our graphs no multiple edges allowed, that means they have no loops. We called the graph with no loops a *simple* graph. We are identifying (α, β) with (β, α). Throughout this thesis, we assume that V is a finite set, and so is E.

Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are *isomorphic* if there exists a matching between their vertices so that two vertices are connected by an edge in Γ_1 if and only if corresponding vertices are connected by an edge in Γ_2.

A graph $\Gamma' = (V', E')$ is a *subgraph* of $\Gamma = (V, E)$ if

$$V' \subseteq V, \text{ and } E' \subseteq V' \times V' \subseteq E.$$

The subgraph $\Gamma' = (V', E')$ is called a *full subgraph of* Γ if

$$(\alpha, \beta) \in E \Longleftrightarrow (\alpha, \beta) \in E', \text{ for all } \alpha, \beta \in V'.$$

A graph $\Gamma = (V, E)$ is a *discrete graph (null graph)* if E is empty. On the other hand, if $(\alpha, \beta) \in E$ for all $\alpha, \beta \in V$, then the graph Γ is called *complete*. A *connected graph* is a graph $\Gamma = (V, E)$ such that for any $\alpha, \beta \in V$, there is a path which connects α and β. A *path* is a sequence of edges. Therefore, it is clear that any complete graph is connected graph but the converse is not true.

We say a graph $\Gamma = (V, E)$ is *disjoint union of subgraphs* $\Gamma_i = (V_i, E_i)$, $1 \leq i \leq m$, if $V = \dot{\cup} V_i$ and $E = \dot{\cup} E_i$.

2.4.2 Graph products of semigroups

The graph product is an operator mixing direct and free products. Graph products of groups were introduced by Green in 1990 in her thesis [55]. This concept was studied by many authors, such as Hermiller and Meier [60]. The graph product of monoids is defined in the same way as the graph product of groups and has been studied specifically by Veloso da Costa, Fohry and Kuske [13], [12], [38]. Fountain and Kambites in 2008 [42] were able to show that the graph product of cancellative monoids is cancellative.

In this section we define the graph product of *semigroups*, this is a different concept than the graph product of monoids and groups. For this purpose, we introduce some relations.

Let $\Gamma = \Gamma(V, E)$ be a graph. Let S_α be a semigroup for each $\alpha \in V$. We assume that $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha \neq \beta \in V$.

Let $\mathscr{S} = \mathscr{S}(V) = \{S_\alpha : \alpha \in V\}$ and put

$$X = X(\Gamma, \mathscr{S}) = \bigcup_{\alpha \in V} S_\alpha.$$

We write a word in the free semigroup X^+ as $x_1 \circ \ldots \circ x_n$ and also use $\circ$ for the operation in X^+.

We define some congruences on the free semigroup X^+,

$$\rho = \rho(\Gamma, \mathscr{S}) = \langle H \rangle$$

and

$$H = H(\Gamma, \mathscr{S}) = H_1 \cup H_2,$$

where

$$H_1 = H_1(\Gamma, \mathscr{S}) = \{(x \circ y, xy) : x, y \in S_\alpha, \alpha \in V\}$$

and
$$H_2 = H_2(\Gamma, \mathscr{S}) = \{(x \circ y, y \circ x) : x \in S_\alpha, y \in S_\beta, (\alpha, \beta) \in E\}.$$
We also let $\rho_1 = \langle H_1 \rangle$. It is clear from the above notations that $\mathscr{S}$, X, H_1 and ρ_1 depend only on V.

Definition 2.4.2. Let $\Gamma = (V, E)$ be a graph and let $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ be a set of semigroups. The *(semigroup) graph product* $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$ of $\mathcal{S}$ with respect to Γ is defined by
$$\mathscr{GP} = X^+/\rho.$$

We say the graph Γ is *the underlying of* $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$. Throughout this thesis, as we only consider semigroup graph products, we will drop (semigroup) and just say graph products. The semigroups S_α are called the *components* of $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$ or, the *vertex semigroups*, as we will show a component S_α, $\alpha \in V$ embeds naturally in $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$.

Note that if $V = \{\alpha\}$, is a singleton, then the graph product $\mathscr{GP}$ is isomorphic to S_α. It is also worth remarking that if the semigroups S_α are all monoids, then similarly to the case for free products, unless V is a singleton, the resulting graph product is not a monoid, since the identities remain distinct.

If Γ is null, then the graph product of the semigroups $\{S_\alpha : \alpha \in V\}$ is exactly the free product $\Pi^\star S_\alpha$ of the given semigroups and we write $\mathscr{FP} = \mathscr{FP}(\Gamma, \mathscr{S})$. However, if the underlying graph is complete, then the graph product is the direct product of the given semigroups.

Note 2.4.3. *Let* $H_2' = H_2'(\Gamma, \mathscr{S}) = \{(x\rho_1, y\rho_1) : (x, y) \in H_2\}$ *and* $\rho_2' = \langle H_2' \rangle$. *We have that* $\mathscr{GP} \simeq \mathscr{FP}/\langle \rho_2' \rangle$.

We use the convention
$$[x_1 \circ \ldots \circ x_n] = x_1 * \ldots * x_n$$
and $*$ for the multiplication in the graph product, so that
$$[x_1 \circ \ldots \circ x_n] * [y_1 \circ \ldots \circ y_m] = [x_1 \circ \ldots \circ x_n \circ y_1 \circ \ldots \circ y_m]$$
can be written as
$$(x_1 * \ldots * x_n) * (y_1 \ldots * y_m) = x_1 * \ldots * x_n * y_1 * \ldots y_m.$$

For any element $w = x_1 \circ \ldots \circ x_n \in X^+$, we say x_i and x_{i+1} can be *squashed* in w if $x_i, x_{i+1} \in S_k$, for some k. Then we can write as $x_1 \circ x_i x_{i+1} \circ \ldots \circ x_n$.

We say a word $x_1 \circ \ldots \circ x_n \in X^+$ is a *reduced form* for $[w] \in \mathscr{F}\mathscr{P}$ if $[w] = [x_1 \circ \ldots \circ x_n]$ and $\sigma(x_r) \neq \sigma(x_{r+1})$, for all $1 \leq r \leq n-1$, that means no more squashed can do. It is obvious that the reduced form of $[w]$ is the shortest length and unique form as $\big(\sigma(x_r), \sigma(x_{r+1})\big) \notin E$ for $1 \leq r \leq n-1$. In Chapter 7, we will generalize this concept and define this form of $[w] \in \mathscr{G}\mathscr{P}$ also we will give more special forms of the elements of the graph product.

Example 2.4.4. Let $\mathscr{G}\mathscr{P} = \mathscr{G}\mathscr{P}(\Gamma, \mathscr{S})$ be a graph product underlying the graph $\Gamma = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6\}$, and

$$E = \{(1,2), (1,3), (1,4), (2,3), (2,4), (2,5), (3,4), (4,5), (5,6)\}.$$

The graph Γ is given by the following figure:

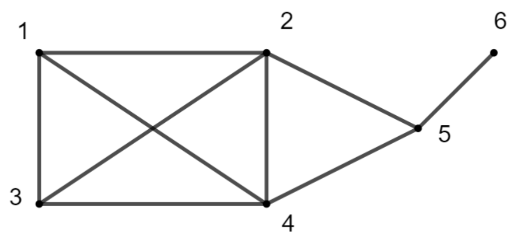


Figure 2.5

Let $u, v \in S_1$, $w, x \in S_2$, $y \in S_3$, $z \in S_4$, $s \in S_5$ and $t \in S_6$. Let

$$[w \circ y \circ z \circ v \circ x \circ u \circ s \circ t] \in \mathscr{G}\mathscr{P}.$$

We write this element in a reduced form as follows

$$\begin{aligned}
[w \circ y \circ z \circ v \circ x \circ u \circ s \circ t] &= [w \circ y \circ z \circ x \circ v \circ u \circ s \circ t] && (\text{as } (2,1) \in E,\ x \circ v = v \circ x\ ) \\
&= [w \circ y \circ x \circ z \circ v \circ u \circ s \circ t] && (\text{as } (2,4) \in E,\ x \circ z = z \circ x\ ) \\
&= [w \circ x \circ y \circ z \circ v \circ u \circ s \circ t] && (\text{as } (2,3) \in E,\ x \circ y = y \circ x\ ) \\
&= [(w \circ x) \circ y \circ z \circ (v \circ u) \circ s \circ t] && \\
&= [(wx) \circ y \circ z \circ (v \circ u) \circ s \circ t] && (\text{as } x,\ w \in S_2) \\
&= [wx \circ y \circ z \circ vu \circ s \circ t] && (\text{as } u,\ v \in S_1).
\end{aligned}$$

The above form is the reduced form as there are no more elements that can be squashed.

We will prove in Chapter 7 that the reduced form of any word w of X^+ is the shortest form of $[w]$ in $\mathscr{GP}$.

The graph product of some semigroups in the following example can be written as free product of direct products.

Example 2.4.5. Let $\Gamma_1 = (V_1, E_1)$, $V_1 = \{1,2,3\}$, $E_1 = \{(1,2),(2,3),(1,3)\}$, $\Gamma_2 = (V_2, E_2)$, $V_2 = \{4,5\}$, $E_2 = \{(4,5)\}$, and $\Gamma_3 = (V_3, E_3)$ where $V_3 = \{6,7\}$, $E_3 = \{(6,7)\}$. Let $\mathscr{GP}_1$, $\mathscr{GP}_2$ and $\mathscr{GP}_3$ be the graph products of semigroup underlying the graphs Γ_1, Γ_2, Γ_3, respectively. Then the graph product of semigroup $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$ which corresponds to the graph $\Gamma = (V, E)$, where

$$V = V_1 \cup V_2 \cup V_3,\ E = E_1 \cup E_2 \cup E_3,$$

is given by the following figure,

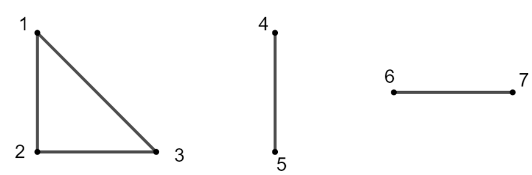


Figure 2.6

which can be written as

$$(\mathscr{GP}_1) * (\mathscr{GP}_2) * (\mathscr{GP}_3).$$

This means that the above graph product is the free product of the graph products $\mathscr{GP}_1$, $\mathscr{GP}_2$ and $\mathscr{GP}_3$.

Chapter 3

Preliminaries III: (Weakly) abundant semigroups

In this chapter, we introduce two sets of binary relations, as analogues of the well known Green's relations. The notion of a (weakly) abundant semigroup is introduced using these relations. More details related to the content of this chapter can be found in [41], [39], [40] and [68].

3.1 Green's $*$-relations and abundant semigroups

Let S be a semigroup and $E = E(S)$ be the set of all idempotents of S. Fountain in [40] defined the relation $\mathcal{L}^*$ on S by the rule that for any $a, b \in S$, $a \,\mathcal{L}^*\, b$ if and only if $a \,\mathcal{L}\, b$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. By the definitions of $\mathcal{L}^*$ and $\mathcal{R}^*$, it is clear that

$$\mathcal{L} \subseteq \mathcal{L}^* \text{ and } \mathcal{R} \subseteq \mathcal{R}^*.$$

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations on S. Moreover, $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. The $\mathcal{D}^*$ relation is the join of $\mathcal{L}^*$ and $\mathcal{R}^*$, while their intersection is denoted by $\mathcal{H}^*$. It is worth pointing out that $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$, [40, Example 1.11]. For details about the relations $\mathcal{H}^*$, $\mathcal{D}^*$ and $\mathcal{J}^*$ we refer the reader to [40]. We denote the $\mathcal{L}^*$-class, $\mathcal{R}^*$-class, $\mathcal{D}^*$-class and $\mathcal{H}^*$-class of an element a of S by L_a^*, R_a^*, D_a^* and H_a^*, respectively.

Let a, b be regular elements of S. Then $a \,\mathcal{L}^*\, b$ if and only if $a \,\mathcal{L}\, b$, dually for the relation $\mathcal{R}^*$. Hence if S is a regular semigroup, we get that $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$ [40].

In 1976, Fountain [41] introduced another characterization of $\mathcal{L}^*$, given as follows.

Lemma 3.1.1. *Let S be a semigroup with $a, b \in S$. Then the following conditions are equivalent:*

(i) $a \, \mathcal{L}^* \, b$;

(ii) for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

The next result is a useful consequence of the above lemma.

Lemma 3.1.2. *[40] Let S be a semigroup with $a \in S$ and $e \in E$. Then the following statements are equivalent:*

(i) $a \, \mathcal{L}^* \, e$;

(ii) $ae = a$ and for any $x, y \in S^1$, $ax = ay$ implies $ex = ey$.

By the above lemma it is clear that an idempotent $e \in E$ acts as a right identity within its $\mathcal{L}^*$-class L_e^*. Both the above results have duals for $\mathcal{R}^*$.

It is well known that a semigroup S is regular if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains an idempotent of S. Corresponding to the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, we introduce the concept of an *abundant semigroup*, as a generalization of the concept of a regular semigroup [40].

Definition 3.1.3. (i) A semigroup S is *right abundant* if each $\mathcal{L}^*$-class of S contains an idempotent of S;

(ii) S is *left abundant* if each $\mathcal{R}^*$-class of S contains an idempotent of S;

(iii) S is *abundant* if it is both left and right abundant.

Note that a semigroup may be left but not right abundant, as it is easily seen by considering a right but not left cancellative monoid. We say a semigroup S is *abundant* if each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class contain an idempotent, unlike the case of $\mathcal{L}$ and $\mathcal{R}$ in a regular semigroup, where if every $\mathcal{L}$-class of a semigroup S contains an idempotent, then so does every $\mathcal{R}$-class of S and vice versa. It is not the case for abundance.

We say that an element a in S is *abundant* if $e\,\mathcal{R}^*\,a\,\mathcal{L}^*\,f$ for some $e, f \in E(S)$. It is clear that any regular element is abundant. It is obvious that a semigroup is abundant if every element of S is abundant.

In view of the definition of abundant semigroups, regular semigroups are abundant, whereas not all abundant semigroups are regular. For example, any cancellative monoid is abundant but not necessary regular. Moreover, a semilattice of abundant semigroups need not be abundant, [40, Example 1.3].

3.2 Green's $\sim$-relations and weakly abundant semigroups

In this section we present a third set of relations. These are $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$, which were introduced by Lawson [68]. These relations extend the starred versions of Green's relations.

For any $a, b \in S$ we define the relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ by

$$a\,\widetilde{\mathcal{L}}\,b \iff (\forall e \in E)(ae = a \iff be = b)$$

and

$$a\,\widetilde{\mathcal{R}}\,b \iff (\forall e \in E)(ea = a \iff eb = b).$$

For details about the associating relations $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{J}}$, we refer the reader to [68]. It follows from the definitions of $\mathcal{L}$, $\mathcal{L}^*$, $\widetilde{\mathcal{L}}$, $\mathcal{R}$, $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ that

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$$

and

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}.$$

If S is abundant then $\widetilde{\mathcal{L}} = \mathcal{L}^*$ and $\widetilde{\mathcal{R}} = \mathcal{R}^*$.

We end this section with some results which will be used later in this thesis. The following results follow immediately from the definition of $\widetilde{\mathcal{L}}$. Both the following results have duals for $\widetilde{\mathcal{R}}$.

Lemma 3.2.1. *[68] Let S be a semigroup with $a \in S$ and $e \in E$. Then the following statements are equivalent:*

(i) $a\,\widetilde{\mathcal{L}}\,e$*;*

(ii) $ae = a$ *and for any* $f \in E$*,* $af = a$ *implies* $ef = e$*.*

An easy observation for the above result gives the following useful lemma.

Lemma 3.2.2. *Let* S *be a semigroup with* $e, f \in E$*. Then* $e\,\mathcal{L}\,f$ *if and only if* $e\,\widetilde{\mathcal{L}}\,f$*.*

From the above result it is clear that if S is regular, then $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R} = \mathcal{R}^* = \widetilde{\mathcal{R}}$.

Corollary 3.2.3. *Let* S *be a semigroup and let* $a \in S$*,* $f \in E$ *be such that* $a\,\widetilde{\mathcal{L}}\,f$ *but* a *is not* $\mathcal{L}^*$*-related to* f*. Then* a *is not* $\mathcal{L}^*$*-related to any idempotent of* S*.*

Proof. Let $a\,\widetilde{\mathcal{L}}\,f$ and a be not $\mathcal{L}^*$-related to f. Suppose that $a\,\mathcal{L}^*\,e$ for some idempotent $e \in E$. Then $a\,\widetilde{\mathcal{L}}\,e$, as $\mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$. By assumption we get that $e\,\widetilde{\mathcal{L}}\,f$, and by Lemma 3.2.2 it is clear that $e\,\mathcal{L}\,f$. As $\mathcal{L} \subseteq \mathcal{L}^*$, we get $a\,\mathcal{L}^*\,f$, a contradiction. $\square$

An abundant semigroup is generalization of a regular semigroup in terms of the relations $\mathcal{R}^*$ and $\mathcal{L}^*$. Similarly, we define a weakly abundant semigroup, which is generalization of an abundant semigroup in terms of the relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$.

Definition 3.2.4. (i) A semigroup S is *right weakly abundant* if each $\widetilde{\mathcal{L}}$-class of S contains an idempotent of S;

(ii) S is *left weakly abundant* if each $\widetilde{\mathcal{R}}$-class of S contains an idempotent of S;

(iii) S is *weakly abundant* if it is both left and right weakly abundant.

It is known that the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on S are always left and right congruences, respectively. However, this is not always true for the relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$. Hence we say that a weakly abundant semigroup S satisfies the *congruence condition* if $\widetilde{\mathcal{R}}$ is a left congruence and $\widetilde{\mathcal{L}}$ is a right congruence on S. Therefore, any regular or abundant semigroup satisfies the congruence condition, but not all weakly abundant semigroups satisfy this condition, [80, Example 2.2.4]. It is clear that if S is a semigroup with no idempotents, then S is not weakly abundant.

Yang in [80, Example 2.2.4.], gives an example show that $\widetilde{\mathcal{R}} \neq \mathcal{R}^*$.

We end this section by the following result which is taken from [16].

Lemma 3.2.5. *Let* S *be a weakly abundant semigroup with* $a \in S$ *and* $e \in E$ *such that* $a\,\widetilde{\mathcal{L}}\,e$*. Then* $a\,\mathcal{L}^*\,e$ *if and only if for any* $x, y \in S$*,* $ax = ay$ *implies that* $ex = ey$*.*

Chapter 4

Abundancy and weak abundancy of external direct and free products

This chapter aims to prove some straightforward results that will be frequently used in the whole thesis. In the first section, we show the abundance of external direct products of abundant semigroups. In the second section, we show the abundance of free products of abundant semigroups.

4.1 Abundancy of external direct products

In this section, we show that the external direct product of finitely many abundant semigroups is always an abundant semigroup. Moreover, the external direct product of finitely many weakly abundant semigroups is a weakly abundant semigroup. These results are used in the proof of our main theorems in this thesis.

Let S and T be semigroups. Let (e, f) be an element of the external direct product $E(S) \times E(T)$. Then

$$(e, f)^2 = (e, f)(e, f) = (e^2, f^2) = (e, f).$$

Hence $(e, f) \in E(S \times T)$. Let $(x, y) \in E(S \times T)$, then

$$(x, y)^2 = (x, y)(x, y) = (x^2, y^2) = (x, y).$$

Hence $x \in E(S)$ and $y \in E(T)$. This proves that

$$E(S \times T) = E(S) \times E(T).$$

Note that for any semigroups S and T that are not monoids, it is clear that $S^1 \times T^1 \neq (S \times T)^1$, as $(1_S, 1_T) \in S^1 \times T^1$, where 1_S is the identity adjoined to S, and 1_T is the identity adjoined to T, but $(1_S, 1_T) \notin (S \times T)^1$. However, if S and T are monoids, then so is $S \times T$ with identity $(1_S, 1_T)$, where 1_S and 1_T are the identities of S and T, respectively, and indeed $S \times T = S^1 \times T^1 = (S \times T)^1$.

Lemma 4.1.1. *Let S and T be semigroups. Then the external direct product, $S \times T$, of S and T is a left abundant semigroup if and only if S and T are left abundant semigroups.*

Proof. Let S and T be left abundant semigroups. Let (s,t) be an element of $S \times T$. It is enough to prove that (s,t) is $\mathcal{R}^*$-related to an idempotent of $E(S \times T)$.

Since S and T are left abundant semigroups, we have

$$s\,\mathcal{R}^*\,e \text{ in } S, \text{ for some } e \in E(S),$$

and

$$t\,\mathcal{R}^*\,f \text{ in } T, \text{ for some } f \in E(T).$$

This means $es = s$ and the equality $xs = x's$ implies that $xe = x'e$ for any $x, x' \in S^1$. Also, $ft = t$ and the equality $yt = y't$ implies that $yf = y'f$ for any $y, y' \in T^1$. Then it is clear that

$$(e,f)(s,t) = (es, ft) = (s,t).$$

Our aim is to show that $(s,t)\,\mathcal{R}^*\,(e,f)$ in $S \times T$. Let $(x,y),\ (x',y') \in S \times T$. Suppose $(x,y)(s,t) = (x',y')(s,t)$, which implies

$$xs = x's \ \text{ and } \ yt = y't.$$

As $s\,\mathcal{R}^*\,e$ and $t\,\mathcal{R}^*\,f$, $xe = x'e$ and $yf = y'f$ so that $(x,y)(e,f) = (x',y')(e,f)$.

Let $(x,y) \in S \times T$, and suppose that $(x,y)(s,t) = (s,t)$. Then $xs = s$ and $yt = t$, so that $xe = e$ in S and $yf = f$ in T, and so $(x,y)(e,f) = (e,f)$. Therefore, by Lemma 3.1.2, we get that $(s,t)\,\mathcal{R}^*\,(e,f)$.

Conversely, let $S \times T$ be left abundant. Our aim is to show that S is left abundant. Fix $t \in T$, let $s \in S$ and choose $(e,f) \in E(S) \times E(T)$ such that $(s,t)\,\mathcal{R}^*\,(e,f)$. Then we get that $(e,f)(s,t) = (s,t)$, which implies $es = s$ and $ft = t$. Now let $x,\ y \in S$ be such that $xs = ys$. Then $(x,f)(s,t) = (y,f)(s,t)$, which implies $(x,f)(e,f) = (y,f)(e,f)$, hence $xe = ye$.

Suppose now that $xs = s$ for some $x \in S$. Then $(x, f)(s, t) = (s, t)$, this implies that $(x, f)(e, f) = (e, f)$. Hence $xe = e$, this proves that $s\,\mathcal{R}^*\,e$ in S. Therefore, S is a left abundant semigroup. Similarly, we prove that T is a left abundant semigroup. $\square$

As a generalisation of the above lemma we have the following result.

Corollary 4.1.2. *Let $S_1, \ldots, S_n$ be semigroups. Then the direct product,*

$$S_1 \times \ldots \times S_n,$$

of $S_1, \ldots, S_n$ is a left abundant semigroup if and only if $S_1, \ldots, S_n$ are left abundant semigroups.

It is clear that left-right dual of Lemma 4.1.1 and Corollary 4.1.2, and hence we have the following result.

Lemma 4.1.3. *Let S and T be semigroups. Then the external direct product, $S \times T$, of S and T is an abundant semigroup if and only if S and T are abundant semigroups.*

As a generalisation of the above Lemma 4.1.3 we have the next result.

Corollary 4.1.4. *Let $S_1, \ldots, S_n$ be semigroups. Then the direct product,*

$$S_1 \times \ldots \times S_n,$$

of $S_1, \ldots, S_n$ is an abundant semigroup if and only if $S_1, \ldots, S_n$ are abundant semigroups.

In the following result we show that the external direct product of any two semigroups is left weakly abundant if and only if these semigroups are left weakly abundant.

Lemma 4.1.5. *Let S and T be semigroups. Then S and T are left weakly abundant semigroups if and only if the external direct product, $S \times T$, is a left weakly abundant semigroup.*

Proof. Let S and T be left weakly abundant semigroups. Let $(s, t) \in S \times T$. It is enough to show that (s, t) is $\widetilde{\mathcal{R}}$-related to some idempotent of $E(S \times T)$. For any $s \in S$ there is some $e \in E(S)$ such that

$$s\,\widetilde{\mathcal{R}}\,e \text{ in } S,$$

and for any $t \in T$ there is some $f \in E(T)$ such that

$$t\,\widetilde{\mathcal{R}}\,f \text{ in } T.$$

The above imply $es = s$ and for any $e' \in E(S)$, the equality $e's = s$ implies $e'e = e$. Also, $ft = t$ and for any $f' \in E(T)$, the equality $f't = t$ implies $f'f = f$. Hence it is clear that $(e, f)(s, t) = (es, ft) = (s, t)$, and for any $(e', f') \in E(S \times T)$ such that

$$(e', f')(s, t) = (e's, f't) = (s, t),$$

we get that $(e', f')(e, f) = (e, f)$ in $S \times T$. Therefore, $(s, t)\,\widetilde{\mathcal{R}}\,(e, f)$ as required.

Conversely, let $S \times T$ be a left weakly abundant semigroup. Our aim is to show that S is a left weakly abundant semigroup. Fix $t \in T$, let $s \in S$ such that $(s, t)\,\widetilde{\mathcal{R}}\,(e, f)$, where $(e, f) \in E(S \times T)$. Hence we get that $(e, f)(s, t) = (s, t)$, this implies $es = s$ and $ft = t$. Let $e' \in E(S)$ such that $e's = s$. Then $(e', f)(s, t) = (s, t)$. As $(s, t)\,\widetilde{\mathcal{R}}\,(e, f)$ and $(e', f) \in E(S \times T)$, we get that $(e', f)(e, f) = (e, f)$, this implies $e'e = e$. Then $s\,\widetilde{\mathcal{R}}\,e$. Therefore, S is a left weakly abundant semigroup. Similarly, we prove that T is left weakly abundant. $\square$

As a generalisation of the above lemma we have the following result.

Corollary 4.1.6. *Let $S_1, \dots, S_n$ be semigroups. Then the external direct product, $S_1 \times \dots \times S_n$, of $S_1, \dots, S_n$ is a left weakly abundant semigroup if and only if $S_1, \dots, S_n$ are left weakly abundant semigroups.*

It is clear that left-right dual of Lemma 4.1.5 and Corollary 4.1.6 and hence we have the following result.

Lemma 4.1.7. *Let S and T be semigroups. Then S and T are weakly abundant semigroups if and only if the external direct product, $S \times T$, is a weakly abundant semigroup.*

As a generalisation of the Lemma 4.1.7 we have the next result.

Corollary 4.1.8. *Let $S_1, \dots, S_n$ be semigroups. Then the external direct product, $S_1 \times \dots \times S_n$, of $S_1, \dots, S_n$ is a weakly abundant if and only if $S_1, \dots, S_n$ are weakly abundant.*

4.2 Abundancy of free products

The aim of this section is to show that the free product of abundant semigroups is abundant. Further, we show that the free product of weakly abundant semigroups is also weakly abundant.

We start this section with the following result that gives the characterisation of the idempotents in the free product, $\mathscr{F}\mathscr{P} = \Pi^{\star}\{S_i : i \in I\}$, of a set of pairwise disjoint semigroups $\{S_i : i \in I\}$.

Lemma 4.2.1. *Let $\{S_i : i \in I\}$ be a set of pairwise disjoint semigroups. Let $[e]$ be an element of the free product $\mathscr{F}\mathscr{P} = \Pi^{\star}\{S_i : i \in I\}$. Then $[e] \in E(\mathscr{F}\mathscr{P})$ if and only if $e \in E(S_i)$ for some $i \in I$.*

Proof. Let $e \in E(S_i)$ for some $i \in I$. Then $[e] \in \mathscr{F}\mathscr{P}$. As e is an idempotent, we get that $[e] \in E(\mathscr{F}\mathscr{P})$.

Conversely, let $[e] \in E(\mathscr{F}\mathscr{P})$. If $|e| \geq 2$ and $[e]$ ends in an element of S_i and begins with an element of S_j where $i \neq j$, then $e \circ e$ is reduced, and $|e \circ e| = 2|e| > |e|$, a contradiction, as $[e] = [e \circ e]$. If $|e| > 2$ and $[e]$ has form $[e_1 \circ \ldots \circ e_n]$ where $e_1, e_n \in S_i$, then if f is a reduced form of $e \circ e$ we have $|f| = 2|e| - 1 > |e|$, a contradiction. Therefore, the length of e must equal 1, that implies $e \in E(S_i)$ for some $i \in I$.

$\square$

Lemma 4.2.2. *Let $\{S_i : i \in I\}$ be a set of pairwise disjoint semigroups. Then the free product of these semigroups, $\mathscr{F}\mathscr{P} = \Pi^{\star}\{S_i : i \in I\}$, is left abundant if and only if S_i is left abundant for all $i \in I$.*

Proof. Suppose that S_i is left abundant for all $i \in I$. Our aim is to prove that the free product of these semigroups, $\mathscr{F}\mathscr{P} = \Pi^{\star}\{S_i : i \in I\}$, is left abundant. Let w be a reduced form of $[w] \in \mathscr{F}\mathscr{P}$, where w begins with an element $s_1 \in S_i$ for some $i \in I$, $s_j \in S_j$, for all $2 \leq j \leq n$ and w has the form

$$w = s_1 \circ s_2 \circ \ldots \circ s_n.$$

As S_i is a left abundant semigroup, there is some $e \in E(S_i)$ such that $e \,\mathcal{R}^* \, s_1$. Let $x = x_1 \circ \ldots \circ x_m$ and $y = y_1 \circ \ldots \circ y_p$ be reduced forms of the elements $[x]$ and $[y]$ of $\mathscr{F}\mathscr{P}$, respectively. Suppose that $[x] \star [w] = [y] \star [w]$. Our aim is to prove that $[x] \star [e] = [y] \star [e]$. We check the possible cases:

1. Let $[x]$ and $[y]$ end in elements x_m, y_p of S_i, which implies

   $$[x_1 \circ \ldots \circ x_{m-1} \circ x_m] \star [s_1 \circ s_2 \circ \ldots \circ s_n] = [y_1 \circ \ldots \circ y_{p-1} \circ y_p] \star [s_1 \circ s_2 \circ \ldots \circ s_n],$$

   giving

   $$[x_1 \circ \ldots \circ x_{m-1} \circ x_m s_1 \circ s_2 \circ \ldots \circ s_n] = [y_1 \circ \ldots \circ y_{p-1} \circ y_p s_1 \circ s_2 \circ \ldots \circ s_n].$$

Hence we get that

$$x_1 \circ \ldots \circ x_{m-1} = y_1 \circ \ldots \circ y_{p-1}, \tag{4.1}$$

and

$$x_m\ s_1 = y_p\ s_1.$$

As $s_1\, \mathcal{R}^*\, e$, $x_m, y_p \in S_i$ and $x_m\ s_1 = y_p\ s_1$, we get that $x_m e = y_p e$. Now we have

$$\begin{aligned}
[x] \star [e] &= [x_1 \circ \ldots \circ x_{m-1} \circ x_m] \star [e] \\
&= [x_1 \circ \ldots \circ x_{m-1} \circ x_m e] \\
&= [y_1 \circ \ldots \circ y_{p-1} \circ y_p e] && \text{(by 4.1, and as } x_m e = y_p e\text{)} \\
&= [y] \star [e].
\end{aligned}$$

Therefore, in this case we get that $[x] \star [e] = [y] \star [e]$.

2. Let $[x]$ end in an element x_m of S_h for $h \neq i$ and $[y]$ end in y_p of S_k, for $k \neq i$, which implies

$$[x_1 \circ \ldots \circ x_m] \star [s_1 \circ s_2 \circ \ldots \circ s_n] = [y_1 \circ \ldots \circ y_p] \star [s_1 \circ s_2 \circ \ldots \circ s_n],$$

giving

$$[x_1 \circ \ldots \circ x_{m-1} \circ x_m \circ s_1 \circ s_2 \circ \ldots \circ s_n] = [y_1 \circ \ldots \circ y_{p-1} \circ y_p \circ s_1 \circ s_2 \circ \ldots \circ s_n].$$

Hence we get that

$$x_1 \circ \ldots \circ x_{m-1} \circ x_m = y_1 \circ \ldots \circ y_{p-1} \circ y_p, \tag{4.2}$$

which implies that $h = k$, and $[x] = [y]$. Therefore, $[x] \star [e] = [y] \star [e]$.

3. Let $[x]$ end in an element x_m of S_i and $[y]$ ends in $y_p \in S_k$, for $k \neq i$, which implies

$$[x_1 \circ \ldots \circ x_{m-1} \circ x_m] \star [s_1 \circ s_2 \circ \ldots \circ s_n] = [y_1 \circ \ldots \circ y_p] \star [s_1 \circ s_2 \circ \ldots \circ s_n],$$

giving

$$[x_1 \circ \ldots \circ x_{m-1} \circ x_m s_1 \circ s_2 \circ \ldots \circ s_n] = [y_1 \circ \ldots \circ y_p \circ s_1 \circ s_2 \circ \ldots \circ s_n].$$

Hence we have

$$x_1 \circ \ldots \circ x_{m-1} \circ x_m s_1 = y_1 \circ \ldots \circ y_p \circ s_1,$$

which implies

$$x_1 \circ \ldots \circ x_{m-1} = y_1 \circ \ldots \circ y_p, \tag{4.3}$$

and

$$x_m\ s_1 = s_1.$$

As $s_1\,\mathcal{R}^*\,e$, $x_m \in S_i$ and $x_m\ s_1 = s_1$, we have $x_m e = e$. Therefore,

$$\begin{aligned}[x] \star [e] &= [x_1 \circ \ldots \circ x_{m-1} \circ x_m] \star [e] \\ &= [x_1 \circ \ldots \circ x_{m-1} \circ x_m e] \\ &= [y_1 \circ \ldots \circ y_p \circ e] \qquad \text{(by 4.3, and as } x_m e = e) \\ &= [y] \star [e].\end{aligned}$$

4. Let $[x]$ end in an element x_m of S_i, so we get that

$$[x_1 \circ \ldots \circ x_{m-1} \circ x_m s_1 \circ s_2 \circ \ldots \circ s_n] = [s_1 \circ s_2 \circ \ldots \circ s_n],$$

giving

$$x_1 \circ \ldots \circ x_{m-1} \circ x_m s_1 = s_1.$$

This implies that x must be an element of length 1, $x = x_m$ and $x_m s_1 = s_1$. It is clear that $x_m e = e$. Hence $[x] \star [e] = [e]$.

Note that we have one remaining case that is impossible.

5. Let $[x]$ end in an element x_m of S_k and $k \neq i$, so we get that

$$[x_1 \circ \ldots \circ x_{m-1} \circ x_m \circ s_1 \circ s_2 \circ \ldots \circ s_n] = [s_1 \circ s_2 \circ \ldots \circ s_n].$$

This case is impossible as w is in a reduced form of $[w]$.

From the above discussion, we can deduce that $[w]\,\mathcal{R}^*\,[e]$. Note that in the four above cases we focused on the beginning letter of $[w]$ and the ending letter of $[x]$ and $[y]$, as the rest of these words never affect our work. Moreover, $[x]$ starts with an element of S_i, $i \in I$ if and only if $[y]$ does.

Conversely, let $\mathscr{F}\mathscr{P}$ be left abundant. Our aim is to show that S_i for all $i \in I$ is left abundant. Let $s \in S_i$ for some $i \in I$, then $[s] \in \mathscr{F}\mathscr{P}$. As $\mathscr{F}\mathscr{P}$ is left abundant, there is some idempotent $[e]$ of $\mathscr{F}\mathscr{P}$ such that $[s]\,\mathcal{R}^*\,[e]$. As $[s]\,\mathcal{R}^*\,[e]$, we get $[e] \star [s] = [s]$ which implies that $[e \circ s] = [s]$. By Lemma 4.2.1, we get that $e \in E(S_i)$ and $es = s$.
Now let $x,\, y \in S_i^1$ such that $xs = ys$. Hence $[x],\, [y] \in \mathscr{F}\mathscr{P}$ and

$$[xs] = [x] \star [s] = [y] \star [s] = [ys].$$

As $[s]\,\mathcal{R}^*\,[e]$, and $s, e \in S_i$, $[x] \star [e] = [xe] = [y] \star [e] = [ye]$. This implies that $xe = ye$, as required. Therefore, S_i, $i \in I$ is a left abundant semigroup.

$\square$

It is clear that left-right dual of the above lemma where we show that $[w]\,\mathcal{L}^*\,[e]$, where $w_n\,\mathcal{L}^*\,e$ in S_n. Hence we get the following result.

Lemma 4.2.3. *Let $\{S_i \colon i \in I\}$ be a set of pairwise disjoint semigroups. Then the free product of these semigroups, $\mathscr{F}\mathscr{P} = \Pi^\star\{S_i \colon i \in I\}$, is abundant if and only if S_i is abundant for all $i \in I$.*

It is worth pointing out that the corresponding result does not hold for regular semigroups, for example the free semigroup product of two trivial groups (which is a free idempotent generated semigroup) is not always regular.

In the following result we show that the free product of finitely many pairwise disjoint semigroups is weakly abundant if and only if the semigroups $\{S_i \colon i \in I\}$ are weakly abundant.

Lemma 4.2.4. *Let $\{S_i \colon i \in I\}$ be a set of pairwise disjoint semigroups. Then the free product of these semigroups, $\mathscr{F}\mathscr{P} = \Pi^\star\{S_i \colon i \in I\}$, is left weakly abundant if and only if S_i is left weakly abundant for all $i \in I$.*

Proof. Let S_i be left weakly abundant semigroup for all $i \in I$. Let $w = w_1 \circ w_2 \circ \ldots \circ w_n$ be a reduced form of $[w] \in \mathscr{F}\mathscr{P}$, where $w_i \in S_i$, for all $1 \leq i \leq n$. We aim to show that $[w]$ is $\widetilde{\mathcal{R}}$-related to some idempotent of $E(\mathscr{F}\mathscr{P})$. As S_i is left weakly abundant for all $1 \leq i \leq n$, there is some $e \in E(S_1)$ such that $e\,\widetilde{\mathcal{R}}\,w_1$. It is clear that

$$[e] \star [w] = [e] \star [w_1 \circ w_2 \circ \ldots \circ w_n] = [ew_1 \circ w_2 \circ \ldots \circ w_n] = [w_1 \circ w_2 \circ \ldots \circ w_n] = [w].$$

Let $[f] \in E(\mathscr{F}\mathscr{P})$ such that $[f] \star [w] = [w]$. By Lemma 4.2.1, we get that $f \in E(S_i)$ for some $i \in I$. As $[w]$ begins with an element $w_1 \in S_1$, the equality $[f] \star [w] = [w]$ happens only if $f \in E(S_1)$ and $fw_1 = w_1$. This implies that $[f] \star [e] = [e]$. Hence $[w]\,\widetilde{\mathcal{R}}\,[e]$.

Conversely, Let the free product of the semigroups $\{S_i \colon i \in I\}$, $\mathscr{F}\mathscr{P}$, be left weakly abundant. Let $s \in S_i$ for some $i \in I$. We aim to show that s is $\widetilde{\mathcal{R}}$-related to some idempotent of $E(S_i)$. As $\mathscr{F}\mathscr{P}$ is left weakly abundant and $[s] \in \mathscr{F}\mathscr{P}$, then $[s]\,\widetilde{\mathcal{R}}\,[e]$ for some $[e] \in E(\mathscr{F}\mathscr{P})$, which implies that $[e] \star [s] = [s]$. This equality happens only if $e \in E(S_i)$ and $es = s$ in S_i.

Moreover, for any $[f] \in E(\mathscr{F}\mathscr{P})$ such that $[f] \star [s] = [s]$, then we get that $[f] \star [e] = [e]$. It is clear that $f \in E(S_i)$, then we get $fe = e$ in S_i. Therefore, we get that $s \, \widetilde{\mathcal{R}} \, e$ in S_i, for any $i \in I$. Therefore, S_i is a left weakly abundant semigroup for all $i \in I$.

Dually, we prove that $[w] \, \widetilde{\mathcal{L}} \, [e]$, where $e \in E(S_n)$ and $e \, \widetilde{\mathcal{L}} \, w_n$.

$\square$

We remark here that a dual result holds for right weakly abundant semigroups. Hence we get the following result.

Lemma 4.2.5. *Let $\{S_i \colon i \in I\}$ be a set of pairwise disjoint semigroups. Then the free product of these semigroups, $\mathscr{F}\mathscr{P} = \Pi^{\star}\{S_i \colon i \in I\}$, is weakly abundant if and only if S_i is weakly abundant for all $i \in I$.*

Chapter 5

Free idempotent generated semigroups

Nambooripad in the early 1970s began the study of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$ over a biordered set $\mathcal{E}$ in his seminal work [72]. In 2014, Gould and Yang [80] investigated the general structure of the free idempotent generated semigroup. In 2015, they investigated the structure of $\mathrm{IG}(\mathcal{B})$ over a band B. They showed that the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$, over a biordered set with trivial products, is always abundant. This chapter aims to present all the results and properties of $\mathrm{IG}(\mathcal{E})$, which will be frequently used in our work.

This chapter is divided into three sections. In Section 5.1, we define a biordered set $\mathcal{E}$, which is the generating set of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$. In Section 5.2, we define idempotent generated semigroups and present the importance of these semigroups. We divide Section 5.3 into four subsections. In the first subsection, we define the free idempotent generated semigroups and present properties of the general structure of $\mathrm{IG}(\mathcal{E})$ with respect to Green's relations. In Subsections 5.3.2 and 5.3.3 we focus on the recent results of the free idempotent generated semigroups over bands and normal bands, respectively. Moreover, we describe the form of the elements of $\mathrm{IG}(\mathcal{B})$, where B is a band and a normal band, respectively. In the last subsection of this chapter, we investigate the word problem of the free idempotent generated semigroup.

Throughout the following chapters we will use the notation $E = E(S)$ for the set for all idempotents in some semigroup S and $\mathcal{E} = \mathcal{E}(S)$ for the biordered set that is associated with $E = E(S)$.

We recommend [80], [72], [16], [75] and [29] as references for Chapter 5.

5.1 Biordered sets

In 1979, Nambooripad introduced the concept of a *biordered set* [72]. The study of biordered sets of idempotents of any semigroup is closely related to the study of idempotent generated semigroups [29]. The importance of biordered sets is to describe the structure of the set of idempotents of semigroups. Nambooripad defined a biordered set as a partial algebra with some binary relations, satisfying six axioms. This section aims to recall these axioms that define biordered sets.

Let $\mathcal{E} = (E, *)$ be a *partial algebra*, that is a set with a partial binary operation, and D_E be the domain of this partial binary operation, so that

$$(e, f) \in D_E \Longleftrightarrow e * f \text{ is defined in } E.$$

Now we define two binary relations ω^r and ω^l on $\mathcal{E}$ by the following rules. For any $e, f \in E$:

$$e\ \omega^r\ f \Longleftrightarrow (e, f) \in D_E \text{ and } f * e = e$$

and

$$e\ \omega^l\ f \Longleftrightarrow (e, f) \in D_E \text{ and } e * f = e.$$

Then we define three binary relations ω, R and L on $\mathcal{E}$ by using the above relations,

$$R = \omega^r \cap (\omega^r)^{-1},\ L = \omega^l \cap (\omega^l)^{-1},\ \omega = \omega^r \cap \omega^l.$$

Next we introduce the simple notation $\omega^r(e)$ and $\omega^l(e)$ where $e \in E$:

$$\omega^r(e) = \{f \in E : f\ \omega^r\ e\},$$

$$\omega^l(e) = \{f \in E : f\ \omega^l\ e\}.$$

We can now define a biordered set.

Definition 5.1.1. [80] A partial algebra $\mathcal{E}$ with the five binary relations ω^r, ω^l, ω, R and L, defined as above, is called a *biordered set* if it satisfies axioms (1), (2), (3), (4), (5) and (6) and their duals, for any $e,\ f,\ g,\ h \in E$:

(1) ω^r and ω^l are pre-orders on $\mathcal{E}$ such that $D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$.

(2) $f \in \omega^r(e)$ implies $f \; R \; f * e \; \omega \; e$.

(3) $f, g \in \omega^r(e)$ and $g \; \omega^l \; f$ imply $g * e \; \omega^l \; f * e$.

(4) $g \; \omega^r \; f \; \omega^r \; e$ implies $g * f = (g * e) * f$.

(5) $f, g \in \omega^r(e)$ and $g \; \omega^l \; f$ imply $(f * g) * e = (f * e)(g * e)$.

Let $M(e, f) = \omega^l(e) \cap \omega^r(f)$, for any $e, f \in E$. The quasi-order relation $\prec$ on $M(e, f)$ is defined by the rule

$$g \prec h \Longleftrightarrow e * g \, \omega^r \, e * h \quad \text{and} \quad g * f \, \omega^l \, h * f.$$

The definition of $\prec$ gives that $(M(e, f), \prec)$ is a quasi-ordered set. The *sandwich set of* e *and* f in that order is the set

$$S(e, f) = \{h \in M(e, f) : g \prec h \;\text{ for all } g \in M(e, f)\}.$$

(6) $f, g \in \omega^r(e)$ implies $S(f, g)e = S(f * e, g * e)$.

A *regular biordered set* $\mathcal{E}$ is a biordered set such that for any $e, f \in E$, we have that $S(e, f) \neq \emptyset$.

Let $\mathcal{E}$ and $\mathcal{F}$ be biordered sets. A map $\theta \colon \mathcal{E} \longrightarrow \mathcal{F}$ is called *a biordered set morphism* if for any $e, f \in E$ and $(e, f) \in D_E$, then

(i) $(e\theta, f\theta) \in D_F$,

(ii) $(e\theta)(f\theta) = (ef)\theta$.

If the morphism is bijective, then we say θ is *a biordered set isomorphism.*

We now explain how (regular) biordered sets are precisely sets of idempotents of (regular) semigroups.
Let S be a semigroup with a set $E = E(S)$ of idempotents. A pair (e, f) of $E \times E$ satisfying the condition $\{e, f\} \cap \{ef, fe\} \neq \emptyset$ is called *basic pair*; we call fe and ef *basic products.* It is worth noting that if $(e, f) \in E \times E$ is a basic pair, then the basic products ef and fe are idempotents. As if $ef = e$, we get that

$$(fe)^2 = f(ef)e = f(e)e = fe.$$

We say that the biordered set $\mathcal{E}$ has *trivial basic products* if for any basic pair (e, f) of E, we have $ef, fe \in \{e, f\}$.

Note 5.1.2. *Let S and T be semigroups and let φ be a morphism from S to T. If (a, b) is a basic pair in S, then $(a\varphi, b\varphi)$ is a basic pair in T, as any morphism preserve the Green's relations.*

Nambooripad in [72] showed that the set of idempotents E of any semigroup S forms a partial algebra with domain

$$D_E = \{(e, f) : (e, f) \text{ is a basic pair, }\}$$

where for any $(e, f) \in D_E$, $e * f$ is defined to be the product ef in S. It is clear $ef \in E$. Furthermore, there are two pre-orders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ defined on S.

It is easy to check that with ω^r and ω^l being the notation of $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ to $E \times E$, the set E forms a biordered set, under this partial operation, we denote this by $\mathcal{E}(S) = (E(S), *)$.

In 1979, Nambooripad gave a sufficient condition for any biordered set to be regular in [72].

Theorem 5.1.3. *Let $\mathcal{E} = (E, *)$ be regular biordered set with associated domain D_E and operations ω^r and ω^l. Then there is a regular semigroup S such that $E = E(S)$.*

In 1985, Easdown [31] showed that every biordered set $\mathcal{E}$ occurs as $E(S)$ for some semigroup S. Hence we lose nothing by assuming that a biordered set $\mathcal{E}$ is of the form $E(S)$ for a semigroup S.

Theorem 5.1.4. *Let $\mathcal{E} = (E, *)$ be a biordered set with associated domain D_E and operations ω^r and ω^l. Then there is a semigroup S such that $E = E(S)$.*

We end this section by fixing some notation. The following is the explanation of this notation, which will be frequently used in the whole thesis.

(1) We use B to denote a band and $\mathcal{B}$ to denote the biordered set of B; in particular, we use Y for a semilattice and $\mathcal{Y}$ for the associated biordered set.

(2) We use $B = \bigcup_{\alpha \in Y} B_\alpha$ to denote a band which is a semilattice Y of rectangular bands $B_\alpha (\alpha \in Y)$.

(3) We use $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ to denote a normal band which is a strong semilattice Y of rectangular bands $B_\delta (\delta \in Y)$, with connecting morphisms $\phi_{\delta,\sigma}$ for all $\delta, \sigma \in Y$ with $\delta \geq \sigma$.

(4) Let Y be a semilattice and let $\alpha, \beta \in Y$. We use $\alpha \sqcap \beta$ to denote the set of all common upper bounds of α and β; of course, $\alpha \sqcap \beta$ may be empty.

(5) Let Y be a semilattice and let $\alpha, \beta \in Y$. We use the notation $\alpha \perp \beta$ to denote the situation where α and β are incomparable, that is, neither $\alpha \leq \beta$ nor $\alpha \geq \beta$ hold.

Note that for any $\alpha, \beta \in Y$, if $\alpha \sqcap \beta = \emptyset$, then $\alpha \perp \beta$, as if $\alpha \leq \beta$ (or $\alpha \geq \beta$), $\alpha \sqcap \beta = \beta$ (or α).

At times we will use this notation without specific comment.

5.2 Idempotent generated semigroups

Let S be a semigroup and denote by $\langle E \rangle$ the subsemigroup of S generated by the set of idempotents of S, $E = E(S)$. If $S = \langle E \rangle$, then we say that S is an *idempotent generated*.

The significance of such semigroups was evident when J.M. Howie in 1966 [62], showed that every semigroup may be embedded into one that is idempotent generated. Moreover, he proved that any finite semigroup is embeddable in a finite regular idempotent generated semigroup. Then many authors extended this result when they have studied the structure of idempotent generated semigroups in many different ways.

5.3 Free idempotent generated semigroups

One of our main goals in this thesis is to study the general structure of a free idempotent generated semigroup. The study of this semigroup began with the seminal work of Nambooripad in the 1970s. Given a biordered set $\mathcal{E}$, that is a set of idempotents of some semigroup S, the *free idempotent generated semigroup*, is an initial object in the category of semigroups that are generated by $\mathcal{E}$, denoted by $\mathrm{IG}(\mathcal{E})$.

In the following sections, we give an overview of $\mathrm{IG}(\mathcal{E})$ and several pleasant properties, particularly with respect to Green's relations. Moreover, we recall the results that have been obtained so far in the current research direction of this area.

5.3.1 Basic definitions and preliminary results

Let S be a semigroup and $E = E(S)$. Let $S' = \langle E \rangle$ be the idempotent generated semigroup with biordered set of idempotents $\mathcal{E}$. It is an important step towards understanding the class of semigroups with a fixed biordered set of idempotents $\mathcal{E}$ is to study the initial object $\mathrm{IG}(\mathcal{E})$.

Throughout this thesis, we denote an element w in $IG(\mathcal{E})$ by $\overline{w} = \overline{e_1} \dots \overline{e_n}$, while $w = e_1 \circ e_2 \dots \circ e_n$ is a word in a free semigroup E^+. We denote the *length* of a word w by $|w|$.

The *free idempotent generated semigroup over* $\mathcal{E}$, where $\mathcal{E}$ is a biordered set of E, is the initial object in the category of all idempotent generated semigroups whose biordered set of idempotents is isomorphic to the biordered set $\mathcal{E}$. In particular, the biordered set of $\mathrm{IG}(\mathcal{E})$ is exactly the given biordered set $\mathcal{E}$. In fact, it is remarkable that $\mathrm{IG}(\mathcal{E})$ exists. We obtain $\mathrm{IG}(\mathcal{E})$ via a presentation as follows:

$$\mathrm{IG}(\mathcal{E}) = \langle E \colon e \circ f = e * f,\ (e, f) \text{ basic pair, } e, f \in E \rangle$$

where $e \circ f$ is the word of length 2 with letters e and f, while $e * f$ denotes the partial multiplication in E considered as a partial algebra. Thus $\mathrm{IG}(\mathcal{E})$ is a quotient semigroup of the free semigroup E^+ over the congruence ρ_E on E^+ generated by $\{(e \circ f, e * f) \colon e, f \in E,\ (e, f) \text{ basic pair}\}$. So we can write

$$\mathrm{IG}(\mathcal{E}) = E^+/\rho_E.$$

There is a (natural) morphism from the free semigroup $\mathcal{E}^+$ to $\mathrm{IG}(\mathcal{E})$, where $\overline{w}$ is the image of $w \in \mathcal{E}^+$ in $\mathrm{IG}(\mathcal{E})$. Clearly, $\overline{e}\,\overline{f} = \overline{e \circ f}$ holds for any $e, f \in \mathcal{E}^+$. We remark that caution is required in $\mathcal{E}$, since for $e, f \in \mathcal{E}$ we write $\overline{e \circ f}$ for the image in $\mathrm{IG}(\mathcal{E})$ of the two letter word $e \circ f \in \mathcal{E}^+$. In our work we supposed that $\mathcal{E}$ is a set of idempotents of some semigroup S, then we have $e * f = ef$. Hence we obtain $\mathrm{IG}(\mathcal{E})$ via a presentation as follows:

$$\mathrm{IG}(\mathcal{E}) = \langle E \colon e \circ f = ef,\ (e, f) \text{ basic pair, } e, f \in E \rangle.$$

If (e, f) is basic, then we have $\overline{e}\,\overline{f} = \overline{e \circ f} = \overline{ef}$ in $\mathrm{IG}(\mathcal{E})$. However, if (e, f) is not basic but ef is an idempotent, then we do not necessarily have $ef = e \circ f$. We say that $e_1 \circ e_2 \circ \dots \circ e_n \in E^+$ is in *normal form* if (e_i, e_{i+1}) is not basic, for all $1 \leq i \leq n - 1$.

For any two words $w, u \in E^+$ we write $w \sim_E u$ if the word w can be obtained from the word u by single *splitting step* or *squashing step*, these are applications of a defining relation of $\mathrm{IG}(\mathcal{E})$. For example

$$w = x(e \circ f)y, \; u = x(ef)y$$

or

$$w = x(ef)y, \; u = x(e \circ f)y,$$

where (e, f) is a basic pair. So the relation ρ_E is the reflexive, transitive closure of $\sim_E$. Where E is clear we write $\sim$ and ρ for $\sim_E$ and ρ_E, respectively.

Let e be an idempotent of a semigroup S. The set eSe is a submonoid of S and is the largest submonoid whose identity is e. The group of units G_e of eSe, is the group of elements of eSe that have two-sided inverses with respect to e is the largest subgroup of S whose identity is e. We called this subgroup the *maximal subgroup of S containing e.*

Maximal subgroups of free idempotent generated semigroups have been of interest for some time. The investigation of the maximal subgroups of $\mathrm{IG}(\mathcal{E})$ was a popular theme to study the structure of $\mathrm{IG}(\mathcal{E})$.

It was thought from the 1970s that all subgroups of a free idempotent generated semigroup would be free, but this conjecture was not true. Although this conjecture had been believed for more than 30 years, it has been disproved in 2009 by a counter-example provided by Brittenham, Margolis and Meakin in [6].

If $\mathcal{E}$ has trivial basic products, then from [73, Theorem 3], and [6, Theorems 3.6 and 4.2], the maximal subgroups of $\mathrm{IG}(\mathcal{E})$ are all free groups. Moreover, if B is a band, then every maximal subgroup of $\mathrm{IG}(\mathcal{B})$ is free [26].

In the following lemma we list some classical properties of $\mathrm{IG}(\mathcal{E})$, with respect to Green's relations, taken from [37],[72], [32], [19], [54]. These properties will be used frequently in this thesis.

Lemma 5.3.1. *Let S be a semigroup, $E = E(S)$, $\mathcal{E}$ be a biordered set of E, $S' = \langle E \rangle$ be any idempotent generated semigroup with biordered set of idempotents $E = E(S)$, and $\mathrm{IG}(\mathcal{E})$ be the free idempotent generated semigroup.*

$(IG1)$ *There is a natural morphism*

$$\varphi\colon \mathrm{IG}(\mathcal{E}) \longrightarrow S,$$

where

$$\overline{e}\varphi = e.$$

The morphism φ is onto on S'.

$(IG2)$ *The restriction of φ to the set of idempotents of $\mathrm{IG}(\mathcal{E})$ is an isomorphism onto the given biordered set $\mathcal{E}$.*

$$\varphi\colon E(\mathrm{IG}(\mathcal{E})) \longrightarrow \mathcal{E}$$

$(IG3)$ *The morphism φ induces a bijection between the set of all $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in the $\mathcal{D}$-class $D_{\overline{e}}$ of an element $\overline{e}$ of $\mathrm{IG}(\mathcal{E})$ and the set of all $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in the $\mathcal{D}$-class D_e of an element e in S'.*

$(IG4)$ *The restriction of φ to the maximal subgroup of $\mathrm{IG}(\mathcal{E})$, containing $e \in \mathcal{E}$ $\big($the $\mathcal{H}$-class of $\overline{e}$ in $\mathrm{IG}(\mathcal{E})\big)$ $H_{\overline{e}}$, is a morphism onto the maximal subgroup of S' containing e, H_e, that is,*

$$\overline{\varphi}\colon H_{\overline{e}} \longrightarrow H_e.$$

The assertion $(IG1)$ is obvious and follows directly from the definition of $\mathrm{IG}(\mathcal{E})$; $(IG2)$ is proved in [72] and [32]; $(IG3)$ is a corollary of [37]. The property $(IG4)$ follows from $(IG2)$.

Nambooripad in [72], proved that the restriction of φ to the set of idempotents set of $\mathrm{IG}(\mathcal{E})$ is an isomorphism onto $E = E(S)$ in the case when E is regular. This was done for arbitrary biordered sets by Easdown in [31].

If S is a regular semigroup with the set of idempotents $E = E(S)$, then we know that $\mathcal{E}$ is a regular biordered set and for any pair (e, f) of E, we get that $S(e, f) \neq \emptyset$ by Theorem 5.1.3. The *free regular idempotent generated semigroup* on E is the homomorphic image of $\mathrm{IG}(\mathcal{E})$ obtained by adding to the presentation of $\mathrm{IG}(\mathcal{E})$, the following relation

$$ehf = e \circ f, \quad \text{for all } e, f \in \mathcal{E},\ h \in S(e, f).$$

We denote *the free regular idempotent generated semigroup* on E by $RIG(\mathcal{E})$. There is a natural morphism from $\mathrm{IG}(\mathcal{E})$ to $RIG(\mathcal{E})$. It is easy to see that $RIG(\mathcal{E})$ satisfies the above properties of $\mathrm{IG}(\mathcal{E})$. There is more important properties of $RIG(E)$

($RIG1$) The semigroup $RIG(\mathcal{E})$ is regular.

($RIG2$) The natural morphism from $\mathrm{IG}(\mathcal{E})$ to $RIG(\mathcal{E})$ induces an isomorphism between the maximal subgroups of any $e \in \mathcal{E}$ in $\mathrm{IG}(\mathcal{E})$ and $RIG(\mathcal{E})$.

($RIG1$) is taken from [72] and in [6, Theorem 3.6]. ($RIG2$) is taken from [72] and [6, Theorem 3.6].

A popular approach to investigating the structure of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$ was studying the behavior of its maximal subgroups. From the 1970s, it was conjectured that the maximal subgroups of $\mathrm{IG}(\mathcal{E})$ are always free [70]. In 2009, Brittenham, Margolis, and Meakin provided a counter example [6]. Prompted by this significant result, in 2012, Gray and Ruškuc [54] showed that any group occurs as the maximal subgroup of some $\mathrm{IG}(\mathcal{E})$.

Theorem 5.3.2. *[54] Every group is a maximal subgroup of some free idempotent generated semigroup.*

In 2014, Gould and Yang [80] gave an alternative proof in a rather transparent way.

In the same paper [6] of Gray and Ruškuc they obtain the following significant result.

Theorem 5.3.3. *[54] Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.*

The theorem above provides a complete characterisation of groups appearing as maximal subgroups of free idempotent generated semigroups arising from finite semigroups. The structure of maximal subgroups of free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$ is still an open area of investigation.

5.3.2 Free idempotents generated semigroups over bands

In this section, we define some special forms of the elements of $\mathrm{IG}(\mathcal{B})$, where B is a band, and we present some useful results about this semigroup. All the definitions and the results in this section are taken from [16].

Let B be a band. We write $B = \bigcup_{\alpha \in Y} B_\alpha$ as a semilattice Y of rectangular bands B_α. Then we have the following morphisms.

($IG4$) There is a morphism

$$\theta \colon B \longrightarrow Y$$

defined by the rule

$$x \longmapsto \alpha,$$

where $x \in B_\alpha$. This is an onto morphism with the kernel classes which are the rectangular bands B_α.

$(IG5)$ There is a morphism

$$\psi \colon \mathcal{B}_\alpha \longrightarrow \mathcal{B},$$

defined by

$$e\psi = e,$$

for all $e \in \mathcal{B}_\alpha$.

$(IG6)$ There is a morphism

$$\overline{\psi} \colon \mathrm{IG}(\mathcal{B}_\alpha) \longrightarrow \mathrm{IG}(\mathcal{B}),$$

defined by

$$(\overline{e_1} \dots \overline{e_n})\overline{\psi} = \overline{e_1} \dots \overline{e_n},$$

for any $\overline{e_1} \dots \overline{e_n} \in \mathrm{IG}(\mathcal{B}_\alpha)$, and for all $\alpha \in Y$.

$(IG4)$ is proved in [16]; $(IG5)$ is obvious as B_α is subsemigroup of B; $(IG6)$ is clear from the presentations of $\mathrm{IG}(\mathcal{B}_\alpha)$ and $\mathrm{IG}(\mathcal{B})$.

Recall that the biordered set $\mathcal{E}$ has *trivial basic products* if for any basic pair (e, f) of E, we have $ef, fe \in \{e, f\}$.

In the rest of this section we focus on the properties of the structure $\mathrm{IG}(\mathcal{E})$ over a biordered set $\mathcal{E}$ that has trivial basic products.

Example 5.3.4. Any semilattice Y is an example of biordered set with trivial basic products as for any $e, f \in Y$ where (e, f) is a basic pair, we get that

$$ef = fe \in \{e, f\}.$$

Moreover, let B be a rectangular band and (e, f) be a basic pair in B. Then $\{e, f\} \cap \{ef, fe\} \neq \emptyset$. If $ef = f$, then by multiplying both sides by e we get that

$$e = (ef)e = fe.$$

Similarly, if $ef = e$, then by multiplying both sides by f we get that

$$f = f(ef) = fe.$$

Therefore, a rectangular band B is an example of biordered set with trivial basic products.

Theorem 5.3.5. *Let $\mathcal{E}$ be a biordered set with trivial basic products. Then every element of* $\mathrm{IG}(\mathcal{E})$ *has a unique normal form.*

The above shows that if B is a semilattice or a rectangular band, then every element of $\mathrm{IG}(\mathcal{B})$ has a unique normal form. However, that is not be true for an arbitrary band B, nor normal band, [16, Example 4.6].

For a semilattice Y, and $\mathcal{Y}$ the biordered set of Y, any element $\overline{\alpha} = \overline{\alpha_1} \ldots \overline{\alpha_n}$ of $\mathrm{IG}(\mathcal{Y})$ is in *normal form* if and only if $\alpha_i \perp \alpha_{i+1}$, for all $1 \leq i \leq n-1$, by the uniqueness of normal forms in $\mathrm{IG}(\mathcal{Y})$. It is clear that any two elements of $\mathrm{IG}(\mathcal{Y})$ are equal if and only if the corresponding normal forms of them are identical word in Y^+.

The following result says that $\mathrm{IG}(\mathcal{E})$ is abundant for any biordered set $\mathcal{E}$ with trivial products. Moreover, it gives us exactly how $\mathcal{R}^*$ and $\mathcal{L}^*$-classes in $\mathrm{IG}(\mathcal{E})$ behave for any $e \in E$.

Theorem 5.3.6. *Let $\mathcal{E}$ be a biordered set with trivial basic products. Then the free idempotent generated semigroup* $\mathrm{IG}(\mathcal{E})$ *is abundant.*

If B is a rectangular band, then it has trivial basic products, and by Theorem 5.3.6 we get that $\mathrm{IG}(\mathcal{B})$ is abundant. Early investigations of Pastijn [75] showed that if B is a rectangular band, then the corresponding $\mathrm{IG}(\mathcal{B})$ is a completely simple semigroup.

Theorem 5.3.7. *[75] Let B be a rectangular band. Then* $\mathrm{IG}(\mathcal{B})$ *is a completely simple semigroup.*

Theorem 5.3.7 tells us if $\overline{x_1}\ldots\overline{x_n} \in \mathrm{IG}(\mathcal{B})$ where B is a rectangular band, then we have $\overline{x_1}\,\mathcal{R}\,\overline{x_1}\ldots\overline{x_n}\,\mathcal{L}\,\overline{x_n}$. However, this result is not true for semilattices [16, Example 3.3]) is an example of semilattice Y such that $\mathrm{IG}(\mathcal{Y})$ is not regular.

The next result is clear, as the elements of B_δ for any $\delta \in Y$ generate a completely simple subsemigroup of $\mathrm{IG}(\mathcal{B})$.

Corollary 5.3.8. *Let B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then for any $x_1, \ldots, x_n \in B_\alpha$, we have $\overline{x_1}\ \mathcal{R}\ \overline{x_1}\ldots\overline{x_n}\ \mathcal{L}\ \overline{x_n}$ in $\mathrm{IG}(\mathcal{B})$. Consequently, $\overline{x_1}\ldots\overline{x_n}$ is a regular element of $\mathrm{IG}(\mathcal{B})$.*

In the following we define the *left to right significant indices* of the elements of $\mathrm{IG}(\mathcal{B})$ as the following.

Let $x_1 \circ \ldots \circ x_n \in B^+$ with $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Then a set of numbers $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ with $i_1 < \ldots < i_r$ is called *the left to right significant indices* of $\overline{x_1}\ldots\overline{x_n}$, if these numbers are picked out in the following manner:

- i_1 : the largest number such that $\alpha_1, \ldots, \alpha_{i_1} \geq \alpha_{i_1}$;
- k_1 : the largest number such that $\alpha_{i_1} \leq \alpha_{i_1}, \alpha_{i_1+1}, \ldots, \alpha_{k_1}$.

Remark that $\alpha_{i_1}, \alpha_{k_1+1}$ are incomparable, since if $\alpha_{i_1} \leq \alpha_{k_1+1}$, then we add 1 to k_1, contradicting the choice of k_1; also if $\alpha_{i_1} > \alpha_{k_1+1}$, then $\alpha_1, \ldots, \alpha_{i_1}, \ldots, \alpha_{k_1} \geq \alpha_{k_1+1}$, contradicting the choice of i_1. Now we continue our process:

- i_2 : the largest number such that $\alpha_{k_1+1}, \ldots,\ \alpha_{i_2} \geq \alpha_{i_2}$;
- k_2 : the largest number such that $\alpha_{i_2} \leq \alpha_{i_2},\ \alpha_{i_2+1}, \ldots,\ \alpha_{k_2}$;

  .

  .

  .

- i_r : the largest number such that $\alpha_{k_{r-1}+1}, \ldots,\ \alpha_{i_r} \geq \alpha_{i_r}$;
- $k_r = n$: here we have $\alpha_{i_r} \leq \alpha_{i_r},\ \alpha_{i_r+1}, \ldots,\ \alpha_n$. We may have $i_r = k_r = n$.

The following *Hasse diagram* depicts the relationship among $\alpha_{i_1}, \ldots,\ \alpha_{i_r}$:

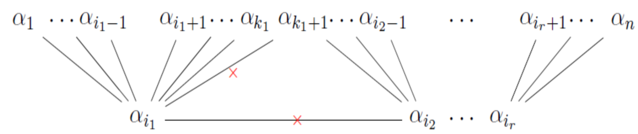


Figure 5.1: Hasse diagram illustrating significant indices

Dually we define the *right to left significant indices* as the following. Let $x_1 \circ \ldots \circ x_n \in B^+$ with $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Then a set of numbers $\{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\}$ with $j_1 < \ldots < j_l$ is called *the right to left significant indices* of $\overline{x_1} \ldots \overline{x_n}$, if these numbers are picked out in the following manner:

- j_1 : the smallest number such that $\alpha_n, \ldots, \alpha_{j_1} \geq \alpha_{j_1}$;
- k_1 : the smallest number such that $\alpha_{j_1} \leq \alpha_{j_1}, \alpha_{j_1-1}, \ldots, \alpha_{k_1}$.

Note that $\alpha_{j_1}, \alpha_{k_1-1}$ are incomparable. Now we continue our process:

- j_2 : the smallest number such that $\alpha_{k_1-1}, \ldots, \alpha_{j_2} \geq \alpha_{j_2}$;
- k_2 : the smallest number such that $\alpha_{j_2} \leq \alpha_{j_2}, \alpha_{j_2-1}, \ldots, \alpha_{k_2}$;

.
.
.

- j_l : the smallest number such that $\alpha_{k_{l-1}-1}, \ldots, \alpha_{j_l} \geq \alpha_{j_l}$;
- $k_l = 1$: here we have $\alpha_{j_l} \leq \alpha_{j_l}, \alpha_{j_l-1}, \ldots, \alpha_1$. We may have $j_l = k_l = 1$.

Let $w = x_1 \circ x_2 \ldots \circ x_n \in B^+$ where $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Suppose that w has left to right significant indices $i_1, \ldots, i_r$. The natural number r is called the *Y-length* of the word w. We denote the *Y-length* of w by $|w|_Y$. The *ordered Y-components* of $\overline{w}$ in $\mathrm{IG}(\mathcal{B})$ is the sequence $\alpha_{i_1}, \ldots, \alpha_{i_r}$, this sequence is depends on $\overline{w}$, not on w. This follows from the next result.

Lemma 5.3.9. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a band. Then there is a well defined epimorphism $\boldsymbol{T}$ from $\mathrm{IG}(\mathcal{B})$ onto $\mathrm{IG}(\mathcal{Y})$ given by*

$$\overline{x_1} \ldots \overline{x_n} \mapsto \overline{\alpha_1} \ldots \overline{\alpha_n}, \text{ if } x_i \in B_{\alpha_i} \text{ for all } 1 \leq i \leq n.$$

In [18], Gould and Yang introduced another notion called *almost normal form* and its associated with *Y-trace*.

In the following we present the definition of *almost normal form* for any word in B^+.

Definition 5.3.10. A word $x_1 \circ x_2 \ldots \circ x_n \in B^+$ is said to be in *almost normal form* if there exists a sequence

$$1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq n$$

such that

$$\{x_1, \ldots, x_{i_1}\} \subseteq B_{\alpha_1}, \{x_{i_1+1}, \ldots, x_{i_2}\} \subseteq B_{\alpha_2}, \ldots, \{x_{i_{r-1}+1}, \ldots, x_n\} \subseteq B_{\alpha_r}$$

where $\alpha_i \perp \alpha_{i+1}$, for all $1 \leq i \leq r-1$. Further, we call the *n-triple* $(\alpha_1, \ldots, \alpha_r)$ the *Y-trace* of $x_1 \circ \ \ldots \ \circ x_n$.

Let $w = x_1 \circ \cdots \circ x_n \in B^+$ be in almost normal form with Y-trace $(\alpha_1, \ldots, \alpha_r)$. Then we can write $w = w_1 \circ \cdots \circ w_r$, where

$$w_1 = x_1 \circ \cdots \circ x_{i_1},\, w_2 = x_{i_1+1} \circ \cdots \circ x_{i_2},\, \ldots,\, w_r = x_{i_{r-1}} \circ \cdots \circ x_n.$$

It is clear that $w_p \in B^+_{\alpha_p}$, for all $1 \leq p \leq r$.

The next result shows that every element of $\mathrm{IG}(\mathcal{B})$ can be written in almost normal form.

Lemma 5.3.11. *[16] Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a band. Then every element of* $\mathrm{IG}(\mathcal{B})$ *can be written as some $\overline{w_1} \ldots \overline{w_n} \in \mathrm{IG}(\mathcal{B})$, where $w_1 \circ \ \ldots \ \circ w_n$ is in almost normal form.*

It is worth noting that a word in almost normal form need not be in normal form, as we do not insist in the above expression that the pair (x_j, x_{j+1}) is not basic for $x_j, x_{j+1} \in B_{\alpha_i}$, $1 \leq i \leq r$. On the other hand, a word being in normal form does not imply that it is in almost normal form. For example, if $x \in B_\alpha$, $y \in B_\beta$ where $\alpha > \beta$ and (x, y) not a basic pair, then $x \circ y$ is in normal form but not almost normal form [16]. However, as $\alpha = \alpha_1 \circ \ldots \circ \alpha_n \in Y^+$ is the normal form of $\overline{\alpha}$ in $\mathrm{IG}(\mathcal{Y})$ if $\alpha_i \perp \alpha_{i+1}$, for all $1 \leq i \leq n-1$, then it is clear that the normal form of $\overline{\alpha}$ is also an almost normal form of $\overline{\alpha}$. Also, any almost normal form of $\overline{\alpha} \in \mathrm{IG}(\mathcal{Y})$ is a normal form of this element. In general, an almost normal form of an element $\overline{w} \in \mathrm{IG}(\mathcal{B})$ is still not necessarily unique, but any almost normal form of $\overline{w}$ has the same Y-trace.

Lemma 5.3.12. *Let $B = \bigcup_{\alpha\in Y} B_\alpha$ be a band. Let*

$$u_1 \circ \ldots \circ u_n, \quad v_1 \circ \ldots \circ v_m$$

be two almost normal forms in B^+ with $u_i \in B^+_{\alpha_i}$ for all $1 \leq i \leq n$ and $v_i \in B^+_{\beta_i}$ for all $1 \leq i \leq m$. Then $\overline{u_1} \ldots \overline{u_n} = \overline{v_1} \ldots \overline{v_m}$ in $\mathrm{IG}(\mathcal{B})$ *implies that*

- *they have the same Y-length, that is, $n = m$;*
- *they have the same ordered Y-components, $\alpha_i = \beta_i$ for all $1 \leq i \leq n$.*

Consequently, any element of $\mathrm{IG}(\mathcal{B})$ *is associated with a unique Y-trace.*

Note that any two words in almost normal forms in B^+ may have different significant indices.

Let $(A_i)_{i\in I}$ be a family of pairwise disjoint rectangular bands, and let the index I be linearly order by some relation $\leq$. Define a multiplication on $A = \bigcup_{i\in I} A_i$ as follows: in each A_i, if $a_1 \in A_i$ and $a_2 \in A_j$, where $i < j$, then

$$a_1a_2 = a_2a_1 = a_1.$$

Hence A become a band. Such semigroups said to be a special kind of *chain of rectangular bands*, [1]. It is easy to see that a band A is a chain of rectangular bands if and only if for any $x, y \in A$

$$xyx = x \text{ or } y.$$

In general, a chain is a special case of a semilattice.

Proposition 5.3.13. *Let $B = \bigcup_{\alpha\in Y} B_\alpha$ be a chain Y of rectangular bands B_α, $\alpha \in Y$. Then* $\mathrm{IG}(\mathcal{B})$ *is a regular semigroup.*

Theorem 5.3.14. *Let B be a band. Then* $\mathrm{IG}(\mathcal{B})$ *is a weakly abundant semigroup with the congruence condition.*

Notice that for a band B, $\mathrm{IG}(\mathcal{B})$ is not always abundant, [16, Example 4.14].

5.3.3 Free idempotent generated semigroups over normal bands

In the previous section, we present the result which shows that the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over a band B is not always abundant. Gould and Yang in [80], gave an example of a normal band B, where $\mathrm{IG}(\mathcal{B})$, is not abundant. All the definitions and the results in this section are taken from Gould and

Yang [16].

It is known that if B is a semilattice or a rectangular band, then every element of $\mathrm{IG}(\mathcal{B})$ has a unique normal form, but this is not true in general for an arbitrary band B, even if B is a normal band.

Let B be a strong semilattice of rectangular bands. We define the equality of two words in B^+ and $\mathrm{IG}(\mathcal{B})$ as the following. Let $e_1,\ e_2,\ldots,\ e_n,\ f_1,\ f_2,\ldots,\ f_m$ be elements of B, so $e_1 \circ e_2 \circ \ldots \circ e_n$ and $f_1 \circ f_2 \circ \ldots \circ f_m$ are two words of B^+. Now we say these two words are equal in B^+ and write

$$e_1 \circ e_2 \circ \ldots \circ e_n = f_1 \circ f_2 \circ \ldots \circ f_m$$

if $n = m$ and $e_i = f_i$, for each $1 \leq i \leq n$ in B. However, the equality of any two elements $\overline{w}$ and $\overline{u}$ of $\mathrm{IG}(\mathcal{B})$ is much more complicated to determine, where the equality in $\mathrm{IG}(\mathcal{B})$ of $\overline{w}$ and $\overline{u}$ means that $w\ \rho\ u$.

Lemma 5.3.15. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $\overline{x}_1 \ldots \overline{x}_n \in \mathrm{IG}(\mathcal{B})$ with $x_i \in B_{\alpha_i}$ and $\alpha_i \geq \alpha$ for all $1 \leq i \leq n$. Suppose that $\overline{y}_1 \ldots \overline{y}_m \in \mathrm{IG}(\mathcal{B})$ with $y_i \in B_{\beta_i}$ for all $1 \leq i \leq m$ and*

$$\overline{x}_1 \ldots \overline{x}_n \sim \overline{y}_1 \ldots \overline{y}_m.$$

Then $\beta_i \geq \alpha$, for all $1 \leq i \leq m$ and in $IG(\mathcal{B}_\alpha)$ we have

$$\overline{x_1\varphi_{\alpha_1,\alpha}} \ldots \overline{x_n\varphi_{\alpha_n,\alpha}} = \overline{y_1\varphi_{\beta_1,\alpha}} \ldots \overline{y_m\varphi_{\beta_m,\alpha}}.$$

The next result is key in understanding $\mathrm{IG}(\mathcal{B})$ for a band B, and it is related to Lemma 5.3.11.

Proposition 5.3.16. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $\overline{x_1} \ldots \overline{x_n} \in \mathrm{IG}(\mathcal{B})$ be such that $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $1 \leq i \leq n$. Then in $\mathrm{IG}(\mathcal{B})$ we have*

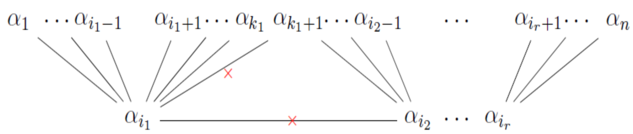
$$\overline{x_1} \ldots \overline{x_n}\ \overline{y} = \overline{x_1\phi_{\alpha_1,\beta}} \ldots \overline{x_n\phi_{\alpha_n,\beta}}\,\overline{y},$$

and

$$\overline{y}\ \overline{x_1} \ldots \overline{x_n} = \overline{y}\ \overline{x_1\phi_{\alpha_1,\beta}} \ldots \overline{x_n\phi_{\alpha_n,\beta}}.$$

Note 5.3.17. *If $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is a normal band, $x \in B_\alpha,\ y \in B_\beta$ and (x, y) is a basic pair, then α and β are comparable. In addition, if $\alpha \geq \beta$, then we have $yx\,\mathcal{R}\,y$, as there exist $x, y \in B^1$ such that $(yx)y = y$ and $(y)x = yx$. Hence (y, yx) is a basic pair.*

Let $w = x_1 \circ \ldots \circ x_n$ be a word of B^+, where $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$, and $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ be the left to right significant indices of $\overline{w}$,



By the definition of the left to right significant indices of $\overline{w}$, we get that $\alpha_{i_1} \leq \alpha_1, \cdots, \alpha_{i_1-1}, ,\alpha_{i_1+1}, \ldots, \alpha_{k_1}$, similarly until $\alpha_{i_r} \leq \alpha_{k_{r-1}+1}, \ldots, \alpha_n$. Hence by Proposition 5.3.16 we write

$$\begin{aligned}
\overline{w_1} &= \overline{x_1} \cdots \overline{x_{i_1}} \cdots \overline{x_{k_1}} \\
&= \overline{x_1 \phi_{\alpha_1, \alpha_{i_1}}} \cdots \overline{x_{i_1}} \cdots \overline{x_{k_1} \phi_{\alpha_{k_1}, \alpha_{i_1}}}, \\
&\vdots \\
\overline{w_r} &= \overline{x_{k+1}} \cdots \overline{x_{i_r}} \cdots \overline{x_n} \\
&= \overline{x_{k_{r-1}+1} \phi_{\alpha_{k_{r-1}+1}, \alpha_{i_r}}} \cdots \overline{x_{i_r}} \cdots \overline{x_{k_1} \phi_{\alpha_n, \alpha_{i_r}}}.
\end{aligned}$$

Now the word $w = w_1 \circ \cdots \circ w_r$, where $r \leq n$ is in an almost normal form of $\overline{w}$.

The following results will be useful in our later work.

Lemma 5.3.18. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band, and let $x \in B_\beta$, $y \in B_\gamma$ with $\beta, \gamma \geq \alpha$. Then (x, y) is a basic pair implies $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair and*

$$(x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}) = (xy)\phi_{\delta,\alpha}$$

where δ is the minimum of β and γ, that is $\delta = \beta\gamma$.

Lemma 5.3.19. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $x, y \in B$ where $x \in B_\alpha$ and $y \in B_\beta$. Then $\alpha \leq \beta$ implies that*

$$\overline{x}\ \overline{y} = \overline{x}\ \overline{y\phi_{\beta,\alpha}} \text{ and } \overline{y}\ \overline{x} = \overline{y\phi_{\beta,\alpha}}\ \overline{x}.$$

The following result is not true for arbitrary bands [16, Example 6.4]

Corollary 5.3.20. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be elements of B_α. Then $\overline{x_1} \ldots \overline{x_n} = \overline{y_1} \ldots \overline{y_m}$ in $IG(\mathcal{B}_\alpha)$ if and only if the equality holds in* $\mathrm{IG}(\mathcal{B})$.

In the following we define a condition (P), then present some results about the free idempotent generated semigroup which satisfies this condition.

Definition 5.3.21. We say that the semigroup $\mathrm{IG}(\mathcal{B})$ satisfies Condition (P) if for any two almost normal forms $u = u_1 \circ \cdots \circ u_n$, $v_1 \circ \cdots \circ v_m = v \in B^+$ of $\overline{u} = \overline{v} \in \mathrm{IG}(\mathcal{B})$ with Y-length r, left to right significant indices $i_1, \ldots, i_r = n$ and $l_1, \ldots, l_r = m$, respectively, then the following statements (with $i_0 = l_0 = 0$) hold:

- $u_{i_s} \; \mathcal{L} \; v_{l_s}$ implies $\overline{u}_1 \ldots \overline{u}_{i_s} = \overline{v}_1 \ldots \overline{v}_{l_s}$, for all $s \in [1, r]$;
- $u_{i_t+1} \; \mathcal{R} \; v_{l_t+1}$ implies $\overline{u}_{i_{t+1}} \ldots \overline{u}_n = \overline{v}_{l_t+1} \ldots \overline{v}_m$, for all $t \in [0, r-1]$.

The following result gives the condition on $\mathrm{IG}(\mathcal{B})$ over a normal band to be abundant.

Proposition 5.3.22. *[16] Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band for which* $\mathrm{IG}(\mathcal{B})$ *satisfies Condition (P). Then* $\mathrm{IG}(\mathcal{B})$ *is an abundant semigroup.*

We end this section with the following table that lists the properties of $\mathrm{IG}(\mathcal{E})$ over different kinds of biordered sets. The results are taken from [16], [75] and [80].

Table 5.1: Idempotent generated semigroups IG($\mathcal{B}$)

Semigroup	Biordered set $\mathcal{E}$	Idempotent generated semigroup IG($\mathcal{E}$)	Ref
	Finite	- Weakly abundant - Satisfies the congruence condition.	[75]
Band		-Weakly abundant -Satisfies the congruence condition. -Any $\overline{w} \in$ IG($\mathcal{B}$) can be written in almost normal form.	[16]
Semilattice	$\mathcal{E}$ with trivial basic products	- Abundant - Any $\overline{w} \in$ IG($\mathcal{B}$) has a unique normal form. - If IG($\mathcal{B}$) satisfies the condition (P) then IG($\mathcal{B}$) is abundant.	[16] [16]
Normal band		- Weakly abundant - Satisfies the congruence condition.	[16] [16]
Trivial normal band		- Abundant - Satisfies condition condition (P).	[80] [16]
Simple normal band		- Abundant - Satisfies condition condition (P). - Any $\overline{w} \in$ IG($\mathcal{B}$) has a normal form.	[80]
Chain Y of rectangular bands		- regular.	[16]

5.3.4 The word problem for free idempotent generated semigroups

Let S be a semigroup presented by $\langle A|R\rangle$, where A is a set of letters and R is a set of pairs $(u_i, v_i), i \in I$, of words over A. Let $S \cong A^+/\rho_R$, where A^+ is the free semigroup of A and ρ_R is the congruence generated by R.

Note that, if $\mathrm{IG}(\mathcal{E}) = E^+/\rho$, we can write this as a presentation,

$$\mathrm{IG}(\mathcal{E}) = \langle\ E : (e \circ f, ef) \text{ whenever } (e, f) \text{ is a basic pair}\,\rangle.$$

There is a natural morphism $\phi\colon E^+ \longrightarrow \mathrm{IG}(\mathcal{E})$, defined by $w\phi = \overline{w}$. The kernel relation of ϕ is precisely ρ. Therefore, the *word problem* for IG($\mathcal{E}$) asks: given two words $u, v \in E^+$ decide if $\overline{u} = \overline{v}$ holds in IG($\mathcal{E}$). We say the word problem of

$\mathrm{IG}(\mathcal{E})$ is *decidable* if for any $u = v$ in E^+, we get that $\overline{u} = \overline{v}$ in $\mathrm{IG}(\mathcal{E})$.

We end this section with the word problem in $\mathrm{IG}(\mathcal{E})$, if E is finite.

Theorem 5.3.23. *[18] Let $\mathcal{E}$ be a biordered set with trivial basic products. If E is finite, then* $\mathrm{IG}(\mathcal{E})$ *has decidable word problem.*

We refer the reader to Dolinka and Ruškuc work [29] for further details of the word problem for $\mathrm{IG}(\mathcal{E})$.

Chapter 6

Free idempotent generated semigroups over iso-normal bands

In 2014, Gould and Yang [80] showed that for an arbitrary band B, the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ is a weakly abundant semigroup with the congruence condition, but not necessarily abundant. Moreover, they constructed a 10-element normal band B for which $\mathrm{IG}(\mathcal{B})$ is not abundant. Following this example of Gould and Yang, an interesting question was asked by Gould: what kind of normal bands are such that $\mathrm{IG}((\mathcal{B})$ is abundant? The aim of this chapter is to investigate the general structure of $\mathrm{IG}(\mathcal{B})$ for a special kind of normal band B.

We proceed as follows. In Section 6.1, we show some specific basic properties concerning the structure of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over a normal band B and describe the general structure of $\mathrm{IG}(\mathcal{B})$. In Section 6.2, we introduce a class of bands, called *iso-normal bands* and define some morphisms on $\mathrm{IG}(\mathcal{B})$ over an iso-normal band B, which we use to prove our main result. In Section 6.3, we prove two special cases of the main result in this chapter. We prove that if $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ is an iso-normal band, where Y is a *fan* semilattice or a *diamond* semilattice, then $\mathrm{IG}(\mathcal{B})$ is always abundant. We use two different strategies to prove these cases. In Section 6.4, it is known that the normal form of any element of $\mathrm{IG}(\mathcal{B})$, where B is semilattice or rectangular band, is unique. However, we may lose the uniqueness of normal forms of $\mathrm{IG}(\mathcal{B})$ if B is not semilattice nor rectangular band. To overcome this problem, the concepts of the complete form and double normal form are introduced. Further, we show that the word problem of $\mathrm{IG}(\mathcal{B})$ is solvable if B is a finite iso-normal band. In

Section 6.5, we prove our main result that for an arbitrary iso-normal band B, $\mathrm{IG}(\mathcal{B})$ is an abundant semigroup.

6.1 Properties of free idempotent generated semigroups over normal bands

A subsemigroup S of a semigroup T is a *retract of* T (via θ) if there exists an epimorphism θ from T onto S such that $\theta_{|_S} = I_S$, where I_S is the identity map on S.

It is known that if S is a subsemigroup of a semigroup T with $E = E(S)$ and $F = E(T)$, then there is a natural morphism from $\mathrm{IG}(\mathcal{E})$ to $\mathrm{IG}(\mathcal{F})$ [80].

Corollary 6.1.1. *[19] Let S be a retract of T via θ, with $E = E(S)$ and $F = E(T)$. Then $\mathrm{IG}(\mathcal{E})$ is a subsemigroup of $\mathrm{IG}(\mathcal{F})$. Further, $\mathrm{IG}(\mathcal{E})$ embeds into $\mathrm{IG}(\mathcal{F})$.*

Definition 6.1.2. Let $\mathcal{E}$, $\mathcal{E}'$ be biordered sets and $\theta\colon \mathcal{E} \longrightarrow \mathcal{E}'$ be a map. Then θ is called *a bimorphism* if it satisfies the following

(M) $(e, f) \in D_{\mathcal{E}}$, then $(e\theta, f\theta) \in D_{\mathcal{E}'}$ and $(ef)\theta = (e\theta)(f\theta)$.

The map θ is called *a regular bimorphism* if, furthermore,

(RM1) $S(e, f)\theta \subseteq S'(e\theta, f\theta)$

(RM2) $S(e, f) \neq \emptyset \Longleftrightarrow S'(e\theta, f\theta) \neq \emptyset$, for all $e, f \in \mathcal{E}$, where $S'(e\theta, f\theta)$ denotes the sandwich set in $\mathcal{E}'$.

It is worth pointing out here if $\mathcal{E}$ is a regular biordered set, then the bimorphism $\theta\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is regular if θ satisfies $(RM1)$. Further, if (e, f) is a basic pair in $\mathcal{E}$, then $(e\theta, f\theta)$ is basic in $\mathcal{E}'$.

If θ_1 and θ_2 are two bimorphisms, then $\theta_1\theta_2$ is also bimorphism. Moreover, if they are regular bimorphisms, then $\theta_1\theta_2$ is a regular bimorphism.

In the following result we prove that, in general, for any two biordered sets $\mathcal{E}$ and $\mathcal{E}'$ if there is a bimorphism θ between $\mathcal{E}$ and $\mathcal{E}'$, then there is a morphism from $\mathrm{IG}(\mathcal{E})$ to $\mathrm{IG}(\mathcal{E}')$.

Lemma 6.1.3. *[80] Let $\mathcal{E}$ and $\mathcal{E}'$ be biordered sets. Then any bimorphism from $\mathcal{E}$ to $\mathcal{E}'$ induces a morphism from $\mathrm{IG}(\mathcal{E})$ to $\mathrm{IG}(\mathcal{E}')$.*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\theta"] \arrow[d] & \mathcal{E}' \arrow[d] \\
\mathcal{E}^+ \arrow[r, "\theta"] \arrow[d, "\rho^\natural_{\mathcal{E}}"'] \arrow[dr, "\tau"] & \mathcal{E}'^+ \arrow[d, "\rho^\natural_{\mathcal{E}'}"] \\
IG(\mathcal{E}) \arrow[r, "\overline{\theta}"'] & IG(\mathcal{E}')
\end{tikzcd}$$

Figure 6.1: the commutative diagram of Lemma 6.1.3

Proof. Let $\theta : \mathcal{E} \longrightarrow \mathcal{E}'$ be a bimorphism. We define a morphism

$$\tau : \mathcal{E} \longrightarrow \mathrm{IG}(\mathcal{E}') \;\text{ by } e \mapsto \overline{e\theta}.$$

So we can define

$$\tau : \mathcal{E}^+ \longrightarrow \mathrm{IG}(\mathcal{E}')$$

by

$$(e_1 \circ \ldots \circ e_n)\tau = \overline{e_1\theta} \ldots \overline{e_n\theta}.$$

A basic pair (g, h) in $\mathcal{E}$ induces a basic pair $(g\theta, h\theta)$ in $\mathcal{E}'$. To see this let $g \circ h$ be a word of length 2 in $\mathcal{E}^+$,

$$\begin{aligned}
(g \circ h)\tau &= \overline{(g \circ h)\theta} && \text{(by the definition of } \tau) \\
&= \overline{(g\theta) \circ (h\theta)} && \text{(as } \theta \text{ is a bimorphism)} \\
&= \overline{(g\theta)(h\theta)} && \text{(as } (g\theta, h\theta) \text{ is a basic pair)} \\
&= (gh)\tau && \text{(by the definition of } \tau).
\end{aligned}$$

Then we get that $(g \circ h, gh) \in \ker\,\tau$. Hence $\rho_{\mathcal{E}} \subseteq \ker\,\tau$, where $\rho_{\mathcal{E}}$ is the congruence determining the quotient semigroup $\mathrm{IG}(\mathcal{E})$. Therefore, there exists a morphism

$$\overline{\theta} \colon\, \mathrm{IG}(\mathcal{E}) \longrightarrow \mathrm{IG}(\mathcal{E}')$$

define by the rule

$$\begin{aligned}(\overline{e_1}\dots\overline{e_n})\overline{\theta} &= (e_1 \circ \dots \circ e_n)\tau \\ &= \overline{(e_1\theta)}\dots\overline{(e_n\theta)}.\end{aligned}$$

□

In the following we prove the isomorphism between two free idempotent generated semigroups.

Lemma 6.1.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be biordered sets. Let*

$$\theta\colon \mathcal{E} \longrightarrow \mathcal{E}'$$

be an bijective bimorphism. Then there is an isomorphism

$$\overline{\theta}\colon\ \mathrm{IG}(\mathcal{E}) \longrightarrow \mathrm{IG}(\mathcal{E}').$$

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be biordered sets. Let $\theta\colon \mathcal{E} \longrightarrow \mathcal{E}'$ be an bijective bimorphism. Then there exists an bijective bimorphism $\theta^{-1}\colon \mathcal{E}' \longrightarrow \mathcal{E}$. Define a morphism

$$\mu : \mathcal{E} \longrightarrow \mathrm{IG}(\mathcal{E}')$$

by

$$e\mu = \overline{e\theta}.$$

Then we can define a morphism $\mu : \mathcal{E}^+ \longrightarrow \mathrm{IG}(\mathcal{E}')$ by

$$\begin{aligned}(e_1 \circ \dots \circ e_n)\mu &= \overline{(e_1 \circ \dots \circ e_n)\theta} \\ &= (\overline{e_1\theta})\dots(\overline{e_n\theta}),\end{aligned}$$

where $e_1 \circ \dots \circ e_n \in \mathcal{E}^+$. By Lemma 6.1.3, there is a morphism

$$\overline{\theta}\colon\ \mathrm{IG}(\mathcal{E}) \longrightarrow \mathrm{IG}(\mathcal{E}')$$

defined by

$$\begin{aligned}(\overline{e_1}\dots\overline{e_n})\overline{\theta} &= \overline{(e_1 \circ \dots \circ e_n)\mu} \\ &= \overline{(e_1\mu) \circ \dots \circ (e_n\mu)} && \text{(as } \mu \text{ is a bimorphism)} \\ &= (\overline{e_1\theta})\dots(\overline{e_n\theta}).\end{aligned}$$

Similarly define a morphism $\nu\colon \mathcal{E}'^+ \longrightarrow \mathrm{IG}(\mathcal{E})$ by

$$\begin{aligned}(f_1 \circ \dots \circ f_n)\nu &= \overline{f_1 \circ \dots \circ f_n)\theta^{-1}} \\ &= (\overline{f_1\theta^{-1}})\dots(\overline{f_n\theta^{-1}}).\end{aligned}$$

By Lemma 6.1.3, there is a morphism

$$\overline{\theta^{-1}} \colon \operatorname{IG}(\mathcal{E}') \longrightarrow \operatorname{IG}(\mathcal{E})$$

defined by

$$(\overline{f_1} \ldots \overline{f_n})\overline{\theta^{-1}} = (\overline{f_1\theta^{-1}}) \ldots (\overline{f_n\theta^{-1}}).$$

Let $\overline{e_1} \cdots \overline{e_n} \in \operatorname{IG}(\mathcal{E})$, we get that

$$\begin{aligned}
(\overline{e_1} \ldots \overline{e_n})(\overline{\theta} \circ \overline{\theta^{-1}}) &= \Big((\overline{e_1} \ldots \overline{e_n})\overline{\theta}\Big)\overline{\theta^{-1}} \\
&= \Big((\overline{e_1\theta}) \ldots (\overline{e_n\theta})\Big)\overline{\theta^{-1}} \\
&= \Big((\overline{e_1\theta})\overline{\theta^{-1}} \ldots (\overline{e_n\theta})\overline{\theta^{-1}}\Big) \\
&= \Big(\overline{(e_1\theta)\theta^{-1}}\Big) \ldots \Big(\overline{(e_n\theta)\theta^{-1}}\Big) \\
&= \Big(\overline{e_1\theta\theta^{-1}}\Big) \ldots \Big(\overline{e_n\theta\theta^{-1}}\Big) \\
&= \overline{e_1} \ldots \overline{e_n}.
\end{aligned}$$

Hence $\overline{\theta} \circ \overline{\theta^{-1}} = I_{IG(\mathcal{E})}$, where $I_{IG(\mathcal{E})}$ is the identity morphism on $\operatorname{IG}(\mathcal{E})$. Similarly, we get that $\overline{\theta^{-1}} \circ \overline{\theta} = I_{IG(\mathcal{E}')}$. Therefore, $\overline{\theta}$ and $\overline{\theta^{-1}}$ are isomorphisms. $\square$

A biordered subset $\mathcal{E}'$ of a biordered set $\mathcal{E}$ is a biordered set which is a partial subalgebra in the usual. For further details, we refer the reader to [72].

Note 6.1.5. *Let S be a subsemigroup of T and let e, $f \in S$. Then (e, f) is basic in S if and only if (e, f) is basic in T. Since if (e, f) is basic in S (or T), then one of the following equalities hold*

$$ef = e,\ ef = f,\ \ fe = e,\ fe = f.$$

Hence (e, f) is basic in T (or S). So if $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ is a normal band, then any normal form $w = x_1 \circ \ldots \circ x_n \in B_\alpha^+$ of $\overline{w}$ in $\operatorname{IG}(\mathcal{B}_\alpha)$ is a normal form of $\overline{w}$ in $\operatorname{IG}(\mathcal{B})$.

If S and T are semigroups and $\varphi \colon S \longrightarrow T$ is a morphism, then from the morphism $\theta = \varphi_{|E(S)} \colon E(S) \longrightarrow E(T)$, it is easy to define a bimorphism

$$\theta \colon \mathcal{E}(S) \longrightarrow \mathcal{E}(T).$$

Consequently, it is easy to deduce the following result from Lemma 6.1.3. However, we prefer to give a direct proof.

Lemma 6.1.6. *Let S and T be semigroups and $\theta\colon S \longrightarrow T$ be a morphism. Let $E = E(S)$, $F = E(T)$, $\mathcal{E} = \mathcal{E}(S)$ and $\mathcal{F} = \mathcal{F}(T)$. Then there is a morphism*

$$\overline{\psi}\colon \mathrm{IG}(\mathcal{E}) \longrightarrow \mathrm{IG}(\mathcal{F})$$

Proof. Let S and T be semigroups and let $\theta\colon S \longrightarrow T$ be a morphism. We write the free idempotent generated semigroup $\mathrm{IG}(\mathcal{E})$ as

$$\mathrm{IG}(\mathcal{E}) = \langle E\colon e \circ f = ef,\ e, f \in E,\ (e, f) \text{ basic pair } \rangle = E^+/\rho_E.$$

Let $\overline{w}$ be the ρ_E-class of $w \in \mathcal{E}^+$, where ρ_E is the congruence determining the quotient semigroup $\mathrm{IG}(\mathcal{E})$. Also, we write $\mathrm{IG}(\mathcal{F})$ as

$$\mathrm{IG}(\mathcal{F}) = \langle F\colon e \circ_F f = ef,\ e, f \in F,\ (e, f) \text{ basic pair } \rangle = F^+/\rho_F.$$

Let $\overline{\overline{w}}$ be the ρ_F-class of $w \in \mathcal{F}^+$, where ρ_F is the congruence determining the quotient semigroup $\mathrm{IG}(\mathcal{F})$. Define the map $\psi\colon \mathcal{E} \longrightarrow \mathrm{IG}(\mathcal{F})$ by $b\psi = \overline{\overline{b\theta}}$. So we can define $\psi\colon \mathcal{E}^+ \longrightarrow \mathrm{IG}(\mathcal{F})$ by

$$(b_1 \circ \ldots \circ b_n)\psi = \overline{\overline{b_1\theta}} \ldots \overline{\overline{b_n\theta}},$$

where $b_1 \circ \ldots \circ b_n \in \mathcal{E}^+$. Then the map ψ is a morphism, as for any basic pair (e, f) in $\mathcal{E}$, we have

$$e \leq_{\mathcal{R}} f,\ e \leq_{\mathcal{L}} f,\quad f \leq_{\mathcal{R}} e \text{ or } f \leq_{\mathcal{L}} e.$$

As θ is a morphism, θ preserves $\mathcal{R}$ and $\mathcal{L}$. Then we get

$$e\theta \leq_{\mathcal{R}} f\theta,\ e\theta \leq_{\mathcal{L}} f\theta,\ f\theta \leq_{\mathcal{R}} e\theta \text{ or } f\theta \leq_{\mathcal{L}} e\theta.$$

Hence $(e\theta, f\theta)$ is a basic pair in $\mathcal{F}$. Therefore, in $\mathrm{IG}(\mathcal{F})$ we can write that

$$\overline{\overline{e\theta \circ_F f\theta}} = \overline{\overline{e\theta f\theta}}. \tag{6.1}$$

Now let (e, f) be a basic pair in $\mathcal{E}$, which implies that $(ef, e \circ f)$ is a generator of ρ_E and $(e\theta, f\theta)$ is a basic pair in $\mathcal{F}$. We can write

$$\begin{aligned}
(e \circ f)\psi &= \overline{\overline{(e\theta) \circ_F (f\theta)}} \\
&= \overline{\overline{e\theta f\theta}} && \text{(by 6.1)} \\
&= \overline{\overline{(ef)\theta}} && \text{(as } \theta \text{ is a morphism)} \\
&= (ef)\psi && \text{(by the definition of } \psi,\ ef \text{ word of length 1 in } \mathcal{E}^+).
\end{aligned}$$

This implies that $(ef, e \circ f) \in \ker\psi$. Hence $\rho_E \subseteq \ker\psi$. Therefore, there exists a morphism

$$\overline{\theta}\colon \operatorname{IG}(\mathcal{E}) \longrightarrow \operatorname{IG}(\mathcal{F}),$$

given by $\big(\overline{b_1}\dots\overline{b_n}\big)\overline{\theta} = \overline{\overline{b_1\theta}}\dots\overline{\overline{b_n\theta}}$, where $\overline{b_1}\dots\overline{b_n} \in \operatorname{IG}(\mathcal{E})$.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\theta"] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathcal{E}^+ \arrow[r] \arrow[d, "\rho^\natural_{\mathcal{E}}"'] \arrow[dr, "\psi"] & \mathcal{F}^+ \arrow[d, "\rho^\natural_{\mathcal{F}}"] \\
IG(\mathcal{E}) \arrow[r, "\overline{\theta}"'] & IG(\mathcal{F})
\end{tikzcd}$$

Figure 6.2: The commutative diagram of Lemma 6.1.6

$\square$

If S is a subsemigroup of T, then S is embedded into T. Hence from Lemma 6.1.6, there is a morphism

$$\overline{\psi}\colon \operatorname{IG}(\mathcal{E}) \longrightarrow \operatorname{IG}(\mathcal{F}),$$

where $\mathcal{E} = \mathcal{E}(S)$ and $\mathcal{F} = \mathcal{E}(T)$.

Lemma 6.1.7. *Let Y be a semilattice, and let Y_0 be a subsemigroup of Y. Then* $\operatorname{IG}(\mathcal{Y}_0)$ *embedded into* $\operatorname{IG}(\mathcal{Y})$*.*

Proof. Define a map

$$\varphi\colon \operatorname{IG}(\mathcal{Y}_0) \longrightarrow \operatorname{IG}(\mathcal{Y}),$$

for $\overline{\alpha_1}\,\overline{\alpha_2}\dots\overline{\alpha_n} \in \operatorname{IG}(\mathcal{Y}_0)$, by

$$(\overline{\alpha_1}\,\overline{\alpha_2}\dots\overline{\alpha_n})\varphi = \overline{\alpha_1}\ \overline{\alpha_2}\dots\overline{\alpha_n}.$$

By Lemma 6.1.6, the map above is a morphism. Now for any $\overline{\alpha_1}\,\overline{\alpha_2}\dots\overline{\alpha_n}$, $\overline{\beta_1}\,\overline{\beta_2}\dots\overline{\beta_m}$ of $\operatorname{IG}(\mathcal{Y}_0)$, where $\alpha_1 \circ \dots \circ \alpha_n$ and $\beta_1 \circ \dots \circ \beta_m$ are words in the normal form of Y_0^+. Suppose that

$$\overline{\alpha_1}\ \overline{\alpha_2}\dots\overline{\alpha_n} = (\overline{\alpha_1}\,\overline{\alpha_2}\dots\overline{\alpha_n})\varphi = (\overline{\beta_1}\,\overline{\beta_2}\dots\overline{\beta_m})\varphi = \overline{\beta_1}\ \overline{\beta_2}\dots\overline{\beta_m}.$$

in $\mathrm{IG}(\mathcal{Y})$. As both $\alpha_1 \circ \ldots \circ \alpha_n$ and $\beta_1 \circ \ldots \circ \beta_m$ are words in the normal form of Y_0^+, then $\alpha_1 \perp \alpha_2 \perp \ldots \perp \alpha_n$ and $\beta_1 \perp \beta_2 \perp \ldots \perp \beta_m$ in Y_0. This also true in Y. Hence we get that $m = n$ and $\alpha_i = \beta_i$ for all $1 \leq i \leq n$. Then we get the equality

$$\overline{\alpha_1}\ \overline{\alpha_2} \ldots \overline{\alpha_n} = \overline{\beta_1}\ \overline{\beta_2} \ldots \overline{\beta_m}$$

in $\mathrm{IG}(\mathcal{Y}_0)$.

□

Lemma 6.1.8. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of rectangular bands B_α, $\alpha \in Y$, where δ is the lower bound of Y. Then the map $f_\delta \colon B \longrightarrow B_\delta$ defined by*

$$t_\mu \longmapsto t_\mu \phi_{\mu,\delta},$$

where $t_\mu \in B_\mu$, and $\mu \in Y$, is a morphism.

Proof. First, it is clear that the map f_δ is well defined. To prove that f_δ is a morphism, let $t_\mu, s_\nu \in B$, where $t_\mu \in B_\mu$ and $s_\nu \in B_\nu$. Then

$$\begin{aligned}
(t_\mu s_\nu) f_\delta &= \big((t_\mu \phi_{\mu,\mu\nu})(s_\nu \phi_{\nu,\mu\nu})\big) f_\delta \\
&= \big((t_\mu \phi_{\mu,\mu\nu})(s_\nu \phi_{\nu,\mu\nu})\big) \phi_{\mu\nu,\delta} && \text{(by the definition of } f_\delta) \\
&= (t_\mu \phi_{\mu,\mu\nu}\ \phi_{\mu\nu,\delta})(s_\nu \phi_{\nu,\mu\nu}\ \phi_{\mu\nu,\delta}) && \text{(as } \phi_{\mu\nu,\delta} \text{ is morphism)} \\
&= (t_\mu \phi_{\mu,\delta})(s_\nu \phi_{\nu,\delta}) \\
&= (t_\mu f_\delta)(s_\nu f_\delta).
\end{aligned}$$

Therefore, f_δ is a morphism.

□

Corollary 6.1.9. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of rectangular bands B_α, $\alpha \in Y$, where δ is the lower bound of Y. Then a bimorphism*

$$f_\delta \colon \mathcal{B} \longrightarrow \mathcal{B}_\delta.$$

The following results are immediate from using 6.1.9 and 6.1.6.

Lemma 6.1.10. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of rectangular bands B_α, $\alpha \in Y$, where δ is the lower bound of Y. Then the map*

$$F_\delta \colon\ \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_\delta)$$

defined by

$$(\overline{x}_1 \ldots \overline{x}_n) F_\delta = (\overline{x_1 f_\delta}) \ldots (\overline{x_n f_\delta})$$

is a morphism, where $\overline{x}_1 \ldots \overline{x}_n \in \mathrm{IG}(\mathcal{B})$.

The next result is clear by using Corollary 6.1.1 and Lemma 6.1.9.

Lemma 6.1.11. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of rectangular bands B_α, $\alpha \in Y$. Let δ be the lower bound of Y. Then there is an embedding of* $\mathrm{IG}(\mathcal{B}_\delta)$ *into* $\mathrm{IG}(\mathcal{B})$.

Proof. By Lemma 6.1.9 we get that there is a morphism $f_\delta \colon B \longrightarrow B_\delta$ defined by

$$t_\mu \longmapsto t_\mu \phi_{\mu,\delta},$$

where $t_\mu \in B_\mu$, and $\mu \in Y$. It is clear that θ is an onto morphism and $\theta_{|B_\alpha} = I_{B_\alpha}$. Hence B_α is a retract of B via θ. Therefore, by Corollary 6.1.1, $\mathrm{IG}(\mathcal{B}_\alpha)$ embeds into $\mathrm{IG}(\mathcal{B})$. □

6.2 Basic definitions and properties of free idempotent generated semigroups over iso-normal bands

In this section, we introduce the concept of an *iso-normal band.* We give some properties of this special kind of normal band.

Recall that a band $B = \bigcup_{\alpha \in Y} B_\alpha$ is called *normal* if for all α, β in Y with $\alpha \geq \beta$ there exists a morphism $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ such that:

(i) for all $\alpha \in Y$, $\phi_{\alpha,\alpha} = 1_{B_\alpha}$;

(ii) for all $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$;

(iii) for all $\alpha, \beta \in Y$ and $x \in B_\alpha, y \in B_\beta$,

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}).$$

We consider a stronger condition on a strong semilattice of semigroups to obtain an iso-normal band, is defined by the following.

Definition 6.2.1. A normal band $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is an *iso-normal band* if $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ is an isomorphism for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$.

We now work towards an alternative description of an iso-normal band. For an iso-normal band $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$, we may define an isomorphism between any two rectangular bands B_α and B_β, $\phi_{\alpha,\beta}$, by the following:

$$\phi_{\alpha,\beta} = \phi_{\alpha,\alpha\beta}(\phi_{\beta,\alpha\beta})^{-1}. \tag{6.2}$$

For all $\alpha \in Y$ we have

$$\phi_{\alpha,\alpha} = \phi_{\alpha,\alpha}\phi_{\alpha,\alpha}^{-1}.$$

If $\alpha \leq \beta$, then

$$\begin{aligned}\phi_{\alpha,\beta} &= \phi_{\alpha,\alpha}\phi_{\beta,\alpha}^{-1}\\ &= \phi_{\beta,\alpha}^{-1}.\end{aligned}$$

Let ν be a common lower bound of α and β in Y and let $\gamma = \alpha\beta$. Then

$$\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}\phi_{\beta,\gamma}^{-1} = (\phi_{\alpha,\gamma}\phi_{\gamma,\nu})(\phi_{\gamma,\nu}^{-1}\phi_{\beta,\gamma}^{-1}) = (\phi_{\alpha,\gamma}\phi_{\gamma,\nu})(\phi_{\beta,\gamma}\phi_{\gamma,\nu})^{-1} = \phi_{\alpha,\nu}\phi_{\beta,\nu}^{-1}.$$

For the remainder of this section we assume that $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is an iso-normal band.

The following result show that transitivity holds in the set of connecting morphisms in any iso-normal band.

Corollary 6.2.2. *Let B be an iso-normal band. Then for any $\alpha, \beta, \gamma \in Y$, we have that*

$$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}.$$

Proof. Let $\kappa \in Y$ be a common lower bound of α, β, γ in Y. Then

$$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = (\phi_{\alpha,\kappa}\phi_{\beta,\kappa}^{-1})(\phi_{\beta,\kappa}\phi_{\gamma,\kappa}^{-1}) = \phi_{\alpha,\kappa}\phi_{\gamma,\kappa}^{-1} = \phi_{\alpha,\gamma}.$$

$\square$

Notice that $B \cong B_\alpha \times Y$, $\alpha \in Y$, under the mapping $x \longrightarrow (x\phi_{\beta,\alpha}, \beta)$, where $x \in B_\beta$.

Proposition 6.2.3. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band. Then B is isomorphic to the direct product $B_\tau \times Y$ for any chosen τ in Y.*

Conversely, any direct product $R \times Z$, where R is a rectangular band and Z is a semilattice is isomorphic to an iso-normal band.

Proof. Suppose that $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is an iso-normal band. For any α and β of Y, recall that $\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}\phi_{\beta,\gamma}^{-1}$, for any lower bound γ of α and β.

Moreover, it is known that the map $\psi : B \longrightarrow Y$ which is defined by

$$b_\alpha\psi = \alpha, \quad b_\alpha \in B_\alpha.$$

is well defined morphism.

Now choose $\tau \in Y$ and define a map $\phi : B \to B_\tau \times Y$ by

$$b_\alpha \phi = (b\phi_{\alpha,\tau}, b_\alpha \psi), \quad b_\alpha \in B_\alpha.$$

As both ψ and $\phi_{\alpha,\tau}$, for all $\tau \in Y$ are well defined maps, ϕ is a well defined map.

Let $b_\alpha \in B_\alpha$ and $b_\beta \in B_\beta$. We get that

$$\begin{aligned}(b_\alpha b_\beta)\phi &= \big((b_\alpha \phi_{\alpha,\tau})(b_\beta \phi_{\beta,\tau}), \alpha\beta\big)\\ &= (b_\alpha \phi_{\alpha,\tau}, \alpha)(b_\beta \phi_{\beta,\tau}, \beta)\\ &= (b_\alpha \phi)(b_\beta \phi).\end{aligned}$$

Hence ϕ is a morphism.

Second, let $x, y \in B$, where $x \in B_\alpha,\ y \in B_\beta$ and suppose that $x\phi = y\phi$. Then we get that

$$x\phi = (x\phi_{\alpha,\tau}, \alpha) = (y\phi_{\beta,\tau}, \beta) = y\phi,$$

which implies $\alpha = \beta$. If $\tau = \alpha = \beta$, then

$$x_\alpha \phi_{\alpha,\tau} = x_\alpha,$$

and

$$y_\alpha \phi_{\alpha,\tau} = y_\alpha.$$

So we get that

$$x_\alpha \phi_{\alpha,\tau} = x_\tau = y_\tau = y_\alpha \phi_{\alpha,\tau}.$$

As for any $\alpha, \tau \in Y$, we have $B_\alpha \cong B_\tau$. Hence $x = y$. Therefore, ϕ is an injective morphism.

Finally, it is clear that ϕ is a surjective morphism, as for any $(b, \alpha) \in B_\tau \times Y$, where $b \in B_\tau$ and $\alpha \in Y$ we have

$$b\phi = (b\phi_{\alpha,\tau}, \alpha).$$

From the above we get that ϕ is an isomorphism.

Conversely, consider the direct product $R \times Z$, where R is a rectangular band and Z is a semilattice.

For any $\alpha \in Z$, let $R_\alpha = R \times \{\alpha\}$. Note that each R_α is a rectangular band, as it is clear that $R_\alpha \cong R$.

For any $\alpha, \beta \in Z$, where $\alpha \geq \beta$ we define $\psi_{\alpha,\beta} : R_\alpha \to R_\beta$ by

$$(r, \alpha)\psi_{\alpha,\beta} = (r, \beta).$$

The map $\psi_{\alpha,\beta}$ is well defined map as for any $(r_1, \alpha), (r_2, \alpha) \in R_\alpha$, where $(r_1, \alpha) = (r_2, \alpha)$, which means that $r_1 = r_2$. Then

$$(r_1, \alpha)\psi_{\alpha,\beta} = (r_1, \beta) = (r_2, \beta) = (r_2, \alpha)\psi_{\alpha,\beta}.$$

Moreover, for any $(r_1, \alpha), (r_2, \alpha) \in R_\alpha$, where

$$(r_1, \alpha)\psi_{\alpha,\beta} = (r_2, \alpha)\psi_{\alpha,\beta},$$

which means that $(r_1, \beta) = (r_2, \beta)$, then $r_1 = r_2$ in R. Hence $(r_1, \alpha) = (r_2, \alpha)$, which proves that $\psi_{\alpha,\beta}$ is an injective. Also, $\psi_{\alpha,\beta}$ is a morphism as

$$\begin{aligned}
\big((r_1, \alpha)(r_2, \alpha)\big)\psi_{\alpha,\beta} &= (r_1 r_2, \alpha)\psi_{\alpha,\beta} \\
&= (r_1 r_2, \beta) \\
&= (r_1, \beta)(r_2, \beta) \\
&= \big((r_1, \alpha)\psi_{\alpha,\beta}\big)\big((r_2, \alpha)\psi_{\alpha,\beta}\big).
\end{aligned}$$

Finally, it is clear that $\psi_{\alpha,\beta}$ is an onto morphism. Therefore, $\psi_{\alpha,\beta}$ is an isomorphism for any $\alpha, \beta \in Z$.

Now we have $C = \mathscr{B}(Z; R_\alpha, \psi_{\alpha,\beta})$ is an iso-normal band as

- For all $\alpha \in Z$, the morphism $\psi_{\alpha,\alpha} : R_\alpha \longrightarrow R_\alpha$ is the identity map on the rectangular band R_α, hence $\psi_{\alpha,\alpha} = 1_{R_\alpha}$,
- for any $\alpha, \beta, \gamma \in Z$ such that $\alpha \geq \beta \geq \gamma$. Let (r, α) we have

$$\begin{aligned}
(r, \alpha)\psi_{\alpha,\beta}\ \psi_{\beta,\gamma} &= \big((r, \alpha)\psi_{\alpha,\beta}\big)\psi_{\beta,\gamma} \\
&= (r, \beta)\psi_{\beta,\gamma} \\
&= (r, \gamma) \\
&= (r, \alpha)\psi_{\alpha,\gamma}.
\end{aligned}$$

  This proves that $\psi_{\alpha,\beta}\ \psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$.
- For all $\alpha, \beta \in Z$, the map $\psi_{\alpha,\beta}$ is an isomorphism, as we proved above.

It is clear that $C = \bigcup_{\alpha\in Z} R_\alpha$, note that $C = R \times Z$ as a set. Our aim is to show that the multiplication on $R \times Z$ is equal the multiplication on C. Now define a map $I : R \times Z \longrightarrow C$ by

$$(r, \alpha)I = (r, \alpha),$$

where $(r, \alpha) \in R \times Z$. For convenience, denote the multiplication in C by $*$. Let $(r, \alpha), (s, \beta) \in R \times Z$, we have

$$\big((r, \alpha)(s, \beta)\big)I = (rs, \alpha\beta),$$

on the other side we have

$$(r, \alpha)I * (s, \beta)I = (r, \alpha) * (s, \beta) = (r, \alpha)\psi_{\alpha,\alpha\beta}(s, \beta)\psi_{\beta,\alpha\beta}.$$

Therefore, I is an isomorphism.

$\square$

In the following result we show that Lemma 6.1.11 is true for any $\alpha \in Y$, where B is an iso-normal band.

Lemma 6.2.4. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band. Then for any $\alpha \in Y$, there is an embedding of* $\mathrm{IG}(\mathcal{B}_\alpha)$ *into* $\mathrm{IG}(\mathcal{B})$.

Proof. Define a map

$$\theta : B \longrightarrow B_\alpha,$$

by $b\theta = b\phi_{\beta,\alpha}$, where $b \in B_\beta$. This map is a morphism as for any x, $y \in B$, where $x \in B_\beta$ and $y \in B_\gamma$

$$\begin{aligned}
(xy)\theta &= \big((x\phi_{\beta,\beta\gamma})(y\phi_{\gamma,\beta\gamma})\big)\theta && \text{(by the definition of the multiplication on } B\text{)}\\
&= \big((x\phi_{\beta,\beta\gamma})(y\phi_{\gamma,\beta\gamma})\big)\phi_{\beta\gamma,\alpha} && \text{(by the definition of } \theta\text{)}\\
&= (x\phi_{\beta,\beta\gamma}\phi_{\beta\gamma,\alpha})(y\phi_{\gamma,\beta\gamma}\phi_{\beta\gamma,\alpha}) && \text{(as } \phi_{\beta\gamma,\alpha} \text{ is a morphism)}\\
&= (x\theta)(y\theta) && \text{(by the definition of } \theta\text{).}
\end{aligned}$$

It is clear that θ is an onto morphism and $\theta_{|B_\alpha} = I_{B_\alpha}$. Hence B_α is a retract of B via θ. Therefore, by Corollary 6.1.1, $\mathrm{IG}(\mathcal{B}_\alpha)$ embeds into $\mathrm{IG}(\mathcal{B})$.

$\square$

In Lemma 6.1.7 we proved that if Y is a semilattice and $Y_0 \subseteq Y$ is a subsemigroup of Y, then there is embedding of $\mathrm{IG}(\mathcal{Y}_0)$ into $\mathrm{IG}(\mathcal{Y})$. However, the following example shows that even if B_0 is an ideal of an iso-normal band B, there does not always exists a monomorphism from $\mathrm{IG}(\mathcal{B}_0)$ to $\mathrm{IG}(\mathcal{B})$.

Example 6.2.5. Let $B = \mathscr{B}(Y, B_\mu, \phi_{\mu,\nu})$ be an iso-normal band, where $Y = \{\alpha, \beta, \gamma, \delta\}$, and let $B_0 = \mathcal{B}(Y_0, B_\mu, \phi_{\mu,\nu})$ be a strong semilattice Y_0, where $Y = \{\beta, \gamma, \delta\}$. See the figure below

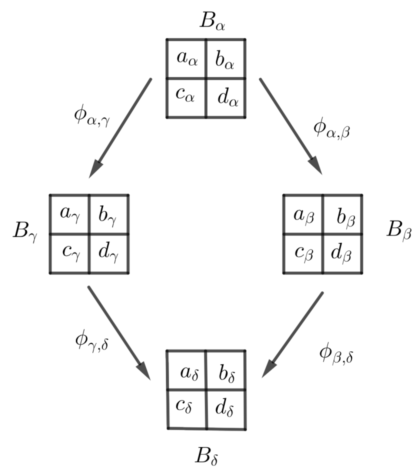


Figure 6.3: The semilattice decomposition structure of example 6.2.5

Let $\varphi : \mathrm{IG}(\mathcal{B}_0) \longrightarrow \mathrm{IG}(\mathcal{B})$, defined as above. Now we consider an element $\overline{a_\beta}\ \overline{d_\gamma} \in \mathrm{IG}(\mathcal{B})$, then we have

$$\overline{a_\beta}\ \overline{d_\gamma} = \overline{a_\beta}\ \overline{d_\beta}\ \overline{d_\gamma} \tag{6.3}$$

in $\mathrm{IG}(\mathcal{B})$, as

$$\begin{aligned} \overline{d_\beta}\ \overline{d_\gamma} &= \overline{d_\alpha}\ \overline{d_\gamma} \\ &= \overline{d_\gamma}\ \overline{d_\gamma} \\ &= \overline{d_\gamma}. \end{aligned}$$

In $\mathrm{IG}(\mathcal{B})$ we have

$$(\overline{a_\beta}\ \overline{d_\gamma})\varphi = \overline{a_\beta}\ \overline{d_\gamma} = \overline{a_\beta}\ \overline{d_\beta}\ \overline{d_\gamma} = (\overline{a_\beta}\ \overline{d_\beta}\ \overline{d_\gamma})\varphi.$$

Now $\overline{a_\beta} \neq \overline{a_\beta}\ \overline{d_\beta}$ in $\mathrm{IG}(\mathcal{B}_\beta)$ by the uniqueness of the normal form in $\mathrm{IG}(\mathcal{B}_\beta)$, so in $\mathrm{IG}(\mathcal{B}_0)$ by Lemma 5.3.20. Then $\overline{a_\beta}\ \overline{d_\gamma} \neq \overline{a_\beta}\ \overline{d_\beta}\ \overline{d_\gamma}$ in $\mathrm{IG}(\mathcal{B}_0)$. Hence φ is not injective morphism.

The following corollary is a special case of Lemma 6.1.9, where if B is an iso-normal band, then Lemma 6.1.9 is true for any $\delta \in Y$.

Corollary 6.2.6. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band. For each $\alpha \in Y$, there is a bimorphism $\boldsymbol{f_\alpha}$ from $\mathcal{B}$ to $\mathcal{B}_\alpha$:*

$$\boldsymbol{f_\alpha} : \mathcal{B} \longrightarrow \mathcal{B}_\alpha,$$

defined by

$$x \mapsto x\boldsymbol{f_\alpha} = x\phi_{\gamma,\alpha}, \quad \textit{if } x \in \mathcal{B}_\gamma.$$

We show that $\boldsymbol{f_\alpha}$ is a bimorphism. Let (g, h) be a basic pair in B with $g \in B_\sigma$ and $h \in B_\delta$. If $\delta \leq \sigma$, then $(g\boldsymbol{f_\delta}, h)$ is a basic pair in B_δ, and hence $(g\boldsymbol{f_\alpha}, h\boldsymbol{f_\alpha})$ is a basic pair in B. Similar arguments hold for the case when $\delta > \sigma$. Therefore, the mapping $\boldsymbol{f_\alpha}$ is a bimorphism.

We will show that for every $w \in B^+$ there is a word $v \in B^+$ having a very particular almost normal form which we will define, and such that $\overline{w} = \overline{v}$.

By Lemma 6.1.3, we have the following result.

Proposition 6.2.7. *For each $\alpha \in Y$, there is an epimorphism $\boldsymbol{f_\alpha} : \mathcal{B} \to \mathcal{B}_\alpha$ given by*

$$e_\beta \boldsymbol{f_\alpha} = e_\beta \phi_{\beta,\alpha}, \quad \textit{where } e_\beta \in \mathcal{B}_\beta$$

and consequently an epimorphism $\boldsymbol{F_\alpha}$ from $\mathrm{IG}(\mathcal{B})$ *onto* $\mathrm{IG}(\mathcal{B}_\alpha)$

$$\boldsymbol{F_\alpha} : \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_\alpha),$$

defined by

$$\overline{e_\beta} \mapsto \overline{e_\beta \boldsymbol{f_\alpha}}, \textit{ where } e_\beta \in \mathcal{B}_\beta.$$

Proof. The map $\boldsymbol{f_\alpha}$ is a bimorphism as for any e_β, $e_\gamma \in \mathcal{B}$, where $e_\beta \in \mathcal{B}_\beta$ and $e_\gamma \in \mathcal{B}_\gamma$, we get that

$$\begin{aligned}
(e_\beta e_\gamma)\boldsymbol{f_\alpha} &= \big((e_\beta\phi_{\beta,\beta\gamma})(e_\gamma\phi_{\gamma,\beta\gamma})\big)\boldsymbol{f_\alpha} && \text{(by definition of the multiplication on } B) \\
&= (e_\beta\phi_{\beta,\beta\gamma}\, e_\gamma\phi_{\gamma,\beta\gamma})\phi_{\beta\gamma,\alpha} && \text{(by definition of } \boldsymbol{f_\alpha}) \\
&= (e_\beta\phi_{\beta,\beta\gamma}\, \phi_{\beta\gamma,\alpha})(e_\gamma\phi_{\gamma,\beta\gamma}\, \phi_{\beta\gamma,\alpha}) && \text{(as } \phi_{\beta\gamma,\alpha} \text{ is a morphism)} \\
&= (e_\beta\phi_{\beta,\alpha})(e_\gamma\phi_{\gamma,\alpha}) && \text{(by Corollary 6.2.2)} \\
&= (e_\beta\boldsymbol{f_\alpha})(e_\gamma\boldsymbol{f_\alpha}).
\end{aligned}$$

By Lemma 6.1.6 we get that $\boldsymbol{F_\alpha}$ is a morphism from. The surjectivity of both, $\boldsymbol{f_\alpha}$ and $\boldsymbol{F_\alpha}$ are clear. $\square$

It is worth noting that $f_\alpha|_{B_\alpha}$ is the identity morphism, with a similar statement for $\boldsymbol{F_\alpha}$, for any $\alpha \in Y$. Further

$$\boldsymbol{f_\beta} = \boldsymbol{f_\alpha}\boldsymbol{f_\beta} \text{ and so } \boldsymbol{F_\alpha}\boldsymbol{F_\beta} = \boldsymbol{F_\beta} \text{ for all } \alpha, \beta \in Y. \tag{6.4}$$

Thus the sets $\{\boldsymbol{f_\alpha} : \alpha \in Y\}$ and $\{\boldsymbol{F_\alpha} : \alpha \in Y\}$ of all such morphisms form right zero bands under composition of mappings (from left to right). Hence, for any $\overline{w}, \overline{p} \in \mathrm{IG}(\mathcal{B})$, $\overline{w}\boldsymbol{F_\alpha} = \overline{p}\boldsymbol{F_\alpha}$ implies that $\overline{w}\boldsymbol{F_\beta} = \overline{p}\boldsymbol{F_\beta}$, by the equation 6.4. Further, it follows from Lemma 5.3.19 that, for any $x, y \in B$ and $\alpha, \beta \in Y$ with $\beta \geq \alpha$, we have

$$\overline{x\boldsymbol{f_\alpha}}\ \overline{y\boldsymbol{f_\beta}} = \overline{x\boldsymbol{f_\alpha}}\ \overline{y\boldsymbol{f_\alpha}} \text{ and } \overline{y\boldsymbol{f_\beta}}\ \overline{x\boldsymbol{f_\alpha}} = \overline{y\boldsymbol{f_\alpha}}\ \overline{x\boldsymbol{f_\alpha}}.$$

6.3 Special cases

It is known that $\mathrm{IG}(\mathcal{B})$ over an iso-normal band is not necessarily regular; for example when B is a semilattice $\{e, f, g\}$ with $g \leq e, f$ and $e \perp f$, $\mathrm{IG}(\mathcal{B})$ is not regular [6, Example 2]. As we stated at the beginning of this chapter that our main aim is to show the abundance of $\mathrm{IG}(\mathcal{B})$, where $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ is an iso-normal band. The proof of this result involved several steps. To help the reader to understand the main proof we are going to present two special cases in this section. One is the case where Y is a *diamond* and the other is the case where Y is a *fan*. A semilattice Y is called a *diamond* if $Y = \{\alpha, \beta, \gamma, \delta\}$, where α and δ are the upper and the lower bounds of Y, respectively, and $\beta \perp \gamma$, see the figure below.

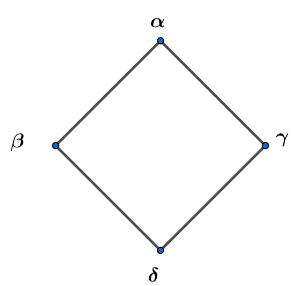


Figure 6.4: Diamond Semilattice

A semilattice Y is called a *fan* if Y has a lower bound δ and for any α_i, $\alpha_j \in Y$ we have $\alpha_i \perp \alpha_j$, $i \neq j$, see the figure below.

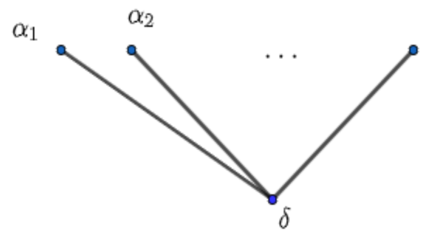


Figure 6.5: Fan Semilattice

An iso-normal band $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$, where Y is a diamond semilattice, is called a *diamond iso-normal band*. Similarly, if Y is a fan semilattice we called B a *fan iso-normal band*.

In order to prove the abundancy of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over a fan iso-normal band B, we introduce the concept of a *complete almost normal form*, where B is a normal band.

Definition 6.3.1. Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a normal band. Let

$$w = w_1 \circ \ldots \circ w_n \in B^+$$

be an almost normal form with $w_i \in B^+_{\alpha_i}$ for $1 \leq i \leq n$ and $\alpha_i \perp \alpha_{i+1}$ for all $1 \leq i \leq n-1$. We say $w = w_1 \circ \ldots \circ w_n$ is a *complete almost normal form* of $\overline{w}$ in $\mathrm{IG}(\mathcal{B})$, if it satisfies the condition that for each $1 \leq i \leq n$, $w_i \in B^+_{\alpha_i}$ is in a normal form of $\overline{w_i} \in \mathrm{IG}(\mathcal{B}_{\alpha_i})$.

In the above definition we call $w_i \in B^+_{\alpha_i}$ for all $1 \leq i \leq n$ the *block* of $\overline{w}$.

By Note 6.1.5, it is clear that w_i is a normal form of $\overline{w_i}$ in $\mathrm{IG}(\mathcal{B})$. From Theorem 5.3.5, we get that a normal form of $w_i \in B_{\alpha_i}$ is unique in $\mathrm{IG}(\mathcal{B}_{\alpha_i})$, as B_{α_i} is a rectangular band, for all $1 \leq i \leq n$. It is clear that the complete almost normal form always exists. Note that we are not saying this form is unique, as the expression of a word over B^+ as an almost normal form may have different blocks.

The following result show that the uniqueness of the the complete almost normal form of any element $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$ over a fan iso-normal band $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$.

Lemma 6.3.2. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a fan iso-normal band, where δ is a lower bound of Y. Then any element $\overline{w} \in \mathrm{IG}(\mathcal{B})$, has a unique complete almost normal form which is one of the following*

- $w = x_1 \circ \ldots \circ x_n \in B_\delta^+$;
- $w = w_1 \circ \ldots \circ w_n \in B^+$, $w_i \in B_{\alpha_i}^+$, *where* $\alpha_i \neq \delta$, *for all* $1 \leq i \leq n$ *and* $\alpha_i \neq \alpha_{i+1}$ *for all* $1 \leq i \leq n-1$.

Proof. Let $\overline{w} = \overline{z_1} \ldots \overline{z_m} \in \mathrm{IG}(\mathcal{B})$, where $z_i \in B_\delta$ for all $1 \leq i \leq m$, then $w = z_1 \circ \ldots \circ z_m \in B_\delta^+$. As B_δ is a rectangular band, by Lemma 5.3.5 there is a unique normal form $x_1 \circ \ldots \circ x_n \in B_\delta^+$ of $\overline{w}$ in $\mathrm{IG}(\mathcal{B}_\delta)$. Hence this form is the unique complete almost normal form with one block of $\overline{w}$ in $\mathrm{IG}(\mathcal{B})$.

Pick $\overline{w} = \overline{z_1} \cdots \overline{z_m} \in \mathrm{IG}(\mathcal{B})$, where $z_j \in B_\delta$ for some $1 \leq j \leq m$. Then by Lemma 6.1.10, we get that $\overline{w} = \overline{w}\mathbf{F}_\delta = (\overline{z_1 \mathbf{f}_\delta}) \ldots \overline{z_j} \ldots (\overline{z_m \mathbf{f}_\delta})$ in $\mathrm{IG}(\mathcal{B}_\delta)$. By Lemma 5.3.5, there is a unique normal form $x_1 \ldots x_n$ of $\overline{w}$ in $\mathrm{IG}(\mathcal{B}_\delta)$, where $x_i \in B_\delta$, for all $1 \leq i \leq n$. Hence $x_1 \ldots x_n$ is the unique complete almost normal form of $\overline{w} \in \mathrm{IG}(\mathcal{B})$.

Therefore, any $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$, that contains at least one letter from B_δ, has a unique complete almost normal form.

Now let $\overline{w} = \overline{w}_1 \cdots \overline{w}_t$ in $IG(\mathcal{B})$, where $w = w_1 \circ \ldots \circ w_t \in B^+$, $w_i \in B_{\alpha_i}^+$ and $\alpha_i \neq \delta$ for all $1 \leq i \leq t$, is a complete almost normal form of $\overline{w} \in \mathrm{IG}(\mathcal{B})$. Let $\overline{w} = \overline{u} = \overline{u}_1 \cdots \overline{u}_s$ in $IG(\mathcal{B})$, where $u = u_1 \circ \ldots \circ u_s \in B^+$, $u_i \in B_{\beta_i}^+$ and $\beta_i \neq \delta$ for all $1 \leq i \leq s$, be another complete almost normal form of $\overline{w} \in \mathrm{IG}(\mathcal{B})$. Since both $w_1 \circ \ldots \circ w_t$ and $u_1 \circ \ldots \circ u_s$ are two almost normal forms in B^+ of $\overline{w} = \overline{u}$, then by Lemma 5.3.12, we get that $s = t$ and $\alpha_i = \beta_i$ for all $1 \leq i \leq t$. Our aim is to show that $w_i = u_i$ in B_{α_i} for all $1 \leq i \leq t$. As $\overline{w} = \overline{u}$, we have

$$w_1 \circ \ldots \circ w_t \,\rho\, u_1 \circ \ldots \circ u_t.$$

If we apply a single relation $(x \circ y, xy)$ to $w_1 \circ \ldots \circ w_t$, then we get that $x, y \in w_i$ for some $1 \leq i \leq t$, as $\alpha_i \perp \alpha_{i+1}$ for all $1 \leq i \leq t-1$ and there is no upper bound. This implies that $w_i \,\rho\, u_i$, then $\overline{w_i} = \overline{u_i}$, for all $1 \leq i \leq t$. Now by the definition of the complete almost normal form we have that w_i and u_i are normal forms of $\overline{w_i} = \overline{u_i}$ of $\mathrm{IG}(\mathcal{B}_{\alpha_i})$. Since B_{α_i} is a rectangular band, then $\overline{w_i} = \overline{u_i}$ has a unique normal form, so $w_i = u_i$, as required.

□

Theorem 6.3.3. *Let $B = \mathcal{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a fan iso-normal band, where δ is the lower bound of Y. Then $\mathrm{IG}(\mathcal{B})$ is abundant.*

Proof. Let $\overline{w} = \overline{w_1}\ \overline{w_2} \dots \overline{w_n} \in \mathrm{IG}(\mathcal{B})$, where $w_i \in B_{\alpha_i}$ for all $1 \leq i \leq n$ and suppose that there is some $1 \leq j \leq n$ such that $w_j \in B_\delta$. Hence

$$\overline{w} = \overline{w}\mathbf{F}_\delta = \overline{w_1\mathbf{f}_\delta}\ \overline{w_2\mathbf{f}_\delta} \dots \overline{w_n\mathbf{f}_\delta},$$

as $\delta \leq \alpha_i$, for all $1 \leq i \leq n$. By the regularity of $\mathrm{IG}(\mathcal{B}_\delta)$, then

$$\overline{w_1\mathbf{f}_\delta}\ \mathcal{R}\ \overline{w}\mathbf{F}_\delta\ \mathcal{L}\ \overline{w_n\mathbf{f}_\delta}$$

in $\mathrm{IG}(\mathcal{B}_\delta)$, so in $\mathrm{IG}(\mathcal{B})$, by Corollary 5.3.8. Then any word that contains at least one letter of B_δ, is regular, so certainly an abundant element of $\mathrm{IG}(\mathcal{B})$.

To prove $\mathrm{IG}(\mathcal{B})$ is abundant, by Lemma 3.2.5 it is enough to show that for any word $\overline{w} = \overline{w_1}\ \overline{w_2} \dots \overline{w_n}$ of $\mathrm{IG}(\mathcal{B})$ and any $\overline{x}, \overline{z}$ of $\mathrm{IG}(\mathcal{B})$, if $\overline{x}\ \overline{w} = \overline{z}\ \overline{w}$, then $\overline{x}\ \overline{w_1} = \overline{z}\ \overline{w_1}$. Let $\overline{x},\ \overline{z} \in \mathrm{IG}(\mathcal{B})$, where

$$\overline{x} = \overline{x_1}\ \overline{x_2} \dots \overline{x_l} \quad \text{and} \quad \overline{z} = \overline{z_1}\ \overline{z_2} \dots \overline{z_m},$$

be the complete almost normal form of $\overline{x}, \overline{z}$, respectively, where $x_i \in B^+_{\gamma_i}$ for all $1 \leq i \leq l$, and $z_j \in B^+_{\beta_j}$ for all $1 \leq j \leq m$ and suppose that $\overline{x}\ \overline{w} = \overline{z}\ \overline{w}$.

Suppose that $w = w_1 \circ \dots \circ w_n \in B^+$, where $w_1 = w_{11} \dots w_{1n_1}$, $w_i \in B^+_{\alpha_i}$, $\alpha_i \neq \delta$, for all $1 \leq i \leq n$, be the complete almost normal form of $\overline{w} \in \mathrm{IG}(\mathcal{B})$, which means that $\overline{w}$ does not contain any letter from B_δ. Note that $\overline{x}$ contains a letter from B_δ if and only if $\overline{z}$ contains a letter from B_δ. Suppose first that both $\overline{x}$ and $\overline{z}$ contain letters from B_δ. Since $\overline{x}\ \overline{w} = \overline{z}\ \overline{w}$, we write

$$(\overline{x_1}\ \overline{x_2} \dots \overline{x_l})\,(\overline{w_1}\ \overline{w_2} \dots \overline{w_n}) = (\overline{z_1}\ \overline{z_2} \dots \overline{z_m})\,(\overline{w_1}\ \overline{w_2} \dots \overline{w_n}).$$

As $\delta \leq \alpha_k$, γ_i, β_j, for all $1 \leq k \leq n$, $1 \leq i \leq l$ and $1 \leq j \leq m$, we can write the above equality as

$$(\overline{x_1\mathbf{f}_\delta}\ \overline{x_2\mathbf{f}_\delta} \dots \overline{x_l\mathbf{f}_\delta})\ (\overline{w_1\mathbf{f}_\delta}\ \overline{w_2\mathbf{f}_\delta} \dots \overline{w_n\mathbf{f}_\delta}) = (\overline{z_1\mathbf{f}_\delta}\ \overline{z_2\mathbf{f}_\delta} \dots \overline{z_m\mathbf{f}_\delta})\ (\overline{w_1\mathbf{f}_\delta}\ \overline{w_2\mathbf{f}_\delta} \dots \overline{w_n\mathbf{f}_\delta})$$

in $\mathrm{IG}(\mathcal{B}_\delta)$, so in $\mathrm{IG}(\mathcal{B})$, by Corollary 5.3.20. As B_δ is rectangular band, $\mathrm{IG}(\mathcal{B}_\delta)$ is a regular semigroup, so $\overline{w_1\mathbf{f}_\delta}\,\mathcal{R}\,\overline{w_1\mathbf{f}_\delta} \dots \overline{w_n\mathbf{f}_\delta}$. Then we get that

$$(\overline{x_1\mathbf{f}_\delta}\ \overline{x_2\mathbf{f}_\delta} \dots \overline{x_l\mathbf{f}_\delta})\ \overline{w_1\mathbf{f}_\delta} = (\overline{z_1\mathbf{f}_\delta}\ \overline{z_2\mathbf{f}_\delta} \dots \overline{z_m\mathbf{f}_\delta})\ \overline{w_1\mathbf{f}_\delta}.$$

Hence $(\overline{x})\mathbf{F}_\delta\ \overline{w_1\mathbf{f}_\delta} = (\overline{z})\mathbf{F}_\delta\ \overline{w_1\mathbf{f}_\delta}$, in $\mathrm{IG}(\mathcal{B}_\delta)$, and by Corollary 5.3.20 we have

$$\overline{x}\,\overline{w_1} = (\overline{x}\mathbf{F}_\delta)(\overline{w_1\mathbf{f}_\delta}) = (\overline{z}\mathbf{F}_\delta)(\overline{w_1\mathbf{f}_{\delta_1}}) = \overline{z}\,\overline{w_1},$$

in IG($\mathcal{B}$). We have shown that if $\overline{w}$ contain a letter from B_δ or $\overline{x}$ (and hence $\overline{z}$ does), then the equality $\overline{x}\,\overline{w} = \overline{z}\,\overline{w}$, implies $\overline{x}\,\overline{w_1} = \overline{z}\,\overline{w_1}$.

Next, let $\alpha_i \neq \delta$, for all $1 \leq i \leq n$, $\gamma_i \neq \delta$, for all $1 \leq i \leq l$, and $\beta_i \neq \delta$, for all $1 \leq j \leq m$, which means that $\overline{w}$, $\overline{x}$ and $\overline{z}$ do not contain any letter form B_δ. Our aim is to show that $\overline{x}\,\overline{w_{11}} = \overline{z}\,\overline{w_{11}}$. We have three possible cases:

(i) If $\gamma_l = \alpha_1 = \beta_m$, then we have

$$\overline{x}\,\overline{w} = \overline{x}_1\overline{x}_2 \ldots (\overline{x_l \circ w_1}) \ldots \overline{w_n} = \overline{z}_1\overline{z}_2 \ldots (\overline{z_m \circ w_1}) \ldots \overline{w_n} = \overline{z}\,\overline{w}.$$

It is clear that both sides of the above equality are complete almost normal forms. By the uniqueness of this form, then both sides have the same number of blocks $l+n-1 = m+n-1$, so $l = m$, $\overline{x_i} = \overline{z_i}$ for all $1 \leq i \leq l-1$ and $\overline{x_l}\,\overline{w_1} = \overline{z_m}\,\overline{w_1}$ in IG($\mathcal{B}_{\alpha_1}$). By the regularity of IG($\mathcal{B}_{\alpha_1}$), we know that $\overline{w}_1\ \mathcal{R}\ \overline{w}_{11}$ in IG($\mathcal{B}_{\alpha_1}$), so in IG($\mathcal{B}$) by Corollary 3.2.2. Then $\overline{w}_1\ \mathcal{R}^*\ \overline{w}_{11}$ in IG($\mathcal{B}$) (as $\mathcal{R} \subseteq \mathcal{R}^*$), and as $\overline{x}_l\overline{w_1} = \overline{z}_m\overline{w_1}$. Then we get that

$$\overline{x}_l\,\overline{w_{11}} = \overline{z}_m\,\overline{w_{11}}, \tag{6.5}$$

Hence we get that $\overline{x}\,\overline{w_{11}} = \overline{z}\,\overline{w_{11}}$.

(ii) If $\gamma_l = \alpha_1 \neq \beta_m$, then we have

$$\overline{x}\,\overline{w} = \overline{x}_1\overline{x}_2 \ldots (\overline{x_l \circ w_1}) \ldots \overline{w_n} = \overline{z}_1\overline{z}_2 \ldots \overline{z}_m\overline{w}_1 \ldots \overline{w_n} = \overline{z}\,\overline{w}.$$

Both sides of the above equality are complete almost normal forms. As this form is unique, both sides have the same number of blocks $l+n-1 = m+n$, so $l-1 = m$, $x_i = z_i$ for all $1 \leq i \leq l-1$ and $x_l w_1 = w_1$ in IG($\mathcal{B}_{\alpha_1}$). By the regularity of IG($\mathcal{B}_{\alpha_1}$), we know that $\overline{w}_1\ \mathcal{R}\ \overline{w}_{11}$, hence $\overline{w}_1\ \mathcal{R}^*\ \overline{w}_{11}$, in IG($\mathcal{B}_{\alpha_1}$), so in IG($\mathcal{B}$) by Corollary 5.3.8. Then the equality $\overline{x}_l\overline{w_1} = \overline{w_1}$, implies $\overline{x}_l\,\overline{w_{11}} = \overline{w_{11}}$. Hence we get that $\overline{x}\,\overline{w_{11}} = \overline{z}\,\overline{w_{11}}$.

(iii) If $\alpha_1 \neq \gamma_l$ and $\alpha_1 \neq \beta_m$, then we write

$$\overline{x}\,\overline{w} = \overline{x}_1\overline{x}_2 \ldots \overline{x_l}\,\overline{w_1} \ldots \overline{w_n} = \overline{z}_1\overline{z}_2 \ldots \overline{z}_m\overline{w}_1 \ldots \overline{w_n} = \overline{z}\,\overline{w}.$$

Both sides of the above equality are complete almost normal forms. As this form is unique, we get that both sides have the same number of blocks $l+n = m+n$, so $l = m$, $x_i = z_i$ for all $1 \leq i \leq l$. Hence it is clear that $\overline{x}\,\overline{w_{11}} = \overline{z}\,\overline{w_{11}}$.

□

As a generalisation of Theorem 6.3.3, we have the following result.

Corollary 6.3.4. *Let Y_1 and Y_2 be fan semilattices, where δ_0 is the lower bound of both Y_1 and Y_2. Let $Y = Y_1 \cup Y_2$, and $Y_1 \cap Y_2 = \{\delta_0\}$. Let B_1 and B_2 be fan iso-normal bands, where*

$$B_1 = \mathscr{B}(Y_1, B_\alpha, \phi_{\alpha,\beta}),$$

and

$$B_2 = \mathscr{B}(Y_2, B_\delta, \psi_{\delta,\gamma}).$$

Then $\mathrm{IG}(\mathcal{B})$ *is abundant, where $B = \mathscr{B}(Y, B_\mu, \theta_{\mu,\nu})$ is an iso-normal band.*

Proof. It is clear that $B = \mathscr{B}(Y, B_\mu, \theta_{\mu,\nu})$ is a fan iso-normal band. Hence by Theorem 6.3.3 we get that $\mathrm{IG}(\mathcal{B})$ is an abundant semigroup. $\square$

With a different strategy, we also prove, as another special case, that $\mathrm{IG}(\mathcal{B})$ is abundant, where B is a diamond iso-normal band. Unlike the case of semilattices and rectangular bands [80], here we lose uniqueness of normal forms in $\mathrm{IG}(\mathcal{B})$. So we introduce a new special form of the elements of $\mathrm{IG}(\mathcal{B})$, where B is a diamond iso-normal band. Then we prove the uniqueness of this special form.

Lemma 6.3.5. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a diamond iso-normal band, where Y has an upper bound α. Then any element $\overline{w} = \overline{x_1} \dots \overline{x_n}$ of* $\mathrm{IG}(\mathcal{B})$*, where $x_i \in B_{\alpha_i}$ and $1 \leq i \leq n$, can be written as*

$$\overline{w} = (\overline{x_1\mathbf{f}_{\alpha_1}}\ \overline{x_2\mathbf{f}_{\alpha_1}} \dots \overline{x_n\mathbf{f}_{\alpha_1}})(\overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_n}}).$$

Proof. By induction on n. If $n = 1$, then it is clear the statement is true as

$$\overline{w} = \overline{x_1} = (\overline{x_1\mathbf{f}_{\alpha_1}})(\overline{x_1\mathbf{f}_{\alpha_1}}).$$

Suppose the statement true for $n-1$, then we can write

$$\overline{x_1}\ \overline{x_2} \dots \overline{x_{n-1}} = \overline{x_1\mathbf{f}_{\alpha_1}}\ \overline{x_2\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}(\overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_{n-1}\mathbf{f}_{\alpha_2}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_{n-1}}}). \quad (6.6)$$

Now we can write $\overline{w}$, by applying the statement to $\left(\overline{x_{n-1}\mathbf{f}_{\alpha_2}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_{n-1}}}\overline{x_n}\right)$, as the following

$$
\begin{aligned}
\overline{w} &= \overline{x_1}\ \overline{x_2} \dots \overline{x_{n-1}}\ \overline{x_n} \\
&= (\overline{x_1}\ \overline{x_2} \dots \overline{x_{n-1}})\overline{x_n} \\
&= \big(\overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}(\overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_{n-1}\mathbf{f}_{\alpha_2}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_{n-1}}})\big)\overline{x_n} && \text{(by 6.6)} \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\big((\overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_{n-1}\mathbf{f}_{\alpha_2}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_{n-1}}})\overline{x_n}\big) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\big((\overline{x_{n-1}\mathbf{f}_{\alpha_2}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_{n-1}}})\overline{x_n}\big) && \text{(as } \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_{n-1}\mathbf{f}_{\alpha_1}} = \overline{x_{n-1}\mathbf{f}_{\alpha_1}}) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\Big((\overline{x_{n-1}\mathbf{f}_{\alpha_2}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_2}})(\overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_n}})\Big) && \text{(by applying the statement)} \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\Big((\overline{x_{n-1}\mathbf{f}_{\alpha_2}})(\overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}})\Big) && \text{(as } \overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_2}} = \overline{x_n\mathbf{f}_{\alpha_2}}) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\Big((\overline{x_{n-1}\mathbf{f}_{\alpha}})(\overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}})\Big) && \text{(as } (\overline{x_{n-1}\mathbf{f}_{\alpha_2}})\mathbf{f}_{\alpha} = \overline{x_{n-1}\mathbf{f}_{\alpha}}) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\Big((\overline{x_{n-1}\mathbf{f}_{\alpha_1}})(\overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}})\Big) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\Big((\overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}})\Big) && \text{(as } \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_{n-1}\mathbf{f}_{\alpha_1}} = \overline{x_{n-1}\mathbf{f}_{\alpha_1}}) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\Big((\overline{x_n\mathbf{f}_{\alpha}}\ \overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}})\Big) && \text{(as } \overline{x_n\mathbf{f}_{\alpha}}\ \overline{x_n\mathbf{f}_{\alpha_2}} = \overline{x_n\mathbf{f}_{\alpha_2}}) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}(\overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}}) && \text{(as } \overline{x_n\mathbf{f}_{\alpha}\mathbf{f}_{\alpha_1}} = \overline{x_n\mathbf{f}_{\alpha_1}}) \\
&= \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_1}}(\overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}}) && \text{(as } \overline{x_n\mathbf{f}_{\alpha_1}} = \overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_1}})
\end{aligned}
$$

So the statement is true for any natural number n. Therefore, any element $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$ can be written as

$$\overline{w} = \overline{x_1}\ \overline{x_2} \dots \overline{x_n} = \overline{x_1\mathbf{f}_{\alpha_1}} \dots \overline{x_{n-1}\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_1}}(\overline{x_n\mathbf{f}_{\alpha_2}} \dots \overline{x_n\mathbf{f}_{\alpha_{n-1}}}\,\overline{x_n\mathbf{f}_{\alpha_n}}).$$

□

Example 6.3.6. Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a diamond iso-normal band (see the figure below),

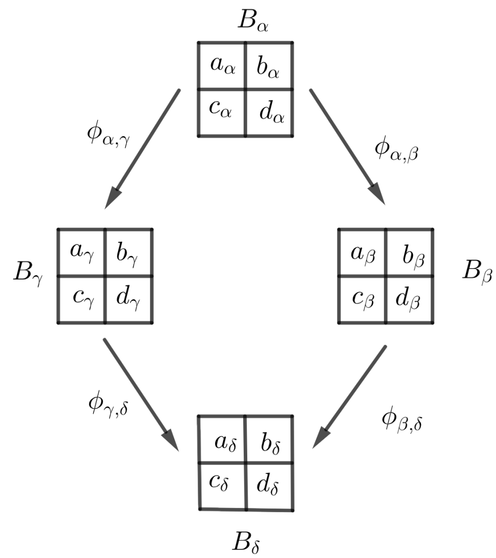


Figure 6.6: The semilattice decomposition structure of Example 6.3.6

and $\phi_{\mu,\nu}\colon B_\mu \longrightarrow B_\nu$ is defined by

$$e_\mu \phi_{\mu,\nu} = e_\nu,$$

for any $\mu \geq \nu$, $\mu, \nu \in Y$ and $e \in \{a, b, c, d\}$.

The element $\overline{w} = \overline{a_\alpha}\,\overline{b_\delta}\,\overline{a_\beta}\,\overline{c_\gamma}$ of $\mathrm{IG}(\mathcal{B})$, can be written as

$$\begin{aligned}
\overline{w} &= a_\alpha\,\overline{b_\delta}\,\overline{a_\beta}\,\overline{c_\gamma} \\
&= (\overline{a_\alpha\mathbf{f}_\alpha}\,\overline{b_\delta\mathbf{f}_\alpha}\,\overline{a_\beta\mathbf{f}_\alpha}\,\overline{c_\gamma\mathbf{f}_\alpha})(\overline{c_\gamma\mathbf{f}_\alpha}\,\overline{c_\gamma\mathbf{f}_\delta}\,\overline{c_\gamma\mathbf{f}_\beta}\,\overline{c_\gamma\mathbf{f}_\gamma}) \\
&= (\overline{a_\alpha \circ b_\alpha \circ a_\alpha \circ c_\alpha})(\overline{c_\delta}\,\overline{c_\beta}\,\overline{c_\gamma}) \\
&= (\overline{a_\alpha \circ c_\alpha})(\overline{c_\delta}\,\overline{c_\beta}\,\overline{c_\gamma}) && (as\ a_\alpha b_\alpha a_\alpha = a_\alpha) \\
&= \overline{a_\alpha}(\overline{c_\delta}\,\overline{c_\beta}\,\overline{c_\gamma}). && (as\ a_\alpha \mathcal{L}\, c_\alpha \text{ in } B_\alpha).
\end{aligned}$$

If $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ is a diamond iso-normal band, then any almost normal form $w = x_1 \circ \ldots \circ x_n \in B^+$ of $\overline{w} \in \mathrm{IG}(\mathcal{B})$, where $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$, can be written as

$$\begin{aligned}
\overline{w} &= \overline{x_1} \ldots \overline{x_n} \\
&= (\overline{x_1 f_{\alpha_1}}\,\overline{x_2 f_{\alpha_1}} \ldots \overline{x_n f_{\alpha_1}})(\overline{x_n f_{\alpha_2}} \ldots \overline{x_n f_{\alpha_n}}) \\
&= (\overline{p_1} \ldots \overline{p_m})(\overline{p_m \mathbf{f}_{\alpha_n}} \ldots \overline{p_m \mathbf{f}_{\alpha_n}}),
\end{aligned}$$

where $p_1 \circ \ldots \circ p_m \in B^+_{\alpha_1}$ is the normal form of the head $\overline{x_1 f_{\alpha_1}}\ \overline{x_2 f_{\alpha_1}} \ldots \overline{x_n f_{\alpha_1}}$ in $\mathrm{IG}(\mathcal{B})$. It is clear that the order $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the *Y-trace* of the element $\overline{w}$ in $\mathrm{IG}(\mathcal{B})$.

In the following we prove the abundancy of the free idempotent generated semigroup over a diamond iso-normal band B.

Theorem 6.3.7. *Let $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a diamond iso-normal band, where Y has an upper and a lower bounds α and δ, respectively. Then $\mathrm{IG}(\mathcal{B})$ is abundant.*

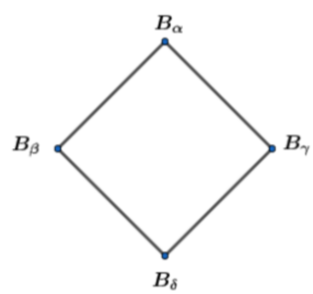


Figure 6.7: The Diamond Semilattice

Proof. To prove the abundance of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$, it is enough to prove that $\mathrm{IG}(\mathcal{B})$ is isomorphic to an abundant semigroup.

By Theorem 5.3.6, we know $\mathrm{IG}(\mathcal{B}_\delta)$ and $\mathrm{IG}(\mathcal{Y})$ are both abundant, where B_δ and Y are rectangular band and semilattice, respectively. Hence the external direct product of them is abundant by Lemma 4.1.3.

Define a map

$$F_\delta \colon\ \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_\delta)$$

by $\overline{w}F_\delta = \overline{x_1 \mathbf{f}_\delta} \ldots \overline{x_n \mathbf{f}_\delta}$, for $\overline{w} = \overline{x_1}\ \overline{x_2} \ldots \overline{x_n} \in \mathrm{IG}(\mathcal{B})$, where $w = x_1 \circ \ldots \circ x_n \in B^+$ and each $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Also, define a map T

$$T \colon\ \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{Y})$$

by $\overline{w}T = \overline{\alpha_1} \ldots \overline{\alpha_n}$. Now define a map

$$\varphi \colon\ \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_\delta) \times \mathrm{IG}(\mathcal{Y}).$$

by

$$(\overline{w})\varphi = (\overline{w}F_\delta, \overline{w}T).$$

It is clear that φ is a morphism, as F_δ and T are both morphisms by Lemma 6.1.10 and Lemma 5.3.9, respectively. Our aim here is to show that φ is an isomorphism.

- By Lemma 5.3.9 and Lemma 6.2.7 we get that T and F_δ are onto morphisms. Then φ is an onto morphism,
- To prove that φ is an injective morphism. Let $w = x_1 \circ \ldots \circ x_n$, $u = y_1 \circ \ldots \circ y_m \in B^+$ be the almost normal forms of $\overline{w}$, $\overline{u} \in \mathrm{IG}(\mathcal{B})$, respectively. Let

$$\begin{aligned}\overline{w} &= \overline{x_1} \ldots \overline{x_n}\\ &= (\overline{x_1\mathbf{f}_{\alpha_1}}\ \overline{x_2\mathbf{f}_{\alpha_1}} \ldots \overline{x_n\mathbf{f}_{\alpha_1}})(\overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_2}} \ldots \overline{x_n\mathbf{f}_{\alpha_n}}),\end{aligned}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the Y-trace of $\overline{w}$, and

$$\begin{aligned}\overline{u} &= \overline{y_1} \ldots \overline{y_m}\\ &= (\overline{y_1\mathbf{f}_{\beta_1}}\ \overline{y_2\mathbf{f}_{\beta_1}} \ldots \overline{y_m\mathbf{f}_{\beta_1}})(\overline{y_m\mathbf{f}_{\beta_1}}\ \overline{y_m\mathbf{f}_{\beta_2}} \ldots \overline{y_m\mathbf{f}_{\beta_m}}),\end{aligned}$$

where $(\beta_1, \beta_2, \ldots, \beta_m)$, is the Y-trace of $\overline{u}$.
Suppose

$$\overline{w}\varphi = (\overline{w}F_\delta, \overline{w}T) = (\overline{u}F_\delta, \overline{u}T) = \overline{u}\varphi,$$

which implies

$$\overline{w}T = \overline{\alpha_1}\ \overline{\alpha_2} \ldots \overline{\alpha_n} = \overline{\beta_1}\ \overline{\beta_2} \ldots \overline{\beta_m} = \overline{u}T.$$

Hence $\overline{w}$ and $\overline{u}$ have the same Y-trace. As α_i and α_{i+1} are incomparable, then $\overline{w}T$ is in normal form in $\mathrm{IG}(\mathcal{Y})$. Similarly, for $\overline{u}T$. Then by uniqueness of normal form in $\mathrm{IG}(\mathcal{Y})$ (as Y is a semilattice), we get that $n = m$, and $\alpha_i = \beta_i$, for all $1 \leq i \leq n$.

As the following equality

$$\begin{aligned}\overline{w}F_\delta &= \overline{x_1\mathbf{f}_\delta} \ldots \overline{x_n\mathbf{f}_\delta}\\ &= \overline{y_1\mathbf{f}_\delta} \ldots \overline{y_m\mathbf{f}_\delta}\\ &= \overline{u}F_\delta\end{aligned}$$

holds in $\mathrm{IG}(\mathcal{B}_\delta)$, so in $\mathrm{IG}(\mathcal{B})$ by Corollary 5.3.20. Further, $\overline{w}F_\mu = \overline{u}F_\mu$ in $\mathrm{IG}(\mathcal{B}_\mu)$, and so in $\mathrm{IG}(\mathcal{B})$, for all $\mu \in Y$.

As $\mathrm{IG}(\mathcal{B}_\mu)$ is a regular for all $\mu \in Y$, then

$$\overline{x_n\mathbf{f}_{\alpha_i}}\ \mathcal{L}\ \overline{w}F_{\alpha_i} = \overline{u}F_{\alpha_i}\ \mathcal{L}\ \overline{y_m\mathbf{f}_{\alpha_i}}, \qquad \text{for all} \quad 1 \leq i \leq n.$$

Hence

$$\begin{aligned}
\overline{w} &= \overline{x_1\mathbf{f}_{\alpha_1}}\ \overline{x_2 f\alpha_1}\ldots\overline{x_n\mathbf{f}_{\alpha_1}}\Big(\overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\overline{y_m\mathbf{f}_{\alpha_1}}\Big(\overline{x_n\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) && (as\ \overline{w}F_{\alpha_1} = \overline{u}F_{\alpha_1}) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\Big(\overline{y_m\mathbf{f}_{\alpha_1}}\ \overline{x_n\mathbf{f}_{\alpha_1}}\Big)\Big(\overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\Big(\overline{y_m\mathbf{f}_{\alpha_1}}\Big)\Big(\overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) && (as\ \overline{y_m\mathbf{f}_{\alpha_1}}\ \mathcal{L}\ \overline{x_n\mathbf{f}_{\alpha_1}}) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\Big(\overline{y_m\mathbf{f}_{\alpha_1}}\ \overline{y_m\mathbf{f}_{\alpha}}\Big)\Big(\overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) && (as\ \overline{y_m\mathbf{f}_{\alpha_1}} = \overline{y_m\mathbf{f}_{\alpha_1}}\ \overline{y_m\mathbf{f}_{\alpha}}) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\Big(\overline{y_m\mathbf{f}_{\alpha_1}}\ \overline{y_m\mathbf{f}_{\alpha_2}}\Big)\Big(\overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) && ((\overline{y_m\mathbf{f}_{\alpha}})\phi_{\alpha,\alpha_2} = \overline{y_m\mathbf{f}_{\alpha_2}}) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\overline{y_m\mathbf{f}_{\alpha_1}}\ \overline{y_m\mathbf{f}_{\alpha_2}}\Big(\overline{x_n\mathbf{f}_{\alpha_2}}\ \overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\overline{y_m\mathbf{f}_{\alpha_1}}\ \Big(\overline{y_m\mathbf{f}_{\alpha_2}}\,\overline{x_n\mathbf{f}_{\alpha_2}}\Big)\overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\overline{y_m\mathbf{f}_{\alpha_1}}\ \Big(\overline{y_m\mathbf{f}_{\alpha_2}}\Big)\overline{x_n\mathbf{f}_{\alpha_3}}\ldots\overline{x_n\mathbf{f}_{\alpha_n}}\Big) && (as\ \overline{y_m\mathbf{f}_{\alpha_2}}\ \mathcal{L}\ \overline{x_n\mathbf{f}_{\alpha_2}}) \\
&\quad\vdots \\
&= \overline{y_1\mathbf{f}_{\alpha_1}}\ \overline{y_2\mathbf{f}_{\alpha_1}}\ldots\overline{y_m\mathbf{f}_{\alpha_1}}\ \Big(\overline{y_m\mathbf{f}_{\alpha_1}}\ \overline{y_m\mathbf{f}_{\alpha_2}}\ldots\overline{y_m\mathbf{f}_{\alpha_n}}\Big) \\
&= \overline{u}.
\end{aligned}$$

Hence φ is an one-to-one morphism. Therefore, $\mathrm{IG}(\mathcal{B})$ is isomorphic to an abundant semigroup, which gives that $\mathrm{IG}(\mathcal{B})$ is abundant.

$\square$

We remark here that Theorem 6.3.3 and Theorem 6.3.7 can also be obtained as a corollaries of Theorem 6.5.10, but for the sake of our readers, we have proved this special case to outline our strategy in a simple case.

6.4 The forms of the elements of $\mathrm{IG}(\mathcal{B})$ over iso-normal bands

Unlike the case of semilattices and rectangular bands, we may lose the uniqueness of normal forms of the elements in $\mathrm{IG}(\mathcal{B})$, where B is an iso-normal band. To overcome this problem, the concepts *complete form* and *double normal form* are introduced. These forms are used in our whole work in this chapter.

The following results show that any element of $\mathrm{IG}(\mathcal{B})$ can be written in these forms, which generalises the result about the forms of $\mathrm{IG}(\mathcal{B})$ over a diamond iso-normal band in Lemma 6.3.5.

Lemma 6.4.1. *Let $\overline{w} = \overline{x_1} \dots \overline{x_n} \in \mathrm{IG}(\mathcal{B})$, where $x_i \in B_{\alpha_i}$ for all $1 \leq i \leq n$ and $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$. Then*

$$\overline{x_1} \dots \overline{x_n} = (\overline{x_1 \mathbf{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_n \mathbf{f}_{\boldsymbol{\alpha_1}}})(\overline{x_n \mathbf{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_n \mathbf{f}_{\boldsymbol{\alpha_n}}}).$$

Proof. We argue by induction on n. Clearly the statement is true for $n = 1$, as $\overline{x_1} = \overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} = \overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}}$. Suppose now that the result is true for all $k < n$. Pick $\alpha \in \alpha_1 \sqcap \alpha_2$. We have

$$\begin{aligned}
\overline{w} &= \overline{x_1} \dots \overline{x_n} \\
&= (\overline{x_1} \dots \overline{x_{n-1}})\ \overline{x_n} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})(\overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_{n-1}}}})\ \overline{x_n} && \text{(by induction)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})(\overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_{n-1}}}}\ \overline{x_n}) && \text{(as } \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}} = \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})(\overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}})(\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(by induction and (6.4))} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ (\overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}})\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ (\overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}})\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} = \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ (\overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha}}})\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} = \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha}}} = \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}} = \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} = \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\,\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha}}}\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} = \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}})\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ (\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_2}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}) && \text{(as } \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha}}} = \overline{x_{n-1} \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\ \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_1}}}\text{)} \\
&= (\overline{x_1 \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_1}}})(\overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_1}}} \dots \overline{x_n \boldsymbol{f}_{\boldsymbol{\alpha_n}}}).
\end{aligned}$$

$\square$

For an element $\overline{w} = \overline{x_1}\ \overline{x_2} \dots \overline{x_n}$ of $\mathrm{IG}(\mathcal{B})$, where $x_i \in B_{\alpha_i}$, the form

$$(\overline{x_1 \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_1}}})(\overline{x_n \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}})$$

is called *the complete form* of $\overline{w}$. The part

$$(\overline{x_n \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}})$$

is called *the tail* of the complete form of $\overline{w}$.

Note that any word $w = x_1 \circ \dots \circ x_n \in B^+$, where $x_i \in B_{\alpha_i}$, and $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$, not necessarily w in almost normal form, we always can find a complete form of $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$ with the above procedure.

Corollary 6.4.2. *Let $\overline{w} = \overline{x_1} \dots \overline{x_n} \in \mathrm{IG}(\mathcal{B})$ be an element defined as in Lemma 6.4.1. Then*

$$\overline{x_1} \dots \overline{x_n} = (\overline{p_1} \dots \overline{p_s})(\overline{p_s \boldsymbol{f_{\alpha_1}}} \dots \overline{p_s \boldsymbol{f_{\alpha_n}}}),$$

where $p_1 \circ \dots \circ p_s \in B^+_{\alpha_1}$ is a normal form of $\overline{x_1 \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_1}}}$ in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$.

Proof. First, as $\overline{p_1} \dots \overline{p_s} = \overline{x_1 \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_1}}}$ in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$, then from Corollary 5.3.8 we get that

$$\overline{x_n \boldsymbol{f_{\alpha_1}}}\ \mathcal{L}\ \overline{x_1 \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_1}}} = \overline{p_1} \dots \overline{p_s}\ \mathcal{L}\ \overline{p_s}$$

in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$, so in $\mathrm{IG}(\mathcal{B})$. By the definition of iso-normal bands, $\phi_{\alpha,\beta}$ is an isomorphism for all $\alpha, \beta \in Y$, so that from the above we get that $\overline{p_s \boldsymbol{f_{\alpha_i}}}\ \mathcal{L}\ \overline{x_n \boldsymbol{f_{\alpha_i}}}$, in $\mathrm{IG}(\mathcal{B}_{\alpha_i})$, for all $1 \leq i \leq n$. We now have

$$\begin{aligned}
\overline{w} &= \overline{x_1} \dots \overline{x_n} \\
&= (\overline{x_1 \boldsymbol{f_{\alpha_1}}} \dots \overline{x_n \boldsymbol{f_{\alpha_1}}})(\overline{x_n \boldsymbol{f_{\alpha_1}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) && \text{(by Lemma 6.4.1)} \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{x_n \boldsymbol{f_{\alpha_1}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{p_s}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) && (\text{ as } \overline{p_s}\ \mathcal{L}\ \overline{x_n \boldsymbol{f_{\alpha_1}}}) \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{p_s \boldsymbol{f_{\alpha_1}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) && (\text{ as } p_s = p_s \boldsymbol{f_{\alpha_1}}) \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{p_s \boldsymbol{f_{\alpha_1}}}\ \overline{p_s \boldsymbol{f_{\alpha}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) && (\text{ as } \overline{p_s \boldsymbol{f_{\alpha_1}}} = \overline{p_s \boldsymbol{f_{\alpha_1}}}\ \overline{p_s \boldsymbol{f_{\alpha}}},\ \alpha \in \alpha_1 \sqcap \alpha_2) \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{p_s \boldsymbol{f_{\alpha_1}}}\ \overline{p_s \boldsymbol{f_{\alpha_2}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) && (\text{ as } \overline{p_s \boldsymbol{f_{\alpha}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}} = \overline{p_s \boldsymbol{f_{\alpha_2}}}\ \overline{x_n \boldsymbol{f_{\alpha_2}}}) \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{p_s \boldsymbol{f_{\alpha_1}}}\ \overline{p_s \boldsymbol{f_{\alpha_2}}}\ \overline{x_n \boldsymbol{f_{\alpha_3}}} \dots \overline{x_n \boldsymbol{f_{\alpha_n}}}) && (\text{ as } \overline{p_s \boldsymbol{f_{\alpha_2}}}\ \mathcal{L}\ \overline{x_n \boldsymbol{f_{\alpha_2}}}) \\
&\ \vdots \\
&= (\overline{p_1} \dots \overline{p_s})(\overline{p_s \boldsymbol{f_{\alpha_1}}} \dots \overline{p_s \boldsymbol{f_{\alpha_n}}}).
\end{aligned}$$

$\square$

In the above proof, notice that $p_s\boldsymbol{f_{\alpha_1}} \circ \ldots \circ p_s\boldsymbol{f_{\alpha_n}} \in B^+$ is a normal form of $\overline{p_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s\boldsymbol{f_{\alpha_n}}}$ in $\mathrm{IG}(\mathcal{B})$ if and only if $\alpha_1 \circ \ldots \circ \alpha_n \in Y^+$ is a normal form of $\overline{\alpha_1} \ldots \overline{\alpha_n}$ in $\mathrm{IG}(\mathcal{Y})$.

Definition 6.4.3. Let B be an iso-normal band. Let

$$w = x_1 \circ x_2 \circ \ldots \circ x_n \in B^+,$$

be an almost normal form of $\overline{w} \in \mathrm{IG}(\mathcal{B})$, where $x_i \in B_{\alpha_i}$, $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$, the form

$$\overline{w} = (\overline{p_1} \ldots \overline{p_s})(\overline{p_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s\boldsymbol{f_{\alpha_n}}})$$

is called *a double normal form* of $\overline{w}$, if $p_1 \circ \ldots \circ p_s \in B^+_{\alpha_1}$ is the normal form of $\overline{x_1\boldsymbol{f_{\alpha_1}}} \ldots \overline{x_n\boldsymbol{f_{\alpha_1}}}$ in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$, and $p_s\boldsymbol{f_{\alpha_1}} \circ \ldots \circ p_s\boldsymbol{f_{\alpha_n}} \in B^+$ is an almost normal form of $(\overline{p_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s\boldsymbol{f_{\alpha_n}}})$ in $\mathrm{IG}(\mathcal{B})$. The parts

$$(\overline{p_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s\boldsymbol{f_{\alpha_n}}}) \text{ and } \overline{p_1} \ldots \overline{p_s}$$

are called *the tail* and the *head* of the double normal form of $\overline{w}$, respectively.

It is clear from the above definition that $(\alpha_1, \ldots, \alpha_n)$ is the Y-trace of $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$.

In the following we prove the uniqueness of the double normal form of the elements of $\mathrm{IG}(\mathcal{B})$, where B is an iso-normal band.

Lemma 6.4.4. *Let $\overline{w} = \overline{x_1} \ldots \overline{x_n} \in \mathrm{IG}(\mathcal{B})$, where $x_i \in B_{\alpha_i}$ for all $1 \leq i \leq n$ and $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$. Then the double normal from of $\overline{w}$ is unique.*

Proof. Let $\overline{w} \in \mathrm{IG}(\mathcal{B})$ and

$$\begin{aligned} \overline{w} &= (\overline{p_1} \ldots \overline{p_s})(\overline{p_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s\boldsymbol{f_{\alpha_n}}}) \\ &= (\overline{q_1} \ldots \overline{q_t})(\overline{q_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{q_t\boldsymbol{f_{\alpha_n}}}) \end{aligned} \tag{6.7}$$

be double normal forms of $\overline{w}$. As there exists a morphism $\boldsymbol{f_{\alpha_1}} : \mathcal{B} \longrightarrow \mathcal{B}_{\alpha_1}$, so by Proposition 6.2.7, there is a morphism

$$\boldsymbol{F_{\alpha_1}} \colon \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_{\alpha_1}).$$

As the equality in (6.7) holds in $\mathrm{IG}(\mathcal{B})$, then the following equality holds in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$

$$\begin{aligned} \big((\overline{p_1} \ldots \overline{p_s})(\overline{p_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s\boldsymbol{f_{\alpha_n}}})\big)\boldsymbol{F_{\alpha_1}} &= \overline{p_1} \ldots \overline{p_s} \\ &= \overline{q_1} \ldots \overline{q_t} \\ &= \big((\overline{q_1} \ldots \overline{q_t})(\overline{q_s\boldsymbol{f_{\alpha_1}}} \ldots \overline{q_t\boldsymbol{f_{\alpha_n}}})\big)\boldsymbol{F_{\alpha_1}}. \end{aligned}$$

As $p_1 \circ \ldots \circ p_s,\ q_1 \circ \ldots \circ q_t \in B^+_{\alpha_1}$ are the normal forms of $\overline{p_1} \ldots \overline{p_s}$ and $\overline{q_1} \ldots \overline{q_t}$ in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$, and by the uniqueness of the normal form in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$, we get that $s = t$ and $\overline{p_j} = \overline{q_j}$, for all $1 \leq j \leq s$ in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$, so in $\mathrm{IG}(\mathcal{B})$ by Corollary 5.3.20. In particular, we have $\overline{p_s} = \overline{q_s}$. As B is an iso-normal band, then $\overline{p_s \boldsymbol{f_{\alpha_i}}} = \overline{q_s \boldsymbol{f_{\alpha_i}}}$, in $\mathrm{IG}(\mathcal{B}_{\alpha_i})$ for all $1 \leq i \leq n$. Therefore, we get

$$\overline{p_s \boldsymbol{f_{\alpha_1}}} \ldots \overline{p_s \boldsymbol{f_{\alpha_n}}} = \overline{q_s \boldsymbol{f_{\alpha_1}}} \ldots \overline{q_s \boldsymbol{f_{\alpha_n}}}.$$

This proves the uniqueness of the double normal form of any $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$, where B is an iso-normal band. $\square$

Lemma 6.4.5. *Let $\overline{w} = \overline{w_1} \ldots \overline{w_n},\ \ \overline{p} = \overline{p_1} \ldots \overline{p_m} \in \mathrm{IG}(\mathcal{B})$. Let*

$$w_1 \circ \ldots \circ w_n,\ p_1 \circ \ldots \circ p_m \in B^+$$

be almost normal forms of $\overline{w}$ and $\overline{p}$, respectively, where $w_i \in B^+_{\alpha_i}$ for all $1 \leq i \leq n$, $\alpha_i \perp \alpha_{i+1}$ for all $1 \leq i \leq n-1$, $(\alpha_1, \ldots, \alpha_n)$ is the Y-trace of $\overline{w}$ and $p_j \in B^+_{\beta_j}$ for all $1 \leq j \leq m$, $\beta_j \perp \beta_{j+1}$ for all $1 \leq j \leq m-1$, $(\beta_1, \ldots, \beta_m)$ is Y-trace of $\overline{p}$. Suppose that $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$ and $\beta_j \sqcap \beta_{j+1} \neq \emptyset$ for all $1 \leq j \leq m-1$. Then the following statements are equivalent:

(i) $\overline{w} = \overline{p}$ in $\mathrm{IG}(\mathcal{B})$;

(ii) $\overline{w_1} \ldots \overline{w_n} = \overline{p_1} \ldots \overline{p_m}$ in $\mathrm{IG}(\mathcal{B})$;

(iii) $n = m$, $\alpha_i = \beta_i$ for all $1 \leq i \leq n$ and $(\overline{w_1} \ldots \overline{w_n})\boldsymbol{F_{\alpha_1}} = (\overline{p_1} \ldots \overline{p_m})\boldsymbol{F_{\alpha_1}}$ in $\mathrm{IG}(\mathcal{B}_{\alpha_1})$.

Proof. The equivalence of (i) and (ii) is by definition.
$(ii) \Rightarrow (iii)$ follows from Lemmas 5.3.9, 6.2.7 and 5.3.6.
To show $(iii) \Rightarrow (ii)$, let $u_1 \circ \ldots \circ u_l \in B^+_{\alpha_1}$ be the normal form of $(\overline{w_1} \ldots \overline{w_n})\boldsymbol{F_{\alpha_1}}(= (\overline{p_1} \ldots \overline{p_m})\boldsymbol{F_{\alpha_1}})$. Then, by Corollary 6.4.2

$$\overline{w_1} \ldots \overline{w_n} = (\overline{u_1} \ldots \overline{u_l})(\overline{u_l \boldsymbol{f_{\alpha_1}}} \ldots \overline{u_l \boldsymbol{f_{\alpha_n}}}) = (\overline{u_1} \ldots \overline{u_l})(\overline{u_l \boldsymbol{f_{\beta_1}}} \ldots \overline{u_l \boldsymbol{f_{\beta_n}}}) = \overline{p_1} \ldots \overline{p_m}.$$

$\square$

In the proof of the abundancy of $\mathrm{IG}(\mathcal{B})$ over a diamond iso-normal band B we proved that

$$\mathrm{IG}(\mathcal{B}) \cong \mathrm{IG}(\mathcal{B}_{\delta_0}) \times \mathrm{IG}(\mathcal{Y}),$$

where δ_0 is the least of Y. In the following we show that if B is an iso-normal band and for all $\beta, \delta \in Y$, we have $\beta \sqcap \delta \neq \emptyset$. Then for any $\alpha \in Y$, we have

$$\mathrm{IG}(\mathcal{B}) \cong \mathrm{IG}(\mathcal{B}_{\alpha}) \times \mathrm{IG}(\mathcal{Y}).$$

Proposition 6.4.6. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band such that $\beta \sqcap \delta \neq \emptyset$ for any $\beta, \delta \in Y$. Let $\alpha \in Y$ be fixed. Then*

$$\mathrm{IG}(\mathcal{B}) \cong \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y}).$$

Proof. We define a mapping $\boldsymbol{\psi}$ from $\mathrm{IG}(\mathcal{B})$ to $\mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$ as follows:

$$\boldsymbol{\psi} : \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y}), \ \ \overline{w} \mapsto (\overline{w}\boldsymbol{F_\alpha}, \overline{w}\boldsymbol{T}).$$

It follows immediately from Lemmas 5.3.9 and 6.2.7 that $\boldsymbol{\psi}$ is a well defined morphism. We now show that $\boldsymbol{\psi}$ is a bijection. Let $\overline{u_1} \dots \overline{u_n}, \overline{v_1} \dots \overline{v_m} \in \mathrm{IG}(\mathcal{B})$. Then, by Lemma 5.3.11, we may assume that both of $u_1 \circ \dots \circ u_n$, $v_1 \circ \dots \circ v_m \in B^+$ are almost normal forms of $\overline{u_1} \dots \overline{u_n}$ and $\overline{v_1} \dots \overline{v_m}$, respectively, in $\mathrm{IG}(\mathcal{B})$, with $u_i \in B^+_{\beta_i}$ for all $1 \leq i \leq n$ and $v_j \in B^+_{\gamma_j}$ for all $1 \leq j \leq m$ such that $\beta_i \perp \beta_{i+1}$ for all $1 \leq i \leq n-1$ and $\gamma_j \perp \gamma_{j+1}$ for all $1 \leq j \leq m-1$.

To show that $\boldsymbol{\psi}$ is one-one, let $(\overline{u_1} \dots \overline{u_n})\boldsymbol{\psi} = (\overline{v_1} \dots \overline{v_m})\boldsymbol{\psi}$. Then we have

$$(\overline{u_1} \dots \overline{u_n})\boldsymbol{F_\alpha} = (\overline{v_1} \dots \overline{v_m})\boldsymbol{F_\alpha} \text{ and } \overline{\beta_1} \dots \overline{\beta_n} = \overline{\gamma_1} \dots \overline{\gamma_m}.$$

Notice that both sides of the second equality are in normal form, giving $n = m$ and $\beta_i = \gamma_i$ for all $1 \leq i \leq n$. By (6.4),

$$(\overline{u_1} \dots \overline{u_n})\boldsymbol{F_{\beta_1}} = (\overline{v_1} \dots \overline{v_m})\boldsymbol{F_{\beta_1}},$$

so that by Lemma 6.4.5 we have $\overline{u_1} \dots \overline{u_n} = \overline{v_1} \dots \overline{v_m}$ in $\mathrm{IG}(\mathcal{B})$.

To show $\boldsymbol{\psi}$ is onto, let $(\overline{x_1} \dots \overline{x_n}, \overline{\sigma_1} \dots \overline{\sigma_m}) \in \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$, where $x_i \in B_\alpha$ for all $1 \leq i \leq n$. Using the remark following Proposition 6.2.7, if $n \geq m$, then

$$(\overline{x_1\boldsymbol{f_{\sigma_1}}} \dots \overline{x_m\boldsymbol{f_{\sigma_m}}}\ \overline{x_{m+1}\boldsymbol{f_{\sigma_m}}} \dots \overline{x_n\boldsymbol{f_{\sigma_m}}})\boldsymbol{\psi} = (\overline{x_1} \dots \overline{x_n}, \overline{\sigma_1} \dots \overline{\sigma_m}).$$

On the other hand, if $n < m$, then

$$(\overline{x_1\boldsymbol{f_{\sigma_1}}} \dots \overline{x_n\boldsymbol{f_{\sigma_n}}}\ \overline{x_n\boldsymbol{f_{\sigma_{n+1}}}} \dots \overline{x_n\boldsymbol{f_{\sigma_m}}})\boldsymbol{\psi} = (\overline{x_1} \dots \overline{x_n}, \overline{\sigma_1} \dots \overline{\sigma_m}).$$

Therefore, $\boldsymbol{\psi}$ is an isomorphism from $\mathrm{IG}(\mathcal{B})$ onto $\mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$.

□

Definition 6.4.7. Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a band and let $w_1 \circ \dots \circ w_n \in B^+$ be an almost normal form with $w_i \in B^+_{\alpha_i}$ for $1 \leq i \leq n$ and $\alpha_i \perp \alpha_{i+1}$ for all $1 \leq i \leq n-1$. Then the set of numbers

$$\{i_1, \dots, i_r\} \subseteq \{1, \dots, n\} \text{ with } i_1 < \dots < i_r$$

is called the *set of breakpoints* of $w_1 \circ \ldots \circ w_n$ if these numbers are picked out in the following manner:

i_1: the largest number such that $\alpha_j \sqcap \alpha_{j+1} \neq \emptyset$ for all $1 \leq j \leq i_1 - 1$ but $\alpha_{i_1} \sqcap \alpha_{i_1+1} = \emptyset$;

i_2: the largest number such that $\alpha_j \sqcap \alpha_{j+1} \neq \emptyset$ for all $i_1 + 1 \leq j \leq i_2 - 1$ but $\alpha_{i_2} \sqcap \alpha_{i_2+1} = \emptyset$;

$\vdots$

i_{r-1}: the largest number such that $\alpha_j \sqcap \alpha_{j+1} \neq \emptyset$ for all $i_{r-2}+1 \leq j \leq i_{r-1} - 1$ but $\alpha_{i_{r-1}} \sqcap \alpha_{i_{r-1}+1} = \emptyset$;

$i_r(= n)$: here we have $\alpha_j \sqcap \alpha_{j+1} \neq \emptyset$ for all $i_{r-1} + 1 \leq j \leq i_r - 1$.

Notice that the breakpoints of any almost normal form are determined by the Y-trace, so for any two almost normal forms representing the same element in $\mathrm{IG}(\mathcal{B})$, their breakpoints must be the same. Moreover, since any two almost normal forms representing the same element $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$ have the same Y-trace, the breakpoints are uniquely determined by $\overline{w}$.

Corollary 6.4.8. *Let $\overline{w} = \overline{w_1} \ldots \overline{w_n}$, $\overline{p} = \overline{p_1} \ldots \overline{p_m} \in \mathrm{IG}(\mathcal{B})$, where $w_1 \circ \ldots \circ w_n$, $p_1 \circ \ldots \circ p_m \in B^+$ are in almost normal form, $w_i \in B^+_{\alpha_i}$ for all $1 \leq i \leq n$, $\alpha_i \perp \alpha_{i+1}$ for all $1 \leq i \leq n-1$ and $p_j \in B^+_{\beta_j}$ for all $1 \leq j \leq m$, $\beta_j \perp \beta_{j+1}$ for all $1 \leq j \leq m-1$. Let $i_1, \ldots, i_r(= n)$ and $j_1, \ldots, j_t(= m)$ be the breakpoints of $w_1 \circ \ldots \circ w_n$ and $p_1 \circ \ldots \circ p_m$, respectively. Then the following statements are equivalent:*

(i) $\overline{w_1} \ldots \overline{w_n} = \overline{p_1} \ldots \overline{p_m}$ in $\mathrm{IG}(\mathcal{B})$;

(ii) $m = n$, $\alpha_i = \beta_i$ for all $1 \leq i \leq n$, $r = t$, $i_k = j_k$ for all $k \in \{1, \ldots, r\}$ and

$$\overline{w_{i_u+1}} \ldots \overline{w_{i_{u+1}}} = \overline{p_{i_u+1}} \ldots \overline{p_{i_{u+1}}}$$

for all $0 \leq u \leq r-1$, where we put $i_0 = 0$;

(iii) $m = n$, $\alpha_i = \beta_i$ for all $1 \leq i \leq n$, $r = t$, $i_k = j_k$ for all $k \in \{1, \ldots, r\}$ and

$$(\overline{w_{i_u+1}} \ldots \overline{w_{i_{u+1}}})\boldsymbol{F_{\alpha_{i_u+1}}} = (\overline{p_{i_u+1}} \ldots \overline{p_{i_{u+1}}})\boldsymbol{F_{\alpha_{i_u+1}}}$$

for all $0 \leq u \leq r-1$, where we put $i_0 = 0$.

Proof. (ii)$\Rightarrow$(iii). Let $m = n$, $\alpha_i = \beta_i$ for all $1 \leq i \leq n$, $r = t$, $i_k = j_k$ for all $k \in \{1, \ldots, r\}$ and let

$$\overline{w'} = \overline{w_{i_u+1}} \ldots \overline{w_{i_{u+1}}} = \overline{p_{i_u+1}} \ldots \overline{p_{i_{u+1}}} = \overline{p'},$$

for all $0 \leq u \leq r-1$, where we put $i_0 = 0$. As $\alpha_q \perp \alpha_{q+1}$ for all $i_u + 1 \leq i_q \leq i_{u+1}$ and $\overline{w'} = \overline{p'}$ in $\mathrm{IG}(\mathcal{B})$, from Lemma 6.4.5 we get that

$$(\overline{w'})\boldsymbol{F_{\alpha_{i_u+1}}} = (\overline{p'})\boldsymbol{F_{\alpha_{i_u+1}}}$$

for all $0 \leq u \leq r-1$.

(iii)$\Rightarrow$(ii). By Lemma 6.4.5, as

$$\overline{w_{i_u+1}} \ldots \overline{w_{i_{u+1}}} \boldsymbol{F}_{\boldsymbol{\alpha_{i_u+1}}} = \overline{p_{i_u+1}} \ldots \overline{p_{i_{u+1}}} \boldsymbol{F}_{\boldsymbol{\alpha_{i_u+1}}}$$

for all $0 \leq u \leq r-1$, we get that

$$\overline{w_{i_u+1}} \ldots \overline{w_{i_{u+1}}} = \overline{p_{i_u+1}} \ldots \overline{p_{i_{u+1}}}$$

for all $0 \leq u \leq r-1$.

(i)$\Rightarrow$(ii). It follows from Lemma 5.3.12 that $m = n$, $\alpha_i = \beta_i$ for all $1 \leq i \leq n$. To show the rest, we put

$$L_w = \{z \in E^+ : \overline{z} = \overline{w}\}.$$

Since w is an almost normal form, we have that $\alpha_1, \ldots, \alpha_n$ is the unique Y-trace of $\overline{w}$ with breaking points $i_1 = j_1, \ldots, i_r = j_r (= n)$. For each fixed $0 \leq u \leq r-1$, we define a map

$$\theta_u : L_w \longrightarrow \mathrm{IG}(\mathcal{B})$$

as follows. Let $x = x_1 \circ \ldots \circ x_v \in L_w$. Suppose that $x_i \in B_{\delta_i}$ for all $1 \leq i \leq v$. Notice that we are not assuming that x is in almost normal form. Recall from [16] that the *left to right significant indices* of $x_1 \circ \ldots \circ x_v$ is defined as a set of numbers

$$\{l_1, \ldots, l_r\} \subseteq \{1, \ldots, v\} \text{ with } l_1 < \cdots < l_r$$

where these numbers are picked out in the following manner:

l_1 : the largest number such that $\delta_1, \ldots, \delta_{l_1} \geq \delta_{l_1}$;

k_1 : the largest number such that $\delta_{l_1} \leq \delta_{l_1}, \delta_{l_1+1}, \ldots, \delta_{k_1}$.

l_2 : the largest number such that $\delta_{k_1+1}, \ldots, \delta_{l_2} \geq \delta_{l_2}$;

k_2 : the largest number such that $\delta_{l_2} \leq \delta_{l_2}, \delta_{l_2+1}, \ldots, \delta_{k_2}$.

$\vdots$

l_n : the largest number such that $\delta_{k_{n-1}+1}, \ldots, \delta_{l_n} \geq \delta_{l_n}$;

$k_n = v$: here we have $\delta_{l_n} \leq \delta_{l_n}, \delta_{l_n+1}, \ldots, \delta_v$. Of course, here we may have $l_n = k_n = v$.

Notice that for all $1 \leq s \leq n-1$, δ_{l_s} and $\delta_{l_{s+1}}$ are incomparable. We can use the following *Hasse diagram* to depict the relationship among $\delta_{l_1}, \ldots, \delta_{l_n}$:

We call the set of numbers $\{k_1, \ldots, k_n\}$ the *adjoint indices* of $x_1 \circ \ldots \circ x_v$. Notice that here we have $\delta_{l_i} = \alpha_i$ for all $1 \leq i \leq n$ and we can use the above

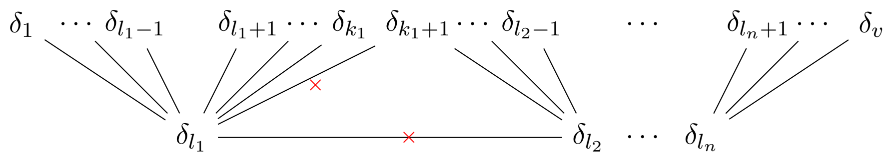


Figure 6.8: Hasse diagram illustrating significant indices

decomposition to obtain an almost normal form q for x by Lemma 5.3.19. We now define

$$x\theta_u = \overline{x_{k_{i_u}+1}} \dots \overline{x_{k_{i_{u+1}}}},$$

where we put $i_0 = 0$. We now claim that

$$\overline{\theta_u} : \overline{w} \mapsto w\theta_u$$

is well defined.

Let $\overline{x} = \overline{z} = \overline{w}$. We know $\overline{z}$ may be obtained from $\overline{x}$ by finitely many steps of squashing or splitting in terms of basic products, but to show $x\theta_u = z\theta_u$, it is sufficient to assume that z is obtained from x by just a single step of squashing (or splitting). Let

$$\overline{x} = \overline{x_1} \cdots \overline{x_{j-1}}\ \overline{x_j}\ \overline{x_{j+1}}\ \overline{x_{j+2}} \cdots \overline{x_s} \text{ and } \overline{z} = \overline{x_1} \cdots \overline{x_{j-1}}\ \overline{x_j \circ x_{j+1}}\ \overline{x_{j+2}} \cdots \overline{x_s}$$

where (x_j, x_{j+1}) is basic. It is impossible that $j = k_{i_s}$ and $j+1 = k_{i_s+1}$ for some $1 \leq s \leq n-1$, for, if this occurred then $\delta_{k_{i_s}}, \delta_{k_{i_{s+1}}}$ would be comparable so that $\delta_{l_{i_s}} \sqcap \delta_{l_{i_{s+1}}} \neq \emptyset$ a contradiction. It follows that we have $x\theta_u = z\theta_u$. Now considering the two almost normal forms w and p of the elements $\overline{w}$ and $\overline{p}$ we are concerned with, we have

$$\overline{w_{i_u+1}} \cdots \overline{w_{i_{u+1}}} = \overline{w}\overline{\theta_u} = \overline{p}\overline{\theta_u} = \overline{p_{i_u+1}} \cdots \overline{p_{i_{u+1}}}$$

for all $0 \leq u \leq r-1$.

□

Theorem 6.4.9. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a finite iso-normal band. Then the word problem of* $\mathrm{IG}(\mathcal{B})$ *is decidable.*

Proof. It follows from [18] that there is an algorithm to get an almost normal form for any element in $\mathrm{IG}(\mathcal{E})$. By Corollary 6.4.8 the word problem of $\mathrm{IG}(\mathcal{B})$ is equivalent to that of $\mathrm{IG}(\mathcal{B}_\alpha)$, $\alpha \in Y$, and $\mathrm{IG}(\mathcal{Y})$, which are decidable by Lemma 5.3.23, and hence $\mathrm{IG}(\mathcal{B})$ has decidable word problem. □

6.5 Abundancy of free idempotent generated semigroups over iso-normal bands

The article [16] begins the study of those bands B such that $\mathrm{IG}(\mathcal{B})$ is abundant. As regards normal bands, Proposition 5.3.22 shows that if a normal band B satisfies the technical Condition (P) for $\mathrm{IG}(\mathcal{B})$, then $\mathrm{IG}(\mathcal{B})$ is abundant. To see that not every iso-normal band satisfies Condition (P), let Y be the diamond semilattice $\{\alpha, \beta, \gamma, \delta\}$, where $\gamma \perp \beta$, α and δ are the upper and the lower bounds of Y, respectively. For each $\mu \in Y$, let $B_\mu = \{a_\mu, b_\mu, c_\mu, d_\mu\}$ be a four-element rectangular band with (a_μ, d_μ) not basic. For the connecting morphisms simply take u_τ to u_ν for any $u \in \{a, b, c, d\}$ and $\tau \geq \nu$ in Y. Let $\overline{a_\alpha}\,\overline{d_\beta}\,\overline{a_\beta}$ be an element of $\mathrm{IG}(\mathcal{B})$. By Lemma 6.3.5, we can write

$$\begin{aligned}\overline{a_\gamma}\,\overline{d_\beta}\,\overline{a_\beta} &= (\overline{a_\gamma}\,\overline{d_\gamma}\,\overline{a_\gamma})(\overline{a_\gamma}\,\overline{a_\beta}\,\overline{a_\beta}) \\ &= (\overline{a_\gamma}\,\overline{d_\gamma}\,\overline{a_\gamma})(\overline{a_\beta}),\end{aligned}$$

in $\mathrm{IG}(\mathcal{B})$. It is clear that the first and last expressions in the above equality are in almost normal form, with Y-length 2, left to right significant indices $i_1 = 1, i_2 = 3$ and $l_1 = 3, l_2 = 4$, respectively. We have $a_{i_1} = a_\alpha \,\mathcal{L}\, a_{l_1} = a_\alpha$ but

$$\overline{a_\gamma} \neq \overline{a_\gamma}\,\overline{d_\gamma}\,\overline{a_\gamma},$$

in $\mathrm{IG}(\mathcal{B}_\gamma)$ by the uniqueness of normal forms, so by Corollary 5.3.20, we have $\overline{a_\gamma} \neq \overline{a_\gamma}\,\overline{d_\gamma}\,\overline{a_\gamma}$ in $\mathrm{IG}(\mathcal{B})$. This shows that $\mathrm{IG}(\mathcal{B})$ does not satisfy condition (P). Moreover, an example is given in [16, Example 6.5] of a normal band B such that $\mathrm{IG}(\mathcal{B})$ is not abundant.

Given an iso-normal band $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$, so that $B \cong B_\alpha \times Y$, by Proposition 6.2.3, one might hope that $\mathrm{IG}(\mathcal{B})$ is isomorphic to $\mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$. If this were to be the case, then since $\mathrm{IG}(\mathcal{B}_\alpha)$ is completely simple, by Theorem 5.3.7, and $\mathrm{IG}(\mathcal{Y})$ is abundant, by Theorem 5.3.6, then by Lemma 4.1.3 we get that $\mathrm{IG}(\mathcal{B})$ is abundant. However, it is only in a special case that we can show $\mathrm{IG}(\mathcal{B}) \cong \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$; in general the situation is more complex, as we show.

The next lemma is needed to determine those bands B such that $\mathrm{IG}(\mathcal{B})$ is isomorphic to $C \times \mathrm{IG}(\mathcal{Y})$ for some completely simple semigroup C and semilattice Y.

Lemma 6.5.1. *For any semilattice Y, Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ are trivial on* $\mathrm{IG}(\mathcal{Y})$.

Proof. We need only give the argument for $\mathcal{R}$. Let $\overline{\alpha} = \overline{\alpha_1} \cdots \overline{\alpha_n}$ and $\overline{\beta} = \overline{\beta_1} \cdots \overline{\beta_m}$ be normal forms of $\mathrm{IG}(\mathcal{Y})$ and suppose that $\overline{\alpha}\, \mathcal{R}\, \overline{\beta}$. Then either $\overline{\alpha} = \overline{\beta}$ or there exist normal forms $\overline{\delta_1} \cdots \overline{\delta_u},\ \overline{\lambda_1} \cdots \overline{\lambda_v} \in \mathrm{IG}(\mathcal{Y})$ such that

$$(\overline{\alpha_1} \cdots \overline{\alpha_n})(\overline{\delta_1} \cdots \overline{\delta_u}) = \overline{\beta_1} \cdots \overline{\beta_m},\ (\overline{\beta_1} \cdots \overline{\beta_m})(\overline{\lambda_1} \cdots \overline{\lambda_v}) = \overline{\alpha_1} \cdots \overline{\alpha_n}$$

giving

$$\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_n}\ \overline{\delta_1} \cdots \overline{\delta_u}\ \overline{\lambda_1} \cdots \overline{\lambda_v}.$$

Let $\overline{\delta_1} \cdots \overline{\delta_u}\ \overline{\lambda_1} \cdots \overline{\lambda_v} = \overline{\mu_1}\ \cdots \overline{\mu_w}$, where $\overline{\mu_1}\ \cdots \overline{\mu_w}$ is a normal form. Notice that for any $1 \leq i \leq u$ we must have $\delta_i \geq \mu_l$ for some $1 \leq l \leq w$ and for any $1 \leq j \leq v$, we must have $\lambda_j \geq \mu_k$ for some $1 \leq k \leq w$. Then

$$\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_n}\ \overline{\mu_1}\ \cdots \overline{\mu_w}.$$

By the uniqueness of normal forms in $\mathrm{IG}(\mathcal{Y})$, we must have α_n is comparable to μ_1. We have the following cases.

Case (i) $\alpha_n \leq \mu_1, \cdots, \mu_s$ and $\alpha_n \perp \mu_{s+1}$ or $s = w$. In this case, we have normal forms

$$\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_n}\ \overline{\mu_{s+1}}\ \cdots \overline{\mu_w}$$

giving $s = w$ by the uniqueness of normal forms in $\mathrm{IG}(\mathcal{Y})$. Notice that, since for any $1 \leq i \leq u$ we have $\delta_i \geq \mu_l$ for some $1 \leq l \leq w$. Then we have $\alpha_n \leq \mu_l \leq \delta_i$, for any $1 \leq i \leq u$, and hence

$$\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_n}\ \overline{\delta_1} \cdots \overline{\delta_u} = \overline{\beta_1} \cdots \overline{\beta_m}.$$

Case (ii) $\mu_1 \leq \alpha_s, \cdots, \alpha_n$ and $\alpha_{s-1} \perp \mu_1$ or $s = 1$. In this case, we have normal forms

$$\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_{s-1}}\ \overline{\mu_1} \cdots \overline{\mu_w}$$

implying $n = s - 1 + w$ and $\alpha_s = \mu_1 \leq \alpha_{s+1}, \cdots, \alpha_n$. To avoid contradiction, we must have $s = n$, so that $n = n - 1 + w$, and so $w = 1$. Therefore, $\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_{n-1}}\ \overline{\mu_1}$, so that $\mu_1 = \alpha_n$. As $w = 1$, we have $\delta_i \geq \mu_1$ for all $1 \leq i \leq u$, so that $\delta_i \geq \mu_1 = \alpha_n$, giving

$$\overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\alpha_1} \cdots \overline{\alpha_n}\ \overline{\delta_1} \cdots \overline{\delta_u} = \overline{\beta_1} \cdots \overline{\beta_m}.$$

□

The next lemma is needed to describe the regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{B})$, where B is a band.

Lemma 6.5.2. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a semilattice of rectangular bands B_α. The regular $\mathcal{D}$-classes of* $\mathrm{IG}(\mathcal{B})$ *are* $\mathrm{IG}(\mathcal{B}_\alpha)$*, where $\alpha \in Y$.*

Proof. It is known that as B_α is rectangular band, $\alpha \in Y$, $\mathrm{IG}(\mathcal{B}_\alpha)$ is regular semigroup, by Theorem 5.3.7. Hence for any $\overline{e}$, $\overline{f} \in \mathrm{IG}(\mathcal{B}_\alpha)$, we get $\overline{e}\,\mathcal{R}\,\overline{e}\overline{f}\,\mathcal{L}\,\overline{f}$, so $\overline{e}\,\mathcal{D}\,\overline{f}$.

Let $\overline{w}$ be a regular element of $\mathrm{IG}(\mathcal{B})$. Our aim is to show that the Y-trace of any regular element $\overline{w}$ of $\mathrm{IG}(\mathcal{B})$, is (α), for some $\alpha \in Y$. Let $w = x_1 \circ \ldots \circ x_n \in B^+$, where $x_i \in B^+_{\alpha_i}$, $\alpha_i \in Y$, is an almost normal form of $\overline{w} \in \mathrm{IG}(\mathcal{B})$. Then the word $\alpha_1 \circ \ldots \circ \alpha_n \in Y^+$ is an almost normal form of $\overline{w}\mathbf{T} = \overline{\alpha_1} \ldots \overline{\alpha_n}$ in $\mathrm{IG}(\mathcal{Y})$. As Y is a semilattice, $\overline{w}\mathbf{T}$ is regular. Hence $\overline{\alpha_1} \ldots \overline{\alpha_n}$ is regular element. By Lemma 6.5.1, we have that $\overline{\alpha} = \overline{\alpha_1} \ldots \overline{\alpha_n}$, some $\alpha \in Y$. This implies that $n = 1$ and $\alpha_1 = \alpha$, as required.

□

Note that the regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{B})$, where $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is an iso-normal band, are $\mathrm{IG}(\mathcal{B}_\alpha)$, $\alpha \in Y$, by the above result. By Lemma 6.5.1, the regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{Y})$ are $\{\overline{\lambda}\}$, $\lambda \in Y$. Hence the regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$ are $\mathrm{IG}(\mathcal{B}_\alpha) \times \{\overline{\lambda}\}$, where $\lambda \in Y$. Notice that a semilattice is always an iso-normal band.

Corollary 6.5.3. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be normal band. Then any element $\overline{w}$ of* $\mathrm{IG}(\mathcal{B})$ *has a single Y-trace if and only if $\overline{w}$ is regular.*

Proof. Let $\overline{w}$ be an element of $\mathrm{IG}(\mathcal{B})$ with single Y-trace, that means $\overline{w} \in \mathrm{IG}(\mathcal{B}_\alpha)$, some $\alpha \in Y$. Hence by Lemma 5.3.7, we get that $\overline{w}$ is a regular element of $\mathrm{IG}(\mathcal{B})$.

Conversely, by Lemma 6.5.2, proved that any regular element has a single Y-trace, (α), for some $\alpha \in Y$.

□

Proposition 6.5.4. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band such that B_α is non-trivial for all $\alpha \in Y$. Then*

$$\mathrm{IG}(\mathcal{B}) \cong \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$$

if and only if for all $\beta, \delta \in Y$, $\beta \sqcap \delta \neq \emptyset$, where α is (any) fixed element in Y.

Proof. Suppose that for all $\beta, \delta \in Y$, $\beta \sqcap \delta \neq \emptyset$. Then by Lemma 6.4.6, there is an isomorphism from $\mathrm{IG}(\mathcal{B})$ onto $\mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$.

Conversely, assume that $\phi : \mathrm{IG}(\mathcal{B}) \longrightarrow \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$ is an isomorphism. Suppose that there exists $\beta, \delta \in Y$ such that $\beta \sqcap \delta = \emptyset$. Since $\mathrm{IG}(\mathcal{B}_\beta)$ and $\mathrm{IG}(\mathcal{B}_\delta)$ are distinct regular $\mathcal{D}$-class of $\mathrm{IG}(\mathcal{B})$, so they must be mapped by ϕ to different regular $\mathcal{D}$-classes $\mathrm{IG}(\mathcal{B}_\alpha) \times \{\overline{\lambda}\}$ and $\mathrm{IG}(\mathcal{B}_\alpha) \times \{\overline{\mu}\}$ by Lemma 6.5.1. Also, since $\beta \sqcap \delta = \emptyset$, we have $\lambda \sqcap \mu = \emptyset$. We pick $g, h \in B_\alpha$ with $g \neq h$. As ϕ is an isomorphism, it takes idempotents to idempotents, so that there exists $e, f \in B_\beta$ and $u, v \in B_\delta$ with $e \neq f$ and $u \neq v$ such that

$$\overline{e}\phi = (\overline{g}, \overline{\lambda}),\ \overline{f}\phi = (\overline{h}, \overline{\lambda}),\ \overline{u}\phi = (\overline{g}, \overline{\mu}),\ \overline{v}\phi = (\overline{h}, \overline{\mu}).$$

Notice that

$$\begin{aligned}
(\overline{e}\ \overline{f}\ \overline{u})\phi &= (\overline{g}, \overline{\lambda})(\overline{h}, \overline{\lambda})(\overline{g}, \overline{\mu}) \\
&= (\overline{g}\,\overline{h}\,\overline{g}, \overline{\lambda}\,\overline{\lambda}\,\overline{\mu}) \\
&= (\overline{g}\,\overline{h}\,\overline{g}, \overline{\lambda}\,\overline{\mu}) && (\text{as } \overline{\lambda}\,\overline{\lambda} = \overline{\lambda}) \\
&= (\overline{g}\,\overline{h}\,\overline{g}, \overline{\lambda}\,\overline{\mu}\,\overline{\mu}) && (\text{as } \overline{\mu}\,\overline{\mu} = \overline{\mu}) \\
&= (\overline{g}, \overline{\lambda})(\overline{h}, \overline{\mu})(\overline{g}, \overline{\mu}) \\
&= (\overline{e}\ \overline{v}\ \overline{u})\phi.
\end{aligned}$$

As ϕ is an isomorphism. Then we get that $\overline{e}\ \overline{f}\ \overline{u} = \overline{e}\ \overline{v}\ \overline{u}$. Since $\beta \sqcap \delta = \emptyset$, hence $\beta \perp \delta$ and by Lemma 6.4.8 we get that

$$(\overline{e}\ \overline{f})\ \overline{u} = \overline{e}\ (\overline{v}\ \overline{u}) \Longleftrightarrow \overline{e}\ \overline{f} = \overline{e} \text{ and } \overline{u} = \overline{v}\ \overline{u}.$$

Notice that as $\mathrm{IG}(\mathcal{B}_\beta)$ is completely simple and $\overline{e}\,\overline{f} = \overline{e}$ in $\mathrm{IG}(\mathcal{B}_\beta)$, $\overline{e}\,\mathcal{R}\,\overline{e}\,\overline{f} = \overline{e}\,\mathcal{L}\,\overline{f}$, would imply $e\ \mathcal{L}\ f$ and so $g\ \mathcal{L}\ h$; similarly, $\overline{u} = \overline{v}\ \overline{u}$ would imply $u\ \mathcal{R}\ v$ and so $g\ \mathcal{R}\ h$. Hence $g\,\mathcal{H}\,h$. As B_α is rectangular band, $g = h$, contradiction. $\square$

The above result is always true if B_α is trivial, since $\mathrm{IG}(\mathcal{B}_\alpha)$ is then trivial.

Corollary 6.5.5. *For any iso-normal band $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ satisfying the property that $\alpha \sqcap \beta \neq \emptyset$ for all $\alpha, \beta \in Y$, we have that $\mathrm{IG}(\mathcal{B})$ is abundant.*

Proof. By Proposition 6.5.4, $\mathrm{IG}(\mathcal{B}) \cong \mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$ where α is any fixed element in Y. Since both $\mathrm{IG}(\mathcal{B}_\alpha)$ and $\mathrm{IG}(\mathcal{Y})$ are abundant by Theorem 5.3.6, it follows from Lemma 4.1.3 that $\mathrm{IG}(\mathcal{B}_\alpha) \times \mathrm{IG}(\mathcal{Y})$ is abundant and hence $\mathrm{IG}(\mathcal{B})$ is abundant. $\square$

Example 6.5.6. Let B be an iso-normal band, $B = \mathscr{B}(Y, B_\alpha, \varphi_{\alpha,\beta})$, where Y is a net semilattice(see the figure below),

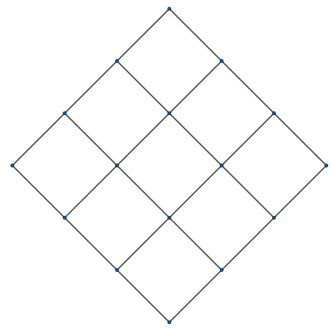


Figure 6.9: Net semilattice

Then $\mathrm{IG}(\mathcal{B})$ is abundant.

Corollary 6.5.7. *For any iso-normal band $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$, where Y has an upper bound. Then $\mathrm{IG}(\mathcal{B})$ is abundant.*

Proof. As α is an upper bound of Y. Hence for any $\beta, \delta \in Y$, we get that $\alpha \in \beta \sqcap \delta$. Then by Corollary 6.5.5, $\mathrm{IG}(\mathcal{B})$ is abundant. $\square$

The free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over a diamond iso-normal band B is an example of the above result.

Proposition 6.5.8. *Let $B = \bigcup_{\alpha \in Z} B_\alpha$ be a normal band satisfying the property that $\mathrm{IG}(\mathcal{B}) \cong C \times \mathrm{IG}(\mathcal{Y})$, where C is a completely simple semigroup and Y is a semilattice. Then $Z = Y$ and B is an iso-normal band.*

Proof. We first note that for any each $\alpha \in Z$,

$$\mathrm{IG}'(\mathcal{B}_\alpha) = \{\overline{w} \in \mathrm{IG}(\mathcal{B}) : \overline{w} = \overline{u_1} \cdots \overline{u_m}, u_i \in B_\alpha \text{ for all } 1 \leq i \leq m\}$$

is a completely simple subsemigroup of $\mathrm{IG}(\mathcal{B})$, as it is morphic image of $\mathrm{IG}(\mathcal{B}_\alpha)$.

Let

$$\psi : \mathrm{IG}(\mathcal{B}) \longrightarrow C \times \mathrm{IG}(\mathcal{Y})$$

be an isomorphism. Since ψ must take regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{B})$ to regular $\mathcal{D}$-classes of $C \times \mathrm{IG}(\mathcal{Y})$, from Lemma 6.5.1 there is a bijection from Z onto Y, $\alpha \mapsto y_\alpha$, induced by $\mathrm{IG}'(\mathcal{B}_\alpha)\psi = C \times \{\overline{y_\alpha}\}$, as the regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{B})$ are $\mathrm{IG}(\mathcal{B}_\alpha)$ where $\alpha \in Z$, and $\mathrm{IG}'(\mathcal{B}_\alpha)$ is isomorphic to $\mathrm{IG}(\mathcal{B}_\alpha)$ by Theorem 5.3.20, we get that $\mathrm{IG}'(\mathcal{B}_\alpha)$ is a regular $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{B})$. Also, the $\mathcal{D}$-classes of $C \times \mathrm{IG}(\mathcal{Y})$, are $C \times \{\overline{y}_\alpha\}$, as C is a completely simple and the $\mathcal{D}$-classes of $\mathrm{IG}(\mathcal{Y})$ are trivial.

We show that B is an iso-normal band by the following 4 steps.

Step (1) We claim $Z \cong Y$ as semilattices. Let $b_\alpha \in B_\alpha, b_\beta \in B_\beta$. Then

$$\begin{aligned}
\alpha \leq \beta \Leftrightarrow\ & \overline{b_\alpha}\ \overline{b_\beta} \in \mathrm{IG}'(\mathcal{B}_\alpha) & \\
\Leftrightarrow\ & (\overline{b_\alpha}\ \overline{b_\beta})\psi = (u, \overline{y_\alpha}) & (\text{ for some } u \in C) \\
\Leftrightarrow\ & (\overline{b_\alpha})\psi\ (\overline{b_\beta})\psi = (u, \overline{y_\alpha}) & (\text{ as } \psi \text{ is a morphism }) \\
\Leftrightarrow\ & (u_1, \overline{y_\alpha})(u_2, \overline{y_\beta}) = (u, \overline{y_\alpha}) & (\text{ where } u = u_1u_2,\ u_1, u_2 \in C) \\
\Leftrightarrow\ & \overline{y_\alpha}\ \overline{y_\beta} = \overline{y_\alpha} & \\
\Leftrightarrow\ & y_\alpha \leq y_\beta &
\end{aligned}$$

so that $Z \cong Y$, and hence, in the remaining proof, we may take $Y = Z$ and without loss of generality assume $y_\alpha = \alpha$ for all $\alpha \in Z$.

Step (2) We claim that $B_\alpha \cong B_\beta$, for all $\alpha, \beta \in Z$.

Let $\alpha, \beta \in Z$, we define an isomorphism

$$\tau_{\alpha,\beta} : C \times \{\overline{\alpha}\} \longrightarrow C \times \{\overline{\beta}\},\ (c, \alpha) \mapsto (c, \beta).$$

Put $\overline{B} = \{\bar{b} : b \in B\} \subseteq \mathrm{IG}(\mathcal{B})$ and define

$$\kappa : \overline{B} \longrightarrow B,\ \bar{b} \mapsto b.$$

Note that κ is the restriction to $\overline{B}$ of the natural map from $\mathrm{IG}(\mathcal{B})$ to B. With the above preparations, we define a map

$$\varphi_{\alpha,\beta} : B_\alpha \longrightarrow B_\beta,\ b_\alpha \longrightarrow \overline{b_\alpha}\psi\tau_{\alpha,\beta}\psi^{-1}\kappa.$$

As ψ, $\tau_{\alpha,\beta}$, ψ^{-1}, and κ are all one-one, we deduce $\varphi_{\alpha,\beta}$ is one-one. Let $b_\beta \in B_\beta$. Then $\overline{b_\beta}\psi = (u, \overline{\beta})$ for some idempotent $u \in C$. By putting $\overline{b_\alpha} = (u, \overline{\alpha})\psi^{-1}$, it is easy check that $b_\alpha\varphi_{\alpha,\beta} = b_\beta$, so that $\varphi_{\alpha,\beta}$ is onto, and hence it is bijective. To show $\varphi_{\alpha,\beta}$ is a morphism, we let $b_\alpha, c_\alpha \in B_\alpha$ with

$$\overline{b_\alpha}\psi = (u, \overline{\alpha}),\ \overline{c_\alpha}\psi = (v, \overline{\alpha}) \text{ and } (\overline{b_\alpha c_\alpha})\psi = (w, \overline{\alpha})$$

for some $u, v, w \in C$. Since $\overline{b_\alpha}\ \mathcal{R}\ \overline{b_\alpha c_\alpha}\ \mathcal{L}\ \overline{c_\alpha}$ and ψ is an isomorphism, we deduce $u\ \mathcal{R}\ w\ \mathcal{L}\ v$. Let

$$(w, \overline{\beta})\psi^{-1} = \overline{l_\beta},\ (u, \overline{\beta})\psi^{-1} = \overline{h_\beta} \text{ and } (v, \overline{\beta})\psi^{-1} = \overline{k_\beta}$$

for some $l_\beta, h_\beta, k_\beta \in B_\beta$. Then $h_\beta\ \mathcal{R}\ l_\beta\ \mathcal{L}\ k_\beta$, implying $l_\beta = h_\beta k_\beta$. We now have

$$(b_\alpha c_\alpha)\varphi_{\alpha,\beta} = \overline{b_\alpha c_\alpha}\psi\tau_{\alpha,\beta}\psi^{-1}\kappa = (w, \overline{\beta})\psi^{-1}\kappa = l_\beta$$

and

$$(b_\alpha\varphi_{\alpha,\beta})(c_\alpha\varphi_{\alpha,\beta}) = (\overline{b_\alpha}\psi\tau_{\alpha,\beta}\psi^{-1}\kappa)(\overline{c_\alpha}\psi\tau_{\alpha,\beta}\psi^{-1}\kappa) = ((u,\overline{\beta})\psi^{-1}\kappa)((v,\overline{\beta})\psi^{-1}\kappa) = h_\beta k_\beta = l_\beta$$

so that

$$(b_\alpha c_\alpha)\varphi_{\alpha,\beta} = (b_\alpha\varphi_{\alpha,\beta})(c_\alpha\varphi_{\alpha,\beta})$$

and hence $\varphi_{\alpha,\beta}$ is an isomorphism from B_α onto B_β.

Step (3) We claim that $\varphi_{\alpha,\alpha} = 1_{B_\alpha}$ and $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for all $\alpha,\beta,\gamma \in Z$.

Let $b_\alpha \in B_\alpha$. Then

$$b_\alpha\varphi_{\alpha,\alpha} = \overline{b_\alpha}\psi\tau_{\alpha,\alpha}\psi^{-1}\kappa = \overline{b_\alpha}\kappa = b_\alpha$$

so that $\varphi_{\alpha,\alpha} = 1_{B_\alpha}$. Further, by putting $\overline{b_\alpha}\psi = (u,\overline{\alpha})$ and $(u,\overline{\beta})\psi^{-1} = \overline{b_\beta}$, where $b_\beta = \overline{b_\alpha}\psi\tau_{\alpha,\beta}\psi^{-1}\kappa$, we have

$$b_\alpha\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = (\overline{b_\alpha}\psi\tau_{\alpha,\beta}\psi^{-1}\kappa)\varphi_{\beta,\gamma} = b_\beta\varphi_{\beta,\gamma} = \overline{b_\beta}\psi\tau_{\beta,\gamma}\psi^{-1}\kappa = (u,\overline{\gamma})\psi^{-1}\kappa$$

and

$$b_\alpha\varphi_{\alpha,\gamma} = \overline{b_\alpha}\psi\tau_{\alpha,\gamma}\psi^{-1}\kappa = (u,\overline{\gamma})\psi^{-1}\kappa$$

so that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

Step (4) We claim that for all $b_\alpha \in B_\alpha$ and $b_\beta \in B_\beta$, $b_\alpha b_\beta = (b_\alpha\varphi_{\alpha,\alpha\beta})(b_\alpha\varphi_{\alpha,\alpha\beta})$.

Let $c_{\alpha\beta} = b_\alpha b_\beta \in B_{\alpha\beta}$. Suppose that

$$\overline{b_\alpha}\psi = (u,\overline{\alpha}), \overline{b_\beta}\psi = (v,\overline{\beta}) \text{ and } \overline{c_{\alpha\beta}}\psi = (w,\overline{\alpha\beta})$$

for some idempotents $u,v,w \in C$. Suppose that

$$(u,\overline{\alpha\beta})\psi^{-1} = \overline{e_{\alpha\beta}}, (v,\overline{\alpha\beta})\psi^{-1} = \overline{f_{\alpha\beta}}$$

for some $e_{\alpha\beta}, f_{\alpha\beta} \in B_{\alpha\beta}$. Then

$$(b_\alpha\varphi_{\alpha,\alpha\beta})(b_\beta\varphi_{\beta,\alpha\beta}) = (\overline{b_\alpha}\psi\tau_{\alpha,\alpha\beta}\psi^{-1}\kappa)(\overline{b_\beta}\psi\tau_{\beta,\alpha\beta}\psi^{-1}\kappa) = (\overline{e_{\alpha\beta}}\kappa)(\overline{f_{\alpha\beta}}\kappa) = e_{\alpha\beta}f_{\alpha\beta}.$$

We now show that $e_{\alpha\beta}f_{\alpha\beta} = c_{\alpha\beta}$. It is easy to see that $\overline{b_\alpha}\ \overline{c_{\alpha\beta}} = \overline{c_{\alpha\beta}}$, implying $(\overline{b_\alpha}\psi)(\overline{c_{\alpha\beta}}\psi) = \overline{c_{\alpha\beta}}\psi$, namely, $(u,\overline{\alpha})(w,\overline{\alpha\beta}) = (w,\overline{\alpha\beta})$, so that $uw = w$. A similar argument gives us $w = wv$, and so $u \mathcal{R} w \mathcal{L} v$. Again, as ψ is an isomorphism, we have $e_{\alpha\beta} \mathcal{R} c_{\alpha\beta} \mathcal{L} f_{\alpha\beta}$, so that $e_{\alpha\beta}f_{\alpha\beta} = c_{\alpha\beta}$. Now we have

$$(b_\alpha\varphi_{\alpha,\alpha\beta})(b_\beta\varphi_{\beta,\alpha\beta}) = e_{\alpha\beta}f_{\alpha\beta} = c_{\alpha\beta} = b_\alpha b_\beta.$$

Therefore, B is an iso-normal band, as required. □

Clearly, in general, the converse of the above result is not true, as shown in Proposition 6.5.4.

Before stating the abundancy result in the general case, we need a simple lemma.

Lemma 6.5.9. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band and let $\overline{w_1}\ \overline{w_2} \in$ $\mathrm{IG}(\mathcal{B})$ be such that $w_1 \in B_\alpha^+$ and $w_2 \in B_\beta^+$ with $\alpha \leq \beta$. Then*

$$\overline{w_1}\ \overline{w_2} = \overline{w_1}\ (\overline{w_2}\boldsymbol{F_\alpha}) \text{ and } \overline{w_2}\ \overline{w_1} = (\overline{w_2}\boldsymbol{F_\alpha})\ \overline{w_1}$$

where the right hand sides of both equalities are almost normal forms of $\overline{w_1}\ \overline{w_2}$ in $\mathrm{IG}(\mathcal{B})$ and $\overline{w_2}\boldsymbol{F_\alpha}$ is regarded as an element in $\mathrm{IG}(\mathcal{B})$.

Proof. Suppose that $w_1 \in B_\alpha^+$ and $w_2 \in B_\beta^+$ with $\alpha \leq \beta$. Then we can write

$$\begin{aligned}
\overline{w_1}(\overline{w_2}\mathbf{F}_\alpha) &= \overline{w_1}(\overline{w_2\mathbf{f}_\alpha}) && \text{(by the definition of } \mathbf{F}_\alpha) \\
&= \overline{w_1}(w_2\phi_{\beta,\alpha}) && \text{(by the definition of } \mathbf{f}_\alpha \text{ and } w_2 \in B_\beta^+) \\
&= \overline{w_1}\ \overline{w_2} && \text{(by Lemma 5.3.19).}
\end{aligned}$$

□

Now we are at the position of stating our main theorem in this chapter.

Theorem 6.5.10. *Let $B = \mathscr{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be an iso-normal band. Then* $\mathrm{IG}(\mathcal{B})$ *is abundant.*

Proof. Let $\overline{w} = \overline{w_1} \cdots \overline{w_n} \in \mathrm{IG}(\mathcal{B})$ be such that $w_1 \circ \ldots \circ w_n \in B^+$ is an almost normal form with $w_i \in B_{\alpha_i}^+$ for all $1 \leq i \leq n$ and $\alpha_i \perp \alpha_{i+1}$ for all $1 \leq i \leq n-1$. We show that

$$\overline{e}\ \mathcal{R}^*\ \overline{w_1} \cdots \overline{w_n}\ \mathcal{L}^*\ \overline{f},$$

where e is the first letter of w_1 and f is the last letter of w_n. It suffices to give the proof for $\mathcal{R}^*$, that for $\mathcal{L}^*$ being dual.

We note first that clearly $\overline{e}\,\overline{w} = \overline{w}$. Thus, if we can show that $\overline{x}\,\overline{w} = \overline{y}\,\overline{w}$, then this implies $\overline{x}\,\overline{e} = \overline{y}\,\overline{e}$ for all $\overline{x}, \overline{y} \in \mathrm{IG}(\mathcal{B})$. Hence it is easy to deduce that $\overline{z}\,\overline{w} = \overline{w}$ implies $\overline{z}\,\overline{e} = \overline{e}$, for any $\overline{z} \in \mathrm{IG}(\mathcal{B})$, and then we get that $\overline{w}\,\mathcal{R}^*\,\overline{e}$.

To the above end, we now suppose that $\overline{p}\,\overline{w} = \overline{q}\,\overline{w}$ where $\overline{p}, \overline{q} \in \mathrm{IG}(\mathcal{B})$. Without loss of generality we write $p = p_1 \circ \cdots \circ p_m$, $q = q_1 \circ \cdots \circ q_s \in B^+$ are almost normal forms of $\overline{p} = \overline{p_1} \cdots \overline{p_m}$ and $\overline{q} = \overline{q_1} \cdots \overline{q_s}$, respectively, so that $p_j \in B_{\beta_j}^+$ for all $1 \leq j \leq m$ and $\beta_j \perp \beta_{j+1}$ for all $1 \leq j \leq m-1$ and $q_l \in B^+_{\delta_l}$ for all $1 \leq l \leq s$ and $\delta_l \perp \delta_{l+1}$ for all $1 \leq l \leq s-1$. Since $\overline{e} \mathrel{\mathcal{R}} \overline{w_1}$ by Lemma 5.3.8, the statement that $\overline{p_1} \cdots \overline{p_m}\, \overline{e} = \overline{q_1} \cdots \overline{q_s}\, \overline{e}$ is equivalent to $\overline{p_1} \cdots \overline{p_m}\, \overline{w_1} = \overline{q_1} \cdots \overline{q_s}\, \overline{w_1}$. We prove the latter, on a case-by-case basis.

Case (i) $\beta_m \perp \alpha_1$ and $\delta_s \perp \alpha_1$. By Lemma 5.3.9, we have

$$\overline{\beta_1} \cdots \overline{\beta_m}\, \overline{\alpha_1} \cdots\ \overline{\alpha_n} = \overline{\delta_1} \cdots \overline{\delta_s}\, \overline{\alpha_1} \cdots \overline{\alpha_n}$$

in $\mathrm{IG}(\mathcal{Y})$. Notice that both sides of this equality are in normal forms and the normal form in $\mathrm{IG}(\mathcal{Y})$ is unique, implying $m = s$ and $\beta_i = \delta_i$ for all $1 \leq i \leq m$. We consider the following two subcases:

(1) $\beta_m \sqcap \alpha_1 = \emptyset$ (so that $\delta_s \sqcap \alpha_1 = \emptyset$). Then by Corollary 6.4.8 we must have $\overline{p_1} \cdots \overline{p_m} = \overline{q_1} \cdots \overline{q_s}$, and hence $\overline{p_1} \cdots \overline{p_m}\, \overline{w_1} = \overline{q_1} \cdots \overline{q_s}\, \overline{w_1}$.

(2) $\beta_m \sqcap \alpha_1 \neq \emptyset$ (so that $\delta_s \sqcap \alpha_1 \neq \emptyset$). Let j be the largest such that $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq j-1$, but $\alpha_j \sqcap \alpha_{j+1} = \emptyset$. Let t be the smallest such that $\beta_l \sqcap \beta_{l+1} \neq \emptyset$ for all $t \leq l \leq m$, but $\beta_{t-1} \sqcap \beta_t = \emptyset$ where we put $\beta_{m+1} = \alpha_1$. Then by (ii) Corollary 6.4.8 we have

$$\overline{p_t} \cdots \overline{p_m}\, \overline{w_1} \cdots \overline{w_j} = \overline{q_t} \cdots \overline{q_s}\, \overline{w_1} \cdots \overline{w_j} \text{ and } \overline{p_1} \cdots \overline{p_{t-1}} = \overline{q_1} \cdots \overline{q_{t-1}}$$

so that, by (iii) Corollary 6.4.8 we get that

$$(\overline{p_t} \cdots \overline{p_m}\, \overline{w_1} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t} \cdots \overline{q_s}\, \overline{w_1} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}}$$

As $(\overline{w_1} \cdots \overline{w_j})\ \boldsymbol{F_{\beta_t}} \mathrel{\mathcal{R}^*} \overline{w_1}\boldsymbol{F_{\beta_t}}$ in $\mathrm{IG}(\mathcal{B}_{\beta_t})$, and $\mathrm{IG}(\mathcal{B}_{\beta_t})$ is completely simple, then by Corollary 5.3.8, we have

$$(\overline{p_t} \cdots \overline{p_m}\, \overline{w_1})\boldsymbol{F_{\beta_t}} = (\overline{q_t} \cdots \overline{q_s}\, \overline{w_1})\boldsymbol{F_{\beta_t}},$$

so that $\overline{p_t} \cdots \overline{p_m}\, \overline{w_1} = \overline{q_t} \cdots \overline{q_s}\, \overline{w_1}$, and both sides have the same Y-trace, by Lemma 6.4.5. Therefore,

$$\overline{p_1} \cdots \overline{p_{t-1}}\, \overline{p_t} \cdots \overline{p_m}\, \overline{w_1} = \overline{q_1} \cdots \overline{q_{t-1}}\, \overline{q_t} \cdots \overline{q_s}\, \overline{w_1}.$$

Case (ii) $\beta_m \leq \alpha_1, \delta_s \perp \alpha_1$. In this case, there must exist a unique $1 < v \leq n+1$ such that $\beta_m \leq \alpha_1, \cdots, \alpha_{v-1}$ and $\beta_m \perp \alpha_v$, or $v = n+1$. Then, by Lemma 6.5.9, we now have

$$\overline{p_1} \cdots \overline{p_{m-1}}\ (\overline{p_m}\, \overline{w_1} \cdots \overline{w_{v-1}})\boldsymbol{F_{\beta_m}}\, \overline{w_v} \cdots \overline{w_n} = \overline{q_1} \cdots \overline{q_s}\, \overline{w_1} \cdots \overline{w_n}$$

both sides are in almost normal forms and have the same Y-trace. By Lemma 5.3.9,

$$\overline{\beta_1} \cdots \overline{\beta_m}\, \overline{\alpha_v} \cdots \overline{\alpha_n} = \overline{\delta_1} \cdots \overline{\delta_s}\, \overline{\alpha_1} \cdots \overline{\alpha_n}.$$

It follows from the uniqueness of normal form in $\mathrm{IG}(\mathcal{Y})$ that $\alpha_{v-1} = \beta_m \leq \alpha_{v-2}$, and hence $v = 2$ (and so $\beta_m = \alpha_1$), to avoid contradiction. Note that $s = m - 1$ and $\delta_i = \beta_i$ for all $1 \leq i \leq s$. Let j be the largest such that $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $v \leq i \leq j-1$, but $\alpha_j \sqcap \alpha_{j+1} = \emptyset$. Let t be the smallest such that $\beta_l \sqcap \beta_{l+1} \neq \emptyset$ for all $t \leq l \leq m-1$, but $\beta_{t-1} \sqcap \beta_t = \emptyset$. Then as $\beta_m \leq \alpha_1$, $(\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\beta_m}} = \overline{p_m}\,\overline{w_1}$ and by Corollary 6.4.8,

$$\overline{p_t}\cdots\overline{p_{m-1}}\,(\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\beta_m}}\,\overline{w_2}\cdots\overline{w_j} = \overline{q_t}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_j} \text{ and } \overline{p_1}\cdots\overline{p_{t-1}} = \overline{q_1}\cdots\overline{q_{t-1}}$$

and hence, by Lemma 6.2.7

$$(\overline{p_t}\cdots\overline{p_{m-1}}\,(\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\beta_m}}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

namely,

$$(\overline{p_t}\cdots\overline{p_{m-1}}\,\overline{p_m}\,\overline{w_1}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

by the remark after Lemma 6.2.7. As $(\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}\,\mathcal{R}\,\overline{w_1}\boldsymbol{F_{\beta_t}}$ in $\mathrm{IG}(\mathcal{B}_{\beta_t})$, we have

$$(\overline{p_t}\cdots\overline{p_{m-1}}\,\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\,\overline{w_1})\boldsymbol{F_{\beta_t}}$$

By Lemma 6.4.5,

$$\overline{p_t}\cdots\overline{p_m}\,\overline{w_1} = \overline{q_t}\cdots\overline{q_s}\,\overline{w_1}$$

so that

$$\overline{p_1}\cdots\overline{p_{t-1}}\,\overline{p_t}\cdots\overline{p_m}\,\overline{w_1} = \overline{q_1}\cdots\overline{q_{t-1}}\,\overline{q_t}\cdots\overline{q_s}\,\overline{w_1}$$

Case (iii) $\beta_m \leq \alpha_1$ and $\delta_s \leq \alpha_1$. In this case, there must exist a unique $1 < u \leq n+1$ such that $\beta_m \leq \alpha_1, \cdots, \alpha_{u-1}$ and $\beta_m \perp \alpha_u$, or $u = n+1$; a unique $1 < v \leq n+1$ such that $\delta_s \leq \alpha_1, \cdots, \alpha_{v-1}$ and $\alpha_v \perp \delta_s$, or $v = n+1$. By Lemma 6.5.9, we have two almost normal forms in $\mathrm{IG}(\mathcal{B})$

$$\overline{p_1}\cdots\overline{p_{m-1}}\,(\overline{p_m}\,\overline{w_1}\cdots\overline{w_{u-1}})\boldsymbol{F_{\beta_m}}\overline{w_u}\cdots\overline{w_n} = \overline{q_1}\cdots\overline{q_{s-1}}\,(\overline{q_s}\,\overline{w_1}\cdots\overline{w_{v-1}})\boldsymbol{F_{\delta_s}}\overline{w_v}\cdots\overline{w_n}$$

implying two normal forms in $\mathrm{IG}(\mathcal{Y})$ by Lemma 5.3.9,

$$\overline{\beta_1}\cdots\overline{\beta_m}\,\overline{\alpha_u}\cdots\overline{\alpha_n} = \overline{\delta_1}\cdots\overline{\delta_s}\,\overline{\alpha_v}\cdots\overline{\alpha_n}.$$

If $v > u$, then $\alpha_{v-1} = \delta_s \leq \alpha_{v-2}$; to avoid contradiction, we must have $v = 2$, implying $u \leq 1$, impossible. Similarly, we cannot have $v < u$, so that $v = u$, giving $m = s$ and $\beta_i = \delta_i$ for all $1 \leq i \leq m$. Let j be the largest such that $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $u \leq i \leq j-1$, but $\alpha_j \sqcap \alpha_{j+1} = \emptyset$. Let t be the smallest such that $\beta_l \sqcap \beta_{l+1} \neq \emptyset$ for all $t \leq l \leq m$, but $\beta_{t-1} \sqcap \beta_t = \emptyset$ where we put $\beta_{m+1} = \alpha_u$. Then,

$$\overline{p_t} \cdots \overline{p_{m-1}}\,(\overline{p_m}\,\overline{w_1} \cdots \overline{w_{u-1}})\boldsymbol{F_{\beta_m}}\overline{w_u} \cdots \overline{w_j} = \overline{q_t} \cdots \overline{q_{s-1}}\,(\overline{q_s}\,\overline{w_1} \cdots \overline{w_{v-1}})\boldsymbol{F_{\delta_s}}\overline{w_v} \cdots \overline{w_j}$$

and $\overline{p_1} \cdots \overline{p_{t-1}} = \overline{q_1} \cdots \overline{q_{t-1}}$. Hence, by Lemma 6.2.7

$$(\overline{p_t} \cdots \overline{p_{m-1}}\,(\overline{p_m}\,\overline{w_1} \cdots \overline{w_{u-1}})\boldsymbol{F_{\beta_m}}\overline{w_u} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t} \cdots \overline{q_{s-1}}\,(\overline{q_s}\,\overline{w_1} \cdots \overline{w_{v-1}})\boldsymbol{F_{\delta_s}}\overline{w_v} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}}$$

namely,

$$(\overline{p_t} \cdots \overline{p_{m-1}}\,\overline{p_m}\,\overline{w_1} \cdots \overline{w_{u-1}}\,\overline{w_u} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t} \cdots \overline{q_{s-1}}\,\overline{q_s}\,\overline{w_1} \cdots \overline{w_{v-1}}\,\overline{w_v} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}}$$

by the remark after Lemma 6.2.7. Again, as $(\overline{w_1} \cdots \overline{w_j})\boldsymbol{F_{\beta_t}}\ \mathcal{R}\ \overline{w_1}\boldsymbol{F_{\beta_t}}$, we have

$$(\overline{p_t} \cdots \overline{p_{m-1}}\ \overline{p_m}\ \overline{w_1})\boldsymbol{F_{\beta_t}} = (\overline{q_t} \cdots \overline{q_{s-1}}\ \overline{q_s}\ \overline{w_1})\boldsymbol{F_{\beta_t}}$$

as $\beta_m, \delta_s \leq \alpha_1$ namely,

$$(\overline{p_t} \cdots \overline{p_{m-1}}\ (\overline{p_m}\ \overline{w_1})\boldsymbol{F_{\beta_m}})\boldsymbol{F_{\beta_t}} = (\overline{q_t} \cdots \overline{q_{s-1}}\ (\overline{q_s}\ \overline{w_1})\boldsymbol{F_{\delta_s}})\boldsymbol{F_{\beta_t}}.$$

By Lemma 6.4.5,

$$\overline{p_t} \cdots \overline{p_{m-1}}\ \overline{p_m}\ \overline{w_1} = \overline{q_t} \cdots \overline{q_{s-1}}\ \overline{q_s}\ \overline{w_1}$$

so that $\overline{p_1} \cdots \overline{p_{m-1}}\ \overline{p_m}\ \overline{w_1} = \overline{q_1} \cdots \overline{q_{s-1}}\ \overline{q_s}\ \overline{w_1}$.

Case (iv) $\beta_m \leq \alpha_1$ and $\delta_s \geq \alpha_1$. There must exist a unique $1 < u \leq n+1$ such that $\beta_m \leq \alpha_1, \cdots, \alpha_{u-1}$ and $\beta_m \perp \alpha_u$, or $u = n+1$; a unique $1 \leq v \leq s$ such that $\delta_v, \cdots, \delta_s \geq \alpha_1$ and $\delta_{v-1} \perp \alpha_1$, or $v = 1$. Then, by Lemma 6.5.9, we have two almost normal forms in $\mathrm{IG}(\mathcal{B})$

$$\overline{p_1} \cdots \overline{p_{m-1}}\ (\overline{p_m}\ \overline{w_1} \cdots \overline{w_{u-1}})\boldsymbol{F_{\beta_m}}\overline{w_u} \cdots \overline{w_n} = \overline{q_1} \cdots \overline{q_{v-1}}\ (\overline{q_v} \cdots \overline{q_s}\ \overline{w_1})\boldsymbol{F_{\alpha_1}}\overline{w_2} \cdots \overline{w_n}.$$

This gives two normal forms in $\mathrm{IG}(\mathcal{Y})$ by Lemma 5.3.9

$$\overline{\beta_1} \cdots \overline{\beta_{m-1}}\ \overline{\beta_m}\ \overline{\alpha_u} \cdots \overline{\alpha_n} = \overline{\delta_1} \cdots \overline{\delta_{v-1}}\ \overline{\alpha_1}\ \overline{\alpha_2} \cdots \overline{\alpha_n},$$

implying $\alpha_{u-1} = \beta_m \leq \alpha_{u-2}$, so that $u = 2$, to avoid contradiction, and hence $\beta_m = \alpha_1$. Note this gives $v = m$ and $\beta_i = \delta_i$ for $1 \leq i \leq v-1$. Let j be the largest such that $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq j-1$, but $\alpha_j \sqcap \alpha_{j+1} = \emptyset$. Let t be the smallest such that $\beta_l \sqcap \beta_{l+1} \neq \emptyset$ for all $t \leq l \leq m-1$, but $\beta_{t-1} \sqcap \beta_t = \emptyset$. Then

$$\overline{p_t} \cdots \overline{p_{m-1}}\ (\overline{p_m}\ \overline{w_1})\boldsymbol{F_{\beta_m}}\overline{w_2} \cdots \overline{w_j} = \overline{q_t} \cdots \overline{q_{v-1}}\ (\overline{q_v} \cdots \overline{q_s}\ \overline{w_1})\boldsymbol{F_{\alpha_1}}\overline{w_2} \cdots \overline{w_j}$$

and $\overline{p_1}\cdots\overline{p_{t-1}} = \overline{q_1}\cdots\overline{q_{t-1}}$. Hence, by Lemma 6.2.7

$$(\overline{p_t}\cdots\overline{p_{m-1}}\,(\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\beta_m}}\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\,(\overline{q_v}\cdots\overline{q_s}\,\overline{w_1})\boldsymbol{F_{\alpha_1}}\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

namely,

$$(\overline{p_t}\cdots\overline{p_{m-1}}\;\overline{p_m}\;\overline{w_1}\;\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\;\overline{q_v}\cdots\overline{q_s}\;\overline{w_1}\;\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

Again, as $(\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}\ \mathcal{R}\ \overline{w_1}\boldsymbol{F_{\beta_t}}$, we have

$$(\overline{p_t}\cdots\overline{p_{m-1}}\;\overline{p_m}\;\overline{w_1})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\;\overline{q_v}\cdots\overline{q_s}\;\overline{w_1})\boldsymbol{F_{\beta_t}}$$

as $\beta_m \leq \alpha_1$, $\overline{p_m}\,\overline{w_1} = (\overline{p_m}\,\overline{w_1})\mathbf{F}_{\beta_t}$. Hence we can write the above equality as follow

$$(\overline{p_t}\cdots\overline{p_{m-1}}\;(\overline{p_m}\;\overline{w_1})\boldsymbol{F_{\beta_m}})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\;(\overline{q_v}\cdots\overline{q_s}\;\overline{w_1})\boldsymbol{F_{\alpha_1}})\boldsymbol{F_{\beta_t}}$$

so that

$$\overline{p_t}\cdots\overline{p_{m-1}}\;\overline{p_m}\;\overline{w_1} = \overline{q_t}\cdots\overline{q_{v-1}}\;\overline{q_v}\cdots\overline{q_s}\;\overline{w_1}$$

and both sides have the same Y-trace, by Lemma 6.4.5, and hence

$$\overline{p_1}\cdots\overline{p_m}\;\overline{w_1} = \overline{q_1}\cdots\overline{q_s}\;\overline{w_1}.$$

Case (v) $\beta_m \geq \alpha_1$ and $\delta_s \geq \alpha_1$. There must exist a unique $1 \leq u \leq m$ such that $\beta_u, \cdots, \beta_m \geq \alpha_1$ and $\beta_{u-1} \perp \alpha_1$, or $u = 1$; a unique $1 \leq v \leq s$ such that $\delta_v, \cdots, \delta_s \geq \alpha_1$ and $\delta_{v-1} \perp \alpha_1$, or $v = 1$. Then, by Lemma 6.5.9, we have two almost normal forms in IG($\mathcal{B}$),

$$\overline{p_1}\cdots\overline{p_{u-1}}\;(\overline{p_u}\cdots\overline{p_m}\;\overline{w_1})\boldsymbol{F_{\alpha_1}}\overline{w_2}\cdots\overline{w_n} = \overline{q_1}\cdots\overline{q_{v-1}}\;(\overline{q_v}\cdots\overline{q_s}\;\overline{w_1})\boldsymbol{F_{\alpha_1}}\overline{w_2}\cdots\overline{w_n}$$

giving two normal forms in IG($\mathcal{Y}$), by Lemma 5.3.9,

$$\overline{\beta_1}\cdots\overline{\beta_{u-1}}\;\overline{\alpha_1}\cdots\overline{\alpha_n} = \overline{\delta_1}\cdots\overline{\delta_{v-1}}\;\overline{\alpha_1}\cdots\overline{\alpha_n},$$

so that $u = v$ and $\beta_i = \delta_i$ for all $1 \leq i \leq v-1$. Let j be the largest such that $\alpha_i \sqcap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq j-1$ and $\alpha_j \sqcap \alpha_{j+1} = \emptyset$. Let t be the smallest such that $\beta_l \sqcap \beta_{l+1} \neq \emptyset$ for all $t \leq l \leq u-1$ and $\beta_{t-1} \sqcap \beta_t = \emptyset$ where we put $\beta_u = \alpha_1$. Then by Lemma 6.4.5 we have

$$\overline{p_t}\cdots\overline{p_{u-1}}\;(\overline{p_u}\cdots\overline{p_m}\;\overline{w_1})\boldsymbol{F_{\alpha_1}}\;\overline{w_2}\cdots\overline{w_j} = \overline{q_t}\cdots\overline{q_{v-1}}\;(\overline{q_v}\cdots\overline{q_s}\;\overline{w_1})\boldsymbol{F_{\alpha_1}}\;\overline{w_2}\cdots\overline{w_j}$$

and

$$\overline{p_1}\cdots\overline{p_{t-1}} = \overline{q_1}\cdots\overline{q_{t-1}}. \tag{6.8}$$

Again, by Lemma 6.4.5 we get that

$$(\overline{p_t}\cdots\overline{p_{u-1}}\,(\overline{p_u}\cdots\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\alpha_1}}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\,(\overline{q_v}\cdots\overline{q_s}\,\overline{w_1})\boldsymbol{F_{\alpha_1}}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

namely,

$$(\overline{p_t}\cdots\overline{p_{u-1}}\,\overline{p_u}\cdots\overline{p_m}\,\overline{w_1}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\,\overline{q_v}\cdots\overline{q_s}\,\overline{w_1}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

Now, as $(\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}\ \mathcal{R}\ \overline{w_1}\boldsymbol{F_{\beta_t}}$ in $\mathrm{IG}(\mathcal{B}_{\beta_t})$,

$$(\overline{p_t}\cdots\overline{p_{u-1}}\,\overline{p_u}\cdots\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\,\overline{q_v}\cdots\overline{q_s}\,\overline{w_1})\boldsymbol{F_{\beta_t}}$$

namely,

$$(\overline{p_t}\cdots\overline{p_{u-1}}\,(\overline{p_u}\cdots\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\alpha_1}})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_{v-1}}\,(\overline{q_v}\cdots\overline{q_s}\,\overline{w_1})\boldsymbol{F_{\alpha_1}})\boldsymbol{F_{\beta_t}},$$

in $\mathrm{IG}(\mathcal{B}_{\beta_t})$, $m+1=s+1$, $\beta_u=\delta_v=\alpha_1$ and $u=v$ and $\beta_i=\delta_i$, $t\leq i\leq u-1$, then by Lemma 6.4.5 we get that

$$\overline{p_t}\cdots\overline{p_{u-1}}\,\overline{p_u}\cdots\overline{p_m}\,\overline{w_1} = \overline{q_t}\cdots\overline{q_{v-1}}\,\overline{q_v}\cdots\overline{q_s}\,\overline{w_1}, \tag{6.9}$$

where both sides have the same Y–trace. So by the equations 6.8 and 6.9 we get that $\overline{p_1}\cdots\overline{p_m}\,\overline{w_1}=\overline{q_1}\cdots\overline{q_s}\,\overline{w_1}$.

Case (vi) $\beta_m\geq\alpha_1$ and $\delta_s\perp\alpha_1$. There must exist $1\leq u\leq m$ such that $\alpha_1\leq\beta_m,\cdots,\beta_u$ and $\alpha_1\perp\beta_{u-1}$, or $u=1$. Then we have two almost normal forms in $\mathrm{IG}(\mathcal{B})$ by Lemma 6.5.9

$$\overline{p_1}\cdots\overline{p_{u-1}}(\overline{p_u}\cdots\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\alpha_1}}\,\overline{w_2}\cdots\overline{w_n} = \overline{q_1}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_n}$$

leading to two normal forms in $\mathrm{IG}(\mathcal{Y})$ by Lemma 5.3.9

$$\overline{\beta_1}\cdots\overline{\beta_{u-1}}\,\overline{\alpha_1}\cdots\overline{\alpha_n} = \overline{\delta_1}\cdots\overline{\delta_s}\,\overline{\alpha_1}\cdots\overline{\alpha_n}$$

giving $s=u-1$ and $\beta_i=\delta_i$ for all $1\leq i\leq s$. Let j be the largest such that $\alpha_i\sqcap\alpha_{i+1}\neq\emptyset$ for all $1\leq i\leq j-1$ and $\alpha_j\sqcap\alpha_{j+1}=\emptyset$. Let t be the smallest such that $\beta_l\sqcap\beta_{l+1}\neq\emptyset$ for all $t\leq l\leq u-1$ and $\beta_{t-1}\sqcap\beta_t=\emptyset$ where we put $\beta_u=\alpha_1$. Then we have

$$\overline{p_t}\cdots\overline{p_{u-1}}(\overline{p_u}\cdots\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\alpha_1}}\,\overline{w_2}\cdots\overline{w_j} = \overline{q_t}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_j}$$

and $\overline{p_1}\cdots\overline{p_{t-1}}=\overline{q_1}\cdots\overline{q_{t-1}}$, so that

$$(\overline{p_t}\cdots\overline{p_{u-1}}(\overline{p_u}\cdots\overline{p_m}\,\overline{w_1})\boldsymbol{F_{\alpha_1}}\,\overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

by Lemma 6.2.7, namely,

$$(\overline{p_t}\cdots\overline{p_{u-1}}(\overline{p_u}\cdots\overline{p_m}\ \overline{w_1})\ \overline{w_2}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\,\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}$$

As $(\overline{w_1}\cdots\overline{w_j})\boldsymbol{F_{\beta_t}}\ \mathcal{R}\ \overline{w_1}\boldsymbol{F_{\beta_t}}$ in $\mathrm{IG}(\mathcal{B}_{\beta_t})$,

$$(\overline{p_t}\cdots\overline{p_{u-1}}(\overline{p_u}\cdots\overline{p_m}\ \overline{w_1}))\boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\ \overline{w_1})\boldsymbol{F_{\beta_t}}$$

namely,

$$(\overline{p_t}\cdots\overline{p_{u-1}}(\overline{p_u}\cdots\overline{p_m}\ \overline{w_1})\boldsymbol{F_{\alpha_1}})\ \boldsymbol{F_{\beta_t}} = (\overline{q_t}\cdots\overline{q_s}\ \overline{w_1})\boldsymbol{F_{\beta_t}}$$

so that

$$\overline{p_t}\cdots\overline{p_{u-1}}\ \overline{p_u}\cdots\overline{p_m}\ \overline{w_1} = \overline{q_t}\cdots\overline{q_s}\ \overline{w_1},$$

where both sides above have the same Y-trace, by Lemma 6.4.5, and hence $\overline{p_1}\cdots\overline{p_m}\ \overline{w_1} = \overline{q_1}\cdots\overline{q_s}\ \overline{w_1}$.

It follows from the above discussion that in any of the possible cases $\overline{p}\,\overline{w_1} = \overline{q}\,\overline{w_1}$ which from earlier remarks suffices to show that $\overline{w_1}\cdots\overline{w_n}\,\mathcal{R}^*\,\overline{e}$. Together with the dual we have shown that $\mathrm{IG}(\mathcal{B})$ is an abundant semigroup, as required. □

We deduce that Theorem 6.3.3 and Theorem 6.3.7 are corollaries of Theorem 6.5.10.

The next example use Theorem 6.5.10 to show the abundancy of $\mathrm{IG}(\mathcal{B})$, over an iso-normal band B, where Y is a combination of a fan semilattice and a diamond semilattice.

Example 6.5.11. Let $B_1 = \mathscr{B}(Y_1, B_\alpha, \phi_{\alpha,\beta})$ be a diamond iso-normal band, where $Y_1 = \{\alpha, \beta, \gamma, \delta_0\}$. Let $B_2 = \mathscr{B}(Y_2, B_\alpha, \phi_{\alpha,\beta})$ be a fan iso-normal band, where $Y_2 = \{\delta_o, \delta_1, \delta_2, \ldots, \delta_n\}$. If $B = \mathscr{B}(Y, B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice $Y = Y_1 \cup Y_2$, where $Y_1 \cap Y_2 = \{\delta_0\}$, then it is clear that B is an iso-normal band. Therefore, by Theorem 6.5.10 we get that $\mathrm{IG}(\mathcal{B})$ is an abundant semigroup.

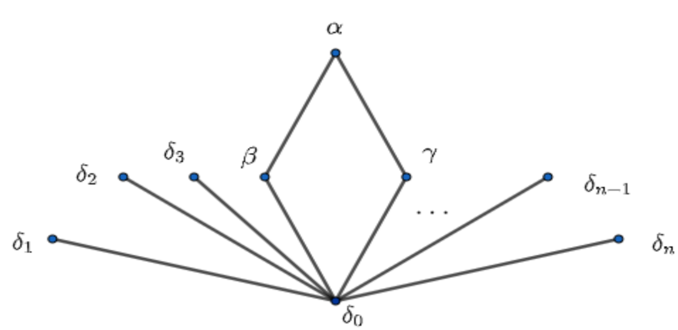


Figure 6.10: The combination of Fan and Diamond Semilattices

Chapter 7

Abundancy of the graph product of abundant semigroups

The notion of a graph product of groups was introduced by Green [55]. *Graph products of monoids* are defined in the same way as for groups [12]. Much of the existing work in graph products of monoids and groups has been to show that various algorithmic or algebraic properties are preserved under graph products. Our work of this chapter follows this stream. During our work to prove the abundancy of the free idempotent generated semigroup $\mathrm{IG}(\mathcal{B})$ over an iso-normal band B, we have proved in Lemma 4.1.3 that the external direct product of semigroups preserves the abundancy property. Moreover, in Lemma 4.2.3 we have proved that the free product of abundant semigroups is an abundant semigroup. It is known that the free products and the (restricted) external direct products are special cases of graph products. Another question comes out very naturally: is the graph product of abundant semigroups abundant? The main result of this chapter is that the graph product of abundant semigroups is always abundant. Moreover, the graph product of weakly abundant semigroups is always weakly abundant.

This chapter is organised as follows. In Section 7.1, we define specific morphisms of the graph product of semigroups which will be frequently used in this chapter. Furthermore, we describe the universal nature of the graph product of semigroups. In Section 7.2, we introduce important forms of the elements of the graph products of semigroups. These forms are used to prove our main result for this chapter. In order to prove the abundancy of the graph product of semigroups, we present a characterization of the idempotents in the graph product of semigroups $\mathscr{GP}$ in Section 7.3. In Section 7.4, we construct three main maps in Lemmas 7.4.1 and 7.4.5, which are used to prove our main result that shows the graph product of abundant semigroups is always abundant. In Section 7.5, we show that the graph product of weakly abundant semigroups is always weakly abundant.

The last two sections of this chapter give the description of the relations, $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ on $\mathscr{GP}$.

7.1 Universal nature of graph products

This section aims to define some useful morphisms of the graph product of semigroups, $\mathscr{GP}$. Further, we describe the universal nature $\mathscr{GP}$.

Recall that if $\Gamma = \Gamma(V, E)$ is a simple graph, S_α is a semigroup for each $\alpha \in V$ and we assume that $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha \neq \beta \in V$. Let

$$\mathscr{S} = \mathscr{S}(\Gamma) = \{S_\alpha : \alpha \in V\}$$

and put

$$X = X(\Gamma, \mathscr{S}) = \bigcup_{\alpha \in V} S_\alpha.$$

The *graph product* $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$ is defined as

$$\mathscr{GP} = X^+/\rho,$$

where the congruence ρ is given by

$$\rho = \rho(\Gamma, \mathscr{S}) = \langle H \rangle,$$

where

$$H = H(\Gamma, \mathscr{S}) = H_1 \cup H_2,\ \ H_1 = H_1(\Gamma, \mathscr{S}) = \{(x \circ y, xy) : x, y \in S_\alpha, \alpha \in V\}$$

and

$$H_2 = H_2(\Gamma, \mathscr{S}) = \{(x \circ y, y \circ x) : x \in S_\alpha, y \in S_\beta, (\alpha, \beta) \in E\}.$$

The following result is a special case of a general isomorphism theorem.

Lemma 7.1.1. *Let S be a semigroup and let $H_1, H_2 \subseteq S \times S$. Let $H = H_1 \cup H_2$ and $\rho = \langle H \rangle$. Let $\rho_1 = \langle H_1 \rangle$ and $\sigma = \langle H_2' \rangle$, where*

$$H_2' = \{(a\rho_1, b\rho_1) : (a, b) \in H_2\} \subseteq (S/\rho_1) \times (S/\rho_1).$$

Then

$$S/\rho \cong (S/\rho_1)/\sigma.$$

Proof. Define a map

$$\psi : S/\rho \longrightarrow (S/\rho_1)/\sigma,\ a\rho \mapsto (a\rho_1)\sigma.$$

We first claim that ψ is well defined. Assume $a\rho = b\rho$ for some $a, b \in S$. Then there exists $n \in \mathbb{N}$, $s_i, t_i \in S^1$ and $(c_i, d_i) \in H = H_1 \cup H_2$, for all $1 \leq i \leq n$, such that

$$a = s_1c_1t_1, s_1d_1t_1 = s_2c_2t_2, \ldots, s_nd_nt_n = b,$$

so that

$$a\rho_1 = (s_1c_1t_1)\rho_1, (s_1d_1t_1)\rho_1 = (s_2c_2t_2)\rho_1, \ldots, (s_nd_nt_n)\rho_1 = b\rho_1.$$

Notice that, for all $1 \leq i \leq n$, $c_i\rho_1 = d_i\rho_1$ if $(c_i, d_i) \in H_1$ and $(c_i\rho_1, d_i\rho_1) \in H_2'$ if $(c_i, d_i) \in H_2$, and hence the above sequence gives $(a\rho_1)\ \sigma\ (b\rho_1)$, namely,

$$(a\rho_1)\sigma = (b\rho_1)\sigma,$$

so that ψ is well defined. Clearly, ψ is an epimorphism. To show ψ is one-one, suppose that $(a\rho_1)\sigma = (b\rho_1)\sigma$. Then exists $n \in \mathbb{N}$, $(u_i\rho_1, v_i\rho_1) \in H_2'$ (and so $(u_i, v_i) \in H_2$) and $s_i\rho_1, t_i\rho_1 \in (S/\rho_1)^1$ for all $1 \leq i \leq n$ such that

$$a\rho_1 = (s_1\rho_1)(u_1\rho_1)(t_1\rho_1), (s_1\rho_1)(v_1\rho_1)(t_1\rho_1) = (s_2\rho_1)(u_2\rho_1)(t_2\rho_1), \ldots, (s_n\rho_1)(v_n\rho_1)(t_n\rho_1) = b\rho_1$$

namely,

$$a\ \rho_1\ s_1u_1t_1, s_1v_1t_1\ \rho_1\ s_2u_2t_2, \ldots, s_nv_nt_n\ \rho_1\ b,$$

where if $s_i\rho_1$(or $t_i\rho_1$) is the adjoined identity of S/ρ_1, we take s_i (or t_i) to be the adjoined identity of S. As $(u_i, v_i) \in H_2 \subseteq H$, $u_i\ \rho\ v_i$. So $s_iu_it_i\ \rho\ s_iv_it_i$ for all $1 \leq i \leq n$, we have

$$a\ \rho\ (s_1u_1t_1)\ \rho\ (s_1v_1t_1)\ \rho\ (s_2u_2t_2)\ \rho \ldots \rho\ (s_nv_nt_n)\ \rho\ b,$$

so that $a\rho = b\rho$, as required. $\square$

As an application of Lemma 7.1.1, we have the following result.

Corollary 7.1.2. *We have that* $\mathscr{GP} \cong \mathscr{FP}/\langle H_2' \rangle$, *where*

$$H_2' = H_2'(\Gamma, \mathscr{S}) = \{([x \circ y], [y \circ x]) : x \in S_\alpha, y \in S_\beta, (\alpha, \beta) \in E\},$$

and $[w]$ *denotes the* ρ_1*-class of* $w \in X^+$.

We now explain the universal nature of graph products of semigroups.

Definition 7.1.3. Suppose that T is a semigroup and we have a collection of morphisms

$$\theta = \{\theta_\alpha : S_\alpha \to T \mid \alpha \in V\}.$$

We say that θ *satisfies the* Γ*-condition* if

$$(s_\alpha \theta_\alpha)(s_\beta \theta_\beta) = (s_\beta \theta_\beta)(s_\alpha \theta_\alpha),$$

for all $(\alpha, \beta) \in E$.

Definition 7.1.4. For each $a \in X$, we write $C(a) = \alpha$ if $a \in S_\alpha$. The *support* $s(x)$ of an element $x = x_1 \circ \ldots \circ x_n \in X^+$ is defined to be the set $\{C(x_i) : 1 \leq i \leq n\}$.

For each fixed $\alpha \in V$, we define a monoid $S_\alpha^{\underline{1}_\alpha} = S_\alpha \cup \{\underline{1}_\alpha\}$ with the multiplication $\cdot$ given by

$$a \cdot b = \begin{cases} ab & \text{if } a, b \in S_\alpha; \\ a & \text{if } b = \underline{1}_\alpha; \\ b & \text{if } a = \underline{1}_\alpha; \\ \underline{1}_\alpha & \text{if } a = b = \underline{1}_\alpha. \end{cases}$$

Notice that we add the identity $\underline{1}_\alpha$ to S_α even if S_α already has its own identity.

Lemma 7.1.5. *For any* $\alpha \in V$ *there is a morphism* $\overline{\tau_\alpha} : \mathscr{GP} \to S_\alpha^{\underline{1}_\alpha}$ *given by*

$$[x_1 \circ \ldots \circ x_n]\overline{\tau_\alpha} = x_{i_1} x_{i_2} \ldots x_{i_k}, \text{ where } C(x_j) = \alpha \Leftrightarrow j = i_h \text{ for some } 1 \leq h \leq k.$$

We interpret the empty product above as being $\underline{1}_\alpha$.

Proof. For each $\alpha \in V$, define a map

$$\tau_\alpha : X^+ \to S_\alpha^{\underline{1}_\alpha}$$

by

$$(x_1 \circ \ldots \circ x_n)\tau_\alpha = x_{i_1} x_{i_2} \ldots x_{i_k} \text{ where } C(x_j) = \alpha \Leftrightarrow j = i_h \text{ for some } 1 \leq h \leq k.$$

We first claim that τ_α is well defined. Assume

$$x = x_1 \circ \ldots \circ x_n,\ y = y_1 \circ \ldots \circ y_m \in X^+$$

and $x = y$. Hence both x and y have the same support, which means

$$s(x) = \{C(x_i) : 1 \leq i \leq n\} = \{C(y_i) : 1 \leq i \leq m\} = s(y).$$

Let $C(x_{i_h}) = \alpha$, for all $1 \leq h \leq k$, then $x_{i_1}, x_{i_2}, \ldots, x_{i_k} \in S_\alpha$. Hence $x_{i_1}x_{i_2}\ldots x_{i_k} \in S_\alpha^{\underline{1_\alpha}}$. Similarly, $C(y_{i_h}) = C(x_{i_h}) = \alpha$, for all $1 \leq h \leq k$, then $y_{i_1}, y_{i_2}, \ldots, y_{i_k} \in S_\alpha$. Hence $y_{i_1}y_{i_2}\ldots y_{i_k} \in S_\alpha^{\underline{1_\alpha}}$. This implies that $(x_1 \circ \ldots \circ x_n)\tau_\alpha = (y_1 \circ \ldots \circ y_m)\tau_\alpha$ in $S_\alpha^{\underline{1_\alpha}}$.

To prove that τ_α is a morphism, assume x and y are elements of X^+ as above. Let $x_i \in S_\alpha$ for all $i \in \{i_1, \cdots, i_k\}$ and $y_j \in S_\alpha$ for $j \in \{j_1, \cdots, j_h\}$. Then we get that

$$(x_1 \circ \ldots \circ x_n)\tau_\alpha = x_{i_1} \circ \cdots \circ x_{i_k}$$

and

$$(y_1 \circ \ldots \circ y_m)\tau_\alpha = y_{j_1} \circ \cdots \circ y_{j_h}.$$

Also we have that

$$\begin{aligned}(x_1 \circ \ldots \circ x_n \circ y_1 \circ \ldots \circ y_m)\tau_\alpha &= x_{i_1} \circ \cdots \circ x_{i_k} \circ y_{j_1} \circ \cdots \circ y_{j_h} \\ &= (x_1 \circ \ldots \circ x_n)\tau_\alpha(y_1 \circ \ldots \circ y_m)\tau_\alpha.\end{aligned}$$

This proved that τ_α is a morphism.

We now claim that $\rho \subseteq \ker \tau_\alpha$, for which it is sufficient to show that $H \subseteq \ker \tau_\alpha$. For generators with a form $(s \circ t, st)$ where $s, t \in S_\alpha$ and $\alpha \in V$,

$$(s \circ t)\tau_\alpha = st = (st)\tau_\alpha.$$

For generators with a form $(s \circ t, t \circ s)$ where $s \in S_\alpha,\ t \in S_\beta,\ (\alpha, \beta) \in E$, then $\alpha \neq \beta$. Hence we get that

$$(s \circ t)\tau_\alpha = s = (t \circ s)\tau_\alpha.$$

Therefore we get that $\rho \subseteq \ker \tau_\alpha$. Then there exists a morphism

$$\overline{\tau_\alpha} : \mathscr{G}\mathscr{P} \to S_\alpha^{\underline{1_\alpha}}$$

defined by

$$[x_1 \circ \ldots \circ x_n]\overline{\tau_\alpha} = (x_1 \circ \ldots \circ x_n)\tau_\alpha.$$

$\square$

The proof of the following result is similar to that of [17, Proposition 2.3]

Lemma 7.1.6. *Let $V' \subseteq V$ and let $\Gamma' = \Gamma(V', E')$ be the resulting full subgraph of Γ. Let $\mathscr{GP}'$ be the corresponding graph product of the semigroups $\mathcal{S}' = \{S_\alpha : \alpha \in V'\}$. Then $\mathscr{GP}'$ is a retract of $\mathscr{GP}$.*

The next result is analogous to [42, Proposition 1.6]

Proposition 7.1.7. *For each $\alpha \in V$,*

(i) there is an embedding

$$\iota_\alpha : S_\alpha \to \mathscr{GP}, \quad s_\alpha \iota_\alpha = [s_\alpha];$$

(ii) the morphism $\iota = \{\iota_\alpha : S_\alpha \to \mathscr{GP} \mid \alpha \in V\}$ satisfies the Γ-condition;

(iii) the graph product $\mathscr{GP}$ is generated by $\{[s_\alpha] : \alpha \in V, s_\alpha \in S_\alpha\}$.

Proof. (i) Clearly, ι_α is a morphism. Let $s_\alpha, t_\alpha \in S_\alpha$ be such that $[s_\alpha] = [t_\alpha]$. Then

$$s_\alpha = [s_\alpha]\overline{\tau_\alpha} = [t_\alpha]\overline{\tau_\alpha} = t_\alpha,$$

where $\overline{\tau_\alpha} : \mathscr{GP} \to S_\alpha^{1_\alpha}$ is the morphism that defined in Lemma 7.1.5. So that ι_α is one to one and hence an embedding morphism.

(ii) For all $\alpha, \beta \in V$ with $(\alpha, \beta) \in E$ and all $s_\alpha \in S_\alpha, s_\beta \in S_\beta$, we have

$$(s_\alpha \iota_\alpha)(s_\beta \iota_\beta) = [s_\alpha][s_\beta] = [s_\alpha \circ s_\beta] = [s_\beta \circ s_\alpha] = [s_\beta][s_\alpha] = (s_\beta \iota_\beta)(s_\alpha \iota_\alpha).$$

Then $\iota = \{\iota_\alpha : S_\alpha \to \mathscr{GP} \mid \alpha \in V\}$ satisfies the Γ-condition.

(iii) It is clear that any element $[s] = [s_1 \circ \ldots \circ s_n]$ of $\mathscr{GP}$ we can write

$$[s] = [s_1 \circ \ldots \circ s_n] = [s_1] \ldots [s_n].$$

Therefore, $\mathscr{GP}$ generated by $\{[s_\alpha] : \alpha \in V, s_\alpha \in S_\alpha\}$.

$\square$

A subset U of a semigroup S is *right unitary* in S if

$$(\forall u \in U)\ (\forall s \in S)\ su \in U \Longrightarrow s \in U.$$

The notation of a *left unitary* is dually, and U is *unitary* in S if it is both right and left unitary.

Corollary 7.1.8. *Each S_α, where $\alpha \in V$, is isomorphic to unitary subsemigroup of $\mathscr{GP}$.*

Proof. Let $[a] \in S'_\alpha$ where $S'_\alpha = \{[a] : a \in S_\alpha\}$ and suppose $[w] \in \mathscr{GP}$ with $[w][a] \in S'_\alpha$. Then $s(w \circ a) = \alpha$. This implies $s(w) = \{\alpha\}$, so $w \in S_\alpha$. Therefore, S'_α is a right unitary in $\mathscr{GP}$. Similarly, S'_α is a left unitary. Hence S'_α is a unitary in $\mathscr{GP}$. It is clear that S_α is isomorphic to S'_α. Therefore, S_α is isomorphic to an unitary subsemigroup of $\mathscr{GP}$. $\square$

Proposition 7.1.9. *Let T be a semigroup and*

$$\psi = \{\theta_\alpha : S_\alpha \to T \mid \alpha \in V\}$$

be a collection of morphisms satisfying the Γ-condition. Then there is a unique morphism

$$\overline{\theta} : \mathscr{GP} \to T$$

such that $\iota_\alpha \overline{\theta} = \theta_\alpha$ for all $\alpha \in V$.

$$\begin{tikzcd}
S_\alpha \arrow[d, "\iota_\alpha"'] \arrow[dr, "\theta_\alpha"] & \\
\mathscr{GP} \arrow[r, "\overline{\theta}"'] & T
\end{tikzcd}$$

Figure 7.1: The commutative diagram of graph product

Proof. Define a map

$$\theta : X^+ \longrightarrow T, \quad s_\alpha \mapsto s_\alpha \theta_\alpha$$

for each $\alpha \in V$ and each $s_\alpha \in S_\alpha$. We now claim that $\rho \subseteq \ker \theta$, for which we need show that $H \subseteq \ker \theta$. For generators with a form $(s \circ t, st)$, where $s,\ t \in S_\alpha$ and $\alpha \in V$,

$$(s \circ t)\theta = (s\theta)(t\theta) = (s\theta_\alpha)(t\theta_\alpha) = (st)\theta_\alpha = (st)\theta.$$

For generators with a form $(s \circ t, t \circ s)$, where $s \in S_\alpha,\ t \in S_\beta$, $(\alpha, \beta) \in E$, by the Γ-condition

$$(s \circ t)\theta = (s\theta)(t\theta) = (s\theta_\alpha)(t\theta_\beta) = (t\theta_\beta)(s\theta_\alpha) = (t\theta)(s\theta) = (t \circ s)\theta.$$

Hence $\rho \subseteq \ker\theta$, this implies that there is a morphism

$$\overline{\theta} : \mathscr{G}\mathscr{P} \to T, \quad [w] \mapsto w\theta,$$

where $[w] \in \mathscr{G}\mathscr{P}$. Further, for all $\alpha \in V$ and all $s_\alpha \in S_\alpha$,

$$s_\alpha \iota_\alpha \overline{\theta} = [s_\alpha]\overline{\theta} = s_\alpha \theta = s_\alpha \theta_\alpha$$

so that $\iota_\alpha \overline{\theta} = \theta_\alpha$. If there is another morphism

$$\overline{\theta'} : \mathscr{G}\mathscr{P} \to T$$

such that $\iota_\alpha \overline{\theta'} = \theta_\alpha$, for all $\alpha \in V$, then for each $\alpha \in V$ and $s_\alpha \in S_\alpha$, we have

$$[s_\alpha]\overline{\theta'} = s_\alpha \iota_\alpha \overline{\theta'} = s_\alpha \theta_\alpha = s_\alpha \theta = [s_\alpha]\overline{\theta}.$$

Therefore, $\overline{\theta} = \overline{\theta'}$, by (iii) of Proposition 7.1.7.

□

We now show that the conditions of Proposition 7.1.7 characterise $\mathscr{G}\mathscr{P}$.

Proposition 7.1.10. *Let U be a semigroup and*

$$\nu = \{\nu_\alpha : S_\alpha \to U \mid \alpha \in V\}$$

be a collection of embeddings satisfying the Γ-condition and U be generated by $\{s_\alpha \nu_\alpha : \alpha \in V, s_\alpha \in S_\alpha\}$. Suppose that U satisfies the condition that for any semigroup T and collection of morphisms

$$\theta = \{\theta_\alpha : S_\alpha \to T \mid \alpha \in V\}$$

satisfying the Γ-condition and there is a unique morphism

$$\psi : U \to T$$

such that $\nu_\alpha \psi = \theta_\alpha$ for all $\alpha \in V$. Then there is an isomorphism

$$\overline{\nu} : \mathscr{G}\mathscr{P} \to U$$

such that $\iota_\alpha \overline{\nu} = \nu_\alpha$ for all $\alpha \in V$.

$$\begin{tikzcd}
 & S_\alpha \arrow[dl, "\iota_\alpha"'] \arrow[d, "\nu_\alpha"] \arrow[dr, "\theta_\alpha"] & \\
\mathscr{G}\mathscr{P} \arrow[r, "\overline{\nu}"'] & U \arrow[r, "\psi"'] & T
\end{tikzcd}$$

Figure 7.2: The commutative diagram of graph product

Proof. The collection of embeddings

$$\iota_\alpha : S_\alpha \to \mathscr{G}\mathscr{P}$$

satisfies the Γ-condition by Proposition 7.1.7 (ii). Then by the assumption, there is a unique morphism $\psi : U \to \mathscr{G}\mathscr{P}$ such that

$$\nu_\alpha \psi = \iota_\alpha$$

for each $\alpha \in V$.

On the other hand, by Proposition 7.1.9, there is a unique morphism

$$\overline{\nu} : \mathscr{G}\mathscr{P} \to U$$

such that

$$\iota_\alpha \overline{\nu} = \nu_\alpha$$

for each $\alpha \in V$.

For any $s_\alpha \in S_\alpha$ we have

$$\begin{aligned}
[s_\alpha]\overline{\nu}\psi &= s_\alpha \iota_\alpha \overline{\nu}\psi && (\text{as } [s_\alpha] = s_\alpha \iota_\alpha) \\
&= s_\alpha \nu_\alpha \psi && (\text{as } \nu_\alpha = \iota_\alpha \overline{\nu}) \\
&= s_\alpha \iota_\alpha && (\text{as } \iota_\alpha = \nu_\alpha \psi) \\
&= [s_\alpha].
\end{aligned}$$

and as $\mathscr{G}\mathscr{P}$ is generated by $\{[s_\alpha] : \alpha \in V, s_\alpha \in S_\alpha\}$, we have that $\overline{\nu}\psi$ is the identity on $\mathscr{G}\mathscr{P}$. The same argument gives that $\psi\overline{\nu}$ is the identity on the subsemigroup of U generated by $\{s_\alpha \nu_\alpha : \alpha \in V, s_\alpha \in S_\alpha\}$, but this is U, so we conclude that ψ and $\overline{\nu}$ are isomorphisms. □

Note that Proposition 7.1.9 and Proposition 7.1.10 are justifying the universal nature of graph products of semigroups.

In the following we explain the relation between the graph product of semigroups and the graph product of monoids.

Let $\mathscr{S}^1 = \mathscr{S}^1(\Gamma) = \{S_\alpha^{1_\alpha} : \alpha \in V\}$ and put

$$Y = Y(\Gamma, \mathscr{S}^1) = \bigcup_{\alpha \in V} S_\alpha^{1_\alpha}$$

Then the *monoid graph product* $\mathscr{G}\mathscr{P}\mathscr{M} = \mathscr{G}\mathscr{P}\mathscr{M}(\Gamma, \mathscr{S}^1)$ *is defined by*

$$\mathscr{G}\mathscr{P}\mathscr{M} = Y^+/\sigma,$$

where $\sigma = \langle L \rangle$ *is such that* $L = L_1 \cup L_2 \cup L_3$ *with*

$$L_1 = \{(x \circ y, xy) : x, y \in S_\alpha^{1_\alpha}\}$$

$$L_2 = \{(x \circ y, y \circ x) : x \in S_\alpha^{1_\alpha}, y \in S_\beta^{1_\beta}, (\alpha, \beta) \in E\}$$
$$L_3 = \{(\underline{1}_\alpha, \underline{1}_\beta) : \alpha, \beta \in V\}.$$

For each $x_1 \circ \ldots \circ x_n \in Y^+$, *we use* $\lfloor x_1 \circ \ldots \circ x_n \rfloor$ *to denote the* σ*-class of* $x_1 \circ \ldots \circ x_n$ *in* $\mathscr{GPM}$.

In 2021, Gould and Yang [17], showed that the graph product of semigroups $\mathscr{S} = \mathscr{S}(\Gamma) = \{S_\alpha : \alpha \in V\}$ *embedded into the graph product of monoids* $\mathscr{S}^1 = \mathscr{S}^1(\Gamma) = \{S_\alpha^{1_\alpha} : \alpha \in V\}$.

Proposition 7.1.11. *Let* $\mathscr{GP}$ *be the graph product of semigroups* $\mathscr{S} = \mathscr{S}(\Gamma) = \{S_\alpha : \alpha \in V\}$ *with respect to* $\Gamma = (V, E)$. *Let* $\mathscr{GPM}$ *be the graph product of monoids* $\mathscr{S}^1 = \mathscr{S}^1(\Gamma) = \{S_\alpha^{1_\alpha} : \alpha \in V\}$ *with respect to* Γ. *The map*

$$\theta : \mathscr{GP} \longrightarrow \mathscr{GPM}, \quad [x_1 \circ \ldots \circ x_n] \mapsto \lfloor x_1 \circ \ldots \circ x_n \rfloor$$

is an embedding.

Note that in general, a monoid graph product is much more complicated than a semigroup graph product, as in a semigroup graph product, the identities remain distinct, which results that if $[w]$, $[v] \in \mathscr{GP}$ with length m and n, respectively, such that $1 \leq m \leq n$, then the length of the product of $[w]$ and $[v]$ is maximal $m + n$, or minimal n. However, in the monoid graph product (of monoids), all the identities of the individual monoids are identified, which results that if $\lfloor w \rfloor$, $\lfloor v \rfloor \in \mathscr{MGP}$ with length m and n, respectively, such that $1 \leq m \leq n$, then the length of the product of $\lfloor w \rfloor$ and $\lfloor v \rfloor$ maybe equal m or less.

7.2 Special forms

The aim of this section is to introduce important forms of the elements of the graph products of some semigroups. These forms are used to prove our main result for this chapter.

Definition 7.2.1. An element $x = x_1 \circ \ldots \circ x_n \in X^+$ is a *reduced form* for $[x] \in \mathscr{GP}$ if $[x] = [x_1 \circ \ldots \circ x_n]$ and for any $1 \leq i, j \leq n$ with $i < j$ and $C(x_i) = C(x_j)$, there must exist $i < k < j$ with $(C(x_i), C(x_k)) \notin E$.

Note that we denote the length of the reduced form of $[x]$ by $|x| = n$.

The proof of the following result follows directly from the definition of the reduced form of the elements of a graph product.

Lemma 7.2.2. *For any $x = x_1 \circ \ldots \circ x_n,\ y = y_1 \circ \ldots \circ y_m \in X^+$,*

(i) $[x] = [y]$ in $\mathscr{GP}$ implies that $s(x) = s(y)$;

(ii) if $s(x)$ is a complete subgraph of $\Gamma(V, E)$, then there exists $z \in X^+$ such that $[x] = [z]$ in $\mathscr{GP}$, where $z = z_1 \circ \ldots \circ z_l$ and $C(z_i) \neq C(z_j)$ for all $i \neq j$, $1 \leq i, j \leq l$; clearly z is a reduced form.

Note that by Lemma 7.2.2 any $[y_1 \circ \cdots \circ y_m]$ of $\mathscr{GP}$, with $C(y_i) = \gamma_i$, $\gamma_i \in V$, for all $1 \leq i \leq m$, can be written in the shortest form $[z_1 \circ \cdots \circ z_n]$ with $C(z_i) = \beta_i$, for all $1 \leq i \leq n$, and $C(z_i) \neq C(z_{i+1})$, for all $1 \leq i \leq n-1$, and $n \leq m$. So it is clear that $\{\beta_1, \cdots, \beta_n\} \subseteq \{\gamma_1, \cdots, \gamma_m\}$.

The next result will be used frequently to prove some results in this chapter.

Lemma 7.2.3. *[17] Let $[x] = [y]$, where $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_n$ are reduced and let $1 \leq m \leq n$. Then $[x_1 \circ \cdots \circ x_m] = [y_1 \circ \cdots \circ y_m]$ if and only if $[x_{m+1} \circ \cdots \circ x_n] = [y_{m+1} \circ \cdots \circ y_n]$.*

For an element $x = x_1 \circ \ldots \circ x_n \in X^+$, if $(C(x_j), C(x_{j+1})) \in E$ for some j, then we may obtain a different expression for x by replacing $x_j \circ x_{j+1}$ by $x_{j+1} \circ x_j$. We call this move a *shuffle*. Two words of X^+ are *shuffle equivalent* if one can be obtained from the other by a sequence of shuffles. However, if $C(x_i) = C(x_{i+1}) = \alpha$, for some $\alpha \in V$, then by *reduction* we can write

$$x = x_1 \circ \ldots \circ x_i x_{i+1} \circ x_{i+2} \circ \ldots \circ x_n,$$

where $x_i x_{i+1} \in S_\alpha$. For more details see [42].

The following result captures how we may shuffle a word to re-order it.

Lemma 7.2.4. *Let $x = x_1 \circ \cdots \circ x_n \in X^+$. Then we can shuffle x to*

$$x' = x_{i_1} \circ \cdots \circ x_{i_n}$$

if and only if for all $1 \leq j < k \leq n$, if $i_k < i_j$ then $(C(x_{i_j}), C(x_{i_k})) \in E$.

Proof. Suppose that we can shuffle x to x'. If $1 \leq j < k \leq n$ and $i_k < i_j$, then in the process we must have changed the order of x_{i_k} and x_{i_j}, then we must have $(C(x_{i_j}), C(x_{i_k})) \in E$.

Conversely, let x' have the property that for all $1 \leq j < k \leq n$, if $i_k < i_j$, then $(C(x_{i_j}), C(x_{i_k})) \in E$. If $n = 1$ the result is clear. Suppose for induction that the result is true for words of shorter length. Then for $1 \leq j < i_1$ we have $x_j = x_{i_{k(j)}}$ where $1 < k(j)$ but $i_{k(j)} < i_1$. By assumption $(C(x_{i_1}), C(x_{i_{k(j)}})) \in E$ so we may shuffle x_{i_1} in

$$x = x_1 \circ x_2 \cdots \circ x_{i_1 - 1} \circ x_{i_1} \circ x_{i_1+1} \circ \cdots x_n$$

to the left to obtain

$$x'' = x_{i_1} \circ x_1 \circ x_2 \cdots \circ x_{i_1-1} \circ x_{i_1+1} \circ \cdots x_n.$$

Considering now the word $x_1 \circ x_2 \cdots \circ x_{i_1-1} \circ x_{i_1+1} \circ \cdots x_n$ and applying our inductive hypothesis (with suitable relabelling) we obtain that x shuffles to x'. $\square$

Note 7.2.5. *Let $x = x_1 \circ \cdots \circ x_n$, $y = y_1 \circ \cdots \circ y_m \in X^+$ be reduced forms. Then $x \circ y$ is* not *reduced exactly if there exist i, j with $1 \leq i \leq n, 1 \leq j \leq m$ such that $C(x_i) = C(y_j)$ and we have $(C(x_i), C(x_h)) \in E$, for all $i < h \leq n$, and $(C(x_i), C(y_k)) \in E$, for all $1 \leq k < j$.*

The following result is the semigroup version of the monoid result [17, Lemma 3.7], and the proof of it is similar to the monoid version.

Lemma 7.2.6. *Let $x \in X^+$. Applying reductions and shuffles leads in a finite number of steps to a reduced word $\overline{x}$ with $[x] = [\overline{x}]$.*

We now recall the definition of *rewriting systems* and their properties [5].

Let X be an alphabet and $\rightarrow$ a binary relation on X. The structure $(X, \rightarrow)$ is called a *rewriting system* and the relation $\rightarrow$ a *rewriting relation.* The reflexive, transitive closure of $\rightarrow$ is denoted by $\xrightarrow{*}$ while $\xleftrightarrow{*}$ denotes the smallest equivalence relation on X that contains $\rightarrow$. We denote the equivalence class of an element $x \in X$ by (x). An element $x \in X$ is said to be *irreducible* if there is no $y \in X$, y not equal to x such that $x \rightarrow y$; otherwise, x is *reducible.* For any $x, y \in X$, if $x \xrightarrow{*} y$ and y is irreducible, then y is a normal form of x. In the following we define two kinds of a rewriting system $(X, \rightarrow)$:

(i) $(X, \rightarrow)$ is called *confluent* if whenever $x, y, z \in X$ are such that $x \xrightarrow{*} y$ and $x \xrightarrow{*} z$, then there is a $v \in X$ such that $y \xrightarrow{*} v$ and $z \xrightarrow{*} v$, as described by the figure below,

$$
\begin{tikzcd}
& x \arrow[dl] \arrow[dr] & \\
y \arrow[dr, "*"'] & & z \arrow[dl, "*"] \\
& v &
\end{tikzcd}
$$

Figure 7.3: Confluence

(ii) $(X, \to)$ is called *noetherian* if there is no infinite sequence $x_0, x_1, \cdots \in X$ such that for all $i \geq 0$, $x_i \to x_{i+1}$.

The next result was originally proven for graph products of monoids in [55]. The argument for semigroups is much simpler, and worth stating.

Proposition 7.2.7. *Every element of the graph product $\mathscr{G}\mathscr{P}$ is represented by a reduced form. Two reduced forms represent the same element of $\mathscr{G}\mathscr{P}$ if and only if they are shuffle equivalent. An element $w \in [x]$ is of minimal length if and only if it is reduced.*

Proof. It follows from Lemma 7.2.6 that for any $[x] \in \mathscr{G}\mathscr{P}$ we have $[x] = [\overline{x}]$ for some reduced word $\overline{x}$.

Next, we show that the set of all shuffle equivalence classes forms a confluent rewriting system, where the rewriting rules are as follows. For convenience we denote by (x) the shuffle equivalence class of $x \in X^+$ and we have rewriting rule $(x) \longrightarrow (y)$ if y is obtained from $x' \in (x)$ by applying a reduction.

Let $x = x_1 \circ \cdots \circ x_n \in X^+$ and pick $x' = x_{i_1} \circ \cdots \circ x_{i_n}$ and $x'' = x_{j_1} \circ \cdots \circ x_{j_n}$ in (x). Suppose that $C(x_{i_k}) = C(x_{i_{k+1}})$ so that we may perform a reduction to obtain

$$y' = x_{i_1} \circ \cdots \circ x_{i_{k-1}} \circ x_{i_k} x_{i_{k+1}} \circ x_{i_{k+2}} \circ \cdots \circ x_{i_n}.$$

Then by Lemma 7.2.4, y' is shuffle equivalent to

$$y = x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots \circ x_n$$

where $p = i_k$ and $q = i_{k+1}$; notice we must have that $p < q$. Applying the same process to x'' results in a word

$$z = x_1 \circ \cdots \circ x_{r-1} \circ x_r x_t \circ x_{r+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n$$

where $r < t$.

Therefore, $(x) \longrightarrow (y)$ and $(x) \longrightarrow (z)$. We now need show that $(y) \overset{*}{\longrightarrow} (v)$ and $(z) \overset{*}{\longrightarrow} (v)$ for some $v \in X^+$, as depicted by the following figure

$$\begin{tikzcd} & (x) \arrow[dl] \arrow[dr] & \\ (y) \arrow[dr, "*"'] & & (z) \arrow[dl, "*"] \\ & (v) & \end{tikzcd}$$

Figure 7.4

Without loss of generality we may assume that $p \leq r$. If $p = r$ then from Lemma 7.2.4 (note that our graphs have no loops), we cannot have $p = r < q < t$ or $p = r < t < q$; we deduce that in this case $q = t$ so that $(y) = (z)$. If $p < r$, then again we cannot have that $r < q$, so that either $q = r$ or $q < r$.

If $q = r$, then $(y) = (y'')$ where y'' is the word

$$x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q \circ x_t \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n$$

and then $(y'') \longrightarrow (v)$ where v is the word

$$x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q x_t \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n.$$

Similarly, $(z) \longrightarrow (v)$.

If $q < r$, then by shuffling and applying a reduction in each case we have $(y) \longrightarrow (u)$ and $(z) \longrightarrow (u)$ where u is the word

$$x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots x_{r-1} \circ x_r x_t \circ x_{r+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n.$$

We have shown that the set of all shuffle equivalence classes forms a confluent rewriting system. It follows that any two reduced forms represent the same element of $\mathscr{GP}$ if and only if they are shuffle equivalent.

Let $w \in [x]$ for some words $w, x \in X^+$. It is clear that if w is of minimal length in $[x]$, then it must be reduced. Finally, if w is reduced then as certainly $[w] = [z]$ for some word z of minimal length in $[x]$, then z is also reduced, giving that w and z are shuffle equivalent, so that they have the same length.

□

Definition 7.2.8. If $x = x_1 \circ \ldots \circ x_k \in X^+$ is a reduced form, where $s(x)$ is complete, we say x is a *complete reduced form* of $[x]$ in $\mathscr{GP}$, with letters $x_i \in X$ for all $1 \leq i \leq k$.

In the following definition we are going to further refine the notation of reduced form.

Definition 7.2.9. Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a reduced form, where each $w_i \in X^+$ is given by

$$w_i = x_{i1} \circ \ldots \circ x_{ip(i)}.$$

We say that w is a *left complete reduced form*, of $[w]$ of $\mathscr{GP}$, with blocks $w_i \in X^+$, $1 \leq i \leq k$, if:

(i) for all $1 \leq i \leq k$, w_i is complete reduced form;

(ii) for $i < k$ and any $j \in s(w_{i+1})$, there is some $\ell \in s(w_i)$ such that $(j, \ell) \notin E$.

Dually we may define the notation of a *right complete reduced form* of an element in X^+. Note that a complete reduced form is precisely a word in left complete reduced form with one block.

It is obvious from the definitions of a left complete reduced form, that the blocks w_i, $1 \leq i \leq k$ of a left complete reduced form $w = w_1 \circ \ldots \circ w_k \in X^+$ are words of X^+, and x_{iq} are letters of X, for all $1 \leq i \leq k$ and $1 \leq q \leq p(i)$. At times we will use this concept without specific comment.

Note 7.2.10. *(i) A word $w = w_1 \circ \cdots \circ w_n$ is a complete reduced form if and only if $s(w)$ is complete and $C(w_i) \neq C(w_j)$ for all $1 \leq i < j \leq n$.*

(ii) If $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_m \in X^+$ are complete reduced forms, then $[x] = [y]$ if and only if x and y are shuffle equivalent if and only if $y_i = x_{i\sigma}$, $1 \leq i \leq n$, for some permutation σ of $\{1, \cdots, n\}$; in particular, $n = m$ and $s(x) = s(y)$.

The fact that every element in $\mathscr{GP}$ has a complete left reduced form is stated in our paper [50]. The result was known for graph monoids [13] and proven via a technique involving cancellation, which we do not have here. We give a full proof in the case of an arbitrary graph product of semigroups; it is essentially the same as that for an arbitrary graph product of monoids, which appears in [17].

The next result shows that any element of $\mathscr{GP}$ may be represented by a left complete reduced form.

Lemma 7.2.11. *Every element in $\mathscr{GP}$ may be represented by a left complete reduced form in X^+.*

Proof. Let $[w] \in \mathscr{GP}$ such that w is a reduced form in X^+. Let $|w| = n$. Then for any $w' \in X^+$ with $[w] = [w']$, w' is a reduced form if and only if $|w| = |w'|$. Let

$Y = \{y \in X^+ : [w] = [y \circ y_2], s(y) \text{ is a complete graph, } y_2 \in X^+ \text{ and } y \circ y_2 \text{ is reduced}\}.$

We pick a $w_1 \in Y$ with maximum length l say; certainly we can do this, since l is bounded by $|w|$. Then $[w] = [w_1 \circ y_2]$ for some $y_2 \in X^+$. Notice that as $w_1 \circ y_2$ is reduced, $n = |w_1| + |y_2|$. We then do the same for y_2, obtaining $[y_2] = [w_2 \circ y_3]$, where $|y_2| = |w_2| + |y_3|$, so that $n = |w_1| + |w_2| + |y_3|$. Continuing this process, we obtain

$$[x] = [w_1 \circ w_2 \circ \ldots \circ w_k]$$

where $[y_j] = [w_j \circ y_{j+1}]$ for $2 \leq j < k$. We now claim that $w_1 \circ \ldots \circ w_k$ is a complete reduced form. By the choice of each w_i, we know $s(w_i)$ is a complete subgraph. Also, it is easy to check that $n = |w_1| + |w_2| + \ldots + |w_k|$, so that $w_1 \circ \ldots \circ w_k$ is a reduced form. Suppose that there exists some $2 \leq j \leq k$ and some $z_l \in w_j$ with $C(z_l) = u \in s(w_j)$ such that $(u, v) \in E$ for all $v \in s(w_{j-1})$. Notice that $u \notin s(w_{j-1})$ as Γ has no loops. Then

$$[y_j] = [(w_{j-1} \circ z_l) \circ w'_j \circ \ldots \circ w_k]$$

where w'_j is w_j with z_l deleted. But $s(w_{j-1} \circ z_l)$ is complete and $(w_{j-1} \circ z_l) \circ w'_j \circ \ldots \circ w_k$ is reduced, since it is shuffle equivalent to the reduced word $w_{j-1} \circ w_j \circ \ldots \circ w_k$. But we have $|s(w_{j-1} \circ z_l)| = |s(w_{j-1})| + 1$, contradiction, and hence $w_1 \circ \ldots \circ w_k$ is a left complete reduced form. □

In the following result we show that a left complete reduced form is unique.

Theorem 7.2.12. *Let $w_1 \circ y_1$ and $w'_1 \circ y'_1$ be left complete reduced forms of $w \in X^+$ such that both w_1 and w'_1 are the first components. Then $s(w_1) = s(w'_1)$ is complete, $[w_1] = [w'_1]$ and $[y_1] = [y'_1]$.*

Proof. We first make the following observation. Let $x_1 \circ \ldots \circ x_n$ and $z_1 \circ \ldots \circ z_n$ be reduced forms of $w \in X^+$. Pick $u \in s(w)$. Let i be the smallest such that $C(x_i) = u$ and l be the smallest such that $C(z_l) = u$. Suppose that there exists some $1 \leq j < i$ such that $C(x_j) = v$ with $(v, u) \notin E$. Then, as $x_1 \circ \ldots \circ x_n$ and $z_1 \circ \ldots \circ z_n$ are shuffle equivalent, there must also exist some $1 \leq k < l$ such that $z_k = x_j$.

Let $w_1 \circ y_1,\ w_1' \circ y_1'$ be reduced forms of $w \in X^+$ defined in the above statement. By the definition of the left complete reduced form we get that $s(w_1),\ s(w_1')$ are complete. We claim that $s(w_1) = s(w_1')$. Let

$$w_1 = a_1 \circ \ldots \circ a_n,\ y_1 = b_1 \circ \ldots \circ b_m,\ w_1' = c_1 \circ \ldots \circ c_s \text{ and } y_1' = d_1 \circ \ldots \circ d_t.$$

Suppose that there exists some $u \in s(w_1)$ but not in $s(w_1')$. Let k be such that $C(a_k) = u$. Then we must have $u \in s(y_1')$. Let j be the smallest such that $C(d_j) = u$. By definition of left complete reduced form there exists some $1 \leq i \leq s$ with $(C(c_i), u) \notin E$ or some $1 \leq l < j$ with $(C(d_l), u) \notin E$. By the above observation, we deduce that there exists $1 \leq r < k$ with $a_r = c_i$ or d_l, but $(C(a_i), u) \notin E$, and $(C(d_l), u) \notin E$, contradiction. Therefore, we have $s(w_1) = s(w_1')$.

We now show that $[w_1] = [w_1']$ and $[y_1] = [y_1']$. Put

$$K_w = \{\text{all reduced forms in } [w]\}.$$

Notice that all elements in K_w must be shuffle equivalent. For each $l \in s(w)$, we define two maps

$$\theta_l : K \longrightarrow \mathscr{G}\mathscr{P}, p = x_1 \circ \ldots \circ x_n \mapsto [p'] \text{ and } \eta_l : K \longrightarrow \mathscr{G}\mathscr{P}, p = x_1 \circ \ldots \circ x_n \mapsto [x_i]$$

as follows. Let i be the smallest such that $C(x_i) = l$. We define

$$p\theta_l = [p'] \text{ and } p\eta_l = [x_i]$$

where p' is obtained by deleting x_i from p. Let $p, q \in K_w$. Then p and q must be shuffle equivalent; q is obtained from p by finitely many steps of shuffle moves. We now show that $[p'] = [q']$. For this, it is sufficient to assume that q is obtained from p by exactly one shuffle move. Let

$$p = x_1 \circ \ldots \circ x_{i-1} \circ x_i \circ x_{i+1} \circ x_{i+2} \circ \ldots \circ x_n$$

and

$$q = x_1 \circ \ldots \circ x_{i-1} \circ x_{i+1} \circ x_i \circ x_{i+2} \circ \ldots \circ x_n$$

Considering p, pick the smallest j such that $C(x_j) = l$. If $1 \leq j \leq i-1$ or $i+2 \leq j \leq n$, then, clearly, $p' = q'$. If $j = i$, then p' is obtained by deleting the letter x_i in the i-th place of p whereas q' is obtained by deleting the letter x_i in the $(i+1)$-th place of q, so that $p' = q'$. A similar argument holds for the case when $j = i+1$. Therefore, we have well defined maps

$$\overline{\theta_l} : \{[w]\} \longrightarrow \mathscr{G}\mathscr{P}, [w] \mapsto z\theta_l \text{ and } \overline{\eta_l} : \{[w]\} \longrightarrow \mathscr{G}\mathscr{P}, [w] \mapsto z\eta_l$$

where $z \in K$. Let $s(w_1) = \{l_1, \cdots, l_m\}$. Then as $s(w_1) = s(w_1')$,

$$[y_1] = [w_1 \circ y_1]\overline{\theta_{l_1}} \dots \overline{\theta_{l_m}} = [w_1' \circ y_1']\overline{\theta_{l_1}} \dots \overline{\theta_{l_m}} = [y_1']$$

and

$$[w_1] = [w_1 \circ y_1]\overline{\eta_{l_1}} \dots [w_1 \circ y_1]\overline{\eta_{l_m}} = [w_1' \circ y_1']\overline{\eta_{l_1}} \dots [w_1' \circ y_1']\overline{\eta_{l_m}} = [w_1']$$

as required. $\square$

Note that if $w = w_1 \circ \cdots \circ w_k \in X^+$ is in left complete reduced form with blocks w_i, $1 \leq i \leq k$, then for any $1 \leq j \leq j' \leq k$ we have $w_j \circ w_{j+1} \circ \cdots \circ w_{j'}$ is also in left complete reduced form, with blocks w_h, $j \leq h \leq j'$.

Corollary 7.2.13. *Let $w = w_1 \circ \ldots \circ w_k$ and $w = w_1' \circ \ldots \circ w_k'$ of X^+ be left complete reduced forms of $[w] \in \mathscr{G}\mathscr{P}$. Then $[w_i] = [w_i']$ for all $1 \leq i \leq k$.*

Proof. Use Theorem 7.2.12 and by induction on the number of the blocks k. $\square$

In the following we introduce the concept of *disjoint form* of the elements of $\mathscr{G}\mathscr{P}$.

Definition 7.2.14. Let $\mathscr{G}\mathscr{P}$ be a graph product of semigroups associated to a graph $\Gamma = (V, E)$, where Γ is the disjoint union of subgraphs $\Gamma_i = (V_i, E_i)$, $i \in I$. Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a reduced form, where each $w_i \in X^+$ is given by

$$w_i = w_{i1} \circ \ldots \circ w_{ip(i)}.$$

We say that w is a *disjoint form* of $[w]$ in $\mathscr{G}\mathscr{P}$ with *D-blocks* $w_i \in X^+$, $1 \leq i \leq k$, if:

(i) for all $1 \leq i \leq k$, the support $s(w_i) \subseteq V_l$ for some $l \in I$,

(ii) if $s(w_i) \subseteq V_l$, then $s(w_{i+1}) \subseteq V_k$, where $l \neq k$ for all $1 \leq i \leq k-1$.

The next result shows that any element of any graph product $\mathscr{G}\mathscr{P}$ as above, may be represented by a disjoint form.

Lemma 7.2.15. *Every element in a graph product as above, may be represented by a disjoint form in X^+.*

Proof. Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be the left complete reduced form of $[w]$, with blocks w_i, $1 \leq i \leq k$. As $s(w_i)$ is complete, $s(w_i) \subseteq V_{l_i}$ for some $l_i \in I$. Let t_1 be the maximum such that

$$V_{l_1} = V_{l_2} = \ldots = V_{l_{t_1}}, \text{ and } V_{l_{t_1}} \neq V_{l_{t_1+1}}.$$

Put $z_1 = w_1 \circ \ldots \circ w_{t_1} \in X^+$, and $z_i = w_{t_{i-1}+1} \circ \ldots \circ w_{t_i} \in X^+$, for all $2 \leq i \leq n$, so we get that

$$w = z_1 \circ \ldots \circ z_n,$$

where $n \leq k$, $s(z_i) \subseteq V_{t_i}$, and z_i for all $1 \leq i \leq n$, is the D-blocks of w. Hence

$$w = z_1 \circ \ldots \circ z_n$$

is a disjoint form of $[w]$.

□

Lemma 7.2.16. *Let $\mathscr{GP}$ be a graph product associated to some disjoint union graph Γ. Let $w = w_1 \circ \ldots \circ w_k$ be disjoint form. If w' is another reduced form and $[w] = [w']$, then $w' = w'_1 \circ \ldots \circ w'_k$, and $[w'_i] = [w_i]$ for all $1 \leq i \leq k$.*

Proof. Suppose w' is obtained from w by n shuffles. If $n = 0$, then it is clear that the result is true. Suppose the result is true for $n-1$, and let w'' be the word obtained from w using the first $n-1$ shuffles, we write

$$w'' = w''_1 \circ \ldots \circ w''_k.$$

By induction w'' is in disjoint form with $[w_i] = [w''_i]$ for all $1 \leq i \leq k$. Now we obtain w' from w'' by single shuffle. This shuffle must be of two elements from w''_i for some i, since $w'' = w''_1 \circ \ldots \circ w''_k$ is in disjoint form. Let w'_i be the result of this shuffle in w''_i, then $[w_i] = [w''_i] = [w'_i]$, and for $j \neq i$, $w'_j = w''_j$. Hence certainly $[w'_p] = [w''_p] = [w_p]$ for all $1 \leq p \leq k$ and $w' = w'_1 \circ \ldots \circ w'_k$ is a disjoint form of $[w]$.

□

In the following we define the concept of *left disjoint form.*

Definition 7.2.17. Let $\mathscr{GP}$ be a graph product of semigroups associated to a graph $\Gamma = (V, E)$, where Γ is the disjoint union of subgraphs $\Gamma_i = (V_i, E_i)$, $i \in I$. Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a disjoint form with blocks w_i, where each $w_i \in X_i^+$ is in left complete reduced form, for all $1 \leq i \leq k$. We say that w is a *left disjoint form* of $[w]$ in $\mathscr{GP}$.

The following result proves the uniqueness of the left disjoint form.

Corollary 7.2.18. *Let $\mathscr{GP}$ be a graph product associated to disjoint union graph Γ, defined as in Definition 7.2.17. Let $w_1 \circ \ldots \circ w_k$ and $w'_1 \circ \ldots \circ w'_k$ be left disjoint forms of $w \in X^+$ with blocks $w_i \in X_i^+$ and $w'_i \in X_i^+$, for all $1 \leq i \leq k$. Then $[w'_i] = [w_i]$ for all $1 \leq i \leq k$.*

Proof. By Lemma 7.2.16, as $w'_1 \circ \ldots \circ w'_k$ is a reduced form, we get that $[w_i] = [w'_i]$, for all $1 \leq i \leq k$. As each w_i and w'_i are in the left complete reduced form, $[w_i] = [w'_i]$ for all $1 \leq i \leq k$, by Corollary 7.2.13,. □

7.3 Idempotents of graph products of abundant semigroups

In order to show the abundancy of the graph product of abundant semigroups, our first step is giving a characterization of the idempotents in the graph product of semigroups $\mathscr{G}\mathscr{P}$.

The next result can be deduced from the work of Da Costa for monoids in [12], and Proposition 7.1.11, but here we give a direct argument.

Lemma 7.3.1. *Let $x = x_1 \circ \ldots \circ x_n,\ y = y_1 \circ \ldots \circ y_n \in X^+$, where $C(x_i) = C(y_i)$ for all $1 \leq i \leq n$ and $C(x_i) \neq C(x_j)$ (and so $C(y_i) \neq C(y_j)$) for all $1 \leq i, j \leq n$ with $i \neq j$. Then*

$$[x_1 \circ \ldots \circ x_n] = [y_1 \circ \ldots \circ y_n] \Longleftrightarrow x_i = y_i \text{ for all } 1 \leq i \leq n.$$

Proof. We define a map

$$\phi_\alpha : X^+ \longrightarrow S_\alpha^{\underline{1}_\alpha}$$

for each $\alpha \in V$ by

$$z\phi_\alpha = \begin{cases} z & \text{if } z \in S_\alpha \\ \underline{1}_\alpha & \text{otherwise.} \end{cases} \tag{7.1}$$

We now claim that $\rho \subseteq \ker \phi_\alpha$, for which we need to show that $H \subseteq \ker \phi_\alpha$. For generators with form $(g \circ h, gh)$ where $g, h \in S_\beta$, $\beta \in V$, if $\beta = \alpha$, then

$$(g \circ h)\phi_\alpha = (g\phi_\alpha)(h\phi_\alpha) = gh = (gh)\phi_\alpha.$$

If $\beta \neq \alpha$, then $g,\ h,\ gh \notin S_\alpha$, and so

$$(g \circ h)\phi_\alpha = (g\phi_\alpha)(h\phi_\alpha) = \underline{1}_\alpha = (gh)\phi_\alpha.$$

For generators with form $(g \circ h, h \circ g)$ with $g \in S_\beta,\ h \in S_\gamma,\ \beta \neq \gamma,\ (\beta, \gamma) \in E$, if $\beta \neq \gamma = \alpha$, then we have

$$(g \circ h)\phi_\alpha = (g\phi_\alpha)(h\phi_\alpha) = \underline{1}_\alpha h = h = (h\phi_\alpha)(g\phi_\alpha) = (h \circ g)\phi_\alpha$$

and similar arguments hold for the case $\alpha = \beta \neq \gamma$. If $\alpha \neq \beta \neq \gamma \neq \alpha$, then

$$(g \circ h)\phi_\alpha = (g\phi_\alpha)(h\phi_\alpha) = \underline{1}_\alpha = (h \circ g)\phi_\alpha.$$

Hence we have $\rho \subseteq \ker \phi_\alpha$, and so there is a morphism

$$\overline{\phi_\alpha} : \mathscr{G}\mathscr{P} \longrightarrow S_\alpha^{1_\alpha}, \quad [x] \mapsto x\phi_\alpha.$$

Let $[x_1 \circ \ldots \circ x_n] = [y_1 \circ \ldots \circ y_n] \in \mathscr{G}\mathscr{P}$ be elements defined in the statement. We have

$$x_i = [x]\overline{\phi_{C(x_i)}} = [y]\overline{\phi_{C(x_i)}} = y_i, \quad \text{for all } 1 \leq i \leq n.$$

The converse of the statement is clear. □

In fact, for any $[x] \in \mathscr{G}\mathscr{P}$, where $x = x_1 \circ \ldots \circ x_n$, if $\alpha \in s(x)$, then

$$[x]\overline{\phi_\alpha} = x_{i_1} x_{i_2} \ldots x_{i_m}$$

and $\{i_1, i_2, \ldots, i_m\}$ is the set of indices such that $C(x_{i_k}) = \alpha$, for all $1 \leq k \leq m$.

Lemma 7.3.2. *Let $x = x_1 \circ \ldots \circ x_n \in X^+$ be a reduced form. Then $[x]$ is an idempotent in $\mathscr{G}\mathscr{P}$ if and only if $s(x)$ is a complete subgraph of $\Gamma(V, E)$ and $x_i = x_i^2$ for all $1 \leq i \leq n$.*

Proof. The sufficiency is clear. To show the necessity, let $x = x_1 \circ \ldots \circ x_n \in X^+$ be such that $[x]$ is an idempotent in $\mathscr{G}\mathscr{P}$. Suppose that $C(x_i) = \alpha_i$ for all $1 \leq i \leq n$, $\alpha_i \in V$. If $s(x)$ is not a complete subgraph of Γ, then there must exist $1 \leq i, j \leq n$, $\alpha_i \neq \alpha_j$ such that $(\alpha_i, \alpha_j) \notin E$. Let $(\{i\} * \{j\})^1$ be the free product of trivial semigroups of $\{i\}$ and $\{j\}$ with identity adjoined. We define a map

$$\theta : X^+ \longrightarrow (\{i\} * \{j\})^1$$

by

$$z\theta = \begin{cases} i & \text{if } z \in S_{v_i} \\ j & \text{if } z \in S_{v_j} \\ 1 & \text{otherwise.} \end{cases} \tag{7.2}$$

We now show that $\rho \subseteq \ker \theta$, for which we need to show that $H \subseteq \ker \theta$. For generators with form $(g \circ h, gh)$ where g, $h \in S_\beta$, $\beta \in V$, it is clear that $gh \in S_\beta$, if $\beta = \alpha_i$, then

$$(g \circ h)\theta = (g\theta)(h\theta) = ii = i = (gh)\theta;$$

similar arguments hold for the case $\beta = \alpha_j$. If $\beta \notin \{\alpha_i, \alpha_j\}$, then

$$(g \circ h)\theta = (g\theta)(h\theta) = 11 = 1 = (gh)\theta.$$

For generators with form $(g \circ h, h \circ g)$, where $g \in S_\beta,\ h \in S_\gamma,\ \beta \neq \gamma,\ (\beta, \gamma) \in E$, if $\alpha_i = \beta$, then $\gamma \notin \{\alpha_i, \alpha_j\}$, so that,

$$(g \circ h)\theta = (g\theta)(h\theta) = i1 = 1i = (h\theta)(g\theta) = (h \circ g)\theta;$$

similar arguments hold for the case $\alpha_j = \beta$ (and so $\gamma \notin \{\alpha_i, \alpha_j\}$); if $\beta, \gamma \notin \{\alpha_i, \alpha_j\}$, then

$$(g \circ h)\theta = (g\theta)(h\theta) = 11 = (h\theta)(g\theta) = (g \circ h)\theta.$$

Hence $\rho \subseteq \ker\theta$, giving a morphism

$$\bar{\theta} : \mathscr{G}\mathscr{P} \longrightarrow (\{i\} * \{j\})^1, \quad [x_1 \circ \ldots \circ x_n] \mapsto (x_1 \circ \ldots x_n)\theta.$$

By assumption $[x] = [x^2]$, so that $x\theta = (x\theta)(x\theta)$. Notice that $x\theta$ must contain letters i and j, so that, if the length of the reduced form of $x\theta$ is l, then $l \geq 2$, so that the length of the reduced form of $(x\theta)(x\theta)$ is either $2l - 1$ or $2l$. By the uniqueness of the length of reduced form of $x\theta = (x\theta)(x\theta)$, we must have $l = 2l$ or $l = 2l - 1$, contradiction, and hence $s(x)$ is a complete subgraph of Γ, so that

$$[x_1 \circ \ldots \circ x_n] = [x_1^2 \circ \ldots \circ x_n^2].$$

Since $x_1 \circ \ldots \circ x_n$ is a reduced form, we have $C(x_i) \neq C(x_j)$ for all $1 \leq i, j \leq n$ with $i \neq j$ by earlier comments, so that $x_i = x_i^2$ for all $1 \leq i \leq n$ by Lemma 7.3.1. $\square$

Let $\mathscr{G}\mathscr{P}$ be a graph product of semigroups associated to a complete graph $\Gamma = (V, E)$, where $V = \{\gamma_1, \cdots, \gamma_m\}$. Then any element $[w] \in \mathscr{G}\mathscr{P}$ in a reduced form can be written as $[w] = [x_1 \circ \cdots \circ x_k]$, where $x_1 \circ \cdots \circ x_k$ is a reduced form and $C(x_i) = \gamma_{j_i}$, for all $1 \leq i \leq k$, and $j_i < j_{i+1}$, for all $1 \leq i \leq k-1$.

Lemma 7.3.3. *Let $\mathscr{G}\mathscr{P}$ be a graph product of semigroups associated to a complete graph $\Gamma = (V, E)$, where $V = \{\gamma_1, \ldots, \gamma_n\}$. Let*

$$T_n = \{[x_1 \circ \cdots \circ x_n] \colon C(x_i) = \gamma_i, 1 \leq i \leq n\}.$$

Then T_n is a subsemigroup of $\mathscr{G}\mathscr{P}$ and

$$T_n \cong S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_n}.$$

Proof. It is obvious that the set $T_n \subseteq \mathscr{G}\mathscr{P}$ is not empty. For any $[x_1 \circ \cdots \circ x_n]$ and $[y_1 \circ \cdots \circ y_n]$ of T_n, then

$$[x_1 \circ \cdots \circ x_n][y_1 \circ \cdots \circ y_n] = [x_1 y_1 \circ \cdots \circ x_n y_n] \in T_n.$$

Thus T_n forms a subsemigroup of $\mathscr{G}\mathscr{P}$.

Define a map

$$\varphi\colon T_n \longrightarrow S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_n}$$

by

$$[x_1 \circ \cdots \circ x_n]\varphi = (x_1, \cdots, x_n),$$

where $C(x_i) = \gamma_i$. Let $[x_1 \circ \cdots \circ x_n] = [y_1 \circ \cdots \circ y_n] \in T_n$, where $C(x_i) = C(y_i) = \gamma_i$, then by Lemma 7.3.1 we get that $x_i = y_i$ for all $1 \leq i \leq n$. Hence it is clear that $[x_1 \circ \cdots \circ x_n]\varphi = [y_1 \circ \cdots \circ y_n]\varphi$, that implies φ is well defined. For $[x_1 \circ \cdots \circ x_n]$ and $[y_1 \circ \cdots \circ y_n]$ of T_n, as Γ is a complete graph, and $C(x_i) = C(y_i) = \gamma_i$, for all $1 \leq i \leq n$, we can write

$$\begin{aligned}
[x_1 \circ \cdots \circ x_n \circ y_1 \circ \cdots \circ y_n]\varphi &= [x_1y_1 \circ \cdots \circ x_ny_n]\varphi \\
&= (x_1y_1, \cdots, x_ny_n) && \text{(by the definition of } \varphi) \\
&= (x_1, \cdots, x_n)(y_1, \cdots, y_n) \\
&= \big([x_1 \circ \cdots \circ x_n]\varphi\big)\big([y_1 \circ \cdots \circ y_n]\varphi\big).
\end{aligned}$$

Then the map φ is a morphism. It is clear for any $[x_1 \circ \cdots \circ x_n]$ and $[y_1 \circ \cdots \circ y_n]$ of T_n such that

$$[x_1 \circ \cdots \circ x_n]\varphi = (x_1, \cdots, x_n) = (y_1, \cdots, y_n) = [y_1 \circ \cdots \circ y_n]\varphi,$$

then $x_i = y_i$, for all $1 \leq i \leq n$, so that

$$[x_1 \circ \cdots \circ x_n] = [y_1 \circ \cdots \circ y_n].$$

Hence φ is one to one map. The morphism φ is clearly onto. Therefore, φ is an isomorphism.

$\square$

Lemma 7.3.4. *Let $\mathscr{G}\mathscr{P}$ be a graph product of semigroups associated to a complete graph $\Gamma = (V, E)$, where $V = \{\gamma_1, \ldots, \gamma_n\}$. For each $\gamma_i \in V$, for all $1 \leq i \leq n$ and for any semigroup S_{γ_i}, let $M_{\gamma_i} = S_{\gamma_i}^{1_{\gamma_i}}$. Then*

$$P_n = \{(s_1, s_2, \ldots, s_n)\colon s_i \in M_{\gamma_i}, 1 \leq i \leq n \text{ and not all } s_i = \underline{1}_i\}$$

is a subsemigroup of $M_{\gamma_1} \times M_{\gamma_2} \times \cdots \times M_{\gamma_n}$ and

$$P_n \cong \mathscr{G}\mathscr{P}.$$

Proof. It is obvious that the set $P_n \subseteq M_{\gamma_1} \times M_{\gamma_2} \times \cdots \times M_{\gamma_n}$ is not empty. For any $(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$ of P_n, then

$$(s_1, s_2, \ldots, s_n)(t_1, t_2, \ldots, t_n) = (s_1t_1, s_2t_2, \ldots, s_nt_n) \in P_n,$$

since if s_i (or t_i) is not $\underline{1}_i$, then $s_i t_i \neq \underline{1}_i$. Thus P_n forms a subsemigroup of $M_{\gamma_1} \times M_{\gamma_2} \times \cdots \times M_{\gamma_n}$.

Define a map

$$\varphi\colon \mathscr{G}\mathscr{P} \longrightarrow P_n,$$

by

$$[x]\varphi = ([x]\overline{\phi_{\gamma_1}}, [x]\overline{\phi_{\gamma_2}}, \ldots, [x]\overline{\phi_{\gamma_n}}),$$

where $\overline{\phi_{\gamma_i}}$, for all $1 \leq i \leq n$ is the morphism defined in Lemma 7.3.1. Hence the map φ is a morphism.

To prove that φ is injective, let $[x] = [x_1 \circ \ldots \circ x_p]$ and $[y] = [y_1 \circ \cdots \circ y_q]$ are elements of $\mathscr{G}\mathscr{P}$ and both in reduced forms, where $C(x_k) = \gamma_{i_k}$, for all $1 \leq k \leq p$ and $C(y_l) = \gamma_{j_l}$ for all $1 \leq l \leq q$. Let

$$[x]\varphi = ([x]\overline{\phi_{\gamma_1}}, \ldots, [x]\overline{\phi_{\gamma_n}}) = ([y]\overline{\phi_{\gamma_1}}, \ldots, [y]\overline{\phi_{\gamma_n}}) = [y]\varphi.$$

Hence we get that $[x]\overline{\phi_{\gamma_i}} = [y]\overline{\phi_{\gamma_i}}$, for all $1 \leq i \leq n$. Since

$$[x] = \big[[x]\overline{\phi_{\mu_1}} \circ \cdots \circ [x]\overline{\phi_{\mu_p}}\big],$$

where $s(x) = \{\mu_1, \ldots, \mu_p\}$. Then we have that

$$\begin{aligned} [x] &= \big[[x]\overline{\phi_{\mu_1}} \circ \cdots \circ [x]\overline{\phi_{\mu_p}}\big] \\ &= \big[[y]\overline{\phi_{\mu_1}} \circ \cdots \circ [y]\overline{\phi_{\mu_p}}\big] \qquad (\text{as } [x]\overline{\phi_{\gamma_i}} = [y]\overline{\phi_{\gamma_i}}, \text{ for all } 1 \leq i \leq n) \\ &= [y]. \end{aligned}$$

This proves that φ is injective as required. Moreover, it is clear that any $(x_1, x_2, \cdots, x_n)$ of P_n, there is an element $[x] = [x_1 \circ \cdots \circ x_n] \in \mathscr{G}\mathscr{P}$ such that $[x]\varphi = (x_1, x_2, \cdots, x_n)$. Therefore, the morphism φ is an isomorphism.

□

Lemma 7.3.5. *Let $\mathscr{G}\mathscr{P}$ be a graph product of semigroups associated to a graph $\Gamma = (V, E)$, where Γ is the disjoint union of subgraphs Γ_i, $1 \leq i \leq m$. Then the graph product $\mathscr{G}\mathscr{P}$ is a free product of the graph products $\mathscr{G}\mathscr{P}_i$ corresponding to Γ_i, $1 \leq i \leq m$.*

Proof. Let $X = \bigcup_{i=1}^{m} X_i$, where

$$X_i = X(\Gamma_i, \mathscr{S}) = \bigcup_{\alpha \in V_i} S_\alpha.$$

Let $\mathcal{S}_i = \{S_\alpha : \alpha \in V_i\}$. Define a map

$$\theta : X \to \mathscr{F}\mathscr{P},$$

where $\mathscr{F}\mathscr{P}$ is the free product of the graph products $\mathscr{G}\mathscr{P}_i$ that is the graph product of $\mathcal{S}_i$ with respect to Γ_i, $1 \leq i \leq m$, by

$$x\theta = [x]_i,$$

where $C(x) \in V_i$, and $[x]_i$ is the equivalence class of x in $\mathscr{G}\mathscr{P}_i$, for some $1 \leq i \leq m$. Then there is a morphism

$$\theta : X^+ \to \mathscr{F}\mathscr{P}$$

by

$$w\theta = (w_1 \circ w_2 \circ \ldots \circ w_n)\theta = [w_1]_{p_1} \star [w_2]_{p_2} \star \ldots \star [w_n]_{p_n},$$

where $w_1 \circ w_2 \circ \ldots \circ w_n \in X^+$ is in disjoint form with D-blocks $w_i \in X_i^+$, $1 \leq i \leq n$. We now claim that $\rho \subseteq \ker \theta$, where ρ is the congruence determining the quotient semigroup $\mathscr{G}\mathscr{P}$, for which we need to show that $H \subseteq \ker \theta$. For generators with a form $(s \circ t, st)$, where $C(s) = C(t) \in V_k$ for some $1 \leq k \leq m$,

$$(s \circ t)\theta = [s \circ t]_k = [st]_k = (st)\theta.$$

For generators with a form $(s \circ t, t \circ s)$, where $C(s) = \beta,\ C(t) = \alpha,\ (\beta, \alpha) \in E$, we must have that $\alpha, \beta \in V_k$ for some $1 \leq k \leq m$. Hence we get that

$$(s \circ t)\theta = [s \circ t]_k = [t \circ s]_k = (t \circ s)\theta.$$

Therefore, we get that $\rho \subseteq \ker \theta$. Then there exists a morphism

$$\overline{\theta} : \mathscr{G}\mathscr{P} \to \mathscr{F}\mathscr{P}$$

defined by

$$[w]\overline{\theta} = w\theta,$$

so that if $w = w_1 \circ \cdots \circ w_n$ is in disjoint form, with D-blocks $w_i \in X_{p_i}^+$, $p_i \in \{1, 2, \cdots, m\}$, for all $1 \leq i \leq n$, and we have

$$[w_1 \circ \ldots \circ w_n]\overline{\theta} = [w_1]_{p_1} \star [w_2]_{p_2} \star \ldots \star [w_n]_{p_n}.$$

It is clear that the map $\overline{\theta}$ is an onto morphism. Let

$$[w]\overline{\theta} = [w_1]_{p_1} \star [w_2]_{p_2} \star \ldots \star [w_n]_{p_n} = [u_1]_{q_1} \star [u_2]_{q_2} \star \ldots \star [u_l]_{q_l} = [u]\overline{\theta},$$

where $w = w_1 \circ \cdots \circ w_n$ and $u = u_1 \circ \cdots \circ u_m$ are in disjoint forms with D-blocks w_i, u_j, respectively, $1 \leq i \leq n$, $1 \leq j \leq l$. Since $[w_1]_{p_1} \star [w_2]_{p_2} \star \ldots \star [w_n]_{p_n}$ and $[u_1]_{q_1} \star [u_2]_{q_2} \star \ldots \star [u_l]_{q_l}$ are reduced elements of a free product, we get that $n = l$, $p_i = q_i$, $1 \leq i \leq n$ and $[w_i]_{p_i} = [u_i]_{p_i}$, $1 \leq i \leq n$. Clearly then $[w] = [u]$ and $\overline{\varphi}$ is an injective morphism. This proves that the morphism θ is an isomorphism.

□

7.4 Abundancy of graph product of semigroups

As we stated at the beginning of this chapter, our main aim is to show the abundance of the graph product $\mathscr{GP}$. In the previous section, we gave the characterization of the idempotents in $\mathscr{GP}$. In this section, we construct three maps in Lemmas 7.4.1 and 7.4.5, which are the key for the proof of the abundancy of $\mathscr{GP}$.

We begin this section by setting up some notation. For each $(\alpha, \beta) \notin E$, where $\alpha \neq \beta$, and for any $z \in X^+$, we obtain the word $z(\alpha, \beta) \in X^+$ by deleting certain z_u from z, where $C(z_u) = \alpha$ by the rule that starting from the right, we delete z_u as long as

(1) there is at least one z_t with $t < u$ such that $C(z_t) = \beta$;

(2) there are no z_s with $u < s$ such that $C(z_s) = \beta$,

Let L be the binary relation on X^+ defined by

$$L = \{(x \circ u \circ y, x \circ v \circ y) : x, y \in X^+, (u, v) \in H\}.$$

Note that ρ is the transitive closure of L.

Lemma 7.4.1. *For each $(\alpha, \beta) \notin E$, where $\alpha \neq \beta$ we define the map*

$$\theta_{\alpha\beta} : X^+ \to \mathscr{GP}, \quad z \mapsto z\theta_{\alpha\beta} = [z(\alpha, \beta)].$$

Then

$$\overline{\theta_{\alpha\beta}} : \mathscr{GP} \to \mathscr{GP}, \quad [w]\overline{\theta_{\alpha\beta}} = w\theta_{\alpha\beta},$$

is well defined.

Proof. We need show that $\rho \subseteq \ker \theta_{\alpha\beta}$. Since ρ is the transitive closure of L, to show $\rho \subseteq \ker \theta_{\alpha\beta}$, we just need show that $L \subseteq \ker \theta_{\alpha\beta}$. We consider the following cases.

Case (i) $(u,v)=(s\circ t, st)$ *where* $s,t\in S_\alpha$. If $\beta\in S(y)$, then clearly

$$(x\circ u\circ y)\theta_{\alpha\beta}=[x\circ u]\,(y\theta_{\alpha\beta})=[x\circ v]\,(y\theta_{\alpha\beta})=(x\circ v\circ y)\theta_{\alpha\beta}.$$

If $\beta\notin s(y)$ and $\beta\notin s(x)$, then

$$(x\circ u\circ y)\theta_{\alpha\beta}=[x\circ u\circ y]=[x\circ v\circ y]=(x\circ v\circ y)\theta_{\alpha\beta}.$$

If $\beta\notin s(y)$ but $\beta\in s(x)$, then

$$(x\circ u\circ y)\theta_{\alpha\beta}=(x\circ y)\theta_{\alpha\beta}=(x\circ v\circ y)\theta_{\alpha\beta}$$

Case (ii) $(u,v)=(s\circ t, st)$ *where* $s,t\in S_\beta$. We have

$$(x\circ u\circ y)\theta_{\alpha\beta}=[x\circ u]\,(y\theta_{\alpha\beta})=[x\circ v]\,(y\theta_{\alpha\beta})=(x\circ v\circ y)\theta_{\alpha\beta}.$$

Case (iii) $(u,v)=(s\circ t, st)$ *where* $s,t\in S_\gamma$ *and* $\gamma\neq\alpha,\beta$. It is clear that

$$(x\circ u\circ y)\theta_{\alpha\beta}=(x\circ v\circ y)\theta_{\alpha\beta}.$$

Case (iv) $(u,v)=(s\circ t, t\circ s)$ *where* $s\in S_\alpha$, $t\in S_\gamma$, $\gamma\neq\alpha,\beta$ *and* $(\alpha,\gamma)\in E$. If $\beta\in s(y)$, then clearly

$$(x\circ u\circ y)\theta_{\alpha\beta}=[x\circ u]\,(y\theta_{\alpha\beta})=[x\circ v]\,(y\theta_{\alpha\beta})=(x\circ v\circ y)\theta_{\alpha\beta}.$$

If $\beta\notin s(y)$ and $\beta\notin s(x)$, then

$$(x\circ u\circ y)\theta_{\alpha\beta}=[x\circ u\circ y]=[x\circ v\circ y]=(x\circ v\circ y)\theta_{\alpha\beta}.$$

If $\beta\notin s(y)$ but $\beta\in s(x)$, then

$$(x\circ u\circ y)\theta_{\alpha\beta}=(x\circ t\circ y)\theta_{\alpha\beta}=(x\circ v\circ y)\theta_{\alpha\beta}.$$

Case (v) $(u,v)=(s\circ t, t\circ s)$ *where* $s\in S_\beta, t\in S_\gamma$, $(\beta,\gamma)\in E$, $\gamma\neq\alpha$. We have

$$(x\circ u\circ y)\theta_{\alpha\beta}=[x\circ u]\,(y\theta_{\alpha\beta})=[x\circ v]\,(y\theta_{\alpha\beta})=(x\circ v\circ y)\theta_{\alpha\beta}.$$

Case (vi) $(u,v)=(s\circ t, t\circ s)$ *where* $s\in S_\mu, t\in S_\gamma, (\gamma,\mu)\in E$, $\gamma,\mu\notin\{\alpha,\beta\}$. It is clear

$$(x\circ u\circ y)\theta_{\alpha\beta}=(x\circ v\circ y)\theta_{\alpha\beta}.$$

The above arguments show that $\rho\subseteq\ker\theta_{\alpha\beta}$, so that there is a map

$$\overline{\theta_{\alpha\beta}}:\mathscr{G}\mathscr{P}\to\mathscr{G}\mathscr{P},\quad [w]\overline{\theta_{\alpha\beta}}=w\theta_{\alpha\beta}.$$

$\square$

The following example justifies the claim that the map $\theta_{\alpha\beta}$ is not a morphism.

Example 7.4.2. Let $(\alpha, \beta) \notin E$, $x \in S_\alpha$, $y \in S_\beta$ and $z = y \circ x = z' \in X^+$. We have

$$z\theta_{\alpha\beta} = (y \circ x)\theta_{\alpha\beta} = [y] = z'\theta_{\alpha\beta},$$

this implies

$$z\theta_{\alpha\beta}z'\theta_{\alpha\beta} = (y \circ x)\theta_{\alpha\beta}\,(y \circ x)\theta_{\alpha\beta} = [y][y] = [y^2],$$

but

$$(z \circ z')\theta_{\alpha\beta} = (y \circ x \circ y \circ x)\theta_{\alpha\beta} = [y \circ x \circ y].$$

Therefore,

$$z\theta_{\alpha\beta}z'\theta_{\alpha\beta} \neq (z \circ z')\theta_{\alpha\beta}.$$

Definition 7.4.3. For each $\alpha \in V$ and each $w = x_1 \circ \ldots \circ x_n \in X^+$, we define a set

$$N_\alpha(w) = \{k \in \{1, \ldots, n\} : C(x_k) = \alpha \text{ and for all } j > k, \text{ either } C(x_j) = \alpha \text{ or } (\alpha, C(x_j)) \in E\}.$$

Of course, $N_\alpha(w)$ may be empty.

Lemma 7.4.4. *Let $\alpha \in V$ and $w = x_1 \circ \ldots \circ x_n \in X^+$. Suppose that $N_\alpha(w) = \{l_1, \ldots, l_r\}$ with $1 \leq l_1 < \ldots < l_r \leq n$. Then*

$$[w] = [w'][x_{l_1} \circ \ldots \circ x_{l_r}],$$

where w' is obtained by deleting all x_{l_i}, $1 \leq i \leq r$, from w.

Proof. Let $p = x_1 \circ \ldots \circ x_{l_1 - 1}$ and $q = x_{l_1} \circ x_{l_1+1} \circ \ldots \circ x_n$. Suppose that q' is obtained by deleting all x_{l_i}, $1 \leq i \leq r$, from q. By the way l_1 is chosen, we must have $C(z) \neq \alpha$ and $(C(z), \alpha) \in E$ for any $z \in q'$, implying that $[q] = [q'][x_{l_1} \circ \ldots \circ x_{l_r}]$, and hence

$$[w] = [p][q] = [p][q'][x_{l_1} \circ \ldots \circ x_{l_r}] = [w'][x_{l_1} \circ \ldots \circ x_{l_r}].$$

$\square$

Lemma 7.4.5. *For each $\alpha \in V$, the maps*

$$\phi_\alpha : X^+ \longrightarrow \mathscr{G}\mathscr{P}^1, \quad \psi_\alpha : X^+ \longrightarrow \mathscr{G}\mathscr{P}^1$$

defined by

$$w\phi_\alpha = [x_{l_1} \circ \ldots \circ x_{l_r}], \quad w\psi_\alpha = [w'],$$

for all $w = x_1 \circ \ldots \circ x_n \in X^+$, where $N_\alpha(w) = \{l_1, \ldots, l_r\}$ with $l_1 < \ldots < l_r$ and w' is the obtained by deleting $x_{l_1}, \ldots, x_{l_r}$ from w, induce maps

$$\overline{\phi_\alpha} : \mathscr{G}\mathscr{P} \longrightarrow \mathscr{G}\mathscr{P}^1, \quad \overline{\psi_\alpha} : \mathscr{G}\mathscr{P} \longrightarrow \mathscr{G}\mathscr{P}^1$$

defined by

$$[w]\overline{\phi_\alpha} = w\phi_\alpha, \quad [w]\overline{\psi_\alpha} = w\psi_\alpha.$$

Further, $[w] = (w\psi_\alpha)\,(w\phi_\alpha) = ([w]\overline{\psi_\alpha})\,([w]\overline{\phi_\alpha})$.

Proof. For this purpose, we need show that $\rho \subseteq \ker \phi_\alpha$ and $\rho \subseteq \ker \psi_\alpha$. Since ρ is the transitive closure of L, to show $\rho \subseteq \ker \phi_\alpha$ and $\rho \subseteq \ker \psi_\alpha$, we just need show that $L \subseteq \ker \phi_\alpha$ and $L \subseteq \ker \psi_\alpha$.

Let $x = x_1 \circ \ldots \circ x_p,\ y = y_1 \circ \ldots \circ y_q \in X^+$ and $(u, v) \in H$. We consider the following cases.

Case (i) $(u, v) = (s \circ t, t \circ s)$, where $s \in S_\beta,\ t \in S_\gamma$ with $(\beta, \gamma) \in E$ and $\beta, \gamma \neq \alpha$. It is easy to see that $N_\alpha(x \circ u \circ y) = N_\alpha(x \circ v \circ y)$ and $p+1,\ p+2 \notin N_\alpha(x \circ u \circ y)$ nor $N_\alpha(x \circ v \circ y)$, so that

$$(x \circ u \circ y)\phi_\alpha = (x \circ v \circ y)\phi_\alpha \ \text{ and } \ (x \circ u \circ y)\psi_\alpha = (x \circ v \circ y)\psi_\alpha. \tag{7.3}$$

Case (ii) $(u, v) = (s \circ t, t \circ s)$ where $s \in S_\beta, t \in S_\alpha$ with $(\beta, \alpha) \in E$ and $\beta \neq \alpha$. We have the following 2 subcases.

(a) $N_\alpha(x \circ u \circ y) = \emptyset$. If $\alpha \notin s(y)$, then there exists y_j with $(C(y_j), \alpha) \notin E$, and hence $N_\alpha(x \circ v \circ y) = \emptyset$; if $\alpha \in s(y)$, then we pick j to be the greatest such that $C(y_j) = \alpha$. As $N_\alpha(x \circ u \circ y) = \emptyset$, there exists k with $j < k \leq q$ such that $C(y_k) \neq \alpha$ and $(\alpha, C(y_k)) \notin E$, so that $N_\alpha(x \circ v \circ y) = \emptyset$. Therefore,

$$(x \circ u \circ y)\phi_\alpha = 1 = (x \circ v \circ y)\phi_\alpha$$

and

$$(x \circ u \circ y)\psi_\alpha = [x \circ u \circ y] = [x \circ v \circ y] = (x \circ v \circ y)\psi_\alpha.$$

So the Equation 7.3 holds.

(b) $N_\alpha(x \circ u \circ y) = \{l_1, \ldots, l_r\}$ where $1 \leq l_1 < \ldots < l_r \leq p + 2 + q$. If $p + 2 < l_1$, then we have $\{l_1, \ldots, l_r\} \subseteq N_\alpha(x \circ v \circ y)$; as $t \in S_\alpha$, there must exist $1 \leq k \leq l'_1 - 1$ where $l'_1 = l_1 - (p+2)$ such that $C(y_k) \neq \alpha$ and $(C(y_k), \alpha) \notin E$, so that $N_\alpha(x \circ v \circ y) = \{l_1, \ldots, l_r\}$, and hence

$$(x \circ u \circ y)\phi_\alpha = (x \circ v \circ y)\phi_\alpha$$

and

$$(x \circ u \circ y)\psi_\alpha = (x \circ v \circ y)\psi_\alpha.$$

so the Equation 7.3 holds.
If $l_1 = p+2$ (similarly if $l_1 = p+1$), then $p+1 \in N_\alpha(x \circ v \circ y)$ and by the definition of $N_\alpha(x \circ u \circ y)$, we deduce that, for any $1 \leq j \leq p$ with $C(x_j) = \alpha$, there exists k with $j < k \leq p$ such that $C(x_k) \neq \alpha$ and $(C(x_k), \alpha) \notin E$. It follows that $N_\alpha(x \circ v \circ y) = \{p+1, l_2, \ldots, l_r\}$, and hence

$$(x \circ u \circ y)\phi_\alpha = (x \circ v \circ y)\phi_\alpha$$

and

$$(x \circ u \circ y)\psi_\alpha = (x \circ v \circ y)\psi_\alpha.$$

Then Equation 7.3 holds.
If $1 \leq l_1 \leq p$, then $p+2 \in N_\alpha(x \circ u \circ y)$, and so $p+1 \in N_\alpha(x \circ v \circ y)$. Also, for any $1 \leq j < l_1$ with $C(x_j) = \alpha$, there must exist k with $j < k \leq l_1$ such that $C(x_k) \neq \alpha$ and $(C(x_k), \alpha) \notin E$, so that $N_\alpha(x \circ v \circ y) = (\{l_1, l_2, \ldots, l_r\} \backslash \{p+2\}) \cup \{p+1\}$, and hence Equation 7.3 holds.

Case (iii) For the cases where $(u, v) = (s \circ t, st)$ *with* $s, t \in S_\beta, \beta \neq \alpha$ *or* $(u, v) = (s \circ t, st)$ *with* $s, t \in S_\alpha$, it is clear that Equation 7.3 holds.

The above arguments show that $\rho \subseteq \ker \phi_\alpha$ and $\rho \subseteq \ker \psi_\alpha$, and hence there are maps

$$\overline{\phi_\alpha} : \mathscr{GP} \longrightarrow \mathscr{GP}^1,\ [w]\overline{\phi_\alpha} = w\phi_\alpha$$

and

$$\overline{\psi_\alpha} : \mathscr{GP} \longrightarrow \mathscr{GP}^1,\ [w]\overline{\psi_\alpha} = w\psi_\alpha.$$

Finally, it follows from Lemma 7.4.4 that $[w] = (w\psi_\alpha)\,(w\phi_\alpha) = ([w]\overline{\psi_\alpha})\,([w]\overline{\phi_\alpha})$. □

With all the above preparations, we are finally at the stage of showing that $\mathscr{GP}$ is left abundant when S_α is left abundant for all $\alpha \in V$. We divide the proof into two steps.

Lemma 7.4.6. *Let* $w = w_1 \circ \ldots \circ w_k \in X^+$ *be a left complete reduced form with blocks* w_i, $1 \leq i \leq k$. *Then, for any* $[x],\ [y] \in \mathscr{GP}^1$, $[x]\,[w] = [y]\,[w]$ *implies that* $[x]\,[w_1] = [y]\,[w_1]$.

Proof. The idea of our proof is to delete letters from the end of w, in the expression $[x \circ w] = [y \circ w]$, until we end with $[x \circ w_1] = [y \circ w_1]$, by using maps defined in Lemma 7.4.1.

Suppose that each $w_t \in X^+$ is given by $w_t = w_{t1} \circ \ldots \circ w_{tp(t)}$ so that, for $1 \leq t \leq k$, $\{C(w_{t1}), \ldots,\ C(w_{tp(t)})\}$ is a complete graph. Now suppose that $[x]\,[w] = [y]\,[w]$. If $[w] = [w_1]$ we are done. Otherwise, let $k \geq 2$ and let $\alpha \in s(w_k)$

and let $\beta \in s(w_{k-1})$, for such that $(\alpha, \beta) \notin E$. Then $\theta_{\alpha\beta}$ act to remove a single element of w_k, w_{kl}, where $C(w_{kl}) = \alpha$, repeat this strategy until we get rid of w_k. Notice that $\beta \notin s(w_k)$ and $\alpha \notin s(w_{k-1})$ so w_{k-1} is unaffected. Then start on w_{k-1}, and continue until we rid w_2. Similarly for $[y \circ w]\overline{\theta_{\alpha\beta}}$. Then from $[x]\,[w] = [y]\,[w]$ we obtain

$$\big((x \circ w)\theta_{\alpha\beta}\big) = [x \circ w]\overline{\theta_{\alpha\beta}} = [y \circ w]\overline{\theta_{\alpha\beta}} = \big((y \circ w)\theta_{\alpha\beta}\big).$$

Notice that if $w_1 \circ \cdots \circ w_k$ is in a left complete reduced form, then so is $w_1 \circ \cdots \circ w_k'$, where w_k' is w_k with letters deleted.

By choosing the right (α, β) we can successively knock off the final elements in $[x \circ w] = [y \circ w]$ to obtain $[x \circ w_1] = [y \circ w_1]$, namely, $[x]\,[w_1] = [y]\,[w_1]$. □

The following result follows immediately from Lemma 7.4.6.

Proposition 7.4.7. *Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a left complete reduced form with blocks $w_i \in X^+$,$1 \le i \le k$. Then $[w]\,\mathcal{R}^*\,[w_1]$.*

Proof. Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a left complete reduced form, where $w_i \in X^+$, for all $1 \le i \le k$. For any $[x], [y] \in \mathscr{GP}^1$, $[x]\,[w] = [y]\,[w]$ implies that $[x]\,[w_1] = [y]\,[w_1]$, by Lemma 7.4.6.

For any $[x], [y] \in \mathscr{GP}^1$, let $[x]\,[w_1] = [y]\,[w_1]$. By multiplying both sides in $[x]\,[w_1] = [y]\,[w_1]$ by $[w_2 \circ \ldots \circ w_k]$, we get that $[x]\,[w] = [y]\,[w]$. □

In the following we establish a connection with the relation $\mathcal{R}^*$ in $\mathscr{GP}$ and the relation $\mathcal{R}^*$ in the vertex semigroups.

Lemma 7.4.8. *Let $z_1, z_1' \in X$ be such that $z_1\,\mathcal{R}^*\,z_1'$ in $S_{C(z_1)}$. Then $[z_1]\,\mathcal{R}^*\,[z_1']$ in $\mathscr{GP}$.*

Proof. Let $x = x_1 \circ \cdots \circ x_m$, $y = y_1 \circ \cdots \circ y_k \in X^+$ be such that $[x][z_1] = [y][z_1]$. We now claim that $[x][z_1'] = [y][z_1']$. Let $C(z_1) = \alpha$. It follows from Lemma 7.4.5 that

$$[x_1 \circ \cdots \circ x_m \circ z_1]\overline{\phi}_\alpha = [y_1 \circ \cdots \circ y_k \circ z_1]\overline{\phi}_\alpha$$

and

$$[x_1 \circ \cdots \circ x_m \circ z_1]\overline{\psi}_\alpha = [y_1 \circ \cdots \circ y_k \circ z_1]\overline{\psi}_\alpha.$$

Suppose that

$$N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1) = \{r_1, \cdots, r_l\}, N_\alpha(y_1 \circ \cdots \circ y_k \circ z_1) = \{s_1, \cdots, s_t\}.$$

Then we must have $r_l = m+1$, $s_t = k+1$ and

$$[x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1] = [y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1].$$

By Proposition 7.1.7 we have $x_{r_1} \cdots x_{r_{l-1}} z_1 = y_{s_1} \cdots y_{s_{t-1}} z_1$ and then since $z_1 \,\mathcal{R}^*\, z_1'$, we deduce $x_{r_1} \cdots x_{r_{l-1}} z_1' = y_{s_1} \cdots y_{s_{t-1}} z_1'$, so that

$$[x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1'].$$

By using $\overline{\psi}_\alpha$, we obtain $[x'] = [y']$, where x' is obtained by deleting all x_{r_j} from x, where $1 \leq j \leq l-1$ and y' is obtained by deleting all y_{s_j} from y, where $1 \leq j \leq t-1$. Using the final part of Lemma 7.4.5 we have

$$[x'][x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [y'][y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1'].$$

Notice that

$$N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1') = N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1)$$

and

$$N_\alpha(y_1 \circ \cdots \circ y_k \circ z_1') = N_\alpha(y_1 \circ \cdots \circ y_k \circ z_1)$$

so that, by Lemma 7.4.4,

$$[x'][x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [x][z_1']$$

Similarly,

$$[y'][y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1'] = [y][z_1'],$$

so that $[x][z_1'] = [y][z_1']$.

Next, let $[x][z_1] = [y][z_1]$. We claim that $[x][z_1'] = [y][z_1']$. Let $C(z_1) = \alpha$. It follows from Lemma 7.4.5 that

$$[x_1 \circ \cdots \circ x_m \circ z_1]\overline{\phi}_\alpha = [z_1]\overline{\phi}_\alpha$$

and

$$[x_1 \circ \cdots \circ x_m \circ z_1]\overline{\psi}_\alpha = [z_1]\overline{\psi}_\alpha.$$

Suppose that

$$N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1) = \{r_1, \cdots, r_l\}, N_\alpha(z_1) = \{s_1\}.$$

We must have $r_l = m+1$, and

$$[x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1] = [z_1].$$

By Proposition 7.1.7 we have $x_{r_1} \cdots x_{r_{l-1}} z_1 = z_1$ and then since $z_1 \; \mathcal{R}^* \; z_1'$, we deduce $x_{r_1} \cdots x_{r_{l-1}} z_1' = z_1'$, so that

$$[x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1'].$$

By using $\overline{\psi}_\alpha$, we obtain $[x'] = 1$, where x' is obtained by deleting all x_{r_j} from x, where $1 \leq j \leq l-1$ and 1 is the adjoin identity of $\mathscr{G}\mathscr{P}^1$. Using the final part of Lemma 7.4.5 we have

$$[x'][x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [z_1'].$$

Notice that

$$N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1') = N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1).$$

By Lemma 7.4.4,

$$[x'][x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [x][z_1']$$

Therefore, we get that $[x][z_1'] = [z_1']$, as required.

□

Lemma 7.4.9. *Let $z = z_1 \circ \cdots \circ z_n \in X^+$ be a complete reduced form. Suppose that $z_k \, \mathcal{R}^* \, z_k'$ in $S_{C(z_k)}$ for $1 \leq k \leq n$ and put $z' = z_1' \circ \cdots \circ z_n'$. Then $[z] \, \mathcal{R}^* \, [z']$ in $\mathscr{G}\mathscr{P}$.*

Proof. We proceed by induction on the length n of z. Clearly, the result holds for the case $n = 1$ by Lemma 7.4.8. Suppose that the result is true for all $k < n$. Then

$$[z_1 \circ \cdots \circ z_{n-1}] \; \mathcal{R}^* \; [z_1' \circ \cdots \circ z_{n-1}'].$$

As $\mathcal{R}^*$ is a left congruence and z is a complete reduced form,

$$[z] = [z_n \circ z_1 \circ \cdots \circ z_{n-1}] \; \mathcal{R}^* \; [z_n \circ z_1' \circ \cdots \circ z_{n-1}'].$$

On the other hand, since $[z_n] \; \mathcal{R}^* \; [z_n']$ and $\mathcal{R}^*$ is a left congruence,

$$[z_n \circ z_1' \circ \cdots \circ z_{n-1}'] = [z_1' \circ \cdots \circ z_{n-1}' \circ z_n] \; \mathcal{R}^* \; [z_1' \circ \cdots \circ z_{n-1}' \circ z_n'] = [z']$$

so that $[z] \, \mathcal{R}^* \, [z']$ in $\mathscr{G}\mathscr{P}$. □

The following result follows immediately from Lemma 7.4.9

Corollary 7.4.10. *Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a complete reduced form of $[w]$ in $\mathscr{G}\mathscr{P}$. Let $w^+ = w_1^+ \circ \ldots \circ w_k^+$, where w_j^+ is an idempotent in the $\mathcal{R}^*$-class of w_j in S_j for all $1 \leq j \leq k$. Then* *(i)* $[w^+]$ *is an idempotent of* $\mathscr{GP}$*;*

(ii) $[w^+]\,[w] = [w]$*;*

(iii) $[w^+]\,\mathcal{R}^*\,[w]$.

Proof. (i) Since $w^+ = w_1^+ \circ \ldots \circ w_k^+$ is a complete reduced form, and w_i^+ is an idempotent in S_i for all $1 \leq i \leq k$, it follows from Lemma 7.3.2 that $[w^+] = [w_1^+ \circ \ldots \circ w_k^+]$ is an idempotent in $\mathscr{GP}$.

(ii) Clearly

$$[w^+][w] = [w_1^+ \circ \ldots \circ w_k^+][w_1 \circ \ldots \circ w_k] = [w_1^+ w_1 \circ \ldots \circ w_k^+ w_k] = [w].$$

(iii) As $w_j^+\ \mathcal{R}^*\ w_j$, by Lemma 7.4.9, we get that $[w^+]\ \mathcal{R}^*\ [w]$.

□

The main result now follows from Lemma 7.4.6 and Corollary 7.4.10.

Theorem 7.4.11. *Let* $\Gamma = \Gamma(V, E)$ *be a graph and let* $\mathscr{S} = \{S_v : v \in V\}$ *be a family of left abundant semigroups. Then the graph product* $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$ *is also left abundant.*

Proof. It is clear that if $w = w_1 \circ \ldots \circ w_k \in X^+$ is a left complete reduced form of $[w]$ in $\mathscr{GP}$ with blocks w_i, $1 \leq k$, then $[w]\,\mathcal{R}^*\,[w_1]$ by Proposition 7.4.7. Also, we know that $[w_1]\,\mathcal{R}^*\,[w_1^+]$ by Corollary 7.4.10. Hence we get that $[w]\ \mathcal{R}^*\ [w_1^+]$.

□

It is clear that the left-right dual of Theorem 7.4.11 holds, and hence we have the following.

Corollary 7.4.12. *Let* $\Gamma = \Gamma(V, E)$ *be a graph and let* $\mathscr{S} = \{S_v : v \in V\}$ *be a family of abundant semigroups. Then the graph product* $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathscr{S})$ *is abundant.*

7.5 Weak abundancy of graph product of semigroups

In this section, we show that a graph product of weakly abundant semigroups is a weakly abundant semigroup.

The following result immediately from Proposition 7.4.7 and the fact that $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$.

Corollary 7.5.1. *Let $w = w_1 \circ \cdots \circ w_n \in X^+$ be a left complete reduced with blocks w_i, for $1 \leq i \leq n$. Then $[w] \widetilde{\mathcal{R}} [w_1]$.*

Lemma 7.5.2. *Let $w = w_1 \circ \ldots \circ w_k \in X^+$ be a complete reduced form, where $C(w_i) = i$, $1 \leq i \leq k$, of $[w]$ in $\mathscr{G}\mathscr{P}$. Let $v_i \in S_i$, $v_i \widetilde{\mathcal{R}} w_i$ in S_i for all $1 \leq i \leq k$ and put $[v] = [v_1 \circ \ldots \circ v_k]$. Then $[v] \widetilde{\mathcal{R}} [w]$.*

Proof. Let $e = e_1 \circ \ldots \circ e_m$ be a reduced word such that $[e] \in E(\mathscr{G}\mathscr{P})$. It is follows from Lemma 7.3.2 that $s(e)$ is complete and $e_i = e_i^2$, for all $1 \leq i \leq m$. Suppose that $[e]\,[w] = [w]$. Hence $s(e \circ w) = s(w)$, that implies that $s(e) \subseteq s(w)$. Without loss of generality, suppose $C(e_1) = C(w_1), \ldots, C(e_m) = C(w_m)$. Then we write

$$[e]\,[w] = [e_1 \circ \ldots \circ e_m \circ w_1 \circ \ldots \circ w_k] = [e_1 w_1 \circ \ldots \circ e_m w_m \circ \ldots \circ w_k] = [w].$$

By (ii) of Note 7.2.10, for all $1 \leq i \leq m$, we get $e_i w_i = w_i$, so $e_i v_i = v_i$, for all $1 \leq i \leq m$. Thus

$$[e]\,[v] = [e_1 \circ \ldots \circ e_m \circ v_1 \circ \ldots \circ v_k] = [e_1 v_1 \circ \ldots \circ e_m v_m \circ \ldots \circ v_k] = [v].$$

Similarly, if $[e]\,[v] = [v]$, then $[e]\,[w] = [w]$. It follows that $[v] \widetilde{\mathcal{R}} [w]$.

□

The next result is useful in our work.

Lemma 7.5.3. *Let $w = w_1 \circ \cdots \circ w_n$ be a complete reduced form of $[w]$ in $\mathscr{G}\mathscr{P}$. Let $w^{++} = w_1^{++} \circ \ldots \circ w_k^{++}$, where w_j^{++} is an idempotent in the $\widetilde{\mathcal{R}}$-class of w_j for all $1 \leq j \leq k$. Then $[w^{++}]$ is an idempotent of $\mathscr{G}\mathscr{P}$ and $[w^{++}]\,[w] = [w]$; further, $[w^{++}] \widetilde{\mathcal{R}} [w]$.*

Proof. Since $w^{++} = w_1^{++} \circ \ldots \circ w_k^{++}$ is a complete reduced form and $w_i^{++} \in E(S_i)$ for all $1 \leq i \leq k$, then by Lemma 7.3.2 we get that $[w^{++}]$ is an idempotent of $\mathscr{G}\mathscr{P}$. If $w_i^{++} \widetilde{\mathcal{R}} w_i$ for all $1 \leq i \leq k$, then it is clear that

$$\begin{aligned}
[w^{++}]\,[w] &= [w_1^{++} \circ \ldots \circ w_k^{++} \circ w_1 \circ \ldots \circ w_k] \\
&= [w_1^{++} w_1 \circ \ldots \circ w_k^{++} w_k] \\
&= [w].
\end{aligned}$$

By Lemma 7.5.2, we get that $[w^{++}] \widetilde{\mathcal{R}} [w]$.

□

The main result in this section follows from Lemma 7.5.3.

Theorem 7.5.4. *Let $\Gamma = \Gamma(V, E)$ be a graph, and let $\mathscr{S} = \{S_v : v \in V\}$ be a family of weakly left abundant semigroups. Then the graph product $\mathscr{G}\mathscr{P} = \mathscr{G}\mathscr{P}(\Gamma, \mathscr{S})$ is also weakly left abundant.*

Proof. It is clear that if $w = w_1 \circ \ldots \circ w_k \in X^+$ is a left complete reduced form of $[w]$ in $\mathscr{G}\mathscr{P}$, by Proposition 7.4.7 we get that $[w] \ \widetilde{\mathcal{R}} \ [w_1]$. From Corollary 7.5.3 we get that $[w_1^{++}] \ \widetilde{\mathcal{R}} \ [w_1]$. Hence it is clear that $[w_1^{++}] \ \widetilde{\mathcal{R}} \ [w]$.

$\square$

The left-right dual of Theorem 7.5.4 holds, resulting in the following.

Corollary 7.5.5. *Let $\Gamma = \Gamma(V, E)$ be a graph, and let $\mathscr{S} = \{S_v : v \in V\}$ be a family of weakly abundant semigroups. Then the graph product $\mathscr{G}\mathscr{P} = \mathscr{G}\mathscr{P}(\Gamma, \mathscr{S})$ is weakly abundant.*

7.6 Description of the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on graph product of semigroups

We proved in Lemma 7.4.9 that if $w = w_1 \circ \ldots \circ w_k \in X^+$ is a complete reduced form where $w_i \in S_i$, and if $v_i \in S_i$ is such that $v_i \ \mathcal{R}^* \ w_i$ for all $1 \leq i \leq k$, then $[v] \ \mathcal{R}^* \ [w]$, where $[v] = [v_1 \circ \ldots \circ v_k]$.

This section gives more description of Green $*$-relations, $\mathcal{R}^*$ (and $\mathcal{L}^*$), on graph products of semigroups. Note that if we say $w \in S_\alpha$, $\alpha \in V$, is right cancellative, then we mean w is right cancellative in S_α.

We start this section by defining the concept of *i-right cancellative.*

Definition 7.6.1. An element w of a semigroup S is *i-right cancellative* if it is right cancellative and there is no $u \in S$ such that $uw = w$.

In other words an element $w \in S$ is i-right cancellative if it is right cancellative in S and does not have left identities.

Lemma 7.6.2. *Let $w \in S_\alpha$, where $\alpha \in V$. Then w is i-right cancellative if and only if $[w]$ is i-right cancellative in $\mathscr{G}\mathscr{P}$.*

Proof. Let $w \in S_\alpha$ be i-right cancellative. Let

$$[x] = [x_1 \circ \cdots \circ x_n],\ [y] = [y_1 \circ \cdots \circ y_m] \in \mathscr{G}\mathscr{P}.$$

Suppose that $[x]\,[w] = [y]\,[w]$, and let

$$N_\alpha(x \circ w) = \{r_1, \ldots, r_l\}, \text{ and } N_\alpha(y \circ w) = \{s_1, \ldots, s_t\}.$$

Then $r_l = n + 1$, $s_t = m + 1$

$$[x \circ w]\overline{\phi_\alpha} = [x_{r_1} \circ \ldots \circ x_{r_{l-1}} \circ w] = [y_{s_1} \circ \ldots \circ y_{s_{t-1}} \circ w] = [y \circ w]\overline{\phi_\alpha}.$$

As w is right cancellative in S_α and by the definition of the map $\overline{\phi_\alpha}$ we have $C(x_{r_j}) = \alpha$, for all $1 \leq j \leq l - 1$, $C(y_{s_j}) = \alpha$, for all $1 \leq j \leq t - 1$, we get that

$$x_{r_1} x_{r_2} \ldots x_{r_{l-1}} w = y_{s_1} y_{s_2} \ldots y_{s_{t-1}} w,$$

in S_α. Hence we get $x_{r_1} x_{r_2} \ldots x_{r_{l-1}} = y_{s_1} y_{s_2} \ldots y_{s_{t-1}}$, so

$$[x_{r_1} \circ \ldots \circ x_{r_{l-1}}] = [y_{s_1} \circ \ldots \circ y_{s_{t-1}}],$$

but

$$[x]\overline{\phi_\alpha} = [x_{r_1} \circ \ldots \circ x_{r_{l-1}}] = [y_{s_1} \circ \ldots \circ y_{s_{t-1}}] = [y]\overline{\phi_\alpha}. \tag{7.4}$$

Moreover, as $[x \circ w] = [y \circ w]$, then we have

$$[x \circ w]\overline{\psi_\alpha} = [y \circ w]\overline{\psi_\alpha}.$$

It is clear that $[x \circ w]\overline{\psi_\alpha}$ and $[y \circ w]\overline{\psi_\alpha}$ do not contain w, then we get that

$$[x]\overline{\psi_\alpha} = [x \circ w]\overline{\psi_\alpha} = [y \circ w]\overline{\psi_\alpha} = [y]\overline{\psi_\alpha}. \tag{7.5}$$

Now by Lemma 7.4.5 and by the equalities (7.4) and (7.5), we get that

$$[x] = [x]\overline{\psi_\alpha}\,[x]\overline{\phi_\alpha} = [y]\overline{\psi_\alpha}\,[y]\overline{\phi_\alpha} = [y],$$

this proves that $[w]$ is a right cancellative element in $\mathscr{GP}$.

Conversely, let $[w]$ be right cancellative in $\mathscr{GP}$. Then for any $[x], [y]$ in $\mathscr{GP}$ such that $[x]\,[w] = [y]\,[w]$, we have that that $[x] = [y]$. Let $x_\alpha, y_\alpha \in S_\alpha$, be such that $x_\alpha w = y_\alpha w$. Then $[x_\alpha w] = [y_\alpha w]$. As $[w]$ is right cancellative in $\mathscr{GP}$, $[x_\alpha] = [y_\alpha]$. Hence by Proposition 7.1.7 we get that $x_\alpha = y_\alpha$. Therefore, w is right cancellative.

Now suppose that w does not have left identities but $[w]$ has in $\mathscr{GP}$. Then there exists some $[x] \in \mathscr{GP}$ such that $[x][w] = [w]$. Without loss of generality, we assume that x is reduced. It is clear that $s(x) \subseteq s(w)$. Hence x must be in $S_{C(w)}$ and we can write $[xw] = [w]$. By Proposition 7.1.7 we get that $xw = w$, a contradiction. $\square$

Lemma 7.6.3. *Let M be a monoid. Let $w \in M$. Then w is a right cancellative element in the monoid M if and only if $w\,\mathcal{R}^*\,1$, where 1 is the identity in M.*

Proof. Let w be a right cancellative in M, and $xw = yw$, where $x, y \in M^1 = M$. Then $x = y$, and it is clear that $x1 = y1$. It is obvious if $x1 = y1$, then $xw = yw$. Therefore, $w\,\mathcal{R}^*\,1$.

Conversely, let $w\,\mathcal{R}^*\,1$ and $xw = yw$, where $x, y \in M^1$. Then $x1 = y1$, so $x = y$, which means that w is right cancellative. □

Lemma 7.6.4. *Let S be a semigroup. Let $w \in S$. Then the following conditions are equivalence:*

(i) $w\,\mathcal{R}^*\,\underline{1}$ *in* $S^{\underline{1}}$;

(ii) w *is a right cancellative element in* $S^{\underline{1}}$;

(iii) w *is i-right cancellative in* S.

Proof. $(i) \Longleftrightarrow (ii)$ It is clear by Lemma 7.6.3.

$(ii) \Longrightarrow (iii)$ Let w be a right cancellative element in $S^{\underline{1}}$. Hence w is cancellative in S and if there exists an element $x \in S$ such that $xw = w$, then we get that $x = \underline{1}$, a contradiction.

$(iii) \Longrightarrow (ii)$ Let w be an i-right cancellative. It is clear that w is a right cancellative in $S^{\underline{1}}$. □

The following result follows immediately from Lemma 7.6.2.

Corollary 7.6.5. *Let $w \in S_\alpha$ and $v \in S_\beta$ such that $[w]\,\mathcal{R}^*\,[v]$ in $\mathscr{G}\mathscr{P}$. Then w is i-right cancellative if and only if v is i-right cancellative.*

Proof. Suppose that $w \in S_\alpha$ is not i-right cancellative, and $v \in S_\beta$ is i-right cancellative, where $\alpha, \beta \in V$. As $w \in S_\alpha$ is not i-right cancellative, there exist $u, u' \in S_\alpha$ such that $u \neq u'$ but $uw = u'w$ in S_α or w has a left identity. Let

$$uw = u'w$$

for some $u, u' \in S_\alpha$ such that $u \neq u'$ in S_α, then

$$[u]\,[w] = [u']\,[w]$$

in $\mathscr{G}\mathscr{P}$. Since $[w]\,\mathcal{R}^*\,[v]$, then we get

$$[u]\,[v] = [u']\,[v].$$

As v is i-right cancellative in S_β, by Lemma 7.6.2 we get $[v]$ is i-right cancellative in $\mathscr{G}\mathscr{P}$. Then we get that $[u] = [u']$. By Proposition 7.1.7 we have that $u = u'$, a contradiction.
If w has a left identity u in S_α, we write that $uw = w$ and so

$$[u]\,[w] = [w]$$

in $\mathscr{G}\mathscr{P}$. Since $[w]\,\mathcal{R}^*\,[v]$, then we get

$$[u]\,[v] = [v],$$

but $[v]$ is an i-right cancellative element in $\mathscr{G}\mathscr{P}$, a contradiction.

$\square$

Note that if w is not right cancellative in S_α, and w^+ is an idempotent in the $\mathcal{R}^*$-class of w in S_α, then w^+ is not right cancellative. As w is not right cancellative in S_α, there is $x \neq y$ and $xw = yw$, but $w\,\mathcal{R}^*\,w^+$, so $xw^+ = yw^+$. If w^+ is right cancellative, then $x = y$, a contradiction.

It is worth remarking for a complete reduced form $w = w_1 \circ \cdots \circ w_n \in X^+$, as $s(w)$ is complete, $[w] = [w_{1\sigma} \circ \cdots \circ w_{n\sigma}]$ for any permutation σ of $\{1, \cdots, n\}$. Without loss of generality we may always assume the i-right cancellative elements succeed the non-i-right cancellative elements in a complete reduced form.

Lemma 7.6.6. *Let $w = w_1 \circ \cdots \circ w_n$ be a complete reduced form of $[w]$ in $\mathscr{G}\mathscr{P}$ and let $w_k, \cdots, w_n$ be i-right cancellative elements in the corresponding vertex semigroups. Let $w_1, \cdots, w_{k-1}$ be elements that are not i-right cancellative. Then*

$$[w]\,\mathcal{R}^*\,[w_1 \circ \cdots \circ w_{k-1}].$$

Proof. As $w_k, \cdots, w_n$ are i-right cancellative in the corresponding vertex semigroups and by Lemma 7.6.2, we get that $[w_k], \cdots, [w_n]$ are i-right cancellative elements in $\mathscr{G}\mathscr{P}$. Let $[x]$, $[y]$ be elements in $\mathscr{G}\mathscr{P}^1$. Suppose that $[x]\,[w] = [y]\,[w]$, then

$$[x]\,[w_1 \circ \cdots \circ w_{k-1}]\,[w_k] \cdots [w_n] = [y]\,[w_1 \circ \cdots \circ w_{k-1}]\,[w_k] \cdots [w_n].$$

As $[w_k], \cdots, [w_n]$ are i-right cancellative elements in $\mathscr{G}\mathscr{P}$,

$$[x]\,[w_1 \circ \cdots \circ w_{k-1}] = [y]\,[w_1 \circ \cdots \circ w_{k-1}].$$

Conversely, if $[x]\,[w_1 \circ \cdots \circ w_{k-1}] = [y][w_1 \circ \cdots \circ w_{k-1}]$,, then by multiplying both sides by $[w_k \circ \cdots \circ w_n]$ we get that

$$[x]\,[w] = [y]\,[w].$$

Let $[x] \in \mathscr{G}\mathscr{P}$ and suppose that

$$[x]\,[w] = [x]\,[w_1 \circ \cdots \circ w_{k-1}]\,[w_k] \cdots [w_n] = [w].$$

As $[w_k], \cdots, [w_n]$ are i-right cancellative elements in $\mathscr{G}\mathscr{P}$,

$$[x]\,[w_1 \circ \cdots \circ w_{k-1}] = [w_1 \circ \cdots \circ w_{k-1}].$$

If $[x]\,[w_1 \circ \cdots \circ w_{k-1}] = [w_1 \circ \cdots \circ w_{k-1}]$, then by multiplying both sides by $[w_k \circ \cdots \circ w_n]$ we get that $[x]\,[w] = [w]$. Therefore, $[w]\,\mathcal{R}^*\,[w_1 \circ \cdots \circ w_{k-1}]$.

$\square$

Lemma 7.6.7. *Let $w = w_1 \circ \cdots \circ w_n, v = v_1 \circ \cdots \circ v_m \in X^+$ be complete reduced forms. Suppose that $w_{k+1}, \cdots, w_n, v_{l+1}, \cdots, v_m$ are i-right cancellative, for some $0 \leq k \leq n$, $0 \leq l \leq m$, and $w_1, \cdots, w_k, v_1, \cdots, v_l$ are not. Then $[w]\,\mathcal{R}^*\,[v]$ in $\mathscr{G}\mathscr{P}$ implies $s(w_1 \circ \cdots \circ w_k) = s(v_1 \circ \cdots \circ v_l)$.*

Proof. Suppose that $[w]\ \mathcal{R}^*\ [v]$. Then by Lemma 7.6.6, we get that

$$[w_1 \circ \cdots \circ w_k]\ \mathcal{R}^*\ [v_1 \circ \cdots \circ v_l].$$

Suppose that $s(w_1 \circ \cdots \circ w_k) \neq s(v_1 \circ \cdots \circ v_l)$. Without loss of generality there exists $1 \leq j \leq k$ such that $C(w_j) = \gamma \notin s(v_1 \circ \cdots \circ v_l)$. By assumption, we have that either w_j is not right cancellative or w_j is right cancellative and has a left identity.

If w_j is not right cancellative, then there must exist $u, z \in S_\gamma$ with $u \neq z$ but $uw_j = zw_j$, giving $[u][w_j] = [z][w_j]$, and so $[u][w] = [z][w]$. Since $[w]\ \mathcal{R}^*\ [v]$, we have $[u][v] = [z][v]$, so that

$$[u][v_1 \circ \cdots \circ v_l] = [z][v_1 \circ \cdots \circ v_l]$$

by Lemma 7.6.6. As $v_1 \circ \cdots \circ v_l$ is reduced and $C(u) = C(z) \notin s(v_1 \circ \cdots \circ v_l)$, by Note 7.2.5 we deduce that $u \circ v_1 \circ \cdots \circ v_l$ and $z \circ v_1 \circ \cdots \circ v_l$ are reduced. It follows from Lemma 7.2.3 that $[u] = [z]$ and so $u = z$ by Proposition 7.1.7, contradiction. Therefore, w_j is right cancellative.

If w_j is right cancellative and there exists $z \in S_{C(w_j)}$ such that $zw_j = w_j$, then $[z][w_j] = [w_j]$, and so $[z][w] = [w]$. Therefore, $[z][v] = [v]$, implying that $C(z) \in s(v)$. As $C(z) = \gamma \notin s(v_1 \circ \cdots \circ v_l)$, $z \circ v$ reduces to $v_1 \circ \cdots \circ v_{i-1} \circ zv_i \circ v_{i+1} \circ \cdots \circ v_n$ for some $l < i \leq m$. It follows from note 7.2.10(ii) that $zv_i = v_i$, and so v_i has a left identity, contradiction. Therefore, $s(w_1 \circ \cdots \circ w_k) = s(v_1 \circ \cdots \circ v_l)$. $\square$

Note that if $w = w_1 \circ \cdots \circ w_n, v = v_1 \circ \cdots \circ v_m \in X^+$ are complete reduced forms with $s(w) = s(v)$, so $n = m$, without loss of generality we may assume $C(w_i) = C(v_i)$ for all $1 \leq i \leq n$.

In the following result we consider some sufficient conditions for any two complete reduced forms $w = w_1 \circ \cdots \circ w_n$ and $v = v_1 \circ \cdots \circ v_n$, where $[w]\ \mathcal{R}^*\ [v]$, to have equal support sets and be such that $w_i\ \mathcal{R}^*\ v_i$ in S_i for all $1 \leq i \leq n$. This result proves the converse to Lemma 7.4.9.

Lemma 7.6.8. *Let $w = w_1 \circ \cdots \circ w_n$ and $v = v_1 \circ \cdots \circ v_m$ be complete reduced forms of $[w]$ and $[v]$ in $\mathscr{GP}$, respectively such that no letters of w or of v are i-right cancellative in the corresponding vertex semigroup. Then $[w]\,\mathcal{R}^*\,[v]$ if and only if $s(w) = s(v)$, $n = m$ and $w_i\,\mathcal{R}^*\,v_i$ in S_i, for all $1 \leq i \leq n$.*

Proof. Suppose that $s(w) = s(v)$ and $w_i\ \mathcal{R}^*\ v_i$ in S_i for all $1 \leq i \leq n$. Then $[w]\ \mathcal{R}^*\ [v]$ by Lemma 7.4.9.

Conversely, let $[w]\,\mathcal{R}^*\ [v]$. By Lemma 7.6.7 we get that $s(w) = s(v)$, $n = m$ and $C(w_i) = C(v_i)$ for all $1 \leq i \leq n$. Our aim here is to show that $w_i\,\mathcal{R}^*\,v_i$ in S_i, for all $1 \leq i \leq n$. Suppose that for some $1 \leq i \leq n$, $x, y \in S^1_{C(w_i)}$ and $xw_i = yw_i$. Then

$$\begin{aligned}[x]\,[w] &= [w_1 \circ \ldots \circ w_{i-1} \circ xw_i \circ w_{i+1} \circ \ldots \circ w_n] \\ &= [w_1 \circ \ldots \circ w_{i-1} \circ yw_i \circ w_{i+1} \circ \ldots \circ w_n] \\ &= [y]\,[w],\end{aligned}$$

As $[w]\,\mathcal{R}^*\ [v]$, $[x]\,[v] = [y]\,[v]$. Then we write

$$[v_1 \circ \ldots \circ v_{i-1} \circ xv_i \circ v_{i+1} \circ \ldots v_m] = [v_1 \circ \ldots \circ v_{i-1} \circ yv_i \circ v_{i+1} \circ \ldots v_m]$$

By Lemma 7.2.3 we get that $[xv_i] = [yv_i]$. Hence it is clear that $xv_i = yv_i$ Proposition 7.1.7. Since no letters of w are i-right cancellative, we cannot have $xw_i = w_i$ for any $x \in S_{C(w_i)}$, it follows that $w_i\ \mathcal{R}^*\ v_i$ for all $1 \leq i \leq n$.

□

Lemma 7.6.9. *Let $w_1 \circ \cdots \circ w_n$ and $v_1 \circ \cdots \circ v_m$ be complete reduced forms of $[w]$ and $[v]$ in $\mathscr{GP}$, respectively. Let $w_k, \cdots, w_n$ be i-right cancellative in the corresponding vertex semigroups, and $v_h, \cdots, v_m$ be i-right cancellative in the corresponding vertex semigroups. Let $w_1, \cdots, w_{k-1}, v_1, \cdots, v_{h-1}$ be elements that are not i-right cancellative. Then $[w]\,\mathcal{R}^*\,[v]$ if and only if $s(w_1 \circ \cdots \circ w_{k-1}) = s(v_1 \circ \cdots \circ v_{h-1})$, $k = l$ and $w_i\,\mathcal{R}^*\,v_i$ in S_i, for all $1 \leq i \leq k-1$.*

Proof. Let $s(w_1 \circ \cdots \circ w_{k-1}) = s(v_1 \circ \cdots \circ v_{h-1})$, $k = l$ and $w_i \mathcal{R}^* v_i$ for all $1 \leq i \leq k-1$. Hence by Lemma 7.4.9 we get that $[w_1 \circ \cdots \circ w_{k-1}] \mathcal{R}^* [v_1 \circ \cdots \circ v_{h-1}]$. By Lemma 7.6.6 we get

$$[w] \mathcal{R}^* [w_1 \circ \cdots \circ w_{k-1}] \mathcal{R}^* [v_1 \circ \cdots \circ v_{h-1}] \mathcal{R}^* [v].$$

Conversely, let $[w] \mathcal{R}^* [v]$. Then by Lemma 7.6.6 we get

$$[w_1 \circ \cdots \circ w_{k-1}] \mathcal{R}^* [w] \mathcal{R}^* [v] \mathcal{R}^* [v_1 \circ \cdots \circ v_{h-1}].$$

This implies that $s(w_1 \circ \cdots \circ w_{k-1}) = s(v_1 \circ \cdots \circ v_{h-1})$, $k-1 = h-1$ and $w_i \mathcal{R}^* v_i$ for all $1 \leq i \leq k-1$, by Lemma 7.6.8.

□

The proofs of the following results follow immediately from Proposition 7.4.7 and Corollary 7.4.10, respectively.

Corollary 7.6.10. *Let $w = w_1 \circ \ldots \circ w_n$, $v = v_1 \circ \cdots \circ v_m \in X^+$ be left complete reduced forms, with blocks w_i, $1 \leq i \leq n$ and v_j, $1 \leq j \leq m$, of $[w]$ and $[v]$ in $\mathscr{G}\mathscr{P}$, respectively. Then $[w] \mathcal{R}^* [v]$ if and only if $[w_1] \mathcal{R}^* [v_1]$.*

Proof. From Proposition 7.4.7 we get that $[w] \mathcal{R}^* [w_1]$ and $[v] \mathcal{R}^* [v_1]$. The result is then clear.

□

Now we state the main result of this section.

Theorem 7.6.11. *Let $[w], [v] \in \mathscr{G}\mathscr{P}$. Let w, v have left complete reduced forms with first blocks $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_m \in X^+$, respectively. Suppose that $x_{k+1}, \cdots, x_n$ and $y_{l+1}, \cdots, y_m$ are i-right cancellative, for some $0 \leq k \leq n, 0 \leq l \leq m$, but $x_1, \cdots, x_k$ and $y_1, \cdots, y_l$ are not. Then $[u] \mathcal{R}^* [v]$ if and only if $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$, $l = k$ and $x_i \mathcal{R}^* y_i$ for all $1 \leq i \leq k$.*

Proof. Suppose that $[w] \mathcal{R}^* [v]$. By Proposition 7.4.7 we get that

$$[x] \mathcal{R}^* [w] \mathcal{R}^* [v] \mathcal{R}^* [y].$$

As x and y are in complete reduced form, where $x_{k+1}, \cdots, x_n$ and $y_{l+1}, \cdots, y_m$ are i-right cancellative, for some $0 \leq k \leq n, 0 \leq l \leq m$, by Lemma 7.6.6 we get

$$[x_1 \circ \cdots \circ x_k] \mathcal{R}^* [x] \mathcal{R}^* [y] \mathcal{R}^* [y_1 \circ \cdots \circ y_l].$$

Then by Lemma 7.6.8 we have $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$, $l = k$ and $x_i \mathcal{R}^* y_i$ for all $1 \leq i \leq k$.

Conversely, let $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$, $l = k$ and $x_i \; \mathcal{R}^* \; y_i$ for all $1 \leq i \leq k$. Then by Lemma 7.6.8, Lemma 7.6.6 and Proposition 7.4.7 we get that

$$[w]\,\mathcal{R}^*\,[x]\,\mathcal{R}^*[x_1 \circ \cdots \circ x_k]\,\mathcal{R}^*\,[y_1 \circ \cdots \circ y_l]\,\mathcal{R}^*\,[y]\,\mathcal{R}^*\,[v].$$

□

7.7 Description of the relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ on graph product of semigroups

This section gives more description of the Green's $\sim$-relations, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$, on graph products of semigroups $\mathscr{GP}$.

Corollary 7.7.1. *Let w and v be elements of S_α, for some $\alpha \in V$. Then $w \; \widetilde{\mathcal{R}} \; v$ in S_α if and only if $[w] \; \widetilde{\mathcal{R}} \; [v]$ in $\mathscr{GP}$.*

Proof. Let $w \, \widetilde{\mathcal{R}} \, v$ in S_α. Let $[e] \in E(\mathscr{GP})$, such that $[e][w] = [w]$. Then $s(e \circ w) = s(w) = \{\alpha\}$, which implies $C(e) = \alpha$. As $[e] \in E(\mathscr{GP})$, $e \in E(S_\alpha)$, $w \, \widetilde{\mathcal{R}} \, v$ and $ew = w$, then $ev = v$. Hence it is clear that $[e]\,[v] = [v]$. Similarly, we prove that if $[e]\,[v] = [v]$, then $[e]\,[w] = [w]$. Therefore, $[w]\,\widetilde{\mathcal{R}}\,[v]$.

Conversely, let $[w]\,\widetilde{\mathcal{R}}\,[v]$ in $\mathscr{GP}$. Let $e \in E(S_\alpha)$ be such that $ew = w$. This implies that $[e][w] = [w]$. As $[w]\,\widetilde{\mathcal{R}}\,[v]$ and $[e]\,[w] = [w]$, we get that $[e]\,[v] = [v]$. Hence by Proposition 7.1.7, we get $ev = v$ in S_α. Therefore, $w \, \widetilde{\mathcal{R}} \, v$ in S_α.

□

It follows from Corollary 7.5.1 that if $w = w_1 \circ \cdots \circ w_n$ and $v = v_1 \circ \cdots \circ v_m$ are two left complete reduced forms with blocks w_i, v_j where $1 \leq i \leq n, 1 \leq j \leq m$, then $[w] \; \widetilde{\mathcal{R}} \; [v]$ if and only if $[w_1] \; \widetilde{\mathcal{R}} \; [v_1]$. Therefore, to characterise $\widetilde{\mathcal{R}}$ in $\mathscr{GP}$, we just need consider the question of when two complete reduced forms are $\widetilde{\mathcal{R}}$-related.

Lemma 7.7.2. *Let $w = w_1 \circ \cdots \circ w_n \in X^+$ be a complete reduced form. Suppose that $w_1, \cdots, w_k$ have idempotent left identities in the corresponding vertex semigroups but $w_{k+1}, \cdots, w_n$ do not, where $0 \leq k \leq n$. Then $[w] \; \widetilde{\mathcal{R}} \; [w_1 \circ \cdots \circ w_k]$.*

Proof. Let $e = e_1 \circ \cdots \circ e_m$ be a reduced word such that $[e] \in E(\mathscr{GP})$. Suppose that $[e][w] = [w]$. Then

$$[e_1 \circ \cdots \circ e_m][w_1 \circ \cdots \circ w_n] = [w_1 \circ \cdots \circ w_n].$$

Then $s(e) \subseteq s(w)$, and since both e and w are reduced we have

$$[e][w] = [z_1 \circ \cdots \circ z_n],$$

where $z_i = w_i$ for $i \in I$ and $z_j = e_{i_j} w_j$ for $j \in J$, with $I \cap J = \emptyset$, $I \cup J = \{1, \cdots, n\}$ and $i \mapsto i_j$ a bijection $\{1, \cdots, m\} \to J$. From Note 7.2.10(ii) we have that $e_{i_j} w_j = w_j$ for $j \in J$, so that $J \subseteq \{1, \cdots, k\}$ and so $[e][w_1 \circ \cdots \cdots \circ w_k] = [w_1 \circ \cdots \cdots \circ w_k]$. The result follows. $\square$

In the above result if $k = 0$, we interpret this result as saying that $[w]$ has no idempotent left identity.

Lemma 7.7.3. *Let $w = w_1 \circ \cdots \circ w_n, v = v_1 \circ \cdots \circ v_m \in X^+$ be complete reduced forms. Suppose that $w_1, \cdots, w_k, v_1, \cdots, v_l$ have idempotent left identities in the corresponding vertex semigroups but $w_{k+1}, \cdots, w_n, v_{l+1}, \cdots, v_m$ do not, for some $0 \leq k \leq n$, $0 \leq l \leq m$. If $[w]\ \widetilde{\mathcal{R}}\ [v]$ in $\mathscr{GP}$, then $s(w_1 \circ \cdots \circ w_k) = s(v_1 \circ \cdots \circ v_l)$ and so $k = l$.*

Proof. By Lemma 7.7.2,

$$[w_1 \circ \cdots \circ w_k]\ \widetilde{\mathcal{R}}\ [v_1 \circ \cdots \circ v_l].$$

Assume that $s(w_1 \circ \cdots \circ w_k) \neq s(v_1 \circ \cdots \circ v_l)$. If $k = l = 0$ we are done. Otherwise, without loss of generality, let $\gamma = s(w_j) \in s(w_1 \circ \cdots \circ w_k)$. Since w_j has an idempotent left identity, there must exist an idempotent $u \in S_\gamma$ such that $uw_j = w_j$, so that

$$[u][w_1 \circ \cdots \circ w_k] = [w_1 \circ \cdots \circ w_{j-1} \circ uw_j \circ w_{j+1} \circ \cdots \circ w_k] = [w_1 \circ \cdots \circ w_k].$$

Since $[w_1 \circ \cdots \circ w_k]\ \widetilde{\mathcal{R}}\ [v_1 \circ \cdots \circ v_l]$, we have $[u][v_1 \circ \cdots \circ v_l] = [v_1 \circ \cdots \circ v_l]$ and so $\gamma = C(u) \in s(v_1 \circ \cdots \circ v_l)$, so we are done. $\square$

Lemma 7.7.4. *Let $w = w_1 \circ \cdots \circ w_n$ and $v = v_1 \circ \cdots \circ v_m$ be a complete reduced form of $[w]$ and $[v]$, respectively, in $\mathscr{GP}$. Let the letters w_i, $1 \leq i \leq n$ and v_j, $1 \leq j \leq m$, have idempotent left identities in the corresponding vertex semigroups. Then $[w]\,\widetilde{\mathcal{R}}\,[v]$ if and only if $n = m$, $s(w) = s(v)$ and $w_i\,\widetilde{\mathcal{R}}\,v_i$.*

Proof. Let $n = m$, $s(w) = s(v)$ and $w_i\,\widetilde{\mathcal{R}}\,v_i$. Then by Lemma 7.5.2 we get that $[w]\,\widetilde{\mathcal{R}}\,[v]$.

Conversely, let $[w]\,\widetilde{\mathcal{R}}\,[v]$ in $\mathscr{GP}$. Then by Lemma 7.7.3 we have that $n = m$ and $s(w) = s(v)$. Let for some $1 \leq i \leq n$ and $u \in E(S_{C(w_i)})$ such that $uw_i = w_i$. Then

$$[u][w] = [w_1 \circ \cdots \circ w_{i-1} \circ uw_i \circ w_{i+1} \circ \cdots \circ w_n] = [w_1 \circ \cdots \circ w_{i-1} \circ w_i \circ w_{i+1} \circ \cdots \circ w_n] = [w]$$

implying $[u][v] = [v]$, so that

$$[v_1 \circ \cdots \circ v_{i-1} \circ uv_i \circ v_{i+1} \circ \cdots \circ v_n] = [v_1 \circ \cdots \circ v_{i-1} \circ v_i \circ v_{i+1} \circ \cdots \circ v_n].$$

By Note 7.2.10 (ii), we get that $uv_i = v_i$. Together with the dual arguments, we have $w_i \widetilde{\mathcal{R}} v_i$.

□

Lemma 7.7.5. *Let $w = w_1 \circ \cdots \circ w_n$ be left complete reduced form for $[w]$ in $\mathscr{GP}$. Let $w_1 = w_{1,1} \circ w_{1,2}$, where $w_{1,1}, w_{1,2} \in X^+$, all the letters of $w_{1,1}$ have idempotent left identities in their vertex semigroups, and all the letters of $w_{1,2}$ do not have idempotent left identities in their vertex semigroups. Then $[w] \widetilde{\mathcal{R}} [w_{1,1}]$.*

Proof. By Corollary 7.5.1, we get that $[w] \widetilde{\mathcal{R}} [w_1]$. Moreover, we have $[w_1] \widetilde{\mathcal{R}} [w_{1,1}]$ by Lemma 7.7.2. Therefore, $[w] \widetilde{\mathcal{R}} [w_1] \widetilde{\mathcal{R}} [w_{1,1}]$.

□

In the above result if $w_{1,1}$ is an empty word, which means w_1 does not contain any letters that have idempotent left identities in their vertex semigroups, then $[w] \widetilde{\mathcal{R}} [w_1]$.

Lemma 7.7.6. *Let $w = w_1 \circ \cdots \circ w_n$ and $v = v_1 \circ \cdots \circ v_m$ be left complete reduced forms for $[w]$ and $[v]$ in $\mathscr{GP}$, respectively. Let $w_1 = w_{1,1} \circ w_{1,2}$, and $v_1 = v_{1,1} \circ v_{1,2}$ defined as in Lemma 7.7.5. Then $[w] \widetilde{\mathcal{R}} [v]$ if and only if $[w_{1,1}] \widetilde{\mathcal{R}} [v_{1,1}]$.*

Proof. By Corollary 7.5.1, we get that $[w] \widetilde{\mathcal{R}} [w_1]$. It is clear that $[w_1] \widetilde{\mathcal{R}} [w_{1,1}]$, by Lemma 7.7.2. Similarly for $[v]$. Therefore, we get that

$$[v_{1,1}] \widetilde{\mathcal{R}} [v_1] \widetilde{\mathcal{R}} [v] \widetilde{\mathcal{R}} [w] \widetilde{\mathcal{R}} [w_1] \widetilde{\mathcal{R}} [w_{1,1}].$$

Conversely, let $[w_{1,1}] \widetilde{\mathcal{R}} [v_{1,1}]$. By Corollary 7.5.1 and Lemma 7.7.2 we get that

$$[v] \widetilde{\mathcal{R}} [v_1] \widetilde{\mathcal{R}} [v_{1,1}] \widetilde{\mathcal{R}} [w_{1,1}] \widetilde{\mathcal{R}} [w_1] \widetilde{\mathcal{R}} [w].$$

□

Now we are in the position to give our characterisation of $\widetilde{\mathcal{R}}$ on $\mathscr{GP}$.

Theorem 7.7.7. *Let $w = w_1 \circ \cdots \circ w_n$ and $v = v_1 \circ \cdots \circ v_m$ be left complete reduced forms for $[w]$ and $[v]$ in $\mathscr{GP}$, respectively. Let $w_1 = w_{1,1} \circ w_{1,2}$, and $v_1 = v_{1,1} \circ v_{1,2}$ defined as in Lemma 7.7.5. Let $w_{1,1} = x_1 \circ \cdots \circ x_k$ and $v_{1,1} = y_1 \circ \cdots \circ y_l$. Then $[w] \widetilde{\mathcal{R}} [v]$ if and only if $s(w_{1,1}) = s(v_{1,1})$, $k = l$ and $x_i \widetilde{\mathcal{R}} y_i$ for all $1 \leq i \leq k$.*

Proof. Let $s(w_{1,1}) = s(v_{1,1})$, $k = l$ and $x_i \mathrel{\widetilde{\mathcal{R}}} y_i$ for all $1 \leq i \leq k$. Then by Lemma 7.7.4 we get that $[w_{1,1}] \mathrel{\widetilde{\mathcal{R}}} [v_{1,1}]$. By Lemma 7.7.2 we know that $[w] \mathrel{\widetilde{\mathcal{R}}} [w_{1,1}]$ and $[v] \mathrel{\widetilde{\mathcal{R}}} [v_{1,1}]$, this implies that

$$[w] \mathrel{\widetilde{\mathcal{R}}} [w_{1,1}] \mathrel{\widetilde{\mathcal{R}}} [v_{1,1}] \mathrel{\widetilde{\mathcal{R}}} [v].$$

Conversely, let $[w] \mathrel{\widetilde{\mathcal{R}}} [v]$. Then by Lemma 7.7.2

$$[w_{1,1}] \mathrel{\widetilde{\mathcal{R}}} [w] \mathrel{\widetilde{\mathcal{R}}} [v] \mathrel{\widetilde{\mathcal{R}}} [v_{1,1}].$$

By Lemma 7.7.4 we have $s(w_{1,1}) = s(v_{1,1})$, $k = l$ and $x_i \mathrel{\widetilde{\mathcal{R}}} y_i$ for all $1 \leq i \leq k$, as all the letters of $w_{1,1}$ and $v_{1,1}$ have idempotent left identities in the corresponding vertex semigroups, $\square$

Chapter 8

A plan for further work

Let me finish the writing of my PhD thesis by giving a some open questions for further work.

In this thesis, we considered a very special kind of biordered set, namely, iso-normal band and it is proven that for any iso-normal band B, $\mathrm{IG}(\mathcal{B})$ is always an abundant semigroup. In 2014, Gould and Yang [80], gave an example of a normal band B, where $\mathrm{IG}(\mathcal{B})$, is not abundant. An interesting and valuable question comes out naturally: for which normal bands is $\mathrm{IG}(\mathcal{B})$ abundant?

It is proven that if B is an iso-normal band, then $\mathrm{IG}(\mathcal{B})$ has decidable word problem. Another interesting question: for which normal bands does $\mathrm{IG}(\mathcal{B})$ have decidable word problem?

Let Y_1 and Y_2 be 0-direct union semilattices, where δ_0 is the lower bound of both Y_1 and Y_2 (which means $Y = Y_1 \cup Y_2$, and $Y_1 \cap Y_2 = \{\delta_0\}$). Let B_1 and B_2 be normal bands, where

$$B_1 = \mathscr{B}(Y_1, B_\alpha, \phi_{\alpha,\beta}),$$

and

$$B_2 = \mathscr{B}(Y_2, B_\delta, \psi_{\delta,\gamma}).$$

If $\mathrm{IG}(\mathcal{B}_1)$ and $\mathrm{IG}(\mathcal{B}_2)$ are abundant semigroups. Is $\mathrm{IG}(\mathcal{B})$ abundant, where $B = \mathscr{B}(Y, B_\mu, \theta_{\mu,\nu})$ is a normal band.

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