Model-theoretic methods for algorithmically tame graph classes

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Abstract

The Johnson graphs $J(n,k)$ and Hamming graphs $H(d,q)$ are well-known families of finite graphs with strong symmetry properties, such as distance-transitivity. In this thesis we explore the model theory of these graphs and their infinite limits. A major focus is on Vapnik-Chervonenkis dimension and density, these are invariants of set systems historically of importance in statistical learning theory and extremal combinatorics, and highly relevant to first order structures which do not have the independence property. We show that the edge relation has VC-dimension 4 and VC-density 2 in the class of Johnson graphs and VC-dimension 3 and VC-density 2 on the class of all Hamming graphs. We also consider the limit theory $T_{\mathcal{J}}$ of the Johnson graphs as $\min(n,k,n-k) \to \infty$. We show that $T_{\mathcal{J}}$ is complete, describe the infinite models and prove that it is ω-stable of Morley rank ω, but not monadically dependent. When k is fixed, the limit theory of $J(n,k)$ is totally categorical of Morley rank k.

We also explore how certain graph operations affect the VC-dimension of the edge relation.

Contents

1 Introduction

In recent years the areas of model theory and graph theory have found a common interest in the topics of ***Vapnik-Chervonenkis dimension*** (VC-dimension) and ***Vapnik-Chervonenkis density*** (VC-density). VC-dimension and VC-density are closely linked and for convenience we will refer to them jointly as VC-characteristics. They give a way to measure the complexity of set systems which can be defined from first order logical formulae in structures such as graphs.

First introduced in the context of statistical learning theory [47], VC-dimension also plays a key role in computational learning [46, 28, 25] as well as in model theory [3], and it has applications in numerous areas, including graph theory [10], computational geometry [12], database theory [41], and graph algorithms and complexity [11, 19].

For fixed $k, m \in \mathbb{N}$ with $k \leq m$, the Johnson graph $J(m, k)$ has vertices that correspond to k-element subsets, of an underlying universe set of cardinality m, where two vertices are adjacent if their corresponding sets intersect in $k-1$ elements. Figure 1 shows the Johnson graph $J(4, 2)$. We let $\mathcal{J} := \{J(m, k) \mid k, m \in \mathbb{N}, k \leq m\}$ denote the class of all finite Johnson graphs, and $\mathcal{J}_k := \{J(m, k) \mid m \in \mathbb{N}\}$, and we let $\overline{\mathcal{J}}$ denote the closure of $\mathcal{J}$ under the induced subgraph relation. A first study of induced subgraphs of Johnson graphs has been done in [37].

Hamming graphs arise from Hamming schemes and they naturally model Hamming distance. For fixed $d, q \in \mathbb{N}$, let S be a set with $|S| = q$. The Hamming graph $H(d, q)$ has vertex set S^d, where two vertices are adjacent if they differ in precisely one coordinate. Figure 2 shows the Hamming graph $H(3, 2)$. We let $\mathcal{H} := \{H(d, q) \mid d, q \in \mathbb{N}\}$ denote the class of all Hamming graphs, and we let $\overline{\mathcal{H}}$ denote the closure of $\mathcal{H}$ under taking induced subgraphs. The class $\overline{\mathcal{H}}$ has been characterized in [32] via certain edge labellings. The classes $\mathcal{J}$ and $\mathcal{H}$ admit arbitrarily

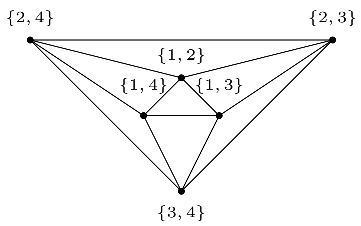


Figure 1: The Johnson graph $J(4, 2)$.

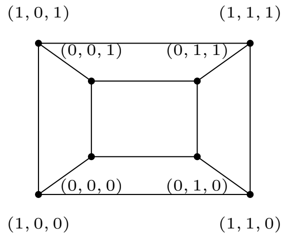


Figure 2: The Hamming graph $H(3, 2)$.

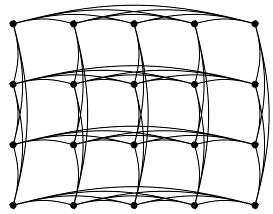


Figure 3: The rook's graph $R(5.4)$.

large cliques as subgraphs.

Johnson graphs and Hamming graphs are graphs of high regularity. They originally arise as the graph-theoretic analogue to the association schemes bearing their name. The relations of the association schemes correspond to fixed distances in the graphs. They feature in different areas of computer science and mathematics, including coding theory, algebraic graph theory and model theory. Johnson graphs also appear in László Babai's algorithm for solving the graph isomorphism problem in quasipolynomial time [5], where they constitute the 'hard case'.

Our motivation for this work is multifaceted largely stemming from algorithmic graph theory, permutation group theory, and model theory as mentioned below. In algorithmic graph theory structural ***tameness*** is often linked to good algorithmic properties.

Many problems on graphs, that are algorithmically hard (e.g. NP-hard) in general, can be solved efficiently on classes of graphs having a ***tame*** structure, such as graphs of bounded tree-width [15], planar graphs, graphs excluding a fixed minor, and nowhere dense classes of graphs [38]. Nowhere dense classes of graphs generalise the previously mentioned classes, and in [24] it was shown that on nowhere dense classes of graphs, every problem expressible in first-order logic is fixed-parameter tractable. All of these classes are sparse. In particular, they cannot contain arbitrarily large cliques. However, intuitively, cliques contain about as much information as independent sets. In [16], clique-width was introduced to address this (the class of all cliques has clique-width 2), and this was further generalised to graph classes of bounded local clique-width. That allowed fixed-parameter tractability for first-order logic [23]. Nowhere dense classes of graphs are closed under taking subgraphs, i. e. if $\mathcal{C}$ is a nowhere dense class of graphs, then the class obtained by closing $\mathcal{C}$ under subgraphs is also nowhere dense. Graph classes of bounded (local) clique-width are closed under taking ***induced*** subgraphs.

So-called ***dependent*** graph classes, i.e. graph classes where every first-order formula has bounded VC-dimension, are a common generalisation of both nowhere dense classes of graphs [1] and classes of bounded local clique-width [26]. We will discuss dependent classes below and we view dependence as an interesting notion of tameness. The classes $\mathcal{J}, \mathcal{J}_k, \overline{\mathcal{J}}, \mathcal{H}$, and $\overline{\mathcal{H}}$ are clearly somewhere dense, as arbitrarily large cliques occur as subgraphs, and they have unbounded local clique-width. Indeed, the open neighbourhood of any vertex of $J(m,k)$ induces a ***rook's graph*** $R(m-k,k)$, cf. Figure 3, and the class of all rook's graphs has unbounded clique-width. Moreover, the open 2-neighbourhood in a Hamming graph $H(d,2)$ induces the 1-subdivision of the complete graph on d vertices, see Corollary 6.1.5, and it is known that the class of 1-subdivisions of complete graphs has unbounded clique-width (cf. e. g. [2]).

Hamming graphs and Johnson graphs are regular and have large vertex transitive automorphism groups making them of particular interest in permutation group theory. The symmetric group S_m is the full automorphism group of the Johnson graph $J(m,k)$ whenever $m \neq 2k$, and the wreath product $S_q \mathrm{wr} S_d$ is the full automorphism group of the Hamming graph $H(d,q)$. In both cases these groups act *distance-transitively*: if (u,v) and (u',v') are pairs of vertices with $d(u,v) = d(u',v')$ then there is an element g in the group with $g(u) = u'$ and $g(v) = v'$. This symmetry is exploited in some of our proofs to reduce the number of cases that need to be checked.

This has also caused Johnson graphs and Hamming graphs to crop up in various areas of model theory. E.g. in [13, IVB] Gregory L. Cherlin, Gary A. Martin, and Daniel Saracino give an upper bound on the arity of a relation symbol needed to get quantifier elimination in $\mathcal{J}_k$, and in the theory of Hamming graphs $H(d, q)$ for a fixed d.

Stability theory is one of the major research themes in model theory. Originating in the 1960's with the seminal work of Michael Morley [36] and continuing on to this day it aims to classify first order theories according to their logical complexity. Stability is a strong tameness condition which has been generalized in various ways to give different weaker tameness conditions. A major research area of stability theory is classifying theories according to which tameness conditions they satisfy. Many of the results in this area can be found on [21].

Straddling both the graph-theoretic and the model-theoretic aspects of VC-characteristics we will be looking at graphs and graph classes from the perspective of their first order theory. The simplicity of the language of graphs, i.e. a language with a single binary relation, gives credence to the notion of a canonically "simplest" formula, namely $\phi(x, y) := Exy$. It can therefore be illuminating to look in detail at that formula e.g. computing its VC-dimension and VC-density.

Graph theory and model theory find themselves at odds when it comes to finitism. In graph theory, graphs are usually assumed to be finite, possibly arbitrarily large but finite. Model theory by contrast is primarily concerned with structures, of which graphs are a special case, that are infinitely large. The first order theory of a finite structure is often seen as simplistic since their entire structure can be captured in a single formula. In this thesis we will reconcile this in two ways.

Firstly we will look at infinite classes of finite graphs, and extend the definitions of VC-characteristics to apply to such classes rather than a single infinite model.

Secondly we will extend the definition of Johnson graphs to the infinite. The natural definition of infinite Johnson graphs will give rise to an additional definition of infinite graphs that behave like Johnson graphs, but fall outside the formal definition. We then unite those definitions in what we call ***Generalized Johnson graphs***.

Some results are already known e.g. that in the limit theory of $\mathcal{J}_k$ a formula $\phi(\bar{x}; \bar{y})$ has VC-density at most $2k|\bar{y}|$ [4, 1.1]. This is similar to a result on nowhere dense graph classes where it has been shown that the VC-density of a formula $\phi(\bar{x}; \bar{y})$ is at most $|\bar{y}|$ [39].

It is also known the limit theory of $H(\aleph_0, 2)$ is ω-stable [3, 4.10]. These results focus on cases where one of the parameters is fixed. Our work however, for the most part, doesn't have such a restriction and works on the entirety of $\mathcal{J}$ and $\mathcal{H}$.

1.1 Main results

The main results of this thesis can be split into two areas. First we compute the VC-dimension and VC-density of the edge relation on classes of finite Johnson graphs.

Theorem (4.2.2 and 4.3.1)
The edge relation has VC-dimension 4 *and VC-density* 2 *on the class of Johnson graphs.*

Theorem (6.2.2 and 6.3.1)
The edge relation has VC-dimension 3 *and VC-density* 2 *on the class of Hamming graphs.*

Secondly, in a deeper dive into the model theory of Johnson graphs, this work focuses on infinite graphs. First we look at the limit theory of $J(n, k)$ for a fixed k and prove:

Theorem (5.2.1 and 5.2.4)
The limit theory of $J(n,k)$ for some fixed k and $n \to \infty$ is a complete theory with Morley rank k.

We also show:

Theorem (5.2.6)
The Johnson graph $J(\aleph_0, 2)$ is not monadically dependent.

A corollary to this, which is relevant to recent research, is that the class $\mathcal{J}_2$, and therefore $\mathcal{J}$, does not have bounded twin-width. This is due to [44, 8.4].

We then let k loose and working toward a grand unified theory of all infinite Johnson graphs. First we prove:

Theorem (5.3.3)
The theory of $J(n,k)$ stabilizes as n and k approach ∞. I.e. for every sentence σ in the language of graphs there exists a $k_\sigma \in \mathbb{N}$ such that if $k, k', n, n', n-k$, and $n'-k'$ are all greater than k_σ then $J(n,k) \models \sigma \iff J(n',k') \models \sigma$.

From this result we define $T_{\mathcal{J}}$ to be the theory of all sentences that are true in $J(n,k)$ as $n, k, n-k \to \infty$, which by Theorem 5.3.3 is a complete theory.

In Definition 5.1.6 we introduce the notion of ***generalized Johnson graph*** and in Chapter 5.4 we give a set of axioms Σ satisfied by all models of $T_{\mathcal{J}}$ and prove the following result.

Theorem (5.4.3)
A model M has $M \models \Sigma$ if and only if M is a generalized Johnson graph.

Armed with the knowledge of what all models of $T_{\mathcal{J}}$ look like, namely that they are generalized Johnson, graphs we then go on to prove the following.

Theorem (5.5.8 and 5.5.7)

1. *$T_{\mathcal{J}}$ has Morley rank ω.*

2. *$T_{\mathcal{J}}$ is ω-stable.*

3. *$T_{\mathcal{J}}$ is dependent.*

Parts (2) and (3) follow directly from (1) but are important enough to merit their own mention.

1.2 Structure of thesis

In this section we will give an outline of the thesis explaining where each piece of work is carried out.

In Chapter 2 we will introduce the technical definitions required for our work along with the relevant theorems and lemmas from the literature.

In Chapter 3 we will start by exploring the VC-dimension of the edge relation and how it changes under the common graph operations i.e. vertex deletion, edge deletion, edge contraction, local complement, and complement. The results are given by a series of examples, rather than formal theorems, and are summarised in Table 1. We will then move on to looking at the VC-dimension of the edge relation on particular graph classes. We begin by looking at trees and graphs that are close to being trees and give a connection between the VC-dimension of the edge relation and tree-width. The results in this chapter closely mimic those by Stéphan Thomassé and Nicolas Bousquet in [10].

In Chapter 4 we will introduce the notion of Johnson graphs and give technical lemmas on how VC-dimension and VC-density behave on the class of Johnson graphs. We will then move on to compute the VC-dimension and VC-density of the edge relation on the class of all Johnson graphs giving us our first main results.

In Chapter 5 we generalize the definition of Johnson graphs to include infinite graphs defined by infinite sets. It is here where we delve deeper into the first order theory looking beyond just the formula $\phi(x, y) := Exy$ and prove that as n and k approach ∞ the theory of Johnson graphs $J(n, k)$ stabilizes, so we have a single complete theory $T_{\mathcal{J}}$ that describes all Johnson graphs. We give axioms for this theory first by giving axioms for any $J(n, k)$ that can be obtained and then focusing on the infinite case. We also define special types of graphs $J'(\kappa, \lambda)$ which emerge as the connected components of infinite Johnson graphs and finally unify them in the concept "generalized Johnson graphs" which captures all graphs that satisfy $T_{\mathcal{J}}$. We then show that $T_{\mathcal{J}}$ is an extremely tame theory, showing that any generalized Johnson graph has Morley rank ω.

That concludes the work on Johnson graphs and in Chapter 6 we move on to Hamming graphs. The chapter is in many ways analogous to Chapter 4 and follows a similar structure but focuses on Hamming graphs rather than Johnson graphs. Note that our work on Hamming graphs focuses on the VC-characteristics of the edge relation and we do not prove general results about the full theory of Hamming graphs.

Finally in Chapter 7 we get into a discussion of possible directions further research could take.

2 Preliminaries

In this chapter we will introduce our notation and present the basic definitions of graph theory and model theory. We will also cover some basic results relating the VC-dimension and VC-density.

2.1 Clarification on notation

In this section we will clarify some notation used throughout this thesis. All of the concepts introduced are basic and notation given without full definitions.

For a set A we will use $P(A)$ to denote the ***powerset*** of A.

For a function f we will use f^{-1} to denote the inverse of f, if it exists.

Let f be a function $X \to Y$ and $A \subseteq X$. We then denote $f[A] := \{f(a)|a \in A\}$.

In all cases $\log(k)$ stands for the base 2 logarithm.

We use the symbols $\omega = \aleph_0 = |\mathbb{N}|$ interchangeably.

For sets A and B will use $A \sqcup B$ to mean the disjoint union of A and B.

2.2 Model-theoretic preliminaries

In this thesis we will be looking at graphs from a model-theoretic perspective. In this section we will give the basic definitions required for our work. In interest of brevity, this is not going to be a complete introduction to model theory. Rather we focus on those parts that are essential to our focus on the model theory of graphs, primarily finite graphs. For a more complete overview, we point the reader to standard model theory text books [34, 45, 29].

Definition 2.2.1

A **language** *L is a set of relation symbols and function symbols, each equipped with an arity, and of constant symbols.*

We note that function symbols and constant symbols are in fact short hands for relation symbols satisfying certain axioms. In the case of a constant symbol c we have a unary relation symbol R_c which is always satisfied by exactly one element. Similarly for an n-ary function symbol f we have a $n+1$-ary relation symbol R_f such that for any n-tuple $\bar{x}$ there exists a unique y such that $R_f(\bar{x}, y)$.

The simple notion of a logical formula can be quite complicated to pin down giving rise to the following three definitions.

Definition 2.2.2
We define a **term** *in a language L, also known as an L-***term***, inductively as one of*

- *x a variable.*
- *c for some constant symbol c in L.*
- *$f((t_i)_{i=1}^n)$ for some function symbol f of arity n in L and some family $(t_i)_{i=1}^n$ of L-terms.*

Definition 2.2.3
An **atomic formula** *in a language L is either*

- *$t_1 = t_2$ or*
- *$R((t_i)_{i=1}^n)$ for some relation symbol R of arity n in L and some family $(t_i)_{i=1}^n$ of L-terms.*

Definition 2.2.4
A **formula** *in a language L, also known as an L-***formula** *is one of*

- *An atomic formula in L.*
- *$\neg\phi$ for some L-formula ϕ.*

- *$\phi \wedge \psi$ for some L-formulae ϕ and ψ.*

- *$\phi \vee \psi$ for some L-formulae ϕ and ψ.*

- *$\forall x\phi$ for some L-formula and some variable x.*

- *$\exists x\phi$ for some L-formula and some variable x.*

In the last two cases we say the quantifier, $\forall$ or $\exists$, **binds** *variable x in ϕ. We say that a variable that is not bound by any quantifier is a* **free variable**.

This is a common definition found in textbooks, although some definitions omit one of the logical connectives and one of the quantifiers defining the others as shorthand. We allow ourselves to be a little vague with the concepts of free and bound variables. There we defer the common practice of only binding free variables and never having a variable occur free in a formula ϕ if it is bound in some subformula of ϕ. In addition to disjunction and conjunction we will use the following connectives

- $p \to q := \neg p \vee q$

- $p \leftarrow q := p \vee \neg q$

- $p \leftrightarrow q := (p \wedge q) \vee (\neg p \wedge \neg q)$

- $p \oplus q := (p \vee q) \wedge \neg(p \wedge q)$

We will sometimes split the free variables of a formula ϕ into two tuples represented by $\phi(\bar{x}; \bar{y})$. We then call $\bar{y}$ the ***parameters*** and $\bar{x}$ the variables.

Definition 2.2.5

The **quantifier depth** *of a formula is defined inductively on its structure as follows:*

$$QD(\phi) = 0 \textit{ if } \phi \textit{ is atomic}$$

$$QD(\neg\phi) = QD(\phi)$$

$$QD(\phi \vee \psi) = \max(QD(\phi), QD(\psi))$$

$$QD(\phi \wedge \psi) = \max(QD(\phi), QD(\psi))$$

$$QD(\forall x\phi) = QD(\phi) + 1$$

$$QD(\exists x\phi) = QD(\phi) + 1$$

Definition 2.2.6

A **sentence** *in language L, also referred to as an* L**-sentence***, is a formula with no free variables.*

In this work the primary role of sentences will be as axioms.

We now give a definition of models that is much simpler than what can be found in standard texts but is adequate for this thesis.

Definition 2.2.7

Let L be a language. A L**-model** *M consists of a universe of objects along with an interpretation of every relation, function and constant symbol in L by a relation on M, function on M, and constant in M respectively.*

For the purpose of this thesis the universe can be considered to be a set (rather than a proper class), although in general this need not be the case, e.g. when considering models of set theory. We use abuse of notation throughout making no distinction between a model and its universe.

Models are also referred to as ***structures*** and for the purposes of this thesis the terms "model" and "structure" will in almost all cases be interchangeable with "graph" introduced in the next section.

Definition 2.2.8

Let M be a model and $A \subseteq M$. We say that the set A is **definable** *if there is a formula $\phi(\bar{x}; \bar{y})$ and a tuple $\bar{b}$ from M such that $\bar{x} \in A$ if and only if $M \models \phi(\bar{x}; \bar{b})$.*

Definition 2.2.9

An L-**theory** *is a consistent set of L-sentences. We say that an L-theory T is* **complete** *if for any L-sentence ϕ either $\phi \in T$ or $\neg\phi \in T$.*

In this thesis we do not require theories to be closed under logical implication. Since our work primarily deals with complete theories, making such assumptions would not affect the results of this thesis. If we have a complete L-theory T and $M \models T$ then T is in essence everything that can be said about M using L.

For a graph class $\mathcal{C}$ we call the set of all sentences true in all but finitely many models in $\mathcal{C}$ the ***limit theory*** of $\mathcal{C}$.

Definition 2.2.10

Let M and N be two L-structures. We say that M and N are **elementarily equivalent** *if for every L-sentence ϕ we have $M \models \phi \Leftrightarrow N \models \phi$.*

Definition 2.2.11

Let M be an L-structure and $N \subseteq M$. We say that N is an **elementary substructure** *of M if for every L-formula $\phi(\bar{x})$ and every $\bar{a} \in N$ we have:*

$$M \models \phi(\bar{a}) \Leftrightarrow N \models \phi(\bar{a})$$

We say that M is an **elementary extension** *of N if N is an elementary substructure of M.*

Definition 2.2.12

Let T be an L-theory and $A \subseteq M \models T$. Let L_A be the language L expanded with an additional constant symbol for each element of A. An n-**type** *over A of T is a* *maximal set S of L_A-formulae with n free variables such that there is an n-tuple $\bar{x}$ of elements from some elementary extension M' of M such that $M' \models \phi(\bar{x})$ for all $\phi \in S$.*

A **type** *over A is an n-type over A for some n.*

A type over a set A in a model M is the model-theoretic analogue to the notion of orbits of the pointwise stabilizer of A in the automorphism group of M.

Definition 2.2.13

Let λ be an infinite cardinal, T be a theory and $M \models T$. We say that T is **λ-stable** *if the number of 1-types of T over any set $A \subseteq M$ such that $|A| = \lambda$ is λ. We say that T is* **stable** *if it is λ-stable for some infinite cardinal λ.*

Stability, in a sense, restricts the possible complexity of a theory. Assuming L has countably many symbols and T is a λ-stable theory, the number of types over a set of size λ is at least λ and at most 2^λ. Stability theory is an active research area of model theory. Stability has been identified as a strong notion of tameness and as such a desirable property for first order theories.

Definition 2.2.14

Let T be a theory. A formula $\phi(\bar{x};\bar{y})$ has the **independence property** *if there exists a model $M \models T$ containing $(\bar{a}_i)_{i\in\mathbb{N}}$ and $(\bar{b_S})_{S\subset\mathbb{N}}$ such that $i \in S \Leftrightarrow M \models \phi(a_i; b_S)$ for all $i \in \mathbb{N}$ and $S \subset \mathbb{N}$. Otherwise we say that ϕ is* **dependent***. We say that a theory T is* **dependent** *if no formula in T has the independence property.*

Dependent theories are also commonly known as NIP (Not the Independence Property). It is another tameness condition from stability theory albeit a weaker one than stability.

Theorem 2.2.15 ([40, 7.3.0])

Let T be a theory. If T has the independence property then T is unstable

Dependence is particularly important due to its direct link to VC-dimension. Namely, a formula $\phi(\bar{x};\bar{y})$ has the independence property if and only if the family of sets it defines has infinite VC-dimension. That makes it important in algorithmic and extremal graph theory and statistical learning theory as well. In the latter VC-dimension has been found to be highly related to the sample complexity in PAC (Probably Approximately Correct) learning algorithms [26, 31]. The sample complexity of a concept is an indication of how hard it is to learn.

We define the following special terms to deal with monadic second order expansion of theories.

Definition 2.2.16

Let T be a theory. We say that T is **monadically stable** *if any expansion of T by uniary predicates is stable. We say that T is* **monadically dependent** *if any expansion of T by uniary predicates is dependent.*

Definition 2.2.17

Let M be a model and $X \subseteq M$ be a definable set defined by formula ϕ. We define the **Morley rank** *of X, or ϕ, recursively with the following steps:*

- *The Morley rank is at least 0 if $X \neq \emptyset$.*
- *The Morley rank is at least $\alpha+1$ for some ordinal α if X contains $\aleph_0$ disjoint definable subsets of Morley rank at least α.*
- *The Morley rank is at least κ for a limit ordinal κ if it is at least α for all $\alpha < \kappa$.*
- *The Morley rank is α for some ordinal α if it is at least α and not at least $\alpha+1$.*

For a type p the Morley rank $MR(p)$ is the minimal Morley rank of any formula in p.

We will employ a slight abuse of notation writing $MR(a)$ for an element a to mean the Morley rank of the type of a.

Definition 2.2.18
Let $M \models T$ be a model and $X \subseteq M$ be a definable set with Morley rank α. The **Morley degree** *of X is the largest $d \in \mathbb{N}$ such that we may write X as the union of d disjoint definable sets each of Morley rank α. If p is a type with $MR(p) = \alpha$, then its Morley degree is the minimal Morley degree of a formula ϕ of p having $MR(\phi) = \alpha$.*

Lemma 2.2.19 ([45, 6.2.3])
Let X and Y be definable sets. Then $MR(X \cup Y) = \max(MR(X), MR(Y))$.

Lemma 2.2.20 ([45, 6.2.11])
Let A be a definable set and $T(A)$ the set of types over A Then $MR(A) = \max\{MR(p) | p \in T(A)\}$.

Definition 2.2.21
Let κ be a cardinal. A theory T is κ**-categorical** *if it has a model of size κ and all models of T of size κ are isomorphic. We say that T is* **totally categorical** *if it is κ-categorical for all $\kappa \geq \omega$.*

Theorem 2.2.22 (Morley's theorem)
Let κ be an uncountable cardinal and L a language with countably many symbols. Then an L-theory T is $\aleph_1$-categorical if and only if T is κ-categorical.

It is important to note a few connections between categoricity and stability conditions.

Theorem 2.2.23 ([36, 3.8])
Let T be a totally categorical theory. Then T is ω-stable.

Theorem 2.2.24

Let T be a complete L-theory, where L is a language containing countably many symbols. Then T is ω-stable if and only if every definable set has ordinal valued Morley rank. I.e. there is some, possibly infinite, bound on the Morley rank of any definable set.

We will use a type of argument from finite model theory known as Ehrenfeucht-Fraïssé games. Ehrenfeucht-Fraïssé game arguments are a way to determine if a concept is expressible in first order logic, or if two structures satisfy the same sentences.

The game is played between two players, usually named Spoiler and Duplicator.

Definition 2.2.25

Let L be a language with no function symbols, $n \in \mathbb{N}$, and M and M' be L-models. The **Ehrenfeucht-Fraïssé game** *$EF_n(M, M')$ is a 2 player game played in following way.*

In each round first Spoiler plays one element from either M or M', then Duplicator plays one element from whichever model Spoiler did not play in. We call the element played in M on the i-th round a_i and the element played in M' we call b_i regardless of which player plays them. After n rounds have been played we have $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$.

Duplicator has won if for any atomic L-formula ϕ and any sequence $(i_j)_{j=1}^k$ with $k \leq n$ we have $M \models \phi((a_{i_j})_{j=1}^k) \Leftrightarrow M' \models \phi((b_{i_j})_{j=1}^k)$, otherwise Spoiler has won.

Ehrenfeucht-Fraïssé games are a finite analogue to so called back and forth arguments common in model theory.

Ehrenfeucht and Fraïssé showed that:

Theorem 2.2.26 (Ehrenfeucht-Fraïssé [18, 2 and 3])

If Duplicator has a winning strategy in $EF_n(M, M')$ then for all L-sentences ϕ of *quantifier depth at most n,* $M \models \phi \Leftrightarrow M' \models \phi$.

We note that if Duplicator has a winning strategy in $EF_n(M, M')$ for all $n \in \mathbb{N}$ that means that the models M and M' are indistinguishable from the perspective of first order logic. This means that any characteristic that M has and M' does not have cannot be expressed in first order logic.

Example Connectivity in graphs is not first order expressible. See [33, 3.19] for details.

Theorem 2.2.27 (Tarski-Vaught test[45, 2.1.2]))
A substructure N *of* M *is an elementary substructure of* M *if and only if for every formula* $\phi(x;\bar{a})$ *with* $\bar{a} \in N$ *we have* $M \models \exists x(\phi(x;\bar{a}))$ *if and only if there exists some* $b \in N$ *such that* $M \models \phi(b;\bar{a})$.

It is immensely useful to be able to talk about what it means for two structures to "be the same". This is commonly referred to as being isomorphic. In the interest of simplicity we will restrict our definition of isomorphisms to graph isomorphism as that is sufficient for our work.

Definition 2.2.28
Let G *and* H *be graphs. An* **isomorphism** *between* G *and* H *is a bijective function* $f : G \mapsto H$ *such that for all vertices* $v, u \in G$ *we have* $G \models Evu \iff H \models Ef(v)f(u)$. *We say that* G *and* H *are* **isomorphic** *if there exists an isomorphism between them. An* **automorphism** *on* G *is an isomorphism between* G *and* G.

The collection of automorphisms on a given graph G form a group with function composition. This group describes the symmetries of G.

Definition 2.2.29
Let G *be a graph. The group whose elements are the automorphisms of* G *with composition as the group operation is called the* **automorphism group** *of* G *denoted*

$Aut(G)$. Let $A \subseteq G$ we then call the set $\{f \in Aut(G) | f[A] = A\}$ the **setwise stabilizer** *of A. We also call the set $\{f \in Aut(G) | \forall a \in A(f(a) = a)\}$ the* **pointwise stabilizer** *of A. We note that for any $A \subseteq G$ both the pointwise and setwise stabilizer of A form subgroups of $Aut(G)$.*

2.3 Graph-theoretic preliminaries

In this section we will give the basic definitions and lemmas from graph theory for our work. This is not meant as a complete introduction to graph theory but rather a brief introduction and clarification on notation. For a more in-depth introduction to graph theory we point the reader to some standard postgraduate texts on the subject [17, 9, 27, 22]. The notation of graph theory can vary a lot from scholar to scholar, textbook to textbook. We have chosen our notation such that it fits nicely with conventions of model theory.

Definition 2.3.1
The **language of graphs** *L is a language with a single binary relation symbol E.*

Throughout this thesis we will be using L to mean the language of graphs unless we specify otherwise.

Definition 2.3.2
A graph G is a model of L satisfying the following axioms:

$$\forall v \neg Evv$$

$$\forall v \forall u (Euv \to Evu)$$

We note for graph-theoretic audiences that our graphs are undirected and have no multiple edges or loops. We will commonly refer to the universe of a graph G as the ***vertex set*** of G denoted V_G in accordance with graph-theoretic convention. We will also, to keep with convention, use E_G to denote the edge relation as interpreted by G i.e. $E_G vu \Leftrightarrow G \models Evu$.

Definition 2.3.3
Let v be a vertex in a graph G. The **neighbourhood** *of v denoted $N(v)$ is:*

$$N(v) := \{u \in G | Evu\}$$

If $u \in N(v)$ we say that u is a **neighbour** *of v.*

Note *Since we have no loops we have for all vertices $v \notin N(v)$.*

The nomenclature for describing two vertices u and v satisfying Euv is vast and the following phrases are equivalent.

- Euv
- u and v are neighbours.
- u and v are adjacent.
- there is an edge between u and v.

Definition 2.3.4
A set $A \subseteq V_G$ is a **clique** *if for all pairwise distinct u and v in A we have Euv. If V_G is a clique then we say that G is a* **complete** *graph. The complete graph with $|V_G| = n$ is denoted K_n.*

Definition 2.3.5
Let G be a graph and $k \in \mathbb{N}$. A **path** *is a sequence $(v_i)_{i=0}^{k}$ of pairwise distinct vertices such that v_i is adjacent to v_{i+1}, and we say that k is the* **length** *of the path. The graph that contains k vertices which form a path and has no other edges is denoted P_k.*

We note that in our definition the indices start at 0 so the length of the path in P_k is $k-1$.

Definition 2.3.6
The **distance** *from vertex v to u in a graph G, denoted $d(v,u)$, is the minimum length of a path from v to u in G.*

Definition 2.3.7 ([8])
A graph G is **distance-transitive** *if for any two pairs of vertices (u,v) and (a,b) such that $d(u,v)=d(a,b)$ there exist an automorphism $f \in Aut(G)$ such that $f(u)=a$ and $f(v)=b$.*

Definition 2.3.8
Let v be a vertex in a graph G. The **ball of radius** r **around** v *is the set of vertices at distance at most r from v. We say that G has* **radius** r *if $r := \min\{x_v | v \in G \wedge x_v := \max\{d(u,v)|u \in G\}\}$. We say that G has* **diameter** k *if $k := \max\{d(u,v)|u,v \in G\}$.*

Definition 2.3.9
Let G and H be graphs with edge relations E_G and E_H respectively and vertex sets V_G and V_H. We say that H is a **subgraph** *of G if $V_H \subseteq V_G$ and $\forall v,u \in V_H(E_H vu \rightarrow E_G vu)$. If E_G and E_H agree on V_H then we say that H is an* **induced** *subgraph of G or is the subgraph induced by V_H. If H is an induced subgraph of G we write $G[V_H] := H$ and say that V_H induces H as a subgraph of G.*

It is important to note for model-theoretic audiences that a subgraph H of G is a substructure of G if and only if it is an induced subgraph of G.

The concepts of subgraphs and induced subgraph of G correspond to those of: graphs that can be obtained from G by using repeated applications of vertex deletion and edge deletion.

A graph H is an induced subgraph of G if H can be obtained by starting with G and then using only the operation of vertex deletion. Similarly H is a subgraph of G if you can obtain H from G by using vertex deletion and edge deletion. There is one more operation common in graph theory which is that of edge contraction. When an edge is contracted the vertices incident at the edge are merged into one vertex, which is adjacent to all the neighbours of both of the original vertices. For a more detailed description of these graph operations see Chapter 3. Edge contraction gives rise to the concept of graph minor.

Definition 2.3.10

Let H and G be graphs. We say that H is a **minor** *of G if there is a subgraph $U \subseteq G$ and an equivalence relation $\sim$ on V_U such that each equivalence class is a connected subgraph of U and $H \cong U/\sim$. Note that in $U/\sim$ two vertices u and v are adjacent if and only if there are vertices $u' \in u$ and $v' \in v$ adjacent in U. I.e. there is at least one edge between a member of u and a member of v.*

We say that H is a depth r minor of G if each equivalence class of $\sim$ has radius at most r as an induced subgraph of G.

Definition 2.3.11

A **cycle** *is a path of length $k > 2$ from u to v where u and v are adjacent. The graph which contains k vertices and contains a cycle and no other edges is called the cycle of length k denoted C_k.*

Definition 2.3.12

A **tree** *is a connected graph that contains no cycles.*

Definition 2.3.13

We say that a set A of vertices in a graph G is **connected** *if for any pair of vertices $u, v \in A$ there exists a path in G from u to v. We say that G is* **connected** *if V_G is connected. A maximal connected set in a graph G is called a* **connected component** *of G.*

Definition 2.3.14

Let S be a set of vertices in a graph G and $k \in \mathbb{N}$. We say that S is a **distance k dominating set** *if for all vertices v in G, $\min\{d(u, v) | u \in S\} \leq k$. We simply say that S is a* **dominating set** *if it is a distance 1 dominating set.*

Definition 2.3.15

Let G be a finite graph with vertex set V. A **tree decomposition** *of G is a tree T whose vertex set is $U \subseteq P(V)$ such that the following holds:*

$$\bigcup U = V$$

$$\forall v \forall u \in V(Euv \rightarrow (\exists A \in U(u \in A \wedge v \in A)))$$

and for every $v \in V$ we have that the set $\{A \in U | v \in A\}$ is connected.

The **width** *of T is the largest size of a set in U. The* **tree-width** *of a graph G is the minimum width among all possible tree decompositions.*

Definition 2.3.16

Let n and k be natural numbers $n \geq k$. The **Johnson graph** *$J(n, k)$ is the graph whose vertices are the k-element subsets of a set X of size n, where two vertices are adjacent if their corresponding sets intersect in all but one element. i.e. their symmetric difference has size 2. We call X the* **underlying set** *of $J(n, k)$.*

It is important to note the distance in a Johnson graph is directly related to the size of the symmetric difference between sets.

Lemma 2.3.17 ([14, §2])

Let u and v be vertices in a Johnson graph. Then the distance $d(u, v) = \frac{|u \Delta v|}{2}$.

This fact will be leveraged heavily in our proofs.

A special class of graphs shows up in our work as the subgraphs induced by $N(v)$ for any vertex v in a Johnson graph.

Definition 2.3.18

Let $m, n \in \mathbb{N}$. The **rook's graph** $R(m, n)$ *is the cartesian product of K_m and K_n. We can represent every vertex by a pair from $R \times C$ where $|R| = m$ and $|C| = n$ and two vertices $(i, j), (k, l)$ are adjacent if and only if $i = k$ or $j = l$.*

The name of these graphs comes from the intuition that if we take a m by n chessboard and place a vertex on each tile then there is an edge between two tiles if a rook can legally be moved from one to the other.

Definition 2.3.19

Let d and q be natural numbers and S a set with $|S| = q$. The **Hamming graph** $H(d, q)$ *is a graph whose vertices correspond to ordered d-tuples of elements from S and two vertices are adjacent if they agree in all but one coordinate.*

Note that $R(n, n) = H(2, n)$.

Definition 2.3.20 ([38, 2.1])

A class $\mathcal{C}$ of graphs is **somewhere dense** *if there exists an integer τ such that the largest clique that is a depth τ minor of some graph in $\mathcal{C}$ is unbounded. Otherwise, if the largest clique that is a depth i minor of some graph in $\mathcal{C}$ is bounded for each integer i, the class $\mathcal{C}$ is* **nowhere dense**.

2.4 Set systems and Vapnik-Chervonenkis characteristics

In this section we will give definitions of set systems and relevant notions required to define VC-dimension and VC-density.

Definition 2.4.1

A **set system** *is a pair $(X, \mathcal{S})$ consisting of a universe set X and a family $\mathcal{S}$ of subsets of X.*

When the underlying universe X is clear we often refer to the family $\mathcal{S}$ as the set system $(X, \mathcal{S})$.

Definition 2.4.2

Let $\phi(\bar{x}; \bar{y})$ be a formula. We call $\bar{x}$ the object variables and $\bar{y}$ the parameter variables of $\phi(\bar{x}; \bar{y})$. A set system for a formula $\phi(\bar{x}; \bar{y})$ with m object variables and n parameter variables in a model M is a set system (M^m, S_ϕ) where:

$$S_\phi = \{\{\bar{a} \in M^m : M \models \phi(\bar{a}; \bar{b})\} : \bar{b} \in M^n\}$$

Example The set system for Exy in a graph G is $(V(G), \mathcal{S}_E)$ where

$$\mathcal{S}_E := \big\{\{x \mid G \models Exy\} \mid y \in V(G)\big\} = \{N(v) \mid v \in V(G)\}.$$

Throughout this thesis we will attribute to a formula ϕ all the characteristics of the set system for ϕ. Furthermore we will attribute to the edge relation all the characteristics of of the formula $\phi(x, y) := Exy$. We will illustrate this better once we have defined some characteristics of set systems.

Definition 2.4.3

Let $(X, \mathcal{S})$ be a set system and $A \subseteq X$ be a set. We say that A is **shattered** *by $\mathcal{S}$ if the class of intersections of sets in $\mathcal{S}$ with A is the full powerset of A i.e.*

$$\forall B \subseteq A\ \exists S \in \mathcal{S}\ B = A \cap S$$

To illustrate our previous point about attributing to formula the characteristics of their set systems, we say that a set A in a model M is shattered by ϕ if A is shattered by the set system for ϕ in M. More specifically we say that a set A in a graph G is ***shattered by the edge relation*** if it is shattered by the set system for the formula "Exy" on G. Namely $\{N(v) : v \in G\}$. This is an important concept in this thesis and gets used heavily throughout.

Lemma 2.4.4

For any formula ϕ and model M we have:

$$VC((M, \mathcal{S}_\phi)) = VC((M, \mathcal{S}_{\neg\phi}))$$

Proof. We will prove a stronger statement, namely that any set shattered by ϕ is also shattered by $\neg\phi$. We note that $\mathcal{S}_\phi = \{\bar{S} : S \in \mathcal{S}_{\neg\phi}$ Assume $A \subseteq M$ is a set shattered by ϕ. So for any $B \subseteq A$ there is an $S \in \mathcal{S}$ such that $B = A \cap S$. But then we know that $A \setminus B = A \cap \bar{S}$ and $\bar{S} \in S_{\neg\phi}$. Since this holds for any $B \subseteq A$ we see that for any $B \subseteq A$ there is a set $S \in \mathcal{S}_{\neg\phi}$ such that $B = S \cap A$. So A is shattered by $\neg\phi$. $\square$

Definition 2.4.5

Let $(X, \mathcal{S})$ be a set system. We define the **shatter function** $\pi_\mathcal{S} : \mathbb{N} \to \mathbb{N}$ *as:*

$$\pi_\mathcal{S}(n) := \max\{|\{S \cap A : S \in \mathcal{S}\}| : A \subseteq X \wedge |A| = n\}$$

Definition 2.4.6

Let $(X, \mathcal{S})$ be a set system with $\mathcal{S} \neq \emptyset$. The **VC-Dimension** *of $(X, \mathcal{S})$ is:*

$$VC((X, \mathcal{S})) = \sup\{n \in \mathbb{N} \cup \{\infty\} : X \text{ has a subset of size } n \text{ shattered by } \mathcal{S}\ \}$$

If $\mathcal{S} = \emptyset$ then we say that $VC(X, \mathcal{S}) = -\infty$.

Lemma 2.4.7

Every set system $(X, \mathcal{S})$ satisfies:

$$VC((X, \mathcal{S})) < n \Leftrightarrow \pi_\mathcal{S}(n) < 2^n$$

The following result by Norbert Sauer [42] and independently by Saharon Shelah [43] states that the shatter function is exponential up to the VC-dimension and if the VC-dimension is finite it behaves like a polynomial of degree at most the VC-dimension after that.

Lemma 2.4.8 (Sauer-Shelah)
If $\mathcal{S}$ has finite VC-dimension d then we have for all n:

$$\pi_{\mathcal{S}}(n) \leq \sum_{i=0}^{d} \binom{n}{i}$$

Definition 2.4.9
Let $(X, \mathcal{S})$ be a set system. Then the **VC-density** *of $(X, \mathcal{S})$ is:*

$$vc(X, \mathcal{S}) = \begin{cases} \inf\{r \in \mathbb{R}^+ : \pi_{\mathcal{S}}(n) \in \mathcal{O}(n^r)\} \ \textit{if}\ VC(\mathcal{S}) < \infty \\ \infty \ \textit{otherwise} \end{cases}$$

VC-density is often regarded as a better measure of complexity than VC-dimension. The two concepts are closely linked. Not only is VC-density bounded from above by VC-dimension, but we also have that VC-dimension is finite if and only if VC-density is finite.

2.5 Classes of finite set systems

Part of the work in this thesis focuses on ***classes*** of finite graphs, but the definitions of VC-dimension and VC-density given above define the concepts for a single structure. In particular the definition of VC-density given above assumes the structure to be infinite. In this section we give definitions that extend the definitions of VC-dimension, shatter function, and VC-density in a natural way to classes of finite models.

Definition 2.5.1

Let $\mathcal{C}$ be a class of finite set systems. The VC-dimension of $\mathcal{C}$ is then said to be:

$$VC(\mathcal{C}) = \sup\{VC(X, \mathcal{S}) : (X, \mathcal{S}) \in \mathcal{C}\}$$

Definition 2.5.2

Let $\mathcal{C}$ be a class of finite set systems. The shatter function of $\mathcal{C}$ is then said to be:

$$\pi_{\mathcal{C}}(n) = \max\{\pi_{\mathcal{S}}(n) \ : (X, \mathcal{S}) \in \mathcal{C}\}$$

Definition 2.5.3

Let $\mathcal{C}$ be a class of finite set systems. Then the **VC-density** *of $\mathcal{C}$ is:*

$$vc(\mathcal{C}) = \begin{cases} \inf\{r \in \mathbb{R}^+ : \pi_{\mathcal{C}}(n) \in O(n^r)\} \textit{ if } VC(\mathcal{C}) < \infty \\ \infty \textit{ otherwise} \end{cases}$$

In order to give the reader a better intuition on how VC-dimension of the edge relation works on classes of graphs we will now give a few theorems in increasing generality, i.e. each is a corollary to it's successor.

Recall that the set system for the edge relation on a given graph is $(V_G, \{N(v) : v \in V_G\})$.

Theorem 2.5.4

The VC-dimension of the edge relation on the class of all finite trees is 2.

Proof. We note that P_6 is a tree and has a shattered set of size 2 as shown in Figure 4. So it is sufficient to show that in no tree does there exist a set of size 3 that can be shattered by ϕ. Assume G is a tree whose vertex set has a subset $A = \{a, b, c\}$ shattered by ϕ. Then there exists a vertex v such that $N(v) \cap A = A$ i.e. v is a neighbour of each of the vertices a, b and c. Since A is shattered there also exists a $w \neq v$ such that $N(w) \cap A = \{a, b\}$ but then the subgraph $\{a, v, b, w\}$ induces a cycle in G in contradiction with G being a tree.

□

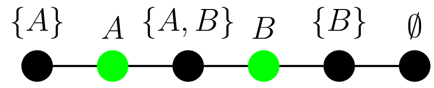


Figure 4: P_6 with the vertices of a shattered set marked green
Black vertices are labeled with the subsets of $\{A, B\}$ that are obtained by intersecting with their neighborhoods.

Theorem 2.5.5

The VC-dimenension of the edge relation on a class of finite graphs with tree-width at most k is at most $k+1$.

Proof. A complete graph K_n has tree-width $n-1$ so by [17, 12.3.6] we have that a class of graphs with tree-width at most k has K_{k+2} as a forbidden minor. It is sufficient to show that for $n \geq 3$, if a graph G contains a set of n vertices which is shattered by ϕ then G has K_n as a minor.

Let G be a graph in $\mathcal{C}$ with vertex set V and $A \subseteq V$ be a vertex set of size n shattered by ϕ.

Then for any $a, b \in A$ there exists a $v \in V$ such that $N(v) \cap A = \{a, b\}$. Now we have two cases either $v \in V \setminus A$ or $v \in A$.

If $v \in V \setminus A$ then by contracting either the edge from a to v or the edge from b to v we get a minor of G where a and b are neighbours.

If $v \in A$ then there exists a vertex w such that $N(w) = A \setminus \{v\}$. Since $N(v) \neq N(w)$ and $w \notin N(w)$ we have $w \in V \setminus A$. By contracting either the edge from a to w or the edge from b to w we get a minor of G where a and b are neighbours.

Note that contracting each of the edges as described above does not contract an edge between two vertices of A so we just need to confirm that all of our selected vertices are pairwise distinct. We observe that pairwise they have distinct neighborhoods and therefore must be distinct. So we end up with a graph minor G' of G where every pair of vertices in A are neighbours i.e. A induces a clique of size n in

G' so K_n is a graph minor of G $\square$

The proof of the following theorem closely mimics the work of Stéphan Thomassé and Nicolas Bousquet [10]. This was an independent discovery and the focus of the theorems is slightly different. We specify the depth at wich K_n appears as a minor whereas Thomassé and Bousquet look at formulae specifying greater distance than 1. In addition they are looking at 2-VC-dimension which is a strictly weaker notion than VC-dimension.

Theorem 2.5.6

Let G be a graph whose edge relation has VC-dimension at least n. Then G has K_n as a depth 1 minor.

Proof. Let G be a graph in $\mathcal{C}$ with vertex set V and $A \subseteq V$ be a vertex set of size n shattered by ϕ. We will show that $A \cup \bigcup_{a \in A} N(a)$ admits K_n as a depth 1 minor.

We will do this by finding, for all pairs of vertices $a, b \in A$ an edge in G such that if it is contracted a and b become neighbours.

We start by observing that for any $a, b \in A$ there exists a $v \in V$ such that $N(v) \cap A = \{a, b\}$ now we have two cases either $v \in V \setminus A$ or $v \in A$

If $v \in V \setminus A$ then by contracting either the edge from a to v or the edge from b to v we get a minor of G where a and b are neighbours.

If $v \in A$ then there exists a vertex w such that $N(w) = A \setminus \{v\}$. Since $N(v) \neq N(w)$ and $w \notin N(w)$ we have $w \in V \setminus A$. By contracting either the edge from a to w or the edge from b to w we get a minor of G where a and b are neighbours.

Note that contracting each of the edges as described above does not contract an edge between two vertices of A so we just need to confirm that all of our selected vertices are pairwise distinct. We observe that pairwise they have distinct neighborhoods and therefore must be distinct. So we end up with a graph minor G' of G

where every pair of vertices in A are neighbours i.e. A induces a clique of size n in G' so K_n is a depth 1 graph minor of G.

$\square$

At first glance the proof of Theorem 2.5.6 is very different than presented in [10] but at closer inspection they follow the same steps in their reasoning.

3 Graph Operations and VC-dimension of the edge relation

In this section we will explore the effects certain graph operations have the VC-dimension of the edge relation. Notably we show that vertex deletion can never increase the VC-dimension of the edge relation but the other operations explored can either increase or decrease the VC-dimension. The results are summarized in Table 1. The chapter is split into several subsections where each subsection contains the discussion about how one of the graph operations affects the VC-dimension of the edge relation.

Operation	Can Increase	Can Decrease
Vertex Deletion	✘	✔
Edge Deletion	✔	✔
Edge Contraction	✔	✔
Complement	✔	✔
Local Complement	✔	✔

Table 1: Different graph operations and what effect they can have on the VC-dimension of the edge relation.

3.1 Vertex deletion

Deleting a vertex u from a graph G gives us the induced subgraph of G on $V_G \setminus \{u\}$. Vertex deletion is a very basic operation on graphs that is equivalent to taking a substructure in the model-theoretic sense.

Lemma 3.1.1

Let G be a graph and $G' := G[V_G \setminus \{u\}]$ be the graph obtained from G by deleting a *single vertex u. Then $VC(G') \leq VC(G)$.*

Proof. For the edge relation we have $\mathcal{S} = \{N(v)|v \in V_G\}$ if we delete a vertex u the edge relation on the resulting subgraph G' will give us the class $\mathcal{S}' = \{N(v)\backslash\{u\}|v \in V_G\backslash\{u\}\}$. Now assume that $VC(G) < VC(G')$. Then there exists a set $A \subseteq V_G\backslash\{u\}$ such that $|A| > VC(G)$ and A is shattered by $\mathcal{S}'$. Since $u \notin A$ we have that $\forall S \subseteq V_G, A \cap S = A \cap (S \setminus \{u\})$. So $P(A) = \{A \cap S | S \in \mathcal{S}'\} \subseteq \{A \cap S | S \in \mathcal{S}\}$ which means that $\mathcal{S}$ shatters A in contradiction with $|A| > VC(G)$. □

We note that deleting vertices can reduce the VC-dimension of the edge relation. For example observe that on a graph on three vertices and just one edge the edge relation has VC-dimension 1 but removing either of the non-isolated vertices gives us a graph where the edge relation has VC-dimension 0.

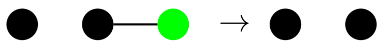


Figure 5: A graph on three vertices and one edge and the result of deleting the vertex coloured green.

3.2 Edge deletion

Deleting an edge from a graph G gives us a subgraph which is not a substructure of G; we simply remove two tuples (u, v) and (v, u) from the edge relation.

In Figure 6 we see an example of a graph where the edge relation has VC-dimension 1 but deleting the edge in green gives us a graph where the edge relation has VC-dimenison 2.

We observe that on a graph on three vertices and just one edge the edge relation has VC-dimension 1 but removing the edge gives us a graph where set system for the edge relation only contains $\emptyset$ and thus has VC-dimension 0.

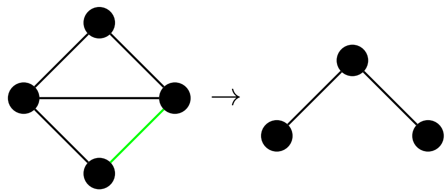


Figure 6: A graph and the result of deleting the edge coloured green.

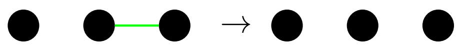


Figure 7: A graph and the result of deleting its edge.

3.3 Edge Contraction

When we contract an edge (u, v) in a graph G we merge the end vertices into one i.e. we remove u and v from G and add in a new vertex w and edges between w and all the neighbours of v, and w and all the neighbours of u. This can result in a substructure of G but will not necessarily do so.

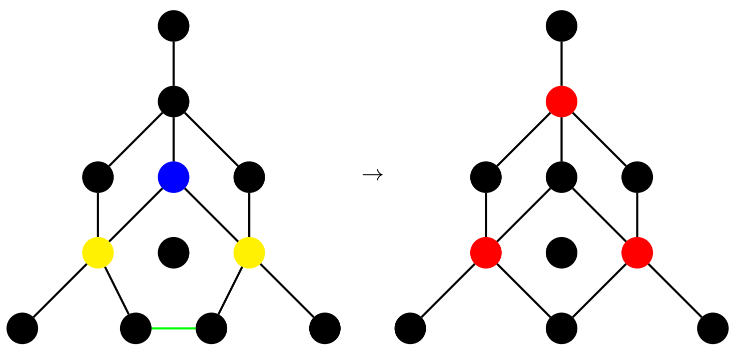


Figure 8: A graph and the result of contracting the edge marked in green.

In Figure 8 we see an example of a graph where the VC-dimension of the edge relation is 2. We can see that it is at most 2 since any set shattered by the edge relation must be contained in the neigbourhood of some vertex, there is only one vertex with degree greater than 2 namely the blue vertex, and his neighbourhood is not shattered since the two yellow vertices have no other neighbours in common. After contracting the edge in green we get a graph where the vertices marked red form a shattered set of size 3, so the edge relation has VC-dimension 3.

We observe that on a graph on three vertices and just one edge the edge relation has VC-dimension 1 since a singleton set containing either of the connected vertices will be shattered, but contracting the edge gives us a graph where the edge relation has VC-dimension 0 since it's set system only contains $\emptyset$. So contracting an edge can also decrease the VC-dimension of the edge relation.

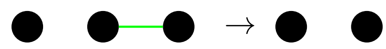


Figure 9: A graph and the result of contracting the edge drawn in green.

3.4 Complement

The operation of taking a graph complement is self inverse so if it can increase the VC-dimension it must also be able to decrease it so it is sufficient to show one direction. At first glance it may seem strange that taking the complement of a graph can change the VC-dimension of the edge relation since by, Lemma 2.4.4, negating a formula doesn't. There is however subtle difference between a graph complement and negating the edge relation.

The key difference is that for any graph G and its complement G' we have that G and G' are both simple undirected graphs without loops so for any vertex $v \in V_G$ we have that $G \models \neg Evv$ and $G' \models \neg Evv$. In Figure 10 we see an example of a graph and its complement in which the edge relation has differing VC-dimension.

In the graph on the left the bottom two vertices form a shattered set of size 2. In the graph on the right there is no shattered set of size 2.

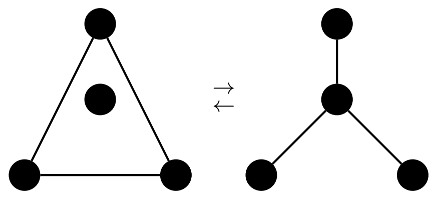


Figure 10: A graph and its complement.

3.5 Local Complement

To take a local complement around a vertex v in a graph G we take the neighbourhood of v, $N(v)$ as an induced subgraph and replace it with its complement [30].

Much like complement taking the local complement is self inverse so it is sufficient to show that a change can occur; we need not worry about its direction.

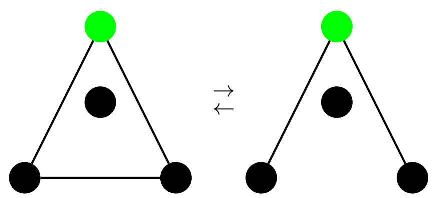


Figure 11: A graph and the result of taking a local complement around the vertex coloured green.

In Figure 11 we see two graphs obtained from one another by taking a local complement. The left one has a VC-dimension of 2 for the edge relation and the one on the right has a VC-dimension of 1 for the edge relation showing that taking local complements can indeed change the VC-dimension of the edge relation.

4 Johnson Graphs

In this chapter we will introduce the notion of a Johnson graph and give a bound on the VC-dimension and VC-density of the edge relation in such graphs. Johnson graphs are a large class of highly symmetrical graphs that arise in a great many applications. For example they provide a good example of a vertex transitive regular graph. The study of Johnson graphs has become more important in recent years as they are recognized as the only barrier to an effective canonical partitioning in László Babai's algorithm for resolving graph isomorphism in quasi-polynomial time [6][5].

4.1 Introduction to Johnson graphs

In this section we will give an introduction to Johnson graphs. We restate their definition and give some lemmas about their local structure to get an intuition about what they look like.

Definition 4.1.1

Let n and k be natural numbers $n \geq k$. The **Johnson graph** *$J(n, k)$ is a graph whose vertices correspond to the k-element subsets of a set X of size n and two vertices are adjacent if their corresponding sets intersect in all but one element. i.e. their symmetric difference has size 2. We call X the* **underlying set** *of $J(n, k)$.*

Johnson graphs are highly symmetric for example they are distance transitive, and their automorphism group is the full symmetric group on their underlying set.

Lemma 4.1.2 ([14, §2])

Let u and v be vertices in a Johnson graph. Then the distance $d(u, v) = \frac{|u \Delta v|}{2}$.

Definition 4.1.3

Let k be a fixed constant. We call the class $\{J(n,k)|n \in \mathbb{N}\}$ the class of Johnson graphs on k-sets, denoted $\mathcal{J}_k$.

In [1] it is shown that the theory of a nowhere dense class of graphs can never have the independence property and thus every formula will have finite VC-dimension.

In any Johnson graph $J(n,k)$ consider a $k-1$ subset S of the underlying set and observe that vertex set $\{v \in V_{J(n,k)}|S \subset v\}$ forms a clique of size $n-k+1$. Therefore it is clear that the class of Johnson graphs on k-sets is somewhere dense for any fixed k.

It is however known that for any fixed k the class of Johnson graphs on k-sets lacks the independence property[3, 1.1]. This is obtained by biinterpretability with a pure set. This relies on a fixed k but most results in this work do not require k to be fixed.

Being somewhere dense and without the independence property makes them particularly interesting classes of graphs to study with regard to VC-dimension and VC-density. Their theory lies somewhere between the notion of nowhere dense and having infinite VC-dimension for some formula.

The following three lemmas give us an insight into what the intersections of neighbourhoods look like. As we have noted before any set shattered by the edge relation must be contained the neighbourhood of some vertex and thus any subset of a set shatterd by edge relation is an intersection of two neighbourhoods.

Lemma 4.1.4

Let v be a vertex in the Johnson graph $J(n,k)$. Then $N(v)$ induces the rook's graph $R(k,n-k)$ as a subgraph of $J(n,k)$.

Proof. Let v be a vertex in the Johnson graph $J(n,k)$ with the underlying set $[1,n] \cap \mathbb{N}$ and without loss of generality assume $v = [1,k] \cap \mathbb{N}$. Every vertex in $N(v)$

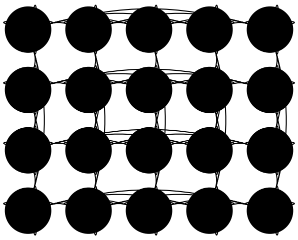


Figure 12: The neighbourhood of $\{1, 2, 3, 4\}$ in $J(10, 4)$.

has the form $(v \setminus \{a\}) \cup \{x\}$ where $a \in v$ and $x \in [k+1, n] \cap \mathbb{N}$. The mapping $(v \setminus \{a\}) \cup \{x\} \mapsto (a, x-k)$ is a graph isomorphism $J(n,k)[N(v)] \to R(k, n-k)$.

$\square$

Lemma 4.1.5

Let v and w be vertices in a Johnson graph with $d(v,w) = 1$. Let a and x be such that $w = (v \setminus \{a\}) \cup \{x\}$. Then we have $u \in N(v) \cap N(w)$ if and only if $u = (v \setminus \{c\}) \cup \{z\}$ with exactly one of $c = a$ or $z = x$.

Proof. Assume $u \in N(v) \cap N(w)$. Then since $d(v,u) = 1$ we must have $u = (v \setminus \{c\}) \cup \{z\}$ for some c and z. Now assume $c \neq a$ and $z \neq x$. Then we have $u \triangle w = \{a, c, x, z\}$ so $|u \triangle v| = 4$ contradicting that $d(u,w) = 1$. So we must have either $c = a$ or $z = x$.

Conversely assume $u = (v \setminus \{a\}) \cup \{z\}$ with $x \neq z$. Then $u \cap v = v \setminus \{a\}$ which has size $k-1$ so $u \in N(v)$. Also $u \cap w = v \setminus \{a\}$ which has size $k-1$ so $u \in N(w)$. Thus we have $u \in N(v) \cap N(w)$.

Assume $u = (v \setminus \{c\}) \cup \{x\}$. Then $u \cap v = v \setminus \{c\}$ which has size $k-1$ so $u \in N(v)$. Also $u \cap w = (v \setminus \{a, c\}) \cup \{x\}$ which has size $k-1$ so $u \in N(w)$. Thus we have $u \in N(v) \cap N(w)$.

Note that if we have both $c = a$ and $z = x$ then $u = w$ in contradiction with Euw. $\square$

Lemma 4.1.6

Let v and w be vertices in a Johnson graph with $d(v, w) = 2$. Let $a, b, x,$ and y be such that $w = (v \setminus \{a, b\}) \cup \{x, y\}$. Then we have $u \in N(v) \cap N(w)$ if and only if $u = (v \setminus \{c\}) \cup \{z\}$ with $c \in \{a, b\}$ and $z \in \{x, y\}$.

Proof. Assume $u \in N(v) \cap N(w)$. Then since $d(v, u) = 1$ we must have $u = (v \setminus \{c\}) \cup \{z\}$ for some $c \in v$ and $z \notin v$.

Now assume $c \notin \{a, b\}$. Then we have $u \triangle w \supseteq \{a, b, c\}$ contradicting that $|u \triangle w| = 2$.

Similarly $z \in \{x, y\}$ as otherwise we have $u \triangle w \supseteq \{x, y, z\}$ in contradiction with $|u \triangle w| = 2$. So we must have $c \in \{a, b\}$ and $z \in \{x, y\}$.

Conversely assume $u = (v \setminus \{c\}) \cup \{z\}$ with $c \in \{a, b\}$ and $z \in \{x, y\}$. Assume without loss of generality $u = (v \setminus \{a\}) \cup \{x\}$. Then $u \cap v = v \setminus \{a\}$ which has size $k - 1$ so $u \in N(v)$. Also $u \cap w = (v \setminus \{a, b\}) \cup \{x\}$ which has size $k - 1$ so $u \in N(w)$. Thus we have $u \in N(v) \cap N(w)$. □

We now gather a key insight obtained from lemmas 4.1.4, 4.1.5, and 4.1.6 into the following lemma.

Lemma 4.1.7

Let u and v be vertices in the Johnson graph $J(m, k)$ then

$$|N(u) \cap N(v)| = \begin{cases} k(m-k) & \text{if } d(u,v) = 0 \\ m-1 & \text{if } d(u,v) = 1 \\ 4 & \text{if } d(u,v) = 2 \\ 0 & \text{if } d(u,v) \geq 3 \end{cases}$$

Proof. When $d(u, v) = 0$ then $u = v$ so $N(u) \cap N(u) = N(u)$ which by Lemma 4.1.4 is the k by $m - k$ rook's graph which has $k(m - k)$ vertices. When $d(u, v) = 1$ then by Lemma 4.1.5 $N(u) \cap N(v)$ is the union of a single row and a single column from the rook's graph that is $N(u)$. They contain k and $m-k$ vertices and intersect in exactly one vertex so their union contain $k+(m-k)-1$ vertices. When $d(u,v)=2$ then by Lemma 4.1.6 there are exactly four vertices in $N(v) \cap N(u)$. Finally, in the case when $d(u,v) \geq 3$, by the definition of distance and neighbourhood in graphs we have that $N(u) \cap N(v) = \emptyset$. $\square$

4.2 VC-dimension of the edge relation

In this section we will prove our main results regarding the VC-dimension of the edge relation on finite Johnson graphs. We start by introducing a lemma that helps us greatly cut down on cases we need to check.

Lemma 4.2.1

Let A be a set of vertices in a Johnson graph shattered by the edge relation and assume $|A| \geq 4$. Then there do not exist three vertices in A pairwise at distance 2 from each other.

Proof. Let v be a vertex such that $A \subseteq N(v)$ and A contains three vertices that are pairwise of distance 2 from each other. That is to say we have $(v \setminus \{a\}) \cup \{x\} \in A$,$(v \setminus \{b\}) \cup \{y\} \in A$,$(v \setminus \{c\}) \cup \{z\} \in A$ where a, b, c, x, y, z are all distinct.

Let w be a vertex such that $N(w) \cap A = \{(v \setminus \{a\}) \cup \{x\}, (v \setminus \{b\}) \cup \{y\}, (v \setminus \{c\}) \cup \{z\}\}$.

If $d(v,w) = 1$ we can write $w = (v \setminus \{a_1\}) \cup \{x_1\}$. By Lemma 4.1.5 we have that $(v \setminus \{a\}) \cup \{x\} \in N(w)$ gives us $a_1 = a$ or $x_1 = x$.

Assume $a_1 = a$. Then since $(v \setminus \{b\}) \cup \{y\} \in N(w)$ we must have $x_1 = y$, so we have $w = (v \setminus \{a\}) \cup \{y\}$. However $(v \setminus \{c\}) \cup \{z\} \notin N((v \setminus \{a\}) \cup \{y\})$ in contradiction to $N(w) \cap A = \{(v \setminus \{a\}) \cup \{x\}, (v \setminus \{b\}) \cup \{y\}, (v \setminus \{c\}) \cup \{z\}\}$.

Alternatively assume $x_1 = x$. Then since $(v \setminus \{b\}) \cup \{y\} \in N(w)$ we have $a_1 = b$

so we have $w = (v \setminus \{b\}) \cup \{x\}$. However $(v \setminus \{c\}) \cup \{z\} \notin N((v \setminus \{b\}) \cup \{x\})$ in contradiction to $N(w) \cap A = \{(v \setminus \{a\}) \cup \{x\}, (v \setminus \{b\}) \cup \{y\}, (v \setminus \{c\}) \cup \{z\}\}$.

So we must have $d(v, w) = 2$ and write $w = (v \setminus \{a_1, a_2\}) \cup \{x_1, x_2\}$. By Lemma 4.1.6 we know that since $(v \setminus \{a\}) \cup \{x\} \in N(w)$ we have $a \in \{a_1, a_2\}$ and $x \in \{x_1, x_2\}$. Without loss of generality we assume $a_1 = a$ and $x_1 = x$.

Similarly, since $(v \setminus \{b\}) \cup \{y\} \in N(w)$, we have $b \in \{a, a_2\}$ and $y \in \{x, x_2\}$ so we have $w = (v \setminus \{a, b\}) \cup \{x, y\}$. But then $(v \setminus \{c\}) \cup \{z\} \notin N(w)$, a contradiction.

□

Theorem 4.2.2

The VC*-dimension of the edge relation in a Johnson graph is at most* 4.

Proof. The proof goes through a series of cases demonstrating that no vertex set of size 5 in a Johnson graph can be shattered. We rely on the fact that every set A shattered by the edge relation must have $A \subseteq N(v)$ for some vertex v and that every subset of a shattered set is also shattered which allows us to drastically reduce the number of cases we need to check.

Observe that in $J(m, k)$ we can pick an element of the underlying set and the set of all vertices not containing that element induces $J(m-1, k)$ as a subgraph of $J(m, k)$ and the set of all vertices containing that element induces $J(m-1, k-1)$. Thus we can assume m and k to be arbitrarily large and since by Lemma 3.1.1 taking induced subgraphs can only decrease the VC-dimension, our argument then holds for all m and k.

We will start by computing the number of configurations that can be obtained by picking 4 vertices out of $N(v)$. Formally the configurations, which we label Case I - Case XVI, are the orbits of the group of automorphisms of $J(m, k)$ fixing v in its action on 4-element subsets of $N(v)$. There are 16 such orbits and out of those 8 are shattered by the edge relation and 8 are not. We will then go through them one by one. For those cases that are not shattered by the edge relation we will give a proof of why they are not shattered, and in the shattered cases, we will demonstrate that whichever way we choose a fifth vertex to add to those collections we will always end up with a set that is not shattered by the edge relation.

Let A be a set of vertices in a Johnson graph with $|A| = 4$, and v be a vertex such that $A \subseteq N(v)$.

Let $v_i = (v \setminus \{a_i\}) \cup \{x_i\}$ for $i \in \{1, 2, 3, 4\}$ be the four vertices of A . Let $\sim_x$ be the equivalence relation $v_i \sim_x v_j$ if and only if $x_i = x_j$ and $\sim_a$ be the equivalence relation $v_i \sim_a v_j$ if and only if $a_i = a_j$. Note that if we have $v_i \sim_x v_j$ and $v_i \sim_a v_j$ then $v_i = v_j$ and by our assumption that the four vertices are distinct we have $i = j$.

There are 5 ways, up to permutation, to split a set of size 4 into equivalence classes. These correspond to the ways of summing up to 4. Not every combination of equivalence classes for $\sim_a$ and $\sim_x$ is possible. We will now look at each of the ways $\sim_x$ can split A and give the available ways for $\sim_a$ to split A. Note that the equivalence classes of $\sim_a$ and $\sim_x$ correspond to the columns and rows of the rook's graph induced by $N(v)$. We now look at each of the different ways of summing up to 4.

4 In this case we have $x_1 = x_2 = x_3 = x_4$ and we therefore must have $a_i \neq a_j$ whenever $i \neq j$. This means $\sim_a$ has 4 equivalence classes of size 1. This gives us Case IX.

$3 + 1$ Without loss of generality we assume $x_1 = x_2 = x_3 \neq x_4$. Then there are two ways for $\sim_a$ to split A into equivalence classes. It can either have $2 + 1 + 1$ or $1 + 1 + 1 + 1$ as the partition. In the former case we can assume without loss of generality that $a_1 = a_4$ and this yields Case X. In the latter we have $a_i \neq a_j$ whenever $i \neq j$ and this gives us Case I.

$2 + 2$ Without loss of generality we assume $x_1 = x_2 \neq x_3 = x_4$. Note that this implies $a_1 \neq a_2$ and $a_3 \neq a_4$. We now have three ways that $\sim_a$ can split A

into equivalence classes.

2 + 2 We assume without loss of generality $a_1 = a_3$ and $a_2 = a_4$, giving us Case II.

2 + 1 + 1 We assume without loss of generality $a_1 = a_3 \neq a_2$, $a_1 \neq a_4$ and $a_2 \neq a_4$. This gives us Case XI.

1 + 1 + 1 + 1 We have $a_i \neq a_j$ whenever $i \neq j$, yielding Case XII.

$2 + 1 + 1$ Without loss of generality we assume $x_1 = x_2 \neq x_3 \neq x_4$ and additionally assume $x_4 \neq x_1$. We can have four ways for $\sim_a$ to split A into equivalence classes.

3 + 1 Without loss of generality we can assume $a_1 = a_3 = a_4 \neq a_2$. This is Case $XIII$.

2 + 2 Without loss of generality we can assume $a_1 = a_3$ and $a_2 = a_4$. This is Case XIV.

2 + 1 + 1 In this instance we have two ways of grouping the vertices with $\sim_a$ that are not equivalent with relabeling.

By making $a_1 = a_3$ we get Case III.

By making $a_3 = a_4$ we get Case IV.

1 + 1 + 1 + 1 We have $a_i \neq a_j$ whenever $i \neq j$, giving us Case V.

$1 + 1 + 1 + 1$ Here we have x_1, x_2, x_3, x_4 all distinct. We can have four ways for $\sim_a$ to split A into equivalence classes.

4 Here we have $a_1 = a_2 = a_3 = a_4$ which is Case XV.

3 + 1 Without loss of generality we may assume $a_1 = a_2 = a_3 \neq a_4$, giving us Case VI.

2 + 2 Without loss of generality we can assume $a_1 = a_2 \neq a_3 = a_4$ which yields Case XVI.

2 + 1 + 1 Without loss of generality we can assume $a_1 = a_2 \neq a_3 \neq a_4$ and $a_1 \neq a_4$ which gives us Case VII.

1 + 1 + 1 + 1 We have $a_i \neq a_j$ whenever $i \neq j$. This gives us Case $VIII$.

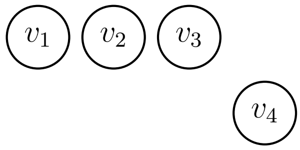
We now have 16 cases and will go through them one by one demonstrating that in each case either A can not be shattered or that adding a fifth vertex to A will always result in a set that cannot be shattered. When proving a configuration does not shatter we have to prove that there exists a subset $B \subseteq A$ such that there exists no w for which $N(w) \cap A = B$. In all cases we will have $B \neq A$ so we have to check the cases $d(v, w) = 1$ and $d(v, w) = 2$.

When we have to add a fifth vertex we will have to check every possible combination of $\sim_a$ and $\sim_x$ between the fifth vertex and the previous four vertices, up to a relabeling of the x_i and a_i. In these subcases we will often simply observe that the fifth vertex along with 3 of the original 4 vertices is identical to a case which is separately proved not to shatter.

We will for each case give a diagram showing those vertices of $N(v)$ we are taking to be in A arranged in rows and columns as they would be in the rook graph induced by $N(v)$. In those cases where we do not give a proof that the four vertices selected cannot form a shattered set we will have a choice of how to pick our fifth vertex to add to A. The fifth vertex we will label with the associated subcase rather than v_5 to avoid confusion and save space on the diagrams. Note that row permutations just correspond to relabeling of the equivalence classes of $\sim_x$ and column permutations correspond to relabeling of equivalence classes of $\sim_a$.

Case I

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$
$$v_3 = (v \setminus \{a_3\}) \cup \{x_1\}$$
$$v_4 = (v \setminus \{a_4\}) \cup \{x_2\}$$



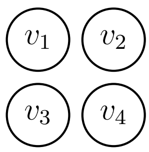
Let w be such that $N(w) \cap A = \{v_2, v_3, v_4\}$. We have 2 cases.

(a) $w = (v \setminus \{a\}) \cup \{x\}$. Since we have to exclude v_1 from $N(w)$ we must by Lemma 4.1.5 have that $a \neq a_1$ and $x \neq x_1$. So in order to have $v_2 \in N(w)$ we must have $a = a_2$ and in order to have $v_3 \in N(w)$ we must have $a = a_3$. But then $a_2 = a_3$ in contradiction with $v_1 \neq v_2$.

(b) $w = (v \setminus \{a, b\}) \cup \{x, y\}$. From Lemma 4.1.6 we get that $v_2 \in N(w)$ yields $a_2 \in \{a, b\}$ and $x_1 \in \{x, y\}$; $v_3 \in N(w)$ yields $a_3 \in \{a, b\}$ and $x_1 \in \{x, y\}$; $v_4 \in N(w)$ yields $a_4 \in \{a, b\}$ and $x_2 \in \{x, y\}$. Thus $\{a_2, a_3, a_4\} \subseteq \{a, b\}$, contradicting that a_2, a_3, a_4 are all distinct.

Case II

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$
$$v_3 = (v \setminus \{a_1\}) \cup \{x_2\}$$
$$v_4 = (v \setminus \{a_2\}) \cup \{x_2\}$$



Let w be such that $N(w) \cap A = \{v_2, v_3, v_4\}$. We have 2 cases.

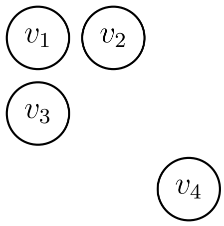
(a) $w = (v \setminus \{a\}) \cup \{x\}$. Since $v_4 \in N(w)$ we have $w \neq v_1$. Since we have to exclude v_1 from $N(w)$, by Lemma 4.1.5 we must have that $a \neq a_1$

and $x \neq x_1$. So in order to have $v_2 \in N(w)$ we must have $a = a_2$ and in order to have $v_3 \in N(w)$ we must have $x = x_2$. But then $w = v_4$ in contradiction with $v_4 \in N(w)$.

(b) $w = (v \setminus \{a, b\}) \cup \{x, y\}$. From Lemma 4.1.6 we get that $v_3 \in N(w)$ yields $a_1 \in \{a, b\}$ and $x_2 \in \{x, y\}$; $v_2 \in N(w)$ yields $a_2 \in \{a, b\}$ and $x_1 \in \{x, y\}$; hence $w = (v \setminus \{a_1, a_2\}) \cup \{x_1, x_2\}$, contradicting that $v_1 \notin N(w)$.

Case III

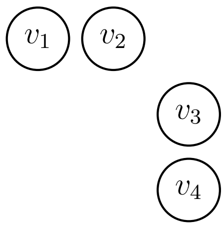
$$
\begin{aligned}
v_1 &= (v \setminus \{a_1\}) \cup \{x_1\} \\
v_2 &= (v \setminus \{a_2\}) \cup \{x_1\} \\
v_3 &= (v \setminus \{a_1\}) \cup \{x_2\} \\
v_4 &= (v \setminus \{a_3\}) \cup \{x_3\}
\end{aligned}
$$



The vertices v_2, v_3, v_4 are at distance 2 from each other so by Lemma 4.2.1 A is not shattered.

Case IV

$$
\begin{aligned}
v_1 &= (v \setminus \{a_1\}) \cup \{x_1\} \\
v_2 &= (v \setminus \{a_2\}) \cup \{x_1\} \\
v_3 &= (v \setminus \{a_3\}) \cup \{x_2\} \\
v_4 &= (v \setminus \{a_3\}) \cup \{x_3\}
\end{aligned}
$$



Let w be such that $N(w) \cap A = \{v_2, v_3, v_4\}$. We have 2 cases.

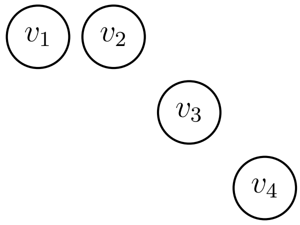
(a) $w = (v \setminus \{a\}) \cup \{x\}$. Since we have to exclude v_1 from $N(w)$ by Lemma 4.1.5 we must have that $a \neq a_1$ and $x \neq x_1$. So in order to have $v_2 \in N(w)$ we must have $a = a_2$ and in order to have $v_3 \in N(w)$ we must have $x = x_2$.

But then $w = (v \setminus \{a_2\}) \cup \{x_2\}$ in contradiction with $v_4 \in N(w)$.

(b) $w = (v \setminus \{a, b\}) \cup \{x, y\}$. From Lemma 4.1.6 we get that $v_2 \in N(w)$ yields $a_2 \in \{a, b\}$ and $x_1 \in \{x, y\}$; $v_3 \in N(w)$ yields $a_3 \in \{a, b\}$ and $x_2 \in \{x, y\}$. Thus $w = (v \setminus \{a_2, a_3\}) \cup \{x_1, x_2\}$ in contradiction with $v_4 \in N(w)$.

Case V

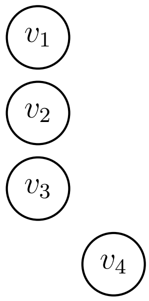
$$
\begin{aligned}
v_1 &= (v \setminus \{a_1\}) \cup \{x_1\} \\
v_2 &= (v \setminus \{a_2\}) \cup \{x_1\} \\
v_3 &= (v \setminus \{a_3\}) \cup \{x_2\} \\
v_4 &= (v \setminus \{a_4\}) \cup \{x_3\}
\end{aligned}
$$



The vertices v_2, v_3, v_4 are at distance 2 from each other so by Lemma 4.2.1 A is not shattered.

Case VI

$$
\begin{aligned}
v_1 &= (v \setminus \{a_1\}) \cup \{x_1\} \\
v_2 &= (v \setminus \{a_1\}) \cup \{x_2\} \\
v_3 &= (v \setminus \{a_1\}) \cup \{x_3\} \\
v_4 &= (v \setminus \{a_2\}) \cup \{x_4\}
\end{aligned}
$$



Let w be such that $N(w) \cap A = \{v_2, v_3, v_4\}$.

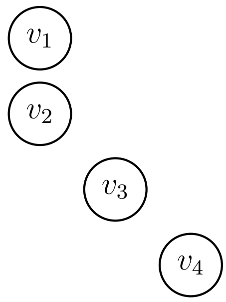
We have 2 cases.

(a) $w = (v \setminus \{a\}) \cup \{x\}$. Since we have to exclude v_1 from $N(w)$ by Lemma 4.1.5 we must have that $a \neq a_1$ and $x \neq x_1$ so in order to have $v_2 \in N(w)$ we must have $x = x_2$ and in order to have $v_3 \in N(w)$ we must have $x = x_3$. But then $x_2 = x_3$, in contradiction with $v_1 \neq v_2$.

(b) $w = (v \setminus \{a, b\}) \cup \{x, y\}$. From Lemma 4.1.6 we get that $v_2 \in N(w)$ yields $a_1 \in \{a, b\}$ and $x_2 \in \{x, y\}$; $v_3 \in N(w)$ yields $a_1 \in \{a, b\}$ and $x_3 \in \{x, y\}$; $v_4 \in N(w)$ yields $a_2 \in \{a, b\}$ and $x_4 \in \{x, y\}$. Thus we have $\{x_2, x_3, x_4\} \subseteq \{x, y\}$, in contradiction with x_2, x_3, x_4 all being distinct.

Case VII

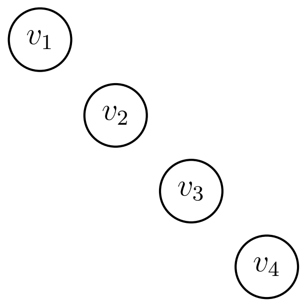
$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_1\}) \cup \{x_2\}$$
$$v_3 = (v \setminus \{a_2\}) \cup \{x_3\}$$
$$v_4 = (v \setminus \{a_3\}) \cup \{x_4\}$$



The vertices v_2, v_3, v_4 are at distance 2 from each other so by Lemma 4.2.1 A is not shattered.

Case VIII

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_2\}$$
$$v_3 = (v \setminus \{a_3\}) \cup \{x_3\}$$
$$v_4 = (v \setminus \{a_4\}) \cup \{x_4\}$$

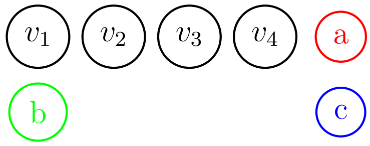


The vertices v_2, v_3, v_4 are at distance 2 from each other so by Lemma 4.2.1 A is not shattered.

The remaining cases shatter, so we look at the different ways a fifth vertex can be added to the collection and demonstrate that the result cannot be a shattered set.

Case IX

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$
$$v_3 = (v \setminus \{a_3\}) \cup \{x_1\}$$
$$v_4 = (v \setminus \{a_4\}) \cup \{x_1\}$$



This case shatters so we take a closer look at what configurations are obtainable by adding a fifth vertex.

a $v_5 = (v \setminus \{a_5\}) \cup \{x_1\}$. Let w be such that $N(w) \cap A = \{v_1, v_2, v_3\}$. Observe that $w \neq v_4$ since $v_5 \in N(v_4)$ so we will need an alternative w. We have 2 cases: either $d(v, w) = 1$ or $d(v, w) = 2$.

Let $w = (v \setminus \{a\}) \cup \{x\}$. Since we have to exclude v_5 from $N(w)$ then by Lemma 4.1.5 we cannot have $x = x_1$. So in order to have $v_1 \in N(w)$ we must have $a = a_1$ but then in order to have $v_2 \in N(w)$ we must have $x = x_1$, a contradiction.

Let $w = (v \setminus \{a, b\}) \cup \{x, y\}$. In order to have $v_1 \in N(w), v_2 \in N(w)$ and $v_3 \in N(w)$, Lemma 4.1.6 gives us $\{a_1, a_2, a_3\} \subseteq \{a, b\}$, a contradiction.

b $v_5 = (v \setminus \{a_1\}) \cup \{x_2\}$. Here v_2, v_3, v_4, v_5 form case I.

c $v_5 = (v \setminus \{a_5\}) \cup \{x_2\}$. Here v_1, v_2, v_3, v_5 form case I.

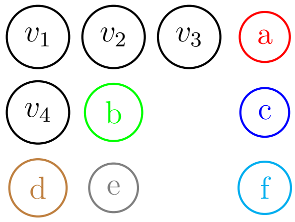
Case X

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$

$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$

$$v_3 = (v \setminus \{a_3\}) \cup \{x_1\}$$

$$v_4 = (v \setminus \{a_1\}) \cup \{x_2\}$$



a $v_5 = (v \setminus \{a_4\}) \cup \{x_1\}$. Then v_2, v_3, v_4, v_5 forms case I.

b $v_5 = (v \setminus \{a_2\}) \cup \{x_2\}$. Then v_1, v_2, v_4, v_5 forms case II.

c $v_5 = (v \setminus \{a_4\}) \cup \{x_2\}$. Then v_1, v_2, v_3, v_5 forms case I.

d $v_5 = (v \setminus \{a_1\}) \cup \{x_3\}$. Then v_2, v_3, v_4, v_5 forms case IV.

e $v_5 = (v \setminus \{a_2\}) \cup \{x_3\}$. Then v_1, v_3, v_4, v_5 forms case III.

f $v_5 = (v \setminus \{a_4\}) \cup \{x_3\}$. Then v_1, v_2, v_4, v_5 forms case III.

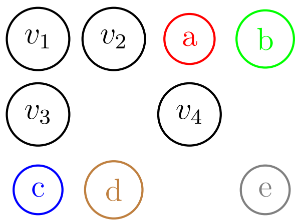
Case XI

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$

$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$

$$v_3 = (v \setminus \{a_1\}) \cup \{x_2\}$$

$$v_4 = (v \setminus \{a_3\}) \cup \{x_2\}$$



a $v_5 = (v \setminus \{a_3\}) \cup \{x_1\}$. Here v_1, v_3, v_4, v_5 form case II.

b $v_5 = (v \setminus \{a_4\}) \cup \{x_1\}$. Here v_1, v_2, v_4, v_5 form case I.

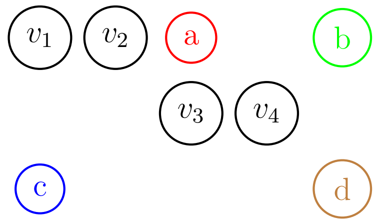
c $v_5 = (v \setminus \{a_1\}) \cup \{x_3\}$. Here v_2, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

d $v_5 = (v \setminus \{a_2\}) \cup \{x_3\}$. Here v_1, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

e $v_5 = (v \setminus \{a_4\}) \cup \{x_3\}$ In this case v_1, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

Case XII

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$
$$v_3 = (v \setminus \{a_3\}) \cup \{x_2\}$$
$$v_4 = (v \setminus \{a_4\}) \cup \{x_2\}$$



a $v_5 = (v \setminus \{a_3\}) \cup \{x_1\}$. Then v_1, v_2, v_4, v_5 form case I.

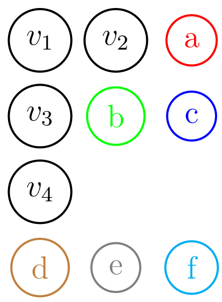
b $v_5 = (v \setminus \{a_5\}) \cup \{x_1\}$. Then v_1, v_2, v_3, v_5 form case I.

c $v_5 = (v \setminus \{a_1\}) \cup \{x_3\}$. In this case v_1, v_3, v_4, v_5 form case IV.

d $v_5 = (v \setminus \{a_5\}) \cup \{x_3\}$. In this case v_1, v_3, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

Case XIII

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$
$$v_3 = (v \setminus \{a_1\}) \cup \{x_2\}$$
$$v_4 = (v \setminus \{a_1\}) \cup \{x_3\}$$



a $v_5 = (v \setminus \{a_3\}) \cup \{x_1\}$. Then v_2, v_3, v_4, v_5 form case IV.

b $v_5 = (v \setminus \{a_2\}) \cup \{x_2\}$. Then v_1, v_2, v_3, v_5 form case II.

c $v_5 = (v \setminus \{a_3\}) \cup \{x_2\}$. In this case v_2, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

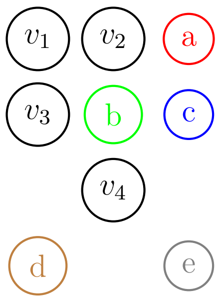
d $v_5 = (v \setminus \{a_1\}) \cup \{x_4\}$. Here v_2, v_3, v_4, v_5 form case VI.

e $v_5 = (v \setminus \{a_2\}) \cup \{x_4\}$. In this case v_1, v_3, v_4, v_5 form case VI.

f $v_5 = (v \setminus \{a_3\}) \cup \{x_4\}$. In this case v_2, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

Case XIV

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_2\}) \cup \{x_1\}$$
$$v_3 = (v \setminus \{a_1\}) \cup \{x_2\}$$
$$v_4 = (v \setminus \{a_2\}) \cup \{x_3\}$$



a $v_5 = (v \setminus \{a_3\}) \cup \{x_1\}$. In this case v_3, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

b $v_5 = (v \setminus \{a_2\}) \cup \{x_2\}$. Then v_1, v_2, v_3, v_5 form case II.

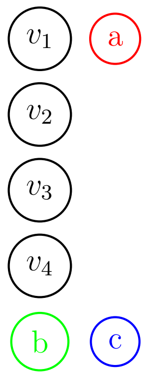
c $v_5 = (v \setminus \{a_3\}) \cup \{x_2\}$. In this case v_1, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

d $v_5 = (v \setminus \{a_1\}) \cup \{x_4\}$. In this case v_1, v_3, v_4, v_5 form case VI.

e $v_5 = (v \setminus \{a_3\}) \cup \{x_4\}$. In this case v_3, v_4, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

Case XV

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_1\}) \cup \{x_2\}$$
$$v_3 = (v \setminus \{a_1\}) \cup \{x_3\}$$
$$v_4 = (v \setminus \{a_1\}) \cup \{x_4\}$$



a $v_5 = (v \setminus \{a_2\}) \cup \{x_1\}$. Here v_2, v_3, v_4, v_5 form case VI.

b $v_5 = (v \setminus \{a_1\}) \cup \{x_5\}$. Let w be such that $N(w) \cap A = \{v_1, v_2, v_3\}$. Observe that $w \neq v_4$ since $v_5 \in N(v_4)$ so we will need an alternative w. We have 2 cases: either $d(v, w) = 1$ or $d(v, w) = 2$.

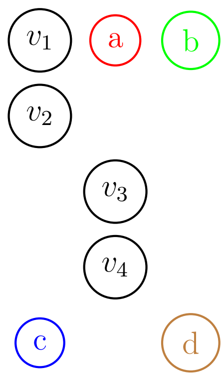
Let $w = (v \setminus \{a\}) \cup \{x\}$. Since we have to exclude v_5 from $N(w)$ then by Lemma 4.1.5 we cannot have $a = a_1$. So in order to have $v_1 \in N(w)$ we must have $x = x_1$ but in order to have $v_2 \in N(w)$ we must have $x = x_2$, a contradiction.

Let $w = (v \setminus \{a, b\}) \cup \{x, y\}$. In order to have $v_1 \in N(w), v_2 \in N(w)$ and $v_3 \in N(w)$ Lemma 4.1.6 gives us we must have $\{x_1, x_2, x_3\} \subseteq \{x, y\}$, a contradiction.

c $v_5 = (v \setminus \{a_2\}) \cup \{x_5\}$ Here v_2, v_3, v_4, v_5 form case VI.

Case XVI

$$v_1 = (v \setminus \{a_1\}) \cup \{x_1\}$$
$$v_2 = (v \setminus \{a_1\}) \cup \{x_2\}$$
$$v_3 = (v \setminus \{a_2\}) \cup \{x_3\}$$
$$v_4 = (v \setminus \{a_2\}) \cup \{x_4\}$$



a $v_5 = (v \setminus \{a_2\}) \cup \{x_1\}$. Here v_2, v_3, v_4, v_5 form case VI.

b $v_5 = (v \setminus \{a_3\}) \cup \{x_1\}$. Here v_2, v_3, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

c $v_5 = (v \setminus \{a_1\}) \cup \{x_5\}$. In this case v_1, v_2, v_3, v_5 form case VI.

d $v_5 = (v \setminus \{a_3\}) \cup \{x_5\}$. Here v_1, v_3, v_5 all have distance 2 from each other and thus by Lemma 4.2.1 A is not shattered.

□

Theorem 4.2.3

The VC*-dimension of the edge relation in the Johnson graph* $J(m, k)$ *is* 4 *if and only if* $1 < k < m - 1$ *and* $|V_{J(m,k)})| = \binom{m}{k} \geq 16$.

Proof. If $|V_{J(m,k)}| < 16 = 2^4$ then the set system induced by the edge relation has fewer than 16 sets. Thus by the pigeonhole principle the VC-dimension of the edge relation is less than 4.

Assume $\binom{m}{k} \geq 16$ and $1 < k < m - 1$. Here we again rely on $J(m-1, k-1)$ and $J(m-1, k)$ being induced subgraphs of $J(m, k)$. We also observe that $J(m, k)$ is isomorphic to $J(m, m-k)$. So since $\binom{m}{k} \geq 16$ then either $J(7, 2)$ or $J(6, 3)$ are induced subgraphs of $J(m, k)$.

Since removing vertices from a graph can only decrease VC-dimension it now suffices to show that the edge relation has VC-dimension 4 in $J(7, 2)$ and $J(6, 3)$. In Figure 13 we show choices for vertices v_1, v_2, v_3, v_4 such that $A = \{v_1, v_2, v_3, v_4\}$ is shattered by the edge relation, along with how each subset of A can be obtained.

So the VC-dimension of the edge relation is at least 4 in both $J(6, 3)$ and $J(7, 2)$. This shows that the VC-dimension of the edge relation is at least 4 in all Johnson graphs $J(m, k)$ where $\binom{m}{k} \geq 16$ and $1 < k < m - 1$. Theorem 4.2.2 shows us that the edge relation has VC-dimension at most 4 in all Johnson graphs so this bound is tight whenever $\binom{m}{k} \geq 16$ and $1 < k < m - 1$. □

$J(7,2)$

$$v_1 = \{1,3\}$$
$$v_2 = \{1,4\}$$
$$v_3 = \{1,5\}$$
$$v_4 = \{1,6\}$$
$$A \cap N(\{2,7\}) = \emptyset$$
$$A \cap N(\{3,7\}) = \{v_1\}$$
$$A \cap N(\{4,7\}) = \{v_2\}$$
$$A \cap N(\{5,7\}) = \{v_3\}$$
$$A \cap N(\{6,7\}) = \{v_4\}$$
$$A \cap N(\{3,4\}) = \{v_1, v_2\}$$
$$A \cap N(\{3,5\}) = \{v_1, v_3\}$$
$$A \cap N(\{3,6\}) = \{v_1, v_4\}$$
$$A \cap N(\{4,5\}) = \{v_2, v_3\}$$
$$A \cap N(\{4,6\}) = \{v_2, v_4\}$$
$$A \cap N(\{5,6\}) = \{v_3, v_4\}$$
$$A \cap N(v_4) = \{v_1, v_2, v_3\}$$
$$A \cap N(v_3) = \{v_1, v_2, v_4\}$$
$$A \cap N(v_2) = \{v_1, v_3, v_4\}$$
$$A \cap N(v_1) = \{v_2, v_3, v_4\}$$
$$A \cap N(\{1,2\}) = A$$

$J(6,3)$

$$v_1 = \{2,3,4\}$$
$$v_2 = \{1,3,4\}$$
$$v_3 = \{1,3,5\}$$
$$v_4 = \{1,2,5\}$$
$$A \cap N(\{4,5,6\}) = \emptyset$$
$$A \cap N(\{2,3,6\}) = \{v_1\}$$
$$A \cap N(\{v_1\}) = \{v_2\}$$
$$A \cap N(\{v_4\}) = \{v_3\}$$
$$A \cap N(\{1,2,6\}) = \{v_4\}$$
$$A \cap N(\{3,4,6\}) = \{v_1, v_2\}$$
$$A \cap N(v_2) = \{v_1, v_3\}$$
$$A \cap N(\{2,4,5\}) = \{v_1, v_4\}$$
$$A \cap N(\{1,3,6\}) = \{v_2, v_3\}$$
$$A \cap N(v_3) = \{v_2, v_4\}$$
$$A \cap N(\{1,5,6\}) = \{v_3, v_4\}$$
$$A \cap N(\{3,4,5\}) = \{v_1, v_2, v_3\}$$
$$A \cap N(\{1,2,4\}) = \{v_1, v_2, v_4\}$$
$$A \cap N(\{2,3,5\}) = \{v_1, v_3, v_4\}$$
$$A \cap N(\{1,4,5\}) = \{v_2, v_3, v_4\}$$
$$A \cap N(\{1,2,3\}) = A$$

Figure 13: Examples of shattered sets of size 4 in $J(7,2)$ and $J(6,3)$.

4.3 VC-density of the edge relation

Recall that the VC-density is bounded above by the VC-dimension so we know that the VC-density of the edge relation on Johnson graphs is at most 4. In this section we will improve on that bound and give an exact value for the VC-density of the edge relation on Johnson graphs.

Theorem 4.3.1
The VC*-density of the edge relation on* $\mathcal{J}$ *is* 2.

Proof. First we show that the VC-density is at least 2. Assume without loss of generality that $m > 2k$ and let X be the underlying set of $J(m,k)$. Fix a vertex $v = \{a_i | 1 \leq i \leq k\}$ in $J(m,k)$, and let $(x_i)_{i=1}^k$ be distinct elements of X such that $x_i \notin v$ for all i. Define $A := \{(v \setminus \{a_i\}) \cup \{x_i\} | 1 \leq i \leq k\}$. Then for any pair of vertices $v_i := (v \setminus \{a_i\}) \cup \{x_i\}$ and $v_j = (v \setminus \{a_j\}) \cup \{x_j\}$ we have that $N((v \setminus \{a_i\}) \cup \{x_j\}) \cap A = \{v_i, v_j\}$. There are $\frac{|A|^2-|A|}{2}$ such pairs so the VC-density of the edge relation on $\mathcal{J}$ is at least 2.

Now we show that the VC-density of the edge relation on $\mathcal{J}$ is at most 2. Let A be a set of vertices in $J(m,k)$, and $\pi(n)$ be the shatter function for the edge relation on $J(m,k)$. Let $|A| = n$ and A be maximally shattered by the edge relation for sets of size n. Let

$$\begin{aligned} S(A) &= \{N(u) \cap A | u \in V_G\}, \\ C_1(A) &= \{N \in S(A) : N \text{ is a clique}\}, \text{ and} \\ C_2(A) &= \{N \in S(A) : N \text{ is not a clique}\}. \end{aligned}$$

By our assumption that A is maximally shattered we have $|S(A)| = \pi(n)$. Note also that $S(A) = C_1(A) \cup C_2(A)$ so we deal with those two cases separately.

$|C_1(A)| \leq \frac{5|A|^2+3|A|}{2}$: There are at most $\frac{|A|^2+|A|}{2}$ cliques of size 2 or less in $S(A)$. There are at most $|A|$ cliques C in $S(A)$ such that $C = A \cap N(v)$ for some $v \in A$.

Now assume we have $C = A \cap N(v)$ for some $v \notin A$ and further assume that $|C| \geq 3$. We want to show that then the clique C is of the form $A \cap Q$ for some maximal clique Q of $J(m, k)$. We then argue that there can be at most $2|A|^2$ maximal cliques of $J(m, k)$ that intersect A in more than one vertex.

Note that in any graph G a maximal clique Q of G is contained in $N(u) \cup \{u\}$ for all $u \in Q$ so $Q \setminus \{u\}$ is a maximal clique in $G[N(u)]$. It is easy to see that the maximal cliques of the rook's graph $R(m, k)$ are the rows and columns. So by Lemma 4.1.4 we find that for every vertex u in $J(m, k)$ the maximal cliques of $J(m, k)$ that u belongs to are of the form $Z \cup \{u\}$ where Z is a row or a column of the rook's graph $J(m, k)[N(u)]$.

Since $|C| \geq 3$ we know by Lemma 4.1.5 the only vertices connected to all vertices in C are v and those vertices that share that row or column with all of C, in the rook's graph induced by $N(v)$, and therefore lie in $N(v)$. It follows that $C = A \cap Q$ for some maximal clique Q of $J(m, k)$.

For every vertex $u \in A$ we have that A intersects at most $|A|$ rows and at most $|A|$ columns of the rook's graph induced by $N(u)$. So u can be a member of at most $2|A|$ maximal cliques of $J(m, k)$ that intersect A in more than two vertices. So the number of maximal cliques of $J(m, k)$ that intersect A in more than two vertices is at most $2|A|^2$.

$|C_2(A)| \leq 4|A|^2$: This holds since every pair of vertices at distance 2 from each other can by Lemma 4.1.7 be contained in the neighbourhood of at most 4 vertices and there are at most $|A|^2$ such pairs.

So we get that $|S(A)| \leq |C_1(A)| + |C_2(A)| \leq \frac{5|A|^2+3|A|}{2} + 4|A|^2 = \frac{13|A|^2+3|A|}{2} \in \mathcal{O}(|A|^2)$. $\square$

5 Theory of Johnson Graphs

In this chapter we will explore the first order theory of Johnson graphs. We start by showing that the limit theory of Johnson graphs $J(n, k)$ as $k \leq \frac{n}{2}$ goes to infinity is a complete theory $T_{\mathcal{J}}$. We then expand our definitions of Johnson graphs to infinity and beyond, defining some useful related class of graphs as well. We will also give axioms for $T_{\mathcal{J}}$ and prove it is ω-stable.

5.1 Introduction to infinite Johnson graphs

In this section we will expand the definition of Johnson graphs to include those having infinite underlying sets. We will introduce some related graphs and give some lemmas that will be helpful when working with them.

Definition 5.1.1

Let κ and λ be cardinals such that $\omega \leq \lambda \leq \kappa$. The Johnson graph $J(\kappa, \lambda)$ is a graph whose vertices are all the size λ subsets of an underlying set of size κ with complement of size κ such that two vertices u and v are adjacent if and only if $|u \setminus v| = |v \setminus u| = 1$.

In the above definiton we fix the size of the complements of vertices in order to make sure that all of the vertices have an equivalently size complements. An equivalent definition is obtained by fixing the size of the vertices to be κ and the complementns of vertices have size λ. The choice between the two definitions is arbitrary, and our choice is equivalent to using the isomorphims between $J(n, k)$ and $J(n, n-k)$ to assume $n \geq 2k$. This means that fixing the size of the complements to be κ rather than having a seperate parameter to specify their size can be without loss of generality for what graphs are being defined.

To gain a deeper understanding of the infinite Johnson graphs we will be looking at what their individual connected components look like.

Definition 5.1.2

Let κ and λ be cardinals such that $\kappa \geq \lambda$. Take sets $A \subseteq X$ such that $|A| = \lambda$, $|X| = \kappa$ and $|A| \leq |X \setminus A|$. We define a graph $J'(\kappa, \lambda)$ in the following way. There are vertices for each set of the form $(A \setminus B) \cup C$ where $B \subset A$, $C \subset X \setminus A$ and $|B| = |C| \in \mathbb{N}$. Note that by choosing $B = C = \emptyset$ we have a vertex for A which we call the central vertex of $J'(\kappa, \lambda)$. Two sets u, v are adjacent if and only if $|u \triangle v| = 2$.

From the definition of $J'(\kappa, \lambda)$ one would be inclined to think that the set A holds a special significance in its structure. It is however important to note that any vertex will have the same characteristics. To demonstrate this we prove the following.

Lemma 5.1.3

The relation R such that aRb if and only if $|a \setminus b| = |b \setminus a| \in \mathbb{N}$ is an equivalence relation.

Proof. For finite sets on one hand this relation is equivalent to equinumerosity, which in know to be an equivalence relation. For infinite sets on the other hand R is more restrictive than equinumerosity so we have some work to do.

R is reflexive: $|a \setminus a| = |a \setminus a| = 0$.

R is symmetric: $|a \setminus b| = |b \setminus a|$ if and only if $|b \setminus a| = |a \setminus b|$.

R is transitive: Assume $|a \setminus b| = |b \setminus a| \in \mathbb{N}$ and $|b \setminus c| = |c \setminus b| \in \mathbb{N}$. We have

$$|a \setminus b| = |(a \setminus b) \setminus c| + |(a \setminus b) \cap c| = |a \setminus (b \cup c)| + |(a \setminus b) \cap c|$$
$$|b \setminus a| = |(b \setminus a) \setminus c| + |(b \setminus a) \cap c| = |b \setminus (a \cup c)| + |(b \setminus a) \cap c|$$
$$|b \setminus c| = |(b \setminus c) \setminus a| + |(b \setminus c) \cap a| = |b \setminus (a \cup c)| + |(b \setminus c) \cap a|$$
$$|c \setminus b| = |(c \setminus b) \setminus a| + |(c \setminus b) \cap a| = |c \setminus (a \cup b)| + |(c \setminus b) \cap a|$$
$$|a \setminus c| = |(a \setminus c) \setminus b| + |(a \setminus c) \cap b| = |a \setminus (b \cup c)| + |(a \setminus c) \cap b|$$
$$|c \setminus a| = |(c \setminus a) \setminus b| + |(c \setminus a) \cap b| = |c \setminus (a \cup b)| + |(c \setminus a) \cap b|$$

$|a \setminus b| = |b \setminus a|$ and $|b \setminus c| = |c \setminus b|$ gives us

$$|b \setminus a| - |b \setminus c| = |a \setminus b| - |c \setminus b|$$

If we expand the left hand side of the equation we get.

$$\begin{aligned}
|b \setminus a| - |b \setminus c| &= |b \setminus (a \cup c)| + |(b \setminus a) \cap c| - |b \setminus (a \cup c)| - |(b \setminus c) \cap a| \\
&= |b \setminus (a \cup c)| - |b \setminus (a \cup c)| + |(b \setminus a) \cap c| - |(b \setminus c) \cap a| \\
&= |(b \setminus a) \cap c| - |(b \setminus c) \cap a| \\
&= |(c \setminus a) \cap b| - |(b \setminus c) \cap a|
\end{aligned}$$

Doing the same on the right hand side gives us.

$$\begin{aligned}
|a \setminus b| - |c \setminus b| &= |a \setminus (b \cup c)| + |(a \setminus b) \cap c| - |c \setminus (a \cup b)| - |(c \setminus b) \cap a| \\
&= |a \setminus (b \cup c)| - |c \setminus (a \cup b)| + |(a \setminus b) \cap c| - |(c \setminus b) \cap a| \\
&= |a \setminus (b \cup c)| - |c \setminus (a \cup b)| + |(a \cap c) \setminus b| - |(a \cap c) \setminus b| \\
&= |a \setminus (b \cup c)| - |c \setminus (a \cup b)|
\end{aligned}$$

So we now have

$$
\begin{aligned}
|(b \setminus a) \cap c| - |(b \setminus c) \cap a| &= |a \setminus (b \cup c)| - |c \setminus (a \cup b)| \\
|(b \setminus a) \cap c| + |c \setminus (a \cup b)| &= |a \setminus (b \cup c)| + |(b \setminus c) \cap a| \\
|(c \setminus a) \cap b| + |c \setminus (a \cup b)| &= |a \setminus (b \cup c)| + |(a \setminus c) \cap b| \\
|c \setminus a| &= |a \setminus c|
\end{aligned}
$$

□

Lemma 5.1.4

Any vertex in $J'(\kappa, \lambda)$ can act as the central vertex.

Proof. Let $M = J'(\kappa, \lambda)$ with central vertex A and underlying set X and B be any other vertex of M. Then let M' be $J'(\kappa, \lambda)$ with central vertex B and underlying set X. By Lemma 5.1.3 we have that M and M' have the same vertex set namely the equivalence class of R that includes A and B. The interpretation of the edge relation does not depend on the selection of central vertex in any way so it's identical in M and M'. □

Similar to our defintion of $J(\kappa, \lambda)$ for infinite κ and λ, the condition $|A| \leq |X \setminus A|$ is added to the above definition to avoid the ambiguity on the size of $X \setminus A$ that arises when $\kappa = \lambda$. This condition is in alignment with the convention of assuming that $2k \leq n$ in the finite case.

Corollary 5.1.5

The Johnson graph $J(\kappa, \lambda)$ is a disjoint union of κ^λ copies of $J'(\kappa, \lambda)$.

We want a more general definiton of infinite graphs that behave like Johnson graphs.

Definition 5.1.6

Let μ be some ordinal. We say that a graph G is a **Generalized Johnson Graph** *if and only if $G = \bigsqcup_{i\in\mu} J'(\kappa_i, \lambda_i)$ where $(\kappa_i)_{i\in\mu}$ and $(\lambda_i)_{i\in\mu}$ are sequences of infinite cardinals such that $\lambda_i \leq \kappa_i$ for all $i \in \mu$.*

We will show that generalized Johnson graphs are indistinguishable from Johnson graphs from the perspective of first order logic and more importantly that any infinite graph that is indistinguishable from a Johnson graph is a generalized Johnson graph.

5.2 Limit theory of $J(n, k)$ for a fixed k

In this section we will take our first steps working with theories of infinite Johnson graphs. We fix some constant k and observe the limit theory of Johnson graphs on k-sets. As we note in the introduction of this thesis some results are already known for this theory, such as Theorem 5.2.4, and Theorem 5.2.2, but they may not appear explicitly in published literature. They stem from the fact that theory of $J(n, k)$ for a fixed set is biinterpretable with that of a pure set. Here we present them with explicit proofs without relying on biinterpretability with a pure set.

Theorem 5.2.1

For any constant k and infinite cardinal κ, duplicator has a winning strategy in the Ehrenfeucht-Fraissé game $EF_{\lfloor \frac{n}{k} \rfloor}(J(n, k), J(\kappa, k))$.

Proof. We will call the vertex played on turn i in $J(\kappa, k)$, a_i regardless of which player plays it. Likewise we will call the vertex played on turn i in $J(\kappa, k)$, b_i regardless of which player plays it.

Duplicator will be building up a partial bijection f from X, the underlying set of $J(n, k)$, to Y the underlying set of $J(\kappa, k)$. We do this by creating a sequence $(f_i)_{i=1}^{\log t}$ where f_1 is the empty map, $f_{\log t} = f$ and f_{i+1} extends f_i. It is through this sequence of functions that duplicator is able to avoid any traps set by spoiler and is guaranteed a move on every turn of the game.

Now we are ready to play the game. By vertex transitivity, the first move of the game is arbitrary both by spoiler and duplicator.

Base case: Duplicator sets f_1 to be a bijection between a_1 and b_1.

For every subsequent move we have:

Induction hypothesis: After $m-1$ moves by both players which grant us the sequences, $(a_i)_{i=1}^{m-1}, (b_i)_{i=1}^{m-1}$ in $J(n,k)$ and $J(\kappa,k)$ respectively, duplicator has formed the function f_{m-1} in such a way that $f_{m-1}[a_i] = b_i$ for any $i < m$.

Inductive step: Assume spoiler plays a_m. Duplicator then chooses $I \subseteq Y \setminus img(f_{m-1})$ such that $|I| = |a_m \setminus dom(f_{m-1})|$. Such an I always exists since $|Y \setminus img(f_{m-1})| = \kappa > k$. He then sets f_m to be an extension of f_{m-1} such that $f_m[a_m \setminus dom(f_{m-1})] = I$. He then plays $b_m = f_m[a_m]$.

Assume spoiler plays b_m. Duplicator then chooses $I \subseteq X \setminus dom(f_{m-1})$ such that $|I| = |b_m \setminus img(f_{m-1})|$. Such an I always exists due to $m \leq \lfloor \frac{n}{k} \rfloor$ giving us $|X \setminus dom(f_{m-1})| \geq n - (m-1)k \geq k$. He then sets f_m to be an expansion of f_{m-1} such that $f_m[I] = b_m \setminus img(f_{m-1})$. He then plays a_m such that $f_m[a_m] = b_m$.

At the end of the game the induction hypothesis is satisfied at $\lfloor \frac{n}{k} \rfloor$ that is to say we have for any $1 \leq i \leq j \leq \lfloor \frac{n}{k} \rfloor$ that $f_{\lfloor \frac{n}{k} \rfloor}[a_i] = b_i$. Recall that in a Johnson graph $d(u,v) = 2|u \triangle v|$ so we have $d(a_i, a_j) = d(b_i, b_j)$ in particular this means that Ea_ia_j if and only if Eb_ib_j so the substructures $A = \{a_i | 1 \leq i \leq \lfloor \frac{n}{k} \rfloor\}$ in $J(n,k)$ and $B = \{b_i | 1 \leq i \leq \lfloor \frac{n}{k} \rfloor\}$ in $J(\kappa, k)$ are isomorphic. □

Corollary 5.2.2

The limit theory of $J(n,k)$ for a fixed k and $n \to \infty$ is a complete theory and is satisfied by $J(\kappa,k)$ for any infinite cardinal κ.

The following lemma is folklore amongst Model theorists, and to some extend considered obvious from the intuition that Morley rank should be the model theoretic notion of dimension. So obvious in fact that it is somewhat difficult to find a proof in literature so we present it here with proof.

Lemma 5.2.3

Let X be an infinite set interpreted as a model of the empty language. Note that equality $'='$ is still interpreted in the empty language. The set X^k has Morley rank k.

Proof. The proof is by induction on k.

Base case: Since any definable subset of X is either finite or cofinite so it has Morley rank 1.

Inductive hypothesis: X^{k-1} has Morley rank $k-1$.

Inductive step: For each $a \in X$ let $X_a = \{(x_i)_{i=1}^k | x_1 = a\}$. The set X_a is in bijection with X^{k-1} so by our induction hypothesis X_a has Morley rank $k-1$. Then $\{X_a | a \in X\}$ is a definable family in bijection with X so it has Morley rank 1. So X^k has a uniformly definable, Morley rank 1, family of disjoint Morley rank $k-1$ subsets. Thus X^k has Morley rank k. $\square$

Theorem 5.2.4

The theory of $J(\aleph_0, k)$, for some fixed k, has Morley rank k.

Proof. Let X be the underlying set of $J(\aleph_0, k)$. Now let $Y_k = \{(x_i)_{i=1}^k | \bigwedge_{1 \le i < j \le k} x_i \neq x_j\}$. $Y_k \subseteq X^k$ so Y_k has Morley rank at most k, in fact exactly k since we have removed sets of smaller Morley rank. We can define an equivalence relation S_k on Y_k as $(x_i)_{i=1}^k S_k (y_i)_{i=1}^k \Leftrightarrow \{x_i | 1 \le i \le k\} = \{y_i | 1 \le i \le k\}$. We can treat the set V

of equivalence classes of S_k as a set of "imaginary elements" of X, for further detail see [45, 8.4]. Each equivalence class of S_k has finite size, namely $k!$, so the set V has Morley rank exactly k.

But V is exactly the vertex set of $J(\aleph_0, k)$ taking X as the underlying set. We note that the edge relation of $J(\aleph_0, k)$ is definable in X so $J(\aleph_0, k)$ has Morley rank k. $\square$

Recall that a theory is *monadically stable* if any expansion of any model of T by unary predicates is stable.

Theorem 5.2.5

The Johnson graph $J(\aleph_0, 2)$ is not monadically dependent.

Proof. We view the Johnson graph as a structure in the language L of graphs (with a single binary relation E).

Let $\mathbb{N}$ be the underlying set of $J(\aleph_0, 2)$. We define the following sets of vertices.

$$A = \{\{2n+1, 0\} : n \in \mathbb{N}\},$$

$$B = \{\{2n+2, 0\} : n \in \mathbb{N}\},$$

$$C = \{\{2n+1, 2m+2\} : m, n \in \mathbb{N}\}.$$

Let $\psi(x, y, z)$ be the formula 'x, y, z are distinct and form a maximal clique of $J(\aleph_0, 2)$'.

$$\Psi(x, y, z) := Exy \wedge Exz \wedge Eyz \wedge \neg\exists w(Exw \wedge Eyw \wedge Ezw)$$

This holds precisely of triangles of the form $\{a, b\}, \{b, c\}, \{a, c\}$.

Let $L^{A,B,C}$ be the expansion of L by unary predicates interpreted by A, B, C. Let $\chi(x, y, z)$ be the $L^{A,B,C}$-formula $\psi(x, y, z) \wedge A(x) \wedge B(y) \wedge C(z)$.

Thus, χ determines a bijection $A \times B \to C$, with χ holding precisely of the triples $(\{2n+1, 0\}, \{2m+2, 0\}, \{2n+1, 2m+2\})$. It follows that each element of C

determines via χ a bipartite graph on $A \cup B$ with a single edge, and each subset C' of C determines a bipartite graph on $A \cup B$, all bipartite graphs arise in this way. In particular, we can define a bipartite graph witnessing the independence property (i.e. the random bipartite graph). $\square$

Corollary 5.2.6

The Johnson graph $J(\aleph_0, 2)$ is not monadically stable.

5.3 Generalized Johnson Graphs

In this section we will begin by showing that we have a complete theory for Johnson graphs $J(n, k)$ as $n, k, n-k$ approach ∞. We then show that generalized Johnson graphs satisfy this theory giving credence to the claim that they generalize the notion of Johnson graphs, justifying our choice of naming them generalized Johnson graphs.

Lemma 5.3.1

$J(n, k)$ with $2 < 2k \leq n$ does not have a distance $\log(k)$ dominating set of size $\log(k)$.

Proof. $J(n, k)$ has diameter k. Take two vertices v, u such that $d(v, u) = k$ then the shortest path between v, u has $k + 1$ vertices and each ball of radius $\log(k)$ can intersect the path in at most $2\log(k) + 1$ vertices, since such a ball cannot contain vertices at distance more than $2\log(k) + 1$ apart. Since $(2\log(k) + 1)\log(k) < k + 1$ this means that the path can not be covered with $\log(k)$ balls of radius $\log(k)$. In particular $J(n, k)$ can not be covered with $\log(k)$ balls of size $\log(k)$. $\square$

Lemma 5.3.2

$J(n, k)$ with $1 < k \leq \frac{n}{2}$ can not be covered by a collection $(B_i)_{i=1}^{\lfloor \log(k) \rfloor}$ of balls where B_i has radius $\lfloor \frac{k}{2^{i+1}} \rfloor - 1$.

Proof. $J(n,k)$ has diameter k. Take two vertices v, u such that $d(v,u) = k$ then the shortest path between v, u has $k+1$ vertices. Since it is a shortest path each ball of radius $\lfloor \frac{k}{2^{i+1}} \rfloor - 1$ can only intersect it in $\frac{k}{2^i}$ elements. So the total number of vertices intersected is:

$$\sum_{i=1}^{\lfloor \log(k) \rfloor} \frac{k}{2^i} = k \sum_{i=1}^{\lfloor \log(k) \rfloor} \frac{1}{2^i} \leq k+1$$

□

This is in essence a graph-theoretic analogue to Zeno's dichotomy paradox of motion [20] which along with the uniformity of Johnson opens up a winning strategy for Duplicator in Ehrenfeucht-Fraïssé games between two Johnson graphs. The strategy relies on keeping track of distances between played vertices, but ignoring those pairs of vertices that are too far to feasibly interact.

Theorem 5.3.3

Duplicator has a winning strategy in the Ehrenfeucht-Fraissé game $EF_{\lfloor \log t \rfloor}(J(n,k), J(n',k'))$ *where* $t = \lfloor \frac{\min(k, n-k, k', n'-k')}{2} \rfloor$.

Proof. Since $J(n,k) \cong J(n, n-k)$ we can assume without loss of generality $n \geq 2k$ and $n' \geq 2k'$. Moreover let us assume that $k' \leq k$. This means that $t = \lfloor \frac{k'}{2} \rfloor$.

As a clarification on notation we will call the vertex played on turn i in $J(n,k)$, a_i regardless of which player plays it. Likewise we will call the vertex played on turn i in $J(n',k')$, b_i regardless of which player plays it.

Duplicator will be building up a partial bijection f from X, the underlying set of $J(n,k)$, to Y the underlying set of $J(n',k')$. We do this by creating a sequence $(f_i)_{i=1}^{\lfloor \log t \rfloor}$ where f_1 is the empty map, $f_{\log t} = f$ and f_{i+1} extends f_i. It is through this sequence of functions that duplicator is able to avoid any traps set by spoiler and is guaranteed a move on every turn of the game.

Now we are ready to play the game.

Base case: By vertex transitivity, the first move of the game is arbitrary both by spoiler and duplicator. Duplicator sets f_1 to be the empty map.

For the second move if spoiler plays a_2 duplicator starts by looking at $a_1 \triangle a_2$. If $|a_1 \triangle a_2| \leq t$ then duplicator plays a b_2 such that $|a_1 \triangle a_2| = |b_1 \triangle b_2|$ then duplicator forms a bijection f_2 from $a_1 \triangle a_2$ to $b_1 \triangle b_2$ such that $x \in a_1 \iff f_2(x) \in b_1$ and $x \in a_2 \iff f_2(x) \in b_2$. If $|a_1 \triangle a_2| > t$ then duplicator plays a b_2 such that $\min(|a_1 \triangle a_2|, 2t) = |b_1 \triangle b_2|$ and sets $f_2 = f_1$ to be the empty map.

Induction hypothesis: After $m-1$ moves by both players which grant us the sequences $(a_i)_{i=1}^{m-1}, (b_i)_{i=1}^{m-1}$ in $J(n,k)$ and $J(n',k')$ respectively, duplicator has formed the function f_{m-1} in such a way that if $|a_i \triangle a_j| \leq \frac{t}{2^{m-1}}$ or $|b_i \triangle b_j| \leq \frac{t}{2^{m-1}}$ then $a_i \triangle a_j \subseteq dom(f_{m-1})$, $b_i \triangle b_j \subseteq img(f_{m-1})$ and

$$x \in a_i \setminus a_j \iff f_{m-1}(x) \in b_i \setminus b_j$$

Note that this implicitly means that for all $i,j < m$ if $d(a_i,a_j) \leq \frac{t}{2^m}$ or $d(b_i,b_j) \leq \frac{t}{2^m}$ then $d(a_i,a_j) = d(b_i,b_j)$.

Inductive step:

Assume spoiler plays a_m. Let $M \subseteq [1, m-1] \cap \mathbb{N}$ be such that $i \in M$ if and only if $d(a_i, a_m) \leq \frac{t}{2^m}$.

If $M = \emptyset$ then duplicator sets $f_m = f_{m-1}$ and plays a b_m such that $d(b_i, b_m) > \frac{t}{2^m}$ for all $i < m$. Such a vertex always exists due to Lemma 5.3.2.

Assume $M \neq \emptyset$. Duplicator has to extend f_{m-1} to f_m with domain

$$dom(f_{m-1}) \cup \bigcup_{i \in M} a_m \triangle a_i$$

in such a way that if $x \in a_i \cap dom(f_m)$ then $f_m(x) \in b_i$ and every element in $(a_m \setminus (\bigcup_{i<m} a_i)) \cap dom(f_m)$ gets mapped to an element that is in the complement of $\bigcup_{i \in M} b_i$.

For every $i, j \in M$ we have that $d(a_i, a_j) \leq \frac{2t}{2^m}$ so by induction hypothesis $a_i \triangle a_j \subseteq dom f_{m-1}$. So

$$(\bigcup_{i \in M} a_m \triangle a_i) \setminus dom(f_{m-1}) \subseteq (a_m \setminus \bigcup_{i \in M} a_i) \cup ((\bigcap_{i \in M} a_i) \setminus a_m)$$

that is to say, every element that duplicator is adding to the domain of f_m is either in only a_m or in every other a_i for $i \in M$. So the elements in the domain of f_m that aren't in the domain of f_{m-1} will be in either $(\bigcap_{i \in M} a_i) \setminus a_m$ or $a_m \setminus \bigcup_{i \in M} a_i$ and thus need to be mapped to $(\bigcap_{i \in M} b_i)$ and $Y \setminus (\bigcup_{i \in M} b_i)$ respectively. In each case we need to show that the domain contains at most as many elements as the image.

Case 1: Mapping $(\bigcap_{i \in M} a_i) \setminus a_m$ to $(\bigcap_{i \in M} b_i)$ is possible since: We know that for all $i \in M$ $|a_i \setminus a_m| \leq \frac{t}{2^m}$ so we get:

$$|(\bigcap_{i \in M} a_i) \setminus a_m| \leq \frac{t}{2^m} \leq t$$

and

$$|(\bigcap_{i \in M} b_i)| \geq k' - \frac{|M|t}{2^m} \geq k' - \frac{tm}{2^m} \geq k' - t \geq t$$

Case 2: We can map $a_m \setminus \bigcup_{i \in M} a_i$ to $Y \setminus (\bigcup_{i \in M} b_i)$ due to the following: Since for $i \in M$ $|a_m \setminus a_i| \leq \frac{t}{2^m}$ we get

$$|a_m \setminus \bigcup_{i \in M} a_i| \leq \frac{t}{2^m} \leq t$$

and

$$|Y \setminus (\bigcup_{i \in M} b_i)| \geq n' - (k' + |M|\frac{t}{2^m}) \geq (n' - k') - m\frac{t}{2^m} \geq k' - \frac{mt}{2^m} \geq k' - t \geq t$$

Duplicator then plays $b_m = f_m[a_m \cap dom(f_m)] \cup \bigcap_{i \in M} b_i$. Note that the set b_m is a vertex, i.e. a set of size k', by the definition of f_m. Note that then we have for all $i \in M$ that $d(a_m, a_i) = d(b_m, b_i)$. So by induction hypothesis we have for $i, j \leq m$ that if $|a_i \triangle a_j| \leq \frac{t}{2^m}$ or $|b_i \triangle b_j| \leq \frac{t}{2^m}$ then $a_i \triangle a_j \subseteq dom(f_m)$, $b_i \triangle b_j \subseteq img(f_m)$ and

$$x \in a_i \setminus a_j \iff f_{m-1}(x) \in b_i \setminus b_j$$

Assume spoiler plays b_m. Let $M \subseteq [1, m-1] \cap \mathbb{N}$ be such that $i \in M$ if and only if $d(b_i, b_m) \leq \frac{t}{2^m}$.

If $M = \emptyset$ then duplicator sets $f_m = f_{m-1}$ and plays an a_m such that $d(a_i, a_m) > \frac{t}{2^m}$ for all $i < m$. Such a vertex always exists due to Lemma 5.3.2.

Assume $M \neq \emptyset$. Duplicator then extends f_{m-1} to f_m with image

$$img(f_{m-1}) \cup \bigcup_{i \in M} b_m \triangle b_i$$

in such a way that if $x \in a_i$ then $f_m(x) \in b_i$ and every element in $b_m \setminus (\bigcup_{i<m} b_i)$ gets mapped to by an element that is in the complement of $\bigcup_{i<m} a_i$.

In this case the elements in the image of f_m that aren't in the image of f_{m-1} will be in either $(\bigcap_{i \in M} b_i) \setminus b_m$ or $b_m \setminus \bigcup_{i \in M} b_i$ and thus need to be mapped onto by $(\bigcap_{i \in M} a_i)$ and $X \setminus (\bigcup_{i \in M} a_i)$ respectively. In each case we need to show that the image contains at most as many elements as the domain.

Case 1: Mapping $(\bigcap_{i \in M} a_i)$ to $(\bigcap_{i \in M} b_i) \setminus b_m$ is possible since:

$$|(\bigcap_{i \in M} b_i) \setminus b_m| \leq \frac{t}{2^m} \leq t$$

and

$$|(\bigcap_{i \in M} a_i)| \geq k - \frac{|M|t}{2^m} \geq k - \frac{tm}{2^m} \geq k - t \geq t$$

Case 2: Mapping $X \setminus (\bigcup_{i \in M} a_i)$ to $b_m \setminus \bigcup_{i \in M} b_i$ is possible since

$$|b_m \setminus \bigcup_{i \in M} b_i| \leq \frac{t}{2^m} \leq t$$

and

$$|X \setminus (\bigcup_{i \in M} a_i)| \geq n - (k + |M|\frac{t}{2^m}) = (n-k) - m\frac{t}{2^m} \geq k - t\frac{m}{2^m} \geq k - t \geq t$$

Duplicator then plays $a_m = f_m^{-1}[b_m \cap img(f_m)] \cup \bigcap_{i \in M} a_i$. Note that the set a_m is a vertex, i.e. a set of size k, by the definition of f_m. Note that then we have for all

$i \in M$ that $d(a_m, a_i) = d(b_m, b_i)$. So by induction hypothesis we have for $i, j \leq m$ that if $[|a_i \triangle a_j| \leq \frac{t}{2^m}$ or $|b_i \triangle b_j| \leq \frac{t}{2^m}$ then $a_i \triangle a_j \subseteq dom(f_m)$, $b_i \triangle b_j \subseteq img(f_m)$ and

$$x \in a_i \setminus a_j \iff f_{m-1}(x) \in b_i \setminus b_j$$

So after m moves by both players, duplicator has formed the function f_m in such a way that if $|a_i \triangle a_j| \leq \frac{t}{2^m}$ then $x \in a_i \setminus a_j \iff f_m(x) \in b_i \setminus b_j$ and $x \in a_j \setminus a_i \iff f_m(x) \in b_j \setminus b_i$. This concludes the inductive step.

At the end of the game we have for any $i, j \leq \lfloor \log t \rfloor$ that if $d(a_i, a_j) \leq \lfloor \frac{t}{2^{\log t}} \rfloor = 1$ or $d(b_i, b_j) \leq \lfloor \frac{t}{2^{\log t}} \rfloor = 1$ then $d(a_i, a_j) = d(b_i, b_j)$. In particular this means that Ea_ia_j if and only if Eb_ib_j so the substructures $A = \{a_i | 1 \leq i \leq \log(t)\}$ in $J(n, k)$ and $B = \{b_i | 1 \leq i \leq \log(t)\}$ in $J(n', k')$ are isomorphic with isomorphism $a_i \mapsto b_i$. $\square$

We now obtain the following definition of a limit theory of $\mathcal{J}$.

Definition 5.3.4

Let $T_{\mathcal{J}}$ be the set of sentences σ such that there is a k_σ such that if $n \geq k \geq k_\sigma$ and $n - k \geq k_\sigma$ then $J(n, k) \models \sigma$.

Corollary 5.3.5

$T_{\mathcal{J}}$ is a complete theory.

Proof. By Theorem 2.2.26 and Theorem 5.3.3 we have that for every sentence σ, that all Johnson graphs $J(n, k)$ with n, k, and $n - k$ all greater than $2^{QD(\sigma)}$ agree on σ. $\square$

This tells us that as n and k grow every sentence σ in the language of graphs eventually becomes true for all arbitrarily large $J(n, k)$ or it eventually becomes false for all arbitrarily large $J(n, k)$.

Now that we have a unified theory $T_{\mathcal{J}}$ that holds for all arbitrarily large Johnson graphs we would like to explore what infinitetly large models of $T_{\mathcal{J}}$ look like.

Theorem 5.3.6

Let $(\kappa_i)_{i\in I}$ and $(\lambda_i)_{i\in I}$ be families of infinite cardinals. Duplicator has a winning strategy in the Ehrenfeucht-Fraïssé game $EF_{\lfloor \log t\rfloor}(J(n,k), \bigsqcup_{i\in I} J'(\kappa_i,\lambda_i))$ where $t = \lfloor \frac{\min(k,n-k)}{2} \rfloor$.

Proof. Duplicator's strategy is more or less identical to the one he uses in the proof of Theorem 5.3.3.

It is worth noting that for the purposes of that strategy there is no difference between two vertices in $\bigsqcup J'(\kappa_i,\lambda_i)$ being in different connected components and merely being further than distance $\frac{t}{2^m}$ apart. Thus the following proof is similar to what we have already presented.

Since $J(n,k) \cong J(n,n-k)$ we can assume without loss of generality $n \geq 2k$. This means that $t = \lfloor \frac{k}{2} \rfloor$.

As a clarification on notation we will call the vertex played on turn i in $J(n,k)$, a_i regardless of which player plays it. Likewise we will call the vertex played on turn i in $\bigsqcup_{j\in I} J'(\kappa_j,\lambda_j)$, b_i regardless of which player plays it.

Without loss of generality we can assume that each of the graphs $J'(\kappa_i,\lambda_i), i \in I$ have underlying sets $Y_i, i \in I$ that are pairwise disjoint. We call their union $\bigcup_{i\in I} Y_i = Y$. Duplicator will be building up a partial bijection f from X, the underlying set of $J(n,k)$, to Y. We do this by creating a sequence $(f_i)_{i=1}^{\lfloor \log t\rfloor}$ where f_1 is the empty map, $f_{\lfloor \log t\rfloor} = f$ and f_{i+1} extends f_i.

Now we are ready to play the game.

Base case: By vertex transitivity, the first move of the game is arbitrary both by spoiler and duplicator.

Duplicator sets f_1 to be the empty map.

For the second move if spoiler plays a_2 duplicator starts by looking at $a_1 \triangle a_2$. If $|a_1 \triangle a_2| \leq t$ then duplicator plays a b_2 such that $|a_1 \triangle a_2| = |b_1 \triangle b_2|$ then duplicator forms a bijection f_2 from $a_1 \triangle a_2$ to $b_1 \triangle b_2$ such that $x \in a_1 \iff f_2(x) \in b_1$ and $x \in a_2 \iff f_2(x) \in b_2$. If $|a_1 \triangle a_2| > t$ then duplicator plays a b_2 such that $\min(|a_1 \triangle a_2|, 2t) = |b_1 \triangle b_2|$ and sets $f_2 = f_1$ to be the empty map.

Induction hypothesis: After $m-1$ moves by both players which grant us the sequences, $(a_i)_{i=1}^{m-1}, (b_i)_{i=1}^{m-1}$ in $J(n,k)$ and $\bigsqcup_{i \in I} J'(\kappa_i, \lambda_i)$ respectively duplicator has formed the function f_{m-1} in such a way that if $|a_i \triangle a_j| \leq \frac{t}{2^{m-1}}$ or $|b_i \triangle b_j| \leq \frac{t}{2^{m-1}}$ then $a_i \triangle a_j \subseteq dom(f_{m-1})$, $b_i \triangle b_j \subseteq img(f_{m-1})$ and

$$x \in a_i \setminus a_j \iff f_{m-1}(x) \in b_i \setminus b_j$$

Note that this implicitly means that for all $i, j < m$ if $d(a_i, a_j) \leq \frac{t}{2^m}$ or $d(b_i, b_j) \leq \frac{t}{2^m}$ then $d(a_i, a_j) = d(b_i, b_j)$.

Inductive step:

Assume spoiler plays a_m. Let $M \subseteq [1, m-1] \cap \mathbb{N}$ be such that $i \in M$ if and only if $d(a_i, a_m) \leq \frac{t}{2^m}$. If $M = \emptyset$ then duplicator sets $f_m = f_{m-1}$ and plays a b_m such that $d(b_i, b_m) > \frac{t}{2^m}$ for all $i < m$. Such a vertex always exists since each connected component of $\bigsqcup_{i \in I} J'(\kappa_i, \lambda_i)$ has infinite diameter.

Assume $M \neq \emptyset$. For every $i, j \in M$ we have that $d(a_i, a_j) \leq \frac{2t}{2^m}$ so by induction hypothesis $a_i \triangle a_j \subseteq dom f_{m-1}$. Note that by our induction hypothesis, for all $i, j \in M$ we have that $d(b_i, b_j)$ is finite and thus b_i and b_j are in the same connected component. Without loss of generality assume it is the y-th component. Duplicator has to extend f_{m-1} to f_m with domain

$$dom(f_{m-1}) \cup \bigcup_{i \in M} a_m \triangle a_i$$

in such a way that if $x \in a_i \cap dom(f_m)$ then $f_m(x) \in b_i$ and every element in $(a_m \setminus (\bigcup_{i<m} a_i)) \cap dom(f_m)$ gets mapped to an element that is in $Y_y \setminus \bigcup_{i \in M} b_i$.

$$(\bigcup_{i\in M} a_m \triangle a_i) \setminus dom(f_{m-1}) \subseteq (a_m \setminus \bigcup_{i\in M} a_i) \cup ((\bigcap_{i\in M} a_i) \setminus a_m)$$

that is to say, every element that duplicator is adding to the domain of f_m is either in only a_m or in every other a_i for $i \in M$. So the elements in the domain of f_m that aren't in the domain of f_{m-1} will be in either $(\bigcap_{i\in M} a_i) \setminus a_m$ or $a_m \setminus \bigcup_{i\in M} a_i$ and thus need to be mapped to $\bigcap_{i\in M} b_i$ and $Y_y \setminus (\bigcup_{i\in M} b_i)$ respectively. In each case we need to show that the domain contains at most as many elements as the image.

Case 1: Mapping $(\bigcap_{i\in M} a_i) \setminus a_m$ to $(\bigcap_{i\in M} b_i)$ is possible since: We know that for all $i \in M$ $|a_i \setminus a_m| \leq \frac{t}{2^m}$ so we get

$$|(\bigcap_{i\in M} a_i) \setminus a_m| \leq \frac{t}{2^m} \leq t$$

and

$$|(\bigcap_{i\in M} b_i)| = \lambda \leq t$$

Case 2: Mapping $a_m \setminus \bigcup_{i\in M} a_i$ to $Y_y \setminus (\bigcup_{i\in M} b_i)$.

Since for $i \in M$ $|a_m \setminus a_i| \leq \frac{t}{2^m}$ we get

$$|a_m \setminus \bigcup_{i\in M} a_i| \leq \frac{t}{2^m} \leq t$$

and since

$$|Y_y \setminus (\bigcup_{i\in M} b_i)| = \kappa \geq t$$

Duplicator then plays $b_m = f_m[a_m \cap dom(f_m)] \cup \bigcap_{i\in M} b_i$. By our definition of f_m, and the fact that $|\bigcap_{i\in M} b_i|$ we get that b_m is in fact a vertex. Note that then we have for all $i \in M$ $d(a_m, a_i) = d(b_m, b_i)$. So by induction hypothesis we have for $i, j \leq m$ that if $|a_i \triangle a_j| \leq \frac{t}{2^m}$ or $|b_i \triangle b_j| \leq \frac{t}{2^m}$ then $a_i \triangle a_j \subseteq dom(f_m)$, $b_i \triangle b_j \subseteq img(f_m)$ and

$$x \in a_i \setminus a_j \iff f_m(x) \in b_i \setminus b_j$$

Assume spoiler plays b_m. Let $M \subseteq [1, m-1] \cap \mathbb{N}$ be such that $i \in M$ if and only if $d(b_i, b_m) \leq \frac{t}{2^m}$.

If $M = \emptyset$ then duplicator sets $f_m = f_{m-1}$ and plays an a_m such that $d(a_i, a_m) > \frac{t}{2^m}$ for all $i < m$. Such a vertex always exists due to Lemma 5.3.2.

Assume $M \neq \emptyset$. Duplicator then extends f_{m-1} to f_m with image

$$img(f_{m-1}) \cup \bigcup_{i \in M} b_m \triangle b_i$$

in such a way that if $x \in a_i$ then $f_m(x) \in b_i$ and every element in $b_m \setminus (\bigcup_{i<m} b_i)$ gets mapped to by an element that is in the complement of $\bigcup_{i<m} a_i$.

In this case the elements in the image of f_m that aren't in the image of f_{m-1} will be in either $(\bigcap_{i \in M} b_i) \setminus b_m$ or $b_m \setminus \bigcup_{i \in M} b_i$ and thus need to be mapped onto by $(\bigcap_{i \in M} a_i)$ and $X \setminus (\bigcup_{i \in M} a_i)$ respectively. In each case we need to show that the image contains at most as many elements as the domain.

Case 1: Mapping $(\bigcap_{i \in M} a_i)$ to $(\bigcap_{i \in M} b_i) \setminus b_m$ is possible since:

$$|(\bigcap_{i \in M} b_i) \setminus b_m| \leq \frac{t}{2^m} \leq t$$

and

$$|(\bigcap_{i \in M} a_i)| \geq k - \frac{|M|t}{2^m} \geq k - \frac{tm}{2^m} \geq k - t \geq t$$

Case 2: Mapping $X \setminus (\bigcup_{i \in M} a_i)$ to $b_m \setminus \bigcup_{i \in M} b_i$ is possible since

$$|b_m \setminus \bigcup_{i \in M} b_i| \leq \frac{t}{2^m} \leq t$$

and

$$|X \setminus (\bigcup_{i \in M} a_i)| \geq n - (k + |M|\frac{t}{2^m}) = (n-k) - m\frac{t}{2^m} \geq k - t\frac{m}{2^m} \geq k - t \geq t$$

By our definition of f_m, we get $f_m^{-1}[b_m \cap img(f_m)]$ has size $k - |\bigcap_{i \in M} a_i|$. Duplicator then plays $a_m = f_m^{-1}[b_m \cap img(f_m)] \cup \bigcap_{i \in M} a_i$. Note that then we have for all

$i \in M$ $d(a_m, a_i) = d(b_m, b_i)$. So by induction hypothesis we have for $i, j \leq m$ that if $|a_i \triangle a_j| \leq \frac{t}{2^m}$ or $|b_i \triangle b_j| \leq \frac{t}{2^m}$ then $a_i \triangle a_j \subseteq dom(f_m)$, $b_i \triangle b_j \subseteq img(f_m)$ and

$$x \in a_i \setminus a_j \iff f_m(x) \in b_i \setminus b_j$$

So after m moves by both players, duplicator has formed the function f_m in such a way that if $|a_i \triangle a_j| \leq \frac{t}{2^m}$ then $x \in a_i \setminus a_j \iff f_m(x) \in b_i \setminus b_j$ and $x \in a_j \setminus a_i \iff f_m(x) \in B_J \setminus B_I$. This concludes the inductive step.

At the end of the game we have for any $i, j \leq \lfloor \log t \rfloor$ that if $d(a_i, a_j) \leq \lfloor \frac{t}{2^{\log t}} \rfloor = 1$ or $d(b_i, b_j) \leq \frac{t}{2^{\log t}} = 1$ then $d(a_i, a_j) = d(b_i, b_j)$. In particular this means that Ea_ia_j if and only if Eb_ib_j so the substructures $A = \{a_i | 1 \leq i \leq \log(t)\}$ in $J(n, k)$ and $B = \{b_i | 1 \leq i \leq \log(t)\}$ in $\bigsqcup_{i \in I} J'(\kappa_i, \lambda_i)$ are isomorphic. □

5.4 Axiomatic theory of Johnson graphs

In this section we will give axioms for $T_{\mathcal{J}}$. We will first give the axioms that are satisfied by any Johnson graph. We then give the axiom schemas that specify the size of the graph i.e. they specify n and k for $J(n, k)$. We are primarily interested in the axiom schemas where n and k are left unbounded. We will then show that a connected graph M is isomorphic to $J'(\kappa, \lambda)$ for some κ and λ if and only if it satisfies those axioms.

Before we begin we will discuss the shorthands we use in the definitions below.

Definition 5.4.1

Let L be the language of graphs i.e. a language with a single binary symbol E.

Let L_c be the language of graphs with 3 additional constant symbols v, a, and b.

We will first give our axioms in L_c. This is done for convenience, and allows us to give the axioms in an intuitive manner that describes what Johnson graphs look like locally. We will later on argue that the axioms apply for any 3 pairwise adjacent vertices so we interpret the constant symbols of L_c as variables in L and get axioms for the theory of Johnson graphs.

We then give the axiom schemas that are specific to each Johnson graph, first the finite ones then $J'(\kappa, \lambda)$ for infinite cardinals κ and λ and finally generalized Johnson graphs.

5.4.1 Definable language extensions

Function symbols Let $\phi(x, y)$ be a formula satisfying

$$\forall x \exists y \phi(x, y)$$

and

$$\forall y \forall z (\exists x \phi(x, y) \wedge \phi(x, z)) \rightarrow y = z$$

Then $\phi(x, y)$ defines a function $f_\phi : M \rightarrow M$ so we can use the shorthand "$f_\phi(x) = y$" for the formula $\phi(x, y)$. We can say that f_ϕ is a bijection by a dual condition (this essentially says the inverse exists).

$$\forall y \exists x \phi(x, y)$$

and

$$\forall x \forall z (\exists y \phi(x, y) \wedge \phi(z, y)) \rightarrow x = z$$

We can change the domain of f_ϕ to a definable subset of M. Let $A \subseteq M$ be a subset defined by ψ. Then we have

$$\forall x (\psi(x) \rightarrow \exists y \phi(x, y))$$

$$\forall y \forall z (\exists x \phi(x, y) \wedge \phi(x, z)) \rightarrow y = z$$

Note that by restricting the domain of the inverse we can talk about bijections between definable subsets of our model.

Sets We can define what it means for a n-tuple to be an ordered set of size n i.e. it contains n distinct elements.

$$Set_n((x_i)_{i=1}^n) := \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} x_i \neq x_j$$

Note that this defines ordered sets. Defining an equivalence between two tuples $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ representing the same set is also possible.

$$SetEquiv_n((x_i)_{i=1}^n, (y_i)_{i=1}) := Set_n((x_i)_{i=1}^n) \wedge Set_n((y_i)_{i=1}^n) \wedge (\bigvee_{p \in S_n} \bigwedge_{i=1}^{n} (x_i = y_{p(i)}))$$

Similarly we can define a formula which holds if and only if the tuples $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ represent sets of size n and their intersection has size m.

$$Intersect_{n,m}((x_i)_{i=1}^n, (y_i)_{i=1}) := Set_n((x_i)_{i=1}^n) \wedge Set_n((y_i)_{i=1}^n) \wedge$$
$$(\bigvee_{p \in S_n} \bigvee_{q \in S_n} SetEquiv_m((x_{p(i)})_{i=1}^m, (y_{q(i)})_{i=1}^m) \wedge Set_{2n-m}((x_{p(i)})_{i=m+1}^n, (y_{q(i)})_{i=m+1}^n))$$

For a fixed $n \in \mathbb{N}$ the set of all n-element subsets of a definable set is definable. Let A be a set defined by ϕ. Then the set of all n-element subsets of A is defined by

$$\psi((x_i)_{i=1}^n) := (\bigwedge_{i=1}^{n} \phi(x_i)) \wedge Set((x_i)_{i=1}^n)$$

Note that this formula is is satisfied by every permutation of every n-element subset of A.

Distance For every $n \in \mathbb{N}$ we define a relation d_n which is satisfied by exactly those vertices that are at distance n from each other.

$$d_0(x, y) := x = y$$

$$d_1(x, y) := Exy$$

$$d_{n+1}(x, y) := \neg d_{n-1}(x, y) \wedge \neg d_n(x, y) \wedge \exists z(Eyz \wedge d_n(x, z))$$

We give a special name to the sets of vertices of a given distance from the constant vertex v.

$$D_n := \{u | d_n(v, u)\}$$

Rook graph structure Recall that in a Johnson graph, for every vertex v, the neighbourhood $N(v)$ induces a rook's graph as a subgraph. We work with the intuitive notions of row and column in the rook graph. We use the constants a and b to identify the difference between two adjacent vertices in $N(v)$ sharing a row on one hand and sharing a column on the other. We say a and b are to be in the same column and the members of that column are representatives of their respective rows. The row that a belongs to is used as the set of representatives for columns.

$$Rep(x) := Evx \land (Exa \lor x = a)$$

$$RRep(x) := Rep(x) \land (Exb \lor x = b)$$

$$CRep(x) := Rep(x) \land (\neg RRep(x) \lor x = a)$$

We then say that two vertices belong to the same row (column) if they are adjacent to the same row (column) representative.

$$Col(x, y) := (Evx \land Evy) \land (Exy \lor x = y) \land \exists z (CRep(z) \land (Exz \lor x = z) \land (Eyz \lor y = z))$$

$$Row(x, y) := (Evx \land Evy) \land (Exy \lor x = y) \land \exists z (RRep(z) \land (Exz \lor x = z) \land (Eyz \lor y = z))$$

Underlying sets We will now give formulae in L_c that we use to capture the underlying set structure of the Johnson graph. Recall that in the rook graph surrounding a vertex A in a Johnson graph with underlying set X, each row contains those vertices that are missing the same element of A and each column those that include the same element from outside A or the inverse. Armed with that knowledge we define

$$\begin{aligned}
\phi_1(x,u,w) &:= CRep(u) \wedge RRep(w) \wedge \\
&((\neg Rep(x) \wedge Exu \wedge Exw \wedge Exv) \vee \\
&(CRep(x) \wedge x = u \wedge Exw) \vee (RRep(x) \wedge Exu \wedge x = w)) \\
\phi_n(x,(u_i)_{i=1}^n,(w_i)_{i=1}^n) &:= d_n(v,x) \wedge Set_n((u_i)_{i=1}^n) \wedge Set_n((w_i)_{i=1}^n) \wedge \\
&\bigwedge_{i=n}^{n}(CRep(u_i) \wedge RRep(w_i)) \wedge \\
&\forall y(d_{n-1}(v,y) \rightarrow (Exy \leftrightarrow (\bigvee_{i=1}^{n}\bigvee_{j=1}^{n}\phi_{n-1}(y,(u'_i)_{\substack{i'=1\\ i'\neq i}}^n,(w'_j)_{\substack{j'=1\\ j'\neq j}}^n))))
\end{aligned}$$

We want ϕ_n to be a formula defining a bijective function $D_n \rightarrow \binom{RRep}{n} \times \binom{CRep}{n}$. For simplicity's sake we call that function $f_n \times g_n$ where $f_n(x) := \{u_i | 1 \leq i \leq n\}$ and $g_n(x) := \{w_i | 1 \leq i \leq n\}$. To explain the intuition behind this notation, vertex x represents the set $(v \setminus f_n(x)) \cup g_n(x)$. I.e. for all vertices x we have:

$$\phi_n((v \setminus f_n(x)) \cup g_n(x), f_n(x), g_n(x))$$

5.4.2 Common Axioms

Here we get into the axioms themselves. We start with the basic axioms of simple undirected graphs

$$\forall x \neg Exx \tag{1}$$

$$\forall x \forall y (Exy \rightarrow Eyx) \tag{2}$$

We also need an axiom stating that our choice of constants v, a, and b form a clique.

$$Eva \wedge Evb \wedge Eab \tag{3}$$

Next we want to give the additional axioms that give us the rook graph structure of $N(v)$.

$$\forall x(d_1(v,x) \wedge \neg Rep(x) \to \forall y \forall z((CRep(y) \wedge CRep(z) \wedge Exy \wedge Exz) \to y = z) \quad (4)$$

$$\forall x(d_1(v,x) \wedge \neg Rep(x) \to \forall y \forall z((RRep(y) \wedge RRep(z) \wedge Exy \wedge Exz) \to y = z) \quad (5)$$

Row is an equivalence relation on the set of neighbours of v:

$$\forall x(d_1(v,x) \to Row(x,x)) \quad (6)$$

$$\forall x \forall y(Row(x,y) \to Row(y,x)) \quad (7)$$

$$\forall x \forall y \forall z((Row(x,y) \wedge Row(y,z)) \to Row(x,z)) \quad (8)$$

Column is an equivalence relation on the set of neighbours of v:

$$\forall x(d_1(v,x) \to Column(x,x)) \quad (9)$$

$$\forall x \forall y(Column(x,y) \to Column(y,x)) \quad (10)$$

$$\forall x \forall y \forall z((Column(x,y) \wedge Column(y,z)) \to Column(x,z)) \quad (11)$$

Adjacent vertices share a row or a column:

$$\forall x \forall y(d_1(v,x) \wedge d_1(v,y)) \to (Exy \to (Column(x,y) \vee Row(x,y))) \quad (12)$$

Two distinct vertices can share a row or column but never both:

$$\forall x \forall y((Column(x,y) \wedge Row(x,y)) \to x = y) \quad (13)$$

Every pair of non-adjacent vertices are the opposing corners of a rectangle. This guarantees that all rows are equinumerous and all columns are equinumerous and they interact in the desired way.

$$\forall x \forall y((d_1(v,x) \wedge d_1(v,y) \wedge \neg Exy) \to$$
$$(\exists u \exists w(Row(x,u) \wedge Row(y,w) \wedge Column(x,w) \wedge Column(y,u)))) \quad (14)$$

The above axioms fully characterize the neighbourhood of v. We now want to give axiom schemas to characterize vertices further away from v. For this purpose we will be relying on the functions f_n and g_n as defined by ϕ_n.

$$\text{The function } f_n \times g_n := D_n \to \binom{RRep}{n} \times \binom{CRep}{n} \text{ is bijective.} \tag{15}$$

In L_c we can express this as:

$$\forall x(d_n(v,x) \to (\exists (u_i)_{i=1}^n \exists (w_i)_{i=1}^n (\bigwedge_{i=1}^n (CRep(u_i) \wedge RRep(w_i)) \wedge$$
$$Set_n((u_i)_{i=1}^n) \wedge Set_n((w_i)_{i=1}^n)$$
$$\phi_n(x,(u_i),w_i) \wedge \forall (u_i')\forall (w_i')(\phi_n(x,(u_i'),(w_i')) \to ((u_i) = (u_i') \wedge (w_i) = (w_i'))))))$$
$$\wedge$$
$$\forall (u_i)\forall (w_i)((\bigwedge_{i=1}^n (CRep(u_i) \wedge RRep(w_i)) \wedge Set_n(u_i) \wedge Set_n(w_i)) \to (\exists x(d_n(v,x)$$
$$\phi_n(x,(u_i),(w_i)) \wedge \forall y(\phi_n(y,(u_i),(w_i)) \to (x = y)))))$$

Let $h_n(x) := f_n(x) \cup g_n(x)$. When x and y are both at distance n from v then Exy if and only if $|h_n(x) \triangle h_n(y)| = 2$. In L_c we can express this by

$$\forall x \forall y((d_n(v,x) \wedge d_n(v,y)) \to$$
$$(Exy \leftrightarrow$$
$$(\exists u \exists u' \exists (u_i)_{i=2}^n \exists w \exists w' \exists (w_i)_{i=2}^n$$
$$(u \neq w' \wedge w \neq u' \wedge (w \neq w' \oplus u \neq u') \wedge$$
$$(\phi_n(x,u,(u_i),w,(w_i)) \wedge \phi_n(y,u',(u_i),w',(w_i)))))))$$

But for simplicity we will write it as

$$\forall x \forall y((d_n(v,x) \wedge d_n(v,y)) \to (Exy \Leftrightarrow |h_n(x) \triangle h_n(y)| = 2)) \tag{16}$$

The above condition fully specifies the distance n vertices so inductively presenting them for each n will give us a full axiomatization of the radius n balls in Johnson graphs.

5.4.3 Size axioms

Here we give the axiom schemas required to specify the size of our Johnson graphs. They interact with our previous axioms namely (15), in such a way that we can give the size of the radius r-ball around v by just how many vertices satisfy $RRep$ and $CRep$. Thus we say for $J(m,k)$:

$$(\exists x_i)_{i=1}^{m}(\bigwedge_{i=1}^{m}\bigwedge_{j=i+1}^{m} x_i \neq x_j) \wedge RRep(x_i) \tag{17}$$

$$(\exists x_i)_{i=1}^{k}(\bigwedge_{i=1}^{k}\bigwedge_{j=i+1}^{k} x_i \neq x_j) \wedge CRep(x_i) \tag{18}$$

For now we focus our attention instead on $J'(\kappa,\lambda)$ which we claim is axiomatized by the above axioms leaving m and k unbounded.

We quickly note that for $\binom{m}{k} < 3$ the Johnson graphs $J(n,k)$ don't satisfy the above axioms as our assumption that the constant symbols v, a and b can be interpreted in our model is false in those cases. This only applies to a couple of very small models whose theory can easily be described by specifying the interpretation of E.

Definition 5.4.2

Let Σ be the set of L_c-axioms (1) through (16), along with the axiom schemas (17) with $m \to \infty$, and (18) with $k \to \infty$.

Theorem 5.4.3

Let κ and λ be infinite cardinals and M be a connected graph. Then $M \models \Sigma$ if and only if $M \cong J'(\kappa,\lambda)$.

Proof. First we show that $J'(\kappa, \lambda) \models \sum$. We then show that a connected model that satisfies the above axioms fits our definition of $J'(\kappa, \lambda)$.

Assume we have $J'(\kappa, \lambda)$ with underlying set X and central vertex A. First we choose vertices for our constant symbols v, a, b. We let $v = A$ and a, b be any two neighbours of A such that $a = (A \setminus \{x\}) \cup \{y\}$ and $b = (A \setminus \{x\}) \cup \{z\}$ where $x \in A$ and $y \neq z \in X \setminus A$.

Second we want to show how the formulas defining $Rep, CRep, RRep, Row, Column$, and d_n will be interpreted. We have for any $n \in \mathbb{N}$ that $d_n(x, y)$ holds if x and y are at distance n from each other and D_n is the set of those vertices that are at distance n from v. Note that since $J'(\kappa, \lambda)$ is a connected graph we have for any vertex u that there exists an $i \in \mathbb{N}$ such that $d_i(v, u)$ holds.

We now observe that the sets of vertices satisfying $Rep, RRep, CRep$ are

$$\{(A \setminus \{i\}) \cup \{j\} | i = x \vee j = y\}$$
$$\{(A \setminus \{x\}) \cup \{j\} | j \in X \setminus A\}$$
$$\{(A \setminus \{i\}) \cup \{y\} | i \in A\}$$

respectively. Note that each of these sets has size at least ω so (17) and (18) hold. This gives us that $Row((A \setminus \{i\}) \cup \{l\}, (A \setminus \{j\}) \cup \{k\}) \iff l = k$ and $Column((A \setminus \{i\}) \cup \{l\}, (A \setminus \{j\}) \cup \{k\}) \iff i = j$. Note that this means that $Row(x, y)$ and $Column(x, y)$ align with our intuition of x and y sharing a row and column in the rook's graph respectively.

We can now go through the axioms in $\sum$ and show that $J'(\kappa, \lambda)$ satisfies each of them with the above interpretations. $J'(\kappa, \lambda)$ is a graph so it satisfies (1) and (2).

Since Row and $Column$ are equivalence relations on the rook graph induced by the neigbourhood of v, $J'(\kappa, \lambda)$ satisfies (6) through (11).

Likewise since the rook's graph has all edges within any given row and all edges within any given column and no other edges (12) holds in $J'(\kappa, \lambda)$.

Since a row and a column always meet in exactly one vertex we have $J'(\kappa, \lambda)$ satisfies (13).

If we take any two such vertices $(A \setminus \{i\}) \cup \{l\}$ and $(A \setminus \{j\}) \cup \{m\}$ with $i \neq j$ and $l \neq m$ we have that they have exactly two neighbours in D_1 in common, namely $(A \setminus \{i\}) \cup \{m\}$ and $(A \setminus \{j\}) \cup \{l\}$. So $J'(\kappa, \lambda)$ satisfies (14).

Finally we check that the axiom schemas hold for $J'(\kappa, \lambda)$.

We write any vertex of $J'(\kappa, \lambda)$ at distance n from v in the form $(A \setminus B) \cup C$ where $B \subseteq A$, $C \subseteq X \setminus A$ and $|B| = n = |C|$. Recall that f_n and g_n are components of a bijective function $D_n \to \binom{RRep}{n} \times \binom{CRep}{n}$. We say that

$$f_n((A \setminus B) \cup C) = \{(A \setminus \{x\}) \cup \{j\} | j \in C\}$$

and

$$g_n((A \setminus B) \cup C) = \{(A \setminus \{i\}) \cup \{y\} | i \in B\}$$

We note that for any n, $f_n \times g_n$ is a bijection $D_n \to \binom{RRep}{n} \times \binom{CRep}{n}$ so the schema (15) holds.

We also note that if we have 2 vertices $(A \setminus B) \cup C$ and $(A \setminus D) \cup F$ adjacent to each other then $|((A \setminus B) \cup C) \triangle ((A \setminus D) \cup F)| = 2$ which implies $|(B \cup C) \triangle (D \cup F)| = 2$. If both are at distance n from v then $|B| = |C| = |D| = |F|$ which gives us $|h_n((A \setminus B) \cup C) \triangle h_n((A \setminus D) \cup F)| = 2$ so the schema (16) holds.

We have now shown how each of the axioms in Σ is satisfied and conclude that $J'(\kappa, \lambda) \models \Sigma$.

Conversely, assume we have a connected model M that satisfies the above axioms. We want to show that M is isomorphic to $J'(\kappa, \lambda)$ for some κ and λ. To do this we will first define a bijection f from sets interpretable in M to X the underlying set of $J'(\kappa, \lambda)$. We will then use f to construct an isomorphism g from M to $J'(\kappa, \lambda)$ as L_c structures.

Since M satisfies the axioms of graph theory M is a graph with E as the edge relation.

We check the cardinality of the sets of representatives, that is $\{x \in M | RRep(x)\}$ and $\{x \in M | CRep(x)\}$. By (17) and (18) we know that $|\{x \in M | RRep(x)\}| \geq \omega$ and $|\{x \in M | CRep(x)\}| \geq \omega$. If $|\{x \in M | RRep(x)\}| \leq |\{x \in M | CRep(x)\}|$ we say $|\{x \in M | RRep(x)\}| = \lambda$ and $|\{x \in M | CRep(x)\}| = \kappa$, otherwise we say $|\{x \in M | RRep(x)\}| = \kappa$ and $|\{x \in M | CRep(x)\}| = \lambda$. We assume $|\{x \in M | CRep(x)\}| \leq |\{x \in M | RRep(x)\}|$, this assumption can not a priori be taken without loss of generality, but the argument when $|\{x \in M | RRep(x)\}| > |\{x \in M | CRep(x)\}|$ is identical with the roles of the two sets reversed. This is justified a posteriori by the observation that it is analogous to swapping $J(n, k)$ and $J(n, n-k)$. In fact the graph $J'(\kappa, \lambda)$ with underlying set X is isomorphic to the graph obtained by replacing each vertex A with $X \setminus A$ and having two vertices adjacent if and only if their symmetric difference has size 2.

We take two disjoint sets A and B in bijection with $\{x \in M | RRep(x)\}$ and $\{x \in M | CRep(x)\}$ respectively and we then say that $X = A \sqcup B$. We can think of A and B as being the set of 'rows' and 'columns' respectively.

So we now have sets X and A with $|X| = \kappa$ and $|A| = \lambda$ just as in the definition of $J'(\kappa, \lambda)$. We then say that v is the vertex corresponding to the set A from the definition of $J'(\kappa, \lambda)$.

We now want a bijection f from representatives to rows and columns: $RRep$ to A and $CRep$ to B. This requires some finessing since a satisfies $RRep$ and $CRep$ and A and B are disjoint but we can simply say that the domain of f is

$$RRep \sqcup CRep := \{(x, y) \in Rep \times \{0, 1\} | (RRep(x) \wedge y = 0) \vee (CRep(x) \wedge y = 1)\}$$

since every vertex $u \neq a$ satisfying Rep only appears in one such pair. We abuse the notation and use u to mean the pair, and for a we use a_r and a_c to distinguish between $(a,0)$ and $(a,1)$. We can now define our mapping f as

$$\begin{aligned} f(a_r) &:= CRep \\ f(a_c) &:= RRep \\ f(u) &:= \{x \in M | x = u \vee (Evx \wedge (\neg Rep(x) \wedge Eux))\} \end{aligned}$$

We note that $f^{-1}[A] = \{x \in M | RRep(x)\}$.

We can now start constructing the isomorphism g. We start with $g(v) = A$ and $g(a) = (A \setminus \{f(a_r)\}) \cup \{f(a_c)\}$. For any vertex $x \neq a$ satisfying $RRep$ we say $g(x) = (A \setminus \{f(a_r)\}) \cup \{f(x)\}$ and any vertex $x \neq a$ other than a that satisfies $CRep(x)$ corresponds to the set $(A \setminus \{f(x)\}) \cup \{f(a_c)\}$.

Our definition of the *Row* and *Column* relations and the axioms stating that they are reflexive on the neighbours of v gives us that every neighbour u of v has both a row- and column-representative denoted r_u and c_u respectively. So we have $g(u) = A \setminus \{f(r_u)\} \cup \{f(c_u)\}$.

Any vertex u in M that has distance n from v satisfies $d_n(v,u)$ so by (15) we can define $g(u) := (A \setminus f[g_n(u)]) \cup (f[f_n(u)])$.

We have now defined a full function g from M to $J'(\kappa, \lambda)$. We note that g is a bijection by our choice of A, B and (15). We now just need to check that it preserves the edge relation to prove that it is an isomorphism.

Consider two adjacent vertices u and w at distance n from v. Then by our axiom about edges at distance n we have that $|(f_n(u) \triangle f_n(w)) \cup (g_n(u) \triangle g_n(w))| = 2$ and since f is bijective that gives us that:

$$|(A \setminus f[f_n(u)]) \cup (f[g_n(u)] \setminus A) \triangle (A \setminus f[f_n(w)]) \cup (f[g_n(w)] \setminus A)| = 2$$

So the sets corresponding to u and w are adjacent in $J'(\kappa, \lambda)$.

Consider two adjacent vertices u and w at distance n and $n-1$ respectively from v. Then by our axiom about edges at distance n we have that $f_{n-1}(w) \subset f_n(u)$.

Since f is bijective that gives us that:

$$|((A \setminus f[g_n(u)]) \cup f[f_n(u)]) \triangle ((A \setminus f[g_{n-1}(w)]) \cup f[f_{n-1}(w)])| = 2$$

So the sets corresponding to u and w are adjacent in $J'(\kappa, \lambda)$.

So g is an isomorphism $M \to J'(\kappa, \lambda)$.

$\square$

So we know Σ describes Johnson graphs locally around a vertex v. We now view Σ as a set of formulae $\phi(v, a, b)$ in L, the language of graphs, by taking v, a, and b to be variables. Then if we bind each variable with a quantifier we obtain a set of sentences in L rather than L_c.

Definition 5.4.4
$\Sigma' := \{\forall v \forall a \forall b((Eva \wedge Evb \wedge Eab) \to \phi(v, a, b)) | \phi \in \Sigma\} \cup \{\forall x \exists y \exists z(Exy \wedge Exz \wedge Eyz)\}$ *is a set of sentences in* L.

For simple referencing we give a number to the new axiom.

$$\forall x \exists y \exists z(Exy \wedge Exz \wedge Eyz) \tag{19}$$

Theorem 5.4.5
Let M *be a connected model and* κ *and* λ *be infinite cardinals. Then* $M \models \Sigma'$ *if and only if* $M \cong J'(\kappa, \lambda)$.

Proof. First assume $M \cong J'(\kappa, \lambda)$. We know by Theorem 5.4.3 that there exist v, a, b pairwise adjacent such that $\phi(v, a, b)$ holds for all $\phi \in \Sigma$. We have for any triangle $v', a', b' \in M$ that there exists automorphism F on M such that $F(v) = v'$, $F(a) = a'$ and $F(b) = b'$. So $M \models \phi(v', a', b')$ for all $\phi \in \Sigma$. For any vertex u in $J'(\kappa, \lambda)$, let $x \in u, y \in X \setminus u$, and $z \in X \setminus u (y \neq z)$. We have that $u \setminus \{x\} \cup \{y\}$ and $u \setminus \{x\} \cup \{z\}$ are two adjacent neighbours of u so (19) holds in $J'(\kappa, \lambda)$. Thus $M \models \Sigma'$.

Now assume that M is a connected model such that $M \models \Sigma'$. Take $v \in M$. Then (19) gives us that v has two adjacent neighbours a and b. Then by taking v, a, b as witnesses for the formulae in Σ', 5.4.3 gives us that $M \cong J'(\kappa, \lambda)$. □

Corollary 5.4.6

$M \models \Sigma'$ if and only if M is a generalized Johnson graph.

Proof. (19) guarantees that every connected component has a triangle and then by Theorem 5.4.5 we have that each connected component is isomorphic to $J'(\kappa, \lambda)$. □

Corollary 5.4.7

Σ' is a complete set of axioms for $T_{\mathcal{J}}$.

This gives us a nice description of the models of $T_{\mathcal{J}}$, namely that they are exactly the generalized Johnson graphs. This is particularly useful since it tells us that an elementary extension of a generalized Johnson graph is also a generalized Johnson graph. In the next section we will leverage this fact to prove the stability of $T_{\mathcal{J}}$.

5.5 Stability of $T_{\mathcal{J}}$

In this section we will prove general results about $T_{\mathcal{J}}$. We know that $T_{\mathcal{J}}$ is a complete theory and we know what every model of $T_{\mathcal{J}}$ looks like, namely that by theorems 5.4.6 and 5.4.7 they are the generalized Johnson graphs.

Lemma 5.5.1

Let $M \models T_{\mathcal{J}}$, and $\phi((c_i)_{i=1}^n, x)$ a formula with $(c_i)_{i=1}^n \in M^n$. Then there is $r = r(\phi)$ such that for any d_1, $d_2 \in M$ such that they have distance greater than r from each c_i then $M \models \phi(\bar{c}, d_1) \leftrightarrow \phi(\bar{c}, d_2)$.

Proof. We will prove this by playing an Ehrenfeucht-Fraïssé game $EF_{n+\lfloor \log(r) \rfloor}(M, M)$. As before we will refer to the move made in the left copy of M in the i-th round as a_i and the move made in the right copy of M as b_i. Assume that for the first n moves $a_i = b_i = c_i$ and $a_{n+1} = d_1$ and $b_{n+1} = d_2$. Now duplicator forms a partial function f_{n+1} such that for any two c_i, c_j such that $d(c_i, c_j) < r$ we have that $c_i \triangle c_j$ is in the domain and image of f_{n+1}. This can be achieved by having f_{n+1} be the identity map everywhere it is defined. We now note that $(a_i)_{i=1}^{n+1}$, $(b_i)_{i=1}^{n+1}$ and f_{n+1} satisfy the induction hypothesis used to prove Theorem 5.3.3. Note that $d_1 \triangle c_i$ and $d_2 \triangle c_i$ are not explicitly added to the domain and image of f_{n+1} for any i. This is due to the fact that if duplicator would be playing the strategy from Theorem 5.3.6 then the set of played vertices sufficiently close to d_1 and d_2 to warrant an update would be empty by our assumption that the distance from either d_1 or d_2 to any c_i is greater than r.

So this is a possible intermediate state of the game where the game is played for $n + \lfloor \log(r) \rfloor$ rounds. So by Theorem 5.3.3 duplicator has a winning strategy. If we now say that $n + \lfloor \log(r) \rfloor$ is at least the quantifier depth of ϕ we know by Theorem 2.2.26 $M \models \phi(\bar{c}, d_1) \leftrightarrow \phi(\bar{c}, d_2)$. □

Corollary 5.5.2

If M is a generalized Johnson graph then all vertices of M have the same 1-type.

Proof. This is the case where there are no c_i so $a_1 = d_1$ and $b_1 = d_2$. □

Theorem 5.5.3

Let M be a generalized Johnson graph, let N be a non-empty union of connected components of M. Then N is an elementary substructure of M.

Proof. Let ϕ and $\bar{c} := (c_i)_{i=1}^{n}$ be such that $M \models \exists x(\phi(x, \bar{c}))$. Let d be a witness i.e. $M \models \phi(d, \bar{c})$. If $d \in N$ we are done. Assume $d \notin N$ then by Lemma 5.5.1 there is an r_ϕ such that for any e such that $\min(d(e, c_i)) \geq r$ we have $M \models \phi(d, \bar{c}) \iff \phi(e, \bar{c})$. Since N has infinite diameter there is such an e in N.

Thus any existential formula with parameters in N has a witness in N. So by the Tarski-Vaught test N is an elementary substructure of M. $\square$

Definition 5.5.4

We say that a set of vertices in a generalized Johnson graph is **small** *if it is contained in a finite union of balls of finite radius. We say that a set is* **large** *if it is not small.*

Lemma 5.5.5

Let B be a ball of radius r in a generalized Johnson graph M. Then B has Morley rank $2r$.

Proof. Since parameters in a different component than B are irrelevant we can assume that $M = J'(\kappa, \lambda)$ for some infinite cardinals κ and λ. Let v be the center of our ball B and $a, b \in N(v)$ such that Eab. From Section 5.4 we know that using those parameters we can define the column containing a in $N(v)$ as a set C and similarly the row containing a is a definable set R. We also know that for all $n \in \mathbb{N}$ we have a definable set $D_n := \{u \in M | d(u, v) = n\}$. Finally we know that we can define a bijective map $f_n \times g_n$ for each $n \in \mathbb{N}$ such that $D_n \mapsto \binom{C}{n} \times \binom{R}{n}$. More importantly both components i.e. f_n and g_n are definable.

Since the domains of the f_n for each n are pairwise distinct and have the same image we define $f = \bigcup_{i \in \mathbb{N}} f_n$ a mapping $\bigcup_{n \in \mathbb{N}} D_n \mapsto P(C)$. Similarly $g = \bigcup_{i \in \mathbb{N}} g_n$ is a mapping $\bigcup_{n \in \mathbb{N}} D_n \mapsto P(R)$.

Let G be the pointwise stabilizer of v, a and b in $Aut(M)$. We then have that G fixes R and C and induces at least $Sym(C \setminus \{a, b\}) \times Sym(R \setminus \{a, b\})$.

Let $F \subseteq M$ be a set of parameters and D be a F-definable subset of C.

Let $X = \bigcup_{c \in F} f(c)$ and $Y = \bigcup_{c \in F} g(c)$. Now take the pointwise stabilizer $G_{X \cup Y}$ of $X \cup Y$. Note that since each element of F has coordinates from $X \cup Y$ we have that $G_{X \cup Y}$ fixes F pointwise and therefore fixes D setwise. We also have that $G_{X \cup Y}$ is transitive on $C \setminus (X \cup \{a, b\})$. Thus D is finite or cofinite.

Thus C has Morley rank 1 and Morley degree 1, i.e. C is strongly minimal. Similarly R is strongly minimal.

Note that by Lemma 2.2.20 we have $MR(B) = \max\{MR(e)|e \in B\}$. Now consider a vertex e such that $d(e, v) = r$ and asume that $(f(e) \cup g(e)) \cap \{a, b\} = \emptyset$. Let $\bar{u}$ and $\bar{v}$ be the tuples from R^r and C^r representing coordinates of e i.e. $\bar{u}$ is a tuple representing $f(e)$ and $\bar{v}$ is a tuple representing $g(e)$.

Then e is definable from $\bar{u}\bar{v}$ and $\bar{u}\bar{v}$ has finitely many conjugates over e. So by [45, 6.4.1] we have that $MR(e) = MR(\bar{u}\bar{v})$.

So we must show $MR(\bar{u}\bar{v}) = 2r$. We do this by showing that $MR(C^t \times R^s) = t+s$ by strong induction on $t + s$.

Base case: We have already shown that C and R are strongly minimal so $MR(R\times C) = 2$.

Induction hypothesis: For all $1 \leq t' \leq t$ and $1 \leq s' \leq s$ such that $t' + s' < t + s$ we have:

$$MR(C^{t'} \times R^{s'}) = t' + s'$$

Inductive step: We want to describe the definable subsets of $C^t \times R^s$.

Any definable subset of $C^t \times R^s$ is a finite union of sets of the form:

$$\{(x_i)_{i=1}^t (y_i)_{i=1}^s | \phi(\bar{x}\bar{y})\}$$

where ϕ is a formula in disjunctive normal form where the atoms are of the form $x_i = \alpha$ for some $\alpha \in C$, $x_i = x_j$, $y_i = y_j$ $y_i = \beta$ for some $\beta \in R$. This follows from quantifier-elimination for the structure induced by $C \cup R$ together with the fact that any parameter definable subset of $C^t \times R^s$ can be defined using parameters from within $C \cup R$ since parameters not in $C \cup R$ can be defined from parameters inside $C \cup R$ using the formula defining f and g.

If some literal of ϕ is positive then it defines a set in bijection with some subset of $C^{t'} \times R^{s'}$ where $t' + s' < t + s$. By our induction hypothesis such a set has Morley rank at most $t' + s'$. Moreover the collection

$$\{\{(x_i)_{i=1}^t (y_i)_{i=1}^s | x_1 = \alpha\} | \alpha \in C\}$$

is a disjoint collection of sets each in bijection with $C^{t-1} \times R^s$ and thus by our induction hypothesis has Morley rank $r + s - 1$ so $C^t \times R^s$ has Morley rank at least $t + s$.

Assume X and Y are two subsets of $C^t \times R^s$ defined by ϕ and ψ respectively, where ϕ and ψ only have negative literals. Let P be the set of all parameters in ϕ and ψ. Then $C \setminus P$, and $R \setminus P$ are infinite sets, in particular they have size greater than $t + s$. Therefore there is a tuple $(\alpha_i)_{i=1}^t (\beta_i)_{i=1}^s$ such that $\alpha_i \notin P$ for all $i \leq t$, $x_i \neq x_j$ for all $i, j \leq t$, $\beta_i \notin P$ for all $i \leq s$, $y_i \neq y_j$ for all $i, j \leq s$. Then $\bar{\alpha}\bar{\beta} \in X \cap Y$ so any two sets defined by only negative literals intersect.

Therefore there do not exist infinitely many disjoint definable subsets of $C^t \times R^s$ with Morley rank at least $t + s$. So the Morley rank of $C^t \times R^s$ is exactly $t + s$, in particular the Morley rank of $C^r \times R^r$ is $2r$. Hence the Morley rank of B is $2r$.

□

Lemma 5.5.6

Let ϕ define a small set. Then ϕ has finite Morley rank.

Proof. Let C be the set defined by ϕ. Since C is small we have $C \subseteq \bigcup_{i=1}^n B_i$ where B_i are balls of finite radius. Let r_i be the radius of B_i. Then by Lemma 5.5.5 B_i has Morley rank $2r_i$. Then 2.2.19 gives us that the Morley rank of C is at most $\max((2r_i)_{i=1}^n)$. □

Theorem 5.5.7

The Morley rank of a generalized Johnson graph is ω.

Proof. We note that generalized Johnson graphs contain a ball of radius r for any $r \in \mathbb{N}$. So by Lemma 5.5.5 such a graph has Morley rank at least ω.

To show that the Morley rank is ω we need to show that there do not exist ω disjoint definable sets each of Morley rank ω. By Lemma 5.5.6 we know that such a collection would need to consist of large sets. So it is sufficient to show that there does not exist such a collection.

Let A and B be large definable sets defined by $\phi((a_i)_{i\in I}, x)$ and $\psi((b_i)_{i\in J}, x)$ respectively. Since A and B are large we get by Lemma 5.5.1 that there are some r_a and r_b such that $\phi(\bar{a}, x)$ holds for any x such that $\min\{d(x, a_i)|i \in I\} \geq r_a$ and $\psi(\bar{b}, x)$ holds for any x such that $\min\{d(x, b_i)|i \in J\} \geq r_b$. Let $r = max(r_a, r_b)$ and note that the set $C := \{x| \min(x, a_i) \geq r \wedge \min(x, b_i) \geq r\}$ is a large set such that $C \subseteq A \cap B$. As this holds for any large definable sets any two large definable sets intersect. In particular there does not exist a collection of ω pairwise disjoint definable large sets. $\square$

Having shown that any generalized Johnson graph has Morley rank ω we get the following important results about $T_{\mathcal{J}}$.

Corollary 5.5.8

The theory $T_{\mathcal{J}}$ is ω-stable so in particular $T_{\mathcal{J}}$ is dependent.

6 Hamming Graphs

In this chapter we will introduce the notion of a Hamming graph and give a bound on the VC-dimension and VC-density of the edge relation in such graphs.

6.1 Introduction to Hamming Graphs

Hamming graphs much like Johnson graphs are a highly symmetrical class of graphs that arise from association schemes. Their definitions are almost identical except Hamming graphs deal with ordered tuples instead of subsets. They are used extensively in coding theory, most notably in the development of error correcting codes.

Definition 6.1.1
Let d and q be natural numbers and S a set with $|S| = q$. The Hamming graph $H(d, q)$ is a graph whose vertices correspond to ordered d-tuples of elements from S where two vertices are adjacent if they agree in all but one coordinate.

We note that Hamming graphs are distance transitive like Johnson graphs. The automorphism group of $H(d, q)$ is the wreath product $S_q \mathrm{wr} S_d$ [35], for more details about wreath products see Chapter 8.4 in [7]. They have primarily been looked at in the case where either d or q are kept constant but our results are obtained allowing both parameters to vary.

Lemma 6.1.2
Let u and v be vertices in the Hamming Graph $H(d, q)$ with $d(u, v) = 1$.

Let $1 \leq i \leq d$ be such that u and v agree on all but the ith coordinate.

Then $w \in N(u) \cap N(v)$ if and only if w agrees with u and v in all but the ith coordinate.

Proof. Since u and v are neighbours we know that they agree in all but but one coordinate the i-th. All vertices that agree with u and v on all coordinates except the i-th will be in $N(u) \cap N(v)$.

Take any vertex w neighboring u that agrees on all coordinates except the j-th for a $j \neq i$. Now we know that w and u agree on the i-th coordinate but since u and v disagree on this coordinate we get that w and v disagree on the i-th and the j-th coordinate and therefore do not have an edge between them. Similarily any vertex neighboring v that agrees with v on the i-th coordinate will not be adjacent to w.

□

Lemma 6.1.3

Let u and v be vertices in the Hamming graph $H(d, q)$ with $d(u, v) = 1$. Then $N(u) \cap N(v)$ induces a clique of size $q - 2$.

Proof. Since u and v are neighbours we know that there is an i such that u and v agree on all but the i-th coordinate. By Lemma 6.1.2 we know that all of the vertices in $N(u) \cap N(v)$ will also agree with u and v on those coordinates. By the definition of $H(d, q)$ there are q vertices that agree on all but one coordinate and those form a clique. Neither u nor v are in $N(u) \cap N(v)$ and therefore $N(u) \cap N(v)$ induces a clique of size $q - 2$. □

Lemma 6.1.4

Let u and v be vertices in the Hamming Graph $H(d, q)$ with $d(u, v) = 2$. Let $1 \leq i < j \leq d$ be such that $u_i \neq v_i$, $u_j \neq v_j$ and $\forall k((k \neq i \wedge k \neq j) \to u_k = v_k)$. Then $w \in N(u) \cap N(v)$ if and only if

$$((w_i = v_i \wedge w_j = u_j) \vee (w_i = u_i \wedge w_j = v_j)) \wedge \forall k((k \neq i \wedge k \neq j) \to w_k = v_k = u_k)$$

Proof. Let w be a vertex in $N(u)$.

Since u and v disagree on both the j-th and the i-th coordinates any vertex $w \in N(u) \cap N(v)$ will have to agree with u on either the i-th or the j-th coordinate and with v on the other one of those. □

Lemma 6.1.4 implies the following.

Corollary 6.1.5

The open 2-neighbourhood in the Hamming Graph $H(d, 2)$ induces the 1-subdivision of the complete graph K_d.

Since $H(d, 2)$ is an induced subgraph of $H(d, q)$ for $q \geq 2$ it follows that $\mathcal{H}$ has unbounded local clique-width as mentioned in the introduction.

Lemma 6.1.6

Let u and v be vertices in the Hamming graph $H(d, q)$ with $d(u, v) = 2$. Then $N(u) \cap N(v)$ contains 2 non-adjacent vertices.

Proof. Let u and v be vertices in the Hamming graph $H(d, q)$ with $d(u, v) = 2$ and $1 \leq i < j \leq d$ be such that $u_i \neq v_i$ and $u_j \neq v_j$. By Lemma 6.1.4 we know that vertices $w \in N(u) \cap N(v)$ satisfy

$$((w_i = v_i \wedge w_j = u_j) \vee (w_i = u_i \wedge w_j = v_j)) \wedge \forall k((k \neq i \wedge k \neq j) \rightarrow w_k = v_k = u_k)$$

Without loss of generality assume $i = 1$ and $j = 2$. It is now sufficient to show that only two vertices can satisfy this formula and that they are not adjacent. Note that $(x_i)_{i=1}^d$ with $x_1 = u_1$ and $\forall i > 1(x_i = w_i)$, and $(y_i)_{i=1}^d$ with $y_1 = w_1$ and $\forall i > 1(y_i = u_i)$ satisfy the formula. They disagree in the first two coordinates so they are not adjacent. $\square$

Lemma 6.1.7

Let u and v be vertices in the Hamming graph $H(d, q)$ then

$$|N(u) \cap N(v)| = \begin{cases} d(q-1) & \text{if } d(u,v) = 0 \\ q-2 & \text{if } d(u,v) = 1 \\ 2 & \text{if } d(u,v) = 2 \\ 0 & \text{if } d(u,v) \geq 3 \end{cases}$$

6.2 VC-dimension in Hamming graphs

Theorem 6.2.1

Let A be a set of vertices in a Hamming graph shattered by the edge relation. Then $|A| \leq 3$.

Proof. Assume there is a set A' with $|A'| > 3$ which is shattered by the edge relation. Then there is a set $A \subseteq A'$ with $|A| = 4$ which is shattered by the edge relation.

Let $A = \{v_1, v_2, v_3, v_4\}$, let v be such that $N(v) \cap A = A$ and w be a vertex such that $N(w) \cap A = \{v_1, v_2, v_3\}$.

Since $v \neq w$ and $|N(v) \cap N(w)| > 2$ we have that $d(v, w) = 1$ but then the intersection of $N(v)$ and $N(w)$ is a clique.

Now we have two cases, either $v_4 = w$ or $v_4 \neq w$.

Assume $v_4 = w$. Then A induces a clique. Let u be such that $N(u) \cap A = \{v_1, v_2\}$. Since A is a clique we know that $u \notin A$. More importantly $d(u, v) = 2$ but then $N(v) \cap N(u)$ by Lemma 6.1.6 has two vertices that are not adjacent in contradiction with A being a clique.

Assume $v_4 \neq w$. Then we know that $v_4 \notin N(v) \cap N(v_1)$ since otherwise it would be in $N(w)$ in contradiction with $N(w) \cap A = \{v_1, v_2, v_3\}$. Then $d(v_4, v_1) = 2$ and similarly $d(v_4, v_2) = 2$. Let u be a vertex such that $N(u) \cap A = \{v_1, v_2, v_4\}$. Since $u \neq v$ and $|N(u) \cap N(v)| > 2$ we have by Lemma 6.1.3 that $N(u) \cap A \subseteq N(u) \cap N(v)$ is a clique in contradiction with $d(v_4, v_1) = 2$.

□

Corollary 6.2.2

The VC-dimension of the edge relation in a Hamming graph $H(d, q)$ is 3 if and only if $d \geq 2$ and $q^d \geq 10$.

Proof. It is clear that the VC-dimension is less than 3 whenever $q^d < 8$. Since $H(1, q) = K_q$, it follows that the VC-dimension is 1 when $d < 2$.

We note that for $d' \leq d$ and $q' \leq q$ we have that $H(d', q')$ is an induced subgraph of $H(d, q)$. It is therefore sufficient to show that $H(2, 3)$ and $H(3, 2)$ have no shattered set of size 3, and $H(2, 4)$, $H(3, 3)$, and $H(4, 2)$ do have a shattered set of size 3 as shown in Figure 14. Note that a set A shattered by the edge relation must have $A \subseteq N(v)$ for some vertex v. Moreover since Hamming graphs are vertex transitive we have that for any vertex u there is an automorphism f such that $u = f(v)$ and $f[A] \subseteq N(u)$. In other words, if there is a shattered set A of size 3 then for any vertex u there is some shattered set A' of size 3 such that $A' \subseteq N(u)$. It is therefore sufficient to show that for a given vertex v, no subset of $N(v)$ of size 3 is shattered.

$H(2, 3)$ The neighbourhood of a vertex v in $H(2, 3)$ has 4 vertices each adjacent to exactly 1 other vertex in $N(v)$. Thus a 3 element subset A of $N(v)$ contains two adjacent vertices u, w and one vertex connected to neither of them. By Lemma 6.1.4 we get that $|N(u) \cap N(w)| = 1$ which gives us $N(u) \cap N(w) = \{v\}$. Since $N(v) \cap A = A$ there is no vertex v' such that $N(v') = \{u, w\}$.

$H(3, 2)$ Since for any vertex v in $H(3, 2)$ we have $|N(v)| = 3$ then a set A of size 3 shattered by the edge relation must have the form $A = N(v)$ for some v. It is therefore sufficient to show that $N(v)$ is not shattered for some v. Without loss of generality assume $v = (a, a, a)$. Observe that $N((a, a, a)) = \{(a, a, b), (a, b, a), (b, a, a)\}$ and any neighbour of (a, a, b) is also a neighbour of (a, b, a) or (b, a, a) thus there does not exists a vertex u such that $N(u) \cap N((a, a, a)) = \{(a, a, b)\}$ so $N((a, a, a))$ is not a shattered set, and thus $H(3, 2)$ has no shattered set of size 3.

$\square$

6.3 VC-density in Hamming graphs

In this section we compute an exact value for the VC-density of the edge relation on Hamming graphs.

Theorem 6.3.1

The VC-density of the edge relation on the class of all Hamming graphs is 2.

Proof. We need to show that the shatter function for the edge relation $\pi(n) \in \mathcal{O}(n^2)$. First we observe that for $d < 1$ a set such that all vertices agree on all but 2 coordinates has the property that $\forall u, v \in A\ \exists w(A \cap N(w) = \{u, v\})$ and since the number of pairs grows quadratically in n, the VC-density is at least 2.

We prove it is at most two by giving a bound on a recursive formula for $\pi(n)$ and showing that it has a $\mathcal{O}(n^2)$ closed form.

Let A be a maximally shattered set of size n in the Hamming graph $H(d, q)$. Let $v \in A$. Let $S_1 = \{A \cap S | S \in \mathcal{S} \wedge v \in S\}$ and $S_2 = \{A \cap S | S \in \mathcal{S} \wedge v \notin S\}$. Note that $|S_1 \cup S_2| = \pi(n)$ and $|S_2| \leq \pi(n-1)$.

Note that every member of S_1 is a neighborhood of neighbour of v. We also note that $N(v)$ induces a disjoint union of d copies of K_{q-1} with no edges between them.

Let

$$
\begin{aligned}
D_0 &= \{v\} \\
D_1 &= A \cap N(v) \\
D_2 &= \{u \in A | d(u, v) = 2\} \\
D_3 &= \{u \in A | d(u, v) > 2\}
\end{aligned}
$$

D_3 intersects no member of S_1 by definition of D_3. Every element of D_2 can be a member of at most 2 sets of S_1 thus the total number of distinct sets containing v and intersecting D_2 is $2|D_2| < 2n$.

Since we have counted all members of S_1 that intersect D_2, and no members of S_1 intersect D_3 we only have left to count those members of S_1 that are subsets of $D_0 \cup D_1$. As mentioned earlier $H(d,q)[N(v)]$ is a disjoint union of d copies of K_{q-1}. Let $(Q_i)_{i=1}^d$ be such that for each i Q_i is all those vertices $u \in D_1$ that disagree with v in the i-th coordinate. Note that $D_1 = \bigcup_{i=0}^d Q_i$ and any element of S_1 which is a subset of $D_0 \cup D_1$ is a subset of $D_0 \cup Q_i$ for some i.

Moreover every subset of $D_0 \cup Q_i$ that is an element of S_1 is either:

$D_0 \cup Q_i \setminus \{u\}$ for some $u \in Q_i$, or $D_0 \cup Q_i$, or $\{v\}$, thus the number of distinct elements of S_1 contained in $D_0 \cup D_1$ is at most

$$\sum_{i=1}^{d} |Q_i| + \min(d,n) + 1 = |D_1| + \min(d,n) + 1 \leq n + \min(d,n) + 1$$

So we have

$$\begin{aligned}
\pi(n) &= |A| \\
&= |S_1| + |S_2| \\
&\leq |S_1| + \pi(n-1) \\
&\leq 2|D_2| + |D_1| + \min(d,n) + 1 + \pi(n-1) \\
&\leq 2n + n + n + 1 + \pi(n-1) \\
&\leq 4n + 1 + \pi(n-1)
\end{aligned}$$

Solving the recurrence relation we get $\pi(n) \leq 4n^2 + n$ and thus $\pi(n) \in \mathcal{O}(n^2)$. This tells us that the VC-density is at most 2.

We have thus demonstrated that the VC-density of the edge relation on the class of all Hamming graphs is at least 2 and at most 2 and conclude it must be 2. $\square$

We do not in this thesis provide a complete model-theoretic account of the limit theory of Hamming graphs but remark that some things are known.

Theorem 6.3.2 ([3, 4.10])

$H(\aleph_0, 2)$ is a ω-stable with Morley rank 1.

$H(2,4)$

$$A = \{(a,a),(a,b),(a,c)\}$$
$$A \cap N((d,d)) = \emptyset$$
$$A \cap N((b,a)) = \{(a,a)\}$$
$$A \cap N((b,b)) = \{(a,b)\}$$
$$A \cap N((b,c)) = \{(a,c)\}$$
$$A \cap N((a,a)) = \{(a,b),(a,c)\}$$
$$A \cap N((a,b)) = \{(a,a),(a,c)\}$$
$$A \cap N((a,b)) = \{(a,a),(a,b)\}$$
$$A \cap N((a,d)) = A$$

$H(3,3)$

$$A = \{(a,a,b),(a,b,a),(b,a,a)\}$$
$$A \cap N((b,b,b)) = \emptyset$$
$$A \cap N((a,a,c)) = \{(a,a,b)\}$$
$$A \cap N((a,c,a)) = \{(a,b,a)\}$$
$$A \cap N((c,a,a)) = \{(b,a,a)\}$$
$$A \cap N((a,b,b)) = \{(a,a,b),(a,b,a)\}$$
$$A \cap N((b,b,a)) = \{(a,a,b),(b,a,a)\}$$
$$A \cap N((b,a,b)) = \{(a,a,a),(a,a,b)\}$$
$$A \cap N((a,a,a)) = A$$

$H(4,2)$

$$A = \{(a,a,a,b),(a,a,b,a),(a,b,a,a)\}$$
$$A \cap N((a,a,a,a)) = A$$
$$A \cap N((b,a,a,b)) = \{(a,a,a,b)\}$$
$$A \cap N((b,a,b,a)) = \{(a,a,b,a)\}$$
$$A \cap N((b,b,a,a)) = \{(b,b,a,a)\}$$
$$A \cap N((a,a,b,b)) = \{(a,a,a,b),(a,a,b,a)\}$$
$$A \cap N((a,b,a,b)) = \{(a,a,a,b),(a,b,a,a)\}$$
$$A \cap N((a,b,b,a)) = \{(a,a,b,a),(a,b,a,a)\}$$
$$A \cap N((b,b,b,b)) = \emptyset$$

Figure 14: Examples of shattered sets of size 4 in $H(2,4)$, $H(3,3)$ and $H(4,2)$.

7 Future work

In this chapter we will give a brief recap of the results of this thesis along with suggestions of research questions that arose from the work and would be natural direction to take further research.

In this thesis we have given values for VC-dimension and VC-density of the edge relation in Johnson and Hamming graphs. We also looked at the limit theory of Johnson graphs, both for a fixed k and letting both parameters vary. Work analogous to what is presented in Chapter 5 can be carried out for Hamming graphs. An interesting thing to note is that for Hamming graphs $H(d, q)$ we can choose to keep either d or q constant and let the other vary to get a limit theory, in addition to the general case where both d and q tend to infinity.

There is a third class of graphs that would be a natural next step to look at.

Definition 7.0.1

Let $n, k \in \mathbb{N}$ and q an integer that is a power of some prime. We define the graph $F_{n,k,q}$ by taking as vertices the k-dimensional subspaces of the n-dimensional vector space $V(n, q)$ over a finite field $\mathbb{F}_q$. Two vertices are adjacent in $F_{n,k,q}$ if and only if their corresponding subspaces intersect in a $k-1$-dimensional subspace.

All of the work done on Johnson and Hamming graphs in this thesis could be carried out on these graphs as well.

First order model checking is known to be tractable on classes of finite twin-width. As we noted in the introduction our Theorem 5.2.6 joined with [44, 8.4] gives us that $\mathcal{J}$ does not have bounded twin-width. But by Theorem 5.5.7 $T_{\mathcal{J}}$ is stable which is a strong tameness condition, this raises the question of tractability for first order model checking on Johnson graphs.

Early on in this thesis we gave some preliminary results on how various graph operations affect the VC-dimension of the edge relation. In most cases we concluded that the operations could, depending on circumstance, lead to an increase or decrease in the VC-dimension. This work could be expanded, for instance to check if there is a limit on how much the VC-dimension can change after each operation.

Due to the simplicity of the language of graphs it would also be interesting to research if it is possible to compute the VC-dimension of a formula from its subformulae. Such an arithmetic would be a powerful tool to assess whole theories just from understanding the VC-dimension of the edge relation.

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