Weighted Simultaneous Diophantine Approximation in a variety of settings

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Abstract

This thesis considers weighted simultaneous Diophantine approximation in a variety of settings, including approximation over real manifolds, p-adic manifolds and p-adic coordinate hyperplanes. In each of these lower bounds on the Hausdorff dimension are obtained via appropriate Mass Transference Principle theorems. Weighted simultaneous approximation sets are often described by $\limsup$ sets of rectangles, so Mass Transference Principles on rectangles are favoured. Examples of these include the Mass Transference Principle from balls to rectangles [112], and the Mass Transference Principle from rectangles to rectangles [111].

Chapters 1 and 2 provide an introduction to real and p-adic Diophantine approximation. Chapter 3 introduces the Mass Transference Principle, given by Beresnevich and Velani [28], and recent variations. These Theorems are vital in the proofs of results in later chapters. In Chapter 4, Diophantine approximation over manifolds is introduced and a survey of recent results is given. It the latter part of the chapter a Dirichlet style Theorem for $\boldsymbol{\tau}$-approximable points over manifolds is proven, which generalises a similar result in [22]. Such result allows us to apply a Mass Transference Principle result and obtain a lower bound on the Hausdorff dimension of weighted $\boldsymbol{\tau}$-approximable points over manifolds. In Chapter 5, a variety of results in p-adic weighted Diophantine approximation are proven. Furthermore, a similar result to that established in Chapter 4 is proven for p-adic approximable points over manifolds. In Chapter 6 the Hausdorff dimension of p-adic approximable points over coordinate hyperplanes is proven. The result relies on a count for the number rational approximations to a p-adic integer, which is proven using p-adic approximation lattices. The thesis is concluded by providing a brief survey on S-arithmetic Diophantine approximation. This is followed by a discussion on how results found throughout the thesis could be replicated in the S-arithmetic setting.

Contents

Declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

Chapter 4 is essentially the contents of:

V. Beresnevich, J. Levesley, B. Ward *A lower bound for the Hausdorff dimension of the set of weighted simultaneously approximable points over manifolds*, to appear in International Journal of Number Theory, ISSN 1793-0421.

Chapter 5 is essentially the contents of:

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Chapter 1

Introduction

1.1 Classical Diophantine Approximation

Diophantine approximation is essentially the study of how well real numbers can be approximated by rational points. Dirichlet [55] proved that for any real number $x \in \mathbb{R}$ and natural number $Q \in \mathbb{N}$ there exists integers $p, q \in \mathbb{Z}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ},$$

where $1 \leq q \leq Q$. This result leads to the corollary that for any $x \in \mathbb{R}$ there are infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| < q^{-2}. \tag{1.1}$$

The question then arises as to whether the approximation on the right of (1.1) is best possible. For example, would the theorem hold when approximated by Cq^{-2} for some arbitrary constant $C > 0$? The answer is no. As proven by Hurwitz, for $C < 1/\sqrt{5}$, there exist real numbers such that there are only finitely many integer pairs (p, q) solving (1.1) with the right hand side replaced with Cq^{-2} [71].

At this point we can classify numbers $x \in \mathbb{R}$ into two different groups, the set of badly approximable numbers, **Bad**, and the set of well approximable numbers. If $x \in \mathbf{Bad}$ then there exists a real number $c > 0$ such that for all $\frac{p}{q} \in \mathbb{Q}$

$$\left| x - \frac{p}{q} \right| \geq cq^{-2}.$$

If $x \notin \mathbf{Bad}$ then x is well approximable. Rather than improving the constant c one can consider improving the exponent of approximation on q. This set is called the set of very well approximable numbers, **VWA**. Concisely, if x has infinitely many rational points $\frac{p}{q}$ solving

$$\left| x - \frac{p}{q} \right| < q^{-2-\varepsilon},$$

for some $\varepsilon > 0$, then x is said to be very well approximable. By the result of Hurwitz we know that **Bad** is non-empty, for example it contains $\frac{1+\sqrt{5}}{2}$. Further, a result from the theory of continued fractions states that for any irrational x with bounded partial coefficients then $x \in$ **Bad**, thus **Bad** is at least countably infinite. Similar results appear for the set **VWA**, as an easy example $\mathbb{Q} \subseteq$ **VWA**. In the following section we will find that both **Bad** and **VWA** are in fact of Lebesgue measure zero.

Motivated by (1.1) we introduce the set of ψ-approximable points, whereby we replace a q^{-1} on the right hand side of inequality (1.1) by a general approximating function $\psi : \mathbb{N} \to \mathbb{R}_+$ to give us the inequality

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}. \tag{1.2}$$

Define the set of ψ-approximable numbers $\mathcal{W}(\psi)$ as follows. Denote by $\mathcal{A}_q(\psi)$ the set

$$\mathcal{A}_q(\psi) := \bigcup_{p=1}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I},$$

where $\mathbb{I} = [0, 1]$ and for $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$ $B(x, y)$ denotes the open ball on $\mathbb{R}$ with centre x and radius y. Then define the set of ψ-approximable points as

$$\mathcal{W}(\psi) := \limsup_{q\to\infty} \mathcal{A}_q(\psi),$$

that is, the set of real numbers in $\mathbb{I}$ that lie in infinitely many balls with rational centres $\frac{p}{q} \in \mathbb{I}$ and radius $\frac{\psi(q)}{q}$. Remark here that we only consider the real numbers contained within the unit interval, however, the setup naturally extends to the real line. The subset $\mathbb{I}$ is chosen here because $\mathcal{W}(\Psi)$ is translation invariant by integers, for example if $x \in \mathbb{I}$ is ψ-approximable then naturally $x + n$ is also ψ-approximable for any integer $n \in \mathbb{Z}$.

In some cases we wish to simplify the set of approximation functions by only considering those of the form $\psi(q) = q^{-\tau}$ for $\tau \in \mathbb{R}_+$. In this case we will use the notation $\mathcal{W}(\tau) = \mathcal{W}(\psi)$ and refer to $\mathcal{W}(\tau)$ as the set τ-approximable points.

1.1.1 Theorems of Khintchine and Duffin-Schaeffer

To understand how well general points in $\mathbb{R}$ can be approximated by some approximation function ψ we appeal to results of metric Diophantine approximation. By Dirichlet's theorem when $\psi(q) = q^{-1}$ we have that $\mathcal{W}(\psi) = \mathbb{I}$. In the previous section we considered whether the approximation on the right hand side of (1.1) could be improved by a constant. A second natural question to ask is whether the exponent of -2 from (1.1) can be improved. For almost all $x \in \mathbb{I}$ the answer is no, as proven by Khintchine [77]. Let λ denote Lebesgue measure, then Khintchine's Theorem reads as follows.

Theorem 1.1.1. *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a monotonic decreasing approximation function. Then*

$$\lambda(\mathcal{W}(\psi)) = \begin{cases} 0 \ \ \textit{if} & \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 \ \ \textit{if} & \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

Probabilistically, this gives us a surprising result. A real number $x \in \mathbb{I}$ is contained in the set $\mathcal{W}(\psi)$ with either probability 1 or probability 0, depending on the approximation function. This result is encompassed by Cassels zero-one law [48], which states that $\lambda(\mathcal{W}(\psi)) \in \{0, 1\}$ for all approximation functions ψ. The zero-one law has many equivalent results in a variety of different setting, see [31] for examples of various zero-one laws and their proofs. What Theorem 1.1.1 shows in particular is that **Bad** and **VWA** are both of Lebesgue measure zero. The fact that $\lambda(\mathbf{VWA}) = 0$ follows immediately by the convergence case, and the fact that $\lambda(\mathbf{Bad}) = 0$ follows from the observation that **Bad** is contained in the compliment of $\mathcal{W}(\psi)$ for $\psi(q) = (q \log q)^{-1}$ which has full measure by the divergence case of Theorem 1.1.1.

Observe that Theorem 1.1.1 is only applicable to monotonic decreasing approximation functions. In order to generalise to all approximation functions we need to slightly alter the set $\mathcal{W}(\psi)$. In particular we wish to only consider points that can be approximated by infinitely many reduced fractions. The following setup construct such a lim sup set. Let

$$\mathcal{A}'_q(\psi) = \bigcup_{\substack{0 \leq p \leq q \\ gcd(p,q)=1}} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I},$$

then we define $\mathcal{W}'(\psi)$ as

$$\mathcal{W}'(\psi) = \limsup_{q \to \infty} \mathcal{A}'_q(\psi).$$

Clearly $\mathcal{W}'(\psi) \subset \mathcal{W}(\psi)$. In 1941 Duffin-Schaeffer [56] conjectured a Khintchine style theorem for $\mathcal{W}'(\psi)$ for ψ a non-monotonic approximation function, and further gave an explicit counterexample as to why the setup $\mathcal{W}(\psi)$ was insufficient to deal with non-monotonic functions. Gallagher [63] proved a zero-one law for this set, that is $\lambda(\mathcal{W}'(\psi)) \in \{0, 1\}$, however the complete conjecture remained unsolved for decades. Recently the conjecture was proven by Maynard and Koukoulopoulos [81].

Theorem 1.1.2. *For $\psi : \mathbb{N} \to \mathbb{R}_+$, if*

$$\sum_{q=1}^{\infty} \frac{\varphi(q)}{q} \psi(q) = \infty,$$

where φ is the Euler phi function, then $\lambda(\mathcal{W}'(\psi)) = 1$, and if the above sum converges then $\lambda(\mathcal{W}'(\psi)) = 0$.

In a similar manner to Khintchine's theorem the convergence case follows by the Borel-Cantelli Lemma. The divergent case is considerably harder, so much so that the proof involved a combination of many areas of mathematics including Graph theory, analysis, and arithmetic combinatorics. Theorem 1.1.2 is a fundamental result in metric number theory. Due to the recent proof of this conjecture many other theorems that hinged on the result of the Duffin-Schaeffer conjecture now also follow. In most of the following sections we will find a Duffin-Schaeffer style result that follows from Theorem 1.1.2.

1.1.2 The Borel-Cantelli Lemmas

For many of the Lebesgue measure statements above a key ingredient in the proofs are the Borel-Cantelli Lemmas from probability theory. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, with measure $\mu(\Omega) < \infty$.

Lemma 1.1.3 (Borel-Cantelli Convergence [42] [47]). *Let $\{E_i\}$ be a family of measurable subsets in Ω and suppose that*

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$

Then,

$$\mu\left(\limsup_{i\to\infty} E_i\right) = 0.$$

A straightforward application of Lemma 1.1.3 provides the convergence case of Theorems 1.1.1-1.1.2 above. The convergent case is the easy part of most Khintchine-style theorems due to the above lemma. The following result, proven by Kochen and Stone [80], compliments Lemma 1.1.3.

Lemma 1.1.4 (Borel-Cantelli Divergence [61]). *Let $\{E_i\}$ be a family of measurable subsets in Ω. Suppose that*

$$\sum_{i=1}^{\infty} \mu(E_i) = \infty.$$

and

$$\limsup_{n\to\infty} \frac{\left(\sum_{i=1}^{n} \mu(E_i)\right)^2}{\sum_{i,j=1}^{n} \mu(E_i \cap E_j)} \geq C \tag{1.3}$$

for some $C > 0$. Then $\mu\left(\limsup_{i\to\infty} E_i\right) \geq C$.

We remark that previous iterations of Lemma 1.1.4 had been proven prior to the result of Kochen and Stone, including versions by Erdos & Renyi [58] and Lamperti [83] to name a few. For a brief history on the developments of Lemma 1.1.4 see [27].

Condition (1.3) is referred to as quasi-independence on average. While showing a set satisfies (1.3) is not always straightforward, there are several results which make it more applicable. Firstly, if the family of measurable subsets $\{M_i\}$ are pairwise independent in the probabilistic sense, that is

$$\mu(E_i \cap E_j) = \mu(E_i)\mu(E_j) \quad \forall i \neq j,$$

then if $\sum_{i=1}^{\infty} \mu(E_i) = \infty$, we have that

$$\mu\left(\limsup_{i\to\infty} E_i\right) = 1.$$

Secondly, if there exists some zero-one law on the measure space, then we would only need to show that

$$\limsup_{n\to\infty} \frac{\left(\sum_{i=1}^{n} \mu(E_i)\right)^2}{\sum_{i,j=1}^{n} \mu(E_i \cap E_j)} > 0$$

in order to prove the divergence case of a Khintchine type theorem.

1.1.3 Hausdorff measure and dimension

Both Theorem 1.1.1 and Theorem 1.1.2 give a complete result in terms of the Lebesgue measure of the set of ψ-approximable points. However, in both cases they fall short when differentiating between two approximating functions where the sum

$$\sum_{q=1}^{\infty} \psi(q)$$

converges. For example, by Theorem 1.1.1 we know that $\lambda(\mathcal{W}(3)) = \lambda(\mathcal{W}(100)) = 0$. Intuitively we would expect the set $\mathcal{W}(100)$ to be far smaller than $\mathcal{W}(3)$. Using Hausdorff measure and Hausdorff dimension in place of Lebesgue measure can provide a more accurate representation of the size of $\mathcal{W}(\psi)$.

We adopt the following conventions when defining Hausdorff measure and Hausdorff dimension. For a locally compact metric space (U, d), a subset $X \subset U$, and $\rho > 0$, define a ρ-cover of X as a sequence of balls $\{B_i\}$ such that $X \subset \bigcup_i B_i$, with all balls $r(B_i) \leq \rho$, where $r(B)$ denotes the radius of the ball B. We define a dimension function $f : \mathbb{R}_+ \to \mathbb{R}_+$ as an increasing function with $f(r) \to 0$ as $r \to 0$. Define

$$\mathcal{H}^f_\rho(X) = \inf \left\{ \sum_i f(r(B_i)) : \{B_i\} \text{ is a } \rho - \text{cover of } X \right\},$$

where the infimum is take over all ρ-covers of X. Clearly, as ρ decreases there are less possible ρ-covers of X, hence the f-Hausdorff measure can be well defined by

$$\mathcal{H}^f(X) = \lim_{\rho\to 0^+} \mathcal{H}^f_\rho(X).$$

The dimension function is usually taken to be $f(x) = x^s$ for some $s \in \mathbb{R}_+$, and denoted by $\mathcal{H}^s$. With this notation we define the Hausdorff dimension as

$$\dim X = \inf \{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

We note several properties of Hausdorff measure and Hausdorff dimension that follow from their definitions. Most of these results and their proofs can be found in Chapter 3 of [59]. Firstly, observe that $\mathcal{H}^s$ is monotonic. That is, for any $E \subset F$ we have that $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$. Secondly, by the definition of $\mathcal{H}^s$, for any single point $x \in \mathbb{R}^n$, we have that $\mathcal{H}^0(x) = 1$ and $\mathcal{H}^s(x) = 0$ for all $s > 0$ (we make a slight abuse of notation here, since $\mathcal{H}^s$ is defined for sets when x a point we write $\mathcal{H}^s(\{x\}) = \mathcal{H}^s(x)$). This further implies that for any countable set X we have that $\mathcal{H}^s(X) = 0$ for all $s > 0$. We also have the following useful Lemma which is very helpful in the application of the Mass Transference Principle (see Chapter 3).

Lemma 1.1.5. *For any subset $X \subset \mathbb{R}^n$ the n-dimensional Hausdorff measure $\mathcal{H}^n(X)$ is equal to the n-dimensional Lebesgue measure $\lambda_n(X)$, up to a constant multiple.*

We can see this result follows clearly on the definitions of both Lebesgue and Hausdorff measure. The following (Proposition 3.1 of [59]) essentially states that the Hausdorff measure behaves well under Lipschitz mappings.

Proposition 1.1.6. *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^n$, such that for all $x, y \in F$,*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

for some constants $c, \alpha > 0$. Then, for each s

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F).$$

Using the above proposition we can obtain many results, one of particular importance to us is that the Hausdorff measure is translation invariant. That is, for any $x \in \mathbb{R}^n$ if we define $F + x = \{a + x : a \in F\}$, then $\mathcal{H}^s(F + x) = \mathcal{H}^s(F)$.

Clearly, by the connection between Hausdorff measure and Hausdorff dimension several of the above results have Hausdorff dimension counterparts. By the monotonicity of $\mathcal{H}^s$, we have that dim is monotonic, that is for any $E \subset F$, $\dim E \leq \dim F$. As mentioned, if X is a countable set of points the $\mathcal{H}^0(X)$ is the cardinality of X, but for $s > 0$ $\mathcal{H}^s(X) = 0$, and so $\dim X = 0$ for countable set X. At the opposite end of the scale, for any $X \subset U = \mathbb{R}^n$ we have that $\dim X \leq \dim \mathbb{R}^n = n$, with equality reached whenever X is an open subset of $\mathbb{R}^n$. The following proposition is the counterpart to Proposition 1.1.6 for Hausdorff dimension [59].

Proposition 1.1.7. *Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$, which satisfies*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

for all $x, y \in F$, where $c, \alpha > 0$ are constants. Then

$$\dim f(F) \leq \frac{1}{\alpha}\dim F.$$

In particular, if f is a bi-Lipschitz transformation, that is

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|,$$

for constants $c_1, c_2 > 0$, then

$$\dim f(F) = \dim F.$$

This Proposition is particularly useful when considering Diophantine approximation on manifolds, as we shall see in Chapters 3-5.

1.1.4 Theorems of Jarnik and Besicovitch

We now return to Diophantine approximation and provide results for both the Hausdorff measure and Hausdorff dimension of $\mathcal{W}(\psi)$. For the Hausdorff dimension of $\mathcal{W}(\psi)$ we have the result by Jarnik [75] and Besicovitch [38] who independently proved the following.

Theorem 1.1.8. *Let $\tau \geq 1$, then*

$$\dim \mathcal{W}(\tau) = \frac{2}{1+\tau}.$$

The condition $\tau \geq 1$ is due to Dirichlet's Theorem, since for $\tau < 1$ we clearly have $\mathcal{W}(\tau) = \mathbb{I}$ and so $\dim \mathcal{W}(\tau) = 1$. At this point we note the usefulness of the Hausdorff dimension. Going back to our example at the beginning of the section we see that $\dim \mathcal{W}(100) = \frac{2}{101} < \frac{2}{4} = \dim \mathcal{W}(3)$ as expected.

For the Hausdorff measure Jarnik proved the following theorem [75].

Theorem 1.1.9. *Let f be a dimension function such that $r^{-1}f(r) \to \infty$ as $r \to 0$ and $r^{-1}f(r)$ is decreasing. Suppose $\psi : \mathbb{N} \to \mathbb{R}_+$ is a monotonic decreasing approximation function with*

$$r^{-1}\psi(r) \ \text{ and } r^2 f\left(\frac{\psi(r)}{r}\right) \ \text{ decreasing }, \quad r\psi(r) \to 0 \ \text{ as } \ r \to \infty. \tag{1.4}$$

Then

$$\mathcal{H}^f(\mathcal{W}(\psi)) = \begin{cases} 0 \ \text{ if } \ \sum_{r=1}^{\infty} rf\left(\frac{\psi(r)}{r}\right) < \infty, \\ \infty \ \text{ if } \ \sum_{r=1}^{\infty} rf\left(\frac{\psi(r)}{r}\right) = \infty. \end{cases}$$

We remark that the conditions (1.4) on ψ were originally imposed by Jarnik, however it was proven in [18] that ψ being monotonic is a sufficient condition.

By setting $f(r) = r^s$ and $\psi(q) = q^{-\tau}$ it can be seen that Theorem 1.1.8 easily follows from Theorem 1.1.9. In fact Theorem 1.1.9 goes one step further and proves that for $s = \frac{2}{1+\tau}$ (with $\tau > 1$) that $\mathcal{H}^s(\mathcal{W}(\tau)) = \infty$. As we shall see in Chapter 3 both of these theorems are implied by Theorem 1.1.1 via the Mass Transference Principle. Further, as proven in [28], the Mass Transference Principle can also be used to prove the Hausdorff measure analogue of Theorem 1.1.2.

1.2 n-dimensional Diophantine approximation

The results of the previous section illustrate that classical Diophantine approximation is largely complete with respect to the Lebesgue and Hausdorff measure. The following section gives a brief layout and overview of results for Diophantine approximation in n-dimensional space. There are several alternative setups to consider in higher dimensions. We begin by defining each setup and the relationships between them, and then discuss the corresponding Lebesgue and Hausdorff measure results.

1.2.1 Weighted simultaneous approximation

The first form of approximation we will be focusing on is simultaneous approximation. We will give a little more detail when defining this setup as it will be the main form of approximation in later Chapters. Let $\Psi = (\psi_1, \dots, \psi_n)$ be an n-tuple of approximation functions $\psi_i : \mathbb{N} \to \mathbb{R}_+$, for $1 \leq i \leq n$, $q \in \mathbb{N}$ and let

$$\mathcal{A}_q^{(n)}(\Psi) := \bigcup_{\substack{0 \leq p_i \leq q \\ i=1,\dots,n}} \left\{ \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{I}^n : \left| x_i - \frac{p_i}{q} \right| < \frac{\psi_i(q)}{q}, \ 1 \leq i \leq n \right\}.$$

Define the set of weighted simultaneously approximable points as

$$\mathcal{W}_n(\Psi) := \limsup_{q \to \infty} \mathcal{A}_q^{(n)}(\Psi).$$

The original use of weighted comes from a slightly different setup where the base approximation function ψ is the same, but a weight vector would be applied (a vector $(t_1, \dots, t_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n t_i = 1$) so that along each ith coordinate axis there would be a weighted approximation function of the form $\psi(q)^{t_i}$, see for example [3, §5.1]. In this case we use it to simply mean that the approximation function in each coordinate axis could be different.

In the special case where the approximation functions are the same in each component (i.e. $\psi = \psi_1 = \cdots = \psi_n$), then this is called simultaneous approximation, we denote this special case by $\mathcal{W}_n(\psi)$. When each approximation function is the same we may use balls to define $\mathcal{A}_q^{(n)}(\psi)$. Let $|\boldsymbol{x}| = \max |x_i|$ denote the sup norm, then for $\boldsymbol{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$ define the n-dimensional open ball as

$$B(\boldsymbol{x}, r) = \left\{ \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n : |\boldsymbol{y} - \boldsymbol{x}| < r \right\}.$$

So for simultaneous approximation we may equivalently define $\mathcal{W}_n(\psi)$ as

$$\mathcal{W}_n(\psi) = \limsup_{q \to \infty} \mathcal{A}_q^{(n)}(\psi) = \limsup_{q \to \infty} \bigcup_{\substack{0 \leq p_i \leq q \\ i=1,\dots,n}} B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q} \right),$$

where $\frac{\mathbf{p}}{q} = \left(\frac{p_1}{q}, \dots, \frac{p_n}{q} \right)$. Since $\mathcal{W}_n(\psi)$ can be described by a lim sup set of balls many results are much easier to prove in comparison to $\mathcal{W}_n(\Psi)$. The reasoning being that many definitions of measures and measure theoretic results are heavily based on covers of balls. The set $\mathcal{W}_n(\Psi)$ is more easily described by a lim sup set of hyperrectangles. This means simple results for $\mathcal{W}_n(\psi)$, such as the convergence case of Khintchine style theorems, become less obvious for $\mathcal{W}_n(\Psi)$. To overcome this issue the following geometrical idea is used. For simplicity assume each approximation function is of the form $\psi_{\tau_i}(q) = q^{-\tau_i}$. Given a vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}_+^n$ with $\tau_1 \geq \cdots \geq \tau_n \geq 0$ and let $\psi_{\boldsymbol{\tau}} = (\psi_{\tau_1}, \dots, \psi_{\tau_n})$ be the n-tuple of approximation functions. For a rational point $\frac{\boldsymbol{p}}{q} \in \mathbb{Q}^n$ define the hyperrectangle

$$R\left(\frac{\mathbf{p}}{q}, \frac{\psi_{\boldsymbol{\tau}}(q)}{q} \right) := \left\{ (y_1, \dots, y_n) \in \mathbb{R}^n : \left| y_i - \frac{p_i}{q} \right| < q^{\tau_i - 1}, \ 1 \leq i \leq n \right\}.$$

Then we have that

$$\mathcal{W}_n(\psi_{\boldsymbol{\tau}}) = \limsup_{q\to\infty} \bigcup_{\substack{0\leq p_i\leq q \\ i=1,\dots,n}} R\left(\frac{\mathbf{p}}{q}, \frac{\psi_{\boldsymbol{\tau}}(q)}{q}\right).$$

Note that $R\left(\frac{\mathbf{p}}{q}, \frac{\psi_{\boldsymbol{\tau}}(q)}{q}\right)$ can be covered by a collection $\mathcal{B}$ of balls of radius $q^{-\tau_j-1}$ for each $1 \leq j \leq n$, where the cardinality of $\mathcal{B}$ is bounded above by q^k, with

$$k = \sum_{i=j}^{n} (\tau_j - \tau_i).$$

and so $\mathcal{A}_q^{(n)}(\psi_{\boldsymbol{\tau}})$ can be covered by q^{n+k} balls of radius $q^{-\tau_j-1}$. This sort of cover is particularly useful when obtaining upper bounds on the Hausdorff measure of $\mathcal{W}_n(\psi_{\boldsymbol{\tau}})$. Similar geometric ideas to this will be used throughout this thesis, in particular in the upper bound proof of Theorem 5.1.4.

Generally, the set $\mathcal{W}_n(\Psi)$ can be thought of as the set of points $x \in \mathbb{I}^n$ that can be approximated by infinitely many rational points. For comparability to later setups we note that $\mathcal{W}_n(\psi)$ can be equivalently written as

$$\mathcal{W}_n(\psi) := \left\{ x \in \mathbb{I}^n : \max_{1\leq i\leq n} |qx_i - p_i| < \psi(q) \text{ for i.m } (\boldsymbol{p}, q) \in \mathbb{Z}^{n+1} \right\}.$$

As with classical Diophantine approximation, our first target is to obtain optimal bounds on the approximation functions such that all $\boldsymbol{x} \in \mathbb{R}^n$ can be approximated. For our Dirichlet-style Theorem for simultaneous and weighted simultaneous approximation we appeal to a theorem from the geometry of numbers.

Theorem 1.2.1 (Minkowski's theorem for systems of linear forms [89])**.** *Given a system of linear inequalities of the form*

$$\begin{cases} |c_{1,1}x_1 + \cdots + c_{1,n}x_n| < Q_1, \\ \quad\vdots \qquad\qquad \vdots \\ |c_{n-1,1}x_1 + \cdots + c_{n-1,n}x_n| < Q_{n-1}, \\ |c_{n,1}x_1 + \cdots + c_{n,n}x_n| \leq Q_n, \end{cases} \tag{1.5}$$

where $c_{i,j} \in \mathbb{R}$ for $i, j \in \{1, \dots, n\}$, and $Q_i \in \mathbb{R}_+$. If

$$|\det(c_{i,j})_{1\leq i,j\leq n}| \leq \prod_{i=1}^{n} Q_i,$$

then there exists a non-zero integer solution $(x_1, \dots x_n) \in \mathbb{Z}^n$ to (1.5).

By considering the system of inequalities

$$\begin{cases} |q_0 x + q_1| < Q^{-\tau_1}, \\ \quad\vdots \qquad\qquad \vdots \\ |q_0 x + q_n| < Q^{-\tau_n}, \\ |q_0| \leq Q, \end{cases}$$

with $\sum_{i=1}^{n} \tau_i = 1$ and each $\tau_i \geq 0$ we can deduce an analogous statement to the corollary of Dirichlet's Theorem. Namely, for any $(x_1, \ldots, x_n) = \boldsymbol{x} \in \mathbb{R}^n$, there exists infinitely many integer vectors $(p_1, \ldots, p_n, q) = (\boldsymbol{p}, q) \in \mathbb{Z}^n \times \mathbb{N}$ such that

$$\left| x_i - \frac{p_i}{q} \right| < q^{-1-\tau_i}, \quad 1 \leq i \leq n,$$

provided $\sum_{i=1}^{n} \tau_i = 1$. An easy corollary to this is that in the simultaneous case we have $\mathcal{W}_n(1/n) = \mathbb{I}^n$. These results lead to the notion of n-dimensional badly approximable points. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_+$ be a weight vector such that

$$\sum_{i=1}^{n} \tau_i = 1. \tag{1.6}$$

Then we may define

$$\mathbf{Bad}_n(\boldsymbol{\tau}) := \left\{ \boldsymbol{x} \in \mathbb{I}^n : \exists\ c > 0 \ \left| x_i - \frac{p_i}{q} \right| \geq c q^{-1-\tau_i},\ 1 \leq i \leq n, \quad \forall\ \frac{\boldsymbol{p}}{q} \in \mathbb{Q}^n \right\}.$$

In the case where we have the weight vector $\boldsymbol{\tau} = (\frac{1}{n}, \ldots, \frac{1}{n})$ we have the set of simultaneously badly approximable points, denoted by $\mathbf{Bad}_n = \mathbf{Bad}_n(\boldsymbol{\tau})$.

As with the classical case we may deduce that $\lambda_n(\mathbf{Bad}_n(\boldsymbol{\tau})) = 0$ via Theorem 1.2.6. Furthermore, it was proven by Jarnik [75] that $\dim \mathbf{Bad}_n(\frac{1}{n}) = n$. This result can be generalised to also show that $\mathbf{Bad}_n(\boldsymbol{\tau})$ is of full dimension for any $\boldsymbol{\tau}$ satisfying (1.6) [82]. The set $\mathbf{Bad}_n(\boldsymbol{\tau})$ is of interest for several reasons, perhaps the most notably due to Schmidt's conjecture [101] which stated that

$$\bigcap_{t=1,2} \mathbf{Bad}_2((\tau_{1_t}, \tau_{2_t})) \neq \varnothing$$

for any pairs $(\tau_{1_1}, \tau_{2_1}), (\tau_{1_2}, \tau_{2_2})$ satisfying (1.6). In 2011 Badziahin, Pollington and Velani [12] proved Schmidt's conjecture to be true, in fact the following much stronger statement was proven [8].

Theorem 1.2.2. *Let $\{(i_t, j_t)\}_{t=1}^{n}$ be a set with each $i_t, j_t > 0$ and $i_t + j_t = 1$ for all $1 \leq t \leq n$. Then*

$$\dim \left(\bigcap_{t=1}^{n} \mathbf{Bad}_2(i_t, j_t) \right) = 2.$$

This landmark theorem has since been developed in a variety of directions. We will not pursue these ideas further so direct the reader to [8, 9, 92, 15] and references therein for more details.

1.2.2 Multiplicative and Dual approximation

The two other forms of n-dimensional approximation that are most widely used are multiplicative and dual approximation. We stress here that this sections is present purely to give a complete picture of n-dimensional approximation. The settings and concepts given in this chapter will not be pursued further. Saying that, it should be remarked that there are still connections between the concepts of the previous section and those that appear here. For example, as seen in the previous section, the set of weighted badly approximable points have connections with Littlewood's conjecture, a statement firmly in Multiplicative approximation. Furthermore Khintchine's transference principle, discussed at the end of this section, gives a clear link between the set of dually approximable points and the set of simultaneously approximable points.

In multiplicative Diophantine approximation, for $\psi : \mathbb{N} \to \mathbb{R}_+$, we consider the set

$$\mathcal{W}_n^{\times}(\psi) := \{\boldsymbol{x} \in \mathbb{R}^n : ||qx_1|| \dots ||qx_n|| < \psi(q) \text{ for i.m } q \in \mathbb{N}\},$$

where $||.||$ denotes the minimum distance to an integer i.e. $||a|| = \min\{|a-n| : n \in \mathbb{Z}\}$. This setup provides many interesting problems. In particular, where $n = 2$ we have the well-known Littlewood's conjecture (see for example §2 of [88]).

Conjecture 1.2.3. *For any pair* $(\alpha, \beta) \in [0,1]^2$,

$$\liminf_{q \to \infty} q||q\alpha||.||q\beta|| = 0.$$

While the conjecture remains unsolved there has been significant steps towards proving the result. Most notably is the result of Einsiedler, Katok and Lindenstrauss [57] who proved that the set of exceptions to Conjecture 1.2.3 has Hausdorff dimension zero. Conjecture 1.2.3 can also be shown to be related to the behaviour of $\mathbf{Bad}_n(\boldsymbol{\tau})$. In particular, it is well known that if

$$\bigcap_{\substack{0<\tau_1,\tau_2<1 \\ \tau_1+\tau_2=1}} \mathbf{Bad}_2((\tau_1,\tau_2)) = \emptyset \quad \text{then} \quad \text{Conjecture 1.2.3 is true, see for example [25].}$$

The second form of Diophantine approximation in n dimensions which we will discuss is dual approximation. Rather that approximating real number by rational points this setup is the approximation of real numbers by rational hyperplanes. Concisely, a point $\boldsymbol{x} \in \mathbb{R}^n$ is said to be dually ψ-approximable if there exists infinitely many $(\mathbf{q}, p) \in \mathbb{Z}^n \times \mathbb{Z}$ satisfying

$$|\mathbf{q}.\boldsymbol{x} - p| < \psi(|\mathbf{q}|),$$

where $\mathbf{q}.\boldsymbol{x} = q_1x_1 + \dots + q_nx_n$, and $|\mathbf{q}| = \max_i |q_i|$ for $1 \leq i \leq n$. The set of dually approximable points is defined as

$$\mathcal{W}_n^*(\psi) := \{\boldsymbol{x} \in \mathbb{R}^n : |\mathbf{q}.\boldsymbol{x} - p| < \psi(|\mathbf{q}|) \text{ for i.m } (\mathbf{q}, p) \in \mathbb{Z}^n \times \mathbb{Z}\}.$$

For a Dirichlet style theorem for dual approximation we can appeal to Theorem 1.2.1 to obtain the immediate corollary.

Corollary 1.2.4. *For any* $(x_1, \dots x_n) \in \mathbb{R}^n$*, there exists infinitely many* $(q_1, \dots, q_n, p) \in (\mathbb{Z}^n \backslash \{0\}) \times \mathbb{Z}$ *such that*

$$|q_1x_1 + q_2x_2 + \cdots + q_nx_n - p| < \left(\max_{1 \leq i \leq n} |q_i| \right)^{-n}.$$

Given this result we have that $\mathcal{W}_n^*(n) = [0,1]^n$.

Both simultaneous and dual approximation can be generalized by the following setup first introduced by Groshev [106]. For a matrix $X = (x_{i,j}) \in \mathbb{R}^{nm}$ we say X is ψ-approximable if for infinitely many $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \backslash \{0\}$,

$$\max_{1 \leq j \leq m} |q_1x_{1,j} + \cdots + q_nx_{n,j} + p_j| < \psi(|q|).$$

For ease of notation we also write

$$||\boldsymbol{q}.\boldsymbol{x} + \boldsymbol{p}|| = \max_{1 \leq j \leq m} |q_1x_{1,j} + \cdots + q_nx_{n,j} + p_j|.$$

The set of ψ-approximable matrices, also called the Groshev approximation set, is defined as

$$\mathcal{G}_{n,m}(\psi) := \left\{ \boldsymbol{x} \in \mathbb{R}^{nm} : ||\boldsymbol{q}\boldsymbol{x} + \boldsymbol{p}|| < \psi(|q|) \text{ for i.m } (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \backslash \{0\} \right\}.$$

We link this setup to both simultaneous and dual approximation by noting that

$$\mathcal{G}_{1,m}(\psi) = \mathcal{W}_m(\psi) \quad \text{and} \quad \mathcal{G}_{n,1}(\psi) = \mathcal{W}_n^*(\psi).$$

Before we begin discussing the metrical results of the above setups we note the following theorem which highlights a relationship between $\mathcal{W}_n(\psi)$ and $\mathcal{W}_n^*(\psi)$. In order to state the result we define the following notation. For any $\boldsymbol{x} \in \mathbb{R}^n$ let

$$s(\boldsymbol{x}) = \sup \left\{ \alpha \in \mathbb{R} : \boldsymbol{x} \in \mathcal{W}_n \left(\frac{1+\alpha}{n} \right) \right\},$$

and

$$d(\boldsymbol{x}) = \sup \left\{ \alpha \in \mathbb{R} : \boldsymbol{x} \in \mathcal{W}_n^*(n+\alpha) \right\}.$$

Khintchine's transference principle (see for example [106, 49]) links these two functions.

Theorem 1.2.5. *For any* $\boldsymbol{x} \in \mathbb{R}^n$*, we have that*

$$\frac{d(\boldsymbol{x})}{n^2 + (n-1)d(\boldsymbol{x})} \leq s(\boldsymbol{x}) \leq d(\boldsymbol{x}).$$

When $d(\boldsymbol{x})$ *is infinite we have*

$$\frac{d(\boldsymbol{x})}{n^2 + (n-1)d(\boldsymbol{x})} = \frac{1}{n-1}.$$

1.2.3 n-dimensional measure results

As with the classical setting, our next step is to give an overview of measure results for the ψ-approximable sets in n dimensions. In particular we provide results for each of the following:

1. A Lebesgue measure statement analogous to Theorem 1.1.1.

2. Hausdorff theory statements equivalent to Theorem 1.1.8 and Theorem 1.1.9.

Corresponding to each form of n-dimensional approximation we have several varieties of Khintchine's Theorem. As we are considering subsets of $\mathbb{R}^n$ we use the n-dimensional Lebesgue measure, denoted λ_n. We begin with $\mathcal{W}_n(\Psi)$, proven by Gallagher in 1962 [62].

Theorem 1.2.6. *Let $\psi_i : \mathbb{N} \to \mathbb{R}^n$ be monotonic decreasing functions for $1 \leq i \leq n$ and $\Psi = (\psi_1, \ldots, \psi_n)$. Then*

$$\lambda_n\left(\mathcal{W}_n(\Psi)\right) = \begin{cases} 0 \;\; \textit{if} \;\; \sum_{q=1}^{\infty} \psi_1(q) \ldots \psi_n(q) < \infty, \\ 1 \;\; \textit{if} \;\; \sum_{q=1}^{\infty} \psi_1(q) \ldots \psi_n(q) = \infty. \end{cases}$$

Clearly this also contains the simultaneous setting, where the result depends on the convergence or divergence of the sum

$$\sum_{q=1}^{\infty} \psi(q)^n. \tag{1.7}$$

As in the one-dimensional setting, the convergence case of this result follows easily from the Borel-Cantelli convergence Lemma (Lemma 1.1.3). In particular, like with classical approximation, we can deduce that $\lambda_n(\mathbf{Bad}_n(\boldsymbol{\tau})) = 0$ for all weight vectors $\boldsymbol{\tau} \in \mathbb{R}^n_+$ with components summing to 1.

In the n-dimensional case the class of non-monotonic approximation functions was solved prior to the proof of Theorem 1.1.2. Using a slightly different setup Gallagher [63] proved that for any ψ, $\lambda_n(\mathcal{W}_n^{'}(\psi))$ is equal to zero or one depending on whether (1.7) converges or diverges respectively. $\mathcal{W}_n^{'}(\psi)$ is defined in the same way as $\mathcal{W}_n(\psi)$ with the additional requirement that the rational points we approximate over are pairwise reduced fractions. That is,

$$\mathcal{W}_n^{'}(\psi) = \limsup_{q\to\infty} \bigcup_{\substack{0\leq p_i \leq q \\ gcd(p_i,q)=1 \\ i=1,\ldots,n}} B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right).$$

In the multiplicative setup we have the following theorem due to Gallagher [62].

Theorem 1.2.7. *Let $\psi : \mathbb{N} \to \mathbb{R}+$ be a monotonic function. Then*

$$\lambda_n(\mathcal{W}_n^{\times}(\psi)) = \begin{cases} 0 \;\; \textit{if} \;\; \sum_{q=1}^{\infty} \psi(q) \log^{n-1} q < \infty, \\ 1 \;\; \textit{if} \;\; \sum_{q=1}^{\infty} \psi(q) \log^{n-1} q = \infty. \end{cases}$$

Lastly we have the following result for our final setup, $\mathcal{G}_{n,m}(\psi)$, which is referred to as the Khintchine-Groshev Theorem. The following version of the theorem which removes monotonicity of the approximation function in all cases except $n = m = 1$ was proven by Beresnevich and Velani in [33]. In the case where $n = m = 1$ Theorem 1.1.2 can be applied.

Theorem 1.2.8. *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ and $mn > 1$. Then*

$$\lambda_{nm}(\mathcal{G}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

Note that this result contains the dual setting, $\mathcal{W}_n^*(\psi)$, stating that if $\sum_{q=1}^{\infty} q^{n-1}\psi(q) < \infty$ then $\mathcal{W}_n^*(\psi)$ has measure zero and when the sum is divergent the set has full measure.

We now consider the second of the questions posed at the start of the section. We will begin with the Hausdorff theory results for $\mathcal{W}_n(\Psi)$, and then give the Hausdorff measure theorem for the Khinthcine-Groshev setup as this encompasses all other Hausdorff theory results. The following result, proven in [18], is the n-dimensional simultaneous generalisation of Theorem 1.1.9.

Theorem 1.2.9. *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a monotonic approximation function. Then*

$$\mathcal{H}^s(\mathcal{W}_n(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} r^{n-s}\psi(r)^s < \infty, \\ \mathcal{H}^s(\mathbb{I}^n) & \text{if } \sum_{r=1}^{\infty} r^{n-s}\psi(r)^s = \infty. \end{cases}$$

We remark that like the classical case this result was originally proven by Jarnik [75] but with additional constraints on the approximation function ψ. Evaluating the sum on the right hand side where it switches from converging to diverging with respect to s we have the following n-dimensional Jarnik-Besicovitch result [75, 38].

Theorem 1.2.10. *Let $\tau \geq \frac{1}{n}$. Then*

$$\dim \mathcal{W}_n(\tau) = \frac{n+1}{\tau+1}.$$

We will prove this theorem in the next section as it provides a clear example of the application of the Mass Transference Principle. We also have the following dimension result, as proven by Rynne [97], for the more general case of weighted simultaneous approximation.

Theorem 1.2.11. *Suppose that $\sum_{i=1}^{n} \tau_i \geq 1$ and assume that $\tau_1 \geq \cdots \geq \tau_n > 0$. Then*

$$\dim \mathcal{W}_n(\boldsymbol{\tau}) = \min_{1\leq k\leq n} \left\{ \frac{n+1+\sum_{i=k}^{n}(\tau_k - \tau_i)}{1+\tau_k} \right\} = s.$$

This result will be proven in Chapter 3 §3.2 as the method illustrates the use of a Mass Transference Principle which will be used in a more complex proof later in the thesis. We note the following unexpected result. If $\tau_1 > \tau_2 > 2$, then we have that

$$\dim \mathcal{W}_2((\tau_1, \tau_2)) = \frac{3}{1+\tau_2}.$$

This means that as τ_1 increases the dimension of $\mathcal{W}_2((\tau_1, \tau_2))$ remains the same. This is unexpected, as it implies that the "size" of $\mathcal{W}_2((\tau_1, \tau_2))$ remains unchanged when τ_1 increases. As shown in [97], Theorem 1.2.11 may be extended to give a dimension result for general approximating functions. Define

$$\tau_i = \lim_{q\to\infty} \frac{-\log \psi_i(q)}{\log q}, \tag{1.8}$$

for $i = 1, \ldots, n$. Suppose for an approximation function $\Psi = (\psi_1, \ldots, \psi_n)$ each limit τ_i exists and is positive finite. Let $\Psi^* = (\tau_1, \ldots, \tau_n)$.

Corollary 1.2.12. *Suppose the n-tuple of approximation functions Ψ have positive finite limits stated above, with $\sum_{i=1}^{n} \tau_i \geq 1$. Then*

$$\dim \mathcal{W}_n(\Psi) = s,$$

where s is the same as in Theorem 1.2.11.

This Corollary follows from Theorem 1.2.11 and noting that the limits (1.8) imply that for any $\epsilon > 0$,

$$q^{-\tau_i-\epsilon} \leq \psi_i(q) \leq q^{-\tau_i+\epsilon}$$

for sufficiently large q. Hence

$$\mathcal{W}_n(\Psi^* + \epsilon) \subseteq \mathcal{W}_n(\Psi) \subseteq \mathcal{W}_n(\Psi^* - \epsilon).$$

Returning to the Hausdorff measure we have the following result from [16] which provides us with the complete theory of n-dimensional Hausdorff measure for ψ-approximable sets.

Theorem 1.2.13. *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a monotonic approximation function and $nm > 1$. Let f and g be dimension functions with $g(r) = r^{-m(n-1)} f(r)$ and $r^{-nm} f(r)$ monotonic. Then,*

$$\mathcal{H}^f(\mathcal{G}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} r^{n+m-1} g\left(\frac{\psi(r)}{r}\right) < \infty, \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{r=1}^{\infty} r^{n+m-1} g\left(\frac{\psi(r)}{r}\right) = \infty. \end{cases}$$

We have the following corollary on the Hausdorff dimension of $\mathcal{G}_{n,m}(\tau)$.

Corollary 1.2.14. *Let $\tau > \frac{m}{n}$, then*

$$\dim \mathcal{G}_{n,m}(\tau) = n(m-1) + \frac{n+m}{1+\tau}.$$

In particular, where $n = 1$ this gives us the Hausdorff dimension of $\mathcal{W}_m^*(\tau)$, namely for $\tau > m$,

$$\dim \mathcal{W}_m^*(\tau) = m - 1 + \frac{m+1}{\tau+1}.$$

1.3 What comes next: an overview of the thesis

The aim of this thesis is to emulate a variety of results displayed in this chapter in other settings, including p-adic approximation and approximation over manifolds. As mentioned earlier in this chapter we will chiefly be focussing on weighted simultaneous approximations in the respective settings.

Prior to proving any new results two more survey chapters are provided. The first gives an introduction to p-adic numbers and p-adic Diophantine approximations. The chapter contains three new results (Theorems 2.2.6 , 2.2.7 and 2.2.12), however, in order to keep the survey succinct these proofs are reserved for Chapter 5. The third and final survey chapter introduces the Mass Transference Principle, a beautiful theorem that enables Hausdorff dimension results to be obtained in a manner more easily than traditional methods.

In the later chapters (Chapters 4-6) the main focus is on the study of weighted simultaneous approximation over manifolds. In Chapter 4 a brief survey on real simultaneous approximation over manifolds is given before a new result (Theorem 4.1.8) on the Hausdorff dimension of simultaneously $\boldsymbol{\tau}$-approximable points over $C^{(2)}$ manifolds is proven. In Chapter 5 the new results stated in Chapter 2 are proven. One result of particular importance is a new Zero-One Law on the set of weighted simultaneously Ψ-approximable p-adic points (Lemma 5.3.5). Similar results to the new Theorem in Chapter 4 are also proven in the p-adic setting (Theorem 5.2.3-5.2.5). In Chapter 6 $\boldsymbol{\tau}$-approximable points over p-adic coordinate hyperplanes are investigated. In particular a new counting result on the set of rational approximations to a p-adic integer is proven (Theorem 6.1.3), which enables a complete Hausdorff dimension result to be proven (Theorem 6.2.1).

Chapter 2

p-adic Diophantine Approximation

In the previous chapter we studied the approximations by rational numbers to the set of real numbers. In this chapter we discuss the approximations by rational numbers to p-adic numbers. In particular we provide a survey of results analogous to those of the previous chapter, highlighting the key similarities and differences.

2.1 p-adic Numbers

We begin with the definition of the p-adic norm and subsequent construction of the p-adic numbers. Most results in this section can be found in a variety of textbooks, for example see [98, 87, 64]. Throughout this chapter we fix some prime number $p \in \mathbb{N}$. For any rational point $\frac{a}{b} \in \mathbb{Q}$ we may rewrite $\frac{a}{b}$ as the reduced fraction

$$\frac{a}{b} = \frac{a'}{b'}p^k,$$

where $gcd(a', p) = gcd(b', p) = 1$ and $k \in \mathbb{Z}$. For any $x \in \mathbb{Q}$ define $\text{ord}_p(x)$ to be the unique $n \in \mathbb{Z}$ such that

$$x = p^n \frac{a}{b} \text{ with } p \nmid a,\ p \nmid b.$$

Conventionally, we take $\text{ord}_p(0) = \infty$. Then for any $x \in \mathbb{Q}$ define the p-adic norm

$$|x|_p = p^{-\text{ord}_p(x)}.$$

Given the p-adic norm we define the set of p-adic numbers, $\mathbb{Q}_p$, to be the completion of $\mathbb{Q}$ by $|.|_p$.

The p-adic norm has several properties that make $\mathbb{Q}_p$ an intriguing space to study. One such property that sets the p-adic norm apart from the Euclidean norm is that $|.|_p$ is isolated for non-zero points. In particular, for any $x \in \mathbb{Q}_p$

$$|x|_p \in \{p^k : k \in \mathbb{Z}\} \cup \{0\}.$$

Secondly, and perhaps most importantly, we have the property that $|.|_p$ satisfies the strong triangle inequality. For any $x, y \in \mathbb{Q}_p$ we have the inequality

$$|x - y|_p \leq \max\{|x|_p, |y|_p\}.$$

In particular, where $|x|_p \neq |y|_p$ we have equality in the above equation. These properties that make $\mathbb{Q}_p$ different from $\mathbb{R}$ lead to several interesting results in p-adic geometry. For any $x \in \mathbb{Q}_p$ and $r \in \mathbb{R}_+$ define the p-adic open ball

$$B(x, r) = \{y \in \mathbb{Q}_p : |x - y|_p < r\}.$$

Due to the strong triangle inequality we have the following lemma on the centres of p-adic balls.

Lemma 2.1.1. *Let $y \in B(x, r)$, then $B(y, r) = B(x, r)$.*

This result follows easily from the strong triangle inequality by noting that for any point $z \in B(x, r)$ we have that

$$|y - z|_p = |y - x + x - z|_p \leq \max\{|y - x|_p, |x - z|_p\} < r,$$

hence $B(x, r) \subset B(y, r)$. The reverse can be shown in the same way. The following lemma generally states that any two p-adic balls are either disjoint, or one is contained within the other.

Lemma 2.1.2. *Let $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ be balls in $\mathbb{Q}_p$ with centres $x_1, x_2 \in \mathbb{Q}_p$ and radii $r_1, r_2 \in \mathbb{R}_+$ respectively. Assume $B_1 \cap B_2 \neq \emptyset$, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.*

Proof. Choose $x_0 \in B_1 \cap B_2$ (we may do this since we assume $B_1 \cap B_2 \neq \emptyset$). Assume that $r_1 \geq r_2$. Then

$$|x_1 - x_2|_p \leq \max\{|x_1 - x_0|_p, |x_0 - x_2|_p\} \leq r_1.$$

Hence $x_2 \in B_1$. Thus for any $x \in B_2$ we have that

$$|x_1 - x|_p \leq \max\{|x_1 - x_2|_p, |x_2 - x|_p\} \leq r_1,$$

so $x \in B_1$ and hence $B_2 \subseteq B_1$. A similar argument can be given to show that $B_1 \subseteq B_2$ if $r_2 \geq r_1$. $\square$

There are many other interesting properties that $\mathbb{Q}_p^n$ has, for example it can be shown that in the p-adic setting at most two points are collinear in the usual sense, or that all p-adic triangles are either isosceles or equilateral. These types of results are trivial to prove, with the key part of the proof being that the p-adic norm satisfies the strong triangle inequality.

As such space is difficult to visualise geometrically in some ways it is easier to determine characteristics of the space algebraically. For some $x \in \mathbb{Q}_p$ we may write the *p-adic expansion* of x uniquely as

$$x = \sum_{i=k}^{\infty} a_i p^i, \tag{2.1}$$

where $a_i \in \{0, \ldots, p-1\}$, $k \in \mathbb{Z}$ and $a_k \neq 0$. Note the condition that $a_k \neq 0$ is added to ensure each expansion is unique. To shorten the notation a p-adic number may also be written as

$$x = \ldots a_2 a_1 a_0 . a_{-1} \ldots a_k,$$

see for example §1.4 of [98]. As an example, in 5-adic space we may write the expansion

$$\frac{15}{7} = \ldots 12040. = 0 \cdot 5^0 + 4 \cdot 5^1 + 0 \cdot 5^2 + 2 \cdot 5^3 + 1 \cdot 5^4 + \ldots.$$

For a method to construct such expansions and more numerical examples see §1.3-1.6 of [87]. By calculating the p-adic expansion of several points we note a few properties. Firstly, for all $x \in \mathbb{Z}$ with corresponding expansion (2.1) we have that $k \geq 0$. Further, if $x \in \mathbb{Z}$, then there exists large $N > 0$ such that $a_n = 0$ for all $n > N$, i.e. the p-adic expansion is finite. Secondly, if $x \in \mathbb{Q}$ then the p-adic expansion of x is eventually periodic (see §1.4 of [87] for a proof of such result). That is, there exists some $j, k, l \in \mathbb{Z}$ such that

$$x = \ldots \overline{a_{j+l} \ldots a_j} a_{j-1} \ldots a_{k-1} a_k,$$

where $a_{j+l} \ldots a_j$ is repeated infinitely. As an example we can continue the p-adic expansion of $\frac{15}{7}$ to find that

$$\frac{15}{7} = \ldots \overline{32412}040.$$

A subset of $\mathbb{Q}_p$ of particular interest is the set of p-adic integers. Define the ring of p-adic integers as

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

We have that $\mathbb{Z}_p$ is an integral domain with 0 and 1 as the additive and multiplicative identities. As noted by the above properties of p-adic expansions we have that

$$\mathbb{Z} \subset \mathbb{Z}_p.$$

In fact, we have the much stronger property that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. As a general proof of such statement observe that for any ball $B(x, r) \subset \mathbb{Z}_p$ we can take p-adic expansion of $x = \sum_{i=0}^{\infty} x_i p^i$ and $t \in \mathbb{N}_0$ such that

$$B(x, r) = B\left(\sum_{i=0}^{\infty} x_i p^i, p^{-t}\right).$$

Then observe that all integers of the form

$$\sum_{i=0}^{t} x_i p^i + p^{t+1}\mathbb{Z} \subset B(x, r),$$

since

$$\left| x - \left(\sum_{i=0}^{t} x_i p^i + p^{t+1} z\right) \right|_p = \left| \sum_{i=t+1}^{\infty} x_i p^i - p^{t+1} z \right|_p < p^{-t}.$$

Further $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, since it is its completion (see Theorems 5.3 and 5.4 of [98] respectively for a thorough proof on the matter). Topologically we also have the properties that $\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact.

2.1.1 Analysis in $\mathbb{Q}_p$

In the following section we note several properties in p-adic analysis that differ from usual analysis in $\mathbb{R}$. In particular, important theorems in real analysis such as Rolle's Theorem, the Mean value Theorem and Taylor's approximation Theorem do not have an immediate p-adic analogue. We begin with a few statements on p-adic series, most of which can be found in [64] with corresponding proofs. Then we give a class of functions that allow us to construct p-adic versions of some results in real analysis. The first lemma shows us that in some instances p-adic series are much easier to work with.

Lemma 2.1.3. *An infinite series*

$$\sum_{n=0}^{\infty} a_n\,, \quad a_i \in \mathbb{Q}_p$$

is convergent if and only if $\lim_{n\to\infty} |a_n|_p = 0$.

Proof. Let $A_n = \sum_{i=0}^{n} a_i$. Then note that

$$\lim_{n\to\infty} |A_n - A_{n-1}|_p = \lim_{n\to\infty} |a_n|_p,$$

so A_n is a Cauchy sequence and hence convergent. The converse direction follows trivially. □

The last line of the proof follows from the fact that, in p-adic space, a sequence $\{a_i\}$ is a Cauchy sequence and hence convergent, if and only if

$$\lim_{n\to\infty} |a_{n+1} - a_n|_p = 0.$$

For a proof of this see Lemma 3.2.2 of [64]. Given this result we have that any power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

converges if and only if $\lim_{n\to\infty} |a_n x^n|_p = 0$. The following lemma (Prop. 5.4.1 of [64]) gives us a p-adic version of the radius of convergence.

Lemma 2.1.4. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$, *and define*

$$\rho = \frac{1}{\limsup \sqrt[n]{|a_n|_p}}.$$

Then,

i) If $\rho = \infty$, *then* $f(x)$ *converges for all* $x \in \mathbb{Q}_p$.

ii) If $0 < \rho < \infty$ *and* $|a_n|_p \rho^n \to 0$ *as* $n \to \infty$, *then* $f(x)$ *converges if and only if* $|x|_p \leq \rho$.

iii) If $0 < \rho < \infty$ *and* $|a_n|_p \rho^n \not\to 0$ *as* $n \to \infty$, *then* $f(x)$ *converges if and only if* $|x|_p < \rho$.

Note at this point another difference to real analysis that makes p-adic analysis easier. By $ii)$ and $iii)$ of Lemma 2.1.4 either all points $|x|_p = \rho$ converge or none. In the real case this is not always true.

The above results imply that p-adic analysis is considerably easier than real analysis. However, the following argument indicates that in many respects this is not the case. In real analysis a key theorem fundamental in many results is the Mean Value Theorem. The following example (found in §5.2.3 of [64]) provides reasoning why we cannot do this for all p-adic functions.

Example 2.1.5

Suppose $f(x)$ is a continuous differentiable function on some $U \subset \mathbb{Q}_p$, and that $|f'(x)|_p \leq M$ for all $x \in U$. Then we would expect a Mean Value Theorem to state that for all $a, b \in U$, with $a \neq b$, we have that

$$\left| \frac{f(a) - f(b)}{a - b} \right|_p \leq M.$$

Considering the following function we see this is false. Take $U = \mathbb{Z}_p$, $f(x) = x^p$, $a = 1$ and $b = 0$. Then

$$|f'(x)|_p = |px^{p-1}|_p \leq p^{-1},$$

for all $x \in \mathbb{Z}_p$. However, we have that

$$\left| \frac{f(1) - f(0)}{1 - 0} \right|_p = 1 > p^{-1},$$

hence the statement is false for this function.

This is not the only function where this statement is false, there are many. Furthermore, things are worse than first appear. The following example, found in [98], shows that even the notion of differentiability in p-adic space can have peculiar implications.

Example 2.1.6

For any $x \in \mathbb{Z}_p$ we can write out its p-adic expansion, say

$$x = \sum_{i=0}^{\infty} a_i p^i.$$

Then define the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ as

$$f(x) = \sum_{i=0}^{\infty} a_i p^{2i}.$$

Such function f is clearly not constant, however calculating the derivative we have that

$$\begin{aligned} |f'(x)|_p =& \left| \lim_{(a,b)\to(x,x)} \frac{f(a) - f(b)}{a - b} \right|_p, \\ =& \lim_{N\to\infty} \frac{|p^{2N}|_p}{|p^N|_p}, \\ =& 0, \end{aligned}$$

for all $x \in \mathbb{Z}_p$.

As well as showing that zero derivative does not imply the function is constant the example also provides reasoning for why there does not exist a general p-adic version of Taylor's Expansion Theorem. One such attempt to find p-adic versions of these results is to only consider small regions locally, that is, to ensure any points under consideration are p-adically close, see for example [96, Section 3.2], [103]. Given the above problem we are motivated to find a set of p-adic functions that satisfy some p-adic versions of results in real analysis. A special class of p-adic functions introduced by Mahler [85] are the set of *normal* functions.

Definition 2.1.7. *A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called a normal function if it can be written as*

$$f(x) = \sum_{n=0}^{\infty} \alpha_n (x - \alpha)^n,$$

where $\alpha, \alpha_n \in \mathbb{Z}_p$ for each n, and $\lim_{n\to\infty} |\alpha_n|_p = 0$.

By Lemma 2.1.4 such functions will converge for all $x \in \mathbb{Z}_p$. Further, the class of functions is quite non-restrictive. For example, given any analytic function $g(z)$ we can find integers r, s such that $p^r g(p^s z)$ is a normal function [2]. Suppose $y \in \mathbb{Z}_p$, then we have that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(y)}{n!}(x - y)^n,$$

is normal, so we have a Taylor series expansion for normal functions. We also note the useful property that if $f(x)$ is normal then $f^{(n)}(x)$ is also normal for any $n \in \mathbb{N}$. To conclude this section we show that normal functions provide a possible p-adic version of the Mean Value Theorem akin to Example 2.1.1.

Lemma 2.1.8. *Let f be a normal function and suppose $|f'(x)|_p \leq M$ for all $x \in U \subset \mathbb{Z}_p$. Then for any $x \in U$ and any $y \in B(x, M) \cap U$ with $y \neq x$ we have that*

$$\left|\frac{f(x) - f(y)}{x - y}\right|_p \leq M.$$

Proof. Since f is normal we can consider the Taylor series expansion of f about any point $y \in B(x, M) \cap U$ to find that

$$f(x) - f(y) = (x - y)\sum_{n=1}^{\infty} \frac{f^{(n)}(y)}{n!}(x - y)^{n-1}.$$

Since $x - y \neq 0$ we may divide through by $x - y$ to obtain

$$\begin{aligned}
\left|\frac{f(x) - f(y)}{x - y}\right|_p &= \left|f'(y) + (x - y)\sum_{n=2}^{\infty} \frac{f^{(n)}(y)}{n!}(x - y)^{n-2}\right|_p, \\
&\leq \max\left\{|f'(y)|_p, \left|(x - y)\sum_{n=2}^{\infty} \frac{f^{(n)}(y)}{n!}(x - y)^{n-2}\right|_p\right\}, \\
&\leq M,
\end{aligned}$$

where the final inequality holds since $|f'(y)|_p \leq M$, $y \in B(x, M) \cap U$ and $\left|\frac{f^{(n)}(y)}{n!}\right|_p \leq 1$ for all $n \in \mathbb{N}$. $\square$

In Chapter 5 we will introduce further definitions and notations for multivariate p-adic manifolds, but for now we return to the main focus of this chapter.

2.2 Diophantine approximation in $\mathbb{Q}_p$

This section is devoted to giving a p-adic analogue of the classical Euclidean results of Diophantine approximation. We start with the following result by Mahler [86], which provides the p-adic version of Dirichlet's theorem.

Theorem 2.2.1. *Let $x \in \mathbb{Z}_p$, then for all $h \in \mathbb{N}$ there exists integer pairs $a_0, a_1 \in \mathbb{Z}$ with* $\max\{|a_0|, |a_1|\} \leq h$ *such that*

$$|a_0 x - a_1|_p \leq h^{-2}. \tag{2.2}$$

In comparison to the Euclidean case, where we had that $|a_0 x - a| < Q^{-1}$ for $a_0 \leq Q$, note that we have an extra exponent of approximation. This is due to the fact that unlike Euclidean approximation the rate of approximation can be increased dramatically by either of the components a_0, a_1. To see this consider the p-adic expansion of $a_0 x$. Let

$$a_0 x = \sum_{i=0}^{\infty} c_i p^i, \qquad c_i \in \{0, \dots, p-1\}.$$

Then choose $a_1 = \sum_{i=0}^{k} c_i p^i$, so we have that

$$|a_0 x - a_1|_p < p^{-k}.$$

If a_1 is unbounded we can let $k \to \infty$ and achieve increasingly close approximations. This observation can be neatly summarised by the previously mentioned statement that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Hence, in order to provide any meaningful results we must bound both integer coefficients. To do this we usually make the approximation function dependent on $\max\{|a_0|, |a_1|\}$, or bound $|a_1| \leq |a_0|$.

The exponent on h in (2.2) is best possible that we can have which allows all $x \in \mathbb{Z}_p$ to have infinitely many rational approximations. As proven by de Weger [54] the Hurwitz-style constant in p-adic space is 1, and so Theorem 2.2.1 is best possible satisfying all $x \in \mathbb{Z}_p$.

Similarly to Chapter 1 we may be inclined to ask whether this approximation function on the right of (2.2) can be improved for almost all points in $\mathbb{Z}_p$. As with the real case we can define the sets of badly approximable points and well approximable points. Let

$$\mathbf{Bad}^{(p)} := \left\{ x \in \mathbb{Z}_p : \exists c(x) > 0 \;\; |q_0 x - q_1|_p \geq c(x) q_0^{-2} \;\forall (q_0, q_1) \in \mathbb{N} \times \mathbb{Z} \text{ with } |q_1| \leq |q_0| \right\},$$

and the set of well approximable points being those that are not in $\mathbf{Bad}^{(p)}$. Define the set of very well approximable points $\mathbf{VWA}^{(p)}$ to be

$$\left\{x \in \mathbb{Z}_p : \exists \epsilon > 0 \ |q_0 x - q_1|_p < q_0^{-2-\epsilon} \ \text{ for i.m. } \ (q_0, q_1) \in \mathbb{N} \times \mathbb{Z} \ \text{ with } |q_1| \leq q_0\right\}.$$

Theorem 2.2.2 below immediately gives us that $\mathbf{Bad}^{(p)}$ and $\mathbf{VWA}^{(p)}$ are both of Haar measure zero. As with the real case both sets are still relatively large. For example the p-adic integer

$$\sum_{n=0}^{\infty} p^{n!}$$

is clearly in $\mathbf{VWA}^{(p)}$ (in fact even more so, it is p-adic Liouville).

We now consider the metric theory of p-adic approximation. In p-adic space we take the associated Haar measure μ_p, normalised by $\mu_p(\mathbb{Z}_p) = 1$. A construction of the Haar measure for $\mathbb{Q}_p$ can be found in [105](Part II, Chapter 1), we highlight below the key properties of μ_p that we will use. Firstly, for any p-adic ball $B(x, p^{-k})$ with centre $x \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$ we have $\mu_p\left(B(x, p^{-k})\right) = p^{-k}$. Secondly, the measure is translation invariant. So for any $x, y \in \mathbb{Q}_p$, $\mu_p\left(B(x, p^{-k})\right) = \mu_p\left(B(y, p^{-k})\right) = p^{-k}$. Lastly, the measure μ_p is doubling, that is, there exists $c > 0$ such that for any ball $B(x, p^{-k})$ we have $\mu_p(B(x, 2p^{-k})) \leq c\mu_p(B(x, p^{-k}))$. More precisely, we have the inequality

$$\mu_p\left(B(x, ap^{-k})\right) \leq p^{\lceil \log_p a \rceil} p^{-k},$$

for any $a \geq 1$.

There are various ways to describe the set of p-adic ψ-approximable points. Initially we provide the construction used by Jarnik in [76] to give a p-adic analogue of Theorem 1.1.1 of Chapter 1. Let

$$\mathfrak{A}_h^*(\psi) = \bigcup_{a=-h}^{h} \left(\left\{ x \in \mathbb{Z}_p : \left| x - \frac{a}{h} \right|_p \leq \psi(h) \right\} \cup \left\{ x \in \mathbb{Z}_p : \left| x - \frac{h}{a} \right|_p \leq \psi(h) \right\} \right),$$

then define

$$\mathfrak{W}^*(\psi) := \limsup_{h \to \infty} \mathfrak{A}_h^*(\psi).$$

Using this setup Jarnik proved the following theorem [76].

Theorem 2.2.2. *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be monotonically decreasing. Then*

$$\mu_p(\mathfrak{W}^*(\psi)) = \begin{cases} 0 & \text{if } \ \sum_{h=1}^{\infty} h\psi(h) < \infty, \\ 1 & \text{if } \ \sum_{h=1}^{\infty} h\psi(h) = \infty. \end{cases}$$

Note the additional power of h in the above summations in comparison to Theorem 1.1.1. This is due to fact that both a_0 and a_1 influence the rate of approximation, as mentioned previously. As with the real case the convergence statement follows almost immediately from the Haar measure of $\mathfrak{A}^*_h(\psi)$ and Lemma 1.1.3.

As shown by the following example in [65] this setup is insufficient when trying to construct a p-adic version of Theorem 1.1.2. As with the real case we know that in order to construct a Duffin-Schaeffer type theorem we need to only consider points approximated by reduced fractions. Let $\mathfrak{W}^{**}(\psi)$ be the subset of $\mathfrak{W}^*(\psi)$ with the added condition that the rational approximations are reduced i.e. $gcd(a,h)=1$.

Example 2.2.3

Consider the function

$$\psi(h) = \begin{cases} p^{-1} & \text{if } p|h, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $x \in \mathfrak{A}^{**}_h(\psi)$ ($\mathfrak{A}^*_h(\psi)$ with the added condition that $gcd(a,h)=1$), we must have that $p|h$. This would imply that $p \nmid a$, so $\left|\frac{a}{h}\right|_p > 1$. Hence we would need x to satisfy

$$\left|x - \frac{h}{a}\right|_p \leq p^{-1},$$

for some $-h \leq a \leq h$. As $p \nmid a$ we have that $|a|_p = 1$ so we may multiply the above equation through by $|a|_p$, and then using the strong triangle inequality we would have that

$$|ax|_p \leq \max\{|ax-h|_p, |h|_p\} \leq p^{-1}.$$

Hence, if $x \in \mathfrak{A}^{**}(\psi)$ then $|x|_p < p^{-1}$. Conversely, for any $x \in p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : |x|_p \leq p^{-1}\}$, and any $h \in \mathbb{N}$ such that $p|h$, i.e. $h \in p\mathbb{N}$, then

$$|ax-h|_p \leq \max\{|ax|_p, |h|_p\} \leq p^{-1}.$$

Thus if $x \in p\mathbb{Z}_p$ then $x \in \mathfrak{A}^{**}_h(\psi)$ for any $h \in p\mathbb{N}$. Combining these we have that

$$\mathfrak{A}^{**}_h(\psi) = p\mathbb{Z}_p$$

for all $h \in p\mathbb{N}$. When $p \nmid h$ then by our choice of ψ, $\mathfrak{A}^{**}_h(\psi)$ is countable so can be ignored. Thus

$$\mu_p(\mathfrak{W}^{**}(\psi)) = p^{-1} \notin \{0,1\}.$$

As the above example shows the setup given by Jarnik has the possibility that $\mu_p(\mathfrak{W}^{**}(\psi)) \notin \{0,1\}$. Hence we need a new setup in order to construct a p-adic equivalent of Theorem 1.1.2. To do this we adopt the construction used by Haynes in [65]. For an approximation function $\psi : \mathbb{N} \to \mathbb{R}_+$ and $h \in \mathbb{N}$ let

$$\mathfrak{A}_h(\psi) = \bigcup_{\substack{|a_0|,|a|\leq h \\ gcd(a_0,a)=1}} \{x \in \mathbb{Z}_p : |a_0x - a|_p < \psi(h)\}.$$

Then define the set of p-adic ψ-approximable points as

$$\mathfrak{W}(\psi) := \limsup_{h\to\infty} \mathfrak{A}_h(\psi).$$

In comparison to the setup by Jarnik, Haynes prove that for any approximation function ψ

$$\mu_p(\mathfrak{W}(\psi)) \in \{0, 1\},$$

thus satisfying a zero-one law (Lemma 1 of [65]). With this setup Haynes proved, modulo the proof of Theorem 1.1.2, the following.

Theorem 2.2.4. *For any prime p and any $\psi : \mathbb{N} \to \mathbb{R}_+$ we have that*

$$\mu_p(\mathfrak{W}(\psi)) = \begin{cases} 0 & \text{if } \sum_{h=1}^{\infty} \mu_p(\mathfrak{A}_h(\psi)) < \infty, \\ 1 & \text{if } \sum_{h=1}^{\infty} \mu_p(\mathfrak{A}_h(\psi)) = \infty. \end{cases}$$

In a similar manner to [28] Theorem 2.2.4 provides us with a Hausdorff measure result for $\mathfrak{W}(\psi)$ via the general MTP (see Theorem 7 of [65]). Given these theorems we have a complete set of results for classical p-adic Diophantine approximation.

2.2.1 n-dimensional approximation

As in Chapter 1 there are a variety of ways we can approximate n-dimensional points. Through this section we will focus on p-adic weighted simultaneous and Groshev-type approximation. Note that p-adic simultaneous and p-adic dual approximation results can both be deduced from the Groshev-type setup provided. We begin with p-adic weighted simultaneous approximation. Let $\boldsymbol{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$, and let $\Psi = (\psi_1, \dots, \psi_n)$ be an n-tuple of approximation functions $\psi_i : \mathbb{N} \to \mathbb{R}_+$. Define

$$\mathfrak{A}_{a_0}(\Psi) = \bigcup_{\substack{(a_1,\dots,a_n)\in\mathbb{Z}^n \\ |a_i|\le a_0\ (1\le i\le n)}} \left\{ \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{Z}_p^n : \left| x_i - \frac{a_i}{a_0} \right|_p < \psi_i(a_0) \ \text{ for all } 1 \le i \le n \right\}.$$

Then define the set of p-adic weighted simultaneously approximable points as

$$\mathfrak{W}_n(\Psi) = \limsup_{a_0\to\infty} \mathfrak{A}^{(n)}_{a_0}(\Psi).$$

As with the real case we adopt the following simplified notation for $\mathfrak{W}_n(\Psi)$ when Ψ is of a special form: $\mathfrak{W}_n(\psi)$ if $\psi_1 = \dots = \psi_n = \psi$; $\mathfrak{W}_n(\boldsymbol{\tau})$ if $\psi_i(q) = q^{-\tau_i}$ for some $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n_+$; and $\mathfrak{W}_n(\tau)$ if furthermore $\boldsymbol{\tau} = (\tau, \dots, \tau)$ for some $\tau > 0$.

In the real case we could divide the whole equation through by the denominator to give us a ball/hyperrectangle with rational centre and radius/side lengths determined by Ψ. In the p-adic case this is a little more complicated. Note that $\left|\frac{1}{a_0}\right|_p \ge 1$ for all $a_0 \in \mathbb{Z}$, so we would be increasing the approximation function in many cases. To overcome this issue we will usually apply the constraint that a_0 and p are coprime, hence leaving both sides of the inequality unaffected when multiplying through by $\left|\frac{1}{a_0}\right|_p = 1$.

In the Groshev setup we use the following construction. For any point $\boldsymbol{y} = (y_1, \dots y_n) \in \mathbb{Z}^n$ let $|\boldsymbol{y}|_p = \max_{1\leq i\leq n} |y_i|_p$, which should not cause confusion. Let $\psi : \mathbb{N} \to \mathbb{R}_+$, $\boldsymbol{q}_0 \in \mathbb{Z}^m$, $\boldsymbol{q} \in \mathbb{Z}^n$ and $h \in \mathbb{N}$. Let

$$\mathfrak{g}_h(\psi) := \bigcup_{|\boldsymbol{q}_0|=h,\ |\boldsymbol{q}|\leq h} \left\{ X \in \mathbb{Z}_p^{mn} : |\boldsymbol{q}_0 X + \boldsymbol{q}|_p < \psi(h) \right\},$$

then define the set of ψ-approximable p-adic integer matrices to be

$$\mathfrak{G}_{n,m}(\psi) := \limsup_{h\to\infty} \mathfrak{g}_h(\psi).$$

When $m = 1$ then $\mathfrak{G}_{n,1}(\psi) = \mathfrak{W}_n(\psi)$, and when $n = 1$ we have the p-adic equivalent of dual approximation which we will denote as $\mathfrak{D}_m(\psi) = \mathfrak{G}_{1,m}(\psi)$.

The following Lemma gives us the p-adic Dirichlet-style theorem for $\mathfrak{W}_n(\boldsymbol{\tau})$.

Lemma 2.2.5. *Let $L_i(\boldsymbol{x})$, with $i = 1, \dots, n$, be linear forms with p-adic integer coefficients. Let $\sum_{i=1}^n \tau_i = n + 1$ for $\tau_i \in \mathbb{R}_+$. Then there exists a non-zero rational integer vector $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ with*

$$\max_{0\leq i\leq n} |x_i| \leq H,$$

satisfying the system of inequalities

$$|L_i(\boldsymbol{x})|_p < pH^{-\tau_i} \quad \text{for } i = 1, \dots, n.$$

For completeness we prove this lemma in Chapter 4. The proof is relatively simple, the key being the choice of sets used to apply the Pigeon-hole principle. Given this lemma we may deduce a simultaneous and weighted simultaneous Dirichlet-style theorem. Namely that $\mathfrak{W}_n(1 + \frac{1}{n}) = \mathbb{Z}_p^n$, or that for any weight vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n \tau_i = n + 1$, then $\mathfrak{W}_n(\boldsymbol{\tau}) = \mathbb{Z}_p^n$. In [105] a dual version of Lemma 2.2.5 had previously been proven. The result (see Lemma 2 of Chapter 2 in [105]) states that for all $\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{Z}_p^m$

$$|a_m x_m + \cdots + a_1 x_1 + a_0|_p < ph^{-m-1},$$

where $h = \max_{0\leq i\leq m} |a_i|$. Similarly to the Euclidean setting, as $h \to \infty$ we note there are infinitely many integer vector solutions, hence we can deduce that $\mathfrak{D}_m(m + 1) = \mathbb{Z}_p^m$.

While not included here we note that a transference principle of a similar flavour to Theorem 1.2.5 exists. Using p-adic approximation lattices (see §5.5.1 for more details) Inoue, Kamada and Naito proved a correspondence between $\mathfrak{W}(\tau)$ and $\mathfrak{D}(\tau)$ [73]. Given such results we move on to the Khintchine-style Theorems for these setups. The following theorem provides a p-adic equivalent of Theorem 1.2.6 of Chapter 1. To ensure that the set satisfies a zero-one law we impose the additional condition that the rational points we consider are reduced fractions. We denote the set of weighted simultaneously Ψ-approximable points by reduced fractions by $\mathfrak{W}_n'(\Psi)$. Denote by $\mu_{p,n}$ the n-dimensional Haar measure, normalised by $\mu_{p,n}(\mathbb{Z}_p^n) = 1$.

Theorem 2.2.6. *Let $\psi_i : \mathbb{N} \to \mathbb{R}_+$ be approximation functions with each $\psi_i(q) < q^{-1}$ for $1 \le i \le n$ and let $\Psi = (\psi_1, \ldots, \psi_n)$. Suppose that $\prod_{i=1}^n \psi_i(q)$ is a monotonic decreasing function as $q \to \infty$. Then*

$$\mu_{p,n}(\mathfrak{W}_n'(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^n \prod_{i=1}^n \psi_i(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^n \prod_{i=1}^n \psi_i(q) = \infty. \end{cases}$$

This is a new theorem within the p-adic setting, a proof is provided in Chapter 5. In tandem with this result we also prove a new zero-one law on $\mathfrak{W}_n(\Psi)$. The proof of the convergence case is immediate upon applying Lemma 1.1.3. The divergence case is proven by showing the limsup set of rectangles satisfies quasi-independence on average, and thus proven by Lemma 1.1.4. Note that Theorem 2.2.6 contains the special simultaneous case $\mathfrak{W}_n(\psi)$ which had previously been proven by Jarnik [76].

In Chapter 5 we also prove a Duffin-Schaeffer style theorem with the monotonicity condition on Ψ removed. As with Theorem 2.2.6 we state the result here and reserve the proof for Chapter 5.

Theorem 2.2.7. *Let $\psi_i : \mathbb{N} \to [0, 1)$ be approximation functions with $\psi_i(q) \ll \frac{1}{q}$ for $1 \le i \le n$ and let $\Psi = (\psi_1, \ldots, \psi_n)$. For φ the Euler phi function suppose that*

$$\limsup_{N \to \infty} \frac{\sum_{q=1}^{N} \varphi(q)^n \prod_{i=1}^n \psi_i(q)}{\sum_{q=1}^{N} q^n \prod_{i=1}^n \psi_i(q)} > 0. \tag{2.3}$$

Then

$$\mu_{p,n}(\mathfrak{W}_n'(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \varphi(q)^n \prod_{i=1}^n \psi_i(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \varphi(q)^n \prod_{i=1}^n \psi_i(q) = \infty. \end{cases}$$

In the Groshev approximation case Lutz [84] proved the following theorem.

Theorem 2.2.8. *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a monotonic decreasing function. Then*

$$\mu_{p,nm}(\mathfrak{G}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{h=1}^{\infty} \psi(h)^n h^{m+n-1} < \infty, \\ 1 & \text{if } \sum_{h=1}^{\infty} \psi(h)^n h^{m+n-1} = \infty. \end{cases}$$

Note that this theorem was proven using a setup in a style similar to the one used by Jarnik as noted at the start of this section. More recently this theorem has been proven via ubiquity (see §12.6 of [18]).

The above theorems provide a complete set of Haar measure results over the various forms of n-dimensional p-adic approximation. However there are still many other areas of interest, for example in [17] a Khintchine-style result is given for the approximation of p-adic numbers with respect to p-adic algebraic numbers. More recently Oliveira proved a variety of Khintchine-style theorems for p-adic simultaneous approximation over various rational subsets, including rational points contained within p-adic balls [93].

2.2.2 Hausdorff theory in $\mathbb{Q}_p$

In this section we will consider the Hausdorff theory of p-adic approximation. We observe that $(\mathbb{Q}_p^n, d)$, where $d(\boldsymbol{x},\boldsymbol{y}) = \max_{1\leq i\leq n}|x_i - y_i|_p$ for $\boldsymbol{x},\boldsymbol{y} \in \mathbb{Q}_p^n$, is a locally compact metric space. If we choose the dimension function $g(x) = x^n$ then g is doubling and Ahlfors regular, whereby we mean that the corresponding measure $\mathcal{H}^g$ is an Ahlfors regular measure i.e. for any ball $B \subseteq \mathbb{Q}_p^n$ of radius $r > 0$ there exists constants $a, b > 0$ such that

$$ar^n \leq \mathcal{H}^g(B) \leq br^n.$$

Further, by Lemma 6 of [65] we have

Lemma 2.2.9.

$$\mu_{p,n} \asymp \mathcal{H}^g.$$

Hence $(\mathbb{Q}_p, d)$ with dimension function g satisfies the conditions for the general MTP (Theorem 3.1.1). With the aid of the general MTP many of the Hausdorff measure results follow from results of the previous section. We begin with the p-adic analogue of Jarnik's Theorem which was proven by Beresnevich, Velani and Dickinson [18].

Theorem 2.2.10. *Let f be a dimension function such that $r^{-n}f(r) \to \infty$ as $r \to \infty$ and $r^{-n}f(r)$ is decreasing. Furthermore suppose $f(r)$ is increasing and let $\psi : \mathbb{N} \to \mathbb{R}_+$. Then*

$$\mathcal{H}^f(\mathfrak{W}_n(\psi)) = \begin{cases} 0 \text{ if } \sum_{h=1}^{\infty} f(\psi(h))h^n < \infty, \\ \infty \text{ if } \sum_{h=1}^{\infty} f(\psi(h))h^n = \infty. \end{cases}$$

Note that Theorem 2.2.10 was proven prior to the general MTP, and used the setup of ubiquitous systems provided in [18]. A clear corollary of the above result is the equivalent Jarnik-Besicovitch Theorem for the dimension of $\mathfrak{W}_n(\psi)$.

Corollary 2.2.11. *For $\tau > \frac{1}{n} + 1$,*

$$\dim \mathfrak{W}_n(\tau) = \frac{n+1}{\tau}.$$

We also prove in Chapter 5 the following result for the set of weighted simultaneously approximable points.

Theorem 2.2.12. *Let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}_+^n$ be a weight vector satisfying $\sum_{i=1}^n \tau_i > n+1$ and $\tau_i > 1$ for each $i = 1, \dots, n$. Then*

$$\dim \mathfrak{W}_n(\boldsymbol{\tau}) = \min_{1 \leq i \leq n} \left\{ \frac{n + 1 + \sum_{j=i}^n (\tau_i - \tau_j)}{\tau_i} \right\}.$$

This theorem requires more work to prove than Corollary 2.2.11 primarily because the set is a lim sup set of hyperrectangles rather than hypercubes. The full proof of this theorem is provided in §4.4.2.

The Hausdorff measure result for $\mathfrak{G}_{n,m}(\psi)$ proven by Beresnevich, Dickinson and Velani in [18] is as follows.

Theorem 2.2.13. *Let f be a dimension function such that $h^{-mn} f(h) \to \infty$ as $h \to 0$ and $h^{-mn} f(h)$ decreasing. Further suppose that $h^{-(m-1)n} f(h)$ is increasing. Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a monotonic decreasing function. Then*

$$\mathcal{H}^f(\mathfrak{G}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{h=1}^{\infty} f(\psi(h)) \psi(h)^{-(m-1)n} h^{m+n-1} < \infty, \\ \infty & \text{if } \sum_{h=1}^{\infty} f(\psi(h)) \psi(h)^{-(m-1)n} h^{m+n-1} = \infty. \end{cases}$$

Clearly, the above theorem implies the n-dimensional p-adic Jarnik-Besicovitch theorem, which was previous proven in [1].

Corollary 2.2.14. *For $\tau > \frac{m+n}{n}$*

$$\dim \mathfrak{G}_{n,m}(\tau) = (m-1)n + \frac{m+n}{\tau}.$$

In particular, for dual p-adic approximation we have that

$$\dim \mathfrak{D}_m(\tau) = (m-1) + \frac{m+1}{\tau}.$$

This concludes the Hausdorff measure and dimension results for the sets of p-adic approximable points.

As a concluding remark to this Chapter note that while there are many differences between real and p-adic approximation they both still follow a general methodology. Namely we require a Dirichlet-style theorem e.g. Lemma 1.2.1, Lemma 2.2.5, which via ubiquity can be used to find Khinthcine-style theorems e.g. Theorem 1.2.6, Theorem 2.2.6, or via MTP-style theorems to construct Hausdorff dimension statements e.g. Theorem 1.2.11, Theorem 2.2.12.

Chapter 3

The Mass Transference Principle

The Mass Transference principle (MTP), first developed by Beresnevich and Velani [28], is an invaluable tool in Diophantine approximation and is now part of the standard machinery for studying many problems in metric Diophantine approximation, see [6] for a survey. The theorem, and following variations, will be used in a variety of settings throughout this thesis. Generally the MTP allows us to turn a full measure statement into a Hausdorff measure statement. We begin by introducing the general MTP and then provide a proof of Theorem 1.2.10 to illustrate how the MTP can be applied. From there we discuss the various forms of MTP from "balls to balls" before moving on to MTP results that provide Hausdorff measure results for $\limsup$ sets of rectangles. Such results, including the MTP from balls to rectangles [112] and the MTP from rectangles to rectangles [111], are crucial in the proofs of the main theorems of Chapters 4-6. Since these results will be used in both the real and p-adic setting we provide these results in full generality.

3.1 From Balls to Balls

Throughout this section let (X, d) be a locally compact metric space. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ to be a doubling function if there exists a constant $\lambda > 1$ such that for all $x > 0$ we have

$$g(2x) \leq \lambda g(x).$$

Suppose there exists constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that

$$c_1 g(r(B)) \leq \mathcal{H}^g(B) \leq c_2 g(r(B)), \tag{3.1}$$

for any ball $B = B(x, r)$ with centre $x \in X$ and $r(B) = r \leq r_0$. Given a dimension function f and a ball $B = B(x, r)$ define

$$B^f = B\left(x, g^{-1}(f(r))\right).$$

Note that $B^g = B$. We may now state the general MTP as given in [28].

Theorem 3.1.1 (General Mass Transference Principle)**.** *Let (X, d) be a locally compact metric space and g a doubling dimension function satisfying (3.1). Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in X with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $f(x)/g(x)$ is monotonic and suppose that for any ball $B \subset X$*

$$\mathcal{H}^g\left(B \cap \limsup_{i\to\infty} B_i^f\right) = \mathcal{H}^g(B). \tag{3.2}$$

Then, for any ball $B \subset X$

$$\mathcal{H}^f\left(B \cap \limsup_{i\to\infty} B_i\right) = \mathcal{H}^f(B). \tag{3.3}$$

We note several properties of this theorem. Firstly, as seen in [28], this theorem is applicable to to the metric space $(\mathbb{I}^n, d)$ where d is the usual Euclidean distance or sup norm. Take the doubling dimension function g to be $g(x) = x^n$, then we have that $\mathcal{H}^n$ is our usual n-dimensional Lebesgue measure up to a constant, by Lemma 1.1.5. Hence (3.2) becomes a Lebesgue measure statement. In turn, for (3.3) we may take any ball $B \subset X$, so by taking a ball B containing our lim sup set, then we have a Hausdorff measure statement on our lim sup set. Similarly Theorem 3.1.1 could be phrased in the p-adic setting by considering a similar argument to that above with Lemma 1.1.5 replaced with Lemma 2 from [65].

To illustrate an application of Theorem 3.1.1 clearly we prove the Jarnik-Besicovitch Theorem for n-dimensional simultaneous approximation (Theorem 1.2.10).

Proof of Theorem 1.2.10. We omit the proof for the upper bound as this follows by taking a standard covering of balls over the set provided in the definition of $\mathcal{W}_n(\tau)$. For the lower bound let

$$B^s = B\left(\frac{\mathbf{p}}{q}, q^{\frac{s}{n}(-1-\tau)}\right),$$

and define

$$\mathcal{A}_q^{(n)}(\tau)^s = \bigcup_{\substack{0\le p_i\le q \\ 1\le i\le n}} B^s.$$

For $s = \frac{n+1}{\tau+1}$ we have that $\mathcal{W}_n\left(\frac{1}{n}\right) = \limsup_{q\to\infty} \mathcal{A}_q^{(n)}(\tau)^s$. By Theorem 1.2.1, $\lambda_n\left(\mathcal{W}_n\left(\frac{1}{n}\right)\right) = 1$, hence by Lemma 1.1.5 for any ball $B \subset \mathbb{I}^n$

$$\mathcal{H}^n\left(B \cap \limsup_{q\to\infty} \mathcal{A}_q^{(n)}(\tau)^s\right) = \mathcal{H}^n(B).$$

Applying Theorem 3.1.1 we have that

$$\mathcal{H}^s\left(B \cap \limsup_{q\to\infty} \mathcal{A}_q^{(n)}(\tau)\right) = \mathcal{H}^s(B).$$

As a requisite for Theorem 1.2.10 is that $\tau > \frac{1}{n}$, then $s < n$ for all τ so taking $B = \mathbb{I}^n$ we have that

$$\mathcal{H}^s\left(\mathcal{W}_n(\tau)\right) = \mathcal{H}^s(\mathbb{I}^n) = \infty,$$

thus $\dim \mathcal{W}_n(\tau) \ge s = \frac{n+1}{\tau+1}$. $\square$

The general MTP also works very well in a variety of other settings, see for example the proof of Theorem 5.2.3 in Chapter 5.

The first generalisation of the MTP was for systems of linear forms established in [29]. In this paper it was conjectured that some of the conditions imposed were unnecessary, in [5] this was shown to be the case. With this MTP for systems of linear forms it's possible to prove the divergence case of Theorem 1.2.13. This proof is beyond the scope of our use of MTP theorems, we focus exclusively on balls/rectangles. For an in depth proof of the claim made above see [4]. Subsequently, Allen and Baker [4] proved a general MTP for sets satisfying certain conditions, these sets included points, linear forms, self similar sets, and smooth compact manifolds amongst many others.

In all the MTP theorems mentioned thus far the sets used in the condition statement and the output result are evenly shrunk over the whole object (e.g. ball/linear form). None of the theorems allow for a varied rate of compression in each coordinate axis. Where this sort of desired theorem would be useful is in providing Hausdorff dimension results for lim sup sets of hyperrectangles, in particular $\mathcal{W}_n(\Psi)$. There are various methods to obtain Hausdorff dimension results for lim sup sets of rectangles via Theorem 3.1.1 (see Chapter 5.3 of [3] for more details) but these methods require an excessive amount of work in comparison to the theorems in the following section.

3.2 From Balls to Rectangles

Here we consider MTP style theorems that provide Hausdorff measure results for lim sup sets of rectangles. These sort of theorems will be of particular use when considering weighted simultaneous Diophantine approximation. We will provide these theorems in the chronological order that they were proven and discuss the advantages, and disadvantages, of each. Such pros and cons will be illustrated by proving the Theorem of Rynne (Theorem 1.2.11) with the two different forms of MTP.

We begin with the following theorem given by Wang, Wu, and Xu [112]. Let $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_+$ be a vector with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then for any ball $B(\boldsymbol{x}, r) \subset \mathbb{R}^n$ define

$$B^{\boldsymbol{a}} = B(\boldsymbol{x}, (r^{a_1}, \dots, r^{a_n})),$$

i.e. a hyperrectangle with sidelenghts $2r^{a_i}$ and centre $\boldsymbol{x}$. The MTP from balls to rectangles is stated as follows.

Theorem 3.2.1. *Let $(\boldsymbol{x}_j)_{j\in\mathbb{N}}$ be a sequence of points in $[0,1]^n$ and $(r_j)_{j\in\mathbb{N}}$ be a sequence of positive real numbers such that $r_j \to 0$ as $j \to \infty$. Let $B_j = B(x_j, r_j)$ and let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a weight vector, with $a_i \in \mathbb{R}_+$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. Suppose that*

$$\lambda_n \left(\limsup_{j\to\infty} B_j \right) = 1.$$

Then

$$\dim\left(\limsup_{j\to\infty} B_j^{\boldsymbol{a}}\right) \geq \min_{1\leq k\leq n}\left\{\frac{n+\sum_{i=k}^{n}(a_k-a_i)}{a_k}\right\} = s.$$

Furthermore, provided $a_1 > 1$, then

$$\mathcal{H}^s\left(\limsup_{j\to\infty} B_j^{\boldsymbol{a}}\right) = \infty.$$

We should note immediately that unlike Theorem 3.1.1 this theorem was proven explicitly for $X = \mathbb{R}^n$, however, it seems possible that the statement could be generalised to "well-behaved" metric spaces[1]. Like the theorems of the previous section Theorem 3.2.1 allows us to obtain a Hausdorff dimension statement from a full measure statement. Also we are still required to start with a full measure statement for a lim sup set of balls, it is only the Hausdorff dimension result which is for lim sup sets of hyperrectangles.

We remark that while this result is incredibly useful in providing Hausdorff dimension results for lim sup sets of rectangles, due to the prerequisites of Theorem 3.2.1 (going from balls to rectangles) we have the condition that the sidelenghts of the rectangles in the output statement are bounded from above by the radius of the balls used in the Lebesgue statement. To show the usefulness, and issues, with Theorem 3.2.1 we provide a proof for a somewhat restricted version of Theorem 1.2.11.

3.2.1 A restricted proof of Theorem 1.2.11 via Theorem 3.2.1

We will prove both the upper and lower bound of this dimension result. While it is only the lower bound that requires the use of Theorem 3.2.1, we still prove the upper bound as it is less straightforward that that of Theorem 1.2.10. As the title of this subsection suggests the following does not prove Theorem 1.2.11 fully. In particular the condition that

$$\sum_{i=1}^{n}\tau_i > 1$$

in Theorem 1.2.11 is replaced by the condition that each $\tau_i > \frac{1}{n}$. Note that this condition is only needed for the lower bound proof.

Upper Bound: Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$. As given in the previous section define $\mathcal{W}_n(\boldsymbol{\tau}) = \limsup_{q\to\infty} \mathcal{A}_q^{(n)}(\boldsymbol{\tau})$ with

$$\mathcal{A}_q^{(n)}(\boldsymbol{\tau}) = \bigcup_{\substack{0\leq p_i\leq q\\ i=1,\ldots,n}} R\left(\frac{\mathbf{p}}{q}, q^{-\boldsymbol{\tau}-1}\right).$$

Here $R\left(\frac{\boldsymbol{p}}{q}, q^{-\boldsymbol{\tau}-1}\right)$ is a rectangle with centre $\frac{\boldsymbol{p}}{q}$ and sidelenghts $2q^{-\tau_i-1}$ along each i-th coordinate axis. By the above setup we clearly have that $\left\{\mathcal{A}_q^{(n)}(\boldsymbol{\tau})\right\}_{q\in\mathbb{N}}$ is a cover for $\mathcal{W}_n(\boldsymbol{\tau})$. Choose a fixed $1 \leq j \leq n$. As

[1] If X is a product metric space with each direction equipped with an Ahlfors regular Hausdorff measure. This was discussed in correspondence with the authors of [112]

mentioned previously, we may cover each hyperrectangle $R\left(\frac{\mathbf{p}}{q}, q^{-\boldsymbol{\tau}-1}\right)$ by a collection of balls $\mathcal{B}_{\frac{\mathbf{p}}{q}}(q^{-1-\tau_j})$ of radius $q^{-1-\tau_j}$, such that

$$\#\mathcal{B}_{\frac{\mathbf{p}}{q}}(q^{-1-\tau_j}) \leq q^{\sum_{i=j}^{n}(\tau_j-\tau_i)}.$$

Let $Q \in \mathbb{Z}$ satisfy $\rho > Q^{-\tau_i}$ for each i. Then, for $q \geq Q$

$$\bigcup_{\substack{0\leq p_i \leq q \\ i=1,\ldots,n}} \mathcal{B}_{\frac{\mathbf{p}}{q}}(q^{-1-\tau_j})$$

is a ρ-cover of $\mathcal{W}_n(\boldsymbol{\tau})$. Hence

$$\mathcal{H}^s\left(\mathcal{W}_n(\boldsymbol{\tau})\right) \leq \sum_{q=Q}^{\infty} q^n q^{\sum_{i=j}^{n}(\tau_j-\tau_i)}(q^{-1-\tau_j})^s$$

which tends to zero as $Q \to \infty$, provided that

$$\sum_{q=0}^{\infty} q^{n+\sum_{i=j}^{n}(\tau_j-\tau_i)-s(\tau_j+1)} < \infty.$$

This sum converges only when

$$s \geq \frac{n+1+\sum_{i=j}^{n}(\tau_j-\tau_i)+\epsilon}{\tau_j+1},$$

for $\epsilon > 0$. As this holds for all $j = 1, \ldots, n$, and letting ϵ tend to zero, we have that

$$\dim \mathcal{W}_n(\boldsymbol{\tau}) \leq \min_{1\leq j\leq n}\left\{\frac{n+1+\sum_{i=j}^{n}(\tau_j-\tau_i)}{\tau_j+1}\right\}.$$

This provides us with our upper bound.

Lower bound:[condition $\tau_i \geq 1/n$ for each $1 \leq i \leq n$, rather than $\sum_{i=1}^{n}\tau_i \geq 1$] By Theorem 1.2.6 we have that

$$\lambda_n\left(\limsup_{q\to\infty}\mathcal{A}_q^{(n)}\left(\frac{1}{n}\right)\right) = 1.$$

If we take $\boldsymbol{a} = (a_1, \ldots, a_n)$ to be the weight vector with coefficients

$$a_i = \frac{n(1+\tau_i)}{1+n} \quad \text{for} \quad 1 \leq i \leq n \tag{3.4}$$

then for the ball $B_{(\mathbf{p}/q)} = B(\frac{\mathbf{p}}{q}, q^{-1-\frac{1}{n}})$,

$$B^{\boldsymbol{a}}_{(\mathbf{p}/q)} = B\left(\frac{\mathbf{p}}{q}, (q^{-1-\tau_1}, \ldots, q^{-1-\tau_n})\right).$$

Using Theorem 3.2.1 we have

$$\begin{aligned}\dim \mathcal{W}_n(\boldsymbol{\tau}) &\geq \min_{1\leq j\leq n}\left\{\frac{n+\sum_{i=j}^{n}\left(\frac{n(1+\tau_j)}{1+n}-\frac{n(1+\tau_i)}{1+n}\right)}{\frac{n(1+\tau_j)}{1+n}}\right\}, \\ &\geq \min_{1\leq j\leq n}\left\{\frac{n+1+\sum_{i=j}^{n}(\tau_j-\tau_i)}{1+\tau_j}\right\}.\end{aligned}$$

$\square$

Note that in order to apply Theorem 3.2.1 we require $a_1 \geq \cdots \geq a_n \geq 1$. This condition forces the requirement that each $\tau_i \geq 1/n$ (to see this combine the condition that each $a_i \geq 1$ and (3.4)). Thus the lower bound dimension result given above is only valid for $\boldsymbol{\tau}$ satisfying $\tau_i \geq 1/n$ rather than the weaker condition $\sum_{i=1}^{n} \tau_i \geq 1$.

3.3 From Rectangles to Rectangles

We now introduce the most recent form of MTP due to Wang and Wu [111] who established a stronger and in a sense more versatile version of the MTP obtained in [112]. This result can truly be seen as a MTP from rectangles to rectangles. In [111] two forms of MTP from rectangles to rectangles were established. As we shall see, the first form has the advantage that we only need a full measure statement to apply the theorem. For the second form a ubiquity hypothesis is required, similar to that of [18], a condition unnecessary in all of the previously stated results.

Prior to the statement of the Theorems we state the notion of local ubiquity for rectangles introduced in [111], which is a generalisation of the notion of local ubiquity for balls introduced in [18]. Fix an integer $n \geq 1$, and for each $1 \leq i \leq n$ let $(X_i, |\cdot|_i, m_i)$ be a bounded locally compact measure-metric space, where $|\cdot|_i$ denotes the metric and m_i denotes a measure over X_i, which will be assume to be a δ_i-Ahlfors regular probability measure. Consider the product space $(X, |\cdot|, m)$, where

$$X = \prod_{i=1}^{n} X_i, \quad m = \prod_{i=1}^{n} m_i, \quad |\cdot| = \max_{1 \leq i \leq n} |\cdot|_i$$

are defined in the usual way. For example, in the setting of Chapter 2 we could take $X_i = \mathbb{Z}_p$, $m_i = \mu_p$ and $|\cdot|_i = |\cdot|_p$ for each $1 \leq i \leq n$ so $X = \mathbb{Z}_p^n$, $m = \mu_{p,n}$, and $|\cdot|$ is the usual sup norm. For any $x \in X$ and $r \in \mathbb{R}_+$ define the open ball

$$B(x, r) = \left\{ y \in X : \max_{1 \leq i \leq n} |x_i - y_i|_i < r \right\} = \prod_{i=1}^{n} B_i(x_i, r),$$

where B_i are the usual open r-balls associated with the i^{th} metric space X_i. Let J be a countably infinite index set, and $\beta : J \to \mathbb{R}_+$, $\alpha \mapsto \beta_\alpha$ a positive function satisfying the condition that for any $N \in \mathbb{N}$

$$\# \{\alpha \in J : \beta_\alpha < N\} < \infty.$$

Let l_n, u_n be two sequences in $\mathbb{R}_+$ such that $u_n \geq l_n$ with $l_n \to \infty$ as $n \to \infty$. Define

$$J_n = \{\alpha \in J : l_n \leq \beta_\alpha \leq u_n\}.$$

Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that $\rho(x) \to 0$ as $x \to \infty$. For each $1 \leq i \leq n$, let $(R_{\alpha,i})_{\alpha \in J}$ be a sequence of subsets in X_i. The family of sets $(R_\alpha)_{\alpha \in J}$ where

$$R_\alpha = \prod_{i=1}^{n} R_{\alpha,i},$$

for each $\alpha \in J$, are called *resonant sets*. For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ define

$$\Delta(R_\alpha, \rho(r)^{\boldsymbol{a}}) = \prod_{i=1}^{n} \Delta'(R_{\alpha,i}, \rho(r)^{a_i}),$$

where for some set $A \subset X_i$ and $b \in \mathbb{R}_+$

$$\Delta'(A, b) = \bigcup_{a \in A} B(a, b)$$

is the union of balls in X_i of radius b centred at all possible points in A.

Definition 3.3.1 (Local ubiquitous system of quasi-rectangles)**.** Call the pair $\big((R_\alpha)_{\alpha \in J}, \beta\big)$ *a local ubiquitous system of rectangles with respect to* $(\rho, \boldsymbol{a})$ if there exists a constant $c > 0$ such that for any ball $B \subset X$

$$\limsup_{n \to \infty} m\left(B \cap \bigcup_{\alpha \in J_n} \Delta(R_\alpha, \rho(u_n)^{\boldsymbol{a}})\right) \geq cm(B).$$

We remark here that the definition is stated as local ubiquitous systems of quasi-rectangles due to fact that the objects $\Delta(R_\alpha, \rho(u_n)^{\boldsymbol{a}})$ may look nothing like rectangles in the usual sense, for example if the resonant sets are lines. In the special case of the resonant sets being points then we could consider Definition 3.2.2 as a ubiquitous system of rectangles. The second property needed to state the Wang-Wu theorem is a local scaling property, which was first introduced in [4], and which is a version of the intersection properties of [18]. In our setting the condition will be satisfied for $k = 0$ and holds trivially. Nevertheless, we include the condition for the sake of completeness.

Definition 3.3.2 (k-scaling property)**.** Let $0 \leq k < 1$ and $1 \leq i \leq n$. The sequence $\{R_{\alpha,i}\}_{\alpha \in J}$ has *k-scaling property* if for any $\alpha \in J$, any ball $B(x_i, r) \subset X_i$ with centre $x_i \in R_{\alpha,i}$, and $0 < \epsilon < r$ then

$$c_2 r^{\delta_i k} \epsilon^{\delta_i(1-k)} \leq m_i\left(B(x_i, r) \cap \Delta(R_{\alpha,i}, \epsilon)\right) \leq c_3 r^{\delta_i k} \epsilon^{\delta_i(1-k)},$$

for some constants $c_2, c_3 > 0$.

Finally, for $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$, define

$$W(\mathbf{t}) = \limsup_{\alpha \in J} \Delta(R_\alpha, \rho(\beta_\alpha)^{\boldsymbol{a}+\mathbf{t}}).$$

We now state the following theorems due to Wang and Wu [111].

Theorem 3.3.3 (Mass Transference Principle from Rectangles to Rectangles with Ubiquity)**.** *Let* $(X, |\cdot|, m)$ *be a product space of* n *bounded locally compact metric spaces* $(X_i, |\cdot|_i, m_i)$ *with* m_i *a* δ_i*-Ahlfors probability measure, for* $1 \leq i \leq n$*. Let* $(R_\alpha)_{\alpha \in J}$ *be a sequence of subsets contained in* X *and assume that* $((R_\alpha)_{\alpha\in J}, \beta)$ *is a local ubiquitous system of rectangles with respect to* $(\rho, \boldsymbol{a})$ *for some* $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_+$*, and that* $(R_\alpha)_{\alpha\in J}$ *satisfies the* k*-scaling property. Then, for any* $\boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{R}^n_+$

$$\dim W(\boldsymbol{t}) \geq \min_{A_i \in A} \left\{ \sum_{j\in K_1} \delta_j + \sum_{j \in K_2} \delta_j + k \sum_{j\in K_3} \delta_j + (1-k) \frac{\sum_{j\in K_3} a_j \delta_j - \sum_{j \in K_3} t_j \delta_j}{A_i} \right\} = s,$$

where $A = \{a_i, a_i + t_i, 1 \leq i \leq n\}$ *and* K_1, K_2, K_3*, dependent on the choice of* A_i*, are a partition of* $\{1, \dots, n\}$ *defined as*

$$K_1 = \{j : a_j \geq A_i\}, \quad K_2 = \{j : a_j + t_j \leq A_i\} \backslash K_1, \quad K_3 = \{1, \dots n\} \backslash (K_1 \cup K_2).$$

Furthermore, for any ball $B \subset X$

$$\mathcal{H}^s(B \cap W(\boldsymbol{t})) = \mathcal{H}^s(B). \tag{3.5}$$

Theorem 3.3.4 (Mass Transference Principle from Rectangles to Rectangles without Ubiquity)**.** *Suppose that each measure* m_i *is* δ_i*-Ahlfors regular and* $R_{\alpha,i}$ *has* k*-scaling property for each* $\alpha \in J$ $(1 \leq i \leq n)$*. Suppose*

$$m \left(\limsup_{\alpha \in J} \Delta(R_\alpha, \rho(\beta_\alpha)^{\boldsymbol{a}}) \right) = m(X).$$

Then

$$\dim W(\boldsymbol{t}) \geq s,$$

where s *is defined in Theorem 3.3.3.*

Note that the full measure statement of Theorem 3.3.4 is far easier to establish than the local ubiquity statement required in Theorem 3.3.3. However this short cut comes at the cost of s-Hausdorff measure statement, which we cannot attain via Theorem 3.3.4. In cases where the ubiquity statement is relatively easy to establish (see for example Theorem 5.1.4) this is not a problem.

3.3.1 A proof of Theorem 1.2.11 via Theorem 3.3.4

We now provide a complete lower bound proof of Theorem 1.2.11. Since Theorem 1.2.11 is a statement purely on the Hausdorff dimension we use Theorem 3.3.4.

Lower bound of Theorem 1.2.11:[condition $\sum_{i=1}^n \tau_i \geq 1$] By Theorem 1.2.6 we have that

$$\lambda_n \left(\limsup_{q\to\infty} \mathcal{A}_q^{(n)}(\boldsymbol{a}) \right) = 1$$

for any n-tuple $\boldsymbol{a} = (a_1, \ldots, a_n)$ such that $a_i > 0$ and $\sum_{i=1}^n a_i = 1$. Without loss of generality we may suppose that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n > 0$. For $0 \leq i \leq n-1$ define a_{n-i} recursively by

$$a_{n-i} = \min\left\{\tau_{n-i}, \frac{1 - \sum_{j=n-i+1}^n a_j}{n-i}\right\}.$$

In the case where $i = 0$ we take the second term to be $\frac{1}{n}$. We claim there exists $0 \leq K \leq n-1$ such that

$$a_u = \frac{1 - \sum_{j=n-K+1}^n a_j}{n-K},$$

for all $1 \leq u \leq n-K$, and

$$a_u = \tau_u$$

for all $n-K+1 \leq u \leq n$. To show this claim is true note that $\tau_{n-(K+1)} \geq \tau_{n-K}$, since $\tau_1 \geq \cdots \geq \tau_n$, and so if

$$\tau_{n-K} = \frac{1 - \sum_{j=n-K+1}^n a_j}{n-K}$$

then clearly

$$\tau_{n-(K+1)} \geq \frac{1 - \sum_{j=n-K+1}^n a_j}{n-K}. \tag{3.6}$$

Furthermore note that

$$\begin{aligned}
\frac{1 - \sum_{j=n-(K+1)+1}^n a_j}{n-(K+1)} &= \frac{1 - \sum_{j=n-K+1}^n a_j - a_{n-K}}{n-(K+1)}, \\
&= \frac{1 - \sum_{j=n-K+1}^n a_j - \left(\frac{1-\sum_{j=n-K+1}^n a_j}{n-K}\right)}{n-(K+1)}, \\
&= \frac{(n-K)\left(1 - \sum_{j=n-K+1}^n a_j\right) - \left(1 - \sum_{j=n-K+1}^n a_j\right)}{(n-K)(n-(K+1))}, \\
&= \frac{1 - \sum_{j=n-K+1}^n a_j}{n-K},
\end{aligned}$$

and so, by the above and (3.6), we have that

$$a_{n-(K+1)} = \min\left\{\tau_{n-(K+1)}, \frac{1 - \sum_{j=n-(K+1)+1}^n a_j}{n-(K+1)}\right\} = \frac{1 - \sum_{j=n-K+1}^n a_j}{n-K}.$$

We should observe at this point that if $K = 0$ then each $a_i = \frac{1}{n}$, and so we would begin with a full measure statement on a lim sup set of balls. Note that the vector $(a_1, \ldots, a_n)$ constructed above satisfies the condition that $\sum_{i=1}^n a_i = 1$, since

$$\sum_{i=1}^n a_i = \sum_{i=1}^{n-K} \left(\frac{1 - \sum_{j=n-K+1}^n a_j}{n-K}\right) + \sum_{j=n-K+1}^n a_j = 1.$$

By construction we have that $a_i \leq \tau_i$ for each $1 \leq i \leq n$, so the n-tuple $\mathbf{t} = (t_1, \ldots, t_n)$ is well defined as

$$t_i = \tau_i - a_i.$$

Note that for each metric space $X_i = \mathbb{I}$ the measure λ is a 1-Ahlfors probability measure. Consider the following three cases:

i) $A_i \in \{a_1+1, \dots, a_{n-K}+1\}$: For these values of A_i we have that

$$K_1 = \{1, \dots, n-K\}, \quad K_2 = \{n-K+1, \dots, n\}, \quad K_3 = \emptyset.$$

In the case where $K = 0$ take $K_2 = \emptyset$. Applying Theorem 3.3.4 we have that

$$\begin{aligned}\dim \mathcal{W}_n(\boldsymbol{\tau}) &\geq \min_{A_i} \left\{ \frac{(n-K)(a_i+1) + (n-(n-K+1)+1)(a_i+1) - \sum_{j=n-K}^{n} t_j}{a_i+1} \right\}, \\ &= \min_{A_i} \left\{ n - \frac{\sum_{j=n-K+1}^{n} t_j}{a_i+1} \right\}.\end{aligned}$$

Since $t_j = 0$ for $n-K+1 \leq j \leq n$ we have that $\dim \mathcal{W}_n(\boldsymbol{\tau}) \geq n$.

ii) $A_i \in \{a_{n-K+1}+1, \dots, a_n+1\}$: For such values of A_i observe that

$$K_1 = \{1, \dots, i\}, \quad K_2 = \{i+1, \dots, n\}, \quad K_3 = \emptyset.$$

Applying Theorem 3.3.3 we have, in this case,

$$\dim \mathcal{W}_n(\boldsymbol{\tau}) \geq \min_{A_i} \left\{ \frac{i(a_i+1) + (n-i)(a_i+1) - \sum_{j=i+1}^{n} t_j}{a_i+1} \right\}.$$

Similarly to the previous case, since $t_j = 0$ for $n-K+1 \leq j \leq n$ the r.h.s of the above equation is n, the maximal dimension of $\mathcal{W}_n(\boldsymbol{\tau})$.

iii) $A_i \in \{\tau_1+1, \dots, \tau_n+1\}$: For $\tau_i = a_i$ with $n-K+1 \leq i \leq n$ ii) covers such result. So we only need to consider the set of $A_i \in \{\tau_1+1, \dots \tau_{n-K}+1\}$. If A_i is contained in such set, then

$$K_1 = \emptyset, \quad K_2 = \{i, \dots, n\}, \quad K_3 = \{1, \dots, i-1\}.$$

Thus, by Theorem 3.3.3, we have that

$$\begin{aligned}\dim \mathcal{W}_n(\boldsymbol{\tau}) &\geq \min_{A_i} \left\{ \frac{(n-i+1)(\tau_i+1) + \sum_{j=1}^{i-1}(a_j+1) - \sum_{j=i}^{n} t_j}{\tau_i+1} \right\}, \\ &= \min_{A_i} \left\{ \frac{(n-i+1)(\tau_i+1) + (i-1)\left(1 + \frac{1-\sum_{j=n-K+1}^{n} a_j}{n-K}\right) - \sum_{j=i}^{n-K}(\tau_j - a_j) - \sum_{j=n-K+1}^{n} t_j}{\tau_i+1} \right\}, \\ &= \min_{A_i} \left\{ \frac{(n-i+1)(\tau_i+1) + (n-K)\left(\frac{1-\sum_{j=n-K+1}^{n}(a_j-1)}{n-K}\right) + (i-1) - \sum_{j=i}^{n-K} \tau_j - \sum_{j=n-K+1}^{n} t_j}{\tau_i+1} \right\}, \\ &= \min_{A_i} \left\{ \frac{n+1+\sum_{j=i}^{n}(\tau_i - \tau_j)}{\tau_i+1} \right\},\end{aligned}$$

since $a_j + t_j = \tau_j$.

These are all possible choices of A_i and thus completes the lower bound of Theorem 1.2.11.

$\square$

There are several remarks to make on this proof. Firstly, note that the general framework of proof remains unchanged between this proof and the one above via Theorem 3.2.1. The first real difference appears when discussing the value of K in the above proof. In the proof via Theorem 3.2.1 this value of K is rigidly fixed at $K = 0$, whereas in the application of Theorem 3.3.4 this need not be the case.

Variations of the recursive formula used above will be applied regularly when using Theorems 3.3.3-3.3.4. The technical details of the formula can be ignored, the key idea being that the formula ensures our original rectangles used in the full measure statement have sufficiently large sidelenghts so that the rectangles in our $\limsup$ set of study (e.g. $\mathcal{W}_n(\boldsymbol{\tau})$ in this case) can fit inside.

As the above results show, when calculating the Hausdorff dimension of $\limsup$ sets the MTP theorems make the lower bound calculation much easier, provided a full measure statement can be proven. In Chapters 4-6 where the lower bound result is difficult to calculate using traditional methods these MTP style theorems are incredibly useful.

Chapter 4

Real Weighted Simultaneous Approximation over manifolds

We start this Chapter with an overview of the latest results in the field of simultaneous approximation over manifolds. In particular we will focus on the more precise measure of Hausdorff measure and Hausdorff dimension. This provides a background for our result proven at the end of the chapter, which is to obtain a general lower bound on the dimension of weighted simultaneously approximable points on manifolds. The contents of this chapter is essentially [23] jointly published with Beresnevich and Levesley. Since the publication of [23] there has been an improvement in Mass Transference Principle results, namely the developments given in [111]. This has subsequently led to an improvement in the range of approximation functions that can be used in the main result of this chapter. See [7] for more details.

4.1 Diophantine approximation on manifolds

When considering manifolds we look at them locally on some open subset $\mathcal{U} \subset \mathbb{R}^d$ and use the following Monge parametrisation without loss of generality

$$\mathcal{M} := \{(\boldsymbol{x}, \mathbf{f}(\boldsymbol{x})) : \boldsymbol{x} \in \mathcal{U}\} \subseteq \mathbb{R}^n,$$

where d is the dimension of the manifold, and $\mathbf{f}$ is a map such that $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$ with $m = n - d$ being the codimension of the manifold. As the manifold is of this form we can consider the approximation of the coordinates $\boldsymbol{x}$ and $\mathbf{f}(\boldsymbol{x})$ separately. We will refer to $\boldsymbol{x}$ as the independent variables, and the codomain of $\mathbf{f}$ as the dependent variables. In the special case of simultaneous approximation on manifolds the approximation functions on both the independent and dependent variables are the same.

Much progress has been made in establishing measure theoretic results for the set $\mathcal{W}_n(\psi) \cap \mathcal{M}$, we highlight some of these results below. Sprindzhuk established many of the foundational results in this area which he referred to as Diophantine approximation on dependent variables [105]. A differentiable manifold is called extremal if almost all points, with respect to the induced Lebesgue measure of the manifold, are extremal, whereby we mean that the Dirichlet approximation exponent of $\mathbb{R}^n$ cannot be improved for almost all points on the manifold. It was first conjectured [107] and later proven by Kleinbock and Margulis [78] that any non-degenerate submanifold of $\mathbb{R}^n$ is extremal, where non-degeneracy is defined as below.

Definition 4.1.1. *A map $\boldsymbol{f}\colon U \to \mathbb{R}^n$ is non-degenerate at $u \in U \subset \mathbb{R}^m$ if there exists some $k \in \mathbb{N}$ such that $\boldsymbol{f}$ is k times continuously differentiable on some sufficiently small ball centred at u, and the partial derivatives of $\boldsymbol{f}$ at u of orders up to k span $\mathbb{R}^n$. The map $\boldsymbol{f}$ is non-degenerate if it is non-degenerate at almost all points $u \in U$, in terms of λ_m. A manifold $\mathcal{M}$, with $\dim \mathcal{M} = m > n$, embedded in $\mathbb{R}^n$ is said to be non-degenerate if it arises from a non-degenerate map $\boldsymbol{f}\colon U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^m$, that is $\mathcal{M} = \boldsymbol{f}(U)$.*

Generally a manifold is non-degenerate if it is sufficiently curved almost everywhere with respect to the induced Lebesgue measure of the manifold. As an example note that any connected analytic manifold not contained within a hyperplane is non-degenerate. For an example of a degenerate space note that any line or hyperplane is degenerate everywhere.

For the Hausdorff dimension of $\mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M}$ note trivially that

$$\dim \mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M} \leq \dim \mathcal{M},$$

with equality if $\sum_{i=1}^{n} \tau_i \leq 1$ by Theorem 1.2.6. One of the first non-trivial advances with respect to the Hausdorff dimension of the set $\mathcal{W}_2(\psi) \cap \mathcal{M}$ was by Beresnevich, Dickinson, and Velani in [19], where they determined the dimension of the set of simultaneously approximable points on sufficiently curved planar curves in $\mathbb{R}^2$. There is also a related paper [30] which uses a similar technique to find the Hausdorff dimension of $\mathcal{W}_2(\boldsymbol{\tau}) \cap \mathcal{M}$ for $\boldsymbol{\tau} = (\tau_1, \tau_2)$ bounded below and above by 0 and 1 respectively. Both papers give an equality for the dimension rather than just a lower bound as presented in this paper. The following is Theorem 4 from [30]. We denote the set of n times continuously differentiable functions by $C^{(n)}$.

Theorem 4.1.2 (Beresnevich et al. [30])**.** *Let f be a $C^{(3)}$ function over an interval $I_0 \subset \mathbb{R}$, and let $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Let $\boldsymbol{\tau} = (\tau_1, \tau_2)$, where τ_1 and τ_2 are positive numbers such that $0 < \min\{\tau_1, \tau_2\} < 1$ and $\tau_1 + \tau_2 \geq 1$. Assume that*

$$\dim \left\{ x \in I_0 : f''(x) = 0 \right\} \leq \frac{2 - \min\{\tau_1, \tau_2\}}{1 + \max\{\tau_1, \tau_2\}}. \tag{4.1}$$

Then

$$\dim \mathcal{W}_2(\boldsymbol{\tau}) \cap \mathcal{C}_f = \frac{2 - \min\{\tau_1, \tau_2\}}{1 + \max\{\tau_1, \tau_2\}}.$$

Theorem 4 from [19] is the simultaneous case where $\tau_1 = \tau_2$. The common approach in both papers is ubiquity, as established in [18], to determine the lower bound. The upper bound is found through a combination of Huxley's estimate [72], which gives an upper estimate on the number of rational points within a specified neighbourhood of the curve, and the property given by (4.1). This result has been further improved by Beresnevich and Zorin [35] who showed that the lower bound dimension result holds for weakly non-degenerate curves (see Theorem 4 of [35]). Upper bound dimension results have been found for various forms of weakly non-degenerate curves [68], but the complete result remains elusive. In the n-dimensional setting Beresnevich et al. [22] proved the following result.

Theorem 4.1.3 (Beresnevich et al. [22])**.** *Let $\mathcal{M}$ be any twice continuously differentiable submanifold of $\mathbb{R}^n$ of codimension m and let*

$$\frac{1}{n} \leq \tau < \frac{1}{m}.$$

Then

$$\dim \mathcal{W}_n(\tau) \cap \mathcal{M} \geq s := \frac{n+1}{\tau+1} - m.$$

Furthermore,

$$\mathcal{H}^s(\mathcal{W}_n(\tau) \cap \mathcal{M}) = \mathcal{H}^s(\mathcal{M}).$$

Remark 4.1.4. In the special case where the submanifold $\mathcal{M}$ is a curve this result has been proven for a wider range of τ. In particular, for any analytic non-degenerate curve $\mathcal{C} \subset \mathbb{R}^n$, if

$$\frac{1}{n} < \tau < \frac{3}{2n-1},$$

then the dimension result of Theorem 4.1.3 still holds, that is

$$\dim \mathcal{W}_n(\tau) \cap \mathcal{C} \geq s := \frac{n+1}{\tau+1} - (n-1).$$

See Theorem 7.2 of [14] for more details. More recently this has been extended to non-degenerate curves [13].

A key result required in the proof of Theorem 4.1.3 is the Mass Transference Principle (Theorem 3.1.1). Recently, Beresnevich et.al. [26] worked on finding an upper bound on the distribution of rational points within a ψ-neighbourhood of manifolds. Using this result they proved the following theorem, giving a corresponding upper bound to Theorem 4.1.3.

Theorem 4.1.5. *Let $\mathcal{M}_f \subset \mathbb{R}^n$ be a manifold defined on an open subset $\mathcal{U} \subset \mathbb{R}^d$, and suppose that*

$$\mathcal{H}^s\left(\left\{\alpha \in \mathcal{U} : \left|\det\left(\frac{\partial^2 f_j}{\partial \alpha_1 \partial \alpha_i}(\alpha)\right)_{1 \leq i,j \leq m}\right| = 0\right\}\right) = 0, \tag{4.2}$$

for

$$s = \frac{n+1}{\tau+1} - m.$$

If

$$d > \frac{n+1}{2}, \quad \text{and} \quad \frac{1}{n} \leq \tau \leq \frac{1}{2m+1},$$

Then

$$\dim \mathcal{W}_n(\tau) \cap \mathcal{M}_f \leq s.$$

This is a simplified version of the main result established in [26]. In particular the convergent Hausdorff measure result was proven for general functions $\psi(q) \geq q^{-1/(2m+1)}(\log q)^{2/(2m+1)}$. Further still, the result was proven for the general case of inhomogeneous simultaneous approximation.

Since the establishment of Theorem 4.1.5 there has been several results that allow for a broader range of manifolds. Recently Simmons relaxed condition (4.2) as follows (see Theorem 2.1 of [104]). Suppose there exists some $k \in \mathbb{N}$ such that

$$s\left(1+\frac{k}{2m+k}\right) > n+1,$$

and

$$rank\left(\boldsymbol{y}.\frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j}(\alpha)\right)_{1\leq i,j\leq d} \geq k, \quad \forall \boldsymbol{y} \in \mathbb{R}^m\backslash\{0\},$$

for almost all $\alpha \in \mathcal{U}$ (w.r.t the Hausdorff $(s-m)$-measure). Then

$$\dim \mathcal{W}_n(\tau) \cap \mathcal{M}_f \leq s-m.$$

Results of this type have also been proven for hypersurfaces. In [69] Huang proved an upper bound on the number of rational points within a small neighbourhood of a general $C^{(l)}$ hypersurface $\mathcal{H} \subseteq \mathbb{R}^n$ with Gaussian curvature bounded above zero. This theorem provided a variety of results, including the following (Theorem 5 of [69]).

Theorem 4.1.6 (Huang [69])**.** *Let $n \geq 3$ be an integer and let*

$$l = \max\left\{\left\lfloor\frac{n-1}{2}\right\rfloor + 5, n+1\right\}.$$

For any approximation function ψ, any $s > \frac{n-1}{2}$, and any $C^{(l)}$ hypersurface $\mathcal{H} \subseteq \mathbb{R}^n$ with non-vanishing Gaussian curvature everywhere except possibly on a set of zero Hausdorff s-measure, we have that

$$\mathcal{H}^s(\mathcal{W}_n(\psi) \cap \mathcal{H}) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi(q)^{s+1} q^{n-1-s} < \infty.$$

In the special case of τ-approximable functions the above theorem implies that

$$\dim \mathcal{W}_n(\tau) \cap \mathcal{H} \leq \frac{n+1}{\tau+1} - 1.$$

Thus we note that when considered against the applicable range of approximation functions from Theorem 4.1.3 this is the best possible upper bound.

Similar results have also been found for simultaneous approximation on affine subspaces. For a matrix $M \in \mathbb{R}^{d\times(n-d)}$, and row vector $\alpha \in \mathbb{R}^{n-d}$ let

$$\mathfrak{A} := \left\{ \left(\boldsymbol{x}, (1, \boldsymbol{x}). \begin{pmatrix} \alpha \\ M \end{pmatrix} \right) : \boldsymbol{x} \in \mathbb{R}^d \right\}.$$

Huang and Liu [70] proved that such affine subspace $\mathfrak{A} \subseteq \mathbb{R}^n$ with bounded Diophantine properties on the matrix $\begin{pmatrix} \alpha \\ M \end{pmatrix} \in \mathbb{R}^{(d+1)\times(n-d)}$, and any approximation function $q^{-\tau}$ with $\tau \geq 1/n$, then

$$\dim \mathcal{S}_n(\tau) \cap \mathfrak{A} \leq \frac{n+1}{\tau+1} - (n-d).$$

For general manifolds Theorem 4.1.3 and Theorem 4.1.5 collectively give the following corollary.

Corollary 4.1.7. *Let $\mathcal{M}_f$ be a manifold satisfying* (4.2) *and $d > \frac{n+1}{2}$. Suppose that*

$$\frac{1}{n} \leq \tau \leq \frac{1}{2m+1},$$

then

$$\dim \mathcal{W}_n(\tau) \cap \mathcal{M}_f = \frac{n+1}{\tau+1} - m.$$

In this chapter we adapt the arguments given in [22] to establish the following result, a weighted simultaneous version of Theorem 4.1.3.

Theorem 4.1.8. *Let $\mathcal{M} := \left\{ (\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \mathcal{U} \subset \mathbb{R}^d \right\}$ where $f : \mathcal{U} \to \mathbb{R}^m$ with $f \in C^{(2)}$. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_{>0}$ with*

$$\tau_1 \geq \cdots \geq \tau_d \geq \max_{d+1\leq i\leq n} \left\{ \tau_i, \frac{1 - \sum_{j=1}^{m} \tau_{j+d}}{d} \right\}, \quad \textit{and} \quad \sum_{i=1}^{m} \tau_{d+i} < 1.$$

Then

$$\dim \left(\mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M} \right) \geq \min_{1\leq j\leq d} \left\{ \frac{n+1+\sum_{i=j}^{n}(\tau_j - \tau_i)}{\tau_j + 1} - m \right\}.$$

Remark 4.1.9. Note that the minimum is taken over only the τ_i for $1 \leq i \leq d$, that is the approximation functions over the independent variables $\boldsymbol{x} \in \mathbb{R}^d$. This condition may only be due to the setup of our proof and the fact that all approximation functions over the independent variables are larger than all the dependent variable approximation functions. While we suspect this to be unnecessary, the mass transference style result used in the proof of Theorem 4.1.8 forces such conditions to be applied.

Remark 4.1.10. We only have a lower bound here rather than equality. This lower bound agrees with both Theorem 4.1.2 and Theorem 4.1.6 so in these cases this is the best lower bound. In order to prove the upper bound result to Theorem 4.1.8 we need an upper bound result on the number of rational points within a Ψ-neighbourhood of a manifold. While there are many various results for counting rational points in the simultaneous case (e.g. [72, 26, 69, 34]), a weighted simultaneous version is yet to be proven. Without such results an upper bound result is currently out of reach.

We would like to generalise Theorem 4.1.8 to more general approximation functions. To achieve this we must apply some constraints on our approximation function. Given a decreasing approximation function $\Psi = (\psi_1, \ldots, \psi_n)$ define the upper order $v(\Psi) = (v_1, \ldots, v_n)$ of Ψ at infinity by

$$v_i := \limsup_{q \to \infty} \frac{-\log \psi_i(q)}{\log q}, \quad 1 \leq i \leq n. \tag{4.3}$$

Given such a function, we can state the following Corollary.

Corollary 4.1.11. *Let $\mathcal{M} := \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \mathcal{U} \subset \mathbb{R}^d\}$ where $f : \mathcal{U} \to \mathbb{R}^m$ with $f \in C^{(2)}$. For any approximation function $\Psi = (\psi_1, \ldots, \psi_n)$ such that (4.3) are positive finite, and*

$$v_1 \geq v_2 \geq \cdots \geq v_d \geq \max_{d+1 \leq i \leq n} \left\{ v_i, \frac{1 - \sum_{i=d+1}^{n} v_i}{d} \right\}, \quad \text{and} \quad \sum_{i=d+1}^{n} v_i < 1.$$

we have that

$$\dim \left(\mathcal{W}_n(\Psi) \cap \mathcal{M} \right) \geq \min_{1 \leq j \leq d} \left\{ \frac{n + 1 + \sum_{i=j}^{n} (v_j - v_i)}{v_j + 1} \right\}.$$

Proof. By properties of the approximation function given by (4.3) we have that, for any $\epsilon > 0$ there exists a $q_0 \in \mathbb{N}$ such that for all $q > q_0$

$$\psi_i(q) \geq q^{-v_i - \epsilon}, \text{for each } \ 1 \leq i \leq n.$$

Using this property, for $\epsilon = (\epsilon, \ldots, \epsilon) \in \mathbb{R}^n_+$ we obtain that

$$\mathcal{W}_n(v(\Psi) + \epsilon) \subset \mathcal{W}_n(\Psi)$$

so by Theorem 4.1.8, and letting $\epsilon \to 0$, we have that

$$\begin{aligned} \dim \left(\mathcal{W}_n(\Psi) \cap \mathcal{M} \right) &\geq \lim_{\epsilon \to 0^+} \dim \left(\mathcal{W}_n(v(\Psi) + \epsilon) \cap \mathcal{M} \right), \\ &\geq \min_{1 \leq i \leq d} \left\{ \frac{n + 1 + \sum_{i=j}^{n} (v_j - v_i)}{v_j + 1} \right\}, \end{aligned}$$

as required. $\square$

Remark 4.1.12. Note that this proof is similar to the proof of Corollary 1 from [97]. However we can use the weaker condition of the lim sup rather than lim as we only need the lower bound rather than equality.

The remainder of this Chapter is laid out as follows. In the following section we recall some key theorems required in the proof of Theorem 4.1.8. One of these key results, the Mass transference principle from balls to rectangles, has already been stated in Chapter 3. In §3.2.1 we prove a Dirichlet style result on weighted simultaneous approximation over manifolds. This result is vital in order to apply the mass transference style theorem. In the final section we combine the results to prove Theorem 4.1.8.

4.2 Preliminary results

As stated above a key result in the proof of Theorem 4.1.8 is the Mass Transference Principle (MTP) from balls to rectangles (Theorem 3.2.1). We refer the reader to Chapter 3 §3.2 for the statement and an application of the Theorem. As observed in Chapter 3 we need two key ingredients in order to apply Theorem 3.2.1. Firstly we need a lim sup set of balls with full Lebesgue measure. Secondly, we need to construct a weight vector $\boldsymbol{a}$ that we can use to transform our set of full Lebesgue measure to our desired lim sup set of hyperrectangles. We will ascertain these results in the following section.

The last measure theoretic result we will be using to prove Theorem 4.1.8 is a lemma from [31], which essentially states that the Lebesgue measure of a lim sup set remains the same when the balls are altered by some fixed constant.

Lemma 4.2.1. *Let $\{B_i\}$ be a sequence of balls in $\mathbb{R}^k$ with $\lambda_k(B_i) \to 0$ as $i \to \infty$, where λ_k is k-dimensional Lebesgue measure. Let $\{U_i\}$ be a sequence of Lebesgue measurable sets such that $U_i \subset B_i$ for all i. Assume that for some $c > 0$, $\lambda_k(U_i) \geq c\mu_k(B_i)$ for all i. Then the sets*

$$\limsup_{i\to\infty} U_i \qquad \text{and} \qquad \limsup_{i\to\infty} B_i$$

have the same Lebesgue measure.

We can use Lemma 4.2.1 to change the radius of the balls used in our construction of the lim sup set by a constant and still ensure we have full Lebesgue measure.

4.2.1 Dirichlet Style Theorem on Manifolds

In order to apply Theorem 3.2.1 we construct a lim sup set of balls with full Lebesgue measure. We achieve this by varying the approximation functions only over the dependent variables, so we can form a lim sup set from the balls centred at certain rational points in the independent variable space. The theorem below constructs such a set.

Theorem 4.2.2. *Let $\mathcal{M} := \left\{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \mathcal{U} \subset \mathbb{R}^d\right\}$ where $f : \mathcal{U} \to \mathbb{R}^m$ with $f \in C^{(2)}$. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m) \in \mathbb{R}^m_{>0}$, and let $\tilde{\tau} = \frac{1}{m}\sum_{i=1}^m \tau_i$. If*

$$\tilde{\tau} m < 1,$$

then for any $\boldsymbol{x} \in \mathcal{U}$ *there is an integer* Q_0 *such that for any* $Q \geq Q_0$ *there exists* $(p_1, \ldots, p_n, q) \in \mathbb{Z}^n \times \mathbb{N}$ *with* $1 \leq q \leq Q$ *and* $(\frac{p_1}{q}, \ldots, \frac{p_d}{q}) \in \mathcal{U}$ *such that*

$$\left| x_i - \frac{p_i}{q} \right| < \frac{4^{m/d}}{q(Q^{1-\tilde{\tau}m})^{1/d}}, \quad 1 \leq i \leq d, \tag{4.4}$$

and

$$\left| f_j\left(\frac{p_1}{q}, \ldots, \frac{p_d}{q}\right) - \frac{p_{d+j}}{q} \right| < \frac{q^{-\tau_j - 1}}{2}. \quad 1 \leq j \leq m. \tag{4.5}$$

Further, for any $\boldsymbol{x} \in \mathcal{U} \backslash \mathbb{Q}^d$ *there exists infinitely many* $(p_1, \ldots, p_n, q) \in \mathbb{Z}^n \times \mathbb{N}$ *with* $(\frac{p_1}{q}, \ldots, \frac{p_d}{q}) \in \mathcal{U}$ *satisfying* (4.5) *and*

$$\left| x_i - \frac{p_i}{q} \right| < 4^{m/d} q^{-1-(1-\tilde{\tau}m)/d}, \quad 1 \leq i \leq d. \tag{4.6}$$

Before proving Theorem 4.2.2 we will state several properties of our manifold $\mathcal{M}$ that we will be using. Given that $\mathcal{M}$ is constructed by a twice continuously differentiable function f we can choose a suitable $\mathcal{U}$ such that, without loss of generality, the following two constants exist:

$$C = \max_{\substack{1 \leq i,k \leq d \\ 1 \leq j \leq m}} \sup_{\boldsymbol{x} \in \mathcal{U}} \left| \frac{\partial^2 f_j}{\partial x_i \partial x_k}(\boldsymbol{x}) \right| < \infty, \tag{4.7}$$

and

$$D = \max_{\substack{1 \leq i \leq d \\ 1 \leq j \leq m}} \sup_{\boldsymbol{x} \in \mathcal{U}} \left| \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) \right| < \infty. \tag{4.8}$$

A brief outline of the proof is as follows; firstly we alter the system of inequalities to a suitable form so Minkowski's Theorem for systems of linear forms can be applied. We then use Taylor's approximation Theorem to return the system of inequalities to the initial form and show that the dependent variable inequalities can be displayed in terms of the independent variable approximation. We finish by concluding that there are infinitely many different integer solutions via a proof by contradiction. The proof given below is a generalisation of the proof of Theorem 4 in [22] to the case of approximations with weights.

Proof. Define

$$g_j := f_j - \sum_{i=1}^{d} x_i \frac{\partial f_j}{\partial x_i}, \quad 1 \leq j \leq m,$$

and consider the system of inequalities

$$\left| q g_j(\boldsymbol{x}) + \sum_{i=1}^{d} p_i \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) - p_{d+j} \right| < \frac{Q^{-\tau_j}}{4}, \quad 1 \leq j \leq m, \tag{4.9}$$

$$|q x_i - p_i| < \frac{4^{m/d}}{Q^{(1-\tilde{\tau}m)/d}}, \quad 1 \leq i \leq d, \tag{4.10}$$

$$|q| \leq Q. \tag{4.11}$$

Taking the product of the right hand side of the above inequalities, and taking the determinant of the matrix

$$A = \begin{pmatrix} g_1 & \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_d} & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ g_2 & \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_d} & 0 & \dots & -1 \\ x_1 & -1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_d & 0 & \dots & -1 & 0 & \dots & 0 \\ 1 & 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix},$$

then by Minkowski's Theorem for systems of linear forms, there exists a non-zero integer solution $(\boldsymbol{p}, q) \in \mathbb{Z}^{n+1}$ to the inequalities (4.9)-(4.11). We now show that this system of inequalities implies inequalities (4.4)-(4.5). Firstly, fix some $\boldsymbol{x} \in \mathcal{U}$ and as $\mathcal{U}$ is open there exists a ball $B(\boldsymbol{x}, r)$ for some $r > 0$ which is contained in $\mathcal{U}$. Define

$$\mathcal{Q} := \left\{ Q \in \mathbb{N} : \left(4^{-m} Q^{1-\tilde{\tau}m}\right)^{-1/d} < \min \left\{ 1, r, \left(\frac{1}{2Cd^2} \right)^{1/2} \right\} \right\},$$

where C is defined by (4.7). As $\tilde{\tau}m < 1$ we have that $\left(4^{-m} Q^{1-\tilde{\tau}m}\right)^{-1/d} \to 0$ as $Q \to \infty$, so there exists an integer Q_0 such that for all $Q \geq Q_0$ we have that $Q \in \mathcal{Q}$. We will show that for any $Q \in \mathcal{Q}$ the solution $(p_1, \dots, p_n, q)$ to the system of inequalities (4.9)-(4.11) is a solution to (4.4)-(4.5).

Suppose $q = 0$. By the definition of the set $\mathcal{Q}$ we have that

$$\left(4^{-m} Q^{1-\tilde{\tau}m}\right)^{-1/d} < 1.$$

By the set of inequalities (4.10) we have that $|p_i| < 1$, hence $p_i = 0$ for all $1 \leq i \leq d$. Further, from (4.9) we can see that

$$|p_{d+j}| < \frac{Q^{-\tau_j}}{4} < 1,$$

for $1 \leq j \leq m$. This would conclude that our solution $(p_1, \dots, p_n, q) = \mathbf{0}$ which contradicts Minkowski's Theorem for systems of linear forms, thus $|q| \geq 1$. Without loss of generality we will assume that $q \geq 1$. Dividing (4.10) by q gives us that $\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) \in B(\boldsymbol{x}, r) \subset \mathcal{U}$, and note that (4.4) is satisfied upon dividing (4.10) by q.

Lastly we need to prove that (4.9)-(4.11) implies (4.5). By Taylor's approximation Theorem

$$f_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) = f_j(\boldsymbol{x}) + \sum_{i=1}^{d} \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) \left(\frac{p_i}{q} - x_i\right) + R_j(\boldsymbol{x}, \hat{\boldsymbol{x}}),$$

for some $\hat{\boldsymbol{x}}$ on the line connecting $\boldsymbol{x}$ and $\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right)$, and

$$R_j(\boldsymbol{x}, \hat{\boldsymbol{x}}) = \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f_j}{\partial x_i \partial x_k}(\hat{\boldsymbol{x}}) \left(\frac{p_i}{q} - x_i\right) \left(\frac{p_k}{q} - x_k\right).$$

We may rewrite (4.9) using Taylor's theorem and our definition of g_j as

$$\left|qg_j(\boldsymbol{x}) + \sum_{i=1}^{d} p_i \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) - p_{d+j}\right| = \left|qf_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) - p_{d+j} - qR_j(\boldsymbol{x}, \hat{\boldsymbol{x}})\right|.$$

Using the triangle inequality and the assumption that

$$|qR_j(\boldsymbol{x}, \hat{\boldsymbol{x}})| < \frac{q^{-\tau_j}}{4}, \tag{4.12}$$

we obtain that

$$\left|qf_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) - p_{d+j}\right| < \frac{Q^{-\tau_j}}{4} + \frac{q^{-\tau_j}}{4}.$$

Noting the monotonicity of the approximation function and dividing by q we obtain

$$\left|f_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) - \frac{p_{d+j}}{q}\right| < \frac{q^{-\tau_j-1}}{2},$$

thus (4.5) is satisfied. To complete the first part of the theorem it remains to show that (4.12) is satisfied for all $Q \in \mathcal{Q}$. Using the definition of $R_j(\boldsymbol{x}, \hat{\boldsymbol{x}})$ we have that

$$\begin{aligned}|qR_j(\boldsymbol{x}, \hat{\boldsymbol{x}})| &= \left|\frac{q}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} \frac{\partial^2 f_j}{\partial x_i \partial x_k}(\hat{\boldsymbol{x}}) \left(\frac{p_i}{q} - x_i\right)\left(\frac{p_k}{q} - x_k\right)\right|, \\ &\leq \frac{Cqd^2}{2}\left(\frac{4^{m/d}}{qQ^{(1-\tilde{\tau}m)/d}}\right)^2,\end{aligned}$$

for $1 \leq j \leq m$. Hence we must show that

$$\frac{Cqd^2}{2}\left(\frac{4^{m/d}}{qQ^{(1-\tilde{\tau}m)/d}}\right)^2 < \frac{q^{-\tau_j}}{4},$$

for $1 \leq j \leq m$. Rearranging the equation we obtain the inequality

$$\left(\frac{4^m}{Q^{(1-\tilde{\tau}m)}}\right)^{1/d} < \left(\frac{q^{1-\tau_j}}{2Cd^2}\right)^{1/2}.$$

Considering that $\tilde{\tau}m < 1$ we have that each $\tau_j < 1$, and so $\inf_{q\in\mathbb{N}} q^{1-\tau_j} = 1$ for each $1 \leq j \leq m$. Thus by the definition of the set $\mathcal{Q}$ the above inequality is satisfied by all $Q \in \mathcal{Q}$, so (4.12) is true for all $1 \leq j \leq m$.

We now prove the second part of the theorem, that is that there is infinitely many integer vector solutions. Suppose that there are only finitely many such q and let A be the corresponding set. As $\boldsymbol{x} \in \mathcal{U}\backslash\mathbb{Q}^d$ there exists some $1 \leq j \leq d$ where $x_j \notin \mathbb{Q}$. Fix such j, then there exists some $\delta > 0$ such that

$$\delta \leq \min_{q\in A,\, p_j\in\mathbb{Z}} |qx_j - p_j|.$$

By (4.4) we now have that

$$\delta \leq |qx_j - p_j| \leq \frac{4^{m/d}}{Q^{(1-\tilde{\tau}m)/d}}.$$

However, as $\mathcal{Q}$ is an infinite set and $Q^{(1-\tilde{\tau}m)/d} \to \infty$ as $Q \to \infty$, we have a contradiction so there are infinitely many different q. Lastly, as $q \leq Q$, we can replace Q by q in (4.4) to obtain (4.6) as desired. $\square$

4.3 Proof of Theorem 4.1.8

We are now in a position to prove Theorem 4.1.8. To do so we construct a lim sup set of balls satisfying the conditions of Theorem 4.2.2. The lim sup set will thus have full Lebesgue measure. Next we choose a suitable weight vector $\boldsymbol{a}$ that we use to transform our lim sup set of balls to a lim sup set of hyperrectangles with a known lower bound for its Hausdorff dimension. The proof is completed by showing that the constructed lim sup set is at least contained within our set $\mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M}$, thus our lower bound is a lower bound for $\mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M}$.

Proof. Take the set

$$N(f,\tau) := \Bigg\{ (\boldsymbol{p},q) \in \mathbb{Z}^n \times \mathbb{N} : \left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) \in \mathcal{U},$$
$$\text{and } \left| f_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) - \frac{p_{d+j}}{q} \right| < \frac{q^{-\tau_{d+j}-1}}{2},\ 1 \le j \le m \Bigg\}.$$

In view of Theorem 4.2.2 we have that for almost all $\boldsymbol{x} \in \mathcal{U}$ there are infinitely many different vectors $(\boldsymbol{p},q) \in N(f,\tau)$ satisfying

$$\left| x_i - \frac{p_i}{q} \right| < 4^{m/d} q^{-1-(1-\tilde{\tau}m)/d}, \quad 1 \le i \le d,$$

where $\tilde{\tau} = \frac{1}{m}\sum_{i=1}^{m} \tau_{d+i}$. By Lemma 4.2.1, we can choose a constant $k > 0$ such that for almost every $\boldsymbol{x} \in \mathcal{U}$ there are infinitely many different vectors $(\boldsymbol{p},q) \in N(f,\tau)$ satisfying

$$\left| x_i - \frac{p_i}{q} \right| < k q^{-1-(1-\tilde{\tau}m)/d}, \quad 1 \le i \le d.$$

Take the ball

$$B_{(\boldsymbol{p},q)} := \left\{ \boldsymbol{x} \in \mathcal{U} : \left| x_i - \frac{p_i}{q} \right| < k q^{-1-(1-\tilde{\tau}m)/d} \quad \text{for} \quad 1 \le i \le d \right\}.$$

By Theorem 4.2.2 and Lemma 4.2.1 we have that

$$\mu_d \left(\limsup_{(\boldsymbol{p},q) \in N(f,\tau)} B_{(\boldsymbol{p},q)} \right) = 1,$$

where μ_d is the d dimensional Lebesgue measure. Let $\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{R}^d_+$ be a weight vector with each

$$a_i = \frac{d(1+\tau_i)}{d+1-\tilde{\tau}m}, \quad 1 \le i \le d. \tag{4.13}$$

Note that by the condition that $\tau_i \ge \frac{1-\tilde{\tau}m}{d}$ for all $1 \le i \le d$, we have that each $a_i \ge 1$. $B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}$ is the hyperrectangle with the following properties:

$$B^{\boldsymbol{a}}_{(\boldsymbol{p},q)} = \left\{ \boldsymbol{x} \in \mathcal{U} : \left| x_i - \frac{p_i}{q} \right| < k^{a_i} q^{-1-\tau_i},\ 1 \le i \le d \right\}.$$

By Theorem 3.2.1 we have that

$$\dim \left(\limsup_{(\boldsymbol{p},q) \in N(f,\tau)} B^{\boldsymbol{a}}_{(\boldsymbol{p},q)} \right) \ge \min_{1 \le j \le d} \left\{ \frac{d + \sum_{i=j}^{d} (a_j - a_i)}{a_j} \right\}.$$

Replacing each a_i with (4.13) we have that

$$\dim\left(\limsup_{(\boldsymbol{p},q)\in N(f,\tau)} B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}\right) \geq \min_{1\leq j\leq d}\left\{\frac{d+\sum_{i=j}^{d}\left(\frac{d(1+\tau_j)}{d+1-\tilde{\tau}m}-\frac{d(1+\tau_i)}{d+1-\tilde{\tau}m}\right)}{\frac{d(1+\tau_j)}{d+1-\tilde{\tau}m}}\right\},$$

$$= \min_{1\leq j\leq d}\left\{\frac{d(d+1-\tilde{\tau}m)+\sum_{i=j}^{d}(d(1+\tau_j)-d(1+\tau_i))}{d(1+\tau_j)}\right\},$$

$$= \min_{1\leq j\leq d}\left\{\frac{d+1-\tilde{\tau}m+\sum_{i=j}^{d}(\tau_j-\tau_i)}{(1+\tau_j)}\right\}.$$

Using the definition of $\tilde{\tau}$ and that $d=n-m$ we may rewrite this as

$$\dim\left(\limsup_{(\boldsymbol{p},q)\in N(f,\tau)} B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}\right) \geq \min_{1\leq j\leq d}\left\{\frac{n-m+1-\sum_{i=d+1}^{n}\tau_i+\sum_{i=j}^{d}(\tau_j-\tau_i)}{1+\tau_j}\right\},$$

$$= \min_{1\leq j\leq d}\left\{\frac{n-m+1+\sum_{i=d+1}^{n}(\tau_j-\tau_i)-m\tau_j+\sum_{i=j}^{d}(\tau_j-\tau_i)}{1+\tau_j}\right\},$$

$$= \min_{1\leq j\leq d}\left\{\frac{n+1-m(1+\tau_j)+\sum_{i=j}^{n}(\tau_j-\tau_i)}{1+\tau_j}\right\},$$

$$= \min_{1\leq j\leq d}\left\{\frac{n+1+\sum_{i=j}^{n}(\tau_j-\tau_i)}{1+\tau_j}-m\right\},$$

as required. We now finish by showing that

$$\dim(\mathcal{W}_n(\boldsymbol{\tau})\cap\mathcal{M}) \geq \dim\left(\limsup_{(\boldsymbol{p},q)\in N(f,\tau)} B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}\right).$$

Note that any $\mathbf{y}\in\mathcal{W}_n(\boldsymbol{\tau})\cap\mathcal{M}$ must have infinitely many solutions $(\boldsymbol{p},q)\in\mathbb{Z}^n\times\mathbb{N}$ to the following system of inequalities

$$|qx_i-p_i|<q^{-\tau_i},\ 1\leq i\leq d, \tag{4.14}$$

$$|qf(\boldsymbol{x})-p_{d+j}|<q^{-\tau_{d+j}},\ 1\leq j\leq m, \tag{4.15}$$

where $\mathbf{y}=(\boldsymbol{x},\mathbf{f}(\boldsymbol{x}))$ for some $\boldsymbol{x}=(x_1,\dots,x_d)\in\mathcal{U}$. Let the set of $\boldsymbol{x}$ satisfying (4.14)-(4.15) be denoted by $\pi_d(\mathcal{W}_n(\boldsymbol{\tau})\cap\mathcal{M})$. This set is the orthogonal projection of $\mathcal{W}_n(\boldsymbol{\tau})\cap\mathcal{M}$ onto $\mathbb{R}^d$. A result of fractal geometry states that a bi-Lipschitz mapping of a set has the same Hausdorff dimension of the original set (see Proposition 3.3 of [59]). As the projection π_d is bi-Lipschitz it is sufficient to prove that

$$\dim\pi_d(\mathcal{W}_n(\boldsymbol{\tau})\cap\mathcal{M}) \geq \min_{1\leq j\leq d}\left\{\frac{n+1+\sum_{i=j}^{n}(\tau_j-\tau_i)}{1+\tau_j}-m\right\}.$$

Let $\boldsymbol{x}\in B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}$ for some $(\boldsymbol{p},q)\in N(f,\tau)$. On using the triangle inequality, the mean-value theorem, and

(4.8) we have that for any $1 \leq j \leq m$,

$$\begin{aligned}
\left|f_j(\boldsymbol{x}) - \frac{p_{d+j}}{q}\right| &\leq \left|f_j(\boldsymbol{x}) - f_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right)\right| + \left|f_j\left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) - \frac{p_{d+j}}{q}\right|, \\
&< \left|\left(\frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_d}{\partial x_d}\right) \cdot \left(\boldsymbol{x} - \left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right)\right)\right| + \frac{q^{-1-\tau_{d+j}}}{2}, \\
&< D\sum_{i=1}^{d}\left|x_i - \frac{p_i}{q}\right| + \frac{q^{-1-\tau_{d+j}}}{2}, \\
&< Dd\max_{1\leq i\leq d}\left|x_i - \frac{p_i}{q}\right| + \frac{q^{-1-\tau_{d+j}}}{2}, \\
&< Ddk^{a_d}q^{-1-\tau_d} + \frac{q^{-1-\tau_{d+j}}}{2}.
\end{aligned}$$

We can choose k sufficiently small, and note that $\tau_d \geq \max_{1\leq j\leq m} \tau_{d+j}$, so that we have

$$\left|f_j(\boldsymbol{x}) - \frac{p_{d+j}}{q}\right| < q^{-1-\tau_j}, \quad 1 \leq j \leq m.$$

We have that

$$B^{\boldsymbol{a}}_{(\boldsymbol{p},q)} \subseteq \left\{\boldsymbol{x} \in \mathcal{U} : \begin{array}{c} \left|x_i - \frac{p_i}{q}\right| < q^{-1-\tau_i}, \quad 1 \leq i \leq d, \\ \text{for i.m } (\boldsymbol{p}, q) \in N(f, \tau) \subset \mathbb{Z}^n \times \mathbb{N} \end{array}\right\}.$$

Hence for any $\boldsymbol{x} \in \limsup_{(\boldsymbol{p},q)\in N(f,\tau)} B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}$, (4.14)-(4.15) are satisfied for infinitely many $(\boldsymbol{p}, q) \in \mathbb{Z}^n \times \mathbb{N}$, thus

$$\dim \pi_d(\mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M}) \geq \dim\left(\limsup_{(\boldsymbol{p},q)\in N(f,\tau)} B^{\boldsymbol{a}}_{(\boldsymbol{p},q)}\right),$$

as required. □

4.4 Concluding remarks on Theorem 4.1.8

Using the arguments above and, principally applying the MTP of [30], we have established a lower bound for $\mathcal{W}_n(\boldsymbol{\tau}) \cap \mathcal{M}$ which coincides with that of $\mathcal{W}_n(\psi) \cap \mathcal{M}$ from Theorem 4.1.3. The natural question is can equality be determined. That is, can an upper bound be found which agrees with our calculated lower bound? Thus achieving a complete analogue of Theorem 4.1.8. In trying to attain an upper bound, it is likely necessary to find an estimate for the number of rational points within a $\boldsymbol{\tau}$-neighbourhood of the manifold. There are a variety of results on the cardinality of rational points within a simultaneous ψ-neighbourhood of curves, manifolds, hypersurfaces, and affine subspaces (see [72], [26], [69], [70], [34] respectively). Unfortunately, no such results have been found for the number of rational points within a weighted Ψ-neighbourhood of such subsets. It may be possible to adapt the proofs of the simultaneous results to give us weighted version of such results, but this is yet to be proven. We suspect such result would lead to a suitable upper bound corresponding to Theorem 4.1.8.

We remark that since the our proof of Theorem 4.1.8 there has been developments in Mass Transference style theorems. In particular, Wang and Wu [111] have developed a Mass Transference Principle from rectangles to rectangles(MTPRR), see §4.2.1 for more details. The MTPRR requires a ubiquity hypothesis which is more restrictive than that required by the MTP. Unfortunately Theorem 4.2.2 is not sufficient to prove the ubiquity hypothesis. While using the MTPRR would likely improve the bound on $\boldsymbol{\tau}$, it will require additional conditions on the manifolds. In this chapter we only require the manifold be twice continuously differentiable, whereas with the ubiquity hypothesis we would expect to need some non-degeneracy condition. We intend to pursue this in a further paper.

Lastly, note that Corollary 4.1.11 whilst being relatively general does not cover all approximation functions. We provide no results for functions with infinite upper order (see (4.3)), and also provide imprecise lower dimension results for approximation functions with different upper and lower orders. For example, an approximation function defined by a step function bounded between two functions $q^{-\tau_1}$ and $q^{-\tau_2}$ would have different upper and lower bounds. It would be of interest to extend the class of approximating functions somehow.

Chapter 5

Simultaneous p-adic Approximation over manifolds

In this chapter we prove the new results stated in Chapter 2. A key result used here is a new zero-one law which is also proven here. We also provided a brief survey on the state of the art of p-adic approximation over manifolds. Within this section we give our new results on the Hausdorff dimension of various manifolds. These results are proven in the latter part of this chapter. The contents of this chapter is essential that of [24] in joint work with Beresnevich and Levesley.

5.1 Weighted simultaneous p-adic approximation

We begin by restating the Theorems of Chapter 2 that we will prove here. Let $\Psi = (\psi_1, \ldots, \psi_n)$ be an n-tuple of approximation functions as in previous chapters, and for each $a_0 \in \mathbb{N}$ let

$$\mathfrak{A}_{a_0}(\Psi) = \bigcup_{\substack{(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ |a_i|\le a_0\ (1\le i\le n)}} \left\{ \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n : \left| x_i - \frac{a_i}{a_0} \right|_p < \psi_i(a_0) \text{ for all } 1 \le i \le n \right\}. \tag{5.1}$$

Define the set of p-adic simultaneously Ψ-approximable points in $\mathbb{Z}_p$ as

$$\mathfrak{W}_n(\Psi) = \limsup_{a_0\to\infty} \mathfrak{A}_{a_0}(\Psi).$$

Similarly let

$$\mathfrak{A}'_{a_0}(\Psi) = \bigcup_{\substack{(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ |a_i|\le a_0\ \&\ (a_i,a_0)=1\ (1\le i\le n)}} \left\{ \boldsymbol{x} \in \mathbb{Z}_p^n : \left| x_i - \frac{a_i}{a_0} \right|_p < \psi_i(a_0) \text{ for all } 1 \le i \le n \right\} \tag{5.2}$$

and define the corresponding limsup set as

$$\mathfrak{W}'_n(\Psi) = \limsup_{a_0\to\infty} \mathfrak{A}'_{a_0}(\Psi).$$

For convenience we restate the theorems of chapter 2 that we will prove in this chapter:

Theorem 5.1.1. *Let $\psi_i : \mathbb{N} \to [0,1)$ be approximation functions with $\psi_i(q) \ll \frac{1}{q}$ for each $1 \leq i \leq n$ and let $\Psi = (\psi_1, \dots, \psi_n)$. Suppose that $\prod_{i=1}^{n} \psi_i$ is monotonically decreasing. Then*

$$\mu_{p,n}(\mathfrak{W}'_n(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^n \prod_{i=1}^{n} \psi_i(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^n \prod_{i=1}^{n} \psi_i(q) = \infty. \end{cases}$$

Theorem 5.1.2. *Let $\psi_i : \mathbb{N} \to [0,1)$ be approximation functions with $\psi_i(q) \ll \frac{1}{q}$ for $1 \leq i \leq n$ and let $\Psi = (\psi_1, \dots, \psi_n)$. For φ the Euler phi function suppose that*

$$\limsup_{N\to\infty} \frac{\sum_{q=1}^{N} \varphi(q)^n \prod_{i=1}^{n} \psi_i(q)}{\sum_{q=1}^{N} q^n \prod_{i=1}^{n} \psi_i(q)} > 0. \tag{5.3}$$

Then

$$\mu_{p,n}(\mathfrak{W}'_n(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \varphi(q)^n \prod_{i=1}^{n} \psi_i(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \varphi(q)^n \prod_{i=1}^{n} \psi_i(q) = \infty. \end{cases}$$

Remark 5.1.3. Note that the condition that each $\psi_i(q) \ll \frac{1}{q}$ is a necessary condition, since the p-adic distance between any two rational integers can be made arbitrarily small. This is in stark contrast to the real case where $\psi(q) < \frac{1}{2}$ is sufficient to ensure rectangles in the same 'layer' are non-intersecting. We remark that the monotonicity condition of Theorem 5.1.1 is only required in the divergence case. This condition is replaced in the Duffin-Schaeffer type theorem [56] of Theorem 5.1.2.

These results, alongside a zero-one law on $\mathfrak{W}'_n(\Psi)$, will be proven in §5.3.1-5.4. In the case of Hausdorff dimension we prove the following Theorem

Theorem 5.1.4. *Let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n_+$ be such that $\sum_{i=1}^{n} \tau_i > n+1$ and $\tau_i > 1$ for each $1 \leq i \leq n$. Then*

$$\dim \mathfrak{W}_n(\boldsymbol{\tau}) = \min_{1 \leq i \leq n} \left\{ \frac{n + 1 + \sum_{\tau_j < \tau_i} (\tau_i - \tau_j)}{\tau_i} \right\}.$$

Remark 5.1.5. We note that the condition on the summation of the exponent vector $\boldsymbol{\tau}$ is present due to the fact that if $\sum_{i=1}^{n} \tau_i \leq n+1$, then, by the p-adic version of Dirichlet's Theorem, we have that $\mathfrak{W}_n(\boldsymbol{\tau}) = \mathbb{Z}_p^n$.

Remark 5.1.6. The condition that each $\tau_i > 1$ may seem unnecessarily restrictive, however, the following reasoning shows why this must be the case. The key reasoning behind the condition is that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ so in any coordinate axis where $\tau_i < 1$ all points along the axis can be approximated, regardless of the choice of a_0 in our approximation sets. If, for example, we considered the approximation set $\mathfrak{W}_2((1-\varepsilon, \tau_2))$ for $\varepsilon > 0$ and $\tau_2 > 2$ then the above argument gives us that $\mathfrak{W}_2((1-\varepsilon, \tau_2)) = \mathbb{Z}_p \times \mathfrak{W}_1(\tau_2)$.

Using well known bounds on the Hausdorff dimension of product spaces (see e.g [109]) we have that

$$\dim \mathfrak{W}_1(\tau_2) + \dim \mathbb{Z}_p \leq \dim \mathfrak{W}_2((1-\varepsilon, \tau_2)) \leq \dim \mathfrak{W}_1(\tau_2) + \dim_B \mathbb{Z}_p,$$

where $\dim_B$ is the box-counting dimension (see [59] for the definition of box-counting dimension and it's relation to the Hausdorff dimension), we have that

$$\dim \mathfrak{W}_2((1-\varepsilon, \tau_2)) = \frac{2}{\tau_2} + 1.$$

However, if Theorem 5.1.4 was applicable we would have that

$$\dim \mathfrak{W}_2((1-\varepsilon, \tau_2)) = \min \left\{ \frac{3 + (\tau_2 - (1-\varepsilon))}{\tau_2}, \frac{3}{1-\varepsilon} \right\} = \frac{2}{\tau_2} + \frac{\tau_2 + \varepsilon}{\tau_2}.$$

We will prove Theorem 5.1.4 in §5.5. An overview of the proof is as follows: Using a standard method in ubiquitous systems we show that the lim sup set of rectangles used to construct $\mathfrak{W}_n(\boldsymbol{\tau})$ is a ubiquitous system of rectangles. Applying the MTP for rectangles to rectangles developed in [111](see Chapter 3 for more details) we obtain the lower bound dimension result. The corresponding upper bound result uses the standard cover of $\mathfrak{W}_n(\boldsymbol{\tau})$ and a similar geometrical argument to that in the real case.

We can further extend this result to general approximation functions. Suppose that the limits

$$v_i = \lim_{q \to \infty} \frac{-\log \psi_i(q)}{\log q}, \tag{5.4}$$

exist and are positive for each $1 \leq i \leq n$. Define the exponents vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n_+$.

Corollary 5.1.7. *Let Ψ be such that the limits* (5.4) *exist and are positive. Suppose that $\sum_{i=1}^n v_i > n+1$ and each $v_i > 1$. Then*

$$\dim \mathfrak{W}_n(\Psi) = \min_{1 \leq i \leq n} \left\{ \frac{n + 1 + \sum_{v_j < v_i}(v_i - v_j)}{v_i} \right\}.$$

Proof. By the condition that each function ψ_i has corresponding positive limit (5.4), for any $\epsilon > 0$ we have that

$$q^{-(v_i + \epsilon)} \leq \psi_i(q) \leq q^{-(v_i - \epsilon)} \quad (1 \leq i \leq n)$$

for all sufficiently large $q \in \mathbb{N}$. Let $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon) \in \mathbb{R}^n_+$. Then, we have that

$$\mathfrak{W}_n(\mathbf{v} + \boldsymbol{\epsilon}) \subseteq \mathfrak{W}_n(\Psi) \subseteq \mathfrak{W}_n(\mathbf{v} - \boldsymbol{\epsilon}).$$

By letting $\epsilon \to 0$, and applying Theorem 5.1.4 we get the required result. □

5.2 p-adic approximation on manifolds

When it comes to p-adic approximation on curves and manifolds, less is known. In [79], Kleinbock and Tomanov, generalized the key results from [78] to the S-arithmetic setting (see chapter 7 for more details on S-arithmetic space), which includes the p-adic setting. In particular, Kleinbock and Tomanov proved that under the natural assumption $\sum_{i=1}^{n} \tau_i > n+1$ the set $\mathfrak{W}_n(\boldsymbol{\tau}) \cap C$ has zero measure on C for a large and natural class of manifolds in $\mathbb{Q}_p^n$. Whilst there are no results relating to the Haar measure of $\mathfrak{W}_n(\Psi) \cap C$ for C a p-adic curve or manifold in the case Ψ is a general n-tuple of approximation functions, there are several results for dual approximation including inhomogeneous setting, see [17, 21, 43, 44, 45, 52, 53, 91]. Regarding the Hausdorff dimension of $\mathfrak{W}_n(\boldsymbol{\tau}) \cap C$, Bugeaud, Budarina, Dickinson, and O'Donnell [46] and more lately Badziahin, Bugeaud and Schleischitz [11] calculated $\dim(\mathfrak{W}_n(\tau) \cap C)$ in the case $C = (x, \dots, x^n)$ for large exponents τ. Apart from these pair of findings nothing else seems to be known. In this paper we obtain a sharp lower bound on the dimension of $\mathfrak{W}_n(\boldsymbol{\tau}) \cap C$ for a natural class of manifolds defined over $\mathbb{Z}_p^d$ and relatively small exponent vector $\boldsymbol{\tau}$. Specifically we will consider manifolds immersed by maps with the following property, which is a multivariable analogue of C^1 functions given in for example [102].

Definition 5.2.1. *A function $f : \mathcal{U} \to \mathbb{Q}_p$ defined on an open set $\mathcal{U} \subset \mathbb{Q}_p^d$ will be referred to as* differentiable with quadratic error *(DQE) at $\boldsymbol{x} \in \mathcal{U}$ if there exists constants $C > 0$ and $\varepsilon > 0$ and p-adic numbers $\partial f(\boldsymbol{x})/\partial x_\ell \in \mathbb{Q}_p$ $(1 \leq \ell \leq d)$, which will be referred to as partial derivatives of f at $\boldsymbol{x}$, such that for any $\boldsymbol{y} \in B(\boldsymbol{x}, \varepsilon) \subset \mathcal{U}$*

$$\left| f(\boldsymbol{y}) - f(\boldsymbol{x}) - \sum_{i=1}^{d} \frac{\partial f(\boldsymbol{x})}{\partial x_i}(y_i - x_i) \right|_p < C \max_{1 \leq i \leq d} |y_i - x_i|_p^2 . \tag{5.5}$$

We will say that a map $\boldsymbol{f} = (f_1, \dots, f_m) : \mathcal{U} \to \mathbb{Z}_p^m$ is DQE at $\boldsymbol{x}$ with if so is each coordinate function f_j. We will say that f (resp. $\boldsymbol{f}$) is DQE on $\mathcal{U}$ if it is DQE at each point $\boldsymbol{x} \in \mathcal{U}$.

Remark 5.2.2. Note that if the right hand side of (5.5) was simply $o\left(\max_{1 \leq j \leq d} |y_j - x_j|_p\right)$ then f would be simply differentiable at $\boldsymbol{x}$. The above definition imposes a stronger condition than differentiability in the sense that the error term in (5.5) is quadratic. It is readily verified that any C^2 function, as defined in [98] (see also [79] for a brief survey of p-adic C^k functions), is DQE at every point. The converse may not be true. Mahler's normal functions are C^∞ and so they are DQE.

We are now in position to state our results for $\boldsymbol{\tau}$-approximable points on $\mathcal{C}_f$, extending Theorem 4.1.8 to the p-adic setting. This can be done in two ways: by stating our results for exactly the set $\mathfrak{W}_n(\boldsymbol{\tau}) \cap \mathcal{C}_f$, or by stating them for the set of $\boldsymbol{x} \in \mathcal{U}$ such that $(\boldsymbol{x}, \mathbf{f}(\boldsymbol{x})) \in \mathfrak{W}_n(\boldsymbol{\tau})$, we opt for the latter since it requires fewer assumptions, albeit the two ways are equivalent if we assume that $\mathbf{f}$ is a Lipschitz map, which follows from Proposition 1.1.7. Thus, our statements will be about the Hausdorff measure and dimension of

$$\mathbf{F}^{-1}\left(\mathfrak{W}_n(\boldsymbol{\tau})\right) := \left\{\boldsymbol{x} \in \mathcal{U} : (\boldsymbol{x}, \mathbf{f}(\boldsymbol{x})) \in \mathfrak{W}_n(\boldsymbol{\tau})\right\}.$$

It is easily seen that this set is subset of the projection of $\mathfrak{W}_n(\boldsymbol{\tau})$ onto the first d coordinates.

Theorem 5.2.3. *Let $\mathcal{C}_f := \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) : \boldsymbol{x} \in \mathcal{U}\} \subset \mathbb{Z}_p^n$, where $\boldsymbol{f} : \mathcal{U} \to \mathbb{Z}_p^m$ is DQE at almost all point of an open set $\mathcal{U} \subseteq \mathbb{Z}_p^d$. Suppose*

$$1 + \frac{1}{n} < \tau < 1 + \frac{1}{m},$$

Then

$$\dim(\mathbf{F}^{-1}\left(\mathfrak{W}_n(\tau)\right) \geq s := \frac{n+1}{\tau} - m\,. \tag{5.6}$$

Furthermore, if $\boldsymbol{f}$ is Lipschitz on $\mathcal{U}$ then

$$\mathcal{H}^s\left(\mathbf{F}^{-1}\left(\mathfrak{W}_n(\tau)\right)\right) = \infty. \tag{5.7}$$

Theorem 5.2.4. *Let $\mathcal{C}_f := \{(x, f_1(x), \ldots, f_{n-1}(x)) : x \in \mathcal{U}\}$ be a curve, where for $i = 1, \ldots, n-1$ the function $f_i : \mathcal{U} \to \mathbb{Z}_p$ is DQE at almost every point of an open subset $\mathcal{U} \subset \mathbb{Z}_p$. Suppose that $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}_+^n$ satisfies the conditions*

$$\tilde{\tau} := \sum_{j=2}^{n} \tau_j < n, \qquad \tau_1 \geq \max_{2 \leq i \leq n}\{\tau_i, n+1-\tilde{\tau}\} \quad \textit{and} \quad \tau_i > 1 \;\; \textit{for} \;\; 2 \leq i \leq n.$$

Then

$$\dim \mathbf{F}^{-1}\left(\mathfrak{W}_n(\boldsymbol{\tau})\right) \geq s := \frac{n+1+\sum_{j=2}^{n}(\tau_1 - \tau_j)}{\tau_1} - (n-1) = \frac{n+1-\tilde{\tau}}{\tau_1}. \tag{5.8}$$

Furthermore, if $\boldsymbol{f}$ is Lipschitz on $\mathcal{U}$ or $\tau_1 > \max_{2 \leq i \leq n} \tau_i$ then

$$\mathcal{H}^s\left(\mathbf{F}^{-1}\left(\mathfrak{W}_n(\boldsymbol{\tau})\right)\right) = \infty. \tag{5.9}$$

Theorem 5.2.5. *Let $\mathcal{C}_f$ be satisfy the conditions of Theorem 5.2.3 and suppose that $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}_+^n$ satisfies the conditions*

$$\tau_i > 1,\ (1 \leq i \leq n), \qquad \sum_{i=1}^{m} \tau_{d+i} < m+1, \qquad \sum_{i=1}^{n} \tau_i > n+1, \quad \textit{and} \; \min_{1 \leq i \leq d} \tau_i \geq \max_{1 \leq j \leq m} \tau_{d+j}.$$

Then

$$\dim\left(\mathbf{F}^{-1}\left(\mathfrak{W}_n(\boldsymbol{\tau})\right)\right) \geq \min_{1 \leq i \leq d}\left\{\frac{n+1+\sum_{\tau_j < \tau_i}(\tau_i - \tau_j)}{\tau_i} - m\right\}. \tag{5.10}$$

Remark 5.2.6. Note that the dimension results of Theorem 5.2.3–5.2.4 are contained within Theorem 5.2.5. However, due to the method of proof we are not able to obtain the Hausdorff measure result in Theorem 5.2.5. Also note that the statements remain true if the assumptions imposed on $\mathbf{f}$ are imposed on a sufficiently small ball $B \subset \mathcal{U}$ instead of $\mathcal{U}$.

Remark 5.2.7. The assumption that the approximations over the independent variables (τ_1 in Theorem 5.2.4 and $\tau_1, \ldots, \tau_d$ in Theorem 5.2.5) are larger than the approximations over the dependent variables is merely technical. Observe that this conditions is not needed amongst the approximations over each respective variable, since we may permute the variables to obtain the desired ordering. However, the other requirements placed on $\boldsymbol{\tau}$ are necessary to allow the result to hold for as general set of manifolds as possible. In particular, the conditions that $\sum_{i=1}^{m} \tau_{d+i} < m+1$ and $\tau_i > 1$ for $1 \leq i \leq n$ ensure that even if the manifold is a hyperplane passing through badly approximable points we will still have an infinite number of rational approximations. If these conditions do not hold a counterexample can be readily obtained on modifying the example of Remark 3 in [22]. It is also easy to see that the lower bound $\tau_1 \geq n+1-\tilde{\tau}$ is necessary for otherwise (5.8) would be false. The upper bound $\tilde{\tau} < n$ on $\tilde{\tau}$ can likely be improved, however this will require imposing additional conditions on the curves such as non-degeneracy (meaning $1, x, f_1(x), \ldots, f_{n-1}(x)$ are linearly independent over $\mathbb{Z}_p$), and will require a different approach such as that of [14]. We plan to address the problem for non-degenerate curves separately in a subsequent publication.

Remark 5.2.8. We expect that the lower bound of Theorem 5.2.3-5.2.5 is sharp and each dimension result should indeed be equality at least for non-degenerate curves. Obtaining the upper bounds represents a challenging open problem even in dimension 2. We would like to stress that there is currently no equivalent to Huxley's estimate [72] in the p-adic setting, let alone the sharper Vaughan-Velani result [110]. The absence of such estimates is the only obstacle when trying to establish the complementary upper bounds.

The remainder of this Chapter is devoted to proofs of the Theorems stated above. Firstly we provide proofs of the Haar measure statement results, followed by a proof of the Hausdorff dimension result in the classical case, and finished by proofs of the statements on approximation over p-adic manifolds. Prior to these we provide a preliminary section on know results that will be used throughout the rest of the chapter. Since many of the results require a lower bound Hausdorff dimension result we will be using a variety of Mass Transference Principles all of which can be found in Chapter 3.

5.3 Auxiliary concepts and results

Before giving the proofs of Theorems 5.1.1, 5.1.4 and 5.2.3-5.2.5, we collect together some auxiliary results and concepts which we will need. We prove the following lemma, which can be considered as the p-adic equivalent of Minkowski's Theorem for systems of linear forms.

Lemma 5.3.1. *Let $L_i(\boldsymbol{x})$, with $i = 1, \ldots, n$, be linear forms in $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ with p-adic integer coefficients. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_+$ satisfy $\sum_{i=1}^{n} \tau_i = n+1$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ satisfy $\sum_{i=1}^{n} \sigma_i = n$. Then there exists $H_{\boldsymbol{\sigma}} > 0$ such that for all integers $H_0, \ldots, H_n \geq 1$ such that $T^{n+1} :=$*

$(H_0+1)\cdots(H_n+1) \geq H_{\boldsymbol{\sigma}}$ *there exists a non-zero rational integer vector* $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ *satisfying*

$$|x_i| \leq H_i \qquad \text{for all } 0 \leq i \leq n \tag{5.11}$$

and

$$|L_i(\boldsymbol{x})|_p \leq p^{\sigma_i} T^{-\tau_i} \quad \text{for all } i = 1, \dots, n\,. \tag{5.12}$$

As we can see, by Lemma 5.3.1, if all τ_i are equal and H_i are equal then we have $\tau_i = 1 + 1/n$, which agrees with the p-adic n-dimensional Dirichlet theorem.

Proof. This is a standard proof using Dirichlet's pigeon-hole principle, which is given here for completeness. To begin with, note that there are T^{n+1} different rational integer vectors $\boldsymbol{x} = (x_0, \dots, x_n)$ satisfying (5.11), subject to the condition that $x_i \geq 0$ for each i. Let $\varepsilon \in (0,1)$ and $T_\varepsilon = T - \varepsilon$. For each $i = 1, \dots, n$ let δ_i be the unique integer such that

$$p^{\delta_i - 1} \leq p^{-\sigma_i} T_\varepsilon^{\tau_i} < p^{\delta_i}\,. \tag{5.13}$$

Assuming $H_{\boldsymbol{\sigma}}$, which can be found explicitly, is sufficiently large we ensure that $\delta_i \geq 0$ for each i. Clearly, for each $\boldsymbol{x} \in \mathbb{Z}^n$ we have that $L(\boldsymbol{x}) := (L_1(\boldsymbol{x}), \dots, L_n(\boldsymbol{x})) \in \mathbb{Z}_p^n$. Split $\mathbb{Z}_p^n$ into the subsets $S(\boldsymbol{a})$ given by

$$S(\boldsymbol{a}) = \prod_{i=1}^{n} \{x_i \in \mathbb{Z}_p : |x_i - a_i|_p \leq p^{-\delta_i}\}$$

for each $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $0 \leq a_i < p^{\delta_i}$. It is readily seen that the sets $S(\boldsymbol{a})$ are disjoint and cover the whole of $\mathbb{Z}_p^n$. Furthermore, using the facts that $\sum_i \tau_i = n+1$, $\sum_i \sigma_i = n$ and (5.13), we find that the number of sets $S(\boldsymbol{a})$ is

$$p^{\sum_i \delta_i} \leq T_\varepsilon^{\sum_i \tau_i} = T_\varepsilon^{n+1} < T^{n+1}\,.$$

Hence, by the pigeon-hole principle, at least one of the sets $S(\boldsymbol{a})$ contains $L(\boldsymbol{x}_i)$ for at least two distinct integer points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as specified above. Let $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. Clearly, (5.11) is satisfied and $\boldsymbol{x}$ is non-zero. Furthermore, for each $i = 1, \dots, n$ we have that

$$|L_i(\boldsymbol{x})| = |L_i(\boldsymbol{x}_1 - \boldsymbol{x}_2)|_p = |L_i(\boldsymbol{x}_1) - L_i(\boldsymbol{x}_2)|_p \leq p^{-\delta_i} \overset{(5.13)}{<} p^{\sigma_i} T_\varepsilon^{-\tau_i}\,. \tag{5.14}$$

Since there are only finitely many integer vectors $\boldsymbol{x} = (x_0, \dots, x_n)$ satisfying (5.11), there is a non-zero $\boldsymbol{x}$ subject to (5.11) satisfying (5.14) for every $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0$ verifies (5.12) and completes the proof. $\square$

We also require a variety of statements given in Chapters 1-3. Chiefly this includes the Borel-Cantelli lemmas (Lemma 1.1.3–1.1.4), certain results in Hausdorff Theory (Proposition 1.1.7, Lemma 2.2.9), and various Mass Transference Principle results (Theorem 3.1.1, Theorem 3.3.3–3.5).

5.3.1 A zero-one law on $\mathfrak{W}'_n(\Psi)$

In what follows we will need a statement showing that, given a sequence of balls, if the radii of the balls are multiplied by some constant, then the Haar measure of the corresponding lim sup set remains unchanged. We establish this lemma in greater generality for arbitrary ultrametric spaces where such a statement may be useful when solving problems of the same ilk, for example, in Diophantine approximation over locally compact fields of positive characteristic.

Lemma 5.3.2. *Let (X,d) be a separable ultrametric space and μ be a Borel regular measure on X. Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of balls in X with radii $r_i \to 0$ as $i\to\infty$. Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of μ-measurable sets such that $U_i \subset B_i$ for all i. Assume that for some $c>0$*

$$\mu(U_i) \geq c\mu(B_i) \qquad \textit{for all } i\,. \tag{5.15}$$

Then the limsup sets

$$\mathcal{U} = \limsup_{i\to\infty} U_i := \bigcap_{j=1}^{\infty} \bigcup_{i\geq j} U_i \qquad \textit{and} \qquad \mathcal{B} = \limsup_{i\to\infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i\geq j} B_i$$

have the same μ-measure.

The $\mathbb{R}^n$ version of this statement is well known and can be found for example in [32, Lemma 1], which proof is originally due to Cassels and uses Lebesgue's density theorem. Below we give a full proof of Lemma 5.3.2 for completeness. Our proof is built on the ideas of [32, Lemma 1] and [105, Lemma 1 in Part II, Ch. 1].

Proof. Let $\mathcal{U}_j := \bigcup_{i\geq j} U_i$ and $\mathcal{D}_j := \mathcal{B}\setminus\mathcal{U}_j$. Then, $\mathcal{D} := \mathcal{B}\setminus\mathcal{U} = \bigcup_j \mathcal{D}_j$ and we need to prove that $\mathcal{D}$ has μ-measure zero. Assume the contrary. Then, since every set $\mathcal{D}_j$ is μ-measurable and $\mathcal{D}_j \subseteq \mathcal{D}_{j+1}$ for all j, by the continuity of μ, there is an $\ell\in\mathbb{N}$ such that $\mu(D_\ell) > 0$. Since μ is Borel regular $\mu(\mathcal{D}_\ell) = \inf\{\mu(A) : \mathcal{D}_\ell \subset A,\ A \text{ is open}\}$. Since X is separable and ultrametric, every open set A can be written as a disjoint countable union of balls. Hence, for any $\varepsilon>0$ there exists a countable collection of disjoint balls (A_i) such that

$$\mathcal{D}_\ell \subset \bigcup_i A_i \quad \text{and} \quad \sum_i \mu(A_i) - \varepsilon \leq \mu(\mathcal{D}_\ell) \leq \sum_i \mu(A_i)\,. \tag{5.16}$$

Let

$$\lambda := \sup\left\{\frac{\mu(A_i\cap\mathcal{D}_\ell)}{\mu(A_i)} : i\in\mathbb{N},\ \mu(A_i)>0\right\}.$$

Note that, since $\mu(\mathcal{D}_\ell)>0$, the above set is non-empty and therefore $\lambda\in[0,1]$. Then, by (5.16), we have that

$$\mu(\mathcal{D}_\ell) = \sum_i \mu(A_i\cap\mathcal{D}_\ell) \leq \lambda\sum_i \mu(A_i) \leq \lambda(\mu(\mathcal{D}_\ell)+\varepsilon)\,.$$

Therefore,

$$\lambda \geq \frac{\mu(\mathcal{D}_\ell)}{\mu(\mathcal{D}_\ell)+\varepsilon}\,.$$

Since $\mu(\mathcal{D}_\ell) > 0$, on taking $\varepsilon > 0$ small enough, we can ensure that $\lambda > 1-c$. Then, by the definition of λ, there exists $i_0 \in \mathbb{N}$ such that $\mu(A_{i_0}) > 0$ and

$$\frac{\mu(A_{i_0} \cap \mathcal{D}_\ell)}{\mu(A_{i_0})} > 1-c\,. \tag{5.17}$$

Take $j \geq \ell$ sufficiently large so that for every $i \geq j$ the radius of B_i is less than the radius of A_{i_0}. Then, since X is ultrametric, for all $i \geq j$ if $B_i \cap A_{i_0} \neq \varnothing$ then $B_i \subset A_{i_0}$. Since $\mathcal{D}_\ell \subset \mathcal{D} \subset \mathcal{B} \subset \bigcup_{i \geq j} B_i$, we have that

$$A_{i_0} \cap \mathcal{D}_\ell \subset \bigcup_{i\geq j,\ B_i \cap A_{i_0} \neq \varnothing} B_i \cap \mathcal{D}_\ell\,. \tag{5.18}$$

Without loss of generality assume the B_i over $i \geq j$ are disjoint, since if not we can take a disjoint sub-collection of $(B_i)_{i\geq j}$ such that the union of balls in this subcollection is again $\bigcup_{i\geq j} B_i$ and so the sub-collection would satisfy (5.18). Such sub-collection is possible to choose since X is ultrametric. Therefore, by (5.18), we have that

$$\mu(A_{i_0} \cap \mathcal{D}_\ell) \leq \sum_{i\geq j,\ B_i \cap A_{i_0} \neq \varnothing} \mu(B_i \cap \mathcal{D}_\ell)\,. \tag{5.19}$$

By construction $\mathcal{D}_i \cap U_i = \emptyset$ for every i. Thus, in view of (5.15) and the fact that $U_i \subset B_i$ we have that

$$\mu(B_i) \geq \mu(U_i \cap B_i) + \mu(\mathcal{D}_i \cap B_i) \geq c\mu(B_i) + \mu(\mathcal{D}_i \cap B_i)$$

and so $\mu(\mathcal{D}_i \cap B_i) \leq (1-c)\ \mu(B_i)$ for all i. In particular, since $\mathcal{D}_i \subset \mathcal{D}_{i+1}$ for all i and $j \geq \ell$ we get that

$$\mu(\mathcal{D}_\ell \cap B_i) \leq \mu(\mathcal{D}_i \cap B_i) \leq (1-c)\ \mu(B_i) \quad \text{for all } i \geq j\,.$$

Hence, by (5.19) and the assumption that the B_i for $i \geq j$ are disjoint, we get that

$$\mu(A_{i_0} \cap \mathcal{D}_\ell) \leq \sum_{i\geq j,\ B_i \cap A_{i_0} \neq \varnothing} (1-c)\mu(B_i) = (1-c)\mu\left(\bigcup_{i\geq j,\ B_i \cap A_{i_0} \neq \varnothing} B_i\right) \leq (1-c)\mu(A_{i_0})\,.$$

This contradicts (5.17). The proof is thus complete. ☐

Note that Lemma 5.3.2 is only applicable to limsup sets contained between two balls with radius varying by some constant. Since many of our sets of interest are lim sup sets of rectangles we make the following extension to Lemma 5.3.2.

Lemma 5.3.3. *Let $n \in \mathbb{N}$. For each $1 \leq j \leq n$ let (X_j, d_j) be a separable ultrametric space equipped with a Borel regular σ-finite measure μ_j, $(B_i^{(j)})_{i\in\mathbb{N}}$ be a sequence of balls in X_j with radii $r_i^{(j)} \to 0$ as $i \to \infty$, $(U_i^{(j)})_{i\in\mathbb{N}}$ be a sequence of μ_j-measurable sets such that $U_i^{(j)} \subset B_i^{(j)}$ for all i and assume that for some $c^{(j)} > 0$*

$$\mu_j\left(U_i^{(j)}\right) \geq c^{(j)}\mu_j\left(B_i^{(j)}\right) \qquad \text{for all } i \in \mathbb{N}. \tag{5.20}$$

Let $X = \prod_{j=1}^n X_j$*,* $d = \max_j d_j$ *be the metric on* X*,* $\mu = \prod_{j=1}^n \mu_j$ *be the product of measure on* X *and for each* $i \in \mathbb{N}$ *let* $B_i = \prod_{j=1}^n B_i^{(j)}$ *and* $U_i = \prod_{j=1}^n U_i^{(j)}$*. Then the limsup sets*

$$\mathcal{U} = \limsup_{i\to\infty} U_i \qquad \textit{and} \qquad \mathcal{B} = \limsup_{i\to\infty} B_i \tag{5.21}$$

have the same μ*-measure.*

The key ingredients in the proof of Lemma 5.3.3 are Lemma 5.3.2 and Fubini's Theorem, which we recall below in the special case of integrating the characteristic function of a measurable set, see [40, p. 233] or [60, §2.6.2].

Theorem 5.3.4 (Fubini's Theorem)**.** *Let* μ_1 *be a* σ*-finite measure over* X *and* μ_2 *be a* σ*-finite measure over* Y*. Then* $\mu_1 \times \mu_2$ *is a regular measure over* $X \times Y$*. Let* $S \subseteq X \times Y$ *be a* $\mu_1 \times \mu_2$ *measurable set and let*

$$S^x := \{y : (x, y) \in S\},$$
$$S_y := \{x : (x, y) \in S\}.$$

Then

$$(\mu_1 \times \mu_2)(S) = \int_Y \mu_1(S^y) d\mu_2 = \int_X \mu_2(S_x) d\mu_1.$$

We now proceed with the proof of Lemma 5.3.3.

Proof. We initially prove that

$$\mu\left(\limsup_{i\to\infty} B_i^{(1)} \times \prod_{j=2}^n B_i^{(j)}\right) = \mu\left(\limsup_{i\to\infty} U_i^{(1)} \times \prod_{j=2}^n B_i^{(j)}\right),$$

and note that Lemma 5.3.3 follows inductively. For ease of notation let

$$\hat{\mu} = \prod_{j=2}^n \mu_j, \quad \hat{B}_i = \prod_{j=2}^n B_i^{(j)}, \quad \hat{X} = \prod_{j=2}^n X_j.$$

For any $y \in \hat{X}$ let

$$I_y = \{i : y \in \hat{B}_i\},$$

and for any $F \subseteq X$ let F_y denote the fiber of F at y, that is

$$F_y = \{x : (x, y) \in F\} \subseteq X_1.$$

Observe that

$$A := \left(\limsup_{i\to\infty} B_i^{(1)} \times \hat{B}_i\right)_y = \limsup_{\substack{i\to\infty \\ i\in I_y}} B_i^{(1)} =: D. \tag{5.22}$$

Indeed, if $x \in A$ then it implies there exists an infinite sequence $\{i_k\}$ such that

$$(x,y) \in B_{i_k}^{(1)} \times \hat{B}_{i_k} \quad \text{for all } i_k.$$

Hence $\{i_k\} \subseteq I_y$ and so $x \in D$.

Conversely, if $x \in D$ then D is non-empty and so I_y must be infinite. By the definition of I_y and the fact that $x \in D$ we have that $x \in B_i^{(1)}$ for infinitely many $i \in I_y$, and so $x \in A$.

Similarly, we have that

$$\left(\limsup_{i\to\infty} U_i^{(1)} \times \hat{B}_i\right)_y = \limsup_{\substack{i\to\infty \\ i\in I_y}} U_i^{(1)}. \tag{5.23}$$

Applying Fubini's Theorem we have that

$$\begin{aligned}
\mu\left(\limsup_{i\to\infty} B_i^{(1)} \times \hat{B}_i\right) &= \int_{\hat{X}} \mu_1\left(\left(\limsup_{i\to\infty} B_i^{(1)} \times \hat{B}_i\right)_y\right) d\hat{\mu}, \\
&\overset{(5.22)}{=} \int_{\hat{X}} \mu_1\left(\limsup_{\substack{i\to\infty \\ i\in I_y}} B_i^{(1)}\right) d\hat{\mu}, \\
&\overset{\text{Lemma 5.3.2}}{=} \int_{\hat{X}} \mu_1\left(\limsup_{\substack{i\to\infty \\ i\in I_y}} U_i^{(1)}\right)_y d\hat{\mu}, \\
&\overset{(5.23)}{=} \int_{\hat{X}} \mu_1\left(\left(\limsup_{i\to\infty} U_i^{(1)} \times \hat{B}_i\right)_y\right) d\hat{\mu}, \\
&= \mu\left(\limsup_{i\to\infty} U_i^{(1)} \times \hat{B}_i\right).
\end{aligned}$$

Note that in the above argument we have not made use of the fact $\hat{B}_i$ are products of balls; we only used the fact that these are measurable sets. Hence, the above argument can be repleaded $n-1$ more times, for $\ell = 2, \ldots, n-1$ each time replacing $B_i^{(\ell)}$ by $U_i^{(\ell)}$ so that at step ℓ we get that

$$\mu\left(\limsup_{i\to\infty} \prod_{j=1}^{\ell-1} U_i^{(j)} \times \prod_{j=\ell}^{n} B_i^{(j)}\right) = \mu\left(\limsup_{i\to\infty} \prod_{j=1}^{\ell} U_i^{(j)} \times \prod_{j=1}^{\ell+1} B_i^{(j)}\right).$$

Putting all these equations for $\ell = 1, \ldots, n$ together we get (5.21) as claimed. □

Lemma 1.1.4 only proves positive measure for a lim sup set. In the context of Theorem 5.1.1 we need a zero-full law. In [65] Haynes proved a zero-full result for the simultaneous case. We adapt this method of proof for the weighted case.

Lemma 5.3.5. *Let $n \in \mathbb{N}$, p be a prime and $\Psi = (\psi_1, \ldots, \psi_n)$ be any n-tuple of approximation functions. Then*

$$\mu_{p,n}(\mathfrak{W}'_n(\Psi)) \in \{0,1\}.$$

We note that Haynes proved this result for the more complex set of S-arithmetic approximation. While we suspect that the same could be proven in this context we only prove the p-adic case since this is the only result we will need in this paper.

Proof. Firstly, note that the sets $\mathfrak{A}'_{a_0}(\Psi)$ used to construct our lim sup set have the property that if $p \mid a_0$, then $\mathfrak{A}'_{a_0}(\Psi) = \emptyset$ or $\mathbb{Z}_p^n$, so assume $p \nmid a_0$. Define the map $\pi : \mathbb{Z}_p \to \mathbb{Z}_p$ as follows. For a p-adic integer $x \in \mathbb{Z}_p$ with p-adic expansion

$$x = \sum_{i=0}^{\infty} a_i p^i, \quad a_i \in \{0, \dots, p-1\},$$

define

$$\pi(x) = \begin{cases} \sum_{i=0}^{\infty} a_{i+1} p^i, & \text{if } a_0 = 0, \\ 1 + \sum_{i=0}^{\infty} a_{i+1} p^i, & \text{otherwise.} \end{cases}$$

Let $\pi_n : \mathbb{Z}_p^n \to \mathbb{Z}_p^n$ be the transformation $(x_1, \dots, x_n) \mapsto (\pi(x_1), \dots, \pi(x_n))$. By using the fact that $p \nmid a_0$, and that each $(a_i, a_0) = 1$, it can be shown that under such mapping

$$\pi_n(\mathfrak{W}'_n(\Psi)) \subseteq \mathfrak{W}'_n(p\Psi),$$

where $p\Psi$ means each component of Ψ has to be multiplied by p. This can be repeated inductively to show that $\pi_n^K(\mathfrak{W}'_n(\Psi)) \subseteq \mathfrak{W}'_n(p^K\Psi)$ for any $K \in \mathbb{N}$. Assuming that $\mu_{p,n}(\mathfrak{W}'_n(\Psi)) > 0$, then by a p-adic version of the Lebesgue Density Theorem (see e.g. Lemma 1 in [105, Part II, Ch. 1]) for any $\epsilon > 0$ there exists integer vector $\boldsymbol{x}_0 \in \mathbb{Z}^n$ and $N \in \mathbb{N}$ such that

$$\mu_{p,n}\left(\{\boldsymbol{x} \in \mathfrak{W}'_n(\Psi) : |\boldsymbol{x} - \boldsymbol{x}_0|_p \le p^{-N}\}\right) \ge (1-\epsilon)p^{-N}.$$

Further, we have that

$$\pi_n^N\left(\{\boldsymbol{x} \in \mathfrak{W}'_n(\Psi) : |\boldsymbol{x} - \boldsymbol{x}_0|_p \le p^{-N}\}\right) \subseteq \mathfrak{W}'_n(p^N\Psi),$$

and so

$$\begin{aligned} \mu_{p,n}\left(\mathfrak{W}'_n(p^N\Psi)\right) &\ge \mu_{p,n}\left(\pi_n^N\left(\{\boldsymbol{x} \in \mathfrak{W}'_n(\Psi) : |\boldsymbol{x} - \boldsymbol{x}_0|_p \le p^{-N}\}\right)\right), \\ &\ge p^N(1-\epsilon)p^{-N}, \\ &= (1-\epsilon). \end{aligned}$$

Since ϵ is arbitrary we have that $\mu_{p,n}(\bigcup_{N=1}^{\infty} \mathfrak{W}'_n(p^N\Psi)) = 1$. Now observe that

$$\mathfrak{W}'_n(\Psi) \subset \mathfrak{W}'_n(p\Psi) \subset \mathfrak{W}'_n(p^2\Psi) \subset \dots$$

and so, by Lemma 5.3.3 with $X = \mathbb{Z}_p^n$, d given by the sup norm, and $\mu = \mu_{p,n}$, we have that $\mu_{p,n}(\mathfrak{W}'_n(\Psi)) = \mu_{p,n}(\mathfrak{W}'_n(p^N\Psi))$ for every $N \in \mathbb{N}$. Hence,

$$\mu_{p,n}(\mathfrak{W}'_n(\Psi)) = \lim_{N\to\infty} \mu_{p,n}\left(\mathfrak{W}'_n(p^N\Psi)\right) = \mu_{p,n}\left(\bigcup_{N=1}^{\infty} \mathfrak{W}'_n(p^N\Psi)\right) = 1\,,$$

thus finishing the proof. □

5.4 Proof of Theorems 5.1.1 & 5.1.2

By Lemma's 1.1.3–1.1.4 it is clear that we need bounds on the measure of $\mathfrak{A}'_{a_0}(\Psi)$ and $\mathfrak{A}'_{a_0}(\Psi)\cap\mathfrak{A}'_{b_0}(\Psi)$ for $a_0, b_0 \in \mathbb{N}$. As we are considering these measures at fixed values of a_0 and b_0 the monotonicity condition of Theorem 5.1.1 does not appear until we consider the summations over the measures of these sets. For that reason Theorems 5.1.1 & 5.1.2 are proven in tandem up to such point.

Since $(a_0, a_i) = 1$ observe that we must have $p \nmid a_0$. If $p \mid a_0$ then the reduced fractions $\frac{a_i}{a_0}$ used in the composition of $\mathfrak{A}'_{a_0}(\Psi)$ would satisfy $\left|\frac{a_i}{a_0}\right|_p > 1$ for any component $1 \leq i \leq n$. And so for sufficiently large a_0 we have that

$$\left\{\boldsymbol{x} \in \mathbb{Z}_p^n : \left|x_i - \frac{a_i}{a_0}\right|_p < \psi_i(a_0),\ 1 \leq i \leq n\right\} = \emptyset,$$

since the components of the approximation vector are less than 1. Hence without loss of generality when considering the measure of $\mathfrak{A}'_{a_0}(\Psi)$ and $\mathfrak{A}'_{a_0}(\Psi) \cap \mathfrak{A}'_{b_0}(\Psi)$ we will assume that $p \nmid a_0, b_0$.

With regards to the condition that each $\psi_i(q) \ll \frac{1}{q}$ note that Lemma 5.3.3 allows us to reduce this to the condition that each $\psi_i(q) < \frac{1}{q}$ for $1 \leq i \leq n$ and the measure results will remain unchanged. Similarly such constants would not effect the convergence or divergence of the summation of interest.

Note that for any $x \in \mathbb{Z}_p$ and $0 < r < 1$ there exists $t \in \mathbb{N}_0$ such that $B(x, r) = B(x, p^{-t})$. For each $1 \leq i \leq n$ define the function $t_i : \mathbb{N} \to \mathbb{N}_0$ with $t_i(a_0)$ satisfying

$$p^{-t_i(a_0)} < \psi_i(a_0) \leq p^{-t_i(a_0)+1}.$$

Then for any $1 \leq i \leq n$ and $a_0 \in \mathbb{N}$ we have that $\psi_i(a_0) \asymp p^{-t_i(a_0)}$ and

$$B\left(x, \psi_i(a_0)\right) = B\left(x, p^{-t_i(a_0)+1}\right).$$

Hence, without loss of generality we could replace the n-tuple of approximation functions Ψ with the function T given by $T(a_0) = \left(p^{-t_1(a_0)+1}, \ldots, p^{-t_n(a_0)+1}\right)$. Thus, we have that $\mu_{p,n}(\mathfrak{W}_n(\Psi)) = \mu_{p,n}(\mathfrak{W}_n(T))$.

For $a_0, b_0 \in \mathbb{N}$ and φ Euler's totient function we prove the following claims

(a) $\mu_{p,n}\left(\mathfrak{A}'_{a_0}(\Psi)\right) \ll \varphi(a_0)^n \prod_{i=1}^n \psi_i(a_0)$,

(b) $\mu_{p,n}\left(\mathfrak{A}'_{a_0}(\Psi)\right) \gg \varphi(a_0)^n \prod_{i=1}^n \psi_i(a_0)$,

(c) $\mu_{p,n}\left(\mathfrak{A}_{a_0}(\Psi) \cap \mathfrak{A}_{b_0}(\Psi)\right) \ll a_0^n b_0^n \prod_{i=1}^n \psi_i(a_0)\psi_i(b_0)$.

Beginning with (a) observe that

$$\mu_{p,n}(\mathfrak{A}'_{a_0}(\Psi)) = \mu_{p,n}\left(\bigcup_{\substack{|a_i| \leq a_0 \\ \gcd(a_i, a_0)=1,\, 1 \leq i \leq n}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, \psi_i(a_0)\right)\right). \tag{5.24}$$

If each rectangle in the above composition is disjoint then

$$\mu_{p,n}(\mathfrak{A}'_{a_0}(\Psi)) = \sum_{\substack{|a_i|\leq a_0 \\ \gcd(a_i,a_0)=1,\, 1\leq i\leq n}} \mu_{p,n}\left(\prod_{i=1}^{n} B\left(\frac{a_i}{a_0}, \psi_i(a_0)\right)\right) \asymp \varphi(a_0)^n \prod_{i=1}^{n} \psi_i(a_0), \tag{5.25}$$

since $\mu_{p,n}$ is the product measure of n copies of μ_p, and so the measure of the product of the balls in the above expression equals the product of their measures. This provides us with an upper bound on $\mu_{p,n}(\mathfrak{A}'_{a_0}(\Psi))$, since any non-empty intersections in the union within (5.24) would only make the measure of the union smaller than their sum given by (5.25).

To prove (b) we simply need to show that the union within (5.24) contains no non-empty intersections. Suppose this is not the case, say

$$\left(\prod_{i=1}^{n} B\left(\frac{b_i}{a_0}, \psi_i(a_0)\right)\right) \bigcap \left(\prod_{i=1}^{n} B\left(\frac{c_i}{a_0}, \psi_i(a_0)\right)\right) \neq \emptyset,$$

for some points $b = (b_1, \ldots, b_n), c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ with $|b_i|, |c_i| \leq a_0$ and $b \neq c$. Then we have that

$$|b_i - c_i|_p \leq \psi_i(a_0), \quad 1 \leq i \leq n,$$

since $|a_0|_p = 1$. Such inequalities would hold if and only if $\psi_i(a_0) \geq \frac{1}{a_0}$ for all $1 \leq i \leq n$ such that $b_j \neq c_j$. However, we have that $\psi_i(q) < \frac{1}{q}$ for all $1 \leq i \leq n$ and $q \in \mathbb{N}$ and thus, by (5.25), we have the required lower bound on $\mu_{p,n}(\mathfrak{A}'_{a_0}(\Psi))$.

To prove (c) define the set

$$Q := \{(a,b) \in \mathbb{Z}^2 : |a| \leq a_0,\, |b| \leq b_0,\, \gcd(a,a_0) = \gcd(b,b_0) = 1\}.$$

Observe that

$$\mu_{p,n}\left(\mathfrak{A}'_{a_0}(\Psi) \cap \mathfrak{A}'_{b_0}(\Psi)\right) \ll \prod_{i=1}^{n} \#\left\{(a_i,b_i) \in Q : \left|\frac{a_i}{a_0} - \frac{b_i}{b_0}\right|_p < \Delta_i\right\} \delta_i. \tag{5.26}$$

where

$$\Delta_i = \max\{\psi_i(a_0), \psi_i(b_0)\} \qquad \text{and} \qquad \delta_i = \min\{\psi_i(a_0), \psi_i(b_0)\}\,.$$

Fix any i and without loss of generality suppose that $\Delta_i = \psi_i(a_0) \geq \psi_i(b_0) = \delta_i$. Note that since $p \nmid a_0, b_0$ then the inequality in the above equation is equivalent to $(a_i, b_i) \in Q$ satisfying

$$|a_i b_0 - b_i a_0|_p < \psi_i(a_0). \tag{5.27}$$

To count solutions satisfying (5.27) we observe that such solutions also solve the congruence

$$b_i a_0 - a_i b_0 \equiv 0 \mod p^{t_i(a_0)}. \tag{5.28}$$

Let $d = \gcd(a_0, b_0)$, and let $a_0' = \frac{a_0}{d}$ and $b_0' = \frac{b_0}{d}$. Suppose that

$$b_i a_0' - a_i b_0' = k,$$

for some integer k, with $|k| \leq 2\frac{a_0 b_0}{d}$. The bounds on k follow on the observation that

$$|b_i a_0 - a_i b_0| \leq 2a_0 b_0,$$

for all $(a_i, b_i) \in Q$. Considering the congruence

$$a_i b_0' \equiv b_i a_0' - k \mod a_0',$$

note that per k there is at most one solution a_i modulo a_0', and so at most $\frac{2a_0}{a_0'} = 2d$ possible a_i with $|a_i| \leq a_0$. Clearly, each b_i is uniquely determined by each a_i and k, so per fixed k there are at most $2d$ possible pairs $(a_i, b_i) \in Q$. To solve (5.28) we must have that

$$k \equiv 0 \mod p^{t_i(a_0)}, \tag{5.29}$$

of which there are at most

$$\frac{4a_0 b_0}{d\, p^{t_i(a_0)}} + 1$$

possible k satisfying $|k| \leq 2a_0 b_0/d$. Note that one such possible value of k satisfying (5.29) is $k = 0$. But this is impossible, since it implies that

$$a_0' b_i = a_i b_0'.$$

Indeed, assuming $a_0 > b_0$, we get that $a_0 \neq 1$ and $\gcd(a_0', a_i) = \gcd(a_0', b_0') = 1$, and so we must have that $b_i a_0' - a_i b_0' \neq 0$. If $b_0 > a_0$ then the argument is similar. Hence there are at most

$$\frac{4a_0 b_0}{d\, p^{t_i(a_0)}}$$

values of k that have corresponding solutions in Q, and so there are at most

$$2d \frac{4a_0 b_0}{d\, p^{t_i(a_0)}} \ll a_0 b_0 \psi_i(a_0)$$

pairs $(a_i, b_i) \in Q$ that solve (5.27). Combining this upper bound with (5.26) we have that

$$\mu_{p,n}(\mathfrak{A}_{a_0}'(\Psi) \cap \mathfrak{A}_{b_0}'(\Psi)) \ll a_0^n b_0^n \prod_{i=1}^{n} \psi_i(a_0)\psi_i(b_0).$$

By (c), we have that

$$\sum_{a_0, b_0 = 1}^{N} \mu_{p,n}(\mathfrak{A}_{a_0}'(\Psi) \cap \mathfrak{A}_{b_0}'(\Psi)) \ll \sum_{a_0, b_0 = 1}^{N} a_0^n b_0^n \prod_{i=1}^{n} \psi_i(a_0)\psi_i(b_0) \ll \left(\sum_{a_0=1}^{N} a_0^n \prod_{i=1}^{n} \psi_i(a_0) \right)^2. \tag{5.30}$$

Now assuming the monotonicity of $\prod_{i=1}^{n} \psi_i(q)$, by (a), (b), we have that

$$\sum_{a_0=1}^{N} \mu_{p,n}(\mathfrak{A}_{a_0}'(\Psi)) \asymp \sum_{a_0=1}^{N} \varphi(a_0)^n \prod_{i=1}^{n} \psi_i(a_0) \asymp \sum_{a_0=1}^{N} a_0^n \prod_{i=1}^{n} \psi_i(a_0)\,. \tag{5.31}$$

Hence (5.31) completes the convergence case of Theorem 5.1.1 via Lemma 1.1.3. In turn, (5.30) and (5.31) together with Lemma 1.1.4 proves that $\mu_{p,n}(\mathfrak{W}'_n(\Psi)) > 0$ and finally applying Lemma 5.3.5 completes the proof of Theorem 5.1.1.

Regarding Theorem 5.1.2, Claim (a), completes the convergence case via Lemma 1.1.3. In the divergence case we note that Claim (b), (5.30) and condition (5.3) imply that

$$\limsup_{N\to\infty}\left(\frac{\sum_{a_0=1}^{N}\varphi(a_0)^n\prod_{i=1}^{n}\psi_i(a_0)}{\sum_{a_0=1}^{N}a_0^n\prod_{i=1}^{n}\psi_i(a_0)}\right)^2 > 0.$$

Hence, Lemma 1.1.4 is applicable and we get that $\mu_{p,n}(\mathfrak{W}'_n(\Psi)) > 0$. Applying Lemma 5.3.5 completes the proof of Theorem 5.1.2.

5.5 Proof of Theorem 5.1.4

As with many Hausdorff dimension results we prove the upper bound and lower bound independently. As we are working with lim sup sets of hyperrectangles defined by (5.1) we will naturally appeal to Theorem 3.3.3 to get the lower bound. We start with the upper bound which takes advantage of a standard cover of $\mathfrak{W}_n(\boldsymbol{\tau})$.

5.5.1 Upper bound result

Recall that $\mathfrak{W}_n(\Psi) = \limsup_{a_0\to\infty}\mathfrak{A}_{a_0}(\Psi)$, where $\mathfrak{A}_{a_0}(\Psi)$ is given by (5.1), that is

$$\mathfrak{A}_{a_0}(\Psi) = \bigcup_{\substack{(a_1,\dots,a_n)\in\mathbb{Z}^n \\ |a_i|\leq a_0\ (1\leq i\leq n)}} R_{a_0,a_1,\dots,a_n}(\Psi)$$

and

$$R_{a_0,a_1,\dots,a_n}(\Psi) = \left\{\boldsymbol{x} = (x_1,\dots,x_n)\in\mathbb{Z}_p^n : \left|x_i - \frac{a_i}{a_0}\right|_p < \psi_i(a_0)\ \text{ for } 1\leq i\leq n\right\}.$$

Throughout this proof $\Psi = (q^{-\tau_1},\dots,q^{-\tau_n})$. Then for every $i\in\{1,\dots,n\}$ we can trivially cover $R_{a_0,a_1,\dots,a_n}(\boldsymbol{\tau}) := R_{a_0,a_1,\dots,a_n}(\Psi)$ by a finite collection $\mathfrak{B}(a_0)$ of balls of radius $a_0^{-\tau_i}$ such that

$$\#\mathfrak{B}(a_0) \ll \prod_{j=1}^{n}\max\left\{1, \frac{a_0^{-\tau_j}}{a_0^{-\tau_i}}\right\} = a_0^{\sum_{\tau_j<\tau_i}(\tau_i-\tau_j)},$$

where the power of a_0 on the R.H.S of the above inequality can be obtained by removing the cases where $\frac{a_0^{-\tau_j}}{a_0^{-\tau_i}} < 1$. Let $s_0 = \frac{n+1+\sum_{\tau_j<\tau_i}(\tau_i-\tau_j)+\delta}{\tau_i}$ for some $\delta > 0$. Then for any $N > 0$

$$\begin{aligned}\mathcal{H}^{s_0}(\mathfrak{W}_n(\boldsymbol{\tau})) &\ll \sum_{a_0\geq N}\sum_{\substack{|a_i|\leq a_0\\1\leq i\leq n}} \#\mathfrak{B}(a_0)a_0^{-s_0\tau_i},\\ &\ll \sum_{a_0\geq N} a_0^{n+\sum_{\tau_j<\tau_i}(\tau_i-\tau_j)-s_0\tau_i},\\ &= \sum_{a_0\geq N} a_0^{-1-\delta} \to 0 \quad \text{as } N\to\infty.\end{aligned}$$

This implies that $\dim \mathfrak{W}_n(\boldsymbol{\tau})) \leq s_0$. The above argument follows for any choice of τ_i, hence we may choose the minimum over the set of all τ_i and so the upper bound for the dimension in Theorem 5.1.4 follows on letting $\delta \to 0$.

5.5.2 Lower bound result

In order to apply Theorem 3.3.3 we need to construct a set of resonant points that we can show are a locally ubiquitous system of rectangles. Let

$$\hat{R}_{a_0,i} = \left\{\frac{a_i}{a_0} \in \mathbb{Q} : |a_i| \leq a_0\right\},$$

for each $1 \leq i \leq n$, and let $\hat{R}_{a_0} = \prod_{i=1}^n R_{a_0,i}$. In line with the notation prior to Theorem 3.3.3 let $J = \mathbb{N}$, and $\beta : J \to \mathbb{R}_+$ be $\beta(a_0) = a_0$. Choose $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ to be $\rho(a_0) = a_0^{-1}$, and choose the two sequences $l_k = M^k$, and $u_k = M^{k+1}$, for some fixed integer $M \geq 2$ to be chosen later, so that

$$J_k = \{a_0 \in \mathbb{N} : M^k \leq a_0 \leq M^{k+1}\}.$$

In order to show such set of resonant points is a local ubiquitous system of rectangles we prove the following proposition.

Proposition 5.5.1. *Let $\hat{R}_{a_0}$, J, β, and ρ be defined as above. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ with each $\alpha_i > 1$ be a vector satisfying*

$$\sum_{i=1}^n \alpha_i = n+1. \tag{5.32}$$

Let $M \geq p^{n+1}$. Then there is a $c_2 > 0$ such that for any ball $B \subset \mathbb{Z}_p^n$

$$\mu_{p,n}\left(B \cap \bigcup_{M^k\leq a_0\leq M^{k+1}} \Delta\left(R_{a_0}, \left(\frac{c_1}{M^{k+1}}\right)^{\boldsymbol{\alpha}}\right)\right) \geq c_2\, \mu_{p,n}(B)$$

for all sufficiently large $k \in \mathbb{N}$.

Proof. Fix some ball $B = B(\boldsymbol{y}, r)$ for some $\boldsymbol{y} \in \mathbb{Z}_p^n$ and $r \in \{p^i : i \in \mathbb{N} \cup \{0\}\}$. We will assume that k is sufficiently large so that $M^k r \geq 1$. In view of (5.32) and the fact that $\alpha_i > 1$ for all i, by Lemma 5.3.1, we have that for any $\boldsymbol{x} = (x_1, \ldots, x_n) \in B$ there exists $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$, satisfying

$$|a_i| \;\leq\; M^k \qquad (1 \leq i \leq n), \qquad 0 < a_0 < M^{k+1}$$

such that

$$|a_0 x_i - a_i|_p < p\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}, \quad 1 \leq i \leq n. \tag{5.33}$$

Since $\alpha_i > 1$ for each $1 \leq i \leq n$, (5.33) combined with $0 < a_0 \leq M^{k+1}$ implies that $|a_i|_p \leq |a_0|_p$ for each $1 \leq i \leq n$, provided that k is sufficiently large. Let λ be the integer such that $|a_0|_p = p^{-\lambda}$. Write $a_0' = a_0 p^{-\lambda}$ and $a_i' = a_i p^{-\lambda}$. Observe that $a_0', a_i' \in \mathbb{Z}$,

$$0 < a_0 \leq p^{-\lambda} M^{k+1}, \quad |a_i| \leq p^{-\lambda} M^k, \tag{5.34}$$

for each $1 \leq i \leq n$ and that

$$\begin{aligned} \left| x_i - \frac{a_i'}{a_0'} \right|_p &= p^{\lambda} |a_0 x_i - a_i|_p \\ &< p^{\lambda+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}, \end{aligned} \tag{5.35}$$

for $1 \leq i \leq n$. We want to remove the a_0' values that are 'too' prime, that is $|a_0'|_p < p^{-\lambda_0}$ for some fixed $\lambda_0 \in \mathbb{N}$ to be chosen later. We consider the integer vectors $(a_0', \ldots, a_n')$ satisfying (5.34) such that

$$\left(\frac{a_1'}{a_0'}, \ldots, \frac{a_n'}{a_0'}\right) \in B(\boldsymbol{y}, r).$$

Considering the congruence equations for a_0' fixed we have that there are

$$\left(2\frac{M^k}{p^\lambda} r + 1\right)^n < \left(3\frac{M^k}{p^\lambda} r\right)^n$$

such points. Hence

$$\begin{aligned} &\mu_{p,n}\left(B \cap \bigcup_{\lambda \geq \lambda_0} \bigcup_{\substack{|a_i| \leq \frac{M^k}{p^\lambda} \\ 0 < a_0' \leq \frac{M^{k+1}}{p^\lambda}}} \bigcup_{\left(\frac{a_1'}{a_0'}, \ldots, \frac{a_n'}{a_0'}\right) \in B} \prod_{i=1}^{n} B\left(\frac{a_i'}{a_0'}, p^{\lambda+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right) \right) \\ &\qquad \leq \sum_{\lambda \geq \lambda_0} \frac{M^{k+1}}{p^\lambda} \left(3\frac{M^k}{p^\lambda} r\right)^n p^{n\lambda + n} M^{-k(n+1)-1}, \\ &\qquad = \sum_{\lambda \geq \lambda_0} \mu_{p,n}(B) 3^n p^{n-\lambda}, \\ &\qquad \leq 3^n \frac{p^{n+1-\lambda_0}}{p-1} \mu_{p,n}(B). \end{aligned}$$

Taking λ_0 sufficiently large, e.g. $p^{\lambda_0} > 2\frac{3^n p^{n+1}}{p-1}$, then we have that

$$\mu_{p,n}\left(B \cap \bigcup_{\lambda \geq \lambda_0} \bigcup_{\substack{|a_i| \leq \frac{M^k}{p^\lambda} \\ 0 < a_0' \leq \frac{M^{k+1}}{p^\lambda}}} \bigcup_{\left(\frac{a_1'}{a_0'},\ldots,\frac{a_n'}{a_0'}\right) \in B} \prod_{i=1}^n B\left(\frac{a_i'}{a_0'}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \leq \frac{1}{2}\mu_{p,n}(B).$$

Then we have that

$$\begin{aligned}
&\mu_{p,n}\left(B \cap \bigcup_{\lambda < \lambda_0} \bigcup_{\substack{|a_i| \leq M^k \\ 0 < a_0 < M^{k+1} : |a_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\qquad \geq \mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i| \leq M^k \\ 0 < a_0 < M^{k+1}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\qquad\qquad - \mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i| \leq M^k \\ 0 < a_0 < M^{k+1} : |a_0|_p < p^{-\lambda_0}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\qquad\qquad \geq \frac{1}{2}\mu_{p,n}(B).
\end{aligned}$$

Similarly to the above we can deduce that

$$\begin{aligned}
&\mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i| \leq M^k \\ M^k < a_0 \leq M^{k+1} : |a_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\qquad \geq \mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i| \leq M^k \\ 0 < a_0 < M^{k+1} : |a_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\qquad\qquad - \mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i| \leq M^k \\ 0 < a_0 \leq M^k : |a_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right).
\end{aligned}$$

Using similar calculations to those of above we have that

$$\mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i| \leq M^k \\ 0 < a_0 \leq M^k : |a_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^n B\left(\frac{a_i}{a_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \leq \frac{3^n p^{n\lambda_0+n}}{M}\mu_{p,n}(B).$$

Thus, provided $M > 2\,3^n p^{n\lambda_0+n}$, then there exists some constant $c_2 > 0$ such that

$$\mu_{p,n}\left(B \cap \bigcup_{\substack{|a_i|\leq M^k \\ M^k < a_0 \leq M^{k+1} : |a_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{a_i}{a_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \geq c_2\,\mu_{p,n}(B).$$

Taking the constant

$$c_1 = \max_{1\leq i\leq n} p^{\frac{\lambda_0+1}{\alpha_i}} M^{1-\frac{1}{n+1}}$$

completes the proof. □

Given Proposition 5.5.1, we have that (R_{a_0}, β) is a local ubiquitous system with respect to $(\rho, \boldsymbol{\alpha})$, provided $\sum_{i=1}^{n} \alpha_i = n+1$. Using the setup provided for Theorem 3.3.3 let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) = (\alpha_1 + t_1, \dots, \alpha_n + t_n) \in \mathbb{R}^n_+$, then $\mathfrak{W}_n(\boldsymbol{\tau}) = W(\boldsymbol{\tau})$. Without loss of generality let $\tau_1 \geq \cdots \geq \tau_n$. Define α_i recursively as

$$\alpha_i = \min\left\{\tau_i, \frac{n+1-\sum_{j=n-i+1}^{n}\alpha_j}{n-i}\right\}.$$

Since $\sum_{i=1}^{n}\tau_i > n+1$ and $\sum_{i=1}^{n}\alpha_i = n+1$ such recursive formula is possible and we have that $\alpha_i \leq \tau_i$ for each $1 \leq i \leq n$, so $\boldsymbol{\tau}$ is well defined. Since $\tau_1 \geq \cdots \geq \tau_n$ we have that $\alpha_1 \geq \cdots \geq \alpha_n$, and furthermore there exists $k \in \{1, \dots, n\}$ such that for all $1 \leq i \leq n-k$

$$\alpha_i = \frac{n+1-\sum_{j=n-k+1}^{n}\alpha_j}{n-k}.$$

Such observation follows by noting that at least

$$\alpha_1 = n+1-\sum_{j=n-1}^{n}\alpha_j$$

by the fact that $\sum_{i=1}^{n}\alpha_i = n+1$. Note that for each metric space $X_i = \mathbb{Z}_p$ the Haar measure μ_p is a 1-Ahlfors probability measure. With reference to Theorem 3.3.3, consider the following three cases:

i) $A_i \in \{\alpha_1, \dots \alpha_{n-k}\}$: For these values of A_i we have that

$$K_1 = \{1, \dots, n-k\}, \quad K_2 = \{n-k+1, \dots, n\}, \quad K_3 = \emptyset.$$

Applying Theorem 3.3.3 we get that

$$\begin{aligned}\dim \mathfrak{W}_n(\boldsymbol{\tau}) &\geq \min_{A_i}\left\{\frac{(n-k)\alpha_i + (n-(n-k+1)+1)\alpha_i - \sum_{j=n-k}^{n} t_j}{\alpha_i}\right\}, \\ &= \min_{A_i}\left\{n - \frac{\sum_{j=n-k+1}^{n} t_j}{\alpha_i}\right\}.\end{aligned}$$

Since $t_i = 0$ for $n-k < i \leq n$ we have that $\dim \mathfrak{W}_n(\boldsymbol{\tau}) \geq n$.

ii) $A_i \in \{\alpha_{n-k+1}, \ldots, \alpha_n\}$: For such values of A_i observe that

$$K_1 = \{1, \ldots, i\}, \quad K_2 = \{i+1, \ldots, n\}, \quad K_3 = \emptyset.$$

Applying Theorem 3.3.3 we have, in this case,

$$\dim \mathfrak{W}_n(\boldsymbol{\tau}) \geq \min_{A_i} \left\{ \frac{i\alpha_i + (n-i)\alpha_i - \sum_{j=i+1}^{n} t_j}{\alpha_i} \right\}.$$

Similarly to the previous case, since $t_j = 0$ for $n-k+1 \leq i \leq n$ the r.h.s of the above equation is n, the maximal dimension of $\mathfrak{W}_n(\boldsymbol{\tau})$.

iii) $A_i \in \{\tau_1, \ldots, \tau_n\}$: Since $\tau_i = \alpha_i$ for $n-k+1 \leq i \leq n$, ii) covers such result. So we only need to consider the set of $A_i \in \{\tau_1, \ldots \tau_{n-k}\}$. If A_i is contained in such set, then

$$K_1 = \emptyset, \quad K_2 = \{i, \ldots, n\}, \quad K_3 = \{1, \ldots, i-1\}.$$

Thus, by Theorem 3.3.3, we have that

$$\begin{aligned}
\dim \mathfrak{W}_n(\boldsymbol{\tau}) &\geq \min_{A_i} \left\{ \frac{(n-i+1)\tau_i + \sum_{j=1}^{i-1} a_j - \sum_{j=i}^{n} t_j}{\tau_i} \right\}, \\
&= \min_{A_i} \left\{ \frac{(n-i+1)\tau_i + (i-1)\left(\frac{n+1-\sum_{j=n-k+1}^{n} a_j}{n-k}\right) - \sum_{j=i}^{n-k}(\tau_j - a_j) - \sum_{j=n-k+1}^{n} t_j}{\tau_i} \right\}, \\
&= \min_{A_i} \left\{ \frac{(n-i+1)\tau_i + (n-k)\left(\frac{n+1-\sum_{j=n-k+1}^{n} a_j}{n-k}\right) - \sum_{j=i}^{n-k} \tau_j - \sum_{j=n-k+1}^{n} t_j}{\tau_i} \right\}, \\
&= \min_{A_i} \left\{ \frac{n+1+\sum_{j=i}^{n}(\tau_i - \tau_j)}{\tau_i} \right\},
\end{aligned}$$

since $a_j + t_j = \tau_j$.

These are all possible choices of A_i. The proof of Theorem 5.1.4 is thus complete.

5.6 Dirichlet-style Theorem on p-adic manifolds

This section provides a full measure statement needed to deploy a Mass Transference Principle for the proofs of Theorems 5.2.3–5.2.5.

Theorem 5.6.1. *Let $\boldsymbol{f} = (f_1, \ldots, f_m) : \mathcal{U} \to \mathbb{Z}_p^m$ be a map defined on an open subset $\mathcal{U} \subseteq \mathbb{Z}_p^d$, $\boldsymbol{x} \in \mathcal{U}$ and suppose that $\boldsymbol{f}$ is DQE at $\boldsymbol{x}$ and let λ be given by*

$$\max \left\{ 1, \max_{\substack{1 \leq i \leq d \\ 1 \leq j \leq m}} \left| \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) \right|_p \right\} = p^{\lambda}. \tag{5.36}$$

Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m) \in \mathbb{R}_+^m$, $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}_+^d$ *and*

$$\begin{array}{ll} \sum_{i=1}^m \tau_i < m+1, & \tau_i > 1,\ (1 \le i \le m), \\ \sum_{i=1}^d v_i = n+1 - \sum_{i=1}^m \tau_i, & v_i > 1,\ (1 \le i \le d)\,. \end{array}$$

Then there exist $H_0 \in \mathbb{N}$ *such that for all* $H > H_0$ *and some* $k \in \mathbb{Z}$ *the following system*

$$\begin{cases} \left| x_i - \frac{a_i}{a_0} \right|_p < p^{(n+m\lambda)/d} p^k H^{-v_i} & (1 \le i \le d), \\ \left| f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) - \frac{a_{d+j}}{a_0} \right|_p < (p^{-k}H)^{-\tau_j} & (1 \le j \le m), \\ \max\limits_{0 \le i \le n} |a_i| \le p^{-k}H \end{cases} \tag{5.37}$$

has a solution $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ *satisfying*

$$(a_0, p) = 1, \qquad \gcd(a_0, \ldots, a_n) = 1 \qquad \textit{and} \qquad \left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) \in \mathcal{U}. \tag{5.38}$$

Proof. By Lemma 5.3.1 with $\boldsymbol{\sigma} = ((n+m\lambda)/d, \ldots, (n+m\lambda)/d, -\lambda, \ldots, -\lambda)$, $H_0 = \cdots = H_n = H$ and $T = H+1$, for any integer $H \ge H_{\boldsymbol{\sigma}}^{1/(n+1)}$ the following system

$$\begin{cases} |b_0 x_i - b_i|_p & < p^{(n+m\lambda)/d} H^{-v_i} \quad (1 \le i \le d), \\ \left| b_0 p^\lambda f_j(\boldsymbol{x}) - \sum\limits_{i=1}^d p^\lambda \dfrac{\partial f_j}{\partial x_i}(\boldsymbol{x})\,(b_0 x_i - b_i) - p^\lambda b_{d+j} \right|_p & < p^{-\lambda} H^{-\tau_j} \quad (1 \le j \le m), \\ \max\limits_{0 \le i \le n} |b_i| & \le H \end{cases} \tag{5.39}$$

has a non-zero integer solution $(b_0, b_1, \ldots, b_n) \in \mathbb{Z}^{n+1}$. Without loss of generality we can assume that $d = \gcd(b_0, b_1, \ldots, b_n)$ is a power of p as otherwise we can divide (5.39) through by any other prime powers in the factorisation of d without affecting (5.39). Let $C > 0$ and $0 < \varepsilon < 1$ be the constants that satisfy Definition 5.2.1 for all f_j simultaneously. In particular, we have that $B(\boldsymbol{x}, \varepsilon) \subseteq \mathcal{U}$. Let $v_{\min} := \min_{1 \le i \le d} v_i$ and $\tau_{\max} := \max_{1 \le j \le m} \tau_j$. Let H_0 be defined as follows

$$H_0 := \max \left\{ \begin{array}{cc} C^{\frac{2}{2v_{\min} - \tau_{\max}}}, & (\alpha 1) \\ C^{\frac{1}{v_{\min}-1}}, & (\alpha 2) \\ (\varepsilon^{-1} p^{(n+m\lambda)/d})^{\frac{1}{v_{\min}}-1}, & (\beta) \\ p^{\frac{n+n\lambda}{d(v_{\min}-1)}}, & (\gamma) \\ H_{\boldsymbol{\sigma}}^{1/(n+1)} & (\delta) \end{array} \right\}.$$

Note that H_0 is a well defined positive real number since $v_{\min} - 1 > 0$ and $2v_{\min} - \tau_{\max} > 0$. The latter follows from the facts that each $\tau_j > 1$ and $\sum_{j=1}^m \tau_j < m+1$ and so $\tau_j < 2$, and the condition that each $v_i > 1$. Note that (γ) implies that $p^{(n+m\lambda)/d} H^{-v_i} < H^{-1}$ whenever $H > H_0$. We will use this observation a few times in this proof.

We will now prove two statements concerning the integer solution $(b_0, b_1, \ldots, b_n)$ to (5.39). First we verify that $b_0 \neq 0$. Suppose the contrary, that is $b_0 = 0$. Then by the first inequality of (5.39) we have that $|b_i|_p < p^{(n+m\lambda)/d} H^{-v_i} < H^{-1}$. As $|b_i| \leq H$ and $H > H_0$, we have that $b_i = 0$ for $1 \leq i \leq d$. Considering the second set of inequalities of (5.39), for each $1 \leq j \leq m$ we have that $|b_{d+j}|_p < H^{-\tau_j}$ which also forces us to conclude that $b_{d+j} = 0$, since $\tau_j > 1$ for each $1 \leq j \leq m$. Thus $(b_0, b_1, \ldots, b_n) = \mathbf{0}$, a contradiction. So we must have that $b_0 \neq 0$.

Now we show that $\frac{b_i}{b_0}$ is a p-adic integer for all $1 \leq i \leq d$. Since $b_0 \neq 0$, we may rewrite the first inequality of (5.39) to get

$$|b_0|_p \left| x_i - \frac{b_i}{b_0} \right|_p < p^{(n+m\lambda)/d} H^{-v_i}, \quad 1 \leq i \leq d.$$

Suppose that $\left| \frac{b_i}{b_0} \right|_p > 1$ for some $1 \leq i \leq d$, then $\left| \frac{b_i}{b_0} \right|_p > |x_i|_p$ since $\boldsymbol{x} \in \mathcal{U} \subseteq \mathbb{Z}_p^d$ so, by the strong triangle inequality, we have that

$$|b_i|_p = |b_0|_p \max \left\{ |x_i|_p, \left| \frac{b_i}{b_0} \right|_p \right\} = |b_0|_p \left| x_i - \frac{b_i}{b_0} \right|_p < p^{(n+m\lambda)/d} H^{-v_i} < H^{-1}$$

for $H > H_0$. Such inequality fails unless $b_i = 0$, since $|b_i| \leq H$. Thus, $\frac{b_i}{b_0} \in \mathbb{Z}_p$ for all $1 \leq i \leq d$.

Now we are ready to construct $(a_0, \ldots, a_n)$ with $(a_0, p) = 1$. Let $k \geq 0$ be the unique integer such that $p^k | b_0$ but $p^{k+1} \nmid b_0$. Then, since $\frac{b_i}{b_0} \in \mathbb{Z}_p$ so we have that $p^k | b_i$ for all $1 \leq i \leq d$. By (5.39), we get that

$$\begin{aligned} |b_{d+j}|_p &\leq \max \left\{ \left| b_0 f_j(\boldsymbol{x}) - \sum_{i=1}^{d} \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) \left(b_0 x_i - b_i\right) - b_{d+j} \right|_p, |b_0 f_j(\boldsymbol{x})|_p, \left| \sum_{i=1}^{d} \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) \left(b_0 x_i - b_i\right) \right|_p \right\} \\ &\leq \max \left\{ H^{-\tau_j}, p^{-k}, p^{\lambda} p^{(n+m\lambda)/d} H^{-v_{\min}} \right\} = p^{-k}, \end{aligned}$$

since $\tau_j > 1$ and $H > H_0$. Therefore, $p^k | b_{d+j}$ and we have that $\frac{b_{d+j}}{b_0} \in \mathbb{Z}_p$ for each $1 \leq j \leq m$. In particular we have that $d = \gcd(b_0, b_1, \ldots, b_n) = p^k$. For $0 \leq i \leq n$ define the numbers $a_i = p^{-k} b_i$, which, by what we have proven above, are all integers satisfying $\gcd(a_0, a_1, \ldots, a_n) = 1$ and, by the choice of k, $(a_0, p) = 1$. By the third inequality of (5.39), we have that $\max_{0 \leq i \leq n} |a_i| \leq p^{-k} H$, which verifies the third inequality in (5.37). Further, using the first set of inequalities of (5.39), we get that

$$|a_0 x - a_i|_p = |p^{-k} b_0 x - p^{-k} b_i|_p = p^k |b_0 x - b_i|_p < p^{(n+m\lambda)/d} p^k H^{-v_i} \tag{5.40}$$

for each $1 \leq i \leq d$, since $v_i > 1$. This verifies the first set of inequalities in (5.37).

By (5.40) and the fact that $p^k \leq H$, we get that

$$\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0} \right) \in B\left(\boldsymbol{x}, p^{(n+m\lambda)/d} H^{-v_{\min}+1} \right) \subseteq B(\boldsymbol{x}, \varepsilon) \subseteq \mathcal{U},$$

where the last inclusion follows from condition (β) on H_0. Thus, $\boldsymbol{y} = \left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) \in \mathcal{U}$ and, in particular, $f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right)$ is well defined and (5.5) is applicable to $f = f_j$ for each $1 \le j \le m$.

Using the fact that each f_j is DQE at $\boldsymbol{x}$ we get that

$$\begin{aligned} \left| f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) - f_j(\boldsymbol{x}) - \sum_{1\le i\le d} \frac{\partial f_j}{\partial x_i}(\boldsymbol{x}) \left(\frac{a_i}{a_0} - x_i\right) \right|_p &< C \max_{1\le i\le d} \left| \frac{a_i}{a_0} - x_i \right|_p^2 \\ &< (p^{-k}H)^{-\tau_j} \end{aligned} \tag{5.41}$$

for each $1 \le j \le m$, where the last inequality follows since

$$\begin{aligned} C \max_{1\le i\le d} \left| \frac{a_i}{a_0} - x_i \right|_p^2 &\overset{(5.40)}{<} Cp^{(2n+2m\lambda)/d} p^{2k} H^{-2v_{\min}} \\ &= Cp^{(2n+2m\lambda)/d} p^{-2k(v_{\min}-1)} (p^{-k}H)^{-2v_{\min}} \\ &\overset{(*)}{\le} (p^{-k}H)^{-\tau_{\max}} \le (p^{-k}H)^{-\tau_j} . \end{aligned}$$

Here $(*)$ follows from condition ($\alpha 1$) on H_0 if $p^k \le H_0^{1/2}$ and it follows from condition ($\alpha 2$) on H_0 if $p^k > H_0^{1/2}$, and we also use the facts that $v_{\min} > 1$ and $2v_{\min} > \tau_{\max}$.

For each $1 \le j \le m$ in the second row of inequalities of (5.39) we may divide through by $p^k = |b_0|_p^{-1}$ and p^λ, and combine with (5.41) to obtain

$$\left| f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) - \frac{a_{d+j}}{a_0} \right|_p < (p^{-k}H)^{-\tau_j}$$

for each $1 \le j \le m$. This verifies the second set of inequalities in (5.37), while the first set of inequalities in (5.37) follows from (5.40). The proof is thus complete. □

In order to use a Mass Transference Principle, namely Theorem 3.3.4, we now establish the following Corollary.

Corollary 5.6.2. *Let $\boldsymbol{f}$, $\boldsymbol{\tau}$ and $\mathbf{v}$ be as in Theorem 5.6.1. Let $\boldsymbol{x} \in \mathcal{U}\backslash\mathbb{Q}^d$ and λ be given by* (5.36)*. Then the following system*

$$\begin{cases} \left| x_i - \frac{a_i}{a_0} \right|_p < p^{(n+m\lambda)/d} h^{-v_i} & (1 \le i \le d), \\ \left| f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) - \frac{a_{d+j}}{a_0} \right|_p < h^{-\tau_j} & (1 \le j \le m), \end{cases} \tag{5.42}$$

where $h = \max\limits_{0\le i\le n} |a_i|$, has infinitely many integer solutions $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ satisfying (5.38)*.*

Proof. First, observe that (5.42) is a consequence of (5.37) since $h = \max_{0\le i\le n} |a_i| \le p^{-k}H$ and $v_i > 1$ for all i. So we only need to verify that there are infinitely many different solutions $(a_0, \ldots, a_n)$ to (5.37) as H varies. Suppose the contrary. Then, since $\boldsymbol{x} \in \mathbb{Z}_p^d \backslash \mathbb{Q}^d$, there is $1 \leq i \leq d$ such that $x_i - \frac{a_i}{a_0} \neq 0$ and so

$$\delta := \min \left| x_i - \frac{a_i}{a_0} \right|_p > 0 \tag{5.43}$$

where the minimum is taken amongst the solutions $(a_0, a_1, \ldots, a_n)$ to (5.37) over all $H \geq H_0$. On the other hand, by (5.37), we have that $\delta < p^{(n+m\lambda)/d} p^k H^{-v_i} \leq p^{(n+m\lambda)/d} H^{-v_i+1} \to 0$ as $H \to \infty$ since $v_i > 1$, giving a contradiction for large H. $\square$

Corollary 5.6.3. *Let $\boldsymbol{f}$, $\boldsymbol{\tau}$ and $\mathbf{v}$ be as in Theorem 5.6.1 and suppose that $\boldsymbol{f}$ is DQE for almost every $\boldsymbol{x} \in \mathcal{U}$. Let $\delta > 0$ be any constant. Then for almost every $\boldsymbol{x} \in \mathcal{U}$ the following system*

$$\begin{cases} \left| x_i - \frac{a_i}{a_0} \right|_p < \delta h^{-v_i} & (1 \leq i \leq d), \\ \left| f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) - \frac{a_{d+j}}{a_0} \right|_p < h^{-\tau_j} & (1 \leq j \leq m), \end{cases} \tag{5.44}$$

where $h = \max\limits_{0 \leq i \leq n} |a_i|$, has infinitely many integer solutions $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ satisfying (5.38).

Proof. Define the set of integer points

$$S_{\boldsymbol{\tau}} = \left\{ (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} : \begin{array}{c} \text{(5.38) holds and for all } 1 \leq j \leq m \\ \left| f_j\left(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0}\right) - \frac{a_{d+j}}{a_0} \right|_p < h^{-\tau_{d+j}}, \\ \text{where } \max\limits_{0 \leq i \leq n} |a_i| = h \end{array} \right\}, \tag{5.45}$$

and for each $\boldsymbol{a} \in S_{\boldsymbol{\tau}}$ and $\delta > 0$ consider the hyperrectangles

$$B_{\boldsymbol{a}}(\boldsymbol{\tau}; \delta) = \left\{ \boldsymbol{x} \in \mathbb{Z}_p^d : \left| x_i - \frac{a_i}{a_0} \right|_p < \delta h^{-\tau_i} \qquad (1 \leq i \leq d) \right\}. \tag{5.46}$$

By Corollary 5.6.2, the set

$$\bigcup_{\delta > 0} \limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\boldsymbol{\tau}; \delta) \tag{5.47}$$

has full measure in $\mathcal{U}$, since the sequence of sets in (5.47) is increasing as δ increases. These are Borel sets and therefore measurable. Hence, by the continuity of measure, we have that

$$\lim_{\delta \to +\infty} \mu_{p,n} \left(\limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\boldsymbol{\tau}; \delta) \right) = \mu_{p,n} \left(\bigcup_{\delta > 0} \limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\boldsymbol{\tau}; \delta) \right) = \mu_{p,n}(\mathcal{U}) \,. \tag{5.48}$$

By Lemma 5.3.3, every limsup set in (5.48) is of the same measure. Hence,

$$\mu_{p,n} \left(\limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\boldsymbol{\tau}; \delta) \right) = \mu_{p,n}(\mathcal{U})$$

for every $\delta > 0$. This is exactly what we have to prove. $\square$

5.6.1 Proof of Theorems 5.2.3–5.2.5

We begin with the following proposition that lays the basis for applying the Mass Transference Principles.

Proposition 5.6.4. *Let* $\boldsymbol{f}:\mathcal{U}\to\mathbb{Z}_p^m$*, where* $\mathcal{U}\subseteq\mathbb{Z}_p^d$ *is an open subset, and for* $\boldsymbol{x}\in\mathcal{U}$ *let* $\mathbf{F}(\boldsymbol{x})=(\boldsymbol{x},\boldsymbol{f}(\boldsymbol{x}))$*. Let* $\mathcal{U}^*$ *be the subset of* $\boldsymbol{x}\in\mathcal{U}$ *such that* $\boldsymbol{f}$ *is DQE at* $\boldsymbol{x}$*. Let* $\boldsymbol{\tau}=(\tau_1,\dots,\tau_n)\in\mathbb{R}_+^n$*. Let* $S_{\boldsymbol{\tau}}$ *and* $B_{\boldsymbol{a}}(\boldsymbol{\tau};\delta)$ *be defined by* (5.45) *and* (5.46) *respectively. Then for any* $0<\delta\le 1$

$$\mathcal{U}^*\cap\limsup_{\boldsymbol{a}\in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\boldsymbol{\tau};\delta)\subset\mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau}))\tag{5.49}$$

provided that

$$\min_{1\le i\le d}\tau_i>\max_{1\le j\le m}\tau_{d+j}\,.\tag{5.50}$$

If

$$\min_{1\le i\le d}\tau_i=\max_{1\le j\le m}\tau_{d+j}\,.\tag{5.51}$$

and $\boldsymbol{f}$ *is a Lipschitz map with the Lipschitz constant* L*, then* (5.49) *holds for any* $0<\delta\le\min\{1,L^{-1}\}$*.*

Proof. Suppose $\boldsymbol{x}\in\mathcal{U}^*\cap B_{\boldsymbol{a}}(\boldsymbol{\tau};\delta)$. Then

$$\begin{aligned}\left|f_j(\boldsymbol{x})-f_j\left(\frac{a_1}{a_0},\dots,\frac{a_d}{a_0}\right)\right|_p&<\max\left\{\max_{1\le i\le d}\left|\frac{\partial f_j(\boldsymbol{x})}{\partial x_i}\right|_p\max_{1\le i\le d}\left|x_i-\frac{a_i}{a_0}\right|_p, C\max_{1\le i\le d}\left|x_i-\frac{a_i}{a_0}\right|_p^2\right\}\\&<\max\left\{\max_{1\le i\le d}\left|\frac{\partial f_j(\boldsymbol{x})}{\partial x_i}\right|_p\delta h^{-\tau_{\min}}, C\delta^2h^{-2\tau_{\min}}\right\}<h^{-\tau_{d+j}}\end{aligned}$$

for any $1\le j\le m$ and all sufficiently large h if (5.50) holds. In turn, if (5.51) holds, we use the fact that $\mathbf{f}$ is Lipschitz:

$$\left|f_j(\boldsymbol{x})-f_j\left(\frac{a_1}{a_0},\dots,\frac{a_d}{a_0}\right)\right|_p<L\max_{1\le i\le d}\left|x_i-\frac{a_i}{a_0}\right|_p<L\delta h^{-\tau_{\min}}\le h^{-\tau_{d+j}}$$

for any $1\le j\le m$ and all sufficiently large h since $0<\delta\le L^{-1}$. In either case, if $\boldsymbol{a}\in S_{\boldsymbol{\tau}}$, then

$$\left|f_j(\boldsymbol{x})-\frac{a_{d+j}}{a_0}\right|_p\le\max\left\{\left|f_j(\boldsymbol{x})-f_j\left(\frac{a_1}{a_0},\dots,\frac{a_d}{a_0}\right)\right|_p,\left|\frac{a_{d+j}}{a_0}-f_j\left(\frac{a_1}{a_0},\dots,\frac{a_d}{a_0}\right)\right|_p\right\}<h^{-\tau_{d+j}}$$

provided that h is sufficiently large. Hence, assuming that $\boldsymbol{x}\in\mathcal{U}^*\cap\limsup_{\boldsymbol{a}\in S_{\boldsymbol{\tau}}}B_{\boldsymbol{a}}(\boldsymbol{\tau};\delta)$ we conclude that the system of inequalities

$$\begin{cases}|a_0x_i-a_i|_p<\delta h^{-\tau_i}\le h^{-\tau_i}, & (1\le i\le d),\\ |a_0f_j(\boldsymbol{x})-a_{d+j}|_p<h^{-\tau_{d+j}} & (1\le j\le m),\\ \max\{|a_0|,\dots,|a_n|\}=h\end{cases}\tag{5.52}$$

holds for infinitely many $\boldsymbol{a}\in\mathbb{Z}^{n+1}$. Therefore, $\boldsymbol{x}\in\mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau}))$ and the proof is complete. □

Proof of Theorems 5.2.3–5.2.4. First of all, note that (5.6) and (5.8) follow from Theorem 5.2.5. Thus we only need to verify the measure part of these theorems, that is (5.7) and (5.9). Consequently, we will assume that $\mathbf{f}$ is Lipschitz on $\mathcal{U}$. Let $0 < \delta \leq \min\{1, L^{-1}\}$, where L is the Lipschitz constant of $\mathbf{f}$. With reference to the Mass Transference Principle from balls to balls (Theorem 3.1.1), take the function $g(x) = x^d$ as our dimension function. Note that g is doubling and that $\mathcal{H}^g \asymp \mu_{p,d}$. For any ball $B = B(x,r)$ and dimension function $f(x) = x^s$, define $B^s = B(x, g^{-1}(x^s))$. Note that in Theorems 5.2.3 and 5.2.4 we have that $\tau_1 = \tau_2 = \cdots = \tau_d$. Therefore the sets $B_{\boldsymbol{a}}(\boldsymbol{\tau}; \delta)$ defined by (5.46) are balls. Let the vector $\mathbf{v} = (v_1, \ldots, v_d)$ be of the form $\mathbf{v} = (v, \ldots, v)$ where

$$v = \frac{n+1-\sum_{i=1}^m \tau_{d+i}}{d}.$$

Note that this $\mathbf{v}$ satisfies the requirements of Theorem 5.6.1 and its corollaries. Let

$$s = \frac{n+1-\sum_{i=1}^m \tau_{d+i}}{\tau_d},$$

Then

$$B^s_{\boldsymbol{a}}(\tau_d; \delta) = \left\{ \boldsymbol{x} \in \mathbb{Z}_p^d : \max_{1 \leq i \leq d} \left| x_i - \frac{a_i}{a_0} \right|_p < \delta^{s/d} h^{-v} \right\},$$

and, by Corollary 5.6.3,

$$\mu_{p,d}\left(\limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B^s_{\boldsymbol{a}}(\tau_d; \delta) \right) = \mu_{p,d}(\mathcal{U}).$$

Hence, for any ball $B \subset \mathcal{U}$,

$$\mathcal{H}^g\left(B \cap \limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B^s_{\boldsymbol{a}}(\tau_d; \delta) \right) = \mathcal{H}^g(B).$$

By the Mass Transference Principle (Theorem 3.1.1), we have that for any ball $B \subseteq \mathcal{U}$,

$$\mathcal{H}^s\left(B \cap \limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B^g_{\boldsymbol{a}}(\tau_d; \delta) \right) = \mathcal{H}^s(B). \tag{5.53}$$

By Proposition 5.6.4 and the choice of δ, we have that (5.49) holds, where $\mathcal{U}^* = \mathcal{U}$. Combining (5.53) and (5.49) gives the required Hausdorff measure results and completes the proof. □

Proof of Theorem 5.2.5. First of all, without loss of generality we can assume throughout this proof that (5.50) holds. Otherwise we could consider $\boldsymbol{\tau'} = (\tau_1 + \varepsilon, \ldots, \tau_d + \varepsilon, \tau_{d+1}, \ldots, \tau_n)$ for a suitably small $\varepsilon > 0$ and note that $\mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau'})) \subset \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau}))$. Hence, the validity of (5.10) for $\boldsymbol{\tau'}$ would give us the bound

$$\dim\left(\mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau}))\right) \geq \dim\left(\mathbf{F}^{-1}\left(\mathfrak{W}_n(\boldsymbol{\tau'})\right)\right) \geq \min_{1 \leq i \leq d} \left\{ \frac{n+1+\sum_{\tau_j < \tau_i}(\tau_i - \tau_j)}{\tau_i + \varepsilon} - m \right\}$$

and on letting $\varepsilon \to 0$ we would get the required result for $\boldsymbol{\tau}$.

Now, since (5.50) holds, by Proposition 5.6.4 with $\delta = 1$, get that

$$\limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\boldsymbol{\tau}; 1) \subset \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau})) \tag{5.54}$$

Corollary 5.6.3 provides us with a full measure statement, which will be the basis for applying the Mass Transference Principle from rectangles to rectangles without Ubiquity (Theorem 3.3.4). With reference to the notation used in Theorem 3.3.4 take

$$J = S_{\boldsymbol{\tau}}, \qquad \rho(q) = q^{-1},$$
$$R_\alpha = \left\{ \left(\tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0} \right) \right\}, \qquad \beta_\alpha = a_0 \qquad \text{for } \alpha = (a_0, \dots, a_n) \in S_{\boldsymbol{\tau}}$$

and so

$$\limsup_{\boldsymbol{a} \in S_{\boldsymbol{\tau}}} B_{\boldsymbol{a}}(\mathbf{v}; 1) = \limsup_{\alpha \in J} \Delta(R_\alpha, \rho(\beta_\alpha)^{-\mathbf{v}}). \tag{5.55}$$

By Corollary 5.6.3 and (5.55), we have that

$$\mu_{p,d}\left(\limsup_{\alpha \in J} \Delta(R_\alpha, \rho(\beta_\alpha)^{-\mathbf{v}}) \right) = \mu_{p,d}(\mathcal{U}) \tag{5.56}$$

for any $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d_+$ satisfying

$$v_i > 1, \quad \sum_{i=1}^{d} v_i = n + 1 - \sum_{j=1}^{m} \tau_j. \tag{5.57}$$

Without loss of generality we will assume that $\tau_1 > \tau_2 > \cdots > \tau_d$. Similarly to what proceeds the proof of Proposition 5.5.1 define each v_i recursively, starting with $r = 0$, by

$$v_{d-r} = \min \left\{ \tau_{d-r}, \frac{n + 1 - \sum_{j=1}^{m} \tau_{d+j} - \sum_{i=d-r+1}^{d} v_i}{d - i} \right\}.$$

Observe that this choice of $\mathbf{v}$ satisfies (5.57). Furthermore, there exists a $1 \leq b \leq d$ such that

$$v_c = \frac{n + 1 - \sum_{j=1}^{m} \tau_{d+j} - \sum_{i=d-b}^{d} v_i}{d - b}$$

for all $1 \leq c \leq d - b$. Define $t_1, \dots, t_d$ from the equations

$$\tau_j = v_j + t_j$$

then note that $\boldsymbol{\tau} = (t_1, \dots, t_d) \in \mathbb{R}^d_{\geq 0}$ and thus satisfies the conditions of Theorem 3.3.4. Thus, the set $W(\boldsymbol{\tau})$, defined in Theorem 3.3.4, is exactly the right hand side of (5.54). Hence, by (5.54), we get that

$$\dim \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau})) \geq \dim W(\boldsymbol{\tau}).$$

Also, in view of (5.56), Theorem 3.3.4 is applicable and so $\dim \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau})) \geq s$, where s is the same as in Theorem 3.3.4. The proof is now split into the following three cases.

i) $A_i \in \{v_1, \dots v_{d-b}\}$: For these values of A_i, which are defined in Theorem 3.3.4, we have that

$$K_1 = \{1, \dots, d - b\}, \quad K_2 = \{d - b + 1, \dots, d\}, \quad K_3 = \emptyset.$$

Applying Theorem 3.3.4 gives

$$\dim \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau})) \geq \dim W(\boldsymbol{\tau}) \geq \min_{1\leq i\leq d-b}\left\{\frac{(d-b)v_i+(d-(d-b+1)+1)v_i-\sum_{j=d-b}^{n}t_j}{v_i}\right\},$$
$$= \min_{1\leq i\leq d-b}\left\{d-\frac{\sum_{j=d-b+1}^{d}t_j}{v_i}\right\}.$$

Since $t_i=0$ for $d-b+1\leq i\leq d$ we have that $\dim \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau}))\geq d$, which is the maximal possible dimension for $\mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau}))$.

ii) $A_i\in\{v_{d-b+1},\dots,v_d\}$: For such values of A_i observe that

$$K_1=\{1,\dots,i\},\quad K_2=\{i+1,\dots,d\},\quad K_3=\emptyset.$$

Then in this case we have that

$$\dim \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau})) \geq \dim W(\boldsymbol{\tau}) \geq \min_{d-b+1\leq i\leq d}\left\{\frac{iv_i+(d-i)v_i-\sum_{j=i+1}^{d}t_j}{v_i}\right\}.$$

Similarly to the previous case, since $t_j=0$ for $d-b+1\leq j\leq d$ the r.h.s of the above equation is d.

iii) $A_i\in\{\tau_1,\dots,\tau_d\}$: Since $\tau_i=v_i$ for $d-b+1\leq i\leq d$, ii) covers such result. So we only need to consider the set of $A_i\in\{\tau_1,\dots\tau_{d-b}\}$. If A_i is contained in such set, then

$$K_1=\emptyset,\quad K_2=\{i,\dots,d\},\quad K_3=\{1,\dots,i-1\}.$$

Thus, by Theorem 3.3.4, we have that

$$\dim \mathbf{F}^{-1}(\mathfrak{W}_n(\boldsymbol{\tau})) \geq \min_{1\leq i\leq d}\left\{\frac{(d-i+1)\tau_i+\sum_{j=1}^{i-1}v_j-\sum_{j=i}^{d}t_j}{\tau_i}\right\},$$
$$= \min_{1\leq i\leq d}\left\{\frac{(d-i+1)\tau_i+(i-1)\left(\frac{n+1-\sum_{j=1}^{m}\tau_{d+j}-\sum_{j=d-b+1}^{d}v_j}{d-b}\right)-\sum_{j=i}^{d-b}(\tau_j-v_j)-\sum_{j=d-b+1}^{d}t_j}{\tau_i}\right\},$$
$$= \min_{1\leq i\leq d}\left\{\frac{(d-i+1)\tau_i+(d-b)\left(\frac{n+1-\sum_{j=1}^{m}\tau_{d+j}-\sum_{j=d-b+1}^{d}v_j}{d-b}\right)-\sum_{j=i}^{d-b}\tau_j-\sum_{j=d-b+1}^{d}t_j}{\tau_i}\right\},$$
$$= \min_{1\leq i\leq d}\left\{\frac{n+1+\sum_{j=i}^{d}(\tau_i-\tau_j)-\sum_{j=1}^{m}\tau_{d+j}}{\tau_i}\right\},$$
$$= \min_{1\leq i\leq d}\left\{\frac{n+1+\sum_{j=i}^{n}(\tau_i-\tau_j)}{\tau_i}-m\right\}.$$

Considering all cases we have that

$$\dim \mathbf{F}^{-1}(\mathfrak{W}(\boldsymbol{\tau})) \geq \dim W(\boldsymbol{\tau}) \geq \min_{1\leq i\leq d}\left\{\frac{n+1+\sum_{j=i}^{n}(\tau_i-\tau_j)}{\tau_i}-m\right\}$$

as required. □

5.7 Final remarks on Theorem 5.2.3–5.2.5

We make several concluding remarks to the results of this paper. As outlined in §5.2 a current obstruction to further results in the p-adic setting over dependent variables is the lack of counting results for rational points near manifolds. Precisely, given a manifold $\mathcal{M}$ with dimension d and codimension m such that $m+d=n$, and functions $f_i : \mathbb{Z}_p^d \to \mathbb{Z}_p$, for $1 \leq i \leq m$, with parametrisation

$$\mathcal{M} := \{(\boldsymbol{x}, f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{Z}_p^d\} \subseteq \mathbb{Z}_p^n,$$

then for an exponent vector $\boldsymbol{\tau} \in \mathbb{R}_+^n$ and fixed $M \in \mathbb{N}$, what bounds can we put on the cardinality of

$$\left\{ (a_0, \dots, a_n) \in \mathbb{Z}^{n+1} : \begin{array}{c} \frac{a_i}{a_0} \in \mathbb{Z}_p \text{ for each } 1 \leq i \leq d, \quad \max_{0 \leq i \leq n} |a_i| \leq M, \\ \left| a_0 f_i\left(\frac{a_1}{a_0}, \dots, \frac{a_d}{a_0}\right) - a_{d+i} \right|_p < M^{-\tau_i}, \ 1 \leq i \leq m \end{array} \right\}. \tag{5.58}$$

In the real simultaneous case such bounds have been found, see [26]. Given an upper bound on (5.58) we would expect the corresponding upper bound of Theorem 5.2.3 to follow.

Secondly, while in this paper we make use of the general MTP and the MTPRR to obtain lower bounds for $\dim \mathfrak{W}(\boldsymbol{\tau}) \cap \mathcal{C}_f$ the stronger ubiquity statement is absent hence we cannot obtain a s-Hausdorff measure result in the full generalised case (Theorem 5.2.5). The main reason being that we do not have a precise enough understanding on the distribution of rational points close to p-adic manifolds. Furthermore, as shown in [14] working from a purely ubiquitous setup can extend the range of applicable τ-approximations to Theorem 5.2.3–5.2.5. While we suspect this would add additional constraints to our set of applicable manifolds we intent to pursue this idea in a further paper.

Chapter 6

Simultaneous p-adic Approximation over coordinate hyperplanes

Recall from the previous chapters, by degenerate we generally mean a curve or surface that is 'flat' for relatively large parts, with respect to the associated measure. In particular any manifold contained within some hyperplane is degenerate. In this chapter we will focus on the special class of degenerate manifold, coordinate hyperplanes. Note from the comments of the previous two chapters, a key notion needed in order to find results in such settings is a bound on the number of rational points close to the manifold. In the p-adic setting such results are few and far between. In this chapter we find bounds on the number of rational points close to n-dimensional p-adic integers by using p-adic approximation lattices. This respectively allows us to obtain a Hausdorff dimension result for almost all coordinate hyperplanes with respect to the Haar measure on $\mathbb{Z}_p$.

6.1 Counting rational points close to p-adic integers

The study of rational points on algebraic varieties, usually called Diophantine geometry, has a wide variety of applications in many areas of mathematics. A variation of this is the study of rational points that lie close to such algebraic varieties. In the setting of $\mathbb{R}^n$ there has been many results of this type, including counts on the number of rational points close to curves [19, 110, 99, 100, 72] and manifolds [14, 26, 69, 70]. In the p-adic setting less is known. In [10, 11] a bound on the number of rational points that lie on the curve $\mathcal{C}_f = \{(x, x^2, \dots, x^n) : x \in \mathbb{Z}_p\}$ was found, but as yet no other results are available. In this paper we provide an upper and lower bound on the number of rational points within a small neighbourhood of a p-adic integer. Such result allows us to find bounds on the number of rational points close to p-adic coordinate hyperplanes.

Fix a prime number $p \in \mathbb{N}$ and let $|.|_p$ denote the p-adic norm. Define the set of p-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with respect to the p-adic norm. Denote by $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the ring of p-adic integers. Let $\boldsymbol{x} \in \mathbb{Z}_p^n$, $N \in \mathbb{N}$, and $\Psi = (\psi_1, \ldots, \psi_n)$ be an n-tuple of approximation functions of the form $\psi_i : \mathbb{N} \to \mathbb{R}_+$, with $\psi_i(q) \to 0$ as $q \to \infty$ for each $1 \leq i \leq n$. We provide bounds on the cardinality of the set

$$\mathcal{Q}(\boldsymbol{x}, \Psi, N) := \left\{ (q_0, q_1, \ldots, q_n) \in \mathbb{Z}^{n+1} : \begin{array}{c} 0 < q_0 \leq N, \\ \max_{1 \leq i \leq n} |q_i| \leq N, \end{array} \quad |q_0 x_i - q_i|_p < \psi_i(N),\ 1 \leq i \leq n \right\}.$$

If the approximation functions ψ_i are of the form $\psi_i(q) = q^{-\tau_i}$ for some vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_{>0}$ we will use the notation $\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}, N)$. Note that to get a result for general $\boldsymbol{x} \in \mathbb{Z}_p^n$ we must apply some conditions. For example, if $\boldsymbol{x} \in \mathbb{Q}^n$ then for sufficiently large $N \in \mathbb{N}$ we have that $\#\mathcal{Q}(\boldsymbol{x}, \Psi, N) \asymp N^2$ for any Ψ. Here $a \asymp b$ means there exists constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that $c_1 b \leq a \leq c_2 b$. Conversely, if $\boldsymbol{x}$ is badly approximable each approximation function satisfies $\psi_i(q) < q^{-1-\frac{1}{n}-\epsilon}$ for some $\epsilon > 0$, then $\#\mathcal{Q}(\boldsymbol{x}, \Psi, N) \ll 1$. In order to obtain good bounds on the cardinality of $\mathcal{Q}(\boldsymbol{x}, \Psi, N)$ we use the Diophantine exponent $\tau(\boldsymbol{x})$ defined as

$$\tau(\boldsymbol{x}) := \sup \left\{ \sum_{i=1}^{n} \tau_i : |q_0 x_i - q_i|_p < Q^{-\tau_i}, \quad \text{for i.m. } Q \in \mathbb{N} \text{ with } |q_i| \leq Q \right\}.$$

By a Theorem of Mahler [105] we have that for all $x \in \mathbb{Z}_p$, $\tau(x) \geq 2$. Further, by a result of Jarnik [76] we have that $\tau(\boldsymbol{x}) = n + 1$ for almost all $\boldsymbol{x} \in \mathbb{Z}_p^n$, with respect to the n-dimensional Haar measure $\mu_{p,n}$ on $\mathbb{Q}_p^n$, normalised by $\mu_{p,n}(\mathbb{Z}_p^n) = 1$.

We have the following result on the cardinality of $\mathcal{Q}(x, \psi, N)$ for general $x \in \mathbb{Z}_p$.

Lemma 6.1.1. *Let $x \in \mathbb{Z}_p$ with Diophantine exponent $\tau(x)$ and let $\psi(q) = q^{-\tau}$ for some $\tau \in \mathbb{R}_+$ with* $\max\{1, \tau(x) - 1\} < \tau < \tau(x)$. *Then for any $\epsilon > 0$ there exists sufficiently large $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$*

$$\#\mathcal{Q}(x, \tau, N) \leq N^{\tau(x) - \tau + \epsilon}.$$

Remark 6.1.2. This result provides us with an analogous result of Huxley's estimate (see §7.2) in the setting of p-adic coordinate hyperplanes. Note by our previous remark on the Diophantine exponent that for almost all $x \in \mathbb{Z}_p$ we have $\tau(x) = 2$, so the above lemma reads that for $\psi(q) = q^{-\tau}$ with $1 < \tau < 2$, then for almost all $x \in \mathbb{Z}_p$

$$\#\mathcal{Q}(x, \psi, N) \leq N^{2-\tau+\epsilon}.$$

While Lemma 6.1.1 gives us an upper bound for all $x \in \mathbb{Z}_p$ provided the approximation function ψ is 'close' to the function related to the Diophantine exponent the bound given has an extra N^ϵ term. The notion of $\tau(x)$ is inherently connected to some ϵ term, and so for the majority of functions the N^ϵ term of Lemma 6.1.1 cannot be removed. However, by bounding the approximation functions away from the $\tau(\boldsymbol{x})$ exponent we can remove such term. The following theorem offers an improvement in this respect.

Theorem 6.1.3. *Let $\boldsymbol{x} \in \mathbb{Z}_p^n$ and suppose that $\tau(\boldsymbol{x}) = n+1$. Let Ψ be an n-tuple of approximation functions with each*

$$q^{-1-\frac{1}{n}+\epsilon} < \psi_i(q) < q^{-1}, \quad 1 \leq i \leq n,$$

for some $\epsilon > 0$. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\#\mathcal{Q}(\boldsymbol{x}, \Psi, N) \leq C_1 N^{n+1} \prod_{i=1}^{n} \psi_i(N),$$

where

$$C_1 = \max\left\{ 3(6\sqrt{n})^n, \frac{(n+2)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)} \right\}.$$

Remark 6.1.4. Akin to the comparison between Lemma 6.1.1 and Huxley's estimate, Theorem 6.1.3 provides the p-adic coordinate hyperplane analogue of the counting result proven by Vaughan and Velani [110](see §7.2 for more details). As with Lemma 6.1.1, we can deduce that the above upper bound is true for almost all $\boldsymbol{x} \in \mathbb{Z}_p^n$. This type of result has already been proven in the real case (see Lemma 6.1 of [20]).

Remark 6.1.5. In the case where the approximation functions are of the form $\psi_i(q) = q^{-\tau_i}$ then the theorem reads: if

$$\sum_{i=1}^{n} \tau_i < n+1, \quad \text{and} \quad \tau_i > 1,$$

then for any $\boldsymbol{x} \in \mathbb{Z}_p^n$ with $\tau(\boldsymbol{x}) = n+1$,

$$\#\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}, N) \leq C_1 N^{n+1-\sum_{i=1}^{n}\tau_i}.$$

Lastly, we have the following lemma which provides a complimentary lower bound to the previous two results.

Lemma 6.1.6. *Let $\boldsymbol{x} \in \mathbb{Z}_p^n$ and*

$$\sum_{i=1}^{n} \tau_i < n+1, \quad \textit{and} \quad \tau_i > 1$$

for each $1 \leq i \leq n$. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have that

$$\#\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}, N) \geq \frac{1}{p} N^{n+1-\sum_{i=1}^{n}\tau_i} - 1.$$

As with Theorem 6.1.3, the equivalent version of this result in $\mathbb{R}^n$ has previously been proven, (see Lemma 3 of [95]). Further, as $\sum_{i=1}^{n}\tau_i < n+1$ we can choose N large enough such that

$$\#\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}, N) \geq \frac{1}{2p} N^{n+1-\sum_{i=1}^{n}\tau_i}.$$

Thus combining this with Theorem 6.1.3 we have the expected result that $\#\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}, N) \asymp N^{n+1-\sum_{i=1}^{n}\tau_i}$.

The proofs of Lemma 6.1.1 and Lemma 6.1.6 use elementary techniques. The proof of Theorem 6.1.3 is more substantial and uses p-adic approximation lattices and lattice counting techniques. Prior to the proofs of these results we give an example of their applications in Diophantine approximation.

6.2 p-adic Diophantine approximation on coordinate hyperplanes

As an application of the main results in the previous section we consider the set of p-adic simultaneously approximable points over coordinate hyperplanes. Recall the set of weighted simultaneously approximable points, as defined by Haynes [65], as follows. For an n-tuple of approximation functions $\Psi = (\psi_1, \ldots, \psi_n)$ and $q_0 \in \mathbb{N}$ let

$$\mathfrak{A}'_{q_0}(\Psi) = \bigcup_{\substack{|q_i| \leq q_0,\ gcd(q_i, q_0)=1 \\ 1 \leq i \leq n}} \left\{ \boldsymbol{x} \in \mathbb{Z}_p^n : \left| x_i - \frac{q_i}{q_0} \right|_p < \psi_i(q_0) \right\},$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$. Define the set of weighted Ψ-approximable p-adic points as

$$\mathfrak{W}'_n(\Psi) := \limsup_{q_0 \to \infty} \mathfrak{A}'_{q_0}(\Psi).$$

Note that we adopt Haynes setting of taking approximations by reduced fractions.

In the previous chapter a lower bound for the Hausdorff dimension was found for general n-dimensional manifolds satisfying the DQE property (see Definition 5.2.1). We remark here that the lower bound dimension result of the following theorem is already proven by Theorem 5.2.5. However, the Hausdorff s-measure result and the upper bound dimension result are new. A key reason the upper bound could not be obtained in the previous chapter was a lack in results on the behaviour of rational points close to p-adic manifolds. The main results of this chapter provide us with a good understanding of the behaviour of rational points close to coordinate hyperplanes and so the upper bound result is achievable. The results of this section are closely related to a variety of results in the real case on Diophantine approximation over coordinate hyperplanes, see [22, 94, 95].

For a p-adic integer $\boldsymbol{\alpha} \in \mathbb{Z}_p^m$ for $1 \leq m \leq n-1$ define the coordinate hyperplane

$$\Pi_{\boldsymbol{\alpha}} := \{(x_1, \ldots, x_d, \boldsymbol{\alpha}) : (x_1, \ldots, x_d) \in \mathbb{Z}_p^d\} \subset \mathbb{Z}_p^n,$$

where $n = d + m$. For the set $\mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$ we have the trivial result that

$$\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} \leq \dim \Pi_{\boldsymbol{\alpha}} = n - m,$$

with equality when $\sum_{i=1}^{n} \tau_i \leq n + 1$. In this paper we prove the following result on the Hausdorff dimension of $\mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$.

Theorem 6.2.1. *Let $\Pi_{\boldsymbol{\alpha}}$ be a coordinate hyperplane of $\mathbb{Z}_p^n$, let $\boldsymbol{\alpha} \in \mathbb{Z}_p^m$ satisfy $\tau(\boldsymbol{\alpha}) = m+1$. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$ be a weight vector with the properties that*

$$m + \frac{m}{n} \leq \sum_{i=1}^{m} \tau_{d+i} < m+1, \quad \sum_{i=1}^{n} \tau_i > n+1, \quad \tau_i > 1,$$

for all $1 \leq i \leq n$. Then

$$\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = \min_{1 \leq i \leq d} \left\{ \frac{n+1-\sum_{i=1}^{m} \tau_{d+i} + \sum_{\substack{\tau_j \leq \tau_i \\ 1 \leq j \leq d}} (\tau_i - \tau_j)}{\tau_i} \right\} = s.$$

Furthermore

$$\mathcal{H}^s\left(\mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}\right) = \infty.$$

Remark 6.2.2. The constraints on $(\tau_{d+1}, \ldots, \tau_n)$ ensure that we can apply Theorem 6.1.3. The condition that $\sum_{i=1}^{n} \tau_i > n+1$ ensures that we do not include the trivial case when $\mathfrak{W}'_n(\boldsymbol{\tau}) = \mathbb{Z}_p^n$, in which case $\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = n - m$.

Remark 6.2.3. In the special case where the approximation functions are the same i.e. ($\boldsymbol{\tau} = (\tau, \ldots, \tau)$), then we have that, for $1 + \frac{1}{n} < \tau < 1 + \frac{1}{m}$,

$$\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = \frac{n+1}{\tau} - m,$$

which is the dimension of the set of $\boldsymbol{\tau}$-approximable points less the codimension of the hyperplane.

Remark 6.2.4. We can use the same style of proof used to prove the upper bound of Theorem 6.2.1, in combination with Lemma 6.1.1 rather than Theorem 6.1.3, to prove that for any $\alpha \in \mathbb{Z}_p$ and approximation exponent $\max\{1, \tau(\alpha) - 1\} < \tau_n < \tau(\alpha)$ we have that

$$\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\alpha} \leq \min_{1 \leq i \leq n-1} \left\{ \frac{n + \tau(\alpha) - 1 - \tau_n + \sum_{\substack{\tau_j \leq \tau_i \\ j \neq n}} (\tau_i - \tau_j),}{\tau_i} \right\}.$$

Proving the corresponding lower bound of this result is currently beyond our reach. This is because we do not have a complimentary lower bound to Lemma 6.1.1.

For general approximation functions $\Psi = (\psi_1, \ldots, \psi_n)$, let

$$v_i = \lim_{q \to \infty} \frac{-\log(\psi(q))}{\log q}. \tag{6.1}$$

Providing the limits exists and are positive and finite for each $1 \leq i \leq n$ then define $\Psi^* = (v_1, \ldots, v_n)$.

Corollary 6.2.5. *Let $\Psi = (\psi_1, \ldots, \psi_n)$ be an n-tuple of approximation functions with each ψ_i having positive finite limit* (6.1). *If Ψ^* satisfy the same conditions as in Theorem 6.2.1, then for all $\boldsymbol{\alpha} \in \mathbb{Z}_p^m$*

with $\tau(\boldsymbol{\alpha}) = m+1$,

$$\dim \mathfrak{W}'_n(\Psi) \cap \Pi_{\boldsymbol{\alpha}} = \min_{1 \leq i \leq d} \left\{ \frac{n+1-\sum_{i=1}^{m} \psi^*_{d+i} + \sum_{\substack{v_j < v_i \\ 1 \leq j \leq d}} (v_i - v_j)}{v_i} \right\}.$$

The corollary easily follows from the observation that by the definition of (6.1) there exists sufficiently large $q \in \mathbb{N}$ such that

$$q^{-v_i - \epsilon_i} \leq \psi_i(q) \leq q^{-v_i + \epsilon_i}$$

for all $1 \leq i \leq n$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n) > 0$ with $\epsilon_i \to 0$ as $q \to \infty$. And so

$$\mathfrak{W}'_n(\Psi^* + \epsilon) \subseteq \mathfrak{W}'_n(\Psi) \subseteq \mathfrak{W}'_n(\Psi^* - \epsilon).$$

Letting $\epsilon \to 0$ we obtain the desired result. Note that while Corollary 6.2.5 provides a result for general Ψ with components satisfying (6.1), there are many functions where such limits do not exist.

As a reference of auxiliary results and concepts used in the proof of Theorem 6.2.1 we refer the reader to §4.2 of the previous chapter for a recap of key results. One of particular importance is the Mass Transference Principle from rectangles to rectangles, which can be found in Chapter 3.

6.3 Proof of Theorem 6.2.1

We split the proof into the upper and lower bound, and solve each case separately. In both cases we will use the following simplified set. Let π be the projection $\pi : \mathbb{Z}_p^n \to \mathbb{Z}_p^{n-m}$, defined by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_d).$$

By a well known theorem of Hausdorff theory (see Proposition 3.3 of [59]) as π is a bi-Lipschitz mapping over $\mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$, we have that

$$\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = \dim \pi(\mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}).$$

Let $\boldsymbol{\tau}_m = (\tau_{d+1}, \dots, \tau_n)$ denote the m-tuple of approximation exponents of $\boldsymbol{\alpha}$ and similarly let $\boldsymbol{\tau}_d = (\tau_1, \dots, \tau_d)$ denote the d-tuple of approximation exponents of the independent variables of $\Pi_{\boldsymbol{\alpha}}$. Consider the set of integers

$$\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m) := \left\{ q_0 \in \mathbb{N} : \left| \alpha_i - \frac{q_{d+i}}{q_0} \right|_p < q_0^{-\tau_{d+i}}, \text{ for some } \begin{array}{c} |q_i| \leq q_0, \\ gcd(q_i, q_0) = 1, \end{array} \ 1 \leq i \leq m \right\},$$

and the union of sets

$$\mathfrak{A}^*_{q_0}(\boldsymbol{\tau}_d) = \bigcup_{\substack{|q_i| \leq q_0, \ gcd(q_i, q_0) = 1 \\ 1 \leq i \leq d}} \left\{ \boldsymbol{x} \in \mathbb{Z}_p^d : \left| x_i - \frac{q_i}{q_0} \right|_p < q_0^{-\tau_i} \right\}.$$

Note that this is essentially the same set as $\mathfrak{A}'_{q_0}(\boldsymbol{\tau})$, except that we are working with the d-dimensional space rather than the whole n-dimensional space, hence the $*$ notation. Then,

$$\pi(\mathcal{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}) = \limsup_{q_0 \in \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m)} \mathfrak{A}^*_{q_0}(\boldsymbol{\tau}_d),$$

hence we only need to find the upper and lower bounds for $\dim \limsup_{q_0 \in \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m)} \mathfrak{A}^*_{q_0}(\boldsymbol{\tau}_d)$.

6.3.1 Upper bound

For the upper bound we take the standard cover of hyperrectangles used in the construction of $\mathfrak{A}^*_q(\boldsymbol{\tau}_d)$. By a standard geometrical argument note that each hyperrectangle, centred at some $\left(\frac{q_1}{q}, \dots, \frac{q_n}{q_0}\right) \in \mathbb{Q}^d$ in the construction of $\mathfrak{A}^*_q(\boldsymbol{\tau}_d)$, can be covered by a finite collection of balls $\mathfrak{B}_q(\tau_i)$ of radius $q^{-\tau_i}$ for $1 \leq i \leq d$. Without loss of generality we can assume that

$$\tau_1 \geq \dots \geq \tau_d,$$

since if not then we could take some bi-Lipschitz mapping to reorder the coordinate axes such that this was the case. Hence for each $j \leq i$,

$$\frac{q^{-\tau_j}}{q^{-\tau_i}} \leq 1.$$

Hence in the product below we only consider the $j \geq i$. By the above argument we have that the cardinality of $\mathfrak{B}_q(\tau_i)$ is

$$\#\mathfrak{B}_q(\tau_i) \ll \prod_{j=i}^{d} \frac{q^{-\tau_j}}{q^{-\tau_i}} = q^{\sum_{j=i}^{d}(\tau_i - \tau_j)}.$$

As each τ_i-approximation function is decreasing as q increases, for each interval $2^k \leq q < 2^{k+1}$ take $q = 2^k$ over such interval. Let

$$\mathcal{Q}'(\boldsymbol{x}, \boldsymbol{\tau}_m, N) := \{q_0 \in \mathbb{N} : (q_0, \dots, q_m) \in \mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}_m, N) \text{ and } gcd(q_i, q_0) = 1 \, \forall \, 1 \leq i \leq n\}.$$

Since each $\tau_i > 1$ for $1 \leq i \leq m$ each q_0 has unique associated $(q_1, \dots, q_m)$ in $\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}_m, N)$ so we have that $\#\mathcal{Q}'(\boldsymbol{x}, \boldsymbol{\tau}_m, N) \leq \#\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\tau}_m, N)$. Further, by the coprimality of each q_i with q_0 note that the inequalities

$$|q_0 x_i - q_i|_p < H^{-\tau_i}, \quad \text{and} \quad \left| x_i - \frac{q_i}{q_0} \right|_p < H^{-\tau_i}$$

are equivalent since $p \nmid q_0$. To check this observe that each $x_i \in \mathbb{Z}_p$ and then use the strong triangle inequality.

Given the above we have that $\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m) \subseteq \bigcup_{k\in\mathbb{N}} \mathcal{Q}'(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, 2^k)$. Hence for any $k_0 \geq 1$

$$\begin{aligned}
\mathcal{H}^s\left(\limsup_{q\in\mathcal{Q}(\boldsymbol{\alpha},\boldsymbol{\tau}_m)} \mathcal{A}_q^*(\boldsymbol{\tau}_d)\right) &\leq \sum_{k=k_0}^{\infty} \sum_{q\in\mathcal{Q}(\boldsymbol{\alpha},\boldsymbol{\tau}_m,2^k)} \varphi(q)^d \#\mathfrak{B}_q(\tau_i).(q^{-\tau_i})^s, \\
&\overset{\text{Theorem 6.1.3}}{\ll} \sum_{k=k_0}^{\infty} 2^{k(m+1-\sum_{i=1}^m \tau_{d+i})}(2^{k+1})^d (2^{k+1})^{\sum_{j=i}^d(\tau_i-\tau_j)}(2^k)^{-\tau_i s}, \\
&\ll \sum_{k=k_0}^{\infty} 2^{k(n+1-\sum_{i=1}^m \tau_{d+i}+\sum_{j=i}^d(\tau_i-\tau_j)-\tau_i s)},
\end{aligned}$$

The above sum converges when

$$s \geq \frac{n+1-\sum_{i=1}^m \tau_{d+i}+\sum_{j=i}^d(\tau_i-\tau_j)}{\tau_i} + \epsilon,$$

for any $\epsilon > 0$. Thus the tail end of the summation must converge to zero i.e. as $k_0 \to \infty$ the above summation calculation tends to zero. This is true for each $1 \leq i \leq d$, and as ϵ is arbitrary, we have that

$$s \geq \min_{1\leq i\leq d}\left\{\frac{n+1-\sum_{i=1}^m \tau_{d+i}+\sum_{j=i}^d(\tau_i-\tau_j)}{\tau_i}\right\},$$

completing the upper bound result. Note that the result of Remark 2.2 can similarly be obtained by replacing Theorem 6.1.3 by Lemma 6.1.1.

6.3.2 Lower bound

In order to use Theorem 3.3.3 to prove the lower bound of Theorem 6.2.1 we need to construct a ubiquitous system of rectangles. In following with the ubiquity setup for Theorem 3.3.3 let

$$\begin{gathered}
J = \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m), \quad R_{q,i} = \left\{\tfrac{q_i}{q} \in \mathbb{Q} : \begin{array}{c} |q_i| \leq q, \\ gcd(q_i, q) = 1 \end{array}\right\}, \qquad R_q = \textstyle\prod_{i=1}^d R_{q,i}, \\
\beta(q) = q, \qquad \rho(q) = q^{-1}, \qquad l_k = M^k,\ u_k = M^{k+1},
\end{gathered}$$

where $M \in \mathbb{N}$ is a fixed integer to be determined later. Then we have that

$$J_k = \{q \in \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m) : M^k \leq q < M^{k+1}\}.$$

Note that $J_k \subseteq \mathcal{Q}'(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, 2^{k+1})$. For a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ let

$$\Delta(R_q, \rho(r)^{\boldsymbol{a}}) = \prod_{i=1}^{n} \bigcup_{q_i\in R_{q,i}} B\left(\frac{q_i}{q}, r^{-a_i}\right).$$

We prove the following.

Proposition 6.3.1. *Let R_q, ρ, and J_k be as above, and let $\tilde{v} = (v_1, \ldots, v_d) \in \mathbb{R}_{>0}^d$ with each $v_i > 1$ and*

$$\sum_{i=1}^{d} v_i = n+1-\sum_{i=1}^{m} \tau_{d+i},$$

for

$$m + \frac{m}{n} \leq \sum_{i=1}^{m} \tau_{d+i} < m+1$$

and each $\tau_i > 1$*. Then for any ball* $B = B(x,r) \subset \mathbb{Z}_p^d$*, with centre* $x \in \mathbb{Z}_p^d$ *and radius* $0 < r < r_0$ *for some* $r_0 \in \mathbb{R}_+$*, there exists a constant* $c > 0$ *such that*

$$\mu_{p,d}\left(B \cap \bigcup_{q \in J_k} \Delta(R_{\boldsymbol{q}}, \rho(u_k)^{\tilde{v}})\right) \geq c\mu_{p,d}(B),$$

provided $M > (3^d C_1)^{\frac{1}{n+1-\sum_{i=1}^{m}\tau_{d+i}}}$.

The proof of this result follows the same style of many similar results in $\mathbb{R}^n$. For example see Theorem 1.3 of [25] for the one dimensional real case, or Proposition 5.1 of [24] for the n-dimensional p-adic case.

Proof. Fix some ball $B = B(\boldsymbol{y}, r)$ for some $\boldsymbol{y} \in \mathbb{Z}_p^n$ and $r \in \{p^i : i \in \mathbb{N} \cup \{0\}\}$. We will assume that k is sufficiently large so that $M^k r \geq 1$. For any $y = (y_1, \ldots, y_d) \in (\mathbb{Z}_p \backslash \mathbb{Q})^d$, consider the system of inequalities

$$\begin{cases} |q_0\alpha_i - q_{d+i}|_p < (M^{k+\frac{1}{n+1}})^{-\tau_{d+i}}, \quad 1 \leq i \leq m, \\ |q_0 y_i - q_i|_p < p^{\frac{n}{d}}(M^{k+\frac{1}{n+1}})^{-v_i}, \quad 1 \leq i \leq d, \\ \max_{1\leq i \leq n}|q_i| \leq M^k, \\ |q_0| \leq M^{k+1}. \end{cases} \tag{6.2}$$

By the condition on $\tilde{v}$ we have, by Lemma 5.3.1, that there exists a non-zero integer solution $(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$ to (6.2) for any $\boldsymbol{y} \in B$. Furthermore, note that if $(q_0, \ldots, q_n)$ solves (6.2) then $q_0 \in \mathcal{Q}'(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^{k+\frac{1}{n+1}})$. We can assume without loss of generality that $q_0 \geq 0$, and furthermore that $q_0 \neq 0$ since each $v_i, \tau_{d+i} > 1$ for all $1 \leq i \leq d$ and $1 \leq j \leq m$. As we wish to have a statement for rectangles, rather than linear forms, we need to divide through by $|q_0|_p$. To ensure we do not divide by a value too large we remove the set of q_0 that are 'too prime', that is all q_0 such that $|q_0|_p \leq p^{-\lambda_0}$ for some fixed $\lambda_0 \in \mathbb{N}$. Since $v_i, \tau_{d+j} > 1$ for each $1 \leq i \leq d$ and $1 \leq j \leq m$, (6.2) combined with $0 < q_0 \leq M^{k+1}$ implies that $|q_i|_p \leq |q_0|_p$ for each $1 \leq i \leq n$, provided that k is sufficiently large. Let λ be the integer such that $|q_0|_p = p^{-\lambda}$. Write $q_0' = q_0 p^{-\lambda}$ and $q_i' = q_i p^{-\lambda}$. Observe that $q_0', q_i' \in \mathbb{Z}$,

$$(q_0', q_{d+1}', \ldots, q_n') \in \mathcal{Q}\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^{\lambda}}\right), \quad 0 < q_0' \leq p^{-\lambda}M^{k+1}, \quad |q_i'| \leq p^{-\lambda}M^k, \tag{6.3}$$

for each $1 \leq i \leq n$ and that

$$\begin{aligned} \left|y_i - \frac{q_i'}{q_0'}\right|_p &= p^{\lambda}|q_0 y_i - q_i|_p \\ &< p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-v_i}, \end{aligned} \tag{6.4}$$

for $1 \le i \le d$. The same is true for the inequalities on $\boldsymbol{\alpha}$. At this point we only want the $\frac{\boldsymbol{q}}{q_0} = \left(\frac{q_1}{q_0}, \dots, \frac{q_d}{q_0}\right) \in R_{q_0}$ such that

$$B \cap \prod_{i=1}^{d} B\left(\frac{q_i}{q_0}, \rho(M^{k+1})^{v_i}\right) \neq \emptyset.$$

This is equivalent to the set of solutions to

$$\left| y_i - \frac{q_i}{q_0} \right|_p < r, \quad 1 \le i \le d. \tag{6.5}$$

For q_0 fixed and each $|q_i| \le q_0$ by congruence classes we have that there are at most

$$(2q_0 r + 1)^d < 3^d M^{kd} r^d$$

suitable values of $\boldsymbol{q}$ solving (6.5). Hence we have that

$$\begin{aligned}
\mu_{p,d}\left(B \cap \bigcup_{\substack{\lambda \ge \lambda_0 \\ 0<a_0' \le \frac{M^{k+1}}{p^\lambda} :\, a_0' \in \mathcal{Q}'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^\lambda}\right)}} \bigcup_{|a_i'| \le \frac{M^k}{p^\lambda}} \bigcup_{\left(\frac{a_1'}{a_0'}, \dots, \frac{a_d'}{a_0'}\right) \in B} \prod_{i=1}^{d} B\left(\frac{a_i'}{a_0'}, p^{\lambda + \frac{n}{d}} \left(M^{k+\frac{1}{n+1}}\right)^{-v_i}\right) \right) \\
\le \sum_{\lambda \ge \lambda_0} \#\mathcal{Q}'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^\lambda}\right) \left(3\frac{M^k}{p^\lambda} r\right)^d p^{d\lambda + n} M^{-k(n+1-\tilde{\tau})-1+\frac{\tilde{\tau}}{n+1}}, \\
\overset{\text{Theorem 6.1.3}}{\le} \sum_{\lambda \ge \lambda_0} C_1 \left(\frac{M^{k+1}}{p^\lambda}\right)^{m+1-\tilde{\tau}} 3^d \left(\frac{M^k}{p^\lambda}\right)^d p^{d\lambda+n} M^{-\left(k(n+1-\tilde{\tau})+1-\frac{\tilde{\tau}}{n+1}\right)} \mu_{p,d}(B) \\
= \sum_{\lambda \ge \lambda_0} \mu_{p,d}(B) C_1 3^d p^{n-\lambda(m+1-\tilde{\tau})} M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}}, \\
\le C_1 3^d M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}} \frac{p^{n+(1-\lambda_0)(m+1-\tilde{\tau})}}{p^{m+1-\tilde{\tau}}-1} \mu_{p,d}(B).
\end{aligned}$$

Take λ_0 sufficiently large, say

$$p^{\lambda_0} > \left(2\frac{C^1 3^d p^{n+m+1-\tilde{\tau}} M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}}}{p^{m+1-\tilde{\tau}}-1}\right)^{\frac{1}{m+1-\tilde{\tau}}}.$$

Observe that since $\tilde{\tau} \ge m + \frac{m}{n}$ the fact that λ_0 is dependent on M is irrelevant since the value decreases as M increases, hence we could replace $M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}}$ by 1. Then we have that

$$\mu_{p,d}\left(B \cap \bigcup_{\substack{\lambda \ge \lambda_0 \\ 0<a_0' \le \frac{M^{k+1}}{p^\lambda} :\, a_0' \in \mathcal{Q}'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^\lambda}\right)}} \bigcup_{|a_i| \le \frac{M^k}{p^\lambda}} \bigcup_{\left(\frac{a_1'}{a_0'}, \dots, \frac{a_d'}{a_0'}\right) \in B} \prod_{i=1}^{d} B\left(\frac{a_i'}{a_0'}, p^{\lambda + \frac{n}{d}} \left(M^{k+\frac{1}{n+1}}\right)^{-v_i}\right) \right) \le \frac{1}{2}\mu_{p,d}(B).$$

and so

$$\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ 0<a_0'\le M^{k+1}:\ a_0'\in\mathcal{Q}'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |a_0|_p\ge p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)$$
$$\ge\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ 0<a_0'\le M^{k+1}:\ a_0'\in\mathcal{Q}'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right)}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)$$
$$-\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ 0<a_0'\le M^{k+1}:\ a_0'\in\mathcal{Q}'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |a_0|_p<p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)$$
$$\ge\frac{1}{2}\mu_{p,d}(B).$$

Similarly to the above we can deduce that

$$\begin{aligned}&\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ M^k<a_0\le M^{k+1}:\ a_0\in\mathcal{Q}'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |a_0|_p\ge p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)\\
&\quad\ge\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ 0<a_0<M^{k+1}:\ a_0\in\mathcal{Q}'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |a_0|_p\ge p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)\\
&\quad-\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ 0<a_0\le M^{k}:\ a_0\in\mathcal{Q}'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k}\right),\ |a_0|_p\ge p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right).\end{aligned}\tag{6.6}$$

Note that we are justified in taking the set $\mathcal{Q}'(\boldsymbol{\alpha},\boldsymbol{\tau},M^k)$ rather than $\mathcal{Q}(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1})$ in the last row of the (6.6) since

$$\left\{0<a_0\le M^k : a_0\in\mathcal{Q}(\boldsymbol{\alpha},\boldsymbol{\tau},M^k)\right\}\supseteq\left\{0<a_0\le M^k : a_0\in\mathcal{Q}(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1})\right\}.$$

Calculating the measure of the 3rd row of (6.6) gives us that

$$\begin{aligned}
&\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ 0<a_0\le M^k:\ a_0\in\mathcal{Q}'(\boldsymbol{\alpha},\boldsymbol{\tau},M^k),\ |a_0|_p\ge p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)\\
&\qquad\le \#\mathcal{Q}'(\boldsymbol{\alpha},\boldsymbol{\tau},M^k)\left(3M^kr\right)^d p^{d\lambda_0+n}M^{-\left(k(n+1-\tilde{\tau})+1-\frac{\tilde{\tau}}{n+1}\right)}\\
&\qquad\overset{\text{Theorem 6.1.3}}{\le} C_1M^{k(m+1-\tilde{\tau})}3^dM^{kd}p^{d\lambda_0+n}M^{-\left(k(n+1-\tilde{\tau})+1-\frac{\tilde{\tau}}{n+1}\right)}\mu_{p,d}(B)\\
&\qquad\le 3^dp^{d\lambda_0+n}M^{-(1-\frac{\tilde{\tau}}{n+1})}\mu_{p,d}(B).
\end{aligned}$$

Taking

$$M\ge\left(c_13^dp^{d\lambda_0+n}\right)^{\frac{n+1}{n+1-\tilde{\tau}}}$$

for some constant $c_1<\frac{1}{2}$ and applying this to (6.6) gives us that

$$\mu_{p,d}\left(B\cap\bigcup_{\substack{|a_i|\le M^k\\ M^k<a_0\le M^{k+1}:\ a_0\in\mathcal{Q}'(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}),\ |a_0|_p\ge p^{-\lambda_0}}}\prod_{i=1}^{n}B\left(\frac{a_i}{a_0},p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right)\ge\left(\frac{1}{2}-c_1\right)\mu_{p,d}(B).$$

To complete the proof take the constant C associated with the function ρ to be

$$C=\max_{1\le i\le n}p^{\frac{\lambda_0+\frac{n}{d}}{\alpha_i}}M^{1-\frac{1}{n+1}}.$$

$\square$

Given Proposition 6.3.1 we have that (R_q,β) is a local ubiquitous system of rectangles with respect to $(\rho,\tilde{v})$, provided $\sum_{i=1}^{d}v_i=n+1-\sum_{i=1}^{m}\tau_{d+i}$. Observe that given this ubiquity result we are essentially at the same stage of proof as §4.3 in the real case, or §5.6.1 of the p-adic case. The following method is essential the same as that given in Chapter 5 (p.94-95). For completeness we give the method here, although in a more streamline way. See the sections mentioned above for a more detailed explanation.

Given $\boldsymbol{\tau}_d=(\tau_1,\dots,\tau_d)\in\mathbb{R}^d_{>0}$ assume without loss of generality that $\tau_1>\tau_2>\cdots>\tau_d$ and define each v_{d-i} recursively by

$$v_{d-i}=\min\left\{\tau_{d-i},\frac{n+1-\sum_{i=1}^{m}\tau_{d+i}-\sum_{j=d-i+1}^{d}v_j}{d-i}\right\}.$$

By the condition on $\boldsymbol{\tau}_d$ of Theorem 6.2.1, there exists a $k\in\{1,\dots,d\}$ such that

$$v_l=\frac{n+1-\sum_{i=1}^{m}\tau_{d+i}-\sum_{j=d-k+1}^{d}v_j}{d-k},$$

for all $1 \leq l \leq d-k$. Clearly each $v_i \leq \tau_i$ for $1 \leq i \leq d$, and so the associated vector $\mathbf{t} = (t_1, \dots t_{n-1}) \in \mathbb{R}^{n-1}_{\geq 0}$ is defined by

$$t_i = \tau_i - v_i, \quad 1 \leq i \leq d.$$

Consider the set

$$A = \{v_1, \dots, v_d, \tau_1, \dots, \tau_d\}.$$

For each $A_i \in A$ observe the following:

i) $A_i \in \{v_1, \dots, v_d\}$: Then we have the sets

$$K_1 = \{1, \dots, \max\{i, d-k\}\}, \quad K_2 = \{\max\{i+1, d-k+1\}, \dots, d\}, \quad K_3 = \emptyset.$$

By Theorem 3.3.3 we have that

$$\begin{aligned}\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} &\geq \min_{1 \leq i \leq d} \left\{ \frac{\max\{i, d-k\}v_i + (d - \max\{i+1, d-k+1\})v_i - \sum_{j=\max\{i+1,d-k+1\}}^{d} t_j}{v_i} \right\}, \\ &= \min_{1 \leq i \leq d} \left\{ \frac{dv_i - \sum_{j=\max\{i+1,d-k+1\}}^{d} t_j}{v_i} \right\}.\end{aligned}$$

Since $t_j = 0$ for $d-k+1 \leq j \leq d$ the above equation gives that $\dim \mathfrak{W}'_n(\boldsymbol{\tau}) = d = n-m$, the maximal dimension of $\mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$.

ii) $A_i \in \{\tau_1, \dots, \tau_d\}$: Since $\tau_i = v_i$ for $d-k+1 \leq i \leq d$ the above argument covers such case, so we only need to consider τ_i for $1 \leq i \leq d-k$. For such τ_i we have the sets

$$K_1 = \emptyset, \quad K_2 = \{i, \dots, d\}, \quad K_3 = \{1, \dots, i-1\}.$$

Applying Theorem 3.3.3 we have

$$\begin{aligned}\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} &\geq \min_{1 \leq i \leq d} \left\{ \frac{(d-i)\tau_i + \sum_{j=1}^{i-1} v_j - \sum_{j=i}^{d} t_j}{\tau_i} \right\}, \\ &= \min_{1 \leq i \leq d} \left\{ \frac{(d-i)\tau_i + (d-k)\left(\frac{n+1-\sum_{i=1}^{m}\tau_{d+i} - \sum_{j=d-k+1}^{d} v_j}{d-k}\right) - \sum_{j=1}^{d-k}\tau_j}{\tau_i} \right\}, \\ &= \min_{1 \leq i \leq d} \left\{ \frac{n+1-\sum_{i=1}^{m}\tau_{d+i} + \sum_{j=i}^{d}(\tau_i - \tau_j)}{\tau_i} \right\}.\end{aligned}$$

Combining i) and ii) we have that

$$\dim \mathfrak{W}'_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} \geq \min_{1 \leq i \leq d} \left\{ \frac{n+1-\sum_{i=1}^{m}\tau_{d+i} + \sum_{j=i}^{d}(\tau_i - \tau_j)}{\tau_i} \right\},$$

completing the proof.

6.4 Proof of the counting results

Recall, we aim to provide bounds on the set

$$\mathcal{Q}(\boldsymbol{x},\Psi,N) := \left\{ (q_0,\dots,q_n)\in\mathbb{Z}^{n+1} : \begin{array}{c} 0<q_0\leq N, \\ \max_{1\leq i\leq n}|q_i|\leq N, \end{array} \quad |q_0x_i-q_i|_p<\psi_i(N),\ 1\leq i\leq n \right\}.$$

We begin with the proof of Lemma 6.1.6. This style of proof is not new and follows a similar method to the proof in the euclidean case (see Lemma 3 of [95]).

Proof of Lemma 6.1.6: Fix $\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{Z}_p^n$ and take $t=(t_1,\dots,t_n)\in\mathbb{N}^n$ to be the integers such that

$$p^{-t_i}\leq N^{-\tau_i}<p^{-t_i+1}, \quad 1\leq i\leq n.$$

Denote by $P=\prod_{i=1}^n p^{t_i}$. Consider a set of open disjoint rectangles $\{R_i\}_{i=1}^P$, each with some centre point $k_i=(k_{i,1},\dots,k_{i,n})\in\mathbb{Z}^n$ and sidelenghts p^{-t_i}. Choose the set of points $\{k_i\}$ such that $\mathbb{Z}_p^n\subseteq\bigcup_{i=1}^P R_i$. Consider the $(N+1)^{n+1}$ set of points of the form

$$(q_0x-q)=(q_0x_1-q_1,\dots,q_0x_n-q_n)\in\mathbb{Z}_p^n,$$

with $q_i\in[0,N]$ for each $0\leq i\leq n$. By the Pigeon-hole principle there exists at least one rectangle, say R_j, containing at least

$$\frac{(N+1)^{n+1}}{P}>\frac{1}{p^n}N^{n+1-\sum_{i=1}^n\tau_i}$$

points. As $\sum_{i=1}^n\tau_i<n+1$ we can choose N sufficiently large enough such that $p^{-n}N^{n+1-\sum_{i=1}^n\tau_i}>2$. Order the points $(q_0,\dots,q_n)$, correspond to the points q_0x-q contained in R_j, by the absolute value of the q_0 component. If the q_0 components are equal then order by q_1 and so on. Suppose that the vector $(m_0,\dots,m_n)$ is the smallest by our ordering. Then for all other vectors $(r_0,\dots,r_n)$ contained in R_j we have that

$$|k_{j,i}-(m_0x_i-m_i)-(k_{j,i}-(r_0x_i-r_i)|_p<p^{-t_i},$$
$$|(r_0-m_0)x_i-(r_i-m_i)|_p<p^{-t_i}\leq N^{-\tau_i}.$$

Hence the vectors $(r_0-m_0,\dots,r_n-m_n)\in\mathbb{Z}^{n+1}$ solve the inequality of $\mathcal{Q}(\boldsymbol{x},\boldsymbol{\tau},N)$. Further $(r_i-m_i)\in[-N,N]$, and by the ordering stated above $r_0-m_0\in[0,N]$. To exclude the case where $r_0-m_0=0$ observe that each $\tau_i>1$ and so we would have that

$$N^{-1}\leq|r_i-m_i|_p<p^{-t_i}<N^{-1}$$

for $1\leq i\leq n$, a contradiction. The above argument yields $p^{-n}N^{n+1-\sum_{i=1}^n\tau_i}-1$ such points, completing the proof. $\square$

Lemma 6.1.1 also has a relatively simple proof. The following method of assuming a contradiction and then using the Pigeon-hole principle to prove otherwise is a well know technique used in a variety of texts [105, 37].

Proof of Lemma 6.1.1: We use a proof by contradiction. Suppose that

$$\#\mathcal{Q}(x,\tau,N) > 2N^{\tau(x)-\tau+\epsilon}. \tag{6.7}$$

We use the following notations. Let $X \in \mathbb{N}$ be an integer such that

$$|x - X|_p < p^{-M},$$

for some suitably large $M \in \mathbb{N}$, in particular we may take

$$X = \sum_{i=0}^{M} x_i p^i,$$

where $(x_i)_{i\in\mathbb{N}}$ is the p-adic expansion of x. . Define V_N^+ and V_N^- to be the sets

$$V_N^+ := \{(q,q_1) \in \mathbb{N}\times\mathbb{Z} : 0 < q \leq N,\ 0 \leq q_1 \leq N,\ \},$$
$$V_N^- := \{(q,q_1) \in \mathbb{N}\times\mathbb{Z} : 0 < q \leq N,\ -N \leq q_1 \leq 0,\ \}.$$

Let $t \in \mathbb{N}$ be the integer such that

$$p^{-t} \leq N^{-\tau} < p^{-t+1},$$

and similarly $k \in \mathbb{N}$ be the integer such that

$$p^{-k} \leq N^{-(\tau(x)+\epsilon)} < p^{-k+1}.$$

Note that as $\tau(x) > \tau$, we have that $k \geq t$, and so $p^{k-t} \in \mathbb{N}$. Further, observe that

$$p^{k-t} < pN^{\tau(x)-\tau+\epsilon}. \tag{6.8}$$

Lastly, by the definition of $\tau(x)$, we have that there exists only finitely many $Q \in \mathbb{N}$ such that

$$|qx - q_1|_p < Q^{-(\tau(x)+\epsilon)},$$

for $0 < q, |q_1| \leq Q$. Hence we may choose a sufficiently large N_0 such that for all $N > N_0$ for any pair $0 < q, |q_1| \leq N$,

$$|qx - q_1|_p \geq N^{-(\tau(x)+\epsilon)}, \tag{6.9}$$

for all $\epsilon > 0$. Consider the set of points in $\mathcal{Q}(x,\tau,N)$. Note that $(q,q_1) \in \mathcal{Q}(x,\tau,N)$ if and only if $(q,q_1) \in V_N^+ \cup V_N^-$, and

$$qX - q_1 \equiv 0 \mod p^t. \tag{6.10}$$

Thus, for all $(q, q_1) \in \mathcal{Q}(x, \tau, N)$ we have that

$$qX - q_1 = \lambda p^t,$$

for some $\lambda \in \mathbb{Z}$. Split the set of points in $\mathcal{Q}(x, \tau, N)$ into two disjoint sets, the set of pairs in V_N^+, and the set of pairs in V_N^-. As there are greater than $2N^{\tau(x)-\tau+\epsilon}$ pairs, at least one of the sets has greater than $N^{\tau(x)-\tau+\epsilon}$ pairs. Without loss of generality assume such set of points belong in V_N^+. Considering the range of values of λp^t for t fixed and λ varying we observe there are p^{k-t} possible values of λp^t modulo p^k. By (6.7) and (6.8) we have, by the Pigeon-hole principle, that there exists at least two pairs, say (a, a_1) and (b, b_1), such that

$$(a - b)X - (a_1 - b_1) \equiv 0 \mod p^k.$$

This is equivalent to

$$|(a - b)x - (a_1 - b_1)|_p \leq p^{-k} \leq N^{-(\tau(x)+\epsilon)},$$

with $(a - b, a_1 - b_1) \in V_N^+ \cup V_N^-$, as $0 < a - b \leq N$ by our choice of ordering of a, b, and $|a_1 - b_1| \leq N$ by the fact that the pairs $(a, a_1), (b, b_1) \in V_N^+$. However, such result contradicts (6.9) which follows from the definition of $\tau(x)$, thus (6.7) must be false. $\square$

6.4.1 p-adic approximation lattices

Prior to the proof of Theorem 6.1.3 we recall some basic definitions and results of geometry of numbers that will be needed. Define a lattice Λ as a discrete additive subgroup of $\mathbb{R}^n$. If $\Lambda \subseteq \mathbb{Z}^n$ the Λ is an integer lattice. A set of linearly independent vectors $b_1, \dots, b_n$ that generate Λ is called a basis of Λ. Let B be a $n \times n$ matrix with columns b_i, then call B a basis matrix. Define the fundamental region as

$$\mathcal{F}(B) := \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in \mathbb{R},\ 0 \leq a_i < 1 \right\}.$$

A standard result of geometry of numbers states that if B is a basis matrix for Λ then $\mathcal{F}(B)$ contains no lattice points other than the origin (see Chapter 3, Lemma 6 of [50]).

The volume of the fundamental region can be found by taking the determinant of the basis matrix, that is $vol(\mathcal{F}(B)) = |\det B|$. A basis matrix is not unique for each Λ, however for any lattice Λ the volume of the fundamental region is the same regardless of choice of basis matrix. For this reason we use the notation $\det \Lambda$ to denote the volume of the fundamental region. If $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix and B_1 is a basis matrix for Λ then $B_2 = B_1 U$ is also a basis matrix for Λ.

The successive minima of a lattice is an incredibly useful notion that allows us to deduce several properties of a lattice. Let $B_n = B(0, 1)$ denote the n-dimensional euclidean unit ball. For $c \in \mathbb{R}_+$ we use the notation $cB_n = B(0,c)$. Define the successive minima of a lattice $\Lambda \subset \mathbb{R}^n$ of rank n as the set of values

$$\lambda_i(\Lambda) := \min\{\lambda > 0 : \dim(\Lambda \cap \lambda B) \geq i\},$$

for $i = 1, \ldots, n$. By Minkowski's inequalities on the successive minima (see e.g. [67]) we have that

$$vol(B_n) \prod_{i=1}^{n} \lambda_i(\Lambda) \leq 2^n \det \Lambda. \tag{6.11}$$

For a count on the number of lattice points within a convex body we have the follow theorem due to Blichfeldt [41].

Theorem 6.4.1. *Let $\Lambda \subset \mathbb{R}^n$ be a lattice of full dimension and let $V \subset \mathbb{R}^n$ be a convex body about the origin such that the span of vectors contained in $\Lambda \cap V$ is $\mathbb{R}^n$. Then*

$$\#(\Lambda \cap V) \leq n! \frac{vol(V)}{\det \Lambda} + n.$$

The constant for such estimate can be excessively large, however in our use of the Theorem the size of such constant is irrelevant.

In 1993 an alternative lattice counting theorem was proven by Betke, Henk and Wills [39], which utilised the properties of the successive minima. This result was further generalised by Henk [66], giving us the following theorem.

Theorem 6.4.2. *Let $n \geq 2$, $B(0,K)$ a n-dimensional ball of radius $K > 0$ centred at the origin and Λ a n-dimensional lattice. Then*

$$\#(\Lambda \cap K) < 2^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2K}{\lambda_i(\Lambda)} + 1 \right\rfloor.$$

We remark that if $rank(\Lambda \cap B(0,M)) < n$ then we must have at least that $\lambda_n(\Lambda) \geq M$. Thus the n^{th} value of the product in Theorem 6.4.2 would be bounded above by 3, a point we make use of later on.

For the proof of Theorem 6.1.3 we use p-adic approximation lattices. Such lattices have been used in p-adic Diophantine approximation regularly, for example De Weger [54] used them to prove a variety of results in classical p-adic Diophantine approximation, including the p-adic analogue of Hurwitz Theorem. Recently n-dimensional forms of p-adic approximation lattices have been used to provide lattice based cryptosystems [73, 74]. In these papers both dual and simultaneous approximation lattices were discussed. In particular Dirichlet-style exponents were proven for simultaneous and dual approximation.

For a n-tuple of approximation functions $\Psi = (\psi_1, \ldots, \psi_n)$, an integer $N \in \mathbb{N}$, and a fixed $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ define the Ψ-approximation lattice $\Lambda_{N,\boldsymbol{x}}$ over R^{n+1} by

$$\Lambda_{N,\boldsymbol{x}} = \{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} : |a_0 x_i - a_i|_p \leq \psi_i(N),\ 1 \leq i \leq n\}.$$

To briefly justify the above claim note that the collection of points are discrete, and that any integer linear combination of such points is also contained within the set due to the strong triangle inequality. Observe that such claim is clearly false if we were to consider the real case analogous to the above setting. Observe that

$$\mathcal{Q}(\boldsymbol{x}, \Psi, N) \subseteq \Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N),$$

since the euclidean ball $B(0, \sqrt{n}N)$ contains all integer points satisfying $\max_{0 \leq i \leq n} |q_i| \leq N$.

For any $\boldsymbol{x} \in \mathbb{Z}_p^n$ we may write each x_j as the p-adic expansion

$$x_j = \sum_{i=0}^{\infty} x_{j,i} p^i, \quad x_{j,i} \in \{0, 1, \ldots, p-1\}.$$

Let $X_{j,N} \in \mathbb{Z}$ be the integer

$$X_{j,N} = \sum_{i=0}^{t_j} x_{j,i} p^i,$$

where each $t_j \in \mathbb{N}$ is the unique value associated with N satisfying

$$p^{-t_j} < \psi_j(N) \leq p^{-t_j+1}. \tag{6.12}$$

Lastly, for each $1 \leq j \leq n$ let $\psi^*_{j,N} = p^{t_j}$. Then the set of vectors

$$B = \left\{ \begin{pmatrix} 1 \\ X_{1,N} \\ \vdots \\ X_{n,N} \end{pmatrix}, \begin{pmatrix} 0 \\ \psi^*_{1,N} \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \psi^*_{n,N} \end{pmatrix} \right\}, \tag{6.13}$$

form a basis for $\Lambda_{N,\boldsymbol{x}}$. To support this claim observe that

$$|x_i - X_{i,N}|_p = \left| \sum_{j=t_i+1}^{\infty} x_{i,j} p^j \right|_p < p^{-t_i} < \psi_i(N),$$

and

$$|0 \cdot x_i - \psi^*_{i,N}|_p = |p^{t_i}|_p = p^{-t_i} < \psi_i(N),$$

for $1 \leq i \leq n$. Hence the span of vectors B at least produces a sublattice of $\Lambda_{N,\boldsymbol{x}}$. To show B is a basis of $\Lambda_{N,\boldsymbol{x}}$ we consider the fundamental region $\mathcal{F}(B)$ and show that the only lattice point contained is $\mathbf{0}$. Suppose there exists a non-zero lattice point in $\mathcal{F}(B)$ of the form

$$\sum_{i=1}^{n+1} c_i b_i$$

where b_i are the vectors of B and each $0 \leq c_i < 1$. Since $\Lambda_{N,\boldsymbol{x}} \subseteq \mathbb{Z}^{n+1}$ and the first vector of B is the only vector with a non-zero entry in the first row we must have $c_1 = 0$. For the remaining vectors observe that each non-zero term is a prime power, so each c_i must be of the form p^{-r} for some $r \in \mathbb{N}$. However, by the construction of each $\psi_{i,N}^*$ we have that

$$|\psi_{i,N}^*|_p < \psi_i(N) \quad \text{and} \quad |p^{-1}\psi_{i,N}^*|_p \geq \psi_i(N),$$

and so the only possible value for each c_i is zero.

Now we have a basis for $\Lambda_{N,\boldsymbol{x}}$ we can calculate

$$|\det \Lambda_{N,\boldsymbol{x}}| = \prod_{i=1}^{n} \psi_{i,N}^* \asymp \left(\prod_{i=1}^{n} \psi_i(N)\right)^{-1},$$

where the implied constants can be easily found using (6.12) to obtain

$$\left(\prod_{i=1}^{n} \psi_i(M)\right)^{-1} \leq |\det \Lambda_{N,\boldsymbol{x}}| \leq p^n \left(\prod_{i=1}^{n} \psi_i(N)\right)^{-1}. \tag{6.14}$$

In the simultaneous case, $\Psi = (\psi, \dots, \psi)$, it was proven in [74] that

$$\lambda_1(\Lambda_{N,\boldsymbol{x}}) \ll \psi(N)^{-\frac{n}{n+1}}.$$

In the following proposition we generalise this result to weighted approximation and find a lower bound result for $\boldsymbol{x} \in \mathbb{Z}_p$ satisfying certain Diophantine exponent properties. It should be remarked that the upper bound result is trivial and was probably known to the authors of [74].

Proposition 6.4.3. *Let $\Lambda_{N,\boldsymbol{x}}$ be defined above with $\tau(\boldsymbol{x}) = n+1$, and suppose that*

$$\prod_{i=1}^{n} \psi_i(N) < N^{-n}.$$

Then for any $\varepsilon > 0$ the exists sufficiently large $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)}\right)^{\frac{1}{n+1}-\varepsilon} \leq \lambda_1(\Lambda_{N,\boldsymbol{x}}) \leq C_2 \left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)}\right)^{\frac{1}{n+1}},$$

where

$$C_2 = 2\left(\frac{\Gamma\left(\frac{n+1}{2}+1\right)p^n}{\pi^{\frac{n+1}{2}}}\right)^{\frac{1}{n+1}}.$$

As will become clear in the proof below the condition that $\tau(\boldsymbol{x}) = n+1$ is only necessary in the lower bound result.

Proof. We prove the upper bound case first. Such proof is a standard application of Minkowski's first Theorem on successive minima and follows almost immediately by the above calculation of $\det(\Lambda_{M,\boldsymbol{x}})$. Concisely, we have that

$$\lambda_1(\Lambda_{N,\boldsymbol{x}})^{n+1} vol(B(0,1)) \leq 2^{n+1} \det(\Lambda_{N,\boldsymbol{x}}).$$

Rearranging for $\lambda_1(\Lambda_{N,\boldsymbol{x}})$, using (6.14), and recalling the volume of an $n+1$-ball we obtain our result.

For the lower bound observe that for any $\boldsymbol{x} \in \mathbb{Z}_p^n$

$$\begin{cases} \prod_{i=1}^n |q_0 x_i - q_i|_p < N^{-(n+1)}, \\ \max_{0 \leq j \leq n} |q_j| \leq N, \end{cases}$$

for infinitely many N (see for example Lemma 5.3.1). Further, since $\tau(\boldsymbol{x}) = n+1$ there exists N_0 such that for all $N \geq N_0$ then any rational integer vectors $(q_0, \ldots, q_n)$ satisfying $\max_{0 \leq i \leq n} |q_i| \leq N$ we have that

$$\prod_{i=1}^n |q_0 x_i - q_i|_p \geq N^{-(n+1+\varepsilon)} \tag{6.15}$$

for some $\varepsilon > 0$. Choose N sufficiently large such that

$$N_0 \leq \left(\prod_{i=1}^n \psi_i(N) \right)^{-\left(\frac{1}{n+1} - \varepsilon\right)}.$$

Such N is possible since $\prod_{i=1}^n \psi_i(N) < N^{-n}$ and so the value on the RHS of the above inequality tends to infinity as $N \to \infty$ for any small ε $(\varepsilon < \frac{1}{n(n+1)})$.

Suppose that $(q_0, \ldots, q_n)$ is a minimum length non-zero vector of $\Lambda_{N,\boldsymbol{x}}$, then note that $\lambda_1(\Lambda_{N,\boldsymbol{x}}) \geq \max_{1 \leq i \leq n} |q_i|$ due to the euclidean nature of $\lambda_1(\Lambda_{N,\boldsymbol{x}})$. Suppose that

$$\max_{1 \leq i \leq n} |q_i| < \left(\prod_{i=1}^n \psi_i(N) \right)^{-\left(\frac{1}{n+1} - \varepsilon\right)}. \tag{6.16}$$

We prove (6.16) to be false. Observe that

$$\prod_{i=1}^n |q_0 x_i - q_i|_p < \prod_{i=1}^n \psi_i(N),$$

since $(q_0, \ldots, q_n) \in \Lambda_{N,\boldsymbol{x}}$. Then

$$\prod_{i=1}^n |q_0 x_i - q_i|_p < \left(\left(\prod_{i=1}^n \psi_i(N) \right)^{-\left(\frac{1}{n+1} - \varepsilon\right)} \right)^{-\frac{n+1}{1-\varepsilon(n+1)}}.$$

But this contradicts (6.15). So we must have that (6.16) is false, and so

$$\lambda_1(\Lambda_{N,\boldsymbol{x}}) \geq \left(\prod_{i=1}^n \psi_i(N) \right)^{-\left(\frac{1}{n+1} - \varepsilon\right)},$$

completing the proof. □

Given Proposition 6.4.3 we can proceed with the following.

Proof of Theorem 6.1.3: For $N \geq N_0$, where N_0 is chosen by Proposition 6.4.3, consider the following two cases:

i) $rank(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) = n+1$: By Theorem 6.4.1 we have that

$$\begin{aligned}
\#(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) &\leq (n+1)!\frac{vol(B(0,\sqrt{n}N))}{\det \Lambda_{N,\boldsymbol{x}}} + n + 1, \\
&\leq \frac{(n+1)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)} N^{n+1}.(\prod_{i=1}^{n} \psi_{i,N}^{*})^{-1} + n + 1, \\
&\leq \frac{(n+2)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)} N^{n+1} \prod_{i=1}^{n} \psi_i(N).
\end{aligned}$$

Note that the last inequality follows since $\prod_{i=1}^{n} \psi_i(N) > N^{-(n+1-\varepsilon)}$. This proves Theorem 6.1.3 for the rank $n+1$ case.

ii) $rank(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) < n+1$: Since $rank(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) < n+1$ we must have $\lambda_{n+1}(\Lambda_{N,\boldsymbol{x}}) > \sqrt{n}N$. Hence, by the remark made previously, the final product on the right of Theorem 6.4.2 is less than or equal to 3. Furthermore, for each $\lambda_i(\Lambda_{N,\boldsymbol{x}})$, $1 \leq i \leq n$ we have that

$$\begin{aligned}
\lambda_n(\Lambda_{N,\boldsymbol{x}}) \geq \cdots \geq \lambda_1(\Lambda_{N,\boldsymbol{x}}) &\overset{\text{Prop. 6.4.3}}{\geq} \left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)}\right)^{\frac{1}{n+1}-\varepsilon}, \\
&\geq \left(\frac{1}{N\prod_{i=1}^{n} \psi_i(N)}\right)^{1/n},
\end{aligned}$$

where the second inequality follows since

$$\begin{aligned}
\left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)}\right)^{\frac{1}{n+1}-\varepsilon} &\geq \left(\left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)}\right)^{n-\varepsilon n(n+1)}\right)^{\frac{1}{n(n+1)}}, \\
&\geq \left(\frac{1}{N^{n+1}\prod_{i=1}^{n} \psi_i(N)}\left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)}\right)^{n}\right)^{\frac{1}{n(n+1)}}.
\end{aligned}$$

combining the two ideas above, and Theorem 6.4.2, we have that

$$\begin{aligned}
\#(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) &< 2^n 3 \prod_{i=1}^{n}\left(\frac{2\sqrt{n}N}{\lambda_1(\Lambda_{N,\boldsymbol{x}})} + 1\right), \\
&< 2^n 3 \left(2\sqrt{n}N^{1+1/n}\left(\prod_{i=1}^{n} \psi_i(N)\right)^{1/n} + 1\right)^{n}, \\
&< 3(6\sqrt{n})^n N^{n+1} \prod_{i=1}^{n} \psi_i(N).
\end{aligned}$$

Thus, in either case $i)$ or $ii)$ we have that

$$\#(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) \leq C_1 N^{n+1} \prod_{i=1}^{n} \psi_i(N),$$

with

$$C_1 = \max\left\{3(6\sqrt{n})^n, \frac{(n+2)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)}\right\}.$$

$\square$

6.5 Concluding remarks on Theorem 6.1.3

Theorem 6.1.3 provides sharp bounds on the number of rational points close to almost all n-dimensional p-adic integers. While this result allows us to find simultaneous p-adic Diophantine approximation results on coordinate hyperplanes, it falls a long way short of providing results for Diophantine approximation sets on curves and manifolds. However, as will be shown in the next chapter, Theorem 6.1.3 can be used to obtain bounds on rational points close to certain classes of submanifolds. It is hoped the techniques used in the proof of Theorem 6.1.3 could be developed further to find counts on the number of rational points close to general manifolds.

Chapter 7

Further Research

This final chapter provides a brief overview and discussion of possible developments to the preceding chapters. We introduce S-arithmetic Diophantine approximation and discuss how the techniques of Chapters 4 and 5 could be implemented to obtain similar results in the S-arithmetic setting.

We also discuss the latest results in counting rational points close to real manifolds, and possible methods to obtain similar results in the p-adic setting. Such results would provide a complimentary upper bound result to Theorem 5.2.3-5.2.5.

7.1 Introduction to S-arithmetic numbers

The S-arithmetic setting is a combination of both real and p-adic numbers. For that reason many of the notions and ideas of the previous chapters can be applied in this setting. Let S be a finite set of valuations on $\mathbb{Q}$ of cardinality k. Then define

$$\mathbb{Q}_S = \prod_{\nu \in S} \mathbb{Q}_\nu,$$

where $\mathbb{Q}_\nu$ is the completion of $\mathbb{Q}$ with respect to the valuation ν. In the case where the Euclidean valuation is contained in S we will say $\infty \in S$, and similarly we will denote $\mathbb{R}$ by $\mathbb{Q}_\infty$. The set $\mathbb{Q}_S$ has the associated norm defined for any $\boldsymbol{x} = (x^{(\nu)})_{\nu \in S} \in \mathbb{Q}_S$ by

$$|\boldsymbol{x}|_S = \max_{\nu \in S} |x^{(\nu)}|_\nu.$$

Hence, for any point $\boldsymbol{y} \in \mathbb{Q}_S$ and real number $r > 0$ we may define the S-arithmetic open ball

$$B_S(\boldsymbol{y}, r) := \left\{ \boldsymbol{x} \in \mathbb{Q}_S : |\boldsymbol{x} - \boldsymbol{y}|_S < r \right\}.$$

Where it is clear we are referring to an S-arithmetic ball we will drop the notation and use $B(\boldsymbol{y}, r)$ to denote a S-arithmetic ball with center $\boldsymbol{y} \in \mathbb{Q}_S$ and radius $r > 0$. As with the p-adic setting, define the ring of S-arithmetic integers $\mathbb{Z}_S \subset \mathbb{Q}_S$ as

$$\mathbb{Z}_S := \{\boldsymbol{x} \in \mathbb{Q}_S : |\boldsymbol{x}|_S \leq 1\}.$$

At this stage we note that many general properties of $\mathbb{Q}_S$ depend on whether $\infty \in S$. For example, if $\infty \in S$ then $\mathbb{Z}^k \not\subset \mathbb{Z}_S$, since the Euclidean norm of any integer (with the exception of ± 1) is greater than one. Conversely, if $\infty \notin S$ then $\mathbb{Z}^k \subset \mathbb{Z}_S$, and further still $\mathbb{Z}^k$ is dense in $\mathbb{Z}_S$ (this follows by using similar ideas to the p-adic case). Further, observe that if $\infty \notin S$ then $|.|_S$ satisfies the strong triangle inequality, a result that is clearly false if $\infty \in S$.

Denote by μ_S the S-arithmetic Haar measure, normalised by $\mu_S(\mathbb{Z}_S) = 1$. Note that μ_S is simply the product measure of measures over each $\mathbb{Q}_\nu$, i.e.

$$\mu_S = \prod_{\nu \in S} \mu_\nu,$$

with $\mu_\infty = \lambda$, the Lebesgue measure.

The notion of Diophantine approximation in $\mathbb{Q}_S$ can be considered in a variety of ways. As with the previous chapters we will focus on simultaneous Diophantine approximation. For a general introduction and a variety of results on dual and Groshev S-arithmetic Diophantine approximation see [79, 90]. For integer vector $(q_0, \boldsymbol{q}) \in \mathbb{Z}^{k+1}$ we will be interested in the quantity

$$|\boldsymbol{x}q_0 - \boldsymbol{q}|_S,$$

where $\boldsymbol{x}q_0$ can be considered as usual scalar multiplication i.e. $\boldsymbol{x}q_0 = (q_0 x^{(\nu)})_{\nu \in S}$. As discussed in Chapter 2 the size of both q_0 and $\boldsymbol{q}$ can greatly influence the rate of approximation for the p-adic norm. The same is clearly true for $|.|_S$. Let $\psi : \mathbb{N} \to \mathbb{R}_+$ with $\psi(r) \to 0$ as $r \to \infty$, then we define a point $\boldsymbol{x} \in \mathbb{Q}_S$ to be ψ-simultaneously approximable if there exists infinitely many integer vectors $(q_0, q_1, \ldots, q_k) = (q_0, \boldsymbol{q}) \in \mathbb{Z}^{k+1}$ that solve

$$\begin{cases} |\boldsymbol{x}q_0 - \boldsymbol{q}|_S < \psi(H), \\ \max\limits_{0 \leq i \leq k} |q_i| \leq H. \end{cases}$$

For each valuation $\nu \in S$ denote by $q^{(\nu)}$ the integer associated with such valuation. Precisely, let $\boldsymbol{q} = (q^{(\nu)})_{\nu \in S} \in \mathbb{Z}^k$, then

$$|\boldsymbol{x}q_0 - \boldsymbol{q}|_S = \max_{\nu \in S} |x^{(\nu)} q_0 - q^{(\nu)}|_\nu.$$

When considering weighted simultaneous approximation we will vary the rate of approximation over each $\nu \in S$. Let $\Psi = (\psi_\nu)_{\nu \in S}$ be a k-tuple of approximation functions. A point $\boldsymbol{x} \in \mathbb{Q}_S$ is said to be Ψ-simultaneously approximable if there exists infinitely many integer vector solutions $(q_0, \boldsymbol{q}) \in \mathbb{Z}^{k+1}$ to

$$\begin{cases} |x^{(\nu)} q_0 - q^{(\nu)}|_\nu < \psi_\nu(H), \quad \nu \in S, \\ \max\limits_{\nu \in S} \{|q_0|, |q^{(\nu)}|\} \leq H. \end{cases}$$

In lim sup form we may describe these set of points as

$$\mathcal{W}_S(\Psi) := \limsup_{h\to\infty} \bigcup_{|q_0|,|q^{(\nu)}|\leq h} \left\{ \boldsymbol{x} \in \mathbb{Z}_S : |x^{(\nu)} q_0 - q^{(\nu)}|_\nu < \psi_\nu(h),\ \nu \in S \right\}.$$

Similarly to the previous chapters we will use the notation $\mathcal{W}_S(\psi)$ for the set of $(\psi, \dots, \psi)$-approximable points and $\mathcal{W}_S(\boldsymbol{\tau})$ for the set of $(q^{-\tau_1}, \dots, q^{-\tau_k})$-approximable points. We will often denote each τ_i by the associated valuation i.e. $\boldsymbol{\tau} = (\tau_\nu)_{\nu\in S} \in \mathbb{R}_+^k$ for notational purposes.

As usual the initial aim is to find a Dirichlet-style theorem. In order to provide an optimal Dirichlet-style result we need to know whether $\infty \in S$. As justification for this note that if $\infty \notin S$ then for any integer vector $(q_0, \boldsymbol{q}) \in \mathbb{Z}^{k+1}$ and $\boldsymbol{x} \in \mathbb{Z}_S$ we have that

$$|\boldsymbol{x}.q_0 - \boldsymbol{q}|_S \leq 1, \tag{7.1}$$

and so $\boldsymbol{x}.q_0 - \boldsymbol{q} \in \mathbb{Z}_S$. If $\infty \in S$ then (7.1) is satisfied only for certain values of $q^{(\infty)}$ dependent on q_0.

For each $\nu \in S\backslash\{\infty\}$ let $p_\nu \in \mathbb{N}$ be such prime associated to the valuation ν. The following lemma provides us with a generalised Dirichlet-style theorem for S-arithmetic approximation. Note that, as with Lemma 5.3.1, similar versions of the lemma below have been proven prior by a number of authors, see for example [79].

Lemma 7.1.1. *For each valuation $\nu \in S$ let $L_\nu : \mathbb{Q}_\nu^{k+1} \to \mathbb{Q}_\nu$ be a linear form with p_ν-adic integer coefficients, or real coefficients in the interval $[0,1]$ if $\nu = \infty$. Let $\boldsymbol{\tau} = (\tau_\nu)_{\nu\in S} \in \mathbb{R}_+^k$ be a weight vector. Suppose*

i) $\infty \notin S$ and $\sum_{\nu\in S} \tau_\nu = k+1$, or

ii) $\infty \in S$, $\sum_{\nu\in S} \tau_\nu = k$ and the coefficients of the linear form

$$L_\infty(\boldsymbol{x}) = a_0 x_0 + \cdots + a_k x_k$$

satisfy

$$\sum_{i=0}^{k} |a_i| \leq 2|a_j|, \tag{7.2}$$

for some $j \in \{0, \dots, k\}$.

Then for any $H \geq 1$ there exists a non-zero rational integer vector $\boldsymbol{x} = (x_0, x_1, \dots, x_k)$ such that

$$\max_{0\leq i\leq k} |x_i| \leq H \tag{7.3}$$

and

$$|L_\nu(\boldsymbol{x})|_\nu < p_\nu H^{-\tau_\nu} \quad \text{for each } \nu \in S, \tag{7.4}$$

where $p_\infty = 1$.

Note that for our purposes the conditions on the coefficients of the linear form over $\mathbb{R}$ is generally unrestrictive, in particular we simply need $x^{(\infty)} \in [0,1]$. Note that the proof of Lemma 7.1.1 follows closely the proof of Lemma 5.3.1.

Proof. Such result depends on whether $\infty \in S$, so for that reason we split the proof into two cases. This is a standard proof using Dirichlet's pigeon-hole principle, which is given here for completeness.

i) $\infty \notin S$: To begin with, note that there are $(H+1)^{k+1}$ different rational integer vectors $\boldsymbol{x} = (x_0, \ldots, x_k)$ satisfying (7.3), subject to the condition that $x_i \geq 0$ for each i. Further for all of these rational integer vectors we have that

$$|L_\nu(\boldsymbol{x})|_\nu \leq 1.$$

ii) $\infty \in S$: Let the linear form over $\mathbb{R}$ be of the form

$$L_\infty(\boldsymbol{x}) = a_0 x_0 + \ldots a_k x_k,$$

for each $a_i \in [0,1]$, and suppose $a_j = \max_{0 \leq i \leq k} a_i$. Dividing L_∞ through by a_j (note that $a_j \neq 0$ since L_∞ is not a constant function) we have another linear form with each coefficient $\frac{a_i}{a_j} \in [0,1]$ for each $0 \leq i \leq k$. We may write

$$\frac{1}{a_j} L_\infty(\boldsymbol{x}) = \left(\frac{1}{a_j} L_\infty(\boldsymbol{x}) - x_j \right) + x_j, \tag{7.5}$$

where the quantity in the brackets satisfies

$$0 \leq \frac{1}{a_j} L_\infty(\boldsymbol{x}) - x_j \leq H$$

for all $\boldsymbol{x} \in \{0, \ldots H\}^{k+1}$ due to (7.2). Thus, by (7.5) we can choose at least one $x_j \in \{0, \ldots, H\}$ such that for any $(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k) \in \{0, \ldots, H\}^k$

$$\left| \frac{1}{a_j} L_\infty(\boldsymbol{x}) \right|_\infty \leq 1.$$

Since $a_j \in [0,1]$ we may multiply through by a_j and obtain that $|L_\infty(\boldsymbol{x})|_\infty \leq 1$ for at least $(H+1)^k$ integer vectors $\boldsymbol{x}$.

To summarize, by i), if $\infty \notin S$ there are $(H+1)^{k+1}$ rational integer vectors such that

$$|L_\nu(\boldsymbol{x})|_\nu \leq 1, \tag{7.6}$$

whereas, if $\infty \in S$ there are only $(H+1)^k$ rational integer vectors that satisfy (7.6). Denote by τ_ν the τ_i associated with the approximation on the $\mathbb{Q}_\nu$ component. For each $\nu \in S \backslash \{\infty\}$ let δ_ν be the unique integer such that

$$p_\nu^{\delta_\nu} \leq H^{\tau_i} < p_\nu^{\delta_\nu + 1}.$$

By (7.6) we have that $L(\boldsymbol{x}) := (L_\nu(\boldsymbol{x}))_{\nu\in S} \in \mathbb{Z}_S$. Split $\mathbb{Z}_S$ into the subsets $S(\boldsymbol{a})$ given by

$$S(\boldsymbol{a}) = \{x^{(\infty)} \in [0,1] : |x^{(\infty)} - a^{(\infty)}|_\infty \leq H^{-\tau_\infty}\} \times \prod_{\nu\in S\setminus\{\infty\}} \{x^{(\nu)} \in \mathbb{Z}_{p_\nu} : |x^{(\nu)} - a^{(\nu)}|_\nu \leq p_\nu^{-\delta_i}\}$$

for each $\boldsymbol{a} = (a^{(\nu)})_{\nu\in S} \in \mathbb{Z}^{k+1}$ with $0 \leq a^{(\nu)} < p_\nu^{\delta_\nu}$ and $a^{(\infty)} \in \left\{\frac{2n-1}{H^{\tau_\infty}}\right\}_{1\leq n\leq \frac{1}{2}H^{\tau_\infty}}$. It is readily seen that the sets $S(\boldsymbol{a})$ are disjoint and cover the whole of $\mathbb{Z}_S$. Furthermore, there are exactly $\frac{1}{2}H^{\tau_\infty} \times \prod_{\nu\in S\setminus\{\infty\}} p_\nu^{\delta_\nu} \leq \frac{1}{2}H^{\sum_{\nu\in S}\tau_\nu}$ of them. Hence, by the pigeon-hole principle, in either case $i)$ or $ii)$ at least one of the sets $S(\boldsymbol{a})$ contains $L(\boldsymbol{x}_i)$ for at least two distinct integer points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as specified in $i)$ or $ii)$ respectively. Let $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. Clearly, (7.3) is satisfied and $\boldsymbol{x}$ is non-zero. Furthermore, for each $\nu \in S$ we have that

$$|L_\nu(\boldsymbol{x}_1 - \boldsymbol{x}_2)|_\nu = |L_\nu(\boldsymbol{x}_1) - L_\nu(\boldsymbol{x}_2)|_{p_\nu} < p_\nu H^{-\tau_\nu},$$

with $p_\infty = 1$. This verifies (7.4) and thus completes the proof. □

As we can see, by Lemma 7.1.1, if all τ_i are equal then we have the following corollary.

Corollary 7.1.2. *Let $\boldsymbol{x} \in \mathbb{Z}_S$, then for any $H \in \mathbb{N}$ there exists an integer solution $(q_0, \boldsymbol{q}) \in \mathbb{Z}^{k+1}$ to the system of inequalities*

$$\begin{cases} |\boldsymbol{x}.q_0 - \boldsymbol{q}|_S < H^{-\omega}, \\ \max\limits_{0\leq i\leq k} |q_i| \leq H, \end{cases}$$

where,

$$\omega = \begin{cases} 1 + \frac{1}{k} & \text{if } \infty \notin S, \\ 1 & \text{otherwise.} \end{cases}$$

For an idea on how one would construct such result for the dual setting see Section 10 of [79].

7.1.1 Metric S-arithmetic approximation

In following with the previous chapters the next step is to provide Haar measure and Hausdorff theory results on the set of approximable points. Note that, to date, there are no results of this kind on the set of weighted simultaneously approximable points. However, it is expected that the results of the previous chapters (in particular chapter 4) could readily be adapted to work in the S-arithmetic setting. For the time being we will focus exclusively on results for the set of simultaneously approximable points.

For a slightly different setup to ours above Jarnik [76] proved a Khintchine-style Theorem for the set $\mathcal{W}_S(\psi)$. More recently Haynes proved the following theorem (Theorem 4 of [65]).

Theorem 7.1.3. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic decreasing approximation function. Then*

$$\mu_S(\mathcal{W}'_S(\psi)) = \begin{cases} 0 & \text{if } \sum_{h=1}^{\infty}(\varphi(h)\psi(h))^k < \infty, \\ 1 & \text{if } \sum_{h=1}^{\infty}(\varphi(h)\psi(h))^k = \infty. \end{cases}$$

Note that, as with the previous chapters $\mathcal{W}'_S(\psi)$ denotes the set of S-arithmetic points that can be approximated by infinitely many reduced fractions ($\mathcal{W}_S(\psi)$ with the added condition that $gcd(q_0, q^{\nu)}) = 1$ for each $\nu \in S$). The result was proven based on the assumption that the Duffin Schaeffer Conjecture was true, which has since been proven (see Theorem 1.1.2), and so Theorem 7.1.3 followed. Recently Oliveira [93] proved a Khintchine style theorem of an S-arithmetic nature. Rather than simultaneous approximation of S-arithmetic points they approximated p_{ν_1}-adic or real points by rationals contained within different p_{ν_2}-adic balls.

For the Hausdorff theory Haynes proved the following result by applying the general MTP to Theorem 7.1.3 [65].

Theorem 7.1.4. *Let $k + r > 1$. Then for any dimension function f with the property that $f(n)/n^{k+r}$ is monotonic, we have*

$$\mathcal{H}^f(\mathcal{W}'_S(\psi)) = \begin{cases} \infty & \text{if } \sum_{n=0}^{\infty} f(\psi(n))\varphi(n)^{k+r} = \infty, \\ 0 & \text{if } \sum_{n=0}^{\infty} f(\psi(n))\varphi(n)^{k+r} < \infty. \end{cases}$$

Oliveira also used the general MTP to provide a corresponding Hausdorff measure statement for approximable points by rationals from balls in other valuations (see Theorem 1.4 of [93]). The above set of results provide a complete view of S-arithmetic simultaneous approximation.

The next step is to find results on the set of S-arithmetic approximable points over curves and manifolds. As with the real and p-adic case there are significantly more results for the set of dual approximable points, see for example [51, 91] and references therein. In the simultaneous setting it has been shown that non-degenerate (see [79] for the precise definition in this setting) manifolds are extremal [79] by using similar methods to the work of Kleinbock and Margulis [78]. Khintchine convergence and divergence results have also been found. In [44, 36] the Khintchine divergence result for simultaneously approximable points over polynomial functions of degree ≥ 3 was proven, and later the corresponding convergence result was proven by the same authors [45].

Currently there are no results on the Hausdorff dimension of simultaneously approximable S-arithmetic points on curves or manifolds. Using the methods of Chapters 4-6 we suspect lower bounds on the Hausdorff dimension of such sets could be obtained. The upper bound, as with the main results of Chapters 4-6 require a knowledge of the distribution of rational points near S-arithmetic manifolds.

7.2 Counting points close to manifolds

As mentioned in Chapters 4, 5, and the end of the previous section, a knowledge of the distribution of rational points close to manifolds is essential in order to obtain corresponding upper bound Hausdorff dimension results of the previous chapters. When such results are available the Hausdorff dimension result follows readily, Chapter 6 gives an easy example of this. Generally, for a manifold $\mathcal{M} \subset \mathcal{F}$ where $\mathcal{F}$ is either $[0,1]^n$, $\mathbb{Z}_p^n$ or $\mathbb{Z}_S$ (with $\#S = n$), a ball $\mathcal{U} \subseteq \mathcal{F}$, a fixed integer $Q \in \mathbb{N}$, and $\psi > 0$ we wish to provide bounds on the cardinality of the sets

$$\mathcal{N}_{\mathcal{M}}(\mathcal{U}, Q, \psi) := \left\{ \frac{a}{b} \in \mathbb{Q}^n : |a|, |b| \leq Q, \, dist\left(\mathcal{M}, \frac{a}{b}\right) < \psi \right\},$$

where $a \in \mathbb{Z}^n$, and $dist\left(\mathcal{M}, \frac{a}{b}\right) := \inf\{r \in \mathbb{R} : |\boldsymbol{x} - \frac{a}{b}|_{\mathcal{F}} < r \ \boldsymbol{x} \in \mathcal{M}\}$ with $|.|_{\mathcal{F}}$ the valuation associated with $\mathcal{F}$. Note that since we are only considering manifolds contained within some ball of radius ≤ 1 we may bound $|a| \leq Q$ with no adverse effect when considering real space.

It should be clear that in order to obtain useful bounds for general manifolds we need to apply some conditions. The most important of which is that the manifold is not flat over large intervals. As an example suppose that $\mathcal{M}$ is contained in some rational hyperplane of $\mathcal{F}$. Then we would have that $\mathcal{N}_{\mathcal{M}}(\mathcal{U}, Q, \psi) \asymp Q^{n-1}$. In order to ensure manifolds are sufficiently curved a non-degeneracy condition is usually applied, see [78, 79]. In the case of planar curves this condition can be simply stated as the second derivative being non-zero for a significantly large portion of the curve. Another condition required is that the value ψ is not too small relative to Q. As an example take the hyperplane $\Pi_{\alpha,n}$ as described in Chapter 5, with $\tau(\alpha) = 2$. If $\psi < Q^{-2-\varepsilon}$ for $\varepsilon > 0$ and Q sufficiently large then there will be no rational approximations in the α component of Π_α, and thus no rational points in the ψ-neighbourhood of Π_α.

In this section we will review several results of this type in the real setting and discuss how such results could be obtained in the p-adic setting.

Rational points close to planar curves

The first of these types of results was due to Huxley [72]. For $\mathcal{U} = [0,1]^2$, and $\mathcal{M}$ a planar curve described by a real twice continuously differentiable function with bounded second derivatives over some fixed interval Huxley proved that

$$\#\mathcal{N}_{\mathcal{M}}(\mathcal{U}, Q, \psi) \ll \psi Q^{3+\varepsilon}$$

for $\varepsilon > 0$ arbitrary and $\psi > Q^{-2}$. Such result was proven using the Swinnerton-Dyer determinant method [108]. It was believed that for ψ bounded the Q^ε term of Huxley's estimate could be removed. In [110] Vaughan and Velani successfully proved such result for C^3 curves, using duality, as used in [72], in combination with harmonic analysis. In [19] the complimentary upper bound was also found, thus for planar curves we have the complete result that

$$\#\mathcal{N}_{\mathcal{M}}(\mathcal{U}, Q, \psi) \asymp \psi Q^3,$$

for $\psi \geq Q^{-2+\varepsilon}$. Note that this has recently been improved by Huang, who proved the asymptotic formula [68].

Rational points close to manifolds

For general manifolds Beresnevich [14] proved the following. For any analytic non-degenerate manifold $\mathcal{M}$ of codimension m and any ψ satisfying

$$CQ^{-\frac{m+1}{m}} < \psi < C^{-1},$$

for some constant $C>0$ dependant on our choice of ball $\mathcal{U}$, then for Q sufficiently large we have that

$$\#\mathcal{N}_{\mathcal{M}}(\mathcal{U},Q,\psi) \gg \psi^m Q^{n+1},$$

where the implied constants are dependent only on the choice of $\mathcal{U}$ (see Corollary 1.5 of [14]). In [14] Beresnevich proves further than this, actually giving results on the distribution of the rational points close to the manifold in the form of a ubiquity statement. In the case of n-dimensional curves the condition on ψ can be relaxed to $\psi \gg Q^{-3/(2n-1)}$. As stated as a remark to Theorem 5.2.3 it is hoped that by applying the notion of ubiquity in a similar manner to as done in [14] the upper bound of τ in Theorem 5.2.3 could be improved to $1+\frac{3}{2n-1}$.

For the manifold $\mathcal{M}$ being a compact C^k hypersurface with Gaussian curvature bounded away from zero, where k is

$$k = \max\left\{ \left\lceil \frac{n-1}{2} \right\rceil + 5,\, n+1 \right\},$$

Huang [69] proved that

$$\mathcal{N}(\mathcal{U},Q,\psi) \ll \psi Q^{n+1}. \tag{7.7}$$

Note that such result had been proven previously by Beresnevich, Vaughan, Velani and Zorin in [26]. The more recent result by Huang improved the bound on the second term of (7.7) (we have omitted such terms in the above bounds, see [69] for more details).

p-adic points close to manifolds

As shown by the previous section there is a variety of results for the set of rational points close to real manifolds or curves. In the p-adic setting this is not the case. As far as the author is aware Theorem 6.1.3 is the first result remotely of this type in the p-adic setting.

Let $\mathbf{f}:\mathbb{Z}_p^d \to \mathbb{Z}_p^m$ with $\mathbf{f}=(f_1,\dots,f_m)$ where $f_i:\mathbb{Z}_p^d\to\mathbb{Z}_p$ for $1\le i\le m$. Let $r=d+m$ and

$$\mathcal{C}_f := \{(\boldsymbol{x},\mathbf{f}(\boldsymbol{x})) : \boldsymbol{x}\in\mathcal{U}\subseteq\mathbb{Z}_p^d\}\subset\mathbb{Z}_p^r.$$

For $\boldsymbol{\tau}=(\tau_1,\dots,\tau_m)\in\mathbb{R}_{>0}^m$ and $N\in\mathbb{N}$ let

$$\mathcal{N}_{\mathbf{f}}(\mathcal{U},N,\boldsymbol{\tau}) := \left\{ (q_0,\dots,q_r)\in\mathbb{Z}^{r+1} : \begin{array}{c} 0<q_0\le N, \\ \frac{q^{(d)}}{q_0}\in\mathcal{U},\ q^{(d)}=(q_1,\dots,q_d), \\ \max_{1\le i\le r}|q_i|\le N, \end{array} \left| f_i\left(\frac{q^{(d)}}{q_0}\right) - \frac{q_{d+i}}{q_0}\right|_p < N^{-\tau_i},\ 1\le i\le m \right\}.$$

As with the real case we would expect certain conditions to be imposed on the manifold $\mathcal{C}_f$. In particular it would be required that $\mathcal{C}_f$ avoids lying in rational hyperplanes for large regions, this would usually be restricted by imposing conditions on the curvature of the manifold. For example, in the real case with curves, the second derivative is bounded away from zero. However this becomes more problematic in the p-adic case where curves can be varying but still have derivative zero, see Example 2.1.1.

A key initial result is this setting would be to obtain the analogous result of Huxley's estimate for p-adic curves. Given such result the corresponding upper bound result of Theorem 5.2.5 for the case of planar curves would follow readily.

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