Optimal Stopping Methods for Multidimensional Problems, Pricing and Hedging of American Options

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This thesis is dedicated to my family.

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Abstract

In Chapter 1, we give an introduction to all subsequent chapters in the thesis.

In Chapter 2, we first introduce the underlying stochastic process, notations and some formulae used in the thesis. We then collect some classical results of optimal stopping and free boundary problems, including the solution of the American option pricing problem under the classical Black and Scholes model.

In Chapter 3, we consider the seller of a perpetual American put option who can hedge her portfolio once, until the underlying stock price leaves a certain range of values (a, b). We determine optimal trading boundaries as functions of the initial stock holding, and an optimal hedging strategy for a bond/stock portfolio. Optimality here refers to minimal variance of the hedging error at the (random) time when the stock leaves the interval (a, b). Our study leads to analytical expressions for both the optimal boundaries and the optimal stock holding, which can be evaluated numerically with no effort.

In Chapter 4, we study pricing of American put options on the Black and Scholes market with a stochastic interest rate and finite-time maturity. We prove that the option value is a C^1 function of the initial time, interest rate and stock price. By means of Itô calculus we rigorously derive the option's early exercise premium formula and the associated hedging portfolio. We prove the existence of an optimal exercise boundary splitting the state space into continuation and stopping region. The boundary has a parametrisation as a jointly continuous function of time and stock price, and it is the unique solution to an integral equation which we compute numerically. Our results hold for a large class of interest rate models including CIR and Vasicek models. We show a numerical study of the option price and the optimal exercise boundary for Vasicek model.

In Chapter 5, we derive a change of variable formula for C^1 functions $U : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ whose second order spatial derivatives may explode and not be integrable in the neighbourhood of a surface $b : \mathbb{R}_+ \times \mathbb{R}^{m-1} \to \mathbb{R}$ that splits the state

space into two sets $\mathcal{C}$ and $\mathcal{D}$. The formula is tailored for applications in problems of optimal stopping where it is generally very hard to control the second order derivatives of the value function near the optimal stopping boundary. Differently to other existing results on similar topics we only require that the surface b be monotonic in each variable and we formally obtain the same expression as the classical Itô's formula.

In Chapter 6, we provide sufficient conditions under which a two dimensional (time-space) optimal stopping surface, arising from a general three dimensional optimal stopping problem, is continuous. We require mild local regularity assumptions on the coefficients of the dynamics of the underlying process, the gain function and the value function. Further, we assume monotonicity of the optimal stopping surface in each variable.

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Chapter 1

Introduction

Optimal stopping problems deal with finding the time at which stopping a stochastic dynamics produces the largest reward or smallest cost according to a certain criterion. Such problems are extensively studied in various research areas. In mathematical finance, a benchmark optimal stopping problem is that of the optimal exercise of an American option. An American option allows its buyer to exercise a certain right, typically by buying or selling the underlying asset, at any time before a given maturity. Such flexibility has given American options popularity in the financial market, but the corresponding pricing and hedging problems are challenging.

In this thesis, we study two problems related to pricing and hedging of American put options. The problems are formulated as optimal stopping problems for diffusion processes under Markovian settings. We apply analytical and probabilistic methods to solve these problems while also developing techniques to handle two challenging questions that arise in general multi-dimensional optimal stopping problems. To be specific, under some mild assumptions, we prove the joint continuity of a two-dimensional time-space optimal stopping boundary, and we derive a change of variable formula for continuous differentiable (C^1) value functions in optimal stopping problems.

In the sequel, we give an introduction to the material covered in each chapter of this thesis. We focus on the main contributions and leave the literature review to the introduction of each individual chapter.

Chapter 2 contains background material needed in the subsequent chapters. We start with an introduction to diffusion processes considered in Chapter 3 and Chapter 4. After presenting some formulae and notations, we establish the convergence of diffusion's hitting and entry times to Borel sets under certain regularity conditions. We then collect key facts from optimal stopping theory and explain the free boundary methods for solving optimal stopping problems.

We finally illustrate the well-known free boundary techniques for pricing the American put option under the classical Black and Scholes model.

In Chapter 3, we construct a hedging strategy in a Black and Scholes market for the seller of a perpetual American put option, who holds a bond/stock hedging portfolio and can rebalance her position only once, until the stock price leaves a predetermined interval (a, b) with $0 < a < b < +\infty$. The aim of the trader is to minimise the variance of the hedging error at the (random) time at which the stock leaves the above interval.

We reduce the problem to an optimal stopping problem corresponding to the timing of a single rebalancing opportunity and an optimisation problem for the choice of the initial portfolio. We prove the existence of two optimal trading boundaries. When the stock price reaches either of the two optimal trading boundaries the hedging portfolio must be rebalanced; we give an analytical formula for the optimal stock holding after the trade, which in general is different from that prescribed by the classical Delta-hedging, i.e. stock holding is the Greek Delta, which is the derivative of the Black-Scholes option value with respect to the underlying stock price (see Chapter 2 Section 2.3.1).

The stopping boundaries can be calculated from analytical formulae up to a solution of algebraic equations. Those algebraic equations cannot be solved explicitly and do not reveal any further properties of the boundaries. Instead, we employ delicate probabilistic arguments to show that those boundaries exhibit monotone and continuous dependence on the initial stock holding.

We prove that the value function V of the stopping problem is a unique solution of a free boundary problem associated with the optimal trading boundaries. We show that V is everywhere continuously differentiable with respect to the initial stock holding and the initial stock price. Furthermore, discontinuities in the second order derivative with respect to the initial stock price occur only at the optimal trading boundaries. We finally compare the performance of our optimal hedging strategy to some frequently used ad-hoc strategies. Our optimal hedging strategy produces the variance of the tracking error which is up to 4 times smaller than the other strategies.

In Chapter 4, we study the pricing of an American put option on a Black and Scholes market with a stochastic interest rate and finite-time maturity. The stock price and the interest rate are driven by (possibly) correlated Brownian motions and we make mild assumptions about the dynamics of the interest rate under the pricing measure. It is worth noticing that CIR model, which does not satisfy these conditions, is also included in our analysis.

The American put option price is given by the value function of a related optimal stopping problem. In our model, this optimal stopping problem has a 3-dimensional state space with 2-dimensional diffusive dynamics (stock price and interest rate) and time. The stopping set, i.e., the set of points (t, r, x) for which it is optimal to exercise the option, is separated from the continuation set, where it is optimal to hold (or sell) the option, by a single surface (the stopping boundary). The value function is a classical solution to a second order parabolic PDE in the interior of the continuation set, i.e., it is twice continuously differentiable in (r, x) and continuously differentiable in t, whereas it coincides with the put payoff in the stopping set.

One of our technical contributions is to establish by means of probabilistic methods that the value function is globally continuously differentiable in all variables. Then, the continuity of the gradient of the value function permits the application of a change of variable formula (a generalisation of Itô's formula which we prove in Chapter 5) and a rigorous derivation of a hedging portfolio. The hedging portfolio invests in three instruments: the money market (savings) account, the zero-coupon bond with maturity equal to the maturity of the option and the stock. We show that the holdings in the bond and the stock are given by partial derivatives of the value function with respect to the interest rate and the stock price. As a further consequence of the change of variable formula we also derive the decomposition of the American option price as the sum of the price of a European put option with the same maturity and the same exercise price, and an *early exercise premium*. This is known in the literature as the *early exercise premium formula*, which corresponds to Doob's decomposition of supermartingales into a martingale and a non-increasing process (applied here to the Snell envelope of the optimal stopping problem).

Our second contribution concerns the continuity properties of the stopping boundary in our model, which have not been established in the literature. We are able to demonstrate that the stopping boundary, when parametrised as a function of (t, x), is continuous. Apart from being of interest in its own right, this enables a characterisation of the stopping boundary as the unique continuous solution of an integral equation arising from the early exercise premium decomposition. When a stopping boundary is known, efficient numerical methods are at disposal for computation of the option price. We finally compute the optimal stopping boundary surface numerically under the Vasicek interest rate model using an iterative scheme.

In Chapter 5, we develop a change of variable formula for C^1 value functions in general multi-dimensional optimal stopping problems. Our result complements existing generalisations of Itô's formula. We have a function $U : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ which can be thought of as the value function of an optimal stopping problem whose underlying stochastic process is a

d dimensional diffusion $\boldsymbol{X}$. We divide the state space $\mathbb{R}_+ \times \mathbb{R}^d$ into two subsets $\mathcal{C}$ and $\mathcal{D}$, whose boundary $\partial\mathcal{C}$ would correspond to the optimal stopping boundary. Our focus is on obtaining a formula that resembles the classical Itô's formula and does not involve either local times or the quadratic covariation between the underlying process $\boldsymbol{X}$ and the spatial gradient $\nabla U(t, \boldsymbol{X})$. This is important, for example, when deriving the dynamics of hedging portfolios for American options on multiple assets or integral equations for optimal stopping boundaries (in the spirit of numerous examples in the book by Peskir and Shiryaev [114]). Since we want to avoid using local times and quadratic covariation, we do require that the spatial gradient ∇U to be a continuous function. However, we require minimal regularity on the second order spatial derivatives of U near the boundary $\partial\mathcal{C}$ and very mild monotonicity properties of the boundary itself. Our assumptions are shown to hold naturally in a very broad class of optimal stopping problems for which existing generalisations of Itô's formula are either technically more involved than ours or not applicable.

In Chapter 6, we prove the joint continuity of the stopping boundary surface for general optimal stopping problems. Considering a finite horizon optimal stopping problem for a two dimensional diffusion, we prove that the optimal stopping boundary parameterised as a function of time and a space variable is continuous. Due to some technical difficulties, our arguments may not be extended beyond three dimensional time-space problems. Nonetheless, our work adds to the many known results of this type that only address scenarios when the boundary is a curve, i.e. a function of time or of a single state variable (e.g. [38], [113]). We take advantage of the local nature of the infinitesimal generator by setting assumptions on an open subset $\mathcal{U}$ of the state space and proving the continuity of the stopping boundary on $\mathcal{U}$. We assume the stopping boundary on $\partial\mathcal{C} \cap \mathcal{U}$ can be represented as a surface with certain monotonicity properties. Some mild and verifiable regularity assumptions of the coefficients of the SDE, the gain function and the value function are required on $\mathcal{U}$.

Chapter 2

Optimal stopping problems and the American put option

This chapter provides a theoretical background for optimal stopping problems as well as an introduction to the American put option under the classical Black and Scholes model. Some important properties and frequently used notations of the underlying stochastic process and stopping times are presented first.

2.1 Properties of the state process

We first introduce the diffusion process as a solution of a Stochastic Differential Equation (SDE). We point out its strong Markov property and the infinitesimal generator. The concept of regularity for one-dimensional diffusions is given as well as an analytical expression of the resolvent, which will be frequently used in Chapter 3. We then provide some useful facts of the first entry and hitting times, which will be used in proving the continuously differentiability of the value functions in Chapter 3 and Chapter 4.

2.1.1 The state process as a solution of SDE

Let $\boldsymbol{B}_t = (B_t^1, ..., B_t^{\hat{d}})$ be a $\hat{d}$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$, we have a d-dimensional diffusion $\boldsymbol{X} := (X^1, ..., X^d)$, which is a solution of the following SDE with coefficients $\boldsymbol{\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ and $\boldsymbol{\sigma} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{\hat{d}}$,

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{\alpha}(\boldsymbol{X}_t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t)\mathrm{d}\boldsymbol{B}_t, \quad \boldsymbol{X}_0 = \boldsymbol{x}. \tag{2.1}$$

Each X_t^i, $i = 1, ..., d$ has dynamics

$$\mathrm{d}X_t^i = \alpha^i(\boldsymbol{X}_t)\mathrm{d}t + \sum_{j=1}^{\hat{d}} \sigma^{ij}(\boldsymbol{X}_t)\mathrm{d}B_t^j, \quad X_0^i = x_i. \tag{2.2}$$

The process $\boldsymbol{X}$ takes values on $\mathcal{O} \subseteq \mathbb{R}^d$. The coefficients $\boldsymbol{\alpha}$, $\boldsymbol{\sigma}$ are assumed to be smooth enough (e.g. Lipschitz continuous), so that (2.1) admits a unique strong solution. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by $\boldsymbol{B}$ satisfying the usual condition, i.e., $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains all P-null sets. When it is needed, we use $\boldsymbol{X}^{\boldsymbol{x}} := (X^{1,\boldsymbol{x}}, ..., X^{d,\boldsymbol{x}})$ to indicate that the process starts from $\boldsymbol{x}$. We also often use notations

$$\mathsf{P}_{\boldsymbol{x}}\left(\,\cdot\,\right) = \mathsf{P}\left(\,\cdot\,|\boldsymbol{X}_0 = \boldsymbol{x}\right), \quad \mathsf{P}_{t,\boldsymbol{x}}\left(\,\cdot\,\right) = \mathsf{P}\left(\,\cdot\,|\boldsymbol{X}_t = \boldsymbol{x}\right),$$

the corresponding expectations are denoted by $\mathsf{E}_{\boldsymbol{x}}$, $\mathsf{E}_{t,\boldsymbol{x}}$.

Definition 2.1.1. *A Markov time τ with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$ is random variable $\Omega \to [0,\infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$. If $\tau < \infty$, $\mathsf{P} - a.s.$, we say τ is a stopping time.*

It is well known that the diffusion $\boldsymbol{X}$ is a time-homogeneous strong Markov process (cf. (2.4)) on the state space $(\mathcal{O}, \mathcal{B}(\mathcal{O}))$, where $\mathcal{B}(\mathcal{O})$ is the Borel σ-algebra of $\mathcal{O}$ ([87, Ch. 5.4], [119, Ch. V Sec. 4], [108, Ch. 7.2]). For a bounded measurable function $f : \mathcal{O} \to \mathbb{R}$, we define

$$(P_t f)(\boldsymbol{x}) := \mathsf{E}_{\boldsymbol{x}}\left[f(\boldsymbol{X}_t)\right]. \tag{2.3}$$

It follows that $\boldsymbol{x} \mapsto (P_t f)(\boldsymbol{x})$ is $\mathcal{B}(\mathcal{O})$ measurable ([17, Ch. 1, Thm. 3.6]). The process $\boldsymbol{X}$ is strong Markov, i.e.,

$$\mathsf{E}_{\boldsymbol{x}}\left[f(\boldsymbol{X}_{t+\tau})|\mathcal{F}_\tau\right] = (P_t f)(\boldsymbol{X}_\tau) = \mathsf{E}_{\boldsymbol{X}_\tau}\left[f(\boldsymbol{X}_t)\right], \quad \mathsf{P}_{\boldsymbol{x}} - a.s. \tag{2.4}$$

for any stopping time τ.

Sometimes it is convenient to consider an equivalent Markov process on the canonical space, so that a shift operator is well defined ([17, Ch. 1.3, 1.4]). Namely, consider a process $\widehat{\boldsymbol{X}} = (\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}_t}, \widehat{\boldsymbol{X}_t}, \theta_t, \widehat{\mathsf{P}}_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{O}}$, where $\widehat{\Omega} = C([0,\infty); \mathbb{R}^d)$, $\widehat{\boldsymbol{X}_t}(\omega) = \omega(t)$ for each $\omega \in \widehat{\Omega}$, $\widehat{\mathcal{F}} \supset \widehat{\mathcal{G}}^0 := \sigma\{\widehat{\boldsymbol{X}_t},\ t \geq 0\}$, $\widehat{\mathcal{F}_t} \supset \widehat{\mathcal{G}}_t^0 := \sigma\{\widehat{\boldsymbol{X}_s},\ s \leq t\}$ and $\widehat{\mathsf{P}}_{\boldsymbol{x}}(\widehat{\boldsymbol{X}_t} \in E) = \mathsf{P}_{\boldsymbol{x}}(\boldsymbol{X}_t \in E)$ for all $E \in \mathcal{B}(\mathcal{O})$. The space $\widehat{\Omega}$ is equipped with a family of shift operator $\theta_t : \widehat{\Omega} \to \widehat{\Omega}$, $t \geq 0$, such that

$$\widehat{\boldsymbol{X}_t}(\theta_s(\omega)) = \widehat{\boldsymbol{X}}_{t+s}(\omega), \quad \forall \omega \in \widehat{\Omega}. \tag{2.5}$$

In most of applications in this thesis, the expression under expectation only depends on the process, we hence assume, without loss of generality, the shift operator also applies to the process $\boldsymbol{X}$ in the same way as in (2.5). Using the shift operator θ, the strong Markov property (2.4) can be expressed as

$$\mathsf{E}_{\boldsymbol{x}}\left[f(\boldsymbol{X}_s)\circ\theta_\tau|\mathcal{F}_\tau\right]=\mathsf{E}_{\boldsymbol{X}_\tau}\left[f(\boldsymbol{X}_s)\right],\quad \mathsf{P}_{\boldsymbol{x}}-a.s. \tag{2.6}$$

More generally, we have

$$\mathsf{E}_{\boldsymbol{x}}\left[H\circ\theta_\tau|\mathcal{F}_\tau\right]=\mathsf{E}_{\boldsymbol{X}_\tau}\left[H\right]. \tag{2.7}$$

where $H:\Omega\to\mathbb{R}$ is a bounded and $\mathcal{F}_\infty^{\boldsymbol{X}}$-measurable random variable.

The infinitesimal generator of $\boldsymbol{X}$ acting on a function $f\in C^2(\mathcal{O})$ takes form

$$\mathcal{L}_{\boldsymbol{X}}f(\boldsymbol{x})=\frac{1}{2}\sum_{i,j=1}^{d}\beta^{ij}(\boldsymbol{x})f_{x_ix_j}(\boldsymbol{x})+\sum_{i=1}^{d}\alpha^i(\boldsymbol{x})f_{x_i}(\boldsymbol{x}), \tag{2.8}$$

where $\beta^{ij}(\boldsymbol{x}):=\sum_{k=1}^{\hat{d}}\sigma^{ik}(\boldsymbol{x})\sigma^{jk}(\boldsymbol{x})$, and $f_{x_i}, f_{x_ix_j}$ denote the first and second order partial derivatives of f with respect to x_i and x_i,x_j. In the one-dimensional case, derivatives are denoted by f_x or f', f''.

We now define some stopping times used in later chapters. Define $\tau_{\mathcal{K}}$ as first entry time of $\boldsymbol{X}$ to a Borel set $\mathcal{K}\subset\mathcal{O}$ such that

$$\tau_{\mathcal{K}}:=\inf\{t\geq 0:\ \boldsymbol{X}_t\in\mathcal{K}\}.$$

As $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and $\boldsymbol{X}$ is continuous and adapted to $\mathcal{F}_t$, it follows that $\tau_{\mathcal{K}}$ is a stopping time (or Markov time) ([118, Ch. III, Thm. 2.17]). If σ is a stopping time, we have

$$\tau_{\mathcal{K}}\circ\theta_\sigma=\inf\{t\geq 0:\ \boldsymbol{X}_{t+\sigma}\in\mathcal{K}\}.$$

The first hitting time to a set $\mathcal{K}\subset\mathcal{O}$ is denoted by $\sigma_{\mathcal{K}}$, and the first entry time to the interior of $\mathcal{K}$ is denoted by $\mathring{\sigma}_{\mathcal{K}}$. That is for $\boldsymbol{x}\in\mathcal{O}$, we set

$$\begin{aligned}\sigma_{\mathcal{K}}&:=\inf\{t>0:\boldsymbol{X}_t\in\mathcal{K}\},\quad \mathsf{P}_{\boldsymbol{x}}-a.s.\\ \mathring{\sigma}_{\mathcal{K}}&:=\inf\{t\geq 0:\boldsymbol{X}_t\in\operatorname{int}(\mathcal{K})\},\quad \mathsf{P}_{\boldsymbol{x}}-a.s.\end{aligned}$$

The above definition includes the special case when $\mathcal{K}=\{y\}$, $y\in\mathcal{O}$, in which the first hitting time is the first time $\boldsymbol{X}$ hits a point. When $\operatorname{int}(\mathcal{K})=\emptyset$, we set $\mathring{\sigma}_{\mathcal{K}}=\infty$. The first exit time from the set $\mathcal{K}$ is

$$\tau_{\mathcal{K}^c}:=\inf\{t\geq 0:\boldsymbol{X}_t\notin\mathcal{K}\},\quad \mathsf{P}_{\boldsymbol{x}}-a.s.$$

We end this subsection by introducing some notations for function spaces.

$$
\begin{aligned}
C(\mathcal{U}) &:= \{f : \mathcal{U} \to \mathbb{R} \mid f \text{ is continuous}\},\\
C^k(\mathcal{U}) &:= \{f : \mathcal{U} \to \mathbb{R} \mid f \text{ is } k \text{ times continuous differentiable}\},\\
L^p(\mathcal{U}) &:= \{f : \mathcal{U} \to \mathbb{R} \mid f \text{ is Lebesgue measurable}, \int_{\mathcal{U}} |f|^p \mathrm{d}x < \infty\},\\
L^\infty(\mathcal{U}) &:= \{f : \mathcal{U} \to \mathbb{R} \mid f \text{ is Lebesgue measurable}, \operatorname*{ess\,sup}_{\mathcal{U}} |f| < \infty\},\\
W^{k,p}(\mathcal{U}) &:= \{f \in L^p(\mathcal{U}) : D^\alpha f \in L^p(\mathcal{K}), \forall\, |\alpha| \le k\},
\end{aligned}
$$

where $k \in \mathbb{N}$, $1 \le p < \infty$ and $D^\alpha f$ is the mixed weak derivative of f with multi-index α. See e.g. [58, Ch. 5] for more detailed definition of Sobolev space $W^{k,p}$ and weak derivatives. We will use notation "ℓoc" if the property holds locally, e.g. we write $f \in L^p_{\ell oc}(\mathcal{U})$ if $f \in L^p(\mathcal{K})$ for any compact subset $\overline{\mathcal{K}} \subset \mathcal{U}$. We will also use notation $C^{i_1,\dots,i_n}$ to indicate the class of functions that have i_jth continuous derivative in the jth argument.

2.1.2 Linear diffusions

Let $d = \hat{d} = 1$ in (2.1) and let us restrict the state space to the interior of a possibly unbounded interval $\mathcal{O} := (l, r)$. This gives us a so called linear diffusion, i.e., there are functions $\alpha : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}_+$ and a one-dimensional Brownian motion B, such that X follows

$$
\mathrm{d}X_t = \alpha(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \quad X_0 = x. \tag{2.9}
$$

The infinitesimal generator of X acting on $f \in C^2(\mathcal{O})$ is

$$
\mathcal{L}_X f(x) = \frac{1}{2}\sigma^2(x) f_{xx}(x) + \alpha(x) f_x(x). \tag{2.10}
$$

The boundary points l, r are classified depending on the behaviour of the diffusion near the boundary. In this thesis, if the boundary point can be attained, we assume it is exit-not-entrance or reflecting. A full classification of boundary points can be found in [18, Ch. II].

Definition 2.1.2 ([119, Ch. V, (45.2)]). *We say a linear diffusion X starting at $x \in \operatorname{int}(\mathcal{O})$ is a regular diffusion if*

$$
\mathsf{P}_x(\sigma_{\{y\}} < \infty) > 0,
$$

for any $y \in \mathcal{O}$.

We only work with regular linear diffusions in this thesis. The following conditions are sufficient to guarantee that X is regular ([37], [88, p. 343-345])

$$\begin{aligned} &\forall\, x \in \mathcal{O}, \quad \sigma^2(x) > 0, \\ &\forall\, x \in \mathcal{O}, \quad \exists\, \varepsilon, \text{ such that } \int_{(x-\varepsilon, x+\varepsilon)} \frac{1+|\alpha(y)|}{\sigma^2(y)} dy < \infty. \end{aligned} \tag{2.11}$$

Lemma 2.1.3 ([119, Ch. V, (46.1)]). *For the regular diffusion X with $X_0 = x$, the following holds*

(i)

$$\mathsf{P}_x\left(\exists\, \varepsilon > 0 \textit{ such that } X_t \le x,\ \forall\, t \le \varepsilon\right) = 0, \tag{2.12}$$

(ii) for $a, b \in \mathcal{O}$, $a \le x \le b$, and all $p > 0$

$$\mathsf{E}_x\left[(\sigma_{\{a\}} \wedge \sigma_{\{b\}})^p\right] < \infty.$$

Note that one can also show that (2.12) holds with $X_t \le x$ replaced by $X_t \ge x$. The statement in Lemma 2.1.3 (i) indicates that for almost every ω, for any $\varepsilon > 0$, whenever $X_0 = x$ we have

$$\sup_{t\in[0,\varepsilon]} X_t(\omega) > x, \qquad \inf_{t\in[0,\varepsilon]} X_t(\omega) < x.$$

As a direct consequence, any boundary point $\partial\mathcal{K} := \{a, b\}$ of an interval $\mathcal{K} := [a, b]$ with $a, b \in \operatorname{int}(\mathcal{O})$ is regular in the sense that

$$\mathsf{P}_x\left(\sigma_{\mathcal{K}^c} > 0\right) = \mathsf{P}_x\left(\mathring{\sigma}_{\mathcal{K}^c} > 0\right) = 0, \quad x \in \partial\mathcal{K}. \tag{2.13}$$

The diffusion X driven by (2.9) with coefficients satisfying (2.11) has speed measure with density $m'(x)$ and scale function $s(x)$ given by ([118, Ch. VII, (3.20)]),

$$m'(x) = \frac{2}{\sigma^2(x) s'(x)}, \quad s(x) = \int_c^x s'(y)\mathrm{d}y, \quad s'(x) = e^{-\int_c^x 2\frac{\alpha(y)}{\sigma^2(y)}\mathrm{d}y}, \tag{2.14}$$

where c is an arbitrary constant on $\mathcal{O}$. Using the speed measure and the scale function, for any $x \in \mathcal{I} = (a, b) \subset \mathcal{O}$, and any bounded measurable function $g : \mathcal{I} \to \mathbb{R}$, the following expression holds (see [18, Ch. II, p. 19] and [3])

$$\mathsf{E}_x\left[\int_0^{\tau_{\mathcal{I}^c}} e^{-\lambda u} g(X_u)\mathrm{d}u\right] = w^{-1}\left(\varphi(x)\int_a^x \psi(z)g(z)m'(z)\mathrm{d}z + \psi(x)\int_x^b \varphi(z)g(z)m'(z)\mathrm{d}z\right). \tag{2.15}$$

Here w is the Wronskian (with the value independent of x)

$$w = \psi'(x)\frac{\varphi(x)}{s'(x)} - \varphi'(x)\frac{\psi(x)}{s'(x)} > 0,$$

and φ, ψ are decreasing and increasing fundamental solutions of

$$\mathcal{L}_X u(x) = \lambda u(x),$$

with boundary conditions depending on the boundary behaviour of X. We will further explain how to use this formula in Chapter 3.

2.1.3 Convergence of stopping times

In this subsection, we come back to the multi-dimensional case. We show that the first entry/hitting time of the state process to suitable sets are continuous with respect to the starting point of the process. Similar results can be found in many textbooks, e.g., [48, p. 32-40], [17, Ch. 1], however, it is convenient to gather them here to provide a direct reference for later applications.

In finite horizon optimal stopping problems (as the situation in Chapter 4) we work with the process $(t+s, \boldsymbol{X}_s)_{s\geq 0}$ on the state space $\widetilde{\mathcal{O}}$, where $\boldsymbol{X}$ is the time-homogeneous diffusion in (2.2) and

$$\widetilde{\mathcal{O}} := [0, T] \times \mathcal{O},$$

$T \in \mathbb{R}_+$ is some fixed constant. In infinite horizon problems where the state variable is only $\boldsymbol{X}$ (as the situation in Chapter 3), the following arguments can be repeated step by step and the main results still hold with obvious changes.

Let $\mathcal{K}$ be a closed subset of $\widetilde{\mathcal{O}}$, i.e., $\overline{\mathcal{K}} \cap \widetilde{\mathcal{O}} = \mathcal{K}$. Let $\sigma_{\mathcal{K}}$ be the first hitting time to $\mathcal{K}$ and $\mathring{\sigma}_{\mathcal{K}}$ be the first entry time to the interior of $\mathcal{K}$, that is, for $(t, \boldsymbol{x}) \in \widetilde{\mathcal{O}}$, $\mathsf{P}_{\boldsymbol{x}}$-a.s.

$$\begin{aligned} \sigma_{\mathcal{K}} &:= \inf\{s > 0 \,:\, (t+s, \boldsymbol{X}_s) \in \mathcal{K}\} \wedge (T-t), \\ \mathring{\sigma}_{\mathcal{K}} &:= \inf\{s \geq 0 \,:\, (t+s, \boldsymbol{X}_s) \in \operatorname{int}(\mathcal{K})\} \wedge (T-t). \end{aligned} \tag{2.16}$$

Both $\sigma_{\mathcal{K}}$ and $\mathring{\sigma}_{\mathcal{K}}$ are stopping times with respect to $\mathcal{F}_s$. It is immediate to see that

$$\sigma_{\mathcal{K}} \leq \mathring{\sigma}_{\mathcal{K}}, \qquad \mathsf{P}_{\boldsymbol{x}} - a.s. \tag{2.17}$$

We write $\sigma_{\mathcal{K}}(t, \boldsymbol{x})$ and $\mathring{\sigma}_{\mathcal{K}}(t, \boldsymbol{x})$ to indicate the starting point of the process.

Denote by $\mathcal{A}$ the complement of $\mathcal{K}$ in $\operatorname{int}(\widetilde{\mathcal{O}})$: $\mathcal{A} := \mathcal{K}^c \cap \operatorname{int}(\widetilde{\mathcal{O}})$ (which is an open set). We make two assumptions which will be verified in each specific case in later chapters.

Assumption 2.1.4 (Regularity)**.** *For* $(t_0, \boldsymbol{x}_0) \in \partial\mathcal{K}$*, we have*

$$\mathsf{P}_{t_0,\boldsymbol{x}_0}(\sigma_{\mathcal{K}} > 0) = \mathsf{P}_{t_0,\boldsymbol{x}_0}(\mathring{\sigma}_{\mathcal{K}} > 0) = 0. \tag{2.18}$$

Assumption 2.1.5 (Continuity of the flow)**.** *For any sequence* $(\boldsymbol{x}_n)_{n\geq 1} := (x_{1,n}, ..., x_{d,n})$ *in* $\mathcal{O}$ *converging to* $\boldsymbol{x} := (x_1, ..., x_d) \in \mathcal{O}$ *as* $n \to \infty$*, it holds that*

$$\lim_{n\to+\infty} \sup_{0\leq t\leq T} \|\boldsymbol{X}_t^{\boldsymbol{x}_n} - \boldsymbol{X}_t^{\boldsymbol{x}}\| = 0, \qquad \mathsf{P} - a.s, \tag{2.19}$$

where $\|\cdot\|$ *is the Euclidean norm.*

Note that (2.18) means the boundary of $\mathcal{A}$ is regular for the set $\mathcal{K}$ with respect to the process $(t, \boldsymbol{X})$. As discussed in the previous subsection, (2.18) holds for regular one dimensional diffusions as a consequence of Lemma 2.1.3. In multi-dimensional case, whether (2.18) holds or not depends on both the geometry of the boundary and the process itself. We will prove a boundary surface is regular for a two dimensional diffusion process in Chapter 4 under suitable conditions. A broader discussion on this topic can be found in [87, Ch. 4].

Lemma 2.1.6. *Under Assumption 2.1.4,* $\mathsf{P}_{t,\boldsymbol{x}}(\sigma_{\mathcal{K}} = \mathring{\sigma}_{\mathcal{K}}) = 1$ *for all* $(t, \boldsymbol{x}) \in \widetilde{\mathcal{O}}$.

Proof. The equality is trivial for $(t, \boldsymbol{x}) \in \text{int}(\mathcal{K})$. Take $(t, \boldsymbol{x})$ in its complement, i.e., in $\overline{\mathcal{A}} \cap \widetilde{\mathcal{O}}$. Since (2.17) holds, we only need to show that $\mathsf{P}_{t,r,x}(\sigma_{\mathcal{K}} < \mathring{\sigma}_{\mathcal{K}}) = 0$. Let us argue by contradiction and assume that $\mathsf{P}_{t,\boldsymbol{x}}(\sigma_{\mathcal{K}} < \mathring{\sigma}_{\mathcal{K}}) > 0$. There exists $\delta > 0$ such that $\mathsf{P}_{t,\boldsymbol{x}}(\mathring{\sigma}_{\mathcal{K}} \geq \sigma_{\mathcal{K}} + \delta) > 0$. By the strong Markov property we get

$$\begin{aligned}\mathsf{P}_{t,\boldsymbol{x}}(\mathring{\sigma}_{\mathcal{K}} \geq \sigma_{\mathcal{K}} + \delta) &= \mathsf{E}_{t,\boldsymbol{x}}\left[\mathsf{E}_{t,\boldsymbol{x}}\left(\mathbb{1}_{\{\sigma_{\mathcal{K}}+\mathring{\sigma}_{\mathcal{K}}\circ\theta_{\sigma_{\mathcal{K}}}\geq\sigma_{\mathcal{K}}+\delta\}}\middle|\mathcal{F}_{\sigma_{\mathcal{K}}}\right)\right] \\ &= \mathsf{E}_{t,\boldsymbol{x}}\left[\mathsf{P}_{t+\sigma_{\mathcal{K}},\boldsymbol{X}_{\sigma_{\mathcal{K}}}}(\mathring{\sigma}_{\mathcal{K}} \geq \delta)\right] = 0,\end{aligned}$$

where the last equality follows by observing that $(t + \sigma_{\mathcal{K}}, \boldsymbol{X}_{\sigma_{\mathcal{K}}}) \in \partial\mathcal{A}$, $\mathsf{P}_{t,\boldsymbol{x}}$-a.s. by the continuity of the path, and Assumption 2.1.4. □

Lemma 2.1.7. *Let* $\widetilde{\mathcal{O}} \ni (t_n, \boldsymbol{x}_n) \to (t, \boldsymbol{x}) \in \widetilde{\mathcal{O}}$ *as* $n \to \infty$*. Then under Assumption 2.1.5* P*-a.s. it holds*

$$\limsup_{n\to\infty} \mathring{\sigma}_{\mathcal{K}}(t_n, \boldsymbol{x}_n) \leq \mathring{\sigma}_{\mathcal{K}}(t, \boldsymbol{x}).$$

Proof. For simplicity, we denote $\mathring{\sigma}_n := \mathring{\sigma}_{\mathcal{K}}(t_n, \boldsymbol{x}_n)$ and $\mathring{\sigma}_{\mathcal{K}} := \mathring{\sigma}_{\mathcal{K}}(t, \boldsymbol{x})$. For P-a.e. $\omega \in \Omega$ we have by (2.19)

$$(t_n + s, \boldsymbol{X}_s^{\boldsymbol{x}_n})(\omega) \to (t + s, \boldsymbol{X}_s^{\boldsymbol{x}})(\omega), \qquad s \in [0, T - t]. \tag{2.20}$$

Fix $\omega \in \Omega$ in the set of P-full measure for which the above holds. If $\mathring{\sigma}_{\mathcal{K}}(\omega) = T - t$ then the result is obvious because $\mathring{\sigma}_n(\omega) \leq T - t_n$. Assume $\mathring{\sigma}_{\mathcal{K}}(\omega) < T - t$. Take any $\delta < T - t$ such that $\mathring{\sigma}_{\mathcal{K}}(\omega) < \delta$. By the continuity of paths and the openness of int$(\mathcal{K})$, there is $\delta' \in (\mathring{\sigma}_{\mathcal{K}}(\omega), \delta)$ such that $(t + \delta', \boldsymbol{X}^{\boldsymbol{x}}_{\delta'})(\omega) \in$ int$(\mathcal{K})$. From (2.20) and the openness of int$(\mathcal{K})$, $(t_n + \delta', \boldsymbol{X}^{\boldsymbol{x}_n}_{\delta'})(\omega) \in$ int$(\mathcal{K})$ for all sufficiently large n, so $\limsup_{n\to\infty} \mathring{\sigma}_n(\omega) \leq \delta'$. As the above argument holds for any $\delta > \mathring{\sigma}_{\mathcal{K}}(\omega)$ and for a.e. $\omega \in \Omega$, we obtain the claim. □

Lemma 2.1.8. *Let $\widetilde{\mathcal{O}} \ni (t_n, \boldsymbol{x}_n) \to (t, \boldsymbol{x}) \in \widetilde{\mathcal{O}}$ as $n \to \infty$. Then, under Assumptions 2.1.4 and 2.1.5,*

$$\liminf_{n\to\infty} \sigma_{\mathcal{K}}(t_n, \boldsymbol{x}_n) \geq \sigma_{\mathcal{K}}(t, \boldsymbol{x}), \qquad \mathsf{P}\text{-}a.s. \tag{2.21}$$

Proof. For $\boldsymbol{y}, \boldsymbol{z} \in \widetilde{\mathcal{O}}$ we denote by $\mathrm{d}(\boldsymbol{y}, \boldsymbol{z})$ their Euclidean distance and by $\mathrm{d}(\boldsymbol{y}, \partial\mathcal{A}) = \inf\{\mathrm{d}(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z} \in \partial\mathcal{A}\}$. Denote $\sigma_n := \sigma_{\mathcal{K}}(t_n, \boldsymbol{x}_n)$ and $\sigma_{\mathcal{K}} := \sigma_{\mathcal{K}}(t, \boldsymbol{x})$. To simplify notation we also set

$$\zeta_s := (t + s, \boldsymbol{X}^{\boldsymbol{x}}_s) \quad \text{and} \quad \zeta^n_s := (t_n + s, \boldsymbol{X}^{\boldsymbol{x}_n}_s).$$

Fix $\omega \in \Omega$ in a set of full P-measure on which trajectories of ζ are continuous and the limit (2.19) holds. If $\sigma_{\mathcal{K}}(\omega) = 0$ then (2.21) holds trivially. Otherwise, we must have $(t, \boldsymbol{x}) \in \mathcal{A}$ by Assumption 2.1.4, so $\mathrm{d}(\zeta_0(\omega), \partial\mathcal{A}) > 0$ since $\mathcal{A}$ is open. For $0 < \varepsilon < \mathrm{d}(\zeta_0(\omega), \partial\mathcal{A})$, define

$$\delta_\varepsilon = \inf\{s \in [0, T - t] \,:\, \mathrm{d}(\zeta_s(\omega), \partial\mathcal{A}) \leq \varepsilon\}.$$

Using the triangle inequality we get $\mathrm{d}(\zeta^n_s(\omega), \partial\mathcal{A})) + \mathrm{d}(\zeta^n_s(\omega), \zeta_s(\omega)) > \varepsilon/2$ for all $s \in [0, \delta_\varepsilon]$. Thanks to Assumption 2.1.5, $\mathrm{d}(\zeta^n_s(\omega), \zeta_s(\omega)) < \varepsilon/4$ for all $s \in [0, \delta_\varepsilon]$ and all sufficiently large n, so $\mathrm{d}(\zeta^n_s(\omega), \partial\mathcal{A})) \geq \varepsilon/4$ for all $s \in [0, \delta_\varepsilon]$ and all sufficiently large n. This gives

$$\liminf_{n\to\infty} \sigma_n(\omega) \geq \delta_\varepsilon.$$

Using the continuity of $t \mapsto \zeta_t(\omega)$ and the fact that $\zeta_{\sigma_{\mathcal{K}}}(\omega) \in \partial\mathcal{A}$, we have $\lim_{\varepsilon\to 0} \delta_\varepsilon = \sigma_{\mathcal{K}}(\omega)$. As this holds for a.e. $\omega \in \Omega$, the proof of (2.21) is complete. □

Lemma 2.1.6, 2.1.7 and 2.1.8 imply the main result of this subsection

Proposition 2.1.9. *Let $\widetilde{\mathcal{O}} \ni (t_n, \boldsymbol{x}_n) \to (t, \boldsymbol{x}) \in \widetilde{\mathcal{O}}$ as $n \to \infty$. Then*

$$\begin{aligned} \lim_{n\to\infty} \mathring{\sigma}_{\mathcal{K}}(t_n, \boldsymbol{x}_n) &= \mathring{\sigma}_{\mathcal{K}}(t, \boldsymbol{x}), \qquad \mathsf{P} - a.s. \\ \lim_{n\to\infty} \sigma_{\mathcal{K}}(t_n, \boldsymbol{x}_n) &= \sigma_{\mathcal{K}}(t, \boldsymbol{x}), \qquad \mathsf{P} - a.s. \end{aligned} \tag{2.22}$$

Remark 2.1.10. *The result in Proposition 2.1.9 means that for each* $(t_n, \boldsymbol{x}_n) \to (t, \boldsymbol{x})$*, we have* $\mathring{\sigma}_{\mathcal{K}}(t_n, \boldsymbol{x}_n) \to \mathring{\sigma}_{\mathcal{K}}(t, \boldsymbol{x})$*,* $\sigma_{\mathcal{K}}(t_n, \boldsymbol{x}_n) \to \sigma_{\mathcal{K}}(t, \boldsymbol{x})$*,* P*-a.s. One should note that the set of full measure depends on the sequence chosen. The convergence is sometimes called "continuity" in later chapters. However, one should be careful that the first hitting time may be a.s. discontinuous with respect to the starting point, as this is the case for Brownian motion (see [118, Prop. 3.8]).*

2.2 Optimal stopping problems

We now formulate an optimal stopping problem for diffusion processes and provide some important results collected from the book of Peskir and Shiryaev [114]. We will elaborate on free boundary methods in solving optimal stopping problems under the Markovian structure. Notice that the processes and functions in the formulation are not as general as in [114], but sufficient for the optimal stopping problems studied in later chapters.

An optimal stopping problem in infinite horizon has the following general form

$$V(\boldsymbol{x}) := \sup_{0 \le \tau < \infty} \mathsf{E}_{\boldsymbol{x}} \left[M(\boldsymbol{X}_\tau) \right], \tag{2.23}$$

where $\boldsymbol{X} = (X_t^1, ..., X_t^d)$ is a d dimensional diffusion on $\mathcal{O} \subset \mathbb{R}^d$ driven by SDE (2.2), τ is a $(\mathcal{F}_t)_{t \ge 0}$ stopping time. We assume that $\mathsf{E}_{\boldsymbol{x}} [M(\boldsymbol{X}_\tau)]$ is well defined for every $\boldsymbol{x}$ and τ, and M is continuous.

We also have the following finite horizon optimal stopping problem where the stopping time is bounded by a constant

$$V(t, \boldsymbol{x}) := \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,\boldsymbol{x}} \left[M(t+\tau, \boldsymbol{X}_{t+\tau}) \right]. \tag{2.24}$$

In the main theorems below, we will only use the formulation (2.23) as the results also apply to (2.24) if we consider the process $\boldsymbol{Z}_t := (t, \boldsymbol{X}_t)$, where $\boldsymbol{Z}$ has state space $[0, T] \times \mathcal{O}$.

The function V is referred to as the value function, and M is referred to as the gain function (we also call it payoff function in the context of option pricing). Let

$$\mathcal{C} := \{\boldsymbol{x} \in \mathcal{O} : V(\boldsymbol{x}) > M(\boldsymbol{x})\}, \tag{2.25}$$

$$\mathcal{D} := \{\boldsymbol{x} \in \mathcal{O} : V(\boldsymbol{x}) = M(\boldsymbol{x})\}. \tag{2.26}$$

We call the set $\mathcal{C}$ the continuation set and its complement $\mathcal{D}$ the stopping set. In the case of finite horizon, the corresponding continuation and stopping set are

$$\mathcal{C} := \{(t, \boldsymbol{x}) \in [0, T] \times \mathcal{O} : V(t, \boldsymbol{x}) > M(t, \boldsymbol{x})\}, \tag{2.27}$$

$$\mathcal{D} := \{(t, \boldsymbol{x}) \in [0, T] \times \mathcal{O} : V(t, \boldsymbol{x}) = M(t, \boldsymbol{x})\}. \tag{2.28}$$

For simplicity, throughout this section, we also assume that V is continuous. When the underlying stochastic process is a diffusion and the gain function is (Lipschitz) continuous, we can usually prove the value function is (Lipschitz) continuous. We will verify the continuity of the value function in each specific case in later chapters. As a consequence of the continuity of both V and M, the continuation set $\mathcal{C}$ is open and the stopping set $\mathcal{D}$ is closed.

Let

$$\tau_{\mathcal{D}} := \inf\{t \geq 0 : \boldsymbol{X}_t \in \mathcal{D}\}, \tag{2.29}$$

be the first entry time of $\boldsymbol{X}$ into the set $\mathcal{D}$, hence a stopping (Markov) time. We now discuss the superharmonic property of the value function and the optimality of $\tau_{\mathcal{D}}$.

Definition 2.2.1 ([114, Ch. I, Definition 2.3])**.** *A measurable function $f : \mathcal{O} \mapsto \mathbb{R}$ is a superharmonic function if for all bounded $(\mathcal{F}_t)_{t\geq 0}$ stopping times τ and all $\boldsymbol{x} \in \mathcal{O}$,*

$$\mathsf{E}_{\boldsymbol{x}}\left[f(\boldsymbol{X}_\tau)\right] \leq f(\boldsymbol{x}).$$

Definition 2.2.2. *For each $\boldsymbol{x} \in \mathcal{O}$, a $(\mathcal{F}_t)_{t\geq 0}$ stopping time τ is optimal for the problem (2.23) if*

$$V(\boldsymbol{x}) = \mathsf{E}_{\boldsymbol{x}}\left[M(\boldsymbol{X}_\tau)\right].$$

Theorem 2.2.3 ([114, Ch. I, Thm. 2.4])**.** *Assume for every $\boldsymbol{x} \in \mathcal{O}$, there exists an optimal stopping time $\tau_*^{\boldsymbol{x}}$ for problem (2.23), i.e.*

$$V(\boldsymbol{x}) = \mathsf{E}_{\boldsymbol{x}}\left[M(\boldsymbol{X}_{\tau_*^{\boldsymbol{x}}})\right].$$

Then the value function V is the smallest superharmonic function that dominates the gain function M on $\mathcal{O}$. Moreover, we have

(i) the stopping time $\tau_{\mathcal{D}}$ defined in (2.29) satisfies $\tau_{\mathcal{D}} \leq \tau_*^{\boldsymbol{x}}$, $\mathsf{P}_{\boldsymbol{x}} - a.s.$, for every $\boldsymbol{x} \in \mathcal{O}$ and it is optimal for (2.23);

(ii) the stopped process $(V(\boldsymbol{X}_{t\wedge\tau_{\mathcal{D}}}))_{t\geq 0}$ is a continuous martingale under $\mathsf{P}_{\boldsymbol{x}}$ for every $\boldsymbol{x} \in \mathcal{O}$.

It is very often that we need to consider the following optimal stopping problem that contains a running cost and a discounting factor:

$$\widetilde{V}(\boldsymbol{x}) := \sup_{0\le\tau<\infty} \mathsf{E}_{\boldsymbol{x}}\left[\int_0^\tau e^{-\Lambda_u} H(\boldsymbol{X}_u)\mathrm{d}u + e^{-\Lambda_\tau} M(\boldsymbol{X}_\tau)\right], \tag{2.30}$$

where

$$\Lambda_u := \int_0^u \lambda(\boldsymbol{X}_s)\mathrm{d}s, \tag{2.31}$$

and H, λ are continuous functions. We now provide an analogue of Theorem 2.2.3 that applies to problem (2.30).

Theorem 2.2.4. *Assume for every $\boldsymbol{x} \in \mathcal{O}$, there exists an optimal stopping time $\tau_*^{\boldsymbol{x}}$ for problem (2.30), i.e.*

$$\widetilde{V}(\boldsymbol{x}) = \mathsf{E}_{\boldsymbol{x}}\left[\int_0^{\tau_*^{\boldsymbol{x}}} e^{-\Lambda_u} H(\boldsymbol{X}_u)\mathrm{d}u + e^{-\Lambda_{\tau_*^{\boldsymbol{x}}}} M(\boldsymbol{X}_{\tau_*^{\boldsymbol{x}}})\right].$$

Then $\widetilde{V}$ is the smallest function that dominates M and such that the process

$$Y_t := \int_0^t e^{-\Lambda_u} H(\boldsymbol{X}_u)\mathrm{d}u + e^{-\Lambda_t}\widetilde{V}(\boldsymbol{X}_t)$$

is a supermartingale under $\mathsf{P}_{\boldsymbol{x}}$. Moreover, we have

(i) the stopping time $\tau_{\mathcal{D}}$, where $\mathcal{D} := \{\boldsymbol{x} \in \mathcal{O} : \widetilde{V}(\boldsymbol{x}) = M(\boldsymbol{x})\}$, satisfies $\tau_{\mathcal{D}} \le \tau_*^{\boldsymbol{x}}$, $\mathsf{P}_{\boldsymbol{x}} - a.s.$, for every $\boldsymbol{x} \in \mathcal{O}$ and it is optimal for (2.30).

(ii) the stopped process $(Y_{t\wedge\tau_{\mathcal{D}}})_{t\ge 0}$ is a continuous martingale under $\mathsf{P}_{\boldsymbol{x}}$ for every $\boldsymbol{x} \in \mathcal{O}$.

While Theorems 2.2.3 and 2.2.4 summarise key properties of the value function and identify $\tau_{\mathcal{D}}$ as the minimal optimal stopping time, we still need to justify the existence of an optimal stopping time. The following corollary is a useful tool. In particular, it indicates that an optimal stopping time always exists in finite horizon problems.

Corollary 2.2.5 ([114, Ch. I, Cor. 2.9])**.** *For the infinite horizon problem (2.23), if for all $\boldsymbol{x} \in \mathcal{O}$*

$$\mathsf{P}_{\boldsymbol{x}}(\tau_{\mathcal{D}} < \infty) = 1,$$

then $\tau_{\mathcal{D}}$ is optimal in (2.23). If $\mathsf{P}_{\boldsymbol{x}}(\tau_{\mathcal{D}} < \infty) < 1$, then there is no optimal stopping time.

For the finite horizon problem (2.24), the stopping time

$$\tau_{\mathcal{D}} := \inf\{s \ge 0 : (t+s, \boldsymbol{X}_s) \in \mathcal{D}\} \wedge (T-t),$$

is optimal.

An optimal stopping problem can be connected to a Partial Differential Equation. Consider the following Dirichlet problem.

$$\begin{aligned} &(\mathcal{L}_{\boldsymbol{X}} - \lambda)U = -H, \qquad \boldsymbol{x} \in \mathcal{C}, \\ &U = \widetilde{V} = M, \qquad \boldsymbol{x} \in \partial\mathcal{C}, \end{aligned} \tag{2.32}$$

where $\mathcal{C}$ is the continuation set of problem (2.30), H and λ are as defined in (2.30). If the coefficients of $\boldsymbol{X}$ are sufficiently smooth, e.g. Hölder continuous, classical PDE results ([69, Ch. 3], [75, Ch. 6]) give that, for any open ball $\mathcal{K} \subset \mathcal{C}$, the problem

$$\begin{aligned} &(\mathcal{L}_{\boldsymbol{X}} - \lambda)U = -H, \qquad \boldsymbol{x} \in \mathcal{K}, \\ &U = \widetilde{V}, \qquad \boldsymbol{x} \in \partial\mathcal{K}, \end{aligned} \tag{2.33}$$

admits a unique solution $U \in C^2(\mathcal{K}) \cap C(\overline{\mathcal{K}})$. Applying Itô 's formula to $U(\boldsymbol{X})$ stopped at $\tau_{\mathcal{K}^c}$ and using the martingale property of the process Y in Theorem 2.2.4, one can show that $U = \widetilde{V}$ on $\mathcal{K}$ (see the complete proof in Chapter 6 Lemma 6.3.2). As this holds for any $\mathcal{K} \subset \mathcal{C}$, we can conclude that the value function $\widetilde{V}$ is the unique solution to (2.32) and $\widetilde{V} \in C^2(\mathcal{C})$. This fact characterises the value function in the continuation set. However, the PDE system (2.32) is not sufficient to uniquely determine the optimal stopping boundary and the value function. The so called smooth-fit principle should also hold, which means the value function should be continuously differentiable at any point $\boldsymbol{x}_0 \in \partial\mathcal{C}$:

$$\lim_{\mathcal{C} \ni \boldsymbol{x} \to \boldsymbol{x}_0 \in \partial\mathcal{C}} \widetilde{V}_{x_i}(\boldsymbol{x}) = \lim_{\mathcal{D} \ni \boldsymbol{x} \to \boldsymbol{x}_0 \in \partial\mathcal{C}} \widetilde{V}_{x_i}(\boldsymbol{x}) = \lim_{\mathcal{D} \ni \boldsymbol{x} \to \boldsymbol{x}_0 \in \partial\mathcal{C}} M_{x_i}(\boldsymbol{x}), \quad i = 1, ..., d. \tag{2.34}$$

The smooth fit condition plays a crucial role in uniquely determining the stopping boundary. We will provide detailed proofs in later chapters. The smooth-fit condition (2.34) together with the PDE system (2.32) form a free boundary problem, where the term "free" indicates that the stopping boundary $\partial\mathcal{C}$ must be found as part of the solution.

This section just covers the fundamentals of an optimal stopping problem. More work is needed to analyse the properties of the value function and characterise the optimal stopping boundary as explicitly as possible. It turns out that methods for solving one and multidimensional problems are significantly different. In the following part, we will use the well-known American put option pricing problem to demonstrate the difference.

2.3 The American put option with constant interest rate

As the American option pricing problem under the classical Black and Scholes model is already extensively studied in the literature, we only present the main results without giving proofs and focus on explaining the core ideas. An American put option written on an risky asset X_t struck at K allows the buyer to exercise the option with payoff $(K - X_t)^+$ at any time before a given maturity date T. We start with the perpetual American put option, i.e. the case when $T = \infty$. Under the Black and Scholes model, the option price is defined by

$$P(x) := \sup_{0 \le \tau < \infty} \mathsf{E}_x \left[e^{-r\tau} \Big(K - X_\tau \Big)^+ \right]. \tag{2.35}$$

Here X is a regular linear diffusion with state space $\mathbb{R}_+$ driven by the following SDE with constant coefficients $r, \sigma > 0$,

$$\mathrm{d}X_t = rX_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \quad X_0 = x \in \mathbb{R}_+. \tag{2.36}$$

This SDE admits an explicit solution given by

$$X_t = xe^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t}, \quad t \ge 0. \tag{2.37}$$

We proceed to solve problem (2.35) via the so-called "guess and verify" approach. Firstly, it is evident that the option buyer should never exercise the option when $X_t > K$. Secondly, as the stock price decreases below K, the payoff increases. However, because of the discounting factor, the longer the buyer waits, the smaller the present value she finally obtains. It is then reasonable to guess that the optimal strategy is to exercise the option at some level $\hat{a} < K$. Based on this guess, one conjectures the stopping set and continuation set of the form

$$\begin{aligned} \mathcal{D} &= \{x \in \mathbb{R}_+ : P(x) = (K - x)^+\} = \{x \in \mathbb{R}_+ : x \le \hat{a}\}, \\ \mathcal{C} &= \{x \in \mathbb{R}_+ : P(x) > (K - x)^+\} = \{x \in \mathbb{R}_+ : x > \hat{a}\}, \end{aligned}$$

where $\hat{a}$ needs to be determined.

By Theorem 2.2.3, the corresponding optimal stopping time is the first entry time into set $\mathcal{D}$,

$$\tau_* = \inf\{t \ge 0 : X_t \le \hat{a}\}. \tag{2.38}$$

and the value function is given by

$$P(x) = \mathsf{E}_x \left[e^{-r\tau_*} (K - X_{\tau_*})^+ \right].$$

From the discussion around equations (2.32)-(2.34), we postulate that the value function P should solve a free boundary problem. We need to identify a couple (U, a) solving

$$\begin{aligned} &(\mathcal{L}_X - r)U(x) = 0, \quad x > a, \\ &(\mathcal{L}_X - r)U(x) \le 0, \quad a.e.\ x \in \mathbb{R}_+, \\ &U(x) \ge (K-x)^+, \quad x \in \mathbb{R}_+ \\ &U(x) = (K-x)^+, \quad x \le a, \\ &U'(x) = -1, \quad x = a. \end{aligned} \tag{2.39}$$

The free boundary system (2.39) allows an explicit solution that reads

$$U(x) = \begin{cases} \frac{1}{d} a^{1+d} x^{-d}, & a \le x < \infty, \\ K - x, & 0 \le x \le a, \end{cases} \tag{2.40}$$

where $d = 2r/\sigma^2$ and

$$a = \frac{K}{1 + \frac{1}{d}}.$$

Notice that the first equation in (2.39) admits a fundamental solution of the form $U(x) = C_1 x^{-d} + C_2 x$. As the value function P is bounded on $\mathbb{R}_+$, we can immediately deduce that $C_2 = 0$. The remaining equations in (2.39) are sufficient to determine C_1 and a. The next step is to verify that U in (2.40) is indeed the solution of the optimal stopping problem (2.35) and $\hat{a} = a$, which is done by the following verification theorem.

Theorem 2.3.1. *The couple $(U(x), a)$ uniquely solves the optimal stopping problem (2.35), that is $P(x) = U(x)$, $\hat{a} = a$ and τ_* in (2.38) is the corresponding optimal stopping time.*

Proof. see [114, Ch. VII, Thm. 25.1] □

The key step of proving Theorem 2.3.1 is to apply Itô–Tanaka–Meyer formula to function U, then the optional sampling theorem and Fatou's lemma to show that $U = P$. This method only applies to one-dimensional problems and requires a simple enough gain function whose properties can be easily analysed. Otherwise, it is difficult to guess the optimal stopping boundary and further derive a candidate value function. This issue appears in Chapter 3.

For optimal stopping problems with one-dimensional diffusions, there is an efficient alternative method developed by Dayanik and Karatzas [37]. Based on the characteristics of linear diffusions, they establish a direct relationship between the value function and the concave majorant of the gain function. In the context of the optimal stopping problem (2.35), let φ, ψ be the decreasing and increasing fundamental solutions of

$$\mathcal{L}_X u(x) - r u(x) = 0,$$

and $F(x) := \psi(x)/\varphi(x)$. Then their results ([37, Proposition 4.2], [37, Proposition 4.3]) conclude that function P/φ is the smallest concave majorant of function

$$\widehat{M}(x) := \frac{(K - F^{-1}(x))^+}{\varphi(F^{-1}(x))},$$

where F^{-1} is the inverse of F. The optimal stopping problem (2.35) is then reduced to a problem of finding a concave majorant of the function $\widehat{M}$. This method, in general, is applicable to any infinite horizon optimal stopping problem for one-dimensional diffusions. However, it still requires the gain function to be simple enough after being transformed via function F. So that its concave majorant can be derived. When the dimension of the state space of the problem is greater or equal than two, the stopping boundary is no longer constant and an explicit candidate solution is no longer available. As a result, neither the "guess and verify" nor the concave majorant characterisation in [37] are applicable.

We now move to the American put option with finite maturity $T < \infty$. In the Black-Scholes model, the option's arbitrage free price at time $t < T$ is given by

$$P(t, x) := \sup_{0 \le \tau \le T-t} \mathsf{E}_x \left[e^{-r\tau} \left(K - X_\tau \right)^+ \right], \tag{2.41}$$

where K is the strike, X is the underlying stock price following Geometric Brownian motion as in (2.36). Note that we still consider X starting at $X_0 = x$ in (2.41) because the discounted payoff function is time-homogeneous. As Theorem 2.2.3 still applies, we have

$$\mathcal{C} = \{(t, x) \in [0, T] \times \mathbb{R}_+ \, : \, P(t, x) > (K - x)^+\},$$
$$\mathcal{D} = \{(t, x) \in [0, T] \times \mathbb{R}_+ \, : \, P(t, x) = (K - x)^+\},$$

and $\tau_* = \inf\{s \ge 0, (t+s, X_s) \in \mathcal{D}\}$ is the optimal stopping time. It is evident that $\tau_* \le T - t$ as $P(T, x) = (K - x)^+$.

We summarize the results of this problem in the theorem below.

Theorem 2.3.2. *The following results hold for the optimal stopping problem* (2.41).

(i) $P(t, x)$ *is non increasing and convex in* x*, and continuous on* $[0, T] \times \mathbb{R}_+$.

(ii) There exists a function $b : [0, T] \mapsto (0, K]$ *such that*

$$\mathcal{C} = \{(t, x) \in [0, T] \times \mathbb{R}_+ \, : \, x > b(t)\}, \tag{2.42}$$
$$\mathcal{D} = \{(t, x) \in [0, T] \times \mathbb{R}_+ \, : \, x \le b(t)\}. \tag{2.43}$$

(iii) $b(t)$ *is continuous, non-decreasing and* $\lim_{t\uparrow T} b(t) = K$.

(iv) $P(t,x)$ *satisfies the spatial smooth fit condition at* $b(t)$*:*

$$\lim_{x\downarrow b(t)} P_x(t,x) = \lim_{x\uparrow b(t)} \partial_x (K-x)^+ = -1$$

(v) The value function P *solves following free boundary problem:*

$$\begin{aligned} &P(t,x) \in C^{1,2}(\mathcal{C}) \cap C^{1,2}(\mathcal{D}), \\ &P_t(t,x) + (\mathcal{L}_X - r)P(t,x) \le 0, \qquad \textit{a.e.}\ (t,x) \in [0,T] \times \mathbb{R}_+, \\ &\lim_{x\downarrow b(t)} P_x(t,x) = -1, \\ &P(t,x) = (K-x)^+, \qquad (t,x) \in \mathcal{D}, \\ &P(t,x) \ge (K-x)^+, \qquad (t,x) \in [0,T] \times \mathbb{R}_+. \end{aligned} \tag{2.44}$$

(vi) $P(t,x)$ *admits the so-called early exercise premium representation:*

$$P(t,x) = rK \int_0^{T-t} e^{-ru} \mathsf{P}_x \left(X_u < b(t+u)\right) \mathrm{d}u + \mathsf{E}_x \left[e^{-r(T-t)} (K - X_{T-t})^+\right], \tag{2.45}$$

(vii) The stopping boundary $b(t)$ *is the unique solution of the following non-linear integral equation in the class of continuous functions,*

$$\begin{aligned} &K - b(t) \\ &= rK \int_0^{T-t} e^{-ru} \mathsf{P}_{b(t)} \left(X_u < b(t+u)\right) \mathrm{d}u + \mathsf{E}_{b(t)} \left[e^{-r(T-t)} (K - X_{T-t})^+\right]. \end{aligned} \tag{2.46}$$

Proof. see [114, Ch. VII, p. 379 - 392] □

Contrary to the perpetual option case where we formulate the free boundary problem (2.39) directly as a guess, in the finite maturity case, we must show that the value function and the stopping boundary solve (2.44) from first principles. To achieve this, we first prove the monotonicity and the continuity of P. Using the monotonicity of P in x, we can show that $(t,x_1) \in \mathcal{C} \implies (t,x_2) \in \mathcal{C}$ for any $x_1 < x_2$ and each t. As a consequence, we can parameterise the stopping boundary b as a function of time and characterise the stopping region as in (2.43). After showing the spatial smooth fit holds at b (also see [80] and subsequent works for the proof), the associated free boundary problem (2.44) can be established using the arguments in the previous section. The early exercise representation (2.45) is obtained by applying a change of variable formula developed in Peskir [110] to the value function P. To further analyse the stopping boundary b, we first prove its continuity and then derive the associated integral equation (2.46) by letting $x = b$ in (2.45). Once we justify the uniqueness of the solution to the integral equation (2.46), various numerical schemes are available to compute b. In Chapter 4, we will adopt similar procedures but use more sophisticated probabilistic methods to study the pricing problem under the stochastic interest rate model.

2.3.1 Delta hedging for American put options

It is worth mentioning that, in the Black and Scholes model, we can derive the hedging portfolio for the American put option using the dynamics of the discounted option's value. Letting $t = 0$ and applying the change of variable formula [110, Thm. 3.1] to $P(s, X_s)$, we have

$$\mathrm{d}e^{-rs}P(s,X_s) = -e^{-rs}rK\mathbb{1}_{\{X_s<b(s)\}}\mathrm{d}s + e^{-rs}\sigma X_s P_x(s,X_s)dB_s. \tag{2.47}$$

The option seller can perfectly hedge her position by continuously trading a self-financing portfolio Π_s with $\Pi_0 = P(0,x)$, and for $s \in [0,T]$

$$\Pi_s = \Pi_0 + \int_0^s \phi_u^{(1)}\mathrm{d}X_u + \int_0^s \phi_u^{(2)}\mathrm{d}\xi_u - C_s,$$

where ξ is a risk-free money market account with dynamics

$$\mathrm{d}\xi_t = r\xi_t\mathrm{d}t, \quad \xi_0 = 1,$$

$\phi^{(1)}, \phi^{(2)}$ are the stock and money market account holding, C is a continuous non-decreasing process that models consumption. The discounted portfolio dynamics reads

$$\mathrm{d}e^{-rs}\Pi_s = e^{-rs}\phi_s^{(1)}\sigma X_s\mathrm{d}B_s - e^{-rs}\mathrm{d}C_s. \tag{2.48}$$

Comparing (2.48) and (2.47), we derive the stock holding and the consumption that replicate the option's payoff

$$\phi_s^{(1)} = P_x(s,X_s), \quad C_s = \int_0^s rK\mathbb{1}_{\{X_u<b(u)\}}\mathrm{d}u. \tag{2.49}$$

Note that the hedging strategy is admissible as it can be shown that $\phi^{(1)} = P_x$ is bounded between -1 and 0 ([114, p. 382]). We need consumption C in the hedging portfolio as the discounted option value is a super-martingale. As indicated in [129, Ch. 8.3], the finance intuition is that as long as the option buyer does not exercise her right when the stock price is below the exercise boundary $b(s)$, the seller's portfolio always has value $K - X_s$, as $\Pi_s = P(s, X_s) = K - X_s$ when $X_s \leq b(s)$. Then the seller can hold amount K in her money market account and consume the interest rate. As the first derivative of the option value with respect to the stock price is called Delta in the Black and Scholes model, the corresponding hedging strategy in (2.49) is referred to as Delta hedging. In Chapter 3, we will illustrate the Delta hedging strategy for the perpetual American put option. In Chapter 4, we will see how the Delta hedging strategy changes under the stochastic interest rate model.

Chapter 3

Optimal hedging of a perpetual American put with a single trade

3.1 Introduction

[1] In this chapter, we analyse an optimal hedging problem for the seller of a perpetual American put option in a Black and Scholes market. We assume the following scenario. An option seller sells a perpetual American put option and constructs a portfolio to hedge her position. The portfolio consists of the underlying stock and a risk-free bond. Unlike the situation in the Black and Scholes model where continuous trading is allowed, our option seller can rebalance her hedging portfolio only *once* until the underlying stock price leaves a predetermined interval (a, b) with $0 < a < b < +\infty$. Our goal is to develop a hedging strategy that minimises the variance of the hedging error at the random time when the stock leaves (a, b). This involves determining the stock holding before and after rebalancing the hedging portfolio and the time at which the portfolio is rebalanced.

Continuous trading as prescribed by the classical delta-hedging strategy in the Black and Scholes model is not viable due to various reasons, e.g. transaction cost. The question of rebalancing portfolios with a limited number of trades has a long history and has been addressed in various ways. In academia, an extensively studied approach is to develop hedging strategies in discrete time models. Work in this area includes [66], [20], [21], [127], [105]. Another approach is to incorporate a certain form of transaction cost and solve a singular stochastic control problem to maximise utility. In this case, the trader only rebalances in the so-called ac-

[1] The results from this chapter form part of the article [28], which was published in *SIAM Journal on Financial Mathematics*.

tion region. Important work of this kind includes [95], [36], [136]. Our setting is more similar to [99] and [131], where the model is set up in continuous time but the trader is only allowed to trade a given number of times. Practitioners have adopted a broad range of simple rules for their rebalancing strategies, e.g., rebalancing at fixed times or rebalancing at fixed values of the underlying asset's price (see, e.g., [130]). The latter strategy, in particular, inspired our work: we determine *optimal* values of the asset price at which a trade should be made and also an optimal trade.

Of course, it would be desirable to extend our setting to allow the trader multiple trades (not just one), as in, e.g., [99], [132] and [131], but such an extension inevitably leads to more abstract results than ours. Indeed, the above papers aim for a general setup and obtain mostly results on the existence of optimal strategies (via viscosity theory, in [99], and martingale methods, in [132]). These results do not allow to determine analytically shapes of the trading regions and to compute efficiently trading strategies. If rebalancing once in the entire lifetime of the option is certainly too restrictive, our assumption of rebalancing once prior to the stock price leaving a given interval (a, b) improves on real-life strategies, where traders set target values of the stock price at which they reassess their position (our a and b). Considering a perpetual option is convenient because it guarantees that the problem is time-homogeneous and allows for explicit calculations. This is also a reasonable approximation for options far from their maturity.

Using the variance of the tracking error as the optimisation criterion is a natural choice in our setting, as we will show that the mean of the tracking error is zero. The variance, or second moment, has been widely used as a minimisation target in the optimal hedging literature. From a mathematical point of view, it motivated important work on the approximation of stochastic integrals. Since the variance of the tracking error for the hedging portfolio is an L^2-distance for stochastic processes, its minimisation is referred to as quadratic hedging. The foundations of quadratic hedging for claims in incomplete markets were laid in the seminal work by Föllmer and Schweizer [67]. As we mentioned earlier, developing hedging strategies in discrete time models is one approach to avoid unrealistic continuous trading. In the context of quadratic hedging, this involves approximation of random variables (representing European claims at maturity) via stochastic integrals for discrete-time processes. The work of Schweizer [127], Schäl [126] and Mercurio and Vorst [105] falls into this category. Although we employ a similar optimisation criterion as in those papers, our work is built on the continuous time Black and Scholes model, and we only limit the number of trades. More recently in the mathematical literature we find numerous papers concerning the asymptotic optimality of discrete-time hedging strategies as the number of hedging opportunities tends to infinity (see, e.g., Fukasawa [71], Gobet and Landon [76], Rosenbaum and Tankov [120], Cai et al. [29]). Those papers also approach the problem by approximating random variables with stochastic integrals for discrete-time processes. Finally, Ekren, Liu and Muhle-Karbe [52] study optimal hedging frequency in the asymptotic limit of small transaction costs for portfolio with multiple assets. The methodology and the nature of the results in those papers are rather far from our work.

In the finance literature we find work by Ahn and Wilmott [1], who illustrate numerically the performance of various hedging strategies with finitely many hedging opportunities. They numerically solve systems of PDEs for the mean and variance of the hedging portfolio with the condition of minimising the variance at each rebalancing time. Given a fixed number of rebalancing opportunities, they compare the performance of the strategy that rebalances at fixed time intervals and the strategy that rebalances at optimised time intervals (the rebalancing times become new variables in the model). Their results show that the variance reduction from the optimised strategy is more pronounced for small numbers of rebalancing chances. In Section 3.7, we compare our optimal hedging strategy with other naive strategies. Our results show that, given only one rebalancing chance, the strategy with the optimised rebalancing time can significantly reduce the variance, which is consistent with Ahn and Wilmott's finding.

Boyle and Emanuel [20] study the distribution of portfolio returns with discrete hedging. Maštinsek [100] studies the error in piecewise constant hedging strategies as a function of the time interval δt between trades in the presence of transaction costs. Mello and Neuhaus [102] research the accumulated hedging error due to discrete rebalancing, extending the work by Figlewski [63] to imperfect markets. The idea of allowing hedging at the time when fixed relative changes in the stock price occur is explored in [116], where the price dynamic (in an incomplete market) is a marked point process.

Finally, it is worth noticing that optimal multiple stopping has been studied in the context of pricing of swing options in the energy market (see, e.g., Lempa [97] for a survey). In particular, optimal boundaries for options with a put payoff are studied analytically in Carmona and Touzi [32] and Carmona and Dayanik [31] in infinite horizon, and De Angelis and Kitapbayev [41] in finite horizon ([32] also consider finite horizon but only numerically). In models of optimal multiple stopping there is normally a minimum time-lag between subsequent admissible stopping times. That is imposed as a constraint on the set of admissible stopping sequences and guarantees that simultaneous use of all the stopping opportunities cannot occur. In the case of discrete hedging, there is no need for such constraint: simultaneous use of all stopping times never occurs because, at each stopping time, the portfolio weights are adjusted and therefore each subsequent stopping problem is of a different nature. Moreover, the optimisation of the portfolio weights leads to extremely convoluted analytical expressions for the subsequent stopping problems, so that the resulting optimal multiple stopping problem is much harder to tackle than those in [31], [32] and [41].

This chapter is organised as follows. In Section 3.2 we set the hedging problem in a rigorous mathematical framework. In Section 3.3 we study the hedging problem for a fixed value of the initial stock holding. We prove continuity and differentiability of the value function with respect to the initial stock price (Theorem 3.3.11). We determine the existence of optimal trading boundaries (Proposition 3.3.10) and we prove that the value function solves a suitable variational problem (Theorem 3.3.13). In Section 3.4 we prove that the optimal trading boundaries are continuous monotonic functions of the initial stock holding (Theorems 3.4.3 and 3.4.4). Section 3.5 gives necessary first order condition which enable computation of an optimal initial stock holding. The study is complemented with an extensive numerical analysis of the properties of the optimal hedging strategies and of the corresponding hedging error.

3.2 Problem formulation and background material

Consider a Black-Scholes economy on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with risk neutral measure P. We have one risky stock X and a risk-free bond ξ, following the dynamics

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t \mathrm{d}B_t, \quad X_0 = x, \tag{3.1}$$

$$\mathrm{d}\xi_t = r\xi_t\mathrm{d}t, \qquad \xi_0 = 1, \tag{3.2}$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion, $r > 0$ is the risk-free rate and $\sigma > 0$ is the stock's volatility. Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by B satisfying the usual conditions.

An option trader sells one perpetual American put option written on the stock X with the strike price K. Such option gives its holder the right but not the obligation to sell one share of the stock X for the price K at any (random) time $\tau \in [0, \infty]$. It is well-known that, if the initial stock price is x, the arbitrage-free price $P(x)$ of the option is given by

$$P(x) = \sup_{\tau} \mathsf{E}\left[e^{-r\tau}(K - X^x_\tau)^+\right], \tag{3.3}$$

where the supremum is taken over all $\mathbb{F}$-Markov times τ and the payoff is defined as 0 when $\tau = \infty$. The explicit form of $P(x)$ is given by (see Chapter 2 2.3)

$$P(x) = \begin{cases} \frac{1}{d}\hat{a}^{1+d}x^{-d}, & \hat{a} \le x < \infty, \\ \\ K - x, & 0 \le x \le \hat{a}, \end{cases} \tag{3.4}$$

where $d := 2r/\sigma^2$ and

$$\hat{a} := \frac{K}{1 + \frac{1}{d}} \tag{3.5}$$

is the exercise boundary; that is, the holder exercises the option optimally according to the stopping rule

$$\tau_{\hat{a}} := \inf\{t \ge 0 : X_t \le \hat{a}\}. \tag{3.6}$$

By a straightforward application of Itô-Tanaka's formula we can derive the dynamics of the discounted option price, that is

$$\mathrm{d}(e^{-rt}P(X_t)) = -e^{-rt}rK\mathbb{1}_{\{X_t<\hat{a}\}}\mathrm{d}t + e^{-rt}\sigma X_t P'(X_t)\mathrm{d}B_t, \tag{3.7}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function. It is immediate to verify that

$$t \mapsto e^{-rt}P(X_t) \text{ is a supermartingale and } t \mapsto e^{-r(t\wedge\tau_{\hat{a}})}P(X_{t\wedge\tau_{\hat{a}}}) \text{ is a martingale.}$$

According to classical theory the seller of the option should use Delta hedging to construct a replicating portfolio for the perpetual American put. The Delta of the option corresponds to the first derivative

$$P'(x) = \max\{-(\hat{a}/x)^{1+d}, -1\},$$

which is an increasing function taking values in $[-1, 0)$ and is strictly increasing on $(\hat{a}, \infty)$. Notice that the Delta appears in the stochastic integral of (3.7). This highlights that, under the classical Black-Scholes model, if the option holder does not exercise the option at $\tau_{\hat{a}}$, the option seller gains instantaneous interests rK with her short position perfectly hedged.

In our problem formulation, we tacitly assume that the option holder exercises the option optimally, hence as soon as X_t falls below $\hat{a}$.

Our trader faces the following hedging scenario: after selling the option, she constructs a self-financing (hedging) portfolio $\Pi = (\Pi_t)_{t\ge0}$ with bond holding $(m_t)_{t\ge0}$ and stock holding $(\theta_t)_{t\ge0}$, that is

$$\Pi_t = \theta_t X_t^x + m_t\xi_t, \qquad t \ge 0;$$

at time $t = 0$, she chooses an initial stock holding $\theta_0 = h$ and bond holding $m_0 = P(x) - hx$. However, in contrast to the classical Delta hedging model, the seller is allowed to rebalance her portfolio only once at a (stopping) time τ of her choosing before the stock price leaves a given interval (a, b). Her goal is to find an admissible trading strategy (in a sense which will be made precise in Definition 3.2.1) so that the variance of the tracking error is minimised (this will also be clarified in a moment). The thresholds a, b can be interpreted as re-assessment price levels set by the option seller. From a practical point of view, the option seller will choose those levels on the grounds of subjective propensity to risk and operational/regulatory constraints.

Since we are assuming that the option holder exercises the option according to the stopping rule (3.6), it is natural to only allow $a \geq \hat{a}$. We also assume that $b < \infty$ and define $\mathcal{I} := (a, b)$ and $\bar{\mathcal{I}} := [a, b]$. The (random) time horizon of our problem is given by

$$\tau_{\mathcal{I}}^x := \inf\{t \geq 0 : X_t^x \notin \mathcal{I}\}.$$

Note that the first exit time from $\mathcal{I}$ should be denoted by $\tau_{\mathcal{I}^c}^x$ following the definition in Chapter 2, but we will use $\tau_{\mathcal{I}}^x$ in this chapter for convenience. Other stopping time notations will still follow the definitions in Chapter 2 unless specified. We often omit the superscript x if it does not lead to ambiguity. For mathematical completeness the case of $b = \infty$ is discussed separately in Section 3.6 as it presents some specific technical features.

In order to formally define admissible trading strategies, we need to introduce some notation. Given an initial stock price $X_0 = x \in \bar{\mathcal{I}}$, we let

$$\mathcal{T}_x := \{\tau : \tau \text{ is a } \mathbb{F}\text{-stopping time such that } \tau \leq \tau_{\mathcal{I}}^x, \text{ P-a.s.}\}$$

and for any $\tau \in \mathcal{T}_x$ we define

$$\mathcal{H}^\tau := \{h_1 : \Omega \to \mathbb{R} \ : \ h_1 \text{ is } \mathcal{F}_\tau\text{-measurable and } \mathsf{E}\left[(h_1)^2\right] < \infty\}. \tag{3.8}$$

Since the seller's optimisation problem ends at the time when the price process leaves the interval $\mathcal{I}$, it is natural to consider an initial stock holding θ_0 which lies in the set

$$\mathcal{H} := [P'(a), P'(b)],$$

where it is worth recalling that $P'(x) = -(\hat{a}/x)^{1+d}$ for $x \geq \hat{a}$.

Definition 3.2.1 (Trading strategy)**.** *For an initial stock price $X_0 = x \in \bar{\mathcal{I}}$, the set of admissible trading strategies $\mathcal{A}_x$ consists of pairs (τ, θ), such that $\tau \in \mathcal{T}_x$ and*

$$\theta_t := \begin{cases} h, & 0 \leq t \leq \tau, \\ h_1, & \tau < t \leq \tau_{\mathcal{I}}^x, \end{cases}$$

where $h \in \mathcal{H}$ is the initial stock holding and $h_1 \in \mathcal{H}^\tau$ is the new stock holding after the trade.

Given a trading strategy $(\tau, \theta) \in \mathcal{A}_x$, the trader's self-financing, hedging portfolio follows the dynamics

$$\mathrm{d}\Pi_t^{\tau,\theta} = \theta_t \mathrm{d}X_t^x + m_t \mathrm{d}\xi_t, \qquad \Pi_0^{\tau,\theta} = hx + m_0 = P(x). \tag{3.9}$$

Then, combining (3.9) with (3.1)–(3.2), it is easy to verify that the discounted portfolio process $t \mapsto e^{-rt}\Pi_t^{\tau,\theta}$ is a local martingale with the dynamics

$$\mathrm{d}(e^{-rt}\Pi_t^{\tau,\theta}) = \theta_t \mathrm{d}(e^{-rt}X_t^x) = e^{-rt}\theta_t \sigma X_t^x \mathrm{d}B_t. \tag{3.10}$$

Finally, we can formulate the optimisation problem for the option seller. As mentioned above, the seller wants to minimise the *variance* of the tracking error (i.e., the difference between the hedging portfolio and the option price) at the terminal time $\tau_{\mathcal{I}}$. It is worth remarking that the choice of the variance is natural since the mean of the tracking error is completely uninformative. Indeed

$$\mathsf{E}_x\left[e^{-r\tau_{\mathcal{I}}}\big(\Pi_{\tau_{\mathcal{I}}}^{\tau,\theta} - P(X_{\tau_{\mathcal{I}}})\big)\right] = 0 \tag{3.11}$$

thanks to the optional sampling theorem, upon recalling that on the stochastic interval $[0, \tau_{\mathcal{I}}]$ the price process X is bounded and $(\theta_t)_{t\geq 0}$ is a square integrable process (cf. (3.8)). Then, given an initial price $X_0 = x$ we are interested in the problem

$$\begin{aligned} \mathcal{V}(x) &= \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathcal{V}ar_x\left[e^{-r\tau_{\mathcal{I}}}\big(\Pi_{\tau_{\mathcal{I}}}^{\tau,\theta} - P(X_{\tau_{\mathcal{I}}})\big)\right] \\ &= \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathsf{E}_x\left[e^{-2r\tau_{\mathcal{I}}}\big(\Pi_{\tau_{\mathcal{I}}}^{\tau,\theta} - P(X_{\tau_{\mathcal{I}}})\big)^2\right], \end{aligned} \tag{3.12}$$

where we use the notation $\mathcal{V}ar_x[\,\cdot\,] = \mathcal{V}ar[\,\cdot\,|X_0 = x]$ and the second equality follows from (3.11).

Remark 3.2.2. *It is assumed above that the option is sold for the price $P(x)$ and the seller invests the proceeds in the hedging portfolio, i.e., $\Pi_0^{\tau,\theta} = P(x)$. However, the seller aware of her trading constraints may sell the option at a premium over the Black-Scholes price, i.e., for $P(x) + \delta$ with $\delta > 0$. Denoting by $(\Pi_t^{\tau,\theta;\delta})_{t\geq 0}$ the associated hedging portfolio, for any trading strategy $(\tau, \theta) \in \mathcal{A}_x$ it follows from (3.9) and (3.10) that $\Pi_t^{\tau,\theta;\delta} = e^{rt}\delta + \Pi_t^{\tau,\theta}$ for all $t \geq 0$. The mean tracking error equals (c.f. (3.11))*

$$\mathsf{E}_x\left[e^{-r\tau_{\mathcal{I}}}\big(\Pi_{\tau_{\mathcal{I}}}^{\tau,\theta;\delta} - P(X_{\tau_{\mathcal{I}}})\big)\right] = \delta$$

and consequently

$$\begin{aligned}\inf_{(\tau,\theta)\in\mathcal{A}_x} \mathcal{V}ar_x\left[e^{-r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta;\delta}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})\right)\right] &= \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathcal{V}ar_x\left[e^{-r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})\right)+\delta\right]\\ &= \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathsf{E}_x\left[e^{-2r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})\right)^2\right] = \mathcal{V}(x).\end{aligned}$$

Hence the problem simplifies to the one studied in this chapter.

One may argue that if all sellers on the market charge a premium on the Black-Scholes price, then the tracking error should be computed accounting for such premium too. As shown in the next remark, if we assume a multiplicative premium we can embed these models in our set-up.

Remark 3.2.3. *Due to trading frictions on real markets, the selling price of the option may be higher than the theoretical Black-Scholes price. Assuming a multiplicative adjustment, the option's selling price is $P(x)(1+\varepsilon)$, where $\varepsilon \geq 0$, and we denote by $(\Pi^{\tau,\theta;\varepsilon}_t)_{t\geq 0}$ the associated hedging portfolio. The trader receives $P(x)(1+\varepsilon)$ at time 0 and tracks the selling price $P(X_t)(1+\varepsilon)$ (so that she can close the position at time $\tau_\mathcal{I}$). In view of (3.11), the mean tracking error is*

$$\mathsf{E}_x\left[e^{-r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta;\varepsilon}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})(1+\varepsilon)\right)\right] = (1+\varepsilon)\mathsf{E}_x\left[e^{-r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta'}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})\right)\right] = 0,$$

where $\theta'_t = \theta_t/(1+\varepsilon)$, $t \geq 0$, is used along with (3.10) to obtain $e^{-rt}\Pi^{\tau,\theta;\varepsilon}_t = (1+\varepsilon)e^{-rt}\Pi^{\tau,\theta'}_t$. Therefore,

$$\begin{aligned}&\inf_{(\tau,\theta)\in\mathcal{A}_x} \mathcal{V}ar_x\left[e^{-r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta;\varepsilon}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})(1+\varepsilon)\right)\right]\\ &= (1+\varepsilon)^2 \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathcal{V}ar_x\left[e^{-r\tau_\mathcal{I}}\left(\Pi^{\tau,\theta}_{\tau_\mathcal{I}} - P(X_{\tau_\mathcal{I}})\right)\right] = (1+\varepsilon)^2\mathcal{V}(x),\end{aligned}$$

and the optimisation problem simplifies to the one studied in this chapter.

Using the integral forms of the dynamics (3.10) and (3.7) and Itô's isometry we obtain a more convenient problem formulation:

$$\begin{aligned}\mathcal{V}(x) &= \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathsf{E}_x\left[\left(\int_0^{\tau_\mathcal{I}} e^{-ru}(\theta_u - P'(X_u))\sigma X_u \mathrm{d}B_u\right)^2\right] \\ &= \inf_{(\tau,\theta)\in\mathcal{A}_x} \mathsf{E}_x\left[\int_0^{\tau_\mathcal{I}} e^{-2ru} f(X_u,\theta_u)\mathrm{d}u\right],\end{aligned} \tag{3.13}$$

where

$$f(x,\theta) := (\theta - P'(x))^2\sigma^2 x^2. \tag{3.14}$$

The final expression in (3.13) highlights the well-known fact that Delta hedging amounts to controlling the difference between θ_t and $P'(X_t)$. In the absence of trading constraints the optimal trading strategy would be the Black-Scholes strategy $\theta_t = P'(X_t)$, which would produce no tracking error with certainty, i.e. $\mathcal{V}(x) = 0$.

Notice that we can rewrite our problem as

$$\mathcal{V}(x) = \inf_{h \in \mathcal{H}} V(x, h), \tag{3.15}$$

where

$$V(x,h) := \inf_{(\tau,h_1)\in\mathcal{T}_x\times\mathcal{H}^\tau} \mathsf{E}_x\left[\int_0^\tau e^{-2ru} f(X_u,h)\mathrm{d}u + \int_\tau^{\tau_\mathcal{I}} e^{-2ru} f(X_u,h_1)\mathrm{d}u\right]. \tag{3.16}$$

In light of this observation we will first proceed with a detailed analysis of the function $V(x,h)$ and subsequently we will determine $\mathcal{V}(x)$. By doing this, we will also obtain an optimal control (τ_*, θ^*).

We close this section recalling some useful facts and some notation. The infinitesimal generator of the process X is denoted by $\mathcal{L}$, and defined by its action on functions $v \in C^2(\mathbb{R}^+)$ as follows:

$$\mathcal{L}v(x) := \tfrac{\sigma^2}{2} x^2 \partial_{xx} v(x) + rx\partial_x v(x).$$

Recalling that $d = 2r/\sigma^2$ we have that

$$q_1 = \frac{1-d+\sqrt{(1-d)^2+8d}}{2} > 0, \qquad q_2 = \frac{1-d-\sqrt{(1-d)^2+8d}}{2} < 0, \tag{3.17}$$

are the roots of

$$q^2 + (d-1)q - 2d = 0.$$

To utilise the formulae of linear diffusion X in Chapter 2, we will need the functions φ and ψ defined, respectively, as the unique (up to multiplication) decreasing and increasing fundamental solutions of the ODE

$$(\mathcal{L} - 2r)v(x) = 0, \quad x \in (a,b). \tag{3.18}$$

As our process X is absorbed at boundary a, b, the boundary points are classified as exit-not-entrance, hence ψ, φ satisfy boundary conditions (see [18, Ch. II])

$$\psi(a+) = 0, \quad \psi'(a+) > 0, \quad \varphi(b-) = 0, \quad \varphi'(b-) < 0. \tag{3.19}$$

The fundamental solutions are conveniently constructed as linear combinations of $\hat{\varphi}(x) := x^{q_2}$ and $\hat{\psi}(x) := x^{q_1}$ by taking (see, e.g., [3])

$$\varphi(x) = \hat{\varphi}(x) - \frac{\hat{\varphi}(b)}{\hat{\psi}(b)}\hat{\psi}(x) \quad \text{and} \quad \psi(x) = \hat{\psi}(x) - \frac{\hat{\psi}(a)}{\hat{\varphi}(a)}\hat{\varphi}(x). \tag{3.20}$$

Finally, using φ and ψ, recall the analytical expression of the resolvent for a one-dimensional diffusion that we stated in (2.15). For the problem in this chapter, we have for any $x \in \mathcal{I}$, and any bounded measurable function $g : \mathcal{I} \to \mathbb{R}$

$$\mathsf{E}_x\left[\int_0^{\tau_\mathcal{I}} e^{-2ru} g(X_u)\mathrm{d}u\right] = w^{-1}\left(\varphi(x)\int_a^x \psi(z)g(z)m'(z)\mathrm{d}z + \psi(x)\int_x^b \varphi(z)g(z)m'(z)\mathrm{d}z\right), \tag{3.21}$$

where w is the Wronskian (with the value independent of x)

$$w = \psi'(x)\frac{\varphi(x)}{s'(x)} - \varphi'(x)\frac{\psi(x)}{s'(x)} > 0,$$

and $s'(x)$ and $m'(x)$ are the densities of the scale function and of the speed measure of $(X_t)_{t\geq 0}$, respectively. They are explicitly given by

$$s'(x) = c\,x^{-d} \qquad \text{and} \qquad m'(x) = 2x^{d-2}/c\,\sigma^2, \tag{3.22}$$

where $c > 0$ is the same constant in both expressions (s' and m' are uniquely defined up to multiplication). For future reference, we notice that the Wronskian w can be also expressed in terms of the Wronskian $\hat{w}$ associated to $\hat{\varphi}$ and $\hat{\psi}$. In particular, it is not hard to check that (recall that $q_2 < 0 < q_1$)

$$w = \hat{w}\left(1 - (a/b)^{q_1 - q_2}\right), \tag{3.23}$$

where

$$\hat{w} := \hat{\psi}'(x)\frac{\hat{\varphi}(x)}{s'(x)} - \hat{\varphi}'(x)\frac{\hat{\psi}(x)}{s'(x)}$$

This observation will be useful when we later consider fundamental solutions of (3.18) on intervals $\mathcal{I}' \neq \mathcal{I}$.

3.3 A one dimensional optimal stopping problem

In this section, we study problem (3.16) for each fixed initial stock holding $h \in \mathcal{H}$. First we find the optimal stock holding h_1 and reduce (3.16) to a standard one dimensional optimal stopping problem, then we solve the optimal stopping problem via associated free boundary problems.

3.3.1 Reduction to a Markovian optimal stopping problem

The first task is to show that it is sufficient to draw $h_1 \in \mathcal{H}^\tau$ from the class of Markovian controls. To this end, we introduce the set of Markovian controls $\mathcal{H}^\tau_m$ defined as

$$\mathcal{H}^\tau_m := \{h_1 \in \mathcal{H}^\tau : h_1 = \ell(X_\tau) \text{ for some measurable } \ell : \bar{\mathcal{I}} \to \mathbb{R}\}.$$

Consider an analogue of problem (3.16) but with the constraint of using Markovian controls and denote its value by

$$\widetilde{V}(x,h) := \inf_{(\tau,h_1)\in\mathcal{T}_x\times\mathcal{H}^\tau_m} \mathsf{E}_x\left[\int_0^\tau e^{-2ru} f(X_u,h)\mathrm{d}u + \int_\tau^{\tau_\mathcal{I}} e^{-2ru} f(X_u,h_1)\mathrm{d}u\right]. \tag{3.24}$$

Next we show the equivalence of (3.16) and (3.24).

Proposition 3.3.1. *For all $(x,h) \in \bar{\mathcal{I}} \times \mathcal{H}$ we have $\widetilde{V}(x,h) = V(x,h)$.*

Proof. Since h_1 is $\mathcal{F}_\tau$ measurable, expanding the square in (3.14) and using the tower property of conditional expectation, we can write (3.16) as

$$\begin{aligned} V(x,h) = \inf_{(\tau,h_1)\in\mathcal{T}_x\times\mathcal{H}^\tau} \mathsf{E}\Bigg[&\int_0^\tau e^{-2ru} f(X_u,h)\mathrm{d}u + h_1^2 \mathsf{E}_x\left(\int_\tau^{\tau_\mathcal{I}} e^{-2ru}\sigma^2 X_u^2 \mathrm{d}u \Big| \mathcal{F}_\tau\right) \\ &- 2h_1 \mathsf{E}_x\left(\int_\tau^{\tau_\mathcal{I}} e^{-2ru} P'(X_u)\sigma^2 X_u^2\mathrm{d}u\Big|\mathcal{F}_\tau\right) \\ &+ \mathsf{E}_x\left(\int_\tau^{\tau_\mathcal{I}} e^{-2ru}(P'(X_u))^2\sigma^2 X_u^2 \mathrm{d}u\Big|\mathcal{F}_\tau\right)\Bigg]. \end{aligned} \tag{3.25}$$

Notice that for any trading time τ, the expression under the expectation E_x is quadratic in h_1. Then the optimal stock holding h_1^* is

$$h_1^* = \frac{\mathsf{E}_x\left(\int_\tau^{\tau_\mathcal{I}} e^{-2ru}P'(X_u)\sigma^2 X_u^2 \mathrm{d}u\Big|\mathcal{F}_\tau\right)}{\mathsf{E}_x\left(\int_\tau^{\tau_\mathcal{I}} e^{-2ru}\sigma^2 X_u^2\mathrm{d}u\Big|\mathcal{F}_\tau\right)} = \frac{\mathsf{E}_{X_\tau}\left[\int_0^{\tau_\mathcal{I}} e^{-2ru}P'(X_u)\sigma^2X_u^2\mathrm{d}u\right]}{\mathsf{E}_{X_\tau}\left[\int_0^{\tau_\mathcal{I}} e^{-2ru}\sigma^2 X_u^2\mathrm{d}u\right]}, \tag{3.26}$$

where the final equality follows from the strong Markov property of the process X. Therefore, the optimal stock holding h_1^* is a measurable function of the stock price X_τ at time τ. Hence it suffices to consider problem (3.24) instead of (3.16). Notice that a similar result was also obtained by [99]. □

Thanks to Proposition 3.3.1, we can apply the strong Markov property of $(X_t)_{t\geq 0}$ to transform (3.16) into a canonical impulse control form:

$$V(x,h) = \inf_{(\tau,h_1)\in\mathcal{T}_x\times\mathcal{H}^\tau_m} \mathsf{E}_x\left[\int_0^\tau e^{-2ru} f(X_u,h)\mathrm{d}u + e^{-2r\tau}\widehat{M}(X_\tau,h_1)\right], \tag{3.27}$$

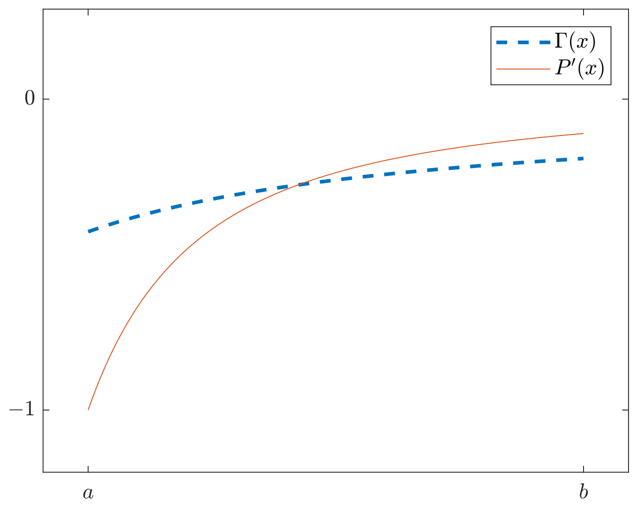


Figure 3.1: Plots of the functions $\Gamma(x)$ and $P'(x)$ using parameters $r = 3\%$, $\sigma = 30\%$, $K = 100$ and $b = 150$. Notice that $a = K/(1 + d^{-1}) = 40$.

where

$$\widehat{M}(x,\zeta) := \mathsf{E}_x\left[\int_0^{\tau_{\mathcal{I}}} e^{-2ru} f(X_u,\zeta)\mathrm{d}u\right], \qquad \zeta \in \mathbb{R}, \quad x \in \bar{\mathcal{I}}. \tag{3.28}$$

Expanding the square in f yields the following representation for $\widehat{M}$

$$\widehat{M}(x,\zeta) = \zeta^2\gamma_1(x) - 2\zeta\gamma_2(x) + \gamma_3(x), \tag{3.29}$$

where

$$\begin{aligned} \gamma_1(x) &= \mathsf{E}_x\left[\int_0^{\tau_{\mathcal{I}}} e^{-2ru}\sigma^2 X_u^2 \mathrm{d}u\right], \qquad \gamma_2(x) = \mathsf{E}_x\left[\int_0^{\tau_{\mathcal{I}}} e^{-2ru}P'(X_u)\sigma^2 X_u^2 \mathrm{d}u\right], \\ \gamma_3(x) &= \mathsf{E}_x\left[\int_0^{\tau_{\mathcal{I}}} e^{-2ru}(P'(X_u))^2\sigma^2 X_u^2 \mathrm{d}u\right]. \end{aligned} \tag{3.30}$$

Direct calculations, using (3.21), lead to explicit formulae for γ_i, $i = 1, 2, 3$,

$$\gamma_1(x) = -x^2 + A_1 D_2 x^{q_1} + A_2 D_1 x^{q_2}, \tag{3.31}$$

$$\gamma_2(x) = -\frac{1}{d}\hat{a}^{1+d}x^{1-d} + A_1 C_2 x^{q_1} + A_2 C_1 x^{q_2}, \tag{3.32}$$

$$\gamma_3(x) = -\frac{1}{d^2}\hat{a}^{2+2d}x^{-2d} + A_1 B_2 x^{q_1} + A_2 B_1 x^{q_2}, \tag{3.33}$$

where, using q_1 and q_2 given in (3.17),

$$\begin{aligned} A_i &:= [a^{q_i - q_{3-i}} - b^{q_i - q_{3-i}}]^{-1}, & B_i &:= d^{-2}\big((\hat{a}/a)^{2+2d}a^{2-q_i} - (\hat{a}/b)^{2+2d}b^{2-q_i}\big), \\ C_i &:= d^{-1}\big((\hat{a}/a)^{1+d}a^{2-q_i} - (\hat{a}/b)^{1+d}b^{2-q_i}\big), & D_i &:= (a^{2-q_i} - b^{2-q_i}). \end{aligned}$$

The expression (3.26) in the proof of Proposition 3.3.1 implies that the optimal stock holding after the rebalancing of the portfolio is a measurable function of the stock price at the rebalancing time. It is a minimiser of $\zeta \mapsto \widehat{M}(x, \zeta)$ which, thanks to the representation (3.29), is unique and given by

$$\Gamma(x) = \arg\min_{\zeta} \widehat{M}(x, \zeta) = \frac{\gamma_2(x)}{\gamma_1(x)}, \qquad x \in \mathcal{I}. \tag{3.34}$$

Notice that the optimal stock holding Γ is defined only on $\mathcal{I}$. If the trader trades at $\tau_{\mathcal{I}}$, her choice of the stock holding becomes irrelevant for the optimisation problem.

Denoting

$$M(x) := \widehat{M}(x, \Gamma(x)) \quad \text{for } x \in \mathcal{I}, \quad \text{and} \quad M(a) = M(b) = 0, \tag{3.35}$$

$V(x, h)$ can be represented as

$$V(x, h) = \inf_{\tau \in \mathcal{T}_x} \mathsf{E}_x \left[\int_0^\tau e^{-2ru} f(X_u, h) \mathrm{d}u + e^{-2r\tau} M(X_\tau) \right]. \tag{3.36}$$

While (3.36) defines a standard optimal stopping problem, the explicit expression of M is extremely convoluted and makes the analysis of our problem very challenging. Indeed, it immediately follows from (3.29) and (3.34) that

$$M(x) = -\frac{\gamma_2^2(x)}{\gamma_1(x)} + \gamma_3(x), \qquad x \in \mathcal{I}. \tag{3.37}$$

However, thanks to the analytical expressions we can easily assert the smoothness of Γ and M in $\mathcal{I}$ and their behaviour at the boundary $\partial\mathcal{I}$.

Proposition 3.3.2. *The optimal stock holding Γ and the payoff function M belong to $C^\infty(\mathcal{I})$. Furthermore,*

(i) Γ is negative, strictly increasing, with bounded first derivative on $\mathcal{I}$. The limits of Γ at a and b satisfy

$$\Gamma(a) := \lim_{x \downarrow a} \Gamma(x) > P'(a) \quad \textit{and} \quad \Gamma(b) := \lim_{x \uparrow b} \Gamma(x) < P'(b). \tag{3.38}$$

(ii) Limits of the derivatives M', M'' at a and b exist and are finite. Moreover,

$$\lim_{x \downarrow a} M(x) = \lim_{x \uparrow b} M(x) = 0. \tag{3.39}$$

Proof. The smoothness of Γ and M on $\mathcal{I}$ can be checked directly from their explicit expressions (3.34) and (3.37).

The monotonicity of Γ in (i) is hard to obtain directly from its analytical expression (3.34) with γ_1, γ_2 as in (3.31)-(3.32). Instead we exploit the probabilistic formulae for γ_i's given in (3.30), combined with (3.21). It can be easily verified that

$$\Gamma(x) = \frac{\varphi(x)p_2(x) + \psi(x)p_4(x)}{\varphi(x)p_1(x) + \psi(x)p_3(x)}, \tag{3.40}$$

where

$$p_1(x) = \int_a^x \psi(z)\sigma^2 z^2 m'(z)\mathrm{d}z, \qquad p_2(x) = \int_a^x \psi(z)P'(z)\sigma^2 z^2 m'(z)\mathrm{d}z,$$
$$p_3(x) = \int_x^b \varphi(z)\sigma^2 z^2 m'(z)\mathrm{d}z, \qquad p_4(x) = \int_x^b \varphi(z)P'(z)\sigma^2 z^2 m'(z)\mathrm{d}z.$$

From (3.40) using simple algebra, we obtain

$$\Gamma'(x) = \frac{ws'(x)}{\big(\varphi(x)p_1(x) + \psi(x)p_3(x)\big)^2}\big(p_1(x)p_4(x) - p_2(x)p_3(x)\big), \tag{3.41}$$

where w is the Wronskian and in the calculations we have used

$$\psi(x)p_4'(x) = -\varphi(x)p_2'(x), \quad \psi(x)p_3'(x) = -\varphi(x)p_1'(x).$$

Since $P'(\,\cdot\,)$ is strictly increasing, we have

$$p_4(x) > P'(x)\int_x^b \varphi(z)\sigma^2 z^2 m'(z)\mathrm{d}z = P'(x)p_3(x),$$
$$p_2(x) < P'(x)\int_a^x \psi(z)\sigma^2 z^2 m'(z)\mathrm{d}z = P'(x)p_1(x).$$

Therefore $p_1(x)p_4(x) > P'(x)p_3(x)p_1(x) > p_2(x)p_3(x)$, which implies that $\Gamma'(x) > 0$. Noticing that

$$p_1(a) = p_2(a) = p_3(b) = p_4(b) = 0,$$
$$p_1'(x)P'(x) = p_2'(x), \quad \text{and} \quad p_3'(x)P'(x) = p_4'(x),$$

and using de L'Hospital's rule for the right-hand side of (3.41), we can compute the limits

$$\lim_{x\downarrow a}\Gamma'(x) = \frac{w(p_4(a) - P'(a)p_3(a))}{\psi'(a+)p_3(a)^2} < \infty,$$
$$\lim_{x\uparrow b}\Gamma'(x) = \frac{w(p_2(b) - P'(b)p_1(b))}{\varphi'(b-)p_1(b)^2} < \infty,$$

$$\lim_{x\downarrow a}\Gamma(x)=\frac{p_4(a)}{p_3(a)}>P'(a)\quad\text{and}\quad\lim_{x\uparrow b}\Gamma(x)=\frac{p_2(b)}{p_1(b)}<P'(b),$$

which, together with (3.41), concludes the proof of (i).

Now we prove (ii). The boundedness of derivatives follows directly from the explicit representation (3.37). Limits at a and b are deduced from (3.31)-(3.33). □

We close this section by proving the Lipschitz continuity of the value function. Since M is continuous on $\bar{\mathcal{I}}$, [109, Thm. 3.4] implies that V is continous and the smallest optimal stopping time is in the standard form, i.e., the first hitting time of the set where V coincides with M (see (3.48) below). However, in the particular case of the optimal stopping problem $V(x,h)$, the Lipschitz continuity, and, therefore, continuity, can be proven directly. Arguments below rely on the Lipschitz continuity of f and M and not on their particular form. Notice that the underlying process is absorbed at a and b which differentiates our setting from results found in the literature.

Proposition 3.3.3. *There exists a constant L such that for any (x,h) and (x',h') in $\bar{\mathcal{I}}\times\mathcal{H}$*

$$|V(x,h)-V(x',h')|\le L(|x-x'|+|h-h'|). \tag{3.42}$$

Proof. Take (x,h) and (x',h') in $\bar{\mathcal{I}}\times\mathcal{H}$. By definition of the value function, for any $\varepsilon>0$, there exists $\tau_1\in\mathcal{T}_x$ such that

$$V(x,h)\ge \mathsf{E}\left[\int_0^{\tau_1} e^{-2ru}f(X_u^x,h')\mathrm{d}u+e^{-2r\tilde{\tau}}M(X_{\tau_1}^x)\right]-\varepsilon,$$

τ_1 is the so-called ε-optimal stopping time for $V(x,h)$. Let $\tilde{\tau}=\tau_1\wedge\tau_{\mathcal{I}}^{x'}$, so that $\tilde{\tau}\in\mathcal{T}_{x'}$. Since $\tilde{\tau}$ is in general sub-optimal for $V(x',h')$, we have

$$\begin{aligned}V(x',h')&\le \mathsf{E}\left[\int_0^{\tilde{\tau}} e^{-2ru}f(X_u^{x'},h')\mathrm{d}u+e^{-2r\tilde{\tau}}M(X_{\tilde{\tau}}^{x'})\right]\\&\le \mathsf{E}\left[\int_0^{\tau_1} e^{-2ru}f(X_u^{x'},h')\mathrm{d}u\right]+\mathsf{E}\left[e^{-2r\tau_1}M(X_{\tau_1}^{x'})\mathbb{1}_{\{\tau_1\le\tau_{\mathcal{I}}^{x'}\}}\right],\end{aligned}$$

where we used that $f\ge 0$ and $M(X^{x'}_{\tau_{\mathcal{I}}^{x'}})=0$ by (3.39). Now, by direct comparison we obtain

$$\begin{aligned}&V(x',h')-V(x,h)\\&\le \mathsf{E}\left[\int_0^{\tau_1} e^{-2ru}(f(X_u^{x'},h')-f(X_u^x,h))\mathrm{d}u\right]\\&\quad+\mathsf{E}\left[e^{-2r\tau_1}\big(M(X_{\tau_1}^{x'})-M(X_{\tau_1}^x)\big)\mathbb{1}_{\{\tau_1\le\tau_{\mathcal{I}}^{x'}\}}-e^{-2r\tau_1}M(X_{\tau_1}^x)\mathbb{1}_{\{\tau_1>\tau_I^{x'}\}}\right]+\varepsilon\\&\le \mathsf{E}\left[\int_0^{\tau_1} e^{-2ru}|f(X_u^{x'},h')-f(X_u^x,h)|\mathrm{d}u\right]+\mathsf{E}\left[e^{-2r\tau_1}|M(X_{\tau_1}^{x'})-M(X_{\tau_1}^x)|\mathbb{1}_{\{\tau_1\le\tau_{\mathcal{I}}^{x'}\}}\right]+\varepsilon.\end{aligned} \tag{3.43}$$

The map $(x,h) \mapsto f(x,h)$ is Lipschitz on $K \times \mathcal{H}$, with $K \subset \mathbb{R}_+$ any compact, and $x \mapsto M(x)$ is also Lipschitz by (ii) in Proposition 3.3.2. Since $X^x_{t\wedge\tau_1} \in \bar{\mathcal{I}}$, for all $t \geq 0$, then

$$X^{x'}_{t\wedge\tau_1} = x'/xX^x_{t\wedge\tau_1} \in [a^2/b, b^2/a] =: K_{a,b}. \tag{3.44}$$

Let L_1, L_2 be the Lipschitz constants for $f(x,h)$ on $K_{a,b} \times \mathcal{H}$ and for $M(x)$ on $\bar{\mathcal{I}}$, respectively. Then, using the explicit expression of X^x_t, we can bound (3.43) with

$$\begin{aligned}
&V(x',h') - V(x,h) \\
&\leq \mathsf{E}\left[\int_0^{\tau_1} e^{-2ru} L_1(|x-x'|X^1_u + |h-h'|)\mathrm{d}u\right] + \mathsf{E}\left[e^{-2r\tau_1} L_2|x-x'|X^1_{\tau_1}\right] + \varepsilon \\
&\leq (L_1 \vee L_2)(|x-x'| + |h-h'|)\left(1 + \int_0^\infty e^{-ru}\mathrm{d}u\right) + \varepsilon,
\end{aligned}$$

where we used $\mathsf{E}[e^{-rt}X^1_t] = 1$ for any $t \geq 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $V(x',h') - V(x,h) \leq (1+1/r)(L_1 \vee L_2)(|x-x'| + |h-h'|)$. A symmetric argument leads to the reverse inequality and (3.42) is proven with $L = (1+1/r)(L_1 \vee L_2)$. □

We note here for future use that

$$V(a,h) = V(b,h) = 0. \tag{3.45}$$

Thanks to the reduction to a standard Markovian setup we can introduce the continuation and stopping set of problem (3.36), denoted respectively by $\mathcal{C}$ and $\mathcal{D}$, and defined as

$$\mathcal{C} := \{(x,h) \in \bar{\mathcal{I}} \times \mathcal{H} : V(x,h) < M(x)\}, \tag{3.46}$$

$$\mathcal{D} := \{(x,h) \in \bar{\mathcal{I}} \times \mathcal{H} : V(x,h) = M(x)\}. \tag{3.47}$$

Obviously, we have $\{a,b\} \times \mathcal{H} \subset \mathcal{D}$ due to (3.45). It is well known (see, e.g., [114, Ch. I, Cor. 2.9]) that the minimal optimal stopping time in (3.36) is

$$\tau^*_{x,h} := \inf\{t \geq 0 : (X^x_t, h) \in \mathcal{D}\}. \tag{3.48}$$

For simplicity, in the rest of this chapter we also use the notation $\tau^*_h = \tau^*_{x,h}$ under P_x.

The slightly odd aspect of (3.48) is that the two dimensional process (X,h) is actually constant in its second coordinate. This motivates introducing the sets

$$\mathcal{C}_h := \{x \in \bar{\mathcal{I}} : V(x,h) < M(x)\},$$
$$\mathcal{D}_h := \{x \in \bar{\mathcal{I}} : V(x,h) = M(x)\},$$

for each $h \in \mathcal{H}$. In terms of these two sets, the optimal stopping time (3.48) reads

$$\tau^*_{x,h} := \inf\{t \geq 0 : X^x_t \in \mathcal{D}_h\}. \tag{3.49}$$

Since functions M, V are continuous, the sets $\mathcal{C}$ and $\mathcal{C}_h$ are open whereas $\mathcal{D}$ and $\mathcal{D}_h$ are closed.

Finally, letting

$$Y^h_t := e^{-2r(t\wedge\tau_\mathcal{I})}V(X_{t\wedge\tau_\mathcal{I}}, h) + \int_0^{t\wedge\tau_\mathcal{I}} e^{-2rs} f(X_s, h)\mathrm{d}s \tag{3.50}$$

we have that, for any $(x, h) \in \mathcal{I} \times \mathcal{H}$, the process $(Y^h_t)_{t\geq 0}$ is a P_x-sub-martingale and

$$\text{the process } (Y^h_{t\wedge\tau^*_h})_{t\geq 0} \text{ is a } \mathsf{P}_x\text{-martingale.} \tag{3.51}$$

3.3.2 A free boundary problem

It is expected that, for each $h \in \mathcal{H}$, the stopping problem (3.36) be linked to an obstacle problem

$$\min\left\{(\mathcal{L} - 2r)u + f, M - u\right\}(x, h) = 0, \quad \text{a.e. } x \in \mathcal{I}, \tag{3.52}$$

$$u(a, h) = u(b, h) = 0. \tag{3.53}$$

This problem can be stated as the following free boundary problem

$$(\mathcal{L} - 2r)u(x, h) + f(x, h) = 0, \quad x \in \{z \in \mathcal{I} : u(z, h) < M(z)\}, \tag{3.54}$$

$$(\mathcal{L} - 2r)u(x, h) + f(x, h) \geq 0, \quad \text{a.e. } x \in \mathcal{I}, \tag{3.55}$$

$$u(x, h) \leq M(x), \quad x \in \mathcal{I}, \qquad u(a, h) = u(b, h) = 0. \tag{3.56}$$

It is also often postulated that the so-called smooth-pasting condition holds, i.e.,

$$\partial_x u(\cdot, h) = M'(\cdot) \quad \text{on} \quad \partial\{z \in \mathcal{I} : u(z, h) < M(z)\}. \tag{3.57}$$

In the literature on one dimensional optimal stopping problems the obstacle problem (3.52) is usually solved in its form (3.54)–(3.56) by first making an educated guess on the shape of the set $\{z \in \mathcal{I} : u(z, h) < M(z)\}$ and then by solving the corresponding boundary value problem (3.54). The solution of the resulting ODE can be often computed explicitly and the smooth pasting (3.57) is used to determine the boundary $\partial\{z \in \mathcal{I} : u(z, h) < M(z)\}$. The latter normally relies on finding roots of nontrivial algebraic equations. Finally, one verifies (3.55)-(3.56).

Since the payoff function $M(x)$ has a very complicated form, the approach sketched above is infeasible, particularly, the verification of (3.55)-(3.56) from the smooth-pasting condition. Instead, we follow a mixed probabilistic/analytic approach. In this section we determine the shape of the continuation set, while in Section 3.3.3 we prove the smoothness of the value function and determine in what sense it solves the obstacle problem (3.52)-(3.53).

It is well-known that one can gain insights into the geometry of the stopping set $\mathcal{D}$ by studying the sign of the function $G : \mathcal{I} \times \mathcal{H} \mapsto \mathbb{R}$ defined as

$$G(x,h) := (\mathcal{L} - 2r)M(x) + f(x,h), \tag{3.58}$$

where $\mathcal{L}M$ is well-defined thanks to Proposition 3.3.2.

Lemma 3.3.4. *For each $h \in \mathcal{H}$,*

$$\{x \in \mathcal{I} : G(x,h) < 0\} \subset \mathcal{C}_h. \tag{3.59}$$

Proof. The proof of (3.59) is standard but we present arguments for the convenience of the reader. For a fixed h, assume there is $\hat{x} \in \mathcal{I}$ such that $G(\hat{x},h) < 0$ and let

$$\tau_0 := \inf\{t \geq 0 : G(X_t,h) \geq 0\} \wedge \tau_{\mathcal{I}}.$$

Then $\tau_0 > 0$, $\mathsf{P}_{\hat{x}}$-a.s., by continuity of G and $t \mapsto X_t$. Since $M \in C^2(\mathcal{I})$ with bounded derivatives (Proposition 3.3.2 (ii)), by an application of Dynkin's formula we have

$$\begin{aligned} V(\hat{x},h) &\leq \mathsf{E}_{\hat{x}}\left[\int_0^{\tau_0} e^{-2ru} f(X_u,h)\mathrm{d}u + e^{-2r\tau_0} M(X_{\tau_0})\right] \\ &= \mathsf{E}_{\hat{x}}\left[\int_0^{\tau_0} e^{-2ru} G(X_u,h)\mathrm{d}u\right] + M(\hat{x}) < M(\hat{x}), \end{aligned}$$

hence $\hat{x} \in \mathcal{C}_h$. □

The following lemma provides an explicit expression for G.

Lemma 3.3.5. *For all $(x,h) \in \mathcal{I} \times \mathcal{H}$ we have*

$$G(x,h) = \sigma^2 x^2 \Big((h - P'(x))^2 - (\Gamma(x) - P'(x))^2 - (\Gamma'(x))^2 \gamma_1(x) \Big). \tag{3.60}$$

Proof. Using (3.37), we obtain

$$\begin{aligned} G(x,h) = (\Gamma(x))^2(\mathcal{L}\gamma_1 - 2r\gamma_1)(x) - 2\Gamma(x)(\mathcal{L}\gamma_2 - 2r\gamma_2)(x) + (\mathcal{L}\gamma_3 - 2r\gamma_3)(x) \\ + \sigma^2 x^2 \Gamma'(x)(\Gamma(x)\gamma_1'(x) - \gamma_2'(x)) + \sigma^2 x^2 (h - P'(x))^2. \end{aligned} \tag{3.61}$$

Recall the probabilistic expressions for γ_1, γ_2 and γ_3 given in (3.34) and (3.33). Hence, for $i = 1, 2, 3$,

$$(\mathcal{L} - 2r)\gamma_i(x) = -g_i(x), \quad \text{on } \mathcal{I}, \tag{3.62}$$

where $g_1(x) = \sigma^2 x^2$, $g_2(x) = \sigma^2 x^2 P'(x)$ and $g_3(x) = \sigma^2 x^2 (P'(x))^2$. Furthermore,

$$\Gamma(x)\gamma_1'(x) - \gamma_2'(x) = -\Gamma'(x)\gamma_1(x), \tag{3.63}$$

since $\Gamma(x) = \gamma_2(x)/\gamma_1(x)$. Finally, inserting (3.62) and (3.63) into (3.61) yields (3.60). $\square$

Next we proceed to prove that the continuation and the stopping sets have non-empty intersection with $\mathcal{I}$ (recall that $\{a, b\} \in \mathcal{D}_h$). For any $h \in \mathcal{H}$, it is convenient to define $x_p(h) \in \bar{\mathcal{I}}$ as the unique root of the equation $P'(x) - h = 0$, that is,

$$x_p(h) = \hat{a}(-h)^{-\frac{1}{1+d}}. \tag{3.64}$$

Notice that for $h \in \text{int}(\mathcal{H})$ we have $x_p(h) \in \mathcal{I}$.

Proposition 3.3.6. *For each $h \in \mathcal{H}$, we have $\mathcal{D}_h \cap \mathcal{I} \neq \varnothing$ and $\mathcal{C}_h \neq \varnothing$.*

Proof. First consider $h \in (P'(a), P'(b))$. Then $x_p(h) \in \mathcal{I}$ and it is immediate to see from (3.60) that $G(x_p(h), h) < 0$. Hence, (3.59) implies $\mathcal{C}_h \neq \varnothing$. If $h = P'(a)$ then the expression (3.60) and the fact that $\Gamma(a) > P'(a)$ (Proposition 3.3.2) imply $G(a, P'(a)) < 0$. By the continuity of G, there is $x \in \mathcal{I}$ with $G(x, P'(a)) < 0$ and an application of (3.59) gives $\mathcal{C}_h \neq \varnothing$. A similar argument applies for $h = P'(b)$.

Assume now that $\mathcal{D}_h \setminus \{a, b\} = \varnothing$ so that $\mathcal{C}_h = \mathcal{I}$. Then for any $x \in \mathcal{I}$ we have

$$M(x) > V(x,h) = \mathsf{E}_x\left[\int_0^{\tau_{\mathcal{I}}} e^{-2ru} f(X_u, h)\mathrm{d}u\right] = \widehat{M}(x,h) \geq \inf_{l \in \mathcal{H}} \widehat{M}(x,l) = M(x),$$

hence a contradiction. $\square$

The subsequent analysis will show that the roots of the map $x \mapsto G(x,h)$ for each $h \in \mathcal{H}$ determine the shape of the continuation and the stopping sets. Due to the complexity of the expression for G, it seems very hard to determine analytically the exact number of zeros of the map $x \mapsto G(x,h)$. However, the exercise is trivial from a numerical point of view, thanks to the fully explicit expression in (3.60). We performed extensive numerical tests and observed only three possible situations displayed in Figure 3.2. It will also follow from the proof of Proposition 3.3.9 that the map $x \mapsto G(x,h)$ has at least one root if $h \in [P'(a), \Gamma(a)] \cup [\Gamma(b), P'(b)]$ and it has at least two roots if $h \in (\Gamma(a), \Gamma(b))$. The following assumption provides a necessary ingredient to determine the exact number of zeros of G and the shape of the stopping set.

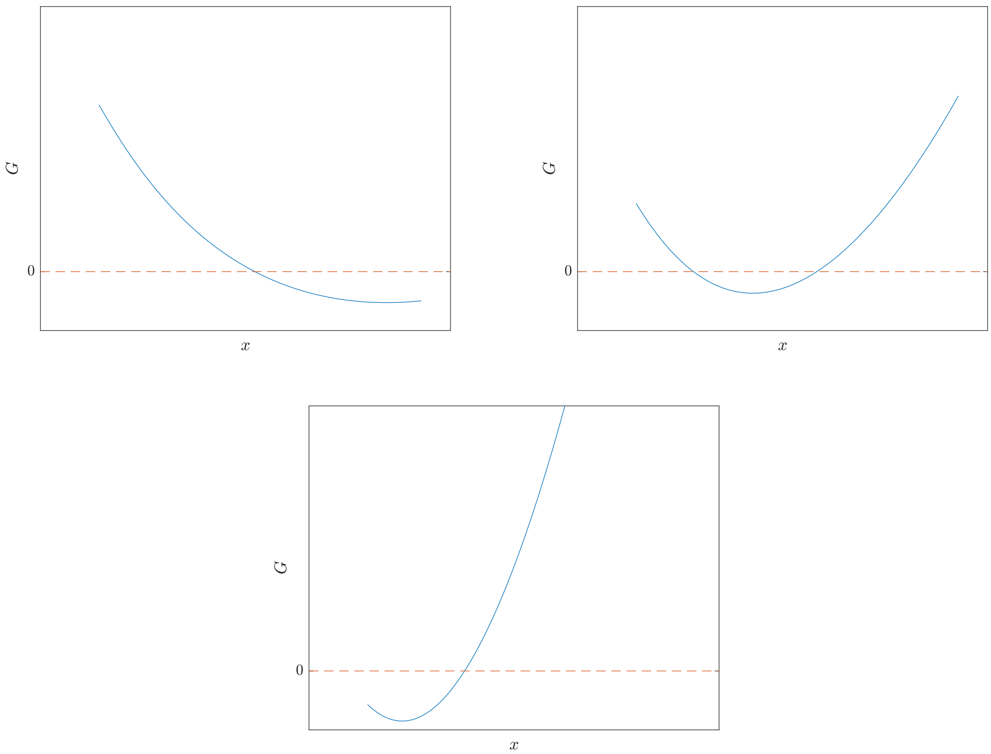


Figure 3.2: Plots of the map $x \mapsto G(x,h)$ for different values of the initial stock holding h using parameters $r = 3\%$, $\sigma = 30\%$, $K = 100$, $b = 150$, and $a = \hat{a} = K/(1 + d^{-1}) = 40$.

Assumption 3.3.7. *For each $h \in \mathcal{H}$, the equation $G(\,\cdot\,, h) = 0$ has at most two roots in $\mathcal{I}$.*

Denoting the roots of $G(\,\cdot\,, h) = 0$ on $\mathcal{I}$ by x_{G_1} and x_{G_2} (when they both exist) consider the following three cases:

(A.1) $G(x, h) > 0$ for $x \in (a, x_{G_1})$ and $G(x, h) < 0$ for $x \in (x_{G_1}, b)$, except possibly at x_{G_2};

(A.2) $G(x, h) > 0$ for $x \in (a, x_{G_1}) \cup (x_{G_2}, b)$ and $G(x, h) < 0$ for $x \in (x_{G_1}, x_{G_2})$;

(A.3) $G(x, h) < 0$ for $x \in (a, x_{G_1})$ and $G(x, h) > 0$ for $x \in (x_{G_1}, b)$, except possibly at x_{G_2}.

Remark 3.3.8. *In (A.1) we mean that, if $G(\,\cdot\,, h)$ has two roots then it must be $\partial_x G(x_{G_2}, h) = 0$. The root x_{G_2} may be on the right or on the left of x_{G_1}. An analogous rationale holds in (A.3).*

It turns out that the above cases (A.1)-(A.3) are uniquely linked to the choice of the initial stock holding h as the following proposition demonstrates.

Proposition 3.3.9. *Under Assumption 3.3.7, we have:*

(i) Condition (A.1) holds if and only if $h \in [\Gamma(b), P'(b)]$*;*

(ii) Condition (A.2) holds if and only if $h \in (\Gamma(a), \Gamma(b))$*;*

(iii) Condition (A.3) holds if and only if $h \in [P'(a), \Gamma(a)]$.

Proof. Assume (A.1) and $h \in \mathcal{H}$. Then $G(b,h) \leq 0$. Using $\gamma_1(b) = 0$ in (3.60), we obtain

$$(h - P'(b))^2 - (\Gamma(b) - P'(b))^2 \leq 0,$$

which yields $h \geq \Gamma(b)$ and completes the proof of the right implication in (i).

Consider now $h \in [\Gamma(b), P'(b)]$. Directly from (3.60) we calculate $G(a,h) > 0$ since $h > \Gamma(a) > P'(a)$ and $\gamma_1(a) = 0$. For $h > \Gamma(b)$ we have $G(b,h) < 0$, which combined with Assumption 3.3.7 and the continuity of G proves (A.1). For $h = \Gamma(b)$ we have to use a different argument because $G(b, \Gamma(b)) = 0$. Rewriting (3.60) yields

$$G(x, \Gamma(b)) = \sigma^2 x^2 \Big((\Gamma(b) - \Gamma(x))(\Gamma(b) + \Gamma(x) - 2P'(x)) - (\Gamma'(x))^2 \gamma_1(x) \Big).$$

The last term in the bracket is non-positive. We have $\Gamma(b) + \Gamma(x) - 2P'(x) < 0$ for $x \in \mathcal{I}$ sufficiently close to b, and, $\Gamma(b) - \Gamma(x) > 0$ by the monotonicity of Γ. Hence, $G(x, \Gamma(b)) < 0$ for $x \in \mathcal{I}$ sufficiently close to b, which immediately proves (A.1).

Assume now (A.2). Using arguments from the beginning of the proof, $G(b,h) > 0$ implies $h < \Gamma(b)$. Analogously, $G(a,h) > 0$ implies $h > \Gamma(a)$. For the left implication in (ii), we note that $G(x_p(h), h) < 0$ for $h \in \text{int}(\mathcal{H})$. The sign of $G(x,h)$ at $x \in \{a, b\}$ is determined by the sign of

$$(h - P'(x))^2 - (\Gamma(x) - P'(x))^2 = (h - \Gamma(x))(h + \Gamma(x) - 2P'(x)).$$

Recalling that $P'(a) < \Gamma(a) < \Gamma(b) < P'(b)$ (c.f. (3.38)), we have $G(a,h) > 0$ and $G(b,h) > 0$ for $h \in (\Gamma(a), \Gamma(b))$. As above, the continuity of G and Assumption 3.3.7 completes the proof of the left implication in (ii).

The proof of (iii) is analogous to (i). □

In light of the above proposition, we will refer to conditions (A.1)–(A.3) as determining the ranges of h as well as the zeros of $G(x,h)$. We now show that they are sufficient to determine shapes of the continuation and the stopping sets $\mathcal{C}_h$ and $\mathcal{D}_h$.

Proposition 3.3.10. *Let Assumption 3.3.7 hold and take* $h \in \mathcal{H}$*. Then we have*

(i) under (A.1) there is $x_1^* \in (a, x_{G_1}]$ *such that* $\mathcal{C}_h = (x_1^*, b)$*;*

(ii) *under (A.2) there exist* $x_1^* \in [a, x_{G_1}]$ *and* $x_2^* \in [x_{G_2}, b]$ *such that* $\mathcal{C}_h = (x_1^*, x_2^*)$*. Moreover, at least one of* x_1^*, x_2^* *is in* $\mathcal{I}$*;*

(iii) *under (A.3) there is* $x_2^* \in [x_{G_1}, b)$ *such that* $\mathcal{C}_h = (a, x_2^*)$*.*

Proof. We only give a full proof of (iii) as the other claims follow by analogous arguments. Assume (A.3) and that the root x_{G_2} exists and is smaller than x_{G_1}. Inclusion (3.59) implies $\mathcal{D}_h \cap \mathcal{I} \subseteq \{x_{G_2}\} \cup [x_{G_1}, b)$. We will show that $x_{G_2} \notin \mathcal{D}_h$. Indeed, for a small $\varepsilon > 0$, let

$$\tau_\varepsilon := \inf\{t \geq 0 : X_t \notin (x_{G_2} - \varepsilon, x_{G_2} + \varepsilon)\}.$$

Since τ_ε is sub-optimal for $V(x_{G_2}, h)$, we have

$$\begin{aligned} V(x_{G_2}, h) &\leq \mathsf{E}_{x_{G_2}}\left[\int_0^{\tau_\varepsilon} e^{-2ru} f(X_u, h)\mathrm{d}u + e^{-2r\tau_\varepsilon} M(X_{\tau_\varepsilon})\right] \\ &= \mathsf{E}_{x_{G_2}}\left[\int_0^{\tau_\varepsilon} e^{-2ru} G(X_u, h)\mathrm{d}u\right] + M(x_{G_2}) < M(x_{G_2}), \end{aligned}$$

where the equality is an application of Dynkin formula for $M(X_{\tau_\varepsilon})$ and the final strict inequality holds because, under (A.3), we have $G(x, h) < 0$ on $(x_{G_2} - \varepsilon, x_{G_2} + \varepsilon) \setminus \{x_{G_2}\}$ for a sufficiently small ε and $G(x_{G_2}, h) = 0$. This shows that $x_{G_2} \notin \mathcal{D}_h$ and therefore $\mathcal{D}_h \cap \mathcal{I} \subseteq [x_{G_1}, b)$. The latter inclusion trivially holds if $x_{G_2} > x_{G_1}$ or when the second root x_{G_2} does not exist.

Next we show that if $x_0 \in [x_{G_1}, b)$ and $x_0 \in \mathcal{D}_h$, then $[x_0, b] \subseteq \mathcal{D}_h$. Arguing by contradiction, assume there exists such an x_0 and an open set $U \subset (x_0, b)$ such that $U \subset \mathcal{C}_h$. For any $x \in U$, we have

$$\tau_{x,h}^* \leq \inf\{t \geq 0 : X_t^x \leq x_0\}, \quad \mathsf{P} - a.s.$$

Applying Dynkin formula, we obtain

$$\begin{aligned} V(x, h) &= \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} f(X_u, h)\mathrm{d}u + e^{-2r\tau_h^*} M(X_{\tau_h^*})\right] \\ &= \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} G(X_u, h)\mathrm{d}u\right] + M(x) \geq M(x), \end{aligned}$$

where the final inequality is due to $G(x, h) \geq 0$ on (x_0, b). Hence a contradiction. Notice that the existence of $x_0 \in [x_{G_1}, b)$ such that $x_0 \in \mathcal{D}_h$ is guaranteed by $\mathcal{D}_h \cap \mathcal{I} \neq \varnothing$ (Proposition 3.3.6). □

The above proposition shows that under (A.1) and (A.3) the shape of the stopping set is unambiguously determined. Only under (A.2), the set $\mathcal{D}_h \cap \mathcal{I}$ may have one or two connected components, depending on the choice of the parameters in the problem.

3.3.3 Solution of the free boundary problem

Showing that the value function V is a solution to the free boundary problem (3.54)-(3.56) is relatively easy. However, this provides little value unless one can further ascertain uniqueness. This is done via a verification argument, which typically requires smooth pasting across stopping boundaries. Smooth pasting is also required for efficient calculation of stopping boundaries via a solution of algebraic equations, see Subsection 3.3.4. In this section we first show that the value function V of (3.36) satisfies $V(\,\cdot\,,h) \in C^1(\mathcal{I})$ for each $h \in \mathcal{H}$ (i.e., smooth pasting), then we use this fact to prove that V solves (3.54)-(3.56) uniquely (Theorem 3.3.13).

We can immediately claim that $V(\,\cdot\,,h) \in C^2(\mathcal{I} \setminus \partial\mathcal{C}_h)$. Indeed, on $\mathcal{D}_h \setminus \partial\mathcal{C}_h$, $V = M$, so the result is trivial by (ii) in Proposition 3.3.2. Instead, on $\mathcal{C}_h$, the result follows by (3.51) and a standard argument [87, Ch. 4.2] (see also [114, Ch. III, Sec. 7]). Hence, for any $h \in \mathcal{H}$, V is a classical solution of

$$(\mathcal{L} - 2r)V(x,h) = -f(x,h), \qquad x \in \mathcal{C}_h, \tag{3.65}$$

with the boundary condition $V(x,h) = M(x)$ for $x \in \partial\mathcal{C}_h$.

We will show the regularity of the value function across the boundary. For that we will revisit the convergence of the stopping time

$$\tau_{\mathcal{K}}^x := \inf\{t \geq 0 : X_t^x \in \mathcal{K}\}$$

in the sense of (3.68), where $\mathcal{K}$ is a closed subset of $\mathbb{R}_+$, i.e. $\mathcal{K} = \overline{\text{int}(\mathcal{K})}$. Also define

$$\mathring{\sigma}_{\mathcal{K}}^x := \inf\{t \geq 0 : X_t^x \in \text{int}(\mathcal{K})\}. \tag{3.66}$$

Using analogous arguments as in Lemma 2.1.6 we have

$$\mathsf{P}(\tau_{\mathcal{K}}^x = \mathring{\sigma}_{\mathcal{K}}^x) = 1, \quad \text{for all } x \in \mathbb{R}_+. \tag{3.67}$$

This fact together with Proposition 2.1.9, which can be adapted to the current setting, implies

$$\tau_{\mathcal{K}}^{x_n} \to \tau_{\mathcal{K}}^{x_0}, \qquad \mathsf{P} - a.s. \tag{3.68}$$

when $(x_n)_{n\geq 0} \subset \mathbb{R}_+$ converges to $x_0 \in \mathbb{R}_+$ as $n \to \infty$. In particular, under Assumption 3.3.7 and using Proposition 3.3.10, this implies that for any sequence $(x_n)_{n\geq 0} \subset \mathcal{C}_h$ converging to $x_0 \in \partial\mathcal{C}_h$ as $n \to \infty$, we have

$$\tau_{x_n,h}^* \to 0, \qquad \mathsf{P} - a.s. \tag{3.69}$$

This is the key tool to the next result, which makes use of an approach developed in [42].

Theorem 3.3.11. *Under Assumption 3.3.7 we have, for each $h \in \mathcal{H}$,*

$$V(\cdot, h) \in C(\bar{\mathcal{I}}) \cap C^1(\mathcal{I}) \cap C^2(\mathcal{I} \setminus \partial\mathcal{C}_h)$$

and for any $x_0 \in \partial\mathcal{C}_h \cap \mathcal{I}$

$$\lim_{\mathcal{C}_h \ni x \to x_0} \partial_{xx} V(x, h) = 2(\sigma x_0)^{-2}\left(-r x_0 M'(x_0) + 2rM(x_0) - f(x_0, h)\right). \tag{3.70}$$

Proof. The continuity of $V(\cdot, h)$ follows from Proposition 3.3.3, whereas (3.70) can be obtained from (3.65) provided that $V(\cdot\,, h) \in C^1(\mathcal{I})$. Hence, it only remains to show that for any $x_0 \in \partial\mathcal{C}_h \cap \mathcal{I}$ it holds

$$\lim_{\mathcal{C}_h \ni x \to x_0} \partial_x V(x, h) = M'(x_0).$$

Fix $x \in \mathcal{C}_h$ and denote $\tau^* := \tau^*_{x,h}$ which is optimal for the problem $V(x, h)$. Fix $\varepsilon > 0$ and notice that the stopping time $\tau^* \wedge \tau_{\mathcal{I}}^{x+\varepsilon} \in \mathcal{T}_{x+\varepsilon}$ is admissible for the problem $V(x+\varepsilon, h)$. We get an upper bound

$$V(x+\varepsilon, h) \le \mathsf{E}\left[\int_0^{\tau^* \wedge \tau_{\mathcal{I}}^{x+\varepsilon}} e^{-2ru} f(X_u^{x+\varepsilon}, h)\mathrm{d}u + e^{-2r(\tau^* \wedge \tau_{\mathcal{I}}^{x+\varepsilon})} M(X^{x+\varepsilon}_{\tau^* \wedge \tau_{\mathcal{I}}^{x+\varepsilon}})\right].$$

Using this and the optimality of τ^* for $V(x, h)$ we obtain

$$\begin{aligned}
&\frac{V(x+\varepsilon, h) - V(x, h)}{\varepsilon}\\
&\le \frac{1}{\varepsilon}\mathsf{E}\left[\int_0^{\tau^* \wedge \tau_{\mathcal{I}}^{x+\varepsilon}} e^{-2ru}\big(f(X_u^{x+\varepsilon}, h) - f(X_u^x, h)\big)\mathrm{d}u + e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}}\right]\\
&\quad - \frac{1}{\varepsilon}\mathsf{E}\left[\left(\int_{\tau_{\mathcal{I}}^{x+\varepsilon}}^{\tau^*} e^{-2ru} f(X_u^x, h)\mathrm{d}u + e^{-2r\tau^*} M(X_{\tau^*}^x)\right)\mathbb{1}_{\{\tau^* > \tau_{\mathcal{I}}^{x+\varepsilon}\}}\right]\\
&\le \frac{1}{\varepsilon}\mathsf{E}\left[\int_0^{\tau^* \wedge \tau_{\mathcal{I}}^{x+\varepsilon}} e^{-2ru}\big(f(X_u^{x+\varepsilon}, h) - f(X_u^x, h)\big)\mathrm{d}u + e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}}\right],
\end{aligned}$$

where in the first inequality we also use $M(X^{x+\varepsilon}_{\tau_{\mathcal{I}}^{x+\varepsilon}}) = 0$, P-a.s., by (3.39), and the second inequality follows from $f \ge 0$ and $M \ge 0$. The final term in the last inequality can be further estimated by

$$\begin{aligned}
&\mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}}\right]\\
&= \mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^* < \tau_{\mathcal{I}}^x\}}\right]\\
&\quad + \mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^* = \tau_{\mathcal{I}}^x\}}\right]\\
&= \mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^* < \tau_{\mathcal{I}}^x\}}\right]\\
&\quad + \mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^* = \tau_{\mathcal{I}}^x\}\cap\{X^x_{\tau_{\mathcal{I}}} = a\}}\right],
\end{aligned}$$

where we use $\{\tau^* \leq \tau_{\mathcal{I}}^{x+\varepsilon}\} \cap \{\tau^* = \tau_{\mathcal{I}}^x\} \cap \{X_{\tau_{\mathcal{I}}}^x = b\} = \varnothing$ in the second equality. Notice that on $\{\tau^* \leq \tau_{\mathcal{I}}^{x+\varepsilon}\} \cap \{\tau^* = \tau_{\mathcal{I}}^x\} \cap \{X_{\tau_{\mathcal{I}}}^x = a\}$ we have

$$
\begin{aligned}
M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x) &\leq (X_{\tau^*}^{x+\varepsilon} - X_{\tau^*}^x) \sup_{z\in[a,b]} |M'(z)| \\
&= ((1+\varepsilon/x)a - a) \sup_{z\in[a,b]} |M'(z)| \leq \varepsilon \sup_{z\in[a,b]} |M'(z)|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^*\leq\tau_{\mathcal{I}}^{x+\varepsilon}\}}\right] \\
&\leq \mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\mathbb{1}_{\{\tau^*\leq\tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^*<\tau_{\mathcal{I}}^x\}}\right] + \varepsilon \sup_{z\in[a,b]} |M'(z)| \mathsf{P}\left(\tau^* = \tau_{\mathcal{I}}^x\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\frac{V(x+\varepsilon,h) - V(x,h)}{\varepsilon} \\
&\leq \mathsf{E}\left[\int_0^{\tau^*\wedge\tau_{\mathcal{I}}^{x+\varepsilon}} e^{-2ru}\big(f(X_u^{x+\varepsilon},h) - f(X_u^x,h)\big)\varepsilon^{-1}\mathrm{d}u\right] \\
&\quad + \mathsf{E}\left[e^{-2r\tau^*}\big(M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)\big)\varepsilon^{-1}\mathbb{1}_{\{\tau^*\leq\tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^*<\tau_{\mathcal{I}}^x\}}\right] + \sup_{z\in[a,b]} |M'(z)|\, \mathsf{P}(\tau^* = \tau_{\mathcal{I}}^x).
\end{aligned}
$$

From (3.68) we obtain that $\tau_{\mathcal{I}}^{x+\varepsilon} \to \tau_{\mathcal{I}}^x$, P-a.s., as $\varepsilon \to 0$. Thus, when $\varepsilon \to 0$ we have

$$
\mathbb{1}_{\{\tau^*\leq\tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^*<\tau_{\mathcal{I}}^x\}} \to \mathbb{1}_{\{\tau^*<\tau_{\mathcal{I}}^x\}}, \quad \mathsf{P}-a.s.
$$

By the smoothness of f on $K_{a,b}$ (defined in (3.44)) and of M on $[a,b]$, we have

$$
\begin{aligned}
&\int_0^{\tau^*\wedge\tau_{\mathcal{I}}^{x+\varepsilon}} e^{-2ru}\frac{|f(X_u^{x+\varepsilon},h) - f(X_u^x,h)|}{\varepsilon}\mathrm{d}u \leq \sup_{z\in K_{a,b}} |\partial_x f(z,h)| \int_0^{\tau^*} e^{-(r+\frac{1}{2}\sigma^2)u+\sigma B_u}\mathrm{d}u, \\
&e^{-2r\tau^*}\frac{|M(X_{\tau^*}^{x+\varepsilon}) - M(X_{\tau^*}^x)|}{\varepsilon}\mathbb{1}_{\{\tau^*\leq\tau_{\mathcal{I}}^{x+\varepsilon}\}\cap\{\tau^*<\tau_{\mathcal{I}}^x\}} \leq \sup_{z\in[a,b]} |M'(z)|.
\end{aligned}
$$

Thus, letting $\varepsilon \to 0$ and applying the dominated convergence theorem we get

$$
\begin{aligned}
\partial_x V(x,h) \leq \mathsf{E}&\left[\int_0^{\tau^*} e^{-2ru}\, \partial_x f(X_u^x,h)\, X_u^1 \mathrm{d}u + e^{-2r\tau^*} M'(X_{\tau^*}^x) X_{\tau^*}^1 \mathbb{1}_{\{\tau^*<\tau_{\mathcal{I}}^x\}}\right] \\
&+ \sup_{z\in[a,b]} |M'(z)|\, \mathsf{P}(\tau^* = \tau_{\mathcal{I}}^x).
\end{aligned} \tag{3.71}
$$

Similar arguments, applied to the stopping time $\tau^* \wedge \tau_{\mathcal{I}}^{x-\varepsilon}$, which is admissible for $V(x-\varepsilon, h)$, allow us to obtain

$$\begin{aligned} \partial_x V(x,h) &= \lim_{\varepsilon\to 0}\frac{V(x,h)-V(x-\varepsilon,h)}{\varepsilon} \\ &\geq \mathsf{E}\left[\int_0^{\tau^*} e^{-2ru}\,\partial_x f(X_u^x,h)\,X_u^1\mathrm{d}u + e^{-2r\tau^*}M'(X_{\tau^*}^x)X_{\tau^*}^1\mathbb{1}_{\{\tau^*<\tau_{\mathcal{I}}^x\}}\right] \\ &\quad - \sup_{z\in[a,b]}|M'(z)|\,\mathsf{P}(\tau^*=\tau_{\mathcal{I}}^x). \end{aligned} \tag{3.72}$$

Recall that $\tau^* = \tau^*_{x,h}$ and then let $\mathcal{C}_h \ni x \to x_0 \in \partial\mathcal{C}_h \cap \mathcal{I}$. From (3.69) we get $\mathsf{P}(\tau^*_{x,h} = \tau_{\mathcal{I}}^x) \to 0$ and $\mathbb{1}_{\{\tau^*_{x,h}<\tau_{\mathcal{I}}^x\}} \to 1$, P-a.s. Then using dominated convergence in (3.71) and (3.72) we obtain

$$\partial_x V(x,h) \to M'(x_0), \quad \text{as } x \to x_0,$$

which concludes the proof. □

Remark 3.3.12. *We could not infer smooth fit across the stopping boundary diectly from [42] because our underlying process is killed at points a, b. Instead, we adapted the line of arguments in the aforementioned and used the particular characteristics of our optimal stopping problem.*

Thanks to the regularity obtained in the theorem above we can rigorously connect the stopping problem (3.36) to the obstacle problem (3.52)–(3.53) (equivalently to the free boundary problem (3.54)–(3.57)).

Theorem 3.3.13. *Let Assumption 3.3.7 hold. For each $h \in \mathcal{H}$ the value function $V(\,\cdot\,,h)$ is the unique solution, in the a.e. sense, of (3.52)–(3.53) (equivalently of (3.54)–(3.57)) in the class of functions $C(\bar{\mathcal{I}}) \cap C^1(\mathcal{I})$ whose second order partial derivative lies in $L^\infty_{\ell oc}(\mathcal{I})$.*

Proof. From Theorem 3.3.11 we know that $V(\,\cdot\,,h)$ has the right regularity. Moreover, we have $V(\,\cdot\,,h) = M(\,\cdot\,)$ on $\mathcal{D}_h$, where $(\mathcal{L}-2r)M \geq -f$ by (3.59). Then, combining these facts with (3.65) we conclude that for any $h \in \mathcal{H}$

$$\min\{(\mathcal{L}-2r)V(x,h)+f(x,h), M(x)-V(x,h)\} = 0, \quad \text{for } x \in \mathcal{I}\setminus\partial\mathcal{C}_h$$

and clearly $V(a,h) = V(b,h) = 0$ (cf. (3.45)). The same argument guarantees that $V(\,\cdot\,,h)$ also solves (3.54)–(3.65).

Uniqueness of the solution follows by a standard verification argument. Let u be another solution of (3.52)–(3.53) in $C(\bar{\mathcal{I}}) \cap C^1(\mathcal{I})$ with $u'' \in L^\infty_{\ell oc}(\mathcal{I})$ (for simplicity of notation we omit $h \in \mathcal{H}$, given and fixed). Then, by Tanaka's formula and using $(\mathcal{L} - 2r)u \geq -f$, we obtain

$$\mathsf{E}_x\left[e^{-2r\tau}u(X_\tau)\right] \geq u(x) - \mathsf{E}_x\left[\int_0^\tau e^{-2rt}f(X_t,h)\mathrm{d}t\right],$$

for any stopping time $\tau \in \mathcal{T}_x$. Rearranging terms and using $u \leq M$ we obtain

$$u(x) \leq \mathsf{E}_x\left[\int_0^\tau e^{-2rt}f(X_t,h)\mathrm{d}t + e^{-2r\tau}M(X_\tau)\right].$$

Hence $u \leq V$. To prove the reverse inequality it is sufficient to choose $\tau = \inf\{t \geq 0 : u(X_t) = M(X_t)\}$ and all the inequalities above become equalities. □

3.3.4 Analytical formulae

Thanks to Proposition 3.3.10 and Theorems 3.3.11 and 3.3.13, for $h \in \mathcal{H}$, the value function $V(\,\cdot\,,h)$ is a classical solution of the system

$$\begin{cases} (\mathcal{L} - 2r)V(x,h) = -f(x,h), & x \in (x_1^*, x_2^*), \\ V(x,h) = M(x), & x \in [a, x_1^*] \cup [x_2^*, b], \\ V_x(x_1^*,h) = M'(x_1^*), & \text{if } x_1^* > a, \\ V_x(x_2^*,h) = M'(x_2^*), & \text{if } x_2^* < b, \\ a \leq x_1^* < x_2^* \leq b, \end{cases} \tag{3.73}$$

where x_1^*, x_2^* are the optimal stopping boundaries, i.e., $\mathcal{C}_h = (x_1^*, x_2^*)$. Conversely, under Assumption 3.3.7, we will prove that for a fixed $h \in \mathcal{H}$, the solution of the above ODE system also solves (3.52)–(3.53), hence the solution is unique. This is proved by Lemma 3.3.16-3.3.17 and Proposition 3.3.18.

To this end, let $(v(\cdot), x_1, x_2)$, $x_1, x_2 \in \mathbb{R}$, be a solution of

$$\begin{cases} (\mathcal{L} - 2r)v(x) = -f(x,h), & x \in (x_1, x_2), \\ v(x) = M(x), & x \in [a, x_1] \cup [x_2, b], \\ v'(x_1) = M'(x_1), & \text{if } x_1 > a, \\ v'(x_2) = M'(x_2), & \text{if } x_2 < b, \\ a \leq x_1 < x_2 \leq b. \end{cases} \tag{3.74}$$

Denote

$$v_m(x) := M(x) - v(x), \quad x \in \mathcal{I}.$$

Remark 3.3.14. *(i) values of x_1, x_2 in (3.74) could be a and b respectively.*

(ii) We need strict inequality $x_1 < x_2$ to prevent trivial solutions. Otherwise, $v = M$ and any $x_1 = x_2 \in \mathcal{I}$ solve (3.74).

We start from a study of properties of a solution to (3.74).

Lemma 3.3.15. *v' is absolutely continuous on $\mathcal{I}$ with $v'' \in L^\infty_{\ell oc}(\mathcal{I})$.*

Proof. From (3.74) we have that $v \in C^2(\mathcal{I} \setminus \{x_1, x_2\})$. Therefore, it suffices to check that v'' is bounded in the neighbourhood of x_1, x_2 for $x_1, x_2 \in \mathcal{I}$. For this, we check the directional derivatives at x_1, x_2. System (3.74) gives

$$v''(x_1^-) = M''(x_1), \quad v_2''(x_1^+) = \frac{2}{\sigma^2 x_1^2}\big(-rx_1 M'(x_1) + 2rM(x_1)\big),$$
$$v''(x_2^+) = M''(x_2), \quad v_2''(x_2^-) = \frac{2}{\sigma^2 x_2^2}\big(-rx_2 M'(x_2) + 2rM(x_2)\big).$$

Proposition 3.3.2 gives that $M \in C^\infty(\mathcal{I})$, so the above terms are finite. Hence $v'' \in L^\infty_{loc}(\mathcal{I})$ and the absolute continuity of v' follows. □

Proofs of the following two lemmas are based on the ideas from [122, Lemma 2.5 and Lemma 2.6].

Lemma 3.3.16. *Let $y \in [x_1, x_2]$, if v_m attains a local maximum at y and $v_m(y) \geq 0$, then $G(y,h) \leq 0$; if v_m attains a local minimum at y and $v_m(y) \leq 0$, then $G(y,h) \geq 0$.*

Proof. Notice that for any $x \in (x_1, x_2)$, function v_m satisfies:

$$(\mathcal{L} - 2r)v_m(x) = G(x,h).$$

Integrate over $(x, x+\epsilon)$ and $(x-\epsilon, x)$ for any $\epsilon > 0$ and $x+\epsilon, x-\epsilon \in (x_1, x_2)$:

$$v_m'(x+\epsilon) = v_m'(x) + \int_x^{x+\epsilon} \frac{2}{\sigma^2 z^2}(2rv_m(z) - rzv_m'(z) + G(z,h))\mathrm{d}z, \tag{3.75}$$
$$v_m'(x-\epsilon) = v_m'(x) + \int_{x-\epsilon}^{x} \frac{2}{\sigma^2 z^2}(-2rv_m(z) + rzv_m'(z) - G(z,h))\mathrm{d}z. \tag{3.76}$$

First assume that y is a local maximum and $v_m(y) \geq 0$. If $y \in (x_1, x_2)$, we take $x = y$ in (3.75) and (3.76). Assume $G(y,h) > 0$. For all sufficiently small ϵ, we have

$$v_m'(y+\epsilon) = \int_y^{y+\epsilon} \frac{2}{\sigma^2 z^2}(2rv_m(z) - rzv_m'(z) + G(z,h))\mathrm{d}z > 0, \tag{3.77}$$
$$v_m'(y-\epsilon) = \int_{y-\epsilon}^{y} \frac{2}{\sigma^2 z^2}(-2rv_m(z) + rzv_m'(z) - G(z,h))\mathrm{d}z < 0. \tag{3.78}$$

If $y = x_1$ (or $y = x_2$), then we only consider (3.75) ((3.76), respectively). Assuming $G(y, h) > 0$, we still obtain (3.77) ((3.78), respectively). In either cases, (3.77) or (3.78) contradicts that y is a local maximum, so we must have $G(y, h) \leq 0$.

Using the same argument for function $-v_m$, one can prove the case when y is a local minimum. □

Lemma 3.3.17. *Under Assumption 3.3.7.*

(i) If (A.1) holds, we have $x_2 = b$ and $x_1 \leq x_{G_1}$.

(ii) If (A.2) holds, we have $x_1 \leq x_{G_1}$ and $x_2 \geq x_{G_2}$.

(iii) If (A.3) holds, we have $x_1 = a$ and $x_2 \geq x_{G_2}$.

Moreover, in any of these three cases we have $v_m \geq 0$ on $[x_1, x_2]$.

Proof. We focus on case (ii), as the same argument can be easily adapted to prove case (i) and (iii). We first show that $x_2 \geq x_{G_2}$ when $x_2 < b$ (it is trivial if $x_2 = b$). Argue by contradiction, assume $x_2 < x_{G_2}$, then we have either (1) $x_1 \leq x_{G_1} < x_2 < x_{G_2}$, (2) $x_{G_1} < x_1 < x_2 < x_{G_2}$ or (3) $x_1 < x_2 \leq x_{G_1}$. Figure 3.3 shows these three cases.

Recall that under (A.2), the sign of $G(x, h)$ for each x is known if we know whether x lies between the zero points x_{G_1}, x_{G_2}, and vice versa. We first study the sign of v_m in small neighbourhood of x_1, x_2, then show that the position of x_1, x_2 in each of cases above contradicts the sign of v_m. Since $v'_m(x_1) = v'_m(x_2) = 0$, for any sufficiently small $\epsilon > 0$ we have

$$v'_m(x_2 - \epsilon) = \int_{x_2-\epsilon}^{x_2} \frac{2}{\sigma^2 z^2}(-2rv_m(z) + rzv'_m(z) - G(z, h))\mathrm{d}z, \tag{3.79}$$

$$v'_m(x_1 + \epsilon) = \int_{x_1}^{x_1+\epsilon} \frac{2}{\sigma^2 z^2}(2rv_m(z) - rzv'_m(z) + G(z, h))\mathrm{d}z. \tag{3.80}$$

By (3.79), because $v_m(x_2) = v'_m(x_2) = 0$, we have $v'_m(x_2 - \epsilon) > 0$ for any sufficiently small $\epsilon > 0$ once $G(x_2, h) < 0$. Hence $v_m < 0$ in $(x_2 - \epsilon, x_2)$ in the case $G(x_2, h) < 0$. Similarly, by (3.80), we have $v'_m(x_1 + \epsilon) > 0$ for any small enough ϵ once $G(x_1, h) > 0$, which implies $v_m > 0$ in $(x_1, x_1 + \epsilon)$ once $G(x_1, h) > 0$.

In case (1), we have $G(x_2, h) < 0, G(x_1, h) > 0$. Thus, we know that $v_m > 0$ in $(x_1, x_1+\epsilon)$ and $v_m < 0$ in $(x_2 - \epsilon, x_2)$ for any small enough ϵ. Hence there exists $z \in (x_1, x_2)$ such that $v_m(z) = 0$. Let $y_1 \in [x_1, z]$, such that $v_m(y_1) = \sup_{x_1 \leq x \leq z} v_m(x)$. Then y_1 is a local maximum and $v_m(y_1) > 0$. By Lemma 3.3.16, we have $G(y_1, h) \leq 0$ and therefore $x_{G1} \leq y_1 < z$. Let $y_2 \in [z, x_2]$, such that $v_m(y_2) = \inf_{z \leq x \leq x_2} v_m(x)$. Then y_2 is a local minimum and $v_m(y_2) < 0$.

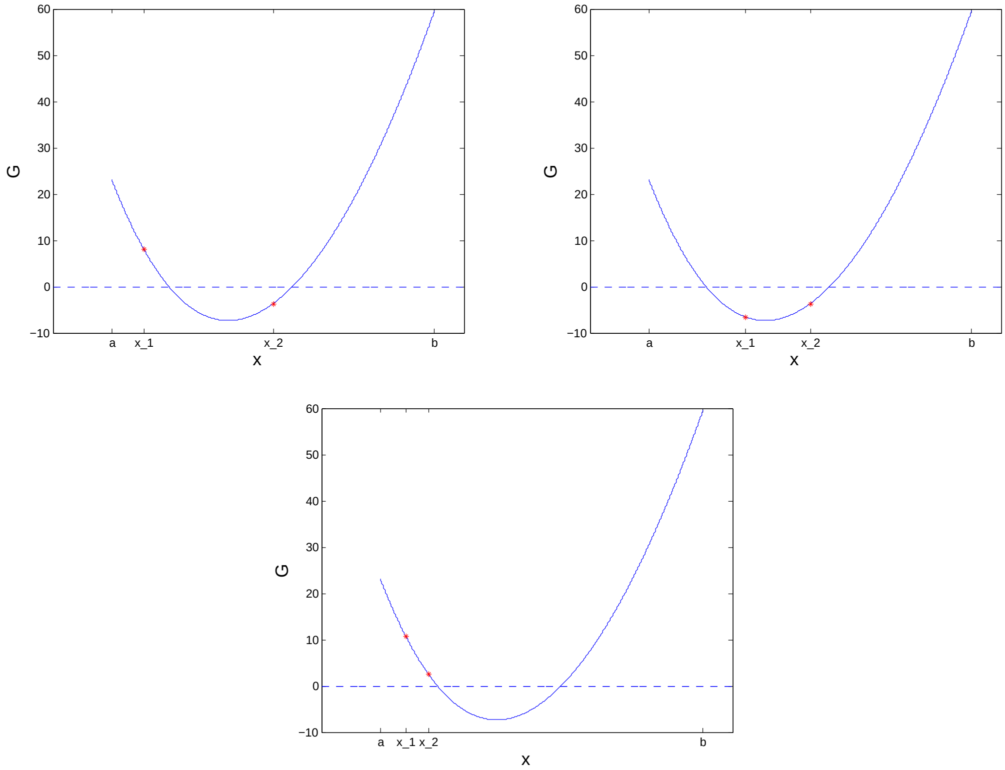


Figure 3.3: x_1, x_2 in case (1) (Top-left), case (2) (Top-right) and case (3) (Bottom)

By Lemma 3.3.16, $G(y_2, h) \geq 0$, and thus we have $z < y_2 \leq x_{G1}$. But we just showed $x_{G1} \leq y_1 < z$, a contradiction.

In case (2), $G(x, h) < 0$ for $x \in [x_1, x_2]$, by the maximum principle, we know that $v_m > 0$ for $x \in (x_1, x_2)$. However, as shown above, we have $v_m < 0$ in $(x_2 - \epsilon, x_2)$ because $G(x_2, h) < 0$, a contradiction.

In case (3), $G(x, h) > 0$ for $x \in [x_1, x_2]$, by the maximum principle, we have $v_m < 0$ for $x \in (x_1, x_2)$. This contradicts that $v_m > 0$ in $(x_1, x_1 + \epsilon)$ since $G(x_1, h) > 0$. Conclude all three cases, we must have $x_2 \geq x_{G_2}$. Apply the same argument above one can show $x_1 \leq x_{G_1}$.

We now prove $v_m \geq 0$ for $x \in [x_1, x_2]$ by showing that v_m is decreasing in $[x_{G_2}, x_2)$ and increasing in $(x_1, x_{G_1}]$. The monotonicity implies that $v_m(x) > 0$ for any $x \in (x_1, x_{G_1}] \cup [x_{G_2}, x_2)$. The rest is just an application of maximum principle for v_m in (x_{G_1}, x_{G_2}). Assume v_m is not decreasing on $[x_{G_2}, x_2)$. By the previous proof, we know that $v_m > 0$ in $(x_2 - \epsilon, x_2)$. Then there exists $z \in (z_1, x_2)$ for some $z_1 > x_{G_2}$, such that $v_m(z) = \sup_{z_1 < x < x_2} v_m(x) > 0$ is a local maximum. By Lemma 3.3.16, $G(z, h) \leq 0$, contradicts that $G(x, h) > 0$ for $x \in (x_{G_2}, x_2)$. With the same argument one can prove that v_m is increasing in $(x_1, x_{G_1}]$. $\square$

Proposition 3.3.18. *Under Assumption 3.3.7, for $h \in \mathcal{H}$, the ODE system (3.74) admits a unique solution $(V(\,\cdot\,,h), x_1^*, x_2^*)$.*

Proof. We show the proof in case (A.2). Cases (A.1) (A.3) are analogous. By the definition of G in (3.58), we have

$$(\mathcal{L}-2r)v(x) = \begin{cases} -f(x,h), & x \in (x_1,x_2), \\ \\ G(x,h)-f(x,h), & x \in [a,x_1] \cup [x_2,b], \end{cases} \tag{3.81}$$

By Lemma 3.3.17 (ii), we have $v_m \geq 0$ on $[x_1,x_2]$ and $G(x,h) \geq 0$ on $[a,x_1] \cup [x_2,b]$, which give us $v(x) \leq M(x)$ and $(\mathcal{L}-2r)v(x) \geq -f(x,h)$ on $\mathcal{I}$. Together with $v(x) = M(x)$ on $[a,x_1] \cup [x_2,b]$, it is evident that $(v(\cdot),x_1,x_2)$ solves the variational inequality (3.52)–(3.53). By Lemma 3.3.15, we also know that $v'' \in L^\infty_{\ell oc}(\mathcal{I})$. Due to Theorem 3.3.13, we conclude that $(v(\cdot),x_1,x_2)$ coincides with $(V(\,\cdot\,,h), x_1^*, x_2^*)$. The uniqueness follows. □

Remark 3.3.19. *An alternative approach to the one we adopted in the section above, consists in proving directly that there exists a (classical) solution (v,x_1,x_2) to the system (3.74). The arguments given above imply that this solution is unique, it is the value function of (3.36), and x_1,x_2 are the optimal stopping boundaries. Since solving v explicitly is infeasible due to the complexity of the equations arising from the explicit form of the function M, we took an alternative route: we used direct methods to obtain the properties of the stopping set and the smoothness of the value function, getting, as a consequence, the existence of a solution to (3.74).*

Having established its existence and uniqueness, computing the solution of system (3.74) is now straightforward. A general solution of the ODE in the first line of the system is of the form

$$v(x) = C_1 x^{q_1} + C_2 x^{q_2} - x^2(h - d^{-1}(\hat{a}/x)^{1+d})^2, \tag{3.82}$$

with d as in (3.4) and q_1,q_2 as in (3.17). The constants C_1,C_2 and the optimal stopping boundaries x_1^*,x_2^* are determined by solving a system of algebraic equations derived from the remaining conditions in (3.74). The existence and uniqueness of those constants follows from the earlier discussion in this subsection. We mention that we could not solve those algebraic equations analytically, so all examples presented in this chapter involve numerical solution of this system of algebraic equations.

Figure 3.4 displays three possible forms of the stopping set and corresponding value functions. The stopping sets are identified by the values where the solid line (the payoff $M(\cdot)$) coincides with the dashed line (the value function $V(\cdot,h)$).

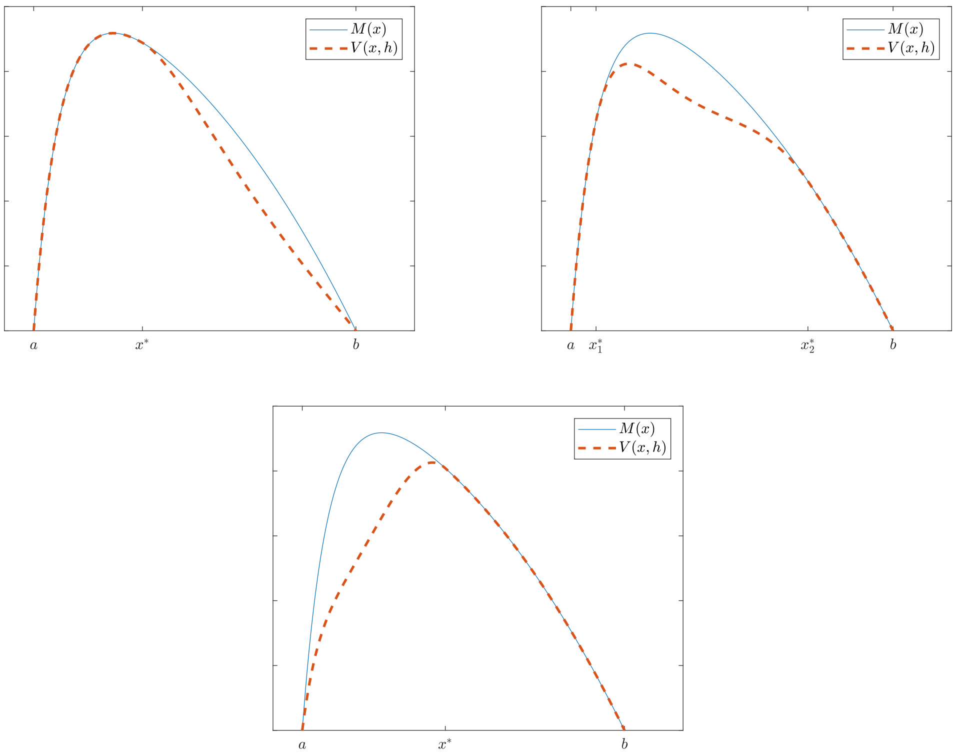


Figure 3.4: Plots of the map $x \mapsto V(x,h)$ and $M(x)$ for different values of the initial stock holding h using parameters $r = 3\%$, $\sigma = 30\%$, $K = 100$, $b = 150$, and $a = \hat{a} = K/(1 + d^{-1}) = 40$.

3.4 Regularity of the stopping boundaries

So far we have studied an optimal control problem for a fixed initial stock holding $h \in \mathcal{H}$. Optimal stopping boundaries x_1^*, x_2^* from Proposition 3.3.10 obviously depend on h; denote them by $x_{1,h}^*$ and $x_{2,h}^*$ with one of them possibly being equal to a or b. We will show that $x_{1,h}^*$ and $x_{2,h}^*$ are non-decreasing and continuous in h. Apart from these results being of interest on their own, they will be instrumental in studying the mapping $h \mapsto V(x,h)$ and, consequently, in determining, in Section 3.5, an optimal initial stock holding h^* in problem (3.13).

Recalling $\tau_{x,h}^*$ from (3.49) we introduce functions $\hat{\gamma}_{h,i}$, $i = 1,2$, and $\widehat{\Gamma}_h$, which are analogues of those in (3.30) and (3.34):

$$
\begin{aligned}
&\hat{\gamma}_{h,1}(x) := \mathsf{E}_x\Big[\int_0^{\tau_{x,h}^*} e^{-2ru}\sigma^2 X_u^2 \mathrm{d}u\Big], \quad \hat{\gamma}_{h,2}(x) := \mathsf{E}_x\Big[\int_0^{\tau_{x,h}^*} e^{-2ru} P'(X_u)\sigma^2 X_u^2 \mathrm{d}u\Big], \\
&\hat{\gamma}_{h,3}(x) := \mathsf{E}_x\left[\int_0^{\tau_{x,h}^*} e^{-2ru} (P'(X_u))^2\sigma^2 X_u^2 \mathrm{d}u\right] \quad \text{and} \quad \widehat{\Gamma}_h(x) := \frac{\gamma_{h,2}(x)}{\gamma_{h,1}(x)}.
\end{aligned}
\tag{3.83}
$$

If we fix the next trading time at $\tau^*_{x,h}$, we can easily see that the value $\widehat{\Gamma}_h(x)$ minimises (3.36). This leads to a fixed point, so that an optimal h^* in (3.13) must satisfy $\widehat{\Gamma}_{h^*}(x) = h^*$. We will later show that $\widehat{\Gamma}_{h^*}(x)$ indeed determines an optimal initial stock holding.

Applying similar arguments as in the proof of Proposition 3.3.2 to $\widehat{\Gamma}_h$ with $x^*_{1,h}$ and $x^*_{2,h}$ in place of a and b we obtain the next result.

Proposition 3.4.1. *For any $h \in \mathcal{H}$, we have $\widehat{\Gamma}_h$ is C^∞ and strictly increasing on $(x^*_{1,h}, x^*_{2,h})$ with*

$$\widehat{\Gamma}_h(x^*_{1,h}) := \lim_{x \downarrow x^*_{1,h}} \widehat{\Gamma}_h(x) > P'(x^*_{1,h}),$$
$$\widehat{\Gamma}_h(x^*_{2,h}) := \lim_{x \uparrow x^*_{2,h}} \widehat{\Gamma}_h(x) < P'(x^*_{2,h}).$$

We now prove a technical lemma which is fundamental for showing the monotonicity of the stopping boundaries. For $h \in (\Gamma(a), \Gamma(b))$, thanks to the monotonicity of Γ (Proposition 3.3.2) we have that there exists a unique point $x_\Gamma(h) \in \mathcal{I}$ such that

$$\Gamma(x_\Gamma(h)) = h. \tag{3.84}$$

Moreover $x_\Gamma(h) \in \mathcal{C}_h$, because $G(x_\Gamma(h), h) < 0$ by (3.60).

Lemma 3.4.2. *Fix $h \in \mathcal{H}$ and let Assumption 3.3.7 hold.*

(i) If $x^*_{1,h} > a$, then $h > \Gamma(x^*_{1,h})$ and $h + \Gamma(x^*_{1,h}) \geq 2\widehat{\Gamma}_h(x^*_{1,h})$.

(ii) If $x^*_{2,h} < b$, then $h < \Gamma(x^*_{2,h})$ and $h + \Gamma(x^*_{2,h}) \leq 2\widehat{\Gamma}_h(x^*_{2,h})$.

(iii) If $h > \Gamma(a)$ and $x^*_{1,h} = a$, then $h + \Gamma(x) > 2\widehat{\Gamma}_h(x)$ for all $x \in (a, x_\Gamma(h))$.

(iv) If $h < \Gamma(b)$ and $x^*_{2,h} = b$, then $h + \Gamma(x) < 2\widehat{\Gamma}_h(x)$ for all $x \in (x_\Gamma(h), b)$.

The statement of the lemma has an intuitive financial interpretation. In (i), if the left stopping boundary $x^*_{1,h}$ is non-trivial, then the optimal trade at $x^*_{1,h}$ is to increase the short position in the stock (recall that Γ is negative). This is consistent with the Delta hedge P' being an increasing function starting from -1 at $\hat{a}$ and increasing to 0 at ∞. Analogously, statement (ii) says that if the right stopping boundary $x^*_{2,h}$ is non-trivial, the optimal trade at $x^*_{2,h}$ is to reduce the short position in the stock. Statements (iii)-(iv) formulate a stronger version of the previous two when the stopping boundaries are trivial.

Proof of Lemma 3.4.2. Recall that $x_p(h) \in \mathcal{I}$ and $G(x_p(h), h) < 0$, where $x_p(h)$ is defined in (3.64). Hence, it must be $x^*_{1,h} < x_p(h) < x^*_{2,h}$ for $h \in \mathcal{H}$. The relative placement of $x^*_{1,h}$, $x^*_{2,h}$, $x_p(h)$ and $x_\Gamma(h)$ will be central in this proof.

Proof of (i)*:* If $x^*_{1,h} > a$, then $G(x^*_{1,h}, h) \geq 0$ by (3.59) and

$$
\begin{aligned}
0 \leq G(x^*_{1,h}, h) &< \sigma^2(x^*_{1,h})^2 \left((h - P'(x^*_{1,h}))^2 - (\Gamma(x^*_{1,h}) - P'(x^*_{1,h}))^2\right) \\
&= \sigma^2(x^*_{1,h})^2(h - \Gamma(x^*_{1,h}))(h + \Gamma(x^*_{1,h}) - 2P'(x^*_{1,h})),
\end{aligned}
\tag{3.85}
$$

where the strict inequality comes from (3.60) upon noting that $\left(\Gamma'(x^*_{1,h})\right)^2\gamma_1(x^*_{1,h}) > 0$ since $x^*_{1,h} \in \mathcal{I}$. Recalling that $x^*_{1,h} < x_p(h)$ and P' is strictly increasing, we have $P'(x^*_{1,h}) < h$, so

$$
h + \Gamma(x^*_{1,h}) - 2P'(x^*_{1,h}) > \Gamma(x^*_{1,h}) - h. \tag{3.86}
$$

If $h - \Gamma(x^*_{1,h}) < 0$, combining (3.85) and (3.86) gives $(h - \Gamma(x^*_{1,h}))^2 < 0$, which is impossible. The equality $h - \Gamma(x^*_{1,h}) = 0$ contradicts (3.85). Hence, $h - \Gamma(x^*_{1,h}) > 0$, which is the first claim in (i).

For the second claim we expand the square in $f(X_u, h)$ and obtain

$$
\mathsf{E}_x\left[\int_0^{\tau^*_h} e^{-2ru} f(X_u, h)\mathrm{d}u\right] = h^2\hat{\gamma}_{h,1}(x) - 2h\hat{\gamma}_{h,2}(x) + \hat{\gamma}_{h,3}(x) \tag{3.87}
$$

with the notation introduced in (3.83). The explicit formulae for $\hat{\gamma}_{h,i}$, $i = 1, 2, 3$, can be derived from (3.21) upon replacing a and b by $x^*_{1,h}$ and $x^*_{2,h}$, and φ and ψ by φ_h and ψ_h. The latter are, respectively, the decreasing and increasing fundamental solutions of the ODE

$$
(\mathcal{L} - 2r)u(x) = 0, \quad x \in (x^*_{1,h}, x^*_{2,h}),
$$

with the boundary conditions

$$
\psi_h(x^*_{1,h}+) = 0, \quad \psi'_h(x^*_{1,h}+) > 0, \quad \varphi_h(x^*_{2,h}-) = 0, \quad \varphi'_h(x^*_{2,h}-) < 0.
$$

Again, these can be calculated explicitly using (3.20). Later we will also use that

$$
\hat{\gamma}'_{h,1}(x^*_{1,h}) = w_h^{-1}\psi'_h(x^*_{1,h})\int_{x^*_{1,h}}^{x^*_{2,h}} \varphi_h(z)\sigma^2 z^2 m'(z)\mathrm{d}z > 0, \tag{3.88}
$$

where $w_h = \hat{w}\left(1 - (x^*_{1,h}/x^*_{2,h})^{q_1 - q_2}\right)$ is the Wronskian (c.f. (3.23)).

From (3.35) and (3.36) we can write the value function V and the stopping payoff M as

$$V(x,h) = \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} f(X_u,h)\mathrm{d}u\right] + \mathsf{E}_x\left[\int_{\tau_h^*}^{\tau_\mathcal{I}} e^{-2ru} f(X_u,\Gamma(X_{\tau_h^*}))\mathrm{d}u\right] \tag{3.89}$$
$$= V_1(x) + V_2(x),$$

$$M(x) = \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} f(X_u,\Gamma(x))\mathrm{d}u\right] + \mathsf{E}_x\left[\int_{\tau_h^*}^{\tau_\mathcal{I}} e^{-2ru} f(X_u,\Gamma(x))\mathrm{d}u\right] \tag{3.90}$$
$$= M_1(x) + M_2(x),$$

where in V_1, V_2, M_1, M_2 we omit the dependence on $h \in \mathcal{H}$ which is fixed. Thanks to the explicit formulae for ψ_h and φ_h, (3.21) and $\Gamma \in C^\infty(\mathcal{I})$ (Proposition 3.3.2) it is not hard to verify that $M_1, V_1 \in C^1([x_{1,h}^*, x_{2,h}^*])$. For V_2, using the strong Markov property we have

$$V_2(x) = \widehat{M}(x_{1,h}^*, \Gamma(x_{1,h}^*))\mathsf{E}_x\left[e^{-2r\tau_{1,h}^*}\mathbb{1}_{\{\tau_{1,h}^*<\tau_{2,h}^*\}}\right] + \widehat{M}(x_{2,h}^*, \Gamma(x_{2,h}^*))\mathsf{E}_x\left[e^{-2r\tau_{2,h}^*}\mathbb{1}_{\{\tau_{1,h}^*>\tau_{2,h}^*\}}\right],$$

where $\tau_{1,h}^*, \tau_{2,h}^*$ denote the first entry time to $[a, x_{1,h}^*]$ and $[x_{2,h}^*, b]$, respectively. It is well-known that the two expected values on the right-hand side of the equation above can be expressed in terms of ψ_h and φ_h ([18, Ch. II, Par. 10]), hence proving $V_2 \in C^1([x_{1,h}^*, x_{2,h}^*])$. An analogous argument applies for M_2.

Since $V = M$ at $x_{1,h}^*$ and the smooth-fit holds we have

$$V_1(x_{1,h}^*) + V_2(x_{1,h}^*) = M_1(x_{1,h}^*) + M_2(x_{1,h}^*), \tag{3.91}$$
$$V_1'(x_{1,h}^*) + V_2'(x_{1,h}^*) = M_1'(x_{1,h}^*) + M_2'(x_{1,h}^*). \tag{3.92}$$

Noticing that $\mathsf{P}_{x_{1,h}^*}(\tau_h^* = 0) = 1$ we have $V_1(x_{1,h}^*) = M_1(x_{1,h}^*) = 0$ and hence,

$$V_2(x_{1,h}^*) = M_2(x_{1,h}^*). \tag{3.93}$$

Using the optimality of $\Gamma(x)$ for $\widehat{M}(x,\cdot)$ and the strong Markov property, for $x \in (x_{1,h}^*, x_{2,h}^*)$ we have

$$V_2(x) = \mathsf{E}_x\left[e^{-2r\tau_h^*}M(X_{\tau_h^*})\right] \le \mathsf{E}_x\left[e^{-2r\tau_h^*}\widehat{M}(X_{\tau_h^*}, \Gamma(x))\right] = M_2(x).$$

Hence, $V_2'(x_{1,h}^*) \le M_2'(x_{1,h}^*)$. Inserting the latter into (3.92) we deduce

$$V_1'(x_{1,h}^*) \ge M_1'(x_{1,h}^*). \tag{3.94}$$

Our task is now to rewrite both sides of (3.94) using (3.87) and (3.37). For an arbitrary $x \in [x_{1,h}^*, x_{2,h}^*]$ we have

$$V_1'(x) = h^2\hat{\gamma}_{h,1}'(x) - 2h\hat{\gamma}_{h,2}'(x) + \hat{\gamma}_{h,3}'(x)$$
$$M_1'(x) = \Gamma^2(x)\hat{\gamma}_{h,1}'(x) - 2\Gamma(x)\hat{\gamma}_{h,2}'(x) + \hat{\gamma}_{h,3}'(x) + 2\Gamma(x)\Gamma'(x)\hat{\gamma}_{h,1}(x) - 2\Gamma'(x)\hat{\gamma}_{h,2}(x).$$

Inserting the above in (3.94) we obtain

$$(h^2 - \Gamma^2(x^*_{1,h}))\hat{\gamma}'_{h,1}(x^*_{1,h}) - 2(h - \Gamma(x^*_{1,h}))\hat{\gamma}'_{h,2}(x^*_{1,h}) \geq 0. \tag{3.95}$$

Since $\hat{\gamma}'_{h,1}(x^*_{1,h}) > 0$ by (3.88) and we have shown above that $h - \Gamma(x^*_{1,h}) > 0$, we can divide both sides of (3.95) by $(h - \Gamma(x^*_{1,h}))\hat{\gamma}'_{h,1}(x^*_{1,h})$, thus obtaining

$$h + \Gamma(x^*_{1,h}) \geq 2\frac{\gamma'_{h,2}(x^*_{1,h})}{\gamma'_{h,1}(x^*_{1,h})} = 2 \lim_{x \downarrow x^*_{1,h}} \widehat{\Gamma}_h(x) =: 2\widehat{\Gamma}_h(x^*_{1,h}),$$

where the first equality follows from d'Hospital's rule (see (3.83)). This concludes the proof of (i).

Proof of (ii): This is analogous to that of (i), hence we omit further details.

Proof of (iii) and (iv): We give a full argument only for (iv) as the case of (iii) can be treated analogously. Fix $h \in \mathcal{H}$ such that $h < \Gamma(b)$ and $x^*_{2,h} = b$.

First we notice that for all $x \in \mathcal{I} \setminus \{x_\Gamma(h)\}$ we have

$$M(x) = \mathsf{E}_x\left[\int_0^{\tau_\mathcal{I}} e^{-2ru} f(X_u, \Gamma(x))\mathrm{d}u\right] < \mathsf{E}_x\left[\int_0^{\tau_\mathcal{I}} e^{-2ru} f(X_u, h)\mathrm{d}u\right], \tag{3.96}$$

where the strict inequality is due to the fact that for each $x \in \mathcal{I}$, the mapping $\zeta \mapsto \widehat{M}(x, \zeta)$ is strictly convex and attains its minimum at $\zeta = \Gamma(x)$.

Now fix an arbitrary point $\hat{x} \in (x_\Gamma(h), b)$. With the notation introduced in (3.90) we rewrite (3.96) as

$$\begin{aligned} M_1(\hat{x}) + M_2(\hat{x}) &< \mathsf{E}_{\hat{x}}\left[\int_0^{\tau^*_h} e^{-2ru} f(X_u, h)\mathrm{d}u\right] + \mathsf{E}_{\hat{x}}\left[\int_{\tau^*_h}^{\tau_\mathcal{I}} e^{-2ru} f(X_u, h)\mathrm{d}u\right] \\ &=: V_1(\hat{x}) + \widetilde{V}_2(\hat{x}). \end{aligned} \tag{3.97}$$

Here we are again omitting the dependence of V_1 and $\widetilde{V}_2$ on h and note that V_1 is the same as in (3.89), whereas $\widetilde{V}_2$ is not. Since $x^*_{2,h} = b$, we have $\{X_{\tau^*_h} = b\} = \{\tau^*_h = \tau_\mathcal{I}\}$, so $\{X_{\tau^*_h} = x^*_{1,h}\} = \{\tau^*_h < \tau_\mathcal{I}\}$. Using this fact and the strong Markov property we obtain

$$\begin{aligned} \widetilde{V}_2(\hat{x}) - M_2(\hat{x}) &= \mathsf{E}_{\hat{x}}\left[e^{-2r\tau^*_h}\mathsf{E}_{X_{\tau^*_h}}\left[\int_0^{\tau_\mathcal{I}} e^{-2ru}\big(f(X_u, h) - f(X_u, \Gamma(\hat{x}))\big)\mathrm{d}u\right]\right] \\ &= \mathsf{E}_{\hat{x}}\left[e^{-2r\tau^*_h}\mathbb{1}_{\{\tau^*_h < \tau_\mathcal{I}\}}\right]\mathsf{E}_{x^*_{1,h}}\left[\int_0^{\tau_\mathcal{I}} e^{-2ru}\big(f(X_u, h) - f(X_u, \Gamma(\hat{x}))\big)\mathrm{d}u\right] \\ &= \mathsf{E}_{\hat{x}}\left[e^{-2r\tau^*_h}\mathbb{1}_{\{\tau^*_h < \tau_\mathcal{I}\}}\right]\big(\widehat{M}(x^*_{1,h}, h) - \widehat{M}(x^*_{1,h}, \Gamma(\hat{x}))\big) < 0, \end{aligned} \tag{3.98}$$

where it remains to justify the final inequality. Since $\hat{x} > x_\Gamma(h) > x^*_{1,h}$, by the monotonicity of Γ (Proposition 3.3.2), we have

$$\Gamma(\hat{x}) > h > \Gamma(x^*_{1,h}). \tag{3.99}$$

Since the mapping $\zeta \mapsto \widehat{M}(x^*_{1,h}, \zeta)$ is strictly convex and attains its minimum at $\Gamma(x^*_{1,h})$, it is strictly increasing for $\zeta > \Gamma(x^*_{1,h})$. Hence the inequality in (3.98) holds and $\widetilde{V}_2(\hat{x}) < M_2(\hat{x})$ upon noticing that $\mathsf{P}_{\hat{x}}(\tau^*_h < \tau_{\mathcal{I}}) > 0$.

Combining (3.97) with (3.98) implies $V_1(\hat{x}) > M_1(\hat{x})$. Rewriting this inequality in terms of the functions $\hat{\gamma}_{h,i}$, $i = 1, 2, 3$, given in (3.83), we obtain

$$(h^2 - \Gamma^2(\hat{x}))\hat{\gamma}_{h,1}(\hat{x}) - 2(h - \Gamma(\hat{x}))\hat{\gamma}_{h,2}(\hat{x}) > 0. \tag{3.100}$$

It is clear from (3.83) that $\hat{\gamma}_{h,1}(\hat{x}) > 0$ since $\hat{x} \in (x^*_{1,h}, x^*_{2,h})$. Then, using also (3.99) we can divide both sides of (3.100) by $(h - \Gamma(\hat{x}))\hat{\gamma}_{h,1}(\hat{x}) < 0$ to obtain

$$h + \Gamma(\hat{x}) < 2\frac{\hat{\gamma}_{h,2}(\hat{x})}{\hat{\gamma}_{h,1}(\hat{x})} = 2\widehat{\Gamma}_h(\hat{x}).$$

□

With Lemma 3.4.2 in place we can now show that the optimal stopping boundaries $x^*_{1,h}$, $x^*_{2,h}$ are non-decreasing in h.

Theorem 3.4.3. *Let Assumption 3.3.7 hold. Then, the mappings $h \mapsto x^*_{1,h}$ and $h \mapsto x^*_{2,h}$ are non-decreasing on $\mathcal{H}$.*

Proof. We only show that $h \mapsto x^*_{2,h}$ is non-decreasing as the arguments for the monotonicity of $h \mapsto x^*_{1,h}$ are analogous. For the clarity of notation let us set $x^*_i(h) = x^*_{i,h}$ for $i = 1, 2$.

Fix $h < \tilde{h}$ in $\mathcal{H}$. If $\tilde{h} \geq \Gamma(b)$ then $x^*_2(\tilde{h}) = b$ by Propositions 3.3.9 and 3.3.10, so trivially $x^*_2(h) \leq x^*_2(\tilde{h})$. Assume now that $\tilde{h} < \Gamma(b)$. We split the proof into two cases.

(*Case 1*). Let us first consider $x^*_2(h) = b$ (this can occur under (A.2); see Proposition 3.3.10). Arguing by contradiction we assume $x^*_2(\tilde{h}) < b$. Then, we have

$$V(x, \tilde{h}) = M(x) > V(x, h), \quad \text{for all } x \in (x^*_2(\tilde{h}) \vee x^*_1(h), b). \tag{3.101}$$

Taking τ^*_h optimal for $V(x, h)$ and noticing that it is also admissible for $V(x, \tilde{h})$, it is easy to check that (3.101) implies

$$\mathsf{E}_x\left[\int_0^{\tau^*_h} e^{-2ru} f(X_u, \tilde{h})\mathrm{d}u\right] > \mathsf{E}_x\left[\int_0^{\tau^*_h} e^{-2ru} f(X_u, h)\mathrm{d}u\right] \quad \text{for all } x \in (x^*_2(\tilde{h}) \vee x^*_1(h), b).$$

Both expected values above can be written using the functions $\hat{\gamma}_{h,i}$, $i = 1, 2, 3$, introduced in (3.83) (see also (3.87)). This gives

$$(\tilde{h}^2 - h^2)\hat{\gamma}_{h,1}(x) - 2(\tilde{h} - h)\hat{\gamma}_{h,2}(x) > 0.$$

Dividing both sides by $(\tilde{h} - h)\hat{\gamma}_{h,1}(x) > 0$ we obtain

$$h + \tilde{h} > 2\frac{\hat{\gamma}_{h,2}(x)}{\hat{\gamma}_{h,1}(x)} = 2\widehat{\Gamma}_h(x). \tag{3.102}$$

Since $x_\Gamma(\tilde{h}) \in \mathcal{C}_{\tilde{h}}$ by (3.84) and Γ is strictly increasing, we have $x_\Gamma(\tilde{h}) < x_2^*(\tilde{h})$ and $\tilde{h} < \Gamma(x)$ for $x \in (x_2^*(\tilde{h}) \vee x_1^*(h), b)$. Hence,

$$\Gamma(x) + h > \tilde{h} + h > 2\widehat{\Gamma}_h(x),$$

which contradicts (iv) in Lemma 3.4.2.

(*Case 2*). Let us now consider $x_2^*(h) < b$. In this case we have $\Gamma(x_2^*(h)) < \Gamma(b)$, which gives rise to two sub-cases.

(*Case 2a*). If $h < \Gamma(x_2^*(h)) \leq \tilde{h} < \Gamma(b)$, by monotonicity of Γ we obtain $x_2^*(h) \leq x_\Gamma(\tilde{h})$. Moreover, using that $x_\Gamma(\tilde{h}) \in \mathcal{C}_{\tilde{h}}$, it must be $x_\Gamma(\tilde{h}) < x_2^*(\tilde{h})$. Hence the claim.

(*Case 2b*). If $h < \tilde{h} < \Gamma(x_2^*(h)) < \Gamma(b)$, we adapt arguments from Case 1 above. Assume, by contradiction, that $x_2^*(\tilde{h}) < x_2^*(h)$. Then, as in (3.102), we have $h + \tilde{h} > 2\widehat{\Gamma}_h(x)$ for all $x \in (x_2^*(\tilde{h}) \vee x_1^*(h), x_2^*(h))$. By assumption $\tilde{h} < \Gamma(x_2^*(h))$, hence

$$h + \Gamma(x_2^*(h)) > h + \tilde{h} \geq 2 \lim_{x \uparrow x_2^*(h)} \widehat{\Gamma}_h(x),$$

which contradicts (ii) in Lemma 3.4.2. □

Theorem 3.4.3 allows us to prove the continuity of the optimal boundaries and the continuity of the optimal stopping time with respect to x and h (jointly). This is needed to prove that $\partial_h V$ exists and it is (jointly) continuous, which will then allow to establish first order conditions for a minimiser in (3.15).

Theorem 3.4.4. *Let Assumption 3.3.7 hold. Then the mappings $h \mapsto x_{1,h}^*$ and $h \mapsto x_{2,h}^*$ are continuous on $\mathcal{H}$. Moreover, $(x, h) \mapsto \tau_{x,h}^*$ is continuous on $\mathcal{I} \times \mathcal{H}$, P-a.s.*

Proof. First we show continuity of the optimal boundaries and then continuity of the stopping times. For the clarity of notation let us set $x_i^*(h) = x_{i,h}^*$ for $i = 1, 2$.

(*Continuity of the boundaries*). We only give full arguments for the upper boundary x_2^* as the case of the lower boundary x_1^* can be handled analogously. First we show that x_2^* is left-continuous using a standard argument (see, e.g., [114, Ch. VII]). Fix $h \in \mathcal{H}$ and consider an increasing sequence $(h_n)_{n\geq 1} \subset \mathcal{H}$ such that $h_n \uparrow h$ as $n \to \infty$. For each $n \geq 1$, we have $(x_2^*(h_n), h_n) \in \mathcal{D}$ and in the limit

$$\lim_{n\to\infty} (x_2^*(h_n), h_n) = (x_2^*(h-), h),$$

where the left limit $x_2^*(h-)$ is well-defined by the monotonicity of x_2^*. Since $\mathcal{D}$ is closed it must be $(x_2^*(h-), h) \in \mathcal{D}$ and then $x_2^*(h-) \geq x_2^*(h)$. However, since $x_2^*(\cdot)$ is increasing we also have $x_2^*(h-) \leq x_2^*(h)$, so that left-continuity follows.

The proof of right-continuity of x_2^* follows ideas contained in [38]. If $x_2^*(h) = b$ the claim is trivial. Consider the case $x_2^*(h) < b$. Arguing by contradiction let us assume that $x_2^*(h+) > x_2^*(h)$. Then we can find x_d and x_u, such that $x_2^*(h) < x_d < x_u < x_2^*(h+)$, and a sufficiently small $\varepsilon > 0$ such that $(x_d, x_u) \times (h, h+\varepsilon] \subset \mathcal{C}$. Recalling (3.65), we have

$$(\mathcal{L} - 2r)V(x, h+\varepsilon) = -f(x, h+\varepsilon), \quad \text{for } x \in (x_d, x_u). \tag{3.103}$$

Take any $\Psi \in C_c^\infty([x_d, x_u])$ (the set of functions of infinitely differentiable with compact support in $[x_d, x_u]$) with $\Psi \geq 0$. Multiplying (3.103) by Ψ, integrating over $[x_d, x_u]$ and using integration by parts we obtain

$$\int_{x_d}^{x_u} V(z, h+\varepsilon)(\mathcal{L}^*\Psi - 2r\Psi)(z)\mathrm{d}z = -\int_{x_d}^{x_u} f(z, h+\varepsilon)\Psi(z)\mathrm{d}z, \tag{3.104}$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$:

$$(\mathcal{L}^* - 2r)\Psi(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\Psi(x)\sigma^2 x^2) - \frac{\partial}{\partial x}(\Psi(x) r x) - 2r\Psi(x).$$

By the continuity of V, we have $\lim_{\varepsilon\to 0} V(z, h+\varepsilon) = V(z, h) = M(z)$ for all $z \in (x_2^*(h), x_2^*(h+))$. Then, using the dominated convergence theorem in (3.104) to pass to the limit as $\varepsilon \to 0$ we get

$$-\int_{x_d}^{x_u} f(z,h)\Psi(z)\mathrm{d}z = \int_{x_d}^{x_u} M(z)(\mathcal{L}^* - 2r)\Psi(z)\mathrm{d}z = \int_{x_d}^{x_u} \Psi(z)(\mathcal{L} - 2r)M(z)\mathrm{d}z.$$

This is equivalent to $\int_{x_d}^{x_u} \Psi(z)G(z,h)\mathrm{d}z = 0$. However, $[x_d, x_u]$ is in the stopping region $\mathcal{D}_h$, so $G(z,h) \geq 0$. Recalling that Ψ is arbitrary and non-negative, we conclude that $G(z,h) = 0$ for almost all $z \in [x_d, x_u]$, which contradicts Assumption 3.3.7.

(*Continuity of optimal stopping times*). The idea is similar to the proofs in section 2.1.3 where we show the convergence of first hitting times (see also, e.g. [42], [103]). As indicated in Remark 2.1.10, continuity here means for any $(x_n, h_n) \to (x, h)$, we have $\tau^*_{x_n,h_n} \to \tau^*_{x,h}$, P-a.s.

Let

$$\hat{\tau}^*_{x,h} := \inf\{t \geq 0 : X^x_t \notin [x^*_1(h), x^*_2(h)]\}$$

and $\Omega^0 = \{\tau^*_{x,h} = \hat{\tau}^*_{x,h}\}$. By (3.67), we have $\mathsf{P}(\Omega^0) = 1$.

Fix $(x, h) \in \mathcal{I} \times \mathcal{H}$ and let $(x_n, h_n)_{n\geq 1}$ be a sequence converging to (x, h) as $n \to \infty$. For any $\omega \in \Omega^0$, if $\tau^*_{x,h}(\omega) = 0$, then lower semi-continuity holds trivially. If $\tau^*_{x,h}(\omega) > 0$, then for any $t > 0$ such that $\tau^*_{x,h}(\omega) > t$, there exists $\varepsilon > 0$ (depending on (t, x, h, ω)) such that

$$\inf_{0\leq u\leq t} d((X^x_u(\omega), h), \partial\mathcal{C}) \geq \varepsilon > 0, \tag{3.105}$$

where we use the standard Euclidean distance

$$d(y, \partial\mathcal{C}) := \inf_{\hat{y}\in\partial\mathcal{C}} d(y, \hat{y}), \quad \text{for } y \in \bar{\mathcal{I}} \times \mathcal{H}.$$

By uniform continuity of $(t, x) \mapsto X^x_t(\omega)$ on compact sets, for n sufficiently large we have

$$\inf_{0\leq u\leq t} d((X^{x_n}_u(\omega), h_n), (X^x_u(\omega), h)) \leq \varepsilon/2. \tag{3.106}$$

Combining (3.105) and (3.106) we obtain

$$\inf_{0\leq u\leq t} d((X^{x_n}_u(\omega), h_n), \partial\mathcal{C}) > \varepsilon/2,$$

for all sufficiently large n. Hence $\tau^*_{x_n,h_n}(\omega) > t$ for all such n. Since $t > 0$ was arbitrary we have

$$\liminf_{n\to\infty} \tau^*_{x_n,h_n}(\omega) \geq \tau^*_{x,h}(\omega).$$

To prove the upper semi-continuity we use $\hat{\tau}^*_{x,h}$ which is identical to $\tau^*_{x,h}$ on Ω^0. Recall that $\mathsf{P}(\hat{\tau}^*_{x,h} < \infty) = 1$ as $\hat{\tau}^*_{x,h}$ is the exit time of a geometric Brownian motion from a bounded interval. For any $\omega \in \Omega^0$, there is $t > \hat{\tau}^*_{x,h}(\omega)$ which is arbitrarily close to $\hat{\tau}^*_{x,h}(\omega)$ such that $X^x_t(\omega) \notin [x^*_1(h), x^*_2(h)]$. By the continuity of $x^*_1(\cdot)$, $x^*_2(\cdot)$ and $x \mapsto X^x_t(\omega)$, we have $X^{x_n}_t(\omega) \notin [x^*_1(h_n), x^*_2(h_n)]$ and $\hat{\tau}^*_{x_n,h_n}(\omega) < t$ for sufficiently large n. Hence $\limsup_{n\to\infty} \hat{\tau}^*_{x_n,h_n}(\omega) \leq \hat{\tau}^*_{x,h}(\omega)$. Combined with the lower semi-continuity proved above, this implies the a.s. continuity of $(x, h) \mapsto \tau^*_{x,h}$. $\square$

Figure 3.5 illustrates the optimal stopping boundaries $x^*_{1,h}$ and $x^*_{2,h}$ when $h \in \mathcal{H}$ is varying. We highlight points h_α and h_β where the continuation region changes from $(a, x^*_{2,h})$ to $(x^*_{1,h}, x^*_{2,h})$ and from $(x^*_{1,h}, x^*_{2,h})$ to $(x^*_{1,h}, b)$, respectively. The three regimes (i)–(iii) of Proposition 3.3.10 are clearly visible on the graph.

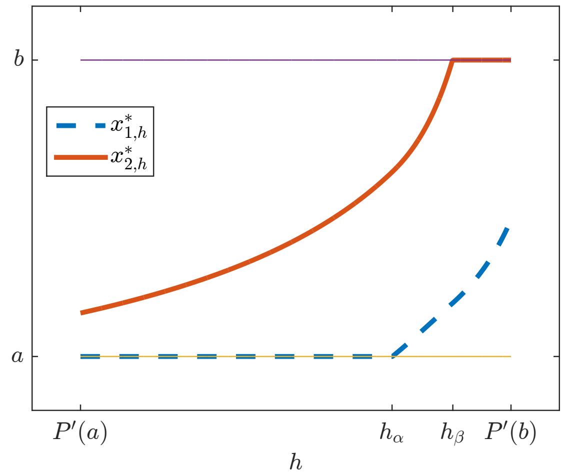


Figure 3.5: Plots of the optimal stopping boundaries $x^*_{1,h}, x^*_{2,h}$ as functions of h using parameters $r = 3\%$, $\sigma = 30\%$, $K = 100$, $b = 150$ and $a = \hat{a} = K/(1 + d^{-1}) = 40$.

3.5 Optimal initial stock holding

The existence of an optimal initial stock holding in (3.15) follows from compactness of $\mathcal{H}$ and continuity of $h \mapsto V(x,h)$. Here we show that the minimum of $V(x,\,\cdot\,)$ is attained in the interior of $\mathcal{H}$. Moreover, although an optimal h^* cannot be obtained explicitly, we show that it must solve a simple algebraic equation whose numerical solution is straightforward.

Proposition 3.5.1. *Under Assumption 3.3.7, we have $V(x,\,\cdot\,) \in C^1(\mathcal{H})$ for all $x \in \mathcal{I}$. Moreover, we have*

$$\partial_h V(x,h) = \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} 2(h - P'(X_u))\sigma^2 X_u^2 du\right], \tag{3.107}$$

and $\partial_h V \in C(\mathcal{I} \times \mathcal{H})$.

Proof. The argument of proof is analogous to the one used to prove Theorem 3.3.11, so we only provide a sketch. Let $\varepsilon > 0$ and denote by $\tau^*_{x,h}$ an optimal stopping time for $V(x,h)$. Since $\tau^*_{x,h}$ is admissible but sub-optimal for $V(x,h+\varepsilon)$, an application of the mean value theorem yields

$$V(x,h+\varepsilon) - V(x,h) \le \varepsilon \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} 2(h_\varepsilon - P'(X_u))\sigma^2 X_u^2 \mathrm{d}u\right],$$

where $h_\varepsilon \in [h, h+\varepsilon]$. Dividing both sides of the inequality by ε and letting $\varepsilon \to 0$, we obtain

$$\limsup_{\varepsilon \to 0} \frac{V(x,h+\varepsilon) - V(x,h)}{\varepsilon} \le \mathsf{E}_x\left[\int_0^{\tau_h^*} e^{-2ru} 2(h - P'(X_u))\sigma^2 X_u^2 \mathrm{d}u\right]. \tag{3.108}$$

For the lower bound we denote by $\tau^*_{x,h+\varepsilon}$ the optimal stopping time for $V(x, h+\varepsilon)$ and arguing as above we get

$$V(x, h+\varepsilon) - V(x, h) \geq \varepsilon \mathsf{E}_x \left[\int_0^{\tau^*_{h+\varepsilon}} e^{-2ru} 2(h_\varepsilon - P'(X_u))\sigma^2 X_u^2 \mathrm{d}u \right].$$

Dividing by ε both sides of the inequality, letting $\varepsilon \to 0$ and recalling the continuity of the map $h \mapsto \tau^*_{x,h}$ (Theorem 3.4.4) we obtain

$$\liminf_{\varepsilon \to 0} \frac{V(x, h+\varepsilon) - V(x, h)}{\varepsilon} \geq \mathsf{E}_x \left[\int_0^{\tau^*_h} e^{-2ru} 2(h - P'(X_u))\sigma^2 X_u^2 \mathrm{d}u \right]. \tag{3.109}$$

Combining (3.109) and (3.108) gives

$$\partial_h^+ V(x, h) = \mathsf{E}_x \left[\int_0^{\tau^*_h} e^{-2ru} 2(h - P'(X_u))\sigma^2 X_u^2 \mathrm{d}u \right],$$

where ∂_h^+ denotes the right partial derivative. The same arguments can be applied to obtain the same expression as above also for the left partial derivative $\partial_h^- V$, hence (3.107) holds.

Continuity of the map $(x, h) \mapsto \partial_h V(x, h)$ is easily deduced from P-a.s. continuity of the maps

$$(x, h) \mapsto (h - P'(X_u^x))\sigma^2 (X_u^x)^2 \quad \text{and} \quad (x, h) \mapsto \tau^*_{x,h},$$

and the dominated convergence theorem. □

Finally, we give our result regarding an optimal initial stock holding h^*.

Theorem 3.5.2. *Under Assumption 3.3.7, for each initial stock price $X_0 = x \in \mathcal{I}$,*

$$\arg\min_{h \in \mathcal{H}} V(x, h) \subseteq (P'(a), P'(b)) = int(\mathcal{H}).$$

Moveover, each minimiser $h^* \in \arg\min_{h\in\mathcal{H}} V(x, h)$ is a solution of the following equation

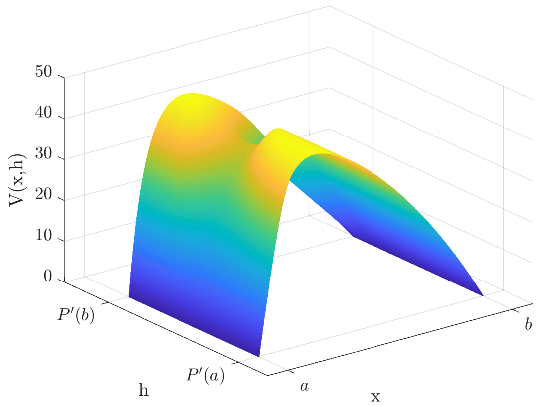
$$h^* = \widehat{\Gamma}_{h^*}(x), \tag{3.110}$$

where $\widehat{\Gamma}_h$ was defined in (3.83).

Proof. Fix $x \in \mathcal{I}$ and let $\mathcal{C}^x := \{h \in \mathcal{H} : V(x, h) < M(x)\}$. We have $\mathcal{C}^x \neq \varnothing$ due to Proposition 3.3.6. Hence $\arg\min_{h\in\mathcal{H}} V(x, h) \subset \mathcal{C}^x$ given that $V \leq M$ and M is independent of h.

Although it is possible that $P'(a)$ or $P'(b)$ are in $\mathcal{C}^x$, we will show that the minimum of $V(x, \cdot)$ cannot be attained there. For that purpose, notice that

$$\partial_h V(x, h) = 2\hat{\gamma}_{h,1}(x)\big(h - \widehat{\Gamma}_h(x)\big)$$



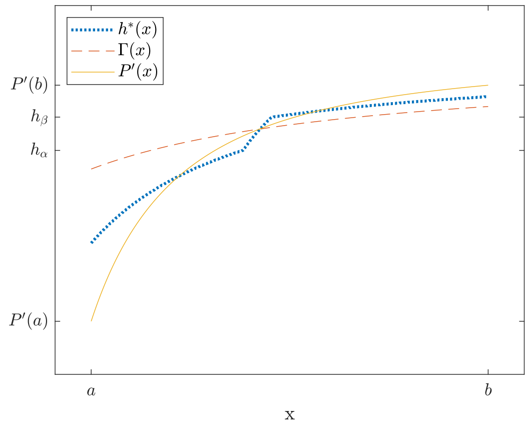


Figure 3.6: Top panel: 3-D plot of the value function $(x, h) \mapsto V(x, h)$. Bottom panel: plot of optimal stock holdings $x \mapsto h^*(x)$, $x \mapsto \Gamma(x)$ and the Black-Scholes Delta $x \mapsto P'(x)$ using parameters $r = 3\%$, $\sigma = 30\%$, $K = 100$, $b = 150$ and $a = \hat{a} = K/(1 + d^{-1}) = 40$.

thanks to (3.107) and with the notation of (3.83). If $P'(a) \in \mathcal{C}^x$, the inequality $\widehat{\Gamma}_h(P'(a)+) > P'(a)$ (see Proposition 3.4.1) implies that $\partial_h V(x, P'(a)+) < 0$. Hence the minimum of $V(x, \cdot)$ is not attained at $P'(a)$. Similarly, if $P'(b) \in \mathcal{C}^x$, then $\partial_h V(x, P'(b)-) > 0$, so the minimum of $V(x, \cdot)$ cannot be attained at $P'(b)$.

Consequently, each minimiser h^* of $V(x, \cdot)$ is in $(P'(a), P'(b))$ and must satisfy $\partial_h V(x, h^*) = 0$, which is equivalent to (3.110). □

We used the first order condition (3.110) to numerically compute the optimal initial stock holding and it turned out that (3.110) admitted a unique solution in all examples we considered.

The top panel of Figure 3.6 displays the three dimensional plot of the value function V. The bottom panel plots the optimal initial stock holding $h^*(x)$, the optimal hedge $\Gamma(x)$ at the rebalance time and the benchmark Black-Scholes Delta $P'(x)$. Notice that the optimal stock holding after rebalancing $\Gamma(x)$ is flatter than the Delta $P'(x)$, thus the constrained trader under/over-hedges, compared to the Black-Scholes benchmark, if the option is in-the-money/out-of-the-money.[1] This reflects the fact that no further trades are possible before $\tau_{\mathcal{I}}$. For example, if rebalancing occurs when the option is out of the money (close to b), there is still a positive probability of reaching the left boundary a before hitting b. Therefore, the optimal stock holding $\Gamma(x)$ strikes a balance between optimal Black-Scholes hedges $P'(b)$ at b and $P'(a)$ at a. This is unnecessary in the Black-Scholes setting because the portfolio can be rebalanced continuously reacting to changes in the underlying price. The optimal initial stock holding $h^*(x)$ exhibits similar flatter characteristics as $\Gamma(x)$ close to boundaries a, b but is steeper than the Black-Scholes hedge $P'(x)$ in the middle of the graph. The kinks in the map $x \mapsto h^*(x)$ correspond to the points h_α, h_β from Figure 3.5. They are the points at which the transition between single and double boundaries is observed. The steep part of the graph of $h^*(x)$ coincides with the region where the rebalancing occurs at two boundaries.

3.6 Remarks on the role of the upper bound b

Before moving on to the numerical illustration, it is worth turning our attention to the question of what happens if we take $b = +\infty$.

In this case, $\tau_{\mathcal{I}} = \inf\{t \geq 0 : X_t \leq a\} =: \tau_a$ and since X_t has a positive drift, we have $\mathsf{P}(\tau_a = \infty) > 0$. The martingale $(e^{-rt}X_t)_{t\geq 0}$ is not uniformly integrable and neither is the one defined by (3.10), for a general admissible trading strategy (τ, θ). Then the derivation of (3.12) via optional sampling is not possible (since (3.11) does not hold) and the whole problem formulation becomes less transparent. We propose here two possible problem formulations and their corresponding solutions. We note that such solutions appear to be structurally different as a consequence of different mathematical ways in which we can interpret the event $\{\tau_a = \infty\}$ in our model.

Thanks to the explicit dynamics of X we can easily derive $\lim_{t\to\infty} e^{-rt}X_t = 0$, P_x-a.s., for all $x \in (0, \infty)$. Then, using a standard convention on the event $\{\tau_a = \infty\}$, we have

$$\begin{aligned} e^{-r\tau_a}X_{\tau_a} &= e^{-r\tau_a}X_{\tau_a}\mathbb{1}_{\{\tau_a<\infty\}} + e^{-r\tau_a}X_{\tau_a}\mathbb{1}_{\{\tau_a=\infty\}} \\ &= e^{-r\tau_a}a1_{\{\tau_a<\infty\}} + \lim_{t\to\infty} e^{-rt}X_t\mathbb{1}_{\{\tau_a=\infty\}} = e^{-r\tau_a}a1_{\{\tau_a<\infty\}}. \end{aligned} \tag{3.111}$$

Analogously, recalling that the put option price is bounded by K we also have

$$e^{-r\tau_a}P(X_{\tau_a}) = e^{-r\tau_a}P(X_{\tau_a})\mathbb{1}_{\{\tau_a<\infty\}} = e^{-r\tau_a}P(a)\mathbb{1}_{\{\tau_a<\infty\}}. \tag{3.112}$$

[1] We are grateful to an anonymous reviewer for this observation.

3.6.1 Zero-mean tracking

With the aim of retaining a zero-mean tracking error analogue to (3.11) we set

$$\tau_n := \inf\{t \geq 0 : X_t \geq n\}, \qquad \text{for } n \in [a, \infty),$$

and, recalling that $\tau_{\mathcal{I}} = \tau_a$, we study the problem

$$\mathcal{V}(x) := \inf_{(\tau,\theta)\in\mathcal{A}_x^\infty} \limsup_{n\uparrow\infty} \mathcal{V}ar_x \left[e^{-r\tau_a\wedge\tau_n}\left(\Pi^{\tau,\theta}_{\tau_a\wedge\tau_n} - P(X_{\tau_a\wedge\tau_n})\right)\right], \tag{3.113}$$

where $\mathcal{A}_x^\infty$ is defined in the same way as Definition 3.2.1 with $\mathcal{I}$ replaced by (a, ∞) and $\tau_{\mathcal{I}}^x$ replaced by τ_a^x. Notice also that we have $h \in \mathcal{H} = [P'(a), 0]$. With this approach the mean tracking error can be computed as

$$\lim_{n\to\infty} \mathsf{E}_x \left[e^{-r\tau_a\wedge\tau_n}\left(\Pi^{\tau,\theta}_{\tau_a\wedge\tau_n} - P(X_{\tau_a\wedge\tau_n})\right)\right] = 0$$

by an application of optional sampling for each $n \geq a$ given and fixed. Clearly for $b < \infty$ problem formulations (3.12) and (3.113) are equivalent since $\tau_{\mathcal{I}} = \tau_{\mathcal{I}} \wedge \tau_n$ for all $n > b$.

As in Sections 3.2 and 3.3.1 (with a slight abuse of notation) we have

$$\mathcal{V}(x) = \inf_{(\tau,\theta)\in\mathcal{A}_x^\infty} \limsup_{n\uparrow\infty} \mathsf{E}_x \left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru} f(X_u, \theta_u)\mathrm{d}u\right] =: \inf_{h\in\mathcal{H}} V(x, h), \tag{3.114}$$

where

$$V(x, h) = \inf_{\tau\leq\tau_a, h_1\in\mathcal{H}_m^\tau} \limsup_{n\uparrow\infty} \mathsf{E}_x \left[\int_0^{\tau\wedge\tau_n} e^{-2ru} f(X_u, h)\mathrm{d}u + e^{-2r(\tau\wedge\tau_n)}\widehat{M}_n(X_{\tau\wedge\tau_n}, h_1)\right], \tag{3.115}$$

and

$$\widehat{M}_n(x, \zeta) := \mathsf{E}_x \left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru} f(X_u, \zeta)\mathrm{d}u\right], \qquad \zeta \in \mathbb{R}, \quad x \in \bar{\mathcal{I}}. \tag{3.116}$$

First, we show that $\lim_{n\uparrow\infty} \widehat{M}_n(x, \zeta) = \infty$ for all $\zeta \neq 0$. Then we will use it to argue that the infimum in (3.115) is attained for $h_1 \equiv 0$.

For each $n > a$ and $x \in (a, n)$ we have

$$\begin{aligned}
&\mathsf{E}_x \left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru} f(X_u, \zeta)\mathrm{d}u\right] \qquad (3.117)\\
&= \zeta^2 \mathsf{E}_x \left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru}\sigma^2 X_u^2 \mathrm{d}u\right] - 2\zeta \mathsf{E}_x \left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru} P'(X_u)\sigma^2 X_u^2 \mathrm{d}u\right]\\
&\quad + \mathsf{E}_x \left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru} (P'(X_u))^2\sigma^2 X_u^2 \mathrm{d}u\right].
\end{aligned}$$

The first term on the right-hand side can be written using (3.21) as

$$\begin{aligned}&\mathsf{E}_x\left[\int_0^{\tau_a\wedge\tau_n} e^{-2ru}\sigma^2 X_u^2 \mathrm{d}u\right] \\ &= w_n^{-1}\left(\varphi_n(x)\int_a^x \psi(z)\sigma^2 z^2 m'(z)\mathrm{d}z + \psi(x)\int_x^n \varphi_n(z)\sigma^2 z^2 m'(z)\mathrm{d}z\right),\end{aligned} \tag{3.118}$$

where ψ and φ_n are, respectively, the increasing and decreasing fundamental solutions to (3.18), with boundary conditions $\psi(a+) = 0$, $\psi'(a+) > 0$ and $\varphi_n(n-) = 0$, $\varphi_n'(n-) < 0$, while w_n is the associated Wronskian. These quantities can be computed explicitly as in (3.20) and (3.23), and they read

$$\psi(x) = x^{q_1} - a^{q_1-q_2}x^{q_2}, \quad \varphi_n(x) = x^{q_2} - n^{q_2-q_1}x^{q_1}, \quad w_n = \hat{w}(1-(a/n)^{q_1-q_2}),$$

where $q_2 < 0 < q_1$ are given in (3.17).

Clearly $w_n \uparrow \hat{w}$ and $\varphi_n(x) \uparrow x^{q_2}$ as $n \to \infty$. Then, the first integral on the right-hand side of (3.118) remains bounded as $n \to \infty$. For the second integral we have, by monotone convergence,

$$\lim_{n\to\infty}\int_x^n \varphi_n(z)\sigma^2 z^2 m'(z)\mathrm{d}z = \int_x^\infty z^{q_2}\sigma^2 z^2 m'(z)\mathrm{d}z = +\infty,$$

where the final equality can be easily obtained by recalling the expression of $m'(z)$ (see, (3.22)) and upon noticing that $q_2 + d + 1 > 0$. Using the same method one can check that the second and third terms on the right-hand side of (3.117) remain finite as $n \to \infty$, due to the damping effect of $P'(x)$ as $x \to \infty$. Then, we have $\lim_{n\uparrow\infty}\widehat{M}_n(x,\zeta) = +\infty$ unless $\zeta \equiv 0$.

For any $\tau \le \tau_a$ and $h_1 \in \mathcal{H}_m^\tau$, using $\widehat{M}_n(n, h_1) = 0$ we obtain

$$\begin{aligned}&\mathsf{E}_x\left[\int_0^{\tau\wedge\tau_n} e^{-2ru} f(X_u,h)\mathrm{d}u + e^{-2r(\tau\wedge\tau_n)}\widehat{M}_n(X_{\tau\wedge\tau_n},h_1)\right] \\ &= \mathsf{E}_x\left[\int_0^{\tau\wedge\tau_n} e^{-2ru} f(X_u,h)\mathrm{d}u + e^{-2r\tau}\widehat{M}_n(X_\tau,h_1)\mathbb{1}_{\{\tau<\tau_n\}}\right].\end{aligned}$$

Since $\tau_n \uparrow \infty$ as $n \to \infty$, $f \ge 0$ and $\widehat{M}_n$ is non-negative and increasing in n, we can apply monotone convergence theorem to pass the limit under expectation. Hence,

$$\begin{aligned}&\limsup_{n\uparrow\infty}\mathsf{E}_x\left[\int_0^{\tau\wedge\tau_n} e^{-2ru} f(X_u,h)\mathrm{d}u + e^{-2r\tau}\widehat{M}_n(X_\tau,h_1)\mathbb{1}_{\{\tau<\tau_n\}}\right] \\ &= \mathsf{E}_x\left[\int_0^{\tau} e^{-2ru} f(X_u,h)\mathrm{d}u + \lim_{n\uparrow\infty} e^{-2r\tau}\widehat{M}_n(X_\tau,h_1)\mathbb{1}_{\{\tau<\tau_n\}}\right].\end{aligned}$$

Recalling that $\lim_{n\uparrow\infty} \widehat{M}_n(x,\zeta) = +\infty$ for $\zeta \neq 0$, we have that the second term above is infinite unless $h_1 = 0$, P_x-a.s. It follows that the infimum in (3.113) must necessarily be attained for $h_1 = 0$, P_x-a.s., and using the tower property we have

$$\limsup_{n\uparrow\infty} \mathsf{E}_x \left[e^{-2r\tau}\widehat{M}_n(X_\tau,0)\mathbb{1}_{\{\tau<\tau_n\}}\right] = \mathsf{E}_x\left[\int_\tau^{\tau_a} e^{-2ru}(P'(X_u))^2\sigma^2 X_u^2 \mathrm{d}u\right] =: \mathsf{E}_x\left[e^{-2r\tau}M(X_\tau)\right].$$

In light of the above, the hedging problem becomes

$$V(x,h) = \inf_{\tau\leq\tau_a} \mathsf{E}_x\left[\int_0^\tau e^{-2ru}f(X_u,h)\mathrm{d}u + e^{-2r\tau}M(X_\tau)\right].$$

In this case, we have

$$G(x,h) = (\mathcal{L}-2r)M(x) + f(x,h) = \sigma^2x^2h(h-2P'(x)),$$

and it is easy to check that, for each $h\in\mathcal{H}$, the map $x\mapsto G(x,h)$ has a unique root $x_G = x_p(h/2)$ on (a,∞) (see (3.64)). It follows that $G(x,h)<0$ for $x\in(a,x_G)$ and $G(x,h)>0$ for $x>x_G$. By the same argument as in the proof of Proposition 3.3.10 we have that $\mathcal{C}_h = (a,x_h^*)$ for some $x_h^*\geq x_G$ that can be found explicitly by solving an analogue of (3.73). The corresponding optimal hedging strategy prescribes to clear the stock position (i.e., $h_1^*\equiv 0$) as soon as the stock price X_t enters the interval $[x_h^*,\infty)$.

3.6.2 Non-zero mean tracking error

We can formulate the problem directly with the random time horizon $\tau_{\mathcal{I}} = \tau_a$. With the same notation as in Section 3.6.1, here we want to solve

$$\mathcal{V}(x) = \inf_{(\tau,\theta)\in\mathcal{A}_x^\infty} \mathcal{V}ar_x\left[e^{-r\tau_a}\left(\Pi_{\tau_a}^{\tau,\theta} - P(X_{\tau_a})\right)\right],$$

and we will indeed produce explicit solutions.

Consider an admissible strategy

$$\tau = 0 \quad \text{and} \quad h_1 = P(a)a^{-1},$$

i.e., the rebalancing is immediate at $t=0$ and the bond holding after the trade is $\bar{m} = P(x) - h_1x$. The discounted portfolio value associated to the above strategy is $\hat{\Pi}_t := e^{-rt}\Pi_t = P(a)a^{-1}e^{-rt}X_t + \bar{m}$. Using (3.111) and (3.112) the tracking error at time τ_a is deterministic and amounts to

$$e^{-r\tau_a}\left(\Pi_{\tau_a} - P(X_{\tau_a})\right) = \bar{m}.$$

Hence, the associated variance is zero and the proposed strategy is optimal.

There is, however, a catch: the hedging portfolio under-replicates the claim. Indeed, recalling the expression for $\bar{m}$ we have

$$e^{-r\tau_a}\big(\Pi_{\tau_a} - P(X_{\tau_a})\big) = \bar{m} = P(x) - (x/a)P(a) < 0, \quad \mathsf{P}_x\text{-a.s.},$$

for all $x > a$, where we used that $P(x) < P(a)$.

One can, however, construct a strategy with a non-negative tracking error (the portfolio value Π_{τ_a} dominates $P(X_{\tau_a})$, P_x-a.s.), but with non-zero variance. This strategy prescribes to initially take a position $h = P'(a)$ in stocks (recall that $P'(a) < 0$ so this is short-selling), and buy $m_0 = P(x) - hx$ bonds. We will show that, on the one hand, if the stock price approaches the boundary a, the value of this portfolio grows and allows us to rebalance to a perfect hedge for the boundary a. On the other hand, if the stock price diverges to ∞ before rebalancing, the discounted portfolio value converges to m_0 thanks to (3.111); it follows from $h < 0$ and $P(x) > 0$ that $m_0 > 0$, so the tracking error is non-negative as required.

Before rebalancing the hedging portfolio evolves according to $e^{rt}m_0 + hX_t$. We choose the rebalancing time so that a perfect hedge can be constructed. We set

$$\tau = \tau^* := \inf\big\{t \geq 0 : e^{rt}m_0 + hX_t = X_t P(a)a^{-1}\big\}, \tag{3.119}$$

and $h_1 = P(a)a^{-1}$ so that the whole portfolio wealth $e^{rt}m_0 + hX_t$ is invested in stocks. The strategy is clearly self-financing and in order to show that it provides a hedge we first show that $\tau^* < \tau_a$, P_x-a.s., i.e., the rebalancing occurs before hitting the boundary a.

First of all the stopping time τ^* can be rewritten as the first time the discounted stock price $\hat{X}_t := e^{-rt}X_t$ falls below a certain threshold:

$$\tau^* = \inf\left\{t \geq 0 : \hat{X}_t \leq \frac{m_0}{a^{-1}P(a) - h}\right\}.$$

Using that $a^{-1}P(a) > x^{-1}P(x)$ for all $x > a$ by the monotonicity of $y \mapsto P(y)$, we have

$$\frac{m_0}{a^{-1}P(a) - h} = \frac{x^{-1}P(x) - h}{a^{-1}P(a) - h}x < x = \hat{X}_0 = X_0.$$

Therefore $\tau^* > 0$, P_x-a.s. Since the mapping $y \mapsto P(y) - hy$ is strictly increasing for $y > a$ (because $P'(y) > P'(a) = h$) we also have

$$\frac{P(x) - hx}{a^{-1}P(a) - h} = \frac{P(x) - hx}{P(a) - ha}a > a,$$

so that $\hat{X}_t = (P(x) - hx)/(a^{-1}P(a) - h)$ implies $X_t > a$ and therefore $\tau_a > \tau^*$, P_x-a.s., as needed.

Denoting by Π_t the portfolio value, we have $\Pi_{\tau_a} = P(a)$ on $\{\tau_a < \infty\}$. Using (3.111) and (3.112) the discounted tracking error at time τ_a amounts to

$$e^{-r\tau_a}\left(\Pi_{\tau_a} - P(X_{\tau_a})\right) = e^{-r\tau_a}\left(\Pi_{\tau_a} - P(X_{\tau_a})\right)\mathbb{1}_{\{\tau^*=\infty\}} = m_0\mathbb{1}_{\{\tau^*=\infty\}}, \quad \mathsf{P}_x\text{-a.s.}$$

Hence, neither the associated variance nor the expectation is zero but the portfolio value dominates the payoff at time τ_a.

The hedging strategies obtained in Section 3.6.2 seem economically unintuitive as they prescribe to take a long position in the stock and null investment in bonds after the rebalance. It should be clear that this is a mathematical artefact due to the infinite-time horizon.

3.7 Numerical comparisons

We assess the performance of our optimal hedging strategy against the performance of ad-hoc strategies inspired by those often used in practice (see [130, Ch. 6]). The quality of each strategy is measured in terms of the variance of the tracking error at $\tau_{\mathcal{I}}$.

We consider five hedging strategies:

(Strategy 1) Our optimal strategy (θ^*, τ^*).

(Strategy 2) Start with an initial stock holding $\theta_0 = P'(x)$ and rebalance at the stopping time

$$\zeta := \inf\left\{t \geq 0 : X_t^x \notin \left(\tfrac{1}{2}(a+x), \tfrac{1}{2}(b+x)\right)\right\},$$

with the classical Delta hedge $\theta_\zeta = P'(X_\zeta^x)$; then hold until $\tau_{\mathcal{I}}$.

(Strategy 3) Start with an initial stock holding $\theta_0 = P'(x)$ and rebalance when the Delta of the current stock price leaves a certain region. That is, let

$$\rho := \inf\left\{t \geq 0 : P'(X_t^x) \notin \left(\tfrac{1}{2}(P'(a) + P'(x)), \tfrac{1}{2}(P'(b) + P'(x))\right)\right\},$$

and rebalance at ρ with the classical Delta hedge $\theta_\rho = P'(X_\rho^x)$; then hold until $\tau_{\mathcal{I}}$.

(Strategy 4) Start with $\theta_0 = \Gamma(x)$ and hold the same amount of stock until $\tau_{\mathcal{I}}$.

(Strategy 5) Start with $\theta_0 = P'(x)$ and hold the same amount of stock until $\tau_{\mathcal{I}}$.

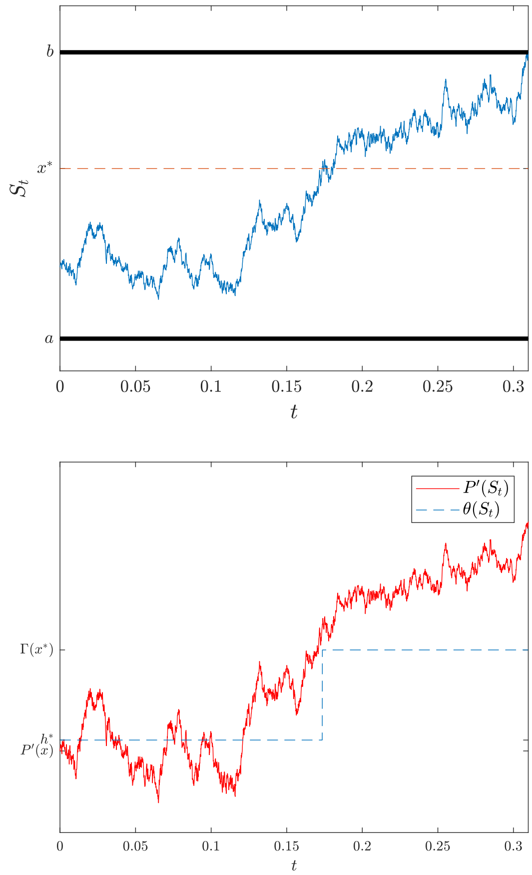


Figure 3.7: A simulation of stock price and the optimal hedging strategy 1 using parameters $r = 3\%, \sigma = 30\%, K = 100, a = 90, x = 100, b = 130$.

An illustration of Strategy 1, which is the optimal one from our analysis, is given in Figure 3.7. Strategies with fixed thresholds, like Strategy 2 and 3 above, are popular in the finance sector (see [130, p. 95]) and have an intuitive meaning: the trader makes the portfolio Delta-neutral when the underlying stock price or the associated Delta diverge by a 'fixed amount' from their initial values. Such an 'amount' of course can be chosen in several different ways; here we only display results for the specific choices made above. However, other specifications of the intervals in the stopping rules ζ and ρ give results qualitatively consistent with those presented in this section.

Strategies 4 and 5 are so-called static hedging strategies. In Strategy 4 the static hedging is optimal in the sense that $\Gamma(x)$ minimises the variance of the tracking error when no other rebalancing is allowed.

We evaluate the performance of these five strategies by conducting three experiments: we

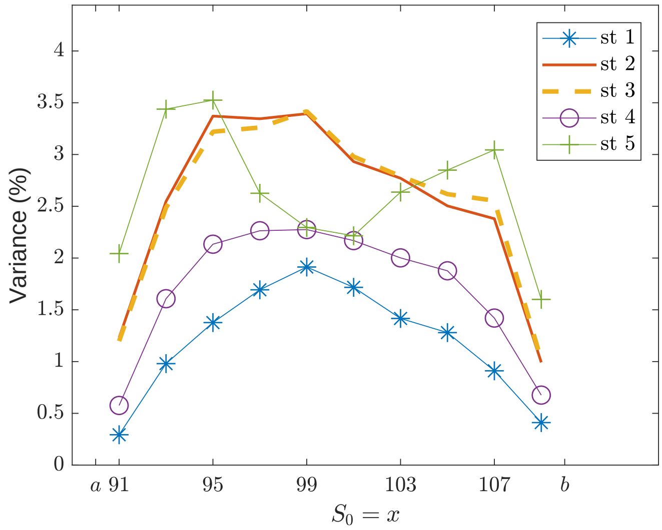


Figure 3.8: Sample variance of the hedging error for different values of X_0, with parameters as in (3.120).

calculate the sample variance of tracking error with different values of initial stock price X_0, volatility σ and upper re-assessment boundary b, respectively, when other parameters are fixed. In all experiments, the estimates are based on the same $N = 1000$ sample paths of the stock price and model parameters are fixed as

$$r = 3\%, \sigma = 30\%, K = 100, X_0 = 100, a = 90, b = 110. \tag{3.120}$$

For the first experiment we consider 10 different values for the initial stock price X_0 evenly spaced in the interval

$$[91, 109]$$

with all other parameters fixed as in (3.120). As shown in Figure 3.8 and Table 3.1, the variance of the tracking error for Strategy 1 is at least 40% lower than the variance for the dynamic strategies 2 and 3, and at least 15% lower than the variance for the static hedging strategies 4 and 5. It is worth noticing that the static strategy 4 outperforms the dynamic strategies 2 and 3.

In the second experiment, we take 10 values of the volatility σ evenly spaced in the interval

$$[20\%, 40\%]$$

with all other parameters fixed as in (3.120). Results are shown in Figure 3.9 and Table 3.2. Our optimal strategy (Strategy 1) produces the variance of the tracking error which is about

Table 3.1: Sample variance of the hedging error (%) for different values of X_0 with parameters as in (3.120).

	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5
$X_0 = 91$	0.29	1.23	1.20	0.58	2.04
$X_0 = 93$	0.98	2.55	2.49	1.61	3.44
$X_0 = 95$	1.38	3.37	3.22	2.13	3.53
$X_0 = 97$	1.69	3.35	3.26	2.26	2.63
$X_0 = 99$	1.91	3.39	3.42	2.27	2.29
$X_0 = 101$	1.72	2.93	2.98	2.17	2.21
$X_0 = 103$	1.42	2.77	2.79	2.00	2.64
$X_0 = 105$	1.28	2.50	2.62	1.88	2.85
$X_0 = 107$	0.91	2.38	2.55	1.42	3.04
$X_0 = 109$	0.41	0.99	1.01	0.68	1.60

$30 - 40\%$ lower than the variance for strategies 2 and 3, and about $15 - 20\%$ lower than the variance for strategies 4 and 5. The relative gap between different strategies does not vary significantly as the volatility changes. Strategies 4 and 5 produce almost the same results; this happens because X_0 is taken as the middle point in (a, b) and therefore the difference between $P'(X_0)$ and $\Gamma(X_0)$ is very small (for example, when $\sigma = 31.11\%$ we have $P'(100) = -0.2110$ and $\Gamma(100) = -0.2115$). Strategies 2 and 3 also give very similar results, but they are outperformed by the static strategies.

The steep decline of all graphs in Figure 3.9 may seem at odds with the intuition that a high volatility corresponds to a risky trading environment. However, a high volatility also causes the option price to change slower as a function of the stock price: the difference $P(a) - P(b)$ is above 0.14 for $\sigma = 20\%$ and less than 0.05 for $\sigma = 40\%$. A larger volatility makes the tracked values closer to each other and, hence, the tracking problem easier. This intution was confirmed by extensive numerical studies with representative results for in-the-money and out-of-the-money options displayed on Figure 3.10.

The third experiment studies the effect of the upper boundary b. We take 10 values of b evenly spaced in the interval

$$[105, 150]$$

with all other parameters fixed as in (3.120). The results are displayed in Figure 3.11 and Table 3.3. For values of b close to X_0, the variance of the tracking error for all strategies is

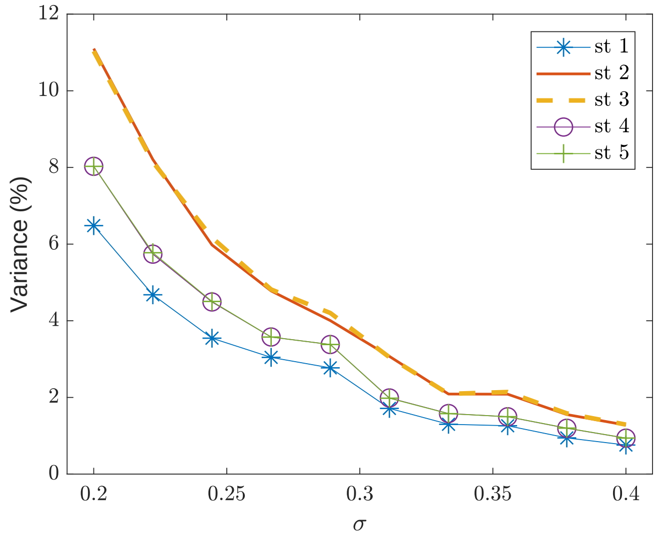


Figure 3.9: Sample variance of the hedging error for different values of σ with parameters as in (3.120).

low because the stock price leaves the interval (a, b) quickly. Observe that when b is large, the dynamic optimal strategy 1 produces the variance which is 40% lower than the second best (Strategy 4). This gap shrinks to about 20% when b is small. This indicates that both dynamic hedging and optimisation are important when one of the re-assessment boundaries is far away from X_0. Quite remarkably, in all the above experiments, the optimised static hedging (Strategy 4) gives a smaller variance of the tracking error than strategies 2 and 3, despite the fact that the latter two allow for one rebalancing opportunity.

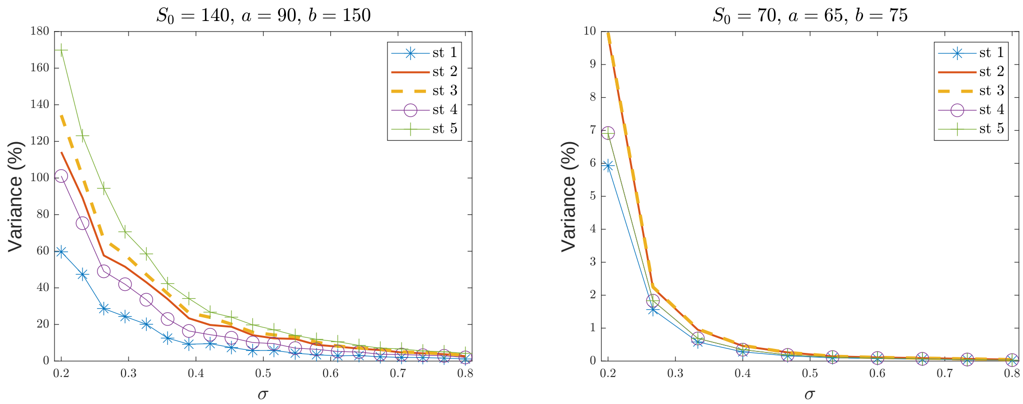


Figure 3.10: Sample variance of the hedging error (%) for different values of σ with parameters $r = 3\%$, $K = 100$ and deep out-of-the-money option (left panel) and deep in-the-money option (the right panel).

Table 3.2: Sample variance of the hedging error (%) for different values of σ with parameters as in (3.120).

	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5
$\sigma = 20.00\%$	6.48	11.10	11.03	8.03	8.03
$\sigma = 22.22\%$	4.68	8.21	8.14	5.74	5.78
$\sigma = 24.44\%$	3.55	5.98	6.17	4.49	4.50
$\sigma = 26.67\%$	3.05	4.78	4.82	3.58	3.57
$\sigma = 28.89\%$	2.77	4.00	4.20	3.38	3.38
$\sigma = 31.11\%$	1.71	3.07	3.05	1.98	1.98
$\sigma = 33.33\%$	1.30	2.09	2.09	1.58	1.58
$\sigma = 35.56\%$	1.26	2.08	2.14	1.49	1.50
$\sigma = 37.78\%$	0.94	1.55	1.58	1.20	1.20
$\sigma = 40.00\%$	0.76	1.28	1.29	0.93	0.94

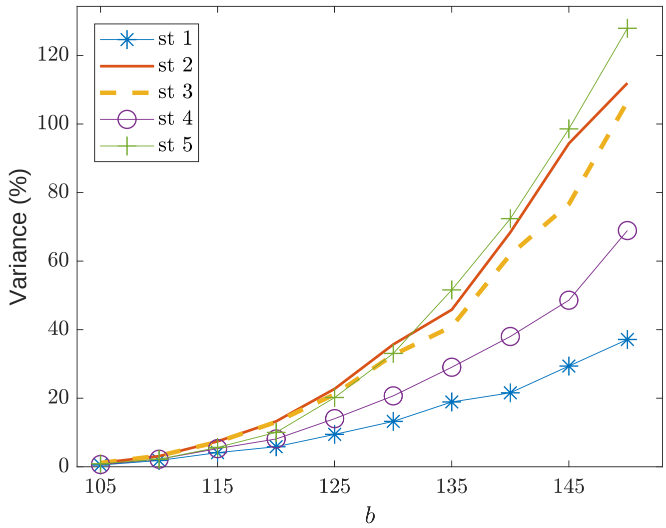


Figure 3.11: Sample variance of the hedging error for different values of b with parameters as in (3.120).

Table 3.3: Sample variance of the hedging error (%) for different values of b with parameters as in (3.120).

	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5
$b = 105$	0.51	1.12	1.14	0.66	0.88
$b = 110$	1.84	3.10	3.21	2.20	2.20
$b = 115$	4.12	7.40	7.24	5.29	5.65
$b = 120$	5.85	13.26	13.06	8.11	10.03
$b = 125$	9.45	22.75	21.14	14.04	20.21
$b = 130$	13.23	35.68	32.63	20.69	33.07
$b = 135$	18.91	45.82	40.91	29.04	51.60
$b = 140$	21.58	68.40	61.91	38.00	72.40
$b = 145$	29.38	94.32	76.60	48.59	98.58
$b = 150$	37.14	111.93	106.53	68.92	127.93

Chapter 4

The American put with finite-time maturity and stochastic interest rate

4.1 Introduction

[1]Pricing of American options is a classical problem in mathematical finance which has attracted continuous attention since the initial work of McKean in 1965 [101]. Its study has also become a benchmark for methodological developments of optimal stopping theory and the associated free boundary problems. In this chapter we contribute to this strand of research by studying the American put option on a Black and Scholes market with a stochastic interest rate and finite-time maturity. The stock price and the interest rate are driven by (possibly) correlated Brownian motions. We assume the interest rate dynamics either follows the CIR model, or more generally has time independent and Lipschitz continuous coefficients.

American option pricing with stochastic interest rates has already attracted a lot of attention in the literature, mainly focussing on approximations and numerical methods. Lattice (tree) based methods are employed by Appolloni, Caramellino and Zanette [6] to price options in Black and Scholes model with CIR interest rate dynamics and by Battauz and Rotondi [12] in a model with Vasicek interest rates. Geske and Johnson's ([73]) approximation of discretely exercised American options prices is adapted by Ho, Stapleton and Subrahmanyam [78] and Chung [35] to a class of stochastic interest rate models that lead to log-normally distributed bond prices. An alternative approximation is provided by Menkveld and Vorst [104]. A framework for option pricing with Heath, Jarrow, Morton's [77] bond market model is developed

[1]The results from this chapter form part of the article [26], which is currently under review.

by Amin and Jarrow [5] with a binomial-tree-based implementation of pricing of foreign exchange options performed in Amin and Bodurtha [4].

Hedging underlies the success of mathematical finance in derivatives markets. A rigorous theory that links hedging of American options with solutions of optimal stopping problems was initiated by Bensoussan [15] using PDE methods and extended by Karatzas [86] to more general models and payoffs thanks to the martingale theory of optimal stopping. A hedging strategy for an American option consists of an investment portfolio and a non-decreasing cumulative consumption process which increases only when the state-time process is in the stopping set. As shown in Chapter 2 Section 2.3.1, in the Black and Scholes model with constant interest rate, the classical Delta hedge is known to replicate the option. Our work seems to be the first to rigorously derive the hedging strategy for American put options on a market with a stochastic interest rate. This is accomplished thanks to the C^1-regularity of the value function that we are able to prove and which did not appear in previous works.

A characterisation of an optimal stopping boundary as solution to a (system of) integral equations has been known since the earliest works (see Van Moerbeke [133]). In more recent works [33, 80, 89, 107] the stopping boundary for the classical Black and Scholes market with constant interest rate is shown to be the unique solution to an uncountable *system* of integral equations arising from the early exercise premium decomposition of the option price. A breakthrough came with the work of Peskir [111] where he shows that the stopping boundary is the unique continuous solution of a single integral equation. The key observation in [111] is that the integral equation only needs to be satisfied for stock prices at the boundary while earlier results required that it does so for all stock prices at and below the boundary. Peskir's [111] integral equation opens doors to side-stepping the computation of the value function in the process of determining the optimal exercise strategy; see numerical methods designed in [90, 98]. Our work extends Peskir's [111] results to the market with a stochastic interest rate and the optimal boundary being a two-dimensional surface. It is also the continuity of the boundary that allows us to establish the uniqueness of solutions to the integral equation. A closely related paper that furthermore motivated our numerical approach is [46] where the authors solve an integral equation for Black and Scholes market with stochastic volatility.

The regularity of the value function in one-dimensional optimal stopping problems is often phrased as smooth-fit. In Chapter 2 Section 2.3 and Chapter 3 Section 3.3.4, we have illustrated the importance of the smooth-fit in uniquely determining explicit solutions to the optimal stopping problems. In a Black and Scholes model with constant interest rate, as stated in Theorem

2.3.2, the smooth-fit for American options with finite-time maturity is understood as continuous differentiability of the value function with respect to the stock price, for each fixed value of the time variable. That is a "directional" derivative and continuity is only considered with respect to one variable. Sobolev space regularity is studied in [82] for American options on multiple assets and *deterministic*, time-dependent discount rate under the assumption of uniform ellipticity of the associated second order differential operator. By Sobolev embedding it is possible to determine continuous differentiability of the value function with respect to the initial values of all the assets but not with respect to time. Continuous differentiability with respect to time and stock price for the value of the American put with finite-time maturity and constant interest rate is obtained in [42] along with other complementary findings about continuous differentiability of the value function for a large class of optimal stopping problems. In this chapter, we refine the arguments from [42] removing global integrability conditions that may not hold in our set-up.

The early exercise premium formula for American options is studied in great generality, in non-Markovian problems beyond the setting of the American put option by Rutkowski [124] with methods from martingale theory. The nature of the methods employed in [124] to derive their main results is such that the emphasis is removed from the optimal boundary, which in fact only appears in specific examples ([124, Sec. 3]) as a time-dependent function. Here instead we derive the early exercise premium formula starting from the analysis of the optimal boundary (and its regularity) as a function of time and one stochastic factor from our two-factor model.

Rutkowski's work [124] is later applied by Detemple and Tian [45], who study the pricing of American call options in a general diffusive model with a d-dimensional Brownian motion. They formulate assumptions under which there is a single exercise surface but without proving its continuity. In a Black and Scholes market model with Vasicek interest rates, they justify the existence of the optimal exercise boundary for the American call option. Using the general early exercise premium formula in [124], they show that this exercise boundary solves an integral equation of the same form as in this chapter. However, the uniqueness of the solution to this integral equation is not discussed. To numerically compute the solution, they truncate the domain of the interest rate and use step functions to approximate the exercise boundary. In section 4.9, we will use a different method based on Picard iteration to compute the exercise boundary for the American put option with Vasicek interest rates.

This chapter is structured as follows. Section 4.2 introduces the market model, main assumptions and notation. In Sections 4.3 and 4.4, under the sole Assumption 4.2.1, we establish continuous differentiability of the value function $v(t, r, x)$ (jointly in all variables), along with its monotonicity in (t, r, x) and convexity in x. We also prove the existence and monotonicity of an optimal exercise boundary and present two possible parametrisations of it. Then, in Sections 4.5—4.8, under a mild additional assumption on α and β (Assumption 4.5.1) we derive continuity of the optimal exercise boundary (as a function of two variables) and an integral equation that uniquely determines it (also under Assumption 4.5.3). We also obtain the early exercise premium formula for the option price and the hedging portfolio that replicates the option's payoff at all times. Finally, a numerical study with interest rates following Vasicek model is presented in Section 4.9 along with a sensitivity analysis.

4.2 Problem formulation

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space carrying two correlated Brownian motions $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ with $\mathsf{E}(W_t B_t) = \rho\, t$ for all $t \geq 0$ and a fixed $\rho \in (-1, 1)$ (here $\mathsf{E}(\,\cdot\,)$ is the expectation under P). We denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by (B, W) augmented with the P-null sets. On this probability space we consider a financial market with one risky asset $(X_t)_{t\geq 0}$ and a bond. The asset and the risk-free (short) rate $(r_t)_{t\geq 0}$ take values, respectively, in intervals $\mathbb{R}_+ := (0, \infty)$ and $\mathcal{I} \subseteq \mathbb{R}$, and follow the dynamics

$$dX_t = r_t X_t dt + \sigma X_t dB_t, \qquad X_0 = x, \tag{4.1}$$

$$dr_t = \alpha(r_t)dt + \beta(r_t)dW_t, \qquad r_0 = r, \tag{4.2}$$

with $\alpha, \beta : \mathcal{I} \to \mathbb{R}$ specified below. The probability measure P is a risk neutral measure for this market. We denote by $T > 0$ a fixed finite trading horizon.

Throughout the chapter we assume $\sigma > 0$ and $\mathcal{I} = (\underline{r}, \overline{r})$ (with $\mathcal{I}$ possibly unbounded). The right boundary $\overline{r}$ is unattainable in a finite time (it is a natural or entrance-not-exit boundary). The left boundary $\underline{r}$ is either unattainable or reflecting. It will become clear later that the exact behaviour of the interest rate process at this boundary is irrelevant for the majority of results and their proofs. For the dynamics of the interest rate our benchmark example is the CIR model, but, with a relatively small additional effort, our results cover other stochastic interest rate models, e.g., Vasicek model. Therefore, we make the following standing assumption:

Assumption 4.2.1. *The coefficients α and β in (4.2) meet one of the conditions below:*

(i) (CIR model) *For $\kappa, \theta, \gamma > 0$ we have $\alpha(r) = \kappa(\theta - r)$ and $\beta(r) = \gamma\sqrt{r}$.*

(ii) α *and* β *are globally Lipschitz and continuously differentiable on bounded subsets of* $\mathcal{I}$ *with* $\beta(r) > 0$ *for all* $r \in \mathcal{I}$*, and* $\overline{r} > 0 \geq \underline{r}$*. For any compact set* $\mathcal{K} \subset \mathcal{I}$*, and any* $p \in [1, p']$ *for some* $p' > 2$ *and* $T > 0$*, there is* $C_1 > 0$ *(depending on* T*,* p *and* $\mathcal{K}$*) such that*

$$\sup_{r\in\mathcal{K}} \mathsf{E}\left[\sup_{0\leq s\leq T} e^{-p\int_0^s r_u du}\Big| r_0 = r\right] \leq C_1. \tag{4.3}$$

The assumption that $\overline{r} > 0$ cannot be relaxed without trivialising the pricing problem. A strictly positive lower boundary $\underline{r}$ could, however, be of interest. For the clarity of presentation, it is omitted but it can be studied with similar methods as those developed in this chapter.

The above assumptions are sufficient to guarantee that (4.2) admits a unique strong solution defined on $\mathcal{I}$. In the case of CIR model, we also have $\kappa\theta > 0$ which implies that the spot rate is non-negative (but not necessarily strictly positive), see e.g. [83, Sec. 6.3.1], so the left boundary $\underline{r} = 0$ is reflecting (also non-attainable if $\kappa\theta > \sigma^2/2$). Hence, the bound (4.3) is satisfied with the constant $C_1 = 1$. The linear growth of α and β in (4.2) guarantees that for each $p \geq 2$ there is $C_2 > 0$ only depending on T and p, such that [93, Thm. 2.5.9]

$$\mathsf{E}\big[\sup_{0\leq s\leq T} |r_s|^p \,\big|\, r_0 = r\big] \leq C_2(1 + |r|^p), \qquad \text{for } r \in \mathcal{I}. \tag{4.4}$$

Under Assumption 4.2.1, the solution of (4.1) may be expressed as

$$X_t = x \exp\left(\sigma B_t + \int_0^t \left(r_s - \tfrac{\sigma^2}{2}\right) ds\right), \qquad \text{for } t \geq 0, \tag{4.5}$$

so that X depends on both initial values r and x. On the contrary, the dynamics of the interest rate does not depend on the initial asset value. The coupling between the processes $(r_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ stems from formula (4.5) and the correlation between the Brownian motions. To keep track of the dependence of the processes on their initial values, in what follows we often use the notation $(r_t^r, X_t^{r,x})_{t\geq 0}$ for the process started at $r_0^r = r$ and $X_0^{r,x} = x$. Also we may sometimes use the notation $\mathsf{P}_{t,r,x}(\,\cdot\,) = \mathsf{P}(\,\cdot\,|r_t = r, X_t = x)$, $\mathsf{P}_{r,x} = \mathsf{P}_{0,r,x}$, and $\mathsf{P}_r(\,\cdot\,) = \mathsf{P}(\,\cdot\,|r_0 = r)$.

The rational price of an American put option with maturity time T, strike price $K > 0$, written on the asset X and evaluated at time $t \in [0, T]$ is given by

$$v(t, r, x) = \sup_{0\leq\tau\leq T-t} \mathsf{E}_{r,x}\left[e^{-\int_0^\tau r_t dt}\Big(K - X_\tau\Big)^+\right], \tag{4.6}$$

where $r \in \mathcal{I}$ and $x \in \mathbb{R}_+$ are, respectively, the values of the spot rate and of the asset at time t, the function $(\,\cdot\,)^+$ denotes the positive part and the optimisation runs over all $(\mathcal{F}_t)_{t\geq 0}$-stopping times bounded by $T - t$. The above is an optimal stopping problem with Markovian structure and a 3-dimensional state space.

Since the process

$$t \mapsto e^{-\int_0^t r_s ds}\Big(K - X_t\Big)^+ \tag{4.7}$$

is non-negative and continuous, and thanks to the integrability condition (4.3), we can rely on standard optimal stopping theory (see, e.g., [88, Appendix D]) to conclude that the smallest optimal stopping time for (4.6) is $\mathsf{P}_{r,x}$-a.s. given by

$$\tau_* := \inf\{s \geq 0 \,:\, v(t+s, r_s, X_s) = (K - X_s)^+\}, \tag{4.8}$$

where we note that $\tau_* \leq T - t$ since $v(T, r, x) = (K - x)^+$. Clearly $\tau_* = \tau_*(t, r, x)$ depends on the initial value (t, r, x) of the 3-dimensional state process $(t + s, r_s, X_s)_{s \geq 0}$.

The form (4.8) of τ^* gives rise to the so-called continuation set $\mathcal{C}$ and its complement, the stopping set $\mathcal{D}$, that is

$$\mathcal{C} := \{(t, r, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}_+ \,:\, v(t, r, x) > (K - x)^+\}, \tag{4.9}$$

$$\mathcal{D} := \{(t, r, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}_+ \,:\, v(t, r, x) = (K - x)^+\}. \tag{4.10}$$

Upon observing the spot rate and the asset value, at each time the option holder must decide whether to hold the option or to exercise it. She should *wait* (possibly trading the option on the market) if $(t, r_t, X_t) \in \mathcal{C}$ since the option value is strictly larger than the payoff of immediate exercise. On the contrary, if $(t, r_t, X_t) \in \mathcal{D}$ the option should be immediately *exercised*. Notice that

$$\{T\} \times \mathcal{I} \times \mathbb{R}_+ \subseteq \mathcal{D}.$$

Remark 4.2.2. *Setting*

$$D_s := \exp\Big(-\int_0^s r_u du\Big), \quad V_s := v(t+s, r_s, X_s) \quad \textit{and} \quad Y_s := D_s V_s$$

(i.e., Y is the discounted option value process), we have that [88, Appendix D]

$$(Y_s)_{s \in [0, T-t]} \textit{ is a right-continuous } \mathsf{P}_{r,x}\textit{-supermartingale,} \tag{4.11}$$

$$(Y_{s \wedge \tau_*})_{s \in [0, T-t]} \textit{ is a right-continuous } \mathsf{P}_{r,x}\textit{-martingale.} \tag{4.12}$$

We will soon show (Proposition 4.3.3) that v is a continuous function, so that Y is a continuous process.

Notation. We set

$$\mathcal{O} := [0, T) \times \mathcal{I} \times \mathbb{R}_+, \tag{4.13}$$

and denote by $\partial\mathcal{C}$ the boundary of $\mathcal{C}$ in $\mathcal{O}$, i.e., $\partial\mathcal{C} := (\overline{\mathcal{C}} \cap \mathcal{O}) \setminus \mathcal{C}$.

For future frequent use we denote by $\mathcal{L}$ the infinitesimal generator of $(r_t, X_t)_{t\geq 0}$, which, for any $f \in C^2(\mathcal{I} \times \mathbb{R})$ reads

$$\mathcal{L}f := \frac{\sigma^2 x^2}{2} f_{xx} + \frac{\beta^2(r)}{2} f_{rr} + \rho\sigma x \beta(r) f_{rx} + r x f_x + \alpha(r) f_r, \tag{4.14}$$

where f_r, f_x and f_{rr}, f_{rx}, f_{xx} denote, respectively, the first and second order partial derivatives of f.

4.3 Optimal stopping boundary

In the classical Black and Scholes model with constant interest rate, the stopping set is determined by a boundary: it is optimal to exercise the option the first time when the stock price drops below this boundary. A similar characterisation of the stopping region $\mathcal{D}$ can be derived in our model with the difference that the stopping boundary is a surface. To this end, we research monotonicity properties of the value function.

Proposition 4.3.1. *The value function v is finite for all $(t, r, x) \in \mathcal{O}$ and it satisfies the following conditions:*

(i) $t \mapsto v(t, r, x)$ *is non increasing for all* $(r, x) \in \mathcal{I} \times \mathbb{R}_+$*,*

(ii) $r \mapsto v(t, r, x)$ *is non increasing for all* $(t, x) \in [0, T] \times \mathbb{R}_+$*,*

(iii) $x \mapsto v(t, r, x)$ *is convex and non increasing for all* $(t, r) \in [0, T] \times \mathcal{I}$*.*

Proof. Finiteness of v follows by (4.3) and boundedness of the put payoff. Monotonicity in (i) is also a trivial consequence of the fact that the discounted put payoff is independent of time. For (ii) we argue as follows: since $r \mapsto r_t^r$ is increasing P-a.s. for all $t \in [0, T]$ (by uniqueness of the trajectories) we get, for any $\varepsilon > 0$

$$\begin{aligned} v(t, r+\varepsilon, x) &= \sup_{0\leq\tau\leq T-t} \mathsf{E}\left[\left(Ke^{-\int_0^\tau r_t^{r+\varepsilon} dt} - xe^{\sigma B_\tau - \frac{\sigma^2}{2}\tau}\right)^+\right] \\ &\leq \sup_{0\leq\tau\leq T-t} \mathsf{E}\left[\left(Ke^{-\int_0^\tau r_t^{r} dt} - xe^{\sigma B_\tau - \frac{\sigma^2}{2}\tau}\right)^+\right] = v(t, r, x) \end{aligned}$$

where we took the discounting inside the positive part and used (4.5).

Finally, monotonicity in (iii) is a simple consequence of monotonicity of (4.5) with respect to x and the fact that $x \mapsto (K-x)^+$ is decreasing. Convexity also follows by standard arguments: fix $\lambda \in (0,1)$, take x and y in $\mathbb{R}_+$ and denote $x_\lambda := \lambda x + (1-\lambda)y$. By the convexity of the put payoff, using that $X^{r,x_\lambda} = \lambda X^{r,x} + (1-\lambda)X^{r,y}$ and that $\sup(f+g) \leq \sup f + \sup g$, it is not hard to verify that $v(t,r,x_\lambda) \leq \lambda v(t,r,x) + (1-\lambda)v(t,r,y)$. □

The monotonicity in t and x and the convexity in x is the same as in the classical Black and Scholes model and the proof is very similar. The dependence on r has financial explanation: larger interest rate implies stronger discounting of future cash flows and, hence, lower present value.

Remark 4.3.2. *In the case $T = +\infty$ (perpetual option) the discounted payoff process (4.7) is still uniformly integrable and continuous. This implies that, letting v_∞ denote the value of the perpetual option, the stopping time*

$$\tau_\infty = \inf\{t \geq 0 \,:\, v_\infty(r_t, X_t) = (K - X_t)^+\}$$

is optimal by standard theory and (4.11)–(4.12) continue to hold in this setting (see, e.g., [128, Ch. 3, Thm. 3]).

Moreover, the proof of Proposition 4.3.1 can be repeated step by step, upon also noticing that

$$\widetilde{X}_t := e^{-\int_0^t r_s ds} X_t\,, \qquad t \geq 0$$

is a continuous martingale with $\lim_{t\to\infty} \widetilde{X}_t = 0$, P-a.s. So $r \mapsto v_\infty(r,x)$ is non-increasing and $x \mapsto v_\infty(r,x)$ is convex and non-increasing.

From the general optimal stopping theory we expect that the value function v be continuous. In our case, we can show that v is actually locally Lipschitz as presented in Proposition 4.3.3 (without relying on the form of the stopping set).

Proposition 4.3.3. (Lipschitz continuity). *For any compact $\mathcal{K} \subset \mathcal{O}$ there exists a constant $L_\mathcal{K} > 0$ such that*

$$|v(t_1,r_1,x_1) - v(t_2,r_2,x_2)| \leq L_\mathcal{K}\big(|t_1-t_2| + |r_1-r_2| + |x_1-x_2|\big) \tag{4.15}$$

for all (t_1,r_1,x_1) and (t_2,r_2,x_2) in $\mathcal{K}$.

Proof. We look separately at Lipschitz continuity in the three variables. Arguments for r and x are quite standard while the main argument for the Lipschitz continuity in t goes back to [82, Thm. 3.6]. However, in our framework the interest rate is random and the coefficients of the underlying process are state dependent, which results in some additional difficulties.

Continuity in x. Fix $(t,r)\in[0,T)\times\mathcal{I}$ and take $x_1\le x_2$ in $\mathbb{R}_+$. Let $\tau_1:=\tau_*(t,r,x_1)$ and note that it is admissible for $v(t,r,x_2)$. Using Proposition 4.3.1(iii), the explicit expression for $X^{r,x}$ in (4.5) and the Lipschitz property of the put payoff, we get

$$0\le v(t,r,x_1)-v(t,r,x_2)\le \mathsf{E}\left[e^{-\int_0^{\tau_1} r^r_s ds}\Big((K-X^{r,x_1})^+-(K-X^{r,x_2})^+\Big)\right]$$
$$\le \mathsf{E}\Big[e^{\sigma B_{\tau_1}-\frac{\sigma^2}{2}\tau_1}\Big](x_2-x_1)=(x_2-x_1),$$

where in the last equality we used Doob's optional sampling theorem.

Continuity in r. Fix $(t,x)\in[0,T)\times\mathbb{R}_+$ and take $r_1\le r_2$ in $\mathcal{I}$ such that $(t,r_1,x)\in\mathcal{K}$. Denote, for simplicity, $r^1:=r^{r_1}$ and $r^2:=r^{r_2}$ and notice that $r^2_t\ge r^1_t$ for all $t\ge 0$ P-a.s. Set $\tau_1:=\tau_*(t,r_1,x)$. From Proposition 4.3.1(ii) and simple estimates we obtain

$$\begin{aligned}0\le v(t,r_1,x)-v(t,r_2,x)&\le K\mathsf{E}\left[e^{-\int_0^{\tau_1} r^1_s ds}-e^{-\int_0^{\tau_1} r^2_s ds}\right]=K\mathsf{E}\left[e^{-\int_0^{\tau_1} r^1_s ds}\Big(1-e^{-\int_0^{\tau_1}(r^2_s-r^1_s)ds}\Big)\right]\\&\le K\mathsf{E}\Big[e^{-\int_0^{\tau_1} r^1_s ds}\int_0^{\tau_1}(r^2_s-r^1_s)ds\Big].\end{aligned}\tag{4.16}$$

To complete the proof we consider separately cases (i) and (ii) in Assumption 4.2.1. Let us start with (i): using that $r^1_t\ge 0$ for $t\ge 0$, and the explicit form of the SDE in the CIR model, we get

$$\begin{aligned}\mathsf{E}\Big[e^{-\int_0^{\tau_1} r^1_s ds}\int_0^{\tau_1}(r^2_s-r^1_s)ds\Big]&\le\int_0^{T-t}\mathsf{E}\left[r^2_s-r^1_s\right]ds\\&=\int_0^{T-t}\mathsf{E}\Big[(r_2-r_1)+\int_0^s\kappa(r^1_u-r^2_u)du\Big]ds\le(T-t)(r_2-r_1),\end{aligned}$$

where we have used the integral equation for (r_t) and that $r^2_t\ge r^1_t$.

If Assumption 4.2.1(ii) holds instead, we apply Hölder inequality:

$$\begin{aligned}\mathsf{E}\Big[e^{-\int_0^{\tau_1} r^1_s ds}\int_0^{\tau_1}(r^2_s-r^1_s)ds\Big]&\le\Big(\mathsf{E}\Big[e^{-2\int_0^{\tau_1} r^1_s ds}\Big]\Big)^{\frac12}\Big(\mathsf{E}\Big[\Big(\int_0^{T-t}(r^2_s-r^1_s)ds\Big)^2\Big]\Big)^{\frac12}\\&\le C_1^{1/2}\Big((T-t)\int_0^{T-t}\mathsf{E}\left[(r^2_s-r^1_s)^2\right]ds\Big)^{\frac12},\end{aligned}\tag{4.17}$$

where $C_1>0$ is the constant from (4.3) which depends on $\mathcal{K}$. To conclude it is sufficient to use moment estimates for SDEs [93, Thm. 2.5.9] which guarantee that

$$\mathsf{E}\left[\sup_{0\le s\le T}(r^2_s-r^1_s)^2\right]\le c'(r_2-r_1)^2\tag{4.18}$$

for some $c' > 0$ only depending on T and the coefficients in (4.2).

Continuity in t. For $t \in [0,T)$, define $r_u^{T-t} := r_{u(T-t)}$ and $X_u^{T-t} := X_{u(T-t)}$ for $u \in [0,1]$. The couple $(r_u^{T-t}, X_u^{T-t})_{u\in[0,1]}$ is a strong solution to (see, e.g., [11, Ch. 1, Prop. 8.6])

$$\begin{aligned} dX_u^{T-t} &= (T-t)r_u^{T-t}X_u^{T-t}du + \sigma X_u^{T-t}d\widetilde{B}_u, \qquad X_0^{T-t} = x,\\ dr_u^{T-t} &= (T-t)\alpha(r_u^{T-t})du + \beta(r_u^{T-t})d\widetilde{W}_u, \qquad r_0^{T-t} = r, \end{aligned}$$

where $(\widetilde{B}_u, \widetilde{W}_u)_{u\in[0,1]} := (B_{u(T-t)}, W_{u(T-t)})_{u\in[0,1]}$. Using these processes, we can rewrite (4.6) as

$$v(t,r,x) = \sup_{0\le\theta\le 1} \mathsf{E}_{r,x}\left[\exp\left\{-(T-t)\int_0^\theta r_u^{T-t}du\right\}\left(K - X_\theta^{T-t}\right)^+\right], \tag{4.19}$$

where for any $(\mathcal{F}_s)_{s\ge 0}$-stopping time τ in $[0,T-t]$ the random variable $\theta := \tau/(T-t)$ is an $(\mathcal{F}_{u(T-t)})_{u\in[0,1]}$-stopping time.

Since the process $(B_{u(T-t)}, W_{u(T-t)})_{u\in[0,1]}$ is identical in law to

$$(\sqrt{T-t}B_u, \sqrt{T-t}W_u)_{u\in[0,1]},$$

with a slight abuse of notation we can identify $(r_u^{T-t}, X_u^{T-t})_{u\in[0,1]}$ with the unique strong solution of

$$dX_u^{T-t} = (T-t)r_u^{T-t}X_u^{T-t}du + \sqrt{T-t}\sigma X_u^{T-t}dB_u, \qquad X_0^{T-t} = x, \tag{4.20}$$

$$dr_u^{T-t} = (T-t)\alpha(r_u^{T-t})du + \sqrt{T-t}\beta(r_u^{T-t})dW_u, \qquad r_0^{T-t} = r, \tag{4.21}$$

and take stopping times $\theta \in [0,1]$ in (4.19) with respect to the filtration $(\mathcal{F}_t)$ generated by (B,W). In what follows we denote by $\theta_* = \theta_*(t,r,x)$ an optimal stopping time for (4.19).

Fix now $0 \le t_1 < t_2 < T$ and set $r^1 := r^{T-t_1}$, $r^2 := r^{T-t_2}$. Let $\theta_1 := \theta_*(t_1,r,x)$ and for $i = 1,2$ denote also

$$R_u^i = (T-t_i)\int_0^u r_s^i ds \quad \text{and} \quad \hat{X}_u^{T-t} = \exp\left(\sqrt{T-t}\,\sigma B_u - (T-t)\frac{\sigma^2}{2}u\right),$$

so that $X_u^{T-t_i} = xe^{-R_u^i}\hat{X}_u^{T-t_i}$. We remark that θ_1 is also admissible for the problem in (4.19) and the underlying dynamics (4.20)–(4.21) with $t = t_2$, because it is an $(\mathcal{F}_s)_{s\ge 0}$-stopping time in $[0,1]$. Indeed the advantage of (4.19) with (4.20)–(4.21) is that the class of admissible stopping times no longer depends on the initial time t.

Recalling Proposition 4.3.1(i) and using Lipschitz continuity of $x \mapsto (x)^+$ we have

$$\begin{aligned} 0 \ge v(t_2,r,x) - v(t_1,r,x) &\ge -\mathsf{E}_r\left[\left|\left(Ke^{-R_{\theta_1}^2} - x\hat{X}_{\theta_1}^{T-t_2}\right)^+ - \left(Ke^{-R_{\theta_1}^1} - x\hat{X}_{\theta_1}^{T-t_1}\right)^+\right|\right]\\ &\ge -K\mathsf{E}_r\left[\left|e^{-R_{\theta_1}^2} - e^{-R_{\theta_1}^1}\right|\right] - x\mathsf{E}\left[\left|\hat{X}_{\theta_1}^{T-t_1} - \hat{X}_{\theta_1}^{T-t_2}\right|\right]. \end{aligned} \tag{4.22}$$

Let us consider the second term on the right hand side of (4.22). By the fundamental theorem of calculus and the explicit formula for $\hat{X}^{T-t}$

$$\begin{aligned}\mathsf{E}\big[\big|\hat{X}^{T-t_1}_{\theta_1}-\hat{X}^{T-t_2}_{\theta_1}\big|\big] &= \mathsf{E}\Big[\Big|\int_{t_1}^{t_2}\hat{X}^{T-t}_{\theta_1}\Big(\frac{\sigma^2}{2}\theta_1-\frac{1}{2\sqrt{T-t}}\sigma B_{\theta_1}\Big)dt\Big|\Big]\\ &\le \int_{t_1}^{t_2}\mathsf{E}\Big[\Big|\hat{X}^{T-t}_{\theta_1}\Big(\frac{\sigma^2}{2}\theta_1-\frac{1}{2\sqrt{T-t}}\sigma B_{\theta_1}\Big)\Big|\Big]dt. \qquad (4.23)\end{aligned}$$

For $t\in(t_1,t_2)$, define a measure $\tilde{\mathsf{P}}$ by $\frac{d\tilde{\mathsf{P}}}{d\mathsf{P}}:=\hat{X}^{T-t}_1$. Then $\tilde{B}_s=B_s-\sigma s\sqrt{T-t}$ is a Brownian motion under $\tilde{\mathsf{P}}$ and

$$\begin{aligned}&\mathsf{E}\Big[\Big|\hat{X}^{T-t}_{\theta_1}\Big(\frac{\sigma^2}{2}\theta_1-\frac{1}{2\sqrt{T-t}}\sigma B_{\theta_1}\Big)\Big|\Big]=\tilde{\mathsf{E}}\left[\Big|\frac{\theta_1}{2}\sigma^2-\frac{\sigma}{2\sqrt{T-t}}(\tilde{B}_{\theta_1}+\sqrt{T-t}\sigma\theta_1)\Big|\right]\\ &=\tilde{\mathsf{E}}\left[\Big|\frac{\sigma}{2\sqrt{T-t}}\tilde{B}_{\theta_1}\Big|\right]\le\left(\tilde{\mathsf{E}}\Big[\frac{\sigma^2\tilde{B}^2_{\theta_1}}{4(T-t)}\Big]\right)^{1/2}\le\frac{\sigma}{2\sqrt{T-t}}\le\frac{\sigma}{2\sqrt{T-t_2}}=:c_1,\end{aligned}$$

where we applied Hölder inequality and used that $\theta_1\le 1$. Inserting the above estimate into (4.23) gives

$$\mathsf{E}\big[\big|\hat{X}^{T-t_1}_{\theta_1}-\hat{X}^{T-t_2}_{\theta_1}\big|\big]\le c_1(t_2-t_1). \qquad (4.24)$$

Next we address the first term on the right hand side of (4.22). This is performed separately in cases (i) and (ii) of Assumption 4.2.1. We start by considering case (ii), i.e., α and β in (4.21) are Lipschitz continuous. Fundamental theorem of calculus and Hölder inequality give

$$\begin{aligned}&\mathsf{E}_r\left[\Big|e^{-R^1_{\theta_1}}-e^{-R^2_{\theta_1}}\Big|\right]\\ &\le\mathsf{E}_r\left[\max_{i=1,2}\Big\{e^{-(T-t_i)\int_0^{\theta_1}r^i_u du}\Big\}\Big|(T-t_1)\int_0^{\theta_1}r^1_u du-(T-t_2)\int_0^{\theta_1}r^2_u du\Big|\right]\\ &\le\mathsf{E}_r\left[\max_{i=1,2}\Big\{e^{-(T-t_i)\int_0^{\theta_1}r^i_u du}\Big\}\Big((t_2-t_1)\Big|\int_0^{\theta_1}r^1_u du\Big|+(T-t_2)\Big|\int_0^{\theta_1}(r^2_u-r^1_u)du\Big|\Big)\right]\\ &\le 2c_2\left[(t_2-t_1)\Big(\mathsf{E}_r\big[\sup_{0\le t\le 1}(r^1_t)^2\big]\Big)^{\frac{1}{2}}+(T-t_2)\Big(\mathsf{E}_r\Big[\int_0^1(r^2_u-r^1_u)^2du\Big]\Big)^{\frac{1}{2}}\right],\end{aligned} \qquad (4.25)$$

where, using (4.3),

$$c_2:=\sup_{(t,r,x)\in\mathcal{K}}\left(\mathsf{E}_r\Big[\sup_{0\le s\le 1}e^{-2(T-t)\int_0^s r^{T-t}_u du}\Big]\right)^{\frac{1}{2}}<\infty.$$

Thanks to (4.4), $c_3:=\sup_{(t,r,x)\in\mathcal{K}}\Big(\mathsf{E}_r\big[\sup_{0\le s\le 1}\big(r^{T-t}_s\big)^2\big]\Big)^{\frac{1}{2}}<\infty$, so it remains to estimate the last term of (4.25). By [93, Thm. 2.5.9] there is a constant c_4 depending only on $\mathcal{K}$ and the

Lipschitz constant for α and β in (4.21) such that

$$\begin{aligned}&\mathsf{E}_r\Big[\sup_{0\le t\le 1}\big(r^1_t-r^2_t\big)^2\Big]\\&\le c_4\,\mathsf{E}_r\Big[\int_0^1\Big(|(T-t_1)\alpha(r^1_u)-(T-t_2)\alpha(r^1_u)|^2+|\sqrt{T-t_1}\beta(r^1_u)-\sqrt{T-t_2}\beta(r^1_u)|^2\Big)\,du\Big]\\&\le c_4(t_2-t_1)^2\,\mathsf{E}_r\Big[\int_0^1|\alpha(r^1_u)|^2du\Big]+c_4\,(t_2-t_1)\,\mathsf{E}_r\Big[\int_0^1|\beta(r^1_u)|^2du\Big],\end{aligned}$$

where for the second inequality we used that $\sqrt{T-t_1}-\sqrt{T-t_2}\le\sqrt{t_2-t_1}$. Notice that by (4.4) and the linear growth of α and β

$$c_5:=\sup_{(r,t,x)\in\mathcal{K}}\mathsf{E}_r\Big[\int_0^1|\alpha(r^{T-t}_u)|^2+|\beta(r^{T-t}_u)|^2du\Big]<\infty.$$

Inserting the above estimates into (4.25) we conclude that there is a constant c_6 such that for any $(t_1,r,x),(t_2,r,x)\in\mathcal{K}$

$$\mathsf{E}_r\left[\left|e^{-R^1_{\theta_1}}-e^{-R^2_{\theta_1}}\right|\right]\le c_6|t_2-t_1|.$$

This and (4.24) feed into (4.22) so that

$$0\ge v(t_2,r,x)-v(t_1,r,x)\ge -c|t_2-t_1| \tag{4.26}$$

for a suitable $c>0$ that depends on $\mathcal{K}$.

Finally, we must estimate the first term on the right hand side of (4.22) under the assumption that $(r_t)_{t\ge0}$ follows the CIR dynamics (Assumption 4.2.1(i)). Let $\hat r^i_u:=r^i_u/(T-t_i)$ for $u\in[0,1]$ and $i=1,2$. The dynamics for $\hat r^i$ reads

$$d\hat r^i_u=\kappa\big(\alpha-(T-t_i)\hat r^i_u\big)du+\beta\sqrt{\hat r^i_u}dW_u,\quad u\in[0,1]. \tag{4.27}$$

Since $\kappa(\alpha-(T-t_1)\hat r)<\kappa(\alpha-(T-t_2)\hat r)$ for $\hat r\ge0$, and $\hat r^1_0=r/(T-t_1)\le r/(T-t_2)=\hat r^2_0$, comparison results for SDEs [87, Prop. 5.2.18] imply

$$\hat r^1_u\le\hat r^2_u\quad\text{for all } u\in[0,1],\ \mathsf{P}\text{-a.s.} \tag{4.28}$$

Using the integral version of (4.27) and the martingale property of the stochastic integral, we obtain

$$\begin{aligned}\mathsf{E}_r\left[\hat r^2_u-\hat r^1_u\right]&=r\Big(\frac{1}{T-t_2}-\frac{1}{T-t_1}\Big)+\mathsf{E}_r\left[\int_0^u\big((T-t_1)\hat r^1_s-(T-t_2)\hat r^2_s\big)\,ds\right]\\&\le r\frac{t_2-t_1}{(T-t_1)(T-t_2)}+(t_2-t_1)\int_0^1\mathsf{E}_r\left[\hat r^1_s\right]ds+(T-t_2)\int_0^u\mathsf{E}_r\left[\hat r^1_s-\hat r^2_s\right]ds.\end{aligned}$$

Due to (4.28), the last term is non-positive, so

$$0 \leq \mathsf{E}_r\left[\hat{r}_u^2 - \hat{r}_u^1\right] \leq (t_2 - t_1)\Big(\frac{r}{(T-t_1)(T-t_2)} + q_1\Big) \quad \text{for all } u \in [0,1] \tag{4.29}$$

where

$$q_1 := \sup_{(t,r,x)\in\mathcal{K}} \frac{1}{T-t}\int_0^1 \mathsf{E}_r\left[r_u^{T-t}\right] du < \infty.$$

We use the inequalities (4.28)–(4.29) and the property that $\hat{r}_u^i \geq 0$, for $i = 1, 2$, to obtain the following estimates

$$\begin{aligned} \mathsf{E}_r\left[\left|e^{-R_{\theta_1}^1} - e^{-R_{\theta_1}^2}\right|\right] &= \mathsf{E}_r\left[\left|e^{-(T-t_1)^2\int_0^{\theta_1}\hat{r}_u^1 du} - e^{-(T-t_2)^2\int_0^{\theta_1}\hat{r}_u^2 du}\right|\right] \\ &\leq \mathsf{E}_r\left[\left|e^{-(T-t_1)^2\int_0^{\theta_1}\hat{r}_u^1 du} - e^{-(T-t_2)^2\int_0^{\theta_1}\hat{r}_u^1 du}\right|\right] \\ &\quad + \mathsf{E}_r\left[\left|e^{-(T-t_2)^2\int_0^{\theta_1}\hat{r}_u^1 du} - e^{-(T-t_2)^2\int_0^{\theta_1}\hat{r}_u^2 du}\right|\right] \\ &\leq q_1\left((T-t_1)^2 - (T-t_2)^2\right) + (T-t_2)^2\int_0^1 \mathsf{E}_r\left[\hat{r}_u^2 - \hat{r}_u^1\right] du \\ &\leq (t_2-t_1)\Big(2Tq_1 + r\frac{T-t_2}{T-t_1} + q_1(T-t_2)^2\Big) \leq c_7(t_2-t_1), \end{aligned} \tag{4.30}$$

where the constant $c_7 > 0$ depends only on $\mathcal{K}$ but not on a specific choice of t_1, t_2, r, x. Hence, as in the case of Assumption 4.2.1(ii), we obtain (4.26). □

The continuity of v means that the continuation set $\mathcal{C}$ is open and the stopping set $\mathcal{D}$ is closed. In view of the monotonicity properties established in Proposition 4.3.1, we can show that there is a surface splitting $\mathcal{C}$ and $\mathcal{D}$.

Proposition 4.3.4. *There exists a function* $c(t,x)$ *on* $[0,T] \times [0,\infty]$*, such that*

$$\mathcal{D} = \{(t,r,x) \in \mathcal{O} \,:\, r \geq c(t,x)\} \cup \left(\{T\} \times \mathcal{I} \times \mathbb{R}_+\right), \tag{4.31}$$

$$\mathcal{C} = \{(t,r,x) \in \mathcal{O} \,:\, r < c(t,x)\}. \tag{4.32}$$

The function $c(t,x)$ *has following properties:*

(i) For any $(t_0,x_0) \in [0,T) \times \mathbb{R}_+$*, the mapping* $t \mapsto c(t,x_0)$ *is right-continuous and non-increasing and the mapping* $x \mapsto c(t_0,x)$ *is left-continuous and non-decreasing.*

(ii) $c(t,x) = \bar{r}$ *for* $(t,x) \in [0,T) \times [K,\infty)$.

(iii) $c(t,x) \geq 0$ *for* $(t,x) \in [0,T) \times \mathbb{R}_+$*, and* $\lim_{x\downarrow 0} c(t,x) = 0$ *for* $t \in [0,T)$.

Proof. The payoff does not depend on (r_t) and v is non-increasing in r by Proposition 4.3.1. Therefore, if $(t,r_1,x)\in\mathcal{D}$ then $(t,r_2,x)\in\mathcal{D}$ for for any $r_2>r_1$. This allows us to represent the stopping region $\mathcal{D}$ via (4.31) with

$$c(t,x):=\inf\{r\in\mathcal{I}:v(t,r,x)=(K-x)^+\},\tag{4.33}$$

with the convention that $\inf\emptyset=\overline{r}$. It is convenient to prove (ii) first.

(ii) Fix $(t,r,x)\in[0,T)\times\mathcal{I}\times[K,\infty)$. If we show that $\mathsf{P}_{r,x}(X_\varepsilon<K)>0$ for some $\varepsilon\in(0,T-t]$, then $v(t,r,x)>0=(K-x)^+$. This means that $(t,r,x)\in\mathcal{C}$ and $c(t,x)=\overline{r}$. Recall that $\rho\in(-1,1)$ is the correlation coefficient between the Brownian motions B and W driving the SDEs for X and r, respectively. Then we can write $B_t=\rho W_t+\sqrt{1-\rho^2}B^0_t$ for some other Brownian motion B^0 independent of W. Letting $(\mathcal{F}^W_t)_{t\geq 0}$ be the filtration generated by W, using the explicit form of the dynamics of X we have

$$\begin{aligned}
&\mathsf{P}_{r,x}(X_\varepsilon<K)\\
&=\mathsf{E}_{r,x}\Big[\mathsf{P}_{r,x}(X_\varepsilon<K|\mathcal{F}^W_\varepsilon)\Big]\\
&=\mathsf{E}_{r,x}\Big[\mathsf{P}_r\Big(\exp\big(\sigma\sqrt{1-\rho^2}B^0_\varepsilon\big)<(K/x)\exp\Big(-\sigma\rho W_\varepsilon-\int_0^\varepsilon r_t dt+\tfrac{\sigma^2}{2}\varepsilon\Big)\Big|\mathcal{F}^W_\varepsilon\Big)\Big]\\
&=\mathsf{E}_{r,x}\Big[\Psi_x\Big(\sigma\rho W_\varepsilon+\int_0^\varepsilon r_t dt-\tfrac{\sigma^2}{2}\varepsilon\Big)\Big],
\end{aligned}\tag{4.34}$$

where

$$\Psi_x(z):=\mathsf{P}\Big(\exp\big(\sigma\sqrt{1-\rho^2}B^0_\varepsilon\big)<(K/x)e^{-z}\Big)$$

and the final equality above holds by the independence of B^0_ε from $\mathcal{F}^W_\varepsilon$ and the fact that $(W_\varepsilon,\int_0^\varepsilon r_t dt)$ is $\mathcal{F}^W_\varepsilon$-measurable. Since $\rho\in(-1,1)$, then $\Psi_x(z)>0$ for any $z\in\mathbb{R}$ and we conclude that $\mathsf{P}_{r,x}(X_\varepsilon<K)>0$.

(i) By the monotonicity of v in t, we have $(t_1,r,x)\in\mathcal{D}\implies(t_2,r,x)\in\mathcal{D}$ for any $t_2>t_1$, hence $c(t,x)$ is non-increasing in t.

Fix $0\leq x_1<x_2<K$ and let $\tau_1:=\tau_*(t,r,x_1)$ be optimal for $v(t,r,x_1)$. Then, using that $X^{r,x_1}\leq X^{r,x_2}$ and recalling (4.5), we obtain

$$\begin{aligned}
v(t,r,x_2)-v(t,r,x_1)\geq&\,\mathsf{E}\left[e^{-\int_0^{\tau_1}r_s ds}\left(\big(K-X^{r,x_2}_{\tau_1}\big)^+-\big(K-X^{r,x_1}_{\tau_1}\big)^+\right)\right]\\
\geq&\,\mathsf{E}\left[e^{-\int_0^{\tau_1}r_s ds}\left(X^{r,x_1}_{\tau_1}-X^{r,x_2}_{\tau_1}\right)\right]\\
=&\,x_1-x_2=(K-x_2)^+-(K-x_1)^+.
\end{aligned}$$

Therefore, if $(t,r,x_1)\in\mathcal{C}$ then $(t,r,x_2)\in\mathcal{C}$, which implies that $c(t,x)$ is non-decreasing in x.

Fix arbitrary $(t,x) \in [0,T) \times \mathbb{R}_+$, let $t_n \downarrow t_0$ as $n \to \infty$, then $c(t_n,x) \uparrow c(t_0+,x)$ as $n \to \infty$, where the limit exists by the monotonicity of $t \mapsto c(t,x)$. Since $(t_n, c(t_n,x), x) \in \mathcal{D}$, then also $(t_0, c(t_0+,x), x) \in \mathcal{D}$ by the closedness of $\mathcal{D}$, hence $c(t_0+,x) \geq c(t_0,r)$ which implies $c(t_0+,r) = c(t_0,r)$. Taking $x_n \uparrow x_0$, a similar argument yields $c(t,x_0-) = c(t,x_0)$.

(iii) Under the CIR model, the positivity follows by the definition of $c(t,x)$. Only under Assumption 4.2.1 (ii) a proof is required. Assume that there exists $(t_0,\hat{x}) \in [0,T) \times (0,K)$ such that $c(t_0,\hat{x}) < 0$. Let $0 > r_2 > r_0 > r_1 > c(t_0,\hat{x})$ and $0 < x_0 < \hat{x}$. Define a stopping time

$$\tau_1 = \inf\{s \geq 0 : (s, r_s^{r_0}, X_s^{x_0,r_0}) \notin [0, T-t_0) \times (r_1,r_2) \times (0,\hat{x})\}.$$

By the monotonicity of $c(t,x)$, we have $(t_0,r_0,x_0) \in \mathcal{D}$. Hence, τ_1 is sub-optimal and

$$K - x_0 = v(t_0,r_0,x_0) \geq \mathsf{E}_{r_0,x_0}\left[e^{-\int_0^{\tau_1} r_s ds}(K - X_{\tau_1})^+\right] \geq K\mathsf{E}_{r_0,x_0}\left[e^{-\int_0^{\tau_1} r_s ds}\right] - x_0, \tag{4.35}$$

where the last inequality follows from the optional sampling theorem and the fact that $(K - X_{\tau_1})^+ \geq K - X_{\tau_1}$. Since $\mathsf{P}_{x_0,r_0}(\tau_1 > 0) = 1$ and $r_s(\omega) < r_2 < 0$ for $s \in [0, \tau_1(\omega))$, we obtain

$$K\mathsf{E}_{r_0,x_0}\left[e^{-\int_0^{\tau_1} r_s ds}\right] - x_0 > K - x_0,$$

which, in conjunction with (4.35), leads to a contradiction.

Finally, we show that $c(t,0+) := \lim_{x\downarrow 0} c(t,x) = 0$ for any $t \in [0,T)$. Assume $c(t,0+) \geq \delta > 0$ for some $t \in [0,T)$. By the monotonicity of $c(t,x)$ and the openness of $\mathcal{C}$ there is $\hat{t} \in [t,T)$ such that

$$[0,\hat{t}) \times (r_1,r_2) \times (0,\infty) \subset \mathcal{C},$$

where $0 < r_1 < r_2 < \delta$. Fix $0 \leq t_0 < \hat{t}$ and $r_0 \in (r_1,r_2)$. Take an arbitrary $x_0 > 0$. Let

$$\tau_2 = \inf\{s \geq 0 : (s, r_s) \notin [0, \hat{t} - t_0) \times (r_1,r_2)\}.$$

By construction $\mathsf{P}_{r_0,x_0}\big((t_0+s, r_s, X_s) \in \mathcal{C} \text{ for } s \leq \tau_2\big) = 1$, so $\tau_2 \leq \tau_*(t_0,r_0,x_0)$ P_{r_0,x_0}-a.s. By the martingale property of the value function we obtain

$$\begin{aligned} K - x_0 < v(t_0,r_0,x_0) &= \mathsf{E}_{r_0,x_0}\left[e^{-\int_0^{\tau_2} r_s ds} v\left(t_0+\tau_2, r_{\tau_2}, X_{\tau_2}\right)\right] \\ &\leq K\mathsf{E}_{r_0,x_0}\left[e^{-r_1\tau_2}\right] = K\mathsf{E}_{r_0}\left[e^{-r_1\tau_2}\right]. \end{aligned} \tag{4.36}$$

A contradiction is obtained by taking the limit $x_0 \downarrow 0$, since $\mathsf{E}_{r_0}\left[e^{-r_1\tau_2}\right]$ is independent of X and strictly smaller than 1. $\square$

Notice that (ii) and (iii) above imply that it is never optimal to exercise the option out of the money or if the interest rate is negative. This is in line with classical financial wisdom.

In models with constant interest rate, an optimal boundary is often defined as function of time which provides a threshold for the process (X_t). An analogous representation of the continuation and stopping sets is valid in our model. The following proposition whose simple proof is omitted gives details of the re-parametrisation of the stopping boundary.

Proposition 4.3.5. *Define*

$$b(t,r) := \inf\{x \in \mathbb{R}_+ : c(t,x) > r\}, \qquad (t,r) \in [0,T) \times \mathcal{I}.$$

The mappings $t \mapsto b(t,r_0)$ *and* $r \mapsto b(t_0,r)$ *are right-continuous and non-decreasing for any* $(t_0,r_0) \in [0,T) \times \mathcal{I}$*. For any* $t \in [0,T)$ *we have* $K > b(t,r) > 0$ *when* $r > 0$*, and* $b(t,r) = 0$ *when* $r < 0$*. Furthermore,*

$$\begin{aligned} \mathcal{D} &= \{(t,r,x) \in \mathcal{O} \,:\, x \le b(t,r)\} \cup \big(\{T\} \times \mathcal{I} \times \mathbb{R}_+\big), \\ \mathcal{C} &= \{(t,r,x) \in \mathcal{O} \,:\, x > b(t,r)\}. \end{aligned}$$

The parametrisation of the stopping boundary via the function $c(t,x)$ will usually be more convenient. In particular, due to technical reasons that will become clearer in later sections, we will be able to prove the continuity of $(t,x) \mapsto c(t,x)$ jointly in both variables (t,x), but not the joint continuity of b in (t,r). However, b is more convenient for numerical computations in Section 4.9 as it admits values in a bounded interval $[0,K]$.

An important consequence of Proposition 4.3.5 is that for $\varepsilon \in (0,x)$

$$(t,r,x) \in \mathcal{D} \implies (t+\varepsilon,r,x), (t,r+\varepsilon,x), (t,r,x-\varepsilon) \in \mathcal{D}.$$

We immediately see that $\partial\mathcal{C}$ enjoys the so-called *cone property* [87, Def. 4.2.18]. Indeed, for any $(t_0,r_0,x_0) \in \partial\mathcal{C}$, there is an orthant $\widehat{C}_0$ with vertex in (t_0,r_0,x_0) (hence a cone with aperture $\pi/4$) that satisfies $\widehat{C}_0 \cap \mathcal{O} \subseteq \mathcal{D}$. This will be used to establish regularity of the boundary $\partial\mathcal{C}$ in the sense of diffusions, which, has important consequences for the smoothness of our value function v, as we shall see below.

Introduce the hitting time to $\mathcal{D}$, denoted $\sigma_{\mathcal{D}}$, and the entry time to the interior of $\mathcal{D}$, denoted $\mathring{\sigma}_{\mathcal{D}}$. That is, for $(t,r,x) \in \mathcal{O}$ we set $\mathsf{P}_{r,x}$-a.s.

$$\begin{aligned} \sigma_{\mathcal{D}} &:= \inf\{s > 0 \,:\, (t+s,r_s,X_s) \in \mathcal{D}\}, \\ \mathring{\sigma}_{\mathcal{D}} &:= \inf\{s \ge 0 \,:\, (t+s,r_s,X_s) \in \mathrm{int}(\mathcal{D})\} \wedge (T-t). \end{aligned} \tag{4.37}$$

Both $\sigma_{\mathcal{D}}$ and $\mathring{\sigma}_{\mathcal{D}}$ are stopping times with respect to the filtration $(\mathcal{F}_t)_{t\ge0}$. We will often write $\sigma_{\mathcal{D}}(t,r,x)$ and $\mathring{\sigma}_{\mathcal{D}}(t,r,x)$ to indicate the starting point of the process.

Proposition 4.3.6 (Regularity of the boundary). *For* $(t_0, r_0, x_0) \in \partial\mathcal{C}$*, we have*

$$\mathsf{P}_{t_0,r_0,x_0}(\sigma_{\mathcal{D}} > 0) = \mathsf{P}_{t_0,r_0,x_0}(\mathring{\sigma}_{\mathcal{D}} > 0) = 0. \tag{4.38}$$

Proof. The proof rests on Gaussian bounds for the transition density of a diffusion and ideas from the proof of well-known analogous results for multi-dimensional Brownian motion, see e.g. [87, Thm. 4.2.19]. It is also worth recalling that $\partial\mathcal{C}$ is the boundary of $\mathcal{C}$ in $\mathcal{O}$, so that it excludes $\{T\} \times \mathcal{I} \times \mathbb{R}_+$.

Fix $(t_0, r_0, x_0) \in \partial\mathcal{C}$ and define $\mathcal{R} := [r_0, \overline{r}] \times [0, x_0]$, where we also recall that $\mathcal{I} = (\underline{r}, \overline{r})$. Since $t \mapsto c(t, x)$ is non-increasing, it is immediate to see that $[t_0, T] \times \mathcal{R} \subseteq \mathcal{D}$. Recalling the notation introduced in (2.16), set

$$\mathring{\sigma}_{\mathcal{R}}(r_0, x_0) := \inf\{s \geq 0 \,:\, (r_s^{r_0}, X_s^{r_0,x_0}) \in \text{int}(\mathcal{R})\}.$$

We have $\mathring{\sigma}_{\mathcal{R}}(r_0, x_0) \geq \mathring{\sigma}_{\mathcal{D}}(t_0, r_0, x_0)$, P-a.s., and $\mathring{\sigma}_{\mathcal{R}}(r_0, x_0) \geq \sigma_{\mathcal{D}}(t_0, r_0, x_0)$ by the continuity of the process (r, X). From now on we omit in the notation the dependence on (t_0, r_0, x_0) since the initial point is fixed throughout the proof.

Take a compact ball $\mathcal{K} \subset \mathcal{I} \times \mathbb{R}_+$ centred at (r_0, x_0). Let $\Sigma(r, x)$ denote the matrix of the diffusion coefficient for (4.1)–(4.2), i.e.

$$\Sigma(r, x) := \frac{1}{2} \begin{pmatrix} \sigma^2 x^2 & \rho\sigma x\beta(r) \\ \rho\sigma x\beta(r) & \beta^2(r) \end{pmatrix}.$$

Since the correlation coefficient $\rho \in (-1, 1)$, there is $\gamma = \gamma_{\mathcal{K}} > 0$ such that

$$\frac{1}{\gamma}\|z\|^2 \leq \langle \Sigma(r, x)z, z\rangle \leq \gamma\|z\|^2, \qquad z \in \mathbb{R}^2 \setminus \{0\},\ (r, x) \in \mathcal{K}, \tag{4.39}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^2$ and $\|\cdot\|$ the corresponding norm.

Define a new process $(\widetilde{r}, \widetilde{X})$ with the dynamics defined on $\mathbb{R}^2$

$$d\widetilde{X}_t = \mu_{\mathcal{K}}(\widetilde{r}_t, \widetilde{X}_t)dt + \sigma_{\mathcal{K}}(\widetilde{X}_t)dB_t, \qquad \widetilde{X}_0 = x_0 \tag{4.40}$$

$$d\widetilde{r}_t = \alpha_{\mathcal{K}}(\widetilde{r}_t)dt + \beta_{\mathcal{K}}(\widetilde{r}_t)dW_t, \qquad \widetilde{r}_0 = r_0 \tag{4.41}$$

such that the coefficients coincide with the coefficients of (4.1)-(4.2) on $\mathcal{K}$, are Lipschitz continuous on $\mathbb{R}^2$ and satisfy the uniform ellipticity condition (4.39) with γ on $\mathbb{R}^2$. Denoting $\tau_{\mathcal{K}} := \inf\{t \geq 0 \,:\, (r_t, X_t) \notin \text{int}(\mathcal{K})\}$ and $\widetilde{\tau}_{\mathcal{K}} := \inf\{t \geq 0 \,:\, (\widetilde{r}_t, \widetilde{X}_t) \notin \text{int}(\mathcal{K})\}$, by the uniqueness of solutions for SDEs we get indistinguishable stopped paths:

$$(r_{t\wedge\tau_{\mathcal{K}}}, X_{t\wedge\tau_{\mathcal{K}}})_{t\geq 0} = (\widetilde{r}_{t\wedge\widetilde{\tau}_{\mathcal{K}}}, \widetilde{X}_{t\wedge\widetilde{\tau}_{\mathcal{K}}})_{t\geq 0} \quad \mathsf{P}_{r_0,x_0}\text{-a.s.}$$

The uniform ellipticity condition (4.39) on $\mathbb{R}^2$ implies that the process $(\widetilde{r}, \widetilde{X})$ admits a transition density $\widetilde{p}(t,(r,x),(r',x'))$ which satisfies the following Gaussian bound (see, e.g., [7, 59]): there exists $m>0$ and $\Lambda>0$ such that

$$\widetilde{p}(t,(r,x),(r',x')) \geq m\, t^{-1} \exp\left(-\Lambda \frac{(r'-r)^2+(x'-x)^2}{t}\right). \tag{4.42}$$

Let $\mathcal{R}''$ be a closed cone with vertex (r_0,x_0) and non-empty interior contained in $\big[(r_0,\infty)\times(-\infty,x_0)\big]\cup\{(r_0,x_0)\}$. Put $\mathcal{R}'=\mathcal{R}''\cap(\mathcal{I}\times\mathbb{R}_+)$. Denote by $\mathring{\sigma}'_{\mathcal{R}}$ the entry time of (r,X) to $\mathrm{int}(\mathcal{R}')$ and by $\mathring{\sigma}''_{\mathcal{R}}$ the entry time of $(\widetilde{r},\widetilde{X})$ to $\mathrm{int}(\mathcal{R}'')$. The next estimate relies on analogous results for multi-dimensional Brownian motion ([87, Thm. 4.2.9]); in particular the second inequality below follows from (4.42):

$$\mathsf{P}_{r_0,x_0}(\mathring{\sigma}''_{\mathcal{R}}\leq t) \geq \mathsf{P}_{r_0,x_0}((\widetilde{r}_t,\widetilde{X}_t)\in\mathcal{R}'') \geq \frac{m}{t}\int_{\mathcal{R}''} \exp\left(-\Lambda\frac{(r-r_0)^2+(x-x_0)^2}{t}\right) dr\, dx.$$

We change variables to $y:=(r-r_0)/\sqrt{t}$ and $z:=(x-x_0)/\sqrt{t}$ and use that $\mathcal{R}''$ is invariant under this transformation up to a shift of its vertex to the origin. Denoting $\mathcal{R}''_0$ the cone with vertex in the origin, we obtain

$$\mathsf{P}_{r_0,x_0}(\mathring{\sigma}''_{\mathcal{R}}\leq t) \geq 2\pi m\int_{\mathcal{R}''_0}\frac{1}{2\pi}e^{-\Lambda(y^2+z^2)}dy\, dz =: q>0. \tag{4.43}$$

For any $t>0$, since $\mathcal{R}'\subset\mathcal{R}$ we have

$$\begin{aligned}
&\mathsf{P}_{r_0,x_0}(\mathring{\sigma}_{\mathcal{R}}\leq t)\\
&\geq \mathsf{P}_{r_0,x_0}(\mathring{\sigma}'_{\mathcal{R}}\leq t) \geq \mathsf{P}_{r_0,x_0}(\mathring{\sigma}'_{\mathcal{R}}\leq t, \tau_{\mathcal{K}}>t)\\
&= \mathsf{P}_{r_0,x_0}(\mathring{\sigma}''_{\mathcal{R}}\leq t, \widetilde{\tau}_{\mathcal{K}}>t) \geq \mathsf{P}_{r_0,x_0}(\mathring{\sigma}''_{\mathcal{R}}\leq t) - \mathsf{P}_{r_0,x_0}(\widetilde{\tau}_{\mathcal{K}}\leq t) \geq q - \mathsf{P}_{r_0,x_0}(\widetilde{\tau}_{\mathcal{K}}\leq t),
\end{aligned}$$

where the last inequality is by (4.43). As $t\downarrow 0$, we have $\mathsf{P}_{r_0,x_0}(\widetilde{\tau}_{\mathcal{K}}\leq t)\to 0$ and $\mathsf{P}_{r_0,x_0}(\mathring{\sigma}_{\mathcal{R}}\leq t)\to\mathsf{P}_{r_0,x_0}(\mathring{\sigma}_{\mathcal{R}}=0)$, which implies that $\mathsf{P}_{r_0,x_0}(\mathring{\sigma}_{\mathcal{R}}=0)\geq q>0$. By the Blumenthal $0-1$ law [87, Thm. 2.7.17] we obtain $\mathsf{P}_{r_0,x_0}(\mathring{\sigma}_{\mathcal{R}}=0)=1$. Recalling that $\mathsf{P}_{t_0,r_0,x_0}(\mathring{\sigma}_{\mathcal{R}}\geq\mathring{\sigma}_{\mathcal{D}})=1$, and $\mathsf{P}_{t_0,r_0,x_0}(\mathring{\sigma}_{\mathcal{D}}\geq\sigma_{\mathcal{D}})=1$, we conclude $\mathsf{P}_{t_0,r_0,x_0}(\mathring{\sigma}_{\mathcal{D}}=0)=\mathsf{P}_{t_0,r_0,x_0}(\sigma_{\mathcal{D}}=0)=1$. □

Remark 4.3.7. *It is worth noticing that the arguments above show the existence of the transition density of the process $(\widetilde{r},\widetilde{X})$ for any compact set $\mathcal{K}\subset\mathcal{I}\times\mathbb{R}_+$ such that $\mathcal{K}=\overline{\mathrm{int}(\mathcal{K})}$. This implies that for each $t\in[0,T]$ also the law of (r_t,X_t) is absolutely continuous with respect to the Lebesgue measure on $\mathcal{I}\times\mathbb{R}_+$, when the boundary of $\mathcal{I}\times\mathbb{R}_+$ is unattainable by (r_t,X_t). Indeed, let $N\subset\mathcal{I}\times\mathbb{R}_+$ be such that $\lambda(N)=0$, with λ denoting the Lebesgue measure on $\mathbb{R}^2$.*

Let $\mathcal{K} \subset \mathcal{I} \times \mathbb{R}_+$ be a compact set such that $\mathcal{K} = \overline{\mathrm{int}(\mathcal{K})}$. Then by the same construction as above

$$\begin{aligned}
\mathsf{P}_{r_0,x_0}\big((r_t, X_t) \in N\big) &= \mathsf{P}_{r_0,x_0}\big((r_t, X_t) \in N, t \le \tau_{\mathcal{K}}\big) + \mathsf{P}_{r_0,x_0}\big((r_t, X_t) \in N, t > \tau_{\mathcal{K}}\big) \\
&= \mathsf{P}_{r_0,x_0}\big((\widetilde{r}_t, \widetilde{X}_t) \in N, t \le \widetilde{\tau}_{\mathcal{K}}\big) + \mathsf{P}_{r_0,x_0}\big((r_t, X_t) \in N, t > \tau_{\mathcal{K}}\big) \\
&\le \mathsf{P}_{r_0,x_0}\big((\widetilde{r}_t, \widetilde{X}_t) \in N\big) + \mathsf{P}_{r_0,x_0}\big(\tau_{\mathcal{K}} < t\big) = \mathsf{P}_{r_0,x_0}\big(\tau_{\mathcal{K}} < t\big),
\end{aligned} \tag{4.44}$$

where the final equality uses that the transition law of $(\widetilde{r}, \widetilde{X})$ is absolutely continuous with respect to λ. Now, letting $\mathcal{K} \uparrow \mathcal{I} \times \mathbb{R}_+$, using that 0 and $+\infty$ are not attainable by X and $\underline{r}$ and $\overline{r}$ are not attainable by r, we can make $\mathsf{P}_{r_0,x_0}\big(\tau_{\mathcal{K}} < t\big)$ arbitrarily small, which proves the claim.

4.4 Smoothness of the value function

It is well-known that v satisfies (in the classical sense)

$$\begin{aligned}
&v_t(t, r, x) + (\mathcal{L} - r)v(t, r, x) = 0, \qquad (t, r, x) \in \mathcal{C}, \\
&v(t, r, x) = (K - x)^+, \qquad (t, r, x) \in \mathcal{D},
\end{aligned} \tag{4.45}$$

where $\mathcal{L}$ is the generator of (r, X) defined in (4.14). Hence, standard arguments assert that v is $C^{1,2,2}$ in $\mathcal{C} \cap \mathrm{int}(\mathcal{D})$. Classical optimal stopping theory identifies the boundary of the set $\mathcal{C}$ by imposing the so-called smooth-fit condition. In the American put problem with constant interest rate this corresponds to proving that $x \mapsto v^{\circ}_x(t, x)$ is continuous for each $t \in [0, T)$ fixed, with v° denoting the value function associated to the option price. In our setting we prove a stronger result and show continuous differentiability of v across the stopping boundary $\partial\mathcal{C}$, i.e., the global continuity of the gradient of v (as a function of all variables) in $\mathcal{O}$. We use ideas similar to those in [42] but we must refine arguments therein and use estimates with 'local' nature since we are not able to directly check the assumptions required in [42].

We start by establishing the following continuity properties of processes r and X.

Lemma 4.4.1. *Let $(r_n, x_n)_{n\ge1}$ be a sequence converging to $(r, x) \in \mathcal{I} \times \mathbb{R}_+$ as $n \to \infty$. Then*

$$\lim_{n\to+\infty} \sup_{0\le t\le T} |r^{r_n}_t - r^r_t| = 0, \qquad \mathsf{P}\text{-a.s.} \tag{4.46}$$

$$\lim_{n\to+\infty} \sup_{0\le t\le T} |X^{r_n,x_n}_t - X^{r,x}_t| = 0, \qquad \mathsf{P}\text{-a.s.} \tag{4.47}$$

Proof. Assume first that $(r_n)_{n\ge1}$ is a monotone sequence. Define $f^n_t := r^{r_n}_t - r^r_t$. Then for a.e. $\omega \in \Omega$, $t \mapsto f^n_t(\omega)$ is continuous and $f^n_t(\omega)$ converges to 0 monotonically as $n \to \infty$ for all $t \in [0, T]$. Hence the convergence is uniform on $[0, T]$ thanks to Dini's theorem and (4.46) holds.

For an arbitrary sequence $(r_n)_{n\geq 1}$ define monotone sequences $\bar{r}_n = \sup_{k\geq n} r_k$ and $\underline{r}_n = \inf_{k\geq n} r_k$. Since $r_t^{\underline{r}_n} - r_t^r \leq r_t^{r_n} - r_t^r \leq r_t^{\bar{r}_n} - r_t^r$, we have

$$0 \leq \sup_{0\leq t\leq T} |r_t^{r_n} - r_t^r| \leq \sup_{0\leq t\leq T} \left|r_t^{\underline{r}_n} - r_t^r\right| + \sup_{0\leq t\leq T} |r_t^{\bar{r}_n} - r_t^r| .$$

By virtue of the first part of the proof, the terms on the right-hand side converge to 0 as $n \to \infty$, which proves (4.46). The verification of (4.47) is easy using the representation formula (4.5) for X and (4.46). ☐

Lemma 4.4.2. *Let $(t_n, r_n, x_n)_{n\geq 1}$ be a sequence in $\mathcal{C}$ converging to $(t, r, x) \in \overline{\mathcal{C}} \cap \mathcal{O}$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \tau_*(t_n, r_n, x_n) = \tau_*(t, r, x), \qquad \mathsf{P}\text{-}a.s.$$

Proof. The proof relies on known facts from the theory of Markov processes, which we summarise in Chapter 2 Section 1.3. Proposition 4.3.6 and Lemma 4.4.1 imply that Assumptions 2.1.4 and 2.1.5 are satisfied for $\mathcal{K} = \mathcal{D} \cap \mathcal{O}$. It is also immediate to see that $\sigma_{\mathcal{D}} = \sigma_{\mathcal{K}}$ P-a.s. with $\sigma_{\mathcal{K}}$ defined in (2.16).

The continuity of trajectories of (r, X) means that the process cannot jump instantaneously to the stopping set $\mathcal{D}$ when starting from $\mathcal{C}$, so $\mathsf{P}_{\hat{t},\hat{r},\hat{x}}(\tau_* = \sigma_{\mathcal{D}}) = 1$ for any $(\hat{t}, \hat{r}, \hat{x}) \in \mathcal{C}$. When $(\hat{t}, \hat{r}, \hat{x}) \in \partial\mathcal{C}$, by construction we have $\tau_*(\hat{t}, \hat{r}, \hat{x}) = 0$, P-a.s., and, using Proposition 4.3.6, $\sigma_{\mathcal{D}}(\hat{t}, \hat{r}, \hat{x}) = 0$, P-a.s. Recalling that $\overline{\mathcal{C}} \cap \mathcal{O} = \mathcal{C} \cup \partial\mathcal{C}$, the claim then follows from Proposition 2.1.9. ☐

Next we provide gradient estimates based on probabilistic arguments.

Proposition 4.4.3. *Let $\mathcal{K} \subset \mathcal{O}$ be a compact set with non-empty interior. There is $L = L(\mathcal{K}) > 0$ such that for any $(t, r, x) \in (\text{int}(\mathcal{K}) \setminus \partial\mathcal{C})$ we have*

$$v_x(t, r, x) = -\mathsf{E}_{t,r,x}\left[\mathbb{1}_{\{X_{\tau_*}\leq K\}} e^{\sigma B_{\tau_*} - \frac{\sigma^2}{2}\tau_*}\right], \tag{4.48}$$

$$0 \geq v_t(t, r, x) \geq -L\, \mathsf{E}_{t,r,x}\left[e^{-\int_0^{\tau_{\mathcal{K}}} r_s^r ds} \mathbb{1}_{\{\tau_{\mathcal{K}}\leq\tau_*\}}\right], \tag{4.49}$$

where $\tau_{\mathcal{K}} := \inf\{s \geq 0 \,:\, (t + s, r_s, X_s) \notin \text{int}(\mathcal{K})\}$.

Proof. Fix $(t, r, x) \in (\text{int}(\mathcal{K}) \setminus \partial\mathcal{C})$. Recall that $\mathcal{D} \subset [0, T] \times \mathcal{I} \times [0, K]$. If $(t, r, x) \in \text{int}(\mathcal{D})$ then (4.48) follows easily from $v(t, r, x) = K - x$ and $v_t(t, r, x) = 0$. Assume $(t, r, x) \in \mathcal{C}$ and notice that $\tau_* = \sigma_{\mathcal{D}}$, $\mathsf{P}_{t,r,x}$-a.s. We split the proof into two parts.

(*Proof of* (4.48)) For all sufficiently small $\varepsilon > 0$ we have $(t, r, x+\varepsilon) \in \mathcal{C}$. From now on, consider such ε. To simplify notation let $\sigma_\mathcal{D} := \sigma_\mathcal{D}(t, r, x)$. Using that $\sigma_\mathcal{D}$ is admissible and sub-optimal for $v(t, r, x+\varepsilon)$ we get

$$\begin{aligned}
&v(t, r, x+\varepsilon) - v(t, r, x) \\
&\geq \mathsf{E}\left[e^{-\int_0^{\sigma_\mathcal{D}} r_s^r ds}\left(\left(K - (x+\varepsilon)X_{\sigma_\mathcal{D}}^{r,1}\right)^+ - \left(K - xX_{\sigma_\mathcal{D}}^{r,1}\right)^+\right)\right] \\
&\geq \mathsf{E}\left[e^{-\int_0^{\sigma_\mathcal{D}} r_s^r ds}\mathbb{1}_{\{X_{\sigma_\mathcal{D}}^{r,x}\leq K\}}\left(xX_{\sigma_\mathcal{D}}^{r,1} - (x+\varepsilon)X_{\sigma_\mathcal{D}}^{r,1}\right)\right] = -\varepsilon\mathsf{E}\left[\mathbb{1}_{\{X_{\sigma_\mathcal{D}}^{r,x}\leq K\}}e^{\sigma B_{\sigma_\mathcal{D}} - \frac{\sigma^2}{2}\sigma_\mathcal{D}}\right].
\end{aligned}$$

Dividing the above expression by ε and taking limits as $\varepsilon \to 0$ we get

$$v_x(t, r, x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(v(t, r, x+\varepsilon) - v(t, r, x)\right) \geq -\mathsf{E}\left[\mathbb{1}_{\{X_{\sigma_\mathcal{D}}^{r,x}\leq K\}}e^{\sigma B_{\sigma_\mathcal{D}} - \frac{\sigma^2}{2}\sigma_\mathcal{D}}\right]. \tag{4.50}$$

For the reverse inequality we use that $\sigma_\mathcal{D}$ is admissible and sub-optimal for $v(t, r, x-\varepsilon)$:

$$\begin{aligned}
v(t, r, x) - v(t, r, x-\varepsilon) &\leq \mathsf{E}\left[e^{-\int_0^{\sigma_\mathcal{D}} r_s^r ds}\left(\left(K - xX_{\sigma_\mathcal{D}}^{r,1}\right)^+ - \left(K - (x-\varepsilon)X_{\sigma_\mathcal{D}}^{r,1}\right)^+\right)\right] \\
&\leq -\varepsilon\,\mathsf{E}\left[\mathbb{1}_{\{X_{\sigma_\mathcal{D}}^{r,x-\varepsilon}\leq K\}}e^{\sigma B_{\sigma_\mathcal{D}} - \frac{\sigma^2}{2}\sigma_\mathcal{D}}\right] \leq -\varepsilon\,\mathsf{E}\left[\mathbb{1}_{\{X_{\sigma_\mathcal{D}}^{r,x}\leq K\}}e^{\sigma B_{\sigma_\mathcal{D}} - \frac{\sigma^2}{2}\sigma_\mathcal{D}}\right],
\end{aligned}$$

where in the last inequality we used that $X_s^{r,x-\varepsilon} < X_s^{r,x}$, $s \geq 0$. Divide the above expression by ε and take limits as $\varepsilon \to 0$:

$$v_x(t, r, x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(v(t, r, x) - v(t, r, x-\varepsilon)\right) \leq -\mathsf{E}\left[\mathbb{1}_{\{X_{\sigma_\mathcal{D}}^{r,x}\leq K\}}e^{\sigma B_{\sigma_\mathcal{D}} - \frac{\sigma^2}{2}\sigma_\mathcal{D}}\right]. \tag{4.51}$$

Now (4.50) and (4.51) imply (4.48).

(*Proof of* (4.49)) The upper bound $v_t(t, r, x) \leq 0$ follows from the monotonicity of v in t (Proposition 4.3.1). For all sufficiently small $\varepsilon > 0$ we have $(t+\varepsilon, r, x) \in \mathcal{K}\cap\mathcal{C}$ and $\tau_\mathcal{K} := \tau_\mathcal{K}(t, r, x) \leq T - t - \varepsilon$. From now on, consider such ε. Denote $\sigma_\mathcal{D} := \sigma_\mathcal{D}(t, r, x)$. Thanks to the choice of ε, the stopping time $\eta := \sigma_\mathcal{D}\wedge\tau_\mathcal{K}$ is admissible for $v(t+\varepsilon, r, x)$. Using the (super)martingale property of v (see (4.11)–(4.12)) we get

$$\begin{aligned}
&v(t+\varepsilon, r, x) - v(t, r, x) \\
&\geq \mathsf{E}\left[e^{-\int_0^{\eta} r_s^r ds}\left(v(t+\varepsilon+\eta, r_\eta^r, X_\eta^{r,x}) - v(t+\eta, r_\eta^r, X_\eta^{r,x})\right)\right] \\
&= \mathsf{E}\left[e^{-\int_0^{\tau_\mathcal{K}} r_s^r ds}\left(v(t+\varepsilon+\tau_\mathcal{K}, r_{\tau_\mathcal{K}}^r, X_{\tau_\mathcal{K}}^{r,x}) - v(t+\tau_\mathcal{K}, r_{\tau_\mathcal{K}}^r, X_{\tau_\mathcal{K}}^{r,x})\right)\mathbb{1}_{\{\tau_\mathcal{K}<\sigma_\mathcal{D}\}}\right],
\end{aligned} \tag{4.52}$$

where the equality follows from $v(t+\varepsilon+\sigma_\mathcal{D}, r_{\sigma_\mathcal{D}}^r, X_{\sigma_\mathcal{D}}^{r,x}) = v(t+\sigma_\mathcal{D}, r_{\sigma_\mathcal{D}}^r, X_{\sigma_\mathcal{D}}^{r,x}) = K - X_{\sigma_\mathcal{D}}^{r,x}$ on $\{\tau_\mathcal{K} \geq \sigma_\mathcal{D}\}$ since $t \mapsto b(t, r)$ is non-decreasing (Proposition 4.3.5). Let $\mathcal{K}^\delta = \{(t+s, r, x) : (t, r, x) \in \mathcal{K}$ and $s \in [0, \delta]\}$. Fix a sufficiently small $\delta > 0$ so that this set is contained in $\mathcal{O}$ and set L equal to the Lipschitz constant for v on $\mathcal{K}^\delta$ (c.f. Proposition 4.3.3). Since

$(t+\tau_{\mathcal{K}}, r^r_{\tau_{\mathcal{K}}}, X^{r,x}_{\tau_{\mathcal{K}}}) \in \partial\mathcal{K}$, we have $(t+\varepsilon+\tau_{\mathcal{K}}, r^r_{\tau_{\mathcal{K}}}, X^{r,x}_{\tau_{\mathcal{K}}}) \in \mathcal{K}^\delta$ for any $\varepsilon < \delta$. Using the Lipschitz continuity of v, we bound (4.52) from below by

$$-\varepsilon\, L\, \mathsf{E}\left[e^{-\int_0^{\tau_{\mathcal{K}}} r^r_s ds}\mathbb{1}_{\{\tau_{\mathcal{K}}<\sigma_{\mathcal{D}}\}}\right].$$

Dividing by ε and taking the limit $\varepsilon \to 0$ completes the proof of (4.49). □

We are now ready to claim that the value function is globally continuously differentiable on $\mathcal{O}$.

Theorem 4.4.4. *We have $v \in C^1(\mathcal{O})$.*

Proof. It suffices to show that the value function has continuous partial derivatives across the stopping boundary, that is

$$\lim_{n\to\infty} v_t(t_n, r_n, x_n) = \lim_{n\to\infty} v_r(t_n, r_n, x_n) = 0, \tag{4.53}$$

$$\lim_{n\to\infty} v_x(t_n, r_n, x_n) = -1, \tag{4.54}$$

for any sequence (t_n, r_n, x_n) in $\mathcal{C}$ converging to $(t_0, r_0, x_0) \in \partial\mathcal{C}$ as $n \to \infty$. Fix such a sequence and denote $\tau_n = \tau_*(t_n, r_n, x_n)$.

Convergence of v_x. Note that $\mathsf{P}_{t_n,r_n,x_n}(X_{\tau_n} = K,\, \tau_n < T-t_n) = 0$ (Proposition 4.3.4) and $\mathsf{P}_{t_n,r_n,x_n}(X_{\tau_n} = K,\, \tau_n = T-t_n) \le \mathsf{P}_{t_n,r_n,x_n}(X_{T-t_n} = K) = 0$ (the final equality can be shown by arguments as in (4.34)). From Proposition 4.4.3 we therefore have

$$v_x(t_n, r_n, x_n) = -\mathsf{E}\left[\mathbb{1}_{\{X^{r_n,x_n}_{\tau_n}<K\}} e^{\sigma B_{\tau_n} - \frac{\sigma^2}{2}\tau_n}\right].$$

From Lemma 4.4.2, we obtain $\lim_{n\to\infty}\tau_n = 0$ P-a.s. We know from $(t_0, r_0, x_0) \in \partial\mathcal{C}$ that $x_0 < K$. Lemma 4.4.1 and the continuity of trajectories of (r, X) imply the convergence $\mathbb{1}_{\{X^{r_n,x_n}_{\tau_n}<K\}} \to \mathbb{1}_{\{x_0<K\}} = 1$ as $n \to \infty$. An application of the dominated convergence theorem completes the proof of (4.54).

Convergence of v_t. Let $\mathcal{K}$ be a closed ball centred on (t_0, r_0, x_0) and contained in $\mathcal{O}$. With no loss of generality (by discarding a finite number of initial elements of the sequence) we assume that $(t_n, r_n, x_n) \in int(\mathcal{K})$ for all $n \ge 1$. Let

$$\tau^n_{\mathcal{K}} := \inf\{s \ge 0 : (t_n + s, r^{r_n}_s, X^{r_n,x_n}_s) \notin \mathcal{K}\}, \quad n \ge 0$$

and notice, in particular, that $\mathsf{P}(\tau^0_{\mathcal{K}} > 0) = 1$. The boundary $\partial\mathcal{K}$ is regular for $\mathcal{O}\setminus\mathcal{K}$ and (t, r, X) by the same reasoning as in the proof of Proposition 4.3.6. Repeating arguments from the proof of Lemma 4.4.2 shows that $\tau_{\mathcal{K}}^n \to \tau_{\mathcal{K}}^0$, P-a.s. Fix $\varepsilon \in (0,1)$. Since $\mathsf{P}(\tau_{\mathcal{K}}^0 > 0) = 1$, there exists $\delta > 0$ such that $\mathsf{P}(\tau_{\mathcal{K}}^0 > \delta) \geq 1 - \varepsilon$. From inequality (4.49), we get

$$\begin{aligned} 0 \geq v_t(t_n, r_n, x_n) &\geq -L\,\mathsf{E}\left[e^{-\int_0^{\tau_{\mathcal{K}}^n} r_s^{r_n} ds} \mathbb{1}_{\{\tau_{\mathcal{K}}^n \leq \tau_n\}}\right] \\ &= -L\,\mathsf{E}\left[e^{-\int_0^{\tau_{\mathcal{K}}^n} r_s^{r_n} ds} \left(\mathbb{1}_{\{\tau_{\mathcal{K}}^n \leq \tau_n\} \cap \{\tau_{\mathcal{K}}^n \geq \delta\}} + \mathbb{1}_{\{\tau_{\mathcal{K}}^n \leq \tau_n\} \cap \{\tau_{\mathcal{K}}^n < \delta\}}\right)\right] \\ &\geq -L\,\mathsf{E}\left[e^{-\int_0^{\tau_{\mathcal{K}}^n} r_s^{r_n} ds} \left(\mathbb{1}_{\{\tau_n \geq \delta\}} + \mathbb{1}_{\{\tau_{\mathcal{K}}^n < \delta\}}\right)\right]. \end{aligned} \tag{4.55}$$

Using that $|r_{t \wedge \tau_{\mathcal{K}}^n}|$ is bounded by some constant $r_{\mathcal{K}}$ for every n, we have

$$0 \geq v_t(t_n, r_n, x_n) \geq -Le^{r_{\mathcal{K}} T} \left(\mathsf{P}\left(\tau_n \geq \delta\right) + \mathsf{P}\left(\tau_{\mathcal{K}}^n < \delta\right)\right). \tag{4.56}$$

Lemma 4.4.2 guarantees that $\tau_n \to 0$ P-a.s., so the first term converges to 0 as $n \to \infty$ by the dominated convergence theorem. Fatou's lemma gives a bound for the second term:

$$\limsup_{n\to\infty} \mathsf{P}\left(\tau_{\mathcal{K}}^n < \delta\right) \leq \mathsf{E}\Big[\limsup_{n\to\infty} \mathbb{1}_{\{\tau_{\mathcal{K}}^n < \delta\}}\Big] \leq \mathsf{E}\left[\mathbb{1}_{\{\tau_{\mathcal{K}^c}^0 \leq \delta\}}\right] \leq \varepsilon,$$

where we used that $\limsup_n 1_{A_n} = 1_{\limsup_n A_n}$ and the convergence of the stopping times. We obtain the convergence of v_t in (4.53) by sending $\varepsilon \to 0$.

Convergence of v_r. Consider a sequence $(t_n, r_n, x_n) \in \mathcal{C}$ converging to $(t_0, r_0, x_0) \in \partial\mathcal{C}$. Since $\partial\mathcal{C}$ is the boundary of $\mathcal{C}$ in $\mathcal{O}$, without loss of generality we can assume that

$$\{(r_n, x_n)\} \subset \text{int}(\mathcal{K}_0), \quad \text{with} \quad \mathcal{K}_0 := [r_a, r_b] \times [x_a, x_b] \subset (\underline{r}, \overline{r}) \times \mathbb{R}_+.$$

Denote $\mathcal{K}_0^T := [t_a, t_b] \times \mathcal{K}_0$. where $t_a = \inf_n t_n \geq 0$ and $t_b = \sup_n t_n < T$.

We know that $v_r \leq 0$ on $\mathcal{C}$ (Proposition 4.3.1). We will now develop a lower bound for v_r on $\mathcal{C} \cap \mathcal{K}_0^T$, which will allow us to show that $v_r(t_n, r_n, x_n) \to 0$ as $n \to \infty$. Let $\mathcal{K}^T = [\bar{t}_a, \bar{t}_b] \times \mathcal{K}$ be a bigger compact, with $\mathcal{K} := [\bar{r}_a, \bar{r}_b] \times [\bar{x}_a, \bar{x}_b] \subset (\underline{r}, \overline{r}) \times \mathbb{R}_+$, such that $\mathcal{K}_0^T \subset \text{int}(\mathcal{K}^T)$. For $(t, r, x) \in \mathcal{C} \cap \mathcal{K}_0^T$ we define

$$\tau_{\mathcal{K}}(t, r, x) := \inf\{s \geq 0 : (r_s^r, X_s^{r,x}) \notin \mathcal{K}\} \wedge (T - t).$$

By the monotonicity of $r \mapsto r_s^r$ and the explicit expression (4.5) for $X^{r,x}$ we have, for all $(r, x) \in \mathcal{K}_0$,

$$r_s^{r_a} \leq r_s^r \leq r_s^{r_b}, \quad \text{and} \quad X_s^{r_a,x} \leq X_s^{r,x} \leq X_s^{r_b,x}, \quad \mathsf{P}\text{-a.s.}$$

from which it is not hard to verify that for all $(t, r, x) \in \mathcal{C} \cap \mathcal{K}_0^T$

$$\tau_{\mathcal{K}}(t, r, x) \geq \widehat{\tau}_{\mathcal{K}} := \tau_{\mathcal{K}}(t_b, r_a, x_a) \wedge \tau_{\mathcal{K}}(t_b, r_a, x_b) \wedge \tau_{\mathcal{K}}(t_b, r_b, x_a) \wedge \tau_{\mathcal{K}}(t_b, r_b, x_b), \quad \mathsf{P}\text{-a.s.},$$

and $\widehat{\tau}_{\mathcal{K}} > 0$, P-a.s., as well.

Take $(t, r, x) \in \mathcal{C} \cap \mathrm{int}(\mathcal{K}_0^T)$. There is $\bar{\varepsilon} > 0$ such that $(t, r + \varepsilon, x) \in \mathcal{C} \cap \mathcal{K}_0^T$ for all $\varepsilon \in (0, \bar{\varepsilon}]$. Denote by τ_* the optimal stopping time for (t, r, x). For any $\varepsilon \in (0, \bar{\varepsilon}]$, we apply the (super)martingale properties of the value function (4.11)-(4.12) with the stopping time $\tau_* \wedge \widehat{\tau}_{\mathcal{K}}$:

$$
\begin{aligned}
0 &\geq v(t, r+\varepsilon, x) - v(t, r, x) \\
&\geq \mathsf{E}\Big[e^{-\int_0^{\tau_* \wedge \widehat{\tau}_{\mathcal{K}}} r_s^{r+\varepsilon} ds} v\big(t + (\tau_* \wedge \widehat{\tau}_{\mathcal{K}}), r^{r+\varepsilon}_{\tau_* \wedge \widehat{\tau}_{\mathcal{K}}}, X^{r+\varepsilon, x}_{\tau_* \wedge \widehat{\tau}_{\mathcal{K}}}\big) \\
&\qquad - e^{-\int_0^{\tau_* \wedge \widehat{\tau}_{\mathcal{K}}} r_s^{r} ds} v\big(t + (\tau_* \wedge \widehat{\tau}_{\mathcal{K}}), r^{r}_{\tau_* \wedge \widehat{\tau}_{\mathcal{K}}}, X^{r, x}_{\tau_* \wedge \widehat{\tau}_{\mathcal{K}}}\big) \Big] \\
&\geq \mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}} \leq \tau_*\}} \left(e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r+\varepsilon} ds} v(t + \widehat{\tau}_{\mathcal{K}}, r^{r+\varepsilon}_{\widehat{\tau}_{\mathcal{K}}}, X^{r+\varepsilon, x}_{\widehat{\tau}_{\mathcal{K}}}) - e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r} ds} v(t + \widehat{\tau}_{\mathcal{K}}, r^{r}_{\widehat{\tau}_{\mathcal{K}}}, X^{r, x}_{\widehat{\tau}_{\mathcal{K}}}) \right) \right] \\
&\quad + \mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}} > \tau_*\}} \left(e^{-\int_0^{\tau_*} r_s^{r+\varepsilon} ds} (K - X^{r+\varepsilon, x}_{\tau_*})^+ - e^{-\int_0^{\tau_*} r_s^{r} ds} (K - X^{r, x}_{\tau_*})^+ \right) \right] \\
&=: E_1 + E_2,
\end{aligned} \tag{4.57}
$$

where for the final inequality we used that $v(t+\tau_*, r^r_{\tau_*}, X^{r,x}_{\tau_*}) = (K - X^{r,x}_{\tau_*})^+$, P-a.s. Recalling that $r_s^{r+\varepsilon} \geq r_s^r$ and v is non-negative we have

$$
\begin{aligned}
E_1 &= \mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}} \leq \tau_*\}} e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r+\varepsilon} ds} \left(v(t + \widehat{\tau}_{\mathcal{K}}, r^{r+\varepsilon}_{\widehat{\tau}_{\mathcal{K}}}, X^{r+\varepsilon, x}_{\widehat{\tau}_{\mathcal{K}}}) - v(t + \widehat{\tau}_{\mathcal{K}}, r^{r}_{\widehat{\tau}_{\mathcal{K}}}, X^{r, x}_{\widehat{\tau}_{\mathcal{K}}}) \right) \right] \\
&\qquad - \mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}} \leq \tau_*\}} \left(e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r} ds} - e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r+\varepsilon} ds} \right) v(t + \widehat{\tau}_{\mathcal{K}}, r^{r}_{\widehat{\tau}_{\mathcal{K}}}, X^{r, x}_{\widehat{\tau}_{\mathcal{K}}}) \right] \\
&\geq -L\, \mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}} \leq \tau_*\}} e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r+\varepsilon} ds} \left(|r^{r+\varepsilon}_{\widehat{\tau}_{\mathcal{K}}} - r^{r}_{\widehat{\tau}_{\mathcal{K}}}| + |X^{r+\varepsilon, x}_{\widehat{\tau}_{\mathcal{K}}} - X^{r, x}_{\widehat{\tau}_{\mathcal{K}}}| \right) \right] \\
&\qquad - K\, \mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}} \leq \tau_*\}} \left(e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r} ds} - e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}} r_s^{r+\varepsilon} ds} \right) \right],
\end{aligned} \tag{4.58}
$$

where the second inequality comes from the local Lipschitz property of the value function ($L > 0$ is the constant from Proposition 4.3.3), and the function v is bounded by the strike price K from above.

We shall now use the differentiability of the diffusion flow (r_s^r) with respect to the parameter r in the sense of [93, Thm. 2.8.6]. Apart from other assumptions, this requires that the coefficients are globally Lipschitz. As we only consider (r, X) in a compact set $\mathcal{K}$, we construct a two dimensional diffusion $(\widetilde{r}, \widetilde{X})$ whose coefficients coincide with the coefficients of (r, X) on $\mathcal{K}$, are globally Lipschitz, continuously differentiable and with a polynomial growth. The process $(\widetilde{r}_s, \widetilde{X}_s)$ is indistinguishable from (r_s, X_s) on $\{s \leq \widehat{\tau}_{\mathcal{K}}\}$, i.e., on the set where it is of interest for the estimation of E_1 and E_2, so for the sake of readability we will write (r, X) in the estimates below (we use an analogous construction in the proof of Proposition 4.3.6, where full details are available).

By [93, Thm. 2.8.6], there is a measurable in (s,ω) process $(y^r_s(\omega))_{s\geq 0}$, depending on r, such that for any $q\geq 1$

$$\lim_{\varepsilon\downarrow 0}\left\|\sup_{s\in[0,T]}\left|\frac{r^{r+\varepsilon}_s-r^r_s}{\varepsilon}-y^r_s\right|\right\|_q=0\quad\text{and}\quad\lim_{\varepsilon\downarrow 0}\left\|\frac{r^{r+\varepsilon}_\cdot-r^r_\cdot}{\varepsilon}-y^r_\cdot\right\|^*_q=0,\tag{4.59}$$

where $\|Z\|_q=(\mathsf{E}[|Z|^q])^{1/q}$ and $\|Y_\cdot\|^*_q=\big(\mathsf{E}\big[\int_0^T|Y_s|^q ds\big]\big)^{1/q}$.

Fix $\frac{1}{p}+\frac{1}{q}+\frac{1}{w}=1$ for some $p\in(1,2]$. Recalling that $r^{r+\varepsilon}_s\geq r^r_s$ and using Hölder inequality yields

$$\begin{aligned}&\frac{1}{\varepsilon}\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^{r+\varepsilon}_s ds}|r^{r+\varepsilon}_{\widehat{\tau}_{\mathcal{K}}}-r^r_{\widehat{\tau}_{\mathcal{K}}}|\right]\\&\leq\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^{r+\varepsilon}_s ds}\left(\left|\frac{1}{\varepsilon}(r^{r+\varepsilon}_{\widehat{\tau}_{\mathcal{K}}}-r^r_{\widehat{\tau}_{\mathcal{K}}})-y^r_{\widehat{\tau}_{\mathcal{K}}}\right|+|y^r_{\widehat{\tau}_{\mathcal{K}}}|\right)\right]\\&\leq C_1^{1/p}\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\left(\left\|\frac{1}{\varepsilon}(r^{r+\varepsilon}_{\widehat{\tau}_{\mathcal{K}}}-r^r_{\widehat{\tau}_{\mathcal{K}}})-y^r_{\widehat{\tau}_{\mathcal{K}}}\right\|_q+\|y^r_{\widehat{\tau}_{\mathcal{K}}}\|_q\right)\underset{\varepsilon\downarrow 0}{\longrightarrow}C_1^{1/p}\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\,\|y^r_{\widehat{\tau}_{\mathcal{K}}}\|_q,\end{aligned}\tag{4.60}$$

where we used the estimate (4.3) in the last inequality and (4.59) to obtain the convergence.

To bound the last term on the right hand side of (4.58), we observe that

$$\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}\left(e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^r_s ds}-e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^{r+\varepsilon}_s ds}\right)\right]\leq\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^r_s ds}\int_0^{\widehat{\tau}_{\mathcal{K}}}(r^{r+\varepsilon}_s-r^r_s)ds\right].$$

We then apply Hölder inequality and the second limit in (4.59):

$$\begin{aligned}&\frac{1}{\varepsilon}\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^r_s ds}\int_0^{\widehat{\tau}_{\mathcal{K}}}(r^{r+\varepsilon}_s-r^r_s)ds\right]\\&\leq\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^r_s ds}\int_0^{\widehat{\tau}_{\mathcal{K}}}\left|\frac{1}{\varepsilon}(r^{r+\varepsilon}_s-r^r_s)-y^r_s\right|+|y^r_s|ds\right]\\&\leq C_1^{1/p}\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\left(\left\|\frac{1}{\varepsilon}(r^{r+\varepsilon}_\cdot-r^r_\cdot)-y^r_\cdot\right\|^*_q+\|y^r_\cdot\|^*_q\right)\underset{\varepsilon\downarrow 0}{\longrightarrow}C_1^{1/p}\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\,\|y^r_\cdot\|^*_q.\end{aligned}\tag{4.61}$$

By the explicit formula (4.5), we have $X^{r,x}_t=e^{\int_0^t r^r_s ds}\hat{X}^x_t$, where $\hat{X}^x_t:=xe^{\sigma B_t-\frac{1}{2}\sigma^2 t}$, and

$$0\leq X^{r+\varepsilon,x}_t-X^{r,x}_t\leq e^{\int_0^t r^{r+\varepsilon}_s ds}\hat{X}^x_t\int_0^t(r^{r+\varepsilon}_s-r^r_s)ds.$$

We proceed similarly as in (4.61) to obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\mathsf{E}\left[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}\leq\tau_*\}}e^{-\int_0^{\widehat{\tau}_{\mathcal{K}}}r^{r+\varepsilon}_s ds}|X^{r+\varepsilon,x}_{\widehat{\tau}_{\mathcal{K}}}-X^{r,x}_{\widehat{\tau}_{\mathcal{K}}}|\right]\\&\leq\|\hat{X}^x_{\widehat{\tau}_{\mathcal{K}}}\|_p\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\left(\left\|\frac{1}{\varepsilon}(r^{r+\varepsilon}_\cdot-r^r_\cdot)-y^r_\cdot\right\|^*_q+\|y^r_\cdot\|^*_q\right)\\&\underset{\varepsilon\downarrow 0}{\longrightarrow}\|\hat{X}^x_{\widehat{\tau}_{\mathcal{K}}}\|_p\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\,\|y^r_\cdot\|^*_q.\end{aligned}\tag{4.62}$$

Similar arguments as above enable us to derive a lower bound for E_2:

$$\begin{aligned}\frac{1}{\varepsilon}E_2 &= \frac{1}{\varepsilon}\mathsf{E}\Big[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}>\tau_*\}}\Big(\Big(Ke^{-\int_0^{\tau_*} r_s^{r+\varepsilon}ds} - \hat{X}^x_{\tau_*}\Big)^+ - \Big(Ke^{-\int_0^{\tau_*} r_s^{r}ds} - \hat{X}^x_{\tau_*}\Big)^+\Big)\Big]\\
&\geq -\frac{1}{\varepsilon}K\,\mathsf{E}\Big[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}>\tau_*\}}\Big(e^{-\int_0^{\tau_*} r_s^{r}ds} - e^{-\int_0^{\tau_*} r_s^{r+\varepsilon}ds}\Big)\Big]\\
&\geq -\frac{1}{\varepsilon}K\,\mathsf{E}\Big[\mathbb{1}_{\{\widehat{\tau}_{\mathcal{K}}>\tau_*\}}e^{-\int_0^{\tau_*} r_s^{r}ds}\int_0^{\tau_*}(r_s^{r+\varepsilon}-r_s^r)ds\Big]\\
&\geq -KC_1^{1/p}\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}>\tau_*)^{\frac{1}{w}}\Big(\Big\|\frac{1}{\varepsilon}(r_\cdot^{r+\varepsilon}-r_\cdot^r)-y_\cdot^r\Big\|_q^* + \Big(\mathsf{E}\Big[\int_0^{\tau_*}|y_s^r|^q ds\Big]\Big)^{1/q}\Big)\\
&\underset{\varepsilon\downarrow 0}{\longrightarrow} -KC_1^{1/p}\mathsf{P}(\widehat{\tau}_{\mathcal{K}}>\tau_*)^{\frac{1}{w}}\Big(\mathsf{E}\Big[\int_0^{\tau_*}|y_s^r|^q ds\Big]\Big)^{1/q},\end{aligned} \tag{4.63}$$

where in the first inequality we used the Lipschitz property of $z\mapsto(z-\hat{X}^x_t(\omega))^+$ for any $\omega\in\Omega$.

Combining (4.60)–(4.63) gives a lower bound for v_r on $\mathcal{C}\cap\mathcal{K}^T$:

$$\begin{aligned}0 &\geq v_r(t,r,x)\\
&\geq -L\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\Big(C_1^{1/p}\|y^r_{\widehat{\tau}_{\mathcal{K}}}\|_q + \|\hat{X}^x_{\widehat{\tau}_{\mathcal{K}}}\|_p\,\|y_\cdot^r\|_q^*\Big) - K\,\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*)^{\frac{1}{w}}\,C_1^{1/p}\|y_\cdot^r\|_q^*\\
&\quad - KC_1^{1/p}\mathsf{P}(\widehat{\tau}_{\mathcal{K}}>\tau_*)^{\frac{1}{w}}\Big(\mathsf{E}\Big[\int_0^{\tau_*}|y_s^r|^q ds\Big]\Big)^{1/q}.\end{aligned} \tag{4.64}$$

By [93, Thm. 2.8.8] and standard diffusion estimates [93, Cor. 2.5.10] the norms of y^r and $\hat{X}^x$ above are bounded uniformly for $(t,r,x)\in\mathcal{K}^T\cap\mathcal{C}$ (recall that $\tau_*=\tau_*(t,r,x)$). Now take $(t,r,x)=(t_n,r_n,x_n)$ in (4.64). Since $\hat{\tau}_{\mathcal{K}}>0$ P-a.s. and $\lim_{n\to\infty}\tau_*(t_n,r_n,x_n)=0$ P-a.s. by Lemma 4.4.2, the dominated convergence theorem gives that the first two terms of (4.64) tend to zero as $n\to\infty$ due to $\mathsf{P}(\widehat{\tau}_{\mathcal{K}}\leq\tau_*(t_n,r_n,x_n))\to 0$ and the last term converges to zero because

$$\lim_{n\to 0}\mathsf{E}\Big[\int_0^{\tau_*(t_n,r_n,x_n)}|y_s^{r_n}|^q ds\Big]=0$$

and the mapping $r\mapsto y^r$ is continuous in the norm $\|\cdot\|_q^*$, see [93, Thm. 2.8.6]. This concludes the proof. □

It is worth noticing that the proof of the above result combines a number of steps that may be of independent interest. In particular, we prove local Lipschitz continuity of v (Proposition 4.3.3) and the regularity of the stopping boundary in the sense of diffusions. The latter gives the continuity of optimal stopping times τ_* as functions of the initial state, which plays a crucial role in the proof of Theorem 4.4.4.

4.5 Continuity of the stopping boundary and Dynkin's formula

Preliminary right/left-continuity properties of the stopping boundary $(t, x) \mapsto c(t, x)$ illustrated above follow from its monotonicity and the closedness of the stopping set $\mathcal{D}$ (see Proposition 4.3.4). However, thanks to the C^1 regularity of the value function v, we can also prove joint continuity of the stopping boundary in both variables. For this we require local Hölder continuity of the derivatives of the coefficients in the dynamics of the short rate r.

Assumption 4.5.1. *The functions α and β in (4.2) have first and second order derivatives, respectively, Hölder continuous on any compact subset of $\mathcal{I}$.*

Note that this assumption is satisfied by CIR model. It strengthens Assumption 4.2.1(ii) by requiring that the derivatives are not only locally continuous but also locally Hölder continuous. This technical requirement is satisfied by many popular short rate models. The joint continuity of optimal stopping boundaries depending on multiple variables has not been proved with probabilistic techniques before, so the next result is of independent mathematical interest.

Proposition 4.5.2. *Under Assumption 4.5.1, the function $c : [0, T) \times \mathbb{R}_+ \to [0, \infty)$ is continuous.*

Proof. Since $c(t, x) = \bar{r}$ on $[0, T) \times [K, \infty)$, it remains to prove the continuity at $(t_0, x_0) \in (0, T) \times (0, K]$. It is known from Proposition 4.3.4 that $t \mapsto c(t, x_0)$ is non-increasing and right-continuous at t_0, and $x \mapsto c(t_0, x)$ is non-decreasing and left-continuous at x_0.

We first show that $x \mapsto c(t_0, x)$ is right continuous at x_0. It is obvious for $x_0 = K$ since $c(t_0, x) = \bar{r}$ for $x \geq K$. We proceed with an argument for $x_0 < K$. Assume, by contradiction, that $c(t_0, x_0+) > c(t_0, x_0)$, so there exist r_1, r_2 such that $c(t_0, x_0+) > r_2 > r_1 > c(t_0, x_0)$. Let $R := (r_1, r_2) \times (x_0, x_1)$ for some $x_1 \in (x_0, K)$ and $R_0 := (r_1, r_2) \times \{x_0\}$. From the monotonicity of $c(t, x)$, we have $\{t_0\} \times R \subset \mathcal{C}$ and $\{t_0\} \times R_0 \subset \mathcal{D}$. Let u be a function defined on $\overline{R}$ and satisfying

$$\begin{aligned} (\mathcal{L} - r)u(r, x) &= -v_t(t_0, r, x), \qquad (r, x) \in R, \\ u(r, x) &= v(t_0, r, x), \qquad (r, x) \in \partial R. \end{aligned} \tag{4.65}$$

Thanks to [69, Thm. 10, p. 72] we know that $(r, x) \mapsto v_t(t_0, r, x)$ is C^1 on R with Hölder continuous derivatives. Since the coefficients of (4.14) have Hölder continuous first derivatives, there is a unique classical solution $u(r, x)$ of the above PDE (which is of elliptic type) and

$u \in C^3(R) \cap C(\overline{R})$ [69, Thm. 19 and 20, p. 87]. From (4.45), the function $(r,x) \mapsto v(t_0,r,x)$ satisfies (4.65), so, by uniqueness, $u = v$ on $\overline{R}$ and $u \in C^1(\overline{R})$ by Theorem 4.4.4.

We differentiate the PDE in (4.65) with respect to r and obtain

$$\frac{1}{2}\sigma^2 x^2 u_{rxx}(r,x) = -\mathcal{L}_1 u_r(r,x) - \mathcal{L}_2 u_x(r,x) - x u_x(r,x) - v_{tr}(t_0,r,x) + u(r,x), \quad (r,x) \in R, \tag{4.66}$$

where

$$\begin{aligned}\mathcal{L}_1 f :=& \frac{1}{2}\beta^2(r) f_{rr} + (\beta(r)\beta'(r) + \alpha(r)) f_r + (\alpha'(r) - r) f \\ \mathcal{L}_2 f :=& \rho\sigma\beta(r) x f_{rr} + (\rho\sigma\beta'(r) + rx) f_r.\end{aligned}$$

Let ϕ be a C^∞ function with compact support on (r_1,r_2) such that $\int_{r_1}^{r_2} \phi(r)dr = 1$ and for $x \in (x_0,x_1)$ define

$$F_\phi(x) = -\int_{r_1}^{r_2} u_{xx}(r,x)\phi'(r)dr.$$

Multiply (4.66) by $\frac{2}{\sigma^2 x^2}\phi(r)$ and integrate over (r_1,r_2):

$$\begin{aligned}\int_{r_1}^{r_2} u_{rxx}(x,r)\phi(r)dr =& -\int_{r_1}^{r_2} \frac{2}{\sigma^2 x^2}\phi(r)\big[\mathcal{L}_1 u_r(r,x) + \mathcal{L}_2 u_x(r,x)\big]dr \\ & -\int_{r_1}^{r_2} \frac{2}{\sigma^2 x}\phi(r) u_x(r,x)dr - \int_{r_1}^{r_2} \frac{2}{\sigma^2 x^2}\phi(r) v_{tr}(t_0,r,x)dr \\ & + \int_{r_1}^{r_2} \frac{2}{\sigma^2 x^2}\phi(r) u(r,x)dr.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}F_\phi(x) =& -\int_{r_1}^{r_2} \frac{2}{\sigma^2 x^2}\big[u_r(r,x)\mathcal{L}_1^*\phi(r)dr + u_x(r,x)\mathcal{L}_2^*\phi(r)\big]dr - \int_{r_1}^{r_2} \frac{2}{\sigma^2 x}\phi(r)u_x(r,x)dr \\ & + \int_{r_1}^{r_2} \frac{2}{\sigma^2 x^2}\phi'(r)v_t(t_0,r,x)dr + \int_{r_1}^{r_2} \frac{2}{\sigma^2 x^2}\phi(r)u(r,x)dr,\end{aligned} \tag{4.67}$$

where $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ are adjoint operators to $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The expression above involves only u and its first derivatives, which are continuous by Theorem 4.4.4. We take the limit $x \to x_0$ in (4.67) and notice that $u_r(r,x_0) = v_r(t_0,r,x_0) = v_t(t_0,r,x_0) = 0$, $u_x(r,x_0) = v_x(t_0,r,x_0) = -1$ and $u(r,x_0) = K - x_0$. Thus,

$$\lim_{x \downarrow x_0} F_\phi(x) = \int_{r_1}^{r_2} \frac{2}{\sigma^2 x_0}\phi(r)dr + \int_{r_1}^{r_2} \frac{2}{\sigma^2 x_0^2}\phi(r)(K - x_0)dr = \frac{2K}{\sigma^2 x_0^2} > 0,$$

where we also use that $\int_{r_1}^{r_2} \mathcal{L}_2^* \phi(r)dr = 0$. Since $x \mapsto F_\phi(x)$ is continuous on (x_0, x_1) and $\lim_{x \downarrow x_0} F_\phi(x) > 0$, we have $F_\phi(x) > 0$ on $(x_0, x_0 + \varepsilon)$ for any sufficiently small $\varepsilon > 0$. Using additionally that u is $C^1(\overline{R})$, we perform the following integration

$$\begin{aligned} 0 < \int_{x_0}^{x_0+\varepsilon} \int_{x_0}^{y} F_\phi(x)dxdy &= -\int_{r_1}^{r_2} \int_{x_0}^{x_0+\varepsilon} \int_{x_0}^{y} u_{xx}(r,x)dxdy\, \phi'(r)dr \\ &= -\int_{r_1}^{r_2} \int_{x_0}^{x_0+\varepsilon} (u_x(r,y)+1)dy\, \phi'(r)dr \\ &= -\int_{r_1}^{r_2} (u(r,x_0+\varepsilon) - (K-x_0) + \varepsilon)\phi'(r)dr \\ &= \int_{r_1}^{r_2} u_r(r,x_0+\varepsilon)\phi(r)dr, \end{aligned}$$

where we have used Fubini's theorem in the first equality, $u_x(r, x_0) = -1$ in the second equality, $u(r, x_0) = K - x_0$ in the third equality, and the integration by parts in the last equality. As the above inequality holds for an arbitrary smooth function ϕ with a compact support in (r_1, r_2), we must have $u_r(r, x_0+\varepsilon) = v_r(t_0, r, x_0+\varepsilon) > 0$ almost everywhere on (r_1, r_2). This contradicts that $r \mapsto v(t_0, r, x_0+\varepsilon)$ is a non-increasing function (see Proposition 4.3.1), hence $x \mapsto c(t, x)$ is continuous.

We turn our attention to the left-continuity of $t \mapsto c(t, x_0)$ at t_0 (the right-continuity has already been established in Proposition 4.3.4). Assume, by contradiction, that the left-continuity fails at t_0. Since $t \mapsto c(t, x_0)$ is non-increasing, there exist r_1, r_2 such that $c(t_0-, x_0) > r_2 > r_1 > c(t_0, x_0)$. By the continuity of $x \mapsto c(t_0, x)$ at x_0 and the monotonicity of $c(t, x)$, there is $x_1 \in (x_0, K)$ such that $r_1 > c(t_0, x_1) \geq c(t_0, x_0)$. Hence, for any sequence $t_n \uparrow t_0$, we have

$$c(t_n, x_1) \geq c(t_n, x_0) \geq c(t_0-, x_0) > r_2 > r_1 > c(t_0, x_1) \geq c(t_0, x_0),$$

so that

$$\begin{aligned} R &:= (t_1, t_0) \times (r_1, r_2) \times (x_0, x_1) \subset \mathcal{C}, \\ R_{t_0} &:= \{t_0\} \times (r_1, r_2) \times (x_0, x_1) \subset \mathcal{D}. \end{aligned}$$

Consider a PDE

$$\begin{aligned} &w_t(t,r,x) + (\mathcal{L} - r)w(t,r,x) = 0, \qquad (t,r,x) \in R, \\ &w(t,r,x) = v(t,r,x), \qquad (t,r,x) \in \partial_p R, \end{aligned} \tag{4.68}$$

where $\partial_p R$ denotes the parabolic boundary of R. By [69, Thm. 6, p. 65], Equation (4.68) admits a unique classical solution w, which coincides with v on $\overline{R}$. This also implies that $w \in C^1(\overline{R})$ by Theorem 4.4.4.

Let ϕ_1 be a C^∞ function with compact support in (x_0, x_1) and ϕ_2 be a C^∞ function with compact support in (r_1, r_2) such that $\int_{x_0}^{x_1} \phi_1(x)dx = \int_{r_1}^{r_2} \phi_2(r)dr = 1$. Fixing $t = t_n \in (t_1, t_0)$ from the sequence $t_n \uparrow t_0$, we multiply (4.68) by $\phi_1(x)\phi_2(r)$ and integrate over $(r_1, r_2) \times (x_0, x_1)$:

$$\int_{r_1}^{r_2}\int_{x_0}^{x_1} \phi_1(x)\phi_2(r)\big\{w_t(t_n, r, x) + (\mathcal{L} - r)w(t_n, r, x)\big\}dxdr = 0.$$

Integration by parts gives

$$\int_{r_1}^{r_2}\int_{x_0}^{x_1} \phi_1(x)\phi_2(r)w_t(t_n, r, x)dxdr + \int_{r_1}^{r_2}\int_{x_0}^{x_1} w(t_n, r, x)(\mathcal{L}^* - r)\phi_1(x)\phi_2(r)dxdr = 0, \tag{4.69}$$

where $\mathcal{L}^*$ is the adjoint operator for $\mathcal{L}$. When $n \to \infty$, the first integral vanishes since $w \in C^1(\overline{R_t})$ and $w_t = v_t = 0$ on R_{t_0}. By the dominated convergence theorem, (4.69) reads

$$0 = \int_{r_1}^{r_2}\int_{x_0}^{x_1} w(t_0, r, x)(\mathcal{L}^* - r)\phi_1(x)\phi_2(r)dxdr = \int_{r_1}^{r_2}\int_{x_0}^{x_1} \phi_1(x)\phi_2(r)(\mathcal{L} - r)(K - x)dxdr$$
$$= \int_{r_1}^{r_2}\int_{x_0}^{x_1} \phi_1(x)\phi_2(r)(-rK)dxdr = \int_{r_1}^{r_2} \phi_2(r)(-rK)dr$$

where we integrate by parts and use that $v(t, r, x) = (K - x)$ on R_{t_0} for the second equality. We obtain a contradiction because the last integral is strictly negative.

Having established the continuity in t and x separately, the monotonicity of c allows us to conclude the continuity of $(t, x) \mapsto c(t, x)$ at (t_0, x_0) (see, e.g., [92]). □

Summarising, we have $v \in C^1(\mathcal{O}) \cap C^{1,2,2}(\mathcal{C}) \cap C^{1,2,2}(\mathcal{D})$, and the optimal stopping boundary c is continuous. This is not sufficient to apply the change of variable formula developed in [112] which is often used in optimal stopping literature to establish Itô's formula for the value function. Indeed, since [112] deals with functions that are not necessarily C^1, it requires that $t \mapsto c(t, X_t)$ be a semi-martingale, so that the local time on the stopping boundary is well-defined. While we were unable to prove it for our optimal boundary, we can instead take advantage of the continuous differentiability of our value function and use a generalisation of Itô's formula from Chapter 5 which only requires the monotonicity of the boundary. Notice that, interestingly, we need not control the second order spatial derivatives near $\partial\mathcal{C}$ in order to apply results from [25]. We do however need to ensure that both boundary points of the set $\mathcal{I}$ are non-attainable, because we have not proven that the derivatives $v_t(t, \underline{r}, x)$, $v_r(t, \underline{r}, x)$ and $v_x(t, \underline{r}, x)$, understood as the limit as $r \to \underline{r}$, are well-defined.

Assumption 4.5.3. *The lower boundary point $\underline{r}$ is non-attainable by the process (r_t). In particular, under Assumptions 4.2.1-(i) we require $k\theta > \sigma^2/2$.*

We will first prove an auxiliary lemma whose assertions are used in the proof of Proposition 4.5.5, and also in Section 4.8 to show admissibility of the hedging strategy.

Lemma 4.5.4. *For any compact set $\mathcal{K} \subset \mathcal{I} \times \mathbb{R}_+$, and $p \in [1,2]$, we have*

$$\sup_{(r,x)\in\mathcal{K}} \sup_{t\in[0,T]} \mathsf{E}_{r,x}\Big[\sup_{0\le s\le T-t} e^{-\int_0^s r_u du} v(t+s, r_s, X_s)\Big] < \infty, \tag{4.70}$$

$$\sup_{(r,x)\in\mathcal{K}} \sup_{s\in[0,T]} \mathsf{E}_{r,x}\Big[e^{-p\int_0^s r_u du}\big|v_x(t+s, r_s, X_s)\big|^p X_s^p\Big] < \infty, \tag{4.71}$$

$$\sup_{(r,x)\in\mathcal{K}} \sup_{s\in[0,T]} \mathsf{E}_{r,x}\Big[e^{-2\int_0^s r_u du}\big(v_r(t+s, r_s, X_s)\big)^2 \beta^2(r_s)\Big] < \infty. \tag{4.72}$$

Proof. From (4.6) we obtain an upper bound for the function v:

$$v(t,r,x) \le K\mathsf{E}_r\Big[\sup_{0\le s\le T-t} e^{-\int_0^s r_u du}\Big]. \tag{4.73}$$

Using this bound, we have $\mathsf{P}_{r,x}$-a.s.

$$\begin{aligned}
e^{-\int_0^s r_u du} v(t+s, r_s, X_s) &\le e^{-\int_0^s r_u du} K\mathsf{E}_{r_s}\Big[\sup_{0\le u\le T-t-s} e^{-\int_0^u r_v dv}\Big] \\
&= e^{-\int_0^s r_u du} K\mathsf{E}_r\Big[\sup_{s\le u\le T-t} e^{-\int_s^u r_v dv}\Big|\mathcal{F}_s\Big] \\
&\le K\mathsf{E}_r\Big[\sup_{0\le u\le T-t} e^{-\int_0^u r_v dv}\Big|\mathcal{F}_s\Big] \le K\mathsf{E}_r\Big[\sup_{0\le u\le T} e^{-\int_0^u r_v dv}\Big|\mathcal{F}_s\Big],
\end{aligned}$$

where in the second equality we employ the Markov property of r. By Doob's maximal inequality applied to the martingale $Y_s = \mathsf{E}_r\Big[\sup_{0\le u\le T} e^{-\int_0^u r_v dv}\Big|\mathcal{F}_s\Big]$, we conclude

$$\begin{aligned}
&\sup_{(r,x)\in\mathcal{K}} \sup_{t\in[0,T]} \mathsf{E}_{r,x}\Big[\sup_{0\le s\le T-t} e^{-\int_0^s r_u du} v(t+s, r_s, X_s)\Big] \\
&\le \sup_{(r,x)\in\mathcal{K}} \sup_{t\in[0,T]} K\,\mathsf{E}_r\Big[\sup_{0\le s\le T-t} Y_s\Big] \le \sup_{(r,x)\in\mathcal{K}} K\,\mathsf{E}_r\Big[\sup_{0\le s\le T} Y_s\Big] \\
&\le \sup_{(r,x)\in\mathcal{K}} 2K\left(\mathsf{E}_r[Y_T^2]\right)^{1/2} = \sup_{(r,x)\in\mathcal{K}} 2K\left(\mathsf{E}_r\Big[\sup_{0\le u\le T} e^{-2\int_0^u r_v dv}\Big]\right)^{1/2} \le 2K(C_1)^{1/2},
\end{aligned}$$

where C_1 is the constant from (4.3). This proves (i).

We now address (4.71). We have

$$e^{-p\int_0^s r_u du}\big|v_x(t+s, r_s, X_s)\big|^p X_s^p = \big|v_x(t+s, r_s, X_s)\big|^p x^p e^{p\sigma B_s - \frac{p}{2}\sigma^2 s} \le x^p e^{p\sigma B_s - \frac{p}{2}\sigma^2 s},$$

where we use $-1 \le v_x \le 0$ in the last inequality, which follows from (4.48). From here, (4.71) is immediate.

It remains to prove (4.72). First we consider the case of Assumption 4.2.1(ii). From (4.16), (4.17) and (4.18), we deduce

$$\big(v_r(t,r,x)\big)^2 \le c_1\,\mathsf{E}_r\Big[\sup_{0\le s\le T-t} e^{-2\int_0^s r_u du}\Big] \tag{4.74}$$

for some constant $c_1>0$ depending only on T and the coefficients of (4.2) (notice in particular that the expected value in the right-hand side above comes from the constant C_1 in (4.17)). Hence

$$\begin{aligned} e^{-2\int_0^s r_u du}\big(v_r(t+s,r_s,X_s)\big)^2\beta^2(r_s) &\le e^{-2\int_0^s r_u du}c_1\,\mathsf{E}_{r_s}\Big[\sup_{0\le u\le T-t-s} e^{-2\int_0^u r_v dv}\Big]\beta^2(r_s)\\ &\le c_1\,\mathsf{E}_r\Big[\sup_{0\le u\le T-t} e^{-2\int_0^u r_v dv}\Big|\mathcal{F}_s\Big]\beta^2(r_s), \end{aligned}$$

where the last inequality is by the same argument as in the proof of (4.70). We take expectation of both sides and apply Hölder inequality with $q=p'/2$ ($p'>2$ is defined in Assumption 4.2.1) and $q'=q/(q-1)$

$$\begin{aligned} &\mathsf{E}_r\Big[e^{-2\int_0^s r_u du}\big(v_r(t+s,r_s,X_s)\big)^2\beta^2(r_s)\Big]\\ &\le c_1\Big(\mathsf{E}_r\Big[\mathsf{E}_r\Big[\sup_{0\le u\le T-t} e^{-2\int_0^u r_v dv}\Big|\mathcal{F}_s\Big]^q\Big]\Big)^{1/q}\big(\mathsf{E}_r[\beta^{2q'}(r_s)]\big)^{1/q'}\\ &\le c_1\Big(\mathsf{E}_r\Big[\sup_{0\le u\le T-t} e^{-p'\int_0^u r_v dv}\Big]\Big)^{1/q}\big(\mathsf{E}_r[\beta^{2q'}(r_s)]\big)^{1/q'}\\ &\le c_1\,C_1^{1/q}\big(\mathsf{E}_r[\beta^{2q'}(r_s)]\big)^{1/q'}, \end{aligned}$$

where the second inequality follows from Jensen's inequality and C_1 is the constant from (4.3). Let L be the Lipschitz constant for β. Then, using triangle inequality for norms,

$$\begin{aligned} \Big(\mathsf{E}_r[(\beta(r_s))^{2q'}]\Big)^{1/q'} &\le \Big(\mathsf{E}_r\Big[\big|\beta(0)+L|r_s|\big|^{2q'}\Big]\Big)^{1/q'} = \bigg(\Big(\mathsf{E}_r\Big[\big|\beta(0)+L|r_s|\big|^{2q'}\Big]\Big)^{1/2q'}\bigg)^2\\ &\le \Big(\beta(0)+L\big(\mathsf{E}_r\big[|r_s|^{2q'}\big]\big)^{1/2q'}\Big)^2 \le \Big(\beta(0)+L\big(C_2(1+|r|^{2q'}\big)^{1/2q'}\Big)^2, \end{aligned}$$

where the last inequality follows from (4.4) and $2q'=p'\ge 2$. Combining the above estimates proves (4.72).

We address the case when r follows the CIR dynamics. From the non-negativity of the process r and from (4.74) we obtain that $\big(v_r(t,r,x)\big)^2\le c_1$ for any $(t,r,x)\in\mathcal{O}$. Hence, we write

$$\mathsf{E}_{r,x}\Big[e^{-2\int_0^s r_u du}\big(v_r(t+s,r_s,X_s)\big)^2\beta^2(r_s)\Big]\le c_1\gamma^2\mathsf{E}_r[|r_s|],$$

where we used the explicit form of β. It remains to recall (4.4) to conclude (4.72). □

Proposition 4.5.5. *Under Assumption 4.5.3, for any* $(t,r,x) \in \mathcal{O}$ *and any stopping time* $\tau \in [0, T-t]$*, the value function satisfies the following Dynkin's formula:*

$$v(t,r,x) = \mathsf{E}_{r,x}\left[\int_0^{\tau} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > c(t+u, X_u)\}} du + e^{-\int_0^{\tau} r_v dv} v(t+\tau, r_\tau, X_\tau)\right]. \tag{4.75}$$

Proof. Let $\mathcal{K}_n$ be an increasing sequence of compact subsets of $\mathcal{O}$ such that $\cup_{n\in\mathbb{N}}\mathcal{K}_n = \mathcal{O}$ and define $\tau_n = \inf\{t \in [0, T-t] : (t+s, r_s, X_s) \notin \mathcal{K}_n\} \wedge (T - t - \frac{1}{n})$ for n large enough so that $\frac{1}{n} \le T - t$. We apply a version of the change of variable formula from Chapter 5 Theorem 5.2.1 (also see [25]). We delay the verification of the assumptions required until the end of the proof. Using that

$$\begin{aligned} &(\partial_t + \mathcal{L} - r)v(t,r,x) = 0, && r < c(t,x),\\ &(\partial_t + \mathcal{L} - r)v(t,r,x) = (\partial_t + \mathcal{L} - r)(K - x) = -rK, && r > c(t,x), \end{aligned} \tag{4.76}$$

we obtain that the dynamics of the discounted value function on $[0, \tau_n]$ is given by

$$\begin{aligned} &e^{-\int_0^{s\wedge\tau_n} r_v dv} v(t + s\wedge\tau_n, r_{s\wedge\tau_n}, X_{s\wedge\tau_n})\\ &= v(t,r,x) - \int_0^{s\wedge\tau_n} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > c(t+u,X_u)\}} du + \int_0^{s\wedge\tau_n} e^{-\int_0^u r_v dv} \sigma X_u v_x(t+u, r_u, X_u) dB_u\\ &\quad + \int_0^{s\wedge\tau_n} e^{-\int_0^u r_v dv} \beta(r_u) v_r(t+u, r_u, X_u) dW_u. \end{aligned} \tag{4.77}$$

Taking expectations and applying the optional sampling theorem we arrive at

$$\begin{aligned} v(t,r,x) = \mathsf{E}_{r,x}\Big[&\int_0^{\tau\wedge\tau_n} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > c(t+u,X_u)\}} du\\ &+ e^{-\int_0^{\tau\wedge\tau_n} r_v dv} v(t + (\tau\wedge\tau_n), r_{\tau\wedge\tau_n}, X_{\tau\wedge\tau_n})\Big]. \end{aligned} \tag{4.78}$$

Using (4.3) and (4.4), Hölder inequality implies

$$\mathsf{E}_{r,x}\left[\int_0^{T-t} e^{-\int_0^u r_v dv} |K r_u| du\right] < \infty.$$

The majorant for the second term of (4.78) follows from (4.70) in Lemma 4.5.4. The dominated convergence theorem proves (4.75), since $\tau_n \uparrow T - t$ upon recalling that the boundary of $\mathcal{I} \times \mathbb{R}_+$ is assumed non-attainable by the process (r_t, X_t).

It remains to verify assumptions of 5.2.1. Identifying $X_t^1 = r_t$ and $X_t^2 = X_t$, we have,

$$\beta^{1,1}(t,r,x) = \beta^2(r), \quad \beta^{1,2}(t,r,x) = \beta^{2,1}(t,r,x) = \sigma\rho\beta(r)x, \quad \beta^{2,2}(t,r,x) = \sigma^2 x^2.$$

By Assumption 4.2.1, $\beta^{i,j}$ is Lipschitz for $i, j = 1, 2$ on every compact set in $\mathcal{O}$. Indeed, it can be directly verified for the CIR process. In case (ii) of Assumption 4.2.1 we use Lipschitz continuity of β. The marginal distribution of the process (r_t, X_t) has density with respect to the Lebesgue measure (Remark 4.3.7), so $(t, r_t, X_t) \notin \partial\mathcal{C}$, $\mathsf{P}_{r,x}$-a.s. for any $t > 0$. This verifies Assumption A.1 of 5.2.1. In the notation of Assumption A.2 in 5.2.1, using (4.76), we have

$$\frac{1}{2}L(t,r,x) = -rxv_x(t,r,x) - \alpha(r)v_r(t,r,x) - v_t(t,r,x) + rv(t,r,x) - 1_{\{(t,r,x)\in\mathcal{D}\}}\, rK.$$

Since $v \in C^1(\mathcal{O})$ and the function $\alpha(r)$ is continuous (see Assumption 4.2.1), L is continuous and bounded on $\mathcal{K}_n \setminus \partial\mathcal{C}$. We finally have that Assumption A.3 in 5.2.1 holds by Proposition 4.3.4. □

In the proof of the above proposition, we show that the discounted value function satisfies for any stopping time $\tau \in [0, T-t]$

$$\begin{aligned}
&e^{-\int_0^\tau r_v dv} v(t+\tau, r_\tau, X_\tau) \\
&= v(t,r,x) - \int_0^\tau e^{-\int_0^s r_v dv} K r_s \mathbb{1}_{\{r_s > c(t+s, X_s)\}} ds + \int_0^\tau e^{-\int_0^s r_v dv} \sigma X_s v_x(t+s, r_s, X_s) dB_s \\
&\quad + \int_0^\tau e^{-\int_0^s r_v dv} \beta(r_s) v_r(t+s, r_s, X_s) dW_s.
\end{aligned} \tag{4.79}$$

This representation will play a fundamental role in deriving a hedging strategy for the American put option in Section 4.8.

4.6 Early exercise premium

Inserting $\tau = T - t$ in (4.75), we obtain a decomposition of the American option price into a sum of the European option price v_e and an *early exercise premium* v_p (see [124] for a derivation of this formula only using general martingale theory):

$$v(t,r,x) = v_p(t,r,x;T,b) + v_e(t,r,x;T), \tag{4.80}$$

where

$$\begin{aligned}
v_e(t,r,x;T) &= \mathsf{E}_{r,x}\left[e^{-\int_0^{T-t} r_v dv}(K - X_{T-t})^+\right], \\
v_p(t,r,x;T,b) &= \mathsf{E}_{r,x}\left[\int_0^{T-t} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > c(t+u, X_u)\}} du\right] \\
&= \mathsf{E}_{r,x}\left[\int_0^{T-t} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{X_u < b(t+u, r_u)\}} du\right].
\end{aligned} \tag{4.81}$$

The last equality follows from $r > c(t,x) \Leftrightarrow x < b(t,r)$ by construction of b as the generalised inverse of c.

4.7 Integral equation for the stopping boundary

Proposition 4.5.5 provides a characterisation of the optimal stopping boundary $c(t,x)$. Indeed, for any $(t,x) \in [0,T) \times \mathbb{R}_+$ such that $c(t,x) \in \mathcal{I}$, inserting $\tau = T-t$ and $r = c(t,x)$ in (4.75) yields an integral equation for c:

$$(K-x)^+ = \mathsf{E}_{c(t,x),x}\left[\int_0^{T-t} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > c(t+u,X_u)\}} du + e^{-\int_0^{T-t} r_v dv}(K - X_{T-t})^+\right]. \tag{4.82}$$

The condition that $c(t,x) \in \mathcal{I}$ is necessary as c can take values $\underline{r}$ and $\overline{r}$ which do not belong do the state space $\mathcal{I}$, and the interest rate process r may not be started from there. Notice also that $c(t,x) \notin \mathcal{I}$ when $x \geq K$ so the left-hand side of (4.82) can be replaced by $(K-x)$. In line with well-known results for American options with constant interest rate [111], it also turns out that c is the unique solution of the integral equation.

Proposition 4.7.1. *Under Assumptions 4.5.1 and 4.5.3, the function c is the unique function $\Phi : [0,T) \times \mathbb{R}_+ \to [0, \overline{r}]$ such that:*

1. *is continuous, non-decreasing in x and non-increasing in t, with $\Phi(t,x) = \overline{r}$ for $x \geq K$,*
2. *Φ satisfies (4.82) (with c therein replaced by Φ) for all $(t,x) \in [0,T) \times \mathbb{R}_+$ for which $\Phi(t,x) \in \mathcal{I}$.*

Proof. The proof follows ideas originally developed in [111]. Assume there exists another continuous function $\tilde{c}$ that satisfies conditions (1) and (2) in the statement of this proposition. Define a function $\mathcal{O} \ni (t,r,x) \mapsto \tilde{v}(t,r,x)$ such that

$$\tilde{v}(t,r,x) = \mathsf{E}_{r,x}\left[\int_0^{T-t} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(t+u,X_u)\}} du + e^{-\int_0^{T-t} r_v dv}(K - X_{T-t})^+\right],$$
$$\tilde{v}(T,r,x) = (K-x)^+, \quad (r,x) \in \mathcal{I} \times \mathbb{R}_+.$$

It is not difficult to prove that $\tilde{v}$ is continuous by the continuity of $\tilde{c}$ and of the flow $(s,r,x) \mapsto (r_s^r, X_s^{r,x})$. By the Markov property of (r,X), one can also check that

$$\tilde{V}_s := \int_0^s e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(t+u,X_u)\}} du + e^{-\int_0^s r_v dv} \tilde{v}(t+s, r_s, X_s), \qquad s \in [0, T-t],$$

is a continuous $\mathsf{P}_{r,x}$-martingale. Hence, for any $(t,r,x) \in \mathcal{O}$ and any stopping time $\tau \leq T-t$, the optional sampling theorem yields

$$\tilde{v}(t,r,x) = \mathsf{E}_{r,x}\big[\tilde{V}_\tau\big] = \mathsf{E}_{r,x}\left[\int_0^{\tau} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(t+u,X_u)\}} du + e^{-\int_0^{\tau} r_v dv} \tilde{v}(t+\tau, r_\tau, X_\tau)\right], \tag{4.83}$$

which is analogous to the formula for v in (4.75).

For an easier exposition of the arguments of proof we proceed in steps. In the first four steps we show the equality $\tilde{c}(t,x) = c(t,x)$ for all $(t,x) \in [0,T) \times \mathbb{R}_+$ such that $\tilde{c}(t,x) \in \mathcal{I}$. Then, in the final step we use monotonicity and continuity of $\tilde{c}$ and c to extend the equality to all $(t,x) \in [0,T) \times \mathbb{R}_+$.

Step 1. We first show that $\tilde{v}(t,r,x) = (K-x)^+$ for any $(t,r,x) \in \mathcal{O}$ such that $r \geq \tilde{c}(t,x)$. Fix $(\hat{t},\hat{r},\hat{x}) \in \mathcal{O}$ such that $\hat{r} > \tilde{c}(\hat{t},\hat{x})$ (the claim for $\hat{r} = \tilde{c}(\hat{t},\hat{x})$ follows by the continuity of $\tilde{v}$). Define a stopping time

$$\tau_1 := \inf\{s \geq 0 : r_s^{\hat{r}} \leq \tilde{c}(\hat{t}+s, X_s^{\hat{r},\hat{x}})\} \wedge (T-\hat{t}).$$

By the continuity of $s \mapsto \tilde{c}(\hat{t}+s, X_s)$ and $s \mapsto r_s$, and the fact that $\underline{r}$ and $\overline{r}$ are unattainable by (r_s), we have $\tilde{c}(\hat{t}+\tau_1, X_{\tau_1}) \in \mathcal{I}$ on $\{\tau_1 < T-\hat{t}\}$. By assumption $\tilde{v}(t,\tilde{c}(t,x),x) = (K-x)^+$ and, consequently, $\tilde{v}\big(\hat{t}+\tau_1, \tilde{c}(\hat{t}+\tau_1, X_{\tau_1}), X_{\tau_1}\big) = (K-X_{\tau_1})^+$ since $\tilde{v}(T,r,x) = (K-x)^+$. In combination with (4.83), this yields

$$\tilde{v}(\hat{t},\hat{r},\hat{x}) = \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_1} e^{-\int_0^u r_v dv} K r_u du + e^{-\int_0^{\tau_1} r_v dv}(K-X_{\tau_1})^+\right], \tag{4.84}$$

where we use that $r_u > c(\hat{t}+u, X_u)$ on $\{u < \tau_1\}$. Applying Tanaka's formula to $(r,x) \mapsto (K-x)^+$ and taking expectation, we get

$$\begin{aligned}
&(K-\hat{x})^+ \\
&= \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_1} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{X_u<K\}} du + e^{-\int_0^{\tau_1} r_v dv}(K-X_{\tau_1})^+ + \frac{1}{2}\int_0^{\tau_1} e^{-\int_0^u r_v dv} dL_u^K(X)\right] \\
&= \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_1} e^{-\int_0^u r_v dv} K r_u du + e^{-\int_0^{\tau_1} r_v dv}(K-X_{\tau_1})^+\right],
\end{aligned}$$

where $L^K(X)$ is the local time of the process X at K. The local time $L^K(X)$ is null until τ_1 since $r_u > \tilde{c}(t+u, X_u) \implies X_u < K$, recalling that $\tilde{c}(t,x) = \overline{r}$ when $x \geq K$. Compare the right-hand side of the above expression to (4.84) to conclude that $\tilde{v}(\hat{t},\hat{r},\hat{x}) = (K-\hat{x})^+$.

Step 2. The next step is to show that $\tilde{v} \leq v$ for $(t,r,x) \in \mathcal{O}$. Since we have already proved $\tilde{v}(t,r,x) = (K-x)^+ \leq v(t,r,x)$ when $r \geq \tilde{c}(t,x)$, we take $(\hat{t},\hat{r},\hat{x}) \in \mathcal{O}$ such that $\hat{r} < \tilde{c}(\hat{t},\hat{x})$. Define a stopping time

$$\tau_2 := \inf\{s \geq 0 : r_s^{\hat{r}} \geq \tilde{c}(\hat{t}+s, X_s^{\hat{r},\hat{x}})\} \wedge (T-\hat{t}).$$

Since $r_u < \tilde{c}(\hat{t}+u, X_u)$ on $\{u < \tau_2\}$, we obtain from (4.83)

$$\tilde{v}(\hat{t},\hat{r},\hat{x}) = \mathsf{E}_{\hat{r},\hat{x}}\left[e^{-\int_0^{\tau_2} r_v dv}(K-X_{\tau_2})^+\right] \leq v(\hat{t},\hat{r},\hat{x}),$$

where the first equality is by $\tilde{v}\big(\hat{t}+\tau_2, \tilde{c}(\hat{t}+\tau_2, X_{\tau_2}), X_{\tau_2}\big) = (K - X_{\tau_2})^+$ and the final inequality holds by the definition of v.

Step 3. Now we show that $\tilde{c}(t,x) \leq c(t,x)$ for any $(t,x) \in [0,T) \times (0,K)$ such that $\tilde{c}(t,x) \in \mathcal{I}$ (it is immediate for $(t,x) \in [0,T) \times [K,\infty)$ as $\tilde{c}(t,x) = c(t,x) = \overline{r}$). Arguing by contradiction, assume that there exists $(\hat{t},\hat{x}) \in [0,T) \times (0,K)$ such that $\mathcal{I} \ni \tilde{c}(\hat{t},\hat{x}) > c(\hat{t},\hat{x})$. Let $\hat{r} > \tilde{c}(\hat{t},\hat{x})$, and define

$$\tau_3 := \inf\{s \geq 0 : r_s^{\hat{r}} \leq c(\hat{t}+s, X_s^{\hat{r},\hat{x}})\} \wedge (T - \hat{t}).$$

By (4.75) and (4.83), we have

$$v(\hat{t},\hat{r},\hat{x}) = \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_3} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > c(\hat{t}+u, X_u)\}} du + e^{-\int_0^{\tau_3} r_v dv} v(\hat{t}+\tau_3, r_{\tau_3}, X_{\tau_3})\right],$$
$$\tilde{v}(\hat{t},\hat{r},\hat{x}) = \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_3} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(\hat{t}+u, X_u)\}} du + e^{-\int_0^{\tau_3} r_v dv} \tilde{v}(\hat{t}+\tau_3, r_{\tau_3}, X_{\tau_3})\right].$$

Since $\tilde{v}(\hat{t},\hat{r},\hat{x}) = (K-\hat{x})^+ = v(\hat{t},\hat{r},\hat{x})$, $r_u > c(\hat{t}+u, X_u)$ on $\{u < \tau_3\}$, and $\tilde{v} \leq v$, the above two equations imply that

$$\mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_3} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(\hat{t}+u, X_u)\}} du\right] \geq \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_3} e^{-\int_0^u r_v dv} K r_u du\right].$$

As the function c is non-negative, $r_u \geq 0$ on $\{u < \tau_3\}$ and we conclude that

$$\mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_3} \mathbb{1}_{\{r_u \leq \tilde{c}(\hat{t}+u, X_u)\}} du\right] = 0. \tag{4.85}$$

The dynamics of (r, X) is non-degenerate on $\mathcal{I} \times \mathbb{R}_+$, so the density of (r_u, X_u) has a full support (on $\mathcal{I} \times \mathbb{R}_+$) for $u > 0$ (this can be inferred by classical Gaussian bounds as those we use in (4.42)). Hence, by the continuity of $\tilde{c}$ and c, for a sufficiently small $\varepsilon > 0$,

$$\mathsf{P}_{\hat{r},\hat{x}}\big(c(\hat{t}+u, X_u) < r_u < \tilde{c}(\hat{t}+u, X_u) \text{ for some } u \in (0,\varepsilon)\big) > 0.$$

Paired with the continuity of trajectories of (r, X), it contradicts (4.85).

Step 4. Next, we prove $\tilde{c} = c$ at all points such that $\tilde{c} \in \mathcal{I}$. Arguing by contradiction, assume $\tilde{c}(\hat{t},\hat{x}) < c(\hat{t},\hat{x})$ for some $(\hat{t},\hat{x}) \in [0,T) \times (0,K)$ such that $\tilde{c}(\hat{t},\hat{x}) \in \mathcal{I}$. Let $\hat{r} \in (\tilde{c}(\hat{t},\hat{x}), c(\hat{t},\hat{x}))$ and define

$$\tau_4 := \inf\{s \geq 0 : r_s^{\hat{r}} \geq c(\hat{t}+s, X_s^{\hat{r},\hat{x}})\} \wedge (T - \hat{t}).$$

By (4.75) and (4.83), we have

$$v(\hat{t},\hat{r},\hat{x}) = \mathsf{E}_{\hat{r},\hat{x}}\left[e^{-\int_0^{\tau_4} r_v dv} v(\hat{t}+\tau_4, r_{\tau_4}, X_{\tau_4})\right],$$
$$\tilde{v}(\hat{t},\hat{r},\hat{x}) = \mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_4} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(\hat{t}+u, X_u)\}} du + e^{-\int_0^{\tau_4} r_v dv} \tilde{v}(\hat{t}+\tau_4, r_{\tau_4}, X_{\tau_4})\right],$$

where in the first expression we used that $\mathbb{1}_{\{r_u > c(\hat{t}+u, X_u)\}} = 0$ on $\{u < \tau_4\}$. Since $\tilde{c}(t,x) \leq c(t,x)$ for $(t,x) \in [0,T) \times (0,K)$, we have $\tilde{v}(\hat{t}+\tau_4, r_{\tau_4}, X_{\tau_4}) = (K - X_{\tau_4})^+ = v(\hat{t}+\tau_4, r_{\tau_4}, X_{\tau_4})$ by step 1. Then recalling that $\tilde{v} \leq v$ and comparing the two equations above give us

$$\mathsf{E}_{\hat{r},\hat{x}}\left[\int_0^{\tau_4} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{r_u > \tilde{c}(\hat{t}+u, X_u)\}} du\right] \leq 0.$$

This is a contradiction since by the continuity of (r, X) and $\tilde{c}$ there is a random variable $\eta > 0$ such that

$$r_u(\omega) > \tilde{c}(\hat{t}+u, X_u(\omega)) \quad \text{for all } u \in [0, \eta(\omega)).$$

Step 5. Here we show that $\tilde{c} = c$ on $[0,T) \times \mathbb{R}_+$. Let (t_n, x_n) be a sequence such that $\tilde{c}(t_n, x_n) \in \mathcal{I}$ and $(t_n, x_n) \to (t_0, x_0)$ with $\tilde{c}(t_0, x_0) = \bar{r}$ (respectively $\tilde{c}(t_0, x_0) = 0$). Since $\tilde{c}(t_n, x_n) = c(t_n, x_n)$ for all n's, by the four steps above, by continuity we also get $c(t_0, x_0) = \tilde{c}(t_0, x_0) = \bar{r}$ (respectively $c(t_0, x_0) = \tilde{c}(t_0, x_0) = 0$). Then, by the monotonicity of both c and $\tilde{c}$ we get $c(t,x) = \tilde{c}(t,x)$ for all $(t,x) \in [0, t_0] \times [x_0, \infty)$ (respectively $(t,x) \in [t_0, T] \times [0, x_0]$). This implies, in particular, that

$$\{(t,x) : \tilde{c}(t,x) \in \mathcal{I}\} = \{(t,x) : c(t,x) \in \mathcal{I}\},$$

which concludes the proof. □

The integral equation (4.82) has an analogue for the function $b(t,r)$ from Proposition 4.3.5. Indeed, for $b(t,r) > 0$, taking $x = b(t,r)$ and $\tau = T - t$ in Proposition 4.5.5 and using $v(t, r, b(t,r)) = K - b(t,r)$ we see that b solves the integral equation:

$$\begin{aligned} K - b(t,r) = \mathsf{E}_{r,b(t,r)}&\left[\int_0^{T-t} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{X_u < b(t+u, r_u)\}} du\right] \\ &+ \mathsf{E}_{r,b(t,r)}\left[e^{-\int_0^{T-t} r_v dv}(K - X_{T-t})^+\right], \end{aligned} \tag{4.86}$$

where we use $\{X_u < b(t+u, r_u)\} = \{r_u > c(t+u, X_u)\}$ which follows from $x > b(t,r) \Leftrightarrow r < c(t,x)$ by construction of b as the generalised inverse of c. This parametrisation of the integral equation extends the one obtained in the classical American put problem with constant interest rate to our two-factor set-up. Once again we can prove uniqueness of the solution to the integral equation but without requiring continuity of b, which is a non-standard result for this type of equations.

Corollary 4.7.2. *Under the assumptions of Proposition 4.7.1, the function b is the unique function $\Psi : [0,T) \times \mathcal{I} \to [0,K)$ such that:*

1. *$t \mapsto \Psi(t,r)$ and $r \mapsto \Psi(t,r)$ are right-continuous and non-decreasing,*

2. *the generalised left-continuous inverse* $\Phi(t,x) := \inf\{r \in \mathcal{I} : \Psi(t,r) \geq x\}$ *is continuous in* (t,x)*, non-decreasing in* x *and non-increasing in* t*,*

3. Ψ *satisfies* (4.86) *with (*b *therein replaced by* Ψ*) for all* $(t,r) \in [0,T) \times \mathcal{I}$ *such that* $\Psi(t,r) > 0$.

Notice that $\Phi(t,x) = \bar{r}$ for $x \geq K$ follows immediately from $\Psi(t,r) < K$.

Proof. We can repeat the same arguments as in the proof of Proposition 4.7.1, always using $x > b(t,r) \iff r < c(t,x)$ to fall back into the exact set-up of steps 1–4 therein. □

Integral equations (4.82) and (4.86) offer a method to compute the optimal stopping boundary without using the value function v. We will demonstrate it in Section 4.9 where we design a numerical method for solving such integral equations. Knowing the stopping boundary b, the decomposition (4.80) can be employed to obtain an efficient numerical estimate of the option value. This offers an alternative to numerical solution of the variational inequality for the value function v, and, subsequently, extraction of the optimal exercise boundary.

4.8 Hedging portfolio

Thanks to the change of variable formula (4.79) we are also able to rigorously construct a hedging portfolio that (super)replicates the option payoff at all times. This is based on the classical delta-hedging ideas in the Black and Scholes model but its rigorous mathematical derivation requires smoothness of the option price function which was not previously established in the literature.

Consider a market comprising three instruments: the money market account $\xi_t := e^{\int_0^t r_u du}$, the risky stock with the dynamics (4.1), and a zero-coupon bond with maturity T. We will construct a hedging portfolio for the American option on this market. We remark that the zero-coupon bond can be replaced by any other financial instrument whose dynamics depends on the Brownian motion W driving the interest rate, see Karatzas [86].

The risk-neutral price of the zero-coupon bond at time $t \in [0,T]$ is given by

$$R(t,r) := \mathsf{E}_r\left[e^{-\int_0^{T-t} r_u du}\right], \quad R(T,r) = 1. \tag{4.87}$$

By standard arguments based on pathwise continuity of the flow $(t,r) \mapsto r_t^r(\omega)$, one can easily show that R is continuous on $[0,T] \times \mathcal{I}$. Then, under Assumption 4.2.1, the classical PDE theory [69, Thm. 9, Ch. 4, Sec. 3] guarantees that R is the unique classical solution of the boundary value problem

$$\begin{aligned} &(\partial_t + \mathcal{L}_r - r)u(t,r) = 0, && (t,r) \in [0,T] \times (a,b), \\ &u(t,r) = R(t,r), && t \in [0,T),\ r \in \{a,b\} \\ &u(T,r) = 1, && r \in [a,b], \end{aligned}$$

where $\mathcal{L}_r = \alpha(r)\partial_r + \beta(r)^2/2\partial_{rr}$ and $(a,b) \subset \mathcal{I}$ is an arbitrary bounded interval. In particular, by arbitrariness of (a,b) we have $R \in C^{1,2}([0,T) \times \mathcal{I})$ and

$$(\partial_t + \mathcal{L}_r - r)R(t,r) = 0, \qquad (t,r) \in [0,T] \times \mathcal{I}.$$

Then, using Itô's formula, the discounted bond price dynamics reads

$$de^{-\int_0^s r_u du} R(s,r_s) = R_r(s,r_s)\beta(r_s)dW_s. \tag{4.88}$$

Denote by $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}$ the holdings in the stock, the bond and the money account, respectively. Let C be a non-decreasing continuous process starting from 0 modelling consumption. A self-financing portfolio starting at time 0 can be constructed by

$$\Pi_s := \phi_s^{(1)} X_s + \phi_s^{(2)} R(s,r_s) + \phi_s^{(3)} \xi_s, \quad s \in [0,T], \tag{4.89}$$

and it holds that

$$\Pi_s = v(0,r,x) + \int_0^s \phi_u^{(1)} dX_u + \int_0^s \phi_u^{(2)} dR(u,r_u) + \int_0^s \phi_u^{(3)} d\xi_u - C_s, \quad s \in [0,T]. \tag{4.90}$$

The portfolio is *admissible* if all integrals above are semimartingales. Taking the money-market account as a numeráire, we obtain from equations (4.90) that the dynamics of the discounted portfolio value reads

$$\begin{aligned} de^{-\int_0^s r_u du}\Pi_s &= \phi_s^{(1)} de^{-\int_0^s r_u du} X_s + \phi_s^{(2)} de^{-\int_0^s r_u du} R(s,r_s) - e^{-\int_0^s r_u du} dC_s \\ &= e^{-\int_0^s r_u du}\phi_s^{(1)}\sigma X_s dB_s + e^{-\int_0^s r_u du}\phi_s^{(2)}\beta(r_s)R_r(s,r_s)dW_s - e^{-\int_0^s r_u du} dC_s. \end{aligned} \tag{4.91}$$

This means that a self-financing portfolio is uniquely determined by the processes $\phi^{(1)}, \phi^{(2)}$ and C.

Comparing (4.91) with (4.79), a candidate for the hedging strategy is given by

$$\phi_s^{(1)} = v_x(s,r_s,X_s), \quad \phi_s^{(2)} = \frac{v_r(s,r_s,X_s)}{R_r(s,r_s)}, \quad C_s = \int_0^s K r_u \mathbb{1}_{\{r_u > c(u,X_u)\}} du. \tag{4.92}$$

Thanks to Lemma 4.5.4, we can indeed prove that such portfolio strategy is admissible and replicates the option's payoff.

Proposition 4.8.1. *Under Assumption 4.5.3 the portfolio $(\phi^{(1)}, \phi^{(2)}, C)$ is admissible and replicates the payoff of the American put option.*

Proof. The admissibility condition can be equivalently written as

$$\int_0^T e^{-2\int_0^s r_u du}\big(\phi_s^{(1)}\sigma X_s\big)^2 ds + \int_0^T e^{-2\int_0^s r_u du}\big(\phi_s^{(2)}\beta(r_s)R_r(s,r_s)\big)^2 ds < \infty, \qquad \mathsf{P}_{r,x}\text{-a.s.} \tag{4.93}$$

Estimates in Lemma 4.5.4 imply

$$\mathsf{E}_{r,x}\Big[\int_0^T e^{-2\int_0^s r_u du}\big(\phi_s^{(1)}\sigma X_s\big)^2 ds + \int_0^T e^{-2\int_0^s r_u du}\big(\phi_s^{(2)}\beta(r_s)R_r(s,r_s)\big)^2 ds\Big] < \infty,$$

which is a stronger condition than (4.93). The fact that the portfolio replicates the option follows from the construction and (4.92). □

4.9 Numerical analysis

In the numerical analysis, we assume that the interest rate r follows Vasicek model. In particular, this means that $\mathcal{I} = \mathbb{R}$ and

$$dr_t = \kappa(\theta - r_t)dt + \beta dW_t, \tag{4.94}$$

whose explicit solution is given by

$$r_s = r_t e^{-(s-t)\kappa} + \theta(1 - e^{-(s-t)\kappa}) + \beta e^{-s\kappa}\int_t^s e^{\kappa u} dW_u, \quad s \geq t \geq 0. \tag{4.95}$$

We first derive a numerical method for computing the optimal stopping boundary using the integral equation from (4.86). Once the boundary is obtained, we use it to also compute the value function via (4.80). Section 4.9.2 contains an analysis of the effect of parameters on the stopping boundary and the value function.

4.9.1 Computational approach

With an abuse of notation, we denote by $R(t,T) = R(t,r,T)$ the time-t price of a zero-coupon bond with maturity T (c.f. (4.87)); the dependence on the initial state r is indicated in the subscript of the expectation operator. Recall the integral equation (4.86) for the boundary b: for $(t,r) \in [0,T) \times \mathcal{I}$ such that $b(t,r) > 0$, we have

$$K - b(t,r) = v_p(t,r,b(t,r);T,b) + v_e(t,r,b(t,r);T), \tag{4.96}$$

where v_e and v_p are stated in (4.81). With the last parameter b of v_p, we emphasise the dependence on the function b:

$$v_p(t,r,x;T,b) = \mathsf{E}_{r,x}\left[\int_0^{T-t} e^{-\int_0^u r_v dv} K r_u \mathbb{1}_{\{X_u<b(t+u,r_u)\}} du\right].$$

In the numerical scheme below, we evaluate v_p for consecutive approximations of b.

We can derive the following formulas for v_e and v_p using well-known properties of the joint law of (r_t, X_t):

$$v_e(t,r,x;T) = R(t,T)K\mathcal{N}(d_1) - x\mathcal{N}(d_2), \tag{4.97}$$

$$v_p(t,r,x;T,b) = \int_t^T KR(t,u)\bigg[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\Big(q(t,u)+y\sqrt{\gamma_2(t,u)}\Big)\mathcal{N}\big(\phi(t,u,y;b)\big)dy\bigg]du, \tag{4.98}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution and other auxiliary quantities are given by (we suppress dependence on r and x for the sake of simplicity):

$$\begin{aligned}
d_1(t,T) :&= \frac{\log\Big(K\frac{P(t,T)}{x}\Big)+\frac{\gamma_1(t,T)}{2}}{\sqrt{\gamma_1(t,T)}}, \qquad d_2(t,T) := d_1(t,T)-\sqrt{\gamma_1(t,T)},\\
q(t,u) :&= e^{-(u-t)\kappa}r+\theta(1-e^{-(u-t)\kappa})-\frac{\beta^2}{2}g(\kappa,u-t)^2,\\
\phi(t,u,y;b) :&= \frac{\log\Big(\frac{P(t,u)}{x}b\big(u,q(t,u)+y\sqrt{\gamma_2(t,u)}\big)\Big)+\frac{\gamma_1(t,u)}{2}-\sqrt{\gamma_1(t,u)}\tilde{\rho}(t,u)y}{\sqrt{(1-\tilde{\rho}(t,u)^2)\gamma_1(t,u)}},\\
\mu(t,u) :&= rg(\kappa,u-t)+\theta\big(u-t-g(\kappa,u-t)\big),\\
\gamma_1(t,u) :&= (u-t)\sigma^2+\frac{2\rho\sigma\beta}{\kappa}\big(u-t-g(\kappa,u-t)\big)\\
&\quad+\frac{\beta^2}{\kappa}\big(u-t-2g(\kappa,u-t)+g(2\kappa,u-t)\big),\\
\gamma_2(t,u) :&= \beta^2 g(2\kappa,u-t),\\
\tilde{\rho}(t,u) :&= \frac{\rho\sigma\beta g(\kappa,u-t)+\frac{\beta^2}{2}g(\kappa,u-t)^2}{\sqrt{\gamma_1(t,u)\gamma_2(t,u)}},\\
g(a,u) :&= \frac{1-e^{-au}}{a}.
\end{aligned} \tag{4.99}$$

Now we give the detailed derivation of (4.97) and (4.98).

Lemma 4.9.1. *For a measurable bounded function $\varphi : (0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $s\geq t$, the function*

$$u(t,s,r,x)=\mathsf{E}_{r,x}\left[e^{-\int_0^{s-t} r_u du}\varphi(X_{s-t},r_{s-t})\right]$$

has an explicit representation

$$u(t,s,r,x)=e^{-\mu(t,s)+\frac{1}{2}\beta^2\int_t^s g(\kappa,s-u)^2du}\int_{\mathbb{R}^2}\varphi\Big(xe^{L(t,s)+\sqrt{\gamma_1(t,s))}z},\ q(t,s)+y\sqrt{\gamma_2(t,s)}\Big)$$
$$e^{-\frac{1}{2(1-\tilde\rho(t,s)^2)}\left(z^2+y^2-2\tilde\rho(t,s)zy\right)}\frac{1}{2\pi\sqrt{1-\tilde\rho(t,s)^2}}dzdy,\quad (4.100)$$

where

$$L(t,s)=\mu(t,s)-\frac{\sigma^2}{2}(s-t)-\int_t^s\left(\beta^2 g(\kappa,s-u)^2+\rho\sigma g(\kappa,s-u)\right)du.$$

Proof. The proof follows the lines of similar computations in the literature, see [13] and [45]. Using the explicit expression of r in (4.95) and stochastic Fubini's theorem [117, Theorem IV.64], we compute

$$\int_t^s r_u du = g(\kappa,s-t)r_t+\theta\left(s-t-g(\kappa,s-t)\right)+\beta\int_t^s g(\kappa,s-u)dW_u \quad (4.101)$$
$$=\mu(t,s)+\beta\int_t^s g(\kappa,s-u)dW_u.$$

Define Z_t implicitly by

$$dB_t=\rho dW_t+\sqrt{1-\rho^2}dZ_t.$$

Then Z is a Brownian motion that is independent of W. Using the explicit expression of X and (4.101), we write u as

$$u(t,s,r,x)=\mathsf{E}_r\Bigg[e^{-\mu(t,s)-\beta\int_0^{s-t}g(\kappa,s-t-u)dW_u}\varphi\Big(x\exp\Big\{\mu(t,s)-\frac{\sigma^2}{2}(s-t)$$
$$+\beta\int_0^{s-t}g(\kappa,s-t-u)dW_u+\int_0^{s-t}\sigma(\rho dW_u+\sqrt{1-\rho^2}dZ_u)\Big\},r_{s-t}\Big)\Bigg]. \quad (4.102)$$

Define a new measure $\widetilde{\mathsf{P}}$ by the Radon-Nikodym density

$$\frac{d\widetilde{\mathsf{P}}}{d\mathsf{P}}:=e^{-\int_0^{s-t}C_u^{(1)}dW_u-\int_0^{s-t}C_u^{(2)}dZ_u-\frac{1}{2}\int_0^{s-t}\left((C_u^{(1)})^2+(C_u^{(2)})^2\right)du},$$

where $C_u^{(1)}=\beta g(\kappa,s-t-u), C_u^{(2)}=0$. Define process $\widetilde{W}$ such that

$$d\widetilde{W}_u=dW_u+\beta g(\kappa,s-t-u)du,$$

then $(\widetilde{W}_u, Z_u)_{u\in[0,s-t]}$ is a two-dimensional Brownian motion under $\widetilde{\mathsf{P}}$.

We write the explicit formula (4.101) for r in terms of $\widetilde{W}$:

$$r_s = r_t e^{-(s-t)\kappa} + \theta(1 - e^{-(s-t)\kappa}) - \frac{\beta^2}{2} g(\kappa, s-t)^2 + \beta \int_t^s e^{-(s-u)\kappa} d\widetilde{W}_u.$$

Denoting by $\widetilde{\mathsf{E}}$ the expectation under $\widetilde{P}$, we obtain from (4.102) (note that we use explicit form of r_s with $s = s - t, t = 0$)

$$u(t,s,r,x) = e^{-\mu(t,s)+\frac{1}{2}\beta^2\int_t^s g(\kappa,s-u)^2du}\widetilde{\mathsf{E}}\Big[\varphi\Big(xe^{L(t,s)+A(t,s)},\ q(t,s) + Y(t,s)\Big)\Big], \qquad (4.103)$$

where

$$\begin{aligned} A(t,s) &:= \int_0^{s-t} (\rho\sigma + \beta g(\kappa, s-t-u)) d\widetilde{W}_u + \int_0^{s-t} \sigma\sqrt{1-\rho^2} dZ_u, \\ Y(t,s) &:= \beta \int_0^{s-t} e^{-(s-t-u)\kappa} d\widetilde{W}_u. \end{aligned}$$

For each fixed $t < s$, the random vector (A, Y) is multivariate Gaussian under $\widetilde{\mathsf{P}}$ with zero mean, and variance and covariance given by

$$\begin{aligned} &Var_{\widetilde{\mathsf{P}}}(A(t,s)) = \gamma_1(t,s), \qquad Var_{\widetilde{\mathsf{P}}}(Y(t,s)) = \gamma_2(t,s), \\ &Cov_{\widetilde{\mathsf{P}}}(A(t,s), Y(t,T)) = \tilde{\rho}(t,s)\sqrt{\gamma_1(t,s)\gamma_2(t,s)}. \end{aligned}$$

Hence, we have an explicit integral representation of (4.103)

$$\begin{aligned} &u(t,s,r,x) \\ &= e^{-\mu(t,s)+\frac{1}{2}\beta^2\int_t^s g(\kappa,s-u)^2du} \int_{\mathbb{R}^2} \varphi\Big(xe^{L(t,s)+z}, \quad q(t,s)+y\Big) \\ &\qquad \times e^{-\frac{1}{2(1-\tilde{\rho}(t,s)^2)}\left(\frac{z^2}{\gamma_1(t,s)} + \frac{y^2}{\gamma_2(t,s)} - \frac{2\tilde{\rho}(t,s)zy}{\sqrt{\gamma_1(t,s)\gamma_2(t,s)}}\right)} \frac{1}{2\pi\sqrt{\gamma_1(t,s)\gamma_2(t,s)}\sqrt{1-\tilde{\rho}(t,s)^2}} dzdy. \end{aligned} \qquad (4.104)$$

A change of variable yields (4.100). □

We now apply the above lemma to derive formulae for $R(t,T)$, v_e and v_p. Taking $\varphi \equiv 1$ in (4.100), we have

$$R(t,s) = \mathsf{E}_r\left[e^{-\int_0^{s-t} r_s ds}\right] = e^{-\mu(t,s)+\frac{1}{2}\beta^2\int_t^s g(\kappa,s-u)^2du} = e^{-\mu(t,s)+\frac{\beta^2}{2\kappa^2}(s-t-2g(\kappa,s-t)+g(2\kappa,s-t))}. \qquad (4.105)$$

This also implies that

$$e^{L(t,s)} = e^{\mu(t,s)-\frac{1}{2}\int_t^s \beta^2 g(\kappa,s-u)^2du}\, e^{-\frac{\sigma^2}{2}(s-t)-\frac{1}{2}\int_t^s \beta^2 g(\kappa,s-u)^2du - \int_t^s \rho\sigma g(\kappa,s-u)du} = \frac{1}{R(t,s)} e^{-\frac{\gamma_1(t,s)}{2}}.$$

Insert this and (4.105) into (4.100) and let

$$z = \tilde{\rho}(t,s)y + \sqrt{1-\tilde{\rho}(t,s)^2}\hat{z}.$$

This transforms the integral in (4.100) into an integral of two independent Gaussian variables

$$u(t,s,r,x) = R(t,s)\int_{\mathbb{R}^2}\varphi\Big(\frac{x}{R(t,s)}e^{-\frac{\gamma_1(t,s)}{2}+\sqrt{\gamma_1(t,s)}\left(\tilde{\rho}(t,s)y+\sqrt{1-\tilde{\rho}(t,s)^2}\hat{z}\right)},$$
$$q(t,s)+\sqrt{\gamma_2(t,s)}y\Big)e^{-\frac{1}{2}(\hat{z}^2+y^2)}\frac{1}{2\pi}d\hat{z}dy. \quad (4.106)$$

Now, letting $\varphi(x,r) = (K-x)^+$ and $\varphi(x,r) = Kr\mathbb{1}_{\{x<b(s,r)\}}$, we obtain (4.97)-(4.98).

Equation (4.96) defines the boundary b as a fixed point of a non-linear mapping. To compute it, we follow an iterative scheme motivated by [46]. We fix $-\infty < r_{min} < r_{max} < \infty$ and discretise the variables (t,r) as follows:

$$\{(t_i,r_j)\in[t,T]\times[r_{min},r_{max}]\},\quad i=1,...,M, j=1,...,N.$$

We specify an initial approximation $b^{(0)}$ of the boundary:

$$b^{(0)}(t_i,r_j) = K,\quad \forall\, i,j.$$

For each $n\geq 1$, we compute the boundary $b^{(n)}$ at points $(t_i,r_j)_{i,j}$ by solving the algebraic equation:

$$K - b^{(n)}(t_i,r_j) - v_e\left(t_i,r_j,b^{(n)}(t_i,r_j);T\right) = v_p\left(t_i,r_j,b^{(n-1)}(t_i,r_j);T,b^{(n-1)}\right). \quad (4.107)$$

The right-hand side, which is difficult to compute, is independent of $b^{(n)}$, while the left-hand side is known in an explicit form. We stop iterations when, for a pre-determined $\varepsilon>0$,

$$\max_{i,j}|b^{(n-1)}(t_i,r_j) - b^{(n)}(t_i,r_j)| < \varepsilon.$$

The numerical evaluation of $v_p\left(t_i,r_j,b^{(n-1)}(t_i,r_j);T,b^{(n-1)}\right)$ requires that the boundary $b^{(n-1)}$ be known for all points (t,r) in the state space while we compute it only on the grid (t_i,r_j). We, therefore, use Matlab interpolation function with the Modified Akima cubic Hermite polynomials ('makima') interpolation method. Integrals are computed using Matlab functions employing standard quadrature methods.

It should be remarked that the stopping boundary b may have a singularity (jump) at $r=0$, which corresponds to a horizontal part of the parametrisation c of the stopping surface: a jump occurs when $c^{-1}(\{0\})\neq[0,T)\times\{0\}$. Furthermore, $b(T-,r) := \lim_{t\uparrow T} b(t,r)$ satisfies $b(T-,r)=0$ for $r<0$ and $b(T-,r)\geq b(0,r)>0$ for $r>0$, see Proposition 4.3.5. This hints at a potential numerical difficulty around $r=0$, particularly for times t close to maturity.

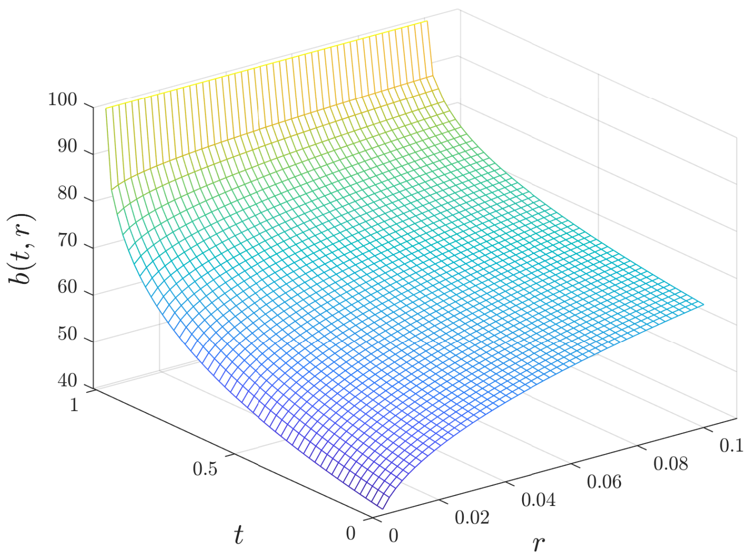


Figure 4.1: Stopping boundary surface $b(t, r)$.

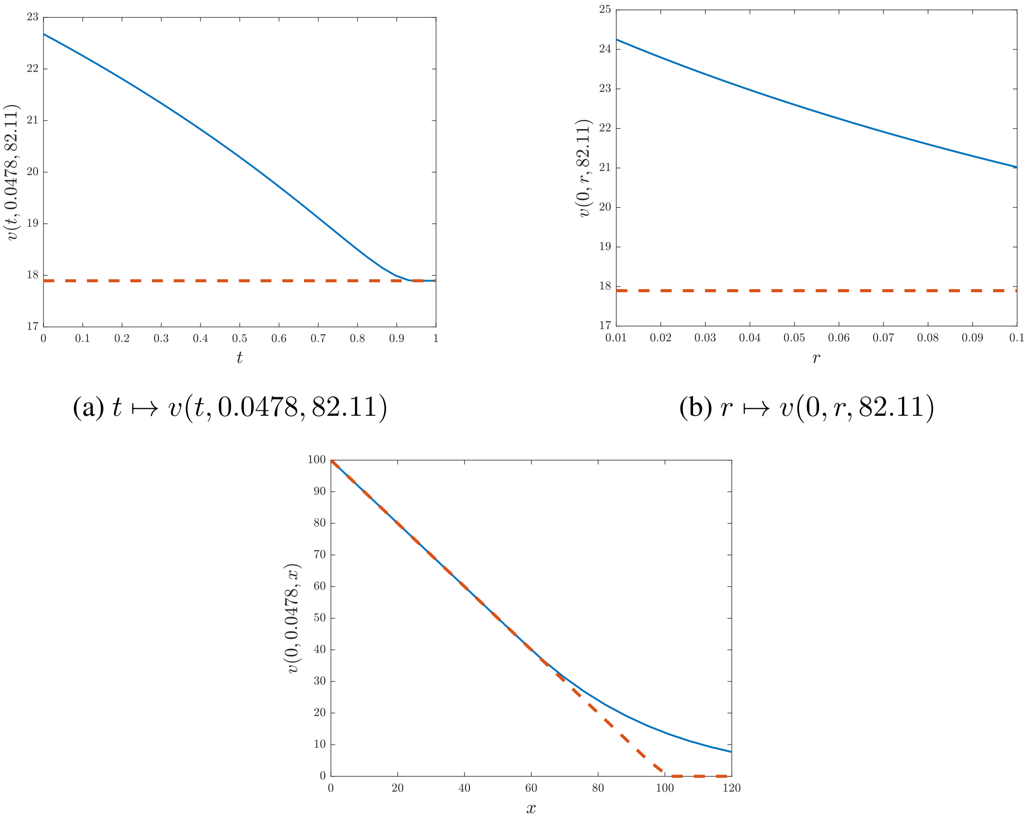
4.9.2 Sensitivity analysis

Unless stated otherwise, numerical results are presented for the parameter values

$$T = 1, K = 100, \sigma = 0.4, \kappa = 0.3, \theta = 0.05, \beta = 0.01, \rho = 0.5, \tag{4.108}$$

and the convergence criterion with $\varepsilon = 0.01$. The magnitude of κ, θ and β is based on empirical findings reported in the literature, c.f. [79, Chapter 31] and [61]. Although main currencies have recently enjoyed much lower interest rates, our choice of θ means that the effects of random interest rate and its parameters on the market dynamics and optimal stopping boundary are more pronounced and graphs more transparent.

Figure 4.1 plots the stopping boundary $b(t, r)$ using parameters (4.108). The optimal stopping boundary increases as t tends to the maturity T and as the interest rate r grows (c.f. Proposition 4.3.5). This behaviour is consistent with the one of the optimal exercise boundary for the American put option in the Black and Scholes model with a constant interest rate [111]. Figure 4.2 illustrates the value function $v(t, r, x)$ via sections in directions of t, r and x rooted at the point $(0, 0.0478, 82.11)$, which illustrates the findings of Proposition 4.3.1. In Panel (a), the value decreases to the value of the immediate exercise as the option is purchased deep in the money.

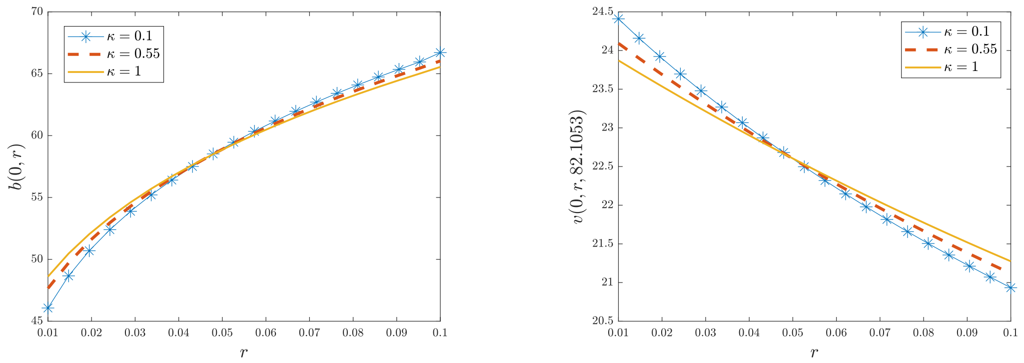


(a) $t \mapsto v(t, 0.0478, 82.11)$

(b) $r \mapsto v(0, r, 82.11)$

(c) $x \mapsto v(0, 0.0478, x)$

Figure 4.2: Sections of the value function $v(t, r, x)$ through the point $(0, 0.0478, 82.11)$. The dashed line displays the payoff $(K - x)^+$.



(a) (b)

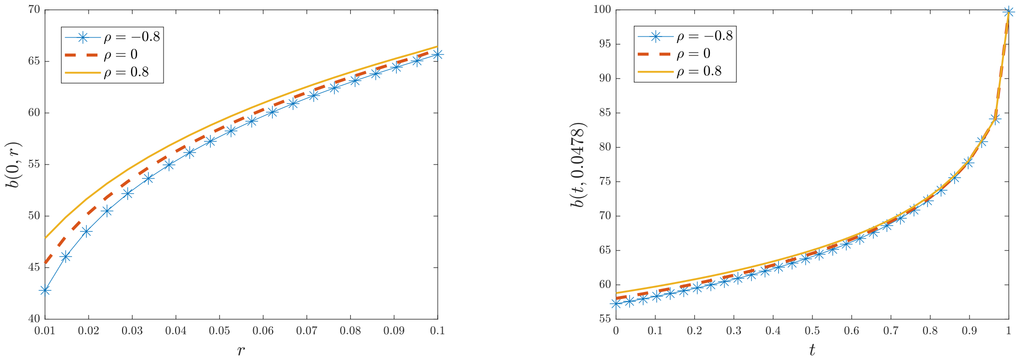
Figure 4.3: The r-sections of the stopping boundary (left panel) and the value function (right panel) for the mean-reversion parameter $\kappa \in \{0.1, 0.55, 1\}$.

Effects of the interest rate. The option price is significantly affected by the initial interest rate (Panel (b)) because the maturity of the option is long (1 year). The effect depends on the mean-reversion coefficient κ and it increases when the mean reversion parameter decreases. Indeed, this tendency is clearly visible in Figure 4.3. A large mean-reversion speed ($\kappa = 1$) means that the interest rate is quickly pulled towards $\theta = 0.05$, diminishing the effect of the initial value. Taking expectation on both sides of (4.95) gives that the expected interest rate at the maturity $T = 1$ is

$$\mathsf{E}_r[r_1] = re^{-\kappa} + \theta(1 - e^{-\kappa}),$$

which, for $\kappa = 1$, means $\mathsf{E}_r[r_1] \approx 0.36\,r + 0.74\,\theta$. On the contrary, we obtain $\mathsf{E}_r[r_1] \approx 0.90\,r + 0.10\,\theta$ for $\kappa = 0.1$ and so the effect of the initial interest rate on the stopping boundary (Figure 4.3a) and the value function (Figure 4.3b) is more pronounced. The optimal strategy for $\kappa = 0.1$ prescribes to be more patient compared to larger values of κ when the interest rate is near 0 and act faster when the interest rate is close to 1. Indeed, with a slow mean-reversion the interest rate stays close to the current value for longer, so the observed behaviour of the stopping boundary and the of value function is akin to that observed by a model with a constant interest rate [22, 111].

Effects of the correlation coefficient. The sensitivity of the stopping boundary with respect to the correlation coefficient ρ between Brownian motions driving the stock price and the interest rate is displayed in Figure 4.4; the value function behaves accordingly and it is not displayed. High positive correlation $\rho = 0.8$ implies that the interest rate and the stock price tend to move together. The increase in the interest rate pushes the stock price up and vice versa,



(a) (b)

Figure 4.4: The r and t-sections of the stopping boundary for the correlation coefficient $\rho \in \{-0.8, 0, 0.8\}$.

resulting in a more unstable environment and an earlier optimal stopping. On the contrary, a strong negative correlation sees the stock price and the interest rate dampening the effect of each other's moves: an increase in the stock price brings a drop in the interest rate, therefore, making longer waiting (lower stopping boundary) more desirable due to effect on the drift of the stock price as well as on the discount factor. Naturally, this effect diminishes the closer one gets to the maturity of the option, see Figure 4.4b.

Effects of the volatility of stock and interest rate. The effect of the diffusion coefficient of the spot rate β on the stopping boundary and on the value function is negligible. We compared results for $\beta \in \{0.005, 0.01, 0.015\}$, the range of values observed in empirical literature mentioned above. We noticed variations in the value function of less than 0.1% and in the stopping boundary of less than 1%.

In line with the financial intuition, the value of American Put option is increasing in σ, see Figure 4.5c and 4.5d. When $\sigma = 0.1$, the optimal stopping boundary is close to the exercise price K (Figure 4.5a), so the option is immediately exercised for the initial stock price $x = 82.1053$ presented on Panel (c), hence the flat graph. For other values of σ, the exercise boundary is below the initial stock price and the effect of the interest rate is clearly visible. The structure of results in Figure 4.5 is, as expected, in line with the findings for the American Put option in the Black and Scholes model with constant interest rate [22, 111].

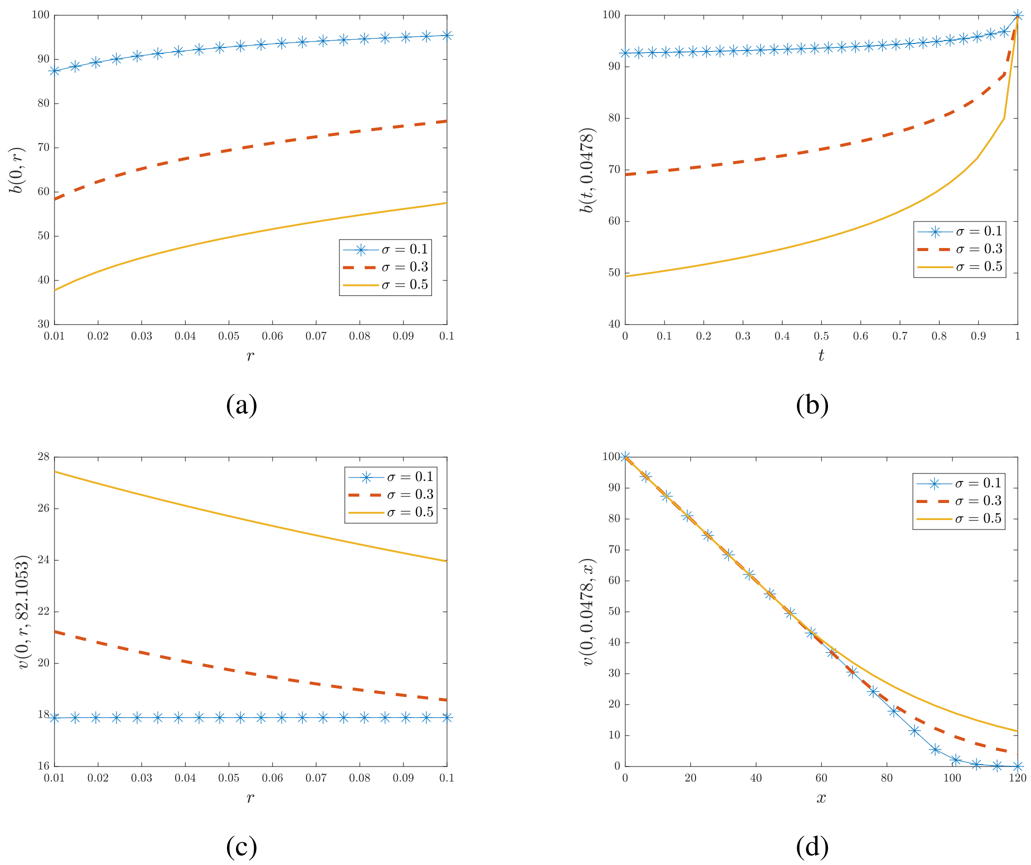


Figure 4.5: Effect of the volatility of the stock price σ. Panels (a) and (b) display the r and t-sections of the stopping boundary $b(t,r)$ and Panels (c) and (d) show the r and x-sections of the value function v for $\sigma \in \{0.1, 0.3, 0.5\}$.

Chapter 5

A change of variable formula with applications to multi-dimensional optimal stopping problems

5.1 Introduction

[1]In this chapter, we provide a version of change of variable formula for the value function $U : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ in a multi-dimensional optimal stopping problem. The result is applied in the proof of Proposition 4.5.5 in Chapter 4. Our formula requires the value function to be continuously differentiable, i.e. the spatial gradient ∇U is a continuous function. However, we require minimal regularity on the second order spatial derivatives of U near the stopping boundary $\partial\mathcal{C}$ and very mild monotonicity properties of the boundary itself.

We now review some of the main results in the field but without the ambition to give a full account of the existing literature, which is vast and branches out in several specialised directions. In order to avoid confusion with our own setting, below we use F to denote the function to which the change of variable formula is applied in the literature that we discuss.

Various change of variable formulae have been developed that do not even require continuity of first order spatial derivatives of F. Perhaps the best known is the so-called Itô-Tanaka-Meyer formula (see, e.g., [117, Thm.IV.7.70]) which applies to functions $F : \mathbb{R} \to \mathbb{R}$ that are a difference of convex functions (see also [8, Sec. 3] for an extension to $F(t, X_t)$ with X a one-dimensional Brownian motion). Relaxing the assumption of convexity is generally difficult but a number of results are known in the literature. An early work in this direction is

[1]The results from this chapter form part of the article [25], which is currently under review.

the one by Bouleau and Yor [19] who establish a formula for functions $F : \mathbb{R} \to \mathbb{R}$ which are absolutely continuous with locally bounded first order derivative and for a fairly broad class of càdlàg semi-martingales. The key idea in that work is that the semi-martingale local time defines a measure on $\mathbb{R}$ via the mapping $a \mapsto L_t^a$ (see, e.g., [117, Thm. IV.7.77] and the subsequent corollary for details). Föllmer and Protter [65] generalise those results to functions $F : \mathbb{R}^d \to \mathbb{R}$ whose first order partial derivatives exist in the weak sense as functions in L^2 and the underlying process is a d-dimensional Brownian motion. Analogous results in the one-dimensional case had been previously obtained by Föllmer, Protter and Shiryaev in [68] (see also Bardina and Jolis [10] for time-space extensions in the case of one-dimensional diffusions with suitable transition density). Those works shift the focus from the use of semi-martingale local times (as in Bouleau and Yor [19]) to the use of quadratic covariation of $\nabla F(X)$ and X. Quadratic covariation appears also in work by Russo and Vallois [123], who require continuous differentiability of the function $F : \mathbb{R}^d \to \mathbb{R}$ but develop change of variable formulae for more general processes than just semi-martingales, thanks to notions of forward and backward integrals they introduce in earlier papers (see also subsequent results by Errami, Russo and Vallois [57]). Further results based on quadratic covariation of $\nabla F(X)$ and X are established by Moret and Nualart [106] when F belongs to the Sobolev class $W^{1,p}_{\ell oc}(\mathbb{R}^d)$ and X is a non-degenerate martingale, using Malliavin calculus techniques. In the case of diffusions associated to uniformly elliptic operators in divergence form Rozkosz [121] establishes a change of variable formula for functions F in the class $W^{1,p}_{\ell oc}(\mathbb{R}^d)$, for $p > 2 \wedge d$, via Stratonovich integrals.

The focus on properties of local times of semi-martingales is central in works by Peskir [110] and [112], which are close in spirit to our work (see also [74] for further results and links to other generalisations of Itô's formula). In particular, in [110] Peskir studies a change of variable formula for processes $F(t, X_t)$ where X is a continuous semi-martingale, $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is such that $F \in C^{1,2}$ on $\overline{\mathcal{C}}$ and $F \in C^{1,2}$ on $\overline{\mathcal{D}}$, with $\mathbb{R}_+ \times \mathbb{R} = \overline{\mathcal{C}} \cup \overline{\mathcal{D}}$ and the sets are separated by the graph of a continuous function $b : \mathbb{R}_+ \to \mathbb{R}$ of bounded variation.

Peskir's formula ([110, Thm. 2.1]) reads as follows.

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \frac{1}{2}\left(F_t(s, X_s+) + F_t(s, X_s-)\right) \mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\left(F_x(s, X_s+) + F_x(s, X_s-)\right) \mathrm{d}X_s \\ &+ \int_0^t \frac{1}{2} F_{xx}(s, X_s) \mathbb{1}_{\{X_s \neq b(s)\}} \mathrm{d}\langle X, X \rangle_s \\ &+ \int_0^t \frac{1}{2}\left(F_x(s, X_s+) - F_x(s, X_s-)\right) \mathbb{1}_{\{X_s = b(s)\}} \mathrm{d}L_s^b(X), \end{aligned} \tag{5.1}$$

where

$$L_s^b(X) = \mathsf{P} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s \mathbb{1}_{\{b(r)-\varepsilon \leq X_s \leq b(r)+\varepsilon\}} \mathrm{d}\langle X, X \rangle_r \tag{5.2}$$

is the local time of X at the curve b for $s \in [0, t]$. The local time is needed in the formula as the spatial derivatives of F need not be continuous across the boundary $\partial\mathcal{C} = \partial\mathcal{D}$ of the two sets. In optimal stopping problems where $\mathcal{C}$ is the continuation set and F is the value function, the assumption $F \in C^{1,2}$ on $\overline{\mathcal{C}}$ could be too restrictive, as the regularity is usually hard to verify at the stopping boundary b. Taking this into account, in [110, Sec. 3], Peskir shows this requirement can be weakened to hold only in the interior of the sets $\mathcal{C}$ and $\mathcal{D}$, separately, if X is a continuous diffusion:

$$\mathrm{d}X_t = \alpha(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t. \tag{5.3}$$

The regularity requirement for F in this case can be replaced by the following assumptions ([110, Thm. 3.1]),

- P.1 $F \in C^{1,2}$ on $\mathcal{C}$ and $F \in C^{1,2}$ on $\mathcal{D}$,
- P.2 $\mathcal{L}F = F_t + \alpha F_x + \frac{1}{2}\sigma^2 F_{xx}$ is locally bounded on $\mathcal{C} \cup \mathcal{D}$,
- P.3 $F_x(s, b(s) \pm \varepsilon) \to F_x(s, b(s))$ uniformly for $s \in [0, t]$ as $\varepsilon \downarrow 0$,
- P.4 $\sup_{0<\varepsilon<\delta} V(F(\,\cdot\,, b(\,\cdot\,) \pm \varepsilon))(t) < \infty$ for some $\delta > 0$, where $V(g)(t)$ denotes the total variation of g on $[0, t]$.

The change of variable formula becomes

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \mathcal{L}F(s, X_s)\mathbb{1}_{\{X_s \neq b(s)\}}\mathrm{d}s + \int_0^t (\sigma F_x)(s, X_s)\mathbb{1}_{\{X_s \neq b(s)\}}\mathrm{d}B_s \\ &+ \int_0^t \frac{1}{2}\left(F_x(s, X_s+) - F_x(s, X_s-)\right)\mathbb{1}_{\{X_s = b(s)\}}\mathrm{d}L_s^b(X). \end{aligned} \tag{5.4}$$

Assumption P.2 is verifiable in optimal stopping. In the corresponding free boundary problem, we often have $\mathcal{L}F = 0$ on $\mathcal{C}$ and $\mathcal{L}F = \mathcal{L}M$ on $\mathcal{D}$ where M is the gain function. Hence, the assumption P.2 holds as long as $\mathcal{L}M$ is locally bounded on $\mathcal{D}$. The second derivative of F may explode when approaching the stopping boundary b from $\mathcal{C}$, but the first integral in (5.4) is well-defined because of the boundedness of function $\mathcal{L}F$. Assumption P.3 and P.4 can be verified by studying the monotonicity and the C^1 regularity (smooth fit) of the value function F. The idea that only having control over function $\mathcal{L}F$ instead of the individual second order derivative F_{xx} inspired our work. We will explain this assumption further in Section 5.3.

In his other paper [112], Peskir extends the result to multi-dimensional, possibly discontinuous semi-martingales $\boldsymbol{X} = (X^1, \ldots, X^d) \in \mathbb{R}^d$ and in this case the sets $\mathcal{C}$ and $\mathcal{D}$ are separated by the graph of a function $b : \mathbb{R}_+ \times \mathbb{R}^{d-1} \to \mathbb{R}$ that is continuous and such that the process $b^X := b(X^1, \ldots X^{d-1})$ is a semi-martingale. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $F \in C^{i_1,\ldots,i_n}$ on $\overline{\mathcal{C}}$ and $F \in C^{i_1,\ldots,i_n}$ on $\overline{\mathcal{D}}$, where i_j is 1 if X^j is of bounded variation and 2 if X^j is not. Then, if $\boldsymbol{X}$ is a continuous semi-martingale, the following change of variable formula for $F(\boldsymbol{X}_t)$ holds ([112, Thm. 2.1]):

$$
\begin{aligned}
F(\boldsymbol{X}_t) = F(\boldsymbol{X}_0) &+ \sum_{i=1}^{d} \int_0^t \frac{1}{2} \left(F_{x_i}(X_s^1, \ldots, X_s^d +) + F_{x_i}(X_s^1, \ldots, X_s^d -) \right) \mathrm{d}X_s^{(i)} \\
&+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{1}{2} \left(F_{x_i x_j}(X_s^1, \ldots, X_s^d +) + F_{x_i x_j}(X_s^1, \ldots, X_s^d -) \right) \mathrm{d}\left\langle X^i, X^j \right\rangle_s \qquad (5.5) \\
&+ \frac{1}{2} \int_0^t \left(F_{x_d}(X_s^1, \ldots, X_s^d +) - F_{x_d}(X_s^1, \ldots, X_s^d -) \right) \mathbb{1}_{\{X_s^d = b_s^{\boldsymbol{X}}\}} \mathrm{d}L_s^b(\boldsymbol{X}),
\end{aligned}
$$

where

$$
L_s^b(\boldsymbol{X}) = \mathsf{P} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s \mathbb{1}_{\{-\varepsilon \le X_s^d - b^{\boldsymbol{X}}(r) \le \varepsilon\}} \mathrm{d}\left\langle X^d - b^{\boldsymbol{X}}, X^d - b^{\boldsymbol{X}} \right\rangle_r \qquad (5.6)
$$

$L_s^b(\boldsymbol{X})$ is the local time of $\boldsymbol{X}$ on the surface $b^{\boldsymbol{X}}$. The formula can be extended to the discontinuous case where $\boldsymbol{X}$ is a general semi-martingale with jumps ([112, Thm. 3.1, Thm. 3.2]). Furthermore, similar to the one dimensional case, the regularity constraint on F in the closure of $\mathcal{C}$ and $\mathcal{D}$ may be replaced by some easily verifiable conditions in the setting of free boundary problems (see [112, Sec. 4]). However, it is still essential that $b^{\boldsymbol{X}} = b(X^1, \ldots, X^{d-1})$ has to be a semi-martingale so that the local time is well-defined. This may be hard to verify directly in applications to optimal stopping, because the boundary b is not given explicitly, and it was one of the main motivations for our own formula. Elworthy, Truman and Zhao [55] also obtain change of variable formulae for time-space processes where the spatial component is a one-dimensional semi-martingale (for an extension to two-dimensional diffusions see [60]); they require left-derivatives in time and space of the function F to have bounded variation.

Eisenbaum [51] developes change of variable formulae for multi-dimensional Lévy processes when first order partial derivatives of the function F exist and are integrable, without further assumptions on second order derivatives. She relies on a suitable notion of integrals with respect to local time $(a,t) \mapsto L^a_t$, understood as integrator in both variables, and connects her results to all the papers we mentioned so far (see also [49] and [50] for earlier closely related work by the same author). More recently, Wilson [134] also studies integrals with respect to local time as a map $(a,t) \mapsto L^a_t$ (building upon ideas from [51] and [74]). He then uses such integrals in [135] to derive a change of variable formula for functions $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ when the underlying process is a two-dimensional jump diffusion process whose jumps are of bounded variation and with no diffusive part in the second component. Wilson's assumptions on F are in the same spirit as those by Eisenbaum but his change of variable formula draws on [110] and [112]. However, [135] requires that either the boundary $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous or $b^X := b(t,X)$ be of bounded variation. Both assumptions are generally difficult to check in applications to optimal stopping. Finally, under the assumption that smooth-fit holds and with an analogue of our Assumption A.2 in place, [135] obtains a generalisation of Itô's formula without requiring b^X of bounded variation (but still requiring X of bounded variation).

It is worth mentioning that a number of interesting results on generalisations of Itô's formula developed in the early 2000 are collected in the book [47]. There we find for example work by Kyprianou and Surya [94] on a change of variable formula with local times on curves, for one-dimensional Lévy processes of bounded variation. Some of the work by Eisenbaum, Peskir, Russo and Vallois are also contained therein.

In the the theory of stochastic control the most widely used extensions of Itô's formula for time-space diffusion processes (generally admitting smooth transition density), require $F \in W^{1,2,p}_{\ell oc}(\mathbb{R}_+ \times \mathbb{R}^d)$ for $p > 1$ sufficiently large to also guarantee that the spatial gradient ∇F is continuous thanks to Sobolev embedding (see, e.g., [16, Ch. 2.8], [93, Ch. 2 Sec. 10] or [64, Ch. 8]). While our proof is inspired by those results, we remark that our function U does not belong to the Sobolev class $W^{1,2,p}_{\ell oc}(\mathbb{R}_+ \times \mathbb{R}^d)$ because we do not require integrability of second order spatial derivatives in neighbourhoods of the boundary $\partial\mathcal{C}$.

In the context of applications to optimal stopping it is also worth mentioning the work by Alsmeyer and Jaeger [2]. They prove a change of variable formula for functions $F : \mathbb{R}^{d+1} \to \mathbb{R}$

that are continuously differentiable and whose derivative in its first variable (denoted $D_{x_0}F$) is absolutely continuous as a map $z \mapsto D_{x_0}F(z, x_1, \dots, x_d)$ for all $(x_1, \dots, x_d)$ fixed. Differently from our set-up their result applies for processes $X = (M, V^1, \dots, V^d)$ where M is a continuous semimartingale and $(V^1, \dots, V^d)$ is a continuous process of locally bounded variation.

This chapter is organised as follows. In Section 5.2 we present our framework and state our change of variable formula. In Section 5.3 we discuss the applicability of our result in optimal stopping problems for multidimensional processes. In Section 5.4 we prove our change of variable formula.

5.2 Setting and main result

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ we consider a $\hat{d}$-dimensional Brownian motion $\boldsymbol{B} := (B^1_t, \dots, B^{\hat{d}}_t)_{t\geq 0}$ and denote by $\boldsymbol{X} := (X^1, ..., X^d)$ a solution in $\mathbb{R}^d$ of the stochastic differential equation (SDE): for $i = 1, \dots, d$,

$$\mathrm{d}X^i_t = \alpha^i(t, \boldsymbol{X}_{t-})\mathrm{d}t + \sum_{j=1}^{\hat{d}} \sigma^{ij}(t, \boldsymbol{X}_{t-})\mathrm{d}B^j_t + \gamma^i(t, \boldsymbol{X}_{t-})\mathrm{d}A^i_t, \quad X^i_0 = x_i, \tag{5.7}$$

where $\boldsymbol{A} = (A^1, ..., A^d)$ is a càdlàg process of bounded variation. Here we use boldface letters to indicate vectors and denote

$$\beta^{ij}(t, \boldsymbol{x}) := \sum_{k=1}^{\hat{d}} \sigma^{ik}(t, \boldsymbol{x})\sigma^{jk}(t, \boldsymbol{x})$$

and $f_{x_i} = \frac{\partial f}{\partial x_i}$, $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for all $i, j = 1, \dots, d$. The coefficients of the SDE are assumed to be measurable and, for the sake of concreteness, we also assume for all $t \geq 0$ that

$$\int_0^t \sum_{i=1}^d \left|\gamma^i(s, \boldsymbol{X}_{s-})\right|\mathrm{d}|A^i|_s + \int_0^t \Big(\sum_{i=1}^d \left|\alpha^i(s, \boldsymbol{X}_s)\right| + \sum_{i,j=1}^d \left|\sigma^{ij}(s, \boldsymbol{X}_s)\right|^2\Big)\mathrm{d}s < \infty, \qquad \mathsf{P}\text{-a.s.},$$

where we denote by $|A^i|_s$ the total variation process associated to A^i.

We divide the state-space into two subsets, i.e., $\mathbb{R}_+ \times \mathbb{R}^d = \mathcal{C} \cup \mathcal{D}$, with $\mathcal{C}$ open and $\mathcal{D}$ closed. We further assume that such subsets can be described in terms of a surface $b_1 : \mathbb{R}_+ \times \mathbb{R}^{d-1} \to \mathbb{R}$ as

$$\mathcal{C} = \{(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d \,:\, x_1 > b_1(t, x_2, ...x_d)\}, \tag{5.8}$$

$$\mathcal{D} = \{(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d \,:\, x_1 \leq b_1(t, x_2, ...x_d)\}. \tag{5.9}$$

The main aim of this chapter is to prove a change of variable formula for functions $U : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ whose second order spatial derivatives may explode along the boundary $\partial\mathcal{C}$ arbitrarily fast.

Theorem 5.2.1. *Assume the following:*

A.1 The coefficients β^{ij} are locally Lipschitz and $\mathsf{P}((t, \boldsymbol{X}_{t-}) \in \partial\mathcal{C}) = 0$ for a.e. $t \geq 0$;

A.2 A function $U : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is such that $U \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ with $U \in C^{1,2}(\mathcal{C}) \cap C^{1,2}(\mathcal{D})$. Moreover, for any compact subset $K \subset \mathbb{R}_+ \times \mathbb{R}^d$ the function

$$L(t, \boldsymbol{x}) := \sum_{i,j=1}^{d} \beta^{ij}(t, \boldsymbol{x}) U_{x_i x_j}(t, \boldsymbol{x}) \tag{5.10}$$

is bounded for $(t, \boldsymbol{x}) \in K \setminus \partial\mathcal{C}$. That is, for any compact K there exists c_K such that

$$\sup_{(t,\boldsymbol{x}) \in K \setminus \partial\mathcal{C}} |L(t, \boldsymbol{x})| \leq c_K; \tag{5.11}$$

A.3 The mappings $x_i \mapsto b_1(t, x_2, \ldots, x_d)$, $i = 2, \ldots, d$, and $t \mapsto b_1(t, x_2, \ldots, x_d)$ are monotonic.

Then, we have the change of variable formula:

$$\begin{aligned}
U(t, \boldsymbol{X}_t) &= U(0, \boldsymbol{x}) \\
&+ \int_0^t \Big[\Big(U_t + \sum_{i=1}^{d} \alpha^i U_{x_i} \Big)(u, \boldsymbol{X}_{u-}) + \tfrac{1}{2} \sum_{i,j=1}^{d} \mathbb{1}_{\{(u, \boldsymbol{X}_{u-}) \notin \partial\mathcal{C}\}} \big(\beta^{ij} U_{x_i x_j} \big)(u, \boldsymbol{X}_{u-}) \Big] \mathrm{d}u \\
&+ \sum_{i=1}^{d} \int_0^t U_{x_i}(u, \boldsymbol{X}_{u-}) \mathrm{d}A_u^{c,i} + \sum_{u \leq t} \Big(U(u, \boldsymbol{X}_u) - U(u, \boldsymbol{X}_{u-}) \Big) \\
&+ \sum_{i,j=1}^{d} \int_0^t U_{x_i}(u, \boldsymbol{X}_{u-}) \sigma^{ij}(u, \boldsymbol{X}_{u-}) \mathrm{d}B_u^j, \qquad \text{for } t \in [0, \infty),\ \mathsf{P}\text{-a.s.},
\end{aligned} \tag{5.12}$$

where we used the decomposition $A_t^i = A_t^{c,i} + \sum_{s \leq t} \Delta A_s^i$ with $A^{c,i}$ the continuous part of the process A^i.

Since the jumps of the process $\boldsymbol{X}$ only arise from the bounded variation process $\boldsymbol{A}$, the expression for the jump terms in (5.12) is equivalent to the usual one found in textbooks:

$$\begin{aligned}
&\sum_{i=1}^{d} \int_0^t U_{x_i}(u, \boldsymbol{X}_{u-}) \mathrm{d}A_u^{c,i} + \sum_{u \leq t} \Big(U(u, \boldsymbol{X}_u) - U(u, \boldsymbol{X}_{u-}) \Big) \\
&= \sum_{i=1}^{d} \int_0^t U_{x_i}(u, \boldsymbol{X}_{u-}) \mathrm{d}A_u^i + \sum_{u \leq t} \Big(U(u, \boldsymbol{X}_u) - U(u, \boldsymbol{X}_{u-}) - \sum_{i=1}^{d} U_{x_i}(u, \boldsymbol{X}_{u-}) \Delta A_u^i \Big).
\end{aligned}$$

It is worth noticing that Assumption A.2 says that the derivatives $U_{x_i x_j}$ are continuous in the closed set $\mathcal{D}$ but they need not be continuous on the closure of $\mathcal{C}$, i.e., the may explode arbitrarily fast when approaching the boundary $\partial\mathcal{C}$ from inside $\mathcal{C}$. Indeed, in general boundedness of the function L in (5.10) is not sufficient for the boundedness of all second order spatial derivatives.

The need to have some control over the function L in (5.10) was already indicated by Peskir. As presented in the previous section, in one-dimensional case, Assumption P.2 is essential to guarantee the first integral in his change of variable formula (5.4) is well-defined. Peskir et al. [56, Thm. 19] also employ a condition similar to (5.11) to obtain Dynkin's formula (rather than Itô's formula) for a two-dimensional diffusion. Their proof requires different arguments to ours as they need convexity/concavity of their function U and use estimates on the expected value of local times.

Remark 5.2.2 (Degenerate processes)**.** *It is intuitively clear and it can be easily seen from the proof of the theorem that if the i-th coordinate of the process $\boldsymbol{X}$ is of bounded variation (i.e., $\sigma^{ij} \equiv 0$ for all $j = 1, \dots, d$) it is* not *necessary to require existence of the second order partial derivatives $U_{x_i x_j}$ for $j = 1, \dots, d$ in Assumption A.2.*

Remark 5.2.3 (Absolutely continuous laws of the process)**.** *If the law of $\boldsymbol{X}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, then we can relax Assumption A.2. Indeed, the time-derivative and the second order spatial derivatives in (5.12) only need to exist a.e. on $\mathbb{R}_+ \times \mathbb{R}^d$. For the proof of the theorem we then require $U \in C(\mathbb{R}_+ \times \mathbb{R}^d)$, with $U_t \in L^2_{\ell oc}(\mathbb{R}_+ \times \mathbb{R}^d)$, $U_{x_i} \in C(\mathbb{R}_+ \times \mathbb{R}^d)$ and $U_{x_i x_j} \in L^2_{\ell oc}(\mathcal{C}) \cap L^2_{\ell oc}(\mathcal{D})$ for all $i, j = 1, \dots, d$, where $f \in L^2_{\ell oc}(\mathcal{C}) \cap L^2_{\ell oc}(\mathcal{D})$ means that for any compact sets $K_1 \subset \mathcal{C}$ and $K_2 \subseteq \mathcal{D}$ we have*

$$\int_{K_1 \cup K_2} |f(t, \boldsymbol{x})|^2 \mathrm{d}t\, \mathrm{d}\boldsymbol{x} < \infty.$$

Notice that $K_1 \cap \partial\mathcal{C} = \varnothing$, whereas it may be $K_2 \cap \partial\mathcal{C} \neq \varnothing$, since $\mathcal{C}$ is open and $\mathcal{D}$ is closed. We also continue to require that for any compact K there exists c_K such that

$$\sup_{(t,\boldsymbol{x}) \in K \setminus \partial\mathcal{C}} |L(t, \boldsymbol{x})| \leq c_K,$$

with L as in (5.10). Notice that these assumptions are less stringent than the usual requirement $U \in W^{1,2,2}_{\ell oc}(\mathbb{R}_+ \times \mathbb{R}^d)$ since we do not require $U_{x_i x_j} \in L^2_{\ell oc}(\mathbb{R}_+ \times \mathbb{R}^d)$ (in particular, $U_{x_i x_j}$ need not be square integrable in a neighbourhood of the boundary $\partial\mathcal{C}$).

The proof of our theorem remains unchanged: the derivation of (5.27) therein is justified using the fundamental theorem of calculus for absolutely continuous functions; all remaining arguments can be repeated verbatim.

Remark 5.2.4 (Assumptions on the boundary)**.** *Assumption A.3 is much easier to verify in applications to multi-dimensional optimal stopping problems than the assumption on the boundary $\partial\mathcal{C}$ made in Peskir's work [112] (and more recently in [135] but only for two dimensional processes). In his formula (5.5), $\boldsymbol{X}$ is a general semi-martingale and the process $b^X_t = b(t, X^2_t, \ldots, X^d_t)$ must also be a semi-martingale (with b continuous). That is not true in general if only monotonicity of the boundary is known. Of course, we are able to allow for much less stringent conditions on the boundary because, differently to Peskir's work, our focus* is not *on the role of local times on surfaces and we assume continuous differentiability of the function U.*

Remark 5.2.5 (Reflecting diffusions)**.** *We chose to state our theorem including the bounded variation process $\boldsymbol{A}$ in the dynamics (5.7) because we have in mind applications to problems for reflecting diffusions and applications in singular stochastic control. In those cases, the condition $\mathsf{P}((t, \boldsymbol{X}_{t-}) \in \partial\mathcal{C}) = 0$ for a.e. $t \geq 0$ in Assumption A.1 is generally satisfied by Skorokhod's construction of reflecting diffusions.*

5.3 Applications in optimal stopping

Our main motivation for the development of a change of variable formula of the kind in Theorem 5.2.1 is its applicability in optimal stopping problems. Indeed, letting $G : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function and $s \mapsto \Pi^t_s(\boldsymbol{X})$ an additive functional of the process $(s, \boldsymbol{X}_s)_{s\geq t}$, one is often interested in problems of the type

$$U(t, \boldsymbol{x}) = \sup_{t\leq\tau\leq T} \mathsf{E}_{t,\boldsymbol{x}}\Big[\mathrm{e}^{-\Pi^t_\tau(\boldsymbol{X})} G(\tau, \boldsymbol{X}_\tau)\Big], \tag{5.13}$$

where $T \in (0, \infty]$ is a fixed horizon, $t \in [0, T]$, the supremum is taken over stopping times of the underlying filtration $(\mathcal{F}_t)$ and the expectation $\mathsf{E}_{t,\boldsymbol{x}}$ is with respect to the measure $\mathsf{P}_{t,\boldsymbol{x}}(\,\cdot\,) := \mathsf{P}(\,\cdot\,|\boldsymbol{X}_t = \boldsymbol{x})$. In most examples the additive functional Π^t arises from a discount rate, i.e.,

$$\Pi^t_s(\boldsymbol{X}) = \int_t^s r(u, \boldsymbol{X}_{u-})\mathrm{d}u, \tag{5.14}$$

for some measurable functions $r : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. However, there are examples in which Π^t may take the form, e.g., of a local time of the process $\boldsymbol{X}$ (see, e.g., [40]).

Under a set of fairly mild assumptions it is known that an optimal stopping time for the problem above exists and it takes the form (see Chapter 2 Section 2.2, also [114])

$$\tau_* = \inf\{s \in [t, T] : U(s, \boldsymbol{X}_s) = G(s, \boldsymbol{X}_s)\}.$$

From this stems the interest for the study of the continuation and stopping sets, denoted by $\mathcal{C}$ and $\mathcal{D}$, respectively, and defined as

$$\mathcal{C} = \{(t,\boldsymbol{x}) : U(t,\boldsymbol{x}) > G(t,\boldsymbol{x})\} \quad \text{and} \quad \mathcal{D} = \{(t,\boldsymbol{x}) : U(t,\boldsymbol{x}) = G(t,\boldsymbol{x})\}.$$

In particular, parametrisations of the continuation and stopping sets as those presented in (5.8) and (5.9) are widely studied in the literature as they often enable a detailed theoretical analysis of the problem at hand.

Together with the probabilistic results on optimality of τ_* and the super-harmonic property there is also an analytic formulation of problem (5.13), in terms of a free boundary problem (see Chapter 2 Section 2.2). For simplicity let us take $\gamma^i \equiv 0$ in (5.7) and Π^t as in (5.14). Then the free boundary problem solved by the value function reads

$$\begin{aligned} &U_t + \tfrac{1}{2}\sum_{i,j}\beta^{ij}U_{x_ix_j} + \sum_i \alpha^i U_{x_i} - rU = 0, \quad \text{in } \mathcal{C}, \\ &U_t + \tfrac{1}{2}\sum_{i,j}\beta^{ij}U_{x_ix_j} + \sum_i \alpha^i U_{x_i} - rU \le 0, \quad \text{in } \mathcal{D}, \end{aligned} \tag{5.15}$$

with terminal condition $U(T,\boldsymbol{x}) = G(T,\boldsymbol{x})$. It is possible to prove (see [42]) that if $\partial\mathcal{C}$ is regular in the sense of diffusions for the interior of the stopping set, then $U \in C^1([0,T)\times\mathbb{R}^d)$. Moreover, it is clear that $U = G$ on $\mathcal{D}$. If for example $G \in C^{1,2}(\mathcal{D})$, then U inherits such regularity and we have

$$U_t + \tfrac{1}{2}\sum_{i,j}\beta^{ij}U_{x_ix_j} + \sum_i \alpha^i U_{x_i} - rU = G_t + \tfrac{1}{2}\sum_{i,j}\beta^{ij}G_{x_ix_j} + \sum_i \alpha^i G_{x_i} - rG, \quad \text{in } \mathcal{D}.$$

So, by the free boundary formulation we see that the function L from Assumption A.2 reads

$$L(t,\boldsymbol{x}) = \begin{cases} 2(rU - \sum_i \alpha^i U_{x_i} - U_t)(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in \mathcal{C}, \\ \sum_{i,j}\beta^{ij}(t,\boldsymbol{x})G_{x_ix_j}(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in \mathcal{D}. \end{cases} \tag{5.16}$$

It is immediate to see that in this context the bound on L required by Assumption A.2 is satisfied as soon as α^i and r are continuous functions and $G \in C^{1,2}(\mathcal{D})$, provided also that U is continuously differentiable once (which would be implied by regularity of $\partial\mathcal{C}$ in the sense of diffusions). This brief discussion shows that in optimal stopping, it is potentially rather easy to prove that Assumption A.2 holds, whereas obtaining bounds on each of the second derivatives $U_{x_ix_j}$ could be extremely difficult. Likewise, proving geometric properties of the boundary $\partial\mathcal{C}$ beyond the existence of a surface b_1 as in (5.8) and its monotonicity in each variable, is prohibitively difficult in the majority of examples in the literature on multi-dimensional optimal stopping problems. However, monotonicity is often sufficient to prove regularity of $\partial\mathcal{C}$ in the sense of diffusions (see, e.g., [26]) and therefore continuous differentiability of the value function. This discussion shows that our change of variable formula is tailored for applications to the value function U of optimal stopping problems like (5.13).

Remark 5.3.1 (Continuous differentiability of U)**.** *It may appear that the requirement $U \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ be much stronger than the usual smooth-fit condition in optimal stopping. However, the smooth-fit condition is normally proved relying upon convergence of τ_* to zero in the limit as the initial point $\boldsymbol{X}_0 = \boldsymbol{x}$ of the underlying process approaches $\partial\mathcal{C}$ along a direction parallel to the x_1-axis (in the parametrisation of* (5.8)*). Such convergence is essentially equivalent to the concept of 'regularity' of $\partial\mathcal{C}$ in the sense of diffusions, which would also imply continuous differentiability of U as shown in [42].*

Optimal stopping problems on multi-dimensional underlying processes are appearing with increasing frequency in the literature and here we briefly review specific examples that fit within our framework. In the previous chapter ([26]), we study the classical American put option problem under stochastic discounting and we apply directly results from this chapter. A general study of optimal stopping boundaries for multi-dimensional diffusions can be found in [34]. In the context of quickest detection problems, multi-dimensional situations arise for example in [84], [72] and [54]. In problems of singular control (that can be linked to optimal stopping) solved via free boundary methods we find the contributions [44], [9], [62], among others.

5.4 Proof of the main Theorem

We first prove our result in Section 5.4.1, in the case when

$$b_1 \text{ is non-decreasing in } t \text{ and in } x_i, \text{ for } i = 2, \dots, d. \tag{5.17}$$

The remaining cases in Assumption A.3 will be discussed later, in Section 5.4.2, as they only require minor changes to the arguments of proof.

5.4.1 Proof under (5.17)

Proof. We regularise our function U to obtain an approximating sequence

$$(U^n)_{n\geq 1} \subset C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$$

defined by

$$\begin{aligned} U^n(t,\boldsymbol{x}) : &= n^d \int_{x_1}^{x_1+1/n} \dots \int_{x_d}^{x_d+1/n} U(t, z_1, \dots, z_d) \mathrm{d}z_1 \dots, \mathrm{d}z_d \\ &= n^d \int_{\Lambda_n(\boldsymbol{x})} U(t,\boldsymbol{z}) \mathrm{d}\boldsymbol{z}, \end{aligned} \tag{5.18}$$

where $\Lambda_n(\boldsymbol{x}) := \bigtimes_{k=1}^{d}[x_k, x_k{+}1/n]$. To keep the notation simple, below we write $\Lambda_n = \Lambda_n(\boldsymbol{x})$ whenever $\boldsymbol{x}$ is fixed. Since $U \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ it is clear that $U^n \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and its derivatives read

$$U_t^n(t,\boldsymbol{x}) = n^d \int_{\Lambda_n} U_t(t,\boldsymbol{z}) \mathrm{d}\boldsymbol{z}, \tag{5.19}$$

$$U_{x_i}^n(t,\boldsymbol{x}) = n^d \int_{\Lambda_n} U_{x_i}(t,\boldsymbol{z}) \mathrm{d}\boldsymbol{z}, \tag{5.20}$$

$$\begin{aligned} U_{x_i x_j}^n(t,\boldsymbol{x}) &= n^d \int_{\Lambda_n^{-i}} \big[U_{x_j}(t, x_i + 1/n, \boldsymbol{z}_{-i}) - U_{x_j}(t, x_i, \boldsymbol{z}_{-i})\big] \mathrm{d}\boldsymbol{z}_{-i} \\ &= n^d \int_{\Lambda_n^{-j}} \big[U_{x_i}(t, x_j + 1/n, \boldsymbol{z}_{-j}) - U_{x_i}(t, x_j, \boldsymbol{z}_{-j})\big] \mathrm{d}\boldsymbol{z}_{-j}, \end{aligned} \tag{5.21}$$

for any $i, j \in 1, \dots, d$, where we use the notations

$$\Lambda_n^{-i} := \Big(\bigtimes_{k=1}^{i-1}[x_k, x_k + 1/n]\Big) \times \Big(\bigtimes_{k=i+1}^{d}[x_k, x_k + 1/n]\Big) \text{ and } \boldsymbol{z}_{-i} := (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_d). \tag{5.22}$$

Although $U_{x_i x_j}$ fails to be continuous at the boundary $\partial\mathcal{C}$, for each $(t,\boldsymbol{x}) \notin \partial\mathcal{C}$ there is a large enough n such that

$$U_{x_i x_j}^n(t,\boldsymbol{x}) = n^d \int_{\Lambda_n} U_{x_i x_j}(t,\boldsymbol{z}) \mathrm{d}\boldsymbol{z}.$$

Consequently, for $i, j = 1, ..., d$, and for any compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$ we have

$$\begin{aligned} &\lim_{n\uparrow\infty} \sup_{(t,\boldsymbol{x})\in K} \Big(\big|U^n - U\big|(t,\boldsymbol{x}) + \big|U_t^n - U_t\big|(t,\boldsymbol{x}) + \sum_{i=1}^{d} \big|U_{x_i}^n - U_{x_i}\big|(t,\boldsymbol{x})\Big) = 0, \\ &\lim_{n\uparrow\infty} U_{x_i x_j}^n(t,\boldsymbol{x}) = U_{x_i x_j}(t,\boldsymbol{x}), \quad \text{for all } (t,\boldsymbol{x}) \in \big(\mathbb{R}_+ \times \mathbb{R}^d\big) \setminus \partial\mathcal{C}. \end{aligned} \tag{5.23}$$

For $\delta > 0$, let us set

$$V^\delta := [0, 1/\delta] \times [-1/\delta, 1/\delta]^d, \tag{5.24}$$

and

$$\tau_\delta := \inf\{t \geq 0 : (t, \boldsymbol{X}_t) \notin V^\delta\}. \tag{5.25}$$

Applying Itô's formula to $U^n(t, \boldsymbol{X}_{t\wedge\tau_\delta})$, we obtain

$$
\begin{aligned}
U^n(t\wedge\tau_\delta, \boldsymbol{X}_{t\wedge\tau_\delta}) &= U^n(0,\boldsymbol{x}) \\
&+ \int_0^{t\wedge\tau_\delta} \Big[\Big(U^n_t + \sum_{i=1}^d \alpha^i U^n_{x_i}\Big)(u,\boldsymbol{X}_{u-}) + \tfrac{1}{2}\sum_{i,j=1}^d \mathbb{1}_{\{(u,\boldsymbol{X}_{u-})\notin\partial\mathcal{C}\}}\big(\beta^{ij}U^n_{x_ix_j}\big)(u,\boldsymbol{X}_{u-})\Big]\mathrm{d}u \\
&+ \sum_{i=1}^d \int_0^{t\wedge\tau_\delta} U^n_{x_i}(u,\boldsymbol{X}_{u-})\mathrm{d}A^i_u + \sum_{u\le t\wedge\tau_\delta}\Big(U^n(u,\boldsymbol{X}_u) - U^n(u,\boldsymbol{X}_{u-}) - \sum_{i=1}^d U^n_{x_i}(u,\boldsymbol{X}_{u-})\Delta A^i_u\Big) \\
&+ \sum_{i,j=1}^d \int_0^{t\wedge\tau_\delta} U^n_{x_i}(u,\boldsymbol{X}_{u-})\sigma^{ij}(u,\boldsymbol{X}_{u-})\mathrm{d}B^j_u, \qquad \text{for } t\in[0,\infty), \text{ P-a.s.}
\end{aligned}
$$

having also used $\mathsf{P}((t,\boldsymbol{X}_{t-})\in\partial\mathcal{C}) = 0$ for a.e. $t\ge 0$ by Assumption A.1. Since the jumps of the process $\boldsymbol{X}$ only arise from the bounded variation process $\boldsymbol{A}$, we can also simplify the expression above by writing

$$
\begin{aligned}
&\sum_{i=1}^d \int_0^{t\wedge\tau_\delta} U^n_{x_i}(u,\boldsymbol{X}_{u-})\mathrm{d}A^i_u + \sum_{u\le t\wedge\tau_\delta}\Big(U^n(u,\boldsymbol{X}_u) - U^n(u,\boldsymbol{X}_{u-}) - \sum_{i=1}^d U^n_{x_i}(u,\boldsymbol{X}_{u-})\Delta A^i_u\Big) \\
&= \sum_{i=1}^d \int_0^{t\wedge\tau_\delta} U^n_{x_i}(u,\boldsymbol{X}_{u-})\mathrm{d}A^{c,i}_u + \sum_{u\le t\wedge\tau_\delta}\Big(U^n(u,\boldsymbol{X}_u) - U^n(u,\boldsymbol{X}_{u-})\Big),
\end{aligned}
$$

by using the decomposition $A^i_t = A^{c,i}_t + \sum_{s\le t}\Delta A^i_s$ with $A^{c,i}$ the continuous part of the process A^i. Letting $n\to\infty$ (possibly along a subsequence) all terms involving only U^n and its first derivatives (including the stochastic integral and the jump terms) converge to their analogue for the function U, thanks to the uniform convergence in (5.23). Notice indeed that $(u,\boldsymbol{X}_{u-})\in V^\delta$ for $u\in[0,t\wedge\tau_\delta]$ and we use pointwise convergence for the single term $U^n(t\wedge\tau_\delta,\boldsymbol{X}_{t\wedge\tau_\delta})$ in the sum of jumps.

If we can justify the use of dominated convergence to pass limits under the integral for the terms involving the second order spatial derivatives, then using the second limit in (5.23) we obtain (5.12), upon also letting $\delta\downarrow 0$ at the end.

Since U is twice continuously differentiable in space at all points off the boundary $\partial\mathcal{C}$ and given that $\mathsf{P}((t,\boldsymbol{X}_{t-})\in\partial\mathcal{C}) = 0$ for a.e. $t\ge 0$, it is enough to to prove that there exists a constant $C_\delta > 0$ independent of n, such that

$$
\sup_{(t,\boldsymbol{x})\in V^\delta}\Big|\sum_{i,j=1}^d \beta^{ij}(t,\boldsymbol{x})U^n_{x_ix_j}(t,\boldsymbol{x})\Big| \le C_\delta. \tag{5.26}
$$

We accomplish our task in two steps.

Step 1. We show that for any $(t,\boldsymbol{x}) \in V^\delta \setminus \partial\mathcal{C}$ and n fixed, $U^n_{x_ix_j}(t,\boldsymbol{x})$ admits the representation:

$$\begin{aligned} U^n_{x_ix_j}(t,\boldsymbol{x}) =& n^d \int_{\Lambda_n} U_{x_ix_j}(t,\boldsymbol{z}) \mathbb{1}_{\{z_1 \geq b_1^\varepsilon(t,z_2,\dots,z_d)\}} \mathrm{d}\boldsymbol{z} \\ &+ n^d \int_{\Lambda_n} U_{x_ix_j}(t,\boldsymbol{z}) \mathbb{1}_{\{z_1 \leq b_1(t,z_2,\dots,z_d)\}} \mathrm{d}\boldsymbol{z} + F^{n,\varepsilon}_{ij}(t,\boldsymbol{x}), \quad \forall \varepsilon > 0 \end{aligned} \tag{5.27}$$

for any $i,j = 1,...,d$, where $F^{n,\varepsilon}_{ij}$ is a remainder that we will show converges to zero and $b_1^\varepsilon : \mathbb{R}_+ \times \mathbb{R}^{d-1} \mapsto \mathbb{R}$ is defined as

$$b_1^\varepsilon(t,z_2,\dots,z_d) := b_1(t+\varepsilon, z_2+\varepsilon, z_3+\varepsilon,\dots,z_d+\varepsilon) + \varepsilon. \tag{5.28}$$

Recall the compact notation $\boldsymbol{z}_{-i}$ from (5.22). Since we are currently assuming that b_1 is non-decreasing in all variables, the limit:

$$b_1^{0+}(t,\boldsymbol{z}_{-1}) := \lim_{\varepsilon \downarrow 0} b_1^\varepsilon(t,\boldsymbol{z}_{-1}),$$

exists and $b_1^{0+}(t,\boldsymbol{z}_{-1}) \geq b_1(t,\boldsymbol{z}_{-1})$. Using that $\mathcal{D}$ is closed then

$$\mathcal{D} \ni \big(t+\varepsilon, b_1^\varepsilon(t,\boldsymbol{z}_{-1}) - \varepsilon, z_2+\varepsilon,\dots,z_d+\varepsilon\big) \to \big(t, b_1^{0+}(t,\boldsymbol{z}_{-1}), z_2,\dots,z_d\big) \in \mathcal{D},$$

as $\varepsilon \downarrow 0$ and, therefore, $b_1^{0+}(t,\boldsymbol{z}_{-1}) \leq b_1(t,\boldsymbol{z}_{-1}) \leq b_1^{0+}(t,\boldsymbol{z}_{-1})$ by definition of the set $\mathcal{D}$. The reason for introducing the function b_1^ε is that the set

$$\mathcal{C}_1^\varepsilon := \{(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d : x_1 > b_1^\varepsilon(t,\boldsymbol{x}_{-1})\} \tag{5.29}$$

is such that its closure is strictly contained in $\mathcal{C}$ for all $\varepsilon > 0$, i.e.,

$$\overline{\mathcal{C}_1^\varepsilon} \subset \mathcal{C}. \tag{5.30}$$

The latter fact will be used several times, along with the fact that $U_{x_ix_j} \in C(\overline{\mathcal{C}_1^\varepsilon})$.

Let us start with $i = 1$ (or $j = 1$) and using the expression in (5.21), let us re-write the integral by considering separately the cases in which the interval $[x_1, x_1 + 1/n]$ overlaps with the interval $[b_1, b_1^\varepsilon]$. To that aim and recalling the notations Λ_d^{-i} and $\boldsymbol{z}_{-i}$, it is useful to observe that

$$\Lambda_n^{-1}(\boldsymbol{x}) = \Theta_n^\varepsilon(x_1) \cup \Gamma_n^\varepsilon(x_1) \cup \Sigma_n^\varepsilon(x_1), \tag{5.31}$$

where the sets

$$\begin{aligned} \Theta_n^\varepsilon(x_1) &:= \{\boldsymbol{z}_{-1} : x_1 \geq b_1^\varepsilon(t,\boldsymbol{z}_{-1})\} \cup \{\boldsymbol{z}_{-1} : x_1 + \tfrac{1}{n} \leq b_1(t,\boldsymbol{z}_{-1})\}, \\ \Gamma_n^\varepsilon(x_1) &:= \{\boldsymbol{z}_{-1} : x_1 + \tfrac{1}{n} \geq b_1^\varepsilon(t,\boldsymbol{z}_{-1})\} \cap \{\boldsymbol{z}_{-1} : b_1(t,\boldsymbol{z}_{-1}) \geq x_1\}, \end{aligned}$$

and

$$\begin{aligned}\Sigma_n^\varepsilon(x_1) :=& \big\{\boldsymbol{z}_{-1} : x_1 + \tfrac{1}{n} \geq b_1^\varepsilon(t,\boldsymbol{z}_{-1}) > x_1 > b_1(t,\boldsymbol{z}_{-1})\big\}\\ &\cup \big\{\boldsymbol{z}_{-1} : b_1^\varepsilon(t,\boldsymbol{z}_{-1}) > x_1 + \tfrac{1}{n} > x_1 > b_1(t,\boldsymbol{z}_{-1})\big\}\\ &\cup \big\{\boldsymbol{z}_{-1} : b_1^\varepsilon(t,\boldsymbol{z}_{-1}) > x_1 + \tfrac{1}{n} > b_1(t,\boldsymbol{z}_{-1}) \geq x_1\big\}\\ &=: \Sigma_{n,1}^\varepsilon(x_1) \cup \Sigma_{n,2}^\varepsilon(x_1) \cup \Sigma_{n,3}^\varepsilon(x_1)\end{aligned}$$

are disjoint. So the integral (5.21) can be written as

$$\begin{aligned}U^n_{x_1x_j}(t,\boldsymbol{x}) &= n^d \int_{\Lambda_n^{-1}} \big[U_{x_j}(t,x_1+\tfrac{1}{n},\boldsymbol{z}_{-1}) - U_{x_j}(t,x_1,\boldsymbol{z}_{-1})\big]\mathrm{d}\boldsymbol{z}_{-1}\\ &= n^d \int_{\Theta_n^\varepsilon(x_1)} \Big(\int_{x_1}^{x_1+\frac{1}{n}} U_{x_1x_j}(t,z_1,\boldsymbol{z}_{-1})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}\\ &\quad + n^d \int_{\Gamma_n^\varepsilon(x_1)} \big[U_{x_j}(t,x_1+\tfrac{1}{n},\boldsymbol{z}_{-1}) - U_{x_j}(t,x_1,\boldsymbol{z}_{-1})\big]\mathrm{d}\boldsymbol{z}_{-1}\\ &\quad + n^d \int_{\Sigma_n^\varepsilon(x_1)} \big[U_{x_j}(t,x_1+\tfrac{1}{n},\boldsymbol{z}_{-1}) - U_{x_j}(t,x_1,\boldsymbol{z}_{-1})\big]\,\mathrm{d}\boldsymbol{z}_{-1},\end{aligned}\tag{5.32}$$

where we also used that $U_{x_1x_j}(t,\,\cdot\,)$ is continuous on $[x_1, x_1+\frac{1}{n}] \times \Theta_n^\varepsilon(x_1)$. In the first integral (on the set $\Theta_n^\varepsilon(x_1)$) we have

$$\begin{aligned}&n^d \int_{\Theta_n^\varepsilon(x_1)} \Big(\int_{x_1}^{x_1+\frac{1}{n}} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}\\ &= n^d \int_{\Lambda_n^{-1}} \mathbb{1}_{\{x_1\geq b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}}\Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1\geq b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}\\ &\quad + n^d \int_{\Lambda_n^{-1}} \mathbb{1}_{\{x_1+\frac{1}{n}\leq b_1(t,\boldsymbol{z}_{-1})\}}\Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1\leq b_1(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}.\end{aligned}\tag{5.33}$$

In the second integral (on the set $\Gamma_n^\varepsilon(x_1)$) we can add and subtract $U_{x_j}(t, b_1^\varepsilon(t,\boldsymbol{z}_{-1}),\boldsymbol{z}_{-1})$ and $U_{x_j}(t,b_1(t,\boldsymbol{z}_{-1}),\boldsymbol{z}_{-1})$ to obtain

$$\begin{aligned}&n^d \int_{\Gamma_n^\varepsilon(x_1)} \big[U_{x_j}(t,x_1+\tfrac{1}{n},\boldsymbol{z}_{-1}) - U_{x_j}(t,x_1,\boldsymbol{z}_{-1})\big]\mathrm{d}\boldsymbol{z}_{-1}\\ &= n^d \int_{\Gamma_n^\varepsilon(x_1)} \Big(\int_{b_1^\varepsilon(t,\boldsymbol{z}_{-1})}^{x_1+\frac{1}{n}} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}\\ &\quad + n^d \int_{\Gamma_n^\varepsilon(x_1)} \big[U_{x_j}(t,b_1^\varepsilon(t,\boldsymbol{z}_{-1}),\boldsymbol{z}_{-1}) - U_{x_j}(t,b_1(t,\boldsymbol{z}_{-1}),\boldsymbol{z}_{-1})\big]\mathrm{d}\boldsymbol{z}_{-1}\\ &\quad + n^d \int_{\Gamma_n^\varepsilon(x_1)} \Big(\int_{x_1}^{b_1(t,\boldsymbol{z}_{-1})} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}\end{aligned}\tag{5.34}$$

by using that $U_{x_1x_j}$ is continuous in $\overline{\mathcal{C}_1^\varepsilon}$ and in $\mathcal{D}$. In the third integral (on the set $\Sigma_n^\varepsilon(x_1)$) we can also proceed in a similar way taking advantage of the decomposition over $\Sigma_{n,1}^\varepsilon(x_1)$, $\Sigma_{n,2}^\varepsilon(x_1)$ and $\Sigma_{n,3}^\varepsilon(x_1)$. In particular, that gives

$$\begin{aligned} n^d &\int_{\Sigma_{n,1}^\varepsilon(x_1)} \left[U_{x_j}(t, x_1 + \tfrac{1}{n}, \boldsymbol{z}_{-1}) - U_{x_j}(t, x_1, \boldsymbol{z}_{-1})\right] \mathrm{d}\boldsymbol{z}_{-1} \\ &= n^d \int_{\Sigma_{n,1}^\varepsilon(x_1)} \Big(\int_{b_1^\varepsilon(t,\boldsymbol{z}_{-1})}^{x_1+\frac{1}{n}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ &\quad + n^d \int_{\Sigma_{n,1}^\varepsilon(x_1)} \left[U_{x_j}(t, b_1^\varepsilon(t, \boldsymbol{z}_{-1}), \boldsymbol{z}_{-1}) - U_{x_j}(t, x_1, \boldsymbol{z}_{-1})\right] \mathrm{d}\boldsymbol{z}_{-1} \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} n^d &\int_{\Sigma_{n,3}^\varepsilon(x_1)} \left[U_{x_j}(t, x_1 + \tfrac{1}{n}, \boldsymbol{z}_{-1}) - U_{x_j}(t, x_1, \boldsymbol{z}_{-1})\right] \mathrm{d}\boldsymbol{z}_{-1} \\ &= n^d \int_{\Sigma_{n,3}^\varepsilon(x_1)} \left[U_{x_j}(t, x_1 + \tfrac{1}{n}, \boldsymbol{z}_{-1}) - U_{x_j}(t, b_1(t, \boldsymbol{z}_{-1}), \boldsymbol{z}_{-1})\right] \mathrm{d}\boldsymbol{z}_{-1} \\ &\quad + n^d \int_{\Sigma_{n,3}^\varepsilon(x_1)} \Big(\int_{x_1}^{b_1(t,\boldsymbol{z}_{-1})} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}. \end{aligned} \tag{5.36}$$

Let us notice that we can add up the first term on the right-hand side of (5.33), (5.34) and (5.35), which gives

$$\begin{aligned} n^d &\int_{\Lambda_n^{-1}} \mathbb{1}_{\{x_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}}\Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ &+ n^d \int_{\Gamma_n^\varepsilon(x_1)} \Big(\int_{b_1^\varepsilon(t,\boldsymbol{z}_{-1})}^{x_1+\frac{1}{n}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ &+ n^d \int_{\Sigma_{n,1}^\varepsilon(x_1)} \Big(\int_{b_1^\varepsilon(t,\boldsymbol{z}_{-1})}^{x_1+\frac{1}{n}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1}. \end{aligned}$$

The above expression is equal to

$$\begin{aligned} n^d &\int_{\Lambda_n^{-1}} \mathbb{1}_{\{x_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}}\Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ &+ n^d \int_{\Lambda_n^{-1}} \mathbb{1}_{\{x_1+\frac{1}{n} \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1}) > x_1\}}\Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ =& n^d \int_{\Lambda_n^{-1}} \Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ =& n^d \int_{\Lambda_n} \mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t, \boldsymbol{z})\mathrm{d}\boldsymbol{z}, \end{aligned} \tag{5.37}$$

where the first equality uses the fact that on $\{x_1 + \frac{1}{n} < b_1^\varepsilon(t, \boldsymbol{z}_{-1})\}$ the integral with respect to $\mathrm{d}z_1$ vanishes. Similarly, we can now add up the second term on the right-hand side of (5.33) and (5.36) with the third one on the right-hand side of (5.34), to obtain

$$\begin{aligned} n^d \int_{\Lambda_n^{-1}} \mathbb{1}_{\{x_1+\frac{1}{n}\le b_1(t,\boldsymbol{z}_{-1})\}} \Big(\int_{x_1}^{x_1+\frac{1}{n}} \mathbb{1}_{\{z_1\le b_1(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ + n^d \int_{\Gamma_n^\varepsilon(x_1)} \Big(\int_{x_1}^{b_1(t,\boldsymbol{z}_{-1})} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ + n^d \int_{\Sigma_{n,3}^\varepsilon(x_1)} \Big(\int_{x_1}^{b_1(t,\boldsymbol{z}_{-1})} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}z_1\Big)\mathrm{d}\boldsymbol{z}_{-1} \\ = n^d \int_{\Lambda_n} \mathbb{1}_{\{z_1\le b_1(t,\boldsymbol{z}_{-1})\}} U_{x_1x_j}(t,\boldsymbol{z})\mathrm{d}\boldsymbol{z}. \end{aligned} \tag{5.38}$$

Finally, we gather the remaining terms from (5.34), (5.35), (5.36) and the one remaining integral from (5.32) (i.e., the one over $\Sigma_{n,2}^\varepsilon(x_1)$) and denote

$$\begin{aligned} F_{1j}^{n,\varepsilon}(t,\boldsymbol{x}) :=& n^d \int_{\Gamma_n^\varepsilon(x_1)} \big[U_{x_j}(t, b_1^\varepsilon(t,\boldsymbol{z}_{-1}), \boldsymbol{z}_{-1}) - U_{x_j}(t, b_1(t,\boldsymbol{z}_{-1}), \boldsymbol{z}_{-1})\big]\mathrm{d}\boldsymbol{z}_{-1} \\ &+ n^d \int_{\Sigma_{n,1}^\varepsilon(x_1)} \big[U_{x_j}(t, b_1^\varepsilon(t,\boldsymbol{z}_{-1}), \boldsymbol{z}_{-1}) - U_{x_j}(t, x_1, \boldsymbol{z}_{-1})\big]\,\mathrm{d}\boldsymbol{z}_{-1} \\ &+ n^d \int_{\Sigma_{n,2}^\varepsilon(x_1)} \big[U_{x_j}(t, x_1 + \tfrac{1}{n}, \boldsymbol{z}_{-1}) - U_{x_j}(t, x_1, \boldsymbol{z}_{-1})\big]\,\mathrm{d}\boldsymbol{z}_{-1} \\ &+ n^d \int_{\Sigma_{n,3}^\varepsilon(x_1)} \big[U_{x_j}(t, x_1 + \tfrac{1}{n}, \boldsymbol{z}_{-1}) - U_{x_j}(t, b_1(t,\boldsymbol{z}_{-1}), \boldsymbol{z}_{-1})\big]\,\mathrm{d}\boldsymbol{z}_{-1}. \end{aligned} \tag{5.39}$$

Combining (5.37), (5.38) and (5.39) we obtain (5.27) for $i = 1$. Before proving that indeed $F_{1j}^{n,\varepsilon}$ vanishes as $\varepsilon \downarrow 0$ while keeping n fixed, we prove (5.27) for a generic couple i, j.

Fix $i \neq 1, j \neq 1$ and recall that we are currently assuming b_1 non-decreasing in all its arguments. Then, in particular we can define the generalised (left-continuous) inverse of b_1 with respect to x_i:

$$b_i(t, \boldsymbol{x}_{-i}) := \sup\{x_i \in \mathbb{R} : x_1 > b_1(t, x_2, \dots, x_d)\}. \tag{5.40}$$

It is not hard to check that $x_1 > b_1(t, \boldsymbol{x}_{-1}) \iff x_i < b_i(t, \boldsymbol{x}_{-i})$, $x_1 \mapsto b_i(t, \boldsymbol{x}_{-i})$ is non-decreasing, while $x_j \mapsto b_i(t, \boldsymbol{x}_{-i})$ and $t \mapsto b_i(t, \boldsymbol{x}_{-i})$ are non-increasing for all $j \neq \{1, i\}$. Thus, we can parametrise $\mathcal{C}$ and $\mathcal{D}$ as

$$\begin{aligned} \mathcal{C} &= \{(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d : x_i < b_i(t, \boldsymbol{x}_{-i})\}, \\ \mathcal{D} &= \{(t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d : x_i \geq b_i(t, \boldsymbol{x}_{-i})\}, \end{aligned} \tag{5.41}$$

and the analogue of (5.28) in this case is

$$b_i^\varepsilon(t,\boldsymbol{x}_{-i}) := b_i(t+\varepsilon, x_1-\varepsilon, x_2+\varepsilon,\dots,x_{i-1}+\varepsilon,x_{i+1}+\varepsilon,\dots,x_d+\varepsilon)-\varepsilon. \tag{5.42}$$

It is important to notice that thanks to the monotonicity stated above for b_i^ε, the limit:

$$b_i^{0+}(t,\boldsymbol{x}_{-i}) := \lim_{\varepsilon\downarrow 0} b_i^\varepsilon(t,\boldsymbol{x}_{-i})$$

exists and an $b_i^{0+}(t,\boldsymbol{x}_{-i}) \le b_i(t,\boldsymbol{x}_{-i})$. Then, as in the case of b_1^ε above, since $\mathcal{D}$ is closed we have

$$(t,x_1,\dots,x_{i-1},b_i^{0+}(t,\boldsymbol{x}_{-i}),x_{i+1},\dots,x_d)\in\mathcal{D},$$

Hence

$$b_i^{0+}(t,\boldsymbol{x}_{-i}) \le b_i(t,\boldsymbol{x}_{-i}) \le b_i^{0+}(t,\boldsymbol{x}_{-i}). \tag{5.43}$$

Furthermore, letting

$$\mathcal{C}_i^\varepsilon := \{(t,\boldsymbol{x})\in\mathbb{R}_+\times\mathbb{R}^d : x_i < b_i^\varepsilon(t,\boldsymbol{x}_{-i})\} \tag{5.44}$$

we have $\overline{\mathcal{C}_i^\varepsilon}\subset\mathcal{C}$, for all $\varepsilon>0$. Thus, repeating the same estimates as above we obtain

$$\begin{aligned} U^n_{x_ix_j}(t,\boldsymbol{x}) =& n^d\int_{\Lambda_n} U_{x_ix_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_i\le b_i^\varepsilon(t,\boldsymbol{z}_{-i})\}}\mathrm{d}\boldsymbol{z} \\ &+ n^d\int_{\Lambda_n} U_{x_ix_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_i\ge b_i(t,\boldsymbol{z}_{-i})\}}\mathrm{d}\boldsymbol{z} + F^{n,\varepsilon}_{ij}(t,\boldsymbol{x}), \end{aligned} \tag{5.45}$$

where

$$\begin{aligned} F^{n,\varepsilon}_{ij}(t,\boldsymbol{x}) :=& n^d\int_{\Gamma_n^\varepsilon(x_i)}\big[U_{x_j}(t,b_i(t,\boldsymbol{z}_{-i}),\boldsymbol{z}_{-i})-U_{x_j}(t,b_i^\varepsilon(t,\boldsymbol{z}_{-i}),\boldsymbol{z}_{-i})\big]\mathrm{d}\boldsymbol{z}_{-i} \\ &+ n^d\int_{\Sigma^\varepsilon_{n,1}(x_i)}\big[U_{x_j}(t,x_i+\tfrac{1}{n},\boldsymbol{z}_{-i})-U_{x_j}(t,b_i^\varepsilon(t,\boldsymbol{z}_{-i}),\boldsymbol{z}_{-i})\big]\,\mathrm{d}\boldsymbol{z}_{-i} \\ &+ n^d\int_{\Sigma^\varepsilon_{n,2}(x_i)}\big[U_{x_j}(t,x_i+\tfrac{1}{n},\boldsymbol{z}_{-i})-U_{x_j}(t,x_i,\boldsymbol{z}_{-i})\big]\,\mathrm{d}\boldsymbol{z}_{-i} \\ &+ n^d\int_{\Sigma^\varepsilon_{n,3}(x_i)}\big[U_{x_j}(t,b_i(t,\boldsymbol{z}_{-i}),\boldsymbol{z}_{-i})-U_{x_j}(t,x_i,\boldsymbol{z}_{-i})\big]\,\mathrm{d}\boldsymbol{z}_{-i} \end{aligned} \tag{5.46}$$

and we have substituted the sets Γ_n^ε, $\Sigma^\varepsilon_{n,1}$, $\Sigma^\varepsilon_{n,2}$ and $\Sigma^\varepsilon_{n,3}$ from (5.39) with their counterparts in this case:

$$\Theta_n^\varepsilon(x_i) := \{\boldsymbol{z}_{-i} : x_i + \tfrac{1}{n} \le b_i^\varepsilon(t, \boldsymbol{z}_{-i})\} \cup \{\boldsymbol{z}_{-i} : x_i \ge b_i(t, \boldsymbol{z}_{-i})\},$$
$$\Gamma_n^\varepsilon(x_i) := \{\boldsymbol{z}_{-i} : x_i \le b_i^\varepsilon(t, \boldsymbol{z}_{-i})\} \cap \{\boldsymbol{z}_{-i} : x_i + \tfrac{1}{n} \ge b_i(t, \boldsymbol{z}_{-i})\},$$

and

$$\begin{aligned}\Sigma_n^\varepsilon(x_i) :=& \big\{\boldsymbol{z}_{-i} : x_i \le b_i^\varepsilon(t, \boldsymbol{z}_{-i}) < x_i + \tfrac{1}{n} < b_i(t, \boldsymbol{z}_{-i})\big\}\\ &\cup \big\{\boldsymbol{z}_{-i} : b_i^\varepsilon(t, \boldsymbol{z}_{-i}) < x_i < x_i + \tfrac{1}{n} < b_i(t, \boldsymbol{z}_{-i})\big\}\\ &\cup \big\{\boldsymbol{z}_{-i} : b_i^\varepsilon(t, \boldsymbol{z}_{-i}) < x_i < b_i(t, \boldsymbol{z}_{-i}) \le x_i + \tfrac{1}{n}\big\}\\ &=: \Sigma_{n,1}^\varepsilon(x_i) \cup \Sigma_{n,2}^\varepsilon(x_i) \cup \Sigma_{n,3}^\varepsilon(x_i).\end{aligned}$$

Since the sets $\{z_i = b_i^\varepsilon(t, \boldsymbol{z}_{-i})\}$ and $\{z_i = b_i(t, \boldsymbol{z}_{-i})\}$ have zero Lebesgue measure in $\mathbb{R}^d$, it is clear that we can take strict inequalities in the indicator functions in the integrals in (5.45). Then we can also use the equivalences

$$z_i < b_i(t, \boldsymbol{z}_{-i}) \iff z_1 > b_1(t, \boldsymbol{z}_{-1})$$

and

$$\begin{aligned}z_i < b_i^\varepsilon(t, \boldsymbol{z}_{-i}) &\iff z_i + \varepsilon < b_i(t+\varepsilon, z_1 - \varepsilon, z_2 + \varepsilon, \dots, z_d + \varepsilon)\\ \iff z_1 - \varepsilon > b_1(t+\varepsilon, z_2+\varepsilon, z_3+\varepsilon, \dots, z_d+\varepsilon) &\iff z_1 > b_1^\varepsilon(t, \boldsymbol{z}_{-1}),\end{aligned}$$

to rewrite (5.45) as

$$\begin{aligned}U^n_{x_i x_j}(t, \boldsymbol{x}) =& n^d \int_{\Lambda_n} U_{x_i x_j}(t, \boldsymbol{z}) \mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t, \boldsymbol{z}_{-1})\}} \mathrm{d}\boldsymbol{z}\\ &+ n^d \int_{\Lambda_n} U_{x_i x_j}(t, \boldsymbol{z}) \mathbb{1}_{\{z_1 \le b_1(t, \boldsymbol{z}_{-1})\}} \mathrm{d}\boldsymbol{z} + F_{ij}^{n,\varepsilon}(t, \boldsymbol{x}).\end{aligned}$$

This proves (5.27) for arbitrary i, j.

Step 2. Now that we have derived (5.27) we are in a position to find the bound (5.26).

Indeed, we have

$$
\begin{aligned}
&\sum_{i,j=1}^{d} \beta^{ij}(t,\boldsymbol{x})U^{n}_{x_i x_j}(t,\boldsymbol{x}) \\
&= n^d \int_{\Lambda_n} \sum_{i,j=1}^{d} \beta^{ij}(t,\boldsymbol{z})U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_1 \geq b^{\varepsilon}_1(t,\boldsymbol{z}_{-1})\}\cup\{z_1\leq b_1(t,\boldsymbol{z}_{-1})\}}\mathrm{d}\boldsymbol{z} \\
&\quad + n^d \int_{\Lambda_n} \sum_{i,j=1}^{d} \big(\beta^{ij}(t,\boldsymbol{x})-\beta^{ij}(t,\boldsymbol{z})\big)U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_1 \geq b^{\varepsilon}_1(t,\boldsymbol{z}_{-1})\}\cup\{z_1\leq b_1(t,\boldsymbol{z}_{-1})\}}\mathrm{d}\boldsymbol{z} \\
&\quad + \sum_{i,j=1}^{d} \beta^{ij}(t,\boldsymbol{x})F^{n,\varepsilon}_{ij}(t,\boldsymbol{x}).
\end{aligned}
\tag{5.47}
$$

Thanks to Assumption A.2, there exists $c_{1,\delta} > 0$, depending only on the compact V^{δ} in (5.24), such that

$$
\left| n^d \int_{\Lambda_n} \sum_{i,j=1}^{d} \beta^{ij}(t,\boldsymbol{z})U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_1 \geq b^{\varepsilon}_1(t,\boldsymbol{z}_{-1})\}\cup\{z_1\leq b_1(t,\boldsymbol{z}_{-1})\}}\mathrm{d}\boldsymbol{z}\right| \leq n^d \int_{\Lambda_n} c_{1,\delta}\,\mathrm{d}\boldsymbol{z} = c_{1,\delta}.
\tag{5.48}
$$

Moreover, recalling that $\mathcal{D}$ is closed, β^{ij} is continuous and $U \in C^{1,2}(\mathcal{D})$ we also have

$$
\left| n^d \int_{\Lambda_n} \sum_{i,j=1}^{d} \big(\beta^{ij}(t,\boldsymbol{x})-\beta^{ij}(t,\boldsymbol{z})\big)U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_1\leq b_1(t,\boldsymbol{z}_{-1})\}}\mathrm{d}\boldsymbol{z}\right| \leq n^d \int_{\Lambda_n} c_{2,\delta}\,\mathrm{d}\boldsymbol{z} = c_{2,\delta},
\tag{5.49}
$$

for some other constant $c_{2,\delta} > 0$ only depending on V^{δ}.

Next we find a bound for the second integral on the right-hand side of (5.47) on the indicator of the set $\{z_1 \geq b^{\varepsilon}_1(t,\boldsymbol{z}_{-1})\}$. We provide the details for $i \neq 1$, $j \neq 1$, but it will be clear that the same arguments apply for $i = 1$ and/or $j = 1$. Recalling (5.45) and the discussion following that expression we have

$$
\begin{aligned}
&n^d \int_{\Lambda_n} \big(\beta^{ij}(t,\boldsymbol{x})-\beta^{ij}(t,\boldsymbol{z})\big)U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_1 \geq b^{\varepsilon}_1(t,\boldsymbol{z}_{-1})\}}\mathrm{d}\boldsymbol{z} \\
&= n^d \int_{\Lambda_n} \big(\beta^{ij}(t,\boldsymbol{x})-\beta^{ij}(t,\boldsymbol{z})\big)U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_i \leq b^{\varepsilon}_i(t,\boldsymbol{z}_{-i})\}}\mathrm{d}\boldsymbol{z} \\
&= n^d \int_{\Lambda_n^{-1}} 1_{\{x_i \leq b^{\varepsilon}_i(t,\boldsymbol{z}_{-i})\}}\Big(\int_{x_i}^{b^{\varepsilon}_i(t,\boldsymbol{z}_{-i})\wedge(x_i+\frac{1}{n})} \big(\beta^{ij}(t,\boldsymbol{x})-\beta^{ij}(t,\boldsymbol{z})\big)U_{x_i x_j}(t,\boldsymbol{z})\mathrm{d}z_i\Big)\mathrm{d}\boldsymbol{z}_{-i}.
\end{aligned}
\tag{5.50}
$$

By Assumption A.2 we know there is a constant $\kappa_\delta > 0$ such that $\sup_{V^\delta} \sum_{j=1}^d |U_{x_j}| \le \kappa_\delta$. Integrating by parts with respect to z_i and recalling that β^{ij} is locally Lipschitz (hence Lipschitz on V^δ with constant $L_{\beta,\delta} > 0$ which can be taken independent of i, j gives

$$\left| \int_{x_i}^{b_i^\varepsilon(t,\boldsymbol{z}_{-i})\wedge(x_i+\frac{1}{n})} \big(\beta^{ij}(t,\boldsymbol{x}) - \beta^{ij}(t,\boldsymbol{z})\big) U_{x_i x_j}(t,\boldsymbol{z}) \mathrm{d}z_i \right|$$
$$= \left| \Big[\big(\beta^{ij}(t,\boldsymbol{x}) - \beta^{ij}(t,\boldsymbol{z})\big)U_{x_j}(t,\boldsymbol{z})\Big]_{z_i=x_i}^{z_i=b_i^\varepsilon(t,\boldsymbol{z}_{-i})\wedge(x_i+\frac{1}{n})} + \int_{x_i}^{b_i^\varepsilon(t,\boldsymbol{z}_{-i})\wedge(x_i+\frac{1}{n})} \beta^{ij}_{x_i}(t,\boldsymbol{z})U_{x_j}(t,\boldsymbol{z})\mathrm{d}z_i \right|$$
$$\le 2\kappa_\delta L_{\beta,\delta}\frac{\sqrt{d}}{n} + \kappa_\delta L_{\beta,\delta}\frac{1}{n} =: c_{3,\delta}\frac{1}{n},$$

upon using that the Euclidean norm $\|\boldsymbol{x} - \boldsymbol{z}\| \le \sqrt{d}/n$ for all $\boldsymbol{z} \in \Lambda_n$ and, in particular, $|x_i - b_i^\varepsilon(t,\boldsymbol{z}_{-i}) \wedge (x_i + \frac{1}{n})| \le 1/n$.

Pugging the above bound back into (5.50) we obtain

$$n^d \int_{\Lambda_n} \big(\beta^{ij}(t,\boldsymbol{x}) - \beta^{ij}(t,\boldsymbol{z})\big)U_{x_i x_j}(t,\boldsymbol{z})\mathbb{1}_{\{z_1 \ge b_1^\varepsilon(t,\boldsymbol{z}_{-1})\}}\mathrm{d}\boldsymbol{z} \le c_{3,\delta}\, n^{d-1}\int_{\Lambda_n^{-1}} \mathrm{d}\boldsymbol{z}_{-1} = c_{3,\delta}. \tag{5.51}$$

Thanks to (5.47), (5.48), (5.49) and (5.51) we have

$$\left| \sum_{i,j} \beta^{ij}(t,\boldsymbol{x})U^n_{x_i x_j}(t,\boldsymbol{x}) \right| \le c_{1,\delta} + c_{2,\delta} + d^2 c_{3,\delta} + \left| \sum_{i,j} \beta^{ij}(t,\boldsymbol{x})F^{n,\varepsilon}_{ij}(t,\boldsymbol{x}) \right|, \tag{5.52}$$

for all $(t,\boldsymbol{x}) \in V^\delta$. Finally, letting $\varepsilon \downarrow 0$ and using that $U \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ and the convergence of b_i^ε to b_i for all i's (recall (5.43)), we obtain

$$\lim_{\varepsilon\downarrow 0} F^{n,\varepsilon}_{ij}(t,\boldsymbol{x}) = 0.$$

Hence

$$\left| \sum_{i,j} \beta^{ij}(t,\boldsymbol{x})U^n_{x_i x_j}(t,\boldsymbol{x}) \right| \le c_{1,\delta} + c_{2,\delta} + d^2 c_{3,\delta}, \quad \text{for all } (t,\boldsymbol{x}) \in V^\delta.$$

The latter is equivalent to (5.26) with $C_\delta := c_{1,\delta} + c_{2,\delta} + d^2 c_{3,\delta}$, since the constants are independent of $(t,\boldsymbol{x}) \in V^\delta$.

This completes the proof of the theorem in the case (5.17) holds. □

5.4.2 Relaxing condition (5.17)

Proof. The case in which the boundary has different monotonicity in each variable (as allowed by Assumption A.3) can be addressed by the same methods employed above up to some obvious changes. In order to illustrate the main points, fix $2 \le \bar{k} \le d$ and let us assume with no loss of generality that $t \mapsto b_1(t, \boldsymbol{x}_{-1})$ and $x_i \mapsto b_1(t, \boldsymbol{x}_{-1})$ are non-decreasing for $2 \leq i \leq \bar{k}$, while $x_i \mapsto b_1(t, \boldsymbol{x}_{-1})$ are non-increasing for $\bar{k} < i \leq d$. Then, in the first part of step 1 in the proof above we replace (5.28) by

$$b_1^\varepsilon(t, x_2, ...x_d) := b_1(t + \varepsilon, x_2 + \varepsilon, \dots, x_{\bar{k}} + \varepsilon, x_{\bar{k}+1} - \varepsilon, \dots, x_d - \varepsilon) + \varepsilon,$$

so that b_1^ε is decreasing as $\varepsilon \downarrow 0$ and its limit $b_1^{0+}(t, \boldsymbol{x}_{-1})$ equals $b_1(t, \boldsymbol{x}_{-1})$ by closedness of $\mathcal{D}$ and the same argument as in step 1. Also in this case (5.30) continues to hold and we can repeat verbatim the estimates that lead to (5.27) for $i = 1$ in step 1 above. For the second part of step 1, we need the generalised inverse b_i for each i. In particular, for $2 \leq i \leq \bar{k}$ the same definition of b_i as in (5.40) and the parametrisation of $\mathcal{C}$ and $\mathcal{D}$ as in (5.41) continue to hold. However, $x_j \mapsto b_i(t, \boldsymbol{x}_{-i})$ is non-decreasing for $j = 1$ and $\bar{k} < j \leq d$, while $x_j \mapsto b_i(t, \boldsymbol{x}_{-i})$ and $t \mapsto b_i(t, \boldsymbol{x}_{-i})$ are non-increasing for all $2 \leq j \leq \bar{k}$ with $j \neq i$. Then, setting

$$b_i^\varepsilon(t, \boldsymbol{x}_{-i}) := b_i(t + \varepsilon, x_1 - \varepsilon, x_2 + \varepsilon, \dots x_{\bar{k}} + \varepsilon, x_{\bar{k}+1} - \varepsilon, \dots x_m - \varepsilon) - \varepsilon$$

the functions b_i^ε increase as $\varepsilon \downarrow 0$ and in the limit $b_i^{0+}(t, \boldsymbol{x}_{-i})$ equals $b_i(t, \boldsymbol{x}_{-i})$. So we can repeat the same arguments as in step 1 and obtain (5.27) for $2 \leq i \leq \bar{k}$ and any j. Finally, for $\bar{k} < i \leq d$, since $x_i \mapsto b_1(t, \boldsymbol{x}_{-1})$ is non-increasing we define its (left-continuous) generalised inverse as

$$b_i(t, \boldsymbol{x}_{-i}) := \inf\{x_i \in \mathbb{R} : x_1 > b_1(t, \boldsymbol{x}_{-1})\}.$$

Then we have $x_1 > b_1(t, \boldsymbol{x}_{-1}) \iff x_i > b_i(t, \boldsymbol{x}_{-i})$, $t \mapsto b_i(t, \boldsymbol{x}_{-i})$ and $x_j \mapsto b_i(t, \boldsymbol{x}_{-i})$ are non-decreasing for $2 \leq j \leq \bar{k}$, while $x_1 \mapsto b_i(t, \boldsymbol{x}_{-i})$ and $x_j \mapsto b_i(t, \boldsymbol{x}_{-i})$ are non-increasing for $\bar{k} < j \leq d$ with $j \neq i$. The sets $\mathcal{C}$ and $\mathcal{D}$ can be parametrised as

$$\mathcal{C} = \{(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d : x_i > b_i(t, \boldsymbol{x}_{-i})\},$$
$$\mathcal{D} = \{(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d : x_i \leq b_i(t, \boldsymbol{x}_{-i})\},$$

and we can define the functions

$$b_i^\varepsilon(t, \boldsymbol{x}_{-i}) := b_i(t + \varepsilon, x_1 - \varepsilon, x_2 + \varepsilon, \dots, x_{\bar{k}} + \varepsilon, x_{\bar{k}+1} - \varepsilon, \dots, x_d - \varepsilon) + \varepsilon.$$

The latter decrease as $\varepsilon \downarrow 0$ and converge to $b_i(t, \boldsymbol{x}_{-i})$ by closedness of $\mathcal{D}$. Since once again $\overline{\mathcal{C}_i^\varepsilon} \subset \mathcal{C}$, we can repeat the arguments from step 1 and arrive at (5.27) also for all j's and $i \neq 1$.

This completes the analogy with step 1. Step 2 can be repeated verbatim. Thus the theorem holds under the generality of Assumption A.3 concerning the boundary. $\square$

Chapter 6

Continuity of the optimal stopping surface

6.1 Introduction

[1]In this chapter, we provide sufficient conditions under which optimal stopping boundaries of the form $(t, y) \mapsto x_*(t, y)$ are continuous. We set mild regularity assumptions on the gain function and the value function of the optimal stopping problem, as well as on the coefficients of the dynamics of the underlying process. The idea is similar to the proof of Proposition 4.5.2 in Chapter 4, but the argument is different as we now formulate the optimal stopping problem in a more general setting.

There is a long history of studying the regularity of the free boundary in obstacle type problems. Famous results derived from PDE methods in this area are found in [23], [24], [30], [70], [125], [115], among others. In their work, the continuity, Lipschitz continuity and even continuous differentiability of the free boundary can be established in certain classes of obstacle problems. However, their results are not tailored for optimal stopping problems, hence are somewhat difficult to apply and generalise. In optimal stopping, people still prefer to study the regularity of the free boundary on a case-by-case basis. In two dimensional finite horizon optimal stopping problems, where the space variable is a solution of one-dimensional SDE and the stopping boundary is of the form $t \mapsto b(t)$, i.e. function of time, the continuity or the differentiability of b can often be proved, as in [91], [133], [53], [80], [14], [96] among others. It is also feasible to prove the continuity of the stopping boundary in some multidimensional optimal stopping problems, when the boundaries are of the the form $y \mapsto b(y)$, where y is a space variable (e.g. [41], [84], [85]). Our work extends those results to the case where the stopping boundary depends on both time and another state variable.

[1]The results from this chapter form part of the article [27], which is currently under review.

Recently, more systematic approaches to study the regularity of the free boundary in optimal stopping problems have emerged. De Angelis [38] provides sufficient conditions under which the free boundary curve in a general finite-horizon optimal stopping problem is continuous. His work focuses on the problem with an underlying process solving a one-dimensional SDE. The corresponding free boundary is characterised as a function of time $b(t)$. The continuity of $b(t)$ is proved by a contradiction argument utilising the local properties of the gain function and infinitesimal generator near $b(t)$. Some degree of smoothness of the coefficients of the process is required, but the smooth-fit between the value function and the gain function is not. Similar contradiction argument is also found in the early work of Lamberton and Mikou [96], where they prove the continuity of the free boundary $b(t)$ in the American put option pricing problem in the exponential Lévy model. Peskir [113] establishes the continuity of the optimal stopping boundary in the problem with a two-dimensional diffusion process whose infinitesimal generator can be parabolic or elliptic. The stopping boundary in this case is still a curve $b(y)$, but it is not necessarily a function of time as the infinitesimal generator can be elliptic. In contrast to [38], the role of the smooth fit is emphasised in [113]. Using the boundary and interior regularity results of PDE, he shows that discontinuity of first kind of $b(y)$ leads to a contradiction if smooth fit holds. On the other hand, using the local time space calculus, he shows that the jump portion of $b(y)$ implies the smooth fit. Thus $b(y)$ is continuous as long as it has no discontinuity of the second kind. Another work that tackles high dimensional problems is done by De Angelis and Stabile [43], who show the local Lipschitz continuity of the stopping boundary in an optimal stopping problem with state space $[0, T] \times \mathbb{R}^d$. Their method relies on a probabilistic representation of the derivatives of the value function and an application of the implicit function theorem. The prominence of their work is that they do not need the uniform ellipticity of the diffusive operator, which is needed in many PDE methods.

The spirit of our work is similar to [38], [96] and [113]. We still pursue a contradiction arising from a jump portion in the stopping boundary, but we complement their work by extending their argument to cover three dimensional optimal stopping problems where the stopping boundary is a surface parametrised as a function $x_*(t, y)$. We prove our result under mild assumptions exploiting the local nature of the infinitesimal generator. While we do require the value function to be C^1 in the neighbourhood of the stopping boundary, we have less restrictive assumptions for the SDE coefficients than many PDE methods, e.g. uniform ellipticity is not necessary in our work.

This chapter is organised as follows. In the first section, we introduce our state process and formulate a generic optimal stopping problem. The optimal stopping boundary is then characterised. In the second section, we make the assumptions and present the main theorem with the proof.

6.2 Settings

On a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ we consider a two-dimensional time-inhomogeneous diffusion process (X, Y). The process (X, Y) is assumed to be the unique solution of the stochastic differential equation

$$\mathrm{d}X_s = \alpha_1(s, X_s, Y_s)\mathrm{d}s + \sqrt{2\beta_1(s, X_s, Y_s)}\mathrm{d}B_s, \quad X_t = x, \tag{6.1}$$

$$\mathrm{d}Y_s = \alpha_2(s, X_s, Y_s)\mathrm{d}s + \sqrt{2\beta_2(s, X_s, Y_s)}\mathrm{d}W_s, \quad Y_t = y, \tag{6.2}$$

for $s \geq t$, where W, B are correlated Brownian motions with correlation coefficient $\rho \in [-1, 1]$, α_i, β_i are (deterministic) Borel-measurable functions. The Brownian motion (B, W) is adapted to $(\mathcal{F}_t)$ and if the solution only exists in the weak sense, then uniqueness is understood in law. The state space of (X, Y) is denoted by $\mathcal{O} \subseteq \mathbb{R}^2$.

We avoid making specific (global) regularity assumptions on the coefficients α_i, β_i, $i = 1, 2$, but we will later require some local properties thereof. We use the notation $\mathsf{P}_{t,x,y}(\,\cdot\,) := \mathsf{P}(\,\cdot\,|X_t = x, Y_t = y)$ and $\mathsf{E}_{t,x,y}[\,\cdot\,]$, for the expectation under $\mathsf{P}_{t,x,y}$.

The infinitesimal generator $\mathcal{L}$ of (X, Y) is defined via its action on any sufficiently smooth functions f and it is given by

$$(\mathcal{L}f)(t, x, y) := \big(\beta_1\partial_{xx}f + \beta_2\partial_{yy}f + 2\bar{\beta}\partial_{xy}f + \alpha_1\partial_x f + \alpha_2\partial_y f\big)(t, x, y), \tag{6.3}$$

with $\bar{\beta}(t, x, y) = \rho\sqrt{\beta_1(t, x, y)\beta_2(t, x, y)}$.

For future use we also introduce a second order differential operator $\mathcal{G}$ defined as

$$\begin{aligned}(\mathcal{G}f)(t, x, y) := \big(\partial_x\beta_1\partial_{xx}f + &\partial_x\beta_2\partial_{yy}f + 2\partial_x\bar{\beta}\partial_{xy}f \\ &+ \partial_x\alpha_1\partial_x f + \partial_x\alpha_2\partial_y f\big)(t, x, y),\end{aligned} \tag{6.4}$$

whenever the partial derivatives in x of α_i, β_i, $\bar{\beta}$, $i = 1, 2$ exist.

Letting $T \in (0, \infty]$ be the time horizon, for $(t, x, y) \in [0, T] \times \mathcal{O}$ we are interested in optimal stopping problems of the form

$$v(t, x, y) = \sup_{t \leq \tau \leq T} \mathsf{E}_{t,x,y}\left[\mathrm{e}^{-\int_t^\tau r(s, X_s, Y_s)\mathrm{d}s} g(\tau, X_\tau, Y_\tau)\right], \tag{6.5}$$

where the supremum is taken over all $(\mathcal{F}_t)$-stopping times. The discount rate $r : [0,T] \times \mathcal{O} \to \mathbb{R}$ and the gain function $g : [0,T] \times \mathcal{O} \to \mathbb{R}$ are Borel-measurable functions and we will require further assumptions later on as necessary.

We assume that the problem is *well-posed*, in the sense that the value function is finite for all $(t,x,y) \in [0,T] \times \mathcal{O}$, the stopping time

$$\tau_* := \inf\{s \in [t,T] : v(s,X_s,Y_s) = g(s,X_s,Y_s)\}, \quad \mathsf{P}_{t,x,y} - a.s.$$

is optimal for all $(t,x,y) \in [0,T] \times \mathcal{O}$ and the process $(Z_s)_{s\in[t,T]}$ defined as

$$Z_s := \mathrm{e}^{-\int_t^s r(u,X_u,Y_u)\mathrm{d}u} v(s,X_s,Y_s), \quad \mathsf{P}_{t,x,y} - a.s.$$

is a super-martingale, whereas the stopped process $(Z_{s\wedge\tau_*})_{s\in[t,T]}$ is a martingale for all $(t,x,y) \in [0,T] \times \mathcal{O}$. Existence of an optimal stopping time and the (super-)martingale property of the value process are discussed in Chapter 2 Section 2.2 (also see [88, Appendix D]), while finiteness of the value is generally easy to prove in specific examples (see [114]).

Remark 6.2.1 (Gain function and underlying dynamics)**.** *It will be clear from the analysis below that adding a running cost/profit of the form*

$$\int_t^\tau \mathrm{e}^{-\int_t^s r(u,X_u,Y_u)du} h(s,X_s,Y_s)\mathrm{d}s$$

in the optimisation criterion leads to no additional difficulty. Moreover, this could always be absorbed in the formulation of (6.5)*, by an application of Dynkin's formula, as soon as* $h = (\partial_t + \mathcal{L} - r)\tilde{g}$ *for some* $\tilde{g}$*.*

In some applications of optimal stopping (see, e.g., [9], [39], [40]) it may be necessary to consider more general underlying dynamics of the form

$$\begin{aligned} \mathrm{d}X_t &= \alpha_1(t,X_t,Y_t)\mathrm{d}t + \sqrt{2\beta_1(t,X_t,Y_t)}\mathrm{d}B_t + \mathrm{d}A_t, \quad X_0 = x, \\ \mathrm{d}Y_t &= \alpha_2(t,X_t,Y_t)\mathrm{d}t + \sqrt{2\beta_2(t,X_t,Y_t)}\mathrm{d}W_t + \mathrm{d}C_t, \quad Y_0 = y, \end{aligned}$$

where the processes (A_t) *and* (C_t) *are of bounded variation and take the form of additive functionals of the triple* (t,X,Y) *(e.g., local times). Likewise, we may add a running cost/profit and a more general discount factor as in*

$$v(t,x,y) = \sup_{t\le\tau\le T} \mathsf{E}_{t,x,y}\left[\int_t^\tau \mathrm{e}^{-\Lambda_s} h(s,X_s,Y_s)\mathrm{d}(s+G_s) + \mathrm{e}^{-\Lambda_\tau} g(\tau,X_\tau,Y_\tau)\right],$$

where $\Lambda_s := \int_t^s r(u,X_u,Y_u)du + H_s$ *and the processes* (G_t) *and* (H_t) *are again of bounded variation and in the form of additive functionals of* (t,X,Y)*.*

In this case Theorem 6.3.3 continues to hold but we additionally require that $dA_t = dC_t = dG_t = dH_t = 0$ *a.s. in a neighbourhood of the stopping boundary (this is precisely the situation of [40], [42], [9]).*

As usual we denote the continuation set by

$$\mathcal{C} := \big\{(t,x,y) \in [0,T] \times \mathcal{O} : v(t,x,y) > g(t,x,y)\big\}$$

and the stopping set by $\mathcal{D} = \big([0,T] \times \mathcal{O}\big) \setminus \mathcal{C}$. We assume that $\mathcal{C} \neq \varnothing$ and $\mathcal{D} \neq \varnothing$.

6.3 The main theorem

The operator $\mathcal{L}$ has a local nature that allows us to prove our results under mild assumptions which are also of local nature. For that reason we will often use the notation $\mathcal{U}$ to indicate a generic open bounded subset of $[0,T) \times \mathcal{O}$ of the form

$$\mathcal{U} = (t,s) \times U \tag{6.6}$$

where $U \subseteq \mathcal{O}$ is bounded open with a C^1 boundary.

The main result of the section is the continuity of the optimal stopping boundary, i.e., $\partial\mathcal{C}$, in any subset $\mathcal{U}$ of the form above in which certain regularity conditions are verified. Letting $\mathcal{U}$ be any such subset, we make four standing assumptions. The first one says that $\partial\mathcal{C}$ can be locally represented as a surface with certain monotonicity properties, the second and third ones clarify the regularity required for the coefficients of the SDE and the gain function, the fourth one concerns regularity of the value function.

Assumption 6.3.1. *Let* $\mathcal{U}$ *be such that* $v \in C(\mathcal{U})$, $\mathcal{U} \cap \mathcal{C} \neq \varnothing$, $\mathcal{U} \cap \mathrm{int}(\mathcal{D}) \neq \varnothing$ *and the following conditions hold:*

(i) (The boundary) *There exists a function* $(t,y) \mapsto x_*(t,y)$*, such that*

$$\mathcal{C} \cap \mathcal{U} = \{(t,x,y) \in \mathcal{U} : x > x_*(t,y)\} \tag{6.7}$$

and both $t \mapsto x_*(t,y)$ *and* $y \mapsto x_*(t,y)$ *are monotonic on their respective domains in* $\mathcal{U}$.

(ii) (Coefficients of the SDE) *For the coefficients of the SDE and the discount rate it holds*

$$\alpha_i,\ \beta_i,\ r,\ \partial_x\alpha_i,\ \partial_x\beta_i,\ \partial_x r,\ \partial_{xx}\beta_i \in C(\mathcal{U}),$$

for $i = 1,2$. *Moreover,* $\beta_2 > 0$ *on* $\mathcal{U}$.

(iii) (The gain function) *We have* $g \in C^{1,2}(\mathcal{U})$ *and setting* $h := (\partial_t + \mathcal{L} - r)g$ *we have* $h \neq 0$ *on* $\mathcal{U}$*,* $\partial_x h \in C(\mathcal{U})$ *and* $\frac{\partial}{\partial x}(h/\beta_2) \geq \delta$ *on* $\mathcal{U}$ *for some* $\delta > 0$*.*

(iv) (The value function) *We have* $v \in C^1(\mathcal{U})$ *with* $\partial_x(v - g) \geq 0$ *on* $\mathcal{U}$*. Moreover,* v *satisfies the boundary value problem*

$$(\partial_t + \mathcal{L} - r)f = 0 \ \text{ in } \mathcal{C} \cap \mathcal{U}, \text{ with } f = v \text{ on } \partial(\mathcal{C} \cap \mathcal{U}) \tag{6.8}$$

and $\partial_x v$ *satisfies*

$$(\partial_t + \mathcal{L} - r)\partial_x v = -(\mathcal{G} - \partial_x r)v \quad \text{in } \mathcal{C} \cap \mathcal{U}, \tag{6.9}$$

with all derivatives understood in the classical sense.

Notice that if $\beta_1, \beta_2 > 0$ on $\overline{\mathcal{U}}$ and $\rho^2 < 1$ the operator $\mathcal{L}$ is uniformly elliptic $\overline{\mathcal{U}}$, so that we can rely upon the next well-known lemma that guarantees (6.8).

Lemma 6.3.2. *Let* $\mathcal{U} \subset [0, T) \times \mathcal{O}$ *be defined as in* (6.6)*, assume that*

$$\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \bar{\beta},\ r,$$

are Hölder continuous in $\mathcal{U}$*,* $\beta_1, \beta_2 > 0$ *in* $\overline{\mathcal{U}}$ *and* $\rho \in (-1, 1)$*. If* v *is continuous then* $v \in C^{1,2}(\mathcal{C} \cap \mathcal{U})$ *and it solves the boundary value problem in* (6.8)*.*

Proof. For any cylinder $D := (t_0, t_1) \times B \subset \mathcal{C} \cap \mathcal{U}$, where B is an open ball, consider a terminal boundary value problem

$$(\partial_t + \mathcal{L} - r)f = 0 \text{ on } D, \quad \text{ with } f = v \text{ on } \partial D, \tag{6.10}$$

where $\partial D := ([t_0, t_1] \times \partial B) \cup (\{t_1\} \times B)$ denotes the parabolic boundary. This problem admits a unique classical solution (see [69, Ch. 3, Cor. 2, p. 71]).

Now, let $(D_n)_{n \in \mathbb{N}}$ be an increasing sequence of cylinders contained in D and such that $D_n \uparrow D$ as $n \to \infty$. Let τ_D be the first exit time of (t, X, Y) from D and τ_{D_n} the first exit time from D_n. An application of Itô's formula gives

$$f(t, x, y) = \mathsf{E}_{t,x,y}\left[\mathrm{e}^{-\int_t^{\tau_{D_n}} r(s, X_s, Y_s)\mathrm{d}s} f(\tau_{D_n}, X_{\tau_{D_n}}, Y_{\tau_{D_n}})\right].$$

Let $n \to \infty$. Using the uniform ellipticity of $\mathcal{L}$ on $\overline{D}$, we obtain that $\tau_{D_n} \uparrow \tau_D$ almost surely as $n \to \infty$. Since D is bounded and f is continuous, by the dominated convergence theorem we obtain

$$\begin{aligned} f(t, x, y) &= \mathsf{E}_{t,x,y}\left[\mathrm{e}^{-\int_t^{\tau_D} r(s, X_s, Y_s)\mathrm{d}s} f(\tau_D, X_{\tau_D}, Y_{\tau_D})\right] \\ &= \mathsf{E}_{t,x,y}\left[\mathrm{e}^{-\int_t^{\tau_D} r(s, X_s, Y_s)\mathrm{d}s} v(\tau_D, X_{\tau_D}, Y_{\tau_D})\right] = v(t, x, y), \end{aligned}$$

for all $(t,x,y) \in D$, where the second equality follows from $(\tau_D, X_{\tau_D}, Y_{\tau_D}) \in \partial D$ and the final equality is by the martingale property of the value function. Hence, v is a unique classical solution of (6.10). As D is arbitrary in $\mathcal{C} \cap \mathcal{U}$, we conclude that v solves (6.8). □

Under Assumption 6.3.1 the function $u = v - g$ solves

$$\big(\partial_t + \mathcal{L} - r\big)u = -h, \quad \text{on } \mathcal{C} \cap \mathcal{U}, \tag{6.11}$$

with boundary conditions

$$u = \partial_t u = \partial_x u = \partial_y u = 0, \quad \text{on } \partial\mathcal{C} \cap \mathcal{U}. \tag{6.12}$$

Taking the partial derivative with respect to x of (6.11) we obtain a PDE for $\partial_x u$:

$$\big(\partial_t + \mathcal{L} - r\big)\partial_x u = -\partial_x h - \big(\mathcal{G} - \partial_x r\big)u, \quad \text{on } \mathcal{C} \cap \mathcal{U}, \tag{6.13}$$

Some comments on the assumptions above are in order. Continuity of the value function (at least locally) is generally not difficult to prove and there are numerous papers addressing this question in broad generality (see, e.g., [114]). If $v \in C(\mathcal{U})$, then the well-posedness (in the sense above) of the optimal stopping problem usually leads to higher smoothness of the value function (e.g., as in Lemma 6.3.2). The existence of an optimal boundary is normally proved on a case by case basis and it is known that there are several possible sufficient conditions that guarantee it (see, e.g. [114], [81]). Therefore, rather than providing an inevitably incomplete list of such sufficient conditions we directly assume that the boundary exists. Also assuming that the continuation set lies above the boundary is with no loss of generality and the results of this chapter carry over to the case in which $\mathcal{C}$ lies below the boundary, up to obvious changes to the arguments of proof. Requiring local monotonicity of the boundary is necessary to avoid pathological examples of boundaries with infinite local variation. In practice, monotonicity is also checked on a case by case basis and sufficient conditions are known that would imply it[1].

Local regularity of the coefficients of the SDE, the discount rate and the gain function are non-restrictive and hold in virtually all examples addressed in the optimal stopping literature. The condition $\partial_x(h/\beta_2) \geq \delta$ is slightly more technical but it is in line with the fact that $\mathcal{C}$ lies above the optimal boundary. Indeed, notice that if $\partial_x \beta_2 = 0$, the condition is equivalent to

[1]For example, if $T < \infty$ and if g, r and the coefficients of the SDE are independent of time, one immediately obtains that $t \mapsto (v - g)(t,x,y)$ is non-increasing. So, if (6.7) holds the boundary is increasing in time.

$\partial_x h > 0$. In many cases the latter, is sufficient to prove $\partial_x v \geq \partial_x g$ which then implies the existence of the boundary as in (6.7)[1].

Sufficient conditions that guarantee $v \in C^1(\mathcal{U})$ are provided in [42] and numerous extensions have been developed in specific examples (see, e.g., [26], [9] and [84] for multi-dimensional optimal stopping problems). It is not hard to check that the requirement $\partial_x(v - g) \geq 0$ is equivalent to (6.7), since $\mathcal{U}$ can be chosen arbitrarily small around a point of the boundary $\partial\mathcal{C}$. Despite this slight redundancy we prefer to add the condition as part of our assumptions for clarity of exposition. Finally, (6.8) and (6.9) hold under very mild conditions that are satisfied in all examples we are aware of. There are many sufficient conditions on the coefficients of the SDE that would guarantee (6.8) and (6.9) (see, e.g., Lemma 6.3.2 above) but we decided to state the assumptions in broader generality to also cover some degenerate cases as, e.g., $\beta_1 \equiv 0$ (as in the American Asian option [114, Sec.27]) or even $\alpha_1 = \beta_1 \equiv 0$ where y only enters as a parameter in connection to singular stochastic control problems (see, e.g., [41]).

Theorem 6.3.3. *Under Assumption 6.3.1, the optimal stopping boundary $(t, y) \mapsto x_*(t, y)$ is continuous on its domain in $\mathcal{U}$.*

Proof. Since the maps $t \mapsto x_*(t, y)$ and $y \mapsto x_*(t, y)$ are monotonic it is sufficient to show that they are also continuous. Then the map $(t, y) \mapsto x_*(t, y)$ is continuous by a simple result from Calculus (see, e.g., [92]). In the rest of the proof we focus on showing continuity of $t \mapsto x_*(t, y)$ and $y \mapsto x_*(t, y)$.

We start by proving our claim in the case when x_* is non-decreasing in both t and y. By continuity of v on $\mathcal{U}$ we know that $\mathcal{C} \cap \mathcal{U}$ is an open set and $\mathcal{D} \cap \mathcal{U}$ is closed relatively to $\mathcal{U}$. Then we can conclude that $x_*(t, y)$ is right continuous in y for each t and right continuous in t for each y. Indeed, fix (t, y) such that $(t, x_*(t, y), y) \in \mathcal{U}$ and let $y_n \downarrow y$ as $n \to \infty$. With no loss of generality we may assume $(t, x_*(t, y_n), y_n) \in \mathcal{U}$ for all n's. Then

$$\mathcal{D} \ni \lim_{n\to\infty} (t, y_n, x_*(t, y_n)) = (t, y, x_*(t, y+)) \implies x_*(t, y+) \leq x_*(t, y), \tag{6.14}$$

[1]For example, take $g \in C^{1,2}([0, T] \times \mathcal{O})$ and $r(t, x, y) \equiv r > 0$ then by an application of Dynkin's formula

$$(v - g)(t, x, y) = \sup_{t \leq \tau \leq T} \mathsf{E}_{t,x,y}\Big[\int_t^\tau e^{-rs} h(s, X_s, Y_s) ds\Big].$$

Assume $\partial_x \alpha_2 = \partial_x \beta_2 = 0$ and that (X, Y) is a strong solution. Denote by $(X^{t,x,y}, Y^{t,x,y})$ the process with the initial condition $(X_t, Y_t) = (x, y)$. For $x' > x$ we have, almost surely, $X_s^{t,x,y} \leq X_s^{t,x',y}$ and $Y_s^{t,x,y} = Y_s^{t,x',y}$ for all $s \geq t$ by pathwise comparison. If $\partial_x h > 0$ then $(v - g)(t, x', y) \geq (v - g)(t, x, y)$, which implies that the mapping $x \mapsto (v - g)(t, x, y)$ is increasing.

where the limit exists by monotonicity, it lies in $\mathcal{D}$ by closedness and the implication holds by definition of the optimal boundary. Since $x_*(t, y_n) \geq x_*(t, y)$ for all n's we conclude that $x_*(t, y+) = x_*(t, y)$. An analogous argument holds for the right-continuity in time.

Next we prove left-continuity of $y \mapsto x_*(t, y)$. Let us fix (t_0, y_0) such that the boundary point $(t_0, y_0, x_*(t_0, y_0))$ lies in $\mathcal{U}$ and, arguing by contradiction, let us assume $x_*(t_0, y_0-) < x_*(t_0, y_0)$. Then we can fix $x_*(t_0, y_0-) < x_1 < x_2 < x_*(t_0, y_0)$ such that $\{t_0\} \times [x_1, x_2] \times \{y_0\} \subset \mathcal{U}$. By the assumed monotonicity of the boundary we have

$$\Sigma := (\tilde{t}, t_0) \times (x_1, x_2) \times (\tilde{y}, y_0) \subset \mathcal{C} \cap \mathcal{U} \quad \text{and} \quad \Sigma_{t_0,y_0} := \{t_0\} \times (x_1, x_2) \times \{y_0\} \subset \mathcal{D} \cap \mathcal{U}$$

for some $\tilde{y} < y_0$ and $\tilde{t} < t_0$ sufficiently close to (t_0, y_0).

By (iii) and (iv) in Assumption 6.3.1 we have that $u := v - g$ satisfies the PDE

$$(\partial_t u + \mathcal{L}u - ru) = -h, \qquad \text{in } \Sigma.$$

In particular, since $\Sigma_{t_0} := \{t_0\} \times (x_1, x_2) \times (\tilde{y}, y_0) \subset \mathcal{C} \cap \mathcal{U}$ we also have

$$(\partial_t u + \mathcal{L}u - ru)(t_0, x, y) = -h(t_0, x, y), \qquad \text{for } (t_0, x, y) \in \Sigma_{t_0}. \tag{6.15}$$

Thanks to (6.9) in Assumption 6.3.1 and (6.13) we can also write

$$\big[(\partial_t + \mathcal{L} - r)\partial_x u\big](t_0, x, y) = -\big[\partial_x h + \big(\mathcal{G} - \partial_x r\big)u\big](t_0, x, y), \qquad \text{for } (t_0, x, y) \in \Sigma_{t_0}. \tag{6.16}$$

On the right hand side of (6.16) above we have a term of the form $\partial_x \beta_2 \partial_{yy} u$ which, using (6.15) we can express as

$$\partial_x \beta_2 \partial_{yy} u = -\frac{\partial_x \beta_2}{\beta_2}\big[h + \partial_t u + \beta_1 \partial_{xx} u + 2\bar{\beta}\partial_{xy} u + \alpha_1 \partial_x u + \alpha_2 \partial_y u - ru\big].$$

Plugging the expression above back into (6.16) and defining $\Gamma\psi := \partial_x \psi - \frac{\partial_x \beta_2}{\beta_2}\psi$ for any $\psi \in C(\mathcal{U})$ with $\partial_x \psi \in C(\mathcal{U})$, we obtain

$$\begin{aligned}
&\big[(\partial_t + \mathcal{L} - r)\partial_x u\big](t_0, x, y) \\
&= -\big[\Gamma h + \Gamma\beta_1\, \partial_{xx} u + 2\Gamma\bar{\beta}\, \partial_{xy} u + \Gamma\alpha_1\, \partial_x u + \Gamma\alpha_2\, \partial_y u - \Gamma r\, u\big](t_0, x, y),
\end{aligned}$$

for $(t_0, x, y) \in \Sigma_{t_0}$. Finally, we can express $\partial_{yyx} u$ that appears on the left-hand side of the above equation as

$$\begin{aligned}
&\partial_{yyx}\, u(t_0, x, y) \\
&\quad = -\beta_2^{-1}\big[\Gamma h + \Gamma\beta_1\, \partial_{xx} u + 2\Gamma\bar{\beta}\, \partial_{xy} u + \Gamma\alpha_1\, \partial_x u + \Gamma\alpha_2\, \partial_y u - \Gamma r\, u\big](t_0, x, y) \\
&\qquad - \beta_2^{-1}\big[\partial_{tx} u + \beta_1 \partial_{xxx} u + 2\bar{\beta}\partial_{xxy} u + \alpha_1 \partial_{xx} u + \alpha_2 \partial_{xy} u - r\partial_x u\big](t_0, x, y)
\end{aligned}$$

for all $(t_0, x, y) \in \Sigma_{t_0}$.

Next, let us pick an arbitrary $\varphi \in C_c^\infty(x_1, x_2)$, with $\varphi \geq 0$ and $\int_{x_1}^{x_2} \varphi(x)\mathrm{d}x = 1$. Then, from the expression above and using integration by parts we obtain

$$\begin{aligned}
F_\varphi(y) := \int_{x_1}^{x_2} \partial_{yyx} u(t_0, x, y)\varphi(x)\mathrm{d}x \qquad (6.17)\\
= -\int_{x_1}^{x_2} \beta_2^{-1}\big[\Gamma h + \Gamma\alpha_1\,\partial_x u + \Gamma\alpha_2\,\partial_y u - \Gamma r\, u - r\partial_x u\big](t_0, x, y)\varphi(x)\mathrm{d}x \\
+ \int_{x_1}^{x_2} \big[\partial_x(\varphi\,\tfrac{\Gamma\beta_1}{\beta_2})\,\partial_x u + 2\partial_x(\varphi\,\tfrac{\Gamma\bar{\beta}}{\beta_2})\,\partial_y u\big](t_0, x, y)\mathrm{d}x \\
+ \int_{x_1}^{x_2} \big[\partial_x(\beta_2^{-1}\varphi)\,\partial_t u + \partial_x(\varphi\beta_2^{-1}\alpha_1)\,\partial_x u + \partial_x(\varphi\beta_2^{-1}\alpha_2)\,\partial_y u\big](t_0, x, y)\mathrm{d}x \\
- \int_{x_1}^{x_2} \big[\partial_{xx}(\varphi\beta_2^{-1}\beta_1)\,\partial_x u + 2\partial_{xx}(\varphi\beta_2^{-1}\bar{\beta})\,\partial_y u\big](t_0, x, y)\mathrm{d}x
\end{aligned}$$

Letting $y \uparrow y_0$, we have $(t_0, x, y) \to (t_0, x, y_0) \in \Sigma_{t_0, y_0} \subset \mathcal{D} \cap \mathcal{U}$ for all $x \in (x_1, x_2)$. Hence, $\partial_x u, \partial_y u, \partial_t u, u \to 0$ by C^1 regularity of u. By dominated convergence we then obtain

$$\begin{aligned}
\lim_{y \to y_0} F_\varphi(y) = -\int_{x_1}^{x_2} \tfrac{\Gamma h}{\beta_2}(t_0, x, y_0)\varphi(x)\mathrm{d}x \qquad (6.18)\\
= -\int_{x_1}^{x_2} \partial_x(h/\beta_2)(t_0, x, y_0)\varphi(x)\mathrm{d}x \leq -\delta,
\end{aligned}$$

where the final inequality is by (iii) in Assumption 6.3.1, since $\varphi \geq 0$ and integrates to one. In particular, this shows that the function $F_\varphi \in C\big([\tilde{y}, y_0)\big)$ admits a continuous extension to $[\tilde{y}, y_0]$.

Since $\delta > 0$, there exists $\hat{y} \in (\tilde{y}, y_0)$ such that

$$F_\varphi(y) = \int_{x_1}^{x_2} \partial_{xyy} u(t_0, x, y)\varphi(x)\mathrm{d}x < -\frac{\delta}{2}, \quad \text{for all } y \in [\hat{y}, y_0].$$

Then, for any $\varepsilon > 0$, integrating over y twice and using Fubini's theorem we have

$$\begin{aligned}
&-\tfrac{\delta}{4}(y_0 - \varepsilon - \hat{y})^2 > \int_{\hat{y}}^{y_0-\varepsilon}\int_{y}^{y_0-\varepsilon}\int_{x_1}^{x_2} \partial_{xyy} u(t_0, x, \zeta)\varphi(x)\mathrm{d}x\,\mathrm{d}\zeta\,\mathrm{d}y \\
&= (y_0 - \varepsilon - \hat{y})\int_{x_1}^{x_2} \partial_{xy} u(t_0, x, y_0 - \varepsilon)\varphi(x)\mathrm{d}x - \int_{x_1}^{x_2} \varphi(x)\big(\partial_x u(t_0, x, y_0 - \varepsilon) - \partial_x u(t_0, x, \hat{y})\big)\mathrm{d}x \\
&= -(y_0 - \varepsilon - \hat{y})\int_{x_1}^{x_2} \partial_y u(t_0, x, y_0 - \varepsilon)\partial_x \varphi(x)\mathrm{d}x \\
&\quad - \int_{x_1}^{x_2} \varphi(x)\big(\partial_x u(t_0, x, y_0 - \varepsilon) - \partial_x u(t_0, x, \hat{y})\big)\mathrm{d}x.
\end{aligned}$$

Letting $\varepsilon \to 0$ and using dominated convergence and the fact that $\partial_x u = \partial_y u = 0$ on Σ_{t_0,y_0} we obtain

$$-\tfrac{\delta}{4}(y_0-\hat{y})^2 \geq \int_{x_1}^{x_2} \partial_x u(t_0,x,\hat{y})\varphi(x)\mathrm{d}x \geq 0, \tag{6.19}$$

where the final inequality is by (iv) in Assumption 6.3.1. Hence a contradiction and $y \mapsto x_*(t,y)$ is left-continuous (and therefore continuous).

We can now use continuity of the map $y \mapsto x_*(t,y)$ to prove that $t \mapsto x_*(t,y)$ is also continuous. For (t_0,y_0) as above we argue by contradiction and assume that $x_*(t_0-,y_0) < x_*(t_0,y_0)$ and fix $x_*(t_0-,y_0) < x_1 < x_2 < x_*(t_0,y_0)$. By continuity of $y \mapsto x_*(t_0,y)$, we can find $\tilde{y} < y_0$ and $\tilde{t} < t_0$ close enough to (t_0,y_0) such that

$$x_*(t_0,y_0) \geq x_*(t_0,\tilde{y}) > x_2 > x_1 > x_*(t_0-,y_0) \geq x_*(s,\zeta), \quad \text{for all } (s,\zeta) \in (\tilde{t},t_0)\times(\tilde{y},y_0),$$

where the final inequality holds thanks to the monotonicity of x_*. Thus,

$$\Sigma = (\tilde{t},t_0)\times(x_1,x_2)\times(\tilde{y},y_0) \subset \mathcal{C}\cap\mathcal{U},$$
$$\Sigma_{t_0} := \{t_0\}\times(x_1,x_2)\times(\tilde{y},y_0) \subset \mathcal{D}\cap\mathcal{U}.$$

Notice that without continuity of $y \mapsto x_*(t_0,y)$ we would not have been able to guarantee the inclusion $\Sigma_{t_0} \subset \mathcal{D}\cap\mathcal{U}$, which is important for the rest of this proof.

Let us take two arbitrary functions $\varphi \in C_c^\infty(x_1,x_2)$ and $\psi \in C_c^\infty(\tilde{y},y_0)$, such that $\varphi,\psi \geq 0$ with $\int_{x_1}^{x_2}\varphi(x)dx = 1$ and $\int_{\tilde{y}}^{y_0}\psi(y)dy = 1$. We multiply (6.15) by $\varphi(x)\psi(y)$ and integrate over $[x_1,x_2]\times[\tilde{y},y_0]$ to obtain, for all $t \in (\tilde{t},t_0)$

$$\begin{aligned}&\int_{x_1}^{x_2}\int_{\tilde{y}}^{y_0}\varphi(x)\psi(y)\left(\partial_t u + \mathcal{L}u - ru\right)(t,x,y)\mathrm{d}y\,\mathrm{d}x\\ &= -\int_{x_1}^{x_2}\int_{\tilde{y}}^{y_0}\varphi(x)\psi(y)h(t,x,y)\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Using integration by parts for the terms of $\mathcal{L}$ involving second derivatives we get

$$\begin{aligned}&\int_{x_1}^{x_2}\int_{\tilde{y}}^{y_0}\varphi(x)\psi(y)\big(\partial_t u + \alpha_1\partial_x u + \alpha_2\partial_y u - ru\big)(t,x,y)\mathrm{d}y\,\mathrm{d}x\\ &-\int_{x_1}^{x_2}\int_{\tilde{y}}^{y_0}\big[\psi\left(\partial_x(\varphi\beta_1)\partial_x u + 2\partial_x(\varphi\bar{\beta})\partial_y u\right) + \varphi\,\partial_x(\psi\beta_2)\partial_y u\big](t,x,y)\mathrm{d}y\,\mathrm{d}x\\ &= -\int_{x_1}^{x_2}\int_{\tilde{y}}^{y_0}\varphi(x)\psi(y)h(t,x,y)\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

for all $t \in (\tilde{t},t_0)$. Letting $t \uparrow t_0$, by dominated convergence and the assumed C^1 regularity of u we obtain

$$0 = \int_{x_1}^{x_2}\int_{\tilde{y}}^{y_0}\varphi(x)\psi(y)h(t_0,x,y)\mathrm{d}y\,\mathrm{d}x.$$

Since φ, ψ are arbitrary, the latter implies that $H(t_0, x, y) = 0$ for all $(x, y) \in (x_1, x_2) \times (\tilde{y}, y_0)$, which contradicts (iii) in Assumption 6.3.1. Therefore $t \mapsto x_*(t, y)$ is continuous.

Next we consider a case with different monotonicity of $t \mapsto x_*(t, y)$ and $y \mapsto x_*(t, y)$. Let us assume for example that $t \mapsto x_*(t, y)$ is non-decreasing and $y \mapsto x_*(t, y)$ is non-increasing. By analogous arguments to those in (6.14) we conclude that $y \mapsto x_*(t, y)$ is left-continuous and $t \mapsto x_*(t, y)$ is right continuous thanks to the monotonicity of x_* and the closedness of $\mathcal{D}$. Then, arguing by contradiction we assume that $x_*(t_0, y_0) > x_*(t_0, y_0+)$ and, slightly abusing the notation, we consider

$$\Sigma := (\tilde{t}, t_0) \times (x_1, x_2) \times (y_0, \tilde{y}) \subset \mathcal{C} \cap \mathcal{U} \quad \text{and} \quad \Sigma_{t_0,y_0} := \{t_0\} \times (x_1, x_2) \times \{y_0\} \subset \mathcal{D} \cap \mathcal{U}$$

for some $\tilde{t} < t_0$ and $\tilde{y} > y_0$ sufficiently close to (t_0, y_0). Using (6.15) and (6.16) (with the 'new' Σ_{t_0}) and repeating verbatim the arguments of proof employed above, when we let $y \downarrow y_0$ we obtain once again (6.18). Then, there exists $\hat{y} \in (y_0, \tilde{y})$ such that

$$\int_{x_1}^{x_2} \partial_{xyy} u(t_0, x, y)\varphi(x)\mathrm{d}x < -\frac{\delta}{2}, \quad \text{for all } y \in [y_0, \hat{y}].$$

Hence, integrating this expression twice we get

$$\begin{aligned} -\tfrac{\delta}{4}(\hat{y} - y_0 - \varepsilon)^2 &> \int_{y_0+\varepsilon}^{\hat{y}} \int_{y_0+\varepsilon}^{y} \int_{x_1}^{x_2} \partial_{xyy} u(t_0, x, \zeta)\varphi(x)\mathrm{d}x \,\mathrm{d}\zeta \,\mathrm{d}y \\ &= (\hat{y} - y_0 - \varepsilon) \int_{x_1}^{x_2} \partial_y u(t_0, x, y_0 + \varepsilon)\partial_x \varphi(x)\mathrm{d}x \\ &\quad + \int_{x_1}^{x_2} \varphi(x)\big(\partial_x u(t_0, x, \hat{y}) - \partial_x u(t_0, x, y_0 + \varepsilon)\big)\mathrm{d}x. \end{aligned}$$

Letting $\varepsilon \to 0$ and using that $\partial_x u = \partial_y u = 0$ at $\partial\mathcal{C}$ we arrive to a contradiction with (iv) in Assumption 6.3.1.

The remaining two cases, in which $t \mapsto x_*(t, y)$ is non-increasing and $y \mapsto x_*(t, y)$ is either non-increasing or non-decreasing, can be treated analogously. □

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