Large Cardinals in Weakened Axiomatic Theories

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Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

The University of Leeds
School of Mathematics

April 2021

Acknowledgements

Producing this thesis has been a long journey which I could not have completed without the support of many people. First and foremost among them, I would like to thank Michael Rathjen and Andrew Brooke-Taylor for the support and guidance you have given me over the course of my PhD. Thank you both for patiently listening to my, sometimes crazy, ideas and then sharing your wealth of knowledge which has allowed some of those ideas to come to fruition and has also got me out of many a dilemma.

Of all the people who have had an impact on my time in Leeds, a special thanks goes to John Howe, Bea Adam-Day, Giovanni Soldà and Stamatis Dimopoulos. Thank you for your friendship, all the discussions we have had about logic and for patiently listening to, and then correcting my many early iterations of this work. And John, thank you for all those long conversations (and arguments) we have had over the years. I don't think I will ever be able to repay you for all the help (and alcohol) you have provided. Also, I apologise to your housemates for all those Monday night seminars in your kitchen.

Next, I want to express my gratitude to the rest of the Leeds Logic group for creating such a friendly, welcoming environment and also to the rest of the mathematics department. I will not try and name you all here because there are so many of you that I am thankful to and if I try to, I will certainly miss several important people out. However, I would like to make a special mention of the mathematics reception team for all the hard work they have done throughout my PhD and especially for doing so much over the last few months to allow me to use the department.

There are also many set theorists whose suggestions have significantly improved my work. Some people to thank in particular are: Philipp Schlicht, for the many discussions we have had and the large amount you have taught me about class forcing; Johannes Schürz, for the suggestion to look at amorphous proper classes, it may not be the answer we expected but it was definitely interesting; Asaf Karagila, for all the help, pointers and general comments about symmetric submodels; And Vika Gitman, for everything you have taught me about Collection Principles and models where Power Set fails. Also, I am grateful to Joel David Hamkins and Paul Shafer for examining this thesis and all the comments and corrections that you have given, in particular the observations from Joel about the importance of Induction in the expanded language for many of the proofs to go through, your suggestions have significantly improved this body of work. All of your help has been invaluable and so, for everything, thanks.

I further express my appreciation for everyone else who has had an impact on my life over the last few years, starting with my family. Thank you for all of your support and for not asking me to try and explain what it is I actually do.

Also, thanks to; Elizabeth, you are the reason I became a mathematician in the first place and words cannot express how grateful I am for that; Marco and Desmond, for keeping me grounded in some realm of normality; the Arcadia quiz team, for all those enjoyable Monday nights; and the Fenton, so that after a difficult day I could sit with friends and enjoy a good pint of beer.

Not all acknowledgements should be positive, and in that vein I would like to acknowledge the global pandemic. Most of this work was written up over the course of the last year and the various lockdowns have given me much more free time than I expected (or possible wanted!) to devote to my studies and reflect on my overall PhD. On the other hand, those restrictions have also hindered my ability to engage with my friends, family and colleagues which has almost certainly affected the quality of this work. Therefore, any and all errors that appear in this thesis are solely the fault of the COVID-19 pandemic.

Abstract

We study the notion of non-trivial elementary embeddings of the form $j\colon \mathrm{V} \to \mathrm{V}$ under the assumption that V satisfies various classical and intuitionistic set theories. In particular, we investigate what consequences can be derived if V is only assumed to satisfy Kripke Platek set theory, set theory without Power Set or intuitionistic set theory.

To do this, we construct the constructible universe in Intuitionistic Kripke Platek without Infinity and use this to find lower bounds for such embeddings. We then study the notion of definable embeddings before giving some initial bounds in terms of the standard large cardinal hierarchy. Finally, we give sufficient requirements for there to be no non-trivial elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ in ZFC without Power Set.

As a by-product of this analysis, we also study Collection Principles in ZFC without Power Set. This leads to models witnessing the failure of various Dependent Choice Principles and to the development of the theory of the Respected Model, a generalisation of symmetric submodels to the class forcing context.

Contents

List of Figures

Chapter 1

Introduction

At its core, axiomatic set theory is the study of the structure of the mathematical universe and the question of which statements we can prove from a given collection of assumptions. By positing the existence of large cardinals, which are infinite sets satisfying some interesting properties that we cannot prove to exist from the axioms of standard set theory, we are able to strengthen our assumptions and thus derive a more complex set-theoretic structure.

In this thesis, we shall explore large cardinals from the perspective of weak fragments of the standard axioms of Zermelo Fraenkel with Choice. In particular, we shall look at theories such as:

- Kripke Platek,
- Set theory without Power Set,
- Intuitionistic Set Theory.

Many of the larger large cardinals can be expressed using non-trivial elementary embeddings of the form

$$j \colon \mathrm{V} \to \mathrm{M}$$

where M is a subclass of the universe, V. M can be thought of as an approximation to the full universe, with the motivation being: the better the approximation, the stronger the resulting large cardinal axiom is.

A natural conclusion to this process is to assume that there is an elementary embedding from V to itself, which is a concept first proposed by Reinhardt in his PhD thesis, [Rei67].

Question (Reinhardt)**.** Is there an elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$?

A landmark result in set theory, proven by Kunen in [Kun71], is that under the standard axioms of Zermelo Fraenkel with Choice there is a limit to how close this approximation can be. Namely, the existence of such a Reinhardt embedding is inconsistent.

In weaker theories it is unclear whether there should still be such a hard ceiling to our theory of large cardinals. In particular, the question of Reinhardt embeddings in the context without Choice has been a subject of much study over the last fifty years. In this thesis, we shall take a different approach by looking at alternative, weaker set theories. In doing so, we shall explore many interesting aspects of these theories and come up against the multitude of limitations that occur from working with deficient base theories.

A conclusion of this process will be a hierarchy of Reinhardt embeddings depending on the underlying theory (Figure 1.1), from some seeming to have a relatively weak strength to other theories in which one can re-derive Kunen's famous inconsistency.

1.1 Structure and Main Results

We now proceed to outline the structure of this thesis. Interspersed through this guide are the main results which appear in this body of work. At the end of this chapter, we also include our *hierarchy of Reinhardt Embeddings* which is a hierarchy of Reinhardt embeddings over various subtheories of ZFC. We shall give the results as they will be stated, however we shall not define all of the terms here. Instead one should refer to where the Theorem appears in the main text.

After giving some preliminaries in Chapter 2, Chapter 3 concerns the notion of *big classes*.

Definition 3.1.1. A proper class is said to be *big* if it surjects onto every non-zero ordinal.

Next, after giving some simple consequences of this concept in various theories, a specific theory is isolated, namely ZF without Power Set but with the Scheme of Dependent Choices of length μ for μ an infinite cardinal.

Theorem 3.2.8. *Suppose that* $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$*-Scheme for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, definable over* V*, there is a subset of* $\mathcal{C}$ *of cardinality* μ*.*

Corollary 3.2.10. *Suppose that* $\mathrm{V} \models \mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$*. Then, for any proper class* $\mathcal{C}$ *which is definable over* V *and any non-zero ordinal* γ*, there is a definable surjection of* $\mathcal{C}$ *onto* γ*.*

As a continuation of the above concept, we prove that it is possible, in ZFC without Power Set, for the full Scheme of Dependent Choices to fail. This will be done by examining a model first constructed by Zarach. In the original model we shall see that the $\mathrm{DC}_{\aleph_2}$-Scheme fails.

Corollary 3.3.17. *For any model* M *of* $\mathrm{ZFC} + \mathrm{CH}$ *there is a model* $\mathcal{N} = \langle \mathrm{N}, \in, \mathrm{M} \rangle$ *with* $\mathrm{N} \supseteq \mathrm{M}$ *which has the same cardinals and cofinalities as* M *and*

$$\mathcal{N} \models \mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{\aleph_2}\text{-Scheme.}$$

Finally, in a joint result with Victoria Gitman, we show that the level of failure can be improved to the $\mathrm{DC}_{\aleph_1}$-Scheme.

Theorem 3.3.19. *Suppose* $\mathrm{V} = \mathrm{L}$ *is a model of* ZFC *and that* $\mathbb{J}$ *is a minimal forcing which adds a real and whose finite support product preserves* $\aleph_1$*. Let* $\mathbb{P} = \prod_\omega^{(\omega)} \mathbb{J}$*. Then, using the notation of Theorem 3.3.7, if* G *is* $\mathbb{P}$*-generic over* L *then* $\mathcal{N} \models \mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{\aleph_1}$*-Scheme.*

In Chapter 4 we begin with what was an attempt to prove that classes need not be big in ZF^- if one does not assume any amount of Choice. This is done by taking the symmetric submodel of a pretame class forcing. While this approach will be shown to fail, it will lay the groundwork for the rest of the chapter.

Theorem 4.1.4. *Over* $\mathrm{GB} + \mathrm{AC}$*, it is consistent that the symmetric submodel of a pretame class forcing does not satisfy* ZF^-*.*

Having deduced that the Collection Scheme need not hold in this model we investigate what axioms can be proven to hold. Due to the difficulty in achieving this, we are led to defining the Respected Model, which is a generalisation of the symmetric submodel to the class forcing context. It is shown that this is indeed the correct model to work with. In this section we work over a fourth-order version of Kelley-Morse set theory which we denote by $\mathrm{KM}^-_{(4)}$.

Theorem 4.3.2. *Working over* $\mathrm{KM}^-_{(4)}$*, suppose that* $\mathbb{P}$ *is a pretame class forcing and* $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *is a tenacious symmetric system. Then the Respected model,* $\mathcal{N}$*, is a model of* KM−*.*

Theorem 4.3.7. *Working over* $\mathrm{KM}^-_{(4)}$*, suppose that* $\mathbb{P}$ *is a tame class forcing. Then* $\mathcal{N}$ *is a model of* KM*.*

We then end the chapter by comparing the Respected Model to the symmetric one and discussing when one can prove that the Collection Scheme holds in this model.

In Chapter 5 we explore the constructible universe in constructive contexts. Building on work of Lubarsky, we show that one can construct L in Intuitionistic Kripke Platek without infinity. This is done by introducing an expanded selection of *fundamental operations* and inspecting the universe constructed as the closure of said operations.

Theorem 5.3.6. *For every axiom* φ *of* $\mathrm{IKP}^{\text{-}Inf}$*,* $\mathrm{IKP}^{\text{-}Inf} \vdash \varphi^{\mathrm{L}}$*. Moreover,*

$$\mathrm{IKP}^{\text{-}Inf} + \text{“Strong Infinity”} \vdash (\text{Strong Infinity})^{\mathrm{L}}.$$

Theorem 5.3.7. $\mathrm{IKP}^{\text{-}Inf} \vdash (\mathrm{V} = \mathrm{L})^{\mathrm{L}}$*.*

Following this initial investigation, we further explore the properties of the fundamental operations. From this we are able to generalise the theorem that M is an inner model of ZF if and only if it is almost universal and closed under the fundamental operations to an intuitionistic context. This is done by relaxing what one requires of an inner model.

Theorem 5.4.8. *Suppose that* V *is a model of* IZF *and* $\mathrm{M} \subseteq \mathrm{V}$ *is a definable, transitive proper class with an external cumulative hierarchy. Then* M *is a model of* IZF *iff* M *is closed under the fundamental operations and is almost universal.*

The need to relax the requirement that an inner model must have the same class of ordinals as the universe is explored next. It is shown that this relaxation is necessary for the equivalence because it is consistent for there to be an ordinal in a model of IZF which is not in its constructible universe. This answers a question originally posed by Lubarsky at the end of [Lub93].

Theorem 5.5.1. *Starting from a model of* ZFC*, it is consistent to have a model of* IZF *such that*

$$\textsc{Ord} \cap \mathrm{V} \neq \textsc{Ord} \cap \mathrm{L}.$$

The second half of the thesis concerns large cardinals that can be defined via elementary embedding characterisations in weak set theories. In Chapter 6 we define the necessary concepts for the rest of the chapters. This is the chapter where we formalise what we mean by an elementary embedding whose domain is a class in set theories where this is difficult to define in a first-order way. Having established the environment that we will be working in, we begin to deduce some basic consequences of elementary embeddings. For example we give sufficient conditions to prove the existence of a critical point and further discuss when such a set exists.

We end the chapter by giving a literature review of some of the results on large cardinals in weak set theories, giving particular emphasis to those concerning elementary embedding characterisations.

Chapter 7 is where we begin an in-depth study of elementary embeddings, starting with intuitionistic theories. Firstly, we show that one has to be careful with how one wants to define a non-trivial, elementary embedding if one wants to be able to conclude that the definition has large consistency strength.

Theorem 7.1.9. $\mathrm{W} := \mathrm{L}^{\mathrm{V}(\mathbb{P})}$ *is a model of* IZF *plus* $\mathrm{W} = \mathrm{L}$ *and* π *is a non-trivial automorphism of* W *which moves an ordinal and is definable in* W*.*

After identifying the problem, we study the notion of a *critical set* which is the intuitionistic version of a critical point. Assuming that such a set exists we study the strength of the resulting embedding in Intuitionistic Kripke Platek.

Theorem 7.3.2. *Suppose that* $\mathrm{V} \models \mathrm{IKP}$ *and* $j\colon \mathrm{V} \to \mathrm{M}$ *is a* Σ-$\textsc{Ord}$*-inary, elementary embedding with witnessing ordinal* κ*. Then*

$$\mathrm{L}_{\kappa^{\#}} \models \mathrm{IZF}.$$

Theorem 7.3.10. *Suppose that* $\mathrm{V} \models \mathrm{IKP} + \forall\alpha \in \textsc{Ord}\ \exists x\ (x = \mathrm{V}_\alpha)$ *and* $j\colon \mathrm{V} \to \mathrm{M}$ *is a* Σ-$\textsc{Ord}$*-inary, elementary embedding with critical ordinal* κ *such that for any ordinal* α*,* $j(\mathrm{V}_\alpha) = (\mathrm{V}_{j(\alpha)})^{\mathrm{M}}$*. Then* $\mathrm{V}_{\kappa^{\#}}$ *is an inaccessible set so, in particular,*

$$\mathrm{V}_{\kappa^{\#}} \models \mathrm{IZF}.$$

We then end Chapter 7 by giving bounds for the existence of elementary embeddings of weak theories in terms of the standard large cardinal hierarchy over ZFC.

Corollary 7.4.11.

$$\mathrm{KP} + \exists j\colon \mathrm{V} \to \mathrm{M} \text{ which is a } \Sigma\text{-elementary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{ZFC} + \text{a proper class of totally indescribable cardinals}).$$

Corollary 7.4.36.

$$\mathrm{IKP} + \exists j\colon \mathrm{V} \to \mathrm{M} \text{ which is a } \Sigma\text{-}\textsc{Ord}\text{-inary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{ZFC} + \text{a proper class of weakly compact cardinals}).$$

Chapter 8 initiates a return to the classical setting. We begin by exploring Suzuki's Theorem, that there is no non-trivial, elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ which is definable from parameters, concluding that the proof goes through without Power Set.

Theorem 8.1.4 (Suzuki)**.** *Assume that* $\mathrm{V} \models \mathrm{ZF}^-$. *Then there is no non-trivial, cofinal, elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *which is definable from parameters.*

Having commented that the proof does not work in the much weaker system of Kripke Platek we study the consequences of definable embeddings in this theory. Using techniques from this study we show that, under appropriate assumptions, one can apply an elementary embedding to itself. This is proven in both the context that the embedding is definable from parameters and the more general case where we expand the language to include a predicate.

Theorem 8.2.3. *Let* V *be a model of* KP *and suppose that* $j\colon \mathrm{V} \to \mathrm{V}$ *is a cofinal,* Σ*-elementary embedding such that* $\mathrm{V} \models \mathrm{KP}_j$. *Then for each* $n \in \omega$ *there is a class function* $j_{(n)}$, *definable from* $\langle \mathrm{V}, j \rangle$, *such that:*

1. $j_{(n)}$ *is a total function from* V *to* V,
2. $j_{(n)}$ *is cofinal,*
3. $j_{(n)}$ *is a* Σ*-elementary embedding,*
4. $\mathrm{crit}_{j_{(n)}} = j^n(\mathrm{crit}_j)$,
5. $\mathrm{V} \models \mathrm{KP}_{j_{(n)}}$.

We then conclude this chapter by using the developed techniques to give a necessary condition on the rank of any such parameters which define an elementary embedding.

Theorem 8.2.13. *Assume that* $\mathrm{V} \models \mathrm{KP}$. *If* $\tau(\cdot,\cdot,p)$ *is a* Σ*-formula defining a cofinal,* Σ_0*-elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *then* $\mathrm{rank}(p) \geq \sup\{j^n(\mathrm{crit}_{\tau(\cdot,\cdot,p)}) \mid n \in \omega\}$.

The final two chapter contain the strongest results of this thesis. In particular we give a thorough examination of both Kunen's original proof of his famous inconsistency and

Woodin's later proof. To begin with, we show that Kunen's proof goes through in a very weak fragment of ZFC.

Theorem 9.1.7. *There is no non-trivial, Σ-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that*

- $\mathrm{V} \models \mathrm{ZBQW}_j$,
- *The supremum of the critical sequence exists.*

Corollary 9.1.11. *There is no non-trivial, Σ-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that* $\mathrm{V} \models (\mathrm{KP}(\mathcal{P}) + \mathrm{W})_j$.

Next, we conclude that Reinhardt embeddings of Kripke Platek with a rank hierarchy of sets have very high consistency strength.

Theorem 9.2.4. *Suppose that* $\mathrm{V} \models \mathrm{KP}^+$ *and there exists a non-trivial, Σ-elementary embedding $j\colon \mathrm{V}_\lambda \to \mathrm{V}_\lambda$ for some limit ordinal λ for which* $\mathrm{V} \models \mathrm{KP}^+_j$. *Then* $\langle \mathrm{V}_\lambda, j\rangle \models \mathrm{ZF} + \mathrm{WA}_\infty$.

Afterwards, we begin a study of Reinhardt embeddings of ZFC without Power Set. This is done by giving an equivalent characterisation to I_1, which is one of the strongest large cardinal assumptions not known to be incompatible with the Axiom of Choice.

Theorem 9.3.2. *Over* ZFC, *there exists an elementary embedding $k\colon \mathrm{V}_{\lambda+1} \to \mathrm{V}_{\lambda+1}$ if and only if there exists an elementary embedding $j\colon \mathrm{H}_{\lambda^+} \to \mathrm{H}_{\lambda^+}$.*

Where, for μ a regular cardinal, $\mathrm{H}_\mu := \{x \mid |\mathrm{trcl}(x)| < \mu\}$. Using the coding idea that appears in the proof of the above theorem we finally conclude that there is no non-trivial, cofinal, elementary embedding in this theory.

Theorem 10.2.1. *There is no non-trivial, elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that* $\mathrm{V} \models \mathrm{ZFC}^-_j$ *and* $(\sup\{j^n(\mathrm{crit}(j)) \mid n \in \omega\})^+ \in \mathrm{V}$.

Theorem 10.2.3. *There is no non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that* $\mathrm{V} \models \mathrm{ZFC}^-_j$ *and* $\mathrm{V}_{\mathrm{crit}(j)} \in \mathrm{V}$.

Corollary 10.3.2. *There is no non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that $\mathrm{V} \models (\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}})_j$.*

While examining the proof of both Theorem 10.2.3 and Corollary 10.3.2 we note the importance of the assumption that $\mathrm{V}_{\mathrm{crit}(j)}$ is a set. This leads to the question of whether or not one can actually prove that this is always the case, which we answer negatively using ideas from Chapter 3.

Corollary 10.4.2. *Assuming the consistency of ZFC plus a measurable cardinal, it is consistent to have $\mathrm{M} \subseteq \mathrm{V}$ and $j\colon \mathrm{V} \to \mathrm{M}$ such that j is a non-trivial, elementary embedding, $\mathrm{V} \models (\mathrm{ZFC}^-_{Ref})_{j,\mathrm{M}}$ and $\mathcal{P}(\omega)$ is a proper class.*

Following this, we study the consistency strength of a measurable cardinal in ZFC without Power Set. Using ideas provided by Victoria Gitman we give quite tight upper and lower bounds using weakened variants of measurable cardinals.

Theorem 10.5.7. *Working in ZFC, if there is a locally measurable cardinal then the theory $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$ plus a V-critical cardinal is consistent.*

Theorem 10.5.9. *Working in ZFC, suppose that $\mathrm{M} \subseteq \mathrm{V}$ and $j\colon \mathrm{V} \to \mathrm{M}$ is a non-trivial, elementary embedding with critical point κ such that*

$$\mathrm{V} \models (\mathrm{ZFC}^-)_{j,\mathrm{M}} + \exists z\ (z = \mathrm{V}_\kappa).$$

Then V_κ is a model of a proper class of baby measurable cardinals.

Finally, we conclude this body of work by considering the consequences of various reflections of stationary sets on the possibility that there could be a Reinhardt in ZFC without Power Set. Here we introduce a new stationary reflection concept, study its consequences and examine models of ZFC in which this principle holds.

Some of the work in this thesis has previously appeared in the author's publication [Mat20]. These are Section 4.1 where a brief outline of the argument was mentioned, Section 8.1, Section 9.3, Section 10.1, Section 10.2 and Section 10.3.

The results of Section 3.3.2 are from joint work of the author with Victoria Gitman, and the bounds obtained in Section 10.5 arose from conversations with Gitman based on the preprint [GS21] by Gitman and Schlicht. Otherwise, all results are due to the author unless stated.

The majority of the Chapters exhibit a large degree of independence from one another and can in general be read separately. The main places where this is not the case are:

- Chapter 6, where the notation for the later chapters is laid out alongside observations regarding the critical point of an elementary embedding and the theory needed to prove that the critical sequence is total,
- The use of the constructible universe in Chapter 7. How one should precisely formulate this intuitionistically is the topic of Chapter 5,
- Theorem 10.2.3, which makes essential use of Section 8.1 and a coding which is introduced in Section 9.3,
- Section 10.4, which uses a model of Zarach that is the focus of study in Section 3.3.

1.2 Hierarchy of Reinhardt Embeddings in Weak Theories

Here we provide a hierarchy of Reinhardt Embeddings in various subtheories of ZFC which constitutes much of the work in this thesis. We denote by T_j ($\mathrm{T}_{\Sigma\text{-}j}$) the theory that there is a non-trivial, (Σ-)elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that V satisfies the theory T expanded to include a predicate for j, as defined in Convention 2.2.1. The numbers indicate where an implication is proved in this thesis and most of the background theories will be formally stated in the next chapter. The definition of *cofinality* appears in Definition 6.2.13 and the concept of a V-*critical* cardinal can be found in Convention 6.2.2. Also, solely for this figure, we will denote by $\mathrm{T}_{\textsc{Ord}\text{-}j}$ the existence of an Ord-*inary* elementary embedding as defined in Definition 6.1.1.

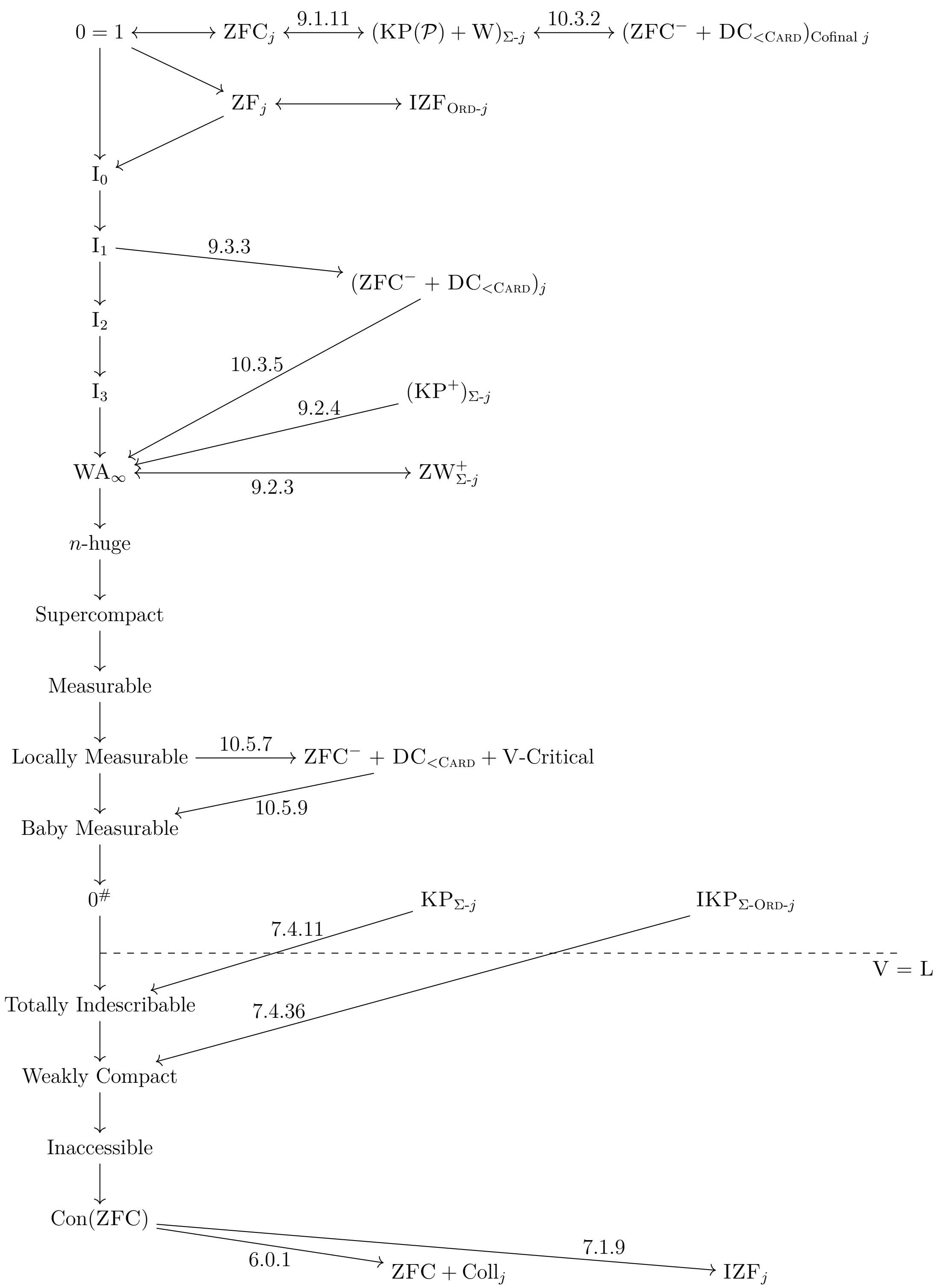


Figure 1.1: Hierarchy of Reinhardt Embeddings in Weak Theories

1.3 Classes and the Metatheory

A substantial portion of this body of work deals with statements such as:

There exists an elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$*,*

and thus it is worthwhile to briefly clarify a few of the metamathematical issues that arise in working with such objects.

The first issue is that j is a class object which begs the question as to what a class is. In standard, first-order, set theory a *class* is simply a collection of the form

$$\{x \mid \varphi(x)\}$$

where φ is a first-order formula in the language of set theory, $\mathcal{L} = \{\in\}$. Taking this approach, j should be definable by a formula φ, possibly with parameters. However, as shown by Suzuki in [Suz99] and discussed in Chapter 8, there is a relatively simple proof that there is no definable embedding $j\colon \mathrm{V} \to \mathrm{V}$ in ZF. Notably, this proof is much simpler than Kunen's proof of inconsistency in ZFC and crucially, it does not require the assumption of the Axiom of Choice. However, Suzuki's Theorem does not go through in a full second-order theory such as Gödel-Bernays or Kelley-Morse, the second of which is the theory in which Kunen originally formulated his inconsistency. Therefore to only deal with the definable case constrains the full power of Kunen's theorem.

There are then two options to allow us to explore Kunen's result in all its glory. The first is to consider the Kunen Inconsistency as a claim in a second-order theory such as Gödel-Bernays.[1] In this theory one distinguishes between the first-order objects which are the *sets*, and denoted by lower case letters, and the second-order objects which are the *classes*, and denoted by upper case letters. A class is then said to be *proper* if it is not a set.

[1] In Section 1 of [HKP12] the authors explain how one can formalise the existence of an elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ in GB and why the Kunen Inconsistency still goes through in this theory. They also include a long discussion on metamathematical issues, many of which will also come up in this work.

For completeness, we let GB be the theory with the following axioms[2]:

- Set Axioms: ZF;
- Every set is a class;
- If $X \in Y$, then X is a set;
- Class Axioms:
  - Class Extensionality: $\forall u(u \in X \leftrightarrow u \in Y) \to X = Y$;
  - Class Replacement: If F is a class function and x is a set, then $\{F(z) \mid z \in x\}$ is a set;
  - First-order Class Comprehension: $\forall Z\ \exists Y\ Y = \{x \mid \varphi(x, Z)\}$ where φ is a formula in which only set variables are quantified;

Moreover, we let GBC be GB plus the existence of a well-order of V of order type Ord. An important remark about GB is that we only allow Comprehension over formulae with set parameters. This is different from the theory of Kelley-Morse, KM, which can be formulated as GBC plus full second-order Comprehension, where we allow quantification over proper classes. One benefit of working over GB is that it is a conservative extension of ZF, that is any property of sets which is provable in GB is already provable in ZF.

The second option is to work in the fragment of GB that we shall denote by ZF_j. This will be explained fully using Convention 2.2.1 and Definition 6.1.1 but essentially the idea is to work in the extended language $\{\in, j\}$ and then expand all of the axiom schemes to include j as a predicate. Notably, we add to ZF; $\mathrm{Separation}_j$, $\mathrm{Replacement}_j$ and $\mathrm{Induction}_j$. This turns out to be the simplest theory in which the power of Kunen's result still goes through while also allowing us to stay in as close to a first-order theory as we can. Therefore, because part of the work is to explore what background theory is necessary to prove the Kunen Inconsistency, this is the theory we shall usually work in. In particular, this means we have the following convention:

[2]We will follow the convention from [GHK21] in not including any choice principles in our formulation of GB.

Convention 1.3.1. Given a theory T and a class predicate A, when we work in the theory T_A as defined in Convention 2.2.1 we will consider a *class* to be a collection of the form

$$\{x \mid \varphi(x)\}$$

where φ is a formula in the language augmented by A.

There is one significant place where we will diverge from our goal of sticking to as close to a first-order theory as possible, which is when we consider *Class Forcing* in Section 2.4 and Chapter 4. One could work in the purely first-order setting where one only considers definable class forcings, however one then loses a significant amount of the power of Class Forcing. Instead, we follow the convention from works such as [Fri00], [HKL+16], [HKS18], [HKS19] and [GHH+20] in defining class forcings over GB or KM[3]. Therefore, when doing Class Forcing we work in a two-sorted setting and, from a standpoint of "*what is a class?*", these sections should be considered independent to the rest of the work in this thesis.

The next issue which arises in working with class embeddings is the issue of expressing them. In ZFC, all standard large cardinals, such as inaccessibility, measurability or super-compactness, can be expressed in a first-order way. However, even without the inconsistency, one would not expect the same to be true for the critical point, κ, of a Reinhardt embedding $j\colon \mathrm{V} \to \mathrm{V}$. For if this were possible then we could express the statement

κ is the least ordinal which is the critical point of a Reinhardt embedding, j

and elementarity would lead us to have to accept that $j(\kappa) > \kappa$ was also the least such ordinal, which is obviously a contradiction. Instead this is normally expressed as a scheme in the metatheory, namely that for each natural number n, j is Σ_n-elementary. This is the approach we will necessarily have to take in our weaker systems, and is the usual way one phrases such an embedding in ZF. It is also the approach taken to define

[3] Or more accurately these theories without Power Set.

the *Wholeness Axiom*, see [Cor00] or [Ham01]. In ZFC, this is the axiom asserting the existence of a non-trivial elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that V satisfies Separation in the language expanded to include j but not Replacement. Here, it is necessary to express this as a scheme, namely that V satisfies Σ_n-elementarity in the expanded language.

The issue then occurs when one tries to say that such an embedding does not exist, because this is the negation of a scheme which is not in general expressible even as a scheme. The way one gets around this is by Gaifman's Theorem, which is Part II Theorem 1 of [Gai74] and appears in Theorem 8.1.1. Gaifman's theorem is that, over ZF, an embedding $j\colon \mathrm{V} \to \mathrm{V}$ is elementary if and only if it is cofinal and Σ_0-elementary, the latter statement being expressible by a single sentence in the language expanded to include j. A similar approach to defining fully elementary embeddings is also taken by Kanamori in Chapter 5 of [Kan08]. We shall see that, in certain cases, we are able to get analogous results in weaker theories and one can then observe that for all of the inconsistencies derived in this thesis, such as Theorems 9.1.11 and 10.2.3, the embeddings can be expressed by a single sentence using Gaifman-type results.

Strongly related to the previous issue is the problem of working in the metatheory. Since elementarity is formally expressed as a scheme, many of the results we give should be considered as metatheoretic results. Particular examples are those which use Induction in the language expanded to include j, such as Section 6.3 where we show that in all of the theories we are considering, the function $n \mapsto j^n(\kappa)$ is a total function and then in further chapters where this result is extensively used. We will in general comment on where such instances of Induction are applied but we shall not explicitly mention the fact that the proof takes place in the metatheory. A similar situation will occur when we discuss Σ-elementary embeddings since this is defined for φ which are Σ-formulae in the metatheory. This means that Σ-elementarity is, at least at first glance, a scheme of assertions. However, as previously mentioned, Gaifman's Theorem will allow us to circumvent this fact in those situations where it would otherwise be problematic.

Chapter 2

Preliminaries

2.1 Axiomatic Systems

We begin by introducing the main axiomatic systems that shall be used throughout this thesis. It will be vital to ensure that we take the correct formulation of each of our axioms because, in general, we shall work in theories where different versions of a given axiom are no longer equivalent. All of the theories we shall mention in this section can be viewed as subsystems of the Zermelo-Fraenkel axioms with Choice, ZFC.

To do this, we first list the main axioms we shall use. These will be predominantly grouped in sections of similar axioms and shall be taken from [Kun80] unless otherwise stated. For ease of presentation we shall use common notation in some of the formulations of the axioms. For example, we use $\emptyset$ to denote the (unique) set satisfying the Axiom of Empty Set.

We shall then discuss the various theories that shall be considered in this thesis. At the end of this section is Figure 2.1, which is a table of the theories and some of their consequences. The idea for this table was directly inspired by the table given in [Mat01].

The Axioms

1. **Extensionality:** $\forall x, y\ (\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

2. **Empty Set:** $\exists x\ \forall y \in x\ (y \neq y)$.

3. **Pairing:** $\forall x, y\ \exists z\ (x \in z\ \wedge\ y \in z)$.

4. **Unions:** $\forall a\ \exists x\ \forall y\ (y \in x \leftrightarrow \exists z \in a(z \in x))$.

5. **Power Set:** $\forall a\ \exists x\ \forall y\ (y \in x \leftrightarrow \forall z(z \in y \rightarrow z \in a))$.

6. **Weak Power Set** ([Fri73]): $\forall a\ \exists x\ \forall y\ \exists z \in x\ \forall w\ (w \in z \leftrightarrow (w \in y\ \wedge\ w \in a))$.

7. **Infinity:** $\exists a\ (\exists x\ (x \in a)\ \wedge\ \forall x \in a\ \exists y \in a\ (x \in y))$.

8. **Strong Infinity** ([AR10]): $\exists a\ (\mathrm{Ind}(a)\ \wedge\ \forall b\ (\mathrm{Ind}(b) \rightarrow \forall x \in a\ (x \in b)))$ where $\mathrm{Ind}(a)$ is an abbreviation for $\emptyset \in a\ \wedge\ \forall x \in a\ (x \cup \{x\} \in a)$.

9. **Foundation Scheme:** For any formula φ, $\exists x \varphi(x) \rightarrow \exists x(\varphi(x)\ \wedge\ \forall y \in x\ \neg\varphi(y))$.

10. **Set Induction Scheme:** For any formula φ, $\forall a(\forall x \in a\ \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a)$.

11. **Separation Scheme:** For any formula φ, $\forall a\ \exists x\ \forall y\ (y \in x \leftrightarrow y \in a\ \wedge\ \varphi(y))$.

12. **Replacement Scheme:** For any formula φ,
$\forall a\ \big(\forall x \in a\ \exists! y\ \varphi(x, y) \rightarrow \exists b\ \forall x \in a\ \exists y \in b\ \varphi(x, y)\big)$.

13. **Collection Scheme:** For any formula φ,
$\forall a\ \big(\forall x \in a\ \exists y\ \varphi(x, y) \rightarrow \exists b\ \forall x \in a\ \exists y \in b\ \varphi(x, y)\big)$.

14. **Reflection Principle:** For any formula φ and set a there is a transitive set A such that $a \subseteq A$ and $\varphi \leftrightarrow \varphi^A$. Where A is *transitive* iff $\forall x \in A\ \forall y \in x\ (y \in A)$.

15. **Axiom of Choice:** $\forall X\ (\emptyset \notin X \rightarrow \exists f \colon X \rightarrow \bigcup X\ \forall a \in X\ f(a) \in a)$.

16. **Well-Ordering Principle:** $\forall a\ \exists R\ (R$ well-orders $a)$.

Definition 2.1.1. Let ZF denote the theory consisting of the axioms; Extensionality, Empty set, Pairing, Unions, Power Set, Infinity, Foundation Scheme, Separation Scheme and Collection Scheme. Let ZFC consist of ZF plus the Axiom of Choice.

For completeness, below are some of the equivalences between the above axioms under classical systems.

Proposition 2.1.2.

- ZF $\vdash$ *Weak Power Set,*
- ZF $\setminus$ {*Infinity*} $\vdash$ *Infinity* $\leftrightarrow$ *Strong Infinity,*
- *Classical Predicate Calculus* $\vdash$ *Foundation Scheme* $\leftrightarrow$ *Set Induction Scheme,*
- ZF $\setminus$ {*Collection Scheme*} $\vdash$ *Replacement* $\leftrightarrow$ *Collection* $\leftrightarrow$ *Reflection,*
- ZF $\vdash$ *Axiom of Choice* $\leftrightarrow$ *Well-Ordering Principle.*

We now define some of the subsystems of ZFC that we shall use. The first is the theory ZFC *without Power Set*. Without Power Set many of the usual equivalent ways to formulate the axioms of ZFC break down. For example, we have that:

Theorem 2.1.3. *There are models of "*ZFC *without Power Set" satisfying*

- (*Zarach, [Zar96] Theorem 5.1*) *The Replacement Scheme but not the Collection Scheme,*
- (*Friedman, Gitman, Kanovei, [FGK19] Theorem 11.2*) *The Collection Scheme but not the Reflection Principle,*
- (*Szczepaniak, [Zar82] Theorem III*) *The Axiom of Choice but not the Well-Ordering Principle.*

The consequences of not assuming the Collection Scheme in the formulation of ZFC without Power Set were studied by Zarach in [Zar82] and [Zar96] and then more recently by Gitman, Hamkins and Johnstone in [GHJ16]. For example, it is shown that one can consistently have that ω_1 exists and is singular or that the Łoś ultrapower theorem can fail. Therefore, in [GHJ16], the authors conclude that only formulating the theory with the Replacement Scheme is not the "correct" way to consider it. As a general principle, we shall agree with them and when we work in ZFC without Power Set we shall assume the Collection Scheme. However, in Chapter 4 we shall see natural models of ZFC without Power Set where the Collection Scheme fails and thus it will be necessary to also consider the weaker formulation of this theory. So, using the notation of [GHJ16], we shall define the various versions of *the theory* ZFC *without Power Set* as follows:

Definition 2.1.4. Let ZF– denote the theory consisting of the axioms; Extensionality, Empty set, Pairing, Unions, Infinity, Foundation Scheme, Separation Scheme and Replacement Scheme.

ZF^- denotes the theory ZF– plus the Collection Scheme.

ZFC^- denotes the theory ZF^- plus the Well-Ordering Principle.

ZFC^-_{Ref} denotes the theory ZFC^- plus the Reflection Principle.

We will also use the same notation for their second-order versions GB– / GB^- and KM– / KM^-.

Two other important theories are the axioms of *Kripke-Platek*, KP, and *Zermelo*, Z. Kripke-Platek is a relatively weak theory however it suffices to deduce many useful set-theoretic results such as the existence of Cartesian products, Mostowski's Collapsing Lemma or the theorem that every set is contained in a transitive set. Moreover, it is the basic system in which one performs recursion theory on sets of natural numbers and also in which one can build Gödel's Constructible Universe, L. However, KP is not strong enough to prove that uncountable sets, for example the reals, exist or that the natural strengthening of Mostowski's Theorem (that every extensional well-founded relation has a Mostowski Collapse) holds. Many of the fundamental theorems that can be proven in KP, including those stated above, can be found in [Bar17].

We remark here that, unlike in [Bar17], we will formulate KP *with Infinity*. However, when building L constructively in Chapter 5, we will be clear to specify where this axiom is used.

The second theory, Z, is considered to be the first full attempt to axiomatise set theory, as done by Zermelo [Zer08] at the beginning of the twentieth century. His aim was to formalise some of the work of Cantor while avoiding the paradoxes identified by people such as Russell. An additional motivation was to create a foundation for his Well-Ordering Theorem by exposing its underlying set-theoretic assumptions. A significant limitation in his axiomatisation is the lack of the Axiom of Replacement which allows one to perform transfinite recursions and associate unique ordinals to well-orderable sets. As we shall see in later sections, this inability will have significant repercussions for what can be achieved.

Definition 2.1.5. Let KP denote the theory consisting of the axioms; Extensionality, Empty Set, Pairing, Unions, Infinity, Foundation Scheme and the Schemes of *Bounded Separation* and *Bounded Collection*, which are the schemes restricted to the class of formulae with only bounded quantification allowed.

Definition 2.1.6. Let Z denote the theory consisting of the axioms; Extensionality, Empty Set, Pairing, Unions, Power Set, Infinity, Foundation Scheme and Separation Scheme.

Remark 2.1.7. In weak systems with restricted Collection some authors decide to also restrict the Foundation Scheme. For example, in [Mat01], Mathias only assumes that Foundation holds for Π_1-formulae when formulating KP. For simplicity, we choose to follow Barwise, [Bar17], by assuming the full Foundation Scheme, although most of the results we obtain would go through with only Π_1-Foundation. One reason for doing so is that often we will be working in transitive models. In this case, such models will satisfy the full Foundation Scheme, assuming that some amount holds in the ambient universe.

Secondly, Zermelo's original formulation of Z does not include Foundation. We choose to follow Mathias, [Mat01], by including the full scheme in our axiomatisation.

For a weak system T we fix notation for two variants that we will regularly need to refer to. These will be T *without Power Set* and T *with a rank hierarchy*. The notation for the second case is an extension of that for ZF^-.

Definition 2.1.8. Fix T to be a subsystem of ZFC. Let T^- denote the theory with the same list of axioms as T except with the Power Set Axiom removed (and where we take the Collection version of an axiom scheme rather than the Replacement version when applicable).

Let T^+ denote the theory T plus the assertion

$$\forall u\ \exists \alpha\ \exists v\ (\alpha \in \mathrm{Ord}\ \wedge\ v = \mathrm{V}_\alpha\ \wedge\ u \in v)$$

where "$z = \mathrm{V}_\alpha$" is read as the formula

$$\exists x\Big(x \textit{ is a function } \wedge\ \mathrm{dom}(x) = \alpha + 1\ \wedge$$
$$\forall \beta \in \alpha + 1\ \forall y\Big(y \in x(\beta) \leftrightarrow \exists \gamma \in \beta(y \subseteq x(\gamma))\Big)\ \wedge\ z = x(\alpha)\Big).$$

A weak system that we shall use is the following theory which was defined by Mac Lane. His aim was to define a fragment of set theory which was sufficient to be able to prove all mathematics not directly connected to mathematical logic. For commentary on the success of this endeavour, and what is provable in related systems, see [Mat01].

Definition 2.1.9. Let ZBQ denote the theory of *Zermelo with Bounded Quantification.* This is the theory consisting of the axioms; Extensionality, Empty Set, Pairing, Unions, Power Set, Infinity, Foundation Scheme and Bounded Separation.

Extending our notation, ZBQ^- denotes ZBQ without Power Set, or equivalently KP without Bounded Collection.

Note that the theory dubbed *Mac* by Mathias is ZBQ plus Choice and the assertion of transitive containment.

In order to keep with the standard notation, and our notation from Definition 2.1.4, we shall define ZC (ZBQC) to be Z (ZBQ) plus the *Well-Ordering Principle*, which is the assertion that every set can be well-ordered. We remark here that, over Z, the Axiom of Choice is equivalent to the Well-Ordering Principle.

However, this will not be a strong enough version of choice for some of our purposes without any Replacement. This is because, as commented upon in [HFL12], Replacement is needed to conclude that every well-ordering is isomorphic to the inclusion order on some ordinal. Importantly for our purposes, in Proposition 6.2.8 we shall see that if one only works in the theory ZC then it is possible to obtain an elementary embedding without a critical point. Therefore, it is helpful to work in the further extensions ZW and ZBQW where

Definition 2.1.10. The axiom W denotes the assertion that every set can be well-ordered with a well-ordering order-isomorphic to an ordinal.

Note that, for a given well-ordering, the associated ordinal is necessarily unique.

The next axiomatic system we consider is KP *plus Power Set*. There are two different versions of this theory which have appeared throughout the literature. The first is $\mathrm{KP}^{\mathcal{P}}$ which was studied by Mathias in Chapter 6 of [Mat01] and the second is $\mathrm{KP}(\mathcal{P})$ which was studied by Rathjen in [Rat20]. The difference between these two formulations is that Mathias restricts the Foundation Scheme to a class of formulae which he denotes by $\Pi_1^{\mathcal{P}}$ whereas Rathjen assumes the full Foundation Scheme. We choose to use Rathjen's definition, which we define now.

Definition 2.1.11. Define subset bounded quantification $\exists x \subseteq y\ \varphi(x)$ and $\forall x \subseteq y\ \varphi(x)$ as abbreviations for $\exists x\ (x \subseteq y\ \wedge\ \varphi(x))$ and $\forall x\ (x \subseteq y \rightarrow \varphi(x))$ respectively.

Let $\Delta_0^{\mathcal{P}}$ denote the smallest class of formulae containing the Δ_0-formulae which is closed under $\vee$, $\wedge$, $\rightarrow$, $\neg$ and the quantifiers $\exists x \in a$, $\forall x \in a$, $\exists x \subseteq a$ and $\forall x \subseteq a$.

$\mathrm{KP}(\mathcal{P})$ is the theory whose axioms are: Extensionality, Empty Set, Pairing, Unions, Power Set, Infinity, Foundations Scheme, $\Delta_0^{\mathcal{P}}$-Separation and $\Delta_0^{\mathcal{P}}$-Collection.

Remark 2.1.12 (Mathias)**.** $\mathrm{KP}(\mathcal{P})$ is different from KP plus Power Set. For example, $\mathrm{L}_{\aleph_\omega^{\mathrm{L}}} \models \mathrm{KP} + \textit{Power Set}$ but $\mathrm{L}_{\aleph_\omega^{\mathrm{L}}} \not\models \mathrm{KP}(\mathcal{P})$.

The final theories that we shall consider are *intuitionistic theories* which we shall discuss in more detail in Section 2.3. The intuitionistic theories we shall consider are *Intuitionistic Zermelo-Fraenkel* (IZF) and *Intuitionistic Kripke-Platek* (IKP). These are formulated with the same axioms as ZF and KP except that the underlying logic is intuitionistic. We highlight here that IZF satisfies the Collection Scheme.

	LEM	{}, ⋃, Ind.	Ext.	Inf.	Sep.	Rep.	Coll.	Rank	Power	AC	WO	W	DC
ZBQ^-	✓	✓	✓	✓	Σ_0	✗	✗	✗	✗	✗	✗	✗	✗
ZBQ	✓	✓	✓	✓	Σ_0	✗	✗	✗	✓	✗	✗	✗	✗
Z	✓	✓	✓	✓	✓	✗	✗	✗	✓	✗	✗	✗	✗
ZC	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓	✓	✗	Set
ZW	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓	✓	✓	Set
KP	✓	✓	✓	✓	Δ	Σ	Σ	✓	✗	✗	✗	✗	✗
$KP(\mathcal{P})$	✓	✓	✓	✓	$\Delta^{\mathcal{P}}$	$\Sigma^{\mathcal{P}}$	$\Sigma^{\mathcal{P}}$	✓	✓	✗	✗	✗	✗
$KP(\mathcal{P}) + WO$	✓	✓	✓	✓	$\Delta^{\mathcal{P}}$	$\Sigma^{\mathcal{P}}$	$\Sigma^{\mathcal{P}}$	✓	✓	✓	✓	✓	Set
ZF–	✓	✓	✓	✓	✓	✓	Σ	✓	✗	✗	✗	✗	✗
ZF^-	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗
ZFC^-	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	Set
$ZFC^- + DC_{<\text{Card}}$	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	Scheme
ZF	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗
ZFC	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	Scheme
IKP	✗	✓	✓	✓	Σ_0	Σ	Σ	✓	✗	✗	✗	✗	✗
IZF	✗	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗
$IZF \setminus \{Ext.\}$ (7.4)	✗	✓	✗	✓	✓	✓	✓	✓	Weak Power	✗	✗	✗	✗

Figure 2.1: Chart of Axiomatic Systems

2.2 Adding a Predicate

When working with weak systems it is often necessary to add a predicate to discuss specific classes. For example, following the notation of Rathjen in [Rat20], the theory $\mathrm{KP}(\mathcal{P})$ can be considered as KP with the language expanded to include a primitive function symbol $\mathcal{P}$ along with the axiom

$$\forall x\ \forall y(y \in \mathcal{P}(x) \longleftrightarrow y \subseteq x)$$

and then expanding the Schemes of Bounded Separation and Collection to this new language.

Another instance of adding a predicate that we shall consider throughout this work is when we add predicates M and j along with the axioms asserting that

- M is a transitive subclass of V,
- j is a non-trivial elementary embedding from V to M.

Therefore, in this section, we specify what we mean by adding a predicate and state some of the important axiomatic properties. To do this, it is beneficial to introduce some notation that we shall use throughout this thesis. This notation should be seen as a convention rather than a formal mathematical definition and whose purpose is to simplify later notation.

Convention 2.2.1. For theories T, models M of T and class predicates A, we say that $\mathrm{M} \models \mathrm{T}_A$ if M, augmented with an interpretation for A, satisfies the axiom Schemes of T in the language expanded to include the predicate A.

For example, we shall use the notation $\mathrm{M} \models \mathrm{ZFC}^-_A$ to denote that A is a class predicate and that M satisfies every instance of Collection, Separation and Induction in the language expanded to include A. To be more formal, we should say that $\langle \mathrm{M}, \in, A \rangle \models \mathrm{ZFC}^-_A$ however we shall use our shortened notation to enhance readability.

Remark 2.2.2. It is important that our models of T_A will also satisfy the Induction Schemes of T in the language expanded to include A. In particular, in the weak theories of ZBQ and KP we will assume full induction in this expanded language. This will be important in Section 6.3 where we will need $\text{Induction}_{\text{j}}$ in order to prove that the function giving the critical sequence of an elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ is total.

We now specify some important classes of formulae which are frequently used. This presentation will be slightly different to how one would present this in the standard classical case. This is because, classically, implication and disjunction can be expressed using conjunction and negation. However, these equivalences no longer hold in intuitionistic theories which is why we will close under more conditions.

Definition 2.2.3. The Σ_0-*formulae* are the smallest class of formulae containing the atomic formula and closed under:

- Conjunction,
- Disjunction,
- Implication,
- Negation,
- Bounded quantification.

The Σ_1-*formulae* are the class of formulae of the form $\exists x_1 \ldots \exists x_n \; \varphi(x_1, \ldots, x_n)$ where φ is a Σ_0-formula.

Finally, let the Σ-*formulae* be the smallest class of formulae containing the Σ_0-formulae and closed under:

- Conjunction,
- Disjunction,
- Bounded quantification,
- Unbounded existential quantification.

Given a predicate A we define the classes Σ_0^A, Σ_1^A and Σ^A in an analogous way. The class of Π_1-formulae is defined as the class of formulae of the form $\forall x_1 \dots \forall x_n\ \varphi(x_1, \dots, x_n)$ where φ is a Σ_0-formula. Finally, the Π-formulae are defined analogously to the Σ-formulae, except for replacing the final condition with closure under unbounded universal quantification.

Note that the class Σ appears impoverished because of the restrictions we placed upon implication. Classically, $\varphi \to \psi$ is equivalent to $\psi \vee \neg\varphi$ so for this to be a Σ-formula we need that ψ is a Σ-formula and φ is a Π-formula (working in a theory where the negation of a Π-formula can be seen to be logically equivalent to a Σ-formula). However, since this equivalence need not hold intuitionistically, this construction becomes more difficult in intuitionistic theories and the extra work does not appear to add any benefit to this body of work. Therefore, we shall just work in the impoverished setting.

Using these classes, we can obtain some important consequences of KP. It will be of benefit to state them in the language expanded to include a predicate for A but the proofs are exactly the same as those found in Section I.4 of Barwise [Bar17].

Lemma 2.2.4. *For each Σ^A-formula $\varphi(v)$ the following are logically valid*

1. $\varphi^{(a)}(v)\ \wedge\ a \subseteq b \to \varphi^{(b)}(v)$,
2. $\varphi^{(a)}(v) \to \varphi(v)$,

where $\varphi^{(u)}(v)$ is the result of replacing each unbounded quantifier in $\varphi(v)$ with bounded quantification over u.

Theorem 2.2.5 (The Σ^A-Reflection Principle)**.** *For all Σ^A-formulae $\varphi(v)$ we have that*

$$\mathrm{KP}_A \vdash \varphi(v) \leftrightarrow \exists a\ \varphi^{(a)}(v).$$

Theorem 2.2.6 (The Σ^A-Collection Principle)**.** *For every Σ^A-formula φ the following is a theorem of KP_A: If $\forall x \in a\ \exists y\ \varphi(x, y)$ then there is a set b such that*

$$\forall x \in a\ \exists y \in b\ \varphi(x, y)\ \wedge\ \forall y \in b\ \exists x \in a\ \varphi(x, y).$$

Theorem 2.2.7 (Δ^A-Separation)**.** *For any Σ^A-formula $\varphi(x)$ and Π^A-formula $\psi(x)$, the following is a theorem of KP_A: If $\forall x \in a\ \varphi(x) \leftrightarrow \psi(x)$, then $\{x \in a \mid \varphi(x)\}$ is a set.*

2.3 Intuitionism

Chapters 5 and 7 are devoted to structural properties of intuitionistic theories. This is where we shall work in a logic in which we have removed "*non-constructive*" principles such as the Law of Excluded Middle and Double Negation Elimination. All of the terms and theorems we shall define in this section can be found in [AR10].

It is instructive to begin with the *Brouwer-Heyting-Kolmogorov interpretation.* This is an informal specification for how one should interpret logical connectives intuitionistically. Kolmogorov's opinion was that one should consider propositions as "problems" which can be solved by being broken down into simpler problems. For example, the problem $A \to B$ represents the problem of producing a method which solves the problem B when given a solution to the problem A.

Definition 2.3.1 (The Brouwer-Heyting-Kolmogorov interpretation, [AR10] 2.2)**.**

- p proves $\perp$ is impossible, so there is no proof of $\perp$,
- p proves $\varphi \wedge \psi$ iff p is a pair $\langle q, r\rangle$ where q proves φ and r proves ψ,
- p proves $\varphi \vee \psi$ iff p is a pair $\langle n, q\rangle$ where $n = 0$ and q proves φ or $n = 1$ and q proves ψ,
- p proves $\varphi \to \psi$ iff p is a function which transforms any proof q of φ into a proof $p(q)$ of ψ,
- p proves $\neg\varphi$ iff p proves $\varphi \to\perp$,
- p proves $(\exists x \in A)\varphi(x)$ iff p is a pair $\langle a, q\rangle$ where a is a member of the set A and q is a proof of $\varphi(a)$,
- p proves $(\forall x \in A)\varphi(x)$ iff p is a function such that for each member a of A, $p(a)$ proves $\varphi(a)$.

In particular, this tells us that we should consider $\wedge$, $\vee$ and $\to$ as distinct symbols rather than being interpretable from each other. Also, negation will be interpreted using implication instead of being a distinct symbol.

When working intuitionistically, it is important to specify the axioms correctly in order to avoid inadvertently being able to deduce the Law of Excluded Middle. For example, we have the following classical results. These go through in much weaker theories but, for simplicity, we just state them in IZF.

Theorem 2.3.2 ([AR10] Proposition 10.4.1)**.** *The Foundation Scheme implies the Law of Excluded Middle.*

Theorem 2.3.3 (Diaconescu, [Dia75])**.** *The Axiom of Choice implies the Law of Excluded Middle.*

Avoiding these principles leads to the following axiomatisations of IZF and IKP. It is proven in [FŠ85] that, intuitionistically, the Replacement Scheme does not imply the Collection Scheme, even if one assumes Dependent Choice holds along with every Σ_1-sentence of ZF. Moreover, Replacement alone is insufficient to do many of the constructions we need, so we shall formulate IZF with the Collection Scheme.

Definition 2.3.4. Let IZF denote the theory consisting of the axioms; Extensionality, Empty Set, Pairing, Unions, Power Set, Infinity, Set Induction Scheme, Separation Scheme and Collection Scheme.

Definition 2.3.5. Let IKP denote the theory consisting of the axioms; Extensionality, Empty Set, Pairing, Unions, Strong Infinity, Set Induction Scheme, Σ_0-Separation Scheme and Σ_0-Collection Scheme.

It is worth noting that not all Choice principles result in instances of Excluded Middle. For example, one can add Dependent Choice and, by a result from Chapter 8 of [Bel05], a version of Zorn's Lemma holds in every Heyting-valued model over a model of ZFC. Moreover, many of the basic properties of KP can still be deduced in IKP. For example,

Proposition 2.3.6 ([AR10] 4.1.1 & 19.1.1)**.** *In* IKP *one can prove that:*

- *Let* $\langle a, b \rangle \coloneqq \{\{a\}, \{a, b\}\}$. *Then* $\forall a, b, c, d\ (\langle a, b \rangle = \langle c, d \rangle \rightarrow (a = c\ \wedge\ b = d))$.
- $\forall a, b\ \exists c\ (c = a \times b)$.

We can also still deduce the Σ-Reflection and Σ-Collection Principles which are Theorems 2.2.5 and 2.2.6. However we are not able to deduce the Δ-Separation Principle because the proof uses the fact that classically $\forall x \in a(\varphi \to \psi)$ holds iff $\forall x \in a(\psi(x) \ \vee \ \neg\varphi(x))$ holds, which is not intuitionistically provable.

An important concept is the notion of an ordinal. This is traditionally defined as a transitive set which is well-ordered by the $\in$ relation, however this is not appropriate in our setting. Therefore, we shall follow the definition given in I.3.6 of Barwise [Bar17]:

Definition 2.3.7. An *ordinal* is a transitive set of transitive sets.

Let 0 denote the ordinal $\emptyset$, which is the unique set given by the Axiom of Empty Set, and 1 the ordinal $\{0\}$. An important class of ordinals is the class of *Truth Values*

$$\Omega := \{x \mid x \subseteq 1\}.$$

$\Omega = \{0, 1\}$ is the assertion that every statement is either true or false, which entails the Law of Excluded Middle. Therefore, in general we may have other truth values even if they can't necessarily be determined. For example, these can be ordinals of the form

$$\{0 \in 1 \mid \varphi\}$$

where φ is a formula for which we can neither determine φ nor $\neg\varphi$. We also have the following proposition which describes the difficulty in assigning an order to the ordinals.

Proposition 2.3.8.

$$\mathrm{IKP} \vdash \forall \alpha \in \textsc{Ord}(0 \in \alpha + 1) \to \textit{ Law of Excluded Middle for } \Sigma_0\textit{-formulae.}$$

We end this section with some "*large*" sets, which are the intuitionistic analogue to large cardinals. Such principles were introduced by Friedman and Šc̆edrov in [FŠ84] and then later studied by authors such as Aczel and Rathjen in [AR10]. Because we no longer have that the ordinals are linearly ordered by $\in$, the notion of a cardinal number loses much of its utility. This means that it is often difficult to derive structural consequences from classical formulations of large cardinals, which makes it beneficial to work with an alternative structure. The principle being that often we are really considering V_κ rather than κ, so large sets should be defined using sets with properties similar to V_κ.

Definition 2.3.9. Given sets a and b we define the class of *multi-valued functions*, $\mathrm{mv}({}^a b)$, to be the collection of all sets $R \subseteq a \times b$ such that

$$\forall x \in a \; \exists y \in b \; \langle x, y \rangle \in R.$$

A set C is said to be *full* in $\mathrm{mv}({}^a b)$ if $C \subseteq \mathrm{mv}({}^a b)$ and

$$\forall R \in \mathrm{mv}({}^a b) \; \exists S \in C \; (S \subseteq R).$$

Definition 2.3.10 (Aczel)**.** A transitive set C is said to be *regular* if

- C is inhabited, that is $\exists u \; (u \in C)$,
- $\forall u \in C \; \forall R \in \mathrm{mv}({}^u C) \; \exists v \in C$ $\big(\forall x \in u \; \exists y \in v \; \langle x, y \rangle \in R \;\wedge\; \forall y \in v \; \exists x \in u \; \langle x, y \rangle \in R\big)$.

Definition 2.3.11 (Rathjen)**.** A transitive set C is said to be *functionally regular* if

- C is inhabited, that is $\exists u \; (u \in C)$,
- $\forall u \in C \; \forall f \colon u \to C \;\; \mathrm{ran}(f) \in C$.

Definition 2.3.12 (Rathjen [RGP98] and Friedman, Ščedrov [FŠ84])**.** A set I is said to be *inaccessible*, denoted $\mathrm{Inacc}(I)$, if it is a regular set which satisfies:

- $\omega \in I$, where ω is the unique set given by Strong Infinity,
- $\forall a \in I \; \bigcup a \in I$,
- $\forall a \in I \; (\exists u \; (u \in a) \to \bigcap a \in I)$,
- $\forall a \in I \; \exists b \in I \; \forall x \; (x \subseteq a \to x \in b)$.

Theorem 2.3.13 (Rathjen)**.** *Under* ZFC*, I is an inaccessible set iff there exists a strongly inaccessible cardinal κ such that $I = \mathrm{V}_\kappa$.*

Remark 2.3.14. In the original definition by Rathjen, the final condition of inaccessibility is replaced by the intuitionistically weaker condition of closure under fullness: $\forall a, b \in I \; \exists c \in I \; I \models$ "*C is full in* $\mathrm{mv}({}^a b)$". This is because Rathjen works in a theory without Power Set. However, because we will be working with power sets, we have chosen to take the above definition, which is equivalent to that found in [FŠ84].

2.4 Class Forcing

In Chapter 4 we shall investigate symmetric submodels of class forcings. Therefore we now introduce the notion of class forcing, followed by a brief introduction to symmetric submodels of set forcings in the next section. We begin by fixing some notation which we shall use throughout this thesis. More details and how to formalise the theory of class forcing can be found in references such as [Fri00], [HKL$^+$16] and [HKS18]. We first state an important class of forcings in its most general terms which shall be one of the main examples of forcings we shall use.

Definition 2.4.1. For classes X and Y, $\mathrm{Add}(X, Y)$ is the collection of set partial functions $p\colon Y \times X \to 2$ such that $\mathrm{dom}(p)$ can be well-ordered and injects into X.

As in set forcing, when trying to formalise the theory of class forcing one often works in a countable, transitive model of some second-order theory such as GB^-. Such a model will be of the form $\mathbb{M} = \langle \mathrm{M}, \mathcal{C}\rangle$ where M denotes the *sets* of the model and $\mathcal{C}$ the *classes*. However, primarily for ease of notation, we shall repeatedly only talk about the first-order part of the theory, noting that if M is a set model of ZF^- and $\mathcal{C}$ is the collection of classes definable over M then $\langle \mathrm{M}, \mathcal{C}\rangle$ is a model of GB^-. We shall say that a class $\dot\Gamma$ is a $\mathbb{P}$-*name* if every element of $\dot\Gamma$ is of the form $\langle \dot x, p\rangle$ where $\dot x$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. We then define $\mathrm{M}^{\mathbb{P}}$ to be the collection of $\mathbb{P}$-names which are elements of M and define $\mathcal{C}^{\mathbb{P}}$ as those names which are in $\mathcal{C}$.

The following piece of notation was introduced by Karagila. Given a collection of $\mathbb{P}$-names, it gives us a simple way to transform it into a single $\mathbb{P}$-name.

Notation 2.4.2 (Karagila)**.** For $I \subseteq \mathrm{M}^{\mathbb{P}}$ a collection of $\mathbb{P}$-names, $I^\bullet := \{\langle \dot x, \mathbb{1}\rangle \mid \dot x \in I\}$.

Essentially, the question is which properties of set forcing are still true when the partial order is assumed to be a proper class. One immediately runs into a problem when trying to prove the forcing theorem which comprises of two parts; *truth* and *definability*. The *definability lemma* is the assertion that the forcing relation is definable in the ground model and the *truth lemma* is that anything true in the generic extension is forced to be true by an element of the generic. However, as shown in [HKL+16], given any countable, transitive model of GB^- there is a class forcing notion which does not satisfy the forcing theorem for atomic formulae. In fact, it is shown in [GHH+20] that, over GB with a global well-order, the statement that the forcing theorem holds for any class forcing is equivalent to elementary transfinite recursion for recursions of length Ord.

Moreover, even if a class forcing satisfies the forcing theorem it is not always the case that GB^- will be preserved in any generic extension. The simplest such example is $\mathrm{Col}(\omega, \mathrm{Ord})$ which generically adds a function collapsing the ordinals onto ω. However, there is a well-known collection of class forcings which both satisfy the forcing theorem and preserve all of the axioms of GB^-: *pretameness*. For the next definition, we shall formulate it as in Section 2 of [HKS18]. This can easily be seen to be equivalent to the definitions found in [Fri00] but has the benefit of avoiding having to consider incompatible conditions, which will make the arguments in Chapter 4 much cleaner.

Definition 2.4.3 ([HKS18] Definition 2.1)**.** For p, q compatible conditions we say that $D \subseteq \mathbb{P}$ is dense below $p \wedge q$ if for every $r \leq p, q$ there is some $s \leq r$ such that $s \in D$.

A set d is said to be *predense below* $p \wedge q$ if for every $r \leq p, q$ there is some $s \in d$ which is compatible with r (compatibility will also be written as $s \parallel r$).

A notion of class forcing $\mathbb{P}$ is *pretame* if for every $p \in \mathbb{P}$ and any sequences $\langle s_i \mid i \in I \rangle \in M$ and $\langle D_i \mid i \in I \rangle$ of classes, where $I \in M$ and each D_i is dense below $p \wedge s_i$, there is a $q \leq p$ and a sequence $\langle d_i \mid i \in I \rangle \in \mathrm{M}$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense below $q \wedge s_i$.

Remarks 2.4.4.

- We are not assuming that $\mathbb{P}$ is closed under meets and thus should only consider $p \wedge q$ as an abbreviation for the collection of conditions below both p and q.
- We use the convention that if $p \perp q$ then only the empty set is predense below $p \wedge q$ while any set is dense below $p \wedge q$.
- When using pretameness to obtain $\langle d_i \mid i \in I \rangle$, without loss of generality we shall assume that for each $i \in I$ every element of d_i is below q.

Theorem 2.4.5 (Stanley)**.** *Suppose* $\mathbb{M}$ *is a model of* GB^- *and* $\mathbb{P}$ *is a pretame class forcing notation. Then* $\mathbb{P}$ *satisfies the forcing theorem and for any generic filter* G*,* $\mathbb{M}[G]$ *satisfies* GB^-*.*

The main class forcing we shall consider is $\mathrm{Add}(\omega, \mathrm{Ord})$, the forcing to add a proper class of Cohen reals. It can be proven that this satisfies the forcing theorem because it is *ordinal approachable by projections*,[1] that is to say it can be written as a continuous, increasing union of set-sized forcings, $\mathrm{Add}(\omega, \mathrm{Ord}) = \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Add}(\omega, \alpha)$.

Theorem 2.4.6 ([HKL+16] Theorem 6.4)**.** *If* $\mathbb{P}$ *is ordinal approachable by projections then* $\mathbb{P}$ *satisfies the forcing theorem.*

As remarked in Lemma 6.7 of [HKL+16], a version of the forcing to add a cofinal function from ω to the ordinals witnesses that this notion alone does not ensure a forcing is pretame. However, it will allow us to prove pretameness for $\mathrm{Add}(\omega, \mathrm{Ord})$ by using an equivalent characterisation of pretameness from [HKS18]:

Theorem 2.4.7 ([HKS18] Lemma 2.6)**.** *Suppose that* $\mathbb{M} = \langle \mathrm{M}, \mathcal{C} \rangle$ *is a model of* GB *and* $\mathbb{P}$ *is a class forcing notion for* $\mathbb{M}$ *which satisfies the forcing theorem. Then* $\mathbb{P}$ *is pretame if and only if there is no set* $a \in \mathrm{M}$*, name* $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ *and condition* $p \in \mathbb{P}$ *such that* $p \Vdash$ "$\dot{F} \colon \check{a} \to \mathrm{Ord}$ *is cofinal* ".

Theorem 2.4.8. *Suppose that* $\mathbb{M} = \langle \mathrm{M}, \mathcal{C} \rangle$ *is a model of* $\mathrm{GB} + \mathrm{AC}$ *and* $\mathbb{P}$ *is a class forcing notion for* $\mathbb{M}$ *which satisfies the forcing theorem. If* μ *is an uncountable cardinal in* $\mathbb{M}$ *and* $\mathbb{P}$ *satisfies the* μ*-cc then* $\mathbb{P}$ *is pretame.*

Proof. This will be proven by a variation on a standard set forcing result which uses the μ-cc to approximate functions in the extension:

[1] In [HKL+16] the authors call this property *approachability by projections*. However we will follow the terminology of [HKS19] where they give a generalised definition of being approachable by projections. They then comment that this is perhaps how approachability should have been defined in the first place and that it is more natural to rename the original definition as being ordinal approachable by projections.

Claim 2.4.9. *Suppose that* $a \in \mathrm{M}$ *and* $F\colon a \to \textsc{Ord}$ *is a class function in* $\mathbb{M}[G]$. *Then there exists some* $f \in \mathrm{M}$ *such that for all* $x \in a$, $F(x) \in f(x)$ *and* $(|f(x)| \in [\textsc{Ord}]^{<\mu})^{\mathbb{M}}$.[2]

To prove the claim, let $\dot{F}$ be a name for F and take p such that $p \Vdash \dot{F}\colon \check{a} \to \textsc{Ord}$. Next, for each $x \in a$, let

$$f(x) = \{\alpha \in \textsc{Ord} \mid \exists q \leq p\ (q \Vdash \dot{F}(\check{x}) = \check{\alpha})\}.$$

Now suppose that for some $x \in a$, $f(x)$ did not have cardinality less than μ. By using Collection and set sized Choice, we can choose a subset Y of $f(x)$ of size μ. Then for each $\alpha \in Y$ we can choose $q_\alpha \leq p$ such that $q_\alpha \Vdash \dot{F}(\check{x}) = \check{\alpha}$. But then this must be an antichain of size μ contradicting the assumption of μ-cc.

Using the claim, we have that if $a \in \mathrm{M}$, $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ are such that $p \Vdash \dot{F}\colon \check{a} \to \textsc{Ord}$ then, taking f from the conclusion of the claim, $\delta = \sup\{\sup(f(x)) \mid x \in a\}$ is an ordinal. Therefore

$$p \Vdash \text{"}\textit{the image of } \dot{F} \textit{ is contained in } \delta\text{"}.$$

So, in particular, p cannot force the function to be cofinal which implies that $\mathbb{P}$ is pretame by Theorem 2.4.7. □

Corollary 2.4.10. $\mathrm{Add}(\omega, \textsc{Ord})$ *is a pretame class forcing.*

2.5 Symmetric Submodels

It is well-known that if M is a model of ZFC and G is a generic filter over a set forcing then $\mathrm{M}[G]$ is a model of ZFC. Furthermore, alongside his original forcing argument, Cohen also presented a method to use his forcing theory to produce a model of ZF in which Choice may fail. The idea was to take an intermediate model between M and $\mathrm{M}[G]$ by restricting the new sets added to be some of those which are definable from G but, importantly, not G itself.

[2] where $[\textsc{Ord}]^{<\mu}$ denotes the collection of sets of ordinals of cardinality strictly less than μ.

Such a model is now known as the *Symmetric Submodel* and we outline the main construction here. Much fuller explanations, which also contain proofs, can be found in either Section 8.12 of [DS96], using partial orders, or the end of Chapter 15 of [Jec03], using Boolean-valued models.

So fix M to be a model of ZF. We call a triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \in \mathrm{M}$ a *symmetric system* if it satisfies the following:

- $\mathbb{P} \in \mathrm{M}$ is a set forcing poset,
- $\mathcal{G} \in \mathrm{M}$ is a group of automorphisms of $\mathbb{P}$,
- $\mathcal{F} \in \mathrm{M}$ is a normal filter of subgroups of $\mathcal{G}$

where

Definition 2.5.1. For a given group of permutations $\mathcal{G}$ on a set S, $\mathcal{F}$ is called a *normal filter of subgroups* of the group $\mathcal{G}$ if

- $\mathcal{G} \in \mathcal{F}$,
- If $\mathcal{H} \in \mathcal{F}$ and $K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$,
- If $H \in \mathcal{F}$ and $H \subseteq K$ then $K \in \mathcal{F}$,
- (*Normality*) If $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

Next, given an automorphism π of $\mathbb{P}$, we can extend π to act on $\mathbb{P}$-names by recursion as

$$\pi\dot{x} \coloneqq \{\langle \pi\dot{y}, \pi p\rangle \mid \langle \dot{y}, p\rangle \in \dot{x}\}.$$

Then, we say that a $\mathbb{P}$-name is $\mathcal{F}$*-symmetric* if

$$\mathrm{sym}(\dot{x}) \coloneqq \{\pi \in \mathcal{G} \mid \pi(\dot{x}) = \dot{x}\} \in \mathcal{F}$$

and we say that $\dot{x}$ is *hereditarily $\mathcal{F}$-symmetric*, written $\dot{x} \in \mathrm{HS}_{\mathcal{F}}$, if

$$\mathrm{sym}(\dot{x}) \in \mathcal{F} \text{ and for any } \langle \dot{y}, p \rangle \in \dot{x},\ \dot{y} \in \mathrm{HS}_{\mathcal{F}}.$$

Finally, given a generic $G \subseteq \mathbb{P}$, we define the *symmetric submodel* given by $\mathcal{F}$ as

$$\mathrm{N} := \{\dot{x}^G \mid \dot{x} \in \mathrm{HS}_{\mathcal{F}}\}.$$

We then have the following two standard results:

Lemma 2.5.2 (Symmetry Lemma, [DS96] Lemma 8.12.3)**.** *For any $p \in \mathbb{P}$, formula $\varphi(v)$, $\mathbb{P}$-name $\dot{x}$ and permutation $\pi \in \mathcal{G}$,*

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi \dot{x}).$$

Theorem 2.5.3 ([DS96] Theorem 18.12.2)**.** *Let* M *be a model of* ZF*. For any symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \in \mathrm{M}$ and any $\mathbb{P}$-generic filter G, the symmetric submodel* N *given by $\mathcal{F}$ is a model of* ZF *and* $\mathrm{M} \subseteq \mathrm{N} \subseteq \mathrm{M}[G]$.

2.6 Kripke Structures

In this section we shall give an outline of what a *Kripke model* is and show how a variation of the forcing technique can be seen as an example of this. Because this method is only needed for the construction of a few models of IZF containing pathological examples of badly behaved ordinals we shall not spend too long on the details of Kripke models. Also, to aid the presentation and readability of this section, the presentation shall be slightly different to the standard presentations. Our presentation is based on that of van Dalen, as given in Section 5.3 of [VD94] along with some of the ideas given in [Lub18]. In the latter work, Lubarsky gives a much more general definition of Kripke models which is more than what we need to use, therefore we shall adapt the simpler definition given by van Dalen. The variation of forcing that we shall present is taken from the work of Lipton, [Lip95].

2.6.1 Kripke Models

In short, a Kripke model is a collection of "*possible worlds*" along with a binary relation which gives us some information as to how the worlds are related to one another. An alternative explanation is that a Kripke model is a collection of "*states of knowledge*" and p is related to q indicates that if we know p then it is possible that we shall know q at a later stage. This gives us our first indication of a relation between Kripke models and forcing partial orders because a stronger condition is one that gives us more information and to say "*p is stronger than q*" is to say that, given p it is possible that we can "*extend*" it to q.

For simplicity, we shall assume that we are working in a language without any additional predicate or functional symbols. However, the definition can be extended by expanding the definition of $\Vdash$ to interpret these symbols in the obvious way. It should also be noted that the definition we give below is what van Dalen defines to be a *modified Kripke model*.

Definition 2.6.1. A (modified) *Kripke model* is an ordered quadruple $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ where $\mathcal{K}$ is a non-empty set of "*nodes*", $\mathcal{D}$ is a function on $\mathcal{K}$, $\mathcal{R}$ is a binary, reflexive relation between elements of $\mathcal{K}$, and ι is a set of functions $\iota_{p,q}$ for each pair $p, q \in \mathcal{K}$ with $p\mathcal{R}q$, such that the following hold:

- For each $p \in \mathcal{K}$, $\mathcal{D}(p)$ is an inhabited class structure.
- If $p\mathcal{R}q$ then $\iota_{p,q} \colon \mathcal{D}(p) \to \mathcal{D}(q)$ is a homomorphism.
- If $p\mathcal{R}q$ and $q\mathcal{R}r$ then $\iota_{p,r} = \iota_{q,r} \circ \iota_{p,q}$.

Next, for atomic formulae φ, let $p \Vdash_{\mathscr{K}} \varphi$ denote that $\mathcal{D}(p) \models \varphi$. Then $\Vdash_{\mathscr{K}}$ can be extended to arbitrary formulae by the prescription:

- For no p do we have $p \Vdash_{\mathscr{K}} \bot$,
- $p \Vdash_{\mathscr{K}} \varphi \wedge \psi$ iff $p \Vdash_{\mathscr{K}} \varphi$ and $p \Vdash_{\mathscr{K}} \psi$,
- $p \Vdash_{\mathscr{K}} \varphi \vee \psi$ iff $p \Vdash_{\mathscr{K}} \varphi$ or $p \Vdash_{\mathscr{K}} \psi$,

- $p \Vdash_{\mathscr{K}} \varphi \to \psi$ iff for any $r \in \mathcal{K}$ with $p\mathcal{R}r$, if $r \Vdash_{\mathscr{K}} \varphi$ then $r \Vdash_{\mathscr{K}} \psi$,
- $p \Vdash_{\mathscr{K}} \forall x\ \varphi(x)$ iff whenever $p\mathcal{R}q$ and $d \in \mathcal{D}(q)$, $q \Vdash_{\mathscr{K}} \varphi(d)$,
- $p \Vdash_{\mathscr{K}} \exists x\ \varphi(x)$ iff there is some $d \in \mathcal{D}(p)$ such that $p \Vdash_{\mathscr{K}} \varphi(d)$.

Note that we shall drop the subscript $\mathscr{K}$ when it is clear from the context that we are working with Kripke models rather than forcing posets. We will also define $\mathscr{K}^p$ to be the *truncation* of the Kripke model to $\mathcal{K}^p := \{q \in \mathcal{K} \mid p\mathcal{R}q\}$. So $\mathcal{K}^p$ is the cone of nodes which are related to p. For example, that this means $p \Vdash_{\mathscr{K}} \neg\varphi$ if and only if for any $r \in \mathcal{K}^p$, $r \nVdash_{\mathscr{K}} \varphi$.

It is worth making a few more remarks on the above definition.

Remarks 2.6.2.

- In van Dalen, a Kripke model is defined as above except that one asserts that $\mathcal{D}(p) \subseteq \mathcal{D}(q)$ whenever $p\mathcal{R}q$. The idea being that later nodes contain more information that one can use. The modified Kripke model is then introduced as an alternative way to view this.
- In the standard definitions of Kripke models, $\mathcal{D}(p)$ is just taken to be an inhabited set. We choose to adopt our alternative definition, which is also the one in [Lub18], in order to allow us to take the entire universe at each node.
- In general, the structures $\mathcal{D}(p)$ need not satisfy any particular theory. However, for our contexts it will be easier to assume that, for each node, $\mathcal{D}(p)$ is a model of ZF. Note that we will be using a classical metatheory since it is much easier to manipulate our intended model when we have a strong logical foundation to use.

Definition 2.6.3. Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model and $p \in \mathcal{K}$.

- A formula φ is said to be *valid at* p iff $p \Vdash_{\mathscr{K}} \varphi$.
- A formula φ is *valid in the full Kripke model*, written $\mathscr{K} \Vdash_{\mathscr{K}} \varphi$, iff for every $p \in \mathcal{K}$, $p \Vdash_{\mathscr{K}} \varphi$.

Furthermore, for Γ a set of sentences and φ a formula, $\Gamma \Vdash_{\mathscr{K}} \varphi$ iff in every Kripke model $\mathscr{K}$ satisfying that for all $\psi \in \Gamma$, $\mathscr{K} \Vdash_{\mathscr{K}} \psi$, it is also true that $\mathscr{K} \Vdash_{\mathscr{K}} \varphi$.

Theorem 2.6.4 ([VD94] Theorem 5.3.6)**.** (Soundness and Completeness of Intuitionistic Predicate Logic)**.** *For* Γ *a set of sentences and* φ *a formula,*

$$\Gamma \vdash \varphi \iff \Gamma \Vdash_{\mathscr{K}} \varphi.$$

It is worthwhile giving the following simple example which shows that the full Kripke model need not necessarily satisfy excluded middle. This example appears as 5.13 in [TvD88].

Example 2.6.5. Suppose that a and b are distinct sets and we have the following Kripke model,

$$\mathscr{K} \quad = \quad \begin{array}{cl} 1\bullet & \mathcal{D}(1) = \{a, b\} \text{ and } \mathcal{D}(1) \models a = b \\ \Big| & \\ 0\bullet & \mathcal{D}(0) = \{a, b\} \end{array}$$

One can see that $\mathscr{K} \nVdash_{\mathscr{K}} a = b \ \vee \ a \neq b$.

We now seek to use the intuitionistic structure given by the full Kripke model to provide a new structure over which one can define an interpretation of IZF. This method is taken from Section 3 of [HL16] and we use their terminology in also calling this the *full model*. The idea is to produce a new Kripke model with a stronger set-theoretic structure and whose underlying logic comes from the original full model. In order to do this, we need to define what the sets are for our required model along with an interpretation for equality and element-hood.

Fix $\mathscr{K}$ to be a Kripke model. We shall assume that, for each node p, $\mathcal{D}(p)$ is a model of ZF. For simplicity, let us further suppose that $\text{Ord}^{\mathcal{D}(p)} = \text{Ord}^{\mathcal{D}(q)}$ for every $p, q \in \mathcal{K}$. We shall simultaneously define the set of objects at p, $\mathrm{M}^p := \bigcup_\alpha \mathrm{M}^p_\alpha$, inductively through the ordinals.

So suppose that $\{\mathrm{M}^p_\beta \mid p \in \mathcal{K}\}$ has been defined for each $\beta \in \alpha$ along with transition functions $k_{p,q} \colon \mathrm{M}^p_\beta \to \mathrm{M}^q_\beta$ for each pair $p\mathcal{R}q$. The objects of M^p_α are then the collection of functions g such that

- $\mathrm{dom}(g) = \mathcal{K}^p$,
- $g \restriction \mathcal{K}^q \in \mathcal{D}(q)$,
- $g(q) \subseteq \bigcup_{\beta \in \alpha} \mathrm{M}^q_\beta$,
- If $h \in g(q)$ and $q\mathcal{R}r$ then $k_{q,r}(h) \in g(r)$.

Finally, extend $k_{p,q}$ to M^p_α by setting $k_{p,q}(g) := g \restriction \mathcal{K}^q$. Then the objects at node p are $\bigcup_\alpha \mathrm{M}^p_\alpha$. This allows us to define a notion of truth at node p by:

- $p \Vdash_{\mathscr{K}} g \in h \iff g \restriction \mathcal{K}^p \in h(p)$,
- $p \Vdash_{\mathscr{K}} g = h \iff g \restriction \mathcal{K}^p = h \restriction \mathcal{K}^p$,
- For logical connectives and quantifiers we use the rules for $\Vdash_{\mathscr{K}}$ which were given after Definition 2.6.1.

Definition 2.6.6. The *full model* $\mathrm{V}(\mathscr{K})$ is the model with the underlying class $\bigcup\limits_{p \in \mathcal{K}} \mathrm{M}^p$ and truth defined by

$$\mathrm{V}(\mathscr{K}) \models \varphi \iff \textit{for every } p \in \mathcal{K},\ p \Vdash \varphi.$$

Theorem 2.6.7 ([HL16] Theorem 3.1)**.** *The full model satisfies* IZF*.*

As an illustration of working with this model, we shall show that $\mathrm{V}(\mathscr{K})$ satisfies a weak instance of excluded middle that shall be useful when talking about ordinals in these structures. But first, in order to do this, we introduce a way to interpret sets from our previous nodes in the full model:

Definition 2.6.8. Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model. We define x^p for $x \in \mathcal{D}(p)$ recursively as follows:

$$x^p \text{ is the function with domain } \mathcal{K}^p \text{ such that for each } q \in \mathcal{K}^p,\ x^p(q) = \{y^q \mid y \in x\}.$$

To give an example of such names, consider how one would prove that the *Axiom of Infinity* holds in $\mathrm{V}(\mathscr{K})$, assuming that it holds at each node. To do this, given a node p one would have to define some n^p such that p believes that $\{n^p \mid n \in \omega\}$ denotes the set of natural numbers. For this to hold, we want 0^p to be the function with domain $\mathscr{K}^p$ which outputs $\emptyset$ on every input and, for $n \in \omega$, n^p should be the function with domain $\mathscr{K}^p$ such that for any $q \in \mathscr{K}^p$,

$$n^p(q) = \{m^q \mid m \in n\}.$$

It is then clear to see that if $q \in \mathscr{K}^p$ then $q \Vdash n^p = n^q$ and, for $m \in n$, $q \Vdash m^p \in n^p$.

Lemma 2.6.9. *Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model and suppose that $\mathcal{K}$ has initial node $\mathbb{1}$. Then for any $I \in \mathcal{D}(\mathbb{1})$,*

$$\mathrm{V}(\mathscr{K}) \models \forall \{a_\alpha \mid \alpha \in I^{\mathbb{1}}\} \; \forall x \; \Big(\Big(x \in \bigcup_{\alpha} a_\alpha \;\wedge\; x \notin \bigcup_{\alpha \neq \gamma} a_\alpha \Big) \to x \in a_\gamma \Big). \tag{$\star$}$$

Proof. Fix $q \in \mathcal{K}^p$ and functions $\{\dot{a}_\alpha \mid \alpha \in I\} \subseteq \mathrm{M}^q$ naming the set in $\mathrm{V}(\mathscr{K})$. Now suppose that

$$q \Vdash x \in \bigcup_{\alpha} a_\alpha \quad \textit{and} \quad q \Vdash x \notin \bigcup_{\alpha \neq \gamma} a_\alpha.$$

By definition, this means that there is some $\alpha \in I$ such that $q \Vdash x \in \dot{a}_{\alpha^q}$ and for all $\beta \in I$,

$$q \Vdash (\beta^q \neq \gamma^q) \to x \notin \dot{a}_{\beta^q}.$$

Now note that, by our interpretation, $q \Vdash \beta^q = \gamma^q$ iff $\beta = \gamma$ so, since we are working in a classical metatheory, we must have that

$$q \Vdash x \in \dot{a}_{\gamma^q}$$

which, by construction, is equivalent to saying that $q \Vdash x \in a_\gamma$. □

2.6.2 Intuitionistic Forcing Models

A specific example of a Kripke frame that we will make use of is one developed by Lipton in Section 4 of [Lip95]. In essence, this is the model constructed by taking a partial order $\mathbb{P}$ and considering the forcing model one produces by asserting that a statement is true iff it is forced by *every* condition. We will then see how this can naturally be viewed as a Kripke model.

As before, we will assume that ZF holds in the background universe. Let $\mathbb{P}$ be a partially ordered set with maximal element $\mathbb{1}$. To fit with the forcing notation, we shall use the standard forcing order here rather than the Kripke ordering, so $q \leq p$ will denote that q is *stronger* than p.

As in the classical forcing case, we will begin by defining the class of *forcing names*. However, because we are not using a generic, unlike in the standard case, we will not define how to evaluate said names. Instead we just work with the formal forcing relation. This means that instead of having V a subclass of V($\mathbb{P}$) we will only have that it is interpretable in the extension using canonical names. Define the class V($\mathbb{P}$) as follows:

$$\mathrm{V}_\alpha(\mathbb{P}) = \bigcup \{\mathcal{P}(\mathrm{V}_\beta(\mathbb{P}) \times \mathbb{P}) \mid \beta \in \alpha\},$$
$$\mathrm{V}(\mathbb{P}) = \bigcup_\alpha \mathrm{V}_\alpha(\mathbb{P}).$$

We will now define the forcing relation recursively on formula complexity. This will be very similar to the classical case except for minor modifications due to the fact that not all of the classical equivalences for predicate calculus hold. It is worth remarking that this definition is very similar to how we defined the corresponding notion for Kripke models.

Definition 2.6.10. Let a, b be names in $\mathrm{V}(\mathbb{P})$ and we will let our quantifiers range over members of $\mathrm{V}(\mathbb{P})$. For sentences φ and nodes $p \in \mathbb{P}$, $p \Vdash \varphi$ is defined inductively as follows:

$$
\begin{array}{lll}
p \Vdash a = b & \text{iff} & \forall \langle c, q\rangle \in a \; \forall r \leq p, q \quad r \Vdash c \in b \\
 & & \text{and } \forall \langle c, q\rangle \in b \; \forall r \leq p, q \quad r \Vdash c \in a \\
p \Vdash a \in b & \text{iff} & \exists c \; \exists q \geq p \; (\langle c, q\rangle \in b \text{ and } p \Vdash a = c) \\
p \Vdash \varphi \wedge \psi & \text{iff} & p \Vdash \varphi \text{ and } p \Vdash \psi \\
p \Vdash \varphi \vee \psi & \text{iff} & p \Vdash \varphi \text{ or } p \Vdash \psi \\
p \Vdash \varphi \to \psi & \text{iff} & \forall q \leq p \quad q \Vdash \varphi \Rightarrow q \Vdash \psi \\
p \Vdash \forall x \; \varphi(x) & \text{iff} & \forall a \; \forall q \leq p \quad q \Vdash \varphi(a) \\
p \Vdash \exists x \; \varphi(x) & \text{iff} & \exists a \quad p \Vdash \varphi(a)
\end{array}
$$

Remark 2.6.11. Some of the cases in the above definition are stronger than their classical forcing counterparts. Notably the cases for $a \in b$, disjunction, implication and existential quantifiers are normally defined over some dense set of conditions. Here it seems to be appropriate to take the stronger definitions we gave above because they fit better with the Brouwer-Heyting-Kolmogorov interpretation given in Definition 2.3.1. For example, to say that $p \Vdash a \in b$ is to say that we can find some element which is in b and which is forced to be equal to a. Namely, some $\langle c, q\rangle \in b$ such that $p \Vdash a = c$.

The fact that we are taking this approach will not cause problems in our situation because we will only be looking for those sentences which are forced by every condition. In the classical case, if $p \Vdash a \in b$ and p is in some generic then, by density, there is some stronger condition r in the generic and some $\langle c, q\rangle \in b$ such that $q \geq r$ and $r \Vdash a = c$. But in our intuitionistic case, for $a \in b$ to hold we will need that it is forced by every condition and therefore we will want that p itself forces $a = c$.

Remark 2.6.12. Recall that the formula $\neg\varphi$ is interpreted as $\varphi \to \bot$ and we will assume that no condition can force a contradiction. Therefore

$$p \Vdash \neg\varphi \quad \text{iff} \quad \forall q \leq p \; q \nVdash \varphi.$$

Moreover, $p \Vdash \neg\neg\varphi$ iff $\forall q \leq p \; \exists r \leq q \; r \Vdash \varphi$.

We then have the following results which are from Section 4 of [Lip95].

Lemma 2.6.13 ([Lip95] Lemma 4.1)**.** (Monotonicity)**.**

$$(p \Vdash \varphi \;\wedge\; q \leq p) \quad \longrightarrow \quad q \Vdash \varphi.$$

Definition 2.6.14 ([Lip95] Definition 4.2))**.** We say that $\mathrm{V}(\mathbb{P}) \models \varphi$ whenever we have that $\forall p \in \mathbb{P} \; p \Vdash \varphi$.

Theorem 2.6.15 ([Lip95] Theorem 4.3)**.** (Soundness of IZF)**.** *For any formula* φ*,*

$$\mathrm{IZF} \vdash \varphi \Longrightarrow \mathrm{V}(\mathbb{P}) \models \varphi.$$

Remark 2.6.16. There are easy canonical names for elements of the ground model, unordered pairs and ordered pairs:

- For $x \in \mathrm{V}$, $\check{x}$ is defined recursively as $\{\langle \check{y}, \mathbb{1} \rangle \mid y \in x\}$.
- A name for the unordered pair of names x and y is $\mathrm{up}(x, y) := \{\langle x, \mathbb{1} \rangle, \langle y, \mathbb{1} \rangle\}$.
- A name for the ordered pair of names x and y is $\mathrm{op}(x, y) := \mathrm{up}(\mathrm{up}(x, x), \mathrm{up}(x, y))$.

We now show how this can be viewed as a Kripke model. At each node p we can define an equivalence relation by

$$x \sim_p y \quad \equiv \quad p \Vdash x = y.$$

We can consider each node as having a domain $\mathrm{V}(\mathbb{P})(p) = \{[x]_p \mid x \in \mathrm{V}(\mathbb{P})\}$ where $[x]_p$ is the equivalence class modulo $\sim_p$. Then, for $q \leq p$ we can define maps

$$\iota_{p,q} \colon \mathrm{V}(\mathbb{P})(p) \to \mathrm{V}(\mathbb{P})(q)$$

by $\iota_{p,q}([x]_p) = [x]_q$. This allows us to view $\mathrm{V}(\mathbb{P})$ as a Kripke model.

We end this section by remarking that one can easily prove that the weak instance of excluded middle, $(\star)$, from Lemma 2.6.9 also holds in $\mathrm{V}(\mathbb{P})$ for any partial order $\mathbb{P}$.

Lemma 2.6.17. *Let* V *be a model of* ZF *and* $\mathbb{P}$ *be a partial order. Then for any set* I *in* V*,*

$$\mathrm{V}(\mathbb{P}) \models \forall \{a_\alpha \mid \alpha \in \check{I}\}\ \forall x \left(\left(x \in \bigcup_\alpha a_\alpha \ \wedge\ x \notin \bigcup_{\alpha \neq \gamma} a_\alpha\right) \to x \in a_\gamma\right). \qquad (\star)$$

Chapter 3

When are Proper Classes Big

3.1 The Easy Cases

In later chapters we will try to obtain the consistency strength of elementary embeddings over weak theories in terms of the large cardinal hierarchy over ZFC. To find a lower bound one often works with a "sufficiently large" fragment of the universe, typically of the form V_κ for some regular cardinal κ. However, without Power Set this may not necessarily be a set. For example, we shall see in Corollary 10.4.2 that one can have embeddings over ZFC^- in which $V_{\omega+1}$ is a proper class.

In this chapter, we shall isolate a notion which will be sufficient for our later purposes; *big classes*. We shall see in Lemma 10.3.1 that if j is an elementary embedding which is the identity on V_α then V_α cannot be a big proper class. Therefore the assumption that every proper class is big, which holds in many natural models of weak theories, will be sufficient to prove that V_α is a set.

Definition 3.1.1. A proper class is said to be *big* if it surjects onto every non-zero ordinal.

It is easy to see that this property holds over any model of ZF.

Proposition 3.1.2. *Under* ZF*, there is a surjection from any proper class onto any non-zero ordinal.*

Proof. Given a proper class $\mathcal{C}$, define

$$S := \{\gamma \in \text{Ord} \mid \exists x \in \mathcal{C} \ \text{rank}(x) = \gamma\}.$$

Since the rank hierarchy is a hierarchy of sets, S must be unbounded in the ordinals. So, given an ordinal α, we can take the first α many elements of S, $\{\gamma_\beta \mid \beta \in \alpha\}$. Then

$$f(x) = \begin{cases} \beta, & \textit{if } \text{rank}(x) = \gamma_\beta \\ 0, & \textit{otherwise} \end{cases}$$

defines a surjection of $\mathcal{C}$ onto α. □

On the other hand, it is easy to see that this property need not hold in KP.

Proposition 3.1.3. $\mathrm{L}_{\aleph_\omega^{\mathrm{L}}}$ *is a model of* KP *containing a proper class which is not big.*

Proof. From Section II.3 of [Bar17], every cardinal is admissible and therefore $\mathrm{L}_{\aleph_\omega^{\mathrm{L}}}$ is a model of KP. Now consider the proper class Card. Externally, this only has cardinality ω and therefore there is no surjection of Card onto $\aleph_1^{\mathrm{L}}$ in L. Thus there is no such surjection in $\mathrm{L}_{\aleph_\omega^{\mathrm{L}}}$. □

Using a similar idea, we can achieve the same result for the theory ZFC−. To do this, we will use the following theorem from [GHJ16]:

Theorem 3.1.4 ([GHJ16] Theorem 6)**.** *Suppose that* $\mathrm{V} \models \mathrm{ZFC} + \mathrm{CH}$*,* κ *is a regular cardinal with* $2^\omega < \aleph_\kappa$ *and that* $G \subseteq \mathrm{Add}(\omega, \aleph_\kappa)$ *is* V*-generic. If* $\mathrm{W} = \bigcup_{\gamma<\kappa} \mathrm{V}[G_\gamma]$ *where* $G_\gamma = G \cap \mathrm{Add}(\omega, \aleph_\gamma)$*, (that is* G_γ *is the first* $\aleph_\gamma$ *many of the Cohen reals added by* G*) then* $\mathrm{W} \models \mathrm{ZFC}-$ *has the same cardinals as* V *but the Reflection Principle fails.*

Proposition 3.1.5. W *is a model of* ZFC− *containing a proper class which is not big.*

Proof. Since V will have the same cardinals as $\mathrm{V}[G]$, that is the full extension by all $\aleph_\kappa$ many reals, in $\mathrm{V}[G]$ $2^\omega = \aleph_\kappa$ and therefore there is no surjection of $\mathcal{P}(\omega)$ onto $\aleph_{\kappa+1}$. Hence there is no such surjection in W, so W is a model of ZFC− in which $\mathcal{P}(\omega)$ is a proper class which is not big. □

So it seems natural to ask which sub-theories of ZFC also prove the feature that every proper class is big, and this will be the focus of this chapter. We shall prove that every proper class is big under ZFC^- with the Dependent Choice Scheme for every cardinal. Moreover, we shall then show that Dependent Choice is necessary for this result. We shall first give a failed attempt to do this because it will lead to other interesting results. Then we shall give a general method, derived from work of Zarach, to produce models of ZFC^- containing proper classes which are not big.

3.2 Classes are Big with Dependent Choice

The aim of this section is to show that, in a mild extension of ZFC^-, we do have the property that every proper class is big. The additional axioms we need are *the Schemes of Dependent Choices*. Under ZFC, this is a consequence of the Well-Ordering Principle but, as shown in [FGK19], it need not hold in every model of ZFC^-. The principle itself is a natural strengthening of *Set Dependent Choice* to the class version where we allow definable relations, and has been previously considered in [GHJ16] and [FGK19].

For μ an infinite cardinal, the DC_μ-*Scheme* is the assertion that any definable class tree of height μ, which has no maximal element and is closed under sequences of length less than μ, has a branch of order type μ. Equivalently, and with a more formal definition,

Definition 3.2.1 (The DC_μ-Scheme)**.** Let φ and ψ be formulae and u be a set such that for some y, $\psi(y,u)$ holds and for every $\alpha \in \mu$ and every α-sequence $s = \langle x_\beta \mid \beta \in \alpha\rangle$ satisfying $\psi(x_\beta, u)$ for each β, there is a z satisfying $\psi(z,u)$ and $\varphi(\alpha, s, z, u)$. Then there is a function f with domain μ such that for each $\alpha \in \mu$, $\psi(f(\alpha), u)$ and $\varphi(\alpha, f \restriction \alpha, f(\alpha), u)$ hold.

Notation 3.2.2. For $\mu = \aleph_0$, we shall call the above the DC-*Scheme*.

The DC-Scheme is known to be equivalent to a very natural strengthening of Collection,

Theorem 3.2.3 ([GHJ16])**.** *Over* ZFC^-*, the* DC*-Scheme is equivalent to the Reflection Principle.*

Remark 3.2.4. To see that the Reflection Principle implies the DC-Scheme, let T be some definable tree of height ω with no maximal element. This statement can be reflected to a transitive set M using the Reflection Principle which can then be well-ordered to provide the witnesses for an infinite branch.

However, this argument does not go through for the DC_μ-Scheme when μ is an uncountable cardinal. The issue here is that the transitive set M need not be closed under infinite sequences definable in the full universe. Therefore, when we reflect the tree it need no longer be the case that the tree is closed under sequences of length less than μ. As we shall see in Corollary 3.3.3, it will in fact be possible to separate different Dependent Choice Schemes over ZFC^-.

Theorem 3.2.5 ([FGK19] Theorem 11.2)**.** *The Reflection Principle is not provable in* ZFC^-*.*

Remark 3.2.6. In [GHJ16], the authors prove that the model of ZFC− produced in Theorem 3.1.4 also satisfies the DC_α-Scheme for every cardinal $\alpha \in \kappa$ but that the DC_κ-Scheme fails. This shows that, without the Collection Scheme, the DC-Scheme does not prove the Reflection Principle and that the $\mathrm{DC}_{<\kappa}$-Scheme does not imply that every proper class is big.

For the sake of completeness and ease of reference, the first proposition that we shall prove is that, as with the set case, if $\delta < \mu$ are infinite cardinals and the DC_μ-Scheme holds then so does the DC_δ-Scheme. This will be proved using a similar technique to the proof of the set version found in [Jec73].

Proposition 3.2.7 ([Lév64a] Theorem 7)**.** *If* $\delta < \mu$ *are infinite cardinals then the* DC_μ*-Scheme implies the* DC_δ*-Scheme.*

Proof. Let φ and ψ be formulae and u be a set such that for some y, $\psi(y, u)$ holds and for every $\alpha \in \delta$ and every α-sequence $s = \langle x_\beta \mid \beta \in \alpha \rangle$ satisfying $\psi(x_\beta, u)$ for each β, there is a z satisfying $\psi(z,u)$ and $\varphi(\alpha,s,z,u)$. We define a new formula ϑ extending φ to apply to any α-sequence, s, for $\alpha\in\mu$ by

$$\vartheta(\alpha,s,z,u,\delta)\equiv\Big(\alpha<\delta\ \wedge\ \forall\beta\in\alpha\ \psi(s(\beta),u)\ \wedge\ \varphi(\alpha,s,z,u)\Big)\ \vee\ \Big(\alpha\geq\delta\ \wedge\ \psi(z,u)\Big).$$

By, for $\alpha\geq\delta$, taking some fixed z such that $\psi(z,u)$ holds, it is clear that for every $\alpha\in\mu$ and applicable α-sequence s there is some z satisfying both $\psi(z,u)$ and $\vartheta(\alpha,s,z,u,\delta)$. Then for any function f with domain μ witnessing this instance of the DC_μ-Scheme, $f\restriction\delta$ witnesses that the DC_δ-Scheme holds for ψ and φ. □

An important consequence of the DC_μ-Scheme for us is that it gives us a useful necessary condition for a class to be proper.

Theorem 3.2.8. *Suppose that* $\mathrm{V}\models\mathrm{ZF}^-+\mathrm{DC}_\mu$*-Scheme for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, definable over* V*, there is a subset of* $\mathcal{C}$ *of cardinality* μ*.*

Proof. Let $\mathcal{C}=\{x\mid\psi(x,u)\}$ be a proper class. We shall in fact prove the equivalent statement that for any $\nu\leq\mu$ there is a subset, b, of $\mathcal{C}$ and a bijection between b and ν. Suppose for a contradiction that this were not the case and let γ be the least ordinal for which no such subset of size γ exists. It is obvious that γ must be an infinite cardinal. Let $\varphi(\alpha,s,y)$ be the statement that s is a sequence of length α and $\mathrm{ran}(s)\cup\{y\}$ is a subset of $\mathcal{C}$ with $y\notin\mathrm{ran}(s)$. Then, by assumption, for every $\alpha\in\gamma$ there is a sequence of elements of $\mathcal{C}$ of length α. Also, since $\mathcal{C}$ is a proper class, if s is an α length sequence from $\mathcal{C}$ then there is some $y\in\mathcal{C}$ which is not in s, so the hypothesis of the DC_γ-Scheme is satisfied. Therefore, by DC_γ, there is a function f with domain γ and whose range gives a subset of $\mathcal{C}$ of cardinality γ, giving us our desired contradiction. □

This leads to an extension of ZFC^- that we will use extensively in the study of elementary embeddings:

Definition 3.2.9. Let $\mathrm{ZFC}^-+\mathrm{DC}_{<\mathrm{Card}}$ denote the theory $\mathrm{ZFC}^-+\forall\mu\in\mathrm{Card}\ \mathrm{DC}_\mu$.

Corollary 3.2.10. *Suppose that* $\mathrm{V}\models\mathrm{ZFC}^-+\mathrm{DC}_{<\mathrm{Card}}$*. Then, for any proper class* $\mathcal{C}$ *which is definable over* V *and any non-zero ordinal* γ*, there is a definable surjection of* $\mathcal{C}$ *onto* γ*.*

3.3 Classes need not be Big without Dependent Choice

In this section we shall prove that it need not be the case that every proper class is big in a model of $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$. The method we shall use to prove this is one that was originally considered by Zarach in [Zar82], which is constructed as a union of ZF^- models. It shall be shown that the resulting model fails to satisfy the $\mathrm{DC}_{\mu^{++}}$ -Scheme for a given cardinal μ.

Following a review of this model we shall look at ways to generalise it. In particular, we shall see that, with an appropriately chosen forcing, it is possible to produce models of ZFC^- in which the DC-Scheme holds but the $\mathrm{DC}_{\aleph_1}$-Scheme does not. This construction will then be generalised to ensure that we still satisfy the DC_{μ}-Scheme for a given regular cardinals.

The main theorem from Zarach that we shall make use of is Theorem 4.1 of [Zar82]. Before giving the theorem, we recall the definition of *Hartog's number*. Under ZFC, the Hartog's number of a set a is $|a|^+$. However, this definition still makes sense in weaker theories without the Axioms of Choice or Power Set.

Definition 3.3.1. The *Hartog's number* of a set x is

$$\aleph(x) := \{\alpha \in \mathrm{Ord} \mid \exists f\ (\mathrm{func}(f)\ \wedge\ f\colon \alpha \to x \textit{ is injective})\}.$$

Theorem 3.3.2 (Zarach, [Zar82] Theorem 4.1)**.** *If* ZFC *is consistent then so is*

$$\mathrm{ZFC}^-_{Ref} + \exists \kappa, \mu \left(\mathcal{P}(\kappa) \textit{ is a proper class } \wedge\ \forall x \subseteq \mathcal{P}(\kappa)\ (|x| \leq \mu)\right)$$
$$+ \exists x \subseteq \mathcal{P}(\kappa)\ (|x| = \mu) + \forall x\ \aleph(x) \in \mathrm{Ord}.$$

When combined with Theorem 3.2.8, we obtain the failure of the DC_{μ^+}-Scheme in the above model.

Corollary 3.3.3. *In any model satisfying the conclusion of Theorem 3.3.2, the* DC_{μ^+}*-Scheme fails.*

Proof. Fix cardinals κ and μ such that $\mathcal{P}(\kappa)$ is a proper class and

$$\forall x \subseteq \mathcal{P}(\kappa)\ (|x| \leq \mu)\ \wedge\ \exists x \subseteq \mathcal{P}(\kappa)\ (|x| = \mu).$$

We shall show that μ^+ exists. This will mean that $\mathcal{P}(\kappa)$ is a proper class with no subset of size μ^+ so, by Theorem 3.2.8, the DC_{μ^+}-Scheme must fail.

To see that μ^+ exists, fix $x \subseteq \mathcal{P}(\kappa)$ of size μ. It is easy to see that for any z, $\aleph(z)$ is a cardinal and furthermore $\aleph(x) > \mu$. So, since by assumption $\aleph(x)$ is a set, μ^+ must exist. □

3.3.1 Union of ZF^- models

We shall now give Zarach's original model. Because of the style in which it was presented, we have decided to rewrite his construction using modern notation. This has the benefit of making it more readable and shortens some of the proofs. To begin with, we introduce a general notation for the product of μ many copies of a forcing with support less than δ.

Definition 3.3.4. Let $\langle \mathbb{P}, \leq \rangle$ be a forcing notion (with maximal element $\mathbb{1}_{\mathbb{P}}$) and let $\delta \leq \mu$ be regular cardinals. Let $\prod_{\mu}^{(\delta)} \mathbb{P}$ denote the product forcing of μ many copies of $\mathbb{P}$ with $< \delta$ support, where the support of a condition is the cardinality of the set of conditions in the sequence which are not $\mathbb{1}_{\mathbb{P}}$.

Moreover, for $e \subseteq \delta$, let $e(\mathbb{P})$ be the restriction of $\prod_{\mu}^{(\delta)} \mathbb{P}$ to those conditions whose support is a subset of e.

Remark 3.3.5. We will treat conditions of $\prod_{\mu}^{(\delta)} \mathbb{P}$ as partial functions from μ into $\mathbb{P}$ of cardinality at most δ. Using this presentation, given a set $X \subseteq \prod_{\mu}^{(\delta)} \mathbb{P}$ and $e \subseteq \mu$, we let $X \upharpoonright e$ denote those conditions in X whose support is a subset of e.

One of the important qualities of such forcings is that they satisfy *weak homogeneity* where

Definition 3.3.6. A forcing $\mathbb{P}$ is Weakly Homogeneous iff for any conditions $p, q \in \mathbb{P}$, there is an automorphism, i, of $\mathbb{P}$ such that $i(p)$ is compatible with q.

One consequence of this it that for any formula φ and any condition p,

$$p \Vdash \varphi \iff \mathbb{1}_{\mathbb{P}} \Vdash \varphi.$$

The aim of this section is to prove the following theorem:

Theorem 3.3.7 (Zarach, [Zar82] Theorem C)**.** *Let* M *be a model of* ZF *and* $\mathbb{P} \in \mathrm{M}$*. Suppose that* $h \colon \mathbb{P} \cong \prod_{\omega}^{(\omega)} \mathbb{P}$ *is an order-isomorphism for some* $h \in \mathrm{M}$*. Let* G *be* $\mathbb{P}$*-generic over* M *and* $H = h``G$ *be the corresponding* $\prod_{\omega}^{(\omega)} \mathbb{P}$*-generic. Let* $G_n = H \restriction \{n\}$ *be the* $\mathbb{P}$*-generic produced by restricting* H *to its* n^{th} *co-ordinate and let* $\mathrm{M}_n = \mathrm{M}[G_0 \times \cdots \times G_{n-1}]$*.*

Then if $\mathrm{N} = \bigcup_n \mathrm{M}_n$*,* $\mathcal{N} = \langle \mathrm{N}, \in \mathrm{M} \rangle$ *is a model of* ZF^-_{Ref}*.*

To prove this theorem will involve multiple stages, primarily to set up the required notation and technical lemmas. After making some basic observations, we shall see that the difficulty arises in proving that $\mathcal{N}$ is a model of the Collection Scheme and the Reflection Principle. This will be proven by showing that $\mathcal{N}$ can be written as the union of a chain of models, each of which are isomorphic to $\mathcal{N}$. The theorem itself will then follow from two model theoretic results; Theorems 3.3.15 and 3.3.16.

Remark 3.3.8. It is worth remarking here that the theorem only works when we can use a definable predicate for M in N. The issue is that N is not a forcing extension of M and therefore there is no reason to believe that one should be able to, in general, define M in the union model. We also point out that even if N were a forcing extension, we wouldn't immediately get the definability. This is because, as shown in [GJ14], the definability of grounds does not always hold over ZF^-_{Ref}.

Remark 3.3.9. We begin by noting some basic properties of N. Firstly, it is clear that N satisfies Extensionality, Empty Set, Pairing, Unions, Infinity, Foundation Scheme and Separation Scheme. Furthermore, N can never satisfy Power Set because $\mathcal{P}(\mathbb{P}) \notin \mathrm{N}$. The reason for this is that, while $G_n \in \mathrm{N}$ for every n, the construction yields that H itself can never be in N.

Proposition 3.3.10. *If* M *also satisfies the Well-Ordering Principle then so does* N.

Proof. Let $x \in \mathrm{N}$. Then $x \in \mathrm{M}_{n+1}$ for some n. Now, if M_n satisfies Well-Ordering then so does $\mathrm{M}_n[G]$ for any set generic G. Therefore, in M_{n+1} there is a well-ordering of x and therefore a well-ordering exists in N. □

Using the notation from Theorem 3.3.7, let M be a model of ZF and $\mathbb{P} \in \mathrm{M}$. Suppose that $h\colon \mathbb{P} \cong \prod_{\omega}^{(\omega)} \mathbb{P}$ is an order-isomorphism for some $h \in \mathrm{M}$. Let G be $\mathbb{P}$-generic over M and $H = h``G$ be the corresponding $\prod_{\omega}^{(\omega)} \mathbb{P}$-generic. Let $G_n = H \restriction \{n\}$ be the $\mathbb{P}$-generic produced by restricting H to its n^{th} co-ordinate and let $\mathrm{M}_n = \mathrm{M}[G_0 \times \cdots \times G_{n-1}]$.

Since G_n is $\mathbb{P}$-generic, we can use the isomorphism h again to obtain further $\mathbb{P}$-generics

$$G_{n,j} \coloneqq h``G_n \restriction \{j\}$$

for any $j \in \omega$. We then define $\mathrm{M}_{n,j} \coloneqq \mathrm{M}_n[G_{n,0} \times \cdots \times G_{n,j-1}]$ and set

$$\mathrm{N}_n \coloneqq \bigcup\nolimits_j \mathrm{M}_{n,j}.$$

Remark 3.3.11. For any n and j,

$$\mathrm{M}_n = \mathrm{M}_{n,0} \subset \mathrm{M}_{n,1} \subset \cdots \subset \mathrm{M}_{n,j} \subset \cdots \subset \bigcup\nolimits_j \mathrm{M}_{n,j} = \mathrm{N}_n \subset \mathrm{M}_{n+1}.$$

In order to be able to use these constructions we now show that N is a definable subclass of $\mathrm{M}[H]$. Now, N is determined by M and $\{G_{\{n\}} \mid n \in \omega\}$ using the following specification:

$$x \in \mathrm{N} \longleftrightarrow \exists y \in \mathrm{M}\ \exists v\ \Big(v \textit{ is a product of finitely many elements of } \{G_{\{n\}} \mid n \in \omega\} \textit{ and } x = \mathrm{K}_v(y)\Big)$$

where, given a set $v \subseteq \prod_{\omega}^{(\omega)} \mathbb{P}$, $\mathrm{K}_v(y) = \{\mathrm{K}_v(z) \mid \exists p \in v\ (\langle z,p\rangle \in y)\}$ is the restriction of the evaluation map to the set v. Therefore, it suffices to specify a name for $\{G_{\{n\}} \mid n \in \omega\}$.

To do this we fix the following notation:

Notation 3.3.12.

- $\dot{G}_{\{n\}} = \{\langle \check{p}, \{\langle n,p\rangle\}\rangle \mid p \in \mathbb{P}\}$ *is a name for* $G_{\{n\}}$,
- $\dot{\mathcal{G}} = \{\langle \dot{G}_{\{n\}}, \mathbb{1}_{\mathbb{P}}\rangle \mid n \in \omega\}$ *is a name for* $\{G_{\{n\}} \mid n \in \omega\}$,
- $\mathbb{M}(\cdot)$ *is a predicate for* M,
- $\mathrm{Seq}(u,Z)$ *denotes that* u *is a sequence of elements of* Z,
- $$KS(x,Z) \equiv \exists y,u,v,k\ \Big(\mathbb{M}(y) \ \wedge\ \mathrm{Seq}(u,Z) \ \wedge\ \mathrm{dom}(u) = k+1 \\ \wedge\ v = u_0 \times \cdots \times u_k \ \wedge\ x = \mathrm{K}_v(y)\Big),$$
- Finally, if $\varphi(x)$ is a formula then $\varphi^{\star}_{Z}(x)$ is the relativisation of φ to $KS(\cdot,Z)$.

Note that $KS(x,Z)$ says that x is named by some element of the evaluation map restricted to finitely many elements from Z, which is what we required in our specification for being an element of N.

Since, for any permutation π of ω, $\pi\dot{G}_{\{n\}} = \dot{G}_{\{\pi n\}}$ and $\pi\dot{\mathcal{G}} = \dot{\mathcal{G}}$, an easy application of the Symmetry Lemma (2.5.2) gives us,

Lemma 3.3.13. *For* $\mathbb{P} \in \mathrm{M}$ *and* $e \subseteq \omega$, *if* $\varphi(x)$ *is a formula and* $a \in \mathrm{M}^{e(\mathbb{P})}$ *then*

$$p \Vdash \varphi^{\star}_{\dot{\mathcal{G}}}(a) \iff p \restriction e \Vdash \varphi^{\star}_{\dot{\mathcal{G}}}(a).$$

Therefore $\mathrm{N} \models \varphi(a) \iff \mathrm{M}[G] \models \varphi^{\star}_{\dot{\mathcal{G}}}(a)$.

Key Lemma 3.3.14. *For every* $n \in \omega$, $\mathrm{N}_n \prec \mathrm{N}$.

Proof. We shall begin by finding $\prod_{\omega}^{(\omega)} \mathbb{P}$-generics T^j_n and $T^{n,j}$ such that $\mathrm{M}_{n,j}[T^j_n] = \mathrm{M}_{n+1}$ and $\mathrm{M}_{n,j}[T^{n,j}] = \mathrm{M}[G]$. This is done by noting that

$$\mathrm{M}_{n+1} = \mathrm{M}_n[h``G_n] = \mathrm{M}_n[h``G_n \restriction j \times h``G_n \restriction (\omega \setminus j)] = \mathrm{M}_{n,j}[h``G_n \restriction (\omega \setminus j)].$$

Then T^j_n will the obvious translation of $h``G_n \restriction (\omega \setminus j)$ to a $\prod_{\omega}^{(\omega)} \mathbb{P}$-generic.

By a similar argument,

$$\mathrm{M}[H] = \mathrm{M}[H \restriction n \times H \restriction (\omega \setminus n)] = \mathrm{M}_{n,j}[h``G_n \restriction (\omega \setminus j) \times H \restriction (\omega \setminus n)].$$

Thus $T^{n,j}$ will the obvious translation of $h``G_n \restriction (\omega \setminus j) \times H \restriction (\omega \setminus n)$ to a $\prod_{\omega}^{(\omega)} \mathbb{P}$-generic.

Now for any formula $\varphi(v)$ and $a \in \mathrm{M}_{n,j}$ we have that

$$\begin{aligned} \mathrm{N}_n \models \varphi(a) &\iff \mathrm{M}_{n,j}[T^j_n] \models \varphi^\star_{\dot{\mathcal{G}}}(a) \\ &\iff \exists f \in T^j_n \ \mathrm{M}_{n,j} \models \text{``} f \Vdash \varphi^\star_{\dot{\mathcal{G}}}(\check{a})\text{''} \\ &\iff \mathrm{M}_{n,j} \models \text{``} \mathbb{1}_{\mathbb{P}} \Vdash \varphi^\star_{\dot{\mathcal{G}}}(\check{a})\text{''} \\ &\iff \mathrm{N} \models \varphi(a). \end{aligned}$$

Where the third equivalence follows from the fact that the finite support product of ω many copies of any forcing is weakly homogeneous and, since $\mathbb{P} \cong \prod_{\omega}^{(\omega)} \mathbb{P}$, so is $\mathbb{P}$ □

Therefore, using the above result, the proof of Theorem 3.3.7 will follow from the following pair of more general model theoretic theorems.

Theorem 3.3.15 (Zarach, [Zar82] Theorem M)**.** *Let* $\mathcal{M}_{n+1} = \langle \mathrm{M}_{n+1}, \in \ \mathrm{N}_n \rangle$ *and* $\mathcal{N}_n = \langle \mathrm{N}_n, \in \rangle$ *for* $n \in \omega$. *Suppose that* $\mathrm{M}_n \subseteq \mathrm{N}_n \subseteq \mathrm{M}_{n+1}$, $\mathrm{N} = \bigcup_n \mathrm{M}_n$ *and* $\mathcal{N} = \langle \mathrm{N}, \in \rangle$. *If for every* $n \in \omega$, $\mathcal{M}_{n+1} \models \mathrm{ZF}^-$ *and* $\mathcal{N}_n \prec \mathcal{N}$ *then* $\mathcal{N} \models \mathrm{ZF}^-$.

Proof. By the above analysis and comments about the construction it suffices to show that the Collection Scheme holds in $\mathcal{N}$. To do this, let $a, u \in \mathrm{N}$ and suppose that

$$\mathcal{N} \models \forall x \in a \ \exists y \ \varphi(x, y, u).$$

Fix n such that $a, u \in \mathrm{M}_n$. Then, since $\mathrm{M}_n \subseteq \mathrm{N}_n$ and $\mathcal{N}_n \prec \mathcal{N}$,

$$\mathcal{N}_n \models \forall x \in a \ \exists y \ \varphi(x, y, u).$$

Since, by assumption, N_n is a definable subset of M_{n+1} and $\mathcal{M}_{n+1}$ is a model of ZF^-, there exists some $b \in \mathrm{M}_{n+1}$ with $b \subseteq \mathrm{N}_n$ satisfying

$$\forall x \in a \ \exists y \in b \ \mathcal{N}_n \models \varphi(x, y, u).$$

Therefore, since $\mathcal{N}_n \prec \mathcal{N}$ and $b \in \mathrm{M}_{n+1} \subseteq \mathrm{N}$, $\mathcal{N} \models \forall x \in a \ \exists y \in b \ \varphi(x, y, u)$. □

Theorem 3.3.16 (Zarach)**.** *Let* $\mathcal{M}_{n+1} = \langle \mathrm{M}_{n+1}, \in, \mathrm{N}_n \rangle$ *and* $\mathcal{N}_n = \langle \mathrm{N}_n, \in \rangle$ *for* $n \in \omega$. *Suppose that* $\mathrm{M}_n \subseteq \mathrm{N}_n \subseteq \mathrm{M}_{n+1}$, $\mathrm{N} = \bigcup_n \mathrm{M}_n$ *and* $\mathcal{N} = \langle \mathrm{N}, \in \rangle$. *If for every* $n \in \omega$, $\mathcal{M}_{n+1} \models \mathrm{ZF}^- + \mathrm{DC}$*-Scheme and* $\mathcal{N}_n \prec \mathcal{N}$ *then* $\mathcal{N} \models \mathrm{ZF}^- + \mathrm{DC}$*-Scheme.*

Proof. Suppose that in $\mathcal{N}$ we have formulae φ and ψ and a set u such that for some y, $\psi(y, u)$ holds and for any m-sequence $s = \langle x_i \mid i \in m \rangle$ satisfying $\psi(x_i, u)$ for each i, there is some z such that $\psi(z, u)$ and $\varphi(m, s, z, u)$ hold. Fix n such that $u \in \mathrm{M}_{n+1}$. Then, by elementarity, the above statement, as well as ψ and φ, reflect to $\mathcal{N}_n$.

We first show that $\mathcal{M}_{n+1}$ can define a branch through the tree of approximations to this instance of the DC-Scheme in $\mathcal{N}_n$. Let $s = \langle x_i \mid i \in m \rangle$ be an m-sequence from N_n such that for each i, $\psi^{\mathcal{N}_n}(x_i, u)$ holds. Since $\mathcal{N}_n \prec \mathcal{N}$, there is some $z \in \mathrm{N}_n$ such that $\psi^{\mathcal{N}_n}(z, u)$ and $\varphi^{\mathcal{N}_n}(m, s, z, u)$ hold. Therefore, the tree of approximations can be defined in $\mathcal{M}_{n+1}$ and, by using the DC-Scheme in $\mathcal{M}_{n+1}$, there is a branch through the tree.

Letting $s \in \mathrm{M}_{n+1}$ be such a branch with $s \subseteq \mathrm{N}_n$, we have that

$$\exists s \; \forall n \in \omega \; \psi^{\mathcal{N}_n}(s(n), u) \;\wedge\; \varphi^{\mathcal{N}_n}(n, s \upharpoonright n, s(n), u).$$

So, since $\mathcal{N}_n \prec \mathcal{N}$, s is a witness to the desired instance of the DC-Scheme in $\mathcal{N}$. □

Since adding a single Cohen real is equivalent to a finite support product of adding ω many Cohen reals, Theorem 3.3.7 gives us the following corollary

Corollary 3.3.17. *For any model* M *of* ZFC + CH *there is a model* $\mathcal{N} = \langle \mathrm{N}, \in, \mathrm{M} \rangle$ *with* $\mathrm{N} \supseteq \mathrm{M}$ *which has the same cardinals and cofinalities as* M *and*

$$\mathcal{N} \models \mathrm{ZFC}^-_{Ref} + \neg \mathrm{DC}_{\aleph_2}\text{-Scheme}.$$

Proof. Let M be a model of ZFC + CH, $\mathbb{P} = \mathrm{Add}(\omega, 1)$ be the forcing to add a Cohen real and let $h \colon \mathbb{P} \cong \prod_{\omega}^{(\omega)} \mathbb{P}$ be an order-isomorphism. Let G be $\mathbb{P}$-generic over M and $H = h``G$ be the corresponding $\prod_{\omega}^{(\omega)} \mathbb{P}$-generic. Let $G_n = H \upharpoonright \{n\}$ be the $\mathbb{P}$-generic produced by restricting H to its n^{th} co-ordinate and let $\mathrm{M}_n = \mathrm{M}[G_0 \times \cdots \times G_{n-1}]$. Then the required model is $\mathcal{N} = \langle \mathrm{N}, \in, \mathrm{M} \rangle$ where $\mathrm{N} = \bigcup_n \mathrm{M}_n$. □

3.3.2 Minimal Forcing

In this short section, which is joint work with Victoria Gitman, we shall show that it is possible to produce a model of $\mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{\aleph_1}$. It is not clear how to determine whether or not $\mathrm{DC}_{\aleph_1}$ holds in the model we just produced because $\mathcal{N}_n$ is not closed under ω sequences from $\mathcal{M}_{n+1}$. However by using the same construction, except with a minimal forcing whose finite support product preserves $\aleph_1$, we shall show that $\mathrm{DC}_{\aleph_1}$ can indeed fail.

Definition 3.3.18. A forcing $\mathbb{P}$ is said to be *minimal* if for any $\mathbb{P}$-generic G and set of ordinals $X \in \mathrm{M}[G]$, either $X \in \mathrm{M}$ or $G \in \mathrm{M}[X]$.

The typical example of a minimal forcing is *Sacks Forcing*, however the finite support product of Sacks forcing is known to collapse ω_1. Therefore, it is necessary to use a subposet of Sacks forcing developed by Jensen [Jen70]. The construction of Jensen's forcing poset, $\mathbb{J}$, is quite complicated and therefore we shall neither define it nor discuss it in detail apart from saying that;

- Jensen forcing has the ccc and therefore the finite support iteration preserves ω_1,
- Jensen forcing adds a canonical generic real s over L,
- By Lemma 11 of [Jen70], Jensen forcing is minimal.

Because this section concerns models of ZFC^- in which instances of the full Dependent Choice Scheme fails it is worth mentioning another use of Jensen's forcing. This is in the paper [FGK19] where they take a symmetric submodel of a product of Jensen forcing and show that the hereditarily countable sets in this model satisfy $\mathrm{ZFC}^- + \neg\mathrm{DC}$. It is interesting that both of these models heavily use the minimality of the generic reals.

So let us suppose that $\mathbb{J}$ is a minimal forcing which adds a real s which is generic over L. In general, $\mathbb{J}$ will not be isomorphic to a finite support product of ω many copies of $\mathbb{J}$ so we take $\mathbb{P} := \prod_{\omega}^{(\omega)} \mathbb{J}$. It is then clear that $\mathbb{P} \cong \prod_{\omega}^{(\omega)} \mathbb{P}$. Therefore, by Theorem 3.3.7, if G is $\mathbb{P}$ generic over L and N is defined as before then

$$\mathcal{N} = \langle \mathrm{N}, \in, \mathrm{M} \rangle \models \mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{\aleph_2}\text{-Scheme}.$$

Theorem 3.3.19. *Suppose* $\mathrm{V} = \mathrm{L}$ *is a model of* ZFC *and that* $\mathbb{J}$ *is a minimal forcing which adds a real and whose finite support product preserves* $\aleph_1$. *Let* $\mathbb{P} = \prod_{\omega}^{(\omega)} \mathbb{J}$. *Then, using the notation of Theorem 3.3.7, if* G *is* $\mathbb{P}$*-generic over* L *then* $\mathcal{N} \models \mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{\aleph_1}$*-Scheme.*

Proof. Let $G \subseteq \mathbb{P}$ be generic and $H = h``G$ the corresponding $\prod_{\omega}^{(\omega)}$ $\mathbb{P}$-generic where h is some fixed isomorphism. As before, let $H \restriction \{n\} = G_n = \langle s_{n,m} \mid m \in \omega \rangle$ be the sequence of reals produced by restricting H to its n^{th} co-ordinate.

Let $\varphi(t, z)$ be the statement

$$z \notin \mathrm{L}[t].$$

Then for any $\alpha \in \omega_1$ and any sequence $t \in \mathrm{N}$ of α many reals, there is some $z \in {}^{\omega}2$ such that $\varphi(t, z)$. This is true since if $t \in \mathrm{N}$ then $t \in \mathrm{M}_n = \mathrm{L}[\{s_{k,m} \mid k < n, m \in \omega\}]$ for some n and therefore $s_{n,0}$ must be generic over $\mathrm{L}[t]$. Moreover, it is clear that t can be extended by the ω length sequence $\langle s_{n,m} \mid m \in \omega \rangle$ which shows that there are arbitrarily large countable sequences.

However, there can be no function $f : \omega_1 \to \mathrm{N}$ such that for each $\alpha \in \omega_1$, $\varphi(f \restriction \alpha, f(\alpha))$ because this would produce ω_1 many reals each of which is generic over the previous ones which we shall show cannot be the case. To see this, let X be any set of reals in N of size ω_1. Then fix $n \in \omega$ such that

$$X \subseteq \mathrm{M}_n = \mathrm{L}[\{s_{k,m} \mid k < n, m \in \omega\}].$$

One can see that for some $k < n$ and $m \in \omega$,

$$(\mathrm{M}_{k,m} \setminus \mathrm{M}_{k,m-1}) \cap X$$

is uncountable. But, by the assumption of minimality, for any $r \in \mathrm{M}_{k,m}$,

$$\mathrm{M}_{k,m-1}[r] = \mathrm{M}_{k,m}.$$

Therefore X cannot be a set satisfying the conditions required for $\mathrm{ran}(f)$ as, given any ordering of X, uncountably many of the reals in X will not be generic over all of the previous ones. □

3.3.3 Generalised Union Models

We now generalise the construction to produce models of ZFC^- which satisfy DC_μ but not $\mathrm{DC}_{\mu^{++}}$ for arbitrary regular cardinals μ. However, as before, it will be unclear about the status of DC_{μ^+} in these models. Because many of the proof are essentially the same as those in Section 3.3.1, we shall omit many of the details.

As in the $\prod_\omega^{(\omega)} \mathbb{P}$ case, we obtain the following theorem.

Theorem 3.3.20. *Let* M *be a model of* ZFC*,* $\mathbb{P} \in \mathrm{M}$ *and* $\delta \leq \mu$ *regular cardinals in* M*. Suppose that* $h\colon \mathbb{P} \cong \prod_\mu^{(\delta)} \mathbb{P}$ *is an order-isomorphism for some* $h \in \mathrm{M}$*. Let* G *be* $\mathbb{P}$*-generic over* $\mathcal{M}$ *and* $H = h``G$ *be the corresponding* $\prod_\mu^{(\delta)} \mathbb{P}$*-generic. Let* $G_\alpha = G \restriction \alpha$ *and let* $\mathrm{M}_\alpha = \mathrm{M}[G \restriction \alpha]$*.*

Then if $\mathrm{N} = \bigcup_\alpha \mathrm{M}_\alpha$*,* $\mathcal{N} = \langle \mathrm{N}, \in \mathrm{M} \rangle$ *is a model of* ZFC^-_{Ref}*.*

Moreover, if $\mathbb{P}$ *is* δ*-closed then* $\mathcal{N} \models \mathrm{DC}_\delta$*-Scheme.*

We now sketch the proof of this theorem, again noting that many of the proofs are easy variations of those we have already done. The proof of the main part of the Theorem will follow from Theorem 3.3.25. To prove the moreover part will require the extra result of Theorem 3.3.26 which is a generalisation of the proof of Dependent Choice from Theorem 3.3.16.

Definition 3.3.21. Suppose that $h\colon \mathbb{P} \cong \prod_\mu^{(\delta)} \mathbb{P}$ is an order-isomorphism where $h, \mathbb{P} \in \mathrm{M}$ and $\mathcal{M} = \langle \mathrm{M}, \in \rangle \models \mathrm{ZFC}$. Let H be $\mathbb{P}$-generic over $\mathcal{M}$ and note that $G \coloneqq h``H$ is $\prod_\mu^{(\delta)} \mathbb{P}$-generic over $\mathcal{M}$.

Note that $G_{\{\alpha\}}$ is $\mathbb{P}$-generic over $\mathcal{M}$ and therefore we can define $G_{\alpha,\beta} \coloneqq h``G_{\{\alpha\}} \restriction \beta$ for any $\beta \in \mu$. Let $\mathrm{M}_\alpha = \mathrm{M}[G \restriction \alpha]$ and $\mathrm{M}_{\alpha,\beta} = \mathrm{M}_\alpha[G_{\alpha,\beta}]$. Finally, let $\mathrm{N} = \bigcup_\alpha \mathrm{M}_\alpha$ and $\mathrm{N}_\alpha = \bigcup_\beta \mathrm{M}_{\alpha,\beta}$.

Remark 3.3.22. For any α and β

$$\mathrm{M}_\alpha = \mathrm{M}_{\alpha,0} \subset \mathrm{M}_{\alpha,1} \subset \cdots \subset \mathrm{M}_{\alpha,\beta} \subset \cdots \subset \bigcup_\beta \mathrm{M}_{\alpha,\beta} = \mathrm{N}_\alpha \subset \mathrm{M}_{\alpha+1}.$$

As before, N is determined by M and $\{G_{\{\alpha\}} \mid \alpha \in \mu\}$ by the specification

$$x \in \mathrm{N} \longleftrightarrow \exists y \in \mathrm{M}\ \exists v\ \Big(v \textit{ is a product of less than } \mu \textit{ many elements of } \{G_{\{\alpha\}} \mid \alpha \in \mu\} \textit{ and } x = \mathrm{K}_v(y)\Big)$$

where, given a set $v \subseteq \prod_{\mu}^{(\delta)} \mathbb{P}$, $\mathrm{K}_v(y) = \{\mathrm{K}_v(z) \mid \exists p \in v\ (\langle z, p\rangle \in y)\}$ is the restriction of the evaluation map to the set v. Therefore, it suffices to specify a name for $\{G_{\{\alpha\}} \mid \alpha \in \mu\}$, which can be done in much the same way as before. In particular, by letting $\varphi^{\star}_{Z}(v)$ be the relativisation of φ to $KS(\cdot, Z)$ we obtain that

Lemma 3.3.23. *If $\varphi(v)$ is a formula and $a \in \mathrm{M}^{e(\mathbb{P})}$ for $e \subseteq \mu$ then*

$$p \Vdash \varphi^{\star}_{\dot{\mathcal{G}}}(a) \Longleftrightarrow p \restriction e \Vdash \varphi^{\star}_{\dot{\mathcal{G}}}(a).$$

Therefore $\mathcal{N} \models \varphi(a) \Longleftrightarrow \mathcal{M}[G] \models \varphi^{\star}_{\dot{\mathcal{G}}}(a)$.

We can then prove the generalisation of Key Lemma 3.3.14. The important thing to note is that the $<\delta$-support product of μ many copies of any forcing $\mathbb{P}$ is weakly homogeneous, which is what we used in the third equivalence in the final section of the proof.

Key Lemma 3.3.24. *For every $\alpha \in \mu$, $\mathrm{N}_\alpha \prec \mathrm{N}$.*

Therefore the proof of the first part of Theorem 3.3.20 will follow from the above result combined with the following more general model theoretic theorem which again is the obvious generalisation of Theorem 3.3.15.

Theorem 3.3.25. *Let $\mathcal{M}_{\alpha+1} = \langle \mathrm{M}_{\alpha+1}, \in \mathrm{N}_\alpha\rangle$ and $\mathcal{N}_\alpha = \langle \mathrm{N}_\alpha, \in\rangle$ for $\alpha \in \mu$. Suppose that $\mathrm{M}_\alpha \subseteq \mathrm{N}_\alpha \subseteq \mathrm{M}_{\alpha+1}$, $\mathrm{N} = \bigcup_{\alpha\in\mu} \mathrm{M}_\alpha$ and $\mathcal{N} = \langle \mathrm{N}, \in\rangle$. If for every $\alpha \in \mu$, $\mathcal{M}_\alpha \models \mathrm{ZF}^-$ and $\mathcal{N}_\alpha \prec \mathcal{N}$ then $\mathcal{N} \models \mathrm{ZF}^-$.*

We now arrive at the added step which is to show that, assuming the forcing is δ-closed, N models the DC_δ-Scheme.

Theorem 3.3.26. *Let* M *be a model of* ZFC*,* $\mathbb{P} \in \mathrm{M}$ *and* $\delta \leq \mu$ *regular cardinals in* M*. Suppose that* $h\colon \mathbb{P} \cong \prod_{\mu}^{(\delta)} \mathbb{P}$ *is an order-isomorphism for some* $h \in \mathrm{M}$ *and that* $\mathbb{P}$ *is* δ*-closed. Let* G *be* $\mathbb{P}$*-generic over* $\mathcal{M}$ *and* $H = h``G$ *be the corresponding* $\prod_{\mu}^{(\delta)} \mathbb{P}$*-generic. Let* $G_\alpha = G \restriction \alpha$ *and let* $\mathrm{M}_\alpha = \mathrm{M}[G \restriction \alpha]$*. Then if* $\mathrm{N} = \bigcup_\alpha \mathrm{M}_\alpha$*,* $\mathcal{N} = \langle \mathrm{N}, \in \mathrm{M} \rangle$ *is a model of* $\mathrm{ZFC}^- + \mathrm{DC}_\delta$*-Scheme.*

Proof. It only remains to prove that $\mathcal{N}$ is a model of the DC_δ-Scheme. First note that $\mathcal{N}$ is closed under $<\delta$ sequences defined over $\mathcal{M}[G]$. To see this, let $\gamma \in \delta$ and $\langle x_\alpha \mid \alpha \in \gamma \rangle \subset \mathrm{N}$. Since $\delta \leq \mu$ are regular cardinals, there is some $\beta \in \mu$ such that $\langle x_\alpha \mid \alpha \in \gamma \rangle \subset \mathrm{M}_{\beta+1}$. Then $\mathrm{M}_{\beta+1} = \mathrm{M}_\beta[G_{\{\beta\}}]$ and $G_{\{\beta\}}$ is $\mathbb{P}$-generic, where $\mathbb{P}$ is a δ-closed forcing. A simple forcing argument shows that there is a $\mathbb{P}$-name for $\langle x_\alpha \mid \alpha \in \gamma \rangle$ which shows that this sequence lies in $\mathrm{M}_{\beta+1} \subseteq \mathrm{N}$.

To prove the DC_μ-Scheme, let φ, ψ be formulae, $u \in \mathrm{N}$ and suppose that

$$\begin{aligned} \mathcal{N} \models\ & \exists y\ \psi(y,u)\ \wedge\ \forall \sigma \in \mu\ \forall s \in {}^\sigma\mathrm{N} \\ & \big(\forall \beta \in \sigma\ \psi(s(\beta),u)\ \wedge\ \exists z(\psi(z,u)\ \wedge\ \varphi(\sigma,s,z,u))\big). \end{aligned}$$

As before, let $G_{\alpha,\beta} = h``G_{\{\alpha\}} \restriction \beta$ and let $\mathrm{N}_\alpha = \bigcup_\beta \mathrm{M}_{\alpha,\beta}$. Next, fix α such that $u \in \mathrm{N}_\alpha$. Then, by the previous theorem, the formula reflects to $\mathcal{N}_\alpha := \langle \mathrm{N}_\alpha, \in \rangle$. Note that, since $\mathcal{M}_{\alpha+1}$ is a model of ZFC, it is a model of the DC_δ-Scheme. So, because N_α is a definable subclass, there is a branch in $\mathrm{M}_{\alpha+1}$ witnessing the above instance of the DC_μ-Scheme in $\mathcal{N}_\alpha$. Where for successor steps we use the definability while for limit steps we use the fact that N_α is closed under sequences definable in $\mathcal{M}_{\alpha+1}$ of length at most δ. Finally, since $\mathcal{N}_\alpha$ is elementary in $\mathcal{N}$ and $\mathrm{M}_{\alpha+1} \subseteq \mathrm{N}$, the branch determined above is also a witness to our original instance of the DC_μ-Scheme. □

Corollary 3.3.27. *Let* M *be a model of* $\mathrm{ZFC} + 2^\mu = \mu^+$, *where* μ *is a regular cardinal. Let* $\mathbb{P} = \mathrm{Add}(\mu, 1)$ *be the forcing to add a* μ*-Cohen function from* μ *onto* 2 *and let* $h\colon \mathbb{P} \cong \prod_{\mu}^{(\delta)} \mathbb{P}$ *be an order-isomorphism where* $\delta \leq \mu$ *is a regular cardinal. Let* G *be* $\mathbb{P}$*-generic over* M *and* $H = h``G$ *be the corresponding* $\prod_{\mu}^{(\delta)} \mathbb{P}$*-generic. Let* $G_\alpha = H \restriction \alpha$ *and let* $\mathrm{M}_\alpha = \mathrm{M}[G \restriction \alpha]$. *Then if* $\mathrm{N} = \bigcup_\alpha \mathrm{M}_\alpha$,

$$\mathcal{N} = \langle \mathrm{N}, \in, \mathrm{M} \rangle \models \mathrm{ZFC}^-_{Ref} + \mathrm{DC}_\delta\text{-Scheme} + \neg\mathrm{DC}_{\mu^{++}}\text{-Scheme}.$$

So, in particular, if $\delta = \mu$ then this gives us a model of ZFC^-_{Ref} plus the DC_μ-Scheme in which the $\mathrm{DC}_{\mu^{++}}$-Scheme fails. However, in general, it seems to be difficult to either prove or disprove the DC_{μ^+}-Scheme in models of this type. It would be possible to achieve this is one were able to find a higher analogue to Jensen forcing. Namely,

Question 3.3.28. Let μ be a regular cardinal. Is there a forcing, $\mathbb{S}$, such that

- $\mathbb{S}$ has the μ-cc and the μ-support product preserves μ^+,
- $\mathbb{S}$ adds a distinguished subset of μ,
- $\mathbb{S}$ is minimal?

By recent work of Gitman, the answer to the above question should be yes for μ an inaccessible cardinal using a generalisation of Jensen forcing.

Chapter 4

The Respected Model

We begin this chapter with what was a first attempt to show that classes need not be big in ZF^- by looking at the symmetric submodel of a pretame class forcing. This attempt will ultimately fail which is why we had to use the union model approach in the previous chapter. It will turn out that it is possible for the Collection Scheme to fail in the symmetric submodel of a class forcing even if the forcing itself is pretame. This leads to the challenging question of *what theory does the symmetric submodel necessarily satisfy?* In fact, it will be unclear how one can prove either Separation or Replacement in general because the witnesses one obtains from pretameness need not be closed under permutations. Therefore, in this chapter we shall introduce the *Respected Model*.

This model is influenced by techniques developed by Karagila, notably in [Kar19], and we shall use notation from that paper. Karagila introduces the notion of a *Respected name* in order to perform an iteration of symmetric extensions. The essential idea is, when taking the symmetric extension $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, instead of considering those names $\dot{x}$ such that $\{\pi \in \mathcal{G} \mid \pi\dot{x} = \dot{x}\} \in \mathcal{F}$ we consider the class of names for which

$$\mathrm{resp}(\dot{x}) \coloneqq \{\pi \in \mathcal{G} \mid \mathbb{1} \Vdash \pi\dot{x} = \dot{x}\} \in \mathcal{F}.$$

Karagila then calls a name *hereditarily $\mathcal{F}$-respected* if this property holds hereditarily. By exploring this concept in more depth, it will turn out that the class of hereditarily Respected names does satisfy some of the properties we desire.

4.1 Collection can fail in Symmetric Models

We start with a motivating assertion which turns out to be incorrect. Let M be a model of ZFC and $\mathbb{P} = \text{Add}(\omega, \text{Ord} \times \omega)$ be the standard class forcing to add Ord many ω-blocks of Cohen reals. As shown in Corollary 2.4.10, this is a pretame class forcing so, for any $\mathbb{P}$-generic G, $\text{M}[G] \models \text{ZFC}^-$. Let N be the symmetric submodel of $\text{M}[G]$ in which there is an amorphous proper class, A, where[1]

Definition 4.1.1. A class A is said to be *amorphous* if every subclass is either finite or its complement is.

Since this is the symmetric submodel of a pretame class forcing and the Collection Scheme holds in the full extension, it seems natural to naively assume that Collection also holds in the symmetric submodel. This would also cohere to the idea that ZF^- is the "correct" way to think about ZF without Power Set.

Therefore, we claim that $\text{N} \models \text{ZF}^-$. Now, since A is amorphous, A cannot surject onto ω. This is because, if f were to be a surjection then $\{x \in A \mid f(x) \textit{ is even}\}$ and $\{x \in A \mid f(x) \textit{ is odd}\}$ would be a partition of A into two disjoint infinite sets which is a contradiction. Thus, we have a model of ZF^- with a proper class which is not big.

However, the following theorem shows that there must be a contradiction somewhere in the above argument.

Theorem 4.1.2. *Suppose that $\langle \text{M}, A \rangle$ satisfies;*

1. $\text{M} \models (\text{ZF}-)_A$,
2. $A \subseteq \text{M}$ *and* $\langle \text{M}, A \rangle \models$ *"A is a proper class",*
3. $\langle \text{M}, A \rangle \models$ *"if $B \subseteq A$ is infinite then B is a proper class".*

Then the Collection Scheme fails in $\langle \text{M}, A \rangle$. In fact, $\langle \text{M}, A \rangle$ does not have a cumulative hierarchy and therefore Power Set also fails.

[1] We will formally define this model in Section 4.1.1

Proof. To prove that the Collection Scheme fails consider classes b satisfying

$$\forall n \in \omega\ \exists y \in b\ (|y| = n\ \wedge\ y \subseteq A).$$

Since $\bigcup b \cap A$ is an infinite subclass of A, by the third assumption b must be a proper class. Therefore, while for every $n \in \omega$ there is a y such that $(|y| = n\ \wedge\ y \subseteq A)$ there is no set witnessing this for all n.

For the second part of the theorem, we shall prove the stronger assertion that any well-orderable sequence of sets can only contain finitely many elements of A. To see this, let $\mathcal{C} = \langle C_\alpha \colon \alpha \in \mathrm{I}\rangle$ be an indexed sequence of sets where I is either $\mathrm{O{\small RD}}$ or an infinite ordinal. We shall show that $\bigcup \mathcal{C} \cap A$ is finite and therefore that $\mathcal{C}$ cannot be a hierarchy for the universe. Suppose for a contradiction that $\bigcup \mathcal{C} \cap A$ was in fact infinite. First note that for any $\alpha \in \mathrm{I}$, $C_\alpha \cap A$ must be finite. Now we define a sequence of ordinals $\delta_n \in \mathrm{I}$ inductively as the least ordinal $\alpha \in \mathrm{I}$ such that $(C_\alpha \setminus \bigcup_{m \in n} C_{\delta_m}) \cap A \neq \emptyset$. Such an ordinal must exist by the assumption that $\bigcup \mathcal{C} \cap A$ is infinite and that $\bigcup_{m \in n}(C_{\delta_m} \cap A)$ is a union of finite sets. But then $\bigcup_{n \in \omega} C_{\delta_n} \cap A$ is an infinite set, contradicting the third condition of the theorem. □

There are two plausible places where the contradiction could have arisen. The first is the assertion that we can produce a symmetric submodel with an amorphous proper class and the second is the assertion that Collection held in the model. We shall formally define the required symmetric submodel which shows that the Collection Scheme can fail in the symmetric submodel of a pretame class forcing.

But first it is worthwhile to discuss why the proof that Collection holds in the full extension of a pretame forcing does not translate into a proof in the symmetric submodel. In fact, we shall show that it is unclear if even the weaker Replacement Scheme holds in this model. We shall phrase this in terms of an arbitrary symmetric submodel of a class forcing but, if one wishes to work with a concrete example, one could instead work with either of the two systems that will be defined in the subsequence section.

So suppose that M is a countable transitive model of ZFC and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system, where $\mathbb{P}$ is a pretame class forcing. We consider the Axiom of Replacement in the symmetric submodel of this system.

So, suppose that

$$p \Vdash \dot{f} \textit{ is a total function on } \dot{a}$$

where $\dot{f}$ and $\dot{a}$ are hereditarily symmetric names. We want a symmetric name for the range of $\dot{f}$. Now, using Collection, we can find some set of hereditarily symmetric names, c, containing witnesses to elements being in the range of $\dot{f}$. Then, using pretameness, for each $\langle \dot{x}, r \rangle \in \dot{a}$ we can find some set $d_{\dot{x},r}$ of $\mathbb{P}$-names such that

$$\dot{b} := \{\langle \dot{y}, s \rangle \mid \dot{y} \in c \ \wedge \ \exists \langle \dot{x}, r \rangle \in \dot{a} \ (s \in d_{\dot{x},r} \ \wedge \ s \Vdash \dot{f}(\dot{x}) = \dot{y})\}$$

is a name for the range of $\dot{f}$. We want to conclude that for any $\pi \in \text{sym}(\dot{a}) \cap \text{sym}(\dot{f})$, $\pi\dot{b} = \dot{b}$. However, in general, $\{\pi(\langle \dot{y}, s \rangle) \mid \pi \in \text{sym}(\dot{a}) \cap \text{sym}(\dot{f})\}$ need not be a set. To see this, take $\langle \dot{y}, s \rangle \in \dot{b}$. Instead of $\dot{y}$, c could have been chosen to include some name $\dot{y}'$ such that

$$p \Vdash \dot{y} = \dot{y}'$$

and $\dot{y}'$ contained additional information which was not fixed by $\text{sym}(\dot{a}) \cap \text{sym}(\dot{f})$. For example, one could just take $\dot{y}' = \dot{y} \cup \{\langle \dot{z}, t \rangle\}$, where t is any condition incompatible with s and $\dot{z}$ is an arbitrary $\mathbb{P}$-name. The point is that, since $\text{sym}(\dot{a}) \cap \text{sym}(\dot{f})$ will in general be a proper class, there is no reason why $\{\pi(\langle \dot{z}, t \rangle) \mid \pi \in \text{sym}(\dot{a}) \cap \text{sym}(\dot{f})\}$ should be a set.

4.1.1 Amorphous Classes

We shall now give the details of the argument to add an amorphous proper class. A detailed account of the set version of this forcing can be found as Exercise 8.13.1 of [DS96] which serves as the basis for our construction. Solely for simplicity, let $\mathbb{M}$ be a countable transitive model of GB + AC + CH.

Let $\mathbb{P} = \text{Add}(\omega, \text{Ord} \times \omega)$ be the poset which adds Ord many ω-blocks of Cohen reals and let $\text{supp}(p)$ denote the *support* of p. Now, for π^0 a permutation of Ord and $\{\pi_\alpha \colon \alpha \in \text{Ord}\}$ a collection of permutations of ω, let π be the permutation

$$\pi \colon \text{Ord} \times \omega \to \text{Ord} \times \omega \qquad \pi(\alpha, n) = (\pi^0(\alpha), \pi_\alpha(n))$$

and let $\mathcal{G}$ be the class of permutations defined in this way. This is known as the *wreath product* of the permutations of Ord and the permutations of ω. Next, let $\mathcal{F}$ be the filter generated by $\text{fix}(E) \coloneqq \{\pi \in \mathcal{G} \mid \pi \restriction E = \text{id}\}$, for finite sets $E \subseteq \text{Ord} \times \omega$. We can extend π to $\mathbb{P}$-names by

$$\pi p(\pi(\alpha, n)) = p(\alpha, n).$$

The idea being that elements of $\mathcal{G}$ first permute the ω-blocks of reals and then permute within the blocks.

Now define

- $\dot{t}_{(\alpha,n)} \coloneqq \{\langle \check{m}, p\rangle \mid p \in \mathbb{P} \ \wedge \ p(\alpha, n, m) = 1\}$. This is the canonical name for the Cohen real generated by $\mathbb{P}$ restricted to the co-ordinate (α, n).
- $\dot{T}_\alpha \coloneqq \{\dot{t}_{(\alpha,n)} \mid n \in \omega\}^\bullet$ to be a name for the α^{th} ω-block of reals.
- $\dot{A} \coloneqq \{\dot{T}_\alpha \mid \alpha \in \text{Ord}\}^\bullet$ to be a name for the collection of all Ord many ω-blocks.

It is clear that for $\pi \in \mathcal{G}$, $\pi\dot{t}_{(\alpha,n)} = \dot{t}_{\pi(\alpha,n)}$, $\pi\dot{T}_\alpha = \dot{T}_{\pi^0(\alpha)}$ and $\pi\dot{A} = \dot{A}$. Therefore $\text{sym}(\dot{t}_{(\alpha,n)}) = \text{fix}(\{(\alpha, n)\})$, $\text{sym}(\dot{T}_\alpha) \supseteq \text{fix}(\{(\alpha, 0)\})$ and $\text{fix}(\dot{A}) = \mathcal{G}$ all of which are in $\mathcal{F}$, so each of these names is hereditarily symmetric in M.

We claim that in the symmetric extension, N, $\dot{A}^G$ is an amorphous proper class.

To prove this, suppose that $\dot{B}$ is a symmetric name and for some $p \in \mathbb{P}$, $p \Vdash \dot{B} \subseteq \dot{A}$. Take $E \subseteq \text{Ord} \times \omega$ a finite set such that $\text{fix}(E) \subseteq \text{sym}(\dot{B})$ and let

$$y \coloneqq \{\alpha \in \text{Ord} \mid \exists n \in \omega \ (\alpha, n) \in E \cup \text{supp}(p)\} \in [\text{Ord}]^{<\omega}.$$

Proposition 4.1.3. *For any $q \leq p$, if there is an $\alpha \notin y$ such that $q \Vdash \dot{T}_\alpha \in \dot{B}$, then $q \Vdash \dot{T}_\beta \in \dot{B}$ for any $\beta \notin y$.*

Note that if the above proposition holds then, for any $q \leq p$,

$$q \Vdash \{\dot{T}_\alpha \mid \alpha \in \text{Ord} \setminus y\} \subseteq \dot{B} \ \vee \ \{\dot{T}_\alpha \mid \alpha \in \text{Ord} \setminus y\} \subseteq \dot{A} \setminus \dot{B}$$

so $\dot{A}^G$ will indeed be forced to be amorphous.

Proof. Suppose that for some $q \leq p$ there is an $\alpha \notin y$ such that $q \Vdash \dot{T}_\alpha \in \dot{B}$. Let $\beta \notin y$ with $\beta \neq \alpha$ and let $r \leq q$. Take

$$k > \max\{n \in \omega \mid \exists \gamma \in \text{Ord}\ (\gamma, n) \in \operatorname{supp}(r)\}.$$

Now, let $\pi^0 = (\alpha, \beta)$ and

$$\pi_\gamma = \begin{cases} \prod_{n<k}(n, n+k), & \text{if } \gamma \in \{\alpha, \beta\} \\ \text{id}, & \text{if } \gamma \notin \{\alpha, \beta\}. \end{cases}$$

where, using standard cycle notation for permutations, $\prod_{n<k}(n, n+k)$ denotes the product of the disjoint permutations switching n with $n+k$ for each $n < k$. Since $\alpha, \beta \notin y$, $\pi p = p$ and, since $\pi \in \operatorname{fix}(E)$, $\pi \in \operatorname{sym}(\dot{B})$. Also,

$$\begin{aligned} \operatorname{supp}(\pi q) \ &= \ \operatorname{supp}(q) \setminus \{(\gamma, n) \mid \gamma \in \{\alpha, \beta\}\} \cup \\ &\quad \{(\alpha, n+k) \mid (\beta, n) \in \operatorname{supp}(q)\} \ \cup \ \{(\beta, n+k) \mid (\alpha, n) \in \operatorname{supp}(q)\}. \end{aligned}$$

Moreover, neither of the two latter unions can be in $\operatorname{supp}(r)$ by the definition of k. Thus $\pi q \parallel r$, allowing us to conclude that

$$\forall \beta \notin y\ \forall r \leq q\ \exists \pi \in \mathcal{G}\ \Big(r \parallel \pi q \ \wedge \ \pi q \Vdash \dot{T}_\beta \in \dot{B}\Big).$$

From which we deduce that $q \Vdash \dot{T}_\beta \in \dot{B}$. □

Recalling from Corollary 2.4.10 that $\operatorname{Add}(\omega, \text{Ord} \times \omega)$ is a pretame class forcing, we have shown that it is possible that the symmetric submodel of a pretame class forcing contains an amorphous proper class. However, by Theorem 4.1.2, this means that Collection fails in N, yielding the following Theorem.

Theorem 4.1.4. *Over* GB + AC*, it is consistent that the symmetric submodel of a pretame class forcing does not satisfy* ZF^-*.*

It is worth noting that we have not specified what the symmetric submodel of a pretame class forcing does actually satisfy. This appears to be a difficult question which is why we shall shortly introduce the *Respected Model* as an alternative model.

4.1.2 Dedekind-Finite Classes

It turns out that there is a much easier symmetric submodel which produces a class A satisfying the hypothesis of Theorem 4.1.2, which is the class forcing version of Cohen's first model to produce a Dedekind-finite class of reals.

Definition 4.1.5. A class X is said to be *Dedekind-finite* if there is no injection from ω into X.

Let $\mathbb{P} = \mathrm{Add}(\omega, \mathrm{ORD})$, $\mathcal{G} = \mathrm{sym}(\omega)$ and $\mathcal{F} = \langle \mathrm{fix}(E) \mid E \in [\mathrm{ORD}]^{<\omega} \rangle$. Let $\dot{t}_\alpha = \{\langle p, n \rangle \mid p(\alpha, n) = 1\}$ and let $\dot{A} = \{\dot{t}_\alpha \mid \alpha \in \mathrm{ORD}\}^\bullet$. Let $G \subseteq \mathbb{P}$ be generic and let N be the symmetric submodel. Then one can prove in much the same way as in Cohen's original model that $A = \dot{A}^G$ is Dedekind-finite in N.

Proposition 4.1.6. *Suppose that $B \subseteq A$ is infinite. Then B is a proper class.*

Proof. We need to prove that if p forces $\dot{B}$ to be a hereditarily symmetric name for an infinite subclass of A then for any ordinal α it is dense below p that the $\dot{t}_\beta$ are unbounded in $\dot{B}$. So, fix $\dot{B}$ to be a hereditarily symmetric name and $p \in G$ such that

$$p \Vdash \dot{B} \subseteq \dot{A} \textit{ is infinite.}$$

It suffices to show that for any ordinal α and any condition $q \leq p$ there is some $s \leq q$ and $\beta > \alpha$ such that $s \Vdash \dot{t}_\beta \in \dot{B}$. Since $\dot{B}$ is symmetric, fix a finite set E such that $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{B})$ and let $y = \mathrm{fix}(E) \cup \mathrm{supp}(q)$, then y is a finite set of ordinals. Since $q \Vdash \dot{B}$ *is infinite* there is some $r \leq q$ and $\gamma \notin y$ such that $r \Vdash \dot{t}_\gamma \in \dot{B}$. Now for our fixed α, take $\beta > \alpha$ such that $\beta \notin y \cup \{\gamma\}$ and define $\pi = (\gamma, \beta)$. Then $\pi r \Vdash \dot{t}_\beta \in \dot{B}$ and, since $\mathrm{supp}(q) \subseteq y$, $\pi r \leq q$. □

Corollary 4.1.7. N *is not a model of* ZF^-*.*

We end this section by noting that it is possible to have a Dedekind-finite class without breaking the Collection Scheme. In fact, as proved by Monro [Mon75], one can produce a model of ZF with a Dedekind-finite class that is big.

Theorem 4.1.8 (Monro)**.** *Let* ZF(K) *be the theory with the language of* ZF *plus a one-place predicate* K *and let* M *be a countable transitive model of* ZF*. Then there is a model* N *such that* N *is a transitive model of* ZF(K) *and*

$$\begin{aligned} \mathrm{N} \models\ & \mathrm{K} \textit{ is a proper class which is Dedekind-finite} \\ & \textit{and can be mapped onto the universe.} \end{aligned}$$

A second remark it is worthwhile making here is that it is possible for the symmetric submodel of a pretame class forcing to satisfy the Collection Scheme. Moreover, it is possible to have a pretame class forcing in which Power Set fails in the full extension and yet the symmetric submodel satisfies full ZF. Two such classical examples are Gitik's model [Git80] and the Morris model [Kar20]. It would be interesting future work to explore what conditions one must place on the symmetric system to ensure the preservation of the Collection Scheme and the Power Set.

4.2 Class Symmetric Systems

Recall, that in order to define class forcing we worked in the second-order theory GB^-. However, now we have two additional complications. The first is that we also need to deal with permutations $\pi\colon \mathbb{P} \to \mathbb{P}$ and subgroups of these permutations, which requires us to formally work in a fourth-order set theory. The second is that to ensure we can run the recursive definition of being hereditarily respected we require the ability to perform class length elementary transfinite recursions, which is a principle that need not hold in GB. The leads us to work in the theory we shall denote by $\mathrm{KM}^-_{(4)}$, which is a generalisation of KM^-. This is probably a significantly stronger theory than is necessary (for example we will not need Choice), and we leave it for further work to determine the theory actually needed.

We work with four sorted models of the form $\mathcal{M} = \langle \mathrm{M}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$ where M denotes the sets of $\mathcal{M}$, $\mathcal{C}_1$ denotes the classes of $\mathcal{M}$, $\mathcal{C}_2$ the hyper-classes and $\mathcal{C}_3$ the hyper-hyper-classes. Our typical example of a model of fourth-order set theory over which we will define the Respected Model will be (countable elementary submodels of) $\langle \mathrm{H}_\mu, \mathcal{P}(\mathrm{H}_\mu), \mathcal{P}^2(\mathrm{H}_\mu), \mathcal{P}^3(\mathrm{H}_\mu) \rangle$ where μ is some fixed regular cardinal and for the purpose of this thesis one could take this as our definition of a model of fourth-order set theory. Lastly, we shall extend the use of $\in$ and $\subseteq$ to their obvious definitions in the higher order contexts.

Definition 4.2.1. We denote by $\mathrm{KM}^-_{(4)}$ the theory in the four sorted language of set theory, where the sorts are denoted by being elements of $\mathrm{M}(= \mathcal{C}_0)$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ respectively, with the following axioms:

For any $m \in \{0, 1, 2\}$,

- $\mathrm{M} \models \mathrm{ZFC}^-$,
- If $X \in \mathcal{C}_m$ then $X \in \mathcal{C}_{m+1}$,
- If $X \in \mathcal{C}_{m+1}$ and $Y \in X$ then $Y \in \mathcal{C}_m$,
- (Higher-Order Extensionality)
$$X, Y \in \mathcal{C}_{m+1} \to \Big(\forall Z(Z \in X \leftrightarrow Z \in Y) \to X = Y\Big),$$
- (Higher-Order Comprehension) for any formula φ whose quantified variables are of type $\mathcal{C}_{m+1}$, $\forall Z \in \mathcal{C}_{m+1}\ \exists Y \in \mathcal{C}_{m+1}\ Y = \{X \in \mathcal{C}_m \mid \varphi(X, Z_1, \ldots, Z_n)\}$,
- (Higher-Order Collection) for any formula φ whose quantified variables are of type $\mathcal{C}_{m+1}$ and for any $Z \in \mathcal{C}_{m+1}$,
$$\forall A \in \mathcal{C}_m\ \forall X \in A\ \exists Y \in \mathcal{C}_{m-1}\ \varphi(X, Y, Z)$$
where $\mathcal{C}_{0-1}$ also denotes $\underline{\mathrm{M}}$ $\to \exists B \in \mathcal{C}_m\ \forall X \in A\ \exists Y \in B\ \varphi(X, Y, Z)$

We will shortly define the Respected Model but we note here that the classes $\mathcal{C}_2$ and $\mathcal{C}_3$ are only needed to ensure that $\mathcal{G}$ and $\mathcal{F}$ can be defined. Therefore, for simplicity, the

Respected Model will only be defined as a two sorted model although it could equally be considered in the full, four sorted way.

Now suppose that $\mathcal{M} = \langle \mathrm{M}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$ is a model of $\mathrm{KM}^-_{(4)}$ and $\mathbb{P}$ is a class forcing which satisfies the forcing theorem over $\langle \mathrm{M}, \mathcal{C}_1 \rangle$[2]. Suppose further that $\mathcal{G} \in \mathcal{C}_2$ is some group of automorphisms of $\mathbb{P}$. Namely, $\mathcal{G}$ is such that, for any $\pi \in \mathcal{G}$, $\pi \in \mathcal{C}_1$ is a bijective class function which respects the ordering of $\mathbb{P}$. Let $\mathcal{K} \in \mathcal{C}_3$ denote the collection of subgroups of $\mathcal{G}$. Finally we shall say that $\mathcal{F} \in \mathcal{C}_3$ is a *normal filter of subgroups* of $\mathcal{G}$ if it satisfies the following:

- $\mathcal{F} \subseteq \mathcal{K}$ and $\mathcal{G} \in \mathcal{F}$,
- If $H \in \mathcal{F}$ and $K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$,
- If $H \in \mathcal{F}$ and $H \subseteq K$, where $K \in \mathcal{K}$, then $K \in \mathcal{F}$,
- (*Normality*) If $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

As with the set forcing case, we shall then call the triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a *symmetric system*. Now, given $\pi \colon \mathbb{P} \to \mathbb{P}$ we can extend π to act on $\mathbb{P}$-names in the usual way by recursion as

$$\pi\dot{x} \coloneqq \{\langle \pi\dot{y}, \pi p \rangle \mid \langle \dot{y}, p \rangle \in \dot{x}\}.$$

Following Karagila's terminology, for a class $\mathbb{P}$-name $\dot{x}$, let

$$\mathrm{resp}(\dot{x}) \coloneqq \{\pi \in \mathcal{G} \mid \mathbb{1} \Vdash \pi\dot{x} = \dot{x}\}$$

and we shall call a name *respected* if $\mathrm{resp}(\dot{x}) \in \mathcal{F}$, noting that this is well-defined because $\mathrm{KM}^-_{(4)}$ allows for class length elementary transfinite recursions. It is also worth mentioning that in the set forcing case any respected name will be equal to a symmetric one, modulo $\mathbb{1}$. This shall be addressed more fully at the end of this chapter.

[2]Since $\langle \mathrm{M}, \mathcal{C}_1 \rangle$ is a model of KM^-, any class forcing will satisfy the forcing theorem by the Main Theorem in [GHH+20] however we will explicitly assume our class forcing satisfies this because of its necessity in defining the model.

Then $\dot{x}$ is said to be *hereditarily $\mathcal{F}$-respected*, written $\dot{x} \in \mathrm{HR}_{\mathcal{F}}$, if

$$\mathrm{resp}(\dot{x}) \in \mathcal{F} \textit{ and for any } \langle \dot{y}, p\rangle \in \dot{x},\ \dot{y} \in \mathrm{HR}_{\mathcal{F}}.$$

Let $\mathrm{M}^{\mathrm{HR}_{\mathcal{F}}}$ denote the elements of $\mathrm{M}^{\mathbb{P}}$ which are hereditarily $\mathcal{F}$-respected and $\mathcal{C}_1^{\mathrm{HR}_{\mathcal{F}}}$ those in $\mathcal{C}_1^{\mathbb{P}}$. We can then define the *Respected Model given by $\mathcal{F}$* as $\mathcal{N} = \langle \mathrm{N}, \mathcal{C}\rangle$, where

$$\mathrm{N} \coloneqq \{\dot{x}^G \mid \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge\ \dot{x} \in \mathrm{HR}_{\mathcal{F}}\}$$

and

$$\mathcal{C} \coloneqq \{\dot{\Gamma}^G \mid \dot{\Gamma} \in \mathcal{C}_1^{\mathbb{P}} \ \wedge\ \dot{\Gamma} \in \mathrm{HR}_{\mathcal{F}}\}$$

for G a $\mathbb{P}$-generic filter over $\mathcal{M}$.

An important additional requirement we need to consider is what Karagila calls *tenacity*:

Definition 4.2.2 (Karagila)**.** Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ be a symmetric system. A condition $p \in \mathbb{P}$ is said to be *$\mathcal{F}$-tenacious* if there exists some $H \in \mathcal{F}$ such that for every $\pi \in H$, $\pi p = p$. $\mathbb{P}$ is said to be $\mathcal{F}$-tenacious if there is a dense subset of $\mathcal{F}$-tenacious conditions.

If p is $\mathcal{F}$-tenacious then define $\mathrm{sym}(p) \coloneqq \{\pi \in \mathcal{G} \mid \pi p = p\}$ which will be in $\mathcal{F}$.

We note next the following theorem by Karagila and Hayut which is in the appendix of [Kar19]. This theorem tells us that, for *set* forcings, we have not lost anything by only considering tenacious ones.

Definition 4.2.3 (Karagila)**.** Two symmetric systems $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ and $\langle \mathbb{P}', \mathcal{G}', \mathcal{F}'\rangle$ are *equivalent* if for every generic $G \subseteq \mathbb{P}$ there is a filter $G' \subseteq \mathbb{P}'$ such that $\mathrm{HR}_{\mathcal{F}}^G = \mathrm{HR}_{\mathcal{F}'}^{G'}$ and vice versa.

Theorem 4.2.4 (Karagila and Hayut)**.** *Over* ZFC*, Every* set *symmetric system is equivalent to a tenacious one.*

However the proof uses results concerning the completion of the corresponding Boolean algebra, so it is not clear if it is true for class forcings in general.

Question 4.2.5. Is every class symmetric system, with $\mathbb{P}$ a pretame class forcing, equivalent to a tenacious one?

It may also seem like a strange choice to require that $\{\pi \in \mathcal{G} \mid \mathbb{1} \Vdash \pi(\dot{x}) = \dot{x}\}$ is in the filter while dropping the "$\mathbb{1}$ forces" requirement for the conditions. However, this seems to be necessary for the construction to go through. Also, any symmetric system seems to satisfy this definition of tenacity. For example, both the symmetric system mentioned in Section 4.1.1 and the one from Section 4.1.2 are tenacious.

We end this section by noting a few basic properties of class symmetric systems. Firstly, we can still define the canonical name for elements of M and therefore $\langle \mathrm{N}, \mathcal{C} \rangle$ is an extension of $\langle \mathrm{M}, \mathcal{C}_1 \rangle$. Moreover, unless $\mathcal{F}$ is trivial, $G \notin \mathcal{C}_1^{\mathrm{HS}_\mathcal{F}}$ and therefore $\langle \mathrm{N}, \mathcal{C} \rangle$ will in general be a proper submodel of the full extension $\langle \mathrm{M}[G], \mathcal{C}_1[G] \rangle$. Finally, because the proof of the *Symmetry Lemma* does not require any assumptions about the symmetric groups of our names, the class generalisation goes through with exactly the same proof.

Lemma 4.2.6 (Symmetry Lemma). *For any $p \in \mathbb{P}$, formula $\varphi(v)$, $\dot{x} \in \mathrm{M}^\mathbb{P}$, $\dot{\Gamma} \in \mathcal{C}_1^\mathbb{P}$ and $\pi \in \mathcal{G}$,*

$$p \Vdash \varphi(\dot{x}, \dot{\Gamma}) \iff \pi p \Vdash \varphi(\pi \dot{x}, \pi \dot{\Gamma}).$$

4.3 The Respected Model

We devote this section to deriving which axioms hold in the Respected Model. We shall do this in stages in order to make it clear where each assumption on the class forcing and ground model is used. That is, we shall first show that if any generic for $\mathbb{P}$ preserves the fundamental operations[3] then the Respected model is also closed under the fundamental operations. Then we shall show that if $\mathbb{P}$ is pretame and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is tenacious, then the Respected Model satisfies full KM−. Finally, we shall show that if we assume tameness then, without any assumption about tenacity, the Respected Model satisfies full KM.

[3]These standard operations, otherwise known as Gödel operations, can be found in Definition 13.6 of [Jec03] or Chapter II of [Bar17] and will be discussed in an intuitionistic context in Section 5.2.

For this section, fix a model $\mathcal{M}$ of $\mathrm{KM}^-_{(4)}$ and let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system for which $\mathbb{P}$ is a class forcing which satisfies the forcing theorem. Given a $\mathbb{P}$-generic G, let $\mathcal{N} := \langle \mathrm{N}, \mathcal{C} \rangle$ denote the Respected Model given by the symmetric system.

Theorem 4.3.1. *Over* $\mathrm{KM}^-_{(4)}$*, If, for any generic* G*,* $\mathcal{M}[G]$ *is closed under the fundamental operations then so is* $\mathcal{N}$*.*

Proof. We shall prove that $\mathcal{N}$ is closed under Pairing and Unions, the other cases being handled in a similar manner.

Pairing: Suppose that $\dot{x}, \dot{y} \in \mathrm{M}^{\mathrm{HR}}$, then $\dot{z} := \{\langle \dot{x}, \mathbb{1} \rangle, \langle \dot{y}, \mathbb{1} \rangle\}$ is a name for the unordered pair of $\dot{x}$ and $\dot{y}$. We shall show that $\mathrm{resp}(\dot{z}) \supseteq \mathrm{resp}(\dot{x}) \cap \mathrm{resp}(\dot{y})$ which will show that the pair of $\dot{x}$ and $\dot{y}$ has a hereditarily-respected name. So let π be in this intersection. Then

$$\mathbb{1} \Vdash \pi\dot{x} = \dot{x} \ \wedge \ \pi\dot{y} = \dot{y}$$

and $\pi\dot{z} = \{\langle \pi\dot{x}, \mathbb{1} \rangle, \langle \pi\dot{y}, \mathbb{1} \rangle\}$. It should be clear that $\mathbb{1} \Vdash \pi\dot{z} = \dot{z}$ since

$$\mathbb{1} \Vdash \forall t \ (t \in \dot{z} \leftrightarrow (t = \dot{x} \ \vee \ t = \dot{y})).$$

Unions: Let $a \in \mathrm{N}$ and fix a name $\dot{a} \in \mathrm{M}^{\mathrm{HR}}$ for it. Since, for any generic G,

$$\mathcal{M}[G] \models \exists z \ \forall x \ (x \in z \leftrightarrow \exists y \in a(x \in y)),$$

$$\mathbb{1} \Vdash \exists z \ \forall x \ (x \in z \leftrightarrow \exists y \in \dot{a}(x \in y)).$$

Let $\mathcal{E} = \{\dot{x} \in \mathrm{M}^{\mathrm{HR}} \mid \mathbb{1} \Vdash \exists y \in \dot{a}(\dot{x} \in y)\}$ and let $E_{\dot{x}} := \{\dot{x}' \in \mathcal{E} \mid \mathbb{1} \Vdash \dot{x} = \dot{x}'\}$ be the equivalence classes modulo forcing equality by $\mathbb{1}$. Now, since $\mathbb{1}$ forces that $\bigcup a$ is a set, there can only be set many equivalence classes. So, by Collection in $\mathcal{M}$, fix I to be a set in M such that for any $\dot{x} \in \mathcal{E}$ there is an $\dot{x}' \in I$ such that $\dot{x}' \in E_{\dot{x}}$. Then

$$\mathbb{1} \Vdash \forall w \ (\exists y \in \dot{a}(w \in y) \leftrightarrow w \in I^{\bullet}).$$

Since I is a set of hereditarily respected names, to prove that $I^{\bullet} \in \mathrm{M}^{\mathrm{HR}}$ it suffices to prove that for any $\pi \in \mathrm{resp}(\dot{a})$,

$$\mathbb{1} \Vdash \pi I^{\bullet} = I^{\bullet}.$$

To see this, suppose that $t_0 \Vdash z \in I^\bullet$ and let $t_1 \leq t_0$ be arbitrary. Then we can fix $t_2 \leq t_1$, $\dot{y} \in \mathrm{dom}(\dot{a})$ and $\dot{x} \in I$ such that

$$t_2 \Vdash \dot{x} \in \dot{y} \ \wedge \ \dot{x} = \dot{z}.$$

Then, by symmetry,

$$\pi^{-1}t_2 \Vdash \pi^{-1}\dot{x} \in \pi^{-1}\dot{y} \in \dot{a}.$$

So, by the construction of I, $\pi^{-1}t_2 \Vdash \pi^{-1}\dot{x} \in I^\bullet$. Therefore, take $t_3 \leq \pi^{-1}t_2$ and $\dot{x}' \in I^\bullet$ such that $t_3 \Vdash \pi^{-1}\dot{x} = \dot{x}'$. Then we have that $\pi t_3 \leq t_1$ and

$$\pi t_3 \Vdash z = \dot{x} = \pi\dot{x}'$$

so $t_0 \Vdash z \in \pi I^\bullet$. The reverse implication is done in a similar manner. □

However, in order to show that $\mathcal{N}$ is a model of KM−, we need to additionally assume that the symmetric system is tenacious.

Theorem 4.3.2. *Working over* $\mathrm{KM}^-_{(4)}$*, suppose that* $\mathbb{P}$ *is a pretame class forcing and* $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *is a tenacious symmetric system. Then the Respected model,* $\mathcal{N}$*, is a model of* KM−.

Proof. It is obvious that $\mathcal{N}$ is a model of Extensionality, Foundation and Infinity so it remains to show that $\mathcal{N}$ is a model of second-order Replacement, Separation and Class Comprehension.

Replacement: Suppose that $\mathcal{N} \models f : a \to \mathrm{N}$ and take $p \in G$ such that

$$p \Vdash \dot{f} \text{ is a total function on } \dot{a}$$

where $\dot{f}$ and $\dot{a}$ are hereditarily respected names for f and a. Note that we are allowing f to be a class function in order to prove second-order replacement. For each $\langle \dot{x}, r \rangle \in \dot{a}$ let

$$D_{\dot{x},r} := \{s \in \mathbb{P} \mid s \leq p, r \ \wedge \ \exists \dot{y} \in \mathrm{M}^{\mathrm{HR}} \ (s \Vdash \dot{f}(\dot{x}) = \dot{y})\}.$$

Then for each $\langle \dot{x}, r\rangle$, $D_{\dot{x},r}$ is a class which is dense below $p \wedge r$. So, by pretameness and genericity, we can fix some $q \leq p$ and sequence of sets $\langle d_{\dot{x},r} \mid \langle \dot{x}, r\rangle \in \dot{a}\rangle$ such that $q \in G$ and each $d_{\dot{x},r}$ is predense below $q \wedge r$. Next, by Collection, we can find some set $c \subseteq \mathrm{M}^{\mathrm{HR}}$ such that

$$\forall \langle \dot{x}, r\rangle \in \dot{a}\ \forall s \in d_{\dot{x},r}\ \exists \dot{y} \in c\ (s \Vdash \dot{f}(\dot{x}) = \dot{y})$$

and take $\dot{b} := \{\langle \dot{y}, s\rangle \mid \dot{y} \in c\ \wedge\ \exists \langle \dot{x}, r\rangle \in \dot{a}\ (s \in d_{\dot{x},r}\ \wedge\ s \Vdash \dot{f}(\dot{x}) = \dot{y})\}$. It is known that this is the standard name for the range of $\dot{f}$ and therefore if we can show it is hereditarily respected then we will be done.

Claim 4.3.3. *For any $\pi \in \mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{f}) \cap \mathrm{sym}(q)$, $q \Vdash \pi\dot{b} = \dot{b}$.*

Proof of Claim. Due to the fact that this proof uses several conditions and their permutations by π, we have decided to present the proof of this claim in a list fashion. Moreover, to aid clarity, we also include a tree of the conditions (Figure 4.1) used in the proof.

(i.) Fix $\pi \in \mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{f}) \cap \mathrm{sym}(q)$, $z \in \mathrm{dom}(\dot{b})$, $t_0 \leq q$ and suppose that $t_0 \Vdash z \in \dot{b}$. We shall show that for any condition below t_0, there is a condition below that forcing $z = \pi\dot{y}'$ for some $\dot{y}' \in \dot{b}$. From which we can deduce that $t_0 \Vdash z \in \pi\dot{b}$.

(ii.) Let $t_1 \leq t_0$ be arbitrary.

(iii.) Take $t_2 \leq t_1$ and $\langle \dot{y}, s\rangle \in \dot{b}$ such that $t_2 \leq s$ and $t_2 \Vdash z = \dot{y}$.

(iv.) Fix $\langle \dot{x}, r\rangle \in \dot{a}$ such that $s \in d_{\dot{x},r}$ and $s \Vdash \dot{f}(\dot{x}) = \dot{y}$.

(v.) Since $t_2 \leq s \leq r$, $t_2 \Vdash \dot{x} \in \dot{a} = \pi\dot{a}$.

(vi.) So fix $t_3 \leq t_2$ and $\langle \pi\dot{x}', \pi r'\rangle \in \pi\dot{a}$ such that $t_3 \leq \pi r'$ and $t_3 \Vdash \pi\dot{x}' = \dot{x}$.

(vii.) Since $t_3 \leq q$ and π fixes q, $\pi^{-1}t_3 \leq q$ so, by predensity, fix $s' \in d_{\dot{x},' r'}$ such that s' and $\pi^{-1}t_3$ are compatible.

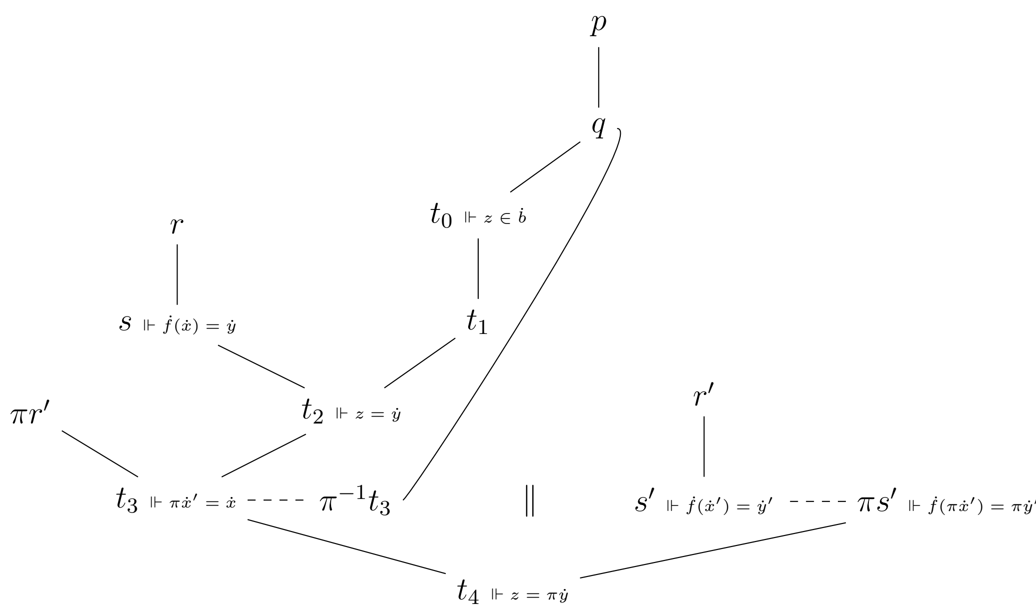


Figure 4.1: Tree of Conditions for Replacement

(viii.) Take $\dot{y}' \in c$ such that $s' \Vdash \dot{f}(\dot{x}') = \dot{y}'$, so $\langle \dot{y}', s' \rangle \in \dot{b}$.

(ix.) Then $\pi s' \Vdash \dot{f}(\pi\dot{x}') = \pi\dot{y}'$.

(x.) Take $t_4 \leq t_3, \pi s'$.

(xi.) Since $t_4 \leq \pi s'$, $t_4 \Vdash \dot{f}(\pi\dot{x}') = \pi\dot{y}'$.

(xii.) Since $t_4 \leq t_3$, $t_4 \Vdash \pi\dot{x}' = \dot{x}$.

(xiii.) Since $t_4 \leq p$, t_4 forces that $\dot{f}$ is a function.

(xiv.) Finally, since t_4 is also below t_2, $t_4 \Vdash z = \dot{y} = \dot{f}(\dot{x}) = \dot{f}(\pi\dot{x}') = \pi\dot{y}'$.

(xv.) Therefore $t_0 \Vdash z \in \pi\dot{b}$.

This proves that $q \Vdash \dot{b} \subseteq \pi\dot{b}$ and the reverse inclusion is proven by a similar argument.

$\square$

Claim 4.3.4. $\mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{f}) \cap \mathrm{sym}(q) \subseteq \mathrm{resp}(\dot{b})$ *and thus* $\dot{b} \in \mathrm{M}^{\mathrm{HR}}$.

Proof of Claim. Fix $\pi \in \mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{f}) \cap \mathrm{sym}(q)$ and $t \in \mathbb{P}$. If $t \parallel q$ then, using the previous claim, for any $t' \leq t, q$ we have that $t' \Vdash \pi\dot{b} = \dot{b}$.

On the other hand, suppose that $t \perp q$. Then every element of $\dot{b}$ is of the form $\langle \dot{y}, s \rangle$ where s is in some $d_{\dot{x},r}$, which is predense below $q \wedge r$. But, by Remark 2.4.4, we can assume that every element of this set is below q, so $t \perp s$ and thus $t \Vdash \dot{b} = \emptyset$. By the same argument $t \perp \pi s$ since πs is also below q. Therefore $t \Vdash \dot{b} = \emptyset = \pi\dot{b}$.

Hence we have that $\mathbb{1} \Vdash \pi\dot{b} = \dot{b}$ so π is in $\mathrm{resp}(\dot{b})$ as required. □

Separation: Let a and Γ be in $\mathcal{N}$ and $\varphi(u, v)$ a formula. We seek a name for $\{x \in a \mid \varphi(x, \Gamma)\}$. To do this, fix names $\dot{a}$ and $\dot{\Gamma}$ for a and Γ. For each $\langle \dot{x}, r \rangle \in \dot{a}$ let

$$D_{\dot{x},r} := \{s \leq r \mid s \Vdash \varphi(\dot{x}, \dot{\Gamma})\}.$$

Then, by pretameness and genericity, we can fix some tenacious condition $p \in G$ and $\langle d_{\dot{x},r} \mid \langle \dot{x}, r \rangle \in \dot{a} \rangle \in \mathrm{M}$ such that each $d_{\dot{x},r}$ is predense below $p \wedge r$. Next, take

$$\dot{b} := \{\langle \dot{x}, s \rangle \mid \exists r \ (\langle \dot{x}, r \rangle \in \dot{a} \ \wedge \ s \in d_{\dot{x},r})\}.$$

Then $\dot{b}$ is a name for $\{x \in a \mid \varphi(x, \Gamma)\}$ so we just need to prove that $\dot{b} \in \mathrm{M}^{\mathrm{HR}}$.

Claim 4.3.5. *For any* $\pi \in \mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{\Gamma}) \cap \mathrm{sym}(p)$, $p \Vdash \dot{b} = \pi\dot{b}$.

Proof of Claim.

(i.) Fix $\pi \in \mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{\Gamma}) \cap \mathrm{sym}(p)$, $z \in \mathrm{dom}(\dot{b})$, $t_0 \leq p$ and suppose that $t_0 \Vdash z \in \dot{b}$.

(ii.) Let $t_1 \leq t_0$ be arbitrary.

(iii.) Take $t_2 \leq t_1$ and $\langle \dot{x}, s \rangle \in \dot{b}$ such that $t_2 \leq s$ and $t_2 \Vdash z = \dot{x}$.

(iv.) Fix r such that $\langle \dot{x}, r \rangle \in \dot{a}$ and $s \in d_{\dot{x},r}$.

(v.) Since $t_2 \leq s \leq r$, $t_2 \Vdash \dot{x} \in \dot{a} = \pi\dot{a}$.

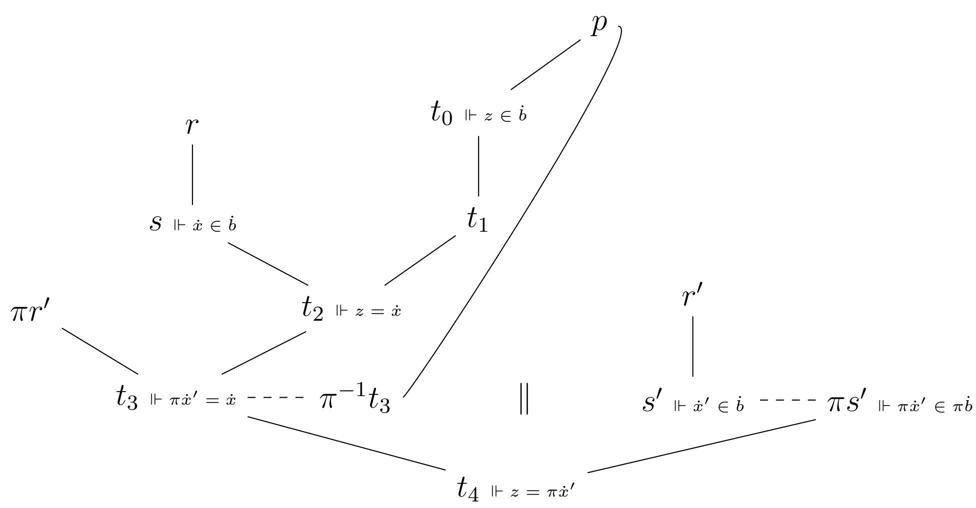


Figure 4.2: Tree of Conditions for Separation

(vi.) So fix $t_3 \leq t_2$ and $\langle \pi\dot{x}', \pi r' \rangle \in \pi\dot{a}$ such that $t_3 \leq \pi r'$ and $t_3 \Vdash \pi\dot{x}' = \dot{x}$.

(vii.) Since $t_3 \leq p$ and π fixes p, $\pi^{-1}t_3 \leq p$ so, by predensity, fix $s' \in d_{\dot{x},'r'}$ such that s' and $\pi^{-1}t_3$ are compatible. Then $\langle \dot{x}', s' \rangle \in \dot{b}$.

(viii.) Take $t_4 \leq t_3, \pi s'$.

(ix.) Since $t_4 \leq \pi s'$ and $\langle \pi\dot{x}', \pi s' \rangle \in \pi\dot{b}$, $t_4 \Vdash \pi\dot{x}' \in \pi\dot{b}$.

(x.) Since $t_4 \leq t_3 \leq t_2$, $t_4 \Vdash \pi\dot{x}' = \dot{x} = z$.

(xi.) Therefore $t_0 \Vdash z \in \pi\dot{b}$.

The implication that for any $z \in \mathrm{dom}(\pi\dot{b})$ and $t_0 \leq p$ if $t_0 \Vdash z \in \pi\dot{b}$ then $t_0 \Vdash z \in \dot{b}$ is proven in a similar way.

□

Hence, by the same argument we used in the argument for Replacement, $\mathbb{1} \Vdash \pi\dot{b} = \dot{b}$.

Class Comprehension: Note that $\dot{\Lambda} = \{\langle \dot{x}, p \rangle \mid p \Vdash \varphi(\dot{x}, \dot{\Gamma})\} \in \mathcal{C}_1^{\mathrm{HR}}$ is a class name for the class $\{x \mid \varphi(x, \dot{\Gamma}^G)\}$. Then for any $\pi \in \mathrm{resp}(\dot{\Gamma})$, if $\langle \dot{x}, p \rangle \in \dot{\Lambda}$ then $\pi p \Vdash \varphi(\pi\dot{x}, \dot{\Gamma})$ so $\langle \pi\dot{x}, \pi p \rangle \in \dot{\Lambda}$, which shows that $\mathrm{resp}(\dot{\Lambda}) \supseteq \mathrm{resp}(\dot{\Gamma})$.

□ *Theorem* 4.3.2

Remark 4.3.6. The reason why the Respected Model of a pretame symmetric system will in general not satisfy Collection is because the argument for the preservation of Replacement made essential use of the fact that f was a function. This meant that when we forced $\pi\dot{y}'$ to be a name for $\dot{f}(\pi\dot{x})$ it was then forced to be equal to $\dot{y}$ for our specified $\dot{y}$ in $\dot{b}$. Without the function assumption, this implication irreparably breaks down. For example, in any of our model where Collection failed in the Symmetric Model it will also fail in the Respected Model.

We further call a pretame class forcing *tame* if it also preserves the Power Set. This notion was defined by Stanley and is studied in [Fri00] where a combinatorial definition can be found. We shall see that if the class forcing poset is tame then the Respected Model preserves full KM. It is worthwhile to note that this theorem will not involve a tenacity assumption because we will only need to prove that $(\mathrm{V}_\alpha)^{\mathcal{N}}$ is a set for every α and this will indeed be forced by $\mathbb{1}$ which is already fixed by any permutation.

Theorem 4.3.7. *Working over* $\mathrm{KM}^-_{(4)}$*, suppose that* $\mathbb{P}$ *is a tame class forcing. Then* $\mathcal{N}$ *is a model of* KM*.*

Proof. It will suffice to show that for any ordinal α, $\mathrm{N}_\alpha \coloneqq \{x \in \mathrm{N} \mid \mathrm{rank}^{\mathrm{N}}(x) < \alpha\}$ is a set. Then, since N is closed under Gödel operations by Theorem 4.3.1, $\mathcal{N}$ will be almost universal in $\mathcal{M}[G]$ and, since Class Comprehension holds in $\mathcal{N}$ by the proof in the previous theorem, therefore a model of KM.

Fix α to be an ordinal. Since $\mathbb{P}$ is tame, the standard cumulative hierarchy exists in $\mathcal{M}[G]$. Therefore, by definition of the forcing relation,

$$\mathbb{1} \Vdash \exists u \, \forall x \, (\mathrm{rank}(x) < \alpha \to x \in u).$$

Let $\mathcal{E} = \{\dot{x} \in \mathrm{M}^{\mathrm{HR}} \mid \mathbb{1} \Vdash \mathrm{rank}(x) < \alpha\}$ and let $E_{\dot{x}} \coloneqq \{\dot{y} \in \mathcal{E} \mid \mathbb{1} \Vdash \dot{x} = \dot{y}\}$ be the equivalence classes modulo forcing equality by $\mathbb{1}$. Now, since $\mathbb{1}$ forces there to be only set many distinct sets of rank at most α, there can only be set many equivalence classes. So, by Collection in $\mathcal{M}$, fix $I \in \mathrm{M}$ such that for any $\dot{y} \in \mathcal{E}$ there is an $\dot{x} \in I$ such that $\dot{y} \in E_{\dot{x}}$. Then

$$\mathbb{1} \Vdash \forall y\ (\mathrm{rank}(y) < \alpha \to y \in I^\bullet).$$

Therefore $I^\bullet \subseteq \mathrm{M}^{\mathrm{HR}}$ is a name for $(\mathrm{V}_\alpha)^{\mathcal{N}}$ so it remains to prove that this name is respected. This is done by proving that for any $\pi \in \mathcal{G}$, $\mathbb{1} \Vdash \pi I^\bullet = I^\bullet$.

To see this, suppose that $t_0 \Vdash z \in I^\bullet$ and fix $t_1 \leq t_0$. We shall find some condition p below t_1 and name $\dot{x}' \in I^\bullet$ such that $p \Vdash z = \pi\dot{x}'$. To do this, take $t_2 \leq t_1$ and $\dot{x} \in \mathrm{I}$ such that $t_2 \Vdash z = \dot{x}$. Since $t_2 \Vdash \mathrm{rank}(\dot{x}) < \check{\alpha}$, we have that $\pi^{-1}t_2 \Vdash \mathrm{rank}(\pi^{-1}\dot{x}) < \check{\alpha}$ and so $\pi^{-1}t_2 \Vdash \pi^{-1}\dot{x} \in I^\bullet$. Hence we can take some $t_3 \leq \pi^{-1}t_2$ and $\dot{x}' \in I^\bullet$ such that $t_3 \Vdash \pi^{-1}\dot{x} = \dot{x}'$ which implies that $\pi t_3 \Vdash \dot{x} = \pi\dot{x}'$. So, since $\pi t_3 \leq t_2$, $\pi t_3 \Vdash z = \pi\dot{x}'$ and thus $\mathbb{1} \Vdash I^\bullet \subseteq \pi I^\bullet$. The reverse implication is done in the same way, as per usual. □

4.4 Symmetric versus Respected

To end this chapter, we give a sufficient condition which implies that the Symmetric Model is equal to the Respected Model. This requirement is that $\{\pi\dot{x} \mid \pi \in H\}$ should form a set for any H in the filter and $\mathbb{P}$-name $\dot{x}$. This proof will not require the tenacity assumption we needed for the pretame case.

Proposition 4.4.1. *Suppose that $\mathcal{M}$ is a model of $\mathrm{KM}^-_{(4)}$. Let $\mathbb{P}$ be a pretame class forcing and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a symmetric system. Suppose further that for any $\dot{x} \in \mathrm{M}^{\mathrm{HR}}$ and any $H \in \mathcal{F}$, $\{\pi\dot{x} \mid \pi \in H\} \in \mathrm{M}$. Then a $\mathbb{P}$-name $\dot{x}$ is in M^{HR} iff there is some name $\dot{y}$ such that*

- $\mathbb{1} \Vdash \dot{x} = \dot{y}$ *and*
- $\{\pi \in \mathcal{G} \mid \pi\dot{y} = \dot{y}\} \in \mathcal{F}$.

Proof. Let $\dot{x} \in \mathrm{M}^{\mathrm{HR}}$. Our desired name shall be

$$\dot{y} := \bigcup \{\pi\dot{x} \mid \pi \in \mathrm{resp}(\dot{x})\}.$$

First note that for any $\sigma \in \mathrm{resp}(\dot{x})$, $\sigma\dot{y} = \dot{y}$ and therefore $\mathrm{resp}(\dot{x}) \subseteq \mathrm{resp}(\dot{y})$ so $\mathrm{resp}(\dot{y})$ is indeed in $\mathcal{F}$ and $\dot{y}$ is hereditarily-respected. We shall now prove that these names are forced to be equal, noting that $\dot{x} \subseteq \dot{y}$. So suppose that $t_0 \Vdash z \in \dot{y}$ and let $t_1 \leq t_0$ be arbitrary. Then we can fix some $t_2 \leq t_1$, $\pi \in \operatorname{resp}(\dot{x})$ and $\langle \dot{u}, s\rangle \in \pi\dot{x}$ such that $t_2 \leq s$ and

$$t_2 \Vdash z = \dot{u} \in \pi\dot{x} = \dot{x}$$

which completes the proof. □

Using this proposition we can show that, under the above assumptions, the Collection Scheme will also hold in the Respected Model.

Theorem 4.4.2. *Suppose that $\mathcal{M}$ is a model of $\mathrm{KM}^-_{(4)}$. Let $\mathbb{P}$ be a pretame class forcing and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ a symmetric system. Suppose further that*

- *for any $\dot{x} \in \mathrm{M}^{\mathrm{HR}}$ and any $H \in \mathcal{F}$, $\{\pi\dot{x} \mid \pi \in H\} \in \mathrm{M}$,*
- *for any $p \in \mathbb{P}$ and $H \in \mathcal{F}$, $\{\pi p \mid \pi \in H\} \in \mathrm{M}$.*

Then $\mathcal{N}$ satisfies second-order Collection.

Proof. Suppose that

$$\mathcal{N} \models \forall x \in a\ \exists y\ \varphi(x, y, \Gamma)$$

where Γ is some class parameter. Using the previous proposition, fix names $\dot{a}$ and $\dot{\Gamma}$ for a and Γ such that $\{\pi \in \mathcal{G} \mid \pi\dot{a} = \dot{a}\}$ and $\{\pi \in \mathcal{G} \mid \pi\dot{\Gamma} = \dot{\Gamma}\}$ are in $\mathcal{F}$. We shall refer to these classes as $\operatorname{sym}(\dot{a})$ and $\operatorname{sym}(\dot{\Gamma})$. Now, take $p \in G$ such that

$$p \Vdash \forall x \in \dot{a}\ \exists y\ \varphi(x, y, \dot{\Gamma}).$$

As we did when proving Replacement earlier, for each $\langle \dot{x}, r\rangle$ in $\dot{a}$ let

$$D_{\dot{x},r} := \{s \in \mathbb{P} \mid s \leq p, r\ \wedge\ \exists \dot{y} \in \mathrm{M}^{\mathrm{HR}}\ (s \Vdash \varphi(\dot{x}, \dot{y}, \dot{\Gamma}))\}$$

and fix $q \in G$ and $\langle d_{\dot{x},r} \mid \langle \dot{x}, r\rangle \in \dot{a}\rangle \in \mathrm{M}$ such that $q \leq p$ and each $d_{\dot{x},r} \subseteq D_{\dot{x},r}$ is predense below q.

Now let $e_{\dot{x},r} \coloneqq \{\pi s \mid s \in d_{\dot{x},r}, \pi \in \operatorname{sym}(\dot{a}) \cap \operatorname{sym}(\dot{\Gamma})\}$ and let

$$E \coloneqq \bigcup \{e_{\dot{x},r} \mid \langle \dot{x}, r\rangle \in \dot{a}\},$$

noting that E is a set by our second assumption. Using Collection in $\mathcal{M}$, fix $c \in \mathrm{M}$ with $c \subseteq \mathrm{M}^{\mathrm{HR}}$ such that

$$\forall \langle \dot{x}, r\rangle \in \dot{a}\ \forall s \in d_{\dot{x},r}\ \exists \dot{y} \in c\ \varphi(\dot{x}, \dot{y}, \dot{\Gamma})$$

and let $\mathcal{K} \coloneqq \{\pi\dot{y} \mid \dot{y} \in c \ \wedge\ \pi \in \operatorname{sym}(\dot{a}) \cap \operatorname{sym}(\dot{\Gamma})\}$. We shall show that a name realising this instance of Collection is the set

$$\dot{b} \coloneqq \{\langle \dot{y}, s\rangle \mid \dot{y} \in \mathcal{K} \ \wedge\ s \in E \ \wedge\ \exists \langle \dot{x}, r\rangle \in \dot{a}\ (s \Vdash \varphi(\dot{x}, \dot{y}, \dot{\Gamma}))\}.$$

Since $\dot{b}$ contains the standard name for the instance of Collection in the full extension, where we take $\dot{y} \in c$ and $s \in d_{\dot{x},r}$, if this name is respected then we will have that

$$\mathcal{N} \models \forall x \in a\ \exists y \in \dot{b}^G\ \varphi(x, y, \Gamma).$$

So let $\langle \dot{y}, s\rangle$ be in $\dot{b}$, π be in $\operatorname{sym}(\dot{a}) \cap \operatorname{sym}(\dot{\Gamma})$ and fix $\langle \dot{x}, r\rangle \in \dot{a}$ such that $s \Vdash \varphi(\dot{x}, \dot{y}, \dot{\Gamma})$. Then $\pi s \Vdash \varphi(\pi\dot{x}, \pi\dot{y}, \dot{\Gamma})$ and $\langle \pi\dot{x}, \pi r\rangle \in \dot{a}$. Moreover, since $\mathcal{K}$ and E are closed under elements of $\operatorname{sym}(\dot{a}) \cap \operatorname{sym}(\dot{\Gamma})$, $\pi\dot{y} \in \mathcal{K}$ and $\pi s \in E$ so $\langle \pi\dot{y}, \pi s\rangle$ is indeed in $\dot{b}$. Therefore $\pi\dot{b} = \dot{b}$ and thus $\operatorname{resp}(\dot{b}) \supseteq \operatorname{sym}(\dot{a}) \cap \operatorname{sym}(\dot{\Gamma})$.

□

Chapter 5

Constructing the Constructible Universe Constructively

5.1 Introduction

The Constructible Universe was developed by Gödel in two influential papers, [Göd39] and [Göd40], in the late 1930s in order to prove the consistency of the Axiom of Choice and the Generalised Continuum Hypothesis with ZF. The constructible universe, denoted by L, is constructed by transfinite induction as $\bigcup_\alpha \mathrm{L}_\alpha$ and there are three main ways to define $\mathrm{L}_{\alpha+1}$, all of which can be undertaken in KP:

1. Syntactically; using the notion of a *"definability operator"* so that $\mathrm{L}_{\alpha+1}$ is the collection of definable subsets of L_α. This is the original approach taken by Gödel in [Göd39] and was formalised in KP by Devlin in [Dev84].
2. By closure under what Barwise calls *"fundamental operations"* or *"Gödel operations"*. This is the approach taken by Gödel in [Göd40] and further studied by Barwise in [Bar17] where he considered the theory KP with urelements.
3. Using *"rudimentary"* functions. This is a modified version of using fundamental operations which was developed by Jensen and further explored by Mathias, leading to his weak system of Provi, the weakest known system in which one can do both set forcing and build L. The details of this theory can be found in [MB15].

The intuitionistic approach to constructing L was first undertaken by Lubarsky in [Lub93] under the assumption that V satisfied IZF. His approach was to show that the syntactic definition of the constructible universe still goes through in intuitionistic logic, with some minor modifications. The main obstacle one has to overcome is that the ordinals are no longer linearly ordered so one has to be more careful as to how one finds witnesses for the collection of definable subsets of some given set X. It also adds complications to proving the Axiom of Constructibility, that is proving that V = L holds in L. Because it is unclear as to why L should contain every ordinal under IZF, there is no reason to assume that

$$\bigcup_{\alpha \in \mathrm{Ord} \cap \mathrm{V}} \mathrm{L}_\alpha = \bigcup_{\alpha \in \mathrm{Ord} \cap \mathrm{L}} \mathrm{L}_\alpha.$$

In order to circumvent this issue, Lubarsky proves the following lemma, under IZF, which we shall reprove later in our weaker context.

Lemma 5.3.8 (Lubarsky)**.** *For every ordinal* α *in* V *there is an ordinal* α^* *in* L *such that* $\mathrm{L}_\alpha = \mathrm{L}_{\alpha^*}$.

The syntactic approach has been further studied by Crosilla, and appears in the appendix to her PhD thesis, [Cro00]. Here she shows that the construction can be carried out in a fragment of constructive set theory, which is equivalent to what we have defined as IKP, by essentially the same proof as found in [Dev84].

The third approach via rudimentary functions has also been explored in constructive contexts by Aczel [Acz13]. Here he defines the weak system of *Rudimentary Constructive Set Theory* and shows that many of Jensen's techniques can be applied in this theory.

In this chapter, we shall be interested in which axioms are sufficient to construct the constructible universe. Because the syntactic approach requires essential use of ω in order to work with arbitrarily long finite sequences, it is not the appropriate method to use in IKP without infinity. Therefore, we shall adapt the second approach and use the fundamental operations. Adapting Barwise's method, we shall show that if one expands the collection of fundamental operations, then one can indeed construct L over the weak system of IKP without infinity. It should be noted that one could also consider urelements, as Barwise does. We have chosen not to undertake this study but this could be done without a significant amount of additional work.

Therefore, the majority of this chapter is just a reproduction of Chapter II of [Bar17] where we have just needed to consider some additional logical operations. These are the operations of conjunction, implication and universal quantification which are not treated in the classical case as distinct cases due to their equivalent definitions using disjunction, negation and existential quantification.

Finally, because we are interested in which axioms are necessary to construct the constructible universe, in this chapter we shall take care to differentiate between IKP *without infinity*, which we will call $\mathrm{IKP}^{-\mathit{Inf}}$, and IKP. We shall therefore regularly refer to IKP as $\mathrm{IKP}^{-\mathit{Inf}} + $ *Strong Infinity* just to make it clear when Strong Infinity is being assumed.

5.2 The Fundamental Operations

Definition 5.2.1. For x an ordered pair, y a set of ordered pairs and z a set, define

- $1^{st}(x) = a$ iff $\exists u \in x \; \exists b \in u \; (x = \langle a, b \rangle)$,
- $2^{nd}(x) = b$ iff $\exists u \in x \; \exists a \in u \; (x = \langle a, b \rangle)$,
- $y``\{z\} := \{u \mid \langle z, u \rangle \in y\}$.

We start by defining the Σ-operations, each of which will comprise of two arguments, which we will use to generate the constructible sets. These are the same as in [Bar17] except for the addition of $\mathcal{F}_{\rightarrow}$ and $\mathcal{F}_{\forall}$. Note that Barwise deduces $\mathcal{F}_{\cap}$ by the classical identity

$$x \cap y = x \setminus (x \setminus y)$$

but this equivalence does not hold intuitionistically.

Definition 5.2.2. The *fundamental operations* are as follows:

($\mathcal{F}_p$) $\mathcal{F}_p(x,y) \coloneqq \{x,y\}$,

($\mathcal{F}_\cap$) $\mathcal{F}_\cap(x,y) \coloneqq x \cap \bigcap y$

($\mathcal{F}_\cup$) $\mathcal{F}_\cup(x,y) \coloneqq \bigcup x$,

($\mathcal{F}_\setminus$) $\mathcal{F}_\setminus(x,y) \coloneqq x \setminus y$,

($\mathcal{F}_\times$) $\mathcal{F}_\times(x,y) \coloneqq x \times y$,

($\mathcal{F}_\to$) $\mathcal{F}_\to(x,y) \coloneqq x \cap \{z \in 2^{nd}(y) \mid y \text{ is an ordered pair } \wedge \ z \in 1^{st}(y)\}$,

($\mathcal{F}_\forall$) $\mathcal{F}_\forall(x,y) \coloneqq \{x``\{z\} \mid z \in y\}$,

($\mathcal{F}_d$) $\mathcal{F}_d(x,y) \coloneqq \mathrm{dom}(x) = \{1^{st}(z) \mid z \in x \ \wedge \ z \text{ is an ordered pair}\}$,

($\mathcal{F}_r$) $\mathcal{F}_r(x,y) \coloneqq \mathrm{ran}(x) = \{2^{nd}(z) \mid z \in x \ \wedge \ z \text{ is an ordered pair}\}$,

($\mathcal{F}_{123}$) $\mathcal{F}_{123}(x,y) \coloneqq \{\langle u,v,w\rangle \mid \langle u,v\rangle \in x \ \wedge \ w \in y\}$,

($\mathcal{F}_{132}$) $\mathcal{F}_{132}(x,y) \coloneqq \{\langle u,w,v\rangle \mid \langle u,v\rangle \in x \ \wedge \ w \in y\}$,

($\mathcal{F}_=$) $\mathcal{F}_=(x,y) \coloneqq \{\langle v,u\rangle \in y \times x \mid u = v\}$,

($\mathcal{F}_\in$) $\mathcal{F}_\in(x,y) \coloneqq \{\langle v,u\rangle \in y \times x \mid u \in v\}$.

Remark 5.2.3. In order to simplify later notation, we shall let $\mathcal{I}$ be the obvious finite set indexing the above operations.

Note that we form n-tuples inductively as

$$\langle x_3, x_2, x_1\rangle \coloneqq \langle x_3, \langle x_2, x_1\rangle\rangle$$

and therefore $\mathrm{ran}(\{\langle x_3, x_2, x_1\rangle\}) = \{\langle x_2, x_1\rangle\}$. The next lemma is adapted from Lemma II.6.1 of [Bar17] which will give us that any instance of Bounded Separation can be written as a sequence of fundamental operations.

Lemma 5.2.4. *For every Σ_0-formula $\varphi(x_1, \ldots, x_n)$ with free variables among $x_1, \ldots, x_n$, there is a term $\mathcal{F}_\varphi$ built up from the operations in 5.2.2 such that*

$$\mathrm{IKP}^{-\mathit{Inf}} \vdash \mathcal{F}_\varphi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 \mid \varphi(x_1, \ldots, x_n)\}.$$

Proof. As in Barwise, we will call a formula $\varphi(x_1, \ldots, x_n)$ a *termed-formula*, or *t-formula*, if there is a term $\mathcal{F}_\varphi$ built from the fundamental operations such that the conclusion of the lemma holds. We shall then proceed by induction on Σ_0-formulae to show that every such formula is a t-formula. Using the proof of [Bar17], Lemma II.6.1, it only remains to consider the following cases:

(i.) If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are t-formulae then so is $\varphi(x_1, \ldots, x_n) \wedge \psi(x_1, \ldots, x_n)$.

(ii.) If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are t-formulae then so is $\varphi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)$.

(iii.) If $\psi(x_1, \ldots, x_{n+1})$ is a t-formula and $\varphi(x_1, \ldots, x_n, b)$ is $\forall x_{n+1} \in b\ \psi(x_1, \ldots, x_{n+1})$, where b is an arbitrary set that does not appear in $\{x_1, \ldots, x_n\}$, then φ is a t-formula.

(iv.) If $\psi(x_1, \ldots, x_{n+1})$ is a t-formula and $\varphi(x_1, \ldots, x_n)$ is $\forall x_{n+1} \in x_j\ \psi(x_1, \ldots, x_{n+1})$, then φ is a t-formula.

Case (i): Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ witness that φ and ψ are t-formulae. First note that for a set z, $\mathcal{F}_p(z, z) = \{z\}$. Then,

$$\begin{aligned}\mathcal{F}_\cap\Big(\mathcal{F}_\varphi(a_1, \ldots, a_n), \mathcal{F}_p\Big(\mathcal{F}_\psi(a_1, \ldots, a_n), \mathcal{F}_\psi(a_1, \ldots, a_n)\Big)\Big)& \\ = \mathcal{F}_\varphi(a_1, \ldots, a_n) \cap \bigcap\{\mathcal{F}_\psi(a_1, \ldots, a_n)\}& \\ = \mathcal{F}_\varphi(a_1, \ldots, a_n) \cap \mathcal{F}_\psi(a_1, \ldots, a_n).&\end{aligned}$$

Therefore, we can define $\mathcal{F}_{\varphi\wedge\psi}(a_1, \ldots, a_n)$ as

$$\mathcal{F}_\cap\Big(\mathcal{F}_\varphi(a_1, \ldots, a_n), \mathcal{F}_p\Big(\mathcal{F}_\psi(a_1, \ldots, a_n), \mathcal{F}_\psi(a_1, \ldots, a_n)\Big)\Big).$$

Case (ii): Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ witness that φ and ψ are t-formulae. For this one, note that

$$\{\langle x_n \ldots, x_1\rangle \in a_n \times \ldots \times a_1 \mid \varphi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)\}$$
$$= (a_n \times \ldots \times a_1) \cap \{z \in \mathcal{F}_\psi(a_1, \ldots, a_n) \mid z \in \mathcal{F}_\varphi(a_1, \ldots, a_n)\}.$$

Also,

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\} = \mathcal{F}_p\Big(\mathcal{F}_p(x, x), \mathcal{F}_p(x, y)\Big)$$

and $a_n \times \ldots \times a_1$ can be defined by repeated used of $\mathcal{F}_\times$ so we can use these constructions. Thus, the above can be expressed as

$$\mathcal{F}_\to\Bigg(a_n \times \ldots \times a_1,\ \Big\langle \mathcal{F}_\varphi(a_1, \ldots, a_n), \mathcal{F}_\psi(a_1, \ldots, a_n)\Big\rangle\Bigg)$$

Giving the required construction of $\mathcal{F}_{\varphi\to\psi}$.

Case (iii): Let $\varphi(x_1, \ldots, x_n, b) \equiv \forall x_{n+1} \in b\ \psi(x_1, \ldots, x_{n+1})$ and let $\mathcal{F}_\psi$ witness that ψ is a t-formula. Then

$$\begin{aligned}
\mathcal{F}_\forall\Big(\mathcal{F}_\psi(a_1, \ldots, a_n, b), b\Big) &= \{\mathcal{F}_\psi(a_1, \ldots, a_n, b)``\{z\} \mid z \in b\} \\
&= \Big\{\Big\{w \mid \langle z, w\rangle \in \mathcal{F}_\psi(a_1, \ldots, a_n, b)\Big\} \mid z \in b\Big\} \\
&= \Big\{\Big\{\langle x_n, \ldots x_1\rangle \mid \langle z, x_n, \ldots, x_1\rangle \in \mathcal{F}_\psi(a_1, \ldots, a_n, b)\Big\} \mid z \in b\Big\} \\
&= \Big\{\operatorname{ran}(\mathcal{F}_\psi(a_1, \ldots, a_n, \{z\})) \mid z \in b\Big\}.
\end{aligned}$$

Therefore $\mathcal{F}_\varphi(a_1, \ldots, a_n, b)$ can be expressed as

$$\Big\{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 \mid \forall x_{n+1} \in b\ \psi(x_1, \ldots x_n)\Big\}$$

$$= (a_n \times \ldots \times a_1) \cap \Big\{w \mid \forall x_{n+1} \in b\ \ \langle x_{n+1}, w\rangle \in \mathcal{F}_\psi\Big(a_1, \ldots, a_n, \{x_{n+1}\}\Big)\Big\}$$

$$= (a_n \times \ldots \times a_1) \cap \bigcap\Big\{\operatorname{ran}(\mathcal{F}_\psi(a_1, \ldots, a_n, \{x_{n+1}\})) \mid x_{n+1} \in b\Big\}$$

$$= \mathcal{F}_\cap\Bigg(a_n \times \ldots \times a_1,\ \mathcal{F}_\forall\Big(\mathcal{F}_\psi(a_1, \ldots, a_n, b), b\Big)\Bigg).$$

Case (iv): Let $\varphi(x_1, \ldots, x_n) \equiv \forall x_{n+1} \in x_j \ \psi(x_1, \ldots, x_{n+1})$. Then

$$\{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 \mid \forall x_{n+1} \in x_j \ \psi(x_1, \ldots, x_{n+1})\}$$

is equal to the following set;

$$\{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 \mid \forall x_{n+1} \in \bigcup a_j \ (x_{n+1} \in x_j \to \psi(x_1, \ldots, x_{n+1}))\}.$$

So, taking $\vartheta(x_1, \ldots, x_n, \bigcup a_j) \equiv \forall x_{n+1} \in \bigcup a_j \ (x_{n+1} \in x_j \to \psi(x_1, \ldots, x_{n+1}))$, φ is a t-formula by cases (ii) and (iii) and the fact that if two formulae are provably equivalent in IKP and one is a t-formula then so is the other [1]. □

Theorem 5.2.5. *For any Σ_0-formula $\varphi(x_1, \ldots, x_n)$ with free variables among $x_1, \ldots x_n$, there is a term $\mathcal{F}_\varphi$ of n arguments built from the operations defined in 5.2.2 such that:*

$$\mathrm{IKP}^{\textit{-Inf}} \vdash \mathcal{F}_\varphi(a, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \{x_i \in a \mid \varphi(x_1, \ldots, x_n)\}.$$

Proof. This follows easily from our lemma since if $\mathcal{F}_\varphi$ is the term built in the previous lemma such that

$$\mathrm{IKP}^{\textit{-Inf}} \vdash \mathcal{F}_\varphi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 \mid \varphi(x_1, \ldots, x_n)\}$$

then our required set can be built from $\mathcal{F}_\varphi(\{x_1\}, \ldots, \{x_{i-1}\}, a_i, \{x_{i+1}\}, \ldots \{x_n\})$ by using $\mathcal{F}_r$ $n - i$ times and then $\mathcal{F}_d$. □

5.3 Defining Definability

In this section we shall define a *definability operator*. The idea being that the definable subsets of b are those sets which can be constructed from b using the fundamental operations. We shall then discuss some of the basic properties one can deduce from this definition and show that the model which one constructs satisfies $\mathrm{IKP}^{\textit{-Inf}}$. To conclude, we will end this section by mentioning other definability operators.

[1]This is statement (b) in the proof of Lemma II.6.1 of [Bar17]

Definition 5.3.1. For a set b:

- $\mathcal{E}(b) := b \cup \{\mathcal{F}_i(x, y) \mid x, y \in b \ \wedge \ i \in \mathcal{I}\}$,
- $\mathcal{D}(b) := \mathcal{E}(b \cup \{b\})$.

The following proposition is then provable using Σ-Collection and the Axiom of Unions in $\mathrm{IKP}^{\text{-}Inf}$.

Proposition 5.3.2. $\mathrm{IKP}^{\text{-}Inf} \vdash \forall b \, \exists x \ (x = \mathcal{E}(b))$.

We would now want to define another operation $\mathrm{Def}(b)$ to be the closure of b under our fundamental operations, that is

$$\mathrm{Def}(b) := \bigcup_{n \in \omega} \mathcal{D}^n(b).$$

This would have the added benefit that if we defined $\mathrm{L}_\alpha := \bigcup_{\beta \in \alpha} \mathrm{Def}(\mathrm{L}_\beta)$, then for each ordinal α, L_α would be transitive. However, this definition requires the Axiom of Infinity, which we are not initially assuming. Therefore, to begin with, we will just use $\mathcal{D}$ to define our universe and use the different script $\mathbb{L}$ to differentiate between the two notions. The relationship between $\mathbb{L}$ and L in the presence of Strong Infinity will be discussed in Lemma 5.3.13.

Definition 5.3.3. For α an ordinal, $\mathbb{L}_\alpha := \bigcup_{\beta \in \alpha} \mathcal{D}(\mathbb{L}_\beta)$ and $\mathbb{L} := \bigcup_{\alpha \in \mathrm{ORD}} \mathbb{L}_\alpha$.

Remark 5.3.4. It is worth mentioning that the hierarchy as defined here does not look like the "standard" one obtained through the syntactic approach. Firstly, $\mathcal{D}(b)$ does not close under the fundamental operations, which is an approach we have chosen to take due to a lack of Strong Infinity in the background universe. Secondly the $\mathbb{L}$ hierarchy does not stratify nicely by rank because $\mathcal{D}(b)$ may potentially add sets that are not subsets of b, for example $x \times y$, and the ordinals of $\mathbb{L}_\alpha$ may not be α, for example we will have that $n \in \mathbb{L}_{2n+1}$. This seems to be a useful deficiency in our weak context because it simplifies some of the proofs and is the approach taken by Barwise. However, we will address alternatives to this approach in Lemma 5.3.13 and Theorem 5.3.16.

We start the analysis by noting some of the basic properties of $\mathbb{L}_\alpha$:

Lemma 5.3.5. (IKP^{-Inf}) *For all ordinals α, β:*

1. *If $\beta \in \alpha$ then $\mathbb{L}_\beta \subseteq \mathbb{L}_\alpha$,*
2. *If $\beta \subseteq \alpha$ then $\mathbb{L}_\beta \subseteq \mathbb{L}_\alpha$,*
3. *$\mathbb{L}_\alpha \in \mathbb{L}_{\alpha+1}$,*
4. *If $x, y \in \mathbb{L}_\alpha$ then for any $i \in I$, $\mathcal{F}_i(x, y) \in \mathbb{L}_{\alpha+1}$,*
5. *If for all $\beta \in \alpha$, $\beta + 1 \in \alpha$ then $\mathbb{L}_\alpha$ is transitive,*
6. *$\mathbb{L}$ is transitive.*

Theorem 5.3.6. *For every axiom of* IKP^{-Inf}, $\mathrm{IKP}^{-Inf} \vdash \varphi^{\mathbb{L}}$. *Moreover,*

$$\mathrm{IKP}^{-Inf} + \text{“Strong Infinity”} \vdash (\text{Strong Infinity})^{\mathbb{L}}.$$

Proof. The axioms of Extensionality and $\in$-induction follow from the fact that $\mathbb{L}$ is a transitive class. Pairing follows from $\mathcal{F}_p$ and Unions from $\mathcal{F}_\cup$. The Axiom of Empty Set holds because

$$\emptyset = \{y \mid y \neq y\} = \mathbb{L}_0 \in \mathbb{L}_1.$$

Bounded Separation follows from Theorem 5.2.5. Therefore it remains to prove Bounded Collection.

Suppose that $\varphi(x, y, z)$ is a Σ_0-formula and, working in IKP^{-Inf}, assume that $a, z \in \mathbb{L}$ and $\forall x \in a\ \exists y \in \mathbb{L}\ (\varphi(x, y, z))^{\mathbb{L}}$. Since bounded formulae are absolute between transitive models, we get that

$$\forall x \in a\ \exists \alpha\ (\exists y \in \mathbb{L}_\alpha\ \varphi^{\mathbb{L}}(x, y, z)).$$

Using the Σ-Collection principle, there is a β such that

$$\forall x \in a\ \exists \alpha \in \beta\ (\exists y \in \mathbb{L}_\alpha\ \varphi^{\mathbb{L}}(x, y, z))$$

which, by property 1 of Lemma 5.3.5, yields that $\forall x \in a\ \exists y \in \mathbb{L}_\beta\ \varphi^{\mathbb{L}}(x, y, z)$. So, setting $b = \mathbb{L}_\beta$ and again using absoluteness, we get that

$$(\forall x \in a\ \exists y \in b\ \varphi(x, y, z))^{\mathbb{L}}.$$

Proving this instance of Bounded Collection.

Now we work in the theory IKP^{-Inf}+ "*Strong Infinity*". We first prove that $\omega \subseteq \mathbb{L}_\omega$. This is done by induction by showing that for all $n \in \omega$, $n \in \mathbb{L}_{2n+1}$. Note that this holds because if $n \in \mathbb{L}_{2n+1}$ then

$$n + 1 = n \cup \{n\} = \mathcal{F}_\cup\Big(n, \mathcal{F}_p(n, n)\Big) \in \mathbb{L}_{2n+1+2}.$$

Then $\omega = \{n \in \mathbb{L}_\omega \mid n = \emptyset \ \vee \ \exists m \in n \ (n = m \cup \{m\})\}$ which will be in $\mathbb{L}$ by Bounded Separation. □

It is worth noting here that we will frequently claim that sets of the form

$$\{z \in \mathbb{L}_\delta \mid \varphi(u, z)\}$$

are in some $\mathbb{L}_{\delta+k}$ for some $k \in \omega$, where $u \in \mathbb{L}_\delta$ and φ is a Σ_0-formula, without computing the required k. This k could be computed by breaking down how φ was built up using the fundamental operations, however this is often an unnecessarily tedious computation. We will also often use formulae of the form $\exists i \in \mathcal{I}\ \varphi(i)$ despite $\mathcal{I}$ not technically being a formally defined set. Since $\mathcal{I}$ is finite this can either be circumvented by taking the obvious indexing or considering $\exists i \in \mathcal{I}\ \varphi(i)$ as an abbreviation for $\bigvee_{i\in\mathcal{I}} \varphi(i)$.

An important property of the constructible universe is the viability of the Axiom of Constructibility; the axiom asserting that $\mathrm{V} = \mathbb{L}$. We shall next prove that this axiom does indeed hold in $\mathbb{L}$, that is:

Theorem 5.3.7. IKP$^{-Inf} \vdash (\mathrm{V} = \mathbb{L})^{\mathbb{L}}$.

Our method of proving this will closely follow the corresponding proof in Lubarsky [Lub93]. In order to prove the theorem it suffices to prove the following lemma:

Lemma 5.3.8 (Lubarsky)**.** *For every ordinal α in* V *there is an ordinal α^* in $\mathbb{L}$ such that $\mathbb{L}_\alpha = \mathbb{L}_{\alpha^*}$.*

In order to do this we define the operation of *hereditary addition* on ordinals. This is necessary because in general it will not be true that $\beta \in \alpha$ implies that $\beta + 1 \in \alpha + 1$:

Definition 5.3.9 (Lubarsky)**.** For ordinals α and γ, hereditary addition is defined inductively on α as

$$\alpha +_H \gamma \coloneqq \left(\bigcup\{\beta +_H \gamma \mid \beta \in \alpha\} \cup \{\alpha\}\right) + \gamma$$

where "+" is the usual ordinal addition. We will also use the notation

$$(\alpha +_H \gamma)^- \coloneqq \left(\bigcup\{\beta +_H \gamma \mid \beta \in \alpha\} \cup \{\alpha\}\right).$$

Proof of Lemma 5.3.8. First note that by using the fundamental operations and Theorem 5.2.5, there is a fixed natural number k such that for any ordinals α and τ,

$$\{\gamma \in \mathbb{L}_\tau \mid \mathcal{D}(\mathbb{L}_\gamma) \subseteq \mathbb{L}_\alpha\} \in \mathbb{L}_{\tau+k}.$$

Therefore we can define α^* as the ordinal

$$\alpha^* \coloneqq \{\gamma \in \mathbb{L}_{(\alpha+_H k)^-} \mid \mathcal{D}(\mathbb{L}_\gamma) \subseteq \mathbb{L}_\alpha\} \in \mathbb{L}_{\alpha+_H k}.$$

Now clearly, for any ordinal $\alpha \in \mathrm{V}$, $\alpha^* \in \mathbb{L}$. We shall prove by induction that for every α, $\mathbb{L}_\alpha = \mathbb{L}_{\alpha^*}$. To this end, observe that if $x \in \mathbb{L}_{\alpha^*}$ then there is some $\gamma \in \alpha^*$ such that $x \in \mathcal{D}(\mathbb{L}_\gamma)$ which is subset of $\mathbb{L}_\alpha$ by construction, and thus $\mathbb{L}_{\alpha^*} \subseteq \mathbb{L}_\alpha$. For the reverse implication, we first prove the following claim:

Claim 5.3.10. *For all $\beta \in \alpha$, $\beta^* \in \alpha^*$*

Proof of Claim. Since $\beta \in \alpha$, $(\beta +_H k) \subseteq (\alpha +_H k)^-$ so, using property 2 of Lemma 5.3.5,

$$\beta^* \in \mathbb{L}_{(\beta+_H k)} \subseteq \mathbb{L}_{(\alpha+_H k)^-}.$$

Moreover, using our inductive hypothesis, $\mathcal{D}(\mathbb{L}_{\beta^*}) = \mathcal{D}(\mathbb{L}_\beta) \subseteq \mathcal{D}(\mathbb{L}_\alpha)$ so $\beta^* \in \alpha^*$ as required. □

Thus,

$$\mathbb{L}_\alpha = \bigcup_{\beta\in\alpha} \mathcal{D}(\mathbb{L}_\beta) = \bigcup_{\beta\in\alpha} \mathcal{D}(\mathbb{L}_{\beta^*}) \subseteq \bigcup_{\gamma\in\alpha^*} \mathcal{D}(\mathbb{L}_\gamma) = \mathbb{L}_{\alpha^*}.$$

□

As mentioned at the beginning of this section, in the presence of infinity we can define the constructible universe using a different definability operator, Def, where $\mathrm{Def}(b) \coloneqq \bigcup_{n\in\omega} \mathcal{D}^n(b)$. This gives us an alternative way to construct the constructible universe, which we now show is equivalent as long as ω exists and is the one we shall, in general, use.

Definition 5.3.11. For α an ordinal, $\mathrm{L}_\alpha \coloneqq \bigcup_{\beta\in\alpha} \mathrm{Def}(\mathrm{L}_\beta)$ and

$$\mathrm{L} \coloneqq \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{L}_\alpha.$$

As before, we can easily observe a few basic properties of this hierarchy:

Proposition 5.3.12. ($\mathrm{IKP}^{-\mathit{Inf}}$ + "Strong Infinity") *For all ordinals α, β:*

1. *If $\beta \in \alpha$ then $\mathrm{L}_\beta \subseteq \mathrm{L}_\alpha$,*
2. $\mathrm{L}_\alpha \in \mathrm{L}_{\alpha+1}$,
3. L_α *is transitive,*
4. L_α *is a model of Bounded Separation,*

It is possible to compare the hierarchies $\mathbb{L}_\alpha$ and L_α via the following correspondence:

Lemma 5.3.13. ($\mathrm{IKP}^{-\mathit{Inf}}$ + "Strong Infinity") *For any ordinal α,* $\mathrm{L}_\alpha = \mathbb{L}_{\omega\cdot\alpha}$.

Proof. We proceed by induction on α. So assume that our claim holds for all $\beta \in \alpha$. First note that for any ordinal α, $\omega \cdot \alpha \coloneqq \{\omega \cdot \gamma + n \mid \gamma \in \alpha \ \wedge n \in \omega\}$ is an additive limit ordinal. Then

$$\begin{aligned} \mathrm{L}_\alpha &= \bigcup_{\beta\in\alpha} \mathrm{Def}(\mathrm{L}_\beta) = \bigcup_{\beta\in\alpha}\bigcup_{n\in\omega} \mathcal{D}^n(\mathrm{L}_\beta) \\ &= \bigcup_{\beta\in\alpha}\bigcup_{n\in\omega} \mathcal{D}^n(\mathbb{L}_{\omega\cdot\beta}) = \bigcup_{\beta\in\alpha}\bigcup_{n\in\omega} \mathbb{L}_{\omega\cdot\beta+n} \\ &= \mathbb{L}_{\omega\cdot\alpha}. \end{aligned}$$

□

Corollary 5.3.14. ($\mathrm{IKP}^{\text{-}Inf}$ + "Strong Infinity") $\mathrm{L} = \mathbb{L}$.

For completeness we briefly discuss how this method relates to the first, syntactic, approach we mentioned at the beginning of the chapter. We shall be sloppy in our presentation of the syntactic definability operator by using the "$\models$" symbol in our definition of definability instead of the more formal way this is presented in the previously mentioned references. We then refer to [Cro00] for the formal way to do this in $\mathrm{IKP}^{\text{-}Inf}$ + *"Strong Infinity"*. We remark here that the syntactic operator is the standard operator we shall use when taking the collection of definable subsets of a given set.

Definition 5.3.15. Say that a set x is definable over a model $\langle \mathrm{M}, \in \rangle$ if there exists a formula φ and $a_1, \ldots, a_n \in \mathrm{M}$ such that

$$x = \{y \in \mathrm{M} \mid \langle \mathrm{M}, \in \rangle \models \varphi[y, a_1, \ldots, a_n]\}.$$

We can then define the collection of definable subsets of M as

$$\mathrm{def}(\mathrm{M}) \coloneqq \{x \subseteq \mathrm{M} \mid x \textit{ is definable over } \langle \mathrm{M}, \in \rangle\}.$$

The constructible hierarchy can then be defined iteratively as

$$W_\alpha \coloneqq \bigcup\nolimits_{\beta \in \alpha} \mathrm{def}(W_\beta).$$

Clearly, given $x, y \in \mathrm{M}$ and $i \in \mathcal{I}$, $\mathcal{F}_i(x, y)$ is a definable subset of M. Moreover, one can define the notion of being definable over $\langle \mathrm{M}, \in \rangle$ using only the fundamental operations, so the two universes they produce will be the same. To see the relationship between the two hierarchies, one can perform a careful analysis of the standard proof, for example Lemma VI.1.17 of [Dev84], which yields;

Theorem 5.3.16. *For every transitive set* M*:*

$$\begin{aligned} \mathrm{def}(\mathrm{M}) &= \mathrm{Def}(\mathrm{M}) \cap \mathcal{P}(\mathrm{M}) \\ &= \bigcup_{n \in \omega} \mathcal{D}^n(\mathrm{M}) \cap \mathcal{P}(\mathrm{M}). \end{aligned}$$

Therefore, in the theory $\mathrm{IKP}^{\text{-}Inf}$ + "*Strong Infinity*" the two standard formulations of the constructible hierarchy are equivalent. One of the main benefits for using the formulation in terms of the fundamental operations is to avoid the use of Strong Infinity in the construction. The second benefit for using our formulation is because of the versatility of these operations over ZF. A notable example is that it allows us to define when an inner model satisfies ZF. This occurs when the inner model is closed under fundamental operations and it satisfies a property known as *almost universality*. We shall see in the next section that an analogous result holds in IZF.

5.4 External Cumulative Hierarchies

In this section we shall show that if V satisfies IZF then so does L. This could be done by a very similar repetition of the analysis in the IKP case however we will take a different approach here in order to derive further axiomatic properties under IZF.

The main theorem of this section is adapted from Theorem 13.9 of [Jec03]. The essence of the theorem is that if V is a model of ZF and M contains all of the ordinals then M being a model of ZF can be expressed by a single first-order sentence. On the face of it, the theorem we present here will be slightly weaker than this because it will requires the additional assumption that M has an *external cumulative hierarchy*.

Definition 5.4.1. Let $\mathrm{M} \subseteq \mathrm{N}$. We say that M has an *external cumulative hierarchy* (e.c.h.) in N if there exists a sequence $\langle \mathrm{M}_\alpha \mid \alpha \in \textsc{Ord} \cap \mathrm{N} \rangle$, definable in N, such that:

- For every $\alpha \in \textsc{Ord} \cap \mathrm{N}$, $\mathrm{M}_\alpha \in \mathrm{M}$,
- $\mathrm{M} = \bigcup \{\mathrm{M}_\alpha \mid \alpha \in \textsc{Ord} \cap \mathrm{N}\}$,
- If $\beta \in \alpha$ then $\mathrm{M}_\beta \subseteq \mathrm{M}_\alpha$.

We say that M has an e.c.h. when $\mathrm{N} = \mathrm{V}$. It is worth remarking that if M is an inner model of IZF, that is a model of IZF containing all of the ordinals, then M will have an external cumulative hierarchy given by the standard rank hierarchy which can be defined as follows:

Definition 5.4.2. Define the rank of a, $\mathrm{rank}(a)$ recursively as follows:

$$\mathrm{rank}(a) \coloneqq \bigcup\{\mathrm{rank}(x) + 1 \mid x \in a\}$$

where $z + 1 \coloneqq z \cup \{z\}$.

Note that one can easily prove that for any set a, $\mathrm{rank}(a)$ is an ordinal and for any ordinal α, $\mathrm{rank}(\alpha) = \alpha$.

Definition 5.4.3. For α an ordinal, $\mathrm{V}_\alpha \coloneqq \bigcup_{\beta\in\alpha} \mathcal{P}(\mathrm{V}_\beta)$.

Proposition 5.4.4. *For any set a, $a \subseteq \mathrm{V}_{\mathrm{rank}(a)}$.*

Proof. This is formally proved by induction on rank noting that for any $x \in a$, if $x \subseteq \mathrm{V}_{\mathrm{rank}(x)}$ then $x \in \mathcal{P}(\mathrm{V}_{\mathrm{rank}(x)}) \subseteq \mathrm{V}_{\mathrm{rank}(a)}$. □

Therefore, if M is an inner model of IZF, then $\langle \mathrm{V}^{\mathrm{M}}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ defines an e.c.h. which is moreover uniformly definable. Also, by construction, we have that $\langle \mathrm{L}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is an e.c.h. for L even though, as we shall prove in Section 5.5, it is not necessarily the case that L contains all of the ordinals of V.

Definition 5.4.5. Let $\mathrm{M} \subseteq \mathrm{N}$. We say that M is *almost universal* in N if for any $x \in \mathrm{N}$, if $x \subseteq \mathrm{M}$ then there exists some $y \in \mathrm{M}$ such that $x \subseteq y$.

Furthermore, it is worth pointing out that classically, for almost universal models, having an e.c.h. is equivalent to having the same ordinals. This will be stated in ZF for simplicity, but in reality only requires very basic set theory and some amount of separation and replacement with regards to the hierarchy.

Proposition 5.4.6. *Suppose that* $\mathrm{M} \subseteq \mathrm{N}$ *are transitive models of* ZF *and* M *is almost universal in* N*. If* M *has an e.c.h. in* N *then* $\mathrm{Ord} \cap \mathrm{M} = \mathrm{Ord} \cap \mathrm{N}$.

Proof. Let $\langle \mathrm{M}_\alpha \mid \alpha \in \mathrm{Ord} \cap \mathrm{N}\rangle$ be an external cumulative hierarchy. We shall prove inductively that for any ordinal $\gamma \in \mathrm{N}$ there is an ordinal $\beta \in \mathrm{N}$ such that $\gamma \subseteq \mathrm{M}_\beta$. Then, since $\mathrm{M}_\beta \in \mathrm{M}$, either $\gamma = \mathrm{M}_\beta \cap \mathrm{Ord}$ or $\gamma \in \mathrm{M}_\beta \cap \mathrm{Ord}$. Almost universality allows us to take some transitive set $y \in \mathrm{M}$ covering the set $\mathrm{M}_\beta \cap \mathrm{Ord}$ and then Bounded Separation in M yields that $\gamma \in \mathrm{Ord} \cap \mathrm{M}$.

Working in N, to prove the claim first note that, by induction,

$$\forall \alpha \in \gamma \; \exists \tau_\alpha \in \mathrm{N} \; \alpha \in \mathrm{M}_{\tau_\alpha}.$$

So, by Collection and the assumption that the hierarchy is cumulative, there is some ordinal β such that for all $\alpha \in \gamma$, $\alpha \in \mathrm{M}_\beta$ and therefore $\gamma \subseteq \mathrm{M}_\beta$. □

Remark 5.4.7. For clarity, the point where we used excluded middle in the above proof was when we asserted that $\gamma \subseteq \mathrm{M}_\beta$ implies that $\gamma \in \mathrm{M}$. If $\gamma = \mathrm{M}_\beta \cap \textsc{Ord}$ then γ will be in $\textsc{Ord} \cap \mathrm{M}$ as

$$\gamma = \{\alpha \in \mathrm{M}_\beta \mid \alpha \textit{ is a transitive set of transitive sets}\}.$$

However, if γ is a proper subset of $\mathrm{M}_\beta \cap \textsc{Ord}$ then we require linearity of the ordinals to conclude that γ is an element of this set. To see an example of this, consider an arbitrary truth value $x \subseteq 1$. It is quite plausible for such an ordinal to be a proper subset of some $\mathrm{M}_\beta \cap \textsc{Ord}$ since this set could contain 1 but there is no reason to assume that x itself is a member of this set.

We now present our adaptation of the theorem from Jech. The assumption of an e.c.h. is not necessary for the right-to-left implication of this theorem but it is needed in order to prove that our model M is almost universal in V. This is because, while we can use the rank hierarchy of M to show that if a set $a \in \mathrm{V}$ is a subset of M there is some $\beta \in \textsc{Ord}$ such that $a \subseteq \bigcup_{\alpha \in \beta} \mathrm{V}_\alpha^{\mathrm{M}}$, it does not seem possible to show that this union is in fact a set in M because there is no reason why β should be in M.

However, having an e.c.h. seems to be a reasonable additional assumption since in most cases our model M will be built up iteratively over the ordinals in V, which gives a very natural hierarchy. Notably, we have that $\mathrm{L} := \bigcup_{\alpha \in \textsc{Ord}} \mathrm{L}_\alpha$.

Theorem 5.4.8. *Suppose that* V *is a model of* IZF *and* $\mathrm{M} \subseteq \mathrm{V}$ *is a definable, transitive proper class with an external cumulative hierarchy. Then* M *is a model of* IZF *iff* M *is closed under the fundamental operations and is almost universal.*

Proof. For the left-to-right implication, first see that, if M is a model of IZF, then M is certainly closed under the fundamental operations because they are Σ-definable. For almost universality, let $\langle \mathrm{M}_\alpha \mid \alpha \in \mathrm{ORD} \rangle$ be an e.c.h. and suppose that $a \in \mathrm{V}$ with $a \subseteq \mathrm{M}$. Then we have that

$$\forall x \in a \; \exists \alpha \in \mathrm{ORD} \; (x \in \mathrm{M}_\alpha).$$

So, by Collection in V, there is some set $b \in \mathrm{V}$ such that

$$\forall x \in a \; \exists \alpha \in b \; (x \in \mathrm{M}_\alpha).$$

Taking $\beta = \mathrm{trcl}(b \cap \mathrm{ORD})$ we can then conclude that $a \subseteq \mathrm{M}_\beta$, which is a set in M by the assumption that the M_α's form an external cumulative hierarchy.

For the reverse implication, since M is transitive it is a model of Extensionality and ∈-induction. Also, Pairing and Unions follow from M being closed under $\mathcal{F}_p$ and $\mathcal{F}_\cup$ while the Axiom of Infinity will follow from the proof of Theorem 5.3.6 along with an instance of Bounded Separation. We now proceed to prove the other axioms.

Bounded Separation: This follows from the same argument as given in Theorem 5.2.5 because any Σ_0-formula can be expressed using the fundamental operations.

Power Set: Let a be a set and note that $\mathcal{P}(a) \cap \mathrm{M}$ is a set in V, and a subset of M. So, by almost universality, we can fix some $b \in \mathrm{M}$ such that $\mathcal{P}(a) \cap \mathrm{M} \subseteq b$. But then

$$\mathcal{P}^{\mathrm{M}}(a) = \{x \in b \mid x \subseteq a\}$$

which is a set in M by Bounded Separation in M.

Collection: Let a be in M and suppose that, in M, $\forall x \in a \; \exists y \in \mathrm{M} \; \varphi(x, y, u)$. By Collection in V we can fix some set b' such that

$$\forall x \in a \; \exists y \in b' \; \varphi^{\mathrm{M}}(x, y, u)$$

and, by almost universality, we can fix $b \in \mathrm{M}$ such that $b \supseteq b'$ yielding, in M,

$$\forall x \in a \; \exists y \in b \; \varphi(x, y, u).$$

Separation: This is shown by induction on the complexity of the formula φ. Σ_0-formulae and all cases except for those involving quantifiers follow immediately from the consequences of the fundamental operations.

So suppose that $\varphi(x, u) \equiv \exists v\ \psi(x, v, u)$. Using Separation in V, define a' to be

$$a' \coloneqq \{x \in a \mid \varphi^{\mathrm{M}}(x, u)\}.$$

Then,

$$\forall x \in a'\ \exists v \in \mathrm{M}\ \psi^{\mathrm{M}}(x, v, u).$$

So, by Collection in V, there exists some set $b' \subseteq \mathrm{M}$ such that

$$\forall x \in a'\ \exists v \in b'\ \psi^{\mathrm{M}}(x, v, u).$$

By almost universality, take $b \in \mathrm{M}$ such that $b' \subseteq b$. Then

$$\forall x \in a'\ \exists v \in b\ \psi^{\mathrm{M}}(x, v, u).$$

Now, by the inductive hypothesis, we have that

$$y \coloneqq \{\langle x, v\rangle \in a \times b \mid \psi^{\mathrm{M}}(x, v, u)\} \in \mathrm{M},$$

and thus, using $\mathcal{F}_d$,

$$z = \mathrm{dom}(y) = \{x \in a \mid \exists v \in b\ \psi^{\mathrm{M}}(x, v, u)\} \in \mathrm{M}.$$

For the final case, suppose that $\varphi(x, u) \equiv \forall v\ \psi(x, v, u)$. For $r \in \mathrm{M}$ let

$$y_r \coloneqq \{x \in a \mid \forall v \in r\ \psi^{\mathrm{M}}(x, v, u)\} \in \mathrm{V}.$$

Then, by using the inductive hypothesis and the proof that Separation holds for bounded universal quantifiers, $y_r \in \mathrm{M}$. Also, it is obvious that if $s \subseteq r$ then $y_r \subseteq y_s$. We aim to show that y_{M}, which is defined in the same way, is in M by showing that it is equal to y_r for some $r \in \mathrm{M}$. To do this, we begin by defining

$$Y \coloneqq \{z \in \mathcal{P}(a) \mid \exists r \in \mathrm{M}\ (z = y_r)\}.$$

So

$$\forall z \in Y \ \exists r \in \mathrm{M} \ (z = y_r).$$

Therefore, by Collection in V, there is some set $d' \subseteq \mathrm{M}$ such that

$$\forall z \in Y \ \exists r \in d' \ (z = y_r).$$

Taking the transitive closure if necessary, by almost universality fix a transitive set $d \in \mathrm{M}$ such that $d' \subseteq d$. We claim that $y_d = y_{\mathrm{M}}$. Firstly, since $d \subseteq \mathrm{M}$, $y_{\mathrm{M}} \subseteq y_d$. For the reverse direction, let $x \in y_d$ and let $v \in \mathrm{M}$. Then $y_{\{v\}} \in Y$ so we can fix $r \in d$ such that $y_{\{v\}} = y_r$. Therefore, since $r \subseteq d$, $y_d \subseteq y_r = y_{\{v\}}$ so $x \in y_{\{v\}}$ and, by construction, $\psi^{\mathrm{M}}(x, v, u)$. Finally, since v was arbitrary, $x \in y_{\mathrm{M}}$ as required. □

Corollary 5.4.9. *For every axiom φ of* IZF*,* IZF $\vdash \varphi^{\mathrm{L}}$*.*

5.5 The Ordinals of the Constructible Universe

In this section we shall answer a question of Lubarsky from the end of [Lub93] about the ordinals in the constructible universe. In particular, in this section we shall prove that:

Theorem 5.5.1. *Starting from a model of* ZFC*, it is consistent to have a model of* IZF *such that*

$$\textsc{Ord} \cap \mathrm{V} \neq \textsc{Ord} \cap \mathrm{L}.$$

To begin with we recall the following lemma from Section 2.6;

Lemma 2.6.9. *Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model and suppose that $\mathcal{K}$ has initial node $\mathbb{1}$. Then for any $I \in \mathcal{D}(\mathbb{1})$,*

$$\mathrm{V}(\mathscr{K}) \models \forall \{a_\alpha \mid \alpha \in I^{\mathbb{1}}\} \ \forall x \ \Big(\big(x \in \bigcup_\alpha a_\alpha \ \wedge \ x \notin \bigcup_{\alpha \neq \gamma} a_\alpha \big) \rightarrow x \in a_\gamma \Big). \qquad (\star)$$

The property $(\star)$ will be useful because it will allow us to satisfy the first condition for the next theorem, which tells us that if we can find a set of ω many incomparable ordinals then we can code every real by a unique ordinal. This means that if two models

M $\subseteq$ N have the same ordinals, then they must have the same reals. Therefore, if V contains a real which is not in L, then the two models cannot possibly have the same ordinals.

Theorem 5.5.2. *Suppose that* M $\subseteq$ N *are models of* IZF *such that* M *satisfies the following weak incidence of excluded middle:*

$$\text{for any set } \{a_n \mid n \in \omega\} \text{ of distinct ordinals, if we have } x \text{ such that}$$
$$x \in \bigcup_n a_n \text{ and for some } k,\ x \notin \bigcup_{n \neq k} a_n \text{ then } x \in a_k.$$

Further suppose that in M *there is an ordinal* α *such that* $\alpha \notin \omega$ *and* $\omega \not\subseteq \alpha$*. Then*

$$\textsc{Ord} \cap \mathrm{N} = \textsc{Ord} \cap \mathrm{M} \Longrightarrow ({}^{\omega}2)^{\mathrm{N}} = ({}^{\omega}2)^{\mathrm{M}}.$$

Proof. Fix α to be an ordinal in M which is incomparable with ω. By the absoluteness of ω, α is still incomparable with ω in N. This gives us that $(\alpha+1) \not\subseteq \omega$, so $\{n \cup (\alpha+1) \mid n \in \omega\}$ is a set of ω many pairwise incomparable ordinals. Now, take $f \in ({}^{\omega}2)^{\mathrm{N}}$ and define

$$\delta_f := \bigcup\nolimits_{n\in\omega}(n \cup (\alpha+1)) + f(n).$$

So δ_f is an ordinal in N and therefore, by hypothesis, an ordinal in M. Now we can define a function $g\colon \omega \to 2$ in M by asserting that $g(k) = 1$ if and only if $(k \cup (\alpha+1))$ is in δ_f. Then, working in N, we have that $g = f$ because

$$f(k) = 1 \longleftrightarrow (k \cup (\alpha+1)) \in \delta_f.$$

Note that the backward implication holds because if $(k\cup(\alpha+1))$ is in δ_f then for some $n \in \omega$, $(k \cup (\alpha+1)) \in (n \cup (\alpha+1)) + f(n)$. But, since the ordinals are incomparable, $(k \cup (\alpha+1)) \notin (n \cup (\alpha+1)) + f(n)$ for $n \neq k$. Therefore, the only option is that $(k \cup (\alpha+1)) \in (k \cup (\alpha+1)) + f(k)$ which implies that $f(k) = 1$. □

Before giving the proof of Theorem 5.5.1, we shall outline a plausible scenario which shows that L can feasibly have a very fragile structure and that doing standard forcing over an intuitionistic model could have some unforeseen consequences. In particular, if the scenario were correct it would mean that L is not absolute under forcing extensions. We currently do not have an actual model of when this could happen but it will provide the motivation for the model that proves Theorem 5.5.1.

Example 5.5.3. Suppose that V is a model of IZF, $\mathbb{P} \in \mathrm{L}$ is a partial order and that there exists some set $\{\alpha_p \mid p \in \mathbb{P}\} \subseteq \mathcal{P}(1)$ such that for all $p, q \in \mathbb{P}$:

- $\alpha_p \neq 0$ (that is $\neg(\forall x \in \alpha_p\ (x \neq x))$),
- If $p \neq q$ then $\alpha_p \neq \alpha_q$,
- $\mathrm{L}_{\alpha_p} = \alpha_p$.

Suppose further that property $(\star)$ from Lemma 2.6.9 holds in L.

Now let $G \subseteq \mathbb{P}$ be a generic. In the classical case one shows that $G \notin \mathrm{L}$ because forcing does not add new ordinals and the definability operator is absolute between transitive models. However, we seek to show that there is no reason to believe this is the case intuitionistically because there could be new ordinals. First we see that $\mathrm{L}_{\alpha_p \cup \{\alpha_p\}} = 1 \cup \alpha_p \cup \{\alpha_p\}$. Now define the ordinal δ_G as

$$\delta_G \coloneqq 1 \cup \{\alpha_p \mid p \in G\}$$

and consider L_{δ_G};

$$\mathrm{L}_{\delta_G} = \bigcup_{\gamma \in \delta_G} \mathrm{def}(\mathrm{L}_\gamma) = \mathrm{L}_1 \cup \bigcup_{p \in G} \mathrm{def}(\mathrm{L}_{\alpha_p}) = \bigcup_{p \in G} 1 \cup \alpha_p \cup \{\alpha_p\}.$$

But $\alpha_p \in \mathrm{L}_{\delta_G} \iff p \in G$ and both L_{δ_G} and $\mathbb{P}$ are sets in L. Therefore

$$G = \{p \in \mathbb{P} \mid \alpha_p \in \mathrm{L}_{\delta_G}\} \in \mathrm{L}.$$

Remark 5.5.4. As can be seen in the work of Lubarsky [Lub02], and will be discussed later, if we define a Kripke frame using the partial order $\mathbb{P}$ then the first point and the third point of the above example can be consistently true. However it is unclear how to have a model which verifies the second point as well.

We shall now prove Theorem 5.5.1 while showing how we can avoid the issue that arose in the previous example. Part of the proof will involve defining a new subset of 1, or truth value, in our model. This will be defined as a set which "*looks like*" 0 at one node and 1 at another. Such an ordinal is defined by Lubarsky in Section 4.1.3 of [Lub02] and, for completeness, we shall state it here and prove that it has the required properties. This will be done in a much more general framework than is needed for the proof.

Definition 5.5.5. Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model. For $p \in \mathcal{K}$ define $1_p \in \mathrm{V}(\mathscr{K})$ by

$$1_p \colon \mathcal{K} \to 2 \qquad 1_p(s) = \begin{cases} 1^s, & \text{if } s \in \mathcal{K}^p \\ 0^s, & \text{otherwise.} \end{cases}$$

where 1^s and 0^s are the canonical names for 1 and 0 as defined in Definition 2.6.8.

The idea is that 1_p looks like 1 at any node above p and 0 otherwise.

Proposition 5.5.6. *In* $\mathrm{V}(\mathscr{K})$, $1_p \subseteq 1$ *is an ordinal with* $\mathrm{L}_{1_p} = 1_p$.

Proof. To prove that $1_p \subseteq 1$ we need to show that for any $s \in \mathcal{K}$, if $s \Vdash x \in 1_p$ then $s \Vdash x \in 1$. Using our classical metatheory, there are two cases: $s \in \mathcal{K}^p$ and $s \notin \mathcal{K}^p$. For the first case, suppose that $s \Vdash x \in 1_p$. Then, by definition, $x \restriction \mathcal{K}^s \in 1_p(s) = 1^s$. Therefore, $x \restriction \mathcal{K}^s = 0^s$ which gives us that $s \Vdash x \in 1$.

For the second case, if $s \notin \mathcal{K}^p$ then $s \Vdash x \in 1_p$ if and only if $x \restriction \mathcal{K}^s \in 1_p(s) = 0^s$. But this set is empty and therefore there can be no such x. Hence $s \nVdash x \in 1_p$, so the implication vacuously holds.

For the second claim we will again split it into the same two cases after noting that

$$s \Vdash x \in \mathrm{L}_{1_p} \implies \textit{for some } \beta,\ s \Vdash x \in \mathrm{def}(\mathrm{L}_\beta) \textit{ and } \beta \in 1_p.$$

Now, if $s \in \mathcal{K}^p$, then $s \Vdash \beta \in 1_p$ if and only if $s \Vdash \beta = 0^s = \mathrm{L}_\beta$. So,

$$s \Vdash x \in \mathrm{def}(0) \qquad \implies \qquad s \Vdash x = 0 \qquad \implies \qquad s \Vdash \mathrm{L}_{1_p} = 1_p.$$

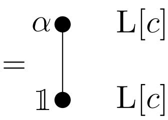
For the second case, if $s \notin \mathcal{K}^p$ then $s \not\Vdash \beta \in 1_p$ for any β and thus $s \not\Vdash x \in \mathrm{L}_{1_p}$ for any x. Therefore

$$s \Vdash \mathrm{L}_{1_p} = 1_p$$

giving us our required result. □

Proof of Theorem 5.5.1. The desired model will be the full model of the Kripke model $\mathscr{K}$ where $\mathcal{K}$ is the two node Kripke structure $\{\mathbb{1}, \alpha\}$ and $\mathcal{D}(\mathbb{1}) = \mathcal{D}(\alpha) = \mathrm{L}[c]$, for c a Cohen real over L, and $\iota_{0,1}$ is the identity.

$\mathcal{K} \;=$



Let c^p be the interpretation of c at node p for $p \in \{\mathbb{1}, \alpha\}$, as given in Definition 2.6.8, and note that, since $\mathcal{D}(p)$ is a model of ZFC,

$$p \Vdash c^p \notin \mathrm{L}.$$

Therefore $\mathrm{V}(\mathscr{K}) \models c \notin \mathrm{L}$. Let 1_α be the new ordinal subset of 1 which is derived from the node α, using Definition 5.5.5.

Working in $\mathrm{V}(\mathscr{K})$, define δ_c to be an ordinal encoding c, for example,

$$\begin{aligned}
\delta_c &= \bigcup_{n\in\omega} (\alpha \cup n) + c(n) \\
&= \{\alpha \cup n \mid c(n) = 0\} \cup \{\alpha \cup n \cup \{\alpha \cup n\} \mid c(n) = 1\} \\
&= \{\alpha \cup n \mid n \in \omega\} \cup \{\{\alpha \cup n\} \mid c(n) = 1\}.
\end{aligned}$$

Then $c(n) = 1$ if and only if $(\alpha \cup n) \in \delta_c$ which means that $c \in \mathrm{L} \longleftrightarrow \delta_c \in \mathrm{L}$, because 1_α is in L. Thus $\delta_c \notin \mathrm{L}$ and $\mathrm{Ord} \cap \mathrm{L} \subsetneq \mathrm{Ord} \cap \mathrm{V}$, completing the proof.

□

Remark 5.5.7. We do not have the same contradiction which arose in Example 5.5.3 because $\{\alpha \cup m \mid m \in n\} \subseteq \mathrm{def}(\mathrm{L}_{\alpha\cup n})$. Now, since c is a Cohen real, $\{n \mid c(n) = 1\}$ is unbounded in ω. This gives us that $\{\alpha \cup n \mid n \in \omega\} \subseteq \mathrm{L}_{\delta_c}$. Therefore, L_{δ_c} loses the definition of c because it contains all of the sets used for the coding and thus δ_c is not definable from L_{δ_c}.

Chapter 6

An Introduction to Elementary Embeddings

The second half of this thesis concerns elementary embeddings of various subsystems of ZFC. Large cardinal axioms are the principal method we use to measure the consistency strength of set-theoretic statements, and many of these axioms can be defined using elementary embeddings from the universe into some inner model M. Therefore, there are two natural questions to consider when working with the larger of the large cardinals over some weak system T:

1. What is the consistency strength of T plus a non-trivial, elementary embedding from the universe to some "inner model" in terms of the ZFC large cardinal hierarchy?
2. What are the consequences for the structure of the universe given such an embedding?

Before we can attempt to answer either of these questions it is necessary to discuss how to define such embeddings. Recall that, over ZFC, a measurable cardinal is the critical point of a non-trivial, elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ where $\mathrm{M} \subseteq \mathrm{V}$ is an inner model. At first sight, j is a proper class so this definition is not first-order definable. However, it is well-known that κ is measurable if and only if there is a non-principal, κ-complete ultrafilter on $\mathcal{P}(\kappa)$, an assertion which is first-order definable.

Moreover, if κ is the critical point of j then

$$U \coloneqq \{X \in \mathcal{P}(\kappa) \mid \kappa \in j(X)\}$$

is such an ultrafilter.

Our first issue is that this first-order way to define measurability breaks down immediately when one tries to weaken the theory. Notably:

- Under ZF, the existence of a non-principal, κ-complete ultrafilter does not imply the existence of a non-trivial, elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ with critical point κ. For example, by Theorem 21.16 of [Jec03], ω_1 can have such an ultrafilter, however ω_1 can never be the critical point of an embedding which is definable in V and has $\mathrm{M} \subset \mathrm{V}$.[1]
- Proofs using ultrafilters appear to need essential instances of excluded middle and so it is unclear what can be achieved in an intuitionistic setting. For example, consider the claim that if there is a non-principal, κ-complete ultrafilter then κ is a regular cardinal.[2] How one proves that κ is regular is to assume it is singular and that there is a partition of κ into α many sets each of which has size less than κ. Then one of these small sets must be in the ultrafilter, from which one can derive a contradiction. This only shows the "negative" result that κ is not a singular cardinal and there seems to be no obvious way to translate this into a "positive" result. We shall see in Chapter 7 that if there is an elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ where V is a model of IKP and j moves an ordinal, then there is a set which is regular, inaccessible and much more.
- The fact that ultrafilters give rise to elementary embeddings makes essential use of *Łoś's Theorem* which can consistently fail in either ZF [3] or ZFC− ([GHJ16]).

[1] If we weaken the hypothesis to allow an elementary embedding $j\colon \mathrm{V} \to \mathrm{M} \subseteq \mathrm{V}[G]$ which is definable in some set generic extension, then it is in fact possible for such an embedding to have critical point ω_1. An example of this can be found in Theorem 10.2 of [Cum10].

[2] Technically, in an intuitionistic context one should be working with an ultrafilter over a "large set" and trying to prove that the set is *regular*, as defined at the end of Section 2.3, but we will ignore this issue here.

[3] In [How75] it is proven that, over ZF, Łoś's Theorem plus the Boolean Prime Ideal Theorem implies the Axiom of Choice.

The existence of a class function from the universe to itself is a second-order claim and therefore it is not apriori definable in a first-order context. There are many ways to circumvent this definability issue in particular circumstances. We shall discuss some of them here and give more details where appropriate throughout the rest of this thesis. However a fuller, more self-contained discussion about the metamathematical preliminaries one should consider when discussing elementary embedding can be found in the introduction to [HKP12].

The first method is to only work in a set sized fragment of the universe. For example, in [HK20], Hayut and Karagila define a cardinal κ to be a *critical cardinal* if it is the critical point of an elementary embedding

$$j\colon \mathrm{V}_{\kappa+1} \to \mathrm{M}.$$

This definition ensures that j is a set while being equiconsistent over ZFC to a measurable cardinal. The embedding itself also still gives us many of the useful consequences we would want, for example one can prove that κ is a regular cardinal which is a limit of cardinals that satisfy a specific formulation of weak compactness. On the other hand, this definition runs into issues if one does not assume the Power Set axiom holds. In particular, we shall later see examples of embeddings over ZFC^- in which $\mathcal{P}(\omega)$ and therefore V_κ are not sets.

The second way to deal with definability is to assert that j is a class which is definable from a parameter. That is, we assume that there is a formula $\varphi(\cdot,\cdot,p)$ with fixed parameter p such that

$$\varphi(x,y,p) \longleftrightarrow j(x) = y.$$

This is perhaps the most natural way to consider class embeddings in a purely first-order context. However, as we shall discuss further in Chapter 8, it is perhaps too restrictive. This is due to Suzuki's Theorem which rules out definable embeddings from the universe to itself under the assumption that V satisfies ZF. Kunen's inconsistency result, on the other hand, is a much stronger claim which rules out many more embeddings than just those definable by a first-order formula.

The third way is to either work in a full, second-order theory such as GB or to expand the language by adding a predicate for j, along with axioms asserting it is an elementary embedding, and then work in the theory T_j as defined in Convention 2.2.1. The idea of working in T_j has been extensively studied by Corazza, for example in [Cor00] and [Cor06], and is the approach we will in general adopt after formalising precisely what we mean by it in Section 6.1. Although this does have its own drawbacks, it allows us to work in as close to a first-order setting as possible, while also allowing j to retain some of its natural class properties.

It is well-known that without expanding the language in some way there is no reason why the existence of such an embedding should have a large consistency strength. For example, it is possible to obtain an embedding of the universe from just the consistency of ZFC. This follows from standard model-theoretic results on indiscernibility (see Theorems 3.3.10 and 3.3.11(d) of [CK73]) and is stated as Proposition 2.3 of [Cor06]. It is important to note that in this next theorem, elementarity will only be with respect to E-formulae and that M does not satisfy the schemes of Replacement or Separation in the language expanded to include j.

Theorem 6.0.1 ([CK73])**.** *If there is a model of* ZFC *then there is a model* $\langle \mathrm{M}, \mathrm{E}, j\rangle$ *of* ZFC *such that* $j\colon \mathrm{M} \to \mathrm{M}$ *is an elementary embedding and for some* $x \in \mathrm{M}$, $x \neq j(x)$.

Perhaps an easier, more set-theoretic, example of this is that $0^\#$ gives rise to elementary embeddings from L to itself. A full explanation can be found in Chapter 9 of [Kan08] but, very briefly, from $0^\#$ we can obtain a sequence of "*indiscernible ordinals*" $\langle \gamma_n \mid n \in \omega\rangle$ and then the map $\gamma_n \mapsto \gamma_{n+1}$ can be used to generate an elementary embedding $j\colon \mathrm{L} \to \mathrm{L}$.

In this case, it is clear that Collection_j holds in L because we have Collection in the full universe V. This allows us to consider the least rank of each witness, from which we can find some L_α which contains at least one witness for every element of the domain. However, it is possible to show that Separation_j already fails in L for any such embedding because the reals of L will only be a countable set in the full universe.

6.1 Formalisation

In order to give a uniform presentation of elementary embedding characterisations in weak theories, this section is devoted to stating precisely what we mean. To do this we shall work in some weak theory T, for example ZBQC or IKP, and slowly add more assumptions as necessary. An important point to note is that when we consider elementary embeddings

$$j\colon \mathrm{V} \to \mathrm{M}$$

it is useful to not only add a predicate for j but also to add a predicate for M. This is because, without being able to construct ultrapowers, it is hard to see how M should otherwise be interpreted. It is also important to remark that the next definition will technically be a metatheoretic axiomatic scheme because, at least naively, it is not expressible in the language of set theory. This necessity similarly arises in [Cor00] and [Ham01] when one tries to express the *Wholeness Axiom* over ZF.

Definition 6.1.1. Let T be a "*sufficient*" theory[4] over $\mathcal{L} = \{\in\}$ and suppose that V is a model of T. Let $\mathcal{L}_{j,\mathrm{M}}$ be the extension of $\mathcal{L}$ to include a unary predicate symbol M and a unary function symbol j, with interpretations M^{V} and j^{V}. Then we say that $j\colon \mathrm{V} \to \mathrm{M}$ is an *elementary embedding* if, in the structure $\langle \mathrm{V}, \in, \mathrm{M}^{\mathrm{V}}, j^{\mathrm{V}} \rangle$, we have:

(i.) $\mathrm{V} \models \mathrm{T}_{j,\mathrm{M}}$,

(ii.) M is transitive. That is $\forall x\ (\mathrm{M}(x) \to \forall y \in x\ \mathrm{M}(y))$,

(iii.) $\mathrm{M}^{\mathrm{V}} \models \mathrm{T}$,

(iv.) $\exists x\ x \neq j(x)$,

(v.) $\mathrm{M}(\omega)$,

(vi.) For any $\mathcal{L}$-formula $\varphi(v_0, \ldots, v_n)$ and sets $a_0, \ldots, a_n$ in V,

$$\varphi(a_0, \ldots, a_n) \longleftrightarrow \varphi^{\mathrm{M}}(j(a_0), \ldots, j(a_n)),$$

where φ^{M} is the result of restricting all quantifiers in φ to M.

[4]Here by sufficient we mean any of the axiomatic theories considered in this thesis, such as ZBQ, IKP or their extensions.

We further call j an $\textsc{Ord}$-*inary* elementary embedding if

$$\exists\kappa \in \textsc{Ord}\ \forall\alpha \in \kappa\ \Big(j(\alpha) = \alpha\ \wedge\ \kappa \in j(\kappa)\Big).$$

In essence, $\textsc{Ord}$-inary embeddings are those which are non-trivial on the ordinals and, as we shall discuss in Section 6.2, in many natural base theories it will just follow from non-triviality. At first sight, calling such an embedding "$\textsc{Ord}$-inary" may seem strange because it is just talking about a single ordinal, κ. However, many of the first consequences one deduces about ordinals, for example those in Chapter 5 of [Kan08], will only really require this property. Notably, over suitably weak base theories, we will be able to show that; κ is a regular limit cardinal (6.2.10), κ is a limit of weakly compact cardinals (and much more, 7.4.11), one can define an ultrafilter U over κ (even if we can't prove that U is necessarily a set, Section 10.5) and that the critical sequence $\langle j^n(\kappa) \mid n \in \omega\rangle$ exists (6.3.3). It also appears to be a crucial property in allowing us to deduce either Suzuki's (8.1.4) or Kunen's Inconsistency results (9.1.11 and 10.2.3). Therefore it is somewhat natural to call such an embedding ordinary.

On occasion we may abuse the notation given in Convention 2.2.1 by associating our class predicate with some predefined set of axioms. Notably, we will sometimes write $\mathrm{V} \models \mathrm{T}_j$ to indicate that j is an elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ as defined above. In such a case it should be clear from the context how this would be formally phrased.

We shall shortly see that in the classical case any non-trivial elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ is $\textsc{Ord}$-*inary* as long as V is a model of at least $\mathrm{KP}_{j,\mathrm{M}}$. However, we shall then later show that it need not be the case intuitionistically. It will follow that there are many different ways to express non-triviality and we shall explore this concept further in Section 7.2. The definition that we have given here; that for some x, $x \neq j(x)$, is the weakest natural one but we shall see it is sometimes too weak for our purposes.

Using Gaifman's Theorem 8.1.1 we shall see that in a sufficiently strong system full elementarity can be expressed by a single sentence. But in weaker systems, such as KP, it is often natural to restrict the elementarity to the subclass of Σ-formulae. As before, to begin with this should be seen as a metatheoretic definition.

Definition 6.1.2. Let T be a "*sufficient*" theory over $\mathcal{L} = \{\in\}$ and suppose that V is a model of T. Let $\mathcal{L}_{j,\mathrm{M}}$ be the extension of $\mathcal{L}$ to include a unary predicate symbol M and a unary function symbol j. Then $j\colon \mathrm{V} \to \mathrm{M}$ is called a Σ_0-Ord-inary (Σ-Ord-inary) elementary embedding if it satisfies points ($i.$) to ($v.$) of Definition 6.1.1 as well as the following two assertions:

- there exists an ordinal κ satisfying $\forall \alpha \in \kappa\ j(\alpha) = \alpha\ \wedge\ \kappa \in j(\kappa)$.
- for any Σ_0-formula (Σ-formula) $\varphi(v_0, \ldots, v_n)$ of $\mathcal{L}$ and sets $a_0, \ldots, a_n$ in V,

$$\varphi(a_0, \ldots, a_n) \longleftrightarrow \varphi^{\mathrm{M}}(j(a_0), \ldots, j(a_n)).$$

Notation 6.1.3. For classes M and N, let $\mathrm{M} \prec_\Sigma \mathrm{N}$ denote the assertion that there exists a Σ-elementary embedding from M into N.

Over the next few chapters, we shall see that Σ-elementarity will be enough to prove many of the basic results we desire, and Gaifman's Theorem will show that, under ZF, Σ-elementarity suffices to deduce full elementarity. However, as stated Σ-elementarity is still a scheme because it refers to metatheoretic formulae. This means that, naively at least, it is still not first-order expressible in the language of set theory. In Part II Theorem 2 of [Gai74], Gaifman uses the result that the satisfaction relation can be expressed by a single formula to give a finite set of formulae which will suffice to deduce Σ_0-elementarity when working in Z^+ (with additional assumptions). Using this result, after proving Gaifman's Theorem we will remark that this allows us to express enough elementarity for the situations we need by a single formula.

It is also worth noting that Σ-elementarity is a minimal theory one would wish to use to develop large cardinals. To see why this is, consider an elementary embedding

$$j\colon \mathrm{V} \to \mathrm{M}$$

arising from a measurable cardinal where V is a model of ZFC. By composing this with the identity embedding

$$\iota\colon \mathrm{M} \to \mathrm{V}$$

we obtain a non-trivial Σ_0-elementary embedding from V to itself. In fact, because Σ_1-statements are upwards absolute and their negations are Π_1-statements which are downwards absolute, this embedding is Δ_1-elementary.

A final important fact that we shall regularly use, without further mentioning it, is the existence of a *satisfaction predicate*. Introduced at the end of Section III.1 of [Bar17], there is, derivable in KP, a Σ-operation $\mathrm{Sat}(a, \varphi)$, such that

$$\mathrm{Sat}(a, \varphi) \longleftrightarrow \varphi \textit{ is a sentence which is true in } \langle a, \in \rangle.$$

Moreover, one can express the predicate $a \models \varphi(u)$ in a Δ-way. Therefore, if we are in the situation where j is a Σ-elementary embedding and

$$a \models \varphi(u)$$

for some formula φ, then, using the satisfaction predicate, we can deduce that

$$j(a) \models \varphi(j(u)).$$

6.2 Critical Points and Cofinality

While Σ-elementarity is needed for many of our results, as long as we satisfy some basic classical set theory, Σ_0-elementarity suffices to show that there will be an ordinal which is not fixed by j.

Proposition 6.2.1. *Let* $\mathrm{N} \subseteq \mathrm{M}$ *be transitive class models of* KP. *Suppose that* $j \colon \mathrm{M} \to \mathrm{N}$ *is a non-trivial,* Σ_0*-elementary embedding and* $\mathrm{M} \models \mathrm{KP}_{j,\mathrm{N}}$. *Then there exists an ordinal* α *such that* $j(\alpha) > \alpha$.

Proof. First recall that Σ_0-elementarity implies elementarity for any formula which is, provably in KP, Δ-definable. Then, since being an ordinal is Σ_0-definable, if α is an ordinal then so is $j(\alpha)$. Next, since $\emptyset$ is definable as the unique set z such that $\forall y \in z\ (y \neq y)$, which is a Σ_0-formula, $j(\emptyset) = \emptyset$. Now, by induction, we have that for every ordinal α, $j(\alpha) \geq \alpha$. So, let x be a set of least rank such that $j(x) \neq x$ and let $\delta = \operatorname{rank}(x)$. Then for all $y \in x$, $y = j(y) \in j(x)$ so $x \subseteq j(x)$. Thus there will be some $z \in j(x) \setminus x$. Supposing that $\operatorname{rank}(j(x))$ were to equal δ, we must have that $j(z) = z \in j(x)$ so, by elementarity, $z \in x$ which yields a contradiction. Hence, since the following is provably in KP Δ-definable, we have that $j(\delta) = \operatorname{rank}(j(x)) > \delta$. □

Notation 6.2.2. When such an ordinal exists, we shall denote by $\operatorname{crit}(j)$ the least such ordinal and call it the *Critical Point* of j.

An alternative name we shall use for the critical point of an embedding $j\colon \mathrm{V} \to \mathrm{M}$ is a V-*critical* Ordinal. This notation comes from Schlutzenberg, [Sch20a] Definition 5.1.

Furthermore, if $\mathrm{M} = \mathrm{V}$ then we call the critical point a *Reinhardt Ordinal*. To be more formal, by $\mathrm{M} = \mathrm{V}$ we mean adding the assertion

$$\forall x\ \mathrm{M}(x)$$

to our definition of ORD-inary embeddings in Definition 6.1.1. We will also call the associated embedding a *Reinhardt Embedding*.

Remark 6.2.3. The terminology V-critical is adopted from [Sch20a]. This is because in the paper [HK20], the authors define a *critical cardinal* to only have domain $\mathrm{V}_{\operatorname{crit}(j)+1}$ in order to be working solely with sets. We choose to take this slightly stronger definition because we will be concerned with theories for which the above is not a set and thus having the domain the full universe is a more natural concept to work with.

Remark 6.2.4. We may on occasion abuse notation by calling an ordinal κ V-critical *if* there exist predicates M and j for which there is an ORD-inary embedding despite it being unclear how to formally express this in the set theory we are working in. In ZFC this is not an issue because κ being V-critical is equivalent to there being a κ-complete non-principal ultrafilter on $\mathcal{P}(\kappa)$. In general, it will either be clear what j and M should be in this case (for example in Theorem 10.5.7) or the mention of a V-critical ordinal will be part of a motivational discussion, in which case one can also assume that they are given witnessing j and M. It is also worth noting that, in Theorem 5.8 of [Sch20a], Schlutzenberg proves that there is a first-order definition of V-critical in the theory ZF plus a proper class of what he calls *weakly Löwenheim-Skolem cardinals*.

The above proof makes use of taking a set of "*least rank*", rank being a Σ-recursive function that uses the linearity of the ordinals. However, when working in a weak system without any assumption of replacement or rank such an argument no longer works. An example of this is the theory Z. As witnessed by Mathias' Model 13 from [Mat06], it is possible to have a model of Zermelo in which the rank function is not everywhere defined. Moreover, using such a model we can find a non-trivial elementary embedding without a critical point.

Definition 6.2.5. A class M is said to be *supertransitive* if for any $x \in \mathrm{M}$, $\mathcal{P}(x) \subseteq \mathrm{M}$.

Definition 6.2.6 (Mathias, [Mat06] Model 13)**.** Let λ be a limit ordinal. Define

$$\mathbf{A}_{13,\lambda} \coloneqq \{u \mid \bigcup u \subseteq u \ \wedge \ \sup(u \cap \lambda) < \lambda\}; \qquad \mathbf{M}_{13,\lambda} \coloneqq \bigcup \mathbf{A}_{13,\lambda}.$$

Proposition 6.2.7 (Mathias)**.** *Under* ZFC*, for any limit ordinal* λ*,* $\mathbf{M}_{13,\lambda}$ *is a supertransitive model of* $\mathrm{ZC} + \forall x\ \exists y\ (\mathrm{Trans}(y) \ \wedge \ x \in y)$ *in which the rank function is not everywhere defined.*

Proposition 6.2.8. *Suppose that* V *is a model of* ZFC *and* κ *is a measurable cardinal. Then there exist sets* $\mathrm{N} \subseteq \mathrm{W}$ *and a non-trivial elementary embedding* $j \colon \mathrm{W} \to \mathrm{N}$ *such that* $\mathrm{W} \models \mathrm{ZC}_j$ *but* j *does not have a critical point.*

Proof. Let $j \colon \mathrm{V} \to \mathrm{M}$ be an elementary embedding with critical point κ arising from a normal measure on κ and fix $\delta < \kappa$ to be an uncountable cardinal. Next, fix $\mu > \kappa$ to also be a strong limit cardinal such that $j(\mu) = \mu$, and let

$$\mathrm{W} \coloneqq \mathbf{M}_{13,\delta} \cap \mathrm{H}_\mu.$$

Then, clearly, W is a supertransitive model of ZC since $\mathbf{M}_{13,\delta}$ is. Now, define N as

$$\mathrm{N} \coloneqq j(\mathrm{W}) = \bigcup \{u \mid \bigcup u \subseteq u \ \wedge \ u \cap \delta < \delta \ \wedge \ (u \in \mathrm{H}_\mu)^{\mathrm{M}}\} = (\mathbf{M}_{13,\delta} \cap \mathrm{H}_\mu)^{\mathrm{M}}.$$

Therefore, we have that $\mathrm{N} \subseteq \mathrm{W}$ and $j \restriction \mathrm{W}$ is an elementary embedding. Moreover, it is clear that $\mathrm{W} \cap \mathrm{ORD} = \delta$, so $j \restriction \mathrm{W}$ does not move an ordinal.

Next, to show that $\mathrm{W} \models \mathrm{ZC}_j$ it suffices to prove that W models Separation$_j$. To see this, let φ be a formula in the language expanded to include $j \restriction \mathrm{W}$ as a predicate and let $a \in \mathrm{W}$. Then, since Separation$_j$ holds in V, $b \coloneqq \{x \in a \mid \varphi^{\mathrm{W}}(x)\}$ is a subset of a so, by supertransitivity, $b \in \mathrm{W}$.

Finally, to see that $j \restriction \mathrm{W}$ is non-trivial, define $u_0 \coloneqq \{0, \{0\}, \{\{0\}\}\}$. Then we can define a sequence of sets inductively as

$$u_{\nu+1} = u_\nu \cup \{u_\nu\}; \qquad u_\gamma = \bigcup_{\nu \in \gamma} u_\nu \textit{ for } \gamma \textit{ a limit.}$$

It is easy to see that each u_ν is in $\mathbf{A}_{13,\delta} \cap \mathrm{H}_\mu$ and if $\nu < \nu'$ then $u_\nu \in u_{\nu'}$. Thus $u_\kappa, u_{j(\kappa)} \in \mathrm{W}$ and $j(u_\kappa) = u_{j(\kappa)}$, so the embedding is indeed non-trivial.[5] □

To circumvent this insufficiency, we can use a different proof of Proposition 6.2.1 which was given by Hamkins on MathOverflow, [Hama]. This will replace the existence of a total rank function by Axiom W; that every set is well-ordered by a well-ordering isomorphic to an ordinal.

Proposition 6.2.9. *Let* $\mathrm{N} \subseteq \mathrm{M}$ *be transitive class models of* ZBQW^-. *Suppose that* $j \colon \mathrm{M} \to \mathrm{N}$ *is a non-trivial,* Σ_0*-elementary embedding and* $\mathrm{M} \models \mathrm{ZBQW}^-_{j,\mathrm{N}}$. *Then there exists an ordinal* α *such that* $j(\alpha) > \alpha$.

Proof. As before, we first note that for every ordinal α, $\alpha \leq j(\alpha)$. So suppose, for a contradiction, that j fixes every ordinal. Using $\in$-induction, take a to be an $\in$-minimal set with $a \neq j(a)$. Then, by definition, for any $x \in a$, $x = j(x)$ so $a \subseteq j(a)$.

Using Well-Ordering, we can fix an ordinal γ and a set f such that

$$f \textit{ is a bijection between } \gamma \textit{ and } a.$$

Since this can be expressed by a Σ_0-formula, it follows that $j(f)$ is a bijection between $j(\gamma) = \gamma$ and $j(a)$. Moreover, since j fixes every element of a by minimality, $f(\beta) = j(f)(j(\beta))$. Now, by assumption, $\beta = j(\beta)$ and therefore $f(\beta) = j(f)(\beta)$ for every $\beta \in \gamma$. But this means that $j(f) = f$, so

$$j(a) = j(f``\gamma) = f``\gamma = a,$$

yielding our contradiction. □

[5] In fact any set in W whose actual rank in V is κ will witness non-triviality. What we have given is an example of the seemingly stronger assertion that for some set x we have $x \in j(x)$.

There are two further necessary assumptions in the above propositions worth noting. The first is that $\mathrm{N} \subseteq \mathrm{M}$. While this assumption is not needed over ZFC ([Kan08], Proposition 5.1) it is required when working without choice because, over ZF, it is possible to have non-trivial elementary embeddings $j\colon \mathrm{M} \to \mathrm{N}$ which fix every ordinal. A proof of this can be found in [Cai03] or alternatively on MathOverflow, [Cai].

Secondly, the use of classical logic is an important point to raise here, in particular, the ability to choose an *x of least rank*. Without the law of excluded middle, the ordinals are not linearly ordered and therefore it is not possible to define a "*least*" ordinal moved by the embedding and it is possible to make embeddings with multiple "critical points". Moreover, the existence of an ordinal α such that $\alpha \neq j(\alpha)$ won't necessarily give us that $\alpha \in j(\alpha)$ as we shall see in Section 7.1.

Proposition 6.2.10. *Let* $\mathrm{M} \subseteq \mathrm{V}$ *be transitive class models of* ZBQ^-. *Suppose that* $j\colon \mathrm{V} \to \mathrm{M}$ *is a non-trivial,* Σ_0*-elementary embedding with critical point* κ *and* $\mathrm{V} \models \mathrm{ZBQ}^-_{j,\mathrm{M}}$. *Then* κ *is a regular cardinal.*

Proof. The conclusion of the proposition will follow from proving that

for any $\alpha \in \kappa$ and any function $f\colon \alpha \to \kappa$ we can find an ordinal $\beta \in \kappa$ such that the class $\mathrm{ran}(f)$ is contained in β.

To see this, fix $\alpha \in \kappa$ and let $f\colon \alpha \to \kappa$ be a function. Then, since α is fixed by j and being a function is Σ_0-definable, $j(f)\colon \alpha \to j(\kappa)$. Next, for any $\gamma \in \alpha$ we have

$$j(f)(\gamma) = j(f)(j(\gamma)) = j(f(\gamma)) = f(\gamma)$$

from which we can conclude that $j(f) = f$. So, since the range of f is contained in κ, we have that

$$\mathrm{M} \models \exists \beta \in j(\kappa)\ \forall \gamma \in \alpha\ j(f)(\gamma) \in \beta,$$

which, by elementarity, gives us that

$$\mathrm{V} \models \exists \beta \in \kappa\ \forall \gamma \in \alpha\ f(\gamma) \in \beta.$$

Hence the claim, and therefore the proposition, is true. □

Remark 6.2.11. It is worth remarking that, so far, at no point in this section have we explicitly used the Axiom of Infinity. In particular, if we removed condition (v.) from Definition 6.1.1 and κ was the critical point of a Σ-elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ where V is a model of ZBQ^- formulated without infinity, then κ is still a regular cardinal. Therefore, the above assumptions prove the Axiom of Infinity. That is,

$$\mathrm{ZBQ}^- \setminus \{\mathrm{Infinity}\} + \exists \kappa \; (\kappa \textit{ is the critical point of } j\colon \mathrm{V} \to \mathrm{M}) \vdash \mathrm{Infinity}$$

Moreover, one can see that in this case $\mathrm{M}(\omega)$ holds by elementarity and the absoluteness of ω. The decision to include condition (v.) comes from Definition 3 of [FŠ84] where they are defining elementary embeddings in IZF. This ensures that the true ordinal ω is in M which is not obviously true when working in an intuitionistic setting.

A crucial property that need not hold if we only assume Σ_0-elementarity is that

$$j(\mathcal{P}(x)) = \mathcal{P}(j(x)).$$

For example, consider the earlier Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ given by an ultrafilter $U \subseteq \kappa$ over ZFC. It is then easy to see that U is not a set in the ultrapower M and so

$$j(\mathcal{P}^{\mathrm{V}}(\kappa)) \neq \mathcal{P}^{\mathrm{V}}(j(\kappa)).$$

However, by expanding to Σ-elementarity this issue can be overcome.

Proposition 6.2.12. *If $j\colon \mathrm{V} \to \mathrm{V}$ is a Σ-elementary embedding then, for any set x,*

$$j(\mathcal{P}(x)) = \mathcal{P}(j(x)).$$

Proof. To see this, let $\varphi(x, y)$ be the formula

$$\forall s, t \; (s \in y \longleftrightarrow (t \in s \to t \in x)).$$

Then it is clear that $\varphi(x, y)$ holds if and only if $y = \mathcal{P}(x)$. Since j is Σ-elementary, or more accurately Π-elementary, $\varphi(x, y) \longleftrightarrow \varphi(j(x), j(y))$, which yields

$$\mathcal{P}(j(x)) = j(y) = j(\mathcal{P}(x)).$$

□

Using a similar argument, if j is a Σ-elementary embedding of V to itself then, for any sets a and b,

$$j({}^a b) = {}^{j(a)} j(b).$$

We end this section with an important property of elementary embeddings that we will use in our analysis of embeddings of the universe in weak systems: the notion of *cofinality.*

Definition 6.2.13. An elementary embedding $j \colon \mathrm{M} \to \mathrm{N}$ is said to be *cofinal* if for every $y \in \mathrm{N}$ there is some $x \in \mathrm{M}$ such that $y \in j(x)$.

Proposition 6.2.14. *Suppose that* $j \colon \mathrm{M} \to \mathrm{N}$ *is an elementary embedding,* $\mathrm{M} \models \mathrm{ZF}_j$*,* $\mathrm{N} \subseteq \mathrm{M}$ *and* $\textsc{Ord} \cap \mathrm{M} = \textsc{Ord} \cap \mathrm{N}$*. Then* j *is cofinal.*

Proof. Since M and N have the same ordinals, we have that for any $\beta \in \mathrm{N}$ there is some $\alpha \in \mathrm{M}$ such that $\beta \in j(\alpha)$. Now fix $y \in \mathrm{N}$. Then, y has rank less than β for some β, so

$$y \in (\mathrm{V}_\beta)^{\mathrm{N}} \subseteq (\mathrm{V}_{j(\alpha)})^{\mathrm{N}} = j((\mathrm{V}_\alpha)^{\mathrm{M}}).$$

□

6.3 The Critical Sequence

One of the most important properties of an $\textsc{Ord}$-inary embedding shall be its *critical sequence* which is the sequence $\langle \mathrm{crit}(j), j(\mathrm{crit}(j)), j^2(\mathrm{crit}(j)), \dots \rangle$. In this section we shall explore the definability of this sequence. It will be shown that $\mathrm{Induction_j}$ is a necessary condition to ensure that the function $n \mapsto j^n(\mathrm{crit}(j))$ is total by looking at what Corazza calls Hatch's Model in [Cor06]. This is a model with non-standard natural numbers in which the function defined above is not provably total. Most of the ideas in this section come from the work of Corazza and we are grateful to Joel David Hamkins for pointing out the essential use of $\mathrm{Induction_j}$ for the arguments we give in later chapters. It is worth remarking that we will in fact only require Σ^j-Induction to prove totality of the function and therefore the results will go through in the weaker versions of KP and ZBQ as defined by Mathias and mentioned in Remark 2.1.7.

Definition 6.3.1. Suppose that $\mathrm{N} \subseteq \mathrm{M}$ and $j\colon \mathrm{M} \to \mathrm{N}$ is a non-trivial, Σ_0-elementary embedding with critical point κ. Then the *critical sequence* of j is the sequence $\langle j^n(\kappa) \mid n \in \omega \textit{ and } j^n(\kappa) \textit{ exists}\rangle$.

When $j^n(\kappa)$ exists we will alternatively refer to it as κ_n.

In order to show that $j^n(\kappa)$ exists for every $n \in \omega$ we require the Σ^j-definable functions from the end of Section 2 of [Cor06]. Because in our very weak systems we will only be discussing the critical sequence of an embedding $j\colon \mathrm{V} \to \mathrm{V}$, we will only consider this situation rather than the more general $j\colon \mathrm{V} \to \mathrm{M}$ case. We will also work in the base theory of ZBQ^-.

Definition 6.3.2 (Corazza, [Cor06])**.** Working over ZBQ^-, suppose that $j\colon \mathrm{V} \to \mathrm{V}$ is a non-trivial $\textsc{Ord}$-inary embedding such that $\mathrm{V} \models \mathrm{ZBQ}^-_j$. For this section, let Γ, Υ and Φ denote the following three Σ^j-definable formulae:

$$\begin{aligned}
\Gamma(f,n,x,y) &\equiv \mathrm{func}(f) \ \wedge \ \mathrm{dom}(f) = n+1 \ \wedge \ f(0) = x \ \wedge \\
&\qquad \forall i \ \Big(0 < i \leq n \to f(i) = j(f(i-1))\Big) \ \wedge \ f(n) = y, \\
\Upsilon(n,x,y) &\equiv n \in \omega \to \exists f \ \Gamma(f,n,x,y), \\
\Phi(n,y) &\equiv \exists x \in y \ \exists z \ \Big(x \neq z \ \wedge \ \Upsilon(n,x,y) \ \wedge \ x \in \textsc{Ord} \ \wedge \\
&\qquad \forall \alpha \in x \ (j(\alpha) = \alpha) \ \wedge \ j(x) = z\Big).
\end{aligned}$$

Explaining what these functions signify: $\Gamma(f,n,x,y)$ denotes that f is the function with domain $n+1$ computing $y = f(n) = j^n(x)$. Then $\Upsilon(n,x,y)$ says that when n is a natural number we can find such an f. Finally, $\Phi(n,y)$ holds whenever $y = j^n(x)$ for some ordinal x which is not fixed by j but whose elements are all fixed by j. Namely, $y = j^n(\mathrm{crit}(j))$.

We shall now prove that the formulae Γ and Φ define class functions with domain ω.

The following Theorem and proof originally appears in Proposition 4.4 of [Cor06] where Corazza works in the theory ZFC+BTEE (the scheme of axioms asserting the existence of a fully elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ with a critical point) $+\Sigma^j$-Induction. The proof will be the same as Corazza's but we include it here for completeness.

Theorem 6.3.3 (Corazza, [Cor06] Proposition 4.4)**.** *Working over* ZBQ^-*, suppose that* $j\colon \mathrm{V} \to \mathrm{V}$ *is a non-trivial,* $\textsc{Ord}$*-inary embedding and* $\mathrm{V} \models \mathrm{ZBQ}^-_j$*. Then:*

1. *For all* n, x, y *there is at most one* f *for which* $\Gamma(f, n, x, y)$ *holds. That is*
$$\forall n \in \omega\ \forall x, y, f, g\ \Big(\Gamma(f, n, x, y)\ \wedge\ \Gamma(g, n, x, y) \to f = g\Big).$$
2. $\Upsilon(n, x, y)$ *defines a class function. That is*
$$\forall n \in \omega\ \forall x\ \exists! y\ \Upsilon(n, x, y).$$
3. $\Phi(n, y)$ *defines a class function. That is*
$$\forall n \in \omega\ \exists! y\ \Phi(n, y).$$

Proof. To begin with, we observe that the following statement can be proved by an instance of bounded induction in ZBQ^-:

Suppose $n \in \omega$, f *and* g *are functions with domain* $n+1$, $f(0) = g(0)$ *and* $f \neq g$.
Then there is a least i *with* $1 \leq i \leq n$ *for which* $f(i) \neq g(i)$. $(\star\star)$

So fix n, x and y and suppose that there are $f \neq g$ for which $\Gamma(f, n, x, y)$ and $\Gamma(g, n, x, y)$ both hold. Since, by definition, $f(0) = g(0)$ by $(\star\star)$ we can fix i with $1 \leq i \leq n$ to be the least number for which $f(i) \neq g(i)$. But then, by the definition of Γ,

$$f(i) = j(f(i-1)) = j(g(i-1)) = g(i)$$

contradicting the assumption on i. Thus 1 holds.

For 2, we first establish uniqueness. So fix $n \in \omega$ and x and suppose that we have $\langle y_1, f_1\rangle$ and $\langle y_2, f_2\rangle$ for which both $\Gamma(f_1, n, x, y_1)$ and $\Gamma(f_2, n, x, y_2)$ hold. By the same argument as was used in part 1, $f_1 = f_2$. Then, by the definition of f in Γ,

$$y_1 = f_1(n) = f_2(n) = y_2$$

so uniqueness does indeed hold. To show the $\Upsilon(n, x, y)$ is a class function we shall show that for any fixed set a,

$$\forall n \in \omega \ \gamma(n, a)$$

where

$$\gamma(n, a) \coloneqq \exists y \ \Upsilon(n, a, y).$$

This shall be done by an instance of Σ^j-Induction on the formula γ, noting that $\Upsilon(0, a, j(a))$ always holds and therefore so does $\gamma(0, a)$. For the induction step, suppose that z satisfies $\Upsilon(n, a, z)$ as witnessed by the function f with domain $n + 1$. Setting $f' \coloneqq f \cup \{\langle n + 1, j(f(n))\rangle\}$, it is clear that f' witnesses $\Upsilon(n + 1, a, j(z))$. Thus, by Σ^j-Induction, $\forall n \ \gamma(n, a)$ holds from which the proof of 2 follows.

Part 3 will be proved in a similar manner. First note that, since the ordinals are linearly ordered, the critical point is unique. That is, if x_1 and x_2 are ordinals for which

$$\forall \alpha \in x_i \ (j(\alpha) = \alpha) \ \wedge \ j(x_i) \neq x_i$$

then $x_1 = x_2$. So, for uniqueness, suppose that y_1 and y_2 are such that $\Phi(n, y_1)$ and $\Phi(n, y_2)$ hold. By the definition of Φ and the uniqueness of the critical point, this means that there is an x for which $\Upsilon(n, x, y_1)$ and $\Upsilon(n, x, y_2)$ both hold. But then, by part 2, $y_1 = y_2$. To show that $\Phi(n, y)$ is a class function we again use Σ^j-Induction and the remark that $\Phi(0, \mathrm{crit}(j))$ holds. It is then obvious that if $\Phi(n, y)$ holds then so does $\Phi(n + 1, j(y))$. Thus $\forall n \in \omega \ \exists y \ \Phi(n, y)$ holds which proves 3. □

Remark 6.3.4. Observe that in the previous proof the formulae we needed were Σ^j-definable and therefore we only needed to use Σ^j-Induction (or equivalently Π^j-Foundation). Therefore the previous theorem will go through even if we use the version of KP or ZBQ^- as formulated by Mathias in [Mat01], see Remark 2.1.7.

Since $\Phi(n, y)$ defines a class function, if we have Σ^j-Replacement (and Σ^j-Induction) then the critical sequence is provably a set and in particular its supremum exists. Notably, this will go through for elementary embeddings $j \colon \mathrm{V} \to \mathrm{V}$ where $\mathrm{V} \models \mathrm{KP}_j$.

Corollary 6.3.5. *Suppose that* $\mathrm{V} \models \mathrm{KP}$ *and* $j\colon \mathrm{V} \to \mathrm{V}$ *is a non-trivial,* Ord*-inary embedding such that* $\mathrm{V} \models \mathrm{KP}_j$*. Then* $\langle j^n(\mathrm{crit}(j)) \mid n \in \omega \rangle$ *is a set.*

Following a suggestion from Hamkins, we end this section with a brief discussion on the necessity for at least Σ^j-Induction in our formulations of KP_j and ZBQ_j^-. For this, we first define *The Wholeness Axiom* which we will further study in Section 9.2.

The Wholeness Axiom is an axiom proposed by Corazza [Cor00] in order to quantify the extension of ZFC needed to derive the Kunen Inconsistency. The intention was to weaken the access V has to j while still allowing enough access to prove the existence of very large cardinals. This will be done by not allowing instances of j in the Replacement Scheme and restricting where instances of j can appear in the Separation Scheme. Removing Replacement means that we are unable to define the supremum of the critical sequence, which will be an essential component of the proof of the Kunen inconsistency. To be more precise, and using the notation from [Ham01],

Definition 6.3.6 (Hamkins)**.** For $n \in \omega$, let WA_n consist of the following conditions:

1. (Elementarity) For any formula φ and set x, $\varphi(x) \leftrightarrow \varphi(j(x))$,
2. (Σ_n-Separation) All instances of the Σ_n-Separation Scheme in the language expanded to include a predicate for j,
3. (Non-triviality) The axiom $\exists x\ (x \neq j(x))$.

Let WA_∞ be defined as above except condition 2 is expanded to allow all instances of the Separation Scheme in the language expanded to include a predicate for j.

Over ZFC, an alternative, useful, characterisation of WA_0 is that j is a non-trivial, *amenable* elementary embedding where

Definition 6.3.7. An embedding $j\colon \mathrm{M} \to \mathrm{M}$ is said to be *amenable* if for every set $a \in \mathrm{M}$, $j \restriction a \in \mathrm{M}$.

Lemma 6.3.8 (Corazza, [Cor06] Lemma 8.6)**.**

$$\mathrm{ZFC} + \mathrm{BTEE} \vdash \Sigma_0^j\textit{-Separation} \longleftrightarrow \text{“}j \textit{ is amenable”}.$$

As shown by Corazza, the assumption that there is an embedding witnessing Wholeness is very strong. For example, such an embedding has a critical cardinal which will be supercompact, extendible and even super-n-huge for every natural number n. The natural upper bound, over ZFC, for the Wholeness Axiom is an I_3 embedding.

Definition 6.3.9. I_3 is the assertion that there exists a non-trivial, elementary embedding $j\colon V_\lambda \to V_\lambda$.

We can now define the model of ZBQ in which there is a non-trivial, ORD-inary embedding with a critical point but for which the critical sequence is not total. Note that, by Theorem 6.3.3, Σ^j-Induction must necessarily fail in this model. The model we will define is referred to as the *Hatch Model* in [Cor06] and was independently observed by Hamkins.

So, working with a background theory of ZFC, suppose that we have a model of $\mathrm{ZFC}+\mathrm{WA}_0$. By a standard application of the Compactness Theorem, there is a model, $\mathcal{M} = \langle \mathrm{M}, \mathrm{E}, j\rangle$, of $\mathrm{ZFC} + \mathrm{WA}_0$ with critical point κ in which the natural numbers are nonstandard. Next, take the model constructed by cutting M off at the supremum of the critical sequence for standard n. That is, let

$$\mathrm{N} \coloneqq \{x \in \mathrm{M} \mid \exists n \in \omega\ \mathcal{M} \models \mathrm{rank}(x) < j^n(\kappa)\}$$

and take $i \coloneqq j \restriction \mathrm{N}$. By examining the argument at the end of Chapter 10 of [Kun80], one can see that $\mathcal{N} \coloneqq \langle \mathrm{N}, \mathrm{E}, i\rangle$ is a model of ZFC (and therefore ZBQ). Next, it is clear to see that $i\colon \mathrm{N} \to \mathrm{N}$ is a non-trivial, elementary embedding and that

$$\forall x \in \mathrm{N}\ \mathcal{N} \models \exists z\ (z = i \restriction x).$$

Therefore i is an amenable embedding so Σ^i_0-Separation holds in $\mathcal{N}$. However, in $\mathcal{N}$, $j^n(\kappa)$ only exists for *standard* n and so the critical sequence is not total. In particular we have a model of

$$\mathrm{ZBQ} + \text{“}i\colon \mathrm{V} \to \mathrm{V}\ \textit{is an elementary embedding}\text{”} + \Sigma^i_0\textit{-Separation} + \neg\Sigma^i_1\textit{-Induction}$$

in which $\Phi(n, y)$ does not define a total class function.

6.4 A Historical Overview

Having introduced the main ideas necessary to formalise the notion of an elementary embedding, we end this chapter with a brief literature review of some of the results concerning large cardinals in weak systems. These results will not be given in chronological order but rather presented in such a way as to try and give a coherent outline as to what has been done before.

Much of the work on large cardinals over the last sixty years can be traced back to the seminal paper *Strong axioms of infinity and elementary embeddings* [SRK78] by Solovay, Reinhardt and Kanamori. This introductory paper outlined many of the large cardinal notions which still preoccupy a large amount of current work in set theory. Their guiding principal was to study various ways to strengthen the notion of *measurability*, with the motivation being that the closer the inner model, M, was to V, the stronger the resulting large cardinal. It then turned out that this basic framework led to a hierarchy of principles, which could in general be linearly ordered by consistency strength. For example, one could:

- Close M under arbitrarily large segments of the cumulative hierarchy of V, that is $\mathrm{V}_{\mathrm{crit}(j)+\gamma} \subseteq \mathrm{M}$. This leads to the notion of the critical point being γ-*strong*.
- Close M under arbitrarily large sequences, that is ${}^{\gamma}\mathrm{M} \subseteq \mathrm{M}$. This leads to the notion of the critical point being γ-*supercompact*.
- Restricting the domain of the elementary embedding to some stage of the cumulative hierarchy and having it embed into a larger segment. This leads to the notion of *extendibility*.

Definition 6.4.1. A cardinal κ is said to be η-*extendible* if there is a γ and an elementary embedding $j\colon \mathrm{V}_{\kappa+\eta} \to \mathrm{V}_{\gamma}$ with $\mathrm{crit}(j) = \kappa$ and $\eta < j(\kappa)$.

A cardinal κ is said to be *extendible* if it is η-extendible for every $\eta > 0$.

- Close M under sequences of length $j^n(\text{crit}(j))$ for n a fixed natural number. This leads to the notion of *n-hugeness*.

The natural conclusion to this process, which is first briefly mentioned at the end of Reinhardt's thesis [Rei67], is the existence of a *Reinhardt embedding*. That is, an embedding

$$j\colon \mathrm{V} \to \mathrm{V}.$$

However, soon after this "*ultimate*" large cardinal assumption was proposed it was shown to be inconsistent with the axioms of second-order ZFC by Kunen. The proof of this was first given in [Kun71] using Jónsson functions and was later included in [SRK78]. There are a variety of alternative proofs of the Kunen inconsistency, notably Woodin's proof using Solovay's Lemma on splitting stationary sets and Harada's using more of the structural properties of the resulting ultrafilter. All three of these proofs can be found in Section 23 of [Kan08]. It is also worth mentioning a further proof by Zapletal, [Zap96], using results from PCF theory.

While Kunen formally worked in the full second-order theory of Kelley Morse it could be easily seen that what was required was the second-order fragment ZFC_j.

Theorem 6.4.2 (Kunen, [Kun71])**.** *Over* ZFC*, There is no non-trivial elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ for which $\mathrm{V} \models \mathrm{ZFC}_j$.*

After stating this, Kunen, following a suggestion of the referee, briefly remarks that the Axiom of Choice is necessary for this result and that it is unknown if Reinhardt embeddings are consistent with ZF. This was then formally stated as open question 1.13 in [SRK78] and has become one of the leading focal questions in all of set theory.

An in-depth study of Reinhardt-type cardinals in ZF without Choice was undertaken by Bagaria, Koellner and Woodin in [BKW19]. Here, they identify a series of strengthening of Reinhardt cardinals such as *Super Reinhardt*, *Totally Reinhardt* and *Berkeley* cardinals and this is further developed by Cutolo [Cut18]. A lower bound for the consistency strength of ZF + DC plus a Reinhardt cardinal was later isolated by Goldberg [Gol20], where he obtained that

Theorem 6.4.3 (Goldberg, [Gol20] Theorem 6.16)**.** *Over* ZF + DC*, the existence of a* Σ*-elementary embedding from* $V_{\lambda+3}$ *to* $V_{\lambda+3}$ *implies* $\mathrm{Con}(\mathrm{ZFC} + \mathrm{I}_0)$*, where* I_0 *is the assertion that for some* δ *there is an elementary embedding* $j\colon L(V_{\delta+1}) \to L(V_{\delta+1})$ *with critical point less than* δ*.*

Theorem 6.4.4 (Goldberg, [Gol20] Theorem 6.20)**.** *Over* ZF + DC*,* $\mathrm{AC} + \mathrm{I}_0$ *is equiconsistent with the following statement;*

For some ordinal λ*, there is an elementary embedding* $j\colon V_{\lambda+2} \to V_{\lambda+2}$*.*

Next, recall that in Section 6.1 the second way to formalise elementary embeddings was to assert that j was a class which is definable from a parameter. While this idea had been known for a long time, it was first explicitly studied by Suzuki in [Suz99]. There, Suzuki showed that there is no definable Reinhardt embedding over ZF.

The notion of definable embeddings has recently been studied further by Goldberg and Schlutzenberg in a series of papers, [Sch20a], [Sch20b], [Gol20] and [GS20]. Here, the authors study the structure of rank-to-rank embeddings over ZF, proving, amongst other things, that given an elementary embedding $j\colon V_{\alpha+1} \to V_{\alpha+1}$, it is definable from parameters over $V_{\alpha+1}$ if and only if $\alpha + 1$ is an odd ordinal.

By examining Kunen's proof that Reinhardt cardinals are inconsistent with ZFC one can refine the result in a way which is first-order.

Theorem 6.4.5 ([Kan08] Corollary 23.14)**.** *Assume that* V *is a model of* ZFC*. Then, for any ordinal* δ*, there is no non-trivial, elementary embedding* $j\colon V_{\delta+2} \to V_{\delta+2}$*.*

This refinement has led to a series of large cardinals known as $\mathrm{I}_0 - \mathrm{I}_3$. $\mathrm{I}_1 - \mathrm{I}_3$ were originally considered by Gaifman [Gai74] and were later studied in [SRK78], while I_0 was introduced by Woodin to study the Axiom of Determinacy. A full account of these axioms along with many of their interesting structural consequences can be found in Chapter 24 of [Kan08] or the survey article [Dim18] by Dimonte.

A second way to refine Kunen's result can be found in work of Eskew and Friedman, [EF19].

Theorem 6.4.6 (Eskew, Friedman [EF19] Theorem 1)**.** *Suppose that* $j\colon \mathrm{M} \to \mathrm{N}$ *is a non-trivial, elementary embedding between two transitive models of* ZFC *with the same class of ordinals* Ω*. Then at least one of the following holds:*

1. *The critical sequence* $\langle j^n(\mathrm{crit}(j)) \mid n \in \omega\rangle$ *is cofinal in* Ω*,*
2. *For some* $\alpha \in \Omega$*,* $j``\alpha \notin \mathrm{N}$*,*
3. *For some* $\alpha \in \Omega$*,* α *is regular in* M *and singular in* N*.*

In this paper the authors study the possibility of having various combinations of these three possibilities. With particular reference to the study of Reinhardt cardinals, if M is equal to N then Theorem 6.4.6 tells us that the closure of M, as measured in V, must run out at some point. However, assuming the existence of a supercompact cardinal, they notably prove that this closure can be arbitrarily high up in the universe.

Theorem 6.4.7 (Eskew, Friedman [EF19] Theorem 2)**.** *Suppose that* $\kappa \leq \lambda$ *are regular cardinals. Then* κ *is* λ*-supercompact iff there exists a transitive class* M *with* ${}^{\lambda}\mathrm{M} \subseteq \mathrm{M}$ *and a non-trivial, elementary embedding* $j\colon \mathrm{M} \to \mathrm{M}$ *with critical point* κ*.*

A related approach involving embeddings between transitive class models of ZFC is taken by Hamkins, Kirmayer and Perlmutter in [HKP12] where the authors show that, for any set generic G, there can be no non-trivial, elementary embedding from V to V$[G]$ (Theorem 7) or from V$[G]$ to V (Theorem 5). A similar result holds between any two set forcing extensions, or between V and HOD. Such results are proven by adapting Woodin's version of the Kunen Inconsistency.

A final way to consider Kunen's inconsistency is in the language with a predicate for j. Then the theorem tells us that the universe cannot satisfy the full fragment ZFC_j, that is, there must be an instance of separation or replacement in the language expanded to include j which does not hold. The precise amount of separation and replacement needed to derive an inconsistency has been determined by Corazza in [Cor00] and [Cor06]. In these papers he also develops the *Wholeness Axioms*, which was further studied by Hamkins in [Ham01] and which we shall discuss in more detail in Section 9.2.

To date there has been very little work on large cardinals in weak subtheories of ZFC as a central topic, with most of the results appearing as minor comments alongside traditional large cardinal study. Examples of this include [GHJ16] where the authors study the effect of only assuming the Replacement Scheme rather than the stronger Collection Scheme over ZFC without Power Set. They prove that consistently one can have normal ultrapowers over models of ZFC− which are well-founded but whose ultrapower map is not elementary. Moreover, they construct elementary embeddings $j\colon \mathrm{M} \to \mathrm{N}$ which are Σ_1-elementary and cofinal but not Σ_2-elementary.

Another example appears in the work of Holmes, Forster and Libert [HFL12]. In the final chapter they mention that ZFC plus the Wholeness Axiom should be considered as a Reinhardt embedding over a model of Zermelo with the Axiom of Choice and a rank hierarchy. This is an idea that we shall discuss further in Section 9.2.

A related branch of work is the study of Ramsey-like cardinals and various weakenings of measurability, notable examples of which include [Git11], [GW11], [HL16], [BM19] and [GS21]. In these papers, the authors study models of the form $\langle \mathrm{M}, \in, \mathcal{U} \rangle$ where M is a model of ZFC^-, $\mathcal{U}$ is a normal ultrafilter over M and M has a restricted amount of access to $\mathcal{U}$. One can then obtain a hierarchy of principles between a Ramsey cardinal and a measurable cardinal by, for example, asserting that M satisfies a larger fragment of the Separation and Collection schemes in the language expanded to include a predicate for $\mathcal{U}$. Alternative ways to increase the strength are to assume that $\mathcal{U}$ is a set in M, rather than a proper class, or that M is an elementary submodel of some H_θ.

There are two notable exceptions where the authors have decided to explicitly work in some very weak fragment. The first is [Agu20] where Aguilera discusses the Axiom of Determinacy in models of Kripke Platek plus the assumption that the reals constitute a set. The author shows that, from a model of ZFC plus the existence of ω^2 many Woodin cardinals with a measurable on top, one can produce a model of the above strengthening of KP which moreover satisfies the Axiom of Determinacy.

The second instance is [Tzo16] where Tzouvaras studies Vopěnka's Principle, VP, over a fragment he calls *Elementary Set Theory*. EST consists of the axioms; Extensionality, Empty Set, Pairing, Union, Cartesian Product, Σ_0-Separation Scheme and Induction along ω. It is then proven that $\mathrm{EST} + \mathrm{VP}$ proves the Axiom of Infinity as well as the full schemes of Replacement and Separation. From this one can deduce that $\mathrm{EST} +$ *Foundation Scheme* $+ \mathrm{VP}$ is the same theory as $\mathrm{ZF} + \mathrm{VP}$, and also the corresponding result when one also assumes AC. To give an outline as to why this could be true, we begin by recalling Vopěnka's Principle.

Definition 6.4.8. VP is the assertion that for any formula φ in the language of set theory, if $X_\varphi = \{x \mid \varphi(x)\}$ is a proper class of $\mathcal{L}_\in$-structures, for some first-order language $\mathcal{L}$, then there are distinct $\mathcal{M}, \mathcal{N} \in X_\varphi$ such that $\mathcal{M} \prec \mathcal{N}$.

As commented upon by Tzouvaras, this can be seen as a set existence principle because if no two such structures embed into one another then the associated class must be a set. Therefore, by a careful examination of the axioms of ZFC, this allows us to deduce each instance in turn. For example, to prove that $\{X \mid X \subseteq A\}$ is a set, one works in the language

$$\mathcal{L}_A = \{\mathbb{U}\} \cup \{c_a \mid a \in A\}$$

where $\mathbb{U}$ is a unary predicate symbol. Then, for each $X \subseteq A$, one considers the $\mathcal{L}_A$-structure $\mathcal{M}_X = \langle A, X, \mathrm{id}_A \rangle$ where $\mathbb{U}$ will be interpreted as X. Next, an analysis of the class $\{\mathcal{M}_X \mid X \in \mathcal{P}(A)\}$ shows that it does not satisfy VP and thus it must be a set, from which one can deduce that $\mathcal{P}(A)$ is also a set.

To conclude this overview, we examine the literature concerning large cardinals in intuitionistic theories. The first major study of such principles was undertaken by Friedman and Ščedrov in [FŠ84] where they add many large cardinal assumptions to IZF. In particular, using a double negation translation, they show that IZF enhanced with axioms such as Inaccessible sets, Mahlo sets, Measurable embeddings, Supercompact embeddings and Huge embeddings are equiconsistent with their classical counterparts. Their methods are used to find a lower bound for a Reinhardt embedding over IKP in Section 7.4.

As previously mentioned in the preliminaries, versions of regular and inaccessible sets in constructive set theories have been investigated in depth by authors such as Aczel and Rathjen and many of their results can be found in [AR10]. Many of these ideas were then further extended in the theses of Gibbons [Gib02] and Ziegler [Zie14] where they begin to look into what properties one can deduce the critical point of an elementary embedding must satisfy. We shall continue this line of work throughout the next chapter when we study ORD-inary embeddings of IKP.

A final, very recent, exploration into the strength of such embeddings was by Jeon in [Jeo21] where the author studied Reinhardt embeddings over a theory known as CZF. Constructive Zermelo Fraenkel is a constructive variant of ZF that was specified by Aczel and Rathjen and the details of which can be found in [AR10]. This is a theory which is equiconsistent with KP but, if one adds the Law of Excluded Middle, then one in fact regains the entirety of ZF. This makes it a very useful constructive theory to work with. Furthermore, an analysis by Gambino in [Gam06] proves that one can interpret ZF^- in a topological version of the double negation interpretation of a model of CZF plus full Separation. Moreover, in Theorem 9.37 of [Zie14], Ziegler proves that any Reinhardt embedding over CZF is cofinal.

As a counterpoint to the investigation within this thesis, Jeon shows that an ORD-inary Reinhardt embedding over CZF with full Separation interprets a Reinhardt embedding over ZF^-. Moreover, if $j\colon \mathrm{V} \to \mathrm{V}$ is such an embedding and there is an inaccessible set K such that $K \in j(K)$, then one is able to find an interpretation of a model of $\mathrm{ZF} + \mathrm{WA}_0$. This is a much stronger bound than we will be able to derive at the end of Section 7.4. One reason for this is that, while both our results and the theorems of Jeon rely on double negation interpretations, IKP appears to be a much more impoverished theory which severely weakens what one can deduce.

Chapter 7

Intuitionistic Embeddings

In this chapter we shall explore the properties of *intuitionistic elementary embeddings* from the universe to some transitive class. In the classical case, such embeddings have very large consistency strength and imply important structural consequences for the set-theoretic universe. For example, under ZFC, the existence of a non-trivial elementary embedding

$$j \colon \mathrm{V} \to \mathrm{M}$$

implies, among other things, that $\mathrm{V} \neq \mathrm{L}$. We shall see that neither the consistency strength nor the structural results need be the case under IZF, even when $\mathrm{M} = \mathrm{V}$.

This is because, as we shall see in Theorem 7.1.9, it is consistent for there to be a non-trivial, elementary embedding $j \colon \mathrm{V} \to \mathrm{V}$ which moves an ordinal α, as long as we only require that $\alpha \neq j(\alpha)$ rather than $\alpha \in j(\alpha)$. In the classical case, this assertion is unnecessary by Theorem 6.2.1, because for any ordinal we have $\alpha \leq j(\alpha)$. The issue is that this assertion makes essential use of the linearity of the ordinals, which is a highly non-constructive principle.

As shown in [FŠ84], being Ord-inary is important for the consistency strength. They show that, by a double negation translation, if both V and M are models of IZF then the existence of an Ord-inary embedding is equiconsistent with ZFC plus a measurable cardinal.

Later on in this chapter we shall look at the consequences of elementary embeddings under the assumption that V is a model of IKP. It shall be shown that, even under this weak base theory, this definition has many interesting structural consequences, particularly concerning the constructible universe. Namely, in Theorem 7.3.2 we shall obtain that there is some ordinal $\kappa^{\#}$ such that

$$\mathrm{L}_{\kappa^{\#}} \models \mathrm{IZF}.$$

We shall then use a double negation translation to obtain a lower bound for the existence of such embeddings using the standard large cardinal hierarchy under ZFC.

Corollary 7.4.36.

$$\mathrm{IKP} + \exists j \colon \mathrm{V} \to \mathrm{M} \textit{ which is a } \Sigma\text{-}\textsc{Ord}\textit{-inary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{ZFC} + \textit{a proper class of weakly compact cardinals}).$$

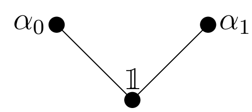
However, before doing this we shall show the importance of the embedding being Ord-inary in order for there to be an increase in consistency strength. This is done by showing that, if we just take the standard definition of non-triviality, the existence of such an embedding from a model of IZF to itself is consistent relative to IZF.

7.1 Automorphisms of the Universe

In this section we show that, consistently from a model of ZFC, one can produce a model of IZF + V = L with a non-trivial, definable, automorphism of the universe which moreover moves an ordinal. That is, we shall find some structure W which models IZF + W = L and some automorphism $\pi \colon \mathrm{W} \to \mathrm{W}$, definable in W, such that for some ordinal α, $\pi(\alpha) \neq \alpha$. This shows that, without making assumptions on the nature of the "*critical ordinal*", intuitionistically it is possible to have definable, non-trivial, elementary embeddings of the universe which is very unlike the classical case.

We begin by sketching the idea behind the argument before spending the rest of the section formalising it. So suppose that one were to have two distinct ordinals $\alpha_0, \alpha_1 \subseteq 1$ which are neither 0 nor 1. Further suppose that these ordinals have no discernible differences between them, that is $\varphi(\alpha_0)$ iff $\varphi(\alpha_1)$ for any formula φ. Then one could think of this as a model of IZF with "*atoms*". Now we know that there are non-trivial automorphisms of universes with atoms which are defined by permuting the atoms and then defining $\pi(x) = \{\pi(y) \mid y \in x\}$ at later stages. So therefore, using the fact that these two sets are indistinguishable from each other, the same map will give us a permutation of our model of IZF.

The approach we are going to take to do this is to produce a forcing model in the style of Lipton [Lip95] which was introduced in Section 2.6.2. Let V = L and let $\mathbb{P}$ be the partial order consisting of two nodes with one node below them both. That is,

$\mathbb{P} =$ 

where we let α_n denote the node given by $\{\langle 0, n\rangle\}$ and $\mathbb{1}$ denote the bottom element.

We now define our two new indistinguishable subsets of 1. Using the same tactic we employed in Definition 5.5.5, where we added an ordinal not in L, we let 1_{α_n} be the set that looks like 0 at α_n and 1 at α_{1-n}. That is $1_{\alpha_0} = \{\langle 0, \alpha_1\rangle\}$ and $1_{\alpha_1} = \{\langle 0, \alpha_0\rangle\}$.

Lemma 7.1.1. *In* V($\mathbb{P}$) *both* 1_{α_0} *and* 1_{α_1} *are ordinals.*

Proof. We shall prove this for 1_{α_0}, the other case being symmetric. The main thing to prove is;

$$\mathbb{1} \Vdash \forall x\ \forall y\ (x \in 1_{\alpha_0} \wedge\ y \in x \longrightarrow y \in 1_{\alpha_0}).$$

This, when combined with the monotonicity of $\Vdash$ from Lemma 2.6.13, will give us that V($\mathbb{P}$) believes that 1_{α_0} is a transitive set. The fact that it is a set of transitive sets follows by essentially the same argument. To see the main claim, let a and b be elements of V($\mathbb{P}$) and $p \in \mathbb{P}$, we need to show that

$$\text{if } p \Vdash a \in 1_{\alpha_0} \text{ and } p \Vdash b \in a \text{ then } p \Vdash b \in 1_{\alpha_0}.$$

There are 3 cases, one for each node:

Case 1 $p = \alpha_1$: In this case, $\alpha_1 \Vdash a \in 1_{\alpha_0}$ if and only if $\alpha_1 \Vdash a = \emptyset$. Therefore, for no $b \in \mathrm{V}(\mathbb{P})$ do we have that $\alpha_1 \Vdash b \in a$ so the conclusion holds.

Case 2 $p = \alpha_0$: In this case, $\alpha_0 \Vdash 1_{\alpha_0} = \emptyset$ and therefore for no $a \in \mathrm{V}(\mathbb{P})$ do we have that $\alpha_0 \Vdash a \in 1_{\alpha_0}$.

Case 3 $p = \mathbb{1}$: Note that $\mathbb{1} \Vdash a \in 1_{\alpha_0}$ if and only if we can find $c \in \mathrm{V}(\mathbb{P})$ and $q \geq \mathbb{1}$ such that $\langle c, q \rangle \in 1_{\alpha_0}$ and $q \Vdash a = c$. But clearly no such pair exists and therefore $\mathbb{1} \nVdash a \in 1_{\alpha_0}$ for any a.

Since we have proven the claim for each condition in $\mathbb{P}$, we do indeed have that 1_{α_0} is an ordinal. □

Note that, by Theorem 5.5.1, we cannot simply assert that these ordinals are in L. So we show that next. First it is easy to see that, since 1_{α_0} and 1_{α_1} are ordinals in $\mathrm{V}(\mathbb{P})$, $\mathrm{L}_{1_{\alpha_0}}$ and $\mathrm{L}_{1_{\alpha_1}}$ are well-defined sets.

Lemma 7.1.2. *For $n \in \{0, 1\}$, $\mathrm{V}(\mathbb{P}) \models 1_{\alpha_n} = \mathrm{L}_{1_{\alpha_n}}$. Hence $1_{\alpha_0}, 1_{\alpha_1} \in \mathrm{L}$.*

Proof. Again, we shall just prove this for α_0. If $p \Vdash x \in \mathrm{L}_{1_{\alpha_0}}$ then there is some β such that $p \Vdash x \in \mathrm{def}(\mathrm{L}_\beta) \ \wedge \ \beta \in 1_{\alpha_0}$. As before we have the same 3 cases: If $p = \alpha_1$ then $\alpha_1 \Vdash \beta \in 1_{\alpha_0}$ if and only if $\alpha_1 \Vdash \beta = \emptyset$. So, since $\mathbb{1} \Vdash \mathrm{L}_0 = \emptyset \ \wedge \ \mathrm{def}(\mathrm{L}_0) = \{\emptyset\}$, $\alpha_1 \Vdash x \in \mathrm{L}_{1_{\alpha_0}}$ if and only if $\alpha_1 \Vdash x = \emptyset$ which means that $\alpha_1 \Vdash \mathrm{L}_{1_{\alpha_0}} = 1_{\alpha_0}$.

For the second case, where $p = \alpha_0$, α_0 forces both 1_{α_0} and $\mathrm{L}_{1_{\alpha_0}}$ to be empty. The final case with $p = \mathbb{1}$ follows from the fact that $\mathbb{1} \nVdash a \in 1_{\alpha_0}$ for any $a \in \mathrm{V}(\mathbb{P})$. Because of this, we have that for any β, $\mathbb{1} \nVdash (a \in \mathrm{def}(\mathrm{L}_\beta) \wedge \beta \in 1_{\alpha_0})$ so

$$\mathbb{1} \nVdash a \in 1_{\alpha_0} \text{ and } \mathbb{1} \nVdash (a \in \mathrm{def}(\mathrm{L}_\beta) \wedge \beta \in 1_{\alpha_0})$$

and the conclusion follows. □

Lemma 7.1.3. $V(\mathbb{P}) \models 1_{\alpha_0} \neq 1_{\alpha_1}$.

Proof. We need to show that any node can be extended to a node which does not force 1_{α_0} to be equal to 1_{α_1}. So it just suffices to prove that

$$\alpha_0 \Vdash 1_{\alpha_0} \neq 1_{\alpha_1}.$$

But this is the case since $\langle \emptyset, \alpha_0 \rangle \in 1_{\alpha_1}$ and $\alpha_0 \Vdash \emptyset \notin 1_{\alpha_0}$ since 1_{α_0} is empty at node α_0. □

Remark 7.1.4. We remark here that the above lemma holds for the given partial order and not for a partial order with more than two nodes attached directly to the base. This is because in that case, if α_n, α_m and α_k are distinct nodes then α_k will believe that 1_{α_n} and 1_{α_m} are both equal to 1. But then $\mathbb{1}$ can either be extended to a condition where 1_{α_n} and 1_{α_m} are equal or to one where they are not equal. This leads to the situation that the forcing model can prove no useful information about whether or not 1_{α_n} and 1_{α_m} are equal.

We now identify a set which will play the role of ω in $V(\mathbb{P})$. This will just be the standard forcing name for ω, namely $\check{\omega} := \{\langle \check{n}, \mathbb{1} \rangle \mid n \in \omega\}$ where we have used the standard forcing notation from Remark 2.6.16. It can then be proven using standard methods that

$$V(\mathbb{P}) \models \check{\omega} = \omega.$$

Remark 7.1.5. $V(\mathbb{P}) \models \neg\neg(1_{\alpha_0} \in \omega)$.

Now let $\pi \colon \mathbb{P} \to \mathbb{P}$ be the automorphism which switches α_0 and α_1 so $\pi(\alpha_0) = \alpha_1$, $\pi(\alpha_1) = \alpha_0$ and $\pi(\mathbb{1}) = \mathbb{1}$. We can extend π to names by the following recursion:

$$\pi(x) = \{\langle \pi(y), \pi(p) \rangle \mid \langle y, p \rangle \in x\}.$$

As in the set forcing case, we have the symmetry lemma which can be proven using a simple inductive argument. The proof is in essence the same as the classical proof, as example of which can be found in Lemma 8.12.3 of [DS96], so we shall omit it.

Lemma 7.1.6. *Let $\varphi(v)$ be any formula of the forcing language, p be a node in $\mathbb{P}$ and x an element of* $\mathrm{V}(\mathbb{P})$*. Then*

$$p \Vdash \varphi(x) \iff \pi(p) \Vdash \varphi(\pi(x)).$$

Theorem 7.1.7. $\mathrm{V}(\mathbb{P})$ *believes that π defines a non-trivial automorphism of the universe.*

Proof. First note that $\mathrm{V}(\mathbb{P}) \models \pi(1_{\alpha_0}) = 1_{\alpha_1}$ so $\mathrm{V}(\mathbb{P})$ believes that π is non-trivial. Now, let φ be a formula. Then

$$\mathrm{V}(\mathbb{P}) \models \varphi(x)$$

if and only if, for any node p,

$$p \Vdash \varphi(x).$$

By the symmetry lemma, this holds if and only if for every node p,

$$\pi(p) \Vdash \varphi(\pi(x)).$$

So, since π is an automorphism of $\mathbb{P}$, this holds if and only if

$$\mathrm{V}(\mathbb{P}) \models \varphi(\pi(x)).$$

□

Remark 7.1.8. In fact, since $\mathrm{V}(\mathbb{P})$ believes that both 1_{α_0} and 1_{α_1} are in L, $\mathrm{V}(\mathbb{P})$ believes that π defines an automorphism of L.

Theorem 7.1.9. $\mathrm{W} := \mathrm{L}^{\mathrm{V}(\mathbb{P})}$ *is a model of* IZF *plus* W = L *and π is a non-trivial automorphism of* W *which moves an ordinal and is definable in* W*.*

Since our current methods do not allow us to produce a model with more than two subsets of 1 which are "*independent*" of one another we have the natural question:

Question 7.1.10. Is it consistent with IZF that there is some class $\mathrm{M} \subset \mathrm{V}$, with $\mathrm{M} \neq \mathrm{V}$, and a non-trivial, elementary embedding $j \colon \mathrm{V} \to \mathrm{M}$?

This could plausibly be achieved if one could produce a model containing a set of ω many independent ordinals $\{\alpha_n \mid n \in \omega\}$. That is, some set satisfying conditions such as:

- $\forall n \in \omega\ \alpha_n \subseteq 1$,
- $\alpha_0 \notin \mathrm{L}[\langle \alpha_n \mid n \geq 1 \rangle]$.

If this were to be the case then one could take $\mathrm{V} = \mathrm{L}[\langle \alpha_n \mid n \in \omega \rangle]$, $\mathrm{M} = \mathrm{L}[\langle \alpha_n \mid n \geq 1 \rangle]$ and j generated by the map

$$j \colon \alpha_n \mapsto \alpha_{n+1}.$$

7.2 Complications of Critical Sets

In this section, we look at the difficulties in defining the critical point of an elementary embedding. Given an embedding $j \colon \mathrm{V} \to \mathrm{M}$ there are many ways to express non-triviality. For example:

(i.) $\exists K\ (K \neq j(K)\ \wedge\ \forall x \in K\ j(x) = x)$,

(ii.) $\exists \kappa\ (\kappa \in \textsc{Ord}\ \wedge\ \kappa \neq j(\kappa)\ \wedge\ \forall \alpha \in \kappa\ j(\alpha) = \alpha)$,

(iii.) $\exists K\ (K \in j(K)\ \wedge\ \forall x \in K\ j(x) = x)$,

(iv.) $\exists K\ (K \in j(K)\ \wedge\ \forall x \in \mathrm{trcl}(K)\ j(x) = x)$,

(v.) $\exists K\ (\mathrm{Trans}(K)\ \wedge\ K \in j(K)\ \wedge\ \forall x \in K\ j(x) = x)$,

(vi.) $\exists \kappa\ (\kappa \in \textsc{Ord}\ \wedge\ \kappa \in j(\kappa)\ \wedge\ \forall \alpha \in \kappa\ j(\alpha) = \alpha)$,

(vii.) $\exists K\ (\mathrm{Inacc}(K)\ \wedge\ K \in j(K)\ \wedge\ \forall x \in K\ j(x) = x)$.

The consequences of these definitions have been studied in places such as [FŠ84], [Gib02] and [Zie14]. In particular, Ziegler looks in detail at various consequences of some of these definitions in Chapter 9 of his thesis. For example, he shows that condition (v) implies condition (vi).

As shown in Theorem 7.1.9, the first two expressions are not very strong under IZF and we shall now discuss the other definitions. To being with, we shall discuss why condition (iii) could be a difficult one to work with. This is done by starting with a measurable cardinal in ZFC and constructing a set which satisfies condition (iii) but not (iv).

While it will be easy to still show that an ordinal in moved from a classical perspective, the idea is to indicate why it could potentially require a non-trivial step in the intuitionistic setting.

Example 7.2.1. Let $\mathrm{V} \models \mathrm{ZFC}$ and $j\colon \mathrm{V} \to \mathrm{M}$ be a non-trivial, elementary embedding. Let κ be the critical point and λ the first ordinal above κ which is fixed by j. We shall define a sequence of sets $\{S_\alpha \mid \alpha \in j(\kappa)\}$ such that $j(S_\alpha) = S_{j(\alpha)}$ and if $\alpha \in \beta$ then $S_\alpha \in S_\beta$. To do this, let

$$\begin{aligned} S_0 &:= \emptyset, \\ S_{\alpha+1} &:= \{S_\beta \mid \beta \in \alpha\} \cup \{\{\lambda + \beta \mid \beta \in \alpha\}\}, \\ S_\gamma &:= \bigcup \{S_\alpha \mid \alpha \in \gamma\} \textit{ for } \gamma \textit{ a limit ordinal.} \end{aligned}$$

Since we are only working with ordinals that are fixed by j, it is clear that $S_\kappa \in j(S_\kappa) = S_{j(\kappa)}$ and any $x \in S_\kappa$ is either of the form S_α or $\{\lambda + \beta \mid \beta \in \alpha\}$, both of which are fixed by j.

In the above example, the rank of S_κ is $\lambda + \kappa$ and it is obvious that any κ length sequence of fixed ordinals could also be used to produce a similar set. So, in particular,

Proposition 7.2.2. *If γ is an ordinal of cofinality κ which is a limit of ordinals fixed by j then there is a K of rank γ such that*

$$K \in j(K) \;\wedge\; \forall x \in K \; x = j(x).$$

Examples of such γ are the ordinals $\delta+\kappa$, $\delta\cdot\kappa$, δ^{κ} and the cardinal $\delta^{+\kappa}$ for δ a fixed point. However, it is not clear that "$\mathrm{cf}(\gamma)=\kappa$" is a necessary condition to obtain such a K of rank γ. For example, it might be possible that there is some new ordinal μ, much greater than λ, such that the behaviour of $j(\mu)$ is not determined by $j\restriction\lambda$. Such an ordinal could then be used to produce a K and it would be unclear how one could recover κ from said K.

It turns out that definitions (iv), (v) and (vi) are equivalent, as we shall see next. This was originally claimed by Ziegler [Zie14] as Theorem 9.1 however there is a small gap in the proof which we fill now. The gap is caused by only assuming that every element of a is fixed and not every element of $\mathrm{trcl}(a)$ in Proposition 7.2.3. Without this, taking a to be the set containing the first non-trivial fixed ordinal, $a=\{\lambda\}$, would satisfy the assumptions but not the conclusions.

Proposition 7.2.3. $\forall a\ \Big(\big(\forall b\in\mathrm{trcl}(a)\ j(b)=b\big)\rightarrow\forall\beta\in\mathrm{rank}(a)\ j(\beta)=\beta\Big).$

Proof. This is proved by set induction. So take $\beta\in\mathrm{rank}(a)$. Then there is some $z\in a$ such that either $\beta=\mathrm{rank}(z)$ or $\beta\in\mathrm{rank}(z)$. In the first case we have

$$j(\beta)=j(\mathrm{rank}(z))=\mathrm{rank}(j(z))=\mathrm{rank}(z)=\beta.$$

And in the second case, since $\mathrm{trcl}(z)\subseteq\mathrm{trcl}(a)$, for any $b\in\mathrm{trcl}(z)$, $j(b)=b$. Therefore, by the induction hypothesis, every ordinal in $\mathrm{rank}(z)$ is fixed. In particular, $j(\beta)=\beta$. □

Lemma 7.2.4. *Suppose that* V *is a model of* IKP*,* M *is a predicate for a transitive subclass of* V *and* j *is a predicate for a non-trivial,* Σ*-elementary embedding* $j\colon \mathrm{V}\rightarrow\mathrm{M}$*. Then the following are equivalent:*

(iv.) $\exists K\ (K\in j(K)\ \wedge\ \forall x\in\mathrm{trcl}(K)\ j(x)=x)$,

(v.) $\exists K\ (\mathrm{Trans}(K)\ \wedge\ K\in j(K)\ \wedge\ \forall x\in K\ j(x)=x)$,

(vi.) $\exists\kappa\ (\kappa\in\mathrm{Ord}\ \wedge\ \kappa\in j(\kappa)\ \wedge\ \forall\alpha\in\kappa\ j(\alpha)=\alpha)$,

Proof. First note that all of the reverse implications are immediate so we only need to prove that (iv) implies (vi). Secondly, transitivity, the transitive closure and rank are all Σ-definable and therefore the previous proposition holds under the assumption that the embedding is Σ-elementary. To prove (iv) implies (vi), fix K satisfying the assumptions and let $\kappa \coloneqq \text{rank}(K)$. Then, by Proposition 7.2.3, for any $b \in K$ and any $\beta \in \text{rank}(b)$, $j(\beta) = \beta$. Therefore, since any $\alpha \in \kappa$ is either equal to, or in $\text{rank}(b)$ for some $b \in K$, $j(\alpha) = \alpha$. Now, $j(\kappa) = j(\text{rank}(K)) = \text{rank}(j(K))$ so, since $K \in j(K)$, $\kappa \in j(\kappa)$. Completing the proof. □

Under IKP, it does not seem possible to show that definition (vii) is equivalent to (vi) because of the difficulty in producing inaccessible sets without a rank hierarchy of sets. This can also be seen in the classical case, which we shall discuss in more detail in Section 10.4. Recall that, classically, a set z is inaccessible if and only if it is equal to V_δ for some weakly inaccessible cardinal δ. Now, suppose that $\mathrm{V} \models \mathrm{KP}$ and $j \coloneqq \mathrm{V} \to \mathrm{M}$ was a non-trivial, Σ-elementary embedding with critical point κ. Then it is not clear why V_κ should be a set and therefore why there should be sets which are truly inaccessible. However, we can circumvent this issue by showing that the two conditions are equivalent in L.

7.3 Ord-inary Embeddings

Due to the non-linearity of the ordinals, it is certainly possible that we could be in a situation where we have a critical ordinal κ which doesn't contain 0 or ω. This means it could be difficult to prove that L_κ satisfies IZF in the way we have formulated it. Therefore, in order to ensure that the structure we will build from an elementary embedding satisfies the Axioms of Empty Set and Strong Infinity, it is useful to define a way to take any ordinal α to an ordinal $\alpha^{\#}$ which contains $\omega + 1$ as a subset. We do this by using a technique from Lubarsky, [Lub93]:

Definition 7.3.1 (Lubarsky)**.** We define the class $\{\alpha^\# \mid \alpha \in \textsc{Ord}\}$ inductively as

$$\alpha^\# := \bigcup\{\beta^\# \mid \beta \in \alpha\} \cup (\omega + 1).$$

Note that, by induction, for each ordinal α, $\alpha^\#$ is an ordinal with $\omega + 1 \subseteq \alpha^\#$. In order for us to build a model of IZF from our elementary embedding we want to define a well-behaved set sized structure for each ordinal α. Since we do not have access to the V_α hierarchy as a hierarchy of sets, we have to make do with the L-hierarchy, which is why we will only be able to get the consistency strength of a proper class of weakly compact cardinals as a lower bound.

Theorem 7.3.2. *Suppose that* $\mathrm{V} \models \mathrm{IKP}$ *and* $j \colon \mathrm{V} \to \mathrm{M}$ *is a* Σ*-*$\textsc{Ord}$*-inary, elementary embedding with witnessing ordinal* κ*. Then*

$$\mathrm{L}_{\kappa^\#} \models \mathrm{IZF}.$$

Before we give the proof, we first note some easy observations:

Remarks 7.3.3.

- $j(\alpha) = \alpha$ for every $\alpha \in \omega + 1$. This can be proved inductively by noting that $\emptyset$ is the unique set a satisfying $\forall x \in a (x \neq x)$.
- $\forall \alpha \in \textsc{Ord}\ j(\alpha^\#) = j(\alpha)^\#$. This is because

$$j(\alpha^\#) = j\Big(\bigcup\{\beta^\# \mid \beta \in \alpha\} \cup (\omega + 1)\Big) = \bigcup\{\beta^\# \mid \beta \in j(\alpha)\} \cup (\omega + 1) = j(\alpha)^\#.$$

- Due to the uniform, absolute definition of L, if $x \in \mathrm{L}_\alpha$ then $j(x) \in \mathrm{L}_{j(\alpha)}$ and $j(\mathrm{L}_\alpha) = \mathrm{L}_{j(\alpha)}$.

Definition 7.3.4. An ordinal δ is a weak additive limit if

$$\forall \alpha \in \delta\ \exists \beta \in \delta\ (\alpha \in \beta).$$

δ is an additive limit if

$$\forall \alpha \in \delta\ (\alpha + 1 \in \delta).$$

Claim 7.3.5. *For δ a weak additive limit,* $\mathrm{L}_\delta = \bigcup_{\alpha\in\delta}\mathrm{L}_\alpha$.

Proof. First note that $\bigcup_{\alpha\in\delta}\mathrm{L}_\alpha \subseteq \bigcup_{\alpha\in\delta}\mathrm{def}(\mathrm{L}_\alpha) = \mathrm{L}_\delta$ for any δ. So let $x \in \bigcup_{\alpha\in\delta}\mathrm{def}(\mathrm{L}_\alpha)$ and fix $\beta \in \delta$ such that $x \in \mathrm{def}(\mathrm{L}_\beta)$. Now take $\gamma \in \delta$ such that $\beta \in \gamma \in \delta$. Then, since $\mathrm{L}_\gamma = \bigcup_{\alpha\in\gamma}\mathrm{def}(\mathrm{L}_\alpha)$, $x \in \mathrm{L}_\gamma \subseteq \bigcup_{\alpha\in\delta}\mathrm{L}_\alpha$. □

Claim 7.3.6. *$\kappa^\#$ is a weak additive limit.*

Proof. Let $\beta \in \kappa^\#$, then $\beta \in \bigcup\{\alpha^\# \mid \alpha \in \kappa\} \cup (\omega + 1)$. Since for any $\alpha \in \kappa$, $j \restriction \alpha^\#$ is the identity and for any $\alpha \in \omega + 1$, $j(\alpha) = \alpha$, we have that $j(\beta) = \beta$. Therefore $\beta \in j(\kappa^\#)$ and $\mathrm{M} \models j(\beta) \in \kappa^\# \in j(\kappa^\#)$ so

$$\mathrm{M} \models \exists\gamma \in j(\kappa^\#)\ (j(\beta) \in \gamma).$$

Hence, using elementarity, $\mathrm{V} \models \exists\gamma \in \kappa^\#\ (\beta \in \gamma)$. □

Lemma 7.3.7. *j restricted to $\mathrm{L}_{\kappa^\#}$ is the identity and therefore* $(\mathrm{L}_{\kappa^\#})^{\mathrm{V}} \in \mathrm{M}$.

Proof. We begin by showing that for any $\alpha \in \kappa^\#$, $j \restriction \mathrm{def}(\mathrm{L}_\alpha)$ is the identity. This is proven by induction, so assume that for every $\beta \in \alpha$, $j \restriction \mathrm{def}(\mathrm{L}_\beta) = \mathrm{id}$. We first see that $j \restriction \mathrm{L}_\alpha = \mathrm{id}$ since if x is in L_α then $x \in \mathrm{def}(\mathrm{L}_\beta)$ for some $\beta \in \alpha$ and thus $j(x) = x$. Next, we have that $j(\mathrm{L}_\alpha) = \mathrm{L}_{j(\alpha)} = \mathrm{L}_\alpha$, from which one can see that for any $t \subseteq \mathrm{L}_\alpha$, $j(t) = t$. Thus, for any $x \in \mathrm{def}(\mathrm{L}_\alpha)$, $j(x) = x$ so $j \restriction \mathrm{def}(\mathrm{L}_\alpha)$ is indeed the identity.

Finally, let $x \in \mathrm{L}_{\kappa^\#}$ and fix $\alpha \in \kappa^\#$ such that $x \in \mathrm{def}(\mathrm{L}_\alpha)$. Since $j \restriction \mathrm{def}(\mathrm{L}_\alpha)$ is the identity, $j(x) = x$ as required. □

Corollary 7.3.8. $\mathrm{L}_{\kappa^\#} \subseteq \mathrm{M}$ *and therefore* $\mathrm{M} \models \mathrm{L}_{\kappa^\#} \in \mathrm{L}_{j(\kappa^\#)}$.

We now prove that $\mathrm{L}_{\kappa^\#}$ is a model of each axiom of IZF. To begin with, note that $\emptyset, \omega \in \kappa^\#$ so $\mathrm{L}_{\kappa^\#}$ is a model of the Axioms of Empty Set and Infinity. Also, because it is a transitive set, $\mathrm{L}_{\kappa^\#}$ is a model of Extensionality and $\in$-induction. Lastly, for a formula φ, recall that $\varphi^{\mathrm{L}_{\kappa^\#}}$ is the formula defined from φ by replacing all instances of the unbounded quantifiers $\forall x$ and $\exists y$ by $\forall x \in \mathrm{L}_{\kappa^\#}$ and $\exists y \in \mathrm{L}_{\kappa^\#}$ respectively.

Pairing and Unions: Let $a, b \in \mathrm{L}_{\kappa^\#}$. Since $\mathrm{def}(x)$ is the closure of x under the fundamental operations, $\{a, b\}$ and $\bigcup a$ are sets in $\mathrm{def}(\mathrm{L}_{\kappa^\#}) \subseteq \mathrm{L}_{j(\kappa^\#)}$. Therefore $\mathrm{M} \models \{a, b\},\ \bigcup a \in \mathrm{L}_{j(\kappa^\#)}$.

So, using elementarity, the fact that $j(a) = a$, $j(b) = b$ and the absoluteness of these definitions, $\mathrm{V} \models \{a, b\},\ \bigcup a \in \mathrm{L}_{\kappa^\#}$.

Separation: Let φ be a formula and $a \in \mathrm{L}_{\kappa^\#}$. Then $\varphi^{\mathrm{L}_{\kappa^\#}}$ is Σ_0 so, by Bounded Separation in L, there exists a $b \in \mathrm{L}$ such that $b = \{x \in a \mid \varphi^{\mathrm{L}_{\kappa^\#}}(x)\}$ and $b \in \mathrm{def}(\mathrm{L}_{\kappa^\#}) \subseteq \mathrm{L}_{j(\kappa^\#)}$. We claim that

$$b \cap \mathrm{L}_{\kappa^\#} = j(b) \cap \mathrm{L}_{j(\kappa^\#)}.$$

To see this, fix $x \in j(b) \cap \mathrm{L}_{j(\kappa^\#)}$. Since $a = j(a)$ and $b \subseteq a$, we have that $j(b) \subseteq a$ so $x = j(x)$ and $j(x) \in j(b) \cap \mathrm{L}_{j(\kappa^\#)}$. Then elementarity gives us that $x \in b \cap \mathrm{L}_{\kappa^\#}$.

Similarly, if $x \in b \cap \mathrm{L}_{\kappa^\#}$ then $x \in j(b) \cap \mathrm{L}_{j(\kappa^\#)}$ so $b \cap \mathrm{L}_{\kappa^\#} = j(b) \cap \mathrm{L}_{j(\kappa^\#)}$ as claimed.

From this we can see that

$$\mathrm{M} \models j(b) \cap \mathrm{L}_{j(\kappa^\#)} \in \mathrm{L}_{j(\kappa^\#)}.$$

Thus,

$$\mathrm{V} \models b = b \cap \mathrm{L}_{\kappa^\#} \text{ and } \mathrm{V} \models b \in \mathrm{L}_{\kappa^\#}.$$

This gives us that

$$\mathrm{V} \models \exists b \in \mathrm{L}_{\kappa^\#}\ \forall x \in \mathrm{L}_{\kappa^\#}\ (x \in b \leftrightarrow x \in a\ \wedge\ \varphi^{\mathrm{L}_{\kappa^\#}}(x)),$$

that is

$$\mathrm{L}_{\kappa^\#} \models \exists b\ \forall x\ (x \in b \leftrightarrow x \in a\ \wedge\ \varphi(x))$$

so this instance of Separation holds in $\mathrm{L}_{\kappa^\#}$.

Collection: Let φ be a formula, $a, u \in \mathrm{L}_{\kappa^\#}$ and suppose that

$$\mathrm{V} \models \forall x \in a \; \exists y \in \mathrm{L}_{\kappa^\#} \; \varphi^{\mathrm{L}_{\kappa^\#}}(x, y, u).$$

Now for each $x \in a$, if $\mathrm{V} \models \varphi^{\mathrm{L}_{\kappa^\#}}(x, y, u)$ then $\mathrm{M} \models \varphi^{\mathrm{L}_{j(\kappa^\#)}}(j(x), j(y), j(u))$. However, we have assumed that $x, y, u \in \mathrm{L}_{\kappa^\#}$ so they are fixed by j. Therefore we have that $\mathrm{M} \models \varphi^{\mathrm{L}_{j(\kappa^\#)}}(x, y, u)$, and thus

$$\mathrm{M} \models \forall x \in a \; \exists y \in \mathrm{L}_{\kappa^\#} \; \varphi^{\mathrm{L}_{j(\kappa^\#)}}(x, y, u).$$

So, by taking $b = \mathrm{L}_{\kappa^\#}$,

$$\mathrm{M} \models \exists b \in \mathrm{L}_{j(\kappa^\#)} \; \forall x \in j(a) \; \exists y \in b \; \varphi^{\mathrm{L}_{j(\kappa^\#)}}(x, y, j(u)).$$

Elementarity then gives

$$\mathrm{V} \models \exists b \in \mathrm{L}_{\kappa^\#} \; \forall x \in a \; \exists y \in b \; \varphi^{\mathrm{L}_{\kappa^\#}}(x, y, u)$$

which proves this instance of Collection.

Lemma 7.3.9. *For any $a, b \in \mathrm{L}_{\kappa^\#}$ there is a $z \in \mathrm{L}_{\kappa^\#}$ such that*

$$\mathrm{L}_{\kappa^\#} \models z = \mathrm{mv}({}^a b).$$

Proof. Let $a, b \in \mathrm{L}_{\kappa^\#}$ and let $R \in \mathrm{mv}({}^a b) \cap \mathrm{L}_{j(\kappa^\#)}$. First note that j restricted to $a \times b$ is the identity and $a \times b \in \mathrm{L}_{\kappa^\#}$. Then

$$\forall \langle x, y\rangle \in a \times b \; \Big(\langle x, y\rangle \in R \iff j(\langle x, y\rangle) \in j(R) \iff \langle x, y\rangle \in j(R)\Big)$$

so $j(R) = R$. Therefore $j(R) \in \mathrm{L}_{j(\kappa^\#)}$ which implies that $R \in \mathrm{L}_{\kappa^\#}$. This means that $\mathrm{mv}({}^a b) \cap \mathrm{L}_{\kappa^\#} = \mathrm{mv}({}^a b) \cap \mathrm{L}_{j(\kappa^\#)}$. But

$$\begin{aligned} \mathrm{mv}({}^a b) \cap \mathrm{L}_{\kappa^\#} &= \{R \in \mathrm{L}_{\kappa^\#} \mid R \subseteq a \times b \;\wedge\; \forall x \in a \; \exists y \in b \; (\langle x, y\rangle \in R)\} \\ &\in \mathrm{def}(\mathrm{L}_{\kappa^\#}) \subseteq \mathrm{L}_{j(\kappa^\#)} \end{aligned}$$

so $\mathrm{M} \models \exists\delta \in j(\kappa^{\#})\ \mathrm{mv}({}^{j(a)}j(b)) \cap \mathrm{L}_{j(\kappa^{\#})} \in \mathrm{def}(\mathrm{L}_\delta)$ and thus

$$\mathrm{V} \models \exists\delta \in \kappa^{\#}\ \mathrm{mv}({}^{a}b) \cap \mathrm{L}_{\kappa^{\#}} \in \mathrm{def}(\mathrm{L}_\delta),$$

yielding $\mathrm{L}_{\kappa^{\#}} \models \exists z\ \mathrm{mv}({}^{a}b) = z$. $\square$

Finally, it is proven in Proposition 5.1.6 of [AR10] that over ECST, which is a sub-system of IZF without Power Set, the Axiom of Power Set is equivalent to the assertion that $\mathrm{mv}({}^{a}b)$ is a set for all a, b. Therefore we have shown that every axiom of IZF hold in $\mathrm{L}_{\kappa^{\#}}$, completing the proof. $\square_{\textit{Theorem } 7.3.2}$

We recall here that we chose to work in the constructible universe because it is not in general true that the V_α's will be sets. This creates the additional problem that, in general, $\mathrm{L}_{\kappa^{\#}}$ is not *functionally regular* as defined in Definition 2.3.11.

To see this, note that if there is a measurable cardinal κ in ZFC then $\mathcal{P}^{\mathrm{L}}(\omega)$ is countable. So we can fix some real $f \in {}^{\omega}\omega$ which is not in L. But then, by defining $g \colon \omega \to \omega$ as

$$n \mapsto \langle n, f(n) \rangle$$

we have that $\mathrm{dom}(g)$ is in L_κ but $\mathrm{ran}(g)$ is not.

On the other hand, if we add in the additional assumption that V_α is a set for every ordinal α, then the above proof gives us the much stronger result of full inaccessibility in V.

Theorem 7.3.10. *Suppose that* $\mathrm{V} \models \mathrm{IKP} + \forall\alpha \in \mathrm{Ord}\ \exists x\ (x = \mathrm{V}_\alpha)$ *and* $j \colon \mathrm{V} \to \mathrm{M}$ *is a* Σ-Ord*-inary, elementary embedding with critical ordinal* κ *such that for any ordinal* α, $j(\mathrm{V}_\alpha) = (\mathrm{V}_{j(\alpha)})^{\mathrm{M}}$*. Then* $\mathrm{V}_{\kappa^{\#}}$ *is an inaccessible set so, in particular,*

$$\mathrm{V}_{\kappa^{\#}} \models \mathrm{IZF}.$$

The additional assumption that $j(\mathrm{V}_\alpha) = (\mathrm{V}_{j(\alpha)})^{\mathrm{M}}$ is needed because it is unclear how to show that $j(\mathcal{P}(x)) = \mathcal{P}(j(x))$ from just Σ-elementarity here, as this is a Π statement. However, using Proposition 6.2.12, if we strengthen j to assume that it is a $(\Sigma \cup \Pi)$-elementary embedding then the conclusion would follow.

We end this section with a note that while $\kappa^{\#}$ is provably a weak additive limit, it need not be an additive limit. Specifically, there is no reason why we should have that $\omega + 1 \in \kappa^{\#}$. To see this, we first remark that for an ordinal λ, L_λ being closed under ordinal successors does not imply that λ is an additive limit.

Remark 7.3.11. IZF $\nvdash \forall \alpha, \delta \in \mathrm{ORD}\ \Big((\alpha \in \lambda\ \wedge\ \alpha + 1 \in \mathrm{L}_\lambda) \to \alpha + 1 \in \lambda\Big)$.

To see this, note that 0 and 1 are in the ordinal $\mathcal{P}(1)$, so

$$2 = \{0, 1\} \in \mathrm{def}(\mathrm{L}_{\mathcal{P}(1)}) = \mathrm{L}_{\mathcal{P}(1) \cup \{\mathcal{P}(1)\}}.$$

Therefore if IZF could prove that

$$\Big(\alpha \in \mathcal{P}(1) \cup \{\mathcal{P}(1)\}\ \wedge\ \alpha + 1 \in \mathrm{L}_{\mathcal{P}(1)\cup\{\mathcal{P}(1)\}}\Big) \to \alpha + 1 \in \mathcal{P}(1) \cup \{\mathcal{P}(1)\}$$

then IZF would prove that $2 \in \mathcal{P}(1) \cup \{\mathcal{P}(1)\}$, implying that $\mathcal{P}(1) = 2$ which is not provable in IZF.

Proposition 7.3.12. *In general $\kappa^{\#}$ need not be an additive limit.*

We shall not formally prove the above proposition, but just sketch a proof of it. To do this we shall take a *realizability model* without formally defining what this is. This is a standard technique to produce models where classical logic fails and can be found in many texts on intuitionistic logic. We refer the reader to Chapters 2 and 3 of the thesis of McCarty [McC85] as one such reference.

Suppose that $\mathrm{V} \models \mathrm{ZFC}$ and $j\colon \mathrm{V} \to \mathrm{M}$ is an elementary embedding with critical point κ. Take a realizability model in which there is a new "*small*" ordinal, μ, and show that the embedding still exists in this model. For example, as shown by Ziegler in Section 9.5 of his thesis [Zie14], if one takes the standard Kleene realizability model $\mathrm{V}(Kl)$ then one can show that the embedding lifts to some

$$j\colon \mathrm{V}(Kl) \to \mathrm{M}(Kl)$$

where $\mathrm{M}(Kl) \coloneqq \mathrm{V}(Kl) \cap \mathrm{M}$ is the realizability structure relativised to M.

Here by "*small*" we mean an ordinal that will be fixed by j. In this case we could take ω^0 which is defined as follows:

$$\overline{n} \coloneqq \{\langle m, \overline{m}\rangle \mid m \in n\},$$
$$\omega^0 \coloneqq \{\langle 0, \overline{n}\rangle \mid n \in \omega\}.$$

It is a standard proof that, in $\mathrm{V}(Kl)$, $\overline{\omega} \coloneqq \{\langle n, \overline{n}\rangle \mid n \in \omega\}$ is the standard name for ω, ω^0 is an ordinal with $\overline{\omega}$ as a subset, but

$$\mathrm{V}(Kl) \models \overline{\omega} \neq \omega^0 \ \wedge \ \overline{\omega} \notin \omega^0 \ \wedge \ \omega^0 \notin \overline{\omega}.$$

Now, we want to take the κ^{th} *additive successor* of ω^0, that is we take

$$\eta \coloneqq \omega^0 + \kappa$$

where $+$ is the standard ordinal addition. Note that η is not the same as $\omega^0 \cup \kappa$.

Then $\eta^{\#}$ will still be a critical ordinal in our realizability model with $\omega \in \eta$ but $\omega + 1 \notin \eta$ so η cannot be an additive limit.

7.4 The Strength of Ord-inary Embeddings

7.4.1 Preserving Power Set

In this section we aim to derive a lower bound for IKP plus an Ord-inary elementary embedding. Because we can prove that $\mathrm{L}_{\kappa^{\#}}$ is a model of IZF the idea is that we should be able to deduce the consistency of ZFC plus a proper class of every large cardinal axiom below a measurable which is consistent with L. To this end we shall show that we can obtain the consistency of ZFC plus a proper class of weakly compact cardinals. The proof we give can then be viewed as a template for similar consistency statements where the main difficulty will arise from redefining the classical large cardinal axiom as an intuitionistic large set axiom whose double negation interpretation can then be taken. For the next three propositions, our background theory is IKP.

To begin with, we show that the power set of $\mathrm{L}_{\kappa^\#}$ is a set in L.

Proposition 7.4.1. *Suppose that $X \subseteq \mathrm{L}_{\kappa^\#}$. Then $X = j(X) \cap \mathrm{L}_{\kappa^\#}$.*

Proof. For the first direction, suppose that $z \in X$. Since $X \subseteq \mathrm{L}_{\kappa^\#}$ and j restricted to $\mathrm{L}_{\kappa^\#}$ is the identity, $j(z) = z$ so $z = j(z) \in j(X) \cap \mathrm{L}_{\kappa^\#}$. For the reverse direction, if $z \in j(X) \cap \mathrm{L}_{\kappa^\#}$ then we again have that $z = j(z)$ so z is indeed in X. □

Proposition 7.4.2. $\mathrm{V} \models \exists x\ \forall t\ (t \in x \longleftrightarrow (t \in \mathrm{L}\ \wedge\ t \subseteq \mathrm{L}_{\kappa^\#}))$.

Proof. Since V proves that $\mathrm{L}_{\kappa^\#}$ satisfies Power Set, M believes that $\mathrm{L}_{j(\kappa)^\#}$ satisfies this also. This means that

$$\mathrm{M} \models \exists x\ \forall t\ (t \in x \longleftrightarrow (t \in \mathrm{L}\ \wedge\ t \subseteq \mathrm{L}_{\kappa^\#}))$$

Therefore all we are required to prove is that $(\mathcal{P}(\mathrm{L}_{\kappa^\#}))^{\mathrm{V}\cap\mathrm{L}} = (\mathcal{P}(\mathrm{L}_{\kappa^\#}))^{\mathrm{M}\cap\mathrm{L}}$. Since $\mathrm{M} \subseteq \mathrm{V}$ the reverse inclusion is immediate. Now for the forward inclusion, suppose that

$$\mathrm{V} \models t \in \mathrm{L}\ \wedge\ t \subseteq \mathrm{L}_{\kappa^\#}.$$

Then $j(t) \in \mathrm{M} \cap \mathrm{L}$ and, by the previous proposition, $t = j(t) \cap \mathrm{L}_{\kappa^\#} \in \mathrm{M} \cap \mathrm{L}$. □

It does not, in general, seem to be possible to prove that $\mathrm{V} \cap \mathrm{L}$ is equal to $\mathrm{M} \cap \mathrm{L}$ because there is no reason to believe that V and M share the same ordinals. However, if α is an ordinal of M then we can prove that $(\mathrm{L}_\alpha)^{\mathrm{V}} = (\mathrm{L}_\alpha)^{\mathrm{M}}$. We shall state this in terms of the original definability operator we defined, which was just a single step closure under fundamental operations, for simplicity.

Proposition 7.4.3. *For any $\alpha \in \mathrm{M}$, $\mathbb{L}_\alpha^{\mathrm{M}} = \mathbb{L}_\alpha^{\mathrm{V}}$.*

Proof. This is proven by induction on α using the assumption that M is a transitive subclass of V to allow the inductive hypothesis to go through. So suppose that the claim holds for every ordinal β in α. Since M is a model of IKP and the fundamental operations are absolute, we have that for every $i \in I$ and $x, y \in \mathrm{M}$, $\mathcal{F}_i(x, y) \in \mathrm{M}$. Therefore, for each $\beta \in \alpha$, $(\mathcal{D}(\mathbb{L}_\beta))^{\mathrm{M}} = (\mathcal{D}(\mathbb{L}_\beta))^{\mathrm{V}}$ and

$$\mathbb{L}_\alpha^{\mathrm{M}} = (\bigcup_{\beta\in\alpha} \mathcal{D}(\mathbb{L}_\beta))^{\mathrm{M}} = \bigcup_{\beta\in\alpha} (\mathcal{D}(\mathbb{L}_\beta))^{\mathrm{M}} = \bigcup_{\beta\in\alpha} \mathcal{D}(\mathbb{L}_\beta) = \mathbb{L}_\alpha^{\mathrm{V}}.$$

□

7.4.2 A Lower Bound

Many traditional large cardinal notions have several different characterisations which, while equivalent over ZFC, imply unwanted structure in weaker systems. For example, they may imply instances of choice, excluded middle or power set, all of which we wish to avoid over IKP. This restricts the cardinal notions which we can describe in our settings and thus the lower bounds we are able to obtain.

However, one concept that is somewhat amenable is that of *indescribability*.

Definition 7.4.4. Let D be a structure over a fixed language $\mathcal{L}$. A variable is said to be of type $m+1$ over D if it ranges over $\mathcal{P}^m(D)$.

Furthermore, we call a first-order formula φ a *formula of higher type over* D if for each variable of φ there is some $m \in \omega$ such that the variable is of type $m+1$ over D.

Definition 7.4.5. A formula is said to be Π^m_n iff it consists of a block of universal quantifiers of type $m+1$ followed by a block of existential quantifiers of type $m+1$ and continues alternating at most n times, followed afterwards by a formula containing variables of type at most $m+1$ and quantified variables of type at most m. The class Σ^m_n is defined analogously with the order of the initial universal and existential quantifiers switched.

Remark 7.4.6. Over ZF, one can prove that any formula of higher type over D is equivalent to either a Π^m_n or Σ^m_n formula over D for some $m, n \in \omega$. However this is not necessarily true over weaker systems.

Example 7.4.7. We give here a few simple examples of higher type formulae. Firstly, over ZFC, the statement that κ is a measurable cardinal is Σ^2_1 over κ because it suffices to say $\exists U \in \mathcal{P}^2(\kappa)$ "*U is an ultrafilter on κ*". Secondly, the axioms of a topology can be seen to be third order. For example, to say that a topology τ is closed under unions is Π^3_1 over arithmetic via the formula

$$\forall \mathcal{U} \in \mathcal{P}^2(\mathbb{N})\Big(\forall U \in \mathcal{P}(\mathbb{N})\ (U \in \mathcal{U} \to U \in \tau) \\ \longrightarrow \exists V \in \mathcal{P}(\mathbb{N})\ \forall x\Big(\big(x \in V \leftrightarrow \exists U \in \mathcal{U}(x \in U)\big) \wedge V \in \tau\Big)\Big).$$

Definition 7.4.8. A set z is said to be *totally indescribable* if it is inaccessible and for any $R \subseteq z$ and sentence φ of higher type over z, if $\langle z, \in, R\rangle \models \varphi$ then there is an inaccessible set $v \in z$ such that $\langle v, \in, R \cap v\rangle \models \varphi$.

Theorem 7.4.9. *Suppose that* $j \colon \mathrm{V} \to \mathrm{M}$ *is a* Σ-Ord*-inary, elementary embedding with witnessing ordinal* κ*. Then*

$$\mathrm{L}_{\kappa^\#} \models \mathrm{IZF} + \forall x\ \exists z\ (x \in z\ \wedge\ \text{“}z \textit{ is totally indescribable”}).$$

Proof. Working in M, we begin by showing that, in $\mathrm{L}_{j(\kappa)^\#}$, $\mathrm{L}_{\kappa^\#}$ is a totally indescribable set. To do this, let φ be a sentence of higher type over $\mathrm{L}_{\kappa^\#}$, $R \subseteq \mathrm{L}_{\kappa^\#}$ and suppose that $\langle z, \in, R\rangle \models \varphi$. Recall that $\mathrm{L}_{j(\kappa)^\#}$ satisfies Power Set so, in particular, $\mathcal{P}^m(\mathrm{L}_{\kappa^\#}) \cap \mathrm{L}_{j(\kappa)^\#}$ is a set for each $m \in \omega$. This means that

$$\mathrm{L}_{j(\kappa)^\#} \models \exists z(\langle z, \in, R \cap z\rangle \models \varphi).$$

Since this can all be expressed in a Σ way in M, using the fact that everything is bounded by $\mathrm{L}_{j(\kappa)^\#}$, by elementarity we have that

$$\mathrm{L}_{\kappa^\#} \models \exists z(\langle z, \in, R \cap z\rangle \models \varphi)$$

holds in V and hence $\mathrm{L}_{\kappa^\#}$ is a totally indescribable set.

Now, fix $x \in \mathrm{L}_{\kappa^\#}$ and recall that $j(x) = x$. Then, taking $z = \mathrm{L}_{\kappa^\#}$,

$$\mathrm{L}_{j(\kappa)^\#} \models \exists z(j(x) \in z\ \wedge\ \text{“}z \textit{ is totally indescribable”}).$$

As before, this reflects by elementarity so

$$\mathrm{L}_{\kappa^\#} \models \exists z(x \in z\ \wedge\ \text{“}z \textit{ is totally indescribable”}).$$

□

Corollary 7.4.10.

$$\mathrm{IKP} + \exists j \colon \mathrm{V} \to \mathrm{M} \textit{ which is a } \Sigma\text{-}\mathrm{O{\scriptsize RD}}\textit{-inary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{IZF} + \textit{a proper class of totally indescribable sets}).$$

Corollary 7.4.11.

$$\mathrm{KP} + \exists j \colon \mathrm{V} \to \mathrm{M} \textit{ which is a } \Sigma\textit{-elementary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{ZFC} + \textit{a proper class of totally indescribable cardinals}).$$

We now seek a lower bound for IKP with an ORD-inary elementary embedding in terms of the traditional large cardinal hierarchy over ZFC. This appears to be difficult to do because of the complexities of performing a double negation translation. In particular, it is unknown how to express total indescribability in a way for which it is suitable to take the translation.

However, using the work of Friedman and Ščedrov [FŠ84], one can obtain some weak lower bounds. The first will be a proper class of Mahlo cardinals, directly using their results. The second, more involved bound, will be a proper class of weakly compact cardinals.

In order to do this, it is necessary to rephrase the large cardinal notions in a more constructive manner. The definitions we take will be mild variants of standard definitions of these large cardinals using higher types.

Proposition 7.4.12. *Over* ZFC*:*

- *(Lévy, [Kan08] Proposition 6.2) A cardinal κ is* Mahlo *iff for any $R \subseteq \mathrm{V}_\kappa$ there is an inaccessible cardinal $\alpha < \kappa$ such that $\langle \mathrm{V}_\alpha, \in, R \cap \mathrm{V}_\alpha \rangle \prec \langle \mathrm{V}_\kappa, \in, R \rangle$.*
- *(Hanf-Scott, [Kan08] Theorem 6.4) A cardinal κ is* weakly compact *iff it is Π^1_1-indescribable.*

Definition 7.4.13 ([FŠ84], Definition 2)**.** A set z is said to be *Mahlo* if it is inaccessible and for each $u \in z$ and each set t, there exists an inaccessible set $v \in z$ with $u \in v$ and:

$$\forall x \in v \ (\exists y \in z(\langle x, y \rangle \in t) \to \exists y \in v(\langle x, y \rangle \in t)).$$

Theorem 7.4.14 ([FŠ84], Theorem 3.1)**.** $\mathrm{IZF} + \forall x \ \exists z \ (x \in z \ \wedge$ "z *is Mahlo*"$)$ *is equiconsistent with* ZF *plus a proper class of Mahlo cardinals.*

Since it is clear that any totally indescribable set is Mahlo, we obtain that if $j\colon \mathrm{V} \to \mathrm{M}$ is a Σ-Ord-inary embedding then

$$\mathrm{L}_{\kappa^{\#}} \models \mathrm{IZF} + \forall x\ \exists z\ (x \in z\ \wedge\ \text{“}z \textit{ is Mahlo}\text{”}).$$

Therefore, we can bound the above intuitionistic large cardinal by ZFC plus a proper class of Mahlo cardinals.

Corollary 7.4.15.

$$\mathrm{IKP} + \exists j\colon \mathrm{V} \to \mathrm{M} \textit{ which is a } \Sigma\text{-}\mathrm{Ord}\textit{-inary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{ZFC} + \textit{a proper class of Mahlo cardinals}).$$

We now seek to obtain a slightly better lower bound, which will be a proper class of *weakly compact cardinals*. This shall be done by using the notion of *2-strongness* which was first introduced by Rathjen in [Rat17]. The definition below is essentially that of being Π^1_1-indescribable written out in a more verbose way and further details of this concept, along with proofs of some of its elementary properties, can be found in [Gib02].

Definition 7.4.16. A set K is called *2-strong* if it is inaccessible and $\Theta(K)$ holds where $\Theta(K)$ is the statement:

For any set S,

$$\forall R \in \mathrm{mv}(K,K)\ \forall u \in K\ \exists x \in K\ \exists v \in K\ (x \subseteq R\ \wedge\ \langle x,u,v\rangle \in S) \longrightarrow$$
$$\exists I \in K\ (\mathrm{Inacc}(I)\ \wedge\ \forall R \in \mathrm{mv}(I,I)\ \forall u \in I\ \exists x \in I\ \exists v \in I\ (x \subseteq R\ \wedge\ \langle x,u,v\rangle \in S)).$$

We then have two theorems which show how 2-strongness relates to other large set notions. The second theorem follows from the fact that being 2-strong is a weakening of being a totally indescribable set.

Theorem 7.4.17 (Rathjen, [Gib02] Lemma 6.10)**.** *Under* ZFC*, for any ordinal κ,* V_κ *is 2-strong iff κ is weakly compact.*

Theorem 7.4.18. (IKP) *Suppose that* $j\colon \mathrm{V} \to \mathrm{M}$ *is a* Σ-Ord*-inary elementary embedding with witnessing ordinal κ. Then*

$$\mathrm{L}_{\kappa^{\#}} \models \mathrm{IZF} + \forall x\ \exists z\ (x \in z\ \wedge\ z \textit{ is 2-strong}).$$

Corollary 7.4.19.

$$\mathrm{IKP} + \exists j\colon \mathrm{V} \to \mathrm{M} \textit{ which is a } \Sigma\text{-}\textsc{Ord}\textit{-inary embedding} \vdash$$
$$\mathrm{Con}(\mathrm{IZF} + \textit{every set is contained in a 2-strong set}).$$

We shall now perform a double negation translation of the above extension of IZF in order to obtain that ZFC plus a proper class of weakly compact cardinals is a lower bound for the theory IKP plus a Σ-Ord-inary embedding. To do this, we begin by fixing some notation:

- Let $\mathcal{IT}$ be the theory IZF plus $\forall x\ \exists z\ (x \in z\ \wedge\ \Theta(z))$.
- Let $\mathcal{T}$ be $\mathcal{IT}$ with classical logic.
- Let $\mathcal{IS}$ be $\mathcal{IT}$ without the Axiom of Extensionality and *weak Power Set* (w.Power) in the place of Power Set.
- Let $\mathcal{S}$ be $\mathcal{IS}$ with classical logic.

Remarks 7.4.20.

- $\mathcal{T}$ can easily be seen to be equivalent to ZFC with a proper class of weakly compact cardinals.
- In $\mathcal{IT}$ it is important that the theory is defined with the Collection Scheme instead of the Replacement Scheme. This is because, by work of Scott [Sco61], Z is equiconsistent with ZFC minus Extensionality assuming that this has only been formulated with the Replacement Scheme. Replacement holds in Scott's model because it is a very weak principle in this context and, as noted by Lévy [Lév64b] in his review of Scott's paper, there are very few formulae for which one can prove that there is a unique y for every x.

We shall then prove that there exist some interpretations φ^* and φ^- such that

1. If $\mathcal{IT} \vdash \varphi$ then $\mathcal{IS} \vdash \varphi^\star$ and similarly for $\mathcal{T}$ and $\mathcal{S}$.
2. If $\mathcal{S} \vdash \psi$ then $\mathcal{IS} \vdash \psi^-$.

This is then sufficient to conclude that all four of these theories are equiconsistent because if $\mathcal{T}$ were to derive a contradiction, then, by the first interpretation, $\mathcal{S}$ would also and so, by the second interpretation, $\mathcal{IS}$ will. Finally, since $\mathcal{IS}$ is a subsystem of $\mathcal{IT}$, we must be able to derive a contradiction in $\mathcal{IT}$. The basic properties of the interpretations we shall use, and the fact that they prove that ZF is equiconsistent with IZF, can be found in [Fri73], with the extension to inaccessible sets coming from [FŠ84].

Therefore, it remains to check that the additional axiom $\forall x\ \exists z\ (x \in z\ \wedge\ \Theta(z))$ is preserved and extend the two interpretations.

7.4.3 The First Interpretation

To begin with, we briefly mention here what the interpretation * is and point the reader to [Fri73] or [FŠ84] for further details. When taking the double negation translation, it is convenient to not have to worry about the Axiom of Extensionality. This interpretation then tells us how to "*simulate*" extensionality through a new definable relation, $\sim$. We will not define the relation here but essentially $a \sim b$ indicates that there is some equivalence relation E with $\langle a, b\rangle \in E$ and for any $x \in \mathrm{trcl}(a)$ there is some $y \in b$ such that $\langle x, y\rangle \in E$ and vice versa. So two sets are related if there is a single equivalence relation witnessing that; a is related to b, every element of a is related to an element of b, every element of b is related to an element of a, and so on.

We then define a relation $\in^\star$ by

$$a \in^\star b \longleftrightarrow \exists x \in b\ (x \sim a).$$

Using this relation one can then define the * interpretation by:

Definition 7.4.21. For φ a formula of $\mathcal{L}_\in$, let $\varphi^\star$ be the formula which is abbreviated by the result of replacing each instance of $\in$ in φ with $\in^\star$ and $=$ by $\sim$.

We can then obtain the following pair of results, which are Lemma 22 and Theorem 1 of [Fri73].

Theorem 7.4.22 (Friedman, [Fri73]).

- *If* φ *is an axiom of* IZF *then* IZF $\setminus \{Ext. + Power\} + W.Power \vdash \varphi^\star$.
- *If* IZF $\vdash \varphi$ *then* IZF $\setminus \{Ext. + Power\} + W.Power \vdash \varphi^\star$.

We will also use the work of Friedman and Ščedrov [FŠ84] where they extend the interpretation to inaccessible sets, which we denote by $\mathrm{Inacc}^\star(z)$. Now, in order to complete the interpretations, it will help to modify $\Theta(z)$. To do this it is beneficial to introduce some auxiliary formulae, the first of which is the following presentation of ordered pairs which is Definition 1 of [Fri73]. In plain language, $\mathbf{P}(a, b, c)$ will hold whenever $c = \langle a, b\rangle$.

Notation 7.4.23.

$$\begin{aligned}\mathbf{P}(a,b,c) \equiv\ & a \in c \ \wedge\ \exists x \Big(x \in c \ \wedge\ a \in x \ \wedge\ b \in x \ \wedge \\ & \forall y\ (y \in x \to (y = a \ \vee\ y = b)) \ \wedge\ \forall z\ (z \in c \to (z = a \ \vee\ z = x))\Big).\end{aligned}$$

Next we give the formula denoting the statement that R is a multi-valued function and then introduce a formula which will cover a variant of the antecedent of the implication in $\Theta(z)$ from Definition 7.4.16. In particular, the formula $\Psi_\varphi(z, w)$ will be the antecedent of $\Theta(z)$ whenever $\varphi(x, u, v, w)$ is taken to be the formula $\langle x, u, v\rangle \in w$. For taking the interpretations, it will be more insightful to work with formulae instead of the sets S:

Notation 7.4.24. Let $\mathbf{MV}(R, z)$ denote the formula

$$\forall a \in z\ \exists b \in z\ \exists c\ (c \in R \ \wedge\ \mathbf{P}(a, b, c)).$$

Then, for φ a formula with free variables x, u, v and w, let $\Psi_\varphi(z, w)$ be the statement

$$\begin{aligned}\forall R\ \forall u\ \Big(&(\mathbf{MV}(R,z) \ \wedge\ u \in z) \longrightarrow \\ & \exists x, v\ \Big(x \in z \ \wedge\ v \in z \ \wedge\ \forall t(t \in x \to t \in R) \ \wedge\ \varphi(x,u,v,w)\Big)\Big).\end{aligned}$$

We can then give the following equivalent characterisation of 2-strongness which was first stated by Rathjen as Corollary 6.16 in [Rat17]. A proof, in CZF, can be found as Lemma 6.6 of [Gib02].

Lemma 7.4.25 (Rathjen)**.** *If z is 2-strong then for any formula φ and set w,*

$$\Psi_\varphi(z,w) \longrightarrow \exists I\Big(I \in z \ \wedge \ \mathrm{Inacc}(I) \ \wedge \ \Psi_\varphi(I,w)\Big).$$

We are now in a position to give the $\in^\star$-translations of the above statements:

$$\begin{aligned}\mathbf{P}^\star(a,b,c) \equiv \ & a \in^\star c \ \wedge \ \exists x \ \Big(x \in^\star c \ \wedge \ a \in^\star x \ \wedge \ b \in^\star x \ \wedge \\ & \forall y \ (y \in^\star x \rightarrow (y \sim a \ \vee \ y \sim b)) \ \wedge \ \forall z \ (z \in^\star c \rightarrow (z \sim a \ \vee \ z \sim x))\Big).\end{aligned}$$

$$\mathbf{MV}^\star(R,z) \equiv \forall a \in^\star z \ \exists b \ \exists c \ (b \in^\star z \ \wedge \ c \in^\star R \ \wedge \ \mathbf{P}^\star(a,b,c)).$$

Finally, for any formula φ which has already been translated so that it contains $\in^\star$ and $\sim$ in place of $\in$ and $=$:

$$\begin{aligned}\Psi^\star_\varphi(z,w) \ \equiv \ & \forall R \ \forall u \ \Big((\mathbf{MV}^\star(R,z) \ \wedge \ u \in^\star z) \longrightarrow \\ & \exists x, v \ \big(x \in^\star z \ \wedge \ v \in^\star z \ \wedge \ \forall t(t \in^\star x \rightarrow t \in^\star R) \ \wedge \ \varphi(x,u,v,w)\big)\Big).\end{aligned}$$

Using the fact that $\mathcal{IS} \vdash x \in z \rightarrow x \in^\star z$, it is easy to see that

$$\mathcal{IS} \vdash \mathbf{P}(a,b,c) \rightarrow \mathbf{P}^\star(a,b,c) \quad \wedge \quad \mathbf{MV}(R,z) \rightarrow \mathbf{MV}^\star(R,z).$$

This will give us the following lemma, which we note here only holds for a formula that has already been translated to only contain $\in^\star$ and $\sim$:

Lemma 7.4.26. *For any formula φ, written in terms of $\in^\star$ and $\sim$, $\mathcal{IS} \vdash \Psi_\varphi(z,w) \rightarrow \Psi^\star_\varphi(z,w)$. Therefore, taking $\varphi(x,u,v,S)$ to be $(\langle x,u,v\rangle \in S)^\star$, we can see that*

$$\mathcal{IS} \vdash \Theta(z) \rightarrow \Theta^\star(z).$$

Concluding this first interpretation, we have that

$$\mathcal{IS} \vdash \forall x \ \exists z (x \in^\star z \ \wedge \ \Theta^\star(z)).$$

So, since the * interpretation preserves deductions, we have proven the first of our two required statements, namely

Theorem 7.4.27. *If $\mathcal{IT} \vdash \varphi$ then $\mathcal{IS} \vdash \varphi^\star$.*

7.4.4 The Second Interpretation

For the second interpretation we take a $\neg\neg$-*translation*, the details of which can be found in Section 81 of Kleene [Kle52]. This is a process initially investigated by Kolmogorov and later independently by both Gödel and Gentzen to study the relationship between classical and intuitionistic arithmetic. The idea is to define a translation that takes a classically valid proposition into one which is classically equivalent but still valid intuitionistically. Therefore if the classical theory derives a contradiction so will its intuitionistic version. Let CPC denote classical predicate calculus and HPC denote intuitionistic predicate calculus.

Definition 7.4.28 ([Fri73])**.** We define the translation φ^- over formulae in the language $\mathcal{L}_\in$ inductively as:

a. $(a \in b)^- \equiv \neg\neg(a \in b)$,

b. $(\varphi \wedge \psi)^- \equiv \varphi^- \wedge \psi^-$,

c. $(\varphi \vee \psi)^- \equiv \neg\neg(\varphi^- \vee \psi^-)$,

d. $(\varphi \to \psi)^- \equiv \varphi^- \to \psi^-$,

e. $(\neg\varphi)^- \equiv \neg(\varphi^-)$,

f. $\forall x\ \varphi(x,u) \equiv \forall x\ \varphi^-(x,u)$,

g. $\exists x\ \varphi(x,u) \equiv \neg\neg\exists x\ \varphi^-(x,u)$.

We then have four fundamental lemmas about this translation,

Lemma 7.4.29.

1. *([Kle52] Section XV.81, Theorem 60) If* $\mathrm{CPC} \vdash \varphi$ *then* $\mathrm{HPC} \vdash \varphi^-$*,*

2. *([Kle52] Section XV.81, Lemma 43a)* $\mathrm{HPC} \vdash (\neg\neg\varphi^-) \leftrightarrow \varphi^-$*,*

3. *([Fri73] Lemma 2.10) If φ is an axiom of* $\mathrm{ZF} \setminus \{Ext. + Power\} + W.Power$*, then* $\mathrm{IZF} \setminus \{Ext. + Power\} + W.Power \vdash \varphi^-$*,*

4. *([Fri73] Theorem 2.2) If* $\mathrm{ZF} \setminus \{Ext. + Power\} + W.Power \vdash \varphi$*, then* $\mathrm{IZF} \setminus \{Ext. + Power\} + W.Power \vdash \varphi^-$*.*

As before, this interpretation is further extended in [FŠ84] to inaccessible sets which we denote by $\mathrm{Inacc}^-(z)$. Therefore, it only remains to extend it to $\Theta(z)$. In order to simplify the notation we shall use the following two abbreviations:

$$a \mathrel{\not\in} b \equiv \neg\neg(a \in b),$$
$$a \mathrel{\not=} b \equiv \neg\neg(a = b).$$

Then the translation of **P** and **MV** are:

$$\mathbf{P}^-(a,b,c) \equiv \ a \notin c \ \wedge \ (\neg\neg\exists x) \Big(x \notin c \ \wedge \ a \notin x \ \wedge \ b \notin x \ \wedge \forall y \ (y \notin x \rightarrow \neg\neg(y \neq a \ \vee \ y \neq b)) \ \wedge \ \forall z (z \notin c \rightarrow \neg\neg(z \neq a \ \vee \ z \neq x))\Big).$$

$$\mathbf{MV}^-(R,z) \equiv \forall a \Big(a \notin z \rightarrow (\neg\neg\exists b) \ (\neg\neg\exists c) \ (b \notin z \ \wedge \ c \notin R \ \wedge \ \mathbf{P}^-(a,b,c))\Big).$$

From this, it is clear that the translation of Ψ_φ is

$$\Psi^-_\varphi(z,w) \ \equiv \ \forall R \, \forall u \ \Big((\mathbf{MV}^-(R,z) \ \wedge \ u \notin z) \longrightarrow (\neg\neg\exists x)(\neg\neg\exists v) \big(x \notin z \ \wedge \ v \notin z \ \wedge \ \forall t (t \notin x \rightarrow t \notin R) \ \wedge \ \varphi^-(x,u,v,w)\big)\Big).$$

Lemma 7.4.30. $\mathcal{IS} \vdash \Psi_\varphi(z,w) \rightarrow \Psi^-_{\varphi^-}(z,w)$.

Note that in the above lemma, when taking the translation of Ψ we also need to ensure that the index, φ, is translated. Now, we will show that we can take the double negation of the expression that every set is contained in a 2-strong set. But first, we will translate the existence of a 2-strong set as this is the main element of the proof. For this, recall that $\neg\neg\exists x \neg\neg\varphi(x) \leftrightarrow \neg\neg(\exists x \varphi(x))$.

Lemma 7.4.31. $\mathcal{IS} \vdash \Theta(z) \rightarrow \Theta^-(z)$.

Proof. Throughout this proof, we work in $\mathcal{IS}$. So, suppose z was a 2-strong set. We shall actually prove the stronger claim that for any formula φ,

$$\forall w \Big(\Psi^-_{\varphi^-}(z,w) \rightarrow (\neg\neg\exists I)(I \notin z \ \wedge \ \mathrm{Inacc}^-(I) \ \wedge \ \Psi^-_{\varphi^-}(I,w))\Big).$$

To do this, fix a formula φ. Then, since z is 2-strong,

$$\forall w\Big(\Psi_{\varphi^-}(z,w) \to \exists I(I \in z \ \wedge \ \text{Inacc}(I) \ \wedge \ \Psi_{\varphi^-}(I,w))\Big).$$

Next, since HPC $\vdash (\psi \to \vartheta) \to (\neg\neg\psi \to \neg\neg\vartheta)$, we have that

$$\forall w\Big(\neg\neg\Psi_{\varphi^-}(z,w) \to \neg\neg(\exists I(I \in z \ \wedge \ \text{Inacc}(I) \ \wedge \ \Psi_{\varphi^-}(I,w)))\Big).$$

Now, clearly, $I \in z \to I \notin z$. Secondly, using the proof of Theorem 3.1 in [FŠ84], $\text{Inacc}(z) \to \text{Inacc}^-(z)$. Finally, we have $(\Psi_{\varphi^-}(I,w))^- \to \Psi^-_{\varphi^-}(I,w)$. Here we use the fact that HPC $\vdash \psi^- \leftrightarrow (\neg\neg\psi^-)$ to ensure the formula appearing in the index of Ψ translates correctly. Thus the whole statement translates, proving the claim. □

Having defined the translation, it is clear that we have the following:

Lemma 7.4.32. $\mathcal{IS} \vdash \forall x \ (\neg\neg\exists z) \ (x \notin z \ \wedge \ \Theta^-(z))$.

This leads us to our desired conclusion:

Theorem 7.4.33. *If* $\mathcal{S} \vdash \varphi$ *then* $\mathcal{IS} \vdash \varphi^-$.

Combining this with Theorem 7.4.27 gives ZF plus a proper class of weakly compact cardinals is equiconsistent with IZF plus a proper class of weakly compact sets.

Theorem 7.4.34. *If* $\mathcal{T} \vdash \varphi$ *then* $\mathcal{IS} \vdash (\varphi^\star)^-$.

Corollary 7.4.35. $\mathcal{T}$ *and* $\mathcal{IT}$ *are equiconsistent.*

Corollary 7.4.36.

$$\text{IKP} + \exists j\colon \text{V} \to \text{M} \textit{ which is a } \Sigma\text{-}\textsc{Ord}\textit{-inary embedding} \vdash$$
$$\text{Con}(\text{ZFC} + \textit{a proper class of weakly compact cardinals}).$$

Proof. Let $j\colon \text{V} \to \text{M}$ be a Σ-ORD-inary embedding with witnessing ordinal κ. Then, by Theorem 7.4.18, $\text{L}_{\kappa^\#}$ is a model of IZF $+ \forall x \ \exists z \ (x \in z \ \wedge \ \Theta(z))$. So, by the above corollary, we obtain the consistency of ZF *plus a proper class of weakly compact cardinals*. Finally, since being weakly compact is absolute when moving to L, we get the desired result with Choice also holding. □

Chapter 8

Definable Embeddings

We begin this chapter by discussing Suzuki's proof [Suz99] that there are no definable, cofinal Reinhardt embeddings over ZF, and will show that this theorem also goes through in ZF^-. This will be done by first giving Gaifman's result that, in a sufficient fragment of ZF, elementarity can be defined by a single sentence.

After this, we shall explore some of the technical results concerning Reinhardt embeddings, with particular emphasis on *application* of elementary embeddings. We shall show that, under suitable circumstances, one can apply one embedding to itself and compose embeddings. This will be done in the context of KP where the embedding is either definable by a formula or when we work with a predicate for the embedding. To end with, we will discuss the Axiom of Constructibility in relation to V-critical cardinals.

8.1 Definable Embeddings in ZF^-

Gaifman's original proof, [Gai74], was completed under the assumption that M is a model of Z^+. He then commented that the assumption of Power Set can be replaced by the Collection Scheme plus the existence of Cartesian Products. This was then formally done by Gitman, Hamkins and Johnstone in [GHJ16] and we reproduce their proof now.

Theorem 8.1.1 (Gaifman, [Gai74] Part II Theorem 1 and [GHJ16])**.** *Suppose that* M *is a model of* ZF^- *and* $j\colon M \to N$ *is a cofinal,* Σ_0*-elementary embedding. Then* j *is fully elementary.*

It is worth remarking how little we are assuming about M, N and j:

Remarks 8.1.2.

- This theorem does not require any assumptions on N.
- The models M and N need not be transitive.
- This theorem does not require M or N to satisfy any of the axioms of ZF^- where j appears as a parameter.

Proof. We first prove that N will satisfy the axiom of ordered pairs: Using cofinality, for $a, b \in N$, fix $x, y \in M$ such that $a \in j(x)$ and $b \in j(y)$. Then

$$M \models \forall u \in x \; \forall v \in y \; \exists w \in x \times y \; (w = \langle u, v\rangle).$$

So, since this is a Σ_0-formula,

$$N \models \forall u \in j(x) \; \forall v \in j(y) \; \exists w \in j(x \times y) \; (w = \langle u, v\rangle)$$

which yields $\langle a, b\rangle \in N$. This means that we can contract like quantifiers in formulae to a single quantifier in N.

We now proceed by induction on the number of unbounded quantifiers to show that

$$\textit{if } M \models \varphi(z) \textit{ then } N \models \varphi(j(z)). \qquad (\star\star\star)$$

The cases of conjunction, disjunction and implication are obvious while negation will follow from the other cases combined with the observation that $\neg(\forall v \psi(v))$ is logically equivalent to $\exists v \neg\psi(v)$. Therefore, we only consider the quantifier cases in detail.

To do this, first suppose that $\varphi(z) \equiv \forall v \; \psi(v, z)$ where ψ is a Σ_0-formula and fix $y \in N$. By cofinality, there is some $x \in M$ such that $y \in j(x)$. Now suppose that

$$\mathrm{M} \models \forall v \in x\ \psi(v, z).$$

Since this is Σ_0,

$$\mathrm{N} \models \forall v \in j(x)\ \psi(v, j(z)),$$

yielding

$$\mathrm{N} \models \psi(y, j(z)).$$

For the second case, where $\varphi(z) \equiv \exists v\ \psi(v, z)$, suppose that $(\star\star\star)$ has been proven for $\psi(v, z)$ and that $\mathrm{M} \models \exists v\ \psi(v, z)$. Then there is some $t \in \mathrm{M}$ such that $\mathrm{M} \models \psi(t, z)$. So, by the inductive hypothesis, $\mathrm{N} \models \psi(j(t), j(z))$ which yields $\mathrm{N} \models \exists v\ \psi(v, j(z))$.

For the final case, it remains to prove that if $\psi(u, v, z)$ has fewer than n unbounded quantifiers and $(\star\star\star)$ is assumed for any formula with at most n unbounded quantifiers, then

$$\textit{if } \mathrm{M} \models \forall u\ \exists v\ \psi(u, v, z) \quad \textit{then} \quad \mathrm{N} \models \forall u\ \exists v\ \psi(u, v, j(z)).$$

To do this, fix $y \in \mathrm{N}$ and, by cofinality, $a \in \mathrm{M}$ such that $y \in j(a)$. Then, by Collection in M, we can find some set $b \in \mathrm{M}$ such that

$$\mathrm{M} \models \forall u \in a\ \exists v \in b\ \psi(u, v, z).$$

Now, by Separation and Cartesian products in M, we can find some set c such that

$$\mathrm{M} \models \forall u \in a\ \exists v \in b\ \langle u, v\rangle \in c\ \wedge\ \forall u\ \forall v\ \Big(\langle u, v\rangle \in c \to \psi(u, v, z)\Big).$$

Since the first half of this formula is Σ_0 and the second half is logically equivalent to a statement with at most n unbounded quantifiers, by the inductive hypothesis,

$$\mathrm{N} \models \forall u \in j(a)\ \exists v \in j(b)\ \langle u, v\rangle \in j(c)\ \wedge\ \forall u\ \forall v\ \Big(\langle u, v\rangle \in j(c) \to \psi(u, v, j(z))\Big),$$

which implies that $\mathrm{N} \models \exists v\ \psi(y, v, j(z))$ as required. □

In order to obtain a version of Suzuki's Theorem on the non-definability of embeddings, [Suz99], in the context of ZF^- we will use the fact that being Σ_0-elementary is definable by a single formula.

Remark 8.1.3. In Part II Theorem 2 of [Gai74], Gaifman constructs a finite set of formulae Ψ and proves that if $\mathrm{M} \models \mathrm{Z}$ and $j\colon \mathrm{M} \to \mathrm{N}$ is a cofinal embedding which is elementary for Ψ then j is Σ_0-elementary. Essentially Ψ is chosen such that the following hold:

1. All subformulae of the following formulae are in Ψ:
$$\exists v\ (v = \emptyset), \qquad \forall u, v\ \exists w\ (w = \langle u, v\rangle), \qquad \forall u\ \exists v\ (v = \bigcup u),$$
2. The formula expressing the satisfaction relation "$x \models \varphi(z)$" along with all sentence expressing the usual recursion conditions for satisfaction are in Ψ. A formal exposition of this formula and these sentences can be found in Section III of [Bar17].

Gaifman further remarks that the definition of Ψ suffices to prove Σ_0-elementarity in substantially weaker theories than Z, and his proof can be easily seen to go through in a theory such as KP. Then, while Theorem 8.1.1 doesn't go through for KP due to the theory's lack of full collection, it is easy to see that the proof shows that if $j\colon \mathrm{M} \to \mathrm{N}$ is cofinal and elementary for the sentence Ψ then it is Σ-elementary.

Moreover, working with either KP and Σ-elementarity or ZF^- and full elementarity, since Ψ can be seen to be a standard sentence, Theorem 8.1.1 implies that our level of elementarity can be expressed by a single formula. Therefore, while our elementarity was defined for metatheoretic φ, we will also have elementarity for object-theoretic formulae.

Theorem 8.1.4 (Suzuki)**.** *Assume that* $\mathrm{V} \models \mathrm{ZF}^-$*. Then there is no non-trivial, cofinal, elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *which is definable from parameters.*

Proof. Formally, this is a theorem scheme asserting that for each formula φ there is no parameter p for which $\varphi(\cdot, \cdot, p)$ defines a non-trivial, cofinal, elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$. Using Theorem 8.1.1, it suffices to show that for no parameter p are we able to define a non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ by

$$j(x) = y \longleftrightarrow \varphi(x, y, p) \text{ holds.}$$

So, seeking a contradiction, let $\sigma(p)$ be the sentence asserting that $\varphi(\cdot,\cdot,p)$ defines a Σ_0-elementary embedding and let $\psi(p)$ assert that $\varphi(\cdot,\cdot,p)$ defines a total function which is non-trivial, cofinal and Σ_0-elementary. That is,

$$\psi(p) \equiv \forall u \exists! v\ \varphi(u,v,p)\ \wedge\ \exists w\ \neg\varphi(w,w,p)\ \wedge\ \forall x\, \exists y,z\ (\varphi(y,z,p)\ \wedge\ x \in z)\ \wedge\ \sigma(p).$$

Let $\vartheta(p,\kappa)$ postulate that κ is the critical point of j. So,

$$\vartheta(p,\kappa) \equiv \kappa \in \textsc{Ord}\ \wedge\ \forall \alpha \in \kappa\ \varphi(\alpha,\alpha,p)\ \wedge\ \neg\varphi(\kappa,\kappa,p).$$

Then, by Proposition 6.2.1,

$$\mathrm{V} \models \psi(p) \rightarrow \exists! \kappa\ \vartheta(p,\kappa).$$

So denote by crit_p the (unique) κ for which $\vartheta(p,\kappa)$ holds. Now fix p such that crit_p is as small as possible, that is such that

$$\mathrm{V} \models \psi(p)\ \wedge\ \forall w\ (\psi(w) \rightarrow \mathrm{crit}_p \leq \mathrm{crit}_w).$$

Then, by elementarity,

$$\mathrm{V} \models \exists s\ \varphi(p,s,p)\ \wedge\ \psi(s)\ \wedge\ \forall w(\psi(w) \rightarrow \mathrm{crit}_s \leq \mathrm{crit}_w).$$

But, $\mathrm{V} \models \mathrm{crit}_p < \mathrm{crit}_s$ because the critical point of the embedding defined by $\varphi(\cdot,\cdot,s)$ must be $j(\mathrm{crit}_p)$, yielding a contradiction. □

We remark here that the main element of the proof was that being fully elementary can be expressed by a single sentence. Moreover, while Proposition 6.2.8 showed that without a total rank function we can't assume that a non-trivial, elementary embedding has a critical point, recall that there will be one in ZW by Proposition 6.2.9. Therefore, using Gaifman's original theorem, since the Collection Scheme also wasn't used in the proof of Theorem 8.1.4, the above proof also shows that there is no non-trivial, cofinal, Reinhardt embedding over ZW^+ which is definable from parameters. It can also be used to show that there is no non-trivial, cofinal ORD-*inary* embedding over Z^+ which is definable from parameters.

Theorem 8.1.5. *Assume that* $\mathrm{V} \models \mathrm{ZW}^+$. *Then there is no non-trivial, cofinal, elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *which is definable from parameters.*

There are two obvious questions which appear here about whether or not the assumptions of cofinality and collection were necessary in the proof that there is no definable embedding,

Question 8.1.6. Are either of the following statements consistent:

1. There exists a non-trivial, elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ which is definable from parameters where $\mathrm{V} \models \mathrm{ZF}^-$?
2. There exists a non-trivial, cofinal, elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ which is definable from parameters where $\mathrm{V} \models \mathrm{ZF}-$?

Remark 8.1.7. For the above questions it may be unclear what we are formally asking for because we would not expect elementarity to be first-order expressible, especially by a single sentence. Otherwise a similar trick to the one used in the proof of Suzuki's Theorem could plausibly be used to derive a contradiction. Instead these questions should be viewed from a more motivational perspective as asking for a class embedding whose elementarity is proved in the metatheory. That is, we would want a scheme such that, for every (metatheoretic) n, the embedding defined is Σ_n-elementary.

It has been proven in [GHJ16] that one can have cofinal, Σ_1-elementary embeddings of ZF− which are not Σ_2-elementary. That is to say that Gaifman's Theorem can fail without the Collection Scheme. Therefore proving Suzuki's Theorem in this context would involve a different approach. We shall also see in Theorem 10.2.3 that the existence of a definable, non-trivial, *non-cofinal*, Reinhardt embedding in ZF^- is intimately linked with the large cardinal axiom I_1. Namely that, assuming the consistency of $\mathrm{ZFC} + \mathrm{I}_1$, the first question has a positive answer.

8.2 Definable Embeddings in KP

In this section, we develop some of the techniques needed to work with the structural properties of elementary embeddings of KP. A large number of the ideas for this work come from observations by Corazza in [Cor06], with the primary focus here being to show that Corazza's observations transfer to our weaker context.

There are two significant issues one comes across when trying to develop properties of elementary embeddings in KP. The first is whether or not the supremum of the critical sequence exists. In general, KP cannot deduce that the supremum of a sequence of cardinals exists because being a cardinal is a Π_1 property. In fact, as shown in Section II.3 of [Bar17], every uncountable cardinal in admissible. However, as observed by Corazza and discussed in Section 6.3, there is a function defining the critical sequence which can be shown to be total by an instance of Σ_1^j-Induction and whose supremum then exists by Σ_1^j-Replacement, so it must exist when working in the theory KP_j.

The second issue is *application* of elementary embeddings. Given an embedding $j \colon \mathrm{V}_\lambda \to \mathrm{V}_\lambda$ with critical point κ it is often beneficial to be able to consider a new embedding with critical point $j(\kappa)$. This is usually obtained by defining $j \cdot j$ as

$$j \cdot j \coloneqq \bigcup_\alpha j(j \restriction \mathrm{V}_\alpha).$$

This is a problematic definition in KP because V_α need not be a set. Moreover, elementarity may not be definable by a single sentence of the right complexity, causing difficulties with the naive assumption that elementarity is preserved. We shall shortly see how to circumvent this issue by defining $j \cdot j$ in a more careful manner using the assumption of cofinality.

We will then end this section with an application of the introduced methods by giving a necessary condition for the embedding $j \colon \mathrm{V} \to \mathrm{V}$ to be definable by a formula with set parameters. This condition will essentially be that the rank of any parameter must be above the supremum of the critical sequence.

To accommodate later sections and avoid repetition, we will simultaneously work with an embedding definable by a formula τ and an embedding constructed using a predicate for it. For the definable case, it is natural to restrict our attention to only those definitions which are Σ-definable in order to be able to use the Σ_0-Separation and Σ-Collection schemes given by KP. This is in some ways quite a significant restriction because it does not reveal anything about an elementary embedding which is definable by a formula of complexity at least Σ_2, since the theory lacks the axiomatic schemes to do large amounts of set theory with this formula.

In particular, from this perspective, one could potentially have the scenario that there is some admissible L_α and some elementary embedding of the form $j\colon \mathrm{L}_\alpha \to \mathrm{L}_\alpha$ with a definition of high complexity. However, this should not be considered a definable embedding in L_α because some instance of Separation$_j$ must fail, even if this set cannot be proven to exist from the axioms of KP.

Definition 8.2.1. Let V be a model of KP. A formula $\tau(\cdot,\cdot,p)$ defines a cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ (via the parameter p) if the following hold:

- $\forall x, y \left(j(x) = y \longleftrightarrow \tau(x,y,p) \right)$,
- (Cofinality) $\forall z\ \exists x, y \left(\tau(x,y,p) \ \wedge\ z \in y \right)$,
- For any Σ_0-formula $\psi(v)$ and set a, there is some b such that $\tau(a,b,p)$ and $\psi(a) \longleftrightarrow \psi(b)$,

So, given a model of KP, suppose that $\tau(\cdot,\cdot,p)$ is a Σ-formula which defines a cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$. We can then mimic Corazza's functions from Definition 6.3.2 to show that both the critical sequence and its supremum are Σ-definable, noting again the use of Σ-Induction to prove that Φ defines a total class function on ω.

$$\begin{aligned}
\Gamma(f,n,x,y) &\equiv \operatorname{func}(f) \ \wedge \ \operatorname{dom}(f)=n+1 \ \wedge \ f(0)=x \ \wedge \\
&\quad \forall i \left(0<i\leq n \to \tau\Big(f(i-1),f(i),p\Big)\right) \ \wedge \ f(n)=y, \\
\Upsilon(n,x,y) &\equiv n\in\omega \to \exists f \ \Gamma(f,n,x,y), \\
\Phi(n,y) &\equiv \exists x\in y \ \exists z \ \Big(x\neq z \ \wedge \ \Upsilon(n,x,y) \ \wedge \ x\in \textsc{Ord} \ \wedge \\
&\qquad \forall\alpha\in x \ \tau(\alpha,\alpha,p) \ \wedge \ \tau(x,z,p)\Big).
\end{aligned}$$

The next thing we shall show is how to *apply* one embedding to another. We shall state it separately for both our definable context and our Σ^j context. However, Theorems 8.2.2 and 8.2.3 will be proved simultaneously.

Theorem 8.2.2. *Let* V *be a model of* KP*. Suppose that* $\tau(\cdot,\cdot,p)$ *is a* Σ*-formula which defines a cofinal,* Σ_0*-elementary embedding* $j\colon \mathrm{V}\to\mathrm{V}$*. Then for each* $n\in\omega$ *there is a* Σ*-formula* $\tau_{(n)}(\cdot,\cdot,p)$ *satisfying:*

1. $\tau_{(n)}$ *defines a total function,*
2. $\tau_{(n)}$ *defines a cofinal function,*
3. $\tau_{(n)}$ *defines a* Σ*-elementary embedding,*
4. $\mathrm{crit}_{\tau_{(n)}}=j^n(\mathrm{crit}_\tau)$*, where* $\mathrm{crit}_{\tau_{(n)}}$ *denotes the critical point defined by* $\tau_{(n)}$*.*

Theorem 8.2.3. *Let* V *be a model of* KP *and suppose that* $j\colon\mathrm{V}\to\mathrm{V}$ *is a cofinal,* Σ*-elementary embedding such that* $\mathrm{V}\models\mathrm{KP}_j$*. Then for each* $n\in\omega$ *there is a class function* $j_{(n)}$*, definable from* $\langle\mathrm{V},j\rangle$*, such that:*

1. $j_{(n)}$ *is a total function from* V *to* V*,*
2. $j_{(n)}$ *is cofinal,*
3. $j_{(n)}$ *is a* Σ*-elementary embedding,*
4. $\mathrm{crit}_{j_{(n)}}=j^n(\mathrm{crit}_j)$*,*
5. $\mathrm{V}\models\mathrm{KP}_{j_{(n)}}$*.*

Remark 8.2.4. Note that in Theorem 8.2.3 we assume that our universe satisfies Σ_0^j-Separation. However, if j was defined by a Σ-formula τ then Σ_0^j-Separation would equate to Σ-Separation which is not an assumed axiom scheme of KP. Therefore when our proofs use Σ_0^j-Separation we will make an additional comment of how to get around this issue in order to prove the equivalent statement for 8.2.2. One of the primary tools that will allow us to do this is the Σ-*Reflection Principle* from Theorem 2.2.5.

For notational clarity, in the proofs of most of the following claims we shall work with the function j rather than its definition τ and when we refer to τ we will occasionally drop the parameter p. It should be easy to see how to transfer between the two of them if one wishes to express everything in terms of definable functions, but to do so would make the proofs significantly less readable. However, we shall continue to express the claims themselves in terms of τ and other related functions.

As previously stated, if one works in $\mathrm{ZF}_{j,k}$, where j and k are cofinal Reinhardt embeddings, then we can define j *applied to* k by

$$j \cdot k := \bigcup_{\alpha} j(k \restriction \mathrm{V}_\alpha).$$

Now, since j is an elementary embedding, we obviously have that if $x \in j(a)$ and $x \in j(b)$ then $j(k \restriction a)(x) = j(k \restriction b)(x)$. Therefore, if $x \in j(a)$ and $c = k \restriction a$ then $j \cdot k(x) = j(c)(x)$. This leads to the following way to define $j \cdot k$:

$$j \cdot k(x) = y \Leftrightarrow \exists a, b, c, d \left(b = j(a) \ \wedge \ x \in b \ \wedge \ c = k \restriction a \ \wedge \ d = j(c) \ \wedge \ y = d(x) \right).$$

So, the embedding $j_{(k)}$ which will witness Theorem 8.2.3 is defined inductively as $j_{(0)} = j$ and $j_{(k+1)} = j \cdot j_{(k)}$.

While this construction works for the definition of $j \cdot j$ over KP, we immediately run into issues when we try to define $j \cdot (j \cdot j)$. This is because, in the construction we require that $k \restriction a$ is a set for a given a, but this is not obviously true. In particular, at first sight, the assertion that $(j \cdot j) \upharpoonright a$ is a set requires $\Sigma_1^{j \cdot j}$-Separation. Indeed Corazza [Cor06] states that it is not clear how one can iterate the procedure in the theory ZFC plus only Σ_0^j-Separation. However, by using cofinality and Σ^j-Collection we shall be able to circumvent this issue.

Proposition 8.2.5. (KP_j) *For every set t and $n \in \omega$, $j_{(n)} \upharpoonright t$ is a set.*

Proof. Firstly, we observe that for any set t, $j \upharpoonright t$ is a set by Σ_0^j-Separation. Now the proposition will be proved using Σ^j-Induction on n. So, for a given n, fix t and assume that $j_{(n)} \upharpoonright t$ is a set. We now consider the class

$$j_{(n+1)} \upharpoonright t \coloneqq \{\langle u, j_{(n+1)}(u)\rangle \mid u \in t\}.$$

Using cofinality, for each $u \in t$ there is some set a_u such that $u \in j(a_u)$. So, using Σ^j-Collection, we can fix s such that

$$\forall u \in t\ \exists a_u \in s\ (u \in j(a_u)).$$

Therefore, for each $u \in t$, $a_u \subseteq \mathrm{trcl}(s)$ and

$$u \in j(a_u) \subseteq j(\mathrm{trcl}(s)).$$

This means that $j_{(n+1)}(u) = j(j_{(n)} \upharpoonright \mathrm{trcl}(s))(u)$ and, importantly, the definition of s was independent of n. Thus, if given t we fix s, then

$$\begin{aligned} j_{(n+1)} \upharpoonright t &= \{\langle u, j(j_{(n)} \upharpoonright \mathrm{trcl}(s))(u)\rangle \mid u \in t\} \\ &= \{x \in j(j_{(n)} \upharpoonright \mathrm{trcl}(s)) \mid 1^{st}(x) \in t\} \end{aligned}$$

will be a set by our inductive hypothesis and Σ_0-Separation. □

For the definability version, observe that the only place we required Σ_0^j-Separation in the above proof was to deduce that $j \upharpoonright t$ was a set. Therefore, we need to reprove this in the context using τ.

Proposition 8.2.6. (KP) *For every set t, $\{\langle u,v\rangle \mid u\in t \;\wedge\; \tau(u,v,p)\}$ forms a set.*

Proof. Since τ defines a total function,

$$\forall u\in t\;\exists w\;\Big(\tau(u,1^{st}(w),p)\;\wedge\;2^{nd}(w)=\langle u,1^{st}(w)\rangle\Big).$$

Observe that each w is of the form $\langle v,\langle u,v\rangle\rangle$ where v is the unique set for which $\tau(u,v,p)$ holds. Next, by Σ-Reflection we can fix a set c such that

$$\forall u\in t\;\exists w\in c\;\Big(\tau^{(c)}(u,1^{st}(w),p)\;\wedge\;2^{nd}(w)=\langle u,1^{st}(w)\rangle\Big).$$

where $\tau^{(c)}$ is the result of bounding each unbounded quantifier in τ by c. Since $\tau^{(c)}$ is a Σ_0 formula, let

$$b:=\Big\{2^{nd}(w)|w\in c\;\wedge\;\exists u\in t\Big(\tau^{(c)}(u,1^{st}(w),p)\;\wedge\;2^{nd}(w)=\langle u,1^{st}(w)\rangle\Big)\Big\}.$$

Now, since $\tau^{(c)}(u,v,p)\to\tau(u,v,p)$ we have that

$$\forall u\in t\;\exists y\in b\;\exists v\;\Big(y=\langle u,v\rangle\;\wedge\;\tau(u,v,p)\Big)$$

and, by definition of b,

$$\forall y\in b\;\exists u\in t\;\exists v\;\Big(y=\langle u,v\rangle\;\wedge\;\tau(u,v,p)\Big).$$

□

Therefore, if τ and σ are two Σ-formulae which define cofinal, Σ_0-elementary embeddings, we can fix a formula $\Xi_{\tau\cdot\sigma}$ defining $\tau\cdot\sigma$ by

$$\Xi_{\tau\cdot\sigma}(x,y)\;\equiv\;\exists a,b,c,d\;\Big(\tau(a,b)\;\wedge\;x\in b\;\wedge\;\text{func}(c)\;\wedge\;\text{dom}(c)=a$$
$$\wedge\;\forall w\in a\;\sigma(w,c(w))\;\wedge\;\tau(c,d)\;\wedge\;y=d(x)\Big).$$

Then, if we let $\tau_{(0)}=\tau$ and $\tau_{(k+1)}=\Xi_{\tau\cdot\tau_{(k)}}$, one can see that $\tau_{(k)}$ remains a Σ-formula. Importantly, if τ was defined with parameter p, then $\Xi_{\tau\cdot\tau}$ can also be defined from the same parameter.

We now prove Theorems 8.2.2 and 8.2.3 using the Σ / Σ^j Induction scheme as applicable. So assume that we have proven each of the four conditions for $\tau_{(n)}$ and $j_{(n)}$. To ease notation, we will also use k to denote $j_{(n)}$.

Claim 8.2.7 (Condition 1)**.** *$\tau_{(n+1)}$ and $j_{(n+1)}$ are total functions.*

Proof. Let x be an arbitrary set and, by the cofinality assumption, fix a such that $x \in j(a)$. Using either Σ_0^j-Separation or the previous proposition, let $c = k \restriction a$. Then, by elementarity, $j(c)(x) = j(k \restriction a)(x)$ is a set and $\Xi_{\tau \cdot \tau_{(n)}}(x, j(c)(x))$ holds.

To prove uniqueness, suppose that $j \cdot k(x) = y$ and $j \cdot k(x) = y'$. Again, fix a such that $x \in j(a)$. Using the definition of application we have that $y = j(k \restriction a)(x) = y'$ so $j_{(n+1)}$ is indeed a function. $\square$

It is useful to remark on how composition and application relate to one another for Reinhardt embeddings. In the standard case, we have the identity

$$j \circ k = (j \cdot k) \circ j$$

and this will transfer to the definable case as well. To see this, we first give a function $\Gamma_{\tau \circ \sigma}$ which will denote the composition of τ and σ:

$$\Gamma_{\tau \circ \sigma}(x, y) \equiv \exists w \ \big(\sigma(x, w) \ \wedge \ \tau(w, y)\big).$$

We can then define the function $\tau^{(n)}$ inductively on the natural numbers such that $\tau^{(n)}(x, y)$ holds if and only if $j^{n+1}(x) = y$. To do this, let $\tau^{(0)} = \tau$ and set

$$\tau^{(n+1)}(x, y) \equiv \Gamma_{\tau \circ \tau^{(n)}}(x, y).$$

Claim 8.2.8 (Condition 2)**.** *$\tau_{(n+1)}$ and $j_{(n+1)}$ are cofinal functions.*

Proof. We begin by proving that

$$j \circ k = (j \cdot k) \circ j$$

To do this, let x be arbitrary and, using the assumption that k is cofinal, fix a such that $x \in k(a)$. Then

$$j \circ k(x) = j\Big((k \upharpoonright a)(x)\Big) = j(k \upharpoonright a)\Big(j(x)\Big) = j \cdot k(j(x)).$$

Next note, that by taking the transitive closure of a, we can assume that for every set x there is a *transitive* set a such that $x \in j(a)$. Now let x be arbitrary and fix transitive sets a and u such that $x \in j(a)$ and $a \in k(u)$. This yields that

$$x \in j(a) \in j \circ k(u).$$

Therefore, $x \in j \cdot k(j(u))$ so $j(u)$ witnesses this instance of cofinality. □

In order to prove Condition 3 for the definable case, it is insightful to include parameters, which we have suppressed so far. Therefore, if we were to consider $\tau(\cdot, \cdot, p)$ then formally $\tau_{(1)}$ should be written as $\Xi_{\tau\cdot\tau}(\cdot, \cdot, p)$. With this being the case, we can see that the map defined by $\tau_{(1)}$ is the same as the map defined by $\tau(\cdot, \cdot, j(p))$.

Claim 8.2.9. $\forall x, y \ \Big(\Xi_{\tau\cdot\tau}(x, y, p) \leftrightarrow \exists q \ \big(\tau(p, q, p) \ \wedge \ \tau(x, y, q)\big)\Big).$

Proof. For the left-to-right direction, fix a, b, c and d witnessing $\Xi_{\tau\cdot\tau}(x, y, p)$. Then in particular we have that $x \in b$ and

$$\tau(a, b, p) \ \wedge \ \text{func}(c) \ \wedge \ \text{dom}(c) = a \ \wedge \ \forall w \in a \ \tau(w, c(w), p).$$

So, by elementarity and the fact that $j(a) = b$,

$$\tau(b, j(b), j(p)) \ \wedge \ \text{func}(j(c)) \ \wedge \ \text{dom}(j(c)) = b \ \wedge \ \forall w \in b \ \tau(w, j(c)(w), j(p)).$$

Since $x \in b$, we have that $\tau(x, j(c)(x), j(p))$ and, by definition,

$$j(c)(x) = j(j \upharpoonright a)(x) = y.$$

So $\tau(x, y, j(p))$ holds, as required.

Similarly, if $\tau(x, y, j(p))$ holds then we obtain that $j(c)(x) = d(x) = y$. So, again since $\Xi_{\tau\cdot\tau}$ defines a total function, $\Xi_{\tau\cdot\tau}(x, y, p)$ holds. □

By a similar argument, we have that for any n, $\tau_{(n)}(\cdot, \cdot, p)$ defines the same map as $\tau(\cdot, \cdot, j^n(p))$.

Claim 8.2.10 (Condition 3)**.** *$\tau_{(n+1)}$ and $j_{(n+1)}$ are Σ-elementary embeddings.*

Proof. It suffices to show Σ_0-elementarity because cofinality and the second quantifier case in the proof of Theorem 8.1.1 will then give us Σ-elementarity.

Let $\varphi(u)$ be a Σ_0-formula with parameter u. Using cofinality of k, fix a such that $u \in k(a)$. Since k is a Σ-elementary embedding, so is the restriction $k \restriction a$. Moreover, the statement "*$k \restriction a$ is Σ_0-elementary*" can be defined is a Σ way using a truth predicate. Therefore $j(k \restriction a)$ is a Σ_0-elementary embedding with domain $j(a)$. Therefore

$$\varphi(u) \leftrightarrow \varphi(j(k \restriction a)(u)).$$

But, by definition, $\varphi(j(k \restriction a)(u))$ holds if and only if $\varphi(j \cdot k(u))$ holds. Thus $\varphi(u) \leftrightarrow \varphi(j \cdot k(u))$ so $j \cdot k$ is indeed Σ_0, and hence Σ, elementary. □

Since j_{n+1} was definable from j (but not vice versa) we immediately get that V satisfies the Bounded Separation and Collection schemes in the language expanded to include $j_{(n+1)}$.

Claim 8.2.11. V *is a model of* $\mathrm{KP}_{j_{(n+1)}}$.

Claim 8.2.12 (Condition 4)**.** $\mathrm{crit}_{\tau_{(n+1)}} = j^{n+1}(\mathrm{crit}_\tau)$ *and* $\mathrm{crit}_{j_{(n+1)}} = j^{n+1}(\mathrm{crit}(j))$. *Moreover,* $j_{(n+1)}(j^{n+1}(\mathrm{crit}(j))) = j^{n+2}(\mathrm{crit}(j))$.

Proof. Let $\kappa = \mathrm{crit}(j)$. Using the induction hypothesis that $\mathrm{crit}_{j_{(n)}} = j^n(\kappa)$ and $j_{(n)}(j^n(\kappa)) = j^{n+1}(\kappa)$,

$$j \cdot j_{(n)}(j^{n+1}(\kappa)) = j \cdot j_{(n)}(j(j^n(\kappa)) = j \circ j_{(n)}(j^n(\kappa)) = j(j^{n+1}(\kappa)) = j^{n+2}(\kappa).$$

Thus it suffices to prove that for all $\alpha \in j^{n+1}(\kappa)$, $j_{(n+1)}(\alpha) = \alpha$. But this just follows from the fact that $j_{(n)} \restriction j^n(\kappa)$ is the identity, which implies that so is

$$j(j_{(n)} \restriction j^n(\kappa)) \colon j^{n+1}(\kappa) \to j^{n+2}(\kappa).$$

□

□*Proof of Theorems* 8.2.2 *and* 8.2.3

As an application of the methods we have just developed, we show that if $\tau(\cdot,\cdot,p)$ defines a cofinal, Σ-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ then the rank of p must be at least as big as the supremum of the critical sequence.

Theorem 8.2.13. *Assume that* $\mathrm{V} \models \mathrm{KP}$. *If* $\tau(\cdot,\cdot,p)$ *is a* Σ*-formula defining a cofinal,* Σ_0*-elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *then* $\mathrm{rank}(p) \geq \sup\{j^n(\mathrm{crit}_{\tau(\cdot,\cdot,p)}) \mid n \in \omega\}$.

Proof. Suppose towards a contradiction that $\tau(\cdot,\cdot,p)$ defined a cofinal, $\Sigma_0^{\tau(\cdot,\cdot,p)}$-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ with

$$j^k(\mathrm{crit}_{\tau(\cdot,\cdot,p)}) \leq \mathrm{rank}(p) < j^{k+1}(\mathrm{crit}_{\tau(\cdot,\cdot,p)})$$

for some $k \in \omega \cup \{-1\}$ where we define $j^{-1}(\mathrm{crit}_{\tau(\cdot,\cdot,p)})$ to be $\emptyset$. Then, by Theorem 8.2.3, $\tau_{(k+1)}(\cdot,\cdot,p)$ defines a $\Sigma^{\tau(\cdot,\cdot,p)}$-elementary embedding $i\colon \mathrm{V} \to \mathrm{V}$ satisfying $i(p) = p$.

Let $\eta = j^{k+1}(\mathrm{crit}_{\tau(\cdot,\cdot,p)})$ denote the critical point of i. Then

$$\mathrm{V} \models \eta \in i(\eta) \ \wedge \ \tau_{(k+1)}(\eta, i(\eta), p) \ \wedge \ \forall \alpha \in \eta \ \tau_{(k+1)}(\alpha,\alpha,p)$$

so, by elementarity,

$$\mathrm{V} \models i(\eta) \in i^2(\eta) \ \wedge \ \tau_{(k+1)}(i(\eta), i^2(\eta), i(p)) \ \wedge \ \forall \alpha \in i(\eta) \ \tau_{(k+1)}(\alpha,\alpha,i(p)).$$

However this leads to a contradiction, since $p = i(p)$ gives us that

$$\mathrm{V} \models \tau_{(k+1)}(\eta, i(\eta), p) \ \wedge \ \tau_{(k+1)}(\eta,\eta,p).$$

□

8.3 Non-Constructibility

One of the first results that one deduces from ZFC plus a measurable cardinal is that the Axiom of Constructibility does not hold, that is $\mathrm{V} \neq \mathrm{L}$. This result still goes through in very weak systems by showing that, in general, if there is a Reinhardt embedding then the universe cannot have a definable, global well-order which is respected by j. This result makes essential use of the fact that the supremum of the critical sequence is Σ_1^j-definable and therefore a set. The observation below can be found as Theorem 21 of [HKP12], where they show that if $\mathrm{V} = \mathrm{HOD}$ then there are no Reinhardt embeddings. This proposition will rule out well-orderings of $[\lambda]^\omega$ which are definable without reference to the parameter j and definable, global well-orderings such as the one given by L.

Proposition 8.3.1. *There is no non-trivial, elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *with* $\mathrm{V} \models \mathrm{KP}_j$ *for which there is a well-order,* $\prec$*, of the class* $[\sup\{j^n(\mathrm{crit}(j)) \mid n \in \omega\}]^\omega$ *which is fixed by* j*. That is, for any* a *and* b *in the class,* $a \prec b$ *if and only if* $j(a) \prec j(b)$*.*

Proof. Suppose for a contradiction that such a situation were possible and let κ denote the critical point of j. Let $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$ and note that $j(\lambda) = \lambda$. Now, given the well-order, $\prec$, of the class $[\lambda]^\omega$ which is fixed by j, let s be the $\prec$-least element of this class. Since $\prec$ was a definable ordering which is fixed by j, by elementarity $j(s)$ is the $\prec$-least element of this class, namely $j(s) = s$.

In particular, this gives us that for every $n \in \omega$, $s(n) = j(s)(n)$ which is to say that every element of s is fixed by j. However, by construction, λ is the first ordinal above κ which is fixed by j which implies that s is not cofinal in λ, yielding the contradiction. □

Corollary 8.3.2. *If there is a non-trivial, elementary embedding* $j\colon \mathrm{V} \to \mathrm{M}$ *where* $\mathrm{M} \subseteq \mathrm{V}$ *and* $\mathrm{V} \models \mathrm{KP}_{j,\mathrm{M}}$ *then* $\mathrm{V} \neq \mathrm{L}$*.*

Proof. Suppose that $\mathrm{V} = \mathrm{L}$ and $j\colon \mathrm{V} \to \mathrm{M}$ was an elementary embedding. Then, by absoluteness of definability, $\mathrm{L}^\mathrm{M} = \mathrm{L}$ and thus $\mathrm{M} = \mathrm{L}$. However, L has a Σ_1-definable global well-ordering so, by the above proposition, the embedding must be trivial. □

Chapter 9

Initial Bounds for Reinhardt Cardinals in Weak Systems

9.1 Embeddings of Weak Theories with Power Set

To begin this chapter, we shall derive our first variant of the Kunen inconsistency. That is, we shall show that there is no non-trivial elementary embedding from V to itself in essentially any theory which satisfies the Well-Ordering Principle and has the function space ${}^{\lambda}\lambda$ as a set. This will be done by following Kunen's original argument using Jónsson functions while keeping a close check on the complexity of each statement we make.

In order to do this, we begin by proving some basic consequences that can be derived from Power Set. Our background theory in this section shall be ZBQW. For clarity, we recall here how we are defining ordered pairs and the Cartesian product.

Definition 9.1.1. Given sets a, b we let the *ordered pair* of a and b be

$$\langle a, b\rangle \coloneqq \{\{a\}, \{a, b\}\}$$

and we define the Cartesian product of a and b by

$$a \times b \coloneqq \{\langle x, y\rangle \mid x \in a \ \wedge \ y \in b\}.$$

Proposition 9.1.2. ZBQW $\vdash \forall a, b\ \exists c, d\ (c = a \times b\ \wedge\ d = {}^{a}b)$.

Proof. To prove Cartesian products, let $w = \mathcal{P}(\mathcal{P}(a \cup b))$. Then we have that for any $x \in a$ and $y \in b$, $\langle x, y\rangle \in w$. Therefore, the required set is

$$c = \{t \in w \mid \exists x \in a\ \exists y \in b\ t = \langle x, y\rangle\}$$

which is a set by Bounded Separation.

To prove exponentiation, the set of functions from a to b is

$${}^{a}b := \{f \in \mathcal{P}(a \times b) \mid \mathrm{func}(f)\ \wedge\ \mathrm{dom}(f) = a\}$$

which is also a set by Bounded Separation. □

Definition 9.1.3. For sets a and b, let $\mathrm{Inj}({}^{a}b)$ denote the set of injections from a to b.

Note that being an injection is Σ_0-definable and therefore $\mathrm{Inj}({}^{a}b)$ is a set. We also verify that the canonical well-ordering of the class of ordinals, which is defined in Chapter 3 of [Jec03], still goes through in ZBQW. This will give us that for any infinite cardinal μ, $\mu \times \mu$ has cardinality μ.

Definition 9.1.4. Define the *canonical ordering*, $\prec$, on $\textsc{Ord} \times \textsc{Ord}$ as follows:

$$\begin{aligned} \langle \alpha, \beta\rangle \prec \langle \gamma, \delta\rangle \quad \longleftrightarrow \quad & \textit{either } \max\{\alpha, \beta\} < \max\{\gamma, \delta\}, \\ & \textit{or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \textit{ and } \alpha < \gamma, \\ & \textit{or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\},\ \alpha = \gamma \textit{ and } \beta < \delta, \end{aligned}$$

Given α and β, $\{\langle \xi, \eta\rangle \mid \langle \xi, \eta\rangle \prec \langle \alpha, \beta\rangle\}$ constitutes a well-orderable set. So, by axiom W, let $\Gamma(\alpha, \beta)$ denote the ordinal this set is isomorphic to.

As stated in [Jec03], $\prec$ is a Σ_0 relation on $\textsc{Ord} \times \textsc{Ord}$ and Γ can be defined in a Δ_1 way. Moreover, for an ordinal α we let the canonical ordering of $\alpha \times \alpha$ be the restriction of $\prec$ to $\alpha \times \alpha$. We shall prove that for any infinite cardinal μ, $\Gamma(\mu, \mu) = \mu$, which will give us that $\mu \times \mu$ has cardinality μ. Note that, under ZF, this holds for any *indecomposable ordinal* however to prove this over ZBQW would lead to unnecessary notation for our purposes.

Proposition 9.1.5. *Over* ZBQW*, for any infinite cardinal* μ*,* $\Gamma(\mu, \mu) = \mu$*.*

Proof. We first remark that $\gamma(\alpha) = \Gamma(\alpha, \alpha)$ is an increasing function, so $\Gamma(\alpha, \alpha) \geq \alpha$ for any α, and $\Gamma(\omega, \omega) = \omega$. So suppose that the claim is false and let μ be the least cardinal witnessing this. Since $\prec$ is a well-ordering of $\mu \times \mu$ of order type greater than μ, we can fix ordinals α and β less than μ such that $\Gamma(\alpha, \beta) = \mu$. Now take $\delta = \max\{\alpha, \beta\} + 1 < \mu$. Then, $\langle \alpha, \beta \rangle \prec \langle \delta, \delta \rangle$ so

$$\mu \subseteq \Gamma(\delta, \delta)$$

which gives us that $|\delta \times \delta| \geq \mu$. But, by definition,

$$|\delta \times \delta| = ||\delta| \cdot |\delta|| = |\delta| < \mu$$

where the last equality follows from the minimality of μ. Yielding our contradiction. $\square$

We now define a variation of a function being ω-Jónsson. We have decided to use this variant rather than the original because, in such a weak system, it is more natural to work with the set of functions from ω to x rather than the set of ω sized subsets of x.

Definition 9.1.6. For any cardinal α and function $\mathcal{H}$, $\mathcal{H}$ satisfies $*_\alpha$ if $\mathcal{H} \colon \operatorname{Inj}({}^\omega\alpha) \to \alpha$ and

$$\forall g \in \operatorname{Inj}({}^\alpha\alpha)\ \forall \gamma \in \alpha\ \exists t \in \operatorname{Inj}({}^\omega\alpha)\ \mathcal{H}(g \circ t) = \gamma.$$

Explaining the above definition in more detail, under ZFC a function F is said to be ω-Jónsson for α if for any set $x \subseteq \alpha$ of size α, $F``([x]^\omega) = \alpha$. For the definition given above, we replace subsets of size $\alpha$ by injections $g$ from $\alpha$ into $\alpha$. Then the claim is that $\mathcal{H}``\{g \circ t \mid t \in \operatorname{Inj}({}^\omega\alpha)\} = \alpha$.

Next, note that, given an ordinal α, the claim that $\mathcal{H}$ satisfies $*_\alpha$ is Σ-definable with the parameters α, ${}^\omega\alpha$ and ${}^\alpha\alpha$ by the formula,

$$\begin{aligned} \mathcal{H} \in \mathcal{P}({}^\omega\alpha \times \alpha) \quad &\wedge \quad \mathcal{H} \textit{ is a function} \ \wedge\ \operatorname{dom}(\mathcal{H}) = \operatorname{Inj}({}^\omega\alpha) \ \wedge\ \operatorname{ran}(\mathcal{H}) = \alpha \\ &\wedge \quad \forall g \in \operatorname{Inj}({}^\alpha\alpha)\ \forall \gamma \in \alpha\ \exists t \in \operatorname{Inj}({}^\omega\alpha)\ \mathcal{H}(g \circ t) = \gamma. \end{aligned}$$

We now show that Reinhardt Cardinals are inconsistent with ZBQW_j if we also assume a single instance of Σ^j-replacement. This is the instance which gives us that the supremum of the critical sequence is a set. Note that this instance is necessary for the proof because, as we shall discuss in more detail in Section 9.2, the Wholeness Axiom gives us a Reinhardt Cardinal over Zermelo set theory.

Theorem 9.1.7. *There is no non-trivial, Σ-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that*

- $\mathrm{V} \models \mathrm{ZBQW}_j$,
- *The supremum of the critical sequence exists.*

Suppose that $j\colon \mathrm{V} \to \mathrm{V}$ is a non-trivial Σ-elementary embedding and, using Proposition 6.2.9, let κ be the critical point. First observe that, using $\mathrm{Induction}_j$, for any set x the function $n \mapsto j^n(x)$ is provably total by the same argument as used in Theorem 6.3.3. So let $\kappa_n \coloneqq j^n(\kappa)$ and let $\lambda \coloneqq \sup\{\kappa_n \mid n \in \omega\}$.

Lemma 9.1.8. *There exists a function $\mathcal{H}$ satisfying $*_\lambda$.*

Proof. By using the assumption that every set can be well-ordered by an ordinal, enumerate $\mathrm{Inj}({}^\lambda\lambda) \times \lambda$ as $\{\langle x_\alpha, \gamma_\alpha\rangle \mid \alpha \in \delta\}$ where δ is a cardinal. Given x_α, define

$$Y_\alpha \coloneqq \{x_\alpha \circ r \mid r \in \mathrm{Inj}({}^\omega\lambda)\}$$

and note that $Y_\alpha \subseteq \mathrm{Inj}({}^\omega\lambda)$ is a set by Σ_0-separation.

Claim 9.1.9. *There is an injection from $\mathrm{Inj}({}^\lambda\lambda) \times \lambda$ into $\mathrm{Inj}({}^\omega\lambda)$.*

Assuming the claim, first fix a well-ordering of $\mathrm{Inj}({}^\omega\lambda)$. Then, for $\alpha \in \delta$, choose s_α to be the least element of $\mathrm{Inj}({}^\omega\lambda)$ according to this well-order such that $s_\alpha \in Y_\alpha$ and for any $\beta \in \alpha$, $s_\alpha \neq s_\beta$.

Finally, define $\mathcal{H}\colon \mathrm{Inj}({}^\omega\lambda) \to \lambda$ by

$$\mathcal{H}(t) = \begin{cases} \gamma_\alpha, & \textit{if } t = s_\alpha \textit{ for some } \alpha \in \delta \\ 0, & \textit{otherwise} \end{cases}$$

To see that this satisfies $*_\lambda$, take $g \in \mathrm{Inj}({}^\lambda\lambda)$ and $\tau \in \lambda$. Then $\langle g, \tau \rangle = \langle x_\alpha, \gamma_\alpha \rangle$ for some $\alpha \in \delta$. So s_α is defined and $s_\alpha = x_\alpha \circ r = g \circ r$ for some $r \in \mathrm{Inj}({}^\omega\lambda)$. Therefore $\mathcal{H}(g \circ r) = \mathcal{H}(s_\alpha) = \gamma_\alpha = \tau$.

Proof of Claim 9.1.9: We shall show that the following chain of injections are all provable in ZBQW,

$$\begin{aligned} \mathrm{Inj}({}^\lambda\lambda) \times \lambda \ &\overset{①}{\prec}\ \mathrm{Inj}({}^\lambda\lambda) \times \mathrm{Inj}({}^\lambda\lambda) \overset{②}{\prec} \mathrm{Inj}({}^\lambda(\lambda \times \lambda)) \\ &\overset{③}{\prec}\ \mathrm{Inj}({}^\lambda\lambda) \overset{④}{\prec} {}^\lambda({}^\lambda 2) \overset{⑤}{\prec} {}^{\lambda\times\lambda}2 \\ &\overset{⑥}{\prec}\ {}^\lambda 2 \overset{⑦}{\prec} {}^\omega\lambda \overset{⑧}{\prec} \mathrm{Inj}({}^\omega(\lambda \times \lambda)) \\ &\overset{⑨}{\prec}\ \mathrm{Inj}({}^\omega\lambda). \end{aligned}$$

- ①: $\langle g, \alpha \rangle \mapsto \langle g, (\beta \mapsto \alpha + \beta) \rangle$.
- ②: $\langle g, h \rangle \mapsto (\alpha \mapsto (g(\alpha), h(\alpha)))$.
- ③: via the canonical bijection between λ and $\lambda \times \lambda$.
- ④:

$$g \mapsto \left(\alpha \mapsto h_\alpha \colon \lambda \to 2, \ \textit{where } h_\alpha(\beta) = \begin{cases} 1, & \textit{if } \beta = g(\alpha) \\ 0, & \textit{otherwise} \end{cases} \right)$$

- ⑤: $g \mapsto (\langle \alpha, \beta \rangle \mapsto g(\alpha)(\beta))$.
- ⑥: via the canonical bijection between λ and $\lambda \times \lambda$.
- ⑧: $g \mapsto (n \mapsto \langle n, g(n) \rangle)$.
- ⑨: via the canonical bijection between λ and $\lambda \times \lambda$.

In order to prove ⑦ we need to fix a series of injections from ${}^{\kappa_n}2$ into the interval $[\kappa_n, \kappa_{n+1})$.

Claim 9.1.10. *For every $\gamma \in \kappa$ there is an injection from ${}^{\gamma}2$ into κ.*

Proof. Suppose this is not the case. Then, using well-ordering, we can fix some $\gamma \in \kappa$ and an injection $t \colon \kappa \to {}^{\gamma}2$. So, by elementarity, $j(t)$ is an injection from $j(\kappa)$ into $j({}^{\gamma}2)$, which is equal to ${}^{\gamma}2$ by Proposition 6.2.12, since γ was fixed by j. Now consider

$$j(t)(\kappa) \colon \gamma \to 2.$$

This is fixed by j because $j\Big(j(t)(\kappa)\Big)$ is another function from γ to 2 such that

$$j\Big(j(t)(\kappa)\Big)(\alpha) = j\Big(j(t)(\kappa)\Big)(j(\alpha)) = j\Big(j(t)(\kappa)(\alpha)\Big) = j(t)(\kappa)(\alpha).$$

Therefore $j\Big(j(t)(\kappa)\Big) \in \operatorname{ran}(j(t))$ so, by elementarity, $j(t)(\kappa)$ is in the range of t. But this means that for some $\delta \in \kappa$, $j(t)(\kappa) = t(\delta)$ which, by the previous argument, is equal to

$$j(t(\delta)) = j(t)(j(\delta)) = j(t)(\delta),$$

contradicting the assumption that $j(t)$ is injective. □

Next, by Σ-elementarity, we have that for every $\gamma \in j(\kappa)$ there is an injection from ${}^{\gamma}2$ into $j(\kappa)$. In particular, this gives us that there is an injection from ${}^{\kappa}2$ into $j(\kappa)$. So, since $j(\kappa)$ is bijective with $[\kappa, j(\kappa))$ we can fix an injection

$$g \colon {}^{\kappa}2 \to [\kappa, j(\kappa)).$$

Finally, by Induction$_j$ and elementarity, for any $n \in \omega$, $j^n(g)$ is an injection from ${}^{\kappa_n}2$ into the interval $[\kappa_n, \kappa_{n+1})$.

Therefore, to prove ⑦, define $\Phi \colon {}^{\lambda}2 \to {}^{\omega}\lambda$ by

$$\Phi(f)(n) = j^n(g)(f \restriction \kappa_n).$$

□ *Claim* 9.1.9 □ *Lemma* 9.1.8

Proof of Theorem 9.1.7. Fix $\mathcal{H}$ satisfying $*_\lambda$. Now, by elementarity, $j(\mathcal{H})$ also satisfies $*_\lambda$ because this was Σ-definable with the parameters λ, ${}^\omega\lambda$ and ${}^\lambda\lambda$, all of which are fixed by j. Since $f\colon \lambda \to \lambda$, $\alpha \mapsto j(\alpha)$ is an injection, we shall now reach a contradiction by showing that for any $t \in \text{Inj}({}^\omega\lambda)$, $j(\mathcal{H})(f \circ t) \neq \kappa$.

To do this, fix $t \in \text{Inj}({}^\omega\lambda)$. Then, for any $n \in \omega$, $f \circ t(n) = j(t(n))$. So,

$$\begin{aligned} j(\mathcal{H})(f \circ t) &= j(\mathcal{H})\Big(\{\langle n, f \circ t(n)\rangle \mid n \in \omega\}\Big) \\ &= j(\mathcal{H})\Big(\{\langle n, j(t(n))\rangle \mid n \in \omega\}\Big) \\ &= j(\mathcal{H})\Big(j(\{\langle n, t(n)\rangle \mid n \in \omega\})\Big) \\ &= j\Big(\mathcal{H}(\{\langle n, t(n)\rangle \mid n \in \omega\})\Big). \end{aligned}$$

Therefore the range of $j(\mathcal{H})(f \circ t)$ must be contained in the Σ_0^j-definable set

$$j``\lambda \coloneqq \{\alpha \in \lambda \mid \exists \beta \in \lambda \ (\alpha = j(\beta))\}.$$

and $\kappa \neq j(\mathcal{H})(f \circ t)$, completing the proof. □

Since the supremum of the critical sequence is Σ^j-definable, the above analysis allows us to conclude that Reinhardt embeddings are inconsistent with $\text{KP}(\mathcal{P})$ plus well-ordering.

Corollary 9.1.11. *There is no non-trivial, Σ-elementary embedding $j\colon \text{V} \to \text{V}$ such that* $\text{V} \models (\text{KP}(\mathcal{P}) + \text{W})_j$.

9.2 The Wholeness Axiom

Section 9.1 can be seen as proving that, over a weak system, the Kunen inconsistency follows from the function space ${}^\lambda\lambda$ being a well-orderable set. In this section we want to show that, if we have a cumulative hierarchy of sets, then Reinhardt embeddings have large consistency strength. In particular, we will show that, again over a weak base theory, V_λ will be a model of ZF plus *the Wholeness Axiom* which we defined in Definition 6.3.6.

As we shall shortly see, if λ witnesses an I_3 embedding then V_λ is a model of ZFC plus WA_∞. In fact, this will even hold when we substantially weaken the theory we assume V to satisfy. In order to do this, we need to use the *Tarski-Vaught Test*. This is a very model theoretic property and so will hold assuming only a very modest background theory, which we will take to be the theory ZBQ^-. We also state here a special case of the Tarski-Vaught Test where we restrict to Σ-formulae, using a proof from Barwise.

Theorem 9.2.1 (Tarski-Vaught Test, [Bar17] Lemma V.7.7)**.** *Suppose that* $\mathrm{M} \subseteq \mathrm{N}$ *are two class substructures of a model of* ZBQ^- *(with the language expanded to include predicates for these). Then:*

1. M *is a* Σ_1*-elementary substructure of* $\mathrm{N} \iff$ *For every* Σ_0*-formula* $\varphi(u, v)$ *and* $a \in \mathrm{M}$*, if* $\mathrm{N} \models \exists x\ \varphi(x, a)$ *then there exists some* $b \in \mathrm{M}$ *such that* $\mathrm{N} \models \varphi(b, a)$*.*
2. M *is an elementary substructure of* $\mathrm{N} \iff$ *For every formula* $\varphi(u, v)$ *and* $a \in \mathrm{M}$*, if* $\mathrm{N} \models \exists x\ \varphi(x, a)$ *then there exists some* $b \in \mathrm{M}$ *such that* $\mathrm{N} \models \varphi(b, a)$*.*

Proof. For the left to right implication of the first claim, suppose that M is a Σ_1-elementary substructure of N, let $\varphi(u, v)$ be a Σ_0-formula and a be in M. Next, suppose that $\mathrm{N} \models \exists x\ \varphi(x, a)$. Since this is a Σ_1-formula, the statement transfers to M so we can fix some $b \in \mathrm{M}$ such that

$$\mathrm{M} \models \varphi(b, a)$$

and this transfers back to N by the definition of being an elementary substructure. The left to right implication of the second claim follows by the same argument.

For the reverse direction of the first claim, we first prove that M is a Σ_0-elementary substructure of N. That is, for any Σ_0-formula $\psi(v)$ and $a \in \mathrm{M}$,

$$\mathrm{M} \models \psi(a) \iff \mathrm{N} \models \psi(a).$$

Note that since we are not assuming the structures are transitive we have to do slightly more work than just using absoluteness. The atomic cases follow from the assumption that M ⊆ N while the connective cases are immediate. Moreover, we can write bounded existential quantifiers as

$$\exists x(x \in a \ \wedge \ \psi(x)),$$

and bounded universal quantifiers as

$$\neg\exists x(x \in a \ \wedge \ \neg\psi(x)).$$

Then the left implication of these cases follow from M ⊆ N while the right implication follows from the criterion we are assuming and the assumption that $x \in a \ \wedge \ \psi(x)$ was a Σ_0-formula.

We now proceed to prove that M is a Σ_1-elementary substructure of N. To do this, let $\varphi(u, v)$ be a Σ_0-formula and suppose that $\mathrm{N} \models \exists x\varphi(x, a)$ for some $a \in \mathrm{M}$. Then, by the assumed criterion, there is some $b \in \mathrm{M}$ such that $\mathrm{N} \models \varphi(b, a)$. Since M is a Σ_0-elementary substructure of N, this gives us $\mathrm{M} \models \varphi(b, a)$ and thus $\mathrm{M} \models \exists x \ \varphi(x, a)$.

On the other hand, if $\mathrm{M} \models \exists x \ \varphi(x, a)$ then we can fix some $b \in M$ such that $\mathrm{M} \models \varphi(b, a)$. Again, by Σ_0-elementarity, $\mathrm{N} \models \varphi(b, a)$ yielding $\mathrm{N} \models \exists x \ \varphi(x, a)$. As before, the reverse direction of the second claim follows by the same argument. □

Observe that in the first claim of the previous theorem we only obtained that M was a Σ_1-elementary substructure. Over ZBQ^- this is weaker than being a Σ-elementary substructure because we required Bounded Collection to prove that every Σ-formula is equivalent to a Σ_1-formula. However, the following is immediate in KP:

Corollary 9.2.2. *Suppose that* M ⊆ N *are two class substructures of a model of* KP *(with the language expanded to include predicates for these). Then:*

M *is a* Σ*-elementary substructure of* N $\Longleftrightarrow$ *For every* Σ_0*-formula* $\varphi(u, v)$ *and* $a \in \mathrm{M}$*, if* $\mathrm{N} \models \exists x \ \varphi(x, a)$ *then there exists some* $b \in \mathrm{M}$ *such that* $\mathrm{N} \models \varphi(b, a)$*.*

We end this section by giving lower bounds for Reinhardt embeddings in two systems for which V_α is a set for every α. The first of these will be the theory ZW^+ while the second will be KP^+, where we recall Definition 2.1.8 which defined the theory T^+ to be T plus the assertion that V_α constitutes as set for every α.

In Section 8.2 of [HFL12], the authors claim that the correct way to think about $\mathrm{ZFC} + \mathrm{WA}_\infty$ is as a Reinhardt over a model of ZW with a cumulative hierarchy of sets. They give their version of the axioms of this theory and then state, without proof, consequences of such an embedding. Since their interpretation of a Reinhardt embedding over a model of ZW^+ is different to that we have presented, it is instructive to state this in a manner cohering to our presentation.

Theorem 9.2.3. *The following two theories are equiconsistent:*

(i) $\mathrm{ZFC} + \mathrm{WA}_\infty$*,*

(ii) ZW^+ *plus the existence of a non-trivial,* Σ*-elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *for which* $\mathrm{V} \models \mathrm{ZW}^+_j$*.*

Proof. For the first implication, suppose that V satisfies ZFC plus Wholeness, as witnessed by j. Then, by definition, V must satisfy every instance of $\mathrm{Separation}_j$ so V satisfies ZW^+_j.

For the second implication, let j again denote the elementary embedding with critical point κ. By Theorem 9.1.7, the supremum of the critical sequence cannot exist and therefore, using $\mathrm{Induction}_j$, we can assume that for every x there is some n such that $x \in \mathrm{V}_{j^n(\kappa)}$. This can be done by restricting ourselves to the definable class

$$\{x \mid \exists n \in \omega \ x \in \mathrm{V}_{j^n(\kappa)}\}$$

which one can easily see is a model of ZW^+_j since V was. Therefore, j is a cofinal Σ-elementary embedding which, by Gaifman's Theorem, gives us that it is fully elementary. We shall now show that V_κ is an elementary substructure of V.

To see this, let $\varphi(u, v)$ be a formula, $a \in \mathrm{V}_\kappa$ and suppose that

$$\mathrm{V} \models \exists x \ \varphi(x, a).$$

Take b witnessing $\varphi(b, a)$ and n such that $b \in \mathrm{V}_{j^n(\kappa)}$. Since $a \in \mathrm{V}_\kappa$, we have that $j^n(a) = a$. So, $\mathrm{V} \models \exists x \in j^n(\mathrm{V}_\kappa)\ \varphi(x, j^n(a))$ and, by using elementarity n times,

$$\mathrm{V} \models \exists x \in \mathrm{V}_\kappa\ \varphi(x, a).$$

Thus, by the Tarski-Vaught Test, our claim holds.

Finally, by the same argument as in Theorem 7.3.10, it is clear that $\mathrm{V}_\kappa \models \mathrm{ZFC}$ which gives us that V is also a model of ZFC, completing the proof. □

Using a very similar idea, we can obtain a lower bound in the theory KP^+.

Theorem 9.2.4. *Suppose that* $\mathrm{V} \models \mathrm{KP}^+$ *and there exists a non-trivial,* Σ*-elementary embedding* $j \colon \mathrm{V}_\lambda \to \mathrm{V}_\lambda$ *for some limit ordinal* λ *for which* $\mathrm{V} \models \mathrm{KP}^+_j$*. Then* $\langle \mathrm{V}_\lambda, j \rangle \models \mathrm{ZF} + \mathrm{WA}_\infty$*.*

Before beginning the proof it is worth remarking that, by Theorem 9.1.7, if such an embedding exists for any ordinal δ then δ is either the supremum of the critical sequence or the supremum plus one. This is because ${}^\delta\delta$ will be a set in $\mathrm{V}_{\delta+2}$.

Proof. Fix λ and j such that $j \colon \mathrm{V}_\lambda \to \mathrm{V}_\lambda$ is a non-trivial, Σ-elementary embedding with critical point κ. Using Theorem 9.1.7 we must have that $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$. We shall first prove that V_λ is a model of Z^+_j. All of the axioms apart from Separation (both in the original language $\{\in\}$ and the language expanded to include j) are immediate so we shall focus on Separation$_j$. To prove this, let $\varphi(u, v)$ be a formula in the expanded language $\mathcal{L}_j = \{\in, j\}$ and a, v be sets in V_λ. Using Σ^j_0-Separation, we have that

$$b := \{x \in a \mid \varphi^{\mathrm{V}_\lambda}(x, v)\}$$

is a set in V. Fixing $n \in \omega$ such that $a \in \mathrm{V}_{j^n(\kappa)}$, it is clear that $\mathrm{rank}(b) < j^n(\kappa)$ and therefore b is also a set in $\mathrm{V}_{j^n(\kappa)} \subseteq \mathrm{V}_\lambda$.

But now, the proof of Theorem 9.2.3 gives us that j is a fully elementary embedding via Gaifman's Theorem and that $\mathrm{V}_\kappa \prec \mathrm{V}_\lambda$. Thus $\mathrm{V}_\lambda \models \mathrm{ZF}$ which, when combined with Separation$_j$, yields

$$\langle \mathrm{V}_\lambda, j \rangle \models \mathrm{ZF} + \mathrm{WA}_\infty.$$

□

9.3 I_1 and embeddings of ZFC$^-$

In this section we give an upper bound for the existence of a Reinhardt embedding over ZFC$^-$. This is done by finding an assertion which is equivalent, over ZFC, to the axiom I_1.

Definition 9.3.1. I_1 is the assertion that for some ordinal λ there exists a non-trivial, elementary embedding $k\colon \mathrm{V}_{\lambda+1} \to \mathrm{V}_{\lambda+1}$.

I_1 is considered one of the strongest large cardinal axioms that is not known to be inconsistent. The following result is adapted from a folklore result which gives an alternate characterisation of 1-extendible cardinals, a proof of which can be found in [BT07]. This theorem shows an equivalent way of considering I_1 embeddings as embeddings of H_{λ^+}, a set with much more structure than $\mathrm{V}_{\lambda+1}$.

For convenience, we note here that the Kunen inconsistency can be proved in $\mathrm{H}_{\lambda^{++}}$, so there is no non-trivial, elementary embedding from $\mathrm{H}_{\lambda^{++}}$ to itself. This can be seen by carefully analysing the second proof of Theorem 23.12 in [Kan08] or, equivalently, it will follow from Theorem 10.2.1.

Theorem 9.3.2. *Over* ZFC*, there exists an elementary embedding* $k\colon \mathrm{V}_{\lambda+1} \to \mathrm{V}_{\lambda+1}$ *if and only if there exists an elementary embedding* $j\colon \mathrm{H}_{\lambda^+} \to \mathrm{H}_{\lambda^+}$*.*

Proof. $(\Leftarrow)$: By the Kunen inconsistency, λ must be the supremum of the critical sequence $\langle \kappa_n \mid n \in \omega \rangle$ of j, where κ_0 is the critical point and $\kappa_{n+1} = j(\kappa_n)$. Then each κ_n is an inaccessible cardinal and thus $2^{<\lambda} = \lambda = \beth_\lambda$. Therefore $\mathrm{V}_\lambda = \mathrm{H}_\lambda$ and $|\mathrm{V}_\lambda| = \lambda$ so $\mathrm{V}_\lambda \in \mathrm{H}_{\lambda^+}$. This means that $\mathrm{V}_{\lambda+1} = \{x \in \mathrm{H}_{\lambda^+} \mid x \subseteq \mathrm{V}_\lambda\}$, so $\mathrm{V}_{\lambda+1}$ is a definable class in H_{λ^+}. Moreover, working in H_{λ^+}, any formula φ can be relativised to $\mathrm{V}_{\lambda+1}$ so $j \restriction \mathrm{V}_{\lambda+1}\colon \mathrm{V}_{\lambda+1} \to \mathrm{V}_{\lambda+1}$ is elementary.

$(\Rightarrow)$: We begin by defining a standard way to code elements of H_{λ^+} by elements of $\mathrm{V}_{\lambda+1}$. This will be done by coding $\mathrm{trcl}(\{x\})$ by some subset of $\lambda \times \lambda$ whose Mostowski collapse is again $\mathrm{trcl}(\{x\})$. However, since we will be working with $\mathrm{trcl}(\{x\})$ rather than x itself, it is necessary to do a simple, preliminary coding.

So, let $\hat{\mathrm{H}} \coloneqq \{\mathrm{trcl}(\{x\}) \mid x \in \mathrm{H}_{\lambda^+}\}$. For $\mathrm{trcl}(\{x\}), \mathrm{trcl}(\{y\}) \in \hat{\mathrm{H}}$ define the relation $\hat{\in}$ by $\mathrm{trcl}(\{x\}) \mathrel{\hat{\in}} \mathrm{trcl}(\{y\})$ if and only if $x \in y$ and similarly for $\hat{=}$. It is then clear that any first-order statement, φ, about H_{λ^+} is equivalent to a formula, $\hat{\varphi}$, over $\hat{\mathrm{H}}$ by the obvious coding.

Next, note that $\mathrm{rank}(\lambda \times \lambda) = \lambda$ and so any subset of $\lambda \times \lambda$ has rank at most λ. So, for any $x \in \mathrm{H}_{\lambda^+}$ and bijection $f \colon |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$ let

$$C_{x,f} \coloneqq \{\langle \alpha, \beta \rangle \in \lambda \times \lambda \mid f(\alpha) \in f(\beta)\} \in \mathrm{V}_{\lambda+1}.$$

Then the Mostowski collapse of $C_{x,f}$, $\mathrm{coll}(C_{x,f})$, is $\mathrm{trcl}(\{x\})$. Let $\tilde{H}$ denote the definable class in $\mathrm{V}_{\lambda+1}$ of all subsets of $\lambda \times \lambda$ which code an element of $\hat{\mathrm{H}}$ in this way. That is $X \in \tilde{H}$ iff X is a well-founded, extensional, binary relation on λ with a single maximal element and $\mathrm{dom}(X) \cup \mathrm{ran}(X)$ is a cardinal which is at most λ.

For $Z \in \tilde{H}$, let $\mathrm{fld}(Z)$ be $\mathrm{dom}(Z) \cup \mathrm{ran}(Z)$ and define $\max(Z)$ to be the unique element of $\mathrm{fld}(Z)$ which is maximal with respect to the relation on Z. Now, for $X, Y \in \tilde{H}$ define relations $\tilde{=}$ and $\tilde{\in}$ by:

$$X \mathrel{\tilde{=}} Y \longleftrightarrow \exists g \colon \lambda \to \lambda \Big(g \textit{ is a bijection } \wedge \forall \alpha, \beta \in \lambda \\ \big(\langle \alpha, \beta \rangle \in X \leftrightarrow \langle g(\alpha), g(\beta) \rangle \in Y \big) \Big)$$

$$X \mathrel{\tilde{\in}} Y \longleftrightarrow \exists g \colon \lambda \to \lambda \Big(g \textit{ is injective } \wedge \langle g(\max(X)), \max(Y) \rangle \in Y \\ \wedge \forall \alpha, \beta \in \mathrm{fld}(X) \big(\langle \alpha, \beta \rangle \in X \leftrightarrow \langle g(\alpha), g(\beta) \rangle \in Y \big) \Big).$$

Then $\tilde{=}$ and $\tilde{\in}$ are definable in $\mathrm{V}_{\lambda+1}$, $X \mathrel{\tilde{=}} Y \Leftrightarrow \mathrm{coll}(X) = \mathrm{coll}(Y)$ and $X \mathrel{\tilde{\in}} Y \Leftrightarrow \mathrm{coll}(X) \in \mathrm{coll}(Y)$. Moreover, any first-order statement, $\hat{\varphi}$, about $\hat{\mathrm{H}}$ is equivalent to a formula, $\tilde{\varphi}$, over $\mathrm{V}_{\lambda+1}$ which is defined by the following coding:

- Replace any parameter $\mathrm{trcl}(\{x\})$ occurring in $\hat{\varphi}$ with $C_{x,f}$ for some (any) bijection $f\colon |\,\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$.
- Replace any instance of $\hat{=}$ with $\tilde{=}$ and $\hat{\in}$ with $\tilde{\in}$.
- Replace any unbounded quantification by the same quantifier taken over $\tilde{H}$.

Then, by the elementarity of k,

$$X \mathrel{\tilde{=}} Y \quad \longleftrightarrow \quad k(X) \mathrel{\tilde{=}} k(Y)$$
and
$$X \mathrel{\tilde{\in}} Y \quad \longleftrightarrow \quad k(X) \mathrel{\tilde{\in}} k(Y).$$

Also, since $\tilde{\mathrm{H}}$ is a definable class in $\mathrm{V}_{\lambda+1}$ the restriction of the embedding $k \restriction \tilde{\mathrm{H}}\colon \tilde{\mathrm{H}} \to \tilde{\mathrm{H}}$ is still elementary.

So we can define $j\colon \mathrm{H}_{\lambda^+} \to \mathrm{H}_{\lambda^+}$ by setting $j(x)$ to be the unique element of $\mathrm{coll}(k(C_{x,f}))$ of maximal rank for some bijection $f\colon |\,\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$. Furthermore, j is elementary since

$$\begin{aligned}
\mathrm{H}_{\lambda^+} \models \varphi(x_1, \ldots, x_n) &\iff \hat{\mathrm{H}} \models \hat{\varphi}(\mathrm{trcl}(\{x_1\}), \ldots, \mathrm{trcl}(\{x_n\})) \\
&\iff \tilde{\mathrm{H}} \models \tilde{\varphi}\Big(C_{x_1,f_1}, \ldots, C_{x_n,f_n}\Big) \\
&\iff \tilde{\mathrm{H}} \models \tilde{\varphi}\Big(k(C_{x_1,f_1}), \ldots, k(C_{x_n,f_n})\Big) \\
&\iff \hat{\mathrm{H}} \models \hat{\varphi}\Big(\,\mathrm{coll}(k(C_{x_1,f_1})), \ldots, \mathrm{coll}(k(C_{x_n,f_n}))\Big) \\
&\iff \mathrm{H}_{\lambda^+} \models \varphi(j(x_1), \ldots, j(x_n)).
\end{aligned}$$
□

Since H_{λ^+} is always a model of $\mathrm{ZFC}^- + \mathrm{DC}_{<\lambda^+}$, the next Corollary is immediate.

Corollary 9.3.3. $\mathrm{ZFC} + \mathrm{I}_1$ *implies the consistency of a Reinhardt embedding under* $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$.

The above theorem shows that the existence of a Reinhardt embedding under ZFC^- is weaker than I_1, however it does not show that the embedding one obtains has any useful structure with respect to j. What we shall show in Chapter 10 is that this embedding must fail to have one of the most useful fundamental characteristics, that of *cofinality*.

Chapter 10

Taking Reinhardt's Power Away

In the previous chapter we proved that, under ZFC, I_1 implies the consistency of a Reinhardt embedding over $(\mathrm{ZFC}^- + \mathrm{DC}_{<\textsc{Card}})_j$. In this chapter, we will explore this theory in much greater depth and consider sufficient additional conditions to render a Reinhardt embedding over ZFC^-_j inconsistent. In particular, we shall see that the embedding can not be cofinal; culminating in Theorem 10.2.3.

Theorem 10.2.3. *There is no non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that $\mathrm{V} \models \mathrm{ZFC}^-_j$ and $\mathrm{V}_{\mathrm{crit}(j)} \in \mathrm{V}$.*

The proof of the above theorem will make essential use of the assumption that $\mathrm{V}_{\mathrm{crit}(j)}$ was a set, leading to the question of whether or not this is always true. A question that we shall answer negatively.

Corollary 10.4.2. *Assuming the consistency of* ZFC *plus a measurable cardinal, it is consistent to have $\mathrm{M} \subseteq \mathrm{V}$ and $j\colon \mathrm{V} \to \mathrm{M}$ such that j is a non-trivial, elementary embedding, $\mathrm{V} \models (\mathrm{ZFC}^-_{Ref})_{j,\mathrm{M}}$ and $\mathcal{P}(\omega)$ is a proper class.*

We end this chapter with two independent sections discussing various concepts concerning elementary embeddings without Power Set. The first of these sections is to obtain bounds for the existence of a V-critical embedding in ZFC^- in terms of the standard ZFC large cardinal hierarchy.

Theorem 10.5.7. *Working in* ZFC*, if there is a locally measurable cardinal then the theory* $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{CARD}}$ *plus a* V*-critical cardinal is consistent.*

Theorem 10.5.9. *Working in* ZFC*, suppose that* $\mathrm{M} \subseteq \mathrm{V}$ *and* $j\colon \mathrm{V} \to \mathrm{M}$ *is a non-trivial, elementary embedding with critical point* κ *such that*

$$\mathrm{V} \models (\mathrm{ZFC}^-)_{j,\mathrm{M}} + \exists z\ (z = \mathrm{V}_\kappa).$$

Then V_κ *is a model of a proper class of baby measurable cardinals.*

In the final section, we analyse the stationary partition that was needed to derive the Kunen Inconsistency. This leads us to isolate a new type of stationary reflection and investigate models of ZFC in which this principle is true.

10.1 Choosing from Classes

To begin this chapter, we mention how one can apply choice to set-length sequences of classes using the Collection Scheme. The standard way to do this in full ZFC is by using Scott's trick to replace each class by the set of elements of least rank of that class. However, if V_α is not a set for each α, then this may not be possible so we have to be slightly more careful in our approach.

Let μ be an ordinal and suppose that we have a sequence of non-empty classes $\langle \mathcal{C}_\alpha \mid \alpha \in \mu \rangle$ which are uniformly defined. This allows us to fix a formula $\varphi(v_0, v_1)$ saying that $v_1 \in \mathcal{C}_{v_0}$. Then, for each $\alpha \in \mu$ there is some set x such that $\varphi(\alpha, x)$. So, by Collection, there is some set b such that for each $\alpha \in \mu$ there is some $x \in b$ such that $\varphi(\alpha, x)$. By well-ordering b, there is some cardinal τ and bijection $h\colon \tau \leftrightarrow b$. So we can define a choice function by taking $x_\alpha \in \mathcal{C}_\alpha$ to be $h(\gamma)$ for the least ordinal $\gamma \in \tau$ such that $\varphi(\alpha, h(\gamma))$.

For example, suppose that $S \subseteq \mu$ were a stationary set which was partitioned into $\tau < \mathrm{cf}(\mu)$ many sets $\langle S_\alpha \mid \alpha \in \tau \rangle$ and one wanted to show that for some $\alpha \in \tau$, S_α was stationary. Arguing for a contradiction, suppose that none of the S_α were stationary. Then for each $\alpha \in \tau$ we define $\mathcal{C}_\alpha$ to be the non-empty class of clubs $D \subseteq \mu$ for which

$D \cap S_\alpha$ is empty. By the above argument, we can then choose a sequence of clubs $\langle D_\alpha \mid \alpha \in \tau \rangle$ such that for each α, $D_\alpha \in \mathcal{C}_\alpha$. Finally, $\bigcap_{\alpha\in\tau} D_\alpha \cap S = \emptyset$, yielding the required contradiction.

Using this idea we are able to prove many useful classical results without much change from their standard proofs. For completeness, we give here two such ZFC$^-$ results which will then be used in our proof of the Kunen inconsistency.

Definition 10.1.1. A function $f\colon S \to \textsc{Ord}$ is *regressive* if for any non-zero $\alpha \in S$, $f(\alpha) < \alpha$.

Theorem 10.1.2 (Fodor)**.** *Let μ be a regular cardinal, $S \subseteq \mu$ stationary and f a regressive function on S. Then there exists some stationary set $T \subseteq S$ and $\gamma \in \mu$ such that for all $\alpha \in T$, $f(\alpha) = \gamma$.*

Proof. Assume for a contradiction that for each $\gamma \in \mu$ the set $\{\alpha \in S \mid f(\alpha) = \gamma\}$ was non-stationary. Using the above comments, for each $\gamma \in \mu$ choose a club D_γ such that for each α in $D_\gamma \cap S$, $f(\alpha) \neq \gamma$. Let

$$D = \Delta_{\gamma\in\mu} D_\gamma \coloneqq \{\alpha \mid \forall\beta \in \alpha\ (\alpha \in D_\beta)\}$$

and note that this is club in μ. Therefore $S \cap D$ is stationary, so in particular non-empty, and for any $\alpha \in S \cap D$ and $\gamma \in \alpha$, $f(\alpha) \neq \gamma$. So $f(\alpha) \geq \alpha$, contradicting the assumption that f was regressive. □

Definition 10.1.3. For cardinals $\delta < \mu$ let $S^\mu_\delta = \{\alpha < \mu \mid \mathrm{cf}(\alpha) = \delta\}$.

Theorem 10.1.4 (Solovay)**.** *Suppose that μ is an uncountable, regular cardinal and $S \subseteq S^\mu_\omega$ is stationary. Then there is a partition of S into μ many disjoint stationary sets.*

Proof. First note that for each $\alpha \in S$ there is some increasing sequence of ordinals $\langle t_n \mid n \in \omega \rangle$ cofinal in α. Therefore, by the comments at the beginning of this section, for each $\alpha \in S$ choose an increasing sequence $\langle a^\alpha_n \mid n \in \omega \rangle$ cofinal in α. Then, as in the usual proof, using our first example and the regularity of μ we can fix some $n \in \omega$

such that for each $\sigma \in \mu$, $\{\alpha \in S \mid a^\alpha_n \geq \sigma\}$ is stationary in μ. Now define a regressive function $f\colon S \to \mu$ by $f(\alpha) = a^\alpha_n$. Using Fodor's Theorem, for each $\sigma \in \mu$ fix some S_σ stationary and $\gamma_\sigma \geq \sigma$ such that for all $\alpha \in S_\sigma$, $f(\alpha) = \gamma_\sigma$. Then if $\gamma_\sigma \neq \gamma_{\sigma'}$, $S_\sigma \cap S_{\sigma'} = \emptyset$ and, by the regularity of μ, $|\{S_\sigma \mid \sigma \in \mu\}| = \mu$, which gives the required partition. □

10.2 Non-existence of embeddings

We are now in the position to prove that there is no non-trivial, cofinal Reinhardt embedding j of ZFC$^-$ with $\mathrm{V}_{\mathrm{crit}(j)} \in \mathrm{V}$. This shall be done in two parts; first we shall show that Woodin's proof of the Kunen inconsistency, which is the second proof of Theorem 23.12 in [Kan08], goes through in ZFC$^-$ under the additional assumption that $(\sup\{j^n(\mathrm{crit}(j)) \mid n \in \omega\})^+ \in \mathrm{V}$. Then we shall show, by modifying the coding from Theorem 9.3.2, that no cofinal embedding can exist in any model that sufficiently resembles H_{λ^+}.

Theorem 10.2.1. *There is no non-trivial, elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that $\mathrm{V} \models \mathrm{ZFC}^-_j$ and $(\sup\{j^n(\mathrm{crit}(j)) \mid n \in \omega\})^+ \in \mathrm{V}$.*

Proof. Suppose for a contradiction that $j\colon \mathrm{V} \to \mathrm{V}$ was a non-trivial elementary embedding with critical point κ and let $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$. Then $j(\lambda) = \lambda$ and, since λ^+ is definable as the least cardinal above λ, $j(\lambda^+) = \lambda^+$. Now, using Theorem 10.1.4, let $\langle S_\alpha \mid \alpha \in \kappa\rangle$ be a partition of $S^{\lambda^+}_\omega$ into κ many disjoint stationary sets and let $S = \{\langle \alpha, S_\alpha\rangle \mid \alpha \in \kappa\}$. Then $j(S) = \{\langle \alpha, T_\alpha\rangle \mid \alpha \in j(\kappa)\}$ and, by elementarity, $\langle T_\alpha \mid \alpha \in j(\kappa)\rangle$ is a partition of $S^{\lambda^+}_\omega$ into disjoint sets such that for each α

$$T_\alpha \textit{ is a stationary subset of } \lambda^+.$$

Also, we have that for each $\alpha \in \kappa$, $j(S_\alpha) = T_\alpha$. We claim that there is some $\beta \in \kappa$ such that $T_\kappa \cap S_\beta$ is stationary. For suppose not, then by our comments on choosing from set many classes, for each α we can fix a club C_α such that $T_\kappa \cap S_\alpha \cap C_\alpha = \emptyset$.

Letting $C = \bigcap_{\alpha \in \kappa} C_\alpha$ we must have that

$$\emptyset = T_\kappa \cap C \cap \bigcup_{\alpha\in\kappa} S_\alpha = T_\kappa \cap C,$$

contradicting the assumption that T_κ was stationary. So fix β such that $T_\kappa \cap S_\beta$ is stationary. Now, let

$$U = \{\gamma \in \lambda^+ \mid \gamma = j(\gamma)\}$$

and note that U contains all of its limit points of cofinality ω. Therefore there exists some $\sigma \in U \cap T_\kappa \cap S_\beta$. But then $\sigma = j(\sigma) \in j(S_\beta) = T_\beta$, contradicting the assumption that the T_α were disjoint. Hence no such embedding can exist. □

Remark 10.2.2. The above theorem did not require any assumption about j being cofinal or that $\mathrm{V}_{\mathrm{crit}(j)}$ was a set.

Theorem 10.2.3. *There is no non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that $\mathrm{V} \models \mathrm{ZFC}_j^-$ and $\mathrm{V}_{\mathrm{crit}(j)} \in \mathrm{V}$.*

Proof. Suppose for a contradiction that $j\colon \mathrm{V} \to \mathrm{V}$ was a non-trivial, cofinal, Σ_0-elementary embedding with critical point κ and let $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$. Recall that, by an instance of Replacement with the parameter j, λ is a set in V. Now there are two cases:

- **Case 1:** λ^+ exists.
- **Case 2:** For all $x \in \mathrm{V}$, there is an injection $f\colon x \to \lambda$.

Case 1: The fact that no such embedding exists is just a special case of Theorem 10.2.1.

Case 2: First note that, by elementarity, since $\mathrm{V}_\kappa \in \mathrm{V}$ so is $\mathrm{V}_{j^n(\kappa)}$ for each $n \in \omega$ and therefore $\mathrm{V}_\lambda = \bigcup_{n\in\omega} \mathrm{V}_{j^n(\kappa)} \in \mathrm{V}$. Note also that $\lambda \times \lambda \in \mathrm{V}$ and, by the Well-Ordering Principle, for each $x \in \mathrm{V}$ there is a bijection

$$f\colon |\operatorname{trcl}(\{x\})| \to \operatorname{trcl}(\{x\}).$$

Moreover, since there is an injection of x into λ, we must have that $|\operatorname{trcl}(\{x\})| \leq \lambda$ for each $x \in \mathrm{V}$.

Now, for any such function f, let

$$C_{x,f} \coloneqq \{\langle \alpha, \beta \rangle \in \lambda \times \lambda \mid f(\alpha) \in f(\beta)\}.$$

Then $C_{x,f} \in \mathrm{V}$ and therefore so is its Mostowski collapse, with $\mathrm{coll}(C_{x,f}) = \mathrm{trcl}(\{x\})$. Now, since

$$C_{x,f} \subseteq \lambda \times \lambda \subseteq V_\lambda,$$

so is $j(C_{x,f})$. Thus,

$$j(C_{x,f}) = j(C_{x,f}) \cap V_\lambda = \bigcup_{\alpha\in\lambda} j(C_{x,f}) \cap V_{j(\alpha)} = \bigcup_{\alpha\in\lambda} j(C_{x,f} \cap V_\alpha).$$

This means that for any x and bijection $f\colon |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$,

$$\begin{aligned} j\Big(\mathrm{trcl}(\{x\})\Big) &= j\Big(\mathrm{coll}(C_{x,f})\Big) = \mathrm{coll}\Big(j(C_{x,f})\Big) \\ &= \mathrm{coll}\Big(\bigcup_{\alpha<\lambda} j(C_{x,f} \cap \mathrm{V}_\alpha)\Big) \\ &= \mathrm{coll}\Big(\bigcup_{\alpha<\lambda} j \restriction \mathrm{V}_\lambda(C_{x,f} \cap \mathrm{V}_\alpha)\Big). \end{aligned}$$

That is, j is completely determined by its construction up to V_λ. Now, let $i \coloneqq j \restriction \mathrm{V}_\lambda$ and note that, since $\mathrm{V}_\lambda \times \mathrm{V}_\lambda \in \mathrm{V}$, so is

$$i = \{\langle x, y\rangle \in \mathrm{V}_\lambda \times \mathrm{V}_\lambda \mid j(x) = y\}.$$

Therefore, by defining $\varphi(\cdot,\cdot,i,\lambda)$ as

$$\begin{aligned} \varphi(x,y,i,\lambda) \;\equiv\; \exists f, z, C_{x,f} \Big(&\text{“}\mathrm{dom}(f) \textit{ is a cardinal}\text{”} \wedge\ \mathrm{ran}(f) = \mathrm{trcl}(\{x\}) \\ &\wedge \text{“}f \textit{ is a bijection}\text{”} \wedge C_{x,f} \coloneqq \{\langle\alpha,\beta\rangle \in \lambda\times\lambda \mid f(\alpha)\in f(\beta)\} \\ &\wedge\ z = \mathrm{coll}\Big(\bigcup_{\alpha<\lambda} i(C_{x,f} \cap \mathrm{V}_\alpha)\Big) \\ &\wedge\ \text{“}y \textit{ is the element of } z \textit{ of maximal rank}\text{”}\Big), \end{aligned}$$

we have that $\varphi(x,y,i,\lambda)$ holds if and only if $j(x) = y$ so j is definable from the parameters i and λ, both of which lie in V, contradicting Theorem 8.1.4. □

10.3 Removing the Assumption that $V_{\text{crit}(j)} \in V$

Assuming that V satisfies the additional assumption of *Dependent Choice of length* μ for every infinite cardinal μ, we are able to remove the assumption that $V_{\text{crit}(j)} \in V$. This will be done using Corollary 3.2.10 which showed that in this theory every proper class must surject onto any given non-zero ordinal. In particular, for $V_{\text{crit}(j)}$ to be a proper class it is necessary for $V_{\text{crit}(j)}$ to surject onto $j(\kappa)$ which we shall show cannot happen. Note that, in the standard ZFC case, the cardinality of $V_{\text{crit}(j)}$ is $\text{crit}(j)$.

Lemma 10.3.1. *Suppose that* $V \models \text{ZFC}^- + \text{DC}_{<\text{Card}}$, $M \subseteq V$ *and* $j\colon V \to M$ *is a non-trivial, elementary embedding with critical point* κ. *Then for any* $\alpha \in \kappa + 1$, $V_\alpha \in V$.

Proof. This is proven by induction on $\alpha \in \kappa + 1$. Clearly limit cases follow by an instance of Collection so it suffices to prove that for $\alpha \in \kappa$, if $V_\alpha \in V$ then so is $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. First note that j fixes every set of rank less than κ so $j \restriction V_{\alpha+1}$ is the identity. Now, suppose for sake of a contradiction that $V_{\alpha+1}$ were a proper class. Then, by Corollary 3.2.10, we could fix a set $b \subseteq V_{\alpha+1}$ and a surjection

$$h\colon b \twoheadrightarrow \kappa.$$

So, by elementarity, there is a surjection

$$j(h)\colon j(b) \twoheadrightarrow j(\kappa)$$

in M. However, since $b \subseteq V_{\alpha+1}$, $j(b)$ is also a subset of $(V_{j(\alpha+1)})^M = V_{\alpha+1}$ and for any $x \in j(b)$, $j(x) = x$. Therefore,

$$x \in j(b) \longleftrightarrow j(x) \in j(b) \longleftrightarrow x \in b$$

and hence $b = j(b)$. Then, for any $x \in b$,

$$j(h)(x) = j(h)(j(x)) = j(h(x)) = h(x)$$

so $j(h) = h$. But this then contradicts the assumption that $j(h)$ was a surjection onto $j(\kappa)$. Hence $V_{\alpha+1}$ must be a set in V as required. $\square$

Combining this result with Theorem 10.2.3 gives the Kunen inconsistency for the theory $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$.

Corollary 10.3.2. *There is no non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that $\mathrm{V} \models (\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}})_j$.*

However this leaves open the question as to whether or not this result is provable without relying on the Dependent Choice Schemes, namely;

Question 10.3.3. Is the existence of a non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$ such that $\mathrm{V} \models \mathrm{ZFC}^-_j$ inconsistent?

One stumbling block in answering this question is whether $\mathrm{V}_{\mathrm{crit}(j)}$ is a set without assuming the $\mathrm{DC}_{\mathrm{crit}(j)}$-Scheme. This question appeared in [Mat20] and we shall answer it negatively in the next section using the techniques from Section 3.3.

Question 10.3.4. Suppose that $\mathrm{V} \models \mathrm{ZFC}^-$, $\mathrm{M} \subseteq \mathrm{V}$ and $j\colon \mathrm{V} \to \mathrm{M}$ is a non-trivial elementary embedding. Is $\mathcal{P}(\omega) \in \mathrm{V}$? Is $\mathrm{V}_{\mathrm{crit}(j)} \in \mathrm{V}$?

A second conclusion we can achieve from Lemma 10.3.1 is that ZFC plus Wholeness is a lower bound for a Reinhardt embedding over $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$. This is because we can observe that the proof of Theorem 9.2.4 did not really require a cumulative hierarchy of sets but only that V_λ was a proper class.

Corollary 10.3.5. *Suppose that $j\colon \mathrm{V} \to \mathrm{V}$ is a non-trivial, elementary embedding such that $\mathrm{V} \models (\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}})_j$ and let λ be the supremum of the critical sequence. Then $\langle \mathrm{V}_\lambda, j\rangle \models \mathrm{ZFC} + \mathrm{WA}_\infty$.*

10.4 Embeddings with $\mathcal{P}(\omega)$ a proper class

In this section we shall show that having an embedding $j\colon \mathrm{V} \to \mathrm{M}$ where $\mathrm{V} \models \mathrm{ZFC}^-_j$ does not imply that $\mathcal{P}(\omega)$ is a set. This in turn suggests that answering Question 10.3.3 may be very difficult and to prove it is inconsistent would involve a very different technique to the one we employed for the theory $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Card}}$.

A first attempt one could try would be to use the forcing $\mathrm{Add}(\omega, \mathrm{O{\small RD}})$. If V is a model of ZFC and $j\colon \mathrm{V} \to \mathrm{M}$ is an elementary embedding with critical point κ then, after forcing with $\mathrm{Add}(\omega, \mathrm{O{\small RD}})$, $\mathcal{P}(\omega)$ is obviously a proper class. The issue is with lifting the elementary embedding. To see this, consider the first κ many Cohen reals, $G \restriction \kappa$. By elementarity, we would need $j^+(G \restriction \kappa)$ to be a set of $j(\kappa)$ many Cohen reals in M, where j^+ is the lift of j. However, there is no way to decide which Cohen reals these are after the first κ many. In fact, since $\mathrm{DC}_{<\mathrm{C{\small ARD}}}$ holds in the extension and $\mathcal{P}(\omega)$ is a proper class, Lemma 10.3.1 implies that no embedding with critical point κ can exist!

Instead, we shall use a different approach which is to consider Zarach's union model from Section 3.3. To do this, we first need to derive a model theoretic counterpart to the lifting of elementary embeddings in the same style as Theorems 3.3.15 and 3.3.16.

Theorem 10.4.1. *Fix structures*

$$\mathcal{M}_{n+1} = \langle \mathrm{M}_{n+1}, \in \mathrm{N}_n \rangle,\ \mathcal{N}_n = \langle \mathrm{N}_n, \in \rangle,\ \mathcal{M}'_{n+1} = \langle \mathrm{M}'_{n+1}, \in \mathrm{N}'_n \rangle \text{ and } \mathcal{N}'_n = \langle \mathrm{N}'_n, \in \rangle$$

and suppose that

$$\mathrm{M}_n \subseteq \mathrm{N}_n \subseteq \mathrm{M}_{n+1} \text{ and } \mathrm{M}'_n \subseteq \mathrm{N}'_n \subseteq \mathrm{M}'_{n+1}$$

for each $n \in \omega$. Define

$$\mathrm{N} = \bigcup_n \mathrm{M}_n, \quad \mathcal{N} = \langle N, \in \rangle, \quad \mathrm{N}' = \bigcup_n \mathrm{M}'_n \quad \text{and} \quad \mathcal{N}' = \langle N', \in \rangle.$$

Suppose further that for each $n \in \omega$, $\mathcal{N}_n \prec \mathcal{N}$, $\mathcal{N}'_n \prec \mathcal{N}'$ and there is an elementary embedding $j_n\colon \mathcal{M}_n \to \mathcal{M}'_n$ such that whenever $m \leq n$, $j_n \restriction \mathrm{M}_m = j_m$. Then there is an elementary embedding $j\colon \mathcal{N} \to \mathcal{N}'$.

$$\begin{array}{ccccccccccccccccc}
\mathcal{M}_0 & \subseteq & \mathcal{N}_0 & \subseteq & \mathcal{M}_1 & \subseteq & \cdots & \subseteq & \mathcal{M}_n & \subseteq & \mathcal{N}_n & \subseteq & \mathcal{M}_{n+1} & \subseteq & \cdots & \subseteq & \mathcal{N} \\
{\scriptstyle j_0}\downarrow & & & & {\scriptstyle j_1}\downarrow & & & & {\scriptstyle j_n}\downarrow & & & & {\scriptstyle j_{n+1}}\downarrow & & & & \downarrow{\scriptstyle \exists j} \\
\mathcal{M}'_0 & \subseteq & \mathcal{N}'_0 & \subseteq & \mathcal{M}'_1 & \subseteq & \cdots & \subseteq & \mathcal{M}'_n & \subseteq & \mathcal{N}'_n & \subseteq & \mathcal{M}'_{n+1} & \subseteq & \cdots & \subseteq & \mathcal{N}'
\end{array}$$

Figure 10.1: Union Chain with Elementary Embeddings

Proof. The desired embedding will be $j \coloneqq \bigcup_n j_n$. To see that this is elementary, fix $a \in \mathrm{N}$, a formula $\varphi(u)$ and suppose that

$$\mathcal{N} \models \varphi(a).$$

Then we can fix $n \in \omega$ such that $a \in \mathrm{M}_n \subseteq \mathrm{N}_n$. Since $\mathcal{N}_n \prec \mathcal{N}$, φ reflects down to $\mathcal{N}_n$ so

$$\mathcal{M}_{n+1} \models \varphi^{\mathcal{N}_n}(a).$$

Using the elementarity of j_{n+1}, which will interpret N_n as N'_n, we have that in $\mathcal{M}'_{n+1}$,

$$\mathcal{N}'_n \models \varphi(j_{n+1}(a)).$$

Since $\mathcal{N}'_n$ is an elementary substructure of $\mathcal{N}'$ this reflects up to $\mathcal{N}'$ so, by definition of j,

$$\mathcal{N}' \models \varphi(j(a))$$

which proves elementarity. □

So let us suppose that V is a model of ZFC+CH and κ is a measurable cardinal which is the critical point of some embedding $j\colon \mathrm{V} \to \mathrm{M}$. We shall consider the model produced in Corollary 3.3.17. Recalling the notation, let $\mathbb{P} = \mathrm{Add}(\omega, 1)$ be the forcing to add a Cohen real, $\prod_{\omega}^{(\omega)} \mathbb{P}$ the product of ω many copies of $\mathbb{P}$ with finite support, and let $h\colon \mathbb{P} \cong \prod_{\omega}^{(\omega)} \mathbb{P}$ be an order-isomorphism which is fixed by j. Let G be $\mathbb{P}$-generic over M and $H = h``G$ be the corresponding $\prod_{\omega}^{(\omega)} \mathbb{P}$-generic. Let $G_n = H \restriction \{n\}$ be the $\mathbb{P}$-generic produced by restricting H to its n^{th} co-ordinate and let $\mathrm{N} = \bigcup_n \mathrm{V}[G_0 \times \cdots \times G_n]$. Then

$$\mathcal{N} \coloneqq \langle \mathrm{N}, \in \mathrm{M} \rangle \models \mathrm{ZFC}^-_{Ref} + \neg \mathrm{DC}_{\aleph_2}.$$

Since $j``H$ is the identity we can lift each of the restricted embeddings to get

$$j_n\colon \mathrm{V}[G_0 \times \cdots \times G_{n-1}] \to \mathrm{M}[G_0 \times \cdots \times G_{n-1}].$$

Letting $\mathrm{N}' = \bigcup_n \mathrm{M}[G_0 \times \cdots \times G_n]$, it is clear that we satisfy the hypothesis of Theorem 10.4.1, giving us the following corollary which answers Question 10.3.4 negatively.

Corollary 10.4.2. *Assuming the consistency of* ZFC *plus a measurable cardinal, it is consistent to have* $\mathrm{M} \subseteq \mathrm{V}$ *and* $j\colon \mathrm{V} \to \mathrm{M}$ *such that* j *is a non-trivial, elementary embedding,* $\mathrm{V} \models (\mathrm{ZFC}^-_{Ref})_{j,\mathrm{M}}$ *and* $\mathcal{P}(\omega)$ *is a proper class.*

Remark 10.4.3. By using the methods of Section 3.3.3 we can also have that our universe satisfies the DC_μ-Scheme for μ an arbitrary uncountable cardinal less than κ.

It is worth commenting that, in the above models, the ultrafilter

$$\mathcal{U} := \{x \subseteq \kappa \mid \kappa \in j(x)\}$$

is also a proper class. This makes it unclear how one can work in a sub-universe of the form $\mathrm{L}[\mathcal{U}]$. In turn, this means it will be difficult to obtain accurate lower bounds for the consistency of such embeddings using the traditional ZFC large cardinal hierarchy.

Finally, the following is technically an open question. This is because the only known method to produce models where the DC-Scheme fails is by using Jensen forcing over L and measurable cardinals imply $\mathrm{V} \neq \mathrm{L}$. However, this seems to be more of a lack in technology rather than a question of fundamental structure.

Question 10.4.4. Is it consistent to have $\mathrm{M} \subseteq \mathrm{V}$ and $j\colon \mathrm{V} \to \mathrm{M}$ such that j is a non-trivial elementary embedding, $\mathrm{V} \models (\mathrm{ZFC}^- + \neg\mathrm{DC})_j$ and $\mathcal{P}(\omega) \notin \mathrm{V}$?

10.5 Local Measurability

In this section we discuss some bounds for a V-critical cardinal over ZFC^- in terms of the ZFC large cardinal hierarchy. This will be done by identifying two weakenings of measurability: *baby measurability* and *local measurability*. We will show that if κ is a locally measurable cardinal then there is a V-critical embedding of a model of ZFC^- with critical point κ and, for any such embedding, if V_κ is a set then it is a model of ZFC with a proper class of baby measurable cardinals. This section is derived from suggestions made by Gitman who proposed to the author that a locally measurable cardinal would be the appropriate bound. Moreover, many of the ideas for the proof come from recent work of Gitman and Schlicht, which will appear in [GS21], and we are grateful to the authors for being able to see an advance copy of this work.

We now define the large cardinals that we shall study in this section. Firstly, *baby measurables* were introduced by Bovykin and McKenzie in [BM19] to measure the consistency strength of a certain strengthening of New Foundations with Urelements, denoted NFUM, which was introduced by Holmes [Hol01]. Gitman and Schlicht have since identified various interesting modifications on the original definition, some of which we give now. Note that in the following definitions, $\mathcal{P}^{\mathrm{M}}(\kappa)$ may be a proper class over M and therefore, in general, there is no need to assume that the ultrafilter is a set in M.

Definition 10.5.1. A filter $\mathcal{U}$ on a cardinal κ is called *uniform* if it contains all tail sets (α, κ) for $\alpha \in \kappa$.

$\mathcal{U}$ is called *normal* if for any sequence $\langle X_\alpha \mid \alpha \in \kappa \rangle$ of elements of $\mathcal{U}$, the diagonal intersection $\triangle_{\alpha\in\kappa} X_\alpha \coloneqq \{\beta \in \kappa \mid \beta \in \bigcap_{\alpha\in\beta} X_\alpha\}$ is in $\mathcal{U}$.

Definition 10.5.2 ([GS21])**.** Assume that $\mathrm{V} \models \mathrm{ZFC}$. A cardinal κ is *very weakly baby measurable* if every $A \subseteq \kappa$ is an element of some transitive set $\mathrm{M} \models \mathrm{ZFC}^-$ such that; M has cardinality κ, κ and V_κ are sets in M and there exists some $\mathcal{U} \subseteq \mathcal{P}^{\mathrm{M}}(\kappa)$ such that

$$\langle \mathrm{M}, \in, \mathcal{U} \rangle \models \mathrm{ZFC}^-_{\mathcal{U}} + \mathcal{U} \textit{ is a uniform normal ultrafilter.}$$

κ is said to be *weakly baby measurable* if it is very weakly baby measurable and M and $\mathcal{U}$ may be chosen such that the ultrapower of M by $\mathcal{U}$ is well-founded.

Finally, κ is said to be *baby measurable* if it is weakly baby measurable and M may be chosen such that $\mathrm{M}^{<\kappa} \subseteq \mathrm{M}$.

The other large cardinal notion we shall need is that of a *locally measurable* cardinal. This was introduced by Holy and Lücke in [HL21] as a cardinal notion which is weaker than measurability but still above the Ramsey-like cardinals. Again, we take the formalisation given in [GS21] rather than the original one.

Measurable
|
Locally Measurable
|
Baby Measurable
|
Weakly Baby Measurable
|
Very Weakly Baby Measurable
|
κ-Ramsey
|
Super Ramsey
|
Strongly Ramsey
|
α-Ramsey
|
Ramsey

Figure 10.2: The Small Embeddings Hierarchy

Definition 10.5.3 ([HL21])**.** Assume that $\mathrm{V} \models \mathrm{ZFC}$. A cardinal κ is said to be *locally measurable* if every $A \subseteq \kappa$ is an element of some transitive set $\mathrm{M} \models \mathrm{ZFC}^-$ such that; M has cardinality κ, κ and V_κ are sets in M and there exists some $\mathcal{U} \subseteq \mathcal{P}^{\mathrm{M}}(\kappa)$, with $\mathcal{U} \in \mathrm{M}$, which is a normal ultrafilter on κ.

While we shall not discuss the consistency strength of these principles in detail, Figure 10.5 shows where these cardinals fit in in terms of the large cardinal hierarchy of ZFC. All of the proofs can be found in [GS21].

So suppose that we wanted to find the consistency strength of a V-critical cardinal over ZFC^-. Taking a naive approach, this could entail finding some sets $\mathrm{N} \subseteq \mathrm{M}$ for which there is a non-trivial elementary embedding $j\colon \mathrm{M} \to \mathrm{N}$ with critical point κ such that $\mathrm{M} \models \mathrm{ZFC}_j^-$. We now see what properties can be derived about M from these assumptions.

To being with, we can define $\mathcal{U} \subseteq \mathrm{M}$ as

$$\mathcal{U} := \{X \in \mathrm{M} \mid X \subseteq \kappa \ \wedge \ \kappa \in j(X)\}.$$

Despite $\mathcal{P}^{\mathrm{M}}(\kappa)$ possibly being a proper class over M, it is easy to see that, over M, $\mathcal{U}$ is a uniform, normal ultrafilter which is well-founded.

Proposition 10.5.4. *j is κ-power set preserving. That is, $\mathcal{P}^{\mathrm{M}}(\kappa) = \mathcal{P}^{\mathrm{N}}(\kappa)$.*

Proof. Let $X \in \mathcal{P}^{\mathrm{M}}(\kappa)$. Then $j(X) \in \mathrm{N}$ and

$$z \in X \ \longrightarrow\ z = j(z) \ \longrightarrow\ z \in j(X) \cap \kappa,$$

so $X = j(X) \cap \kappa \in \mathrm{N}$. Hence $\mathcal{P}^{\mathrm{M}}(\kappa) \subseteq \mathcal{P}^{\mathrm{N}}(\kappa)$ and the other inclusion follows from the assumption that $\mathrm{N} \subseteq \mathrm{M}$. □

Lastly, since we assumed that M satisfies full Separation and Collection with respect to j, we have that

$$\langle \mathrm{M}, \in, \mathcal{U} \rangle \models \mathrm{ZFC}^-_{\mathcal{U}}.$$

This therefore implies that the consistency of a V-critical cardinal in ZFC^- should give us at least a baby measurable cardinal. Note that the only things we don't automatically have are that $\mathrm{V}_\kappa \in \mathrm{M}$ and that such an embedding exists for every $A \subseteq \kappa$. By Corollary 10.4.2, it is consistent for $(\mathrm{V}_\kappa)^{\mathrm{M}}$ to be a proper class and so it is difficult to see how one can produce the required embedding.

However, if we add in the assumption that M believes $(\mathrm{V}_\kappa)^{\mathrm{M}}$ is a set then we shall see that this allows us to show that the consistency strength is in-between a locally measurable cardinal and a proper class of baby measurable ones.

To see that local measurability suffices, we use the following proposition from [GS21].

Definition 10.5.5. Following the notation of [GS21], let ZFC^-_n be the theory ZFC^- where the Collection and Separation Schemes have been restricted to only Σ_n-formulae.

Proposition 10.5.6 ([GS21])**.** *Assume that $\mathrm{V} \models \mathrm{ZFC}$. If $\langle \mathrm{M}, \in, \mathcal{U} \rangle \models \mathrm{ZFC}^-_n$, then the Łoś Theorem holds for Σ_n and Π_n assertions in the extended language.*

Theorem 10.5.7. *Working in* ZFC, *if there is a locally measurable cardinal then the theory* $\mathrm{ZFC}^- + \mathrm{DC}_{<\textsc{Card}}$ *plus a* V*-critical cardinal is consistent.*

Proof. Suppose that κ is a locally measurable cardinal and fix a model M of ZFC^- and a normal ultrafilter $\mathcal{U} \in \mathrm{M}$. Working in M, build the model $\mathrm{M}' := \mathrm{L}[\mathcal{U}]$ which one can easily see is a model of $\mathrm{ZFC}^- + \mathrm{DC}_{<\textsc{Card}}$ with a rank hierarchy. Defining $\hat{\mathcal{U}} = \mathcal{U} \cap \mathrm{M}' \in \mathrm{M}'$, it is easy to see that,

$$\mathrm{M}' \models \hat{\mathcal{U}} \textit{ is a normal ultrafilter on } \kappa.$$

Now, working in M′, one is able to define the ultrapower. This is because, for $f\colon \kappa \to \mathrm{M}'$ we can take the equivalence classes modulo the ultrafilter,

$$[f] := \{g \mid f =^* g \ \wedge \ \forall h(h =^* f \to \mathrm{rank}_{\mathrm{M}'}\, g \leq \mathrm{rank}_{\mathrm{M}'}\, h)\}.$$

Where $f =^* g$ is defined to hold whenever $\{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\} \in \hat{\mathcal{U}}$ and $\mathrm{rank}_{\mathrm{M}'}$ is the obvious rank defined on $\mathrm{L}[\mathcal{U}]$.

Since this is all definable in M′ and $\mathrm{L}_\alpha[\mathcal{U}]$ is a set in M′, $[f]$ is a set for each f and therefore the ultrapower $\mathrm{Ult}(\mathrm{M}',\hat{\mathcal{U}})$ in definable in M′. Using the fact that, in M, $\mathcal{U}$ is countably complete, the ultrapower is well-founded so we can take its collapse to find a transitive class $\mathrm{N} \subseteq \mathrm{M}'$. Finally, we have that the derived embedding $j\colon \mathrm{M}' \to \mathrm{N}$ is fully elementary and M′ satisfies ZFC^-_j, which witnesses that κ is indeed a V-critical cardinal in M′. □

Remark 10.5.8. In order to work in $\mathrm{L}[\mathcal{U}]$ we needed that $\mathcal{U}$ was a set. This means that the proof will no longer go through if we only consider $\mathcal{U}$ to be a definable class over M. To see this, consider a measurable cardinal κ in a model of ZFC. Then the ultrafilter is a class over H_{κ^+} but the collapse of the ultrapower will not be a subclass of H_{κ^+}.

We end this section by showing that a lower bound for a V-critical cardinal under $\mathrm{ZFC}^- + \mathrm{DC}_{<\textsc{Card}}$ is a proper class of baby measurable cardinals. It is worth noting that we do not require full $\mathrm{DC}_{<\textsc{Card}}$ in the proof but only that $\mathrm{V}_{\mathrm{crit}(j)}$ is a set. This theorem is a modification of one in [GS21] where they show that a different variant of baby measurability, which they call *κ-game baby measurability*, is a limit of baby measurable cardinals.

Theorem 10.5.9. *Working in* ZFC*, suppose that* $\mathrm{M} \subseteq \mathrm{V}$ *and* $j \colon \mathrm{V} \to \mathrm{M}$ *is a non-trivial, elementary embedding with critical point* κ *such that*

$$\mathrm{V} \models (\mathrm{ZFC}^-)_{j,\mathrm{M}} + \exists z\ (z = \mathrm{V}_\kappa).$$

Then V_κ *is a model of a proper class of baby measurable cardinals.*

Proof. It suffices to show that κ is a baby measurable cardinal in M since elementarity will then give us that this reflects unboundedly below κ in V. We begin by showing that $\mathcal{P}(\kappa)$ is a set in V. Since V_κ is a set, $(\mathrm{V}_{j(\kappa)})^{\mathrm{M}}$ is a set in M which gives us that $\mathcal{P}^{\mathrm{M}}(\kappa) \in \mathrm{M}$. But, by Proposition 10.5.4, $\mathcal{P}^{\mathrm{V}}(\kappa) = \mathcal{P}^{\mathrm{M}}(\kappa)$. So, since $\mathrm{M} \subseteq \mathrm{V}$, $\mathcal{P}^{\mathrm{V}}(\kappa)$ is indeed a set in V. Next, by the same argument, we have that $(\mathrm{V}_{\kappa+1})^{\mathrm{M}} = (\mathrm{V}_{\kappa+1})^{\mathrm{V}} \in \mathrm{V}$.

Using the same coding as in Theorem 9.3.2, we can now show that H_{κ^+} is a set in V. First, since $\mathcal{P}(\kappa)$ is well-ordered by a cardinal greater than κ, we have that κ^+ exists. Let $\tilde{\mathrm{H}}$ be the definable set of subsets of $\kappa \times \kappa$ which are well-founded, extensional relations on κ with a single maximal element. Then it is clear that

$$\mathrm{H}_{\kappa^+} = \bigcup \{\mathrm{coll}(X) \mid X \in \tilde{\mathrm{H}}\} \in \mathrm{V}.$$

Using the coding, it is moreover clear that $(\mathrm{H}_{\kappa^+})^{\mathrm{V}} = (\mathrm{H}_{\kappa^+})^{\mathrm{M}}$. Finally, working in M, we have that the ultrafilter $\mathcal{U} = \{X \in \mathcal{P}(\kappa) \mid \kappa \in j(X)\}$ is a definable class and

$$\langle \mathrm{H}_{\kappa^+}, \in, \mathcal{U}\rangle \models \mathrm{ZFC}^-_{\mathcal{U}} + \mathcal{U} \textit{ is a uniform normal ultrafilter.}$$

So, given $A \subseteq \kappa$ we can take an elementary submodel of H_{κ^+} in M which witnesses this instance of baby measurability, proving that κ is baby measurable in M. ☐

Using a similar argument to a later theorem of Gitman and Schlicht, it is possible to show that $\mathrm{V}_{\mathrm{crit}(j)}$ satisfies that there is a proper class of cardinals μ which are μ-game baby measurable. However, this would involve defining games and their winning strategies which we have not done in this work.

10.6 Stationary Reflection

We end this chapter with some comments on the consequences for I_1 which we can derive from the proof of Theorem 10.2.1. This section can be seen as an initial unsuccessful attempt to try and prove that I_1 was inconsistent. An important component of the proof was that we could find a partition S of $S^{\lambda^+}_\omega$ into κ many disjoint stationary sets such that $j(S)$ was a partition of $S^{\lambda^+}_\omega$ into $j(\kappa)$ many disjoint stationary sets. When λ^+ is no longer assumed to exist, it is unclear if one can do this, however it will turn out that if we assume enough "*reflection*" then we are in fact able to derive the required result for our contradiction.

We say that a stationary set $S \subseteq \kappa$ *reflects* if there exists some ordinal α such that $S \cap \alpha$ is stationary in α. This concept has been extensively studied and the notion that there are a lot of stationary sets which reflect can be seen as a type of compactness principle with high consistency strength. For example, there is the following Theorem by Magidor, a sketch of which can be found in [Kan08].

Theorem 10.6.1 (Magidor, [Kan08] Theorem 23.23)**.** *The following are equiconsistent:*

- *The existence of a weakly compact cardinal,*
- *Every stationary set $S \subseteq S^{\aleph_2}_{\aleph_0}$ reflects at almost all $\alpha \in S^{\aleph_1}_{\alpha_0}$.*

In this section we will look at a specific instance of stationary reflection rather than asking for every set to reflect. The idea is that if one could prove in ZFC that there was a partition, S, of $S^{\lambda^+}_\omega$ into κ many disjoint stationary sets which simultaneously reflected almost everywhere then $j(S)$ would be a partition into stationary sets and we could obtain the same inconsistency we derived earlier. In particular this would show that $\mathrm{ZFC} + \mathrm{I}_1$ was inconsistent.

To be more precise, suppose that $j\colon \mathrm{H}_{\lambda^+} \to \mathrm{H}_{\lambda^+}$ witnesses I_1 and let $\langle S_\alpha \mid \alpha \in \kappa\rangle$ be a partition of $S^{\lambda^+}_\omega$ into κ many disjoint stationary sets. Next, for $\gamma \in \lambda^+$, let $f_\gamma\colon \alpha \mapsto S_\alpha \cap \gamma$ be the function restricting the partition to γ. Then we can define a new partition $\langle T_\alpha \mid \alpha \in j(\kappa)\rangle$ of $S^{\lambda^+}_\omega$ by

$$T_\sigma = \bigcup_{\gamma \in \lambda^+} j(f_\gamma)(\sigma).$$

It is easy to see that this is a partition of $S^{\lambda^+}_\omega$ into disjoint sets but there is no reason to believe that stationarity should be preserved. In fact, using the proof of the following proposition, this can never be the case.

Proposition 10.6.2. *Suppose that there is a non-trivial, fully elementary embedding* $j\colon \mathrm{V} \to \mathrm{V}$ *such that* $\mathrm{V} \models \mathrm{GB}^-_j$*. Then there is no partition of*

$$S^{\textsc{Ord}}_\omega = \{\beta \in \textsc{Ord} \mid \mathrm{cf}(\beta) = \omega\}$$

into $\mathrm{crit}(j)$ *many disjoint stationary sets,* $\langle S_\alpha \mid \alpha \in \mathrm{crit}(j)\rangle$*, for which there is a regular cardinal* $\mu \geq \mathrm{crit}(j)$ *such that for each* $\alpha \in \kappa$*,*

$$\{\beta \in S^{\textsc{Ord}}_\mu \mid S_\alpha \cap \beta \textit{ is stationary in } \beta\} \textit{ is } \mu\textit{-closed and unbounded in } \textsc{Ord}.$$

Proof. Suppose, for a contradiction, that $j\colon \mathrm{V} \to \mathrm{V}$ were such an embedding with critical point κ and let λ denote the supremum of the critical sequence. We note that, by Theorem 10.2.1, λ^+ does not exist. Let $\langle S_\alpha \mid \alpha \in \kappa\rangle$ be such a partition and define $\langle T_\alpha \mid \alpha \in j(\kappa)\rangle$ as above. Since μ is a regular cardinal above κ it is clear that, by intersecting the appropriate classes, there is a μ-closed unbounded subclass of Ord on which the S_α simultaneously reflect. Denoting this class by C, and using the fact that being μ-closed and unbounded is first-order definable, we have that

$$j(C) := \bigcup_{\alpha \in \textsc{Ord}} (C \cap \alpha)$$

is a $j(\mu)$-closed, unbounded class. By construction, we have that for each $\gamma \in C$,

$$\forall \alpha \in \kappa \; \forall E \; (E \textit{ is club in } \gamma \to (S_\alpha \cap \gamma) \cap E \neq \emptyset).$$

So, by elementarity, for every $\gamma \in j(C)$ and $\sigma \in j(\kappa)$,

$$T_\sigma \cap \gamma \textit{ is stationary in } \gamma.$$

Now, let $D \subseteq \textsc{Ord}$ be a class club and D' its club of limit points, that is

$$D' := \{\alpha \in D \mid D \cap \alpha \textit{ is club in } \alpha\}.$$

Then we can fix some $\gamma \in C \cap D'$. Since $T_\sigma \cap \gamma$ is stationary in γ, $T_\sigma \cap (D \cap \gamma) \neq \emptyset$ which gives us that T_σ intersects the given club.

But this means that $\langle T_\sigma \mid \sigma \in j(\kappa)\rangle$ is a partition of S^{Ord}_ω into $j(\kappa)$ many disjoint stationary sets and the inconsistency derived in Theorem 10.2.1 will go through. □

It should be remarked upon that it is unclear as to whether such a partition of S^{Ord}_ω should exist in the first place. This is because the proof of Solovay's splitting theorem we gave required the use Fodor's Lemma and would almost certainly require some sort of class choice principle. Furthermore, as shown in [GHK21], it is consistent that Fodor's Lemma consistently fails over a model of KM which makes it unclear how to produce such a partition.

However, as a weak indication that such a partition could consistently exist, note that if one works in a model of $\text{ZFC}^- + \text{DC}_{<\text{Card}}$ then it is possible to force to have a partition $\langle S_\alpha \mid \alpha \in \kappa\rangle$ such that the S_α simultaneously reflect on a stationary class. To see this, consider the class forcing whose conditions are partitions of S^γ_ω, for some ordinal γ of cofinality μ, into κ many disjoint stationary sets, ordered by extension. It can be shown by standard arguments that the generic class is a partition of S^{Ord}_ω into κ many disjoint stationary sets which will simultaneously reflect on a stationary subclass of S^{Ord}_μ. On the other hand, it does not seem possible to extend this forcing to get the partition to simultaneously reflect on a μ-closed and unbounded class.

So let us consider the set version of this problem over ZFC.

Definition 10.6.3. For regular cardinals $\kappa \leq \mu < \delta$ let $\dagger(\kappa, \mu, \delta)$ be the principle that there is a partition of S^δ_ω into κ many disjoint stationary sets which simultaneously reflect on a μ-closed unbounded set.

This principle is different to standard principles concerning stationary reflection because we are only asking for a single instance of reflection rather than wanting every stationary set to reflect on a μ-closed unbounded set. Therefore, we shall see that it is possible for this principle to hold in certain circumstances even if there are also stationary subsets of δ which do not reflect anywhere. For example, it will hold in L even though it is well known that, in the constructible universe, S^δ_ω contains a stationary set which does not reflect on any ordinal of uncountable cofinality.

We have not been able to find an example where $\dagger(\kappa, \mu, \delta)$ does in fact fail. However, using Theorem XI.1.3 of [She17], we do know that the stronger result where the partition reflects on *every* point of cofinality μ can consistently fail for $\mu = \aleph_2$. This theorem should be seen as a counterpoint to Magidor's Theorem from 10.6.1.

Theorem 10.6.4 (Shelah)**.** *If* ZFC *plus a Mahlo cardinal is consistent then so is* ZFC *plus every stationary subset of* $S^{\aleph_2}_{\aleph_0}$ *contains a closed copy of* ω_1*.*

On the other hand, we shall show that global square implies that for any $\kappa \leq \mu < \delta$, $\dagger(\kappa, \mu, \delta)$ holds. This is an easy generalisation of an argument by Jensen which can be found in a MathOverflow answer given by Hamkins, [Hamb].

Definition 10.6.5 (Jensen)**.** A global $\square$-sequence is a sequence $\langle C_\alpha \mid \alpha \in \textsc{Ord},\ \operatorname{cf}(\alpha) < \alpha \rangle$ such that for each α

- C_α is club in α,
- $\operatorname{otp}(C_\alpha) < \alpha$,
- (Coherence) If $\beta \in \operatorname{acc}(C_\alpha)$ then $\operatorname{cf}(\beta) < \beta$ and $C_\beta = C_\alpha \cap \beta$.

For δ an ordinal, a $GS(\delta)$-sequence is a sequence $\langle C_\alpha \mid \alpha \in \delta,\ \operatorname{cf}(\alpha) < \alpha \rangle$ satisfying the three above conditions.

Remarks 10.6.6.

1. In L, there is a definable global $\square$-sequence.
2. A $GS(\delta)$-sequence is an approximation to a global $\square$-sequence.
3. If there is a $GS(\delta)$-sequence then $\square_\kappa$ holds for all cardinals $\kappa < \delta$.
4. If κ is κ^+-subcompact then $\square_\kappa$ fails, and therefore so does $GS(\delta)$ for all $\delta > \kappa$.

If $j\colon \mathrm{H}_{\lambda^+} \to \mathrm{H}_{\lambda^+}$ is an I_1 embedding with critical point κ then one can show that κ is κ^+-subcompact. Therefore, the next lemma will not be useful in our contexts. However it is an insightful construction because it shows the consistency of our desired principle. It should also be remarked upon that square sequences are more commonly used to show that there is a stationary set which does not reflect rather than our stationary sets which will reflect almost everywhere.

Lemma 10.6.7 ([Hamb])**.** *Suppose that $GS(\lambda)$ holds and $\omega \leq \kappa < \lambda$ where κ and λ are regular cardinals. Then there exists a partition of S^λ_ω into κ many disjoint stationary sets which simultaneously reflect on every point of cofinality κ.*

Proof. Let $\langle C_\alpha \mid \alpha \in \lambda,\ \mathrm{cf}(\alpha) < \alpha\rangle$ witness $GS(\lambda)$ and fix a partition of S^κ_ω into κ many disjoint sets $\langle T_\nu \cap \kappa \mid \nu \in \kappa\rangle$. We define elements of T_ν inductively by the following rule:

$$\alpha \in T_\nu \longleftrightarrow \mathrm{otp}(C_\alpha) \in T_\nu$$

We claim that for each $\nu \in \kappa$ and each δ of cofinality κ, $T_\nu \cap \delta$ is stationary. Note that if this holds then on any ordinal of cofinality greater than κ, the T_ν will simultaneously reflect.

By construction, the claim holds for $\delta = \kappa$. So let $\delta > \kappa$ have cofinality κ and let $D \subseteq \delta$ be a club. Fix π to be the increasing enumeration of C_δ,

$$\pi\colon\ \mathrm{otp}(C_\delta) \to C_\delta.$$

Then, since $\mathrm{acc}(C_\delta)$ is club in δ and π is a bijection we immediately have the following claim:

Claim 10.6.8. *π^{-1}“$(D \cap \mathrm{acc}(C_\delta))$ is club in $\mathrm{otp}(C_\delta)$.*

Next, fix $\nu \in \kappa$. Now, by our inductive hypothesis, $T_\nu \cap \mathrm{otp}(C_\delta)$ is stationary in $\mathrm{otp}(C_\delta)$ so we can fix some $\gamma \in T_\nu \cap \pi^{-1}$“$(D \cap \mathrm{acc}(C_\delta))$. Then $\pi(\gamma) \in \mathrm{acc}(C_\delta)$ so $C_\delta \cap \pi(\gamma) = C_{\pi(\gamma)}$. Since π is the enumeration, $\mathrm{otp}(C_{\pi(\gamma)}) = \gamma$, which gives us that $\pi(\gamma) \in T_\nu \cap D$ and $T_\nu \cap \delta$ is indeed stationary. □

Corollary 10.6.9. *Suppose that $GS(\lambda)$ holds and $\omega \leq \kappa < \lambda$ where κ and λ are regular cardinals. Then $\dagger(\kappa, \mu, \lambda)$ holds for all $\kappa \leq \mu < \lambda$.*

Remark 10.6.10. The use of some amount of global $\square$ seems to be essential in this argument to ensure that the inductive argument works. Note that, by Theorem 6 of [ST17], if I_1 holds then it is consistent to have an elementary embedding $j\colon \mathrm{V}_{\lambda+1} \to \mathrm{V}_{\lambda+1}$ and for $\square_\lambda$ to hold. This is because the standard forcing to add a $\square_\lambda$-sequence is $< \lambda^+$ strategically closed and therefore does not change $\mathrm{V}_{\lambda+1}$.

The difference between $GS(\lambda^+)$ and $\square_\lambda$ is the second condition. If we have such a $\square_\lambda$ sequence, $\langle C_\alpha \mid \alpha \in \lambda^+, \mathrm{cf}(\alpha) < \alpha \rangle$, then we only require that for β of cofinality less than λ, $|C_\beta| < \lambda$.

An alternative way to achieve instances of $\dagger$ which avoids $\square$ principles is by using collapses. We conclude by giving the following example of this which can be easily generalised.

Proposition 10.6.11. $\dagger(\aleph_0, \aleph_1, \aleph_2)$ *consistently holds.*

Proof. Let $\mathrm{Col}(\omega, \aleph_n)$ denote the Lévy collapse which adds a surjection collapsing $\aleph_n$ onto ω. It is a standard fact that $S^{\lambda}_{\aleph_n^{\mathrm{V}}}$ remains a stationary set in the extension for all $\lambda > \aleph_n^{\mathrm{V}}$. Let $\mathrm{V}[G]$ be the extension where we have collapsed $\aleph_n$ for each n in ω. Then, in $\mathrm{V}[G]$, for each $n \in \omega$

$$S^{\aleph_2^{\mathrm{V}[G]}}_{\aleph_n^{\mathrm{V}}} \subseteq S^{\aleph_2^{\mathrm{V}[G]}}_{\omega}$$

is a stationary set which reflects at every point of cofinality $\aleph_1^{\mathrm{V}[G]}$. Hence the stationary partition $\left\langle S^{\aleph_2^{\mathrm{V}[G]}}_{\aleph_n^{\mathrm{V}}} \mid n \in \omega \right\rangle$ witnesses that $\dagger(\aleph_0, \aleph_1, \aleph_2)$ holds. $\square$

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Notation

Index