Calibrating the complexity of combinatorics: reverse mathematics and Weihrauch degrees of some principles related to Ramsey's theorem

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The candidate confirms that the work submitted is his/her/their own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

The Chapters based on work from jointly authored publications are the following, and the details of the relative publications, are the following:

- Chapter 2: all of the results in this chapter come from the paper *An inside-outside Ramsey theorem and recursion theory*, authored by Marta Fiori Carones, Paul Shafer and Giovanni Soldà (see [27]), currently submitted for publication.

  The main results of the paper are the following: Theorem 3.5, Theorem 3.6, Theorem 4.22, Theorem 4.23, Corollary 5.12, Theorem 6.11, and Theorem 6.14. I gave an essential contribution for the proofs of Theorem 3.6, Theorem 4.22, Theorem 4.23, Theorem 6.11, and Theorem 6.14, and the results leading to these proofs. For the proofs of the other results, the contribution of the other authors is equally distributed.

  We point out that the given numbering of the results refers to the current version of the paper, namely arXiv:2006.16969v1.

- Chapter 3: many of the results in this chapter come from the arXiv paper *(Extra)ordinary equivalences with the ascending/descending sequence principle*, authored by Marta Fiori Carones, Alberto Marcone, Paul Shafer and Giovanni Soldà (see [26]).

The main results of the paper are the following: Proposition 3.8, Theorem 4.9, Theorem 5.6, Theorem 5.9, Lemma 6.3, Theorem 7.2, and Theorem 7.8. I gave an essential contribution for the proofs of Proposition 3.8, Theorem 4.9, Theorem 5.6, Theorem 5.9, and Theorem 7.2, and the results leading to these proofs. For the proofs of the other results, the contribution of the other authors is equally distributed.

We point out that the given numbering of the results refers to the current version of the paper, namely arXiv:2107.02531v1.

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Divento stupido,
mi sento cubico,
mi pento subito.

Abstract

In this thesis, we study the proof-theoretical and computational strength of some combinatorial principles related to Ramsey's theorem: this will be accomplished chiefly by analyzing these principles from the points of view of reverse mathematics and Weihrauch complexity.

We start by studying a combinatorial principle concerning graphs, introduced in [59] as a form of "inside-outside" Ramsey's theorem: we will determine its reverse mathematical strength and present the result characterizing its Weihrauch degree. Moreover, we will study a natural restriction of this principle, proving that it is equivalent to Ramsey's theorem.

We will then move to a related result, this time concerning countable partial orders, again introduced in [59]: we will give a thorough reverse mathematical investigation of the strength of this theorem and of its original proof. Moreover, we will be able to generalize it, and this generalization will itself be presented in the reverse mathematical perspective.

After this, we will focus on two forms of Ramsey's theorem that can be considered asymmetric. First, we will focus on a restriction of Ramsey's theorem to instances whose solutions have a predetermined color, studying it in reverse mathematics and from the point of view of the complexity of the solutions in a computability theoretic sense. Next, we move to a classical result about partition ordinals, which will undergo the same type of analysis.

Finally, we will present some results concerning a recently introduced operator on the Weihrauch degrees, namely the first-order part operator: after presenting an alternative characterization of it, we will embark on the study the result of its applications to jumps of Weak Kőnig's Lemma.

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Introduction

In 1972, Carl Jockusch, in the seminal paper [41], analyzed the computational content of Ramsey's theorem for n-tuples and k colors, which from now on we shorten to RT^n_k. In many ways, that paper can be seen to be the starting point of the analysis of the logical strength of principles from infinite combinatorics: this area of research has since then greatly expanded, and this expansion has led to important developments in the parts of mathematical logic that were used to study it, chiefly among them reverse mathematics, proof theory and computability theory.

This thesis can be seen as a contribution to that research field: essentially the entirety of this document is dedicated to the investigation of the strength of principles that are somehow related to classical Ramsey's theorem, as we will explain in due course.

We will start by introducing, in Chapter 1, the main tools that will be used in the course of this analysis: they can be broadly divided into two groups, namely reverse mathematics and computability theory. We point out that it is, of course, very reductive to describe these two fields as "tools": although there is no way we can do them justice in this comparatively short document, it must be said that they are very active areas of research, of great interest from both the mathematical and the philosophical point of view.

To begin with, we will give a brief introduction to some of the so-called "big five", namely RCA_0, WKL_0, ACA_0 and Π^1_1-CA_0. These are important subsystems of second-order arithmetic, linearly ordered with respect to logical strength, with the very interesting property that much of classical mathematics can be proved to be equivalent to one of them. As noticed in the literature, there is an element of irony in the fact that it is Ramsey's theorem for pairs, a theorem which can be seen as asserting the impossibility of absolute chaos, that ruins this very tame picture: RT^2_2 does not prove, nor is proved by, WKL_0 over RCA_0, and we will see that several other combinatorial principles related to RT^2_2 have the same behavior.

We then move to see some more specific topics concerning the use of computability theory to gauge the strength of principles: other than seeing some classical results from computability theory and their interplay with reverse mathematics, we will also introduce what may be considered the most recently added measure of logical strength for principles, namely Weihrauch reducibility and its variants.

In Chapters 2 and 3, we will put the instruments introduced in Chapter 1 to use to study two combinatorial principles introduced by Ivan Rival and Bill Sands in [59], one of them concerning graphs and the other concerning partial orders: they both stem from the idea of finding Ramsey-like principles that, given a certain structure, predicate the existence of a substructure that is not only nice on its own, as RT^2_2 does, but also has some nice properties with respect to the larger structure we started with.

We will give a thorough reverse mathematical analysis of the principle relative to graphs, and we will limit ourselves to state without proofs the main results relative to its position in the Weihrauch degrees (all the proofs and much more can be found in our paper [27]). We will instead focus on a weakening of that principle, which turns out to be equivalent to RT^2_2, and study it in the Weihrauch degrees.

In the case of the principle relative to partial orders, we will see that the reverse mathematical analysis is much less streamlined, and in particular it will be convenient to have different formulations of that principle in second-order arithmetic, not equivalent to each other over RCA_0. Although we will not manage to classify all of these new principles, some interesting phenomena emerge: one of the formulations turns out to be equivalent to ADS over RCA_0, and thus provides, to the best of our knowledge, the first example of a statement of genuine mathematical interest to be equivalent to ADS. Again, for further details and discussions, we refer to our paper [26].

In Chapter 4, we move to the study of principles that can be considered "asymmetric" instances of Ramsey's theorem, in the sense that all the instances are coloring with codomain 2 such that every solution has color 0. We will start with the reverse mathematical analysis of the principles bRT^n_k, which state that for every coloring $f : [\mathbb{N}]^n \to 2$ such that every f-homogeneous set for color 1 has size less than k, then there is an infinite f-homogeneous set (obviously, for color 0). This is joint (and ongoing) work with Emanuele Frittaion. After noticing that the number k is not very relevant, we will see that the case $n = 2$ can be proved over RCA_0 plus some induction, whereas the case $n = 4$ is equivalent to ACA_0. Although we did not find the precise strength of the case $n = 3$, we will provide some bounds, which require the use of rather advanced machinery recently introduced by Ludovic Patey in [58].

After this, we will move to another asymmetric Ramseyan principle, namely the theorem asserting that ω^2 is a partition ordinal. In this case as well, we will not find the precise strength of this principle, but we will give some bounds and some initial estimates on the complexity of the solutions of its computable instances.

Finally, Chapter 5 will be devoted to the study, carried out in joint work with Manlio Valenti, of a newly introduced operation on the Weihrauch degrees, namely the first-order part operator, defined by Damir Dzhafarov, Reed Solomon and Keita Yokoyama. We will focus on the case that the problem whose first-order part is being considered is the parallelization of a first-order problem (we refer to Definitions 1.2.8 and 5.1.1 for the meaning of these expressions): after providing an alternative characterization of the Weihrauch degree of the first-order part of such problems, we will explicitly compute the degree of ${}^1(\mathsf{WKL}^{(n)})$, i.e. the (strong) Weihrauch degree of the first-order part of the nth jump of WKL. In keeping with the rest of the thesis, we will conclude the Chapter noticing that this result can be seen as relevant in the study of problems associated to combinatorial principles, in that it yields the Weihrauch degree of the first-order part of $\widehat{\mathsf{RT}^n_2}$, the parallelization of Ramsey's theorem for n-tuples.

1. Background material

In this Chapter, we briefly review the background material that will be needed in the following Chapters. It consists of two parts, namely Section 1.1 and Section 1.2.

The first part, Section 1.1, deals with the basics of reverse mathematics: in this Section, we define and sketch some important features of the main subsystems of second-order arithmetic that will be used in the rest of the thesis. It is itself divided into two parts: in the first, Subsection 1.1.1, we focus on some of the so-called "big five", very important and natural subsystems that are fundamental characters of what might be called classical reverse mathematics. In the second, Subsection 1.1.2, we focus on systems whose strength corresponds to either some form of induction or to some Ramseyan principle. Other than for the results we state, this Section is important because in it we define large swaths of the *notation* that we will use in the rest of the thesis.

The second part, Section 1.2, deals with the interplay between computability theory and the study of the strength of mathematical principles. Again, the Section develops along two main axes: one of them, Subsection 1.2.4, deals with some classical results and notions from computability and their impact in reverse mathematics. The other one, corresponding to Subsections 1.2.2 and 1.2.3, introduces a different perspective in the analysis of the strength of principles: based on the fact that many mathematical theorems are Π^1_2 statements, it relies on seeing principles as partial multifunctions. We will formalize this idea and give a few basic results that will allow us to apply this perspective to the problems we will deal with in the next Chapters.

1.1. Reverse mathematics preliminaries

Reverse Mathematics is an ongoing research program started in the 1970s by Harvey Friedman (see for instance [28]; we also recommend [63] for a historical introduction to the topic): its goal is to investigate the strength of theorems, or principles (we will use these two terms interchangeably) of "ordinary mathematics", i.e. the areas of mathematics in which characteristic elements of set theory do not play a crucial role: examples are number theory, geometry, real and complex analysis. This is mainly accomplished in the following way: we search for the minimal set existence axiom A necessary to prove a theorem B. The fact that a candidate axiom is indeed the best possible is often proved by "reversing" the usual mathematical process and deducing the axiom A assuming the theorem B.

Here, we will mainly focus on two aspects of this field. In Section 1.1.1 we introduce RCA_0, WKL_0, ACA_0 and Π^1_1-CA_0, four of the so-called "big five", the main subsystems of second-order arithmetic that turn out to be equivalent to large swaths of ordinary mathematics: they are very natural systems from many points of view, and constitute standard benchmarks for the strength of theorems. Then, in Section 1.1.2, we turn our attention to systems whose strength lies between RCA_0 and ACA_0. These theories arise in two ways: either they are obtained by adding some amount of induction to RCA_0, or they capture the strength of some combinatorial principle related to the study of Ramsey's theorem over RCA_0 (we shall call these theorems *Ramseyan principles*, even if this is not a rigorous definition).

1.1.1. The main subsystems

In this Section, we recall the definitions and some of the main features of the main systems of reverse mathematics that we are going to use in the rest of this thesis. A standard reference for this topic (and for reverse mathematics in general) is [66].

The language of most of the theories that we are going to introduce is L_2, the language of second-order arithmetic. The constant, function, and relation symbols are 0, 1, $<$,

$+$, $\cdot$, and $\in$. It is a two-sorted language: objects of the first sort, the so-called *first-order elements* or *numbers* are thought of as natural numbers, and will in general be denoted by lower-case letters, whereas objects of the second sort, the *second-order elements* or *sets of numbers*, are thought of as sets of natural numbers, and are usually denoted by upper-case letters.

Special care must be taken when considering the symbol $=$, which is defined as a relation on the elements of the first sort, i.e. between numbers. Equality between sets (which we still denote by the same symbol) is defined by $\forall X, Y(X = Y \leftrightarrow \forall x(x \in X \leftrightarrow x \in Y))$.

The models M of the L_2-theories we are going to introduce are given by the tuple

$$M = (\mathbb{N}_M, \mathcal{S}_M, 0_M, 1_M, <_M, +_M, \cdot_M, \in_M),$$

where $\mathbb{N}_M$ denotes the set of first-order elements of the model and $\mathcal{S}_M$ denotes the set of second-order elements of the model. If $\mathbb{N}_M = \omega$, we say that M is an *ω-model*.

As usual, when it is clear which structure is currently being considered, we will dispense with the use of the subscript M.

Related to L_2 is L_1, the language of first-order arithmetic, which consists of the constant symbols 0 and 1, the relation symbol $<$ and the binary functions $\cdot$ and $+$: it is a one-sorted language, whose objects are called *numbers*, and an L_1-structure N is a tuple $N = (\mathbb{N}_N, 0_N, 1_N, <_N, +_N, \cdot_N)$.

Although L_1-theories will not play a prominent role in the rest of the thesis *per se*, we will sometimes have something to say on the *first-order part* of L_2-theories.

Definition 1.1.1. For every L_2-theory T, the *first-order part of the theory T* is the L_1-theory whose theorems are the L_1-formulas that are theorems of T.

It is currently an area of great interest in reverse mathematics to determine the first-order parts of theories coming from the study of principles related to Ramsey's theorem.

Remark 1.1.2. We now make explicit a notational aspect of the definitions given above that might otherwise cause some confusion: throughout the thesis, we will reserve the symbol ω for the natural numbers *of the metatheory*, which we always assume to be ZFC. As suggested in the previous paragraph, the symbol $\mathbb{N}$ will instead be reserved for the set of natural numbers *of the theory under consideration*: we will always make sure that there is no possible confusion as to which theory that is meant to be. In particular, when we are not working in subsystems of second-order arithmetic, $\mathbb{N} = \omega$ holds.

We now introduce the first of the subsystems of second-order arithmetic that we are going to use.

Definition 1.1.3. RCA_0 (for *recursive comprehension axiom*) is the L_2-theory consisting of the following axioms:

- a first-order sentence expressing that the numbers form a discretely ordered commutative semi-ring with cancellation; the collection of these sentences is often called P^-.
- the Σ^0_1 *induction scheme* (denoted $\mathsf{I}\Sigma^0_1$), which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form
$$(\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n \varphi(n),$$
where φ is Σ^0_1; and
- the Δ^0_1 *comprehension scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form
$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$
where φ is Σ^0_1, ψ is Π^0_1, and X is not free in φ.

The intuition behind RCA_0 is that it allows us to build the computable sets (although there are some major caveats when the first-order part is non-standard, as we will see in the next section). This intuition can be made precise with the observation (see [66, Theorem II.1.7]) that a non-empty collection of subsets of ω is the second-order part of an ω-model of RCA_0 if and only if it is a Turing ideal (in particular, $REC = (\omega, \mathcal{C}, 0, 1, < +, \cdot, \in)$, where $\mathcal{C}$ is the set of the computable sets, is a model of RCA_0). This fact lies at the heart of the deep interplay between reverse mathematics and computability theory.

There are several important (and natural) facts that RCA_0 can prove, which makes it a reasonable theory in which to work. One of them, which we will repeatedly use without mentioning it, is the following fact: every infinite set $X \subseteq \mathbb{N}$ has an *enumeration*, i.e. for every infinite set X RCA_0 proves the existence of a function $p_X : \mathbb{N} \to \mathbb{N}$ (also called *principal function* of X) such that $\forall x \in X \exists n \in \mathbb{N}(p_X(n) = x)$ and $\forall n, m \in \mathbb{N}(n < m \to p_X(n) < p_X(m))$. This is Lemma II.3.6 in [66].

Another fact that we are going to use repeatedly without explicitly mentioning it is that RCA_0 is able to implement the usual coding of finite sets and sequences of natural numbers as a single natural number. We will denote by $\langle \cdot \rangle$ the coding operation for every finite sequence of numbers: for instance, the code for the pair of elements $a, b \in \mathbb{N}$ is denoted $\langle a, b \rangle$, and the code for the triple $a, b, c \in \mathbb{N}$ is denoted $\langle a, b, c \rangle$. We refer to [66, Chapter II.2] for more details and properties of the coding of finite sequences.

We point out that infinite sequences of numbers are but functions: in general, every function $f : \mathbb{N} \to \mathbb{N}$ is coded by the set $\{\langle n, f(n) \rangle : n \in \mathbb{N}\}$.

We will use the same symbols to denote coding of sequences of *sets*. Given two subsets of $\mathbb{N}$ $A_0 = \{a_0^0, a_1^0, \dots\}$ and $A_1 = \{a_0^1, a_1^1, \dots\}$, we say that a set A is a *code for the sequence* A_0, A_1, and we denote it by $\langle A_0, A_1 \rangle$, if $A = \{2a_0^0, 2a_1^0, \dots\} \cup \{2a_0^1 + 1, 2a_1^1 + 1, \dots\}$. For any finite sequence of sets $A_0, A_1, \dots, A_n$, we can recursively say when a set A, which we denote by $\langle A_0, A_1, \dots, A_n \rangle$, is a code for that sequence: this happens if $A = \langle A_0, \langle A_1, \dots, A_n \rangle \rangle$. Finally, we notice that a similar procedure allows us to code infinite sequences of sets into just one set: we say that the set A is a code for the sequence $(A_i)_{i \in \mathbb{N}}$, where for every $i \in \mathbb{N}$ $A_i = \{a_0^i, a_1^i \dots\}$, if $A = \bigcup_{i \in \mathbb{N}} \{2^{i+1} a_0^i + 2^i - 1, 2^{i+1} a_1^i + 2^i - 1, \dots\}$. In this case, we denote A by $\langle A_0, A_1, \dots \rangle$.

Notice that this procedure can be used to code infinite sequences of *functions* as well, since functions are coded by sets of numbers.

Regardless of the coding that we chose, the important point of the paragraph above is that we can see sets of numbers as codes for sequences of sets: in particular, RCA_0 has access to some sequences of sets, if they are defined in a sufficiently uniform way. Again, we refer to [66] for a more detailed discussion on this.

We make now explicit a convention that we will use for the rest of the thesis: for the sake of readability, we will, in general[1], *not* refer to sequences via their code. Namely, we will use the notation $(A_i)_{i\in\mathbb{N}}$ to denote the sequence of sets $A_0, A_1, \dots,$ even if, formally speaking, what we actually have while arguing in second-order arithmetic is just a *code* for that sequence. The same goes for finite sequences of sets and of numbers: we will in general prefer the notations $(a_0, a_1, \dots, a_n)$ and $(A_0, A_1, \dots, A_n)$ over $\langle a_0, a_1, \dots, a_n\rangle$ and $\langle A_0, A_1, \dots A_n\rangle$. Similarly, we will denote infinite sequences of numbers $a_0, a_1, \dots$ as $(a_i)_{i\in\mathbb{N}}$.

There will be no confusion in adopting the convention above. We point out, anyway, that as a consequence we will sometimes refer to an element $n \in \mathbb{N}$ as (n), if we want to stress that it has to be seen as a string (typically, this will happen when dealing with trees).

Moreover, for every $k \in \mathbb{N}$ and any set $X \in \mathcal{S}$, RCA_0 proves the existence of $[X]^k$, the set of subsets of X of size k, of X^k, the set of strings (or finite sequences) of elements of X of length k, of $X^{<k}$, the set of strings of length less than k, and of $X^{<\mathbb{N}}$, the set of strings (or finite sequences) of elements of X. All of these facts are essentially obvious (we refer to [66, Chapter II] for the easy proofs and further details), and we stated them explicitly mainly with the end of fixing the notation. On this note, we make explicit a convention that we will use in the rest of the thesis: for every set $N \subseteq \mathbb{N}$, when we write $(x, y) \in [N]^2$ (instead of $\{x, y\}$, as would be appropriate), we mean that $x < y$ and $\{x, y\} \in [N]^2$.

[1] Of course, there will be cases in which we will *have to* use codings of sequences: an example is the definition of the problem lim, where elements of the domain are seen as codes of sequences.

We also notice that, for every three sets $X, Y, Z \subseteq \mathbb{N}$ with $Z \subseteq X$ and every function $f : X \to Y$, RCA_0 proves the existence of the *restriction* $f{\restriction}_Z$ of f to Z, which of course is just the function $g : Z \to Y$ such that, for every $z \in Z$, $f(z) = g(z)$. The same goes if X, Y, Z are subsets of $\mathbb{N}^n$ or of $[\mathbb{N}]^n$ for some n.

Other notational conventions that we will use concerning strings are the following:

- if $\sigma \in X^k$ for some $k \in \mathbb{N}$, then we say that σ *has length* k, and we write $|\sigma| = k$.
- Given two strings $\sigma, \tau \in X^{<\mathbb{N}}$, we say that σ *is a prefix of* τ, and we write $\sigma \sqsubseteq \tau$, if $|\sigma| \leq |\tau|$ and for every $i < |\sigma|$ $\sigma(i) = \tau(i)$. Similarly, for every function $f : \mathbb{N} \to X$, we write $\sigma \sqsubseteq f$ to mean that for every $i < |\sigma|$ $\sigma(i) = f(i)$.
- Given two strings $\sigma, \tau \in X^{<\mathbb{N}}$, we denote by $\sigma^\frown\tau$ the string obtained by concatenating σ and τ: for all $i < |\sigma| + |\tau|$, $\sigma^\frown\tau(i) = \sigma(i)$ if $i < |\sigma|$, and $\sigma^\frown\tau(i) = \tau(i - |\sigma|)$ otherwise.

RCA_0 is the weakest system we will be working with, and hence it is the natural system in which to give definitions. We now list some very standard objects of usual mathematics by defining them over RCA_0. Again, the main goal of this is to fix the notation.

Definition 1.1.4. (RCA_0)

- Let $X \subseteq \mathbb{N}$ be a non-empty set. A *tree (on X)* is a set $T \subseteq X^{<\mathbb{N}}$ such that for every $\tau \in T$ and for every $\sigma \in X^{<\mathbb{N}}$, if $\sigma \sqsubseteq \tau$, then $\sigma \in T$.
- A function $f : \mathbb{N} \to X$ is a *path through* T if for every $k \in \mathbb{N}$ there is a $\sigma \in T \cap X^k$ such that $\sigma \sqsubseteq f$.
- For $r \in \mathbb{N}$, the r^{th} *level* of a tree T is the set $L_r := \{\sigma \in T : |\sigma| = r\}$.
- A tree is *finitely branching* if for every level r there is a k_r such that $|L_r| < k_r$.
- If T is a tree and $\sigma \in T$, we denote by T_σ the set $\{\tau \in \mathbb{N}^{<\mathbb{N}} : \sigma^\frown\tau \in T\}$. It is clear that T_σ is itself a tree.

Remark 1.1.5. It is a convention frequently used in the literature to denote by $[T]$ the set of paths through a tree T. We point out that we cannot give a definition of $[T]$ in any subsystem of second-order arithmetic, since $[T]$ is a third-order object. Hence, while we will use the notation $[T]$ in the rest of this document, we will only be able to do so while arguing in the metatheory.

We are now ready to introduce the second theory that we are going to use. Before we do that, we point out that, in line with most texts in mathematical logic, for every k we will denote the set $\{0, 1, \ldots, k-1\}$ by k.

Definition 1.1.6. WKL_0 is the theory given by RCA_0 plus the statement "for every infinite tree $T \subseteq 2^{<\mathbb{N}}$, there is a path through T.

WKL_0 allows us to carry out many arguments that, in a classical setting, would rely on some form of compactness. We mention an example, namely Dilworth's theorem, regarding partial orders, that will be important in Chapter 3, after giving some relevant definitions that will be used in the rest of the thesis.

Definition 1.1.7. (RCA_0) Let $P \subseteq \mathbb{N}$ be a set and $<_P$ a binary relation on P. We say that the pair $(P, <_P)$ is a *partial order* if $<_P$ is antireflexive, antisymmetric and transitive. We will sometimes refer to them as *posets* for short. We will denote by $\leq_P$ the reflexive closure of $<_P$.

$(L, <_L)$ is a *linear order* if it a partial order and moreover $\forall p, q \in L(p \leq_L q \vee q \leq_L p)$.

Let $(P, <_P)$ be a partial order.

- Given $p, q \in P$, we write $p \between_P q$, if it holds either that $p \leq_P q$ or $q \leq_P p$, and if this happens we say that p and q are *comparable*.
- Given $p, q \in P$, we write $p \,|_P\, q$ if neither $p \leq_P q$ nor $p \geq_P q$ holds. In this case, we say that p and q are *incomparable*.
- A set $A \subseteq P$ such that $\forall a, b \in A(a \neq b \to a \,|_P\, b)$ is called an *antichain of* $(P, <_P)$.

- A set $C \subseteq P$ such that $(C, <_P)$ is a linear order is called a *chain of* $(P, <_P)$.
- For every $k \in \mathbb{N}$, $k \neq 0$, we say that $(P, <_P)$ has *width* k if for every $A \subseteq P$, if A is an antichain then $|A| \leq k$, and there is an antichain $B \subseteq P$ with $|B| = k$.
- For every $k \in \mathbb{N}$, $k \neq 0$, we say that $(P, <_P)$ has *height* k if for every $C \subseteq P$, if C is a chain, then $|C| \leq k$, and moreover there is a chain $D \subseteq P$ with $|D| = k$.

Theorem 1.1.8 ([40], Theorem 3.23)**.** *The following statement is equivalent to* WKL_0 *over* RCA_0*: for every* $k \in \mathbb{N}$ *and for every partial order* $(P, <_P)$*, if* $(P, <_P)$ *has width* k*, then there are sets* $C_0, \dots, C_{k-1}$ *such that* $P = \bigcup_{i<k} C_i$ *and every* C_i *is a chain of* $(P, <_P)$*.*

The next subsystem is ACA_0, which stands for *arithmetical comprehension axiom.*

Definition 1.1.9. ACA_0 is the theory given by RCA_0 plus, for every arithmetical formula[2] $\varphi(n)$ in which the set variable X is not free, the axiom given by the universal closure of the following formula: $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.

An equivalent formulation of ACA_0 will be particularly useful when proving reversals.

Lemma 1.1.10 ([66], Lemma III.1.3)**.** *The following is equivalent to* ACA_0 *over* RCA_0*: if* $f : \mathbb{N} \to \mathbb{N}$ *is an injection, then there is a set* X *such that* $\forall n (n \in X \leftrightarrow \exists s (f(s) = n))$*.*

The Lemma above can be informally stated as saying that ACA_0 is equivalent to the existence of the range $f(\mathbb{N})$ for every function f. Obviously, more generally, for every $k, l \in \mathbb{N}$, every function $f : [\mathbb{N}]^k \to [\mathbb{N}]^l$ and every $X \subseteq [\mathbb{N}]^k$, ACA_0 guarantees the existence of the *f-image of X* $f(X) = \{y \in [\mathbb{N}]^l : \exists x \in X (f(x) = y)\}$. We will sometimes refer to the same set as $\operatorname{ran} f$.

The final subsystem we are going to use is Π^1_1-CA_0.

[2] We recall that an L_2-formula is *arithmetical* if it contains no set quantifiers.

Definition 1.1.11. Π^1_1-CA_0 is the theory given by RCA_0 plus, for every Π^1_1-formula $\varphi(n)$ in which the set variable X is not free, the axiom given by the universal closure of the following formula: $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.

Again, there are some equivalent formulations of the theory above that will come in rather handy in the following chapters.

Definition 1.1.12. (RCA_0) Let $(P, <_P)$ be a partial order, and let $X \subseteq P$.

- We say that X is *well-founded* if it does not contain any infinite descending sequence, i.e. a sequence $(x_i)_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$ $(x_{i+1} <_P x_i)$. If X is not well-founded, it will be said to be *ill-founded*.
- We say that X is *reverse well-founded* if it does not contain any infinite ascending sequence, i.e. a sequence $(x_i)_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$ $(x_i <_P x_{i+1})$. If X is not reverse well-founded, it will be said to be *reverse ill-founded*.

Notice that every tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ can be seen as a partial order $(T, \sqsubseteq)$, i.e. the strings of T are ordered by the extension relation. For historical reasons, though, it is more frequent to see trees as posets ordered by the converse relation, namely $\sqsupseteq$: hence, according to this perspective, if a tree T has a path through it, we say that it is *ill-founded*, otherwise we say that it is *well-founded*.

Lemma 1.1.13 ([66], Lemma VI.1.1, [50], Theorem 6.5)**.** *The following are equivalent over* RCA_0*:*

- Π^1_1-CA_0
- Σ^1_1-CA_0
- *for each sequence of trees* $(T_n)_{n\in\mathbb{N}}$ *with* $T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ *for every* n*, there is a set* $X \subseteq \mathbb{N}$ *such that* $n \in X$ *if and only if* T_n *is well-founded.*
- LPP*, which is the statement "each ill-founded tree* $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *has a* leftmost path, *i.e. a path* $f : \mathbb{N} \to \mathbb{N}$ *through* T *such that for every path* $g : \mathbb{N} \to \mathbb{N}$ *through* T*, it holds that*
$$\forall n(\forall m < n(f(m) = g(m)) \to f(n) \leq g(n))".$$

1.1.2. Intermediate subsystems: bounding, induction and Ramseyan principles

We will now focus on theories whose strength is not captured by any of the big five, but instead lies somewhere between RCA_0 and ACA_0. We start by introducing the bounding and induction axioms schemas.

Definition 1.1.14.

- For every $n \in \omega$, the Σ^0_n *bounding scheme* ($\mathsf{B}\Sigma^0_n$) consists of the universal closures of all L_2-formulas of the form

$$\forall a((\forall n < a)(\exists m)\varphi(n,m) \to \exists b(\forall n < a)(\exists m < b)\varphi(n,m)),$$

where φ is Σ^0_n and a and b are not free in φ. $\mathsf{B}\Pi^0_n$ is defined analogously.

- For every $n \in \omega$, the Σ^0_n *induction scheme* (denoted $\mathsf{I}\Sigma^0_n$) consists of the universal closures of all L_2-formulas of the form

$$(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n\varphi(n),$$

where φ is Σ^0_n. $\mathsf{I}\Pi^0_n$ is defined analogously.

The axioms above are essentially first-order axioms: albeit, by expressing them using L_2-formulas, we are allowing for set parameters, the induction and bounding axioms are very interesting objects even when restricted to L_1-formulas, and have been in fact thoroughly studied in the analysis of models of subsystems of first-order arithmetic, a setting in which they arise quite naturally. We see now an example of this naturality: $\mathsf{I}\Sigma^0_n$ is equivalent to the Σ^0_n- and Π^0_n-last number principles over RCA_0. We point out that, technically, in [55] the equivalences are proved over $P^- + \mathsf{I}\Sigma^0_1$, which is weaker than RCA_0.

Definition 1.1.15. For every $n \in \omega$, the Σ^0_n-*least number principle scheme*, denoted $\mathsf{L}\Sigma^0_n$, consists of the universal closures of all L_2-formulas of the form

$$\exists x\varphi(x) \to \exists x(\varphi(x) \wedge \forall y < x \neg\varphi(y)),$$

where φ is Σ^0_n. $\mathsf{L}\Pi^0_n$ is defined analogously.

Lemma 1.1.16 ([55]). *For every $n \in \omega$, $\mathsf{RCA}_0 \vdash \mathsf{I}\Sigma^0_n \leftrightarrow \mathsf{I}\Pi^0_n \leftrightarrow \mathsf{L}\Sigma^0_n \leftrightarrow \mathsf{L}\Pi^0_n$.*

There is another form of induction that will be of use in the following chapters.

Definition 1.1.17. For every $k \in \omega$, the *bounded Σ^0_k-comprehension scheme* consists of the universal closures of all L_2-formulas of the form

$$\forall n \exists X \forall i (i \in X \leftrightarrow (i < n \wedge \varphi(i))),$$

where φ is Σ^0_n and X is not free in φ.

Lemma 1.1.18 ([32]). *For every $k \in \omega$, RCA_0 proves that $\mathsf{I}\Sigma^0_k$ is equivalent to the bounded Σ^0_k-comprehension scheme.*

As pointed out in [66], the Lemma above is quite interesting, since it allows to see induction as a set-existence axiom.

As we already mentioned, it is clear that $\mathsf{RCA}_0 \vdash \mathsf{I}\Sigma^0_1$, and moreover that for every $n \in \omega$ $\mathsf{ACA}_0 \vdash \mathsf{I}\Sigma^0_n \wedge \mathsf{B}\Sigma^0_n$. We will take care of the other cases in the next Lemma, which sums up results that can be essentially found in [40, Chapter 6] and [55].

Lemma 1.1.19.

- *For every $n > 0$, $\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_{n+1} \leftrightarrow \mathsf{B}\Pi^0_n$.*
- *For every $n > 0$, $\mathsf{RCA}_0 \vdash \mathsf{I}\Sigma^0_{n+1} \rightarrow \mathsf{B}\Sigma^0_{n+1} \rightarrow \mathsf{I}\Sigma^0_n$.*
- *$\mathsf{WKL} \nvdash \mathsf{B}\Sigma^0_2$.*

Of the axioms above, $\mathsf{B}\Sigma^0_2$ turns out to be particularly relevant for the study of infinitary combinatorics, since, as we will see in the next Theorem, $\mathsf{B}\Sigma^0_2$ turns out to be itself a Ramseyan principle.

Theorem 1.1.20 ([40], Theorem 6.4). *The following are equivalent over RCA_0:*

- $\mathsf{B}\Sigma^0_2$

- $\mathsf{RT}^1_{<\infty}$, *which is the statement "for every* $k \in \mathbb{N}$ *and for every function* $f : \mathbb{N} \to k$, *there is an infinite set* H *and an* $i < k$ *such that* $f(H) = i$*".*

All the axioms seen in this section so far have the common feature that any model of RCA_0 such that its first-order part is ω is a model of these axioms. We now move to something radically different: the subsystems of second-order arithmetic related to Ramsey's theorem. We start, of course, with Ramsey's theorem itself.

Definition 1.1.21.

- For every $n, l \in \omega \setminus 1$, RT^n_l is the statement "for every function $f : [\mathbb{N}]^n \to l$, there is an infinite set $H \subseteq \mathbb{N}$ and an $i < l$ such that $f([H]^n) = i$.
- For every $n \in \omega \setminus 1$, $\mathsf{RT}^n_{<\infty}$ (or RT^n) is the statement "for every $l \in \mathbb{N}$ and for every function $f : [\mathbb{N}]^n \to l$, there is an infinite set $H \subseteq \mathbb{N}$ and an $i < l$ such $f([H]^n) = i$.

We will often call functions with a finite range *colorings*. Given a coloring $f : [\mathbb{N}]^n \to l$, any set $H \subseteq \mathbb{N}$ such that $|f([H]^n)| = 1$ is said to be f*-homogeneous*.

It is easy to see that for every $n \in \omega \setminus 1$ and for every $l, l' \in \omega \setminus 2$ $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_l \leftrightarrow \mathsf{RT}^n_{l'}$, so the number of colors (if it is a standard number) does not affect the strength of the principles. As an example, we point out that this is one of the cases where we use the fact that RCA_0 proves that every infinite set has an enumeration: for instance, to show that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_4$, given any instance $f : [\mathbb{N}]^2 \to 4$, we define a coloring $f_0 : [\mathbb{N}]^2 \to 2$ as $f_0(x, y) = f(x, y) \mod 2$. Given any infinite f_0-homogeneous set H, we have then to use the fact that H has an enumeration in order to be able to apply RT^2_2 to $f{\restriction}_H$ (and hence find an infinite f-homogeneous set), since RT^2_2 only applies to colorings with domain $[\mathbb{N}]^2$.

It follows from the previous paragraph that it is more interesting to focus on the exponent n. It was implicitly shown in [41] that, for every $n \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \leftrightarrow \mathsf{ACA}_0$. In particular, this implies that for every $n, l \in \omega$, $\mathsf{ACA}_0 \vdash \mathsf{RT}^n_l$, and moreover it is immediately seen that RT^1_l is provable in RCA_0. The case of RT^2_2 is much more difficult to deal with: although it is easily seen that $\mathsf{ACA}_0 \vdash \mathsf{RT}^2_2$ (as follows from the previous paragraph), it was shown in [49] that $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{WKL}_0$, with the use of rather complicated techniques, and in a sense it has driven the development of a large part of reverse mathematics and computability theory for a long period of time. In particular, many principles were introduced and studied reverse mathematically in order to get some insight on the strength of RT^2_2. This process gave rise to the so-called *zoo below* RT^2_2. Many of the principles we will see in this thesis belong to that zoo.

Amongst the historically first new principles of the zoo to be introduced were COH and SRT^2_2, which were defined in the seminal paper [11].

Definition 1.1.22.
- (RCA_0) For every $l \in \mathbb{N}$, a coloring $f : [\mathbb{N}]^2 \to l$ is *stable* if for every $x \in \mathbb{N}$ there exists a $y \in \mathbb{N}$ such that for every $z > y$ $f(x, y) = f(x, z)$ (which can informally be stated as the existence of $\lim_{y\to\infty} f(x, y)$ for every x).
- SRT^2_l (for *stable Ramsey theorem*) is the statement "for every stable coloring $f : [\mathbb{N}]^2 \to l$ there exists an infinite f-homogeneous set.
- (RCA_0) For sets $A, C \subseteq \mathbb{N}$, $C \subseteq^* A$ denotes that $C \setminus A$ is finite, and $A =^* C$ denotes that both $C \subseteq^* A$ and $A \subseteq^* C$.
- (RCA_0) For every set $A \subseteq \mathbb{N}$, we denote by $\overline{A}$ the set $\mathbb{N} \setminus A$, i.e. the *complement of* A.
- (RCA_0) Let $\vec{A} = (A_i)_{i\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$. A set $C \subseteq \mathbb{N}$ is called *cohesive for* $\vec{A}$ (or simply $\vec{A}$*-cohesive*) if C is infinite and for each $i \in \mathbb{N}$, either $C \subseteq^* A_i$ or $C \subseteq^* \overline{A_i}$.
- COH is the statement "for every sequence $\vec{A}$ of subsets of $\mathbb{N}$, there is a set $C \subseteq \mathbb{N}$ that is cohesive for $\vec{A}$".

We list the main results concerning the principles we just introduced.

Theorem 1.1.23.
1. $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow (\mathsf{SRT}^2_2 \wedge \mathsf{COH})$ *([11, Theorem 7.11] and [52, Claim A.1.3])*
2. $\mathsf{RCA}_0 + \mathsf{COH} \nvdash \mathsf{B}\Sigma^0_2$ *and* $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \vdash \mathsf{B}\Sigma^0_2$ *([11, Theorem 9.1 and Theorem 10.5])*

3. $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \nvdash \mathsf{COH}$ *([13, Corollary 2.8])*

Other principles were then introduced, essentially analyzing the consequences of Ramsey's theorem in structures more complicated than just sets (see [39] and [3]). For all the structures we are about to see, we recall that, due to the very nature of formalization in second-order arithmetic, the domain of those structures is a subset of the natural numbers (see e.g. Definition 1.1.7): for example, when dealing with a linear order $(L, <_L)$, it is useful to remember that $L \subseteq \mathbb{N}$. In particular, it makes sense to say, for two elements $x_0, x_1 \in L$, that $x_0 < x_1$, due to this remark.

Definition 1.1.24.

- ADS (for *ascending/descending sequence principle*) is the statement "for every infinite linear order $(L, <_L)$, there is an infinite sequence $(x_i)_{i\in\mathbb{N}}$ such that $\forall i(x_i < x_{i+1})$ and moreover either $\forall i(x_i <_L x_{i+1})$ or $\forall i(x_i >_L x_{i+1})$ holds".

- (RCA_0) A linear order $(L, <_L)$ is said to be a *stable linear order* if every element has either finitely many $<_L$-predecessors or finitely many $<_L$-successors.

- SADS (for *stable* ADS) is the statement "for every infinite stable linear order $(L, <_L)$, there is an infinite sequence $(x_i)_{i\in\mathbb{N}}$ such that $\forall i(x_i < x_{i+1})$ and either $\forall i(x_i <_L x_{i+1})$ or $\forall i(x_i >_L x_{i+1})$."

- CAC (for *chain/antichain principle*) is the statement "every infinite partial order has an infinite chain or an infinite antichain.

- (RCA_0) A partial order $(P, <_P)$ is said to be *stable* if for every element $p \in P$, either

$$\exists i(\forall q \in P(q > i \to q >_P p) \vee \forall q \in P(q > i \to q|_P p)),$$

in which case p is said to be *small*, or

$$\exists i(\forall q \in P(q > i \to q <_P p) \vee \forall q \in P(q > i \to q|_P p)),$$

in which case p is said to be *large*.

- SCAC (for *stable* CAC) is the statement "every infinite stable partial order has an infinite chain or an infinite antichain".
- (RCA_0) Let T be a set and R_T a binary relation on T. The pair (T, R_T) is said to be a *tournament* if R_T is antireflexive and for every $s, t \in T$ with $s \neq t$, exactly one of sR_Tt and tR_Ts hold. A tournament (T, R_T) is *transitive* if the relation R_T is transitive.
- EM (for *Erdős-Moser principle*) is the statement "for every infinite tournament (T, R_T), there is an infinite set T' such that (T', R_T) is a transitive tournament.

Remark 1.1.25. We point out that, by the way we stated them, the principle ADS (and hence SADS as well) only guarantees the existence of a *function* enumerating an ascending or a descending sequence. Anyway, since we are assuming that that function is $<_{\mathbb{N}}$-increasing, RCA_0 proves that the range of that function exists. Hence, when speaking of an ascending (or descending) sequence, we may refer to it as a set $\{x_0 < x_1 < \dots\}$, and we shall often do so.

We now summarize the relationship between the principles of the zoo that we have introduced so far.

Theorem 1.1.26. *1.* $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{CAC} \to \mathsf{ADS} \to \mathsf{SADS} \to \mathsf{B}\Sigma^0_2$ *([39],[12])*

2. $\mathsf{RCA}_0 \vdash \mathsf{ADS} \to \mathsf{COH}$ *([39])*

3. $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{SCAC} \to \mathsf{SADS}$ *([39])*

4. $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow (\mathsf{ADS} \land \mathsf{EM})$ *(essentially [3]).*

1.2. Computable and uniform analysis of problems

The main focus of the previous section has been to show how to classify principles in an essentially proof-theoretic way: the strength of a certain statement was determined by gauging its consequences over a certain base theory.

The fundamental idea underlying the ways of classifying theorems we will see in this section is different: it relies on the fact that many theorems of mathematics are Π^1_2 statements, i.e. they have the shape $\forall X \exists Y(\varphi(X) \to \psi(X, Y))$, for some arithmetical $\varphi(X)$ and $\psi(X, Y)$. For instance, in the case of RT^2_2, the sentence $\varphi(X)$ is "X is a coloring of $[\mathbb{N}]^2$", and $\psi(X, Y)$ is "Y is an infinite X-homogeneous set".

This simple observation allows us to change our perspective in the following way: instead of seeing principles as *statements*, we see them as *functions*, namely, using the notation above, functions associating to the X's such that $\varphi(X)$ holds the Y's such that $\psi(X, Y)$ holds. The strength of these functions will then be given by the complexity of the information that can be coded using them, and what other functions they can compute.

1.2.1. Computability theoretic notation

We now introduce some of the notation coming from computability theory that we will use in the rest of the thesis. As far as we can tell, the notation is standard and follows one of the canonical books on the subject, [67].

In the setting of computability theory, given two subsets A and B of ω, we call the set $\langle A, B\rangle$ (where $\langle\cdot\rangle$ is the coding of sequences of sets as defined in RCA_0) the *join* of A and B. We point out that in a large part of the literature a different notation is used for the join of two sets, namely $A \oplus B$, but the two notations describe exactly the same set.

The Turing degree of a set $A \subseteq \omega$ will be denoted by $\deg_{\mathrm{T}} A$, and Turing degrees will be denoted by boldface lower-case letter, e.g. $\mathbf{d}$.

We recall that an oracle Turing machine Γ can be seen as a partial function $\Gamma :\subseteq \omega^\omega \to \omega^\omega$ that maps an oracle $p \in \omega^\omega$ to the partial function $\Gamma(p) : \omega \to \omega$ such that $n \mapsto \Gamma(p)(n)$ whenever $\Gamma(p)(n)$ converges. According to this perspective, we will call oracle Turing machines *Turing functionals*. Turing functionals will be denoted by upper-case Greek letters, like Φ and Ψ, and we will assume that a recursive enumeration of them is given: the notation Φ_e, for $e \in \omega$, represents the eth functional in this enumeration.

We will be a bit more specific concerning the notation that we will use for Turing functionals, since in Chapter 5 it will be practical to borrow some notational elements from conventions used chiefly in computable analysis. As usual, for $p \in \omega^\omega$ and $e, x, y \in \omega$, by $\Phi_e(p)(x) = y$ we mean that the eth Turing functional with oracle p converges on input x and outputs y. Similarly, for $p, q \in \omega^\omega$ and $e \in \omega$, by $\Phi_e(p) = q$ we mean that $\forall n \in \omega(\Phi_e(p)(n) = q(n))$. Sometimes, for notational ease, we will denote by $\Phi_{e^\frown p}$ the functional $\Phi_e(p)$ (this will happen, for instance, when dealing with operations on problems, where it is more practical to use just elements of ω^ω without specifying if they should be seen as a concatenation of a number and a function). We add this notational convention accordingly: for $r, p, f, q \in \omega^\omega$ and $e, x, y \in \omega$ such that $f = e^\frown p$, by $\Phi_f(r)(x) = y$ we mean that $\Phi_e(\langle p, r\rangle)(x) = y$, and similarly for $\Phi_f(r) = q$.

1.2.2. Theorems as partial multifunctions and reducibilities between them

As mentioned above, the main point of this Section is to study the behavior of theorems when they are seen as functions: although this sentence is a good slogan, it gives an imprecise description of what we are about to do. Suppose, for instance, that we want to see Ramsey's theorem as a function: by definition of function, this means that we should be able to associate to every coloring $f : [\mathbb{N}]^n \to k$ a *unique* infinite homogeneous set. This is too restrictive for what we want to do, as will be clear from the following discussions. Hence, we introduce the concept of partial multifunction, which turns out to better capture the intuitive idea of the "theorems as functions" framework.

Definition 1.2.1.

- Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets. We say that P is a *partial multifunction from $\mathcal{X}$ to $\mathcal{Y}$*, and we write $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, if $\mathsf{P} \subseteq \mathcal{X} \times \mathcal{Y}$.
- Given a partial multifunction $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $x \in \mathcal{X}$, we denote by $\mathsf{P}(x)$ the set $\{y \in \mathcal{Y} : (x, y) \in \mathsf{P}\}$, and by $\mathrm{dom}(\mathsf{P})$ the set $\{x \in \mathcal{X} : \mathsf{P}(x) \neq \emptyset\}$.

In this framework, it is customary to refer to partial multifunctions as *problems*. Every problem $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ will be described in the following way: for every *instance*, or *input* $x \in \operatorname{dom} \mathsf{P}$ of P, we will describe what the *solutions*, or *outputs* are. As an example, we use again RT^2_2:

Definition 1.2.2. Let $\mathcal{C}$ be the set of colorings of pairs of $[\omega]^2$ with two colors, and let $\mathcal{Y} = 2^\omega$. $\widetilde{\mathsf{RT}^2_2}$ is the following problem:

- Input: any element x of $\mathcal{C}$.
- Output: an infinite x-homogeneous set.

When arguing in this setting, given a coloring $f \in \mathcal{C}$, if $H \in \mathcal{Y}$ is infinite and f-homogeneous, we will say that H is a $\widetilde{\mathsf{RT}^2_2}$-solution to f. The same phrasing extends to the other problems.

There is a small issue with the definition above: although the set of colorings of $[\omega]^2$ in two colors is a perfectly well-defined set, we will have to perform computable operations on its members. It is then more handy to see every $f \in \mathcal{C}$ directly as a member of ω^ω. There is a very general way to solve this issue, namely using represented spaces and realizers of problems.

Definition 1.2.3.

- Let $\mathcal{X}$ be set. A *representation* of $\mathcal{X}$ is a surjective partial function $\delta_{\mathcal{X}} :\subseteq \omega^\omega \to \mathcal{X}$. The pair $(\mathcal{X}, \delta_{\mathcal{X}})$ is called a *represented space*.
- For every represented space $(\mathcal{X}, \delta_{\mathcal{X}})$ and every point $x \in \mathcal{X}$, any point $p \in \omega^\omega$ such that $\delta_{\mathcal{X}}(p) = x$ is said to be a *name* of x.
- Let $(\mathcal{X}, \delta_{\mathcal{X}})$ and $(\mathcal{Y}, \delta_{\mathcal{Y}})$ be represented spaces, and let $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ be a partial multifunction. A partial function $P :\subseteq \omega^\omega \to \omega^\omega$ is a *realizer* for P if for every $q \in \operatorname{dom}(\mathsf{P} \circ \delta_{\mathcal{X}})$, it holds that $\delta_{\mathcal{Y}}(P(q)) \in \mathsf{P}(\delta_{\mathcal{X}}(q))$.

Although we will not make a very deep use of them, it must be pointed out that represented spaces are a very general and useful tool in many areas of mathematics, chiefly among them computable analysis. See for instance [69] for an introduction to this subject.

Let us go back to the translation of RT^2_2: it is very easy to define a representation $\delta_{\mathcal{C}} : \omega^\omega \to \mathcal{C}$, for instance we can fix a (computable) bijective enumeration of $[\omega]^2$, say $r : \omega \to [\omega]^2$, and stipulate that, for every $p \in \omega^\omega$, p is a name for the coloring $f_p : [\omega]^2 \to 2$ defined as $f_p(r(n)) = (p(n) \mod 2)$ for every $n \in \omega$.

We notice that the representation $\delta_{\mathcal{C}}$ has some extremely nice properties, namely it is a computable surjection. Considering this, *there would actually be no harm in seeing the problem* RT^2_2 *as having domain equal to* ω^ω, and we shall do so.

Not every problem has the nice property of having domain equal to ω^ω. The fact that this is the case for RT^2_2 plays an important role when studying some of its features: for instance, the fact that the Squashing Theorem ([18, Theorem 2.5]) can be applied to RT^2_2 relies on this property. We will say more on this aspect as we proceed to translate principles into problems (see in particular Section 2.2).

Regardless of the fact that the domain of a problem is equal to ω^ω or is just a subset, *we translate the combinatorial principles introduced in the previous section to problems* $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ *such that* $\delta_{\mathcal{X}} = \delta_{\mathcal{Y}} = \mathrm{id}_{\omega^\omega}$: this has the main advantage of making the various proofs of reducibilities between principles significantly more straightforward. This also corresponds to a tacit convention adopted, to the best of our knowledge, by the vast majority of the literature on the interplay between reverse mathematics and Weihrauch degrees.[3]

We can now give the "official" translations of RT^n_k and SRT^2_k as partial multifunction.

Definition 1.2.4. For every $n, k \in \omega \setminus 1$, let $r_n : \omega \to [\omega]^n$ be a computable bijective enumeration of the n-tuples of elements of ω, and for every $p \in \omega^\omega$ and $x \in \omega$, let $f_{n,k,p} : [\omega]^n \to k$ be defined as $f_{n,k,p}(r_n(x)) = (p(x) \mod k)$.

[3]At this point, one might wonder whether it was really necessary to go through the hurdles of introducing represented spaces at all: as we will see in the next sections, e.g. when we will define the jump of a problem, the answer seems to be affirmative.

- RT^n_k is the following multifunction:
  - Input: any $p \in \omega^\omega$.
  - Output: an infinite $f_{n,k,p}$-homogeneous set.
- SRT^2_k is the following partial multifunction:
  - Input: any $p \in \omega^\omega$ such that $f_{2,k,p}$ is stable.
  - Output: an infinite $f_{2,k,p}$-homogeneous set.

SRT^2_2 is an example of a problem whose domain is not the whole space ω^ω.

We gave a very rigorous definition of the problems RT^n_k and SRT^2_k to give an example of how the process of finding a translation for combinatorial principles works. In many other cases, however, we will give slicker definitions, and rely on the fact that finding (computable) codings such that domains and codomains of problems can be seen as subsets of ω^ω is, in most cases, a very easy task.

Definition 1.2.5.
- COH is the following partial multifunction.
  - Input: A sequence $\vec{A} = (A_i)_{i\in\omega}$ of subsets of ω.
  - Output: An infinite set cohesive for $\vec{A}$.
- WKL is the following partial multifunction:
  - Input: an infinite binary tree $T \subseteq 2^{<\omega}$.
  - Output: an infinite path $f \in [T]$.

As we were saying above, we will not actually care how we chose to code the sequence $\vec{A}$ as an element of ω^ω, or how we present infinite binary trees, as long as the coding is "reasonable".

We can now introduce the notions of reducibilities between problems that we will use in the rest of the thesis. We will give them in full generality, although, as we mentioned before, in most cases we will be able to avoid the explicit use of represented spaces.

Definition 1.2.6. Let $(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$, $(\mathcal{W}, \delta_{\mathcal{W}})$ and $(\mathcal{Z}, \delta_{\mathcal{Z}})$ be represented spaces, and let $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\mathsf{Q} :\subseteq \mathcal{W} \rightrightarrows \mathcal{Z}$ be partial multivalued functions.

- P *computably reduces* to Q (written $\mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$) if for every $p \in \mathrm{dom}(\mathsf{P} \circ \delta_{\mathcal{X}})$ there is a $\widetilde{p} \leq_{\mathrm{T}} p$ with $\widetilde{p} \in \mathrm{dom}(\mathsf{Q} \circ \delta_{\mathcal{W}})$ such that for every $\widetilde{t} \in Q(\widetilde{p})$ there is a $t \leq_{\mathrm{T}} \langle p, \widetilde{t} \rangle$ with $t \in P(p)$, whenever Q is a realizer of Q and P is a realizer of P.

- P and Q are *computably equivalent* (written $\mathsf{P} \equiv_{\mathrm{c}} \mathsf{Q}$) if $\mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$ and $\mathsf{Q} \leq_{\mathrm{c}} \mathsf{P}$. In this case, P and Q are said to have the same *computable degree*.

- P *strongly computably reduces* to Q (written $\mathsf{P} \leq_{\mathrm{sc}} \mathsf{Q}$) if for every $p \in \mathrm{dom}(\mathsf{P} \circ \delta_{\mathcal{X}})$ there is a $\widetilde{p} \leq_{\mathrm{T}} p$ with $\widetilde{p} \in \mathrm{dom}(\mathsf{Q} \circ \delta_{\mathcal{W}})$ such that for every $\widetilde{t} \in Q(\widetilde{p})$ there is a $t \leq_{\mathrm{T}} \widetilde{t}$ with $t \in P(p)$, whenever Q is a realizer of Q and P is a realizer of P.

- P and Q are *strongly computably equivalent* (written $\mathsf{P} \equiv_{\mathrm{sc}} \mathsf{Q}$) if $\mathsf{P} \leq_{\mathrm{sc}} \mathsf{Q}$ and $\mathsf{Q} \leq_{\mathrm{sc}} \mathsf{P}$. In this case, P and Q are said to have the same *strong computable degree*.

- P *Weihrauch reduces* to Q (written $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$) if there are Turing functionals Φ, Ψ such that the functional $p \mapsto \Psi(\langle p, Q(\Phi(p)) \rangle)$ is a realizer for P whenever Q is a realizer for Q, i.e. if

$$\forall q \in \mathrm{dom}(\mathsf{Q} \circ \delta_{\mathcal{W}})(\delta_{\mathcal{Z}}(Q(q)) \in \mathsf{Q}(\delta_{\mathcal{W}}(q))) \rightarrow$$
$$\forall p \in \mathrm{dom}(\mathsf{P} \circ \delta_{\mathcal{X}})(\delta_{\mathcal{Y}}(\Psi(\langle p, Q(\Phi(p)) \rangle)) \in \mathsf{P}(\delta_{\mathcal{X}}(p))).$$

- P and Q are *Weihrauch equivalent* (written $\mathsf{P} \equiv_{\mathrm{W}} \mathsf{Q}$) if $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ and $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}$. In this case, P and Q are said to have the same *Weihrauch degree*.

- P *strongly Weihrauch reduces* to Q (written $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$) if there are Turing functionals Φ, Ψ such that the functional $p \mapsto \Psi(Q(\Phi(p)))$ is a realizer for P whenever Q is a realizer for Q, i.e. if

$$\forall q \in \mathrm{dom}(\mathsf{Q} \circ \delta_{\mathcal{W}})(\delta_{\mathcal{Z}}(Q(q)) \in \mathsf{Q}(\delta_{\mathcal{W}}(q))) \rightarrow$$
$$\forall p \in \mathrm{dom}(\mathsf{P} \circ \delta_{\mathcal{X}})(\delta_{\mathcal{Y}}(\Psi(Q(\Phi(p))) \in \mathsf{P}(\delta_{\mathcal{X}}(p))).$$

- P and Q are *strongly Weihrauch equivalent* (written $\mathsf{P} \equiv_{\mathrm{sW}} \mathsf{Q}$) if $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$ and $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$. In this case, P and Q are said to have the same *strong Weihrauch degree*.

As an easy example, we can look at the relationship between SRT^2_2 and RT^2_2: we immediately have that $\mathsf{SRT}^2_2 \leq_{\mathrm{sW}} \mathsf{RT}^2_2$, just by using $\Phi = \Psi = \mathrm{id}$. On the other hand, it is also easy to see that $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{SRT}^2_2$, since every computable instance of SRT^2_2 has a Δ^0_2 solution, whereas, by [41, Corollary 3.2], there is a computable instance of RT^2_2 without Σ^0_2 solutions. By the next easy lemma, this is enough to determine whether the other reductions hold or not.

Lemma 1.2.7. *For every partial multifunctions* P *and* Q*,*

$$\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q} \Rightarrow \mathsf{P} \leq_{\mathrm{W}} \mathsf{Q} \Rightarrow \mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$$

and

$$\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q} \Rightarrow \mathsf{P} \leq_{\mathrm{sc}} \mathsf{Q} \Rightarrow \mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$$

There is, in general, no relation between $\leq_{\mathrm{sc}}$ and $\leq_{\mathrm{W}}$. For a more detailed discussion on this subject, and on the topic of the interplay between reducibilities for combinatorial principles, we refer for instance to [37].

1.2.3. Operations on problems

A very interesting feature of the principles-as-functions approach is that it allows us to define operations on problems: as we will see, these operations are not only interesting by themselves, but are also a fundamental tool in determining the position of various principles in the computable and Weihrauch degrees. A standard reference for this topic is [6].

Definition 1.2.8. Let $(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$, $(\mathcal{W}, \delta_{\mathcal{W}})$ and $(\mathcal{Z}, \delta_{\mathcal{Z}})$ be represented spaces, and let $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\mathsf{Q} :\subseteq \mathcal{W} \rightrightarrows \mathcal{Z}$ be partial multifunctions.

- The product space of $(\mathcal{X}, \delta_{\mathcal{X}})$ and $(\mathcal{Y}, \delta_{\mathcal{Y}})$ is $(\mathcal{X} \times \mathcal{Y}, \delta_{\mathcal{X}\times\mathcal{Y}})$, where for every $p, q \in \omega^\omega$ we set $\delta_{\mathcal{X}\times\mathcal{Y}}(\langle p, q\rangle) = (\delta_{\mathcal{X}}(p), \delta_{\mathcal{Y}}(q))$.
- $\mathsf{P}\times\mathsf{Q}$, called the *parallel product* of P and Q, is the following partial multifunction $\mathsf{P} \times \mathsf{Q} : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathcal{Y} \times \mathcal{Z}$:
  - Input: a pair $(x, w) \in \operatorname{dom} \mathsf{P} \times \operatorname{dom} \mathsf{Q}$.
  - Output: an element of $\mathsf{P}(x) \times \mathsf{Q}(w)$.
- The space of finite words over $\mathcal{X}$, denoted $(\mathcal{X}^*, \delta_{\mathcal{X}^*})$, is such that $\mathcal{X}^* = \bigcup_{i\in\omega}\{i\} \times \mathcal{X}^i$ and for every $n \in \omega$, $p_0, \dots, p_{n-1} \in \omega^\omega$, $\delta_{\mathcal{X}^*}(n^\frown(p_0, \dots, p_{n-1})) = (n, \delta_{\mathcal{X}}(p_0), \dots, \delta_{\mathcal{X}}(p_{n-1}))$.
- P^*, called the *finite parallelization* of P, is the following problem $\mathsf{P}^* : \mathcal{X}^* \rightrightarrows \mathcal{Y}^*$:
  - Input: a point $(n, x_0, \dots, x_{n-1}) \in (\operatorname{dom}(\mathsf{P}))^*$.
  - Output: an element of $\{n\} \times \mathsf{P}(x_0) \times \cdots \times \mathsf{P}(x_{n-1})$.
- For every represented space $(\mathcal{X}, \delta_{\mathcal{X}})$, we let the representation $\delta_{\mathcal{X}^\omega}$ of $\mathcal{X}^\omega$ be given as follows: for every infinite sequence $(p_i)_{i\in\omega} \in \left(\omega^\omega\right)^\omega$, we let $\delta_{\mathcal{X}^\omega}(\langle p_0, p_1, \dots\rangle) = (\delta_{\mathcal{X}}(p_0), \delta_{\mathcal{X}}(p_1), \dots)$.
- $\widehat{\mathsf{P}}$, the *parallelization* of P, is the following partial multifunction $\widehat{\mathsf{P}} : \mathcal{X}^\omega \rightrightarrows \mathcal{Y}^\omega$:
  - Input: a sequence $(x_i)_{i\in\omega} \in (\operatorname{dom} \mathsf{P})^\omega$.
  - Output: an element of $\mathsf{P}(x_0) \times \mathsf{P}(x_1) \times \dots$

As customary, we will use the shorthand $\mathcal{X}^n$ to mean the space $\overbrace{\mathcal{X} \times \cdots \times \mathcal{X}}^{n \text{ times}}$ with the obvious representation, and P^n to mean the problem $\overbrace{\mathsf{P} \times \cdots \times \mathsf{P}}^{n \text{ times}}$.

In the lemma below we list some properties of the operations above: they can be summarized by saying that the operations are indeed "reasonable", meaning that they behave on degrees as one would expect. We refer to [4] for the proofs and further comments on this.

Lemma 1.2.9.
- *The operators* * *and* $\widehat{\ }$ *are* Weihrauch-degree theoretic*, i.e. for every two problems* P, Q *with* $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$, $\mathsf{P}^* \leq_{\mathrm{W}} \mathsf{Q}^*$ *and* $\widehat{\mathsf{P}} \leq_{\mathrm{W}} \widehat{\mathsf{Q}}$*, and* idempotent*, i.e.* $(\mathsf{P}^*)^* \leq_{\mathrm{W}} \mathsf{P}^*$ *and* $\widehat{\widehat{\mathsf{P}}} \leq_{\mathrm{W}} \widehat{\mathsf{P}}$.
- *The parallel product of problems is associative, commutative and* Weihrauch-monotone in both components*, i.e., for all problems* P, Q, $\widetilde{\mathsf{P}}$ *and* $\widetilde{\mathsf{Q}}$ *with* $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ *and* $\widetilde{\mathsf{P}} \leq_{\mathrm{W}} \widetilde{\mathsf{Q}}$*, it holds that* $\mathsf{P} \times \widetilde{\mathsf{P}} \leq_{\mathrm{W}} \mathsf{Q} \times \widetilde{\mathsf{Q}}$.

Again, we will use these results without explicitly mentioning them.

By using the operations above, we can proceed to define cylinders, i.e. problems that are powerful enough to code their instances in their solutions.

Definition 1.2.10.
- $\mathsf{id} : \omega^\omega \rightrightarrows \omega^\omega$ is the *identity problem*, i.e. the problem such that $\mathsf{id}(p) = p$ for every $p \in \omega^\omega$.
- Given two represented spaces $(\mathcal{X}, \delta_{\mathcal{X}})$ and $(\mathcal{Y}, \delta_{\mathcal{Y}})$ and a partial multifunction $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, we say that P is a *cylinder* if $\mathsf{P} \times \mathsf{id} \leq_{\mathrm{sW}} \mathsf{P}$.
- For every partial multifunction P, its *cylindrification* is the problem $\mathsf{P} \times \mathsf{id}$.

The reason why we care about cylinders is the following:

Lemma 1.2.11 ([6], Proposition 3.5)**.** *For every partial multifunctions* P *and* Q*, if* Q *is a cylinder and* $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ *holds, then* $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$.

The lemma above will be tacitly used many times in the rest of this thesis: every time we will have to prove that $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$, if Q is a cylinder, we will just have to prove that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$.

We now turn to the composition of problems. As we will see, here things seem to work out less smoothly than with the operations we saw above.

Definition 1.2.12. Let $(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$ and $(\mathcal{Z}, \delta_{\mathcal{Z}})$ be represented spaces, and let $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\mathsf{Q} :\subseteq \mathcal{Y} \rightrightarrows \mathcal{Z}$ be partial multifunctions. We let $\mathsf{Q} \circ \mathsf{P}$ be the partial multifunction $\mathsf{Q} \circ \mathsf{P} : \mathcal{X} \rightrightarrows \mathcal{Z}$ defined as follows:

- Input: an $x \in \mathcal{X}$ such that $\mathsf{P}(x) \subseteq \mathrm{dom}(\mathsf{Q})$.
- Output: an element of $\{z \in \mathcal{Z} : \exists y \in \mathcal{Y}(y \in \mathsf{P}(x) \wedge z \in \mathsf{Q}(y))\}$

We notice that the definition above is not simply the result of translating the definition of composition of relations to the case of partial multifunctions, since we are requiring that $\mathsf{P}(x) \subseteq \mathrm{dom}(\mathsf{Q})$. This restriction has the advantage of making it straightforward to find a realizer for $\mathsf{Q} \circ \mathsf{P}$, namely a composition of a realizer for Q after a realizer for P. This would not have been the case if we had gone for the regular composition of relations. Anyway, we also notice that this choice does not affect the result if P and Q are partial functions: hence, we can still see this definition of composition as an extension of the definition of composition of functions.

It is easy to see that the notion of composition above is associative, but it lacks the nice properties the other operators had: most notably, it is not Weihrauch degree theoretic, and is not monotone in either of the components. We refer to [31] for further details on this.

One of the reason why this is the case is that the composition $\mathsf{Q} \circ \mathsf{P}$ is, so to speak, not flexible enough to handle the composition of multifunctions when seen as problems: intuitively, what we are looking for is an operation such that, given an input for P, provides and output $y \in \mathsf{P}(y)$, then, *after possibly performing some computable operations on* y, applies Q to it and gives an output. But by the definition we gave above, there is no obvious way to perform any transformation on y before we feed it to Q, and this is an issue.

Hence, we will need a more nuanced notion of composition between principles. In order to do this, we start by defining what the right *degree* of the composition is.

Definition 1.2.13. Let $(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$ and $(\mathcal{Z}, \delta_{\mathcal{Z}})$ be represented spaces, and let $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\mathsf{Q} :\subseteq \mathcal{Y} \rightrightarrows \mathcal{Z}$ be partial multifunctions. We define the *compositional product* of P and Q to be the following degree:

$$\mathsf{Q} * \mathsf{P} = \max_{\leq_{\mathrm{W}}}\{\deg_{\mathrm{W}}(\widetilde{\mathsf{Q}} \circ \widetilde{\mathsf{P}}) : \widetilde{\mathsf{P}} \leq_{\mathrm{W}} \mathsf{P}, \widetilde{\mathsf{Q}} \leq_{\mathrm{W}} \mathsf{Q}\}$$

There are several things to be said about the definition above. First of all, we are defining taking the max (with respect to the order $\leq_W$) over something that we have not proved to be a set. Secondly, even assuming that $\{\deg_W(\widetilde{\mathsf{Q}} \circ \widetilde{\mathsf{P}}) : \widetilde{\mathsf{P}} \leq_W \mathsf{P}, \widetilde{\mathsf{Q}} \leq_W \mathsf{Q}\}$ is a set, there is no guarantee that it has a $\leq_W$-maximum. We refer the reader to [7] for proofs that these issues can be solved, i.e. that $\{\deg_W(\widetilde{\mathsf{Q}} \circ \widetilde{\mathsf{P}}) : \widetilde{\mathsf{P}} \leq_W \mathsf{P}, \widetilde{\mathsf{Q}} \leq_W \mathsf{Q}\}$ is a set and it has a $\leq_W$-maximum.

Now, the compositional product $\mathsf{Q} * \mathsf{P}$ has the properties we were looking for.

Lemma 1.2.14 ([7]). *$\mathsf{Q} * \mathsf{P}$ is associative and Weihrauch-monotone in both components.*

Finally, we see that the Weihrauch degree $\mathsf{Q} * \mathsf{P}$ actually corresponds to the degree we were looking for, i.e. it corresponds to the intuitive idea of composition of problems that we gave above. To do this, we will find a representative of the degree we define above. In a slight abuse of notation, we will use the same symbol to denote them.

Lemma 1.2.15 (see [7] and [70]). *Let $(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$ and $(\mathcal{Z}, \delta_{\mathcal{Z}})$ be represented spaces, and let $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\mathsf{Q} :\subseteq \mathcal{Y} \rightrightarrows \mathcal{Z}$ be partial multifunctions, and let P and Q be realizers for P and Q, respectively. Let us consider the partial multifunction $\mathsf{Q} * \mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$:*

- *Input: a pair $(x, p) \in \omega^\omega \times \omega^\omega$ such that $x \in \mathrm{dom}(P)$ and for every $y \in P(x)$, $\Phi_p(y) \in \mathrm{dom}(Q)$.*
- *Output: a pair (y, z) such that $y \in P(x)$ and $z \in Q(\Phi_p(y))$.*

Then, the partial multifunction $\mathsf{Q} * \mathsf{P}$ is the compositional product (which, we recall, is a Weihrauch degree) of P and Q.

As one can easily check, the Lemma above confirms the intuition we gave about what the composition of two partial multifunctions should be.

We conclude this section by defining the jump of a problem.

Definition 1.2.16.

- We define the problem $\mathsf{lim} :\subseteq \omega^\omega \to \omega^\omega$ in the following way:
  - Input: a $p \in \omega^\omega$ such that for every $i \in \omega$ $\lim_{n\to\infty} p_i(n)$ exists, where $(p_i)_{i\in\omega} \in (\omega^\omega)^\omega$ is the sequence of elements of ω^ω such that $p = \langle p_0, p_1, \ldots \rangle$.
  - Output: $q \in \omega^\omega$ such that for every $i \in \omega$ $q(i) = \lim_{n\to\infty} p_i(n)$.

  We remark that lim is actually a partial function.

- Let $(\mathcal{X}, \delta_\mathcal{X})$ be a represented space. We define the representation $\delta'_\mathcal{X} :\subseteq \omega^\omega \to \mathcal{X}$, which we call *jump of the representation* $\delta_\mathcal{X}$, as $\delta_\mathcal{X} \circ \mathsf{lim}$. We denote by $\mathcal{X}'$ the space $\mathcal{X}$ when given the representation $\delta'_\mathcal{X}$.

- Let $(\mathcal{X}, \delta_\mathcal{X})$ and $(\mathcal{Y}, \delta_\mathcal{Y})$ be represented spaces, and let P be a partial multifunction. The *jump of* P, denoted P', is the problem $\mathsf{P}' : \mathcal{X}' \rightrightarrows \mathcal{Y}$ with the same inputs and outputs as P. We denote the *nth jump of* P, i.e. the problem obtained from P by applying to it n jumps, by $\mathsf{P}^{(n)}$.

In essence, the jump P' of problem P is the same problem as P if we forget about the fact that we are dealing with represented spaces: the thing that differentiates P' from P is that, for the former, the *names* of the points in the domain are given in a much more complicated way than for the latter.

There are many analogies between the jump operator we just introduced and the “standard” jump of a set in classical computability theory: we refer to [5] for more on this topic. There are, however, many respects in which they do not behave similarly at all: just to mention one, the jump operator is not Weihrauch-degree theoretic, whereas the Turing jump is of course Turing-degree theoretic.

In the following Theorem, we will list the main features of the jump operator that we will use in the rest of the thesis. Proofs for them can be found in [6] and in [5].

Theorem 1.2.17.

1. *For every two problems* P *and* Q *such that* $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$*, it holds that* $\mathsf{P}' \leq_{\mathrm{sW}} \mathsf{Q}'$*. Hence, the jump is strong Weihrauch-degree theoretic.*
2. *For every problem* P*,* $\mathsf{P}' \leq_{\mathrm{W}} \mathsf{P} * \mathsf{lim}$*.*

3. *For every cylinder* P, P' *is a cylinder as well, and* $\mathsf{P}' \equiv_{\mathrm{W}} \mathsf{P} * \mathsf{lim}$.

4. *For every two problems* P *and* Q, *we have that* $(\mathsf{P}\times\mathsf{Q})' \equiv_{\mathrm{sW}} \mathsf{P}'\times\mathsf{Q}'$, $(\widehat{\mathsf{P}})' \equiv_{\mathrm{sW}} \widehat{(\mathsf{P}')}$, *and* $(\mathsf{P}^*)' \equiv_{\mathrm{sW}} (\mathsf{P}')^*$.

1.2.4. Other computability theoretic notions

In this subsection, we introduce several notions coming from classical computability theory that have been seen to be very useful tools in the study of the strength of combinatorial principles.

We start by recalling what low$_n$ sets and degrees are.

Definition 1.2.18. For every $n > 0$, we say that a set A (respectively, a degree $\mathbf{a}$) is *low$_n$* if it holds that $A^{(n)} \equiv_{\mathrm{T}} \emptyset^{(n)}$ (respectively, $\mathbf{a}^{(n)} \equiv_{\mathrm{T}} \emptyset^{(n)}$). Low$_1$ sets and degrees will be called simply *low*, for shortness.

An important property of low$_n$ degrees is that they behave very well under relativization: as one easily checks, a degree that is low$_n$ over a low$_n$ degree is simply low$_n$.

Next, we introduce PA degrees: these are a fundamental topic in computability theory, and the literature on them is vast. We refer, in particular, to [19] and to [65] for more on this topic.

Definition 1.2.19. Given two Turing degrees $\mathbf{a}$ and $\mathbf{b}$, we say that $\mathbf{b}$ *is PA over* $\mathbf{a}$ if every infinite subtree $T \subseteq 2^{<\omega}$ that is computable in $\mathbf{a}$ has a path $f \in [T]$ such that $f \leq_{\mathrm{T}} \mathbf{b}$.

There are many equivalent definitions of PA degrees. A particularly interesting one is the one that gives them their name: a degree is PA (over $\emptyset$) if is the degree of a complete consistent extension of Peano Arithmetic.

It is immediately clear why these degrees are interesting in reverse mathematics: every computable instance of WKL has a solution computable in a PA degree. This is particularly important when combined with the fundamental *Low Basis Theorem* of

Jockusch and Soare (see [42]), which says that there are low PA degrees: by an easy construction, one can use this fact to produce an ω-model of WKL_0 consisting only of low sets.

There is another very useful properties that makes PA degrees a preferred tool for constructions of sets whose jumps have to be controlled, as we will see in Chapter 4.

Lemma 1.2.20. *Let us fix some $n \in \omega$, and let $\mathbf{d}$ be a Turing degree PA over $\emptyset^{(n)}$. Let a certain enumeration of the Π^0_{n+1}-predicates of first-order arithmetic be given, say it is $\{\varphi_0, \varphi_1, \dots\}$, and let $\langle\cdot\rangle$ be a coding of all the finite sequences of numbers: say that for every x, $x = \langle i_0, \dots i_{n_x}\rangle$. Then, there is a partial function $d :\subseteq \omega \to \omega$ computable in $\mathbf{d}$ such that for every x, if at least one of the φ_{i_j} is true, for $j \le n_x$, then $\varphi_{d(x)}$ is true.*

Proof. This is an immediate generalization of [11, Lemma 4.2] □

PA degrees are very strongly related with another class of interesting Turing degrees, namely DNR degrees. We recall that by Φ_e we mean the eth Turing machine, according to some fixed effective enumeration of them.

Definition 1.2.21.

- Given a function $p \in \omega^\omega$, a function $f \in \omega^\omega$ is DNR *relative to p* if, for every $e \in \omega$, $f(e) \neq \Phi_e(p)(e)$. A degree is DNR over $\deg_{\mathrm{T}}(p)$ if it computes a DNR function relative to p.
- Given a function $p \in \omega^\omega$ and a number $k > 1$, we say that a function $f \in \omega^\omega$ is DNR_k *relative to p* if it is DNR relative to p and $\operatorname{ran} f \subseteq \{0, \dots, k-1\}$. A degree is DNR_k relative to $\deg_{\mathrm{T}}(p)$ if it computes a function that is DNR_k relative to p.

It is immediate to see that every DNR_k function or degree is also DNR, whereas one can show that there are DNR degrees that are not DNR_k for any k. Moreover, the following theorem holds:

Theorem 1.2.22 ([43])**.** *For every $k > 2$, a degree $\mathbf{a}$ is DNR_k (relative to $\emptyset$) if and only if it is PA (over $\emptyset$).*

DNR degrees are another useful benchmark for the strength of principles, thanks to their many connections to other well known sets of degrees. In keeping with this, we introduce the problem DNR, which we will use in Chapter 2.

Definition 1.2.23. DNR is the following partial multifunction:

- Input: any function $p \in \omega^\omega$.
- Output: a function $f \in \omega^\omega$ that is DNR relative to p.

Finally, we introduce an important computability theoretic property of problems, namely cone avoidance: this property is a fundamental tool in the study of the reverse mathematics of combinatorial principles (the original proof by Seetapun that $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{ACA}_0$ was actually a proof that RT^2_2 admits cone avoidance) and is a major current focus of the reverse mathematical community (for instance, the recent fundamental papers [58] and [10] on the strength of a large class of Ramseyan principles can be seen as a contribution to the study of cone avoidance). For the sake of readability, we will state it for problems whose domain and codomain is (a subset of) ω^ω: we will only discuss cone avoidance in this setting in the rest of the thesis.

Definition 1.2.24. A problem $\mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$ *admits cone avoidance* if, for every set $Z \subseteq \omega$, every set $C \not\leq_{\mathrm{T}} Z$ and every Z-computable P-instance x, there is a P-solution y to x such that $C \not\leq_{\mathrm{T}} \langle Z, y\rangle$.

As hinted above, one of the main reasons why this property is of interest to reverse mathematicians is that, roughly speaking, for a Π^1_2 L_2-statement P, if the associated partial multifunction $\tilde{\mathsf{P}} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$ admits cone avoidance, then $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{ACA}_0$. To see this, it is enough to notice that, setting $Z = \emptyset$ and $C = \emptyset'$ in the Definition above, cone avoidance allows one to build an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ that does not contain any set that is Turing-equivalent to $\emptyset'$, which is enough to conclude that that model is not a model of ACA_0. We refer to [36] for more details on these kind of arguments.

Finally, we mention that it is also interesting to study a strengthening of the property above, unsurprisingly called *strong cone avoidance*, which is obtained from

Definition 1.2.24 by removing the condition that the P-instance x be Z-computable. Although we mention this property in Chapter 4, we will never actually use it.

2. Rival-Sands theorem for graphs

In their paper [59], Rival and Sands presented what may be called a rather unusual perspective on the celebrated Ramsey's theorem for pairs: they noticed that, when applied to an infinite graph $G = (V, E)$, Ramsey's theorem gives complete information on the *internal* structure of a certain subgraph H of G, but it provides no information on the *external* behavior of this subgraph, namely the relationship between points of H and points of $V \setminus H$. They then set off to amend this, and proved the following Theorem, which they themselves described as a trade-off:

Theorem 2.0.1 ([59], Theorem 1)**.** *Every infinite graph $G = (V, E)$ contains an infinite subset $H \subseteq V$ such that every vertex of G is adjacent to precisely none, one or infinitely many of the vertices of H. Moreover, every vertex of H is adjacent to none or infinitely many of the vertices of H.*

In essence, the Theorem above guarantees the existence of a subset H of V such that it is particularly nice with respect to both its internal and its external structure: in this sense, it can be considered a sort of “inside-outside Ramsey's theorem”. The price to pay for gaining information on the behavior of the points in $V \setminus H$ is that the internal structure of H will not be as regular as the one of the sets whose existence is guaranteed by Ramsey's theorem: in their paper, Rival and Sands show that not much can be done to strengthen the theorem above, and that it is, in a sense, optimal. They do, however, point out that by considering a more restrictive class of graph, namely comparability graphs of partial orders of finite width, then the Theorem above can take a much nicer form: this modification of Theorem 2.0.1 will be the main focus of Chapter 3.

In this Chapter, we focus on the logical strength of Theorem 2.0.1, restricted to countable graphs: we will call it *Rival–Sands theorem for graphs*. In our exposition, we closely follow our paper [27]: we point out that, as in that paper, the content of this Chapter is joint work with Dr. Marta Fiori Carones and Dr. Paul Shafer.

In Section 2.1, we focus on the reverse mathematics of Theorem 2.0.1: when formalized as the principle RSg, the Theorem turns out to be equivalent to ACA_0. Interestingly, a rather natural modification of it, which we call wRSg, turns out to be equivalent to RT^2_2 over RCA_0: we present this result, which is joint work with Jeffry Hirst and Steffen Lempp.

We then set out to determine the position of the problems associated to RSg and wRSg in the Weihrauch lattice: in order to do that, in Section 2.2, we review some known facts about the relationships between problems associated to combinatorial principles, and we prove some new results. In Section 2.3, we state the main result concerning RSg, without proving it (a complete proof can be found in [27]). Finally, in Section 2.4, we focus on the behavior of the problems associated to wRSg and the closely related problem wRSgr in the Weihrauch lattice.

2.1. The reverse mathematics of RSg

We give our reverse mathematical analysis of the Rival–Sands theorem for graphs. As anticipated above, we show that the Rival–Sands theorem for graphs and its refined version are equivalent to ACA_0 over RCA_0 and that these equivalences remain valid when the theorem is restricted to locally finite graphs. We also show that the inside-only weak Rival–Sands theorem for graphs and its refined version are equivalent to RT^2_2 over RCA_0.

Definition 2.1.1.

- (RCA_0) Let $V \subseteq \mathbb{N}$ be a set and E be a subset of $[V]^2$. Then we say that (V, E) is a *graph*.
- (RCA_0) For a graph $G = (V, E)$ and an $x \in V$, $N(x) = \{y \in V : \{x, y\} \in E\}$ denotes the set of *neighbors* of x.

We notice that, by our definition, our graphs will always be countable, undirected and without loops or multiedges.

We now formalize Theorem 2.0.1 in reverse mathematics: as will be apparent from the definition, using the notation of the Theorem, we find it interesting to analyze separately a version of it in which, in a certain sense, only the external structure of the subgraph H is considered, namely the principle RSg. In this sense, the second principle that we introduce, RSgr, is closer to the full statement of Theorem 2.0.1.

Definition 2.1.2.

- *The Rival–Sands theorem for graphs* (RSg) is the statement "for every infinite graph $G = (V, E)$, there is an infinite $H \subseteq V$ such that for every $x \in V$, either $H \cap N(x)$ is infinite or $|H \cap N(x)| \leq 1$".
- *The Rival–Sands theorems for graphs, refined* (RSgr) is the following statement: "for every infinite graph $G = (V, E)$, there is an infinite $H \subseteq V$ such that
  - for every $x \in V$, either $H \cap N(x)$ is infinite or $|H \cap N(x)| \leq 1$; and moreover
  - for every $x \in H$, either $H \cap N(x)$ is infinite or $H \cap N(x) = \emptyset$."

As we pointed out at the beginning of this Chapter, RSg and RSgr can be seen as a sort of a trade-off: we give up on some internal structure of the set H in order to gain information on the relationship between H and $V \setminus H$. But how much structure are we exactly giving up on? In order to try to answer this question, we introduce two new principles, wRSg and wRSgr: they are obtained by restricting the claim of RSg and RSgr, respectively, to just the set H.

Definition 2.1.3.

- *The weak Rival–Sands theorem for graphs* (wRSg) is the statement "for every infinite graph $G = (V, E)$, there is an infinite $H \subseteq V$ such that for every $x \in H$, either $H \cap N(x)$ is infinite or $|H \cap N(x)| \leq 1$".
- *The weak Rival–Sands theorem for graphs, refined* (wRSgr) is the following statement: "for every infinite graph $G = (V, E)$, there is an infinite $H \subseteq V$ such that for every $x \in H$, either $H \cap N(x)$ is infinite or $H \cap N(x) = \emptyset$."

We notice that it is immediately clear from the definitions above that

$$\mathsf{RCA}_0 \vdash (\mathsf{RSgr} \to \mathsf{RSg} \to \mathsf{wRSg}) \wedge (\mathsf{RSgr} \to \mathsf{wRSgr} \to \mathsf{wRSg}).$$

We now start with the study of the reverse mathematical strength of these principles. We begin by putting an upper-bound on the strength of RSgr (and hence, all the other principles). The original proof of the Rival–Sands theorem in [59] involves detailed elementary reasoning that can be formalized in ACA_0 with a little engineering. We give a quick new proof using cohesive sets.

Theorem 2.1.4. $\mathsf{ACA}_0 \vdash \mathsf{RSgr}$

Proof. Let $G = (V, E)$ be an infinite graph. Let $F = \{x \in V : N(x) \text{ is finite}\}$, which may be defined in ACA_0. There are two cases, depending on whether or not F is finite. If F is finite, simply take

$$H = V \setminus \bigcup_{x \in F} N(x),$$

and observe that, by $\mathsf{B}\Sigma^0_2$ (which is implied by ACA_0), H contains almost every member of V. Consider an $x \in V$. If $x \in F$, then $H \cap N(x) = \emptyset$. If $x \notin F$, then $N(x)$ is infinite, so $H \cap N(x)$ is also infinite. So in this case, for every $x \in V$, either $H \cap N(x)$ is infinite or $H \cap N(x) = \emptyset$.

Suppose instead that F is infinite. Let p_F be the principal function of F, i.e. the function such that for every n, $p_F(n)$ is the nth element of F. Moreover, for every $x \in V$, let M_x be the set defined by $y \in M_x \leftrightarrow p_F(y) \in N(x)$. Let B be an infinite cohesive set for the sequence $\vec{M} = (M_x)_{x \in V}$, and let $C = p_F(B)$. Then, C is an infinite cohesive set for $(N(x))_{x \in V}$ and a subset of F.

As we work in ACA_0, we may define a function $f \colon V \to \{0, 1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } C \subseteq^* \overline{N(x)} \\ 1 & \text{if } C \subseteq^* N(x). \end{cases}$$

Define $H = \{x_0, x_1, \dots\} \subseteq C \subseteq F$ by the following procedure. Let x_0 be the first element of C. Suppose that $x_0 < x_1 < \cdots < x_n$ have been defined. Let $Y = \bigcup_{i \le n} N(x_i)$, which is finite because each x_i is in F. For each $y \in Y$, if $f(y) = 0$, then $C \subseteq^* \overline{N(y)}$; and if $f(y) = 1$, then $C \subseteq^* N(y)$. By $\mathsf{B}\Sigma^0_2$, which is a consequence of ACA_0, there is a bound b such that for all $y \in Y$ and all $z \in C$ with $z > b$, if $f(y) = 0$ then $z \in \overline{N(y)}$ and if $f(y) = 1$, then $z \in N(y)$. Thus choose x_{n+1} to be the first member of $C \setminus Y$ with $x_n < x_{n+1}$ and such that, for every $y \in Y$, if $f(y) = 0$, then $x_{n+1} \in \overline{N(y)}$; and if $f(y) = 1$, then $x_{n+1} \in N(y)$. This completes the construction.

To verify that H is an RSgr-solution to G, consider a $v \in V$. If $H \cap N(v) \neq \emptyset$, let m be least such that $x_m \in N(v)$ (and hence also least such that $v \in N(x_m)$). If $f(v) = 0$, then every x_n with $n > m$ is chosen from $\overline{N(v)}$, so $|H \cap N(v)| = 1$. If $f(v) = 1$, then every x_n with $n > m$ is chosen from $N(v)$, so $H \cap N(v)$ is infinite. Thus for every $v \in V$, either $H \cap N(v)$ is infinite or $|H \cap N(v)| \le 1$. Furthermore, if $v \in H$, then $H \cap N(v) = \emptyset$ because if $m < n$, then x_n is chosen from $\overline{N(x_m)}$. $\square$

Before giving the reversal for the Rival–Sands theorem, we observe that RCA_0 suffices to prove its refined version for highly recursive graphs.

Definition 2.1.5.
- (RCA_0) For a set $X \subseteq \mathbb{N}$, let $\mathcal{P}_{\mathrm{f}}(X)$ denote the set of (codes for) finite subsets of X.
- (RCA_0) A graph $G = (V, E)$ is *locally finite* if $N(x)$ is finite for each $x \in V$.
- (RCA_0) A graph $G = (V, E)$ is *highly recursive* if it is locally finite, and additionally there is a function $b \colon V \to \mathcal{P}_{\mathrm{f}}(V)$ such that $b(x) = N(x)$ for each $x \in V$.

Every highly recursive graph is locally finite by definition. That every locally finite graph is highly recursive requires ACA_0 in general.

Proposition 2.1.6. $\mathsf{RCA}_0 \vdash$ *The Rival–Sands theorem for highly recursive graphs, refined.*

Proof. Let $G = (V, E)$ be a highly recursive infinite graph, and let $b \colon V \to \mathcal{P}_{\mathrm{f}}(V)$ be such that $b(x) = N(x)$ for all $x \in V$. Define an infinite $H = \{x_0, x_1, \cdots\} \subseteq V$ with

$x_0 < x_1 < \cdots$ as follows. Let x_0 be the first member of V. Given $x_0 < x_1 < \cdots < x_n$, let Y be the finite set

$$Y = \{x_i : i \leq n\} \cup \bigcup_{i \leq n} b(x_i) \cup \bigcup_{\substack{i \leq n \\ y \in b(x_i)}} b(y)$$

consisting of all vertices that are of distance ≤ 2 from an x_i with $i \leq n$. Choose x_{n+1} to be the first member of $V \setminus Y$ with $x_n < x_{n+1}$. Then no two distinct members of H are of distance ≤ 2, so H is a RSgr-solution to G. □

Next, we determine the strength of RSg and RSgr.

Theorem 2.1.7. *The following are equivalent over* RCA_0*.*

1. ACA_0
2. RSg
3. RSgr
4. *The Rival–Sands theorem for locally finite graphs.*
5. *The Rival–Sands theorem for locally finite graphs, refined.*

Proof. Notice that (3) trivially implies (2), (4), and (5). Therefore (1) implies (2)–(5) by Theorem 2.1.4. Notice also that (2), (3), and (5) each trivially imply (4). Thus to finish the proof, it suffices to show that (4) implies (1).

By Lemma 1.1.10, it suffices to show that RSg for locally finite graphs implies that the ranges of injections exist. Thus let $f : \mathbb{N} \to \mathbb{N}$ be an injection. Let $G = (\mathbb{N}, E)$ be the graph where $E = \{(v, s) \in [\mathbb{N}]^2 : f(s) < f(v)\}$, which exists by Δ^0_1 comprehension. To see that G is locally finite, consider a $v \in \mathbb{N}$. The function f is injective, so there are only finitely many $s > v$ with $f(s) < f(v)$. Therefore there are only finitely many $s > v$ that are adjacent to v.

Apply RSg for locally finite graphs to G to get an infinite $H \subseteq \mathbb{N}$ such that $|H \cap N(x)| \leq 1$ for every $x \in H$. Enumerate H in increasing order as $x_0 < x_1 < x_2 < \cdots$. We show that, for any $n \in \mathbb{N}$, if $\exists s(f(s) = n)$, then $(\exists s \leq x_{n+1})(f(s) = n)$. Suppose that $f(s) = n$. Then s is adjacent to all but at most n of the vertices $v < s$. This is because if $v < s$, then $(v, s) \notin E$ if and only if $f(v) \leq f(s)$. The function f is an injection, so there are at most $n = f(s)$ many vertices $v < s$ with $f(v) \leq f(s)$. Thus there are at most n vertices $v < s$ to which s is not adjacent. At most one neighbor of s is in H, and therefore there are at most $n + 1$ many vertices in H that are $< s$. Thus $x_{n+1} \geq s$. Thus n is in the range of f if and only if $(\exists s \leq x_{n+1})(f(s) = n)$. So the range of f exists by Δ^0_1 comprehension. □

We finish this section by showing that both the weak Rival–Sands theorem and its refined version are equivalent to RT^2_2 over RCA_0. This was proved in collaboration with Jeffry Hirst and Steffen Lempp.This is a rather interesting result: it can be read as saying that, although it is true that the internal structure of the set H given by RSg is not combinatorially as nice as the one given by Ramsey's theorem, it does not lose anything from the point of view of coding power.

Theorem 2.1.8 (Fiori-Carones, Hirst, Lempp, Shafer, Soldà)**.** *The following are equivalent over* RCA_0.

1. RT^2_2

2. wRSg

3. wRSgr

Proof. For an infinite graph G, every RT^2_2-solution to G is also a wRSgr-solution to G, so (1) implies (3). Trivially (3) implies (2). It remains to show that (2) implies (1).

We show that $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{SRT}^2_2 \wedge \mathsf{ADS}$, from which it follows that $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{RT}^2_2$ by Theorem 1.1.23 item 1 and Theorem 1.1.26 item 2. We start by showing that $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{ADS}$.

Let $L = (\mathbb{N}, <_L)$ be an infinite linear order. Let $G = (\mathbb{N}, E)$ be the graph where $E = \{(x, y) \in [\mathbb{N}]^2 : x <_L y\}$. Let H be a wRSg-solution to G. Then for every $x \in H$, either $H \cap N(x)$ is infinite or $|H \cap N(x)| \leq 1$.

First suppose that $|H \cap N(x)| \leq 1$ for all $x \in H$. For $x \in H$, let $y_0, y_1 \in H$ be such that $x < y_0, y_1$. Then at most one of (x, y_0) and (x, y_1) is in E, so either $y_0 <_L x$ or $y_1 <_L x$. In particular, this implies that H has no $<_L$-minimum element. We can then define a descending sequence $x_0 >_L x_1 >_L x_2 >_L \cdots$ by choosing x_0 to be the first member of H and by choosing each x_{n+1} to be the first member of H that is $<_L$-below x_n.

Now suppose that $H \cap N(x)$ is infinite for some $x \in H$, but further suppose that $|H \cap N(y)| \leq 1$ for all but finitely many $y \in H \cap N(x)$. Let b be a bound such that $|H \cap N(y)| \leq 1$ whenever $y \in H \cap N(x)$ and $y > b$. Let $y_0 < y_1 < y_2 < \cdots$ enumerate in increasing $<$-order the elements of $H \cap N(x)$ that are $> b$. Then $y_0 >_L y_1 >_L y_2 >_L \cdots$ is a descending sequence in L. This is because if $y_n <_L y_{n+1}$ for some n, then $(y_n, y_{n+1}) \in E$, so both x and y_{n+1} are in $H \cap N(y_n)$, which is a contradiction.

Finally, suppose that there is an $x \in H$ with $H \cap N(x)$ infinite and furthermore that whenever $x \in H$ and $H \cap N(x)$ is infinite, then also $H \cap N(y)$ is infinite for infinitely many $y \in H \cap N(x)$. We define an ascending sequence $x_0 <_L x_1 < x_2 <_L \cdots$ where $x_n \in H$ and $H \cap N(x_n)$ is infinite for each n. Recall that for $x \in H$, $H \cap N(x)$ is infinite if and only if $|H \cap N(x)| \geq 2$ because H is a wRSg-solution to G. Let x_0 be any element of H with $|H \cap N(x)| \geq 2$. Given $x_n \in H$ with $|H \cap N(x_n)| \geq 2$, we know by assumption that there are infinitely many $y \in H \cap N(x_n)$ with $|H \cap N(y)| \geq 2$. Let $\langle y, w, z\rangle$ be the first (code for a) triple where $y \in H \cap N(x_n)$, $x_n < y$, $w \neq z$, and $w, z \in H \cap N(y)$. Then $x_n <_L y$ because $x_n < y$ and $(x_n, y) \in E$, so put $x_{n+1} = y$. This completes the proof of ADS.

Now we show that $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{SRT}^2_2$. Note that $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{B}\Sigma^0_2$. This is because $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{ADS}$ by the above argument and that $\mathsf{RCA}_0 + \mathsf{ADS} \vdash \mathsf{B}\Sigma^0_2$ by Theorem 1.1.26 item 2.

Let $c\colon [\mathbb{N}]^2 \to \mathbb{N}$ be a stable coloring, and let $G = (\mathbb{N}, E)$ be the graph where $E =$

$\{(x,y) \in [\mathbb{N}]^2 : c(x,y) = 1\}$. Let H be a wRSg-solution to G. Thus for $x \in H$, $H \cap N(x)$ is infinite if and only if $|H \cap N(x)| \geq 2$.

First suppose that there are only finitely many $x \in H$ with $H \cap N(x)$ infinite, and let b be a bound such that $|H \cap N(x)| \leq 1$ whenever $x \in H$ and $x > b$. Define a homogeneous set $K = \{x_0, x_1, \dots\}$ for c with color 0, where $x_n \in H$ and $x_n > b$ for each n. Let x_0 be the first member of H with $x_0 > b$. Given $b < x_0 < x_1 < \cdots < x_n$, choose x_{n+1} to be the first member of H with $x_{n+1} > x_n$ and $(\forall i \leq n)(x_{n+1} \notin N(x_i))$. Such an x_{n+1} exists because H is infinite, but $|H \cap \bigcup_{i \leq n} N(x_i)| \leq n+1$ since $|H \cap N(x_i)| \leq 1$ for each $i \leq n$. The set K is homogeneous because if $m < n$, then $(x_m, x_n) \notin E$, so $c(x_m, x_n) = 0$.

Now suppose that there are infinitely many $x \in H$ with $H \cap N(x)$ infinite. Define a homogeneous set $K = \{x_0, x_1, \dots\}$ for c with color 1, where $x_n \in H$ and $|H \cap N(x_n)| \geq 2$ for each n. Let x_0 be any element of H with $|H \cap N(x_0)| \geq 2$. Given $x_0 < x_1 < \cdots < x_n$, let $\langle y, w, z \rangle$ be the first (code for a) triple where $x_n < y$, $(\forall i \leq n)(y \in H \cap N(x_i))$, $w \neq z$, and $w, z \in H \cap N(y)$. Then put $x_{n+1} = y$. To see that such a triple exists, observe that $(\forall i \leq n)(\lim_s c(x_i, s) = 1)$ because c is stable and for each $i \leq n$, there are infinitely many s with $c(x_i, s) = 1$ because $N(x_i)$ is infinite. By $\mathsf{B}\Sigma^0_2$, there is a bound b such that $(\forall i \leq n)(\forall s > b)(c(x_i, s) = 1)$. We assume that there are infinitely many $y \in H$ with $H \cap N(y)$ infinite, so there is a desired $y \in H$ with $y > \max\{b, x_n\}$ and $|H \cap N(y)| \geq 2$. Such a y satisfies $(\forall i \leq n)(y \in H \cap N(x_i))$ because $y > b$, so $c(x_i, y) = 1$ for each $i \leq n$, which means that $(x_i, y) \in E$ for each $i \leq n$. The set K is homogeneous because if $m < n$, then $(x_m, x_n) \in E$, so $c(x_m, x_n) = 1$. This completes the proof of SRT^2_2. □

2.2. Combinatorial principles as partial multifunctions

Before moving to the study of RSg, wRSg and wRSgr in the Weihrauch degrees, we will introduce the problems corresponding to the combinatorial principles that we introduced in Section 1.1. Although many of the translations are trivial, others are rather interesting, in that they highlight the differences existing between the reverse mathematical and the Weihrauch theoretic measurement of the strength of a problem.

Most of the things that we are going to say are (implicitly or explicitly) already known, with one exception: the relationship between Ramsey's theorem for singletons and ADS does not seem to have been studied before. We will give some new results about this at the end of this section.

We have already introduced the problems corresponding to RT^n_k and SRT^2_k in Section 1.2.2, and we have seen how they behave in the various degrees in the case that $n = k = 2$.

The next problem to consider is then COH, which we have already introduced: although there are explicit results relating COH, RT^2_2 and SRT^2_2 (see e.g. [8]), we will get these results as consequences of the relationship of COH with other principles.

We introduce now the problems corresponding to ADS, SADS and CAC.

Definition 2.2.1.
- ADS is the following multivalued function.
  - Input: An infinite linear order $L = (L, <_L)$.
  - Output: An infinite $S \subseteq L$ that is either an ascending sequence in L or a descending sequence in L.
- SADS is the following multivalued function:
  - Input: an infinite stable linear order $L = (L, <_P)$.
  - Output: an infinite set $S \subseteq L$ that is either an ascending sequence in L or a descending sequence in L.
- CAC is the following multivalued function:
  - Input: an infinite partial order $P = (P, <_P)$.
  - Output: an infinite set $S \subseteq P$ that is either an antichain or a chain in P.

As noticed in [1], there is actually another possible way to translate ADS and SADS.

Definition 2.2.2. (RCA_0) Let $(L, <_L)$ be a linear order.

- A set $C \subseteq L$ is an *ascending chain* in L if for every $y \in C$, the set $\{x \in C : x <_L y\}$ is finite.
- A set $C \subseteq L$ is a *descending chain* in L if for every $y \in C$, the set $\{x \in C : x >_L y\}$ is finite.

One could then, as was done in [1], consider the principle ADC, where one only requires, upon being given an infinite linear order, that the solution be an infinite ascending *chain*, and similarly for SADC.

Definition 2.2.3.
- ADC (for the *ascending/descending chain principle*) is the following multivalued function.
  - Input: An infinite linear order $L = (L, <_L)$.
  - Output: An infinite $S \subseteq L$ that is either an ascending chain in L or a descending chain in L.
- SADC (for the *stable ascending/descending chain principle*) is the following multivalued function.
  - Input: An infinite stable linear order $L = (L, <_L)$.
  - Output: An infinite $C \subseteq L$ that is either an ascending chain in L or a descending chain in L.

As it is easy to see, $\mathsf{RCA}_0 \vdash \mathsf{ADS} \leftrightarrow \mathsf{ADC}$ and $\mathsf{RCA}_0 \vdash \mathsf{SADS} \leftrightarrow \mathsf{SADC}$, and it is also easy to prove that $\mathsf{ADS} \equiv_{\mathrm{sc}} \mathsf{ADC}$ and $\mathsf{SADS} \equiv_{\mathrm{sc}} \mathsf{SADC}$. On the other hand, it was proved in [1] that $\mathsf{ADC} <_{\mathrm{W}} \mathsf{ADS}$ and $\mathsf{SADC} <_{\mathrm{W}} \mathsf{SADS}$ (technically, what they showed is a slightly different result, as we will see in a second, but the proof can be easily adapted to the case at hand): the issue is that it is impossible to know in a uniform way whether a ADC-solution S to a linear order L can be refined to an ascending or a descending sequence.

Another thing that should be noticed is that the problems ADS, ADC and so on that we are defining here are not the same problems that are used in [1]: the difference is that, in their case, there is a further condition on the domain of the problems. We will only examine the case of ADS, ADC, SADS and SADC, but similar considerations can be applied to the other problems as well.

For $\mathsf{P} = \mathsf{ADS}, \mathsf{ADC}, \mathsf{SADS}, \mathsf{SADC}$, we define the problems $\mathsf{P}\restriction_{L=\omega}$ as follows:

- Input: a linear order $L = (\omega, <_L)$ such that $L \in \operatorname{dom} \mathsf{P}$.
- Output: a P-solution to L.

The problems defined in [1] (and used in other places in the literature) are those restricted to $L = \omega$ (and similarly for CAC). We will clarify now the relationship between these principles.

Lemma 2.2.4. *For $\mathsf{P} = \mathsf{ADS}, \mathsf{ADC}, \mathsf{SADS}, \mathsf{SADC}$, the following hold:*

1. $\mathsf{P} \equiv_{\mathrm{W}} \mathsf{P}\restriction_{L=\omega}$.
2. P *is a cylinder, and it is actually the cylindrification of* P.
3. $\mathsf{P} \not\leq_{\mathrm{sW}} \mathsf{RT}^2_2$, *but* $\mathsf{P} \leq_{\mathrm{W}} \mathsf{RT}^2_2$.

Item 1 is obvious, and Item 2 is also very easy to prove: the idea is that, without any restriction on the set L, we can use it to code arbitrarily large initial segments of itself in its points. We will see an example of this in Proposition 2.4.3.

The fact that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{RT}^2_2$ is obtained by inspecting the proofs given in reverse mathematics that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{P}$ (we refer to [1] for more details). An interesting way to approach the proof of the non-reduction in Item 3 is to introduce the concept of cardinality of a problem.

Definition 2.2.5 ([6])**.** Let $\mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$ be a partial multifunction. By $\#\mathsf{P}$ we denote the cardinal

$$\sup\{|M| : M \subseteq \omega^\omega \land \forall x, y \in M(\mathsf{P}(x) \cap \mathsf{P}(y) = \emptyset)\}$$

We call $\#\mathsf{P}$ the *cardinality* of P.

The next Lemma explains why this concept is interesting to us.

Lemma 2.2.6. *Let* $\mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$ *and* $\mathsf{Q} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$ *be partial multifunctions. Then, if* $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$*, it holds that* $\#\mathsf{P} \leq \#\mathsf{Q}$*.*

In particular, for every cylinder P, we have that $\mathsf{id} \leq_{\mathrm{sW}} \mathsf{P}$. But since clearly $\#\mathsf{id} = 2^{\aleph_0}$, it follows that $\#\mathsf{P}$ is necessarily $2^{\aleph_0}$ as well. But, as proved in [8], $\#\mathsf{RT}^2_2 = 1$: let f and g be two RT^2_2 instances, and let H_f be an infinite f-homogeneous set. Then, let us consider $g\restriction_{[H_f]^2} : [H_f]^2 \to 2$: every infinite $g\restriction_{[H_f]^2}$-homogeneous set is then an infinite homogeneous set for both f and g. This proves that $\#\mathsf{RT}^2_2 = 1$. Let us now go back to the case $\mathsf{P} = \mathsf{ADS}, \mathsf{ADC}, \mathsf{SADS}, \mathsf{SADC}$: since by Item 2 P is a cylinder, it follows that $\mathsf{P} \not\leq_{\mathrm{sW}} \mathsf{RT}^2_2$.

The results above can be summarized by saying that, in the Weihrauch degrees, one can ignore the difference between P and $\mathsf{P}\restriction_{L=\omega}$, whereas the situation is more complicated for the strong Weihrauch degrees. Since we will focus on the Weihrauch degrees, this will not be hugely important. Anyway, it is maybe noteworthy to notice that, as we will see, RSg represents an exception to this phenomenon.

At the end of this section, we include a picture, Figure 2.1, summarizing the relationships between the Weihrauch degrees relative to the problems we have seen so far. In order for the picture to be complete, we still need some results which do not seem to be explicitly found in the literature: we start with the first, which is the Weihrauch equivalence between COH and CADS.

Definition 2.2.7.

- CADS is the statement "for every infinite linear order $(L, <_L)$, there is an infinite subset $S \subseteq L$ such that $(S, <_L)$ is a stable linear order".
- CADS is the following partial multifunction:
  - Input: an infinite linear order $(L, <_L)$.
  - Output: an infinite set $S \subseteq L$ such that $(S, <_L)$ is stable.

- $\mathsf{CADS}{\restriction}_{L=\omega}$ is the following partial multifunction:
  - Input: an infinite linear order $L = (\omega, <_L)$.
  - Output: an infinite set $S \subseteq \omega$ such that $(S, <_L)$ is stable.

The reason why CADS was introduced in [39] was to give a stable-cohesive decomposition of ADS: it is obvious that $\mathsf{RCA}_0 \vdash \mathsf{ADS} \leftrightarrow (\mathsf{SADS} \wedge \mathsf{CADS})$. In [39, Proposition 2.9], it was proved that $\mathsf{RCA}_0 \vdash \mathsf{COH} \to \mathsf{CADS}$, and in [39, Proposition 4.4] it was shown that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \mathsf{CADS} \to \mathsf{COH}$. We show here that these proofs actually yield that $\mathsf{CADS}{\restriction}_{L=\omega} \equiv_{\mathrm{sW}} \mathsf{COH}$.

Proposition 2.2.8 (See [39], Propositions 2.9 and 4.4)**.** $\mathsf{CADS}{\restriction}_{L=\omega} \equiv_{\mathrm{sW}} \mathsf{COH}$. *Therefore* $\mathsf{CADS} \equiv_{\mathrm{sW}} \mathsf{id} \times \mathsf{CADS}{\restriction}_{L=\omega} \equiv_{\mathrm{sW}} \mathsf{id} \times \mathsf{COH}$.

Proof. We have that $\mathsf{CADS} \equiv_{\mathrm{sW}} \mathsf{id} \times \mathsf{CADS}{\restriction}_{L=\omega}$ by an argument analogous to the proof of Proposition 2.4.3 below. So it suffices to show that $\mathsf{CADS}{\restriction}_{L=\omega} \equiv_{\mathrm{sW}} \mathsf{COH}$.

For $\mathsf{CADS}{\restriction}_{L=\omega} \leq_{\mathrm{sW}} \mathsf{COH}$, given a linear order $L = (\omega, <_L)$, apply COH to the sequence $\vec{A} = (A_i)_{i\in\omega}$ where $A_i = \{n \in \omega : i <_L n\}$. Then any $\vec{A}$-cohesive set C is also a CADS-solution to L.

Hence, we just have to show that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{CADS}{\restriction}_{L=\omega}$. Let $\vec{A} = (A_i)_{i\in\omega}$ be a COH-instance. Define a functional $\Phi(\vec{A})$ computing a linear order $L = (\omega, <_L)$ as follows. Given x and y, define $x <_L y$ if and only if $(A_i(x) : i \leq x) <_{\mathrm{lex}} (A_i(y) : i \leq y)$, where $<_{\mathrm{lex}}$ denotes the lexicographic order on $2^{<\omega}$. Let C be a CADS-solution to L, and let Ψ be the identity functional. We claim that C is $\vec{A}$-cohesive and hence that Φ and Ψ witness that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{CADS}{\restriction}_{L=\omega}$.

To see that C is $\vec{A}$-cohesive, fix n and let $F_n = \{\sigma \in 2^{n+1} : (\exists x \in C)(\sigma \sqsubseteq (A_i(x) : i \leq x))\}$. Let $\sigma_0 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} \sigma_{k-1}$ list the elements of F_n in $<_{\mathrm{lex}}$-increasing order. For each $j < k$, let x_{σ_j} be the $<$-least element of C witnessing that $\sigma_j \in F_n$. Then $x_{\sigma_0} <_L \cdots <_L x_{\sigma_{k-1}}$. The order $(C, <_L)$ is stable, so in C exactly one interval $[-\infty, x_{\sigma_0}], [x_{\sigma_0}, x_{\sigma_1}], \ldots, [x_{\sigma_{k-2}}, x_{\sigma_{k-1}}], [x_{\sigma_{k-1}}, \infty]$ is infinite, where $[-\infty, a]$ and $[a, \infty]$ denote $\{x \in C : x <_L a\}$ and $\{x \in C : a <_L x\}$. If $[x_{\sigma_j}, x_{\sigma_{j+1}}]$ is infinite for some

$j < k-1$, then almost every $y \in C$ satisfies $\sigma_j \sqsubseteq (A_i(y) : i \leq y)$. In particular, $A_n(y) = \sigma_j(n)$ for almost every $y \in C$, so either $C \subseteq^* A_n$ or $C \subseteq^* \overline{A_n}$. Similarly, if $[-\infty, x_{\sigma_0}]$ is infinite then $A_n(y) = \sigma_0(n)$ for almost every $y \in C$; and if $[x_{\sigma_{k-1}}, \infty]$ is infinite, then $A_n(y) = \sigma_{k-1}(n)$ for almost every $y \in C$. Thus C is $\vec{A}$-cohesive. $\square$

Finally, we will focus on the relationship between ADS and $\mathsf{RT}^1_{<\infty}$. We introduce the problem $\mathsf{RT}^1_{<\infty}$, as well as other auxiliary problems that we will use in the proofs below.

Definition 2.2.9.

- $\mathsf{RT}^1_{<\infty}$ is the following partial multifunction:
  - Input: a function $f \in \omega^\omega$ with bounded range.
  - Output: an infinite f-homogeneous set.
- For every $k > 0$, cRT^1_k is the following problem:
  - Input: a function $f : \omega \to k$.
  - Output: an $i < k$ such that $f^{-1}(i)$ is infinite.

It is very easy to see that $\mathsf{RT}^1_k \equiv_{\mathrm{W}} \mathsf{cRT}^1_k$ and $\mathsf{RT}^1_k \not\equiv_{\mathrm{sW}} \mathsf{cRT}^1_k$ for every $k > 0$.

Contrary to what happens in the reverse mathematical setting, we will see that $\mathsf{RT}^1_{<\infty} \not\leq_{\mathrm{W}} \mathsf{ADS}$. We will do this by proving the stronger result that $\mathsf{RT}^1_5 \not\leq_{\mathrm{W}} \mathsf{ADS}$.

Theorem 2.2.10. $\mathsf{RT}^1_5 \not\leq_{\mathrm{W}} \mathsf{ADS}$. *Therefore* $\mathsf{RT}^1_{<\infty} \not\leq_{\mathrm{W}} \mathsf{ADS}$.

Proof. As we mentioned, $\mathsf{RT}^1_5 \equiv_{\mathrm{W}} \mathsf{cRT}^1_5$ and ADS is a cylinder, so it suffices to show that $\mathsf{cRT}^1_5 \not\leq_{\mathrm{sW}} \mathsf{ADS}$. Suppose for a contradiction that Φ and Ψ witness that $\mathsf{cRT}^1_5 \leq_{\mathrm{sW}} \mathsf{ADS}$. We compute a coloring $c\colon \omega \to 5$ such that the ADS-instance $\Phi(c)$ has a solution S for which $c^{-1}(\Psi(S))$ is finite, contradicting that Φ and Ψ witness that $\mathsf{cRT}^1_5 \leq_{\mathrm{sW}} \mathsf{ADS}$.

The computation of c proceeds in stages, where at stage $s+1$ we determine the value of $c(s)$. Thus we compute a sequence of strings $(c_s : s \in \omega)$, where $c_s \in 5^s$ and $c_s \sqsubseteq c_{s+1}$ for each s. The final coloring c is $c = \bigcup_{s \in \omega} c_s$.

For each s, let $L_s = \Phi(c_s)\restriction_s$ denote the partially-defined structure obtained by running $\Phi(c_s)(n)$ for s steps for each $n < s$. Write also $L_s = (L_s, <_{L_s})$. L_s is not necessarily a linear order, but it must be consistent with being a linear order because there are functions $c\colon \omega \to 5$ extending c_s.

For $\sigma \in 2^{<\omega}$, let $\mathrm{set}(\sigma) = \{n < |\sigma| : \sigma(n) = 1\}$ denote the finite set for which σ is a characteristic string.

The computation of c begins in phase I, and it may or may not eventually progress to phase II. The goal of phase I is to identify $s, m_* \in \omega$, $k_{\mathrm{asc}}, k_{\mathrm{dec}} < 5$, and $\sigma_*, \tau_* \in 2^{<\omega}$ such that

- $\mathrm{set}(\sigma_*)$ is an ascending sequence in L_s with $k_{\mathrm{asc}} = \Psi(\sigma_*)\downarrow$;
- $\mathrm{set}(\tau_*)$ is a descending sequence in L_s with $k_{\mathrm{dec}} = \Psi(\tau_*)\downarrow$;
- m_* is both the $<_{L_s}$-maximum element of $\mathrm{set}(\sigma_*)$ and the $<_{L_s}$-minimum element of $\mathrm{set}(\tau_*)$.

Once s, m_*, k_{asc}, k_{dec}, σ_*, and τ_* are found, the computation enters phase II and no longer uses colors k_{asc} and k_{dec}. The point is that, at the end of the construction, if $L = \Phi(c)$ has an ascending sequence above m_*, then it has an ascending sequence S with $\sigma_* \sqsubseteq S$ (by which we mean that, if χ_S is the characteristic function of S, then $\sigma_* \sqsubseteq \chi_S$) and hence with $\Psi(S) = k_{\mathrm{asc}}$. Similarly, if L has a descending sequence below m_*, then it has a descending sequence S with $\tau_* \sqsubseteq S$ and hence with $\Psi(S) = k_{\mathrm{dec}}$. In both cases, S is as desired because $c^{-1}(k_{\mathrm{asc}})$ and $c^{-1}(k_{\mathrm{dec}})$ are finite.

Computation in phase I proceeds as follows. We maintain sequences $\vec{\sigma} = (\langle \sigma_\ell, u_\ell, i_\ell \rangle : \ell < a)$ and $\vec{\tau} = (\langle \tau_\ell, d_\ell, j_\ell \rangle : \ell < b)$ satisfying the following properties at each stage s.

1. For each $\ell < a$, $\mathrm{set}(\sigma_\ell)$ is an ascending sequence in L_s, u_ℓ is the $<_{L_s}$-maximum element of $\mathrm{set}(\sigma_\ell)$, and $\Psi(\sigma_\ell) = i_\ell$.
2. For each $\ell < b$, $\mathrm{set}(\tau_\ell)$ is an descending sequence in L_s, d_ℓ is the $<_{L_s}$-minimum element of $\mathrm{set}(\tau_\ell)$, and $\Psi(\tau_\ell) = j_\ell$.

3. For each $\ell_0 < \ell_1 < a$, $u_{\ell_0} >_{L_s} u_{\ell_1}$.

4. For each $\ell_0 < \ell_1 < b$, $d_{\ell_0} <_{L_s} d_{\ell_1}$.

At stage 0, begin with $c_0 = \emptyset$, $\vec{\sigma} = \emptyset$, and $\vec{\tau} = \emptyset$. At stage $s+1$, let $c_{s+1}(s)$ be the least $i < 5$ that is neither i_{a-1} (if $a > 0$) nor j_{b-1} (if $b > 0$). Next, search for an $\eta \in 2^{<s}$ such that $\Psi(\eta)\downarrow$ and either

(i) $\operatorname{set}(\eta)$ is an ascending sequence in L_s with $<_{L_s}$-maximum element u, and $u <_{L_s} u_{a-1}$ if $a > 0$; or

(ii) $\operatorname{set}(\eta)$ is an descending sequence in L_s with $<_{L_s}$-minimum element d, and $d >_{L_s} d_{b-1}$ if $b > 0$.

If there is such an η, let η be the first one found. If η satisfies ((i)), let $\langle \sigma_a, u_a, i_a \rangle = \langle \eta, u, \Psi(\eta) \rangle$ and append this element to $\vec{\sigma}$. If η satisfies ((ii)), let $\langle \tau_b, d_b, j_b \rangle = \langle \eta, d, \Psi(\eta) \rangle$ and append this element to $\vec{\tau}$. If there is no such η, then do not update $\vec{\sigma}$ or $\vec{\tau}$.

Next, search for a $\theta \in 2^{<s}$ such that $\Psi(\theta)\downarrow$ and either

(a) $\operatorname{set}(\theta) \subseteq \{u_0, \ldots, u_{a-1}\}$ is a descending sequence in L_s or

(b) $\operatorname{set}(\theta) \subseteq \{d_0, \ldots, d_{b-1}\}$ is an ascending sequence in L_s.

If there is such a θ, let θ be the first one found. If θ satisfies ((a)), let u_ℓ be the $<_{L_s}$-minimum element of $\operatorname{set}(\theta)$, which is also the $<_{L_s}$-maximum element of σ_ℓ. Set $\sigma_* = \sigma_\ell$, $\tau_* = \theta$, $m_* = u_\ell$, $k_{\text{asc}} = i_\ell$, and $k_{\text{dec}} = \Psi(\theta)$. If θ satisfies ((b)), let d_ℓ be the $<_{L_s}$-maximum element of $\operatorname{set}(\theta)$, which is also the $<_{L_s}$-minimum element of τ_ℓ. Set $\sigma_* = \theta$, $\tau_* = \tau_\ell$, $m_* = d_\ell$, $k_{\text{asc}} = \Psi(\theta)$, and $k_{\text{dec}} = j_\ell$. Go to stage $s + 2$ and begin phase II. If there is no such θ, go to stage $s + 2$ and remain in phase I.

The phase II strategy is to reset $\vec{\sigma}$ and $\vec{\tau}$ to the σ_*, τ_*, m_*, k_{asc} and k_{dec} found at the end of phase I and then rerun a portion of the phase I strategy. Upon beginning phase II, reset $\vec{\sigma}$ and $\vec{\tau}$ to $\vec{\sigma} = \langle \sigma_0, u_0, i_0 \rangle = \langle \sigma_*, m_*, k_{\text{asc}} \rangle$ and $\vec{\tau} = \langle \tau_0, d_0, j_0 \rangle = \langle \tau_*, m_*, k_{\text{dec}} \rangle$.

Throughout phase II, $\vec{\sigma}$ and $\vec{\tau}$ satisfy the same items (1)–(4) from phase I. Computation in phase II proceeds as follows. At stage $s+1$, let $c_{s+1}(s)$ be the least $i < 5$ not in $\{k_{\text{asc}}, k_{\text{dec}}, i_{a-1}, j_{b-1}\}$. Next, as in phase I, search for an $\eta \in 2^{<s}$ with $\Psi(\eta)\downarrow$ that satisfies either ((i)) or ((ii)). If such an η is found, then update either $\vec{\sigma}$ or $\vec{\tau}$ as in phase I and go to stage $s+2$. If no such η is found, go to stage $s+2$ without updating $\vec{\sigma}$ or $\vec{\tau}$. This completes the computation.

Let $L = \Phi(c)$ and write $L = (L, <_L)$. We find an ADS-solution S to L such that $c^{-1}(\Psi(S))$ is finite, contradicting that Φ and Ψ witness that $\mathsf{cRT}^1_5 \leq_{\text{sW}} \mathsf{ADS}$.

First, suppose that the computation of c never leaves phase I. Then there must be a stage after which no further elements are appended to either $\vec{\sigma}$ or $\vec{\tau}$. This is because if, say, elements are appended to $\vec{\sigma}$ infinitely often, then $u_0 >_L u_1 >_L u_2 >_L \cdots$, which means that there is a descending sequence $D \subseteq \{u_\ell : \ell \in \omega\}$. This D is an ADS-solution to L, so $\Psi(D)\downarrow$. Let $\theta \subseteq D$ be long enough so that $\Psi(\theta)\downarrow$. This θ eventually satisfies item ((a)) of phase I, and the construction eventually finds θ. Thus the computation of c eventually enters phase II, contradicting the assumption that it never leaves phase I. So let s_0 be a stage after which no further elements are appended to $\vec{\sigma}$ or $\vec{\tau}$. Then a, b, i_{a-1} (if $a > 0$), and j_{b-1} (if $b > 0$) do not change after stage s_0, and for every $s > s_0$, $c(s)$ is the least $i < 5$ that is neither i_{a-1} (if $a > 0$) nor j_{b-1} (if $b > 0$). Let A be an ADS-solution to L, and assume that A is ascending (the descending case is symmetric). If $a = 0$ or if $x <_L u_{a-1}$ for all $x \in A$, then let $\eta \subseteq A$ be long enough so that $\Psi(\eta)\downarrow$. This η eventually satisfies item ((i)) of phase I, so the computation adds an element to $\vec{\sigma}$ at some stage after s_0, which is a contradiction. Therefore it must be that $a > 0$ and that $x \geq_L u_{a-1}$ for some $x \in A$. As A is ascending, this means that $x >_L u_{a-1}$ for almost every $x \in A$. Let $S = \text{set}(\sigma_{a-1}) \cup \{x \in A : (x > u_{a-1}) \wedge (x >_L u_{a-1})\}$. Then S is an ascending sequence in L. However, $\sigma_{a-1} \sqsubseteq S$, so $\Psi(S) = i_{a-1}$. We have that $c(s) \neq i_{a-1}$ for all $s > s_0$, so S is as desired.

Now, suppose that the computation of c eventually enters phase II at some stage s_0. Then $c(s)$ is neither k_{asc} nor k_{dec} for all $s > s_0$. Recall that $\vec{\sigma}$ and $\vec{\tau}$ are reset at the beginning of phase II. Suppose that elements are appended to $\vec{\sigma}$ infinitely often in phase II. Then $m_* = u_0 >_L u_1 >_L u_2 >_L \cdots$, so there is a descending sequence

$D \subseteq \{u_\ell : \ell \in \omega\}$. Recall that $\mathrm{set}(\tau_*)$ is a descending sequence with $\leq_L$-minimum element m_* and $\Psi(\tau_*) = k_{\mathrm{dec}}$. Let $S = \mathrm{set}(\tau_*) \cup \{x \in D : (x > m_*) \wedge (x <_L m_*)\}$. Then S is a descending sequence with $\tau_* \sqsubseteq S$. Therefore $\Psi(S) = k_{\mathrm{dec}}$. However, $c(s) \neq k_{\mathrm{dec}}$ for all $s > s_0$, so S is as desired. If instead elements are appended to $\vec{\tau}$ infinitely often in phase II, then a symmetric argument shows that there is an ascending sequence S with $\sigma_* \sqsubseteq S$ and therefore with $\Psi(S) = k_{\mathrm{asc}}$.

Finally, suppose that there is a stage $s_1 > s_0$ after which no further elements are appended to either $\vec{\sigma}$ or $\vec{\tau}$. We argue as in the case in which the computation of c never leaves phase I. Notice that a, b, i_{a-1}, and j_{b-1} do not change after stage s_1, and for every $s > s_1$, $c(s)$ is the least $i < 5$ that is not in $\{k_{\mathrm{asc}}, k_{\mathrm{dec}}, i_{a-1}, j_{b-1}\}$. Let A be an ADS-solution to L, and assume that A is ascending (the descending case is symmetric). If $x <_L u_{a-1}$ for all $x \in A$, then the computation must append an element to $\vec{\sigma}$ at some stage after s_1, which is a contradiction. Otherwise, $x >_L u_{a-1}$ for almost every $x \in A$. Let $S = \mathrm{set}(\sigma_{a-1}) \cup \{x \in A : (x > u_{a-1}) \wedge (x >_L u_{a-1})\}$. Then S is an ascending sequence in L with $\Psi(S) = i_{a-1}$, but $c(s) \neq i_{a-1}$ for all $s > s_1$. Thus S is as desired. □

The theorem above leaves open the question of what can be said about $\mathsf{RT}^1_k \leq_{\mathrm{W}} \mathsf{ADS}$ in the case that $k < 5$. We give a partial answer to this question.

We point out that in the following proof we will speak about order-types of orderings, in a rather liberal way, as is standard in classical mathematics: a linear order L has order-type ω if it is isomorphic to the order of the natural numbers, whereas is has order-type ω^* if it is isomorphic to the reversed order of the natural numbers. Finally, given two orders A and B, $A + B$ is the usual composition of orders such that every element of A is smaller than every element of B. For all of them, we do not give an explicit definition, since the one that we use is the standard one that can be found in most classical books on the subject (see e.g. [60]).

We will not always be able to be this easy-going: see Definition 3.1.6 for a definition of various order-types in RCA_0.

Theorem 2.2.11. *$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADC}$.*

Proof. $\mathsf{RT}^1_3 \equiv_{\mathrm{W}} \mathsf{cRT}^1_3$ and ADC is a cylinder, so it suffices to show that $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADC}$. Let c be a cRT^1_3-instance. Define a functional Φ, where $\Phi(c)$ computes a linear order $L = (\omega, <_L)$ as follows. The computation of L proceeds in stages, where at stage s the order $<_L$ is determined on $\{0, 1, \dots, s\}$. Throughout the computation, we maintain three sets $A_s, M_s, D_s \subseteq \{0, 1, \dots, s\}$, with $\max_{<_L}(A_s) <_L \min_{<_L}(M_s)$ and $\max_{<_L}(M_s) <_L \min_{<_L}(D_s)$, where $\min_{<_L}(X)$ and $\max_{<_L}(X)$ denote the minimum and maximum elements of the finite set X with respect to $<_L$. The sets A_s and D_s are used to build an ascending sequence and a descending sequence in L in order to achieve the following.

- If only two colors $i < j < 3$ occur in the range of c infinitely often, then L has order-type $\omega + k + \omega^*$ for some finite linear order k, with the ω-part of L corresponding to color i and the ω^*-part of L corresponding to color j.
- If only one color $i < 3$ occurs in the range of c infinitely often, then L has either order-type $\omega + k$ or order-type $k + \omega^*$ for some finite linear order k, with the ω-part or the ω^*-part of L corresponding to color i.

To monitor the last two colors seen up to s (or the only color seen so far, if c is constant up to s), let $t < s$ be greatest such that $c(t) \neq c(s)$, let $\mathsf{last}_s = \{c(t), c(s)\}$ if there is such a t, and otherwise let $\mathsf{last}_s = \{c(s)\}$. We assign the least color of last_s to A_s and the other color (if it exists) to D_s.

At stage 0, let $A_0 = \{0\}$, $M_0 = \emptyset$, and $D_0 = \emptyset$. Assign A_0 color $c(0)$ and assign D_0 no color. At stage $s+1$, first check if $\mathsf{last}_{s+1} = \mathsf{last}_s$. If $\mathsf{last}_{s+1} = \mathsf{last}_s$, then color $c(s+1)$ is assigned to either A_s or D_s. If $c(s+1)$ is assigned to A_s, then set $A_{s+1} = A_s \cup \{s+1\}$, $M_{s+1} = M_s$, and $D_{s+1} = D_s$. Extend $<_L$ so that $s+1$ is the $<_L$-maximum element of A_{s+1} and $<_L$-below all elements of M_{s+1} and D_{s+1}. If $c(s+1)$ is assigned to D_s, then set $A_{s+1} = A_s$, $M_{s+1} = M_s$, and $D_{s+1} = D_s \cup \{s+1\}$. Extend $<_L$ so that $s+1$ is the $<_L$-minimum element of D_{s+1} and $<_L$-above all elements of A_{s+1} and M_{s+1}. Assign A_{s+1} the same color as A_s, and assign D_{s+1} the same color as D_s. If $\mathsf{last}_{s+1} \neq \mathsf{last}_s$, then set $M_{s+1} = \{0, 1, \dots, s\}$. If $c(s+1)$ is the least color of last_{s+1}, then set $A_{s+1} = \{s+1\}$, set $D_{s+1} = \emptyset$, extend $<_L$ so that $s+1$ is the $<_L$-minimum element of $\{0, 1, \dots, s+1\}$, assign A_{s+1} color $c(s+1)$, and assign D_{s+1} the other color of last_{s+1}. If $c(s+1)$ is not the least color of last_{s+1}, then set $A_{s+1} = \emptyset$, set $D_{s+1} = \{s+1\}$, extend $<_L$ so that $s+1$ is the $<_L$-maximum element of $\{0, 1, \ldots, s+1\}$, assign D_{s+1} color $c(s+1)$, and assign A_{s+1} the other color of last_{s+1}. This completes the computation of L.

The linear order L is a valid ADC-instance, so let S be an ADC-solution to L. Define a functional $\Psi(\langle c, S\rangle)$ by finding the $<$-least element x_0 of S and outputting $\Psi(\langle c, S\rangle) = c(x_0)$. We show that $c(x_0)$ appears in the range of c infinitely often and therefore that $\Psi(\langle c, S\rangle)$ is a cRT^1_3-solution to c. Thus Φ and Ψ witness that $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADC}$.

If every color $i < 3$ appears in the range of c infinitely often, then $c(x_0)$ appears in the range of c infinitely often. Suppose that exactly two colors $i < j < 3$ appear in the range of c infinitely often. Then there is an s_0 such that $\mathsf{last}_s = \mathsf{last}_{s_0} = \{i, j\}$ for all $s \geq s_0$. In this case, each $s \geq s_0$ with $c(s) = i$ is added to A_s, and each $s \geq s_0$ with $c(s) = j$ is added to D_s. Thus L is a linear order of type $\omega + k + \omega^*$ with ω-part $A = \bigcup_{s \geq s_0} A_s$, ω^*-part $D = \bigcup_{s \geq s_0} D_s$, and k-part M_{s_0}. If S is an ascending chain, then it must be that $S \subseteq A$. We have that $c(x) = i$ for all $x \in A$. In particular, $c(x_0) = i$, which occurs in the range of c infinitely often. If S is a descending chain, then it must be that $S \subseteq D$. We have that $c(x) = j$ for all $x \in D$. Thus $c(x_0) = j$, which occurs in the range of c infinitely often.

Finally, suppose that exactly one color $i < 3$ appears in the range of c infinitely often. Then there is an s_0 such that $c(s) = i$ for all $s \geq s_0$ and hence is also such that $\mathsf{last}_s = \mathsf{last}_{s_0}$ for all $s \geq s_0$. If i is the least color of last_{s_0}, then s is added to A_s for all $s \geq s_0$, and L is a linear order of type $\omega + k$ with ω-part $A = \bigcup_{s \geq s_0} A_s$ and k-part $M_{s_0} \cup D_{s_0}$. It must therefore be that $S \subseteq A$. We have that $c(x) = i$ for all $x \in A$. Thus $c(x_0) = i$, which occurs in the range of c infinitely often. If instead i is not the least color of last_{s_0}, then s is added to D_s for all $s \geq s_0$, and L is a linear order of type $k + \omega^*$ with ω^*-part $D = \bigcup_{s \geq s_0} D_s$ and k-part $A_{s_0} \cup M_{s_0}$. It must therefore be that $S \subseteq D$. We have that $c(x) = i$ for all $x \in D$. Thus $c(x_0) = i$, which occurs in the range of c infinitely often. $\square$

As promised, we summarize the results of this section in Figure 2.1.

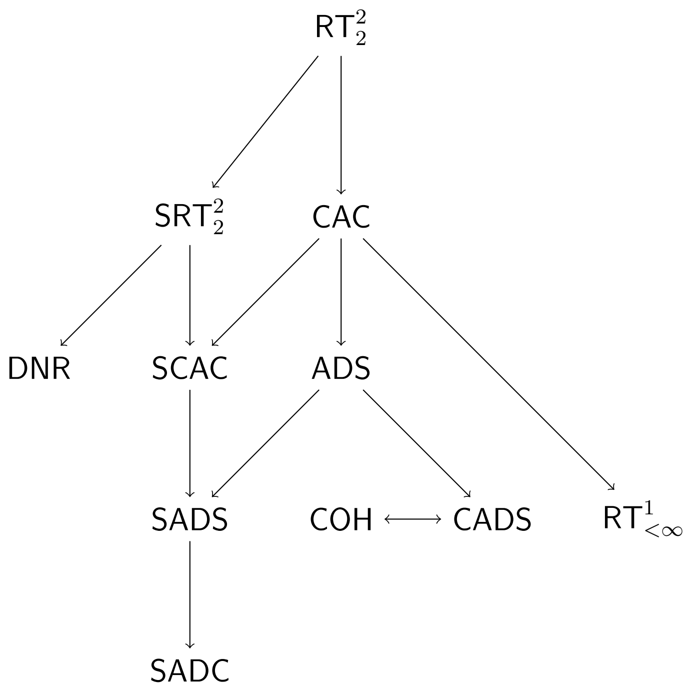

Figure 2.1: Weihrauch reductions and non-reductions in the neighborhood of RT^2_2. An arrow indicates that the target principle Weihrauch reduces to the source principle. No further arrows may be added, except those that may be inferred by following the arrows drawn. No arrows reverse, except the double arrow indicating that $\mathsf{COH} \equiv_{\mathrm{W}} \mathsf{CADS}$. The reductions and non-reductions (often in the form of ω-model separations) not proved here may be found in [1], [8], [22], [37], [38], [39], [48] and [57].

We conclude this section stating explicitly the remaining open question:

Question 2.2.12. Does $\mathsf{RT}^1_4 \leq_{\mathrm{W}} \mathsf{ADS}$ hold?

2.3. RSg in the Weihrauch lattice

We start by defining the problem RSg.

Definition 2.3.1. RSg is the following multivalued function.

- Input: An infinite graph $G = (V, E)$.

- Output: An infinite $H \subseteq V$ such that, for all $v \in V$, either $|H \cap N(v)| = \omega$ or $|H \cap N(v)| \leq 1$.

In the next Theorem, we state the main result concerning the problem RSg. For a proof and further discussions concerning this result, we refer to [27, Section 5].

Theorem 2.3.2 ([27], Corollary 5.12)**.** $\mathsf{WKL}'' \equiv_{\mathrm{sW}} \mathsf{RSg}$

The theorem above allows us to derive many interesting properties that RSg has. We list some of them here.

- A Turing degree $\mathbf{d}$ computes an RSg-solution to the graph (G, E) if and only if $\mathbf{d}$ has PA degree relative to $(G, E)''$: this is a straightforward consequence of a relativization of the Low Basis Theorem.

- RSg has a *universal instance*, i.e. there is a computable RSg-input (G^*, E^*) such that for every RSg-solution H^* to (G^*, E^*) and for every other computable RSg-instance (G, E), there is an RSg-solution H to (G, E) with $H \leq_{\mathrm{T}} H^*$. Again, this follows from known properties of WKL and its jumps.

- Since, by [8, Corollary 4.18], $\mathsf{WKL}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}^n_2}$, it follows that $\mathsf{RSg} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}^2_2}$.

- From the previous Item and the fact that the parallelization operator is idempotent, we have that $\mathsf{RSg} \equiv_{\mathrm{W}} \widehat{\mathsf{RSg}}$. Moreover, since both RSg and its parallelization are cylinders, it follows that $\mathsf{RSg} \equiv_{\mathrm{sW}} \widehat{\mathsf{RSg}}$.

We end this section by mentioning that one could define the problem RSgr analogously to what was done for RSg. As proved in [27], it turns out that $\mathsf{RSg} \equiv_{\mathrm{sW}} \mathsf{RSgr}$, so all the observations we made above extend to RSgr as well.

2.4. Weihrauch and computable reducibility of wRSg and wRSgr

In this section, we compare the Weihrauch degree and the computable degree of the weak Rival–Sands theorem to those of RT^2_2, its consequences, and other familiar benchmarks. The general theme is that although wRSg and RT^2_2 are equivalent over RCA_0, wRSg is much weaker than RT^2_2 in the Weihrauch degrees and in the computable degrees.

Multivalued functions corresponding to the weak Rival–Sands theorem and its refined version are defined as follows.

Definition 2.4.1.

- wRSg is the following multivalued function.
  - Input: An infinite graph $G = (V, E)$.
  - Output: An infinite $H \subseteq V$ such that, for all $v \in H$, either $|H \cap N(v)| = \omega$ or $|H \cap N(v)| \leq 1$.
- wRSgr is the following multivalued function.
  - Input: An infinite graph $G = (V, E)$.
  - Output: An infinite $H \subseteq V$ such that for all $v \in H$, either $|H \cap N(v)| = \omega$ or $|H \cap N(v)| = 0$.

We start noticing that, clearly, $\mathsf{wRSg} \leq_{\mathrm{sW}} \mathsf{wRSgr}$ holds, because given a graph G, every wRSgr-solution to G is also a wRSg-solution to G.

We do not know if this reduction reverses.

Question 2.4.2. Do $\mathsf{wRSgr} \leq_{\mathrm{W}} \mathsf{wRSg}$ or $\mathsf{wRSgr} \leq_{\mathrm{W}} \mathsf{wRSg}$ hold?

We do however show that, from a computable point of view, the two principles are not too different: in Proposition 2.4.5 below, we will prove that $\mathsf{wRSgr} \leq_{\mathrm{c}} \mathsf{wRSg}$.

As a first step in the study of wRSg and wRSgr, we want to determine whether they are cylinders: as we show below, the answer turns out to be affirmative in both cases. A deeper look at this question shows, however, an interesting difference between wRSg, wRSgr and RSg, namely a certain lack of robustness for the first two problems, similarly to what happened for the other principles we saw in Section 2.2: as we shall see, the fact that wRSg and wRSgr are cylinders strongly depends on the conditions one puts on the graph G one feeds them as an input, whereas this is not the case for RSg.

For $\mathsf{P} = \mathsf{wRSg}, \mathsf{wRSgr}, \mathsf{RSg}$, we define the problems $\mathsf{P}{\restriction}_{V=\omega}$ as follows:

- Input: a graph $G = (\omega, E)$ with $G \in \operatorname{dom} \mathsf{P}$.
- Output: a P-solution to G.

Although clearly $\mathsf{wRSg}{\restriction}_{V=\omega} \equiv_{\mathrm{W}} \mathsf{wRSg}$, we will see that $\mathsf{wRSg}{\restriction}_{V=\omega}$ and $\mathsf{wRSg}{\restriction}_{V=\omega}$ are not cylinders. This is in contrast to what happens for $\mathsf{RSg}{\restriction}_{V=\omega}$, which can be shown to be a cylinder.

Lemma 2.4.3.

1. $\mathsf{wRSg}{\restriction}_{V=\omega}$ *and* $\mathsf{wRSgr}{\restriction}_{V=\omega}$ *are not cylinders.*
2. $\mathsf{wRSg} \equiv_{\mathrm{sW}} \mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega}$ *and* $\mathsf{wRSgr} \equiv_{\mathrm{sW}} \mathsf{id} \times \mathsf{wRSgr}{\restriction}_{V=\omega}$, *so* wRSg *and* wRSgr *are cylinders.*
3. $\mathsf{RSg}{\restriction}_{V=\omega}$ *is a cylinder.*

Proof. We prove both items for wRSg. The proofs for wRSgr are analogous.

For item (1), it follows from the discussion following Lemma 2.2.6 that it suffices to prove that every pair of $\mathsf{wRSg}{\restriction}_{V=\omega}$-instances has a common solution: this implies that $\#\mathsf{wRSg} = 1$. It follows that $\mathsf{id} \not\leq_{\mathrm{sW}} \mathsf{wRSg}{\restriction}_{V=\omega}$, so $\mathsf{wRSg}{\restriction}_{V=\omega}$ is not a cylinder. Let $G_0 = (\omega, E_0)$ and $G_1 = (\omega, E_1)$ be two $\mathsf{wRSg}{\restriction}_{V=\omega}$-instances. Let H_0 be an infinite homogeneous set for G_0 (i.e., either an infinite clique or an infinite independent set). Let $G_1{\restriction}H_0 = (H_0, E_1 \cap [H_0]^2)$ be the subgraph of G_1 induced by H_0. Let H be an infinite homogeneous set for $G_1{\restriction}H_0$. Then H is homogeneous for both G_0 and G_1, so it is a $\mathsf{wRSg}{\restriction}_{V=\omega}$-solution to both G_0 and G_1.

For item (2), we first show that $\mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega} \leq_{\mathrm{sW}} \mathsf{wRSg}$. Let $p \in \omega^\omega$, and let $G = (\omega, E)$ be a $\mathsf{wRSg}{\restriction}_{V=\omega}$-instance. Let Φ be the functional given by $\Phi(\langle p, G\rangle) = \widehat{G} = (V, \widehat{E})$, where $V = \{p{\restriction}_n : n \in \omega\}$ and $\widehat{E} = \{(p{\restriction}_m, p{\restriction}_n) : (m,n) \in E\}$. Let $\widehat{H}$ be a wRSg-solution to $\widehat{G}$. Define a functional $\Psi(\widehat{H})$ computing a pair (q, H) as follows. To compute q, given n search for a $\sigma \in \widehat{H}$ with $|\sigma| > n$ and output $q(n) = \sigma(n)$. To compute H, take $H = \{n : q{\restriction}n \in \widehat{H}\}$. The set $\widehat{H}$ consists of infinitely many initial segments of p, so in fact we computed $q = p$ and $H = \{n : p{\restriction}n \in \widehat{H}\}$. Furthermore, H is a $\mathsf{wRSg}{\restriction}_{V=\omega}$-solution to G because the function $n \mapsto p{\restriction}n$ is an isomorphism between G and $\widehat{G}$. Thus Φ and Ψ witness that $\mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega} \leq_{\mathrm{sW}} \mathsf{wRSg}$.

Now we show that $\mathsf{wRSg} \leq_{\mathrm{sW}} \mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega}$. Let $G = (V, E)$ be a wRSg-instance. Let Φ be the functional given by $\Phi(G) = \langle p, \widehat{G}\rangle$, where $p\colon \omega \to V$ enumerates V in increasing order, and $\widehat{G} = (\omega, \widehat{E})$ is the graph with $\widehat{E} = \{(m,n) : (p(m), p(n)) \in E\}$. Then $\langle p, \widehat{G}\rangle$ is a $(\mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega})$-instance. Let $\langle p, \widehat{H}\rangle$ be a $(\mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega})$-solution. Define a functional $\Psi(\langle p, \widehat{H}\rangle)$ computing the set $H = \{v : p^{-1}(v) \in \widehat{H}\}$. Then H is a wRSg-solution to G because p is an isomorphism between $\widehat{G}$ and G. Thus Φ and Ψ witness that $\mathsf{wRSg} \leq_{\mathrm{sW}} \mathsf{id} \times \mathsf{wRSg}{\restriction}_{V=\omega}$.

Item (3) follows from a close inspection of the proof of Lemma 5.9 and Corollary 5.12 of [27], from which one can deduce that actually $\mathsf{RSg}{\restriction}_{V=\omega} \equiv_{\mathrm{sW}} \mathsf{RSg}$, and hence in particular that $\mathsf{RSg}{\restriction}_{V=\omega}$ is a cylinder. See also the remarks at the end of section 5 of the same paper. $\square$

We now turn to comparing wRSg and wRSgr to the Weihrauch and strong Weihrauch degrees of other problems of the zoo below RT^2_2. Many of the arguments in the rest of this section are based on the observations made in the following Lemma.

Lemma 2.4.4. *Let $G = (V, E)$ be an infinite graph.*

1. *If $K \subseteq V$ is an infinite set such that $|K \cap N(x)| < \omega$ for every $x \in K$, then $\langle G, K\rangle$ computes an infinite independent set $C \subseteq K$.*

2. *Let $F = \{x \in V : |N(x)| < \omega\}$.*

   (a) *If F is finite, then $V \setminus F \leq_{\mathrm{T}} G$ is a* wRSgr*-solution to G.*

   (b) *If F is infinite, then G has an infinite independent set $C \leq_{\mathrm{T}} G'$.*

3. *Assume that no $H \leq_{\mathrm{T}} G$ is a* wRSgr*-solution to G.*

   (a) *Then G has an infinite independent set.*

   (b) *Let D be a finite independent set, and let $\sigma \in 2^{<\omega}$ be a characteristic string of D: $|\sigma| > \max(D)$ and $(\forall n < |\sigma|)(\sigma(n) = 1 \leftrightarrow n \in D)$. Then σ extends to the characteristic function of a* wRSgr*-solution to G.*

Proof. (1): Suppose that K is infinite and that $|K \cap N(x)| < \omega$ for every $x \in K$. To compute an infinite independent set $C = \{x_0, x_1, \dots\} \subseteq K$ from $\langle G, K\rangle$, let x_0 be the first element of K, and let x_{n+1} be the first element of K that is $> x_n$ and not adjacent to any of $\{x_0, \dots, x_n\}$.

(2): Let $F = \{x \in V : |N(x)| < \omega\}$. If F is finite, then $I = V \setminus F$ is infinite, $I \leq_{\mathrm{T}} G$, and $|I \cap N(x)| = \omega$ for every $x \in I$. Thus $I \leq_{\mathrm{T}} G$ is a wRSgr-solution to G. Suppose instead that F is infinite. Then there is an infinite $F_0 \subseteq F$ with $F_0 \leq_{\mathrm{T}} G'$ because F is r.e. relative to G'. F_0 satisfies $|F_0 \cap N(x)| < \omega$ for every $x \in F_0$, so there is an infinite independent set $C \leq_{\mathrm{T}} \langle G, F_0\rangle \leq_{\mathrm{T}} G'$ by (1) with $K = F_0$.

(3): Assume that no $H \leq_{\mathrm{T}} G$ is a wRSgr-solution to G. For (3a), if G has no infinite independent set, then there would be a wRSgr-solution $H \leq_{\mathrm{T}} G$ by (2). For (3b), let $\sigma \in 2^{<\omega}$ be a characteristic string of a finite independent set D. Again, let $F = \{x \in V : |N(x)| < \omega\}$ and let $I = V \setminus F$. If I is finite, then F is infinite, $F \leq_{\mathrm{T}} G$, and, by definition, $|F \cap N(x)| < \omega$ for every $x \in F$. Thus by (1), there is an infinite independent $C \leq_{\mathrm{T}} \langle G, F\rangle \equiv_{\mathrm{T}} G$. This contradicts that no $H \leq_{\mathrm{T}} G$ is a wRSgr-solution to G. (In this case, one may alternatively show that σ extends to a wRSgr-solution to G.)

Now suppose that I is infinite. Further suppose that there is an $x \in I$ with $|I \cap N(x)| < \omega$. That is, x has infinitely many neighbors, but only finitely many neighbors of x

have infinitely many neighbors. In this case, let $K = N(x) \setminus I$. Then K is infinite and $|K \cap N(y)| < \omega$ for every $y \in K$. Furthermore, $K \leq_{\mathrm{T}} G$ because $|I \cap N(x)| < \omega$. Thus by (1), there is an infinite independent $C \leq_{\mathrm{T}} \langle G, K\rangle \leq_{\mathrm{T}} G$. This again contradicts that no $H \leq_{\mathrm{T}} G$ is a wRSgr-solution to G.

Finally, suppose that I is infinite and that $|I \cap N(x)| = \omega$ for every $x \in I$. Let n be greater than $|\sigma|$ and the maximum element of $\bigcup_{v \in D\cap F} N(v)$. Let $H = D \cup \{x \in I : x > n\}$. It is clear that $\sigma \subseteq H$. To see that H is a wRSgr-solution to G, consider a $v \in H$. Either $v \in D \cap F$ or $v \in I$. If $v \in D \cap F$, then $|D \cap N(v)| = 0$ because D is independent, and $|\{x \in I : x > n\} \cap N(v)| = 0$ by the choice of n. Hence $|H \cap N(v)| = 0$. If $v \in I$, then $|I \cap N(v)| = \omega$ by assumption, and therefore also $|\{x \in I : x > n\} \cap N(v)| = \omega$. So $|H \cap N(v)| = \omega$. Thus H is a wRSgr-solution to G. □

First, we show that $\mathsf{wRSgr} \leq_{\mathrm{c}} \mathsf{wRSg}$, as promised at the start of the section.

Lemma 2.4.5. $\mathsf{wRSgr} \leq_{\mathrm{c}} \mathsf{wRSg}$. *Hence* $\mathsf{wRSg} \equiv_{c} \mathsf{wRSgr}$.

Proof. Let $G = (V, E)$ be a wRSgr-instance. Then G is also a wRSg-instance, so let H be a wRSg-solution to G. We show that there is a wRSgr-solution $\widehat{H}$ to G with $\widehat{H} \leq_{\mathrm{T}} \langle G, H\rangle$.

Let $I = \{x \in H : |H \cap N(x)| = \omega\}$. Notice that also $I = \{x \in H : |H \cap N(x)| \geq 2\}$ because H is a wRSg-solution to G. Therefore I is r.e. relative to $\langle G, H\rangle$. Now consider three cases.

Case 1: The set I is finite. Let $K = H \setminus I$. Then K is infinite, $K \equiv_{\mathrm{T}} H$, and $|K \cap N(x)| < \omega$ for every $x \in K$. Thus by Lemma 2.4.4 item (1), there is an infinite independent $\widehat{H} \leq_{\mathrm{T}} \langle G, K\rangle \equiv_{\mathrm{T}} \langle G, H\rangle$, which is a wRSgr-solution to G.

Case 2: There is a $v \in I$ with $|I \cap N(v)| < \omega$. Let $K = (H \cap N(v)) \setminus I$. Then K is infinite and $K \leq_{\mathrm{T}} \langle G, H\rangle$ because $H \cap N(v)$ is infinite, $H \cap N(v) \leq_{\mathrm{T}} \langle G, H\rangle$, and $I \cap N(v)$ is finite. Furthermore, $|K \cap N(x)| < \omega$ for every $x \in K$. Thus by Lemma 2.4.4 item (1), there is an infinite independent $\widehat{H} \leq_{\mathrm{T}} \langle G, K\rangle \leq_{\mathrm{T}} \langle G, H\rangle$, which is a wRSgr-solution to G.

Case 3: I is infinite and $|I \cap N(v)| = \omega$ for every $v \in I$. In this case we compute a set $\widehat{H} \leq_{\mathrm{T}} \langle G, H \rangle$ with $\widehat{H} \subseteq I$ and $|\widehat{H} \cap N(x)| = \omega$ for each $x \in \widehat{H}$. This $\widehat{H}$ is thus a wRSgr-solution to G. To compute $\widehat{H} = \{x_0, x_1, \ldots\}$, let x_0 be the first element of I. To find x_{n+1}, decompose n as $n = \langle m, s \rangle$, search for a $y \in I \cap N(x_m)$ with $y > x_n$, and set $x_{n+1} = y$. Such a y exists because $x_m \in I$ and every element of I has infinitely many neighbors in I. The search for y can be done effectively relative to $\langle G, H \rangle$ because I is r.e. relative to $\langle G, H \rangle$. Finally, $|\widehat{H} \cap N(x)| = \omega$ for each $x \in \widehat{H}$ because x_{n+1} is adjacent to x_m whenever n is of the form $\langle m, s \rangle$. $\square$

We may situate wRSg in the computable degrees by combining Lemma 2.4.4 and the proof of Theorem 2.1.8 with established results concerning RT^2_2 and its consequences: this will be done in the following Proposition.

Proposition 2.4.6. *In the computable degrees,* wRSg *is*

- *strictly below* RT^2_2 *and* lim*;*
- *strictly above* ADS *and* SRT^2_2*;*
- *incomparable with* CAC*.*

Proof. Trivially $\mathsf{wRSg} \leq_{\mathrm{sW}} \mathsf{RT}^2_2$, hence $\mathsf{wRSg} \leq_{\mathrm{c}} \mathsf{RT}^2_2$. That $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{wRSg}$ is because every wRSg-instance G has a solution $H \leq_{\mathrm{T}} G'$ by Lemma 2.4.4 item (2), whereas by [41], Theorem 3.1 there are recursive RT^2_2-instances with no solution recursive in $\emptyset'$.

lim is strongly Weihrauch equivalent, hence computably equivalent, to the Turing jump function J. Every wRSg-instance G has a solution $H \leq_{\mathrm{T}} G'$ by Lemma 2.4.4 item (2), so $\mathsf{wRSg} \leq_{\mathrm{c}} \mathsf{lim}$. That $\mathsf{lim} \not\leq_{\mathrm{c}} \mathsf{wRSg}$ follows from the cone-avoidance result for RT^2_2: by [62], Theorem 2.1, every recursive infinite graph has a homogeneous set, hence wRSg-solution, that does not compute $\emptyset'$.

For $\mathsf{ADS} \leq_{\mathrm{c}} \mathsf{wRSg}$ and $\mathsf{SRT}^2_2 \leq_{\mathrm{c}} \mathsf{wRSg}$, see the proof of the $\mathsf{RCA}_0 \vdash \mathsf{wRSg} \rightarrow \mathsf{RT}^2_2$ direction of Theorem 2.1.8. The arguments showing that wRSg implies ADS and SRT^2_2 over RCA_0 describe computable reductions from ADS and SRT^2_2 to wRSg. For the non-reductions, by the results of [39], Section 2, there are ω-models of ADS that are not models of RT^2_2 and therefore not models of wRSg. Hence $\mathsf{wRSg} \nleq_{\mathrm{c}} \mathsf{ADS}$. By impressive recent work of Monin and Patey [53] , there are also ω-models of SRT^2_2 that are not models of RT^2_2 and therefore not models of wRSg. Hence $\mathsf{wRSg} \nleq_{\mathrm{c}} \mathsf{SRT}^2_2$.

That $\mathsf{CAC} \nleq_{\mathrm{c}} \mathsf{wRSg}$ is because again every wRSg-instance G has a solution $H \leq_{\mathrm{T}} G'$, whereas by [35], Theorem 3.1 there are recursive CAC-instances with no solution recursive in $\emptyset'$. That $\mathsf{wRSg} \nleq_{\mathrm{c}} \mathsf{CAC}$ follows from the fact that there are ω-models of CAC that are not models of RT^2_2 and therefore not models of wRSg, as shown in [39], Section 3. □

We remark that Proposition 2.4.6 implies that $\mathsf{COH} <_{\mathrm{c}} \mathsf{wRSg}$ as well because $\mathsf{COH} \leq_{\mathrm{c}} \mathsf{ADS}$ (by Proposition 2.2.8, for example).

We return to the Weihrauch degrees and first show that $\mathsf{SADC} \nleq_{\mathrm{W}} \mathsf{wRSgr}$. As SADC is below both ADS and SRT^2_2 in the Weihrauch degrees (see [1], for example), this implies that the computable reductions $\mathsf{ADS} <_{\mathrm{c}} \mathsf{wRSgr}$ and $\mathsf{SRT}^2_2 <_{\mathrm{c}} \mathsf{wRSgr}$ cannot be improved to Weihrauch reductions. We also show that $\mathsf{DNR} \nleq_{\mathrm{W}} \mathsf{wRSgr}$.

Theorem 2.4.7. *$\mathsf{SADC} \nleq_{\mathrm{W}} \mathsf{wRSgr}$.*

Proof. Suppose for a contradiction that $\mathsf{SADC} \leq_{\mathrm{W}} \mathsf{wRSgr}$ is witnessed by Turing functionals Φ and Ψ. By a well-known result independently of Tennenbaum and Denisov (see [60], Theorem 16.54, for example), there is a recursive linear order $L = (\omega, <_L)$ with $L \cong \omega + \omega^*$ that has no infinite recursive ascending or descending sequence. If $\ell \in L$ has finitely many $<_L$-predecessors, then say that ℓ is in the ω-part of L; and if ℓ has finitely many $<_L$-successors, then say that ℓ is in the ω^*-part of L. Notice that no infinite r.e. set is contained entirely in the ω-part of L, as such a set could be thinned to a recursive ascending sequence. Similarly, no infinite r.e. set is contained entirely in the ω^*-part of L.

The linear order L is a recursive SADC-instance, so $G = \Phi(L)$ is a recursive wRSgr-instance. Write $G = (V, E)$. G cannot have a recursive wRSgr-solution because if there were a recursive solution H to G, then $\Psi(\langle L, H\rangle)$ would be a recursive SADC-solution to L, which would be an infinite recursive set either entirely contained in the ω-part of

L or entirely contained in the ω^*-part of L. Therefore G has an infinite independent set C by Lemma 2.4.4 item (3a). This C is a wRSgr-solution to G, so $\Psi(\langle L, C\rangle)$ is a SADC-solution to L. In particular, $\Psi(\langle L, C\rangle)$ is infinite. Fix any $x \in \Psi(\langle L, C\rangle)$, and assume for the sake of argument that x is in the ω-part of L (the ω^*-part case is symmetric). Let R be the r.e. set

$$R = \{y : \text{there is a finite independent set } D \subseteq V \text{ with } x, y \in \Psi(\langle L, D\rangle)\}.$$

Notice that if $y \in \Psi(\langle L, C\rangle)$, then any sufficiently long initial segment D of C witnesses that $y \in R$. Thus $\Psi(\langle L, C\rangle) \subseteq R$. In particular, R is infinite. However, R is r.e., so it cannot be entirely contained in the ω-part of L. Therefore there must be a $y \in R$ that is in the ω^*-part of L. Let D be a finite independent set witnessing that $y \in R$. By Lemma 2.4.4 item (3b), the characteristic string of D extends to the characteristic function of a wRSgr-solution H to G. However, $x, y \in \Psi(\langle L, H\rangle)$, x is in the ω-part of L, and y is in the ω^*-part of L. Thus $\Psi(\langle L, H\rangle)$ can be neither an infinite ascending chain nor an infinite descending chain. Thus Φ and Ψ do not witness that $\mathsf{SADC} \leq_{\mathrm{W}} \mathsf{wRSgr}$, so $\mathsf{SADC} \nleq_{\mathrm{W}} \mathsf{wRSgr}$. □

Theorem 2.4.8. $\mathsf{DNR} \nleq_{\mathrm{W}} \mathsf{wRSgr}$.

Proof. The proof is similar to the proof of Theorem 2.4.7. Suppose for a contradiction that $\mathsf{DNR} \leq_{\mathrm{W}} \mathsf{wRSgr}$ is witnessed by Turing functionals Φ and Ψ. Let $p \colon \omega \to \omega$ be any recursive function. Then p is a recursive DNR-instance, so $G = \Phi(p)$ is a recursive wRSgr-instance. Write $G = (V, E)$. G cannot have a recursive wRSgr-solution because if there were a recursive solution H to G, then $\Psi(\langle p, H\rangle)$ would be a contradictory recursive DNR-solution to p. Thus G has an infinite independent set C by Lemma 2.4.4 item (3a). This C is a wRSgr-solution to G, so $\Psi(\langle p, C\rangle)$ is DNR relative to p.

Compute a function $g \colon \omega \to \omega$ as follows. On input e, $g(e)$ searches for a finite independent set $D \subseteq V$ such that $\Psi(\langle p, D\rangle)(e)\downarrow$ and outputs the value of $\Psi(\langle p, D\rangle)(e)$ for the first such D found. The function g is total because $\Psi(\langle p, C\rangle)$ is total: for any e, any sufficiently long initial segment D of C is a finite independent set for which $\Psi(\langle p, D\rangle)(e)\downarrow$. The function g is recursive, so it is not DNR relative to p. So there is an e such that $g(e) = \Phi_e(p)(e)$. By the definition of g, there is a finite independent set D such that $\Psi(\langle p, D\rangle)(e) = g(e) = \Phi_e(p)(e)$. By Lemma 2.4.4 item (3b), the characteristic string of D extends to the characteristic function of a wRSgr-solution H to G. Then $\Psi(\langle p, H\rangle)(e) = \Phi_e(p)(e)$, so $\Psi(\langle p, H\rangle)$ is not a DNR-solution to p. Thus Φ and Ψ do not witness that $\mathsf{DNR} \leq_{\mathrm{W}} \mathsf{wRSgr}$, so $\mathsf{DNR} \not\leq_{\mathrm{W}} \mathsf{wRSgr}$. ☐

Remark 2.4.9. We notice that the proofs of Theorems 2.4.7 and 2.4.8 are based on the same strategy: namely, we exploit the fact that wRSgr is not able to produce a non-computable solution and, at the same time, answer another question (what this question is depends on the nature of the non-reduction that is being proved). It is perhaps interesting to point out that a similar result holds in general for RT^2_2: in [23], it was proved that $\mathsf{LPO} \times \mathsf{NON} \not\leq_{\mathrm{W}} \mathsf{RT}^2_2$, where $\mathsf{NON} : \omega^\omega \rightrightarrows \omega^\omega$ is the problem such that, on input f, a g is output such that $g \not\leq_{\mathrm{T}} f$, and LPO, which stands for *limited principle of omniscience*, will be introduced below (see Definition 2.4.13).

On the positive side, we show that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{wRSg}$ and that $\mathsf{RT}^1_{<\infty} \leq_{\mathrm{sW}} \mathsf{wRSg}$.

Theorem 2.4.10. $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{wRSg}$.

Proof. It suffices to show that $\mathsf{CADS} \leq_{\mathrm{W}} \mathsf{wRSg}$ because $\mathsf{CADS} \equiv_{\mathrm{W}} \mathsf{COH}$ by Proposition 2.2.8 and because wRSg is a cylinder by Proposition 2.4.3.

Let $L = (L, <_L)$ be a CADS-instance. Define a functional $\Phi(L)$ computing the graph $G = (V, E)$ where $V = L$ and

$$E = \{(m, n) : (m, n \in V) \wedge (m < n) \wedge (m <_L n)\}.$$

The graph G is a valid wRSg-instance, so let H be a wRSg-solution to G. We define a functional $\Psi(\langle L, H\rangle)$ computing a set $C \subseteq L$ which will be a suborder of L either of type ω^*, of type $1 + \omega^*$, or of type $\omega + k$ for some finite linear order k.

Using Φ, we may compute $\Phi(L) = G$. Using G and H, we may enumerate the set $R = \{x \in H : |H \cap N(x)| \geq 2\}$. We claim that if $|R| \geq 2$, then every $x \in R$ has infinitely many $<_L$-successors in R. To see this, suppose that $|R| \geq 2$, let $x \in R$, and let $z \in R$ be different from x. Then $|H \cap N(x)| \geq 2$ and $|H \cap N(z)| \geq 2$. Therefore $|H \cap N(x)| = \omega$ and $|H \cap N(z)| = \omega$ because H is a wRSg-solution to G. Let w denote the $<_L$-maximum of x and z. Then any sufficiently large $y \in H \cap N(w)$ satisfies $y > x$, $y > z$, $y >_L x$, and $y >_L z$. Thus any such y is in R because $y \in H$ and $x, z \in H \cap N(y)$. Therefore there are infinitely many $y \in R$ with $y >_L x$.

To compute $C = \{x_0, x_1, \dots\}$, first enumerate H in $<$-increasing order as $h_0 < h_1 < h_2 < \cdots$. For each s, let $H_s = \{h_0, \dots, h_s\}$. Take $x_n = h_n$ until possibly reaching an s_0 for which there are distinct $u, v \in H_{s_0}$ with $|H_{s_0} \cap N(u)| \geq 2$ and $|H_{s_0} \cap N(v)| \geq 2$. If such an s_0 is reached, then H_{s_0} witnesses that $u, v \in R$. Thus R is infinite by the claim, so we may switch to computing an ascending sequence in R. Search for a $y \in R$ with $y > x_{s_0 - 1}$ and set $x_{s_0} = y$. Having determined x_s for some $s \geq s_0$, search for a $y \in R$ with $y > x_s$ and $y >_L x_s$, which exists by the claim, and set $x_{s+1} = y$.

We now show that C is a suborder of L either of type ω^*, of type $1 + \omega^*$, or of type $\omega + k$ for some finite linear order k. First suppose that there is an s_0 for which there are distinct $u, v \in H_{s_0}$ with $|H_{s_0} \cap N(u)| \geq 2$ and $|H_{s_0} \cap N(v)| \geq 2$. Then $\{x_n : n \geq s_0\}$ is an ascending sequence in L, so C is a suborder of L of type $\omega + k$ for some finite linear order k. If there is no such s_0, then $C = H$, which in this case is a suborder of L either of type ω^* or of type $1 + \omega^*$. To see this, suppose for a contradiction that there are $a < b$ such that both h_a and h_b have infinitely many $<_L$-successors in H. Then there are infinitely many n with $h_n >_L h_a, h_b$. In particular, there are $n > m > b$ with $h_m >_L h_a, h_b$ and $h_n >_L h_a, h_b$. But then $h_a, h_b \in H_n$; $h_a, h_b \in N(h_m)$; and $h_a, h_b \in N(h_n)$. So for $s_0 = n$ there are $u = h_a$ and $v = h_b$ with $|H_{s_0} \cap N(u)| \geq 2$ and $|H_{s_0} \cap N(v)| \geq 2$, contradicting that there is no such s_0. $\square$

The proof that $\mathsf{RT}^1_{<\infty} \leq_{\mathrm{sW}} \mathsf{wRSg}$ is similar to Hirst's proof that $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{RT}^1_{<\infty}$ from [40].

Proposition 2.4.11. $\mathsf{RT}^1_{<\infty} \leq_{\mathrm{sW}} \mathsf{wRSg}$.

Proof. It suffices to show that $\mathsf{RT}^1_{<\infty} \leq_{\mathrm{W}} \mathsf{wRSg}$ because wRSg is a cylinder by Proposition 2.4.3. Let c be an $\mathsf{RT}^1_{<\infty}$-instance. Define a functional $\Phi(c)$ computing the graph

$G = (\omega, E)$ where $E = \{(m, n) : c(m) = c(n)\}$. The graph G is a valid wRSg-instance, so let H be a wRSg-solution to G. Let $\Psi(\langle c, H\rangle)$ be a functional that computes $G = \Phi(c)$, searches for an $x \in H$ with $|H \cap N(x)| \geq 2$, and outputs the set $H \cap N(x)$ for the first such x found. There must be such an x because G is a disjoint union of finitely many complete graphs (depending on the size of the range of c), and thus H must have infinite intersection with one of these components. The set $H \cap N(x)$ is infinite because H is a wRSg-solution to G, and it is monochromatic because $c(y) = c(x)$ for all $y \in H \cap N(x)$. $\square$

We are ready to summarize the position of wRSg and wRSgr in the Weihrauch degrees. Notice that the uniform computational content of wRSg and wRSgr is considerably less than that of RT^2_2: RT^2_2 is above both DNR and SADC in the Weihrauch degrees, but wRSgr is above neither of these problems.

Theorem 2.4.12. *In the Weihrauch degrees,* wRSg *and* wRSgr *are*

- *strictly below* RT^2_2*;*
- *strictly above* COH *and* $\mathsf{RT}^1_{<\infty}$*;*
- *incomparable with* lim*,* SRT^2_2*,* SADC*, and* DNR*.*

Proof. Trivially $\mathsf{wRSgr} \leq_{\mathrm{sW}} \mathsf{RT}^2_2$. That $\mathsf{RT}^2_2 \nleq_{\mathrm{W}} \mathsf{wRSgr}$ follows from the stronger non-reduction $\mathsf{RT}^2_2 \nleq_{\mathrm{c}} \mathsf{wRSgr}$ of Proposition 2.4.6.

We have that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{wRSg}$ and that $\mathsf{RT}^1_{<\infty} \leq_{\mathrm{sW}} \mathsf{wRSg}$ by Theorem 2.4.10 and Proposition 2.4.11. These reductions are strict (indeed, the corresponding computable reductions are strict) because there are ω-models of COH that are not models of RT^2_2, hence not models of wRSg, by the results of [39], Section 2, for example; and because every recursive $\mathsf{RT}^1_{<\infty}$-instance has a recursive solution.

We now show the incomparabilities. Straightforward arguments show that $\mathsf{SADC} \leq_{\mathrm{sW}} \mathsf{SRT}^2_2$ and that $\mathsf{DNR} \leq_{\mathrm{sW}} \mathsf{lim}$, so it suffices to show that wRSgr is above neither SADC nor DNR and that wRSg is below neither SRT^2_2 nor lim. Theorems 2.4.7 and 2.4.8

$$\begin{tikzcd}
 & & \mathsf{RT}^2_2 \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & \mathsf{SRT}^2_2 \arrow[dl] \arrow[d] & \mathsf{CAC} \arrow[dl] \arrow[d] \arrow[ddrr, bend left] & \mathsf{wRSgr} \arrow[d] & \\
\mathsf{DNR} & \mathsf{SCAC} \arrow[d] & \mathsf{ADS} \arrow[dl] \arrow[dr] & \mathsf{wRSg} \arrow[d] \arrow[dr] & \\
 & \mathsf{SADS} \arrow[d] & \mathsf{COH} \arrow[r, leftrightarrow] & \mathsf{CADS} & \mathsf{RT}^1_{<\infty} \\
 & \mathsf{SADC} & & &
\end{tikzcd}$$

Figure 2.2: Weihrauch reductions and non-reductions in the neighborhood of RT^2_2, including wRSg and wRSgr. An arrow indicates that the target principle Weihrauch reduces to the source principle.

give $\mathsf{SADC} \not\leq_W \mathsf{wRSgr}$ and $\mathsf{DNR} \not\leq_W \mathsf{wRSgr}$. We have that $\mathsf{wRSg} \not\leq_W \mathsf{SRT}^2_2$ because $\mathsf{COH} \leq_{sW} \mathsf{wRSg}$ as mentioned above, but $\mathsf{COH} \not\leq_W \mathsf{SRT}^2_2$ by [22], Corollary 4.5. Finally, $\mathsf{wRSg} \not\leq_W \mathsf{lim}$ because $\mathsf{RT}^1_{<\infty} \leq_{sW} \mathsf{wRSg}$ as mentioned above, but $\mathsf{RT}^1_{<\infty} \not\leq_W \mathsf{lim}$ by [8], Corollary 4.20. □

From Proposition 2.4.11 and Theorem 2.4.12, one may deduce that wRSg and wRSgr are Weihrauch incomparable with a number of other principles, such as ADS, CAC, and its stable version SCAC. Figure 2.2 depicts the position of wRSg and wRSgr relative to a number of principles below RT^2_2 in the Weihrauch degrees.

As $\mathsf{RCA}_0 + \mathsf{wRSg} \vdash \mathsf{RT}^2_2$ but $\mathsf{RT}^2_2 \not\leq_W \mathsf{wRSg}$, it is natural to ask what must be added to wRSg to obtain RT^2_2. In particular, we ask how many applications of wRSg are necessary to obtain RT^2_2. Although we do not give an optimal answer, we give some sensible bounds for it.

We show that an application of two parallel instances of the *limited principle of omniscience* (LPO) suffices to overcome the non-uniformities in the proof that $\mathsf{SRT}^2_2 \leq_{\mathrm{c}} \mathsf{wRSg}$, yielding that $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg}$. In the case of wRSgr, one application of LPO suffices: $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} \mathsf{LPO} * \mathsf{wRSgr}$. As $\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{SRT}^2_2 * \mathsf{COH}$, we conclude that $\mathsf{RT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg} * \mathsf{COH}$. It follows that $\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{wRSg} * \mathsf{wRSg} * \mathsf{wRSg}$ because below we observe that $(\mathsf{LPO} \times \mathsf{LPO}) \leq_{\mathrm{W}} \mathsf{wRSg}$, and $\mathsf{COH} \leq_{\mathrm{W}} \mathsf{wRSg}$ by Theorem 2.4.10. Thus three applications of wRSg suffice to obtain RT^2_2. We do not know if two applications suffice.

A function corresponding to LPO is defined as follows.

Definition 2.4.13. LPO is the following function.

- Input: A function $p \in \omega^\omega$.
- Output: Output 0 if there is an n such that $p(n) = 0$. Output 1 if $p(n) \neq 0$ for every n.

We point out that in the following Theorem we do not strictly use the definition of $\mathsf{Q} * \mathsf{P}$ as given in Lemma 1.2.14, for the sake of readability: namely, instead of describing what the input (x, p) is for the compositional product, we simply describe the procedure that p encodes. The discussion before Lemma 1.2.14 ensures that this is a valid way of proceeding.

Theorem 2.4.14.

1. $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} \mathsf{LPO} * \mathsf{wRSgr}$.

2. $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg}$.

Proof. For (1), let $c\colon [\omega]^2 \to \{0,1\}$ be an SRT^2_2-instance. Using c, compute the graph $G = (\omega, E)$ with $E = \{(n,s) : (n < s) \wedge (c(n,s) = 1)\}$. Let H be a wRSgr-solution to G. We use an application of LPO to determine whether or not H contains two adjacent vertices. Using G and H, uniformly compute a function $p\colon \omega \to \{0,1\}$ by setting $p(n) = 0$ if any two of the least n elements of H are adjacent, and by setting $p(n) = 1$ otherwise. Let $b = \mathsf{LPO}(p)$. If $b = 1$, then H is an independent set and hence an SRT^2_2-solution to c. Thus output H. If $b = 0$, then H contains a pair of adjacent vertices. Notice that if $u \in H$ has a neighbor in H, then $H \cap N(u)$ is infinite because H is a wRSgr-solution to G. Furthermore, such a u is adjacent to almost every vertex in G because c is stable. Compute an infinite clique $K = \{x_0, x_1, \dots\}$ uniformly from G and H as follows. First, search for any $x_0 \in H$ with $|H \cap N(x_0)| \geq 1$. Having determined a finite clique $\{x_0, \dots, x_n\} \subseteq H$, search for the first vertex $x_{n+1} \in H$ that is adjacent to each x_i for $i \leq n$. Such an x_{n+1} exists because each x_i for $i \leq n$ is adjacent to almost every vertex of H. The resulting K is an infinite clique and hence an SRT^2_2-solution to c.

For (2), again let $c\colon [\omega]^2 \to \{0,1\}$ be an SRT^2_2-instance, and again compute the graph $G = (\omega, E)$ with $E = \{(n,s) : (n < s) \wedge (c(n,s) = 1)\}$. Let H be a wRSg-solution to G. Refine H to eliminate *bars*, i.e., pairs of vertices in H where each is the only vertex of H adjacent to the other. To do this, compute an infinite $\widehat{H} \subseteq H$ by skipping the first neighbor of each vertex already added to $\widehat{H}$. Enumerate H in increasing order as $h_0 < h_1 < h_2 < \cdots$. Let $\widehat{H}_0 = \{h_0\}$. Given $\widehat{H}_n$, consider h_{n+1}. If there is a $u \in \widehat{H}_n$ such that h_{n+1} is the least element of $H \cap N(u)$, then skip h_{n+1} by putting $\widehat{H}_{n+1} = \widehat{H}_n$. Otherwise, put $\widehat{H}_{n+1} = \widehat{H}_n \cup \{h_{n+1}\}$. Let $\widehat{H} = \bigcup_{n \in \omega} \widehat{H}_n$, which can be computed uniformly from G and H because at stage n we determine whether or not h_n is in $\widehat{H}$. If $x, y \in H$ are adjacent to each other but to no other vertices of H, then only $\min\{x,y\}$ is in $\widehat{H}$. If $x \in \widehat{H}$ has infinitely many neighbors in H, then it is adjacent to almost every vertex in G because c is stable, and therefore x also has infinitely many neighbors in $\widehat{H}$.

Call a clique of size three a *triangle*. The set $\widehat{H}$ is either an independent set, contains edges but no triangles, or contains triangles. Using G and $\widehat{H}$, uniformly compute two LPO-instances $p, q\colon \omega \to \{0,1\}$ to determine if $\widehat{H}$ contains edges or triangles. Set $p(n) = 0$ if any two of the least n elements of $\widehat{H}$ are adjacent, and set $p(n) = 1$ otherwise. Set $q(n) = 0$ if any three of the least n elements of $\widehat{H}$ form a triangle, and set $q(n) = 1$ otherwise. Let $(a,b) = (\mathsf{LPO} \times \mathsf{LPO})(p,q)$. If $(a,b) = (1,1)$, then $\widehat{H}$

contains no edges; if $(a,b) = (0,1)$, then $\widehat{H}$ contains edges but not triangles; and if $(a,b) = (0,0)$, then $\widehat{H}$ contains triangles. Output $(1,0)$ is not possible because if $\widehat{H}$ contains triangles, then it also contains edges.

If $\widehat{H}$ contains no edges, then it is an independent set and hence an SRT^2_2-solution to c. Thus output $\widehat{H}$.

Suppose that $\widehat{H}$ contains edges but not triangles, and suppose that $x, y \in \widehat{H}$ are adjacent. If neither x nor y has any other neighbors in H, then only one of them would be in $\widehat{H}$. Therefore either x or y has at least two, and therefore infinitely many, neighbors in H. So either x or y has infinitely many neighbors in $\widehat{H}$. Thus there is a $z \in \widehat{H}$ with $\widehat{H} \cap N(z)$ infinite. We can therefore compute an infinite independent set, hence an SRT^2_2-solution to c, uniformly from G and $\widehat{H}$ by searching for a $z \in \widehat{H}$ with $|\widehat{H} \cap N(z)| \geq 2$ and outputting $\widehat{H} \cap N(z)$. We have just seen that such a z exists. If $|\widehat{H} \cap N(z)| \geq 2$, then $|H \cap N(z)| \geq 2$, so $H \cap N(z)$ is infinite, so $\widehat{H} \cap N(z)$ is infinite. Finally, $\widehat{H} \cap N(z)$ is independent because $\widehat{H}$ contains no triangles.

If $\widehat{H}$ contains a triangle, then H contains a triangle, so there are distinct $x, y \in H$ with $|H \cap N(x)| \geq 2$ and $|H \cap N(y)| \geq 2$. Then $H \cap N(x)$ and $H \cap N(y)$ are both infinite because H is a wRSg-solution to G. The coloring c is stable, which means that x and y are adjacent to almost every vertex of G. Thus almost every vertex of H is adjacent to both x and y, and therefore is adjacent to almost every other vertex of H. Compute an infinite clique K as in (1), except this time start by searching for any distinct $x_0, x_1 \in H$ with $|H \cap N(x_0)| \geq 2$ and $|H \cap N(x_1)| \geq 2$. The resulting clique K is an SRT^2_2-solution to c. □

Corollary 2.4.15. $\mathsf{RT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg} * \mathsf{COH}$. *Therefore* $\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{wRSg} * \mathsf{wRSg} * \mathsf{wRSg}$.

Proof. We have that $\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{SRT}^2_2 * \mathsf{COH}$, and $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg}$ by Theorem 2.4.14. Therefore $\mathsf{RT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg} * \mathsf{COH}$. That $\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{wRSg} * \mathsf{wRSg} * \mathsf{wRSg}$ follows because $\mathsf{LPO} \times \mathsf{LPO} \leq_{\mathrm{W}} \mathsf{wRSg}$ and $\mathsf{COH} \leq_{\mathrm{W}} \mathsf{wRSg}$. Theorem 2.4.10 gives us $\mathsf{COH} \leq_{\mathrm{W}} \mathsf{wRSg}$. It is straightforward to show that $\mathsf{LPO} \leq_{\mathrm{W}}$

RT^1_2 and that $\mathsf{RT}^1_2 \times \mathsf{RT}^1_2 \leq_W \mathsf{RT}^1_4$ (see also [18], Proposition 2.1). Therefore

$$\mathsf{LPO} \times \mathsf{LPO} \leq_W \mathsf{RT}^1_2 \times \mathsf{RT}^1_2 \leq_W \mathsf{RT}^1_4 \leq_W \mathsf{RT}^1_{<\infty} \leq_W \mathsf{wRSg},$$

where the last reduction is by Proposition 2.4.11. □

Hence three applications of wRSg (or of wRSgr) suffice to obtain RT^2_2. We do not know if two applications suffice.

Question 2.4.16. Does $\mathsf{RT}^2_2 \leq_W \mathsf{wRSg} * \mathsf{wRSg}$ hold? Does $\mathsf{RT}^2_2 \leq_W \mathsf{wRSgr} * \mathsf{wRSgr}$ hold?

3. Rival-Sands theorem for partial orders

As we said in the introduction to Chapter 2, in their paper [59] Rival and Sands noticed that, by restricting to only considering comparability graphs relative to partial orders with finite width, Theorem 2.0.1 takes a nicer form. We will now explain what we mean by this. What Rival and Sands proved was the following result:

Theorem 3.0.1. *[[59]] Let $(P, <_P)$ be an infinite partial order of finite width. Then, there exists an infinite chain $C \subseteq P$ such that each element of P is comparable with none or with infinitely many elements of C.*

Moreover, if P is countable, C may be chosen so that every element of P is comparable with none or with cofinitely many elements of C.

The first part of the Theorem above can be recast in the language of comparability graphs as follows:

Theorem 3.0.2. *Let $(P, <_P)$ be an infinite partial order of finite width, and let G_P be its comparability graph, i.e. the graph $G_P = (P, E_P)$ such that for every $p, q \in P$, pE_Pq if and only if $p \between_P q$. Then, there is an infinite set $C \subseteq P$ such that C is a complete subgraph of G and for every point $p \in P$, p is adjacent to either none or infinitely many elements of C.*

It is evident that the set C we find in this case is an improved version of the set H provided by Theorem 2.0.1: we know everything about the internal structure of C

(since it is a complete graph), and also the behavior of points of $P \setminus C$ is tamer in this case.

Regardless of the reasons why Rival and Sands proved it, Theorem 3.0.1 is a combinatorial result of independent interest, and its proof has little in common with the proof of Theorem 2.0.1.

In this Chapter, we formalize and study Theorem 3.0.1 in reverse mathematics. As we will see in Section 3.1, there are several subtleties in the formalization of Theorem 3.0.1 to be consider: this gives rise to several reverse mathematical principles. As we will see, it is convenient to fix the width of the poset P we are working with: we will call $\mathsf{RSpo}_k^{\mathsf{CD}}$ and $\mathsf{RSpo}_k^{\mathsf{W}}$ the two formalizations of Theorem 3.0.1 relative to posets of width k that we will work with, and $\mathsf{RSpo}_{<\infty}^{\mathsf{CD}}$ and $\mathsf{RSpo}_{<\infty}^{\mathsf{W}}$ the generalization to posets of every (finite) width.

In Section 3.2, we analyze the original proof by Rival and Sands from a reverse mathematical perspective, and highlight that it requires the strong system Π_1^1-CA_0 to be carried out. In Section 3.3, we provide an easier, although still not optimal, proof of $\mathsf{RSpo}_{<\infty}^{\mathsf{W}}$ in ACA_0: its main merit is to be arguably rather easy to follow, and it introduces the main ideas that will be exploited in order to obtain the optimal proof.

In Section 3.4, we finally determine the strength of $\mathsf{RSpo}_{<\infty}^{\mathsf{CD}}$, of $\mathsf{RSpo}_{<\infty}^{\mathsf{W}}$ and of every $\mathsf{RSpo}_k^{\mathsf{CD}}$ and $\mathsf{RSpo}_k^{\mathsf{W}}$, with the noteworthy exception of $\mathsf{RSpo}_2^{\mathsf{CD}}$ and $\mathsf{RSpo}_2^{\mathsf{W}}$: we show that the former two principles are equivalent to $\mathsf{ADS}+\mathsf{I}\Sigma_2^0$, while the others are equivalent to ADS.

In Section 3.5, we focus on the case of $\mathsf{RSpo}_2^{\mathsf{CD}}$, and we show that it is strictly weaker than the other $\mathsf{RSpo}_k^{\mathsf{CD}}$: we manage to show that, over RCA_0, $\mathsf{RSpo}_2^{\mathsf{CD}}$ is equivalent to SADS.

In Section 3.6, we focus on two principles that are related but different to $\mathsf{RSpo}_k^{\mathsf{CD}}$ and $\mathsf{RSpo}_k^{\mathsf{W}}$: the first is $\mathsf{sRSpo}_2^{\mathsf{CD}}$, a principle obtained by putting more conditions on the solution set we claim exists; the second is $\mathsf{sRSpo}_{\mathbb{N}}$, which in a sense is an extension of Theorem 3.0.1 to posets with no infinite antichains.

Finally, in Section 3.7, we prove a small result on the extendibility of the results of Rival and Sands to posets of higher cardinalities.

We point out that the results of this Chapter are joint work with Marta Fiori Carones, Alberto Marcone and Paul Shafer, and many of them can be found in our paper [26].

3.1. From one principle to many

In this section, we start the study of Theorem 3.0.1 from the perspective of reverse mathematics. We point out that, as usual, this implies in particular that we will have to consider its restriction to countable posets. As we will see, one important aspect in this analysis is that the strength of the theorem strongly depends on how it is formalized in second order arithmetic.

The first element we focus on is what we exactly require of the solution chain C: it is clear that, in our case (i.e., when we only consider countable partial orders), Theorem 3.0.1 can actually be split into two statements, according to the properties we want C to satisfy. In analogy with the notion of homogeneous set used for Ramsey's theorem, we introduce the following definition.

Definition 3.1.1. (RCA_0) Let $(P, <_P)$ be a poset.

- A chain $C \subseteq P$ is a $(0, \infty)$-*homogeneous chain for* $(P, <_P)$ if each $p \in P$ is comparable to none of the elements of C or to infinitely many of them.
- A chain $C \subseteq P$ is a $(0, cof)$-*homogeneous chain for* $(P, <_P)$ if each $p \in P$ is comparable to none of the elements of C or to cofinitely many of them.

For example, the first half of Theorem 3.0.1 can thus be reformulated as the statement "for each infinite partial order $(P, <_P)$ of finite width, there exists an infinite $(0, \infty)$-homogeneous chain C".

The second element we focus on is the requirement about the width of the partial order $(P, <_P)$. Via Dilworth's Theorem, the width $w(P)$ gives us a very valuable piece of information about the partial order: it tells us that it can be decomposed into $w(P)$ many chains.

Unfortunately, as we have already stated in Theorem 1.1.8, Dilworth's Theorem is equivalent to WKL_0: in particular, it can be seen that there is a computable poset of width two that does not admit a computable decomposition into two chains (see [40]). Hence, we cannot use the Theorem freely while arguing in RCA_0.

It is then interesting to look for weaker versions of Dilworth's Theorem that are provable in RCA_0. With a different language, this was done by Kierstead in [44].

Kierstead was interested in extending the algorithmic or constructive content typical of finite combinatorics to countable structures, following the approach of what we would now call on-line combinatorics. His approach with respect to the non computability of solutions of Dilworth's theorem was thus to ask for a bound b such that each computable poset $(P, <_P)$ of width k can be decomposed into at most b computable chains. In [44] the bound b is set to $(5^k - 1)/4$ providing an on-line algorithm to decompose each poset of width k into $(5^k - 1)/4$ chains. The bound has recently been greatly improved in [2].

With the help of Keita Yokoyama, we noticed that Kierstead's proof can actually be formalized in RCA_0.

Theorem 3.1.2 (RCA_0)**.** *For each $k \in \mathbb{N}$ and each poset $(P, <_P)$ of width k, there are 5^k (disjoint) sets $P_0, \dots, P_{5^k-1}$ such that $P = \bigcup_{i<5^k} P_i$ and each P_i is a chain.*

Sketch of the proof. The main idea of the original proof is the following: let P be a given poset of width n, we start out by finding a maximal chain M in it (we will prove in Lemma 3.2.2 that this can be done in RCA_0). Then, using M as a sort of frame of reference, we can define an order $<^*$ on $P' := P \setminus M$, of which $<_P$ is a refinement, such that $(P', <^*)$ has width $n - 1$ or less. In the case where $n = 2$, a convoluted combinatorial argument shows then that every $<^*$-chain can be decomposed into at most 5 recursive $<_P$-chains.

If instead $n > 2$, by induction (of which the case $n = 2$ is the base case) we obtain that this poset can be partitioned into a certain number $f(n)$ of chains dependent on the width, thus obtaining the decomposition $P' = \bigcup_{i<f(n)} C_i$. But then, by applying the case $n = 2$ at most $f(n)$ times to all of the width 2 posets $M \cup C_i$, we obtain the relation $f(n+1) \leq 1 + 5f(n)$: noting that $f(1) = 1$, this relation easily yields that $f(n) \leq \frac{5^n - 1}{4}$, and so in particular $f(n) \leq 5^n$, a suboptimal result that we will use for notational convenience.

Although this is not particularly obvious, the proof above can be formalized in RCA_0: the point is that the construction that we described can be carried out without really using any induction by just building the various orders and chains as the construction proceeds. Namely, the construction can be seen as in terms of an array of size at most $\sum_{i<n+1} 5^i = \frac{5^{n+1}-1}{4}$, listing all of the orders involved that appear in the proof. The last 5^n component actually give the desired decomposition of P. □

The above theorem turned out to be very useful in the study of Theorem 3.0.1: in essentially all of our proofs, all we need is *any* decomposition of P into finitely many chains. In this sense, Dilworth's theorem provides us with too much information, i.e. it gives us an optimal decomposition of P. In the light of this fact, we give the following definition:

Definition 3.1.3. (RCA_0) Let $(P, \leq_P, C_0, \ldots, C_{k-1})$ be a sequence of sets such that $(P, \leq_P)$ is a poset, every C_i is a chain of P and $P = \bigcup_{i<k} C_i$. We say that $(P, \leq_P, C_0, \ldots, C_{k-1})$ has *chain-decomposition-number* k.

In what follows, we will essentially always abuse notation and simply say that P has chain-decomposition-number k: although this is technically wrong (for instance because the same P can have infinitely many chain-decomposition-numbers), the point is that we do not care about the actual decomposition into chains, as long as there is one with the stated number of elements.

Considering this, we can now formulate the different variations of Theorem 3.0.1 that we will consider in the rest of the chapter.

Definition 3.1.4. For every $k \in \mathbb{N}$, $k \neq 0$, we give the following definitions.

- $\mathsf{RSpo}^{\mathsf{W}}_k$ (for *Rival-Sands theorem for posets-Width*) is the statement "for every infinite partial order $(P, <_P)$ of width k there exists an infinite $(0, \infty)$-homogeneous chain C".
- $\mathsf{RSpo}^{\mathsf{W}}_{<\infty}$ stands for $\forall k \mathsf{RSpo}^{\mathsf{W}}_k$.
- $\mathsf{RSpo}^{\mathsf{CD}}_k$ (for *Rival-Sands theorem for posets-Chain Decomposition*) is the statement "for every infinite partial order $(P, <_P)$ with chain-decomposition-number k there exists an infinite $(0, \infty)$-homogeneous chain C".
- $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ stands for $\forall k \mathsf{RSpo}^{\mathsf{CD}}_k$.
- $\mathsf{sRSpo}^{\mathsf{W}}_k$ (for *strong* $\mathsf{RSpo}^{\mathsf{W}}$) is the statement "for every infinite partial order $(P, <_P)$ of width k there exists an infinite $(0, \mathrm{cof})$-homogeneous chain C".
- $\mathsf{sRSpo}^{\mathsf{W}}_{<\infty}$ stands for $\forall k \mathsf{sRSpo}^{\mathsf{W}}_k$.
- $\mathsf{sRSpo}^{\mathsf{CD}}_k$ (for *strong* $\mathsf{RSpo}^{\mathsf{CD}}$) is the statement "for every infinite partial order $(P, <_P)$ of chain-decomposition-number k there exists an infinite $(0, \mathrm{cof})$-homogeneous chain C".
- $\mathsf{sRSpo}^{\mathsf{CD}}_{<\infty}$ stands for $\forall k \mathsf{sRSpo}^{\mathsf{CD}}_k$.

We present some obvious relations between the principles we just introduced.

Lemma 3.1.5. *1.* $\mathsf{RCA}_0 \vdash \forall k(\mathsf{RSpo}^{\mathsf{CD}}_{5^k} \to \mathsf{RSpo}^{\mathsf{W}}_k \to \mathsf{RSpo}^{\mathsf{CD}}_k)$.

2. $\mathsf{RCA}_0 \vdash \mathsf{RSpo}^{\mathsf{W}}_{<\infty} \leftrightarrow \mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$.

3. $\mathsf{WKL} \vdash \forall k(\mathsf{RSpo}^{\mathsf{CD}}_k \leftrightarrow \mathsf{RSpo}^{\mathsf{W}}_k)$.

4. $\mathsf{RCA}_0 \vdash \forall k(\mathsf{sRSpo}^{\mathsf{CD}}_{5^k} \to \mathsf{sRSpo}^{\mathsf{W}}_k \to \mathsf{sRSpo}^{\mathsf{CD}}_k)$.

5. $\mathsf{RCA}_0 \vdash \mathsf{sRSpo}^{\mathsf{W}}_{<\infty} \leftrightarrow \mathsf{sRSpo}^{\mathsf{CD}}_{<\infty}$

6. $\mathsf{WKL} \vdash \forall k(\mathsf{sRSpo}^{\mathsf{CD}}_k \leftrightarrow \mathsf{sRSpo}^{\mathsf{W}}_k)$.

Proof. We will only prove the first three items, the other three are analogous.

Let us fix $k \in \mathbb{N}$. Since every poset of chain-decomposition-number k has width at most k, it follows that $\mathsf{RSpo}^{\mathsf{W}}_k \to \mathsf{RSpo}^{\mathsf{CD}}_k$. Moreover, by Theorem 3.1.2, every poset of width at most k has chain-decomposition-number at most 5^k. This ends the proof of Item 1.

Item 2 follows immediately from Item 1.

Finally, Item 3 follows from the fact that WKL_0 proves Dilworth's Theorem, hence over WKL_0 width and chain-decomposition-number coincide. □

We now make some observations about the *shape* of the solution to the principles above. These remarks are implicit in the original paper [59].

Definition 3.1.6. (RCA_0) Let $(P, <_P)$ be a poset, and let $C \subseteq P$ be a chain.

- We say that C has *order-type* ω if $(C, <_P)$ is an infinite ascending chain (see Definition 2.2.2).
- We say that C has *order-type* ω^* if $(C, <_P)$ is an infinite descending chain (see Definition 2.2.2).
- We say that C has *order-type* ζ if C is an infinite chain such that the following hold:
  - For every $p \in C$, there are $q_0, q_1 \in C$ such that $q_0 <_P p <_P q_1$.
  - For every $p, q \in C$ with $p <_P q$, the set $\{r \in C : p <_P r <_P q\}$ is finite.
- We say that C has *order-type* $\omega + \omega^*$ if C is an infinite chain such that the following hold:
  - Every element of C has either finitely many predecessors or finitely many successors in C.
  - There are infinitely many elements of C with finitely many predecessors in C and there are infinitely many elements of C with finitely many successors in C.

For a certain $c \in C$, we will say that *c is in the ω-part of C* if c has finitely many predecessors, and that *c is in the ω^*-part of C* if it has finitely many successors.

- We say that C has *order-type $\omega + \omega$* if it is the union of two infinite ascending chains C_0 and C_1 such that every element of C_0 is $<_P$-below every element of C_1.

As one can notice, in the definition above we simply recasts the usual definitions of chains of order-type ω, ω^*, $\omega + \omega^*$, $\omega + \omega$ and ζ in the language of second order arithmetic.

Remark 3.1.7. (RCA_0) Let $(P, <_P)$ be an infinite poset. Then the following hold:

1. Any chain $C \subseteq P$ of order-type ζ is $(0, \infty)$-homogeneous.
2. Any chain $C \subseteq P$ of order-type ω or ω^* that is $(0, \infty)$-homogeneous is also $(0, \mathrm{cof})$-homogeneous.

The proofs of both facts are obvious. Nevertheless, the relationship they provide between the shape of a chain and its $(0, \infty)$- and $(0, \mathrm{cof})$-homogeneity will play a rather important role in the following Sections.

We conclude this Section by presenting some other useful consequences of Theorem 3.1.2. If $(P, <_P)$ is an infinite poset of width (or height) k, then it surely contains an infinite chain (resp. antichain). One may wonder if these principles are *computably true*, i.e. if they hold in REC. The answer is positive and Theorem 3.1.2 allows to give a straightforward proof of this.

Hence, we introduce the following principles, that can be seen as weakenings of CAC.

Definition 3.1.8.
- For every $k \in \mathbb{N}$, CC_k is the principle "each infinite poset of width k has an infinite chain".
- $\mathsf{CC}_{<\infty}$ stands for $\forall k \mathsf{CC}_k$.

- For every $k \in \mathbb{N}$, CA_k is the statement "each infinite poset of height k has an infinite antichain".
- $\mathsf{CA}_{<\infty}$ stands for $\forall k \mathsf{CA}_k$.

We now determine the strengths of these principles.

Lemma 3.1.9 (RCA_0). *1. For every $k \in \omega$, $\mathsf{RCA}_0 \vdash \mathsf{CC}_k$.*

2. Over RCA_0, $\mathsf{B}\Sigma^0_2$ and $\mathsf{CC}_{<\infty}$ are equivalent.

3. For every $k \in \omega$, $\mathsf{RCA}_0 \vdash \mathsf{CA}_k$.

4. Over RCA_0, $\mathsf{B}\Sigma^0_2$ and $\mathsf{CA}_{<\infty}$ are equivalent.

Proof. Let a fixed standard k be given, and let $(P, <_P)$ be a partial order of width k. By Theorem 3.1.2, we can decompose P into at most 5^k chains. Since $k \in \omega$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_{5^k}$, and so at least one of the chains in the decomposition has to be infinite. This proves Item 1.

The proof of Item 2 is similar: for any $k \in \mathbb{N}$, given a poset $(P, <_P)$ of width k, Theorem 3.1.2 guarantees that there is a decomposition of P into at most 5^k chains. Since k is now no longer standard, we have to use $\mathsf{B}\Sigma^0_2$ in the form of $\mathsf{RT}^1_{<\infty}$ (see Theorem 1.1.20) to conclude that at least one of the chains is infinite. This proves that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \mathsf{CC}_{<\infty}$.

To see that $\mathsf{CC}_{<\infty}$ implies $\mathsf{B}\Sigma^0_2$, we prove that $\mathsf{RCA}_0 \vdash \mathsf{CC}_{<\infty} \to \mathsf{RT}^1_{<\infty}$. Let $f : \mathbb{N} \to k$ be a coloring, for some $k \in \mathbb{N}$. We define the poset $(P, <_P)$ setting $p <_P q$ whenever $c(p) = c(q)$ and $p < q$: $(P, <_P)$ has width at most k, hence by $\mathsf{CC}_{<\infty}$ it has an infinite chain, which we call C. By construction, C is an infinite f-homogeneous set. This proves Item 2.

Let $(P, <_P)$ be a poset of height k for some fixed $k \in \omega$. We define a coloring $c \colon \mathbb{N} \to k^2$ as follows: for each $n \in \mathbb{N}$, we let $c(n) = \langle |X_n|, |Y_n| \rangle$ where X_n is a chain of maximum length such that it only contains elements both strictly $<_{\mathbb{N}}$-below and strictly $<_P$-below n, and Y_n is a chain of maximum length only containing elements strictly $<_{\mathbb{N}}$-below and strictly $<_P$-above n. By $\mathsf{RT}^1_{k^2}$, which again is provable in RCA_0, we can find an infinite c-homogeneous set, say H. To show that it is an infinite antichain, we just have to show that no two elements of H are comparable: suppose $p, q \in H$ with $p <_{\mathbb{N}} q$. If it was the case that $p <_P q$, then $X_p \cup \{p\}$ would be a chain $<_{\mathbb{N}}$- and $<_P$-below q of length $|X_p| + 1 = |X_q| + 1$, contradicting the c-homogeneity of H. Similarly one can exclude that $q <_P q$. This proves Item 3.

Similarly to what happened for $\mathsf{CC}_{<\infty}$, we can adapt the proof of Item 3 to show that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ proves $\mathsf{CA}_{<\infty}$: one only has to substitute the application of $\mathsf{RT}^1_{k^2}$ for a standard k with $\mathsf{RT}^1_{<\infty}$.

To prove the reverse implication, we again prove that $\mathsf{RCA}_0 \vdash \mathsf{CA}_{<\infty} \rightarrow \mathsf{RT}^1_{<\infty}$. Fix $k \in \mathbb{N}$ and let $c \colon \mathbb{N} \rightarrow k$ be a coloring. We define a poset $(P, <_P)$ as follows: for every $p, q \in \mathbb{N}$, we let $p <_P q$ if and only if $c(p) < c(q)$. It is immediate to check that this is indeed a partial order and that it has height at most k. By $\mathsf{CA}_{<\infty}$, let A be an infinite antichain: one easily checks that A is an infinite c-homogeneous set. This proves Item 4. $\square$

We conclude this section by noticing that the Lemma above can be used to prove what could be considered an extended version of ADS.

Proposition 3.1.10. *The following are equivalent over* RCA_0*:*

1. ADS.
2. *The statement "for every $k \in \mathbb{N}$ and every poset $(P, <_P)$ of width k, P contains either an ascending or a descending sequence".*

Proof. 2 $\Rightarrow$ 1 follows from the fact that linear orders are partial orders of width 1.

Let $(P, <_P)$ be a partial order of width k, for some $k \in \mathbb{N}$. Since $\mathsf{RCA}_0 \vdash \mathsf{ADS} \rightarrow \mathsf{B}\Sigma^0_2$, we can apply $\mathsf{CC}_{<\infty}$ to get an infinite chain $C \subseteq P$. Then, we just have to apply ADS to C. This proves 1 $\Rightarrow$ 2. $\square$

3.2. A reverse mathematical analysis of the original proof

We give a brief analysis of the original proof by Rival and Sands for their result about partial orders. As we will see, the proof is, in a certain sense, suboptimal, in the sense that it seems to make essential use of principles that turn out to be equivalent to Π^1_1-CA_0. Nevertheless, the proof contains many ideas upon which we will expand in the following sections to find shorter and simpler proofs.

Sketch of the original proof of Theorem 3.0.1 in ZFC. Let Let $(P, <_P)$ be a countably infinite partial order of finite width k for some k. Suppose for a contradiction that $(P, <_P)$ contains no infinite $(0, \infty)$-homogeneous chains.

By Proposition 3.1.10, P contains either an infinite ascending sequence or an infinite descending sequence: we assume for simplicity that we are in the first case.

Then, we define a sequence $(S_i, C_i, D_i)_{i \le k+1}$ of triples of subsets of P as follows. Let $S_0 = P$, and let C_0 be a *chain of P that is $\subseteq$-maximal among the chain without a maximum*, and let D_0 be a cofinal ascending sequence in C_0. Suppose now that the triple of sets (S_i, C_i, D_i) is given, we let S_{i+1} be the set of elements of P that are above some elements of D_i and are incomparable with cofinitely many elements of D_i: that such a set is non-empty, and actually infinite, follows from our assumption that P has no infinite $(0, \infty)$-homogeneous chains. Then, we define C_{i+1} and D_{i+1} as in the case $i = 0$.

Using the maximality of the C_j's, one can show that $(\forall j \le i \le k+1)(D_i \subseteq S_j)$. This property allows us to choose an antichain $\{d_1, \dots, d_{k+1}\}$ with $d_i \in D_i$ for each $1 \le i \le k+1$, which contradicts that P has width k. □

Although a large portion of the proof is formalizable in ACA_0, there is one crucial bit that seems not to be, namely, the definition of the C_i: we will see that constructing chains of that kind is equivalent to Π^1_1-CA_0.

Definition 3.2.1. (RCA_0) Call a chain C in a partial order $(P, <_P)$ *max-less* if C has no maximum element: $(\forall x \in C)(\exists y \in C)(x <_P y)$. The *maximal max-less chain principle* (MMLC) is the statement "for every partial order $(P, <_P)$, there is a max-less chain that is $\subseteq$-maximal among the max-less chains of P". That is, there is a max-less chain $C \subseteq P$ for which $C \subseteq D \subseteq P$ implies $C = D$ for all max-less chains D of P. We call such a C a *maximal max-less chain* in P.

First, we give some results and definitions that will be useful in the proof that Π^1_1-CA_0 and MMLC are equivalent over RCA_0.

Lemma 3.2.2. RCA_0 *proves that in every partial order, there is a maximal chain and a maximal antichain.*

Proof. Let $(P, <_P)$ be a partial order. We find a maximal chain $D \subseteq P$. First, if there is a finite maximal chain $F \subseteq P$, then we may simply take $D = F$. So suppose that no finite chain is maximal. Define a $<_{\mathbb{N}}$-increasing sequence $(d_n)_{n \in \mathbb{N}}$ by taking d_0 to be the $<_{\mathbb{N}}$-least element of P, and, for each n, taking d_{n+1} to be the $<_{\mathbb{N}}$-least element p of $P \setminus \{d_0, \dots, d_n\}$ such that $(\forall i \leq n)(p \between_P d_i)$. Such a d_{n+1} always exists because $\{d_0, \dots, d_n\}$ is a finite chain and therefore is not maximal by assumption. It is easy to see that the sequence $(d_n)_{n \in \mathbb{N}}$ is $<_{\mathbb{N}}$-increasing, thus its range $D = \{d_n : n \in \mathbb{N}\}$ exists as a set. The set D is clearly a chain in P. Suppose for a contradiction that D is not maximal. Then there is an $x \in P \setminus D$ that is comparable with every $d \in D$. Let n be maximum such that $d_n <_{\mathbb{N}} x$. Then $x \leq_{\mathbb{N}} d_{n+1}$ and $(\forall i \leq n)(x \between_P d_i)$, so the construction must have chosen $d_{n+1} = x$. Thus $x \in D$, which is a contradiction, and therefore D is a maximal chain in P.

A similar argument with the roles of $<_P$-comparable and $<_P$-incomparable swapped produces a maximal antichain in P. $\square$

Definition 3.2.3.
- (ACA_0) Let $(P, <_P)$ be a partial order, and let $X \subseteq P$. The *downward closure of X in P*, denoted as $X \downarrow_{(P,<_P)}$, is the set $\{p \in P : \exists x \in X(p \leq_P x)\}$.

- (ACA_0) Let $(P, <_P)$ be a partial order, and let $X \subseteq P$. The *upward closure of X in P*, denoted as $X\uparrow_{(P,<_P)}$, is the set $\{p \in P : \exists x \in X(p \geq_P x)\}$.

- (RCA_0) The *Kleene–Brouwer ordering* of $\mathbb{N}^{<\mathbb{N}}$ is the binary relation $<_{\mathrm{KB}}$ on $\mathbb{N}^{<\mathbb{N}}$ such that $\tau <_{\mathrm{KB}} \sigma$ if either τ is a proper extension of σ or τ is to the left of σ. That is,

$$\tau <_{\mathrm{KB}} \sigma \leftrightarrow (\tau \sqsupset \sigma \vee \exists n < \min\{|\sigma|, |\tau|\}\, (\tau(n) < \sigma(n) \wedge (\forall i < n)(\sigma(i) = \tau(i)))).$$

We remark that, in the case $X = \{p\}$ is a singleton, we will abuse notation and indicate the downward and upward closure of $\{p\}$ as, respectively, $p\downarrow_{(P,<_P)}$ and $p\uparrow_{(P,<_P)}$.

We are now ready for the main result of this section.

Theorem 3.2.4. *The following are equivalent over* RCA_0*.*

1. Π^1_1-CA_0.
2. MMLC.
3. MMLC *restricted to linear orders.*

Proof. For 1 $\Rightarrow$ 2, let $(P, <_P)$ be a partial order. Let us consider the set $X = \{p \in P : p\uparrow_{(P,<_P)}$ is reverse ill-founded$\}$: it is easy to see that it is a Σ^1_1 subset of P, and hence we can form it using Π^1_1-CA_0 (see Theorem 1.1.13). We then apply Lemma 3.2.2 to $(X, <_P)$ to obtain a maximal chain C in the partial order $(X, <_P)$.

We first show that C is max-less. To see this, suppose for a contradiction that C has a maximum element m. Then $m \in C \subseteq X$, so $m\uparrow_{(P,<_P)}$ is reverse ill-founded (in P). Thus there is an ascending sequence $\{m <_P a_0 <_P a_1 <_P \cdots\}$ in P. Clearly, for every $i \in \mathbb{N}$, $a_i \in X$, as witnessed by the ascending sequence $\{a_{i+1} <_P a_{i+2} <_P \ldots\}$. Then $C \cup \{a_i : i \in \mathbb{N}\} \subseteq X$ is a chain properly extending C, contradicting that C is a maximal chain in X. Thus C is max-less.

We now show that C is maximal among the max-less chains of P. Suppose that $D \subseteq P$ is a max-less chain with $C \subseteq D$. Let $d \in D$. As D is max-less, we can recursively define an ascending sequence $\{d = d_0 <_P d_1 <_P \cdots\}$ of elements of D by taking d_{n+1} to be the $<_\mathbb{N}$-least element x of D with $d_n <_P x$: thus, $d \in X$. This shows that $D \subseteq X$. That is, C and D are chains in X with $C \subseteq D$. Therefore $C = D$ by the maximality of C in X. Thus C is a maximal max-less chain in P.

It is clear that 2 $\Rightarrow$ 3.

For 3 $\Rightarrow$ 1, we show that MMLC restricted to linear orders implies LPP, which is equivalent to Π^1_1-CA_0 by Theorem 1.1.13. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an ill-founded tree, and apply MMLC for min-less chains instead of max-less chains to the linear order $(T, <_{\mathrm{KB}})$ to obtain a maximal min-less chain C in $(T, <_{\mathrm{KB}})$. Observe that C is $<_{\mathrm{KB}}$-upward-closed, i.e. $C\uparrow_{(T,<_{\mathrm{KB}})} = C$. If $\sigma, \tau \in T$, $\sigma \in C$, and $\sigma <_{\mathrm{KB}} \tau$, then $C \cup \{\tau\}$ is a min-less chain, so it must be that $\tau \in C$ by the maximality of C.

In any linear order, it is easy to see that the union of two min-less chains is a min-less chain. The tree T is ill-founded by assumption, so T has an infinite path h. Then $\{h\restriction_0 >_{\mathrm{KB}} h\restriction_1 >_{\mathrm{KB}} h\restriction_2 >_{\mathrm{KB}} \cdots\}$ is a descending sequence, so $\{h\restriction_n : n \in \mathbb{N}\}$ is a min-less chain. Thus $C \cup \{h\restriction_n : n \in \mathbb{N}\}$ is a min-less chain as well, so $\{h\restriction_n : n \in \mathbb{N}\} \subseteq C$ by the maximality of C. Thus for every n, C contains a string of length n. Define a sequence $(\sigma_n : n \in \mathbb{N})$ by taking σ_n to be the $<_{\mathrm{KB}}$-least (i.e., leftmost) element of C of length n.

We claim that $\sigma_n \sqsubset \sigma_{n+1}$ for all n, which we prove using $\mathsf{I}\Sigma^0_1$. We have that $\sigma_0 = \emptyset$, so $\sigma_0 \sqsubset \sigma_1$. By induction, assume that $\sigma_0 \sqsubset \sigma_1 \sqsubset \cdots \sqsubset \sigma_n$. The chain C has no $<_{\mathrm{KB}}$-minimum element, so there is a $\tau \in C$ with $\tau <_{\mathrm{KB}} \sigma_n$. Let $k = |\tau|$. If $k \leq n$, then $\sigma_n \leq_{\mathrm{KB}} \sigma_k \leq_{\mathrm{KB}} \tau$, where $\sigma_n \leq_{\mathrm{KB}} \sigma_k$ because $\sigma_k \sqsubseteq \sigma_n$ and $\sigma_k \leq_{\mathrm{KB}} \tau$ because σ_k is the $<_{\mathrm{KB}}$-least element of C of length k. This contradicts that $\tau <_{\mathrm{KB}} \sigma_n$. So $k > n$. Furthermore, $\tau\restriction n = \sigma_n$ because $\tau <_{\mathrm{KB}} \sigma_n$, hence $\tau\restriction n \leq_{\mathrm{KB}} \sigma_n$, and σ_n is the $<_{\mathrm{KB}}$-least element of C of length n. So $\tau \sqsupset \sigma_n$. Now consider σ_{n+1}. We have that $\sigma_{n+1} \leq_{\mathrm{KB}} \tau\restriction_{(n+1)} <_{\mathrm{KB}} \sigma_n$ because σ_{n+1} is the $<_{\mathrm{KB}}$-least element of C of length $n+1$. Again, $\sigma_{n+1}\restriction_n = \sigma_n$ because $\sigma_{n+1} <_{\mathrm{KB}} \sigma_n$ and σ_n is the $<_{\mathrm{KB}}$-least element of C of length n. Thus $\sigma_n \sqsubset \sigma_{n+1}$, as desired.

Let $f = \bigcup_n \sigma_n$. Then f is a path through T. In fact, f is the leftmost path through

T. To see this, suppose for a contradiction that g is a path through T that is to the left of f. Then there is an n such that $\forall i < n\,(g(i) = f(i))$ and $g(n) < f(n)$. Then $g\restriction_{(n+1)} <_{\mathrm{KB}} f\restriction_{(n+1)} = \sigma_{n+1}$, and $g\restriction_{(n+1)} \in C$ by the same argument as for h above. This contradicts that σ_{n+1} is the $<_{\mathrm{KB}}$-least element of C of length n. Thus f is the leftmost path through T, which concludes the proof of LPP and hence the proof of the Theorem. □

Remark 3.2.5. By the sketch of the proof of Theorem 3.0.1, we can conclude that it can be formalized in Π^1_1-CA_0. Even without our further analysis, however, it is known from the literature that that proof cannot be optimal, in the sense that there is no hope to find a reversal: no true Π^1_2 statement can be equivalent to Π^1_1-CA_0 over RCA_0 (see for instance [50, Corollary 1.10] for a proof of a stronger result).

3.3. An easy proof of $\mathsf{RSpo}^{\mathsf{W}}_{<\infty}$ in ACA_0

We give a proof of $\mathsf{RSpo}^{\mathsf{W}}_{<\infty}$ in ACA_0. This is not the optimal proof: in Section 3.4 we will show that $\mathsf{RSpo}^{\mathsf{W}}_{<\infty}$ is equivalent to $\mathsf{ADS} + \mathsf{I}\Sigma^0_2$ over RCA_0. Anyway, in a certain sense, the ACA_0 proof seems to strike a good balance between axiomatic simplicity and conceptual simplicity: the proof can be presented in ordinary mathematical language, meaning without reference to relative computability, technical uses of restricted induction, or other technicalities typical in the reverse mathematical approach.

It is based on Dilworth's theorem, the observation made in Remark 3.1.7 that any chain of order-type ζ is automatically $(0,\infty)$-homogeneous, and the observation that a linear order containing no suborder of type ζ can be partitioned into a well-founded part and a reverse well-founded part. This last observation requires the full strength of ACA_0, as shown by Lemma 3.3.4.

In order to complete the proof of the Lemma, we will need a particular linear order with some very nice properies, and whose construction we present in Construction 3.3.2.

Definition 3.3.1. (RCA_0) Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. A number $n \in \mathbb{N}$ is a *true number* if $f(k) > f(n)$ for all $k > n$. Otherwise, n is a *false number*. We say that $n \in \mathbb{N}$ is *true at stage m* if $\forall k\,(n < k \leq m \to f(n) < f(k))$. Otherwise, we say that n is *false at stage m*.

The idea of true numbers appears to have originated with Dekker [17], who called them *minimal*. True numbers are important because the range of f is computable in the join of f with any infinite set of true numbers. In fact, if n is a true number, then one can determine $\operatorname{ran}(f)$ up to $f(n)$ by simply evaluating f on inputs $0, \ldots, n$.

Construction 3.3.2 (RCA_0)**.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. Define a linear order $(L, <_L)$ where $L = \{\ell_n : n \in \mathbb{N}\}$ and for each $n < m$ the following hold:

1. $\ell_n <_L \ell_m$ if $f(k) < f(n)$ for some k such that $n < k \leq m$ (i.e., n is false at stage m),
2. $\ell_m <_L \ell_n$ if $f(n) < f(k)$ for all k such that $n < k \leq m$ (i.e., n is true at stage m).

Given an injection $f\colon \mathbb{N} \to \mathbb{N}$, Construction 3.3.2 produces a stable linear order either of type $\omega + \omega^*$ (if f has infinitely many false numbers) or of type $k + \omega^*$ for some finite k (otherwise). RCA_0 proves that n is true if and only if n is in the ω^*-part of L. Therefore, RCA_0 proves that if there is an infinite subset of the ω^*-part of L, or, equivalently, if there is an infinite descending sequence in L, then the range of f exists. For further details, see the proofs of see [51], Lemma 4.2 and [29], Theorem 4.5.

We now introduce some terminology that will be useful in the rest of the chapter.

Definition 3.3.3. (RCA_0) Let $(P, <_P)$ be a partial order, and let $A, B \subseteq P$.

- We write $A <_P B$ if every element of A is strictly below every element of B: $\forall a \in A\, \forall b \in B\, (a <_P b)$. In the case of singletons, write $a <_P B$ and $A <_P b$ in place of $\{a\} <_P B$ and $A <_P \{b\}$.

- We write $A \leq_{\forall\exists} B$ if every element of A is below some element of B: $\forall a \in A\,\exists b \in B\,(a \leq_P b)$.

- We write $A \mid_P B$ if every element of A is incomparable with every element of B: $\forall a \in A\,\forall b \in B\,(a \mid_P b)$. In the case of singletons, write $a \mid_P B$ and $A \mid_P b$ in place of $\{a\} \mid_P B$ and $A \mid_P \{b\}$.

For a partial order $(P, <_P)$ and nonempty subsets $A, B, C \subseteq P$, RCA_0 suffices to show that $A <_P B <_P C$ implies $A <_P C$ and that $A \leq_{\forall\exists} B \leq_{\forall\exists} C$ implies $A \leq_{\forall\exists} C$. Also, notice that $A \leq_{\forall\exists} B$ simply means that $A \subseteq B{\downarrow}_{(P,<_P)}$ (the existence of which, in general, requires ACA_0 to be proved).

Lemma 3.3.4. *The following are equivalent over* RCA_0*.*

1. ACA_0.

2. *Every linear order* $(L, <_L)$ *with no suborder of type* ζ *can be partitioned as* $L = W \cup R$*, where*

    - $W <_L R$*,*

    - W *is well-founded,*

    - R *is reverse well-founded.*

Proof. For 1 ⇒ 2, let $(L, <_L)$ be a linear order with no suborder of type ζ. First, let $X = \{x \in L : (\forall y <_L x)(\exists z)(y <_L z <_L x)\}$. Intuitively, X is the set of points in L that are the suprema of the points strictly below them. We claim that the downward closure $X{\downarrow}_{(L,<_L)}$ of X is well-founded. To see this, suppose on the contrary that there is a descending sequence $D = (d_n)_{n\in\mathbb{N}}$ in $X{\downarrow}_{(L,<_L)}$. Now define an ascending sequence $A = (a_n)_{n\in\mathbb{N}}$ above d_1 as follows. As $d_0 \in X{\downarrow}_{(L,<_L)}$, fix an $x \in X$ such that $d_0 \leq_L x$. Define

$$a_0 = \min_{<_\mathbb{N}}\{z : d_1 <_L z <_L x\}$$
$$a_{n+1} = \min_{<_\mathbb{N}}\{z : a_n <_L z <_L x\}.$$

Such an a_n exists for each $n \in \mathbb{N}$ because $x \in X$. Then $(D \setminus \{d_0\}) \cup A$ is a suborder of L of type ζ, which is a contradiction. Thus $X{\downarrow}_{(L,<_L)}$ is well-founded.

Let F be the set $F = \{y \in L \setminus X{\downarrow}_{(L,<_L)} : \{z \in L \setminus X{\downarrow}_{(L,<_L)} : z <_L y\}$ is finite$\}$, i.e. the set of elements of $L \setminus X{\downarrow}_{(L,<_L)}$ with only finitely many $<_L$-predecessors in $L \setminus X{\downarrow}_{(L,<_L)}$ (which, we notice, might be empty). We claim that the set $X{\downarrow}_{(L,<_L)} \cup F$ does not contain infinite descending sequences, i.e. it is well-founded: suppose for a contradiction that there exists such a sequence $(d_n : n \in \mathbb{N})$, then there is a $b \in \mathbb{N}$ such that for every $n > b$, $d_n \in X{\downarrow}_{(L,<_L)}$, since by definition the elements of F only have finitely many predecessors in F. But then, the sequence $(d_n : n > b)$ would be an infinite descending sequence in $X{\downarrow}_{(L,<_L)}$, which is a contradiction. We set $W = X{\downarrow}_{(L,<_L)} \cup F$, and let $R = L \setminus W$.

Since we proved that W is well-founded, we just have to prove that R is reverse well-founded. First of all, we notice that if R is empty, then it is also reverse well-founded, so we can suppose that $R \neq \emptyset$. We claim that R has no $<_L$-minimal element. Suppose for a contradiction that $r_0 \in R$ was such an element: then, the existence of r_0 implies that F was infinite, since otherwise r_0 would itself have been an element of F. Hence, since F is an infinite set of elements with only finitely many predecessors, F is a linear order of order-type ω. Then, we claim that $r_0 \in X$, which is a contradiction. To see this, let y be any element of L $<_L$-below r_0: then, $y \in W$, and since we said that F is of order-type ω, it is always possible to find a $z \in F$ such that $y <_L z <_L r_0$, which proves that $r_0 \in X$. This contradiction proves that R has no $<_L$-minimal element. Finally, suppose for a contradiction that there is an infinite ascending sequence $(a_n)_{n\in\mathbb{N}} \subseteq R$: since R has no $<_L$-minimal element, it is possible to build an infinite descending sequence $(d_n)_{n\in\mathbb{N}}$ in R such that $d_0 = a_0$. The union of the range of these sequences then gives an infinite chain of order-type ζ, which contradicts our assumptions on L. This proves that R is reverse well-founded, and hence concludes the proof of 1 $\Rightarrow$ 2.

For 2 $\Rightarrow$ 1, let $f \colon \mathbb{N} \to \mathbb{N}$ be an injection. We show that the true numbers for f form a set. This implies that the range of f exists as a set, which implies ACA_0 by Theorem 1.1.10.

If f has only finitely many false numbers, then the set of all false numbers exists by bounded Σ^0_1 comprehension, in which case the set of true numbers also exists.

Suppose instead that f has infinitely many false numbers. Let $(L, <_L)$ be the linear order defined as in Construction 3.3.2 for f. Recall that in this case $L = \{\ell_n : n \in \mathbb{N}\}$ is a linear order of type $\omega + \omega^*$, where, for each n, ℓ_n is in the ω-part if n is false and ℓ_n is in the ω^*-part if n is true. We modify L by replacing each element in the ω^*-part by an infinite descending sequence and by replacing each element of the ω-part by a finite descending sequence. To do this, let $S = \{s_{n,m} : n, m \in \mathbb{N}$ and n is true at stage $m\}$ (note that if $m \leq n$, then n is true at stage m), and define

$$s_{n_0,m_0} <_S s_{n_1,m_1} \quad \Leftrightarrow \quad (\ell_{n_0} <_L \ell_{n_1}) \vee (\ell_{n_0} = \ell_{n_1} \wedge m_0 >_{\mathbb{N}} m_1).$$

Observe that if n_0 is false and n_1 is true, then $\ell_{n_0} <_L \ell_{n_1}$, so $s_{n_0,m_0} <_S s_{n_1,m_1}$ for every m_0 and m_1. One then sees that no infinite ascending sequence in S can contain an element $s_{n,m}$ where n is true, and no infinite descending sequence in S can contain an element $s_{n,m}$ where n is false. It follows that S cannot contain a suborder of type ζ because such a suborder would have to contain some element $s_{n,m}$, and $s_{n,m}$ is either in no ascending sequence or in no descending sequence, whereas it follows easily from Definition 3.1.6 that in any ordering of order-type ζ has the property that every element belongs to both an ascending sequence and a descending sequence.

We may therefore apply 2 to S, obtaining a partition $S = W \cup R$ where $W <_L R$, W is well-founded, and R is reverse well-founded. We claim that $s_{n,0} \in R$ if and only if n is true. If n is true, then $s_{n,m} \in S$ for every m, and $s_{n,0} >_S s_{n,1} >_S \cdots$ is a descending sequence in S. Thus $s_{n,0}$ cannot be in W as then W would not be well-founded. So $s_{n,0} \in R$. Conversely, if n is false, then, using the assumption that there are infinitely many false numbers, we can define an ascending sequence $\ell_n = \ell_{k_0} <_L \ell_{k_1} <_L \cdots$ in L as follows. Set $k_0 = n$. Given k_i, search for the first pair $\langle k, m \rangle$ where $\ell_{k_i} <_L \ell_k$ and k is false at stage m, and set $k_{i+1} = k$. We then have the corresponding ascending sequence $s_{n,0} = s_{k_0,0} <_S s_{k_1,0} <_S \cdots$ in S. Thus $s_{n,0}$ cannot be in R as then R would not be reverse well-founded. So $s_{n,0} \in R$ if and only if n is true. Therefore

$\{n : s_{n,0} \in R\}$ is the set of true numbers for f, which completes the proof. $\square$

We now move to the promised proof of $\mathsf{RSpo}^{\mathsf{W}}_{<\infty}$ in ACA_0. In order to do that, it will be useful to make some considerations on what it means for an ascending chain *not* to be $(0,\infty)$-homogeneous.

If an ascending sequence $A = \{a_n : n \in \mathbb{N}\}$ in some partial order $(P, <_P)$ is not $(0,\infty)$-homogeneous, then there is a $p \in P$ that is comparable with some elements of P, but only finitely many of them. As A is an ascending sequence, this means that there is an n_0 such that $p >_P a_{n_0}$, but $\forall n > n_0\,(p \mid_P a_n)$. We think of such a p as a *counterexample* to A being $(0,\infty)$-homogeneous. Indeed, p is a counterexample to $\{a_n : n \geq n_0\}$ being $(0,\infty)$-homogeneous.

Definition 3.3.5.

- (RCA_0) Let $(P, <_P)$ be an infinite partial order, and let $A = \{a_n : n \in \mathbb{N}\}$ be an ascending sequence in P. Then $A_{\geq n_0}$ denotes the ascending sequence $\{a_n : n \geq n_0\}$. Sequences of the form $A_{\geq n_0}$ are called *tails* of A.
- (RCA_0) Let $(P, <_P)$ be a partial order, and let $A = \{a_n : n \in \mathbb{N}\}$ be an ascending sequence in P. A $p \in P$ is called a *counterexample to* A if there is an n such that $p >_P a_n$ and $p \mid_P A_{\geq n+1}$.
- (RCA_0) An ascending sequence $B = \{b_\ell : \ell \in \mathbb{N}\}$ is called a *counterexample sequence for* A if B contains counterexamples to infinitely many tails of A: $\forall m\, \exists n > m\, \exists \ell\, (b_\ell >_P a_n \wedge b_\ell \mid_P A_{\geq n+1})$

Again, we remark that, since we are dealing with ascending sequences, we can be quite liberal and deal with what is technically the *range* of the sequences, since that set exists in RCA_0. See also Remark 1.1.25.

Suppose that A is an ascending sequence in a partial order $(P, <_P)$ where no tail of A is $(0,\infty)$-homogeneous. Then for every n, there is a counterexample p to $A_{\geq n}$. The main idea of the next proof is that if P has finite width, then we can make a counterexample sequence out of such counterexamples.

We remark that we will now prove a result that is more general than what we need for the rest of the section: the reason is that we will use it in its full generality in Section 3.6.

Lemma 3.3.6 (ACA_0)**.** *Let $(P, <_P)$ be a partial order with no infinite antichains. Let $A = \{a_n : n \in \mathbb{N}\}$ be an ascending sequence, and assume that no tail of A is $(0, \infty)$-homogeneous for P. Then there is an ascending sequence $B = \{b_n : n \in \mathbb{N}\}$ that is a counterexample sequence for A.*

Moreover, if P can be decomposed into the chains $C_0, \dots, C_{k-1}$, We will be able to find such a B inside C_i, for a certain i.

Proof. Since we are assuming that no tail of A is $(0, \infty)$-homogeneous, for every tail there is a counterexample p to it. For each n, let p_n be the $<_{\mathbb{N}}$-least counterexample to the tail $A_{\geq n}$. Let $\widetilde{P}$ be the set of the p_n we just described. By CAC, $\widetilde{P}$ has an infinite chain, say C, since the whole poset P does not contain infinite antichains. Let X be the set of $n \in \mathbb{N}$ such that $p_n \in C$.

Now, for every $n \in X$, we have that $p_n <_P p_m$ for all sufficiently large $m \in X$. To see this, let $n \in X$. As p_n is a counterexample to $A_{\geq n}$, there is a $c \geq n$ such that $p_n >_P a_c$ and $p_n \mid_P A_{\geq c+1}$. Let $m \in X$ be such that $m > c + 1$, and consider p_m. The chain C contains both p_n and p_m, so $p_n \between_P p_m$. As p_m is a counterexample to $A_{\geq m}$, there is a $d \geq m$ such that $p_m >_P a_d$. Thus we cannot have have $p_m \leq_P p_n$ because this would yield $a_{c+1} <_P a_d <_P p_m \leq_P p_n$, contradicting that $p_n \mid_P a_{c+1}$. Note here that $c + 1 < m \leq d$, so $a_{c+1} <_P a_d$ because A is an ascending sequence. Thus it must be that $p_n <_P p_m$.

We may then define the desired counterexample sequence B as follows. Let n_0 be the $<_{\mathbb{N}}$-least element of X. Given n_ℓ, let $n_{\ell+1}$ be the $<_{\mathbb{N}}$-least element of X with $n_\ell < n_{\ell+1}$ and $p_{n_\ell} <_P p_{n_{\ell+1}}$. Finally, take $b_\ell = p_{n_\ell}$ for each ℓ.

We end the proof by noticing that, if P can be decomposed into the chains $C_0, \dots, C_{k-1}$, then $\widetilde{P} \cap C_i$ is infinite for at least one $i < k$, and so the chain C above can be replaced by $\widetilde{P} \cap C_i$, so that the final B will be a subset of C_i $\square$

Notice that if B is a counterexample sequence to an ascending sequence A in some partial order $(P, <_P)$, then $A \leq_{\forall\exists} B$, but $B \not\leq_{\forall\exists} A$.

Theorem 3.3.7. $\mathsf{ACA}_0 \vdash \mathsf{RSpo}^{\mathsf{W}}_{<\infty}$.

Proof. Let $(P, <_P)$ be an infinite partial order of width k and let $C_0, \ldots, C_{k-1}$ be the decomposition into chains as given by Dilworth's Theorem. Assume for a contradiction that $(P, <_P)$ does not contain a $(0, \infty)$-homogeneous chain. Notice that any chain Z of order type ζ is $(0, \infty)$-homogeneous, as stated in Remark 3.1.7. Indeed, if $p \in P$ is comparable with some $z \in Z$, then it is either comparable with all elements above z or with all elements below z. It follows that C_i, for each $i < k$, does not contain any chains of order type ζ.

Notice that we can apply Lemma 3.3.4 uniformly to all the chains C_i (it is indeed easy to see that the proof of the Lemma can be modified to yield the wanted decomposition for any finite number of chains): we thus get the decompositions $C_i = W_i \cup R_i$, where every W_i is well-founded, R_i is reverse well-founded and $W_i <_P R_i$ for every $i < k$.

We suppose that at least one of the W_i's is infinite. If this is not the case, then at least one of the R_i's is infinite, and we could run an argument essentially identical to the one we are about to present. By changing the enumeration if necessary, let $W_0, \ldots, W_{u-1}$, for some $u < k$, be the infinite W_i's. For every $j < u$, we let W'_j be the subset of W_j formed by the points of W_j with infinitely many successors in W_j, so formally $W'_j := \{p \in W_j : \forall x \exists y >_{\mathbb{N}} x(y \in W_j \wedge y >_P p)$. Since the W_j's are infinite and well founded, so are the W'_j's.

In every W'_j, we can easily find a cofinal sequence of type ω, call it A_j: to do this, simply let a_0 be the $<_{\mathbb{N}}$-minimal element of W'_j, and let a_{n+1} be the $<_{\mathbb{N}}$-least point of W'_j that is $<_P$-above a_n.

By assumption, each tail of every A_j, for each $j \leq u$, is not $(0, \infty)$-homogeneous, so let B_j be the counterexample sequence to A_j given by Lemma 3.3.6. Let $h \colon \{0, \ldots, u\} \to \{0, \ldots, u\}$ be the function such that $B_j \subseteq C_{h(j)}$, for each $j \leq u$. Since B_j is an ascending sequence then it holds, for each $j \leq u$, that $B_j \subseteq W_{h(j)}$.

Notice that, for each $j \leq u$, it holds that $B_j \leq_{\forall\exists} A_{h(j)}$ since $A_{h(j)}$ is cofinal in $W_{h(j)}$. Since it holds that $A_{h(j)} \leq_{\forall\exists} B_{h(j)}$, by choice of $B_{h(j)}$, and since $\leq_{\forall\exists}$ is transitive, it holds that $B_j \leq_{\forall\exists} B_{h(j)}$, for each $j \in \mathbb{N}$. By transitivity we get that $B_{h^n(j)} \leq_{\forall\exists} B_{h^m(j)}$ for each $j \leq u$ and each $n \leq m \in \mathbb{N}$ ($h^m(j)$ stands for the m^{th} iteration of $h(j)$, where $h^0(j) = j$).

We now notice that there are $n < m \leq u$ such that $h^n(0) = h^m(0)$. But then, it follows from the previous paragraph that $B_{h^{n+1}(0)} \leq_{\forall\exists} B_{h^m(0)}$, as we can see applying h $m-n-1$ times. But by assumption on m and n, it follows that $B_{h^{n+1}(0)} \leq_{\forall\exists} B_{h^n(0)}$. This implies that $B_{h^{n+1}(0)} \leq_{\forall\exists} A_{h^{n+1}(0)}$, since $A_{h^{n+1}(0)}$ is cofinal in $W'_{h^{n+1}(0)}$. But this contradicts the fact that $B_{h^{n+1}(0)}$ is a counterexample sequence to $A_{h^{n+1}(0)}$. Hence, we have our contradiction and the theorem is proved. $\square$

3.4. Equivalence with $\mathsf{ADS} + \mathsf{I}\Sigma^0_2$ and ADS

In this section, the proof-theoretic strength of $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ is finally determined: by refining the counterexample-chasing argument already used in the proof of the principle in ACA_0, we will be able to givea proof over RCA_0 of the equivalence of $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ with $\mathsf{ADS} + \mathsf{I}\Sigma^0_2$. Moreover, we will further analyze the argument to show that, for every fixed standard $k \geq 3$, $\mathsf{RSpo}^{\mathsf{CD}}_k$ is equivalent to ADS.

3.4.1. A proof of $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$

We start by proving a combinatorial result on finite trees, which lies at the heart of the proof of $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$.

Lemma 3.4.1. (RCA_0) *Let us fix $k \geq 2$, and let T be a finitely branching tree such that every leaf of T is at level k, for every $\sigma \in T$, $\operatorname{ran}\sigma \subseteq k$ but every node has at most $k-1$ immediate successors. Moreover, let $c : T \to k$ be a coloring of T such that*

- *for every $\sigma \in T$ and $n < k$, if $\tau = \sigma^\frown(n) \in T$, then $c(\sigma) \neq c(\tau)$, and*

- *for every $\sigma \in T$ and $n, m < k$, if $m \neq n$ and $\sigma^\frown(n), \sigma^\frown(m) \in T$, then $c(\sigma^\frown(n)) \neq c(\sigma^\frown(m))$.*

Then, there is a $\sigma \in T$ that is not a leaf such that for every τ immediate extension of σ there is an $\eta \sqsupseteq \tau$ such that $c(\sigma) = c(\eta)$.

Proof. We prove the statement by induction on k: it is clear that this can be done using only $\mathsf{I}\Sigma^0_1$. We start with $k = 2$. In this case, notice that the only possible tree satisfying the requirements above is $T = \{\emptyset, (i), (i, j)\}$, for some $i, j < 2$. Since, by the constraints on c, it holds that $c(\emptyset) = c((i, j))$, it follows that $\emptyset$ is the required string.

Now, assuming that we already proved the statement for k, we prove it for $k + 1$. Let T and c be as in the statement (with k substituted by $k + 1$). If for every $n < k+1$ such that $n \in T$ it holds that there is an extension $\eta_n \sqsupseteq n$ with $c(\emptyset) = c(\eta_n)$, then the conclusion of the Lemma holds, as witnessed by $\emptyset$. So suppose that this is not the case: this means that there is an n such that $T_{(n)}$ (which, we recall, is the tree $\{\sigma \in T : (n) \sqsubset \sigma\}$) does not contain any η with $c(\emptyset) = c((n)^\frown\eta)$. By our assumptions on c, in particular this implies that every node on $T_{(n)}$ has at most $k - 2$ immediate successors. After renaming the strings if necessary, we see that $T_{(n)}$ satisfies the hypotheses of the Lemma. We can thus apply the induction hypothesis to it, and this concludes the proof. □

We now give an informal presentation of how the Lemma above is going to be used in the proof of $\mathsf{ADS} + \mathsf{I}\Sigma^0_2 \vdash \mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$, with the aim of presenting clearly the various concepts that we will introduce formally in the rest of the section. The idea is the following: given a poset P with chain-decomposition-number k, we use ADS to find, say, an ascending sequence in P, and we can assume that it is completely contained in C_0, one of the chains of P. There are two cases: if A is already a $(0, \infty)$-homogeneous chain, then we are done. If not, then, similarly to what we observed for the proof in ACA_0, there must be a counterexample sequences to A. We look at the same time for counterexamples in all of the chains of P: using $\mathsf{I}\Sigma^0_2$, we can determine which chains C_i contain an infinite ascending sequence A_i witnessing that A is not a solution.

Intuitively, this corresponds to building the first level L_1 of the tree T of the lemma above: we color the root with 0, the index of the chain where A is, and we put in L_1 all the indices of the chains containing a counterexample sequence to A, and we let $c((n)) = n$ for every such n (notice that 0 does not appear as a color on L_1, since it is impossible to find a counterexample to A in the same chain where A is). Again, if we do not find any $(0, \infty)$-homogeneous chain among these counterexamples, we repeat the procedure, starting with the A_i instead of A. Again, if no $(0, \infty)$-homogeneous chain is found in the process, we can build a tree of height k, and we can thus apply the Lemma: given the A_σ and the A_η associated to the σ and η's of the Lemma, we will show how to build a chain of order-type $\omega + \omega$ in $C_{c(\sigma)}$ that is a $(0, \infty)$-homogeneous chain, thus concluding the proof of the Theorem.

In order to carry out the proof as just described in a system weaker than ACA_0, we have first to weaken the notion of counterexample sequence.

Definition 3.4.2. (RCA_0) Let P be an infinite poset and $A, B \subset P$ be ascending sequences in P, enumerated as $A = \{a_0 <_P a_1 <_P \dots\}$ and $B = \{b_0 <_P b_1 <_P \dots\}$.

- We say that B is a *local counterexample sequence* to A if it holds that
  1. $\forall n \in \mathbb{N} \exists m \in \mathbb{N}(b_n >_P a_m \wedge b_n \not\geq_P a_{m+1})$, and moreover
  2. $\forall n, m \in \mathbb{N}(b_n \geq_P a_m \rightarrow b_{n+1} \geq a_{m+1})$.
- We say that B is a *strong local counterexample sequence* if it is a local counterexample sequence to A and moreover $\forall n \in \mathbb{N} \exists m \in \mathbb{N}(b_n >_P a_m \wedge b_n |_P a_{m+1})$.

The idea behind the definition above is that it is much easier to look for strong local counterexamples sequences to A than it is to look for counterexamples sequences to A: whereas in the latter case, before we could enumerate an element p in the counterexample sequence, we had to check that p was incomparable to every element of A from a certain point onward, here we essentially just have to find *one* element of A witnessing the incomparability. As one can easily verify, every counterexample sequence is a strong local counterexample sequence, but not viceversa.

The reason why we further weaken the notion of counterexample is that local counterexample sequences have the following property:

Property 3.4.3. (RCA_0) *Let P be a poset, $A = \{a_0, a_1, \dots\}$ an infinite ascending sequence and $B = \{b_0, b_1, \dots\}$ a local counterexample sequence to A. Let A' be an infinite subsequence of A. Then there is an infinite subsequence B' of B such that B' is a local counterexample sequence to A'.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be the increasing function such that $A' = \{a_{f(0)}, a_{f(1)}, \dots\}$. Let $g(n) : \mathbb{N} \to \mathbb{N}$ be the increasing function such that for every n $g(n)$ is the minimal r such that $a_{f(n)} <_P b_r$ (such an r always exists). We claim that $\{b_{g(0)}, b_{g(1)}, \dots\}$ is the B' we are after. Property 1 still holds, since if $b_{g(n)} \not\geq_P a_{f(n)+1}$, then $b_{g(n)} \not\geq_P a_{f(m)}$ for any $m > n$. Moreover, property 2 is obvious from the definition of g. □

The property above does not necessarily hold if we require that B' be a strong local counterexample sequence to A'. However, strong counterexamples sequences do enjoy some nice properties that we will come in handy in the future.

Property 3.4.4. (RCA_0) *Let P be an infinite poset, $A \subset P$ be an ascending sequence in P. Suppose that A' is a subsequence of A and B is a strong local counterexample sequence to A'. Then, it is a local counterexample sequence to A as well.*

Moreover, if P has chain-decomposition-number k, $P = C_0 \cup \dots \cup C_{k-1}$, and B is a strong local counterexample sequence to A, then if $A \subseteq A_i$ and $B \subseteq C_j$, we have that $i \neq j$.

Although this will not play any role in the following, we also notice that the existence of (strong) local counterexample sequences does not, as opposed to counterexample sequences, characterize non-$(0, \infty)$-homogeneous ascending sequences: there may well be $(0, \infty)$-homogeneous ascending sequences that admit strong local counterexample sequences to them.

The next Lemma is essentially a weakening of Lemma 3.3.6: it says that, for a given ascending sequence A of P, if no tail of A is $(0, \infty)$-homogeneous, then we can find a strong local counterexample sequence to A. This weakening has the major advantage of being provable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. For later use, it will be practical to distinguish the case in which our poset P has a standard chain-decomposition-number.

Lemma 3.4.5. *1.* (RCA_0) *Let $k \in \omega$, let P be an infinite poset of chain-decomposition-number k and let $A = \{a_0 <_P a_1 <_P \dots\} \subseteq P$ be an infinite ascending sequence in P. If no tail $A_{\geq m}$ of A is $(0,\infty)$-homogeneous, then there is an ascending chain $B \subseteq P$ that is a strong local counterexample sequence to A.*

2. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ *proves the same statement above for arbitrary $k \in \mathbb{N}$, i.e. if we drop the requirement that $k \in \omega$*

We point out that in this proof, due to the number of orderings involved, we will denote the usual order $<$ on $\mathbb{N}$ by $<_{\mathbb{N}}$.

Proof. We start with the proof of Item 1. The proof is similar to the one of Lemma 3.3.6. We define a function $f : \mathbb{N} \to \mathbb{N}^2$ as follows: $f(0)$ is the pair (p_0, m_0) that is $<_{\mathbb{N}^2}$-least (in some ordering of $\mathbb{N}^2$ of type ω) such that $p_0 >_P a_{m_0}$ and $p|_P a_{m_0+1}$. Such a pair has to exists: if A is not a solution, then there has to be a q comparable with only finitely many elements of A, in particular there is a maximal ℓ such that $q \mid_P a_\ell$. Hence, $q >_P a_\ell$ since A is ascending, and $q|_P a_{\ell+1}$. Such q and ℓ are the p_0 and n_0 we are looking for.

Recursively, we define $f(n+1)$ as the $<_{\mathbb{N}^2}$-least pair (p_{n+1}, m_{n+1}) such that $m_{n+1} >_{\mathbb{N}} m_n$, $p_{n+1} >_P a_{m_{n+1}}$, $p_{n+1}|_P a_{m_{n+1}+1}$ and $p_{n+1} >_{\mathbb{N}} p_n$. The existence of such a pair can be proved in a fashion similar to what has been done above: suppose for a contradiction that no pair (p_{n+1}, m_{n+1}) as above exists, we claim that then a tail of A is $(0,\infty)$-homogeneous. It follows from our assumptions that for every $m >_{\mathbb{N}} m_n$, if there is a $p \in P$ such that $p >_P a_m$ and $p|_P a_{m+1}$, then $p <_{\mathbb{N}} p_n$. Let M be the finite set $M = \{m \in \mathbb{N} : \exists p <_{\mathbb{N}} p_n(a_m <_P p \wedge a_{m+1}|_P p)\}$. Then, M has a $<_{\mathbb{N}}$-maximal element, say $\bar{m}$. It is immediate to check that $A_{\geq \bar{m}+1}$ is a $(0,\infty)$-homogeneous tail of A, which gives us the desired contradiction.

Furthermore, we notice that the set $S = \{p \in P : \exists n((p, n) \in \operatorname{ran} f)\}$ can be shown to exists in RCA_0, since the points of S form an $<_{\mathbb{N}}$-ascending sequence.

By RT^1_k, there is a chain C_i containing infinitely many elements of S. Finally, notice that if $(p, m), (q, n) \in \operatorname{ran} f$ and $p \between_P q$, then $p <_P q$ if $m <_{\mathbb{N}} n$. To see this, notice that if $m <_{\mathbb{N}} n$ then $q \not\leq_P p$, otherwise $p \geq_P q \geq_P a_n \geq_P a_{m+1}$, which is a contradiction. Hence $q >_P p$, and so the first components of $S \cap C_i$ can be seen as an ascending sequence, which will then, thanks to the Property above, be a local counterexample sequence to A. This concludes the proof of Item 1.

Item 2 has the same proof, except for the final step, where we use $\mathsf{RT}^1_{<\infty}$ instead of RT^1_k. □

In the following Lemma, we will show that we can iterate the operation of finding local counterexample sequences in a very tame way, provided we are given enough induction.

Definition 3.4.6. (RCA_0) Let P be an infinite poset of chain-decomposition-number k, and let $\vec{A} = (A_0, \ldots, A_h)$ be a sequence of ascending chains of P. We say that $\vec{A}$ is a *sequence of local counterexamples* if for every $i < h$ A_{i+1} is a local counterexample sequence to A_i, and for every $i \leq h$ there exists $j < k$ such that $A_i \subseteq C_j$.

We recall that $\mathsf{I}\Sigma^0_2$ is equivalent to bounded Π^0_2-comprehension (see Lemma 1.1.18).

Lemma 3.4.7. *($\mathsf{RCA}_0+\mathsf{I}\Sigma^0_2$) Suppose that P is a poset of chain-decomposition-number k, for some $k \in \mathbb{N}$, and suppose that $A \subseteq C_0$ is an ascending sequence in C_0. Then, we can define a tree $T \subseteq k^{<k+1}$ such that $\sigma \in T$ if and only if there is a sequence of ascending sequences $\vec{A} = (A_0, \ldots, A_{|\sigma|})$ such that $A_0 \subseteq A$ and for every $0 < i \leq |\sigma|$, $A_i \subseteq C_{\sigma(i-1)}$ and A_i is a local counterexample sequence to A_{i-1}.*

Proof. We suppose for simplicity that $P = \mathbb{N}$. We define a function $f : k^{<\mathbb{N}} \times \mathbb{N} \to [\mathbb{N} \times \mathbb{N} \times \mathbb{N}]^{<\mathbb{N}}$ by recursion on the number variable, with the following idea: at stage n, we will have a finite set (in fact, this set will have cardinality smaller than n) of triples (d, p, i), which should be read as "p is the d^{th} element of an ascending chain in

$C_{\sigma(i-1)}$", and we only add the triple (d, p, i) to the set of triples if p belongs to A or if it contributes to create a local counterexample sequence to the chain that is being created by the points q such that, for some e, $(e, q, i-1)$ is in $f(\sigma, n)$ (if $i > 0$). In practice, we proceed as follows: given $\sigma \in k^{<\mathbb{N}}$, we start by setting $f(\sigma, 0) = \emptyset$ if $0 \notin A$, otherwise we let $f(\sigma, 0) = \{(0, 0, 0)\}$. Suppose that we have already defined $f(\sigma, n)$. To define $f(\sigma, n+1)$, there are various cases:

- if $f(\sigma, n) = \emptyset$ and $n+1 \notin A$, we let $f(\sigma, n+1) = \emptyset$.
- if $f(\sigma, n) = \emptyset$ and $n+1 \in A$, set $f(\sigma, n+1) = \{(0, n+1, 0)\}$.
- if $f(\sigma, n) \neq \emptyset$, let $b \leq |\sigma| + 1$ be the minimal i such that for no $m < n+1$ $(0, m, i) \in f(\sigma, n)$ holds. For every $i < b$ let d_i be the maximal d such that $(d, m, i) \in f(\sigma, n)$ for some $m < n$, and let p_i be the p such that $(d_i, p, i) \in f(\sigma, n)$. Then:
  - if $n+1 \in A$ and $n+1 >_P p_0$, we set $f(\sigma, n+1) = f(\sigma, n) \cup \{(d_0 + 1, n+1, 0)\}$.
  - if $n+1 \notin A$, we check for every $0 < i < b$ if $n+1 >_P p_i$ and $n+1 \in C_{\sigma(i-1)}$ hold, and if there is a $d < d_{i-1}$ such that, for the $p \in P$ such that $(d, p, i-1) \in f(\sigma, n)$ and the $q \in P$ such that $(d+1, q, i-1) \in f(\sigma.n)$, the following three conditions hold:

    $$p_i \not>_P p, \quad n+1 >_P p, \quad n+1 \ngeq_P q.$$

    Then:
    - If there are such i's, let $\bar{i}$ be the minimal one and let $f(\sigma, n+1) = f(\sigma, n) \cup \{(d_{\bar{i}} + 1, n+1, \bar{i})\}$.
    - If no index i as above is found, and if $b \neq |\sigma| + 1$, we check if $n+1 \in C_{\sigma(b-1)}$ and if there are p, q, d such that $\{(d, q, b-1), (d-1, p, b-1)\} \subseteq f(\sigma, n)$, $n+1 >_P p$ and $n+1 \ngeq_P q$ hold.
      - If this is the case, we set $f(\sigma, n+1) = f(\sigma, n) \cup \{0, p_{n+1}, b)\}$.

· If instead there are no such p, q, d's, or if $b = k + 1$, we let $f(\sigma, n + 1) = f(\sigma, n)$.

Although the construction above might seem complicated, it is just formalizing the obvious recursion used to build a sequence of local counterexamples step by step.

By bounded Π^0_2-comprehension, we can then define the set $T \subset k^{<k+1}$ such that

$$\sigma \in T \leftrightarrow (\sigma \in k^{<k+1} \wedge \forall i \leq |\sigma| \forall m \exists d_i, n_i > m((d_i, n_i, i) \in f(\sigma, n_i + 1))).$$

We claim that the T we just defined is the tree we wanted: $\sigma \in T$ if and only if there is a sequence of local counterexamples $\vec{A} = (A_0, \dots, A_{|\sigma|})$ such that $A_0 \subseteq A$ and $A_i \subseteq C_{\sigma(i-1)}$ for $0 < i \leq |\sigma|$.

We start noting that, given $\sigma \in T$, it is easy to find the corresponding sequence $\vec{A}$: for every $i \leq |\sigma|$, let $A_i = \{n \in P : \exists d \leq n((d, n, i) \in f(\sigma, n + 1))\}$. By the fact that $\sigma \in T$, we have that each one of the A_i's is infinite. Moreover, the construction of f ensures that they all are ascending sequences: for every $i < |\sigma|$, $m, r \in P$ and $d, e \in \mathbb{N}$, if $(d, m, i), (e, r, i) \in f(\sigma, n)$ for some n, then $m <_P r$ if and only if $d < e$. It is also clear from the construction that $A_0 \subseteq A$. Finally, we see that A_i is a local counterexample sequence to A_{i-1} for every $i > 0$: if $i > 0$, then we only add a new triple (d, n, i) if we can find two points $q >_P p$, both in $A_{\sigma(i-1)}$ (or in A_0 if $i = 1$), such that $n >_P p$ and $n \not\geq_P q$, and moreover, if $(d - 1, m, i)$ was also enumerated, in the construction we also require that $m \not>_P p$, which ensures that, if $m >_P r$ and $(e, r, i - 1) \in f(\sigma, n)$, then $n >_P s$ for the s such that $(e + 1, s, i - 1) \in f(\sigma, n)$, as we wanted.

Suppose now that $\vec{A} = (A_0, \dots, A_h)$ is a counterexample sequence such that $A_0 \subseteq A$. Let σ be the string given by $\sigma(i - 1) = j$, where j is such that $A_i \subseteq C_j$. We want to prove that $\sigma \in T$. To do this, we first uniformly refine the A_i's. Let $\alpha_0 : \mathbb{N} \to \mathbb{N}$ be defined as follows: $\alpha_0(0)$ is the $<_\mathbb{N}$-minimal element of A_0, and $\alpha_0(s + 1)$ is the $<_\mathbb{N}$-minimal element of A_0 that is larger than $\alpha_0(s)$ according to both $<_\mathbb{N}$ and $<_P$. Then, for every $i > 0$, we define simultaneously $\alpha_i : \mathbb{N} \to \mathbb{N}$ as follows: $\alpha_i(0)$ is the $<_\mathbb{N}$-minimal element a of A_i such that for some r $\alpha_{i-1}(r) <_P a$ and $\alpha_{i-1}(r + 1) \not<_P a$

(where such an r exists by the fact that $\vec{A}$ is a sequence of local counterexamples and Property 3.4.3), and $\alpha_i(s+1)$ is the $<_{\mathbb{N}}$-minimal $a \in A_i$ such that it is above $\alpha_i(s)$ according to $<_{\mathbb{N}}$ and to $<_P$ and for some r, $\alpha_{i-1}(r+1) <_{\mathbb{N}} a$, $\alpha_{i-1}(r) <_P a$, $\alpha_{i-1}(r+1) \not<_P a$ and $\alpha_{i-1}(r) \not<_P \alpha_i(s)$ hold (again, the fact that such a point exists is guaranteed by our assumptions on $\vec{A}$ and Property 3.4.3). Finally, we define the function

$$g(n) = \{(s, \alpha_i(s), i) : \alpha_i(s) \leq n \wedge i \leq h\}.$$

It is easy to verify that for every n $g(n) = f(\sigma, n)$. Since all of the α_i have infinite range, it follows that $\sigma \in T$. $\square$

Remark 3.4.8. We notice that the construction above is very uniform: in principle, given an *infinite* branch $B \in [k^{<\mathbb{N}}]$, we could extend f to produce for us an infinite sequence $(A_0, A_1, \dots)$ of chains such that $A_0 \subseteq A$, A_{i+1} is a local counterexample sequence to A_i and for $i > 0$ $A_i \subseteq C_{B(i-1)}$.

The final bit of the previous proof is the reason why we had to weaken strong local counterexample sequences to local counterexample sequences: we needed to be able to work with subsequences in order to carry out the verification that T behaves as we want.

As a consequence of this weakening, observe that the T found in the previous proof might contain many strings that are not useful for the proof of $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$: for instance, it might be the case that $0^k \in T$. We will essentially solve this issue by refining T: as we will see, considering the subtree $T' \subseteq T$ of strings σ such that $\sigma(0) \neq 0$ and $\sigma(i) \neq \sigma(i+1)$ contains the right amount of information in order to conclude that $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ holds.

Theorem 3.4.9. $\mathsf{RCA}_0 + \mathsf{ADS} + \mathsf{I}\Sigma^0_2 \vdash \mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$

Proof. Let P be an infinite poset with chain-decomposition-number k. By $\mathsf{B}\Sigma^0_2$, at least one of the chains of the decomposition of P is infinite, and without loss of generality we can suppose that C_0 is infinite. By applying ADS to C_0, we find an infinite ascending or descending sequence A. Again without loss of generality, we can suppose that A is ascending.

By Lemma 3.4.7, we can find the tree $T \subseteq k^{<k+1}$ such that $\sigma \in T$ if and only if we can find a sequence of infinite ascending sequences $(A_0, \dots, A_{|\sigma|})$ such that $A_0 \subseteq A$, A_0 is a local counterexample sequence to A and for every i A_{i+1} is a local counterexample sequence to A_i.

We let T' be the subtree of T defined as follows:

$$\sigma \in T' \leftrightarrow \sigma \in T \wedge \sigma(0) \neq 0 \wedge \forall i < |\sigma| - 1(\sigma(i) \neq \sigma(i+1)).$$

We have two cases:

1. T' has a leaf σ such that $|\sigma| < k$ (notice that this includes the case that T' is empty, since in this case $\emptyset$ is a leaf at level L_0). By Lemma 3.4.7, this means that we can build a sequence $\vec{A} = (A_0, \dots, A_{|\sigma|})$ of local counterexamples such that $A_i \in C_{\sigma(i-1)}$ for all $i > 0$. We claim that a tail of $A_{|\sigma|}$ is a solution. Suppose not, then by Lemma 3.4.5 Item 2 (which we can use since $\mathsf{B}\Sigma^0_2$ is a consequence of $\mathsf{I}\Sigma^0_2$) there is a strong local counterexample sequence $B \subseteq C_i$, for some $i < k$, to $A_{|\sigma|}$. But then, $\sigma^\frown(i)$ should be an element of T', since $A_{|\sigma|} \not\subseteq C_i$. This contradicts the fact that σ is a leaf. Hence, a tail of $A_{|\sigma|}$ is $(0, \infty)$-homogeneous.

2. Every leaf of T' is at level k. In this case, we define a coloring $c : T' \to k$ as follows: if $\sigma = \emptyset$, we put $c(\sigma) = 0$, otherwise we let $c(\sigma) = \sigma(|\sigma| - 1)$. As one can easily check, T' and c satisfy the hypotheses of Lemma 3.4.1. Let $\bar{\sigma}$ be the string given by the Lemma, let $S_{\bar{\sigma}} \subseteq k$ be the set of n's such that $\bar{\sigma}^\frown(n) \in T'$ and, finally, for every $n \in S_{\bar{\sigma}}$, let η_n be the extension of $\bar{\sigma}^\frown(n)$ such that $c(\bar{\sigma}) = c(\eta_n)$, whose existence is guaranteed by the Lemma. For every $\vec{A}_{\eta_n} = (A_0, \dots, A_{|\eta_n|})$, we define $B^n := A_{|\eta_n|}$. Moreover, if $\bar{\sigma} = \emptyset$, we put $B = A$, otherwise, if $\bar{\sigma} \neq \emptyset$ and $A_{\bar{\sigma}} = (A_0, \dots, A_{|\bar{\sigma}|})$, we set $B := A_{|\bar{\sigma}|}$, and enumerate it as $B = \{b_0 <_P b_1 <_P \dots\}$. We claim that there exists an $m \in \mathbb{N}$ such that $S_m := B_{\geq m} \cup \bigcup_{n \in S_{\bar{\sigma}}} B^n_{\geq m}$ (which is a set since every one of the component is) is

a $(0,\infty)$-homogeneous chain.

The fact that S_m is a chain is essentially given by Lemma 3.4.1: since $c(\bar{\sigma}) = c(\eta_n)$ for every $n \in S_{\bar{\sigma}}$, it follows that $B, B^n \subseteq C_{c(\bar{\sigma})}$, for every $n \in S_{\bar{\sigma}}$, and so in particular $B \cup \bigcup_{n \in S_{\bar{\sigma}}} B^n \subseteq C_{c(\bar{\sigma})}$, which implies that every S_m is a chain.

Next, we prove that one of the S_m is $(0,\infty)$-homogeneous, and we suppose towards a contradiction that it is not. We start noticing the following obvious fact: if $p \in C_n$ for some $n \in S_{\bar{\sigma}}$, or if $p \in C_{c(\bar{\sigma})}$, then p is comparable with infinitely many elements of S_m for every $m \in \mathbb{N}$. Suppose then that for some m S_m is not a solution: this means that there is a $p \in P$ such that p is comparable with some, but only finitely many, elements of S_m. What we just observed means that any counterexample to S_m is in $P \setminus \bigcup_{i \in \{c(\bar{\sigma})\} \cup S_{\bar{\sigma}}} C_i$. In particular, if this set is empty, we are done, so we assume that it is non-empty. Moreover, we notice that, if p a counterexample to S_m, then it is a counterexample to $B_{\geq m}$ as well: this follows from the fact that for every $n \in S_{\bar{\sigma}}$ $B \leq_{\forall\exists} B^n$. But then, combining the two previous observations, we can use Lemma 3.4.5 Item 2, applying it to the ascending sequence B and to the poset $P \setminus \bigcup_{i \in S_{\bar{\sigma}}} C_i$: if no S_m is $(0,\infty)$-homogeneous, then there is a local counterexample sequence $D \subseteq P \setminus \bigcup_{i \in S_{\bar{\sigma}}} C_i$. By $\mathsf{B}\Sigma^0_2$, we can assume that $D \subseteq C_j$ for some C_j in the chain decomposition of P. But this is contradiction: since $D \subseteq P \setminus \bigcup_{i \in S_{\bar{\sigma}}} C_i$, $j \notin S_{\bar{\sigma}}$, but since we produced a local counterexample sequence in C_j, this contradicts Lemma 3.4.7. Hence, for some m, S_m is a $(0,\infty)$-homogeneous, as we wanted.

□

We conclude this section with a remark about the "shape" of the chain produced in the Theorem above: whereas in the first case we find a $(0,\infty)$-homogeneous chain of order-type ω, this is not true for the second case. In particular, the argument above does not give a proof of $\mathsf{sRSpo}^{\mathsf{CD}}_{<\infty}$ (see Remark 3.1.7).

Here, the best that we can do is to present a dichotomy: we can always refine S_m to be a $(0,\infty)$-homogeneous chain of type ω or $\omega+\omega$. To see this, notice that instead of $\bigcup_{n \in S_{\bar{\sigma}}} B^n$, the proof would have worked just as well if we had refined it to an ascending chain B' cofinal in $\bigcup_{n \in S_{\bar{\sigma}}} B^n$, which can clearly be found in RCA_0, so that some tail of $B \cup B'$ is a $(0,\infty)$-homogeneous chain. This yields a chain of order-type ω if $B' \leq_{\forall\exists} B$, and a chain of order-type $\omega + \omega$ otherwise.

3.4.2. Reversals

In this subsection, we reverse the implications proved in the previous Theorem. We start by showing that over RCA_0 $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ implies ADS: we will actually prove more, namely that $\mathsf{RSpo}^{\mathsf{CD}}_3$ is already enough to have the implication. In the next section we will see that this result cannot be strengthened: $\mathsf{RSpo}^{\mathsf{CD}}_2$ is strictly weaker than ADS.

Lemma 3.4.10. $\mathsf{RCA}_0 + \mathsf{RSpo}^{\mathsf{CD}}_3 \vdash \mathsf{ADS}$. *So in particular* $\mathsf{RCA}_0 + \mathsf{RSpo}^{\mathsf{CD}}_{<\infty} \vdash \mathsf{ADS}$.

Proof. Let $(L, \leq_L)$ be a linear order and consider $(L \times 3, <_P)$ with the product partial order from $0 <_3 1$ and $2 <_3 1$: i.e., for every $p, q \in P$ and $i, j < 3$, $(p, i) \leq_P (q, j)$ if and only if $p \leq_P q$ and either $i = j$ or $j = 1$. Since $L \times 3$ has clearly width and chain-decomposition-number 3, let C be a $(0,\infty)$-homogeneous chain for $L \times 3$.

For each $i < 3$ set $C_i = C \cap (L \times i)$. By definition of $<_P$ it is easy to see that $C \subseteq C_0 \cup C_1$ or $C \subseteq C_1 \cup C_2$. In fact $(\ell, 0)$ and $(\ell, 2)$ are incomparable for each $\ell \in L$.

We claim that C_1 has no maximum. Suppose on the contrary that $(m, 1)$ is a maximum of C_1 and hence of C. Since $C_0 = \emptyset$ or $C_2 = \emptyset$ and both $(m, 0)$ and $(m, 2)$ are below $(m, 1)$, then at least one between $(m, 0)$ and $(m, 2)$ is comparable with some and finitely many elements of C. This contradicts the assumption that C is $(0,\infty)$-homogeneous. Hence, if $C_1 \neq \emptyset$, we can recursively define an ascending chain in it.

Otherwise, by RT^1_3 at least one between C_0 and C_2 is infinite. In this case either C_0 or C_2 has no minimum, otherwise there would be a point in $(L, 1)$ incomparable with all C but the minimum. It is thus possible to define recursively a descending chain in C_0 or C_2, which is obviously a descending chain in L. $\square$

Since it a known fact that $\mathsf{RCA}_0 + \mathsf{ADS} \vdash \mathsf{B}\Sigma^0_2$ (see for instance [39]), we have the following corollary:

Corollary 3.4.11. $\mathsf{RCA}_0 + \mathsf{RSpo}^{\mathsf{CD}}_{<\infty} \vdash \mathsf{B}\Sigma^0_2$

We can now proceed to the other reversal we need.

Lemma 3.4.12. $\mathsf{RCA}_0 + \mathsf{RSpo}^{\mathsf{CD}}_{<\infty} \vdash \mathsf{I}\Sigma^0_2$

Proof. We will prove that $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ implies the least number principle for a formula φ such that $\varphi(i) \equiv \forall x \exists y \psi(x, y, i)$, where ψ is Δ^0_0: suppose that there is a $k \in \mathbb{N}$ such that $\varphi(k)$ holds, we will find the least i such that $\varphi(i)$ holds. We build a partial order P of chain-decomposition-number $k+1$ as follows: for every triple $(x, y, i) \in \mathbb{N}^2 \times [0, k]$, $(x, y, i) \in P$ if and only if $\forall x' \le x \exists y' \le y \varphi(x', y', i)$ and $\forall y' < y \exists x' \le x \neg\varphi(x', y', i)$ hold, and we set $(x, y, i) \le_P (x', y', j)$ if and only if $(i \ge j \wedge x \le x')$. P can be decomposed into $k+1$ chains: every chain C_i, for $i \le k$, contains the elements of the form (x, y, i). Moreover, P is infinite, since we know that $\forall x \exists y \varphi(x, y, k)$ holds, and so C_k contains infinitely many elements, as can easily be shown using $\mathsf{I}\Sigma^0_1$. Notice that, for every $x \in \mathbb{N}$ and $i \le k$, there is at most one y such that $(x, y, i) \in P$. Finally, we notice that every element of the order is above only finitely many other elements of the order: for every $x \in \mathbb{N}$ and $i \le k$, (x, y, i) can be above at most $x(k+1-i)$ elements.

We apply $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$ to P, thus obtaining an infinite $(0, \infty)$-homogeneous chain S. By $\mathsf{B}\Sigma^0_2$, which is available to us thanks to the Corollary above, there is an $i \le k$ such that $C_i \cap S$ is infinite. We claim that i is minimal such that $\forall x \exists y \varphi(x, y, i)$. First, we show that $\forall x \exists y \varphi(x, y, i)$ holds: if this was not the case, then $\exists \bar{x} \forall y \neg\varphi(\bar{x}, y, i)$ holds. But then, if $(x, y, i) \in C_i$, $x < \bar{x}$, contradicting the hypothesis that $C_i \cap S$, and so in particular C_i, is infinite. Secondly, suppose for a contradiction that there is $j < i$ such that $\forall x \exists y \varphi(x, y, j)$. Let $(x, y, i) \in S$, then there is a $y' \in \mathbb{N}$ such that $(x, y', j) \in P$: then, $(x, y', j) >_P (x, y, i)$, and for every $x' > x$ and $y'' \in \mathbb{N}$, $(x, y', j) |_P (x', y'', i)$. So we only have to prove that there are at most finitely many elements of S above (x, y', j) in order to reach a contradiction. We will do better and prove that actually there are no points of S above (x, y', j): if there was even one, it should necessarily be of the form $(\tilde{x}, y, \tilde{j})$ for some $y \in \mathbb{N}$, $\tilde{x} \ge x$ and $j \ge \tilde{j}$, with at least one inequality strict. Since $S \cap C_i$ is infinite, there are $w, z \in \mathbb{N}$, with $w > \tilde{x}$, such that $(w, z, i) \in S$. But then, $(\tilde{x}, y, \tilde{j})|_P(w, z, i)$, contradicting the assumption that S is a chain. This proves the claim. $\square$

We can now put together the results obtained so far.

Theorem 3.4.13. *The following are equivalent over* RCA_0*:*

1. $\mathsf{I}\Sigma^0_2 + \mathsf{ADS}$*;*
2. $\mathsf{RSpo}^{\mathsf{CD}}_{<\infty}$*;*
3. $\mathsf{RSpo}^{\mathsf{W}}_{<\infty}$.

Proof. $1 \implies 2$ is Theorem 3.4.9, whereas $2 \implies 1$ is given by Lemmas 3.4.10 and 3.4.12.

The fact that $3 \implies 2$ is obvious, since $\mathsf{RCA}_0 \vdash \mathsf{RSpo}^{\mathsf{W}}_k \to \mathsf{RSpo}^{\mathsf{CD}}_k$, and since we also have that $\mathsf{RCA}_0 \vdash \mathsf{RSpo}^{\mathsf{CD}}_{5^k} \to \mathsf{RSpo}^{\mathsf{W}}_k$ by Lemma 3.1.5, the proof is complete. $\square$

3.4.3. A proof in ADS

In this final part of the section, we prove that the proof $\mathsf{RSpo}^{\mathsf{CD}}_k$ can be slightly modified in order to remove the use of induction in the case that k is a standard natural number. Let $(P, <_P)$ be an infinite k-decomposable partial order, where k is a standard integer, and let A be an ascending chain in C_0 (the case of a descending A is of course perfectly symmetric). The main idea of the proof is the following: since we do not have access to Σ^0_2 induction any more, we will not be able to build uniformly the counterexample tree given us by Lemma 3.4.7. But, since we are assuming that k is standard, we can proceed by "exhausting" the chains that can contain a counterexample sequence to A.

In order to do so, we will examine closely the structure of the proof of Theorem 3.4.9. The main idea of the proof is the following: given the ascending sequence A, either A already is a solution, or we can find an ascending chain B such that $A \leq_{\forall\exists} B$ and B can be extended to a solution. This is exactly the sort of statement that we will prove here, but with a different approach. More specifically, implementing also the observation at the end of the previous section about the shape of the solutions we are producing, we aim at proving the following statement for every standard $k > 1$:

($\clubsuit_k$) *Let $(P, <_P)$ be an infinite k-decomposable partial order, and $A \subseteq P$ be an infinite ascending chain. Then, there is a chain $B = B^0 \cup B^1$ such that*

- *B^0 is a chain of order-type ω such that $A \leq_{\forall\exists} B^0$,*
- *B^1 is either empty or a chain of order-type ω such that $B^0 <_P B^1$ (in which case $B^0 \cup B^1$ is a chain of order-type $\omega + \omega$), and*
- *B is a $(0, \infty)$-homogeneous chain for P.*

We will show that $\clubsuit_1$ and $\clubsuit_{k-1} \to \clubsuit_k$ hold: as we will explain in more detail later, this is enough to prove that $\clubsuit_k$ holds for every standard $k > 1$. We also remark that, in the following proof, we will not explicitly use the assumption on the shape of B (essentially, the second bullet point above), but we find it useful to keep in mind what sort of $(0, \infty)$-homogeneous chain we are aiming for.

Proof of $\clubsuit_1$ (RCA_0). If $k = 1$, then P is actually a linear order, so A itself is $(0, \infty)$-homogeneous and we can set $B^0 = A$ and $B^1 = \emptyset$. □

Proof of $\clubsuit_{k-1} \to \clubsuit_k$ (RCA_0). By RT^1_k, there is $i < k$ such that C_i contains infinitely many points of A. After a change of indices if necessary, we can assume that $i = 0$, and we let A^0 be the ascending chain $A \cap C_0$. We describe a procedure lasting at most $k - 1$ stages that is guaranteed to produce a solution: after s stages we have sequences of sets $(A^0, \ldots, A^s)$ and $(F^1, \ldots, F^s)$ such that:

1. for all $i \leq s$, A^i is an ascending sequence contained in $C_{h(i)}$, where $h : s + 1 \to k$ is an injection (with $h(0) = 0$), such that $A^0 \leq_{\forall\exists} A^i$;
2. for every $0 < i \leq s$, F^i is an ascending sequence contained in C^0, and $F^i \leq_{\forall\exists} F^{i+1}$ if $i < s$;

3. $A^i \leq_{\forall\exists} F^i$ for every $0 < i \leq s$, and $A^0 <_P F^1$.

At each stage we may find a $(0, \infty)$-homogeneous chain for P, in which case the construction ends.

We describe the construction of the sequences in stages.

Stage 1. If A^0 or any of its tails is $(0, \infty)$-homogeneous, we are done. So suppose this is not the case. Let A^1 be a local counterexample sequence to A^0, whose existence is given by Lemma 3.4.5 Item 1, and suppose by RT^1_k that it is contained in just one of the chains of the decomposition of P, say C_i: that chain cannot be C_0, so we can set $h(1) = i$. Let us now consider $P_0 := P \setminus C_0$, and apply $\clubsuit_{k-1}$ to this poset and the chain A^1, thus obtaining an infinite $(0, \infty)$-homogeneous (in P_0) chain $B_1 = B^0_1 \cup B^1_1$. If B_1 or one of its tails is a solution for P as well, then $\clubsuit_k$ is proved, since $A^0 \leq_{\forall\exists} A^1 \leq_{\forall\exists} S_1$. If this is not the case, then there are infinitely many points p_i comparable with only finitely many elements of B_1 and, by definition of B_1, they must all belong to C_0. In particular, there exists a local counterexample F^1 to B_1 with $F^1 \subseteq C_0$. Now, if F^1 and A^0 *interleaved*, i.e. if both $F^1 \leq_{\forall\exists} A^0$ and $A^0 \leq_{\forall\exists} F^1$ held, then B_1 would be a solution for P, since every $p \in C_0$ is either below infinitely many points of A^0, and hence below infinitely many points of B_1, or above infinitely many points of F^1, and hence above infinitely many points of B^1, so in either case p is comparable with infinitely many points of B_1, which would mean that B_1 is $(0, \infty)$-homogeneous. So we can assume A^0 and F^1 do not interleave: but then, $A^0 <_P F^1$ (if necessary after removing finitely many points from F^1) since no point of F^1 can be below infinitely many points of A^0. It is clear that the conditions 1, 2 and 3 above are satisfied. This ends stage 1.

Stage $s + 1$. We look for a local counterexample sequence to $A^0 \cup F^s$ in $P_s := P \setminus \bigcup_{i<s+1} C_{h(i)}$, i.e. in the chains not yet containing an A^i: if we cannot find any local counterexample, then in particular there is no real counterexample to $A^0 \cup F^s$ in P_s. But then, $A^0 \cup F^s$ is a solution for P: by construction, for every $i \leq s$, $A^0 \leq_{\forall\exists} A^i \leq_{\forall\exists} F^i \leq_{\forall\exists} F^s$, so every point of $p \in C_{h(i)}$ is above infinitely many points of A^0 (if p happens to be above infinitely many points of A^i) or below infinitely many

points of F^s (if p is below infinitely many points of A^i). Since we are assuming that no (real or local) counterexample to $A^0 \cup F^s$ is to be found in P_s (and obviously $A^0 \leq_{\forall\exists} A^0 \cup F^s$), our claim follows. Hence, we can assume that we can find a local counterexample sequence A^{s+1} in P_s. As before, we can suppose it is completely contained in a chain C_i, and we set $h(s+1) = i$. Similarly to stage 1, by $\clubsuit_{k-1}$ we have a solution B_{s+1} for P_0 such that $A^{s+1} \leq_{\forall\exists} B_{s+1}$, and we look for a local counterexample sequence to it, necessarily in C_0: if we cannot find any, then it means that B_{s+1} is a solution, otherwise we will find a local counterexample sequence $D^{s+1} \subseteq C_0$. Now, from enumerations $D^{s+1} = \{d^{s+1}_0 <_P d^{s+1}_1 <_P \dots\}$ and $F^s = \{f^s_0 <_P f^s_1 <_P \dots\}$, we produce $F^{s+1} = \{f^{s+1}_0 <_P f^{s+1}_1 <_P \dots\}$ by setting $f^{s+1}_i := \max_P\{f^s_i, d^{s+1}_i\}$ (recall that $F^s \subseteq C_0$, which guarantees that F^{s+1} is well-defined): this way, $F^s \leq_{\forall\exists} F^{s+1}$ and $A^{s+1} \leq_{\forall\exists} S_{s+1} \leq_{\forall\exists} F^{s+1}$. This concludes stage $s+1$.

Suppose we never found a $(0,\infty)$-homogeneous chain for P at an intermediate stage, so that we produced sequences $(A^0, \dots, A^{k-1})$ and $(F^1, \dots, F^{k-1})$. We claim that $B = A^0 \cup F^{k-1}$ is a solution for P. To see this, it is enough to notice that every point of every chain is comparable with infinitely many elements of $A^0 \cup F^{k-1}$: suppose $p \in C_i$, then by construction $\exists j < k(h(j) = i)$, so p is either above infinitely many points of A^j or below infinitely many points of A^j. In the first case, p is above infinitely many elements of A^0, whereas in the second p is below infinitely many points of F^j, and so of F^{k-1}. We can then set $B^0 = A^0$ and $B^1 = F^{k-1}$. This concludes the proof. □

Theorem 3.4.14. *For every standard $k \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{ADS} \leftrightarrow \mathsf{RSpo}^{\mathsf{CD}}_k \leftrightarrow \mathsf{RSpo}^{\mathsf{W}}_k$.*

Proof. $\mathsf{ADS} \to \mathsf{RSpo}^{\mathsf{CD}}_k$ was proved in Lemma 3.4.10, and since by Theorem 3.1.2 $\mathsf{RSpo}^{\mathsf{CD}}_{5^k} \to \mathsf{RSpo}^{\mathsf{W}}_k$, considering that if k is standard so is 5^k all we have to do is to show that $\mathsf{RCA}_0 \vdash \mathsf{ADS} \to \mathsf{RSpo}^{\mathsf{CD}}_k$ for standard k. To do so, we actually prove the stronger statements $\clubsuit_k$ for the corresponding k.

We can suppose, by changing indices if necessary, that the chain C_0 in the decomposition of P is infinite (at least one of the chains has to be, since the poset is infinite). Then, by applying ADS, we can find either a ascending or a descending sequence A in

C_0. Suppose that A is ascending, the other case being symmetric, and we let A be the ascending sequence in the statement of $\clubsuit_k$. Then, to prove $\clubsuit_k$, all we have to do is to go through the proof of $\clubsuit_1 \to \clubsuit_2 \to \cdots \to \clubsuit_{k-1} \to \clubsuit_k$, which can be done in RCA_0, since k is standard, and so the number of stages at every step of the construction above is standard: the proof above can be seen as a very long list of possible candidates for a solution, together with a proof that at least one of those candidates is a solution. $\square$

3.5. The case of $\mathsf{RSpo}_2^{\mathsf{CD}}$

In the previous section, we settled the question about the strength of $\mathsf{RSpo}_k^{\mathsf{W}}$ and of $\mathsf{RSpo}_k^{\mathsf{CD}}$ for each $k \geq 3$. As happens with Ramsey's theorem, $\mathsf{RSpo}_2^{\mathsf{W}}$ and $\mathsf{RSpo}_2^{\mathsf{CD}}$ are weaker principles.

3.5.1. Bounded version of SRT^2

To prove the equivalence between $\mathsf{RSpo}_2^{\mathsf{CD}}$ and SADS we will use a weakening of SRT^2_2, which corresponds to put a uniform bound on the number of oscillations of the coloring for every first component. This is made precise in the following definition.

Definition 3.5.1.
- (RCA_0) Let $c \colon [\mathbb{N}]^2 \to k$ be a coloring. We say that c is *n-stable* if for each $x \in \mathbb{N}$ it holds that $|\{y \mid c(x,y) \neq c(x,y+1)\}| \leq n$.
- For every $n, k \in \mathbb{N}$, n-SRT^2_k is the statement "Each n-stable coloring $c \colon [\mathbb{N}]^2 \to k$ contains an infinite homogeneous set".
- For every $n \in \mathbb{N}$, n-$\mathsf{SRT}^2_{\mathbb{N}}$ stands for $\forall k(n\text{-}\mathsf{SRT}^2_k)$.

We now gauge the strength of the principles that we stated above: although, to be precise, only Item 1 will be used in the rest of this section, we find it interesting to say a bit more about these new principles.

Lemma 3.5.2. *1. For each $n, k \in \omega$, RCA_0 proves n-SRT^2_k.*

2. *For each $n \in \omega$, RCA_0 proves that n-$\mathsf{SRT}^2_{\mathbb{N}}$ and $\mathsf{B}\Sigma^0_2$ are equivalent.*

Proof. We prove Item 1 by induction on n. For the base case let $c\colon [\mathbb{N}]^2 \to k$ be 0-stable. For every $j < k$, we define $H_j = \{x \in \mathbb{N} : c(x, x+1) = j\}$ for each $j < k$. To check that H_j is c-homogeneous let $x, y \in H_j$; by definition on H_j it hold that $c(x, x+1) = j$ and $c(y, y+1) = j$, thus $c(x, y) = j$ because c is 0-stable. By RT^1_k there exists $j < k$ such that H_j is infinite.

Now, assume that the statement is true for n-stable colorings and let $c\colon [\mathbb{N}]^2 \to k$ be $(n+1)$-stable.

If there is an x such that $c{\restriction}_{[\mathbb{N}\setminus\{0,\dots,x\}]^2}$ is n-stable, then the coloring $c{\restriction}_{[\mathbb{N}\setminus\{0,\dots,x\}]^2}$ contains an homogeneous set H by induction hypothesis, and clearly H is c-homogeneous as well.

Otherwise, there are infinitely many x such that $|\{y \mid c(x,y) \neq c(x, y+1)\}| = n+1$. Then we can computably find an infinite set of such x's and $n+1$ points $y^x_0, \dots, y^x_n$ such that $c(x, y^x_i) \neq c(x, y^x_{i+1})$, for each $i \leq n$, and such that for no other point this property holds. For every $j < k$, we define $H_j = \{x \in \mathbb{N} : c(x, y^x_n + 1) = j\}$, and by RT^1_k we can find an infinite subset of one of them, call this set $H = \{h_0 < h_1 < \dots\}$. By choice of y^x_n, it holds that $\forall y > y^x_n\, (c(x, y^x_n + 1) = c(x, y))$, so H can be refined to an infinite homogeneous set $\tilde{H}$ for c in the obvious way: at stage 0, enumerate h_0 in $\tilde{H}$, and at stage $s+1$ enumerate the first $h \in H$ such that $h > y^{h_s}_n$. This concludes the proof of Item 1.

Similarly to the proof of Lemma 3.1.9, the fact that $\mathsf{B}\Sigma^0_2$ implies n-$\mathsf{SRT}^2_{\mathbb{N}}$ follows from an inspection of the proof of n-SRT^2_k: all we need to do is to substitute the application of RT^1_k with one of $\mathsf{RT}^1_{<\infty}$, since the number of colors can now be non-standard.

Hence, we just have to prove that 0-$\mathsf{SRT}^2_{\mathbb{N}}$ implies $\mathsf{B}\Sigma^0_2$ over RCA_0, which is immediate: given any coloring $f : \mathbb{N} \to \mathbb{N}$ with range bounded by a certain $k \in \mathbb{N}$, let $c : [\mathbb{N}]^2 \to k$ be defined as $c(x, y) = i$ if and only if $f(x) = i$. Since c is clearly a 0-stable coloring and any c-homogeneous set is also f-homogeneous, we have the desired implication. This concludes the proof of Item 2. $\square$

There are still two principles that one might wish to consider: the first is $\forall n(n\text{-}\mathsf{SRT}^2_{\mathbb{N}})$: it can be seen that it is equivalent to $\mathsf{I}\Sigma^0_2$ over RCA_0, although we do not include a proof of this fact here.

The second principle would be $\forall n(n\text{-}\mathsf{SRT}^2_k)$, for a certain fixed $k \in \omega$. We do not know the precise strength of this principle, but we are able to give some bounds: clearly, it follows from the previous paragraph that it cannot be stronger than $\mathsf{I}\Sigma^0_2$, since $\mathsf{RCA}_0 \vdash \forall k(\forall n(n\text{-}\mathsf{SRT}^2_{\mathbb{N}}) \to \forall n(n\text{-}\mathsf{SRT}^2_k))$. On the other hand, we also have that $\forall n(n\text{-}\mathsf{SRT}^2_k)$ cannot be equivalent to $\mathsf{I}\Sigma^0_2$, since $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to n\text{-}\mathsf{SRT}^2_k$ for every standard k, but $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \nvdash \mathsf{I}\Sigma^0_2$ (see [13]).

We will see another principle with a similar behavior in the next Chapter.

3.5.2. SADS is equivalent to $\mathsf{RSpo}^{\mathsf{CD}}_2$.

We now move to the proof of the equivalence between $\mathsf{RSpo}^{\mathsf{CD}}_2$ and SADS. The proof of $\mathsf{SADS} \to \mathsf{RSpo}^{\mathsf{CD}}_2$ is based on the following observation: the proof of Theorem 3.4.14 makes use of ADS only at the very start, i.e. to produce the ascending sequence (or, equivalently, chain) that is then used in the rest of the argument. But, after we have our ascending sequence, the proof of $\clubsuit_2$ goes through in RCA_0.

The main idea of the following proof is hence to show that, in the case the poset P has chain-decomposition-number 2, we can use SADS instead of ADS.

Theorem 3.5.3 (RCA_0). SADS *implies* $\mathsf{RSpo}^{\mathsf{CD}}_2$.

Proof. Let $(P, <_P)$ be a poset and C_0, C_1 chains such that $P = C_0 \cup C_1$. Let $\{p_n \mid n \in \mathbb{N}\}$ and $\{q_n \mid n \in \mathbb{N}\}$ be enumerations of C_0 and C_1 respectively. Assume that P does not contain $(0, \infty)$-homogeneous chains.

We isolate two combinatorial claims that are used multiple times in the proof.

Claim 3.5.1. *If there exist $D \subseteq \mathbb{N}$ infinite and $n \in \mathbb{N}$ such that for each $d \in D$ and for each $m \geq n$ it holds that $p_d \mid_P q_m$, then P contains a $(0, \infty)$-homogeneous chain.*

Proof. Let $D \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ be as in the statement of the Claim. We define a coloring $f\colon D \to 2^n$ such that $f(d) = \langle b_0, \dots, b_{n-1}\rangle$ for $b_i = 0$ if $p_d \mid_P q_i$ and $b_i = 1$ if $p_d \between_P q_i$, for each $i < n$. By $\mathsf{RT}^1_{<\infty}$, which follows from SADS as proved in [12] (see also Theorem 1.1.26), there exists a set H homogeneous for f.

We claim that $S = \{p_h \mid h \in H\}$ is $(0,\infty)$-homogeneous. Notice that each element of C_0 is comparable with all elements of S, while each element of $C_1 \setminus \{q_0, \dots, q_{n-1}\}$ is incomparable with all elements of S, since $H \subseteq D$. Moreover, for each $i < n$, q_i is either incomparable with all elements of S or comparable with all elements of S, by homogeneity of H and by definition of f. □

Claim 3.5.2. *Suppose $f\colon H \to \mathbb{N}$ is a function such that $H \subseteq \mathbb{N}$ is infinite and $p_h \between_P q_{f(h)}$, for each $h \in H$. If there exists $H' \subseteq H$ infinite such that $f{\restriction}_{H'}$ is injective, then P contains an ascending or descending chain.*

Proof. Let f be a function with the required properties and $H' \subseteq H$ be an infinite set such that $f{\restriction}_{H'}$ is injective. There are either infinitely many $h \in H'$ such that $p_h <_P q_{f(h)}$ or infinitely many $h \in H'$ such that $p_h >_P q_{f(h)}$. Suppose the former is the case and let $\tilde{H} = \{h \in H' \mid p_h <_P q_{f(h)}\}$.

Consider the set $S = \{p_h \mid h \in \tilde{H}\}$. Since $S \subseteq C_0$, S is a linear order. If it is also stable, then SADS finds an ascending or a descending chain in S and so in P. Otherwise, let $n \in \tilde{H}$ be such that $p_n{\downarrow}_{(C_0,<_P)}$ and $p_n{\uparrow}_{(C_0,<_P)}$ are both infinite. We claim that for each $h \in \tilde{H}$ such that $p_n \leq_P p_h$, it holds that $q_{f(h)}{\uparrow}_{(C_1,<_P)}$ is finite. Suppose this does not hold and let $q_{f(h)}{\uparrow}_{(C_1,<_P)}$ be infinite. Then $p_n{\downarrow}_{(C_0,<_P)} \cup q_{f(h)}{\uparrow}_{(C_1,<_P)}$ is a chain, it contains infinitely many elements in both C_0 and C_1 and is thus $(0,\infty)$-homogeneous, contrary to the assumption.

Hence, we have proved that the set $\{q_{f(h)} \mid p_n \leq_P p_h, h \in \tilde{H}\}$ is a descending chain.

If there exist infinitely many $h \in H'$ such that $p_h >_P q_{f(h)}$, an analogous reasoning, with the obvious changes, allows to get the desired conclusion. □

Suppose one of the decomposition chains is finite and name it C_1. By Claim 3.5.1, with $D = \mathbb{N}$ and $n = |C_1|$, P contains a $(0,\infty)$-homogeneous chain, contrary to the assumption.

Suppose now both C_0 and C_1 are infinite. Define a coloring $c\colon [\mathbb{N}]^2 \to 4$ as follows:

$$c(n,m) = \begin{cases} 0 & \text{if } \forall i \leq m\,(p_n \mid_P q_i) \\ 1 & \text{if } \exists i\,(n < i \leq m \wedge p_n <_P q_i) \\ 2 & \text{if } \forall i\,(n < i \leq m \to p_n \not<_P q_i) \wedge \exists i\,(n < i \leq m \wedge p_n >_P q_i) \\ 3 & \text{if } \exists i\,(i \leq n \wedge p_n \between_P q_i) \wedge \forall i\,(n < i \leq m \to p_n \mid_P q_i) \end{cases}$$

Notice that, for each $n \in \mathbb{N}$, $c(n,\cdot)$ changes color at most twice. By 2-SRT^2_4 (available in RCA_0, see Lemma 3.5.2) there exists an infinite homogeneous set H for c. Thanks to H we define an ascending or descending chain in P.

We claim that H is not homogeneous for 0. Suppose on the contrary that it is and let $S = \{p_h \mid h \in H\}$. Clearly each $p \in C_0$ is comparable with S, while each $q \in C_1$ is incomparable with S by the homogeneity of H. It follows that S is $(0,\infty)$-homogeneous contrary to the assumption.

Suppose now that H is c-homogeneous for 1 and consider the set $S = \{p_h \mid h \in H\}$. Let $f\colon H \to \mathbb{N}$ be such that, for each $h \in H$, $f(h)$ is minimum such that $h < f(h)$ and $p_h <_P q_{f(h)}$. It follows straightforwardly from c-homogeneity for 1 that f is total. Moreover, we claim that f is injective. Suppose that $h < k$ and $h,k \in H$. Then, again by c-homogeneity for 1 of H, there exists $i < k$ such that $p_h <_P q_i$, so $f(h) < k$. Now consider $c(k,j)$, for some $j \in H$, $j > k$: by c-homogeneity for color 1, there exists $r > k$ such that $p_k <_P q_r$, so $f(k) > k > f(h)$. By Claim 3.5.2 P contains an ascending or descending chain.

If H is c-homogeneous for color 2, we can reason analogously, so we are left to the case of H being c-homogeneous for color 3.

Notice that if $c(h,k) = 3$, for some $h < k$, $h,k \in H$, then there exists $i \leq h$ such that $p_h \between_P q_i$. We consider two cases depending whether there exists $n \in \mathbb{N}$ such that, for each $h \in H$, if $p_h \between_P q_i$, then $i < n$, or not. If the former is the case, then for each

$h \in H$ and for each $m \geq n$ it holds that $p_h \mid_P q_m$ and by Claim 3.5.1, with $D = H$, we reach a contradiction.

If the latter is the case, it is not difficult to see that, since H is c-homogeneous for 3, for each $n \in \mathbb{N}$ there exist $h > n$, $h \in H$, and $i > n$ such that $i < h$ and $p_h \between q_i$. Define $f\colon H \to \mathbb{N}$ such that $f(h)$ is the minimum i such that $p_h \between_P q_i$, for each $h \in H$. It follows from the assumption that there exists an infinite set $H' \subseteq \mathrm{dom}(f)$ such that $f{\restriction}_{H'}$ is injective. By Claim 3.5.2 P contains an ascending or descending chain.

Suppose P contains an ascending chain. Then $\clubsuit_2$ guarantees that there exists a $(0,\infty)$-homogeneous chain. If P contains a descending sequence D, then $\clubsuit_2$ applied on $(P, >_P)$ and D guarantees that there exists a $(0,\infty)$-homogeneous chain. Thus, we reach a contradiction, and the claim is proved. □

We observe that the proof above could easily be recast to a direct proof (i.e., not a proof by contradiction). We presented it this way because we feel that the *reductio ad absurdum* makes the argument somewhat more streamlined.

We now prove the reversal of the Theorem above. Again, this can be seen as a product of a careful analysis of what happens in the case of $\mathsf{RSpo}_k^{\mathsf{CD}}$ for k larger than 2.

Theorem 3.5.4. *Over* RCA_0, SADS *is equivalent to* $\mathsf{RSpo}_2^{\mathsf{CD}}$.

Proof. We are left to prove the reversal. Let $(L, <_L)$ be an infinite stable linear order. Consider $P = (L \times \{2\}, <_P)$ with the product partial order (from $0 < 1$). Clearly, $L \times \{2\}$ has chain-decomposition-number two. Let C be $(0,\infty)$-homogeneous and set $C_i = C \cap (L \times i)$ for each $i < 2$. By RT_2^1 at least one between C_0 and C_1 is infinite

Suppose C_0 is infinite. If each $(c, 0) \in C_0$ has finitely many predecessors, then it is possible to enumerate computably an ω chain contained in C_0 and hence in L. Otherwise, let $(c, 0) \in C_0$ be such that c has infinitely many predecessors. Notice that since L is stable, c has finitely many successors. We claim that if $(c', 0) \in C_0$, then c' has finitely many successors. Suppose on the contrary that $(c', 0) \in C_0$ has finitely many predecessors. Notice that C_1 must be finite, because $(c, 0)$ has only finitely many successors and $(c,0) \mid_P (d,1)$ for each $d <_L c$ by definition of $<_P$. Then $(c',1)$ is comparable with some and only finitely many elements of C, contrary to the fact that C is $(0,\infty)$-homogeneous. This proves that each element of C_0 has finitely many successors and so it is an infinite ascending chain contained in C_0 and hence in L, which can then be refined to an infinite ascending sequence.

If C_1 is infinite and each element of C_1 has finitely many successors, then C_1 is an infinite descending chain. Otherwise, arguing as in the previous paragraph it is possible to show that C_1 contains an infinite ascending chain. Since, as above, SADC is equivalent to SADS over RCA_0, the Theorem is proved. □

Corollary 3.5.5. *Over* WKL*,* SADS *is equivalent to* $\mathsf{RSpo}^{\mathsf{W}}_2$*.*

Proof. Let $(P,<_P)$ be a poset of width two. By Dilworth's theorem let C_0 and C_1 be chains such that $P = C_0 \cup C_1$. By Theorem 3.5.3 P contains a $(0,\infty)$-homogeneous chain.

Since the partial order $(L \times 2, <_P)$ defined in the proof of Theorem 3.5.4 has width two, the same argument provides a reversal for $\mathsf{RSpo}^{\mathsf{W}}_2$ as well. □

As a consequence of the previous theorem we get that $\mathsf{RSpo}^{\mathsf{W}}_2$ is strictly weaker than ADS, since ADS and $\mathsf{WKL}+\mathsf{SADS}$ are incomparable (see [39], Corollaries 2.16 and 2.28), and not computably true. We do not know whether $\mathsf{RSpo}^{\mathsf{W}}_2$ is equivalent to SADS over RCA_0 as well or whether it lies strictly in between SADS and ADS, although we do know that it has ω-models consisting of low sets, as a consequence of the fact that the theory $\mathsf{WKL}_0 + \mathsf{SADS}$ has such models (again, this follows from results from [39]).

Question 3.5.6. Over RCA_0, is SADS equivalent to $\mathsf{RSpo}^{\mathsf{W}}_2$?

3.6. Beyond $\mathsf{RSpo}^{\mathsf{CD}}$ and $\mathsf{RSpo}^{\mathsf{W}}$

In the previous sections, we were able to characterize the strength of all the principles of type $\mathsf{RSpo}^{\mathsf{CD}}_k$ and $\mathsf{RSpo}^{\mathsf{W}}_k$, for $k \in \mathbb{N} \cup \{<\infty\}$ (with the exception of $\mathsf{RSpo}^{\mathsf{W}}_2$), but we have so far said little about $\mathsf{sRSpo}_k^{\mathsf{CD}}$ and $\mathsf{sRSpo}_k^{\mathsf{W}}$. In this section, we will try to say something more on this subject, although, as we shall see, we will not be able to find satisfactory bounds for the strength of the vast majority of the strong Rival-Sands principles.

We will focus on two different directions: first we analyze the principle $\mathsf{sRSpo}_2^{\mathsf{CD}}$, and show that $\mathsf{RCA}_0 + \mathsf{I}\Sigma_2^0 \vdash \mathsf{ADS} \to \mathsf{sRSpo}_2^{\mathsf{CD}}$ and $\mathsf{RCA}_0 \vdash \mathsf{sRSpo}_2^{\mathsf{CD}} \to \mathsf{ADS}$. Although this is clearly a limited result, we will show that it has some interesting consequences.

Secondly, we consider $\mathsf{sRSpo}_{<\infty}^{\mathsf{CD}}$: although we are unable to provide a proof of it in a system weaker than Π_1^1-CA_0, we will succeed in providing a proof in that system of a more general principle, which we shall call $\mathsf{sRSpo}_{\mathbb{N}}$.

3.6.1. $\mathsf{sRSpo}_2^{\mathsf{CD}}$

We start by proving that $\mathsf{sRSpo}_2^{\mathsf{CD}}$ implies ADS over RCA_0. This is achieved with a proof similar to that of Lemma 3.4.10.

Theorem 3.6.1 (RCA_0)**.** $\mathsf{RCA}_0 \vdash \mathsf{sRSpo}_2^{\mathsf{CD}} \to \mathsf{ADS}$.

Proof. Let $(L, <_L)$ be a linear order and let $P = (L \times \{2\}, <_P)$ the order on the Cartesian product of L, so that $(\ell, i) <_P (m, j) \Leftrightarrow \ell <_L m \wedge i < j$. Such a poset clearly has chain-decomposition-number 2, so let $C \subseteq P$ be $(0, \mathrm{cof})$-homogeneous. For each $i < 2$ set $C_i = C \cap (L \times \{i\})$.

We claim that if C_0 is infinite, then C_0 has no minimum, and can thus be refined to a descending chain. Suppose on the contrary that C_0 is infinite and that $(m, 0)$ is minimum in C_0. By definition of $<_P$ it holds that $(m, 0) <_P (m, 1)$ and $(n, 0) \mid_P (m, 1)$, for each $n >_L m$. It follows that $(m, 1)$ is comparable with some elements of C and incomparable with infinitely many elements of C, contrary to the assumption that C is $(0, \mathrm{cof})$-homogeneous.

Similar reasoning allows us to prove that if C_1 is infinite, then C_1 has no maximum, and hence that L contains an ascending chain. $\square$

This theorem has several interesting consequences.

Corollary 3.6.2. *1. For each $k \leq 2$, $\mathsf{sRSpo}_k^{\mathsf{W}}$ and $\mathsf{sRSpo}_k^{\mathsf{CD}}$ imply ADS over RCA_0.*

2. (RCA_0) Let us fix a $k \in \mathbb{N}$, and let $(P, <_P)$ be a partial order of chain-decomposition-number k. Then, $\mathsf{sRSpo}_k^{\mathsf{CD}}$ implies that P has an infinite ascending sequence that is $(0, \infty)$-homogeneous (and hence $(0, cof)$-homogeneous) for P.

Proof. Item 1 follows immediately from Theorem 3.6.1 and Lemma 3.1.5

We apply $\mathsf{sRSpo}_k^{\mathsf{CD}}$ to the poset P, thus obtaining an infinite $(0, \mathrm{cof})$-homogeneous $C \subseteq P$. Next, we notice that any infinite subset $C' \subseteq C$ is still $(0, \mathrm{cof})$-homogeneous for P. To see this, let us consider any element $p \in P$: if p was comparable with no element of C, then of course p is comparable with no element of C'; if instead p was comparable with cofinitely many elements of C, there were only finitely many elements of C p was not comparable with. Hence, there are at most finitely many elements of C' that are not comparable with p, which proves that C' is $(0, \mathrm{cof})$-homogeneous for P.

Since by Item 1 $\mathsf{sRSpo}_k^{\mathsf{CD}}$ implies ADS, we can find an infinite ascending sequence in C, call it S, and by the previous paragraph this is still an infinite $(0, \mathrm{cof})$-homogeneous chain for P. □

In essence, Item 2 above tells us that we do not lose in generality if we restrict our search for $(0, \mathrm{cof})$-homogeneous chains to ascending chains, which is an interesting fact.

Moreover, we point out, on a more qualitative level, that Theorem 3.6.1 is enough to conclude that $\mathsf{sRSpo}^{\mathsf{CD}}$ and $\mathsf{RSpo}^{\mathsf{CD}}$ are not, so to speak, the same principle: in fact, we proved in the previous section that $\mathsf{RCA}_0 \vdash \mathsf{SADS} \leftrightarrow \mathsf{RSpo}_2^{\mathsf{CD}}$, whereas we now know that $\mathsf{RCA}_0 \vdash \mathsf{sRSpo}_2^{\mathsf{CD}} \rightarrow \mathsf{ADS}$.

Finally, we give an upper bound on the strength of $\mathsf{sRSpo}_2^{\mathsf{CD}}$.

Lemma 3.6.3. $\mathsf{ADS} + \mathsf{I}\Sigma_2^0 \vdash \mathsf{sRSpo}_2^{\mathsf{CD}}$

Proof. Let P be an infinite poset with chain-decomposition-number 2. We assume for the sake of simplicity that every $n \in \mathbb{N}$ is in P. Using ADS, we can find either an ascending or a descending chain A in it: as usual, we suppose that it is ascending, the other case being similar. By refining A if necessary, we can suppose that $A \subseteq C_0$.

We will use the function f defined in Lemma 3.4.7. Let S be the set of strings $\{1, 10, 101, 1010, \dots\}$, and, for $i > 0$, let σ_i be the element of S of length i. There are two cases: either for every i and for every d there is an n such that $(d, n, i) \in f(\sigma_i, n{+}1)$, or not. We will find a $(0, \mathrm{cof})$-homogeneous chain in both cases.

Suppose first that we are in the latter case: then, by $\mathsf{I}\Sigma^0_2$, there is a minimal i such that for some d, for every n it holds that $(d, n, i) \notin f(\sigma_i, n + 1)$. Notice that $i > 0$. Then, let B be the set $\{n : \exists d \leq n((d, p, i - 1) \in f(\sigma_i, n))$. Then, B is an ascending sequence, it is infinite by the definition of i, and a tail of it is $(0, \infty)$-homogeneous by Lemma 3.4.5 Item 1. Hence, that tail is $(0, \mathrm{cof})$-homogeneous.

Next, suppose that for every i and for every d there are a p and an n such that $(d, p, i) \in f(\sigma_i, n)$. In this case, as B we consider the set $\{n : \exists i \leq n((0, n, i) \in f(\sigma_i, n + 1))\}$. The hypotheses of this case (and $\mathsf{I}\Sigma^0_2$) guarantee that B is infinite. Moreover, it is an ascending sequence, since by construction $\{(0, n, i), (0, m, i + 1)\} \in f(\sigma_{i+1}, m + n)$ implies $n <_P m$. Moreover, $B \cap C_0$ and $B \cap C_1$ are both infinite. To show that B is $(0, \mathrm{cof})$-homogeneous we can then argue as in the final part of the previous Theorem: B is ascending and $(0, \infty)$-homogeneous, since there can be no counterexample to it thanks to the fact that $B \cap C_0$ and $B \cap C_1$ are infinite, so B is $(0, \mathrm{cof})$-homogeneous. □

We end this subsection by saying that the result above actually extends to $\mathsf{sRSpo}^{\mathsf{W}}_2$, although we will not give the proof here.

3.6.2. $\mathsf{sRSpo}_{\mathbb{N}}$

So far, we have only studied partial orders $(P, <_P)$ of finite width, i.e. posets such that the size of all the antichains is bounded by a certain number k: after all, it is immediately seen that there are posets of infinite width without infinite antichains, let alone infinite $(0,\infty)$-homogeneous chains.

There is, however, an intermediate case: we could simply ask that the poset P does not have infinite antichains. In this section, we will show that $\mathsf{sRSpo}^{\mathsf{CD}}$ extends to this case as well, and we will prove this in Π^1_1-CA_0.

Definition 3.6.4. $\mathsf{sRSpo}_{\mathbb{N}}$ is the following statement: "for every partial order $(P,<_P)$ without infinite antichains, there is an infinite chain $C \subseteq P$ that is $(0,\mathrm{cof})$-homogeneous for P.

Clearly, $\mathsf{RCA}_0 \vdash \mathsf{sRSpo}_{\mathbb{N}} \to \mathsf{sRSpo}^{\mathsf{W}}_{<\infty}$, hence $\mathsf{sRSpo}_{\mathbb{N}}$ implies, over RCA_0, all the Rival-Sands principles that we have examined so far in this chapter.

In order to prove the result, we will need to introduce some concepts related to the structure of partial orders.

Definition 3.6.5. (RCA_0) Let $(P,<_P)$ be a partial order.

- A set $A \subseteq P$ is said to be a *strong antichain* in P if A is an antichain with the additional property that for every distinct $a_0, a_1 \in A$ there is no $p \in P$ such that $p >_P a_0$ and $p >_P a_1$.
- A set $I \subseteq P$ is an *ideal* of P if $I{\downarrow}_{(P,<_P)} = I$ and for every $i_0, i_1 \in I$ there is $i_2 \in I$ such that $i_2 \geq_P i_0$ and $i_2 \geq_P i_1$.
- We say that P is an *essential finite union of ideals* if there are $k \in \mathbb{N}$ and ideals $I_0, \dots, I_{k-1}$ such that $P = \bigcup_{j<k} I_j$ and moreover $\forall j < k\big(I_j \neq \bigcup_{l<k, l\neq j} I_l\big)$.

We will make use of the following result.

Theorem 3.6.6. *[[29], Lemma 3.3 and Theorem 4.1]* (ACA_0) *Let $(P,<_P)$ be a partial order. Then, the following are equivalent:*

- *P does not contain infinite strong antichains.*

- *P is an essential finite union of ideals.*

We now move to the proof of the main result.

Theorem 3.6.7. $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \mathsf{sRSpo}_{\mathbb{N}}$.

Proof. Let $(P, <_P)$ be a partial order without infinite antichains. By CAC, P contains an infinite chain C, and by ADS applied to C there is an infinite ascending or descending sequence $S \subseteq C$. We assume that S is ascending, the other case being symmetrical.

We then consider the following set $\widetilde{P}$:

$$\widetilde{P} = \{p \in P : p\uparrow_{(P,<_P)} \text{ is reverse ill-founded}\}.$$

Since being reverse ill-founded is a Σ^1_1 condition, we can build the set $\widetilde{P}$ using $\Pi^1_1\text{-}\mathsf{CA}_0$ (see Theorem 1.1.13). Since $S \subseteq \widetilde{P}$, $\widetilde{P}$ is infinite, and in particular non-empty.

Since $\widetilde{P} \subseteq P$, the poset $(\widetilde{P}, <_P)$ does not have infinite antichains, so in particular it does not have infinite strong antichains. Hence, by Theorem 3.6.6 (which we can use since we are working in a system stronger than ACA_0), we can assume to have an essential finite ideal decomposition of $\widetilde{P}$, say given by the ideals $I_0, \dots, I_{k-1}$.

We notice that none of the I_j has a maximal element. Suppose for a contradiction that i_j is a maximal element of I_j, i.e. $\forall i \in I_j(i \leq_P i_j)$. Since $i_j \in \widetilde{P}$, there is an $\widetilde{i} \in \widetilde{P} \setminus I_j$ such that $\widetilde{i} >_P i_j$. Let $l < k$ be such that $\widetilde{i} \in I_l$, then it would follow that $I_j \subseteq I_l$, which contradicts the properties of the ideals we are considering.

From the previous paragraph, it follows that every I_j is infinite. Let us enumerate I_0 as $\{i_0, i_1, \dots\}$. We define an ascending sequence $C := \{c_0 <_P c_1 \dots\}$ as follows: let $c_0 := i_0$ and $c_{n+1} := i_{\min\{l : i_l >_P c_n, i_l >_P i_n\}}$. The fact that a c_{n+1} as we want exists follows from the properties of ideals and the fact that I_0 has no maximal element.

Finally, we claim that at least one tail of C is $(0, \mathrm{cof})$-homogeneous for P. Since C is an ascending sequence, it is enough to verify that at least one tail of C is $(0, \infty)$-homogeneous (see Remark 3.1.7). Suppose for a contradiction that it is not, then by Lemma 3.3.6 there is an infinite ascending sequence $D := \{d_0, d_1, \dots\}$ that is a counterexample sequence to C.

Let l be such that for infinitely many i's $d_i \in I_l$. Then, clearly, $D \subseteq I_l$, since D is a chain. But by the definition o f C, it follows that $I_0 \subseteq D \downarrow_{(\tilde{P}, <_P)}$, which contradicts our assumption that $\tilde{P}$ is the essential union of the I_j's. □

We are not able to precisely gauge the strength of $\mathsf{sRSpo}_{\mathbb{N}}$. Anyway, we can observe that a lower bound to it is given by CAC: given any poset P, it either has an infinite antichain, or it satisfies the hypotheses of $\mathsf{sRSpo}_{\mathbb{N}}$, and hence contains an infinite chain.

3.7. A remark on cardinalities

Up to this point, due to the reverse mathematical approach we stuck to, we have only dealt with countable structures. It is, anyway, legitimate to ask whether there are any analogues to the principles we studied in this chapter and the previous one if we were to drop the requirement that graphs and posets be countable.

These questions were asked, and largely answered, in [30]: for instance, in the case of RSg, the shape that a possible extension to that theorem can have for graphs of cardinality κ strongly depends on the regularity of κ.

Theorem 3.7.1 ([30], Theorems 1 and 2)**.**

- *Let κ be an infinite regular cardinal and let (G, E) be a graph with $|G| = \kappa$. Then, there exists a set $H \subseteq G$ such that $|H| = \kappa$ and such that for every element $g \in G$, there are 0, 1 or κ many elements of H adjacent to g.*

- *If κ is a singular cardinal, the previous result does not hold. However, for every graph (G, E) with $|G| = \kappa$ and for every $\alpha < \kappa$, we can find a set $H \subseteq G$ such that $|H| = \kappa$ and for every $g \in G$, g is adjacent to 0, 1 or at least α many elements of H.*

The situation for $\mathsf{RSpo}^{\mathsf{W}}$ and $\mathsf{sRSpo}^{\mathsf{W}}$ is slightly more complicated: after all, other than removing the limitations on the size of the poset P, one could ask for instance if we can also relax the condition on the width of the poset, or how liberal one can be when it comes to deciding what the analogues of $(0,\infty)$- and $(0,\mathrm{cof})$-homogeneity are in this setting.

To start addressing these questions, we give the following definition,which generalizes the concept of $(0,\infty)$-homogeneity.

Definition 3.7.2. Let κ be an infinite cardinal and let $(P,<_P)$ be a partial order with $|P|=\kappa$. We say that a chain $C \subseteq P$ is $(0,\kappa)$-*homogeneous* for P if every element $p \in P$ is comparable with 0 or at least κ many elements of C.

As one can easily see, $(0,\infty)$-homogeneity is just $(0,\kappa)$-homogeneity when $\kappa=\omega$.

We start by seeing what happens with the "obvious" analogues of $\mathsf{RSpo}^{\mathsf{W}}$: that is, we want to see if, given a poset of size κ but of finite width, we can find a $(0,\kappa)$-homogeneous chain of size κ. Again, regularity seems to play a prominent role.

Theorem 3.7.3 ([30], Theorems 3 and 4)**.**

- *Let κ be an infinite regular cardinal, and let $(P,<_P)$ be an infinite poset of finite width with $|P|=\kappa$. Then there is a chain $C \subseteq P$ such that $|C|=\kappa$ which is $(0,\kappa)$-homogeneous for P.*
- *Let κ be a singular cardinal. Then, there is a poset $(P,<_P)$ of width 2 and with $|P|=\kappa$ such that it has no $(0,\kappa)$-homogeneous chains.*

We now turn our attention to analogues of $\mathsf{sRSpo}^{\mathsf{W}}$: in this case, there are at least two approaches that seem legitimate: given a poset $(P,<_P)$ of size κ, we could look for $(0,\mathrm{cof})$-homogeneous chains of size κ (notice that the definition given in second-order arithmetic still makes sense in this context), or, less restrictively, we could look for a chain C of size κ such that every point $p \in P$ is incomparable with either all the elements of C or less than κ many elements of C (notice that, in this second formulation, we would get $(0,\mathrm{cof})$-homogeneity if we put $\kappa=\omega$). In the next lemma, we show that none of this approaches leads to an interesting principle if $\mathrm{cof}(\kappa)>\omega$.

Lemma 3.7.4. *Let κ be an infinite cardinal such that $\mathrm{cof}(\kappa)>\omega$. There exists a poset $(P,<_P)$ with $|P|=\kappa$ and with $w(P)=2$ such that for every chain $C \subseteq P$ with*

$|C| = \kappa$ such that C is $(0,\kappa)$-homogeneous for P, there is a $p_C \in P$ such that p_C is comparable with κ many points of C and is not comparable with κ many points of C.

Proof. We consider the following partial order $(P, <_P)$: let P be the set $\kappa \times \omega \times 2$, and set $(\alpha, n, i) <_P (\beta, m, j)$ if and only if:

- $i = j$, $n = m$ and $\alpha < \beta$, or
- $i = j$ and $n < m$, or
- $i = 0$, $j = 1$ and $n < m$, or
- $i = 1$, $j = 0$ and $n < m + 1$, or
- $i = 0$, $j = 1$, $n = m$ and $\alpha < \beta$, or
- $i = 1$, $j = 0$, $n = m + 1$ and $\alpha < \beta$.

It is not too difficult to verify that this relation is indeed a partial order. The idea is that $(P, <_P)$ is made up of two interleaving chains, each of order type $\kappa\omega$.

In the following, we will call $P_{n,i}$ the sets $\{(\alpha, n, i) \in P : \alpha < \kappa\}$.

Notice that P contains $(0,\kappa)$-homogeneous chains of size κ: for instance, the set $\{(\alpha, n, i) : \alpha < \kappa, n \in \{0, 1\}, i = 0\}$ is such a chain, as can be easily verified.

Let C be a $(0,\kappa)$-homogeneous chain of size κ. Since we are assuming that $\mathrm{cof}(\kappa) > \omega$, there is at least one $n \in \omega$ and an $i < 2$ such that $|P_{n,i} \cap C| = \kappa$.

Suppose for a contradiction that there is a unique $n \in \omega$ and a unique $i < 2$ such that $|P_{n,i} \cap C| = \kappa$. Let β be the least ordinal γ such that $(\gamma, n, i) \in C$. Then there are two cases:

- if $i = 0$, $(\beta, n, 1) >_P (\beta, n, 0)$ and $(\beta, n, 1)|_P c$ for every other element of $C \cap P_{n,0}$; but then, $(\beta, n, 1)$ is comparable with fewer than κ many elements.
- If $i = 1$, $(\beta, n + 1, 0) >_P (\beta, n, 1)$ and $(\beta, n + 1, 0)|_P c$ for every other element of C; but then, $(\beta, n + 1, 0)$ is comparable with fewer than κ many elements.

Both cases contradict our hypothesis that C is $(0, \kappa)$-homogeneous for P.

We make now an observation that will be useful later: by the definition of P, it holds for all n and i that if $|P_{n,i} \cap C| = \kappa$, then $|P_{n,1-i} \cap C| < \kappa$.

We are now ready to find the point p_C we are looking for. Let $n \in \omega$ be minimal such that there is an $i < 2$ such that $|P_{n,i} \cap C| = \kappa$, and let let m be the minimal number larger than n with the same property: say that $|P_{m,j} \cap C| = \kappa$. Notice that such an m has to exists, thanks to the observation above.

Let β be the minimal γ such that $(\gamma, n, i) \in P_{n,i} \cap C$. Again, there are two cases:

- if $i = 0$, we set $p_C = (\beta, n, 1)$.
- If $i = 1$, we set $p_C = (\beta, n + 1, 0)$.

In both cases, p_C is incomparable with every element (bar one) of $P_{n,i} \cap C$, and is comparable with every element of $P_{m,j} \cap C$. Hence, there are κ many elements of C such that p_C is incomparable with them as well as κ many elements of C that p_C is comparable with. This concludes the proof of the lemma. □

We still do not know whether the result above can be extended to cardinals with cofinality ω.

Finally, we can ask what happens if we relax the requirement on the width of the poset. Again, at least in the case of cardinals with cofinality larger than ω, there seem to be no obvious analogue of $\mathsf{RSpo}^{\mathsf{W}}$.

Theorem 3.7.5 (essentially [30], Theorem 5)**.** *Let κ be an infinite cardinal with $\operatorname{cof}(\kappa) > \omega$. Then there is a poset $(P, <_P)$ of cardinality κ with no infinite antichains that has no $(0, \kappa)$-homogeneous chains of size κ.*

Again, we do not know what happens in the case of cofinality ω.

4. Some asymmetric Ramseyian principles

In this Chapter, we deal with some Ramseyan principles that can be regarded, in some sense, as being asymmetric: the principles we are going to consider have as instances colorings $f : [\mathbb{N}]^n \to 2$ such that we *require* that no infinite f-homogeneous set has color 1. Hence, we can say that there is a strong asymmetry between the color 0 and the color 1.

This Chapter is divided in two parts. The first, corresponding to Section 4.1, deals with what we might consider to be the most fundamental form of an asymmetric Ramsey's theorem: we study the principles bRT^n_k, which are the restrictions of Ramsey's theorem for n-tuples and two colors to the instances f such that the size of the f-homogeneous sets for color 1 is bounded by the number k. We start by studying these principles from the point of view of reverse mathematics: in Subsection 4.1.1, we prove that, if $n > 3$, then bRT^n_k is equivalent to ACA_0 over RCA_0. This leaves open the cases of $n = 2$ and $n = 3$ (since bRT^1_k is easily seen to be provable in RCA_0). We give some bounds for the strength of both: we first prove that every instance of bRT^2_k is provable in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$, which in particular implies that bRT^2_k is computably true, and then show that bRT^3_k implies RT^2_2 but not ACA_0. This last result, which can be found in Subsection 4.1.2, relies on a general framework recently developed by Ludovic Patey in [58], to which we give a minimal introduction. Finally, we focus on the complexity of the solutions for the principles bRT^n_k by analyzing the closely related principle uRT^n. We point out that the results of this Section are joint work with Emanuele Frittaion, with some important contributions by David Belanger and Keita Yokoyama.

In the second part, corresponding to Section 4.2, we analyze another, and arguably historically more relevant, form of asymmetric Ramsey's theorem, namely the result that ω^2 is a partition ordinal. Partition ordinals arise quite naturally in the pursuit to generalize Ramsey theory to ordinals larger than ω, and were studied by combinatorialists of the caliber of Erdős (see e.g. [25]). After a brief, largely historical introduction in Subsection 4.2.1, in Subsection 4.2.2 we give two formalizations in second-order arithmetic of the theorem $\omega^2 \longrightarrow (\omega^2, 3)$ (we will explain this notation in due course), namely SPL_3 and SSPL_3. We then examine two classical proofs of the theorem in Subsection 4.2.3, and see that one of them can be modified to show that SPL_3 and SSPL_3 are both provable in ACA_0. Finally, in Subsection 4.2.4, we give some initial results on the study of the complexity of the solutions of SPL_3 and SSPL_3.

4.1. Bounded Ramsey's theorem

In this section, we will focus on principles that can be seen as forms of RT^n_2 where we put some bounds on the size of the homogeneous sets for one of the two colors. As we pointed out in the introduction to this Chapter, this is joint work with Emanuele Frittaion (with contributions of Keita Yokoyama and David Belanger), and, as a project, can still be considered to be in its initial phases.

4.1.1. Reverse Mathematics of bRT

Let us define the principles we will be studying in this section.

Definition 4.1.1.

- for every $n \geq 2$ and $k \geq n$, bRT^n_k is the statement "for every coloring $f : [\mathbb{N}]^n \to 2$ such that for every f-homogeneous $H \subseteq \mathbb{N}$ with $|H| \geq k$ $f([H]^2) = 0$ holds there is an infinite f-homogeneous set".
- for every n, bRT^n is the statement "for every coloring $f : [\mathbb{N}]^n \to 2$, if for some $k \in \mathbb{N}$ every finite set $H \subseteq \mathbb{N}$ with $|H| = k$ that is f-homogeneous is f-homogeneous for 0, then there is an infinite f-homogeneous set".

We make one remark to the definition above: there would have been no harm in including the case $n = 1$ in the definition, but since both bRT^1 and bRT^1_k are immediately seen to be consequences of RT^1_2, we exclude this case and focus only on the non-trivial principles.

We start by analyzing the behavior of the principles we just introduced in the reverse mathematical context. Some facts are immediately clear, and we present them without proof.

Lemma 4.1.2. *The following can be proved in* RCA_0*:*

1. *for every* $n \geq 2$*,* bRT^n_n *holds.*
2. *for every* $l > k > n \geq 2$*,* $\mathsf{bRT}^n \to \mathsf{bRT}^n_l \to \mathsf{bRT}^n_k$
3. *for every* $n \geq 2$*,* $\mathsf{RT}^n_2 \to \mathsf{bRT}^n$*.*

As we will see (and rather unsurprisingly), the exponent n is of utmost importance when determining the strength of the principles bRT^n_k. We start with the simplest non-trivial case, i.e. that of $n = 2$.

Lemma 4.1.3. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \mathsf{bRT}^2_3$

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be such that there are no f-homogeneous sets for color 1 of size 3. There are two cases:

1. First, we suppose that there is $x \in \mathbb{N}$ such that for for infinitely many y $f(x, y) = 1$. Then, we claim that the set $H := \{y \in \mathbb{N} : f(x, y) = 1\}$ is an infinite f-homogeneous set (for color 0). H is infinite by our assumption on x. Now, suppose for a contradiction that we can find $y_0, y_1 \in H$ such that $f(y_0, y_1) = 1$: then, the set $\{x, y_0, y_1\}$ would be an f-homogeneous set for color 1, which gives us the required contradiction.
2. We can then assume that no x as above exists: hence, for every x, $\lim_{y\to\infty} f(x, y)$ exists and equals 0. Then, it follows from $\mathsf{B}\Sigma^0_2$ that for every finite set F that is

f-homogeneous for color 0, we can find a y such that $F \cup \{y\}$ is f-homogeneous for color 0. Hence we can recursively build an infinite f-homogeneous set $H = \bigcup H_i$ by starting with a pair of numbers a, b such that $f(a,b) = 0$ as H_0 and then setting $H_{s+1} := H_s \cup \{x\}$, where x is minimal such that $H_s \cup \{x\}$ is f-homogeneous for 0.

□

$\mathsf{B}\Sigma^0_2$ is also sufficient to pass from bRT^2_k to bRT^2_{k+1}.

Lemma 4.1.4. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \forall k(\mathsf{bRT}^2_k \to \mathsf{bRT}^2_{k+1})$

Proof. The proof is similar to the one of the Lemma above. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring without f-homogeneous sets of size $k+1$. Suppose at first that there is an $x \in \mathbb{N}$ such that for for infinitely many y $f(x,y) = 1$. Let $X := \{y \in \mathbb{N} : f(x,y) = 1\}$. The set X is infinite by our assumption, and if $H \subset X$ with $|H| = k$ and is f-homogeneous, then it is f-homogeneous for 0: if it was f-homogeneous for 1, then $H \cup \{x\}$ would be f-homogeneous for 1 and would have size $k+1$, which is a contradiction. We can then apply bRT^2_k to $f\restriction_{[H]^2}$ to obtain an infinite f-homogeneous set.

If there is no x as above, then we can repeat the argument of Item 2 of the Lemma above to construct an infinite f-homogeneous set. □

We do not known whether any of the Lemmas above reverses. Moreover, it is also unclear whether $\mathsf{RCA}_0 \vdash \mathsf{bRT}^2_k \to \mathsf{bRT}^2_{k+1}$ holds: the difficulty here is, essentially, Item 2 of Lemma 4.1.3 above, where removing the use of $\mathsf{B}\Sigma^0_2$ does not seem easy.

The two Lemmas above are anyway enough to yield some immediate consequences:

Corollary 4.1.5. *For every standard k, $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \mathsf{bRT}^2_k$. Hence, for every standard k, REC is a model of bRT^2_k.*

We now turn out attention to the more general case of bRT^2. First of all, we show that RCA_0 does not imply it. The combinatorial argument used in the proof will be a major tool in the rest of this section.

Lemma 4.1.6. $\mathsf{RCA}_0 + \mathsf{bRT}^2 \vdash \mathsf{B}\Sigma^0_2$

Proof. Let $g : \mathbb{N} \to \mathbb{N}$ be a function of bounded range, with bound, say, k, for a certain $k \in \mathbb{N}$. By Theorem 1.1.20, it suffices to prove that there exists an infinite g-homogeneous set. We define a function $f : [\mathbb{N}]^2 \to 2$ as follows: for every $x < y$, we put $f(x, y) = 0$ if $g(x) = g(y)$, and $f(x, y) = 1$ otherwise.

We claim that for every finite set H of size $k + 1$, if H is f-homogeneous, then it is f-homogeneous for color 0. Suppose for a contradiction that this is false, and so let $H' = \{h_0, \dots, h_k\}$ be an f-homogeneous set for 1 of size $k+1$. But since RCA_0 proves that for every k there is no injection from $k+1$ to k, there are $i, j \leq k$ with $i \neq j$ such that $g(h_i) = g(h_j)$. Hence $f(h_i, h_j) = 0$, which contradicts our assumption on H'.

Hence, we can apply bRT^2 to f: let H be an infinite f-homogeneous set. Since it is f-homogeneous for 0, by definition of f, it is also an infinite g-homogeneous set. $\square$

Again, we do not know if this implication can be reversed. The best known upper bound on the strength of bRT^2 is given by the following lemma, which we obtained with the help of Keita Yokoyama. In order to do this, we will use the Erdős-Rado tree associated to a coloring f.

Definition 4.1.7. (RCA_0) Let $f : [\mathbb{N}]^n \to k$ be a coloring, for some non-zero $n, k \in \mathbb{N}$. The *Erdős-Rado tree associated to* f is the tree $T^f \subseteq \mathbb{N}^{<\mathbb{N}}$ defined as follows. For every string $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $\sigma \in T^f$ if and only if, the following three conditions hold:

1. $(0, \dots, n-2) \sqsubseteq \sigma$ or $\sigma \sqsubseteq (0, \dots, n-2)$,

and if $|\sigma| > n - 1$, for all $s < |\sigma|$, $\sigma(s)$ is the such that

2. for all $m < s$, $\sigma(m) < \sigma(s)$,

3. for all $m_1 < m_2 < \cdots < m_{n-1} < m \leq s$, $f(\sigma(m_0), \dots, \sigma(m_{n-1}), \sigma(m')) = f(\sigma(m_0), \dots, \sigma(m_{n-1}), \sigma(s))$, and

4. there is no $x < \sigma(s)$ such that for all $m_1 < m_2 < \cdots < m_{n-1} < s$, $f(\sigma(m_0), \ldots, \sigma(m_{n-1}), x) = f(\sigma(m_0), \ldots, \sigma(m_{n-1}), \sigma(s))$.

The fundamental property of T^f is that, if $g \in [T^f]$, then $\operatorname{ran} g$ is an infinite *prehomogeneous set* for f, i.e. an infinite set $P \subseteq \mathbb{N}$ such that for every $a \in [P]^{n-1}$ and every $x, y \in P \setminus [0, \ldots, \max a]$, $f(a \cup \{x\}) = f(a \cup \{y\})$.

We give an intuition of why this is the case under the assumption that the domain of f is $[\mathbb{N}]^2$ (which, on the other hand, is the only case we are going to care about in this thesis): (0) can be regarded as the root of T^f, since it is the only successor of $\emptyset$ by definition. Then, suppose that $f(0,1) = 0$ and $f(0,2) = 1$: by checking the definition, it is clear that $(0,1)$ and $(0,2)$ are both in T^f, but that for instance $(0,1,2)$ is not. Hence, from now on, a number j will be put in the tree T^f above 1 if $f(0,j) = 0$, and above 2 if $f(0,j) = 1$ (although it is maybe not immediately obvious why *every* number should appear in T^f: we will show it in the Lemma below). It is then clear that for every $g \in T^f$ and every $x \in \operatorname{ran} g$, $f(0,x)$ only depends on $g(1)$, namely it only depends on whether g extends $(0,1)$ or $(0,2)$. We could argue in a similar fashion for every level of the tree.

For completeness, we give a proof of the fact that RCA_0 is enough to prove that T^f is infinite and finitely branching.

Lemma 4.1.8 (Essentially [66], Lemma III.7.4 and [36], page 81). (RCA_0) *For every coloring $f : [\mathbb{N}]^n \to k$, T^f is finitely branching and infinite.*

Proof. To show that T^f is finitely branching, it is enough to observe that every string $\sigma \in T^f$ has at most one successor for every function $g : [\operatorname{ran} \sigma]^{n-1} \to k$.

To prove that T^f is infinite, we prove that for every $j \in \mathbb{N}$ there is a string $\sigma \in T^f$ such that $\sigma^\frown(j) \in T^f$. Suppose for a contradiction that this is false. Then, by definition, $j > n-2$. Then, we notice that the string $(0, \ldots, n-2)^\frown(j)$ would satisfy Items 1, 2 and 3 of the Definition above. Hence, since by our assumption $(0, \ldots, n-2)^\frown(j) \notin T^f$, it means that j is not the minimal number satisfying those properties, i.e. there is a $j' < j$ such that $f(0, \ldots, n-2, j') = f(0, \ldots, n-2, j)$.

Let T^f_j be the finite subtree of T^f such that $\sigma \in T^f_j$ if and only if $\sigma \in T^f$ and $\forall i < |\sigma|(\sigma(i) < j)$. Let us enumerate T^f_j as $\{\sigma_0, \ldots, \sigma_{|T^f_j|-1}\}$ in such a way that for every $i, i' < |T^f_j|$, if $\sigma_i \sqsubseteq \sigma_{i'}$, then $i \leq i'$. Let $h < |T^f_j|$ be maximal such that $\sigma_h^\frown(j)$ satisfies Items 1, 2 and 3 of the Definition above. Such an h has to exists by the observations made in the previous paragraph. Then, we claim that $\sigma_h^\frown(j) \in T^f$. To see this, suppose for a contradiction that $\sigma_h^\frown(j) \notin T^f$: since by assumption $\sigma_h^\frown(j)$ satisfies Items 1, 2 and 3, this means that there is a $j' < j$ such that $\sigma_h^\frown(j') \in T^f$ and for all $m_1 < m_2 < \cdots < m_{n-1} < m \leq |\sigma_h|$, $f(\sigma_h(m_0), \ldots, \sigma_h(m_{n-1}), \sigma_h^\frown(j')(m)) = f(\sigma_h(m_0), \ldots, \sigma_h(m_{n-1}), j)$.

Then, this means that $\sigma_h^\frown(j')^\frown(j)$ would be an extension of $\sigma_h^\frown(j')$ satisfying Items 1, 2 and 3: to see that Item 3 is satisfied, notice that for all $m_1 < m_2 < \cdots < m_{n-1} < m \leq |\sigma_h| + 1$:

- if $m < |\sigma_h| + 1$, then by the previous paragraph

$$\begin{aligned} &f(\sigma_h^\frown(j')^\frown(j)(m_0), \ldots, \sigma_h^\frown(j')^\frown(j)(m_{n-1}), \sigma_h^\frown(j')^\frown(j)(m)) \\ &\quad = f(\sigma_h^\frown(j')(m_0), \ldots, \sigma_h^\frown(j')(m_{n-1}), \sigma_h^\frown(j')(m)) \\ &\quad = f(\sigma_h^\frown(j')(m_0), \ldots, \sigma_h^\frown(j')(m_{n-1}), j) \\ &\quad = f(\sigma_h^\frown(j')^\frown(j)(m_0), \ldots, \sigma_h^\frown(j')^\frown(j)(m_{n-1}), j), \end{aligned}$$

  as we wanted, and

- if $m = |\sigma_h| + 1$, then obviously

$$\begin{aligned} &f(\sigma_h^\frown(j')^\frown(j)(m_0), \ldots, \sigma_h^\frown(j')^\frown(j)(m_{n-1}), \sigma_h^\frown(j')^\frown(j)(m)) = \\ &\qquad f(\sigma_h^\frown(j')^\frown(j)(m_0), \ldots, \sigma_h^\frown(j')^\frown(j)(m_{n-1}), j) \end{aligned}$$

This contradicts the minimality of h, and hence proves the Lemma. $\square$

We are now ready to give the upper-bound on the strength of bRT^2.

Lemma 4.1.9. $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 \vdash \mathsf{bRT}^2$

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring such that for a certain number k every f-homogeneous set of size k is f-homogeneous for 0, and let T^f be the Erdős-Rado tree associated to f. We fix an enumeration $\{\sigma_0, \sigma_1, \dots\}$ of it with the property that if $\sigma_i \sqsubseteq \sigma_{i'}$, then $i \leq i'$.

Since by our assumption there are infinitely many s such that for no $i_0, \dots, i_{k-1} < |\sigma_s|$ $\{\sigma(i_0), \dots, \sigma(i_{k-1})\}$ is f-homogeneous for 1, we can use $\mathsf{I}\Sigma^0_2$ (see Theorem 1.1.16) to find the least $h \leq k$ such that for infinitely many s for no $i_0, \dots, i_{h-1} < |\sigma_s|$ $\{\sigma(i_0), \dots, \sigma(i_{h-1})\}$ is f-homogeneous for 1.

From now on, to make the exposition more streamlined, we will use the following convention: for every $s, m \in \mathbb{N}$, with $m > 1$ the formula $\varphi(s, m)$ stands for "for no $i_0, \dots, i_{m-1} < |\sigma_s|$ $\{\sigma(i_0), \dots, \sigma(i_{m-1})\}$ is f-homogeneous for 1" (notice that for $m = 0$ or 1 the formula would make no sense). For instance, then, h above is defined as the least number such that for infinitely many s $\varphi(s, h)$ holds.

Clearly, if $1 < n < m$, $\varphi(s, n)$ implies $\varphi(s, m)$, and RCA_0 is enough to prove this.

Notice that necessarily $h > 1$. On the other hand, if $h = 2$, then we are done: the set S_2 of indices s such that $\varphi(s, 2)$ holds is an infinite Δ^0_1 set, so we can prove its existence in RCA_0. Moreover, by the definition of T^f, for every $s, t \in S_2$ with $s < t$, it holds that $\sigma_s \sqsubseteq \sigma_t$: to see this, suppose this was not the case, and suppose that there are $s, t \in S_2$ with $s < t$ such that $\sigma_s \not\sqsubseteq \sigma_t$: then, $\sigma_t \not\sqsubseteq \sigma_s$ also holds by the way we defined the enumeration of T^f. Let σ_r be the longest segment they have in common (notice that $\sigma_r \neq \emptyset$, since $(0) \sqsubseteq \sigma_r$). But then, by the definition of T^f, we can conclude that $f(\sigma_r(|\sigma_r| - 1), \sigma_s(|\sigma_r|) \neq f(\sigma_r(|\sigma_r| - 1), \sigma_t(|\sigma_r|)$. Hence, at least one of these values is 1, contradicting the definition of S_2. Then, we can define the set $\bigcup \mathrm{ran}(\sigma_i)$ in RCA_0, which is an infinite f-homogeneous set.

Hence, we are left with the case that $h > 2$. By minimality of h, for every $1 < h' < h$ there are only finitely many s such that $\varphi(s, h')$ holds. By $\mathsf{B}\Sigma^0_2$, we can find a t such that for every $s \geq t$ $\neg\varphi(s, h-1)$ holds. Let n be the maximal length of a string σ_r for $r < t$.

Let S_h be the set of numbers $s > t$ such that $|\sigma_s| \geq n+1$ and $\varphi(s, h)$ holds. As for S_2

above, this set is infinite and Δ^0_1. For every $s \in S_h$, we define $g(s) = \tau$, for $\tau \in \mathbb{N}^{n+1}$, if $\tau \sqsubseteq \sigma_s$. Since RCA_0 proves that g takes finitely many values (namely, 2^{n+1}), it also proves that g has bounded range, and so by $\mathsf{B}\Sigma^0_2$ there is an infinite set $H \subseteq \mathbb{N}$ that is g-homogeneous, and let τ' be the string $g(H)$.

Since T^f is a tree, it follows that $\tau' \in T^f$, so let τ' be σ_q for some $q \in \mathbb{N}$. Since $|\tau'| > n$, it follows that $\neg\varphi(q, h-1) \wedge \varphi(q, h)$ holds, and the same holds for every $s \in H$. Hence, similarly to what we did for S_2, by the way T^f is defined, we can conclude that for every $s, r \in H$, if $s < r$ then $\sigma_s \sqsubseteq \sigma_r$. Hence, again similarly to the case $h = 2$, we can conclude that $\bigcup_{s\in H} \operatorname{ran}(\sigma_s) \setminus \operatorname{ran}(\tau')$ is an infinite f-homogeneous set: to see this, recall that, by the discussion right before Lemma 4.1.8, $\bigcup_{s\in H} \operatorname{ran}(\sigma_s)$ is an infinite prehomogeneous set for f. Hence, to refine it to an infinite f-homogeneous set, we have to remove the points $x \in \bigcup_{s\in H} \operatorname{ran}(\sigma_s)$ such that $f(x, y) = 1$ for some $y \in \bigcup_{s\in H} \operatorname{ran}(\sigma_s)$ with $y > x$. But by how we defined τ', all those points are in $\operatorname{ran}(\tau')$. $\square$

We know that the Lemma above admits no reversal: since bRT^2 is a consequence of RT^2_2 and by the results of [15] RT^2_2 does not imply $\mathsf{I}\Sigma^0_2$ over RCA_0, it follows that $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{I}\Sigma^0_2$. We are currently unable to show whether, as seems likely, bRT^2 is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0: we do not know what the precise strength of bRT^2 over RCA_0 is.

There is a rather substantial literature on combinatorial principles weaker than $\mathsf{I}\Sigma^0_2$, and among these principles the so-called Ramsey theorem for singletons on trees, denoted TT^1, is of particular interest: introduced in [24], where it was also proved that $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 \vdash \mathsf{TT}^1$ and $\mathsf{RCA}_0 \vdash \mathsf{TT}^1 \rightarrow \mathsf{B}\Sigma^0_2$, it was shown in [16] that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \nvdash \mathsf{TT}^1$ (and it was later shown, in [14], that TT^1 is also strictly weaker than $\mathsf{I}\Sigma^0_2$). Although it does not seem that the techniques developed for TT^1 are easily applicable to the case of bRT^2, it would be interesting to investigate what the precise link between these two principles is.

However, we have another result concerning the strength of bRT^2.

Theorem 4.1.10. $\mathsf{RCA}_0 \vdash \mathsf{EM} \rightarrow \mathsf{bRT}^2$

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring such that for some k, if F has size k and F is f-homogeneous, then it is f-homogeneous for 0. We define the following binary relation R on $\mathbb{N}$: for numbers $x < y$, we set xRy if $f(x, y) = 0$, and yRx otherwise. $(\mathbb{N}, R)$ is a tournament, since for every pair of points $x, y \in \mathbb{N}$ either xRy or yRx holds, and R is antireflexive. Hence, we can apply EM to obtain an infinite set D on which R is transitive. In particular, (D, R) is a linear order.

Now, we define the binary relation $\prec$ on D as follows: for every $x, y \in D$, we let $x \prec y$ if and only if $x < y$ and xRy (i.e. $f(x, y) = 0$). It is easy to check that $(D, \prec)$ is a partial order.

We notice that D, when seen as a partial order, cannot have antichains of size larger than $k - 1$: suppose for a contradiction that there is an antichain $A = \{a_0 < a_1 < \cdots < a_{k-1}\}$. Then, since $a_i \not\prec a_j$, $f(a_i, a_j) = 1$ for every $i < j < k$, which contradicts our assumption on f. Since, by [45, Proposition 16], $\mathsf{RCA}_0 \vdash \mathsf{EM} \to \mathsf{B}\Sigma^0_2$, we can apply $\mathsf{CC}_{<\infty}$ (which, we recall, is equivalent to $\mathsf{B}\Sigma^0_2$ by Lemma 3.1.9) to get an infinite chain C for the partial order D. C is clearly an infinite f-homogeneous set for color 0. □

Although the result above does not narrow the interval of possible strength of bRT^2 *per se*, it can be seen as a possible new approach to study it.

We now move to the study of bRT^n_k for $n > 2$. We start with an easy result.

Lemma 4.1.11. *For every $n \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{bRT}^{n+1}_{n+2} \to \mathsf{bRT}^n_{n+1}$*

Proof. Let $f : [\mathbb{N}]^n \to 2$ be a coloring such that every set F with size $n + 1$ that is f-homogeneous is f-homogeneous for 0. We define the coloring $g : [\mathbb{N}]^{n+1} \to 2$ by putting

$$g(x_0, \ldots, x_{n-1}, x_n) = f(x_0, \ldots, x_{n-1}).$$

Then given every set $F = \{y_0 < \cdots < y_{n+1}\}$, if F were g-homogeneous for color 1, then $F \setminus \{y_{n+1}\}$ would be f-homogeneous for 1, which is a contradiction. Hence we can apply bRT^{n+1}_{n+2} to g, thus obtaining an infinite g-homogeneous set H, which is clearly also f-homogeneous. □

Next, we move to study the relationship between bRT^{n+1} and RT^n.

Theorem 4.1.12. *For every $n \in \omega$, $\mathsf{RCA}_0 \vdash \mathsf{bRT}^{n+1}_{n+2} \to \mathsf{RT}^n_2$.*

Proof. The proof is by induction on n.

The case $n = 1$ follows from the fact that $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_2$. Alternatively, and more uniformly, it follows from the argument by which Lemma 4.1.6 was proved: given a coloring $f : \mathbb{N} \to 2$, we define the coloring $g : [\mathbb{N}]^2 \to 2$ by setting $g(x_0, x_1) = 0$ if and only if $f(x_0) = f(x_1)$. It is clear that there are no g-homogeneous sets of size 3, and that every infinite g-homogeneous set is also an infinite f-homogeneous set.

Supposing now that we have proved the statement for $n = m - 1$, we will prove it for $n = m$. Let $g : [\mathbb{N}]^m \to 2$ be a coloring. Then, we define the coloring $f : [\mathbb{N}]^{m+1} \to 2$ as follows: for every $m + 1$-tuple $x_0 < x_1 < \cdots < x_m$, we set

$$f(x_0, x_1, \ldots, x_m) = \begin{cases} 0 \text{ if } g(x_0, \ldots, x_{m-2}, x_{m-1}) = g(x_0, \ldots, x_{m-2}, x_m) \\ 1 \text{ otherwise} \end{cases}$$

Suppose that there is an f-homogeneous set F with $|F| = m + 2$, say $F = \{y_0 < y_1 < \cdots < y_m < y_{m+1}\}$. Since it is impossible that $g(x_0, \ldots, x_{m-2}, x_{m-1})$, $g(x_0, \ldots, x_{m-2}, x_m)$ and $g(x_0, \ldots, x_{m-2}, x_{m+1})$ are all pairwise different, it follows that F is f-homogeneous for 0. Hence, we can apply bRT^{m+1}_{m+2} to f: let H be an infinite f-homogeneous set for color 0.

We notice that $g(x_0, \ldots, x_{m-2}, x) = g(x_0, \ldots, x_{m-2}, x')$ for every $\{x_0 < \cdots < x_{m-2} < x < x'\} \subseteq H$. We can then define the coloring $h : [H]^{m-1} \to 2$ by letting $h(x_0, \ldots, x_{m-2}) = g(x_0, \ldots, x_{m-2}, x)$, where x is the minimal element of H larger than x_{m-2}: by the observation above, every infinite h-homogeneous set is also an infinite g-homogeneous set.

By Lemma 4.1.11, we know that $\mathsf{RCA}_0 \vdash \mathsf{bRT}^{m+1}_{m+2} \to \mathsf{bRT}^m_{m+1}$. But by induction hypothesis, we have that $\mathsf{RCA}_0 \vdash \mathsf{bRT}^m_{m+1} \to \mathsf{RT}^{m-1}_2$, hence bRT^{m+1}_{m+2} guarantees the existence of an infinite h-homogeneous set H'. By our considerations above, H' is also a g-homogeneous set, thus proving the Theorem. $\square$

By the fact that for every $m > 2$ and $l \geq 2$ $\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \leftrightarrow \mathsf{RT}^m_l$, we have the following result.

Corollary 4.1.13. *For every $n > 3$, $k > n$, $m > 2$ and $l \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{bRT}^n_k \leftrightarrow \mathsf{RT}^m_l \leftrightarrow \mathsf{bRT}^n \leftrightarrow \mathsf{RT}^m$.*

The case $n = 3$ is not covered by the previous result: although we do not find the precise strength of bRT^3_k or bRT^3, we give an upper bound for it in the next section, by proving that they do not imply ACA_0 over RCA_0.

4.1.2. bRT^3_k admits cone avoidance

As we mentioned, the cases of bRT^3_k and bRT^3 are not covered by the result above, although we can deduce that $\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \to \mathsf{bRT}^3 \to \mathsf{bRT}^3_k \to \mathsf{RT}^2_2$.

In this section, we will show that, perhaps unsurprisingly, bRT^3_k does not imply ACA_0 over RCA_0, and we will then extend the result to show that bRT^3 does not imply ACA_0 either. In order to accomplish this, we will use a very general framework introduced by Patey in [58] to determine which Ramsey principles $\mathsf{RT}^n_k(V, W)$ (which we will introduce below) have (strong) cone-avoidance.

To begin with, we introduce the problems that will be analyzed in this section. As anticipated, we will focus on the case $n = 3$.

Definition 4.1.14.

- For every $n > 1$ and $k > n$, bRT^n_k is the following problem:
  - Input: a coloring $f : [\omega]^n \to 2$ such that, if $H \subseteq \omega$ is f-homogeneous and $|H| \geq k$, then $f([H]^n) = 0$.
  - Output: an infinite f-homogeneous set.
- For every $n > 1$, bRT^n is the following problem:
  - Input: a coloring $f : [\omega]^n \to 2$ such that, for some $k \in \omega$, if $H \subseteq \omega$ is f-homogeneous and $|H| \geq k$, then $f([H]^n) = 0$.
  - Output: an infinite f-homogeneous set.

It is very easy to see that if we manage to show that bRT^3_k admits cone avoidance for every k, then bRT^3 admits cone avoidance as well: this follows from the fact that every instance of bRT^3 is an instance of bRT^3_k, for some sufficiently large k. Hence, we will focus on the problems bRT^3_k.

We now introduce the problems $\mathsf{RT}^n_k(V, W)$.

Definition 4.1.15.
- An RT^n_k*-pattern* P is a finite set of of tuples of the form $\langle v, D\rangle$, where $v < k$ and $D \in [\omega]^n$.

- Given an RT^n_k-pattern $P = \{\langle v_0, D_0\rangle, \dots, \langle v_{l-1}, D_{l-1}\rangle\}$, a coloring $f : [\omega]^n \to k$ and a set of integers $E = \{m_0 < m_1 < \dots < m_{r-1}\}$, we say that E f*-satisfies* P if for every $s < l$, letting $E_s = \{m_i : i \in D_s\}$, $f(E_s) = v_s$ holds.

- A set $H \subseteq \omega$ f*-avoids* P if for no finite $E \subseteq H$ E f-satisfies P.

- Given two collections of RT^n_k-patterns V and W, we denote by $\mathsf{RT}^n_k(V, W)$ the following problem:

  - Input: a coloring $f : [\omega]^n \to k$ such that ω f-avoids every pattern in V.
  - Output: an infinite set $H \subseteq \omega$ such that H f-avoids every pattern in W.

The definition above covers a large class of principles. For our purposes, it is enough to notice that for every $k > 3$, bRT^3_k can be reformulated as $\mathsf{RT}^3_2(V_{\mathsf{bRT}^3_k}, W_{\mathsf{RT}^3_2})$, where

- $V_{\mathsf{bRT}^3_k}$ is the pattern $\{\langle 1, D\rangle : D \in [k]^3\}$: we just have to prevent f-homogeneous sets for 1 of size k from existing.

- $W_{\mathsf{RT}^3_2}$ is the set of patterns $\{\{\langle 0, D^0\rangle, \langle 1, D^1\rangle\} : (D^0, D^1) \in [6]^3 \times [6]^3\}$: we impose that every set of size 6 is f-homogeneous, which is clearly enough to assure that every infinite set H f-avoiding $W_{\mathsf{RT}^3_2}$ is f-homogeneous.

In general, it is easy to see (by adapting the definition above) that for every $n, k \in \omega$ there is a set of RT^n_k-patterns $W_{\mathsf{RT}^n_k}$ such that $\mathsf{RT}^n_k(\emptyset, W_{\mathsf{RT}^n_k})$ is RT^n_k. In the rest of this section, we will refer to this $W_{\mathsf{RT}^n_k}$ without specifying how it is obtained, since it is inessential.

We now state the main result that we will use, and then move to explain its meaning.

Theorem 4.1.16 ([58], Corollary 3.16)**.** *For every $n, k \in \omega$ and V collection of RT^n_k-patterns, $\mathsf{RT}^n_k(V, W_{\mathsf{RT}^n_k})$ admits cone avoidance if and only if $T^n_k(V)$ only contains constant functions.*

We now embark on the task of defining what sort of set $T^n_k(V)$ is. Since we will only need the result above in the case of $n = 3$ and $k = 2$, we can limit ourselves to define $T^3_2(V)$, for the sake of readability.

Definition 4.1.17.

- A function $\mu : \omega \to \omega + 1$ is *strongly increasing left-c.e. function* if there is a uniformly computable sequence of functions $\mu_0, \mu_1, \ldots$ with $\mu_s : \omega \to \omega$ for every $s \in \omega$ and such that:
  - for every $s, x \in \omega$, $\mu_s(x) \leq \mu_{s+1}(x)$;
  - for every $x \in \omega$, $\lim_s \mu_s(x) = \mu(x)$;
  - for every $s \in \omega$ and every $x < y$, $\mu_s(x) \leq \mu_s(y)$ and if $\mu_{s+1}(x) > \mu_s(x)$, then $\mu_s(y) > s$.
- For a function $\mu : \omega \to \omega + 1$, a set $H \subseteq \omega$ is *μ-transitive* if for every $x < y < z$ with $x, y, z \in H$, $\mu(x) > y$ and $\mu(y) > z$ if and only if $\mu(x) > z$.
- Given a strongly increasing left-c.e. function $\mu : \omega \to \omega + 1$ with approximations $\mu_0, \mu_1, \ldots$, and a set D of three points $D = \{x_0 < x_1 < x_2\}$, we let $P_3(\mu, D)$ be the graph $\{\{0, 1, 2\}, E\}$, where $E = \{\{0, 2\}\}$ if $\mu_{x_2}(x_0) > x_1$, and $E = \emptyset$ otherwise.

The definitions above lie at the heart of the approach to the study of problems outlined in [58]: to give a very rough sketch, the main idea of this approach (which builds on similar tools developed in [10]) is to determine what strongly increasing left-c.e.

functions can and cannot be coded into solutions of Ramseyan principles. Combinatorially, this is done by studying graphs that contain enough information to encode such functions.

Definition 4.1.18.
- By P_3 we will denote the set containing the two graphs $G_0 = \{\{0,1,2\},\emptyset\}$ and $G_1 = \{\{0,1,2\},\{\{0,2\}\}\}$.
- CART^3_k is the following problem:
  - Input: a function $f : [\omega]^3 \to k$.
  - Output: an infinite set $H \subseteq \omega$ such that there exist a strongly increasing left c.e. $\mu : \omega \to \omega+1$ such that H is μ-transitive and a coloring $\chi : P_3 \to k$ such that for every $D \in [H]^3$, $f(D) = \chi(P_3(\mu, D))$.
- For every coloring $\chi : P_3 \to k$, $\chi - \mathsf{CART}^3_k$ is the following problem:
  - Input: a function $f : [\omega]^3 \to k$.
  - Output: an infinite set $H \subseteq \omega$ such that there exist a strongly increasing left c.e. $\mu : \omega \to \omega + 1$ such that H is μ-transitive and such that for every $D \in [H]^3$, $f(D) = \chi(P_3(\mu, D))$.
- Given two principles P and Q, we say that $\mathsf{P} \leq_{id} \mathsf{Q}$ if $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$ using the identity functionals in both directions (i.e., every instance f of P is an instance of Q and every Q-solution g to f is also a P solution to f).
- $T^3_k(V) = \{\chi : P_3 \to k : \mathsf{RT}^3_k(\emptyset, V) \leq_{id} \chi - \mathsf{CART}^3_k\}$.

The main feature of the problems CART^n_k is that they are, in a sense, maximal among the principles that admit cone avoidance (we refer to [58] for a rigorous explanation of this sentence): this is also suggested by the fact that a restriction of it is an essential ingredient in the definition of $T^3_2(V)$ which we were after.

They are, however, somewhat difficult to work with, considering how involved their definition is. Fortunately, at least for the case $n = 3$, there is a solution to this issue.

Definition 4.1.19. PACKED_k is the following principle:

- Input: a function $f : [\omega]^3 \to k$.
- Output: an infinite set $H \subseteq \omega$ such that there are two colors (not necessarily distinct) $i_s, i_l < k$ such that $f[H]^3 \subseteq \{i_s, i_l\}$ and, for every 4-tuple $w < x < y < z$ of elements of H, the following hold:
  1. $f(w, x, z) = f(x, y, z) = i_s$ if and only if $f(w, y, z) = i_s$;
  2. if $f(w, x, y) = i_s$, then $f(w, x, z) = i_s$;
  3. if $f(w, x, y) = i_l$ and $f(w, x, z) = i_s$, then $f(x, y, z) = i_s$.

PACKED_k has the following nice property:

Lemma 4.1.20 ([58]). *For every k, $\mathsf{PACKED}_k \equiv_{id} \mathsf{CART}^3_k$.*

We exploit the previous lemma in the next result.

Lemma 4.1.21. *For every collection of RT^3_k-patterns V, if $\mathsf{RT}^3_k(\emptyset, V)$ is such that every instance has at least one solution and $\mathsf{RT}^3_k(\emptyset, V) \not\leq_{id} \mathsf{PACKED}_k$, then $T^3_k(V)$ contains only constant functions.*

Proof. First of all, we notice that, by our assumption that every instance f of $\mathsf{RT}^3_k(\emptyset, V)$ has a solution, it follows that every infinite homogeneous set is a valid solution to f: to see this, for every $j < k$, consider the constant coloring $f_j : [\omega]^3 \to \{j\}$. The only possible solution is an infinite f_j-homogeneous set for color j, which means that the set of patterns V does not prevent a solution from being homogeneous.

From the fact that $\mathsf{RT}^3_k(\emptyset, V) \not\leq_{id} \mathsf{PACKED}_k$, we deduce that $\mathsf{RT}^3_k(\emptyset, V) \not\leq_{id} \mathsf{CART}^3_k$, by Lemma 4.1.20. But by what we said above, it is clear that, if $\chi : P_3 \to k$ is constant, then $\mathsf{RT}^3_k(\emptyset, V) \leq_{id} \chi - \mathsf{CART}^3_k$ (this is easily verified; in any case, it follows from [58, Statement 3.12]). Hence, there must be a non-constant $\chi' : P_3 \to k$ such that $\mathsf{RT}^3_k(\emptyset, V) \not\leq_{id} \chi' - \mathsf{CART}^3_k$. But since every non-constant coloring $\chi : P_3 \to k$ can be obtained from χ' by renaming the colors (since $|P_3| = 2$), it follows that for *every* non-constant coloring $\chi : P_3 \to k$ $\mathsf{RT}^3_k(\emptyset, V) \not\leq_{id} \chi - \mathsf{CART}^3_k$. Hence, $T^3_k(V)$ only contains constant functions. $\square$

Lemma 4.1.22. *For every $k > 3$, $\mathsf{RT}^3_2(\emptyset, V_{\mathsf{bRT}^3_k}) \not\leq_{id} \mathsf{PACKED}_2$.*

Proof. We define the function $f : [\omega]^3 \to 2$ as follows: for every $x < y < z$, we set $f(x, y, z) = 1$ if $y < k - 1$, and $f(x, y, z) = 0$ otherwise.

We claim that ω is a PACKED_2-solution to f with colors $i_s = 1$ and $i_l = 0$. Let us consider four numbers $w < x < y < z$.

1. We verify condition 1: $f(w, x, z) = f(x, y, z) = 1$ if and only if $y < k - 1$ if and only if $f(w, y, z) = 1$, hence the condition is satisfied.

2. We verify condition 2: if $f(w, x, y) = 1$, then $x < k - 1$, hence $f(w, x, z) = 1$, so the condition is satisfied.

3. We verify condition 3: since it never holds that $f(w, x, y) \neq f(w, x, z)$, the condition is vacuously satisfied.

Finally, we notice that the set $\{0, 1, \dots, k - 1\}$ is an f-homogeneous set for color 1. Hence, ω does not avoid $V_{\mathsf{bRT}^3_k}$, and so $\mathsf{RT}^3_2(\emptyset, V_{\mathsf{bRT}^3_k}) \not\leq_{id} \mathsf{PACKED}_2$. □

Thanks to this, we can finally state the result we were after.

Corollary 4.1.23. *For every k, bRT^3_k admits cone avoidance, and so does bRT^3.*

Proof. By Lemma 4.1.22, we have that $\mathsf{RT}^3_2(\emptyset, V_{\mathsf{bRT}^3_k}) \not\leq_{id} \mathsf{PACKED}_2$, which by Lemma 4.1.21 implies that $T^3_2(V_{\mathsf{bRT}^3_k})$ only contains constant functions. Hence, by Theorem 4.1.16, we have that $\mathsf{RT}^3_2(V_{\mathsf{bRT}^3_k}, W_{\mathsf{RT}^3_2})$, which is exactly the problem bRT^3_k, has cone avoidance.

As we already observed, since every instance of bRT^3 is just an instance of bRT^3_k for some k, it follows that bRT^3 has cone avoidance as well. □

As anticipated, this has several reverse mathematical consequences.

Corollary 4.1.24. bRT^3_4 *does not imply* ACA_0 *over* RCA_0*, and this is witnessed by an* ω*-model. Hence,* bRT^3 *does not imply* ACA_0 *over* RCA_0 *either (as witnessed by the same* ω*-model).*

We end this section with a final remark: the framework described above can also be used to show that bRT^2_k admits strong cone avoidance for every k. Since the proof would require the introduction of many other definitions, even if the combinatorial argument would remain essentially unchanged, we will not prove this claim here.

4.1.3. Complexity of the solutions

We conclude the study of the principles bRT by investigating the complexity of the solutions to their instances. In order to do that efficiently, we will introduce another principle.

Definition 4.1.25. For every integer $n \geq 2$, we let uRT^n (for *unbalanced Ramsey theorem*) be the multifunction defined as follows:

- Input: a coloring $f : [\omega]^n \to 2$ such that, if $H \subseteq \omega$ is f-homogeneous and infinite, then $f(H) = 0$.
- Output: an infinite f-homogeneous set.

Contrary to the previous ones, this problem is relatively old: some results about it are contained in [41], which is still an excellent source of information on the subject (we will put some of its ideas into practice in Lemma 4.1.31).

Remark 4.1.26. We point out that, in a certain sense, uRT^n does not have a correspondent problem in reverse mathematics: if we tried to introduce, for instance, the L_2 statement "for every $f : [\mathbb{N}]^n \to 2$, if no infinite $H_1 \subseteq \mathbb{N}$ is f-homogeneous for color 1, then there is an infiniteH_0 that is f-homogeneous for color 0", which seems to be a sensible translation of uRT^n in second-order arithmetic, we obtain something that is logically equivalent to RT^n_2. Nevertheless, as we will see, uRT^n as a problem behaves very differently from RT^n_2, as we will see below.

The following lemma is obvious, but nevertheless quite useful for the rest of this section.

Lemma 4.1.27. *For every $n \geq 2$, $k > n$ and $l > k$, we have that $\mathsf{bRT}^n_k \leq_{\mathrm{sW}} \mathsf{bRT}^n_l \leq_{\mathrm{sW}} \mathsf{bRT}^n \leq_{\mathrm{sW}} \mathsf{uRT}^n$.*

Notice that all the reductions in the previous Lemma are witnessed by the identity functional, so we could have been even more specific and have used the notation $\leq_{id}$, introduced in the previous section, in the place of $\leq_{\mathrm{sW}}$. This is inessential for our purposes, and so we stick to the more standard notions.

Thanks to the previous Lemma, we can use uRT^n to find an upper bound on the complexity of solutions bRT^n_k. As for lower bounds, RT^{n-1}_2 would seem to be the most natural benchmark: after all, in Theorem 4.1.12, we have shown that $\mathsf{RCA}_0 \vdash \mathsf{bRT}^n_k \rightarrow \mathsf{RT}^{n-1}_2$ holds for every $k > n$. Unfortunately, that proof makes a seemingly essential use of induction, and so does not straightforwardly translate to a Weihrauch or computable reduction.

Definition 4.1.28. For every $k > 0$, and every $l > n$, we denote by $R(n, l, k)$ the least number m such that every coloring $c : [m]^n \rightarrow k$, there is a c-homogeneous set of size l.

Lemma 4.1.29. *For every $n \in \omega$ and $k > 0$, we have that $\mathsf{RT}^n_k \leq_{\mathrm{sW}} \mathsf{bRT}^{n+1}_{R(n,n+1,k)}$.*

Proof. Let $f : [\omega]^n \rightarrow k$ be an instance of RT^n_k. We define the coloring $g : [\omega]^{n+1} \rightarrow 2$ by setting, for every $F \in [\omega]^{n+1}$, $g(F) = 0$ if F if f-homogeneous, and $g(F) = 1$ otherwise. Now, we just have to notice that, by the definition of $R(n, n+1, k)$, g is an instance of $\mathsf{bRT}^{n+1}_{R(n,n+1,k)}$, and that every infinite g-homogeneous set is also an infinite f-homogeneous set. $\square$

We remark that an argument similar to the on in the proof above was used, for slightly different purposes, in [41] and in [8].

The combination of Lemma 4.1.29 and [41, Theorem 5.1] yields the following Corollary.

Corollary 4.1.30. *For every $n > 2$, there is an instance f of $\mathsf{bRT}^n_{R(n-1,n,2)}$ without solutions Σ^0_{n-1} in f. Hence, this also holds for bRT^n_k with $k > R(n-1, n, 2)$.*

At present, we do not know if any strengthening of the Corollary above holds.

We now start looking for upper bounds on the complexity of the solutions. Yet again, we will start our analysis with the case $n = 2$. Since in Section 4.1.1 we showed that bRT^2 can be proved in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$, we already know that every computable instance of bRT^2 has computable solutions. In the next Lemma, we will show that this holds for uRT^2 as well.

Lemma 4.1.31 ([41]). *Every instance f of uRT^2 has a solution computable in f.*

Proof. We follow the sketch of proof given in [41]. Given f as in the hypotheses, let $T^f \subseteq \omega^{<\omega}$ be the Erdős-Rado tree associated to f. Let $I \subseteq T^f$ be the set $I = \{\sigma \in T^f : \exists^\infty \tau \in T^f(\sigma \sqsubseteq \tau)\}$ of elements with infinitely many extensions. We claim that there is string $\rho \in I$ such that

$$\forall n > |\rho| \exists \sigma_n \in T^f(|\sigma_n| = n \wedge \rho \sqsubseteq \sigma_n \wedge \forall x, y(|\rho| \leq x < y < |\sigma_n| \rightarrow f(\sigma_n(x), \sigma_n(y)) = 0)).$$

In plain words, the string ρ we want is a string such that for every length $n > |\rho|$, we can find a string $\sigma_n \in T^f$ of length n such that $\operatorname{ran}(\sigma_n) \setminus \operatorname{ran}(\rho)$ is f-homogeneous for color 0.

Suppose for a contradiction that there is no such ρ, then for every $\sigma \in I$ we can find $n_\sigma > |\sigma|$ such that for every τ extending σ of length n_σ there are x_τ and y_τ with $|\sigma| \leq x_\tau < y_\tau < |\tau|$ such that $f(\tau(x_\tau), \tau(y_\tau)) = 1$. But then, by compactness, we can find an infinite sequence $\tau_0 \sqsubset \tau_1 \sqsubset \dots$ of such τ's. Let $g = \bigcup_{n \in \omega} \tau_n$. By the properties of T^f, we have that $\operatorname{ran} g$ is a prehomogeneous set for f. But then, this means that for every $n \in \omega$ and every $m > n$, we have that $f(g(x_{\tau_n}), g(x_{\tau_m})) = f(g(x_{\tau_n}), g(y_{\tau_m})) = 1$. But then, the set $\{g(x_{\tau_n}) : n \in \omega\}$ would be an infinite f-homogeneous set for color 1, contradicting our assumptions on f.

Hence, the exists a ρ as we described above. Given such a ρ, it is clear that we can find an infinite f-homogeneous set H for color 0 computably in f: we simply have to look, for every $n > |\rho|$, for the σ_n as in the definition of ρ. Arguing as in Lemma 4.1.9, one can check that, for every $|\rho| < n < m$, $\sigma_n \sqsubseteq \sigma_m$, which implies that $\bigcup_{n > |\rho|} \sigma_n$ is a branch in T^f. Hence, again similarly to what we did in Lemma 4.1.9, we can conclude that $\bigcup_{n>|\rho|} \operatorname{ran}(\sigma_n) \setminus \operatorname{ran}(\rho)$ is an infinite f-homogeneous set for color 0. $\square$

We now move to the case $n > 2$. A form of the main result that we have about this case, namely Theorem 4.1.33, seems to have been known to Jockusch already in [41] (see the final remarks of the paper). Anyway, no proof of it was given. We give a simple proof of it (which, as far as we know, has not yet appeared in the literature), based on recent results on the complexity of solutions for COH.

Lemma 4.1.32 ([56], Lemma 7.1.1)**.** *Let $\vec{C}$ be an instance of COH such that $\deg_{\mathrm{T}}(\vec{C}) = \mathbf{a}$, for some Turing degree $\mathbf{a}$. Then, a degree $\mathbf{b}$ computes a COH-solution to $\vec{C}$ if and only if $\mathbf{b}'$ has PA degree over $\mathbf{a}'$.*

Theorem 4.1.33. *Let f be an instance of uRT^n, for $n \geq 3$, and let $\mathbf{c}$ be a degree that is PA over $f^{(n-2)}$. Then, f has a uRT^n-solution computable in $\mathbf{c}$.*

Proof. We start with the proof of the case $n = 3$. Let f be a uRT^3-instance, and let $\mathbf{c}$ be PA over f'. We define the following sequence of sets recursively in f: for every pair $\{x_0, x_1\} \in [\omega]^2$, we define $C_{x_0,x_1} = \{x \in \omega : f(x_0, x_1, x) = 0\}$. By the relativized Jump Inversion Theorem (see for instance [47]) there is a degree $\mathbf{d}$ such that $f \leq_{\mathrm{T}} \mathbf{d}$ and $\mathbf{d}' \equiv_{\mathrm{T}} \mathbf{c}$. Letting $\vec{C} = (C_{x_0,x_1} : \{x_0, x_1\} \in [\omega]^2)$ by Lemma 4.1.32, we can find a COH-solution C to $\vec{C}$ recursively in $\mathbf{d}$.

We now consider the coloring $\widetilde{f} : [C]^2 \to 2$ defined as $\widetilde{f}(x_0, x_1) = \lim_{y \in C} f(x_0, x_1, y)$ for every $\{x_0, x_1\} \in [C]^2$: such a limit exists by definition of cohesive set, and is computable in $\mathbf{d}' \equiv_{\mathrm{T}} \mathbf{c}$.

Now, notice that if $H \subseteq C$ was an infinite $\widetilde{f}$-homogeneous set for color 1, then it would also be f-homogeneous for the same color, which contradicts our assumption that f is an instance of uRT^3. Hence, $\widetilde{f}$ is an instance of uRT^2 relativized to $\mathbf{c}$, and by Lemma 4.1.31 if has a solution computable in $\widetilde{f}$, and so in $\mathbf{c}$.

We now move to the inductive step: suppose that the result holds for n, we prove it for $n+1$. Let f be an instance of uRT^{n+1}, and let $\mathbf{c}$ be PA in $f^{(n-1)}$. By the relativized

Low Basis Theorem (see [42]), there is a degree $\mathbf{g}$ that is PA over f' and such that $\mathbf{g}' \equiv_{\mathrm{T}} f''$. Again by the relativized Jump Inversion Theorem, there is a degree $\mathbf{h}$ such that $f \leq_{\mathrm{T}} \mathbf{h}$ and $\mathbf{h}' \equiv_{\mathrm{T}} \mathbf{g}$.

Again, for every $\{x_0, \ldots, x_{n-1}\} \in [\omega]^n$, we define the set $C_{x_0,\ldots,x_{n-1}}$ as $C_{x_0,\ldots,x_{n-1}} = \{x \in \omega : f(x_0, \ldots, x_{n-1}, x) = 0\}$. Letting $\vec{C} = (C_{x_0,\ldots,x_{n-1}} : \{x_0, \ldots, x_{n-1}\} \in [\omega]^2)$, we can find an infinite $\vec{C}$-cohesive set C computably in $\mathbf{h}$ by Lemma 4.1.32, and computably in $\mathbf{g}$ we can find the coloring $\widetilde{f} : [C]^n \to 2$ defined as $\widetilde{f}(x_0, \ldots, x_{n-1}) = \lim_{y \in C} f(x_0, \ldots, x_{n-1}, y)$. As in the case $n = 3$, it is easy to see that any infinite set H that is $\widetilde{f}$-homogeneous for 1 is f-homogeneous for the same color. Hence, $\widetilde{f}$ is an instance of uRT^n relativized to $\mathbf{g}$. But then, we can apply the inductive hypothesis: since $\mathbf{c}$ is PA over $f^{(n-1)} \equiv_{\mathrm{T}} (\mathbf{g}')^{(n-3)} \equiv_{\mathrm{T}} \mathbf{g}^{(n-2)} \geq_{\mathrm{T}} \widetilde{f}^{(n-2)}$, $\mathbf{c}$ is also PA over $\widetilde{f}^{(n-2)}$, and we conclude by induction. □

We point out that the result above is optimal: in [37, Corollary 2.2], it is proved that there exists a computable unbalanced coloring of $[\omega]^3$ such that all of its infinite homogeneous sets have PA degree over $\emptyset'$.

The Theorem above, together with Lemma 4.1.27 and the relativized Low Basis Theorem, immediately yields the next Corollary.

Corollary 4.1.34. *For every $n > 2$ and $k > n$, and for every $\mathbf{c}$ of PA degree over $\emptyset^{(n-2)}$, every computable instance of bRT^n and bRT^n_k has solutions of degree $\mathbf{c}$. In particular, they (and uRT^n) have Δ^0_n solutions.*

4.2. A theorem about partition ordinals

In this section, we will present some results about the reverse mathematics of the theorem, first proved by Specker in [68], that the ordinal ω^2 is a partition ordinal: after a brief general introduction to partitions ordinals, we will see how one of the classical proofs gives a bound on the strength of the principles we are interested in. Finally, we will make some remarks about the complexity of the solutions of these principles.

4.2.1. A brief introduction to the subject

Partition ordinals were introduced in the early stages of the development of what we now call Ramsey theory: the first results and questions about them appeared already in the seminal paper [25] by Erdős and Rado. For a general introduction to this topic, we refer to [33].

To better discuss about this topic, it is practical to introduce the following standard notation.

Definition 4.2.1. Given three ordinals α, β, γ, we write $\alpha \longrightarrow (\beta, \gamma)$ if it is true that for every coloring $f : [\alpha]^2 \to 2$, either there is an f-homogeneous set $H_0 \subseteq \alpha$ such that $f([H_0]^2) = 0$ and the order-type of H_0 is β, or there is an f-homogeneous set $H_1 \subseteq \alpha$ such that $f([H_1]^2) = 1$ and the order-type of H_1 is γ.

The negation of this relation is denoted as $\alpha \not\longrightarrow (\beta, \gamma)$.

For instance, Ramsey's theorem for pairs can be more succinctly restated as $\omega \longrightarrow (\omega, \omega)$.

The notation above has the merit of making clearer what happens when we vary α, β and γ: if we know that $\alpha \longrightarrow (\beta, \gamma)$ holds, then for every $\beta' \leq \beta$ and $\gamma' \leq \gamma$ $\alpha \longrightarrow (\beta', \gamma')$ holds as well. Similarly, if $\alpha' \geq \alpha$, it is easily seen that $\alpha \longrightarrow (\beta, \gamma)$ implies $\alpha' \longrightarrow (\beta, \gamma)$

It is natural to ask for which triple of ordinals (α, β, γ) the relation $\alpha \longrightarrow (\beta, \gamma)$ holds. We will focus on countable ordinals.

It is very easy to see that for a vast class of triples (α, β, γ), the relation $\alpha \longrightarrow (\beta, \gamma)$ cannot hold: denoting by $|\alpha|$ the cardinality of α, and letting $\pi : \alpha \to |\alpha|$ be a bijection, we claim that $\alpha \not\longrightarrow (|\alpha| + 1, \omega)$. To see this, for every two ordinals $x < y < \alpha$, we define a coloring $f : [\alpha]^2 \to 2$ as follows:

$$f(x, y) = \begin{cases} 0 & \text{if } \pi(x) < \pi(y) \\ 1 & \text{otherwise} \end{cases}$$

It is then clear that there are no infinite f-homogeneous sets for color 1, since any such set would give rise to an infinite descending chain of ordinals. Equally, there are no f-homogeneous sets for 0 of order-type $|\alpha| + 1$: for any f-homogeneous set H for color 0 and any $x, y \in H$, we have that $x < y$ if and only if $\pi(x) < \pi(y)$, which implies that the order-type of H can be at most $|\alpha|$.

Thus, thanks to the previous paragraph, we have a complete picture of what happens in the case that α, β, γ are all countable and infinite:

- if $\beta = \gamma = \omega$, then $\alpha \longrightarrow (\beta, \gamma)$ holds, as implied by Ramsey's theorem for pairs;
- in every other case, we have that $\alpha \not\longrightarrow (\beta, \gamma)$ holds.

It becomes then interesting to investigate what happens if we require one of β and γ to be finite. Clearly, if $\gamma = 2$, then $\alpha \longrightarrow (\beta, 2)$ holds for every $\beta \leq \alpha$. But, already for $\gamma = 3$, the problem becomes very interesting.

Definition 4.2.2. We say that a countable ordinal α is a *partition ordinal* if the relation $\alpha \longrightarrow (\alpha, 3)$ holds.

This problem, that might look simple at first, turns out to be very complicated: as a measure of its difficulty, we mention that Erdős himself, in 1987, promised 1000 dollars for a characterization of the partition ordinals.

In this section, we will focus on the simplest result of this area, namely that ω^2 is a partition ordinal, a fact that we will prove in the following subsection. For completeness, we mention that, although a complete characterization of partition ordinals has not yet been given, several other results have been found in this area: just to mention a few, Specker in [68] proved that for all $n \in \omega$ with $n > 2$, $\omega^n \not\longrightarrow (\omega^n, 3)$. Chang, in [9], proved that ω^ω is a partition ordinal. Larson, in [46], gave much simpler proofs of the previous results. Finally, more recently, Schipperus in [61] proved a series of results on what ordinals of the form ω^{ω^α}, where α is a countable ordinal, are partition ordinals.

4.2.2. The principles and some easy results

We now introduce the principles that we will be working with.

Definition 4.2.3.
- For every $k \in \mathbb{N}$, the principle SPL_k (in honor of Specker and Larson) is the L_2-statement "let $\vec{R} = \{R_i : i \in \mathbb{N}\}$ be a sequence of disjoint infinite sets such that $R = \bigcup_i R_i$, and let $f : [R]^2 \to 2$ be a coloring such that there is no f-homogeneous set for color 1 of size k; then, there is an infinite f-homogeneous set $H \subseteq R$ such that for infinitely many i, $H \cap R_i$ is infinite".

- For every $k \in \mathbb{N}$, the principle SSPL_k (for "strong SPL_k") is the L_2-statement "let $\vec{R} = \{R_i : i \in \mathbb{N}\}$ be a sequence of disjoint infinite sets such that $R = \bigcup_i R_i$, and let $f : [R]^2 \to 2$ be a coloring such that there is no f-homogeneous set for color 1 of size k; then, there is an infinite f-homogeneous set $H \subseteq R$ such that for infinitely many i, $H \cap R_i$ is infinite, and such that for every i, if $H \cap R_i \neq \emptyset$, then $H \cap R_i$ is infinite".

The idea behind the two principles above is simple: they both convey the fact that $\omega^2 \longrightarrow (\omega^2, k)$, although in slightly different ways. The initial ordering of type ω^2 is given by the infinite sequence of infinite sets $\vec{R}$. The main difference between SPL_k and SSPL_k is about the shape of the solution H: while for SPL_k we only ask that infinitely many R_i are intersected infinitely often, which classically still gives a solution of order-type ω^2 (essentially because $a+\omega = \omega$ for any finite ordinal a), for SSPL_k we essentially require to be given the list of R_i that are intersected infinitely often by H, which gives us much more information on the solution H.

We do not know much about SPL_k. We summarize some immediate results in the following Lemma.

Lemma 4.2.4. $\mathsf{RCA}_0 \vdash \forall k(\mathsf{SSPL}_k \to \mathsf{SPL}_k \to \mathsf{bRT}^2_k)$. *Hence, by Lemma 4.1.6,* $\mathsf{RCA}_0 + \forall k \mathsf{SPL}_k \vdash \mathsf{B}\Sigma^0_2$.

In particular, it is unclear whether SPL_k is computably true, or even if RCA_0 proves it. We have something more to say about SSPL.

Lemma 4.2.5. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \mathsf{SSPL}_3 \to \mathsf{SRT}^2_2$.

Proof. Let $c : [\mathbb{N}]^2 \to 2$ be a stable coloring, and let $\vec{R}$ be the partition given by $r \in R_i \leftrightarrow \exists i, a \leq r(r = \langle i, a\rangle \wedge i \neq a)$. Let $R = \mathbb{N} \setminus \{r \in \mathbb{N} : \exists i < r(r = \langle i, i\rangle)\}$. We define the coloring $f : [R]^2 \to 2$ as follows: for every pair $\{x, y\} \in [R]^2$ with $x = \langle i, a\rangle$ and $y = \langle j, b\rangle$ for some i, j, a, b, we set

$$f(x,y) = \begin{cases} 0 & \text{if } c(i,a) = c(j,b) \\ 1 & \text{otherwise} \end{cases}$$

There are no f-homogeneous sets for 1 of size 3: suppose for a contradiction that $\{x_0, x_1, x_2\}$ is such a set, and let, for $j < 3$, $x_j = \langle i_j, a_j\rangle$. Then, the three pairs $\{i_j, a_j\}$ would all have different colors according to c, which is a contradiction.

We can then apply SSPL_3 to f: let H be the f-homogeneous set that SSPL_k gives us. By the fact that RCA_0 proves that every infinite Σ^0_1 set has an infinite Δ^0_1 subset, we can find an infinite set $I \subseteq \mathbb{N}$ such that for every $i \in I$, $H \cap R_i \neq \emptyset$.

Then, $c\restriction_{[I]^2}$ is such that for every $i, j \in I$, $\lim_y c(i, y) = \lim_y c(j, y)$. Hence, $\mathsf{B}\Sigma^0_2$ is enough to refine I to a SRT^2_2-solution for c. $\square$

As a corollary of the Lemma above, we have that SSPL_3 is not computably true.

From now on, we focus on SSPL_3, since, by the fact that we saw above, it seems to be the more interesting translation of the theorem that ω^2 is a partition ordinal.

4.2.3. Classical proofs

In this section, we give two proofs of the fact that ω^2 is a partition ordinal. The first one was given by Larson in [46]: it is very short and simple enough to be formalized in second-order arithmetic, where it can be used to see that $\mathsf{ACA}_0 \vdash \mathsf{SSPL}_3$. The other one, which is the original proof given by Specker in [68], is much longer and complex, and we will not formalize it in second-order arithmetic. There are two main reasons for including it in this section: the first is that this proof arguably gives a more interesting combinatorial idea of why ω^2 a partition ordinal; the second is that there seem to be no English translation of the original proof, which is in German.

Theorem 4.2.6 (Larson). $\mathsf{RCA}_0 \vdash \mathsf{RT}^4_{64} \to \mathsf{SSPL}_3$. *Hence,* $\mathsf{ACA}_0 \vdash \mathsf{SSPL}_3$.

Proof. Let $\vec{R}$ be an infinite sequence of infinite disjoint sets R_i with $R = \bigcup_i R_i$, and let us enumerate every R_i as $R_i = \{r^i_0 < r^i_1 < \dots\}$. Let $f : [R]^2 \to 2$ be a coloring with no f-homogeneous sets for color 1 of size 3.

For every quadruple $a < b < c < d$ of elements of $\mathbb{N}$, we define the following coloring $g : [\mathbb{N}]^4 \to 64$:

$$g(a,b,c,d) = 32f(r^a_b, r^c_d) + 16f(r^a_c, r^b_d) + 8f(r^a_d, r^b_c) + 4f(r^a_b, r^a_c) + 2f(r^a_b, r^b_c) + f(r^a_c, r^b_c).$$

Let H be an infinite g-homogeneous set. By the fact that f did not have f-homogeneous sets for color 1 of size 3, it is easy to see that H is g-homogeneous for color 0.

Finally, let $L \subseteq R$ be defined by $r^i_a \in L \leftrightarrow i, a \in H \wedge a > i$. We notice right away that for every $j \in \mathbb{N}$, if $L \cap R_j \neq \emptyset$, then $H \cap R_j$ is infinite. It is then easy to see that L is an infinite f-homogeneous set: for any $\{x, y\} \in L$ we can find $a, b, c, d \in H$ such that $x = r^a_b$, $y = r^c_d$, with $a \leq c$, $a < b$ and $c < d$. Since in the definition of g we have considered any possible configuration of a, b, c, d respecting the three conditions we just mentioned, we can conclude that $f(r^a_b, r^c_d) = 0$. $\square$

Remark 4.2.7. We notice that another proof of the result above could also have been obtained in a slightly different fashion: starting from f, we could have defined the intermediate function $g_0 : [\mathbb{N}]^4 \to 2$ as $g_0(a,b,c,d) = f(r^a_b, r^c_d)$, observing that this is an instance of bRT^4_6. Given an infinite homogeneous set H_0 for g_0, we could have then defined the coloring $g_1 : [H_0]^4 \to 2$ in a similar fashion as before. Continuing like this, we could have found a proof that $\mathsf{RCA}_0 \vdash \mathsf{bRT}^4_6 \to \mathsf{SSPL}_3$. Of course, since $\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \leftrightarrow \mathsf{bRT}^4_6$, there is nothing to be gained from this alternative approach from a reverse mathematical perspective. On the other hand, we will see in the next section that the proof we gave above seems to be a better tool to give an estimate of the complexity of the solutions to computable instances of SSPL_3.

Now, we move to the more convoluted original proof by Specker. As anticipated, we will not formalize it in second-order arithmetic.

Theorem 4.2.8 (Specker)**.** (ZFC) $\omega^2 \longrightarrow (\omega^2, 3)$.

Proof. We identify ω^2 with $\{(i,a) : i, a \in \omega\}$ ordered lexicographically, and for every $i \in \omega$ we call R_i the set $R_i = \{(i,a) : a \in \omega\}$. Let $f : [\omega^2]^2 \to 2$ be a coloring with no f-homogeneous sets for color 1 of size 3. We suppose for a contradiction that there is no infinite f-homogeneous set $H \subseteq \omega^2$ of order-type ω^2.

We will prove the Theorem by proving the following Claim:

Claim 4.2.1. *There are an infinite set $I \subseteq \omega$ and an infinite sequence of infinite sets $\vec{U} = \{U_i : i \in I\}$ such that for every $i \in I$ $U_i \subseteq R_i$ and moreover, if i_0 is the minimal element of I,*

$$\forall u \in U_{i_0} \forall v \in \bigcup_{i \in I} U_i (u \neq v \to f(u,v) = 0).$$

We notice that, if we do this, then we reach our contradiction: we can now use U_{i_0} as the initial segment of length ω of a solution H, and repeat the construction with $\bigcup_{i \in I \setminus \{i_0\}} U_i$ in place of full ω^2 (notice that, under the lexicographical order, $\bigcup_{i \in I \setminus \{i_0\}} U_i$ and ω^2 are isomorphic).

Hence, let us start the proof of Claim 4.2.1.

For every $i \in \omega$, let $\mu_i : \mathcal{P}(R_i) \to 2$ be a *non-atomic finitely additive* $\{0,1\}$*-measure* on R_i, i.e. a finitely additive measure on R_i such that it only takes values 0 or 1 and every finite subset of R_i has measure 0.

Given the measures μ_i, for every $i, j \in \omega$, we define the set

$$B_i^j = \{x \in R_i : \mu_j(\{y \in R_j : f(x,y) = 1\}) = 1\}.$$

In a sense, the B_i^j are the "bad sets" that we will try to eliminate in the rest of the proof.

Claim 4.2.2. *There is an infinite set $N \subseteq \omega$ such that, for every $i, j \in \omega$ with $i \neq j$, $\mu_i(B_i^j) = 0$.*

Proof of Claim 4.2.2. We define the coloring $c_0 : [\omega]^2 \to 2$ as follows: for every pair $\{i, j\} \in [\omega]^2$ with $i < j$, we set $c_0(i, j) = \mu_i(B_i^j)$. Notice that there can be no c_0-homogeneous set for color 1 of size 3: if there were $i < j < k \in \omega$ such that $f(i, j) = f(i, k) = f(j, k) = 1$, then the sets $B_i^j \cap B_i^k$ and B_j^k would have measure 1, and so we could find $x \in B_i^j \cap B_i^k$, $y \in B_j^k$ and $z \in R_k$ such that $f(x, y) = f(x, z) = f(y, z) = 1$, contradicting the hypotheses on f. Hence, any infinite c_0-homogeneous set is c_0-homogeneous for color 0. Let N' be such a set.

Now, we define the coloring $c_1 : [N']^2 \to 2$ as $c_1(i, j) = \mu_i(B_j^i)$, for all $i < j \in N'$. Similarly as for c_0, every infinite c_1-homogeneous set is c_1-homogeneous for 0, so let N be such a set. It is clear that it satisfies the requirements we were looking for. $\square$

Claim 4.2.3. *Let $N = \{n_0 < n_1 < \dots\}$ be the set found in Claim 4.2.2, and so let n_0 be the minimal element of N. Then, we can find an infinite set $L \subseteq N \setminus \{n_0\}$ and, for every $n \in L \cup \{n_0\}$, an infinite subset R_n^* of R_n and a non-atomic $\{0, 1\}$-measure μ_n^* on R_n^* such that the following holds: if we define, for every $i, j \in L \cup \{n_0\}$ with $i \neq j$,*

$$C_i^j = \{x \in R_i^* : \mu_j^*(\{y \in R_j^* : f(x, y) = 1\}) = 1\},$$

then, for every $l \in L$, $C_{n_0}^l = C_l^{n_0} = \emptyset$.

Proof of Claim 4.2.3. We start noticing that for every $x \in R_{n_0}$, there are only finitely many $n \in N$ such that $\mu_n(\{y \in R_n : f(x, y) = 1\}) = 1$: if there were an infinite set N_x of such n's, then we could consider the set

$$H_x = \{y \in \omega^2 : \exists n \in N_x(y \in N_x \wedge f(x, y) = 1)\},$$

which would be an infinite f-homogeneous set of order-type ω^2, thus giving us a contradiction. Hence, for every $x \in R_{n_0}$, there is a $b_x \in N$ such that, for every $n \geq b_x$, $\mu_n(\{y \in R_n : f(x,y) = 1\}) = 0$. In particular, this means that for every $x \in R_{n_0}$ and $n \in N$, if $n > b_x$ then $x \notin B^n_{n_0}$.

We now build the sets $R^*_{n_0}$ and $M \subseteq N$ in stages: as we will see, M is a first approximation of the $L \subseteq N$ we are after. At every stage s, we will have two finite sets, $R^s_{n_0} \subset R_{n_0}$ and $M^s \subset N$, both of cardinality s, and in the end we will let $R^*_{n_0} = \bigcup_s R^s_{n_0}$ and $M = \bigcup_s M^s$. So at stage 0, let $R^0_{n_0} = M^0 = \emptyset$.

Suppose we have the sets $R^s_{n_0}$ and $M^s = \{m_0 < m_1 < \cdots < m_{s-1}\}$, we define the sets $R^{s+1}_{n_0}$ and S^{s+1} as follows: let $x \in R_{n_0}$ be minimal such that

$$x \in R_{n_0} \setminus (B^{m_0}_{n_0} \cup \cdots \cup B^{m_{s-1}}_{n_0} \cup R^s_{n_0}).$$

Notice that such an x exists, since by Claim 4.2.2 $B^{m_0}_{n_0} \cup \cdots \cup B^{m_{s-1}}_{n_0}$ has measure 0 (and $R^s_{n_0}$ is finite by assumption). We let $R^{s+1}_{n_0} = R^s_{n_0} \cup \{x\}$.

Then, we let m_{s+1} be $1 + \sum_{y \in R^{s+1}_{n_0}} b_y$, and we let $M^{s+1} = M^s \cup \{m_{s+1}\}$. This ends the definition of the sets $R^s_{n_0}$ and M^s.

As said above, let $R^*_{n_0} = \bigcup_s R^s_{n_0}$ and $M = \bigcup_s M^s$. By the way in which we have defined $R^*_{n_0}$, we have that, for every $m \in M$,

$$\{x \in R^*_{n_0} : \mu_m(\{y \in R_m : f(x,y) = 1\}) = 1\} = \emptyset,$$

since, for every $x \in R^*_{n_0}$ and $m \in M$, either m is such that $m > b_x$, or $x \notin B^m_{n_0}$.

Now, let $\mu^*_{n_0}$ be any non-atomic $\{0,1\}$-measure on $R^*_{n_0}$. Suppose for a contradiction that there are infinitely many $m \in M$ such that for infinitely many $y \in R_m$

$$\mu^*_{n_0}(\{x \in R^*_{n_0} : f(x,y) = 1\}) = 1.$$

Again, this would lead to a contradiction: let M' be the infinite set of the m's as above, then the set

$$H = \{y \in \omega^2 : \exists m \in M'(y \in R_m \wedge \mu^*_{n_0}(\{x \in R^*_{n_0} : f(x,y) = 1\}) = 1)\}$$

would be an infinite f-homogeneous set of order-type ω^2.

Hence, let L be the infinite (and actually cofinite in M) set of $m \in M \setminus \{n_0\}$ such that there are at most finitely many $y \in R_m$ with $\mu^*_{n_0}(\{x \in R^*_{n_0} : f(x,y) = 1\}) = 1$, and let us set, for ease of notation,

$$F_l = \{y \in R_l : \mu^*_{n_0}(\{x \in R^*_{n_0} : f(x,y) = 1\}) = 1\}.$$

for every $l \in L$: by definition of L, every F_l is a finite set.

Finally, for every $l \in L$, let $R^*_l = R_l \setminus F_l$, and define the finitely additive measure μ^*_l on R^*_l as, for every set $S \subseteq R^*_l$, $\mu^*_l(S) = \mu_l(S)$ (it is clear that μ^*_l is indeed a non-atomic finitely additive $\{0,1\}$-measure; see e.g. [34] for more general results).

It is immediately verified that the R^*_n and L are as we wanted them. $\square$

We now have all the ingredients to prove Claim 4.2.1.

Proof of Claim 4.2.1. Let $I = \{n_0\} \cup L$, and enumerate I as $I = \{i_0 < i_1 < \dots\}$ (hence $n_0 = i_0$). By Claim 4.2.3, we have that for every $i \in I$ with $i \neq i_0$, $C^i_{i_0} = C^{i_0}_i = \emptyset$.

We build the U_i in stages, by defining larger and larger finite approximations U^s_i of the sets U_i in such a way that at every stage s, only a finite number of U^s_i will be non-empty, but in the end, for every $i \in I$, $\bigcup_s U^s_i$ will be infinite.

At stage 0, we have $U^0_i = \emptyset$ for every $i \in I$. Suppose now we have defined the U^s_i, and we will see how to define the sets U^{s+1}_i. There are two cases:

- if $s + 1 = \langle 0, k \rangle$ for some k, then let x be minimal in $R^*_{i_0}$ such that $x \notin U^s_{i_0}$ and, for every $y \in \bigcup_{i \in I \setminus \{i_0\}} U^s_i$ (which is a finite set), $f(x,y) = 0$: such an x exists since $C^{i_0}_i = \emptyset$, and so $\mu^*_{i_0}(\{x \in R^*_{i_0} : f(x,y) = 1\}) = 0$ for every $y \in R^*_i$. Then we set $U^{s+1}_{i_0} = U^s_{i_0} \cup \{x\}$ and $U^{s+1}_i = U^s_i$ for every $i \in I \setminus \{i_0\}$.

- if $s+1 = \langle p, k \rangle$ for some $p \neq 0$, then let x be minimal in $R^*_{i_p}$ such that for every $y \in R^*_{i_0}$ $f(x, y) = 0$. Similarly as above, such an x exists since $C^i_{i_0} = \emptyset$ for every $i \in I$, $i \neq i_0$. We set $U^{s+1}_{i_p} = U^s_{i_p} \cup \{x\}$ and $U^{s+1}_i = U^s_i$ for every $i \in I \setminus \{i_p\}$.

Finally, we set $U'_{i_0} = \bigcup_s U^s_{i_0}$, and, for every $i \in I \setminus \{i_0\}$, we define $U_i = \bigcup_s U^s_i$. The last thing left to do is to refine U'_{i_0} to an f-homogeneous set: let U_{i_0} be the infinite f-homogeneous set obtained by applying bRT^2_3 to U'_{i_0}.

By the way we defined them, it is clear that $\vec{U} = \{U_i : i \in I\}$ is as wanted in the statement of Claim 4.2.1. □

As explained above, this is enough to prove the Theorem. □

4.2.4. Computability theoretic considerations

In this section, we will say something on the complexity of the solutions of SSPL_3. In order to do that, as usual, we first introduce the partial multifunction associated to SSPL_k (the problem associated to SPL_k could be defined in essentially the same way). We will focus on the case $k = 3$.

Definition 4.2.9. For every $k \in \omega$, SSPL_k is the following partial multifunction:

- Input: A pair $(\vec{R}, f)$, where $\vec{R} = \{R_i : i \in \omega\}$ is a partition of ω into infinite disjoints sets (i.e., we assume that every R_i is infinite, $R_i \cap R_j = \emptyset$ for every $i \neq j$, and that $\bigcup_i R_i = \omega$), and f is a coloring $f : [\omega]^2 \to 2$ such that no set of size k is f-homogeneous for color 1.

- Output: an infinite f-homogeneous set H such that, for every $i \in \omega$, $H \cap R_i \neq \emptyset$ implies that $H \cap R_i$ is infinite and such that there are infinitely many $i \in \omega$ such that $H \cap R_i \neq \emptyset$.

We pointed out in Remark 4.2.7 that there would be another way to prove Theorem 4.2.6, which is actually the way in which the proof by Larson was originally presented. Anyway, proving the Theorem the way we did, using one single application of RT^4_{64}, allows us to conclude immediately, using the results from [41], that every computable instance of SSPL_3 has Π^0_4 solutions, and so solutions computable in $\emptyset^{(4)}$. Moreover, a closer inspection of the proof shows that we have actually proved that $\mathsf{SSPL}_3 \leq_{\mathrm{W}} \mathsf{uRT}^4$, since the coloring g we define could very easily be transformed into a coloring $g' : [\omega]^4 \to 2$ with no infinite g'-homogeneous sets for color 1. Hence, using Corollary 4.1.34, we can actually conclude that every computable instance f of SSPL_3 has a SSPL_3-solution computable in any degree $\mathbf{d}$ such that $\mathbf{d}$ is PA over $\emptyset^{(3)}$, and so even has Δ^0_4 solutions.

Of course, this is an upper bound on the complexity of the solutions for computable instances of SPL_3 as well.

In this section, we approach the problem of estimating the complexity of the solutions in a different, and in a certain sense more combinatorial, way. Although we do not succeed in establishing a better upper bound for the complexity of the solutions to the computable SSPL_3 instance $(\vec{R}, f)$, we manage to put some bounds on the complexity of a sets K with the following property: $K \subseteq \omega$ is such that for every i, $R_i \cap K$ is infinite and for every $x \in K$, $\lim_{y \in R_i \cap K} f(x, y)$ exists.

We use the technique known as *first jump control*, introduced in [11]. We point out that the language used in [11] is slightly different than the one we use here, making a more explicit use of *computable Mathias forcing* than us. We refer to [36] and [21] for excellent presentations of this technique (and to [64] for a general introduction to computable forcing).

Lemma 4.2.10. *Let $(\vec{R}, f)$ be a computable instance of SSPL_3. Then, there is a low$_2$ set $K \subseteq \omega$ such that $K \cap R_i$ is infinite for every $i \in \omega$ and such that for every $x \in K$ and $i \in \omega$, $\lim_{y \in K \cap R_i} f(x, y)$ exists.*

Proof. Let $\mathbf{d}$ be a Turing degree that is PA over $\emptyset'$ and such that $\mathbf{d}' \equiv_{\mathrm{T}} \emptyset''$ (we already mentioned in the proof of Theorem 4.1.33 that such degrees exists). We will build the set K in stages, computably in $\mathbf{d}$: at even stages, we will take care of the requirement that for every $x \in K$ and $i \in \omega$, $\lim_{y \in K \cap R_i} f(x, y)$ exists, and in odd stages we will ensure that $K' \equiv_T \mathbf{d}$, which ensures that it is low$_2$. For the first case, we will make essential use of the properties of PA degrees, namely of Lemma 1.2.20.

In what follows, we will call a *condition* a pair (F, L), where $F \subset \omega$ is finite set and $L \subseteq \omega$ is an infinite computable set such that $F < L$ and for every $i \in \omega$, $L \cap R_i$ is infinite. The idea of the construction is that, for every stage s, if we start the stage with the condition (F_s, L_s), we enlarge F_s using elements from L_s, thus obtaining a new finite set F_{s+1}, and then we find an infinite subset L_{s+1} of L_s, so that at the end of stage s we will have another condition (F_{s+1}, L_{s+1}) to pass on to the next stage. The set K we are after will be the union of all the first components of the conditions we build in the construction.

At the start of the construction, we put $F_0 = \emptyset$ and $L_0 = \omega$. Then, at stage s, given the condition (F_s, L_s), we proceed as follows:

- if s is even: let i_s be the maximal i such that $F_s \cap R_i \neq \emptyset$ (unless $s = 0$, in which case we set $i_s = 0$). For every $i \in \omega$, let x_i^s be the minimal element of $(R_i \cap L_s) \setminus F_s$ (recall that we are assuming that $R_i \cap L_s$ is infinite), and let

$$F_{s+1} = F_s \cup \{x_i^s : i \leq i_s + 1\}.$$

  Then, we have to refine L_s to L_{s+1}. For every $i \leq i_s + 1$, $x \in F_{s+1}$ and $k \in 2$, we let $\varphi(i, x, k)$ be the formula "the set $\{y \in R_i : f(x, y) = k\}$ is infinite", which, we notice, is a Π^0_2 formula. Now, let us enumerate F_{s+1} as $F_{s+1} = \{x_0, x_1, \ldots, x_{|F_{s+1}|-1}\}$. For every $i \leq i_s + 1$ and every $\sigma \in 2^{|F_{s+1}|}$, we define the predicate $\psi(i, \sigma)$ as $\bigwedge_{j<|F_{s+1}|} \varphi(i, x_j, \sigma(j))$, which again is Π^0_2. Now, notice that, for every $i \leq i_s + 1$, for at least one $\sigma \in 2^{|F_{s+1}|}$, $\psi(i, \sigma)$ holds: maybe the easiest way of proving this is via a measure-theoretic argument, which we now sketch. Let μ_i be a non-atomic finitely additive $\{0, 1\}$-measure on R_i, and let us call $C_{i,x,k}$ the set $C_{i,x,k} = \{y \in R_i : f(x, y) = k\}$, for every $x \in F_{s+1}$ and $k \in 2$. Since $C_{i,x,k} \cup C_{i,x,1-k}$ is cofinite in R_i, it follows that for some choice of k $\mu_i(C_{i,x,k}) = 1$. Hence, for a certain string $\sigma \in 2^{|F_{s+1}|}$, $\mu_i\big(\bigcap_{j<|F_{s+1}|} C_{i,x_j,\sigma(j)}\big) = 1$, which proves that the intersection is infinite.

By Lemma 1.2.20, we can find one such σ uniformly in $\mathbf{d}$: we call this σ σ_i. We do this for every $i \leq i_s + 1$. Finally, we refine L_s to L_{s+1} as follows:

$$L_{s+1} = \big(L_s \setminus \bigcup_{i \leq i_{s+1}} R_i \quad \cup \bigcup_{i \leq i_{s+1}} \bigcap_{j < |F_{s+1}|} C_{i,x_j,\sigma_i(j)}\big) \setminus [0, \max F_{s+1}].$$

Thus, we have defined the new condition (F_{s+1}, L_{s+1}). We notice that, by the way we have defined L_{s+1}, it still holds that for every $i \in \omega$ $L_{s+1} \cap R_i$ is infinite, and moreover for every $i \leq i_s + 1$ and $x \in F_{s+1}$, $\lim_{y \in R_i \cap L_{s+1}} f(x, y)$ exists.

- if s is odd, say $s = 2e + 1$: we will decide the eth bit of the jump of K. Computably in $\emptyset'$, we check whether there is a finite set $F \subseteq F_s \cup L_s$ such that $F_s \subseteq F$ and $\Phi_e(F)(e)\downarrow$. If such an F exists, then we put $F_{s+1} = F$ and $L_{s+1} = L_s \setminus [0, \max F]$, thus obtaining a new condition (F_{s+1}, L_{s+1}). Otherwise, we let $(F_{s+1}, L_{s+1}) = (F_s, L_s)$.

As anticipated, we let $K = \bigcup_{s \in \omega} F_s$, and we claim that it satisfies the properties we required. It follows easily from an inspection of the even stages that $K \cap R_i$ is infinite for every $i \in \omega$, since at step $2i + 2h$ we make sure that $F_{2i+2h+1} \cap R_i$ has at least h elements. Moreover, for every $x \in K$, if for some even s $x \in F_s$, at stage (at most) $s + 2i$ we make sure that $\lim_{y \in K \cap R_i} f(x, y)$ exists.

Finally, we observe that we can compute K' using $\mathbf{d}$: suppose that, for a certain $e \in \omega$, we want to determine the value of $K'(e)$, i.e. whether $\Phi_e(K)(e)\downarrow$ or not. To do this, we just have to repeat the construction of K up to F_{2e+1}, and notice that by the way we defined F_{2e+2}, $\Phi_e(K)(e)\downarrow \leftrightarrow \Phi_e(F_{2e+2})(e)$, which can be verified computably in $\mathbf{d}$. Hence, this proves that $K' \leq_{\mathrm{T}} \mathbf{d}$, and so that $K'' \equiv_{\mathrm{T}} \emptyset''$. This completes the proof of the Lemma. $\square$

The Lemma above has a nice consequence: the sets $A_{x,k} = \{i \in \omega : \exists^\infty y(f(x, y) = k\}$, which are clearly Π^0_2 sets, are Δ^0_2 relative to K. Hence, when we argue modulo K, we can exploit the reduction in complexity of these sets, as we will do in the next Lemma.

It would now be nice to use this as an initial step to find a bound of the complexity of a set $I_K \subseteq \omega$ such that for every $i \in I_K$ and every $x \in R_i \cap K$, $\lim_{y \in R_j \cap K} f(x,y) = 0$ for all but finitely many $j \in I_k$: again, this set would not be a SSPL_3-solution (nor would it lead to a solution in any obvious way that we could think of), but it seems to be an essential ingredient, for instance, of the original proof given by Specker, where a similar fact is used (see for instance the start of the proof of Claim 4.2.3). Unfortunately, we are unable to do so, although we give a partial result on what we could consider all the finite approximations of such a set in the Lemma below.

Before proving the Lemma, we point out that, for any computable instance $(\vec{R}, f)$ of SPL_3 that does not have computable SPL_3-solutions and for every K as above, ω is a set like the I_K we just described: to see this, suppose for a contradiction that there is an $x \in \omega$ such that for infinitely many $i \in \omega$, there are infinitely many $y \in R_i$ such that $f(x,y) = 1$. Then, the computable set $\{y \in \omega : f(x,y) = 1\}$ is a SPL_3-solution to f, since it is an infinite f-homogeneous set for 0 with infinite intersections with infinitely many R_i. Hence, we can conclude that no x as before exists, which means that for every $x \in \omega$, there are only finitely many $i \in \omega$ such that $\lim_{y \in R_i \cap K} f(x,y) = 1$, which proves that we can take $I_k = \omega$.

Lemma 4.2.11. *Let $(\vec{R}, f)$ be a low_2 instance of SSPL_3 such that for every $x, i \in \omega$ $\lim_{y \in R_i} f(x,y)$ exists. If there is a finite set $F \subseteq \omega$ such that the set $A_F = \{i \in \omega : \forall x \in F(\lim_{y \in R_i} f(x,y) = 0)\}$ does not contain any infinite low_2 set, then f has a low_2 SSPL_3-solution.*

Proof. Suppose that there is a finite set F as in the hypotheses of the Lemma. As we noticed above, for every finite set F, the set A_F is $\Delta_2^{0,(\vec{R},f)}$. By a straightforward relativization of [11, Theorem 3.6], either A_F or the complement of A_F has a infinite subset J that is low_2 relative to $(\vec{R}, f)$, i.e. $(J \oplus (\vec{R}, f))'' \leq_{\mathrm{T}} (\vec{R}, f)''$. So by our assumptions let J be an infinite low_2 (relative to $(\vec{R}, f)$) subset of $\omega \setminus A_F$. Computably in $J \oplus (\vec{R}, f)$, we can define the finite partition $\bigcup_{x \in F} D_x$ of J as follows: first, for every $x \in F$, let $C_x = \{j \in J : \exists^\infty y \in R_j(f(x,y) = 1)\}$. We notice that every C_x is $\Delta_2^{0,J \oplus (\vec{R},f)}$. Then, for every $x \in F$, we set $D_x = C_x \setminus \bigcup_{z \in F, z < x} C_z$. Since the collection of the D_x is a finite partition of J in $\Delta_2^{0,J \oplus (\vec{R},f)}$ sets, by a relativization of [11, Theorem

3.7] there is a $\bar{x} \in F$ such that $D_{\bar{x}}$ contains an infinite set S that is low$_2$ relative to $J \oplus (\vec{R}, f)$, so that $(S \oplus J \oplus (\vec{R}, f))'' \leq_{\mathrm{T}} (J \oplus (\vec{R}, f))''$.

Hence, computably in $S \oplus (\vec{R}, f)$, we can define the set $H = \{y \in \omega : \exists s < y(s \in S \wedge y \in R_s) \wedge f(\bar{x}, y) = 1\}$. The set H is clearly a SSPL_3-solution to $(\vec{R}, f)$. Moreover, since $H \leq_{\mathrm{T}} S \oplus (\vec{R}, f)$, we have that $H'' \leq_{\mathrm{T}} (S \oplus J \oplus (\vec{R}, f))'' \leq_{\mathrm{T}} (J \oplus (\vec{R}, f))'' \leq_{\mathrm{T}} (\vec{R}, f)'' \leq_{\mathrm{T}} \emptyset''$, thus proving that H is low$_2$. $\square$

5. First-order part of problems and parallelization

In this Chapter, we give some results on the *first-order part operator*, an operator on problems recently introduced by Dzhafarov, Solomon and Yokoyama: as we explain in Section 5.1, the first-order part of a problem P correspond to the most complicated problem (with respect to Weihrauch reducibility) that is reducible to P and *first-order*, i.e. with range equal to ω.

Our study will focus on the first-order part of problems that are equivalent to parallelizations of first-order problems. To this end, in Section 5.2, we define a new operator, which we call *unbounded* * *operator*: intuitively, this can be seen as an operation intermediate between the finite parallelization * and the infinite parallelization $\widehat{\ }$. After proving that the unbounded * operator is Weihrauch-degree theoretic, we show that it indeed offers an alternative characterization of the first-order part of the parallelization of a first-order principle P, i.e. ${}^1(\widehat{\mathsf{P}}) = \mathsf{P}^{u*}$.

Finally, in Section 5.3, we will see an example of the computation of the first-order part of a principle: exploiting the known fact that $\mathsf{WKL} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_2}$, we will see that ${}^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^*$. We will then generalize this result and show that, for every $n > 0$, ${}^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)^{(n)}$.

We point out that the result of this Chapter are joint work with Manlio Valenti.

5.1. The first-order part operator

Recently, Dzhafarov, Solomon and Yokoyama [20] introduced the *first-order part* of a problem P: the main idea behind this operator is to produce, in the Weihrauch degrees, something that is reminiscent of the operation of considering the first-order part of a theory in reverse mathematics. One possible approach to do this is to study problems that have ω as codomain, instead of ω^ω. This is precisely the intuition behind Definition 5.1.1.

Before we give the Definition, we remark that in this Chapter we will fully adopt the notational convention described in Subsection 1.2.1: namely, for $p \in \omega^\omega$, we will write Φ_p to mean the Turing functional $\Phi_{p(0)}(p\restriction_{\omega\setminus\{0\}})$, where, as customary, we assume given an enumeration $\Phi_0, \Phi_1, \dots$ of the Turing functionals.

Definition 5.1.1 ([20]).

- We fix the following representation $\delta_\omega :\subseteq \omega^\omega \to \omega$ for the space ω: for every $i \in \omega$, $\delta_\omega(x) = i$ if and only if $x = i^\omega$, i.e. the infinite string outputting i for every input.
- We say that a partial multifunction P is a *first-order problem* if the codomain of P is ω. We denote by $\mathcal{F}$ the set[4] of Weihrauch degrees of problems $\mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega$.
- For every problem $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, the *first-order part* of P is the partial multifunction ${}^1\mathsf{P} :\subseteq \omega^\omega \times \mathcal{X} \rightrightarrows \omega$ defined as follows:
  - Input: a pair $(p, x) \in \omega^\omega \times \mathcal{X}$ such that $x \in \operatorname{dom} \mathsf{P}$ and for every $y \in \mathsf{P}(x)$ and every $q \in \delta_{\mathcal{Y}}^{-1}(y)$, we have that $\Phi_p(q)(0)\downarrow$.
  - Output: an $n \in \omega$ such that for some $y \in \mathsf{P}(x)$ and some $q \in \delta_{\mathcal{Y}}^{-1}(y)$, $\Phi_p(q)(0) = n$.

Although the definition above is rather cumbersome, the degree of the first-order part of a problem admits a nice characterization.

[4] Similarly to what was done to define the set of Weihrauch degrees, $\mathcal{F}$ is technically obtained by only considering the degrees of problems with domain ω^ω.

Theorem 5.1.2 ([20]). *For every problem* P,

$$^1\mathsf{P} \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{\deg_{\mathrm{W}}(\mathsf{Q}) : \deg_{\mathrm{W}}(\mathsf{Q}) \in \mathcal{F} \wedge \mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}\}.$$

We will now focus on the study of the first-order part of a particular class of problems, namely those that can be seen as parallelization of first-order problems, obtaining some general results in Section 5.2. Although this is, in a certain respect, a fairly small class of problems, it contains some important multivalued functions, like lim and WKL. The first-order part of WKL and of its jumps will be thoroughly studied in Section 5.3.

5.2. The unbounded * operator and parallelization

In this Section, we will offer an alternative characterization of the degrees of first-order parts of problems that are parallelizations of first-order problems. This will be done via the introduction of a new operator.

Definition 5.2.1. For every partial multifunction $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, we define the problem $\mathsf{P}^{u*} :\subseteq \omega^\omega \times \mathcal{X}^\omega \rightrightarrows \mathcal{Y}^*$, called the *unbounded * operator*, as follows:

- Input: a pair $(w, (x_i)_{i\in\omega})$ such that $(x_i)_{i\in\omega} \in \operatorname{dom}\widehat{\mathsf{P}}$ and for every $(y_i)_{i\in\omega} \in \widehat{\mathsf{P}}((x_i)_{i\in\omega})$, there is a $k \in \omega$ such that for every $t \in \delta^{-1}_{\mathcal{Y}^k}((y_j)_{j<k})$, $\Phi_w(t)(0)\downarrow$.
- Output: a finite sequence $(k, (y_j)_{j<k})$, for some $k \in \omega$, such that $y_j \in \mathsf{P}(x_j)$ for every $j < k$ and, for every $t \in \delta^{-1}_{\mathcal{Y}^k}((y_j)_{j<k})$, $\Phi_w(t)(0)\downarrow$.

In a certain sense, the operator above corresponds to a form of finite parallelization of the problem P, with one important difference: whereas P^* requires the number of parallel uses of P to be declared in advance, i.e. as part of the input, here we just require that P is used finitely many times. This intuition is corroborated by the following easy Lemma.

Lemma 5.2.2. *For every problem* P, $\mathsf{P}^{u*} \leq_{\mathrm{W}} \widehat{\mathsf{P}}$ *and* $\mathsf{P}^* \leq_{\mathrm{W}} \mathsf{P}^{u*}$.

Proof. Let $(w, (x_i)_{i\in\omega}) \in \operatorname{dom} \mathsf{P}^{u*}$, and let $(y_i)_{i\in\omega} \in \widehat{\mathsf{P}}((x_i)_{i\in\omega})$. In order to find a P^{u*}-solution, we just have to run, in parallel, the computations $\Phi_w(t_n)(0)$, where $t_n \in \delta^{-1}_{\mathcal{Y}^n}((y_i)_{i<n})$: by the assumptions on the domain of P^{u*}, we know that at least one of the computations will converge. Suppose that we find that $\Phi_w(t_k)(0)$ converges, for a certain k, then we just have to output the sequence $(k, (y_i)_{i<k})$. This proves that $\mathsf{P}^{u*} \leq_{\mathrm{W}} \widehat{\mathsf{P}}$.

Now we prove that $\mathsf{P}^* \leq_{\mathrm{W}} \mathsf{P}^{u*}$. Given $(n, x_0, \dots, x_{n-1}) \in \operatorname{dom}(\mathsf{P}^*)$, as a w we choose any index that makes the computation check the first n components of the oracle before converging to 0. Hence, we can define the forward functional as $\Gamma(n, x_0, \dots, x_{n-1}) = (w, ((x_i)_{i\in\omega})$, where $(x_i)_{i\in\omega}$ is obtained by repeating $x_0, \dots, x_{n-1}$ infinitely many times, and the return functional is defined in the obvious way. □

We will say something more about the relationship between the operators *, unbounded * and $\widehat{\ }$ at the end of this section.

Now, we show that the operator unbounded * is rather robust: in the next two Lemmata, we will see that the unbounded * operator is Weihrauch-degree theoretic and idempotent.

Lemma 5.2.3. *For every two problems $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\mathsf{Q} :\subseteq \mathcal{A} \rightrightarrows \mathcal{B}$, if $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$, then $\mathsf{P}^{u*} \leq_{\mathrm{W}} \mathsf{Q}^{u*}$. Hence, the operator unbounded * is Weihrauch-degree theoretic.*

Proof. Suppose that the Weihrauch reduction $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ is witnessed by the pair of functionals Γ, Δ, let $(w, (x_i)_{i\in\omega}) \in \operatorname{dom} \mathsf{P}^{u*}$, and let p_{x_i} be a name of x_i for every $i \in \omega$.

Let $\widetilde{w}$ be the index computed as follows: in $\widetilde{w}$ we encode all the sequence $(p_{x_i})_{i\in\omega}$ and w as well, say that $\widetilde{w}_i = p_{x_i}$ and $\widetilde{w}_{-1} = w$, for notational convenience. We have then $\widetilde{w} = \widetilde{w}(0)^\frown(\widetilde{w}_i)_{i\in\{-1\}\cup\omega}$, where $\widetilde{w}(0)$ is the index for the universal Turing functional Φ such that the following happens: $\Phi_{\widetilde{w}}(\langle\langle((c_i)_{i\in\omega})\rangle\rangle)(0) = \Phi_w(\langle\langle((\Delta(c_i, \widetilde{w}_i)_{i\in\omega})\rangle\rangle)(0)$, for all $(c_i)_{i\in\omega} \in \omega^\omega$ (namely, it is enough to find an index $\widetilde{w}(0)$ that replicates step by step the computation on the right, since we have coded all the necessary data in $\widetilde{w}$).

Now, consider $(\widetilde{w},(\Gamma(p_{x_i}))_{i\in\omega})$, and notice that $\delta_{\omega^\omega\times\mathcal{A}^\omega}(\widetilde{w},(\Gamma(p_{x_i}))_{i\in\omega}) = (\widetilde{w},(\delta_{\mathcal{A}}(\Gamma(p_{x_i})))_{i\in\omega})$, by the way we defined the representations in Section 1.2.3. We claim that $(\widetilde{w},(\delta_{\mathcal{A}}(\Gamma(p_{x_i}))_{i\in\omega})$ is an instance of Q^{u*}. To see this, let $(b_i)_{i\in\omega} \in \widehat{\mathsf{Q}}(\delta_{\mathcal{A}}(\Gamma(p_{x_i}))_{i\in\omega}))$. Then, for every name $(p_{b_i})_{i\in\omega}$ of $(b_i)_{i\in\omega}$, $(\Delta(p_{b_i},p_{x_i}))_{i\in\omega}$ is a name for an element of $\widehat{\mathsf{P}}((x_i)_{i\in\omega})$, so by our assumption on $(w,(x_i)_{i\in\omega})$ there is a $k_{(w,(x_i)_{i\in\omega})}$ such that the first $k_{(w,(x_i)_{i\in\omega})}$ components of $(\Delta(p_{b_i},p_{x_i}))_{i\in\omega}$ make the computation of $\Phi_w((\Delta(p_{b_i},p_{x_i}))_{i\in\omega})(0)$ converge. But since we have that $\Phi_w((\Delta(c_i,p_{x_i}))_{i\in\omega})(0) = \Phi_{\widetilde{w}}(\langle\langle((c_i)_{i\in\omega})\rangle\rangle)(0)$ by our definition of $\widetilde{w}$, then we also have that the first $k_{(w,(x_i)_{i\in\omega})}$ components of $(p_{b_i})_{i\in\omega}$ make $\Phi_{\widetilde{w}}(\langle\langle((p_{b_i})_{i\in\omega})\rangle\rangle)(0)$ converge.

Finally, notice that then it is easy to go from a name of a Q^{u*}-solution to a name for a P^{u*}-solution via the functional Δ. □

Lemma 5.2.4. *For every problem $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, $(\mathsf{P}^{u*})^{u*} \leq_{\mathrm{W}} \mathsf{P}^{u*}$. Hence, the unbounded * operator is idempotent.*

Proof. By unraveling the definition, we get that an instance of $(\mathsf{P}^{u*})^{u*}$ is a sequence $(w,(w_i,(x^i_r)_{r\in\omega})_{i\in\omega})$ such that $(w_i,(x^i_r)_{r\in\omega})_{i\in\omega} \in \operatorname{dom}\widehat{(\mathsf{P}^{u*})}$ and for every $(y_i)_{i\in\omega} \in \widehat{(\mathsf{P})^{u*}}((w_i,(x^i_r)_{r\in\omega})_{i\in\omega})$, there is a $k\in\omega$ such that for every $t \in \delta^{-1}_{(\mathcal{Y}^*)^k}((y_j)_{j<k})$, $\Phi_w(t)(0)\downarrow$, where for every $j<k$, $y_j = (k_j,(z_l)_{l<k_j})$ such that for every $t_j \in \delta^{-1}_{\mathcal{Y}^{k_j}}((z_l)_{l<k_j})$, $\Phi_{w_j}(t_j)(0)\downarrow$.

We define $\widehat{w}\in\omega^\omega$ as the index such that, for every sequence of sequences $(c^i_r)_{i,r\in\omega} \in (\omega^\omega)^\omega$, $\Phi_{\widehat{w}}((c^i_r)_{i,r\in\omega})(0)$ does the following: at computation step s, for every $i<s$, it checks whether for some $k'_i < s$ $\Phi_{w_i,s}((c^i_r)_{r\in\omega})(0)\downarrow$ (we assume to have coded the w_i in $\widehat{w}$). Letting I_s be the set of $i<s$ such that this happens, it then runs $\Phi_{w,s}((k'_i,(c^i_r)_{r\in\omega})_{i\in I_s})(0)$.

We now define the forward functional of the reduction to be given by $(p_w,p_{(w_i,(x^i_r)_{r\in\omega})_{i\in\omega}}) \mapsto (\widehat{w},(p_{(x^i_r)_{r\in\omega}})_{i\in\omega})$ (where we have again used the convention that p_x denotes a name of x). By our assumptions on $(w,(w_i,(x^i_r)_{r\in\omega})_{i\in\omega})$, it is clear that we have defined an instance of P^{u*}, and it is also easy to see how to from a (name for a) P^{u*}-solution of $(\widehat{w},(p_{(x^i_r)_{r\in\omega}})_{i\in\omega})$ to a (name for a) $(\mathsf{P}^{u*})^{u*}$-solution to $(w,(w_i,(x^i_r)_{r\in\omega})_{i\in\omega})$. □

Thanks to the Lemma above, we can conclude that the unbounded * operator is a *closure operator* with respect to Weihrauch reducibility.

We can now move to the relationship between the first-order part and the unbounded * operators.

Theorem 5.2.5. *For every first-order problem Q, we have that ${}^1(\widehat{\mathsf{Q}}) \equiv_{\mathrm{W}} \mathsf{Q}^{u*}$.*

Proof. First, we notice that if $\mathsf{Q} :\subseteq \mathcal{X} \rightrightarrows \omega$ is first-order, then the problem Q^{u*} has codomain $\omega^{<\omega}$: hence, by a minor change of representation, we can see the problem Q^{u*} as a first-order problem. Since, by Lemma 5.2.2, $\mathsf{Q}^{u*} \leq_{\mathrm{W}} \widehat{\mathsf{Q}}$, by Theorem 5.1.2 we can conclude that $\mathsf{Q}^{u*} \leq_{\mathrm{W}} {}^1(\widehat{\mathsf{Q}})$.

Hence, we just have to show that ${}^1(\widehat{\mathsf{Q}}) \leq_{\mathrm{W}} \mathsf{Q}^{u*}$. Let $\mathsf{R} :\subseteq \mathcal{Y} \rightrightarrows \omega$ be a first-order problem such that $\mathsf{R} \leq_{\mathrm{W}} \widehat{\mathsf{Q}}$, as witnessed by the functionals Γ, Δ. Let $y \in \operatorname{dom} \mathsf{R}$, let p_y be a name of y, and let $(z_i)_{i\in\omega} \in \widehat{\mathsf{Q}}(\delta_{\mathcal{X}^\omega}(\Gamma(p_y)))$. Notice that, by our choice of representation of ω, it follows that $(z_i)_{i\in\omega}$ has only one name, $p_{(z_i)_{i\in\omega}}$. Since we know that $\Delta(p_y, p_{(z_i)_{i\in\omega}})(0)$ has to converge, there is a k such that $\Delta(p_y, p_{(z_i)_{i\in\omega}})(0) = \Delta(p_y, p_{(z_i)_{i<k}})(0)$, where $p_{(z_i)_{i<k}}$ is a name of $(z_i)_{i<k}$, and actually just a finite initial segment of $p_{(z_i)_{i<k}}$ is used. Hence, by letting $w_y \in \omega^\omega$ be the index such that $\Phi_{w_y}(\sigma)(0) = \Delta(p_y, \sigma)(0)$ for every $\sigma \in \omega^{<\omega}$, we get that $(w_y, (y_i)_{i\in\omega})$ (where every y_i equals to y) is an instance of Q^{u*}, and from any solution to that it is immediate to compute an R-solution to y. Hence, we have that $\mathsf{R} \leq_{\mathrm{W}} \mathsf{Q}^{u*}$, and hence, by Theorem 5.1.2, that ${}^1(\widehat{\mathsf{Q}}) \leq_{\mathrm{W}} \mathsf{Q}^{u*}$. □

We can now add some considerations on the operator unbounded *: in Lemma 5.2.2, we proved that for every problem P, it holds that $\mathsf{P}^* \leq_{\mathrm{W}} \mathsf{P}^{u*} \leq_{\mathrm{W}} \widehat{\mathsf{P}}$. Using the Theorem above, we are able to show that P^{u*} does not collapse on either of the two other operators: in the next section, we will show that for the problem C_2 (which we will introduce), we have that ${}^1(\widehat{\mathsf{C}_2}) \equiv_{\mathrm{W}} \mathsf{C}_2^*$, so that, by the Theorem above, ${\mathsf{C}_2}^{u*} \equiv_{\mathrm{W}} \mathsf{C}_2^*$, and it is known that $\mathsf{C}_2^* \not\equiv_{\mathrm{W}} \widehat{\mathsf{C}_2}$ (see for instance [6, Section 7]). So, if the operator unbounded * had to collapse on one of the other two, it would have to be the finite parallelization operator. But it is enough to consider ${}^1\mathsf{lim}$ to see that this cannot be the case: it is known that $\mathsf{lim} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}}$ (again, see [6]), so that by the Theorem above ${}^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{LPO}^{u*}$. It is another known result that $\mathsf{C}_\omega \leq_{\mathrm{W}} \mathsf{lim}$ (where C_ω is the closed choice on ω, which we shall not define: we refer to [6] for both a definition of it and a proof of the result above), which, by Theorems 5.1.2 and 5.2.5, implies that $\mathsf{C}_\omega \leq_{\mathrm{W}} \mathsf{LPO}^{u*}$. On the other hand, by [54, Lemma 5], we have that $\mathsf{C}_\omega \not\leq_{\mathrm{W}} \mathsf{LPO}^*$, which proves that $\mathsf{LPO}^{u*} \not\equiv_{\mathrm{W}} \mathsf{LPO}^*$.

5.3. The first-order part of $\mathsf{WKL}^{(n)}$

In this section, we will compute the Weihrauch degree of the first-order part of the jumps of WKL. A very important character in this computation will be the problem C_2, which we now introduce.

Definition 5.3.1. For every $k \in \omega$, C_k is the following partial multifunction:

- Input: an infinite sequence $x \in (k+1)^\omega$ such that $\{0, \dots, k-1\} \not\subseteq \operatorname{ran} x$.
- Output: a point $y \in k$ such that $y \notin \operatorname{ran} x$.

We notice right away that, since C_k can be seen as a partial multifunction with codomain ω, we have that C_k is a first-order problem.

In general, for every represented space $(\mathcal{X}, \delta_{\mathcal{X}})$ which is also a topological space, we could define the problem $\mathsf{C}_{\mathcal{X}}$, called *closed choice on* $\mathcal{X}$, as the problem that, given a non-empty closed set of $\mathcal{X}$, finds a point in that set: these are important and widely studied problems, and we refer for instance to [6] for more on them. In this particular instance, since we do not need these principles in their full generality, we have chosen to limit ourselves to give the definition of closed choice in the case that the space is the finite set k, and the closed set from which to chose a point is given by an enumeration of its complement.

The main reason we are interested in C_2 is given by the following Lemma

Lemma 5.3.2 ([6], Theorem 7.23)**.** $\mathsf{WKL} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2}$

Hence, we could immediately apply the results from the previous section to get that ${}^1\mathsf{WKL} \equiv_{\mathrm{W}} (\mathsf{C}_2^*)^{u*}$. In the next Lemma, we will see that ${}^1\mathsf{WKL}$ admits a more familiar description.

Lemma 5.3.3. ${}^1(\widehat{\mathsf{C}_2}) \equiv_{\mathrm{W}} \mathsf{C}_2^*$

Proof. Since $\mathsf{C}_2^* \leq_{\mathrm{W}} \widehat{\mathsf{C}_2}$ and C_2^* is first-order, it follows that $\mathsf{C}_2^* \leq_{\mathrm{W}} {}^1(\widehat{\mathsf{C}_2})$, hence we just have to show the other reduction.

In order to do this, by Theorem 5.1.2, it is enough to show that for every first-order problem $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \omega$, for some represented space $(\mathcal{X}, \delta_{\mathcal{X}})$, if $\mathsf{P} \leq_{\mathrm{W}} \widehat{\mathsf{C}_2}$, then we have that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{C}_2^*$.

Suppose that the reduction $\mathsf{P} \leq_{\mathrm{W}} \widehat{\mathsf{C}_2}$ is witnessed by the pair of functionals Γ, Δ. Suppose that x is a valid input for P, and let $p_x \in \omega^\omega$ be such that $\delta_{\mathcal{X}}(p_x) = x$.

We now claim that we can determine the number of parallel applications of C_2 that we need to use in order to find a P-solution for x. To do this, we start observing the following thing: since we are assuming that P is first order, for every $\widehat{\mathsf{C}_2}$-solution z to $\Gamma(p_x)$, in order to find a P-solution y to x, the functional Δ will only use a finite amount of the oracle $\langle z, \Gamma(p_x)\rangle$, since y is just an element of ω: that amount will be limited by the use of the computation $\Delta(\langle z, p_x\rangle)(0)$.

We can then proceed in stages as follows. At every stage s, we first examine the set $\Gamma(p_x)(\{0, \dots, s\})$: by our assumptions, this is some initial segment of a sequence of elements of 3^ω. Notice that, by examining $\Gamma(p_x)(\{0, \dots, s\})$, we can exclude some of the possible answers of $\widehat{C}_2$ to $\Gamma(p_x)$: suppose that $\Gamma(p_x)$ is the sequence of sequences $(x_i)_{i\in\omega}$, then if for some $j, m \in \omega$ and $t < s$ we have that $\Gamma(p_x)(t) = x_j(m) = l$, for $l < 2$ then no function $f : \omega \to 2$ such that $f(j) = l$ can be a $\widehat{\mathsf{C}_2}$-solution to $\Gamma(p_x)$. If this happens, we say that the function f has been *excluded at stage s*. We let E_n^s be the set of strings of length n that are initial segments of binary functions that have been excluded at stage s. For every n, we let G_n^s be the set $2^n \setminus E_n^s$: we can see G_n^s as the set of guesses for an initial segment of length n of a $\widehat{\mathsf{C}_2}$-solution to $\Gamma(p_x)$ that are still possible at stage s. Finally, we let C_n^s be the subset of G_n^s such that $\Delta(\langle C_n^s, p_x\rangle)(0)$ converges in less than s steps.

Notice that, for a function $f : \omega \to 2$, if for every s and every n we have that $f\restriction_{\{0,\dots,n-1\}} \notin E^s_n$, then actually $f \in \widehat{\mathsf{C}_2}(\Gamma(p_x))$, by the definition of the problem C_2. We then claim that there are an $\bar{n}$ and an $\bar{s}$ such that every binary string of length $\bar{n}$ is $E^{\bar{s}}_{\bar{n}} \cup C^{\bar{s}}_{\bar{n}}$: if this was not the case, for every n and s there would be a string not in $E^s_n \cup C^s_n$. Then, for every n and s, let $T^s_n = \{f \in 2^\omega : f\restriction_{\{0,\dots,n-1\}} \in 2^n \setminus (E^s_n \cup C^s_n)\}$: every T^s_n is a non-empty closed subset of 2^ω. Since it is easily seen that for every n and s $T^{s+1}_n \subseteq T^s_n$, it follows that for every n $T_n = \bigcap_{s\in\omega} T^s_n$ is a non-empty closed subset of 2^ω. Similarly, one easily sees that, for every n, $T_{n+1} \subseteq T_n$, which means that $T = \bigcap_n T_n$ is a non-empty closed subset of 2^ω. Let us now consider any $f \in T$: as we have just observed, $f \in \widehat{\mathsf{C}_2}(\Gamma(p_x))$ follows from the fact that for every s and n $f\restriction_{\{0,\dots,n-1\}} \notin E^s_n$, but we also have that $\Delta(\langle f, p_x\rangle)(0)\uparrow$ from the fact that for every s and n $f\restriction_{\{0,\dots,n-1\}} \notin C^s_n$. This contradicts the fact that $\mathsf{P} \leq_{\mathrm{W}} \widehat{\mathsf{C}_2}$ via Γ and Δ, and thus proves the existence of $\bar{s}$ and $\bar{n}$ as we want.

Thus, we can prove that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{C}_2^*$ via the following procedure: we start by running the procedure described above until $\bar{s}$ and $\bar{n}$ are found. Then, let $\widetilde{\Gamma}$ be the Turing functional that produces the C_2^*-instance $(\bar{n}, (w_i)_{i<\bar{n}})$, where $(w_i)_{i<\bar{n}}$ are the first $\bar{n}$ sequences in the output of $\Gamma(p_x)$. Then, the pair of functionals $\widetilde{\Gamma}$, Δ witnesses that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{C}_2^*$. □

By Lemma 5.3.2, Lemma 5.3.3 implies that ${}^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^*$.

We now have to take care of the jumps. Doing this is rather delicate, since the jump is not a Weihrauch-degree theoretic operator, as we noticed in Subsection 1.2.3. To do this, we will see that there is a class of first-order problems that has the same nice behavior as the cylinders with respect to the jump.

Definition 5.3.4. Let $\mathsf{P} : \mathcal{X} \rightrightarrows \omega$ be a first-order problem. We say that P is a *first-order cylinder* if for every first-order problem Q, $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P} \Rightarrow \mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$.

Lemma 5.3.5. *For every problem* $\mathsf{P} :\subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, *if* P *is a cylinder, then* ${}^1\mathsf{P}$ *is a first-order cylinder.*

Proof. Suppose that $\mathsf{Q} : \mathcal{Z} \rightrightarrows \omega$ is a first-order problem such that $\mathsf{Q} \leq_{\mathrm{W}} {}^1\mathsf{P}$, as witnessed by functionals Γ, Δ. This means that, for every $z \in \operatorname{dom} \mathsf{Q}$ and every name p_z of it, $\Gamma(p_z)$ is a name for an instance of ${}^1\mathsf{P}$, i.e. a pair (w, p_x) where p_x is the name of some instance x of P. We start modifying Γ to the functional $\widetilde{\Gamma}$ as follows: for every $z \in \operatorname{dom} \mathsf{Q}$, we put $\widetilde{\Gamma}(p_z) = (\langle \widetilde{w}, p_z \rangle, p_x)$, where $\widetilde{w}$ is such that $\Phi_{\langle \widetilde{w}, p_z \rangle}(t)(0) = \Phi_w(t)(0)$ for every $t \in \omega$: in essence, we are just making sure that a name for z is coded in the input for ${}^1\mathsf{P}$.

Since we are assuming that P is a cylinder, there are two functionals Γ_0, Δ_0 witnessing that $\mathsf{id} \times \mathsf{P} \leq_{\mathrm{sW}} \mathsf{P}$. Then, we define our final functional Γ_f as $\Gamma_f(p_z) = (v, \Gamma_0 \circ \widetilde{\Gamma}(p_z))$, where v is an index such that

$$\Phi_v(t)(0) = \Delta(\pi_1(\pi_0(\pi_0(\Delta_0(t)))), \Phi_{\pi_0(\pi_0(\pi_0(\Delta_0(t))))}(\pi_1(t)))(0)$$

for every $t \in \omega^\omega$, where for every $i < 2$, $\pi_i(\langle x_0, x_1 \rangle) = x_i$, i.e π_i is the projection on the ith component. We notice that the computation above is bound to converge by the assumptions on (w, p_z), and that every output of the computation above on input (w, p_z) gives a directly a Q-solution to z.

In general, although the formula above might seem cumbersome, the proof boils down to showing that one can use $\mathsf{id} \times \mathsf{P}$ in the place of P uniformly and without access to the initial inputs. □

We now show that for cylinders, the jump and the first-order part operator commute.

Lemma 5.3.6. *For every cylinder* P, ${}^1(\mathsf{P}') \equiv_{\mathrm{sW}} ({}^1\mathsf{P})'$.

Proof. Since P is a cylinder, by Theorem 1.2.17 Item 3 we have that $\mathsf{P} * \mathsf{lim} \equiv_{\mathrm{W}} \mathsf{P}'$. Hence, $({}^1\mathsf{P})' \leq_{\mathrm{W}} {}^1\mathsf{P} * \mathsf{lim} \leq_{\mathrm{W}} \mathsf{P} * \mathsf{lim} \equiv_{\mathrm{W}} \mathsf{P}'$. It follows that $({}^1\mathsf{P})' \leq_{\mathrm{W}} {}^1(\mathsf{P}')$ by Theorem 5.1.2, and hence $({}^1\mathsf{P})' \leq_{\mathrm{sW}} {}^1(\mathsf{P}')$ by Lemma 5.3.5.

Hence, we just have to show that ${}^1(\mathsf{P}') \leq_{\mathrm{sW}} ({}^1\mathsf{P})'$. We point out that this fact actually holds in general, i.e. it does not depend on P being a cylinder. To see this, by unraveling the definitions, we see that an instance of ${}^1(\mathsf{P}')$ is a pair $(w, (x_i)_{i \in \omega})$ such that the sequence $(x_i)_{i \in \omega}$ converges to an input x for P. But then, we can define the sequence of pairs $(w_i, x_i)_{i\in\omega}$ such that $w_i = w$ for every i, and this is a valid input for $({}^1\mathsf{P})'$, any solution to which is clearly a ${}^1(\mathsf{P}')$-solution to $(w, (x_i)_{i\in\omega})$. Hence, we have that ${}^1(\mathsf{P}') \leq_{\mathrm{sW}} ({}^1\mathsf{P})'$. □

Hence, by the Lemmas above and the fact that WKL is a cylinder, we have that ${}^1(\mathsf{WKL}') \equiv_{\mathrm{sW}} ({}^1\mathsf{WKL})'$. We now show that ${}^1\mathsf{WKL}$ and C_2^* are in the same strong Weihrauch degree. This is equivalent to say that C_2^* is a first-order cylinder, since by the Lemma above we have that $\mathsf{C}_2^* \leq_{\mathrm{sW}} {}^1\mathsf{WKL}$.

Lemma 5.3.7. C_2^* *is a first-order cylinder.*

Proof. The proof will rely on an argument similar to that of the proof of Lemma 5.3.3: namely, a fundamental ingredient will be the fact that for C_2, if a number is not a valid solution for a certain instance, we get to know it in a finite amount of time.

Suppose that $\mathsf{Q} : \mathcal{Z} \rightrightarrows \omega$ is a first-order problem such that $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{C}_2^*$ via the functionals Γ, Δ. We proceed in stages as follows: suppose that n is such that n parallel applications of C_2 suffice to solve $\Gamma(p_z)$ (notice that the bit of information corresponding to the number n has to be produced right away, by the definition of C_2^*). Then, at stage s, for every $\sigma \in 2^n \setminus E_n^s$ (we are using the same terminology of Lemma 5.3.3) we run the first s steps of the computation $\Delta(p_z, \sigma)$.

We claim that, for a certain stage $\bar{s}$, for every $\sigma \in 2^n \setminus E_n^{\bar{s}}$, we have that $\Delta(p_z, \sigma)(0)\downarrow$ in less than $\bar{s}$ steps: if this was not the case, we would have that there exists a C_2^*-solution σ to $\Gamma(p_z)$ for which $\Delta(p_z, \sigma)(0)\uparrow$, which contradicts our assumptions on Δ.

Hence, in order to show that $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{C}_2^*$, we can proceed as follows: we first compute $\bar{s}$ as in the paragraph above. Then, using the obvious fact that $\mathsf{id}_{\omega^{<\omega}} \times \mathsf{C}_2^* \leq_{\mathrm{sW}} \mathsf{C}_2^*$, we use C_2^* to compute an $\mathsf{id}_{\omega^{<\omega}} \times \mathsf{C}_2^*$-solution to $(p_z{\restriction}_{\bar{s}}, \Gamma(p_z))$, say $(p_z{\restriction}_{\bar{s}}, y)$, and use it to compute $\Delta(p_z{\restriction}_{\bar{s}}, y)(0) = \Delta(p_z, y)(0)$. This proves that $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{C}_2^*$. □

Finally, we are ready to compute the first-order part of WKL.

Theorem 5.3.8. *For every $n > 0$, ${}^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)^{(n)}$.*

Proof. In Lemma 5.3.3, we saw that ${}^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^*$. By Lemma 5.3.5, we have that $\mathsf{C}_2^* \leq_{\mathrm{W}} {}^1\mathsf{WKL}$, and by Lemma 5.3.7 we also have that ${}^1\mathsf{WKL} \leq_{\mathrm{sW}} \mathsf{C}_2^*$, so that ${}^1\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_2^*$.

Now, for every $n > 0$, by iterated applications of Lemma 5.3.6 we have that ${}^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{sW}} ({}^1\mathsf{WKL})^{(n)}$. By the previous paragraph and the fact that the jump is strong Weihrauch-degree theoretic, we can conclude that ${}^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)^{(n)}$. $\square$

As a final remark, we notice that, since for every $n, k > 0$ we have that $\widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$ by [8, Corollary 4.18], by the Theorem above we can also conclude that ${}^1(\widehat{\mathsf{RT}_k^n}) \equiv_{\mathrm{W}} (\mathsf{C}_2^*)^{(n)}$.

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