Hamiltonian and Lagrangian structures in integrable hierarchies and covariant field theory

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- In [CS20a], the workload was shared equally between myself, having done most of the calculations, and my supervisor Dr. Vincent Caudrelier providing help and guidance and the original idea of using covariant Poisson brackets to investigate r-matrix structures. Content from [CS20a] is included in Chapter 3 (although reformulated adapting the more general proofs of [CS20b]) and in Chapter 4.
- In [CS20b] the original idea of applying Dickey's procedure to Lagrangian multiforms and its development was mine, and Dr. Caudrelier giving help in overcoming computational difficulties. Content from [CS20b] is included in Chapter 5.
- In [CS21] I developed the Hamiltonian multiform formulation and found the r-matrix in the multi-time Poisson brackets, while Dr. Caudrelier formulated the Lagrangian multiform in generating form. Content from [CS21] is included in Chapter 6 and Appendix B.
- The paper [CSV21a] is a collaboration between Dr. Caudrelier, myself, and Dr. Benoît Vicedo (University of York). Whilst Dr. Vicedo proved the $4d$ Chern-Simons origin of the Zakharov-Mikhailov Lagrangian (included in Appendix A.4), Dr. Caudrelier and myself developed together its covariant Hamiltonian description and its covariant Lax algebra, included in Section 4.5.

I can't do this all on my own
No, I know, I'm no Superman

Superman - Lazlo Bane

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Abstract

Motivated by the recent discoveries of space-time duality of the classical r-matrix, this thesis explores the role of covariant field theory and multi-dimensional consistency for field theories in $1+1$-dimensions.

We obtain for the first time a classical r-matrix in a covariant context for several prototypical examples of integrable field theories. The zero-curvature equations are then reinterpreted as covariant Hamilton's equations for the Lax connection. We propose the notion of Hamiltonian multiforms for integrable hierarchies, which provide the Hamiltonian counterpart of Lagrangian multiforms and encapsulate in a single object an arbitrary number of flows within an integrable hierarchy. This also produces two other important objects: a symplectic multiform and the related multi-time Poisson bracket. This new formulation is applied consistently to three hierarchies, *i.e.* the sine-Gordon hierarchy, the Korteweg-de Vries hierarchy and the Ablowitz-Kaup-Newell-Segur hierarchy, and gives a description of conservation laws in terms of Poisson involutivity with the Hamiltonian multiform. The Ablowitz-Kaup-Newell-Segur hierarchy is analysed in particular detail and a classical r-matrix structure is identified within the multi-time Poisson bracket for the complete hierarchy.

Finally, we study the interplay between the classical Yang-Baxter equation and Lagrangian multiform theory, providing a technique to extract Lagrangians for several hierarchies in terms of a generating formal Laurent series. We demonstrate how to obtain the Lagrangian multiform for the Ablowitz-Kaup-Newell-Segur hierarchy, the Lagrangian for the sine-Gordon equation in light-cone coordinates, and the Lagrangians describing the zero-curvature equation for any Lax pair of Zakharov-Shabat type with rational dependence on the spectral parameter with distinct poles.

Contents

List of abbreviations

- **AKNS:** Ablowitz-Kaup-Newell-Segur (hierarchy)
- **FNR:** Flashka-Newell-Ratiu
- **ISM:** Inverse Scattering Method
- **IST:** Inverse Scattering Transform
- **KdV:** Korteweg-de Vries (equation)
- **mKdV:** modified Korteweg-de Vries (equation)
- **NLS:** Non-Linear Schrödinger (equation)
- **ODE:** Ordinary Differential Equation
- **PDE:** Partial Differential Equation
- **sG:** sine-Gordon (equation)
- **ZM:** Zakharov-Mikhailov

Chapter 1

Introduction

Field theories have provided an exceptional framework to describe the fundamental laws of nature. The standard model, for instance, is a quantum field theory that describes three of the four known fundamental forces of the universe, electromagnetic, strong and weak interactions, and it classifies all known elementary particles. Statistical field theories can describe phase transitions, encompassing models including superconductivity and superfluidity. Classical field theories are described by Partial Differential Equations (PDEs) and include famous examples such as the Einstein equations of gravity and the Navier-Stokes equations, fundamental in the study of fluid dynamics. Within the main theories, the so-called *Integrable Systems* have played a crucial role in providing beautiful theoretical laboratories to understand the mathematical structure of field theories.

The concept of a 'completely integrable system' arose initially in the context of of finite-dimensional classical mechanics in the 19th century. Hamilton reformulated Newton's law of a system with n degrees of freedom in terms of canonically conjugated coordinates, the generalised positions $q_1, \ldots, q_n$ and momenta $p_1, \ldots, p_n$ of the phase space M, and a smooth real-valued function called *Hamiltonian* of the system $H(q_1, \ldots, q_n, p_1, \ldots, p_n)$ [H34]. The power of this reformulation arguably consists in the fact that Hamilton's equation

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.1}$$

for each $i = 1, \ldots, n$ are first order ordinary differential equations (ODEs), whilst Newton's equations $F_i = m\ddot{q}_i$ are second order. If we write $z = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, the equations (1.1) can be written compactly as

$$\dot{z} = -J\nabla H\,, \qquad J = \begin{pmatrix} 0_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0_{n\times n} \end{pmatrix}\,, \tag{1.2}$$

where J is called 'standard symplectic matrix' (or sometimes 'symplectic unity') and is a non-singular and anti-symmetric matrix ($J^T = -J$, $\det J \neq 0$). Any system of ODEs that can be written in the form (1.2) for an anti-symmetric non-singular matrix J and such a function H is said to be Hamiltonian. The origin of the name 'symplectic' is rather interesting, and was first proposed by Weyl in [W46]:

> *The name complex group formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word complex in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective symplectic.*

Hamilton's equations can also famously be reformulated in terms of the *Poisson brackets* $\{F, G\} \coloneqq \nabla F \cdot J\nabla G = \frac{\partial F}{\partial p_1}\frac{\partial G}{\partial q_1} + \ldots + \frac{\partial F}{\partial p_n}\frac{\partial G}{\partial q_n} - \frac{\partial F}{\partial q_1}\frac{\partial G}{\partial p_1} - \ldots - \frac{\partial F}{\partial q_n}\frac{\partial G}{\partial p_n}$ where F and G are two functions on the phase space. Poisson bracket are bilinear, antisymmetric and derivations on both arguments, and satisfy the so-called Jacobi identity, and we remark that $\{p_i, q_j\} = \delta_{ij}$. In fact not only (1.1) can be written

$$\dot{q}_i = \{H, q_i\}\,, \qquad \dot{p}_i = \{H, p_i\}\,, \tag{1.3}$$

for each $i = 1, \ldots, n$, but we can use the Poisson bracket with the Hamiltonian to compute the evolution of any smooth real function F on the phase space along the flow of H as

$$\dot{F} = \{H, F\}. \tag{1.4}$$

Another way to reformulate this is with *symplectic geometry.* In its simplest formulation[1], we pick $M = T^*Q$ as the cotangent bundle of a manifold Q called 'configuration manifold', and we give local coordinates $z = (q^1, \ldots, q^n, p_1, \ldots, p_n)$. We associate to M the following closed non-degenerate 2-form $\omega = \sum_i dp_i \wedge dq^i$, called *symplectic form.* For any function $H\colon T^*Q \to \mathbb{R}$ we can write its Hamiltonian vector field as $\xi_H\colon T^*Q \to TT^*Q$ such that[2] $\xi_H \lrcorner \omega = dH$. Hamilton's equations can then be written as

$$\dot{z} = -\xi_H(z) \tag{1.5}$$

and the Poisson brackets between two $F, G\colon T^*Q \to \mathbb{R}$ as

$$\{F, G\} = -\xi_F \lrcorner dG = \omega(\xi_F, \xi_G)\,. \tag{1.6}$$

If a function F is such that $\{H, F\} = 0$, then we have that $\dot{F} = 0$: the function is constant along the flow of H and it is called a *first integral* (or constant of motion). If in an n-dimensional Hamiltonian system there are n independent first integrals $F_1, \ldots, F_n$ such that $\{F_i, F_j\} = 0$ then the system is said to be *(completely) integrable.* The condition of mutual vanishing Poisson bracket is called involutivity, and we say that F_i and F_j are

[1] The literature is immense, so I am just going to reference [C15a], which is where I personally learnt about this topic.

[2] $(\,)\lrcorner(\,)$ is the inner product between a vector field and a differential form $\xi \lrcorner \omega \coloneqq \omega(\xi, \ldots)$

'in involution', or equivalently that they 'Poisson-commute'. Liouville-Arnold's theorem [L55, A78] ensures that under some circumstances a completely integrable system can be solved in quadratures in terms of the so-called action-angle variables.

Integrable systems, despite being an old concept, have almost laid dormant until the second half of the twentieth century with the discovery by Zabusky and Kruskal [ZK65] of solitons in the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + 6uu_x\,. \tag{1.7}$$

The KdV equation describes the nonlinear evolution of a real-valued field $u(x,t)$, where x is the space and t is the time. It is arguably one of the most famous examples of integrable systems, introduced by Boussinesq in 1877 in a footnote [B77] and then later rediscovered by Korteweg and de Vries in 1895 [KV95] as a mathematical model for shallow water waves. The works of Gardner, Greene, Zabusky, Kruskal and Miura [GGKM67, MGK68, KMGZ70] showed that the KdV has an infinite number of conservation laws and conserved quantities, and mapped the initial value for the KdV Cauchy problem to spectral and scattering data of the Schrödinger operator. The nonlinear evolution of u essentially transforms into the linear evolution of these data, and can be obtained by the inverse transformation, called Inverse Scattering Transform (IST). Zakharov and Faddeev [ZF71] explained that the KdV equation is indeed a completely integrable *infinite-dimensional* system, where the spectral and scattering data can be seen as action-angle variables, and the infinite number of conserved quantities as the first integrals in involution. It is also an infinite-dimensional *Hamiltonian* system in the following sense: the time evolution of the initial datum $u(x,0)$ can be obtained as

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}\,, \qquad H = \int \left(u^3 - \frac{1}{2}(u_x)^2\right) dx \tag{1.8}$$

where $\frac{\delta}{\delta u}$ is called Frechet (or *variational*) derivative. Moreover one can introduce an infinite-dimensional version of the Poisson brackets called *equal-time Poisson brackets* between two functionals of u, F and G

$$\{F,G\} = \int \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta G}{\delta u} dx\,. \tag{1.9}$$

The presence of an infinite number of conservation laws proved that the KdV equation can be seen as a member of an *integrable hierarchy*, all its commuting symmetries being in fact infinite non-linear flows with respect to different time variables $t_1, t_2, \ldots$.

Other examples of integrable systems include the Non-Linear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = 0\,, \tag{1.10}$$

used in both classical and quantum field theories, and applied for instance to Bose-Einstein condensates and nonlinear optics, and many other topics. The 'focusing' version allows the presence of solitons and can be solved using the Inverse Scattering Transform [ZS72], and can be seen as part of the integrable hierarchy called Ablowitz-Kaup-Newell-Segur (AKNS) [AKNS74]. In fact, the next member of the AKNS hierarchy produces the modified Korteweg-de Vries equation

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{2}u^2 u_x = 0\,, \tag{1.11}$$

a modification of the KdV equation where we consider a cubic nonlinear term instead of the usual quadratic one.

Another example of an integrable system is the sine-Gordon (sG) equation

$$u_{\xi\eta} + \sin u = 0 \tag{1.12}$$

(ξ and η are called *light-cone coordinates*), whose name is a pun on the Klein-Gordon equation $u_{\xi\eta} + u = 0$ of which the sine-Gordon is a modification. One of its striking properties is that it is manifestly invariant under spacetime translations and Lorentz boosts. Systems that behave consistently with respect to the theory of relativity can be described as *covariant*. The equal-time description of an integrable system (albeit extremely successful!) is manifestly *not* covariant by construction: while x and t have the same importance in principle in the PDE, we immediately make a distinction between them, promoting t as the 'true time' that generates the Hamiltonian flow, and demoting x as an 'accessory coordinate' that we use to mimic the presence of many degrees of freedom of the finite-dimensional case. This breaks the initial manifest covariance, as it is not possible anymore to perform transformations that mix time and space.

Fortunately, the equal-time formalism is not the only available tool at our hands to describe integrability of a field theory. Lax pairs were introduced by Peter Lax in 1968 [L68] as a general principle to associate nonlinear equations $F(u, u_t, u_x, \dots) = 0$ with linear operators, so that the eigenvalues of the linear operator are conserved quantities of the nonlinear equation. In one of its formulations, due to Zakharov and Shabat [ZS72], we consider a linear system for an auxiliary matrix-valued field $\Psi(u, \lambda)$ that depends on u and its derivatives and a spectral parameter $\lambda \in \mathbb{C}$

$$\begin{cases} \partial_x \Psi(u,\lambda) = U(u,\lambda)\Psi(u,\lambda) \\ \partial_t \Psi(u,\lambda) = V(u,\lambda)\Psi(u,\lambda) \end{cases} \tag{1.13}$$

where U, V are matrices that also depend on u and its derivatives and λ, and are called *Lax pair*. Any nonlinear equation $F(u, \dots) = 0$ that can be expressed as the compatibility condition $\Psi_{xt} = \Psi_{tx}$ of any such auxiliary system is proved to allow an infinite number of conserved quantities[3] (see e.g. [FTR07]). At this stage we have not made any distinction

[3] This is weaker than complete integrability as these conserved quantities may not be in involution.

between 'time' and 'space', but both are treated with equal footing and have the same role: we will refer to any formalism with this characteristic as a *covariant formalism.* The aim of this thesis is to push this approach even further, and initiate the development of a true *covariant* description of integrability for classical field theories, in the above sense.

In a way, the motivation behind this thesis originates from the work [CK15], where surprising properties of space-time duality of the *classical r-matrix* were found for the NLS equation. The classical r-matrix is a solution of one of the fundamental equations in the theory of integrable systems, the classical Yang-Baxter equation (which was introduced first in its quantum version independently by Yang and Baxter), and appears when one takes the equal-time Poisson bracket of the coefficients of the Lax matrix U. It determines the structure, symmetries and solution content of an integrable system, and it has proved to be crucial for canonical quantisation, and the Quantum Inverse Scattering Method [S79, SF78, FST80]. In particular, the authors of [CK15] proved that the classical r-matrix structure remained unchanged when the roles of the space and time were swapped, thus surviving this theoretical distortion and pointing to a possible even deeper role played by this already fundamental object, with respect to a covariant formalism. This belief is also supported by a series of subsequent results [C15b, ACDK16, AC17, F19, DFS19] that elaborate on the space-time duality of the r-matrix.

It is worth noticing that the desire to provide a covariant formulation of Hamiltonian field theory originated early on in the 1930's (possibly even before its non-covariant version) with the works of De Donder [D30] and Weyl [W35]. This formalism followed a less fortunate path that, to the best of our knowledge, never crossed with the theory of integrable systems with the exception of one author, Dickey. His book [D03] provided us with the initial setup of this thesis, *i.e.* the variational bi-complex and his formulation of the multisymplectic form and covariant Hamiltonian, natural generalisations of the respective non-covariant objects, both obtained from the Lagrangian formulation of the PDE. The first question is how to define a covariant Poisson bracket that reproduces the r-matrix structure found in both the equal-time and the equal-space Poisson brackets, the latter being the one obtained swapping the roles of time and space. This problem is tackled successfully in this thesis with content from [CS20a, CSV21a] with the definition of a covariant Poisson bracket that encodes both the equal-time and equal-space one.

Moreover, a natural observation is that the same role should not only be played by space and time, but also by all the other times in the integrable hierarchy, that produce the commuting symmetries that we mentioned above, and a true covariant description of an integrable system should take this into account as well. This requirement is well encoded in the recently developed formalism of *Lagrangian multiforms*, introduced by Lobb and Nijhoff in [LN09] for discrete integrable systems and then extended to the continuous case, to describe integrability in a variational fashion. We will prove that one can use Dickey's procedure adapted to a Lagrangian multiform and obtain in return the covariant Hamiltonian formulation of the complete integrable hierarchy in one strike, called *Hamiltonian multiform description* [CS20b]. One can also naturally define the so-called *multi-time Poisson bracket* that encapsulates all the single-time (*i.e.* the equal-time or space) Poisson brackets of the hierarchy. It will be proved that, for the AKNS hierarchy, this new object will possess a classical r-matrix structure [CS21].

Structure of the thesis We chose to structure this thesis using a 'bottom-up' approach that follows the journey of this PhD. The reader will find a series of results almost in the order that they were discovered, with some of the results that generalise other previous ones. We believe that this approach, despite being admittedly not concise, will improve the understandability of this work. It was decided to keep the formalism as light as possible, in an effort to focus the reader's attention on the new concepts that are introduced. Other people's work will sometimes be reported (and rightly attributed) to keep the thesis as self-consistent as possible, and adapted to the notations and conventions of this thesis. Despite this, from now on we will assume the reader is familiar with some fundamental concepts of Mathematical Physics, most of which can be found in [O93, J99]. In particular we will use classical finite-dimensional Lagrangian and Hamiltonian mechanics (Hamilton equations, symplectic forms, Poisson brackets, first integrals), and $1+1$-dimensional classical field theories to some extent (for instance, what an integrable hierarchy is). We also take for granted some basic knowledge of differential geometry (manifolds, differential forms, vector fields, Lie groups and algebras). Chapters 1-2 are introductory and provide the background and motivations of this thesis. The main results are written in the central Chapters 3-7. Chapter 8 concludes the thesis and describes possible future research directions and perspectives.

- Chapter 2 illustrates the relevant background. We describe the discovery of the space-time duality of the classical r-matrix in [CK15] and the subsequent results. Then we briefly describe the properties of the variational bi-complex [D03, A89], which is the algebraic framework that we work in. Finally, we give a short introduction to Lagrangian multiforms for integrable hierarchies of $1+1$-dimensional field theories.
- In Chapter 3 we extend the work of Dickey [D03], adapting it to the ideas of Kanatchikov [K98], and illustrate how to describe covariantly a $1+1$-dimensional classical field theory. We give the definition of three important objects: the covariant Hamiltonian, the multisymplectic form and covariant Poisson brackets (the latter being original of this thesis, albeit adapted from [K98] and [D03]). The two main original results are Theorem 3.15, which formulates the covariant Hamilton equations in terms of the covariant Poisson brackets, and Proposition 3.17, which relates the covariant Poisson brackets with the single-time Poisson brackets.
- In Chapter 4 we apply the covariant description of a $1+1$-dimensional classical field theory on many archetypal examples of integrable systems: the sine-Gordon equation (in both laboratory and light-cone coordinates), the Non-Linear Schrödinger and

modified Korteweg-de Vries equations, and the Zakharov-Mikhailov action [ZM80]. We consistently find two important results: the presence of an ultra-local classical r-matrix structure within the covariant Poisson brackets for the Lax connection, and the formulation of the zero-curvature equation as a covariant Hamilton equation for the Lax connection under the 'multisymplectic' flow of the covariant Hamiltonian.

- In Chapter 5 we introduce and develop the theory of Hamiltonian multiforms, extending the covariant formulation of Chapter 3 to describe covariantly integrable hierarchies (as opposed to single PDEs). These new objects are introduced with a 'Legendre-like transformation' from the Lagrangian multiforms. We also define the symplectic multiform and the multi-time Poisson bracket, that are respectively the symplectic form and Poisson brackets in a multiform context. They are used systematically to describe the first few flows of the (potential) Korteweg-de Vries hierarchy, the sine-Gordon hierarchy in light-cone coordinates, and the Ablowitz-Kaup-Newell-Segur hierarchy.

- In Chapter 6 we use Hamiltonian multiforms to describe covariantly the whole AKNS hierarchy. We write a Lagrangian multiform in terms of a generating double series, from which we obtain (as generating series) both the symplectic and the Hamiltonian multiforms. We then prove the classical r-matrix structure of the multi-time Poisson bracket, and we reformulate the whole set of zero-curvature equations of the AKNS hierarchy as multi-time Hamilton equations for the complete Lax connection under the flow of the Hamiltonian multiform.

- In Chapter 7 we generalise part of the results of Chapter 6 (*i.e.* the Lagrangian multiform aspects) to describe as generating series several integrable hierarchies. With a uniform approach, we obtain (besides the aforementioned Ablowitz-Kaup-Newell-Segur hierarchy) integrable hierarchies with a rational r-matrix structure (*i.e.* AKNS and the actions included in [D03, Section 20.2] and [ZM80]) and with a trigonometric structure (sine-Gordon).

- In Chapter 8 we summarise the results of this thesis, and write about the possible research outcomes and perspectives.

- Appendix A includes material that may be useful to the reader, that for various reasons we believe would break the natural flow of the thesis. We include a short review of the $s\ell(2,\mathbb{C})$ algebra and the auxiliary spaces notation. We also include the formulation of the Dirac-Poisson brackets for the Non-Linear Schrödinger equation, and the proof of the $4d$ Chern-Simons origin of the Zakharov-Mikhailov action.

- Appendix B contains several of the proofs regarding Chapter 6 that are too long or not necessarily illuminating.

Chapter 2

Background

In this section we provide the necessary background. We start by explaining the properties of space-time duality of the classical r-matrix, adapting the content of the original paper [CK15]. The next section illustrates the construction and general properties of the variational bi-complex, as described by [D03]. Finally, we give a quick overview of the new topic of Lagrangian multiforms for field theories.

2.1 Space-time duality of the classical r-matrix

As mentioned in the introduction, in recent years new surprising properties of space-time duality of the classical r-matrix have been discovered, starting from the paper [CK15]. While studying the presence of integrable defects for the Non-Linear Schrödinger equation, the authors needed to provide a different formulation from the usual one given by the Poisson bracket $\{\ ,\ \}_S$ and the Hamiltonian H_S, but of the same partial differential equation. This was done defining a different Poisson bracket $\{\ ,\ \}_T$, and a different Hamiltonian density H_T, that exchanged the roles of time and space. It is worth noticing that this is not related to a bi-Hamiltonian formulation, as the two Poisson brackets are not compatible, not even living in the same phase space. The surprising property is that both Poisson brackets have the same r-matrix structure (up to an overall sign).

We will consider the (unreduced) Non-Linear Schrödinger(NLS) equation of the form

$$iq_t + \frac{1}{2}q_{xx} - q^2 r = 0\,, \qquad ir_t - \frac{1}{2}r_{xx} + qr^2 = 0 \tag{2.1}$$

for two complex fields q, r dependent on the coordinates (x, t). The actual NLS equation is obtained, in its focusing or defocusing form with the reduction $r = \pm q^*$. It is well known that this equation admits a Lagrangian formulation, and a Lagrangian is

$$L = \frac{i}{2}(q_t r - q r_t) - \frac{1}{2}q_x r_x - \frac{1}{2}q^2 r^2\,. \tag{2.2}$$

Indeed, defining the action $S[q,r]$ as being the integral of L over an appropriate 2-dimensional surface Γ

$$S[q,r] = \int_\Gamma L(q,r,q_x,r_x,q_t,r_t)dxdt\,,$$

we have that its variation reads

$$\begin{aligned}\delta S :=& \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S[q+\varepsilon\delta q, r+\varepsilon\delta r]\\ =& \int_\Gamma \left(\frac{\delta L}{\delta q}\delta q + \frac{\delta L}{\delta r}\delta r\right) dxdt\,.\end{aligned}$$

The quantities

$$\frac{\delta L}{\delta q} := \sum_{\alpha,\beta=0}^{m} (-1)^{\alpha+\beta}\partial_x^\alpha\partial_t^\beta \frac{\partial L}{\partial(\partial_x^\alpha\partial_t^\beta q)}\,, \qquad \frac{\delta L}{\delta r} := \sum_{\alpha,\beta=0}^{m} (-1)^{\alpha+\beta}\partial_x^\alpha\partial_t^\beta \frac{\partial L}{\partial(\partial_x^\alpha\partial_t^\beta r)}\,,$$

are the variational derivatives of L with respect to q and r (we assumed that L only depends on derivatives of the fields up to a finite order m, as it is the case for the NLS). We then have

$$\begin{aligned}\frac{\delta L}{\delta q} =& \frac{\partial L}{\partial q} - \frac{\partial}{\partial x}\frac{\partial L}{\partial q_x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial q_t}\\ =& -\frac{ir_t}{2} - qr^2 + \frac{r_{xx}}{2} - \frac{ir_t}{2}\\ =& -ir_t - qr^2 + \frac{r_{xx}}{2},\end{aligned}$$

so by setting $\frac{\delta L}{\delta q} = 0$ one obtains the second equation of (2.1). Similarly, we get the first equation of (2.1) by setting $\frac{\delta L}{\delta r} = 0$. The Non-Linear Schrödinger is also known to have a Lax pair formulation [ZS72]. Let us consider the following auxiliary problem:

$$\begin{cases}\Psi(x,t,\lambda)_x = U(x,t,\lambda)\Psi(x,t,\lambda)\\ \Psi(x,t,\lambda)_t = V(x,t,\lambda)\Psi(x,t,\lambda)\end{cases} \tag{2.3}$$

where

$$U(x,t,\lambda) = \begin{pmatrix}-i\lambda & q\\ r & i\lambda\end{pmatrix}\,, \quad V(x,t,\lambda) = \begin{pmatrix}-i\lambda^2 - \frac{i}{2}qr & \lambda q + \frac{i}{2}q_x\\ \lambda r - \frac{i}{2}r_x & i\lambda^2 + \frac{i}{2}qr\end{pmatrix} \tag{2.4}$$

is called the *Lax Pair*. Both U and V are 2×2 complex traceless matrices and therefore are $s\ell(2,\mathbb{C})$-valued fields. The compatibility condition $\Psi_{xt} = \Psi_{tx}$ is equivalent to the Non-Linear Schrödinger equation: first we notice that $\Psi_{xt} = (U\Psi)_t = U_t\Psi + U\psi_t = U_t\Psi + UV\Psi$ and similarly $\Psi_{tx} = V_x\Psi + VU\Psi$, so that $\Psi_{xt} = \Psi_{tx}$ is the famous *zero-curvature equation*

$$U_t - V_x + [U,V] = 0\,. \tag{2.5}$$

Remark 2.1: The name is quickly understandable. Let $W = Udx + Vdt$ be the *Lax connection*, then defining its curvature as $F(W) = dW - W \wedge W$ we easily get that $F(W) = 0$ if and only if (2.5) is satisfied:

$$\begin{aligned} d(Udx + Vdt) &= (-U_t + V_x)\, dx \wedge dt\,, \\ W \wedge W &= (UV - VU)\, dx \wedge dt\,, \end{aligned}$$

and so $F(W) = (-U_t + V_x - [U,V])dx \wedge dt$. One could choose another convention for the curvature and define it as $\widetilde{F}(W) = dW + W \wedge W$. The zero curvature then becomes $\widetilde{F}(W) = (-U_t + V_x + [U,V])dx \wedge dt = 0$. The first convention can be recovered by sending $W \to -W$.

Let us compute the curvature: we have

$$\begin{aligned} -U_t &= \begin{pmatrix} 0 & -q_t \\ -r_t & 0 \end{pmatrix}, \\ V_x &= \begin{pmatrix} -\frac{i}{2}(qr_x + q_x r) & \lambda q_x + \frac{i}{2} q_{xx} \\ \lambda r_x - \frac{i}{2} r_{xx} & \frac{i}{2}(qr_x + q_x r) \end{pmatrix}, \\ -[U,V] &= \begin{pmatrix} \lambda^3 - \frac{\lambda}{2} qr + \frac{i}{2} qr_x & -\frac{\lambda}{2} q_x - \frac{i}{2} q^2 r \\ -\frac{\lambda}{2} r_x + \frac{i}{2} qr^2 & \lambda^3 - \frac{\lambda}{2} qr - \frac{i}{2} q_x r \end{pmatrix} \\ &\quad + \begin{pmatrix} -\lambda^3 + \frac{\lambda}{2} qr + \frac{i}{2} q_x r & -\frac{\lambda}{2} q_x - \frac{i}{2} q^2 r \\ -\frac{\lambda}{2} r_x + \frac{i}{2} qr^2 & -\lambda^3 + \frac{\lambda}{2} qr - \frac{i}{2} qr_x \end{pmatrix} \\ &= \begin{pmatrix} \frac{i}{2} qr_x + \frac{i}{2} q_x r & -\lambda q_x - iq^2 r \\ -\lambda r_x + iqr^2 & -\frac{i}{2} q_x r - \frac{i}{2} qr_x \end{pmatrix}. \end{aligned}$$

The diagonal component of the equation reads $0 - \frac{i}{2} qr_x - \frac{i}{2} q_x r + \frac{i}{2} qr_x + \frac{i}{2} q_x r = 0$ identically. The other two components are $-q_t + \lambda q_x + \frac{i}{2} q_{xx} - \lambda q_x - iq^2 r = -q_t + \frac{i}{2} q_{xx} - iq^2 r$, and $-r_t + \lambda r_x - \frac{i}{2} r_{xx} - \lambda r_x + iqr^2 = -r_t - \frac{i}{2} r_{xx} + iqr^2$ that gives (2.1). Therefore, $F(W) = 0$ as a matrix identity is equivalent to the Non-Linear Schrödinger equation.

The usual Hamiltonian formulation is obtained as follows: we take as configuration space the space of smooth functions of x appropriate to our functional-analytic needs[1], and we let

$$p^1(x) = \frac{\partial L}{\partial q_t} = \frac{ir(x)}{2}\,, \quad p^2(x) = \frac{\partial L}{\partial r_t} = -\frac{iq(x)}{2}\,. \tag{2.6}$$

The experienced reader will have noticed that the Lagrangian L is linear in the velocities q_t and r_t, and the usual Legendre transformation is ineffective in obtaining the Hamiltonian formulation because of its lack of invertibility. This makes the system (2.6) a constraint, and therefore we may resort to the *Dirac-Poisson brackets* in order to be able to treat it correctly. We skip this calculation here, but it can be found for instance in [ACDK16,

[1]This amounts, amongst other properties, to the requirement that appropriate conditions at infinity are satisfied to discard the boundary terms after the integration by parts.

Section 3.1], as well as in Appendix A.3, adapted to our notations in this thesis. The result is the following *equal-time* Poisson brackets

$$\{q(x), r(y)\}_S = i\delta(x-y)\,, \tag{2.7a}$$

$$\{q(x), q(y)\}_S = 0\,, \tag{2.7b}$$

$$\{r(x), r(y)\}_S = 0\,, \tag{2.7c}$$

and the Hamiltonian $\mathcal{H}_S = \int H_S\, dx$, where

$$H_S = q_t p^1 + r_t p^2 - L = \frac{1}{2} q_x r_x + \frac{1}{2} q^2 r^2\,. \tag{2.8}$$

The NLS time flow (2.1) is then obtained by the Hamilton's equation (in infinite dimensions) $q_t = \{\mathcal{H}_S, q\}_S$, as shown in Appendix A.3.

Remark 2.2: This should not confuse the reader, as it is similar to what happens with an ODE. In its simplest case, we consider a vector field $X\colon U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^n$ is open. U is called the *phase space* of the ODE. A solution of the ODE defined by X is a curve from an interval $I \subset \mathbb{R} \to U$, $t \mapsto z(t)$ such that $\frac{dz}{dt}(t) = X(z(t))$ $\forall t \in I$. In the same way, for an infinite dimensional system we consider the phase space as being a suitable space of functions $\{\, f(x) \,\}$, on which we inject the time t by considering a curve $t \to f(x,t)$.

In order to see the classical r-matrix structure we need to use the so-called *auxiliary spaces*[2] as explained in Appendix A.2. We calculate the Sklyanin equal-time Poisson bracket between $U(x,\lambda)$ and $U(y,\mu)$. As seen in equation (2.4) $U(x,\lambda)$ is valued in $s\ell(2,\mathbb{C})$, so we use as a basis $\{\,\sigma_3, \sigma_+, \sigma_-\,\}$, to write $U(x,\lambda) = \sum_i u_i(x,\lambda)\,\sigma_i$. We have

$$\begin{aligned} &\{U_{\mathbf{1}}(x,\lambda), U_{\mathbf{2}}(y,\mu)\}_S \\ &\quad = \sum_{i,j=3,+,-} \{u_i(x,\lambda), u_j(y,\mu)\}_S\, \sigma_i \otimes \sigma_j \\ &\quad = \{q(x), r(y)\}_S\, \sigma_+ \otimes \sigma_- + \{r(x), q(y)\}_S\, \sigma_- \otimes \sigma_+ \\ &\quad = i\delta(x-y)(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)\,. \end{aligned} \tag{2.9}$$

It is easy but not straightforward to see that this Poisson bracket can be seen as a commutator of $U_{\mathbf{1}}(x,t,\lambda) + U_{\mathbf{2}}(y,t,\mu)$ with another quantity, called the *rational r-matrix*

$$r_{\mathbf{12}}(\lambda,\mu) = \frac{1}{2}\frac{P_{\mathbf{12}}}{\lambda - \mu}\,, \tag{2.10}$$

where $P_{\mathbf{12}} \coloneqq \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ + \frac{1}{2}(\sigma_3 \otimes \sigma_3 + \mathbb{I} \otimes \mathbb{I})$ is the permutation operator. The

[2] We remark that the indices relative to the auxiliary spaces are denoted in boldface as $\mathbf{1}, \mathbf{2}$.

following identity holds:

$$\{U_{\mathbf{1}}(x,\lambda), U_{\mathbf{2}}(y,\mu)\}_S = \delta(x-y)[r_{\mathbf{12}}(\lambda-\mu), U_{\mathbf{1}}(x,\lambda) + U_{\mathbf{2}}(y,\mu)]\,. \tag{2.11}$$

In fact we have

$$\begin{aligned}
&[r_{\mathbf{12}}(\lambda-\mu), U_{\mathbf{1}}(x,\lambda) + U_{\mathbf{2}}(y,\mu)] \\
&= \frac{1}{2}\frac{1}{\lambda-\mu}[\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+ + \frac{\sigma_3\otimes\sigma_3}{2}, \\
&\qquad (-i\lambda\sigma_3 + q(x)\sigma_+ + r(x)\sigma_-)\otimes\mathbb{I} + \mathbb{I}\otimes(-i\mu\sigma_3 + q(y)\sigma_+ + r(y)\sigma_-)] \\
&= \frac{1}{2}\frac{1}{\lambda-\mu}(-i\lambda[\sigma_+,\sigma_3]\otimes\sigma_- + r(x)[\sigma_+,\sigma_-]\otimes\sigma_- - i\mu\sigma_+\otimes[\sigma_-,\sigma_3] + q(y)\sigma_+\otimes[\sigma_-,\sigma_+] \\
&\quad - i\lambda[\sigma_-,\sigma_3]\otimes\sigma_+ + q(x)[\sigma_-,\sigma_+]\otimes\sigma_+ - i\mu\sigma_-\otimes[\sigma_+,\sigma_3] + r(y)\sigma_-\otimes[\sigma_+,\sigma_-] \\
&\quad + \frac{q(x)}{2}[\sigma_3,\sigma_+]\otimes\sigma_3 + \frac{r(x)}{2}[\sigma_3,\sigma_-]\otimes\sigma_3 + \frac{q(y)}{2}\sigma_3\otimes[\sigma_3,\sigma_+] + \frac{r(y)}{2}\sigma_3\otimes[\sigma_3,\sigma_-])
\end{aligned}$$

We now use the commutation relations of $\sigma_3, \sigma_\pm$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\lambda-\mu}(2i\lambda\sigma_+\otimes\sigma_- + r(x)\sigma_3\otimes\sigma_- - 2i\mu\sigma_+\otimes\sigma_- - q(y)\sigma_+\otimes\sigma_3 \\
&\qquad - 2i\lambda\sigma_-\otimes\sigma_+ - q(x)\sigma_3\otimes\sigma_+ + i\mu\sigma_-\otimes\sigma_+ + r(y)\sigma_-\otimes\sigma_3 + q(x)\sigma_+\otimes\sigma_3 \\
&\qquad - r(x)\sigma_-\otimes\sigma_3 + q(y)\sigma_3\otimes\sigma_+ - r(y)\sigma_3\otimes\sigma_-) \\
&= \frac{1}{2}\frac{1}{\lambda-\mu}(-2i(\mu-\lambda)\sigma_+\otimes\sigma_- + 2i(\mu-\lambda)\sigma_-\otimes\sigma_- + (q(x)-q(y))\sigma_+\otimes\sigma_3 \\
&\qquad + (q(y)-q(x))\sigma_3\otimes\sigma_+ + (r(y)-r(x))\sigma_-\otimes\sigma_3 + (r(x)-r(y))\sigma_3\otimes\sigma_-)\,.
\end{aligned}$$

When multiplied by $\delta(x-y)$ this becomes the desired $i(\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+)$. The equation

$$\{U_{\mathbf{1}}(x,\lambda), U_{\mathbf{2}}(y,\mu)\}_S = \delta(x-y)[r_{\mathbf{12}}(\lambda-\mu), U_{\mathbf{1}}(x,\lambda) + U_{\mathbf{2}}(y,\mu)] \tag{2.12}$$

was first derived by Sklyanin in [S82] and is the starting point of the (quantum) Inverse Scattering Method for solving the Non-Linear Schrödinger equation. In fact, if we introduce the monodromy matrix $M(x,\lambda)$ as the fundamental solution of (2.3) at $t=0$ that is equal to the identity matrix at $x=0$, we get for $x>0$

$$\{M_{\mathbf{1}}(x,\lambda), M_{\mathbf{2}}(x,\mu)\}_S = [r_{\mathbf{12}}(\lambda-\mu), M_{\mathbf{1}}(x,\lambda)M_{\mathbf{2}}(x,\mu)] \tag{2.13}$$

Under specific conditions, this relation is enough to prove Liouville integrability of the Non-Linear Schrödinger equation: roughly, the *transfer matrix*[3] $\operatorname{Tr} M(\lambda) = \sum_i I_i\lambda^{-i}$ commutes for different spectral parameters $\{\operatorname{Tr} M(\lambda), \operatorname{Tr} M(\mu)\}_S = 0$, which means that the coefficients I_i are in involution with each other $\{I_i, I_j\}_S = 0\ \forall i,j$.

Let us now explore the other picture, and exchange the roles of time and space. We choose

[3]The name 'matrix' is a terminology inherited from the quantum case. Here it is just a function.

as configuration space the appropriate space of functions $\{ f(t) \}$ and we look for the flow given by x. We consider the alternative choice of momenta obtained by performing the Legendre transformation with respect to the other independent variable

$$\pi^1(t) = \frac{\partial L}{\partial q_x} = -\frac{1}{2} r_x(t)\,, \quad \pi^2 = \frac{\partial L}{\partial r_x} = -\frac{1}{2} q_x(t)\,, \tag{2.14}$$

and consequently the Hamiltonian $\mathcal{H}_T = \int H_T \, dt$ where

$$H_T = q_x \pi^1 + r_x \pi^2 - L = -\frac{i}{2}(q_t r - q r_t) - \frac{1}{2} q_x r_x + \frac{1}{2} q^2 r^2\,. \tag{2.15}$$

From the expressions of $\pi^{1,2}$ one can canonically construct the *equal-space* Poisson brackets, where the only non-vanishing ones are the following

$$\{q(t), r_x(\tau)\}_T = 2\delta(t-\tau)\,, \tag{2.16a}$$

$$\{r(t), q_x(\tau)\}_T = 2\delta(t-\tau)\,. \tag{2.16b}$$

The Non-Linear Schrödinger equation can then be obtained as for instance[4] $(\pi^2)_x = \{\mathcal{H}_T, \pi^2\}_T$:

$$-\frac{1}{2} q_{xx} = \{\int (-\frac{i}{2}(q_t r - q r_t) - \frac{1}{2} q_x r_x + \frac{1}{2} q^2 r^2)\, d\tau, -\frac{1}{2} q_x\}_T$$

$$= 2iq_t - 2q^2 r \quad \Longrightarrow \quad iq_t + \frac{1}{2} q_{xx} - q^2 r = 0\,.$$

These are two equivalent formulations of the same equation $iq_t + \frac{1}{2} q_{xx} - q^2 r = 0$ that work on two different phase spaces. The first is the usual one and can be called 'equal time' picture. The second can be seen as the 'equal-space' picture.

Remarkably, one can obtain a similar equation to (2.12) for the equal-space bracket $\{\ ,\ \}_T$, with the caveat that one has to use the other Lax matrix V. In fact we have

$$\{V_{\mathbf{1}}(t,\lambda), V_{\mathbf{2}}(\tau,\mu)\}_T = -\delta(t-\tau)[r_{\mathbf{12}}(\lambda-\mu), V_{\mathbf{1}}(t,\lambda) + V_{\mathbf{2}}(\tau,\mu)] \tag{2.17}$$

for the same *rational r-matrix* $r_{\mathbf{12}}(\lambda) = \frac{P_{\mathbf{12}}}{2\lambda}$. This is computed with the same technique as (2.12), but it is more cumbersome because of the less simple expressions of $\{\ ,\ \}_T$ and V. We remark the presence of a minus sign in front of the commutator. In the rest of the thesis we will refer to this property of the classical r-matrix, *i.e.* its presence (up to a minus sign) in both Poisson brackets $\{\ ,\ \}_S$ and $\{\ ,\ \}_T$ as *space-time duality.*

This result of space-time duality of the classical r-matrix for the Non-Linear Schrödinger equation of [CK15] has originated a series of works that explored this direction.
In [C15b] the author proved the same property the sine-Gordon equation in laboratory

[4] Technically, one would also have to consider $q_x = \{\mathcal{H}_T, q\}_T$, which returns the Legendre transformation $\pi^2 = -\frac{1}{2} q_x$.

coordinates $u_{tt} - u_{xx} + \sin u = 0$. This equation is another prototype of integrable model, and is widely recognised as one of the most important examples of integrable relativistic field theory. In [ACDK16] the authors specialised this approach to the Ablowitz-Kaup-Newell-Segur hierarchy, to which the NLS equation belongs, and generalised it to the subsequent level, the modified Korteweg-de Vries equation.
The paper [AC17] gives an algebraic explanation of the space-time duality of the classical r-matrix for the AKNS hierarchy, in the sense that it comes from a Lie-Poisson bracket on a suitable coadjoint orbit of the loop algebra $s\ell(2,\mathbb{C}) \otimes \mathbb{C}(\lambda, \lambda^{-1})$. This is achieved following a series of steps. First the authors choose a time t_n in the hierarchy and restrict the dynamical variables in $Q(\lambda)$ (denoted by L there) to satisfy the n-th time evolution $\partial_n Q(\lambda) = [Q^{(n)}(\lambda), Q(\lambda)]$ where $Q^{(n)}(\lambda) = P_+(\lambda^n Q(\lambda))$ and P_+ is the projector onto the positive loop algebra. With respect to the notations of this chapter, we have

$$Q^{(1)}(\lambda) = U(\lambda)\,, \qquad Q^{(2)}(\lambda) = V(\lambda)\,. \tag{2.18}$$

In this way the Lax matrix $Q^{(n)}(\lambda)$ acquires a natural r-matrix structure with respect to the Poisson bracket $\{\ ,\ \}_n$. Then, they construct an auxiliary problem involving the time t^n and a new time t^k, $k \neq n$, associated to the Lax matrix $Q^{(k)}(\lambda)$. The zero-curvature equation is shown to be Hamiltonian with respect to the Poisson bracket $\{\ ,\ \}_n$. Finally, they swap the roles of n and k, and prove that the zero-curvature equations obtained from the two different choices produce the same set of PDEs and are Hamiltonian with respect to the corresponding Poisson brackets $\{\ ,\ \}_n$ and $\{\ ,\ \}_k$.
Despite investigating the consequences of different choices of 'time' and 'space', these papers still have to make this choice. As introduced in Chapter 1, this thesis succeeds in avoiding this distinction altogether and works in a truly *covariant* fashion, where all the times of the hierarchy are treated with equal footing. This will be done using the framework of the *variational bi-complex*, which will be introduced in the next section.

2.2 Variational bi-complex

Let M be the base manifold with local coordinates x^i, $i = 0, 1, 2, \dots, n$ in a fibered manifold $\pi : E \to M$ whose sections represent the fields of the theory. M will be called *multi-time manifold*. The dimension of M will be $n = 2$ in Chapters 3 and 4, with $x^1 = x$ and $x^2 = t$ hence taking the name *space-time manifold*. The coordinates x^i will be called *horizontal*. The variational bi-complex is a double complex of differential forms defined on the infinite jet bundle of $\pi : E \to M$. One introduces vertical and horizontal differentials δ and d which satisfy

$$d^2 = 0 = \delta^2\,, \qquad d\delta = -\delta d\,, \tag{2.19}$$

so that the operator $d + \delta$ satisfies $(d+\delta)^2 = 0$. Let $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$. Consider the differential algebra with the commuting derivations ∂_i, $i = 0, 1, 2, \dots, n$ generated by the commuting variables $u_k^{(i)}$, $k = 1, \dots, N$, $(i) = (i_0, i_1, i_2, \dots, i_n)$ being a multi-index, and quotiented by the relations

$$\partial_j u_k^{(i)} = u_k^{(i)+e_j}\,, \tag{2.20}$$

where $e_j = (0, \dots, 0, 1, 0, \dots)$ only has 1 in position j. We simply denote $u_k^{(0,0,\dots)}$ by u_k, the fields of the theory which would be the local fibre coordinates mentioned above. We denote this differential algebra by $\mathscr{A}$. The elements of $\mathscr{A}$ will be called *vertical coordinates*, and represent the fields of our theory and their derivatives with respect to the multi-time variables. We will need the notation

$$\partial^{(i)} = \partial_0^{i_0}\partial_1^{i_1}\dots\partial_n^{i_n}.$$

We consider the spaces $\mathscr{A}^{(p,q)}$, $p, q \geq 0$ of *finite* sums of the following form

$$\omega = \sum_{(i),(k),(j)} f^{(i)}_{(k),(j)}\delta u_{k_1}^{(i_1)} \wedge \dots \wedge \delta u_{k_p}^{(i_p)} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q}, \qquad f^{(i)}_{(k),(j)} \in \mathscr{A} \tag{2.21}$$

which are called (p, q)-forms. In other words, $\mathscr{A}^{(p,q)}$ is the space linearly generated by the basis elements $\delta u_{k_1}^{(i_1)} \wedge \dots \wedge \delta u_{k_p}^{(i_p)} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q}$ over $\mathscr{A}$, where $\wedge$ denotes the usual exterior product. For these reasons p will be referred to as the *vertical degree* and q as the *horizontal degree* of ω. We define the operations $d : \mathscr{A}^{(p,q)} \to \mathscr{A}^{(p,q+1)}$ and $\delta : \mathscr{A}^{(p,q)} \to \mathscr{A}^{(p+1,q)}$ as follows. They are graded derivations

$$d(\omega_1^{(p_1,q_1)} \wedge \omega_2^{(p_2,q_2)}) = d\omega_1^{(p_1,q_1)} \wedge \omega_2^{(p_2,q_2)} + (-1)^{p_1+q_1}\omega_1^{(p_1,q_1)} \wedge d\omega_2^{(p_2,q_2)}, \tag{2.22a}$$

$$\delta(\omega_1^{(p_1,q_1)} \wedge \omega_2^{(p_2,q_2)}) = \delta\omega_1^{(p_1,q_1)} \wedge \omega_2^{(p_2,q_2)} + (-1)^{p_1+q_1}\omega_1^{(p_1,q_1)} \wedge \delta\omega_2^{(p_2,q_2)}\,, \tag{2.22b}$$

and on the generators, they satisfy

$$df = \sum \partial_i f\, dx^i = \sum (\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial u_k^{(j)}}\, u_k^{(j)+e_i})dx^i\,, \quad f \in \mathscr{A}\,, \tag{2.23a}$$

$$\delta f = \sum \frac{\partial f}{\partial u_k^{(i)}}\,\delta u_k^{(i)}\,, \quad f \in \mathscr{A}\,, \tag{2.23b}$$

$$\delta(dx^i) = \delta(\delta u_k^{(j)}) = d(dx^i) = 0, \tag{2.23c}$$

$$d(\delta u_k^{(i)}) = -\delta d u_k^{(i)} = -\sum \delta u_k^{(i)+e_j} \wedge dx^j. \tag{2.23d}$$

This determines the action of d and δ on any form as in (2.21). As a consequence, one can show that $d^2 = \delta^2 = 0$ and $d\delta = -\delta d$. For our purpose, it is sufficient to take the following (simplified) definition for the variational bi-complex: it is the space $\mathscr{A}^* = \bigoplus_{p,q} \mathscr{A}^{(p,q)}$ equipped with the two derivation d and δ. Due to the geometrical interpretation of these derivations, d is called *horizontal differential* while δ is called *vertical differential.*
Note that the direct sum over q is finite and runs from 0 (scalars) to n (volume horizontal forms) whereas the sum over p runs from 0 to infinity. Of course, each form in $\mathscr{A}^*$ only contains a finite sum of elements of the form (2.21) for certain values of p and q. The bi-complex $\mathscr{A}^*$ generates an associated complex $\mathscr{A}^{(r)} = \bigoplus_{p+q=r} \mathscr{A}^{(p,q)}$ and derivation

$d+\delta$. It is proved that both the horizontal sequence and the vertical sequence are exact, see e.g. [D03].

Dual to the notion of forms is the notion of vector fields. We consider the dual space of vector fields $T\mathscr{A}$ to the space of one-forms $\mathscr{A}^{(1)}$ with elements of the form

$$\xi = \sum_{k,(i)} \xi_{k,(i)}\, \partial_{u_k^{(i)}} + \sum_i \xi_i^* \,\partial_i\,. \tag{2.24}$$

In the rest of the thesis we will use ∂_i and $\frac{\partial}{\partial x^i}$ interchangeably, and the same with $\partial_{u_k^{(i)}}$ and $\frac{\partial}{\partial u_k^{(i)}}$. The interior product with a form is obtained in the usual graded way together with the rule

$$\partial_i \lrcorner dx^j = \delta_{ij}\,,\quad \partial_{u_k^{(i)}} \lrcorner \delta u_\ell^{(j)} = \delta_{k\ell}\delta_{(i)(j)}\,.$$

where $\delta_{(i)(j)} = \prod_k \delta_{i_k j_k}$. For instance, with $i \neq j$ and $(i) \neq (j)$ or $k \neq l$,

$$\begin{aligned}
&\partial_i \lrcorner (\delta u_k^{(\ell)} \wedge dx^i \wedge dx^j) = -\delta u_k^{(\ell)} \wedge dx^j\,,\\
&\partial_{u_k^{(i)}} \lrcorner (\delta u_\ell^{(j)} \wedge \delta u_k^{(i)} \wedge dx^m) = -\delta u_\ell^{(j)} \wedge dx^m\,.
\end{aligned}$$

We will need the following vertical vector fields

$$\widetilde{\partial}_i = \sum_{k,(j)} u_k^{(j)+e_i} \frac{\partial}{\partial u_k^{(j)}}. \tag{2.25}$$

If $f \in \mathscr{A}$ does not depend explicitly on variables x^i then $\partial_i f = \widetilde{\partial}_i f$. Let us also introduce the notation ∂_i' by $\partial_i = \partial_i' + \widetilde{\partial}_i$, which has the following interpretation:

- ∂_i is the total derivative with respect to the multi-time variable x^i;
- ∂_i' is the partial derivative with respect to x^i, and if $f \in \mathscr{A}$ does not depend explicitly on the space-time variables then $\partial_i' f = 0$;
- $\widetilde{\partial}_i$ is the derivative with respect to x^i only through the fields u_k, and if $f \in \mathscr{A}$ does not depend explicitly on the space-time variables then $\partial_i f = \widetilde{\partial}_i f$

In addition to the vector fields (2.24), in general calculations in the variational bi-complex also require the use of multivector fields of the form $\xi_1 \wedge \cdots \wedge \xi_r$ where each ξ_i is of the form (2.24). In this thesis, we will only need those multivector fields that are linear combination of $\partial_{u_k^{(i)}} \wedge \partial_j$ with coefficients in $\mathscr{A}$ and we may simply call them vector fields as the context should not lead to any confusion. The following example shows the rule for the interior product of such a multivector field, with $(i) \neq (j)$ or $k \neq \ell$,

$$(\partial_{u_k^{(i)}} \wedge \partial_\ell) \lrcorner (\delta u_\ell^{(j)} \wedge \delta u_k^{(i)} \wedge dx^m) = \partial_{u_k^{(i)}} \lrcorner (\partial_\ell \lrcorner (\delta u_\ell^{(j)} \wedge \delta u_k^{(i)} \wedge dx^m)) = -\delta_{\ell m}\, \delta u_\ell^{(j)}\,.$$

Finally, we will need the following useful identity, cf [D03, Corollary 19.2.11].

$$\widetilde{\partial}_i = \delta \widetilde{\partial}_i \lrcorner + \widetilde{\partial}_i \lrcorner \delta \,. \tag{2.26}$$

2.3 Lagrangian multiforms

The notion of *Lagrangian multiforms* was introduced in 2009 by Lobb and Nijhoff [LN09], motivated by the completely open problem of characterising integrability of (partial) differential (or difference) equations purely from a variational/Lagrangian point of view. Initially developed in the realm of fully discrete integrable systems, Lagrangian multiforms provide a framework whereby the notion of multidimensional consistency [N02, BS02], which captures the analog of the commutativity of Hamiltonian flows known in continuous integrable systems, is encapsulated in a generalised variational principle. The latter contains the standard Euler-Lagrange equations for the various equations forming an integrable hierarchy as well as additional equations, originally called *corner equations* which can be interpreted as determining the allowed integrable Lagrangians themselves. The set of all these equations is now called *multiform Euler-Lagrange equations*. The original work of Lobb and Nijhoff [LN09] stimulated a wealth of subsequent developments, first in the discrete realm, see e.g. [LNQ09, LN10, BS10, YLN11, BPS14, BPS15], then progressively into the continuous realm for finite dimensional systems, see e.g. [S13, PS17] and $1+1$-dimensional field theories, see e.g. [XNL11], up to more recent developments in continuous field theory, see e.g. [S16, SV16, V19, SNC19a, PV20], including the first examples in $2+1$-dimensions [SNC19b, SNC21].

Since in this thesis we will only deal with continuous $1+1$-dimensional field theories, we are going to focus on this case. Assume we have a hierarchy of integrable PDEs, such as the Ablowitz-Kaup-Newell-Segur hierarchy to which the previously introduced Non-Linear Schrödinger equation belongs. Suppose we identify $x = x^1$ and call each individual 'time' relative to the n-th flow x^n. For instance, the Non-Linear Schrödinger equation will be relative to the times x^1, x^2. The next equation in the hierarchy (the modified Korteweg-de Vries equation) will be relative the times x^1, x^3 and so on. A generic equation in the hierarchy will be relative to the times x^1, x^n. Suppose each equation has a Lagrangian formulation. The Lagrangian will be a scalar function or a 2-form $\mathscr{L}_{1n}$

$$\mathscr{L}_{1n} = L_{1n}\, dx^1 \wedge dx^n \,, \qquad L_{1n} \in \mathscr{A} \tag{2.27}$$

where L_{1n} depends on the field[5] u and its derivatives, and the action is the result of its integration $\int_{\Gamma_{1n}} \mathscr{L}_{1n}$ over a plane tangent to the x^1 and x^n directions. A solution u of the PDE is required to be a critical value of $\int_{\Gamma_{1n}} \mathscr{L}_{1n}$. In the new multiform approach, we encapsulate all these Lagrangian forms into a *Lagrangian multiform*: a horizontal 2-form

[5] We only consider a scalar field u for simplicity, but this is easily extended to multi-components fields too.

in the larger multi-time manifold with coordinates $(x^1, x^2, x^3, \dots)$

$$\mathscr{L} = \sum_{i<j} L_{ij}\, dx^i \wedge dx^j\,, \qquad L_{ij} \in \mathscr{A}\,. \tag{2.28}$$

The action will be $\mathscr{L}$ integrated over a 2-dimensional surface Γ (which now is not necessarily a plane tangent to any direction), $\int_\Gamma \mathscr{L}$. We remark that the action is a functional of both the fields u and the surface of integration Γ. Of course each $\mathscr{L}_{1n}$ is recovered by specifying $\Gamma = \Gamma_{1n}$, but we remark that not only do we have the coefficients L_{1n} but we also have to introduce L_{ij} for any pair (i, j). We now require that the action is not only stationary with respect to the fields u but that it holds the same critical value for every choice of the surface of integration Γ. This translates in the multiform Euler-Lagrange equations that can be written as (see [SV16])

$$\delta d\mathscr{L} = 0\,, \tag{2.29}$$

where d is the *horizontal* differential and δ is the *vertical* differential that are introduced in Section 2.2. Moreover, the requirement of stationarity with respect to each choice of Γ translates in the closure relation, *i.e.* $d\mathscr{L} = 0$ on shell of the equations $\delta d\mathscr{L} = 0$. We are therefore giving the following definition.

Definition 2.3 *The horizontal* 2*-form*

$$\mathscr{L} = \sum_{i<j} L_{ij}\, dx^{ij} \qquad dx^{ij} = dx^i \wedge dx^j$$

is a Lagrangian multiform *if* $\delta d\mathscr{L} = 0$ *implies* $d\mathscr{L} = 0$.

> **Remark 2.4:** The reader will also find other terminology in literature, which boils down to different interpretations of the closure relation. Usually, when this is considered to be a fundamental property of the variational theory of integrable hierarchies, it is included in the definition (as we do) and the name *Lagrangian multiform* is used. When weaker conditions are assumed, such as $d\mathscr{L} = \text{const}$, the term *pluri-Lagrangian form* is used (e.g. in [BS15, S16, PS17, V19]).

> **Remark 2.5:** Constructing the 'mixed' coefficients L_{ij} is possible although often cumbersome, especially for high values of i and j, and several techniques have been introduced. The paper [SNC19a] writes the coefficient L_{23} for the Ablowitz-Kaup-Newell-Segur hierarchy from the Lagrangians L_{12} and L_{13}, such that they can be taken as coefficients of a well defined Lagrangian multiform $\mathscr{L}_{(123)} = L_{12}\, dx^{12} + L_{23}\, dx^{23} + L_{13}\, dx^{13}$. This coefficient was constructed directly, by forcing the closure relation on the Lagrangian multiform. Other techniques were later introduced using variational symmetries: the papers [SNC19b] and [PV20] (despite different implementations)

use variational symmetries of a Lagrangian L_{12} to compute the coefficients of a new Lagrangian multiform L_{13} and L_{23}. This process could in principle be iterated, if other variational symmetries are known, to construct a $\mathscr{L}_{(1234)}$, a $\mathscr{L}_{(12345)}$, etc.

Given a Lagrangian multiform $\mathscr{L} = \sum_{i<j} L_{ij}\, dx^{ij}$ it is proved [SV16, SNC19a] that the multiform Euler-Lagrange equations $\delta d\mathscr{L} = 0$ are equivalent to the following set of equations

$$\frac{\delta_{(ij)} L_{ij}}{\delta u_{(I)-e_k}} + \frac{\delta_{(jk)} L_{jk}}{\delta u_{(I)-e_i}} + \frac{\delta_{(ki)} L_{ki}}{\delta u_{(I)-e_j}} = 0\,, \quad \forall I, \quad \forall i,j,k. \tag{2.30}$$

Here $(I) = (I_1, I_2, I_3, \dots)$ is a multi-index, and by $(I) - e_k$ we denote the multi-index

$$(I_1, \dots, I_k, \dots) - (0, \dots 0, \underbrace{1}_{k}, 0 \dots) = (I_1, \dots, I_k - 1, \dots)\,.$$

$\delta_{(ij)}/\delta$ is the variational derivative

$$\frac{\delta_{(ij)} F[u]}{\delta u_{(I)}} \coloneqq \sum_{\alpha,\beta\geq 0} (-1)^{\alpha+\beta} \partial_i^\alpha \partial_j^\beta \frac{\partial F[u]}{\partial u_{(I)+\alpha e_i + \beta e_j}}\,. \tag{2.31}$$

Whenever a component of the multi-index (I) is negative, the convention is that $\frac{\delta_{(\ell k)} L_{ij}}{\delta u_{(I)}} = 0$. The subscripts (ℓk), which are not present in the usual formalism, are needed since we must specify in which plane (in the case $\frac{\delta_{(\ell k)}}{\delta}$ it is the x^ℓ, x^k plane) we are taking the variational derivative.
For a Lagrangian multiform $\mathscr{L} = L_{12}\, dx^{12} + L_{23}\, dx^{23} + L_{13}\, dx^{13}$ that is dependent on only one field u and its derivatives up to the second order, they are the following[6]:

- **The usual Euler-Lagrange equations for each L_{ij}:**

$$(I) = (1) \implies \qquad \frac{\delta_{(23)} L_{23}}{\delta u} = 0 \tag{2.32a}$$

$$(I) = (2) \implies \qquad \frac{\delta_{(13)} L_{13}}{\delta u} = 0 \tag{2.32b}$$

$$(I) = (3) \implies \qquad \frac{\delta_{(12)} L_{12}}{\delta u} = 0 \tag{2.32c}$$

Equations (2.32b) and (2.32c) are the usual Euler-Lagrange equation one would get from the variational principles of respectively $\mathscr{L}_{13}$ and $\mathscr{L}_{12}$. Equation (2.32a) is the Euler-Lagrange equation of $\mathscr{L}_{23}$, that has no counterpart in the usual formalism. It is often the case that this equation is a differential consequence of (2.32b)-(2.32c), especially if the Lagrangian L_{23} was constructed a posteriori from the expressions of L_{12} and L_{13}.

[6]We write here, and in the rest of the thesis when it is convenient to do so, $\frac{\partial u}{\partial x^k} \equiv u_k$, $\frac{\partial^2 u}{\partial x^j \partial x^k} \equiv u_{jk}$, etc.

- **The corner equations:**

$$(I)=(11) \implies \frac{\delta_{(23)}L_{23}}{\delta u_1}=0 \tag{2.33a}$$

$$(I)=(12) \implies \frac{\delta_{(23)}L_{23}}{\delta u_2}-\frac{\delta_{(13)}L_{13}}{\delta u_1}=0 \tag{2.33b}$$

$$(I)=(13) \implies \frac{\delta_{(12)}L_{12}}{\delta u_1}+\frac{\delta_{(23)}L_{23}}{\delta u_3}=0 \tag{2.33c}$$

$$(I)=(22) \implies \frac{\delta_{(13)}L_{13}}{\delta u_2}=0 \tag{2.33d}$$

$$(I)=(23) \implies \frac{\delta_{(12)}L_{12}}{\delta u_2}-\frac{\delta_{(13)}L_{13}}{\delta u_3}=0 \tag{2.33e}$$

$$(I)=(33) \implies \frac{\delta_{(12)}L_{12}}{\delta u_3}=0 \tag{2.33f}$$

$$(I)=(111) \implies \frac{\delta_{(23)}L_{23}}{\delta u_{11}}=0 \tag{2.34a}$$

$$(I)=(112) \implies \frac{\delta_{(23)}L_{23}}{\delta u_{12}}-\frac{\delta_{(13)}L_{13}}{\delta u_{11}}=0 \tag{2.34b}$$

$$(I)=(113) \implies \frac{\delta_{(12)}L_{12}}{\delta u_{11}}+\frac{\delta_{(23)}L_{23}}{\delta u_{13}}=0 \tag{2.34c}$$

$$(I)=(122) \implies \frac{\delta_{(23)}L_{23}}{\delta u_{22}}-\frac{\delta_{(13)}L_{13}}{\delta u_{12}}=0 \tag{2.34d}$$

$$(I)=(123) \implies \frac{\delta_{(12)}L_{12}}{\delta u_{12}}+\frac{\delta_{(23)}L_{23}}{\delta u_{23}}-\frac{\delta_{(13)}L_{13}}{\delta u_{13}}=0 \tag{2.34e}$$

$$(I)=(133) \implies \frac{\delta_{(12)}L_{12}}{\delta u_{13}}+\frac{\delta_{(23)}L_{23}}{\delta u_{33}}=0 \tag{2.34f}$$

$$(I)=(222) \implies \frac{\delta_{(13)}L_{13}}{\delta u_{22}}=0 \tag{2.34g}$$

$$(I)=(223) \implies \frac{\delta_{(12)}L_{12}}{\delta u_{22}}-\frac{\delta_{(13)}L_{13}}{\delta u_{23}}=0 \tag{2.34h}$$

$$(I)=(233) \implies \frac{\delta_{(12)}L_{12}}{\delta u_{23}}-\frac{\delta_{(13)}L_{13}}{\delta u_{33}}=0 \tag{2.34i}$$

$$(I)=(333) \implies \frac{\delta_{(12)}L_{12}}{\delta u_{33}}=0 \tag{2.34j}$$

The equations coming from the cases $(I)=(ii)$ and $(I)=(iii)$, $i=1,2,3$ are due to the presence of the so-called *alien derivatives*: if the Lagrangian L_{ij} depends on derivatives of the field u with respect to a time-variable that is 'normal' to the plane x^i, x^j, say u_k, then this would be treated, as far as $\delta_{(ij)}/\delta$ is concerned, as a field variable on its own right (and not a derivative of u). In the usual formalism this is not present, as it would not make sense to introduce derivatives with respect to a variable that is not among the independent variables in consideration. It is of course possible (and often the case) that these are present in a Lagrangian multiform, as we are dealing with all the independent variables at once.

The other equations are called *corner equations*, because of their origin in the fully discrete context where they were formulated on the corner of a cube. These equations are often used as a restriction on the coefficients L_{ij} of a Lagrangian multiform, which is a technique that will not be explained further in this thesis, as we will work with Lagrangian multiforms that are 'ready to use'.

Let us illustrate the notion of Lagrangian multiforms and multiform Euler-Lagrange equations on some examples.

(potential) Korteweg-de Vries hierarchy A Lagrangian multiform that describes the first two levels of the potential Korteweg-de Vries hierarchy is $\mathscr{L} = L_{12}\, dx^{12} + L_{23}\, dx^{23} + L_{13}\, dx^{13}$, where

$$L_{12} = v_1 v_2 \,, \tag{2.35a}$$

$$L_{23} = -3v_1^2 v_2 - v_1 v_{112} + v_{11} v_{12} - v_{111} v_2 \,, \tag{2.35b}$$

$$L_{13} = -2v_1^3 - v_1 v_{111} + v_1 v_3 \,. \tag{2.35c}$$

The multiform Euler-Lagrange equations become the following (we do not report the ones that are trivially satisfied):

$$\begin{aligned}
&\frac{\delta_{(23)} L_{23}}{\delta v} = 6v_1 v_{12} + v_{1112} = 0 \\
&\frac{\delta_{(13)} L_{13}}{\delta v} = -2v_{13} + 12 v_1 v_{11} + 2v_{1111} = 0 \\
&\frac{\delta_{(12)} L_{12}}{\delta v} = -2v_{12} = 0 \\
&\frac{\delta_{(23)} L_{23}}{\delta v_1} = -6v_1 v_2 - 2v_{112} = 0 \\
&\frac{\delta_{(23)} L_{23}}{\delta v_2} - \frac{\delta_{(13)} L_{13}}{\delta v_1} = 3v_1^2 + v_{111} - v_3 = 0 \\
&\frac{\delta_{(12)} L_{12}}{\delta v_1} + \frac{\delta_{(23)} L_{23}}{\delta v_3} = v_2 = 0 \,.
\end{aligned}$$

As the Lagrangian multiform contain also derivatives of the third order, we additionally have to consider $(I) = (1111)$ that brings the identity $\frac{\delta L_{23}}{\delta v_{111}} = v_2 = 0$ and $(I) = (1123)$ that brings $\frac{\delta L_{23}}{\delta v_{112}} - \frac{\delta L_{13}}{\delta v_{111}} = -v_1 + v_1 = 0$. The multiform Euler-Lagrange equations for $\mathscr{L}$ are then summarised as

$$v_2 = 0 \,, \qquad v_3 = v_{111} + 3v_1^2 \tag{2.36}$$

since some of the other multiform Euler-Lagrange equations are seen to be differential consequences of these. The Korteweg-de Vries equation is recovered by taking the differential consequence of the second equation $v_{13} = v_{1111} + 6v_1 v_{11}$ and by taking $u = v_1$

in order to get

$$u_3 = u_{111} + 6uu_1\,. \tag{2.37}$$

sine-Gordon hierarchy A Lagrangian multiform $\mathscr{L} = L_{12}\,dx^{12} + L_{23}\,dx^{23} + L_{13}\,dx^{13}$ for the sine-Gordon hierarchy in light-cone coordinates is

$$L_{12} = \frac{1}{2}u_1u_2 + \cos u, \tag{2.38a}$$

$$L_{13} = \frac{1}{2}u_1u_3 + \frac{1}{2}u_{11}^2 - \frac{1}{8}u_1^4, \tag{2.38b}$$

$$L_{23} = -\frac{1}{2}u_2u_3 + u_{11}u_{12} + u_{11}\sin u - \frac{1}{2}u_1^2\cos u, \tag{2.38c}$$

and produces the equations:

$$\begin{aligned}
\frac{\delta_{(23)}L_{23}}{\delta u} &= u_{23} + u_{11}\cos u + \frac{u_1^2}{2}\sin u = 0\\
\frac{\delta_{(13)}L_{13}}{\delta u} &= -u_{13} + u_{1111} + \frac{3}{2}u_1^2u_{11} = 0\\
\frac{\delta_{(12)}L_{12}}{\delta u} &= -u_{12} - \sin u = 0\\
\frac{\delta_{(23)}L_{23}}{\delta u_1} &= -u_1\cos u - u_{112} = 0\\
\frac{\delta_{(23)}L_{23}}{\delta u_2} - \frac{\delta_{(13)}L_{13}}{\delta u_1} &= -u_3 + \frac{u_1^3}{2} + u_{111} = 0\\
\frac{\delta_{(23)}L_{23}}{\delta u_{11}} &= u_{12} + \sin u = 0\,.
\end{aligned}$$

The multiform Euler-Lagrange equations for $\mathscr{L}$ are then summarised as

$$u_{12} + \sin u = 0\,, \qquad u_3 = u_{111} + \frac{1}{2}u_1^3\,. \tag{2.39}$$

Ablowitz-Kaup-Newell-Segur hierarchy We start from the Lagrangian multiform found in [SNC19b]

$$\mathscr{L} = L_{12}\,dx^{12} + L_{13}\,dx^{13} + L_{23}\,dx^{23}\,, \tag{2.40}$$

where

$$L_{12} = \frac{i}{2}(rq_2 - qr_2) - \frac{1}{2}q_1r_1 - \frac{1}{2}q^2r^2, \tag{2.41a}$$

$$L_{13} = \frac{i}{2}(rq_3 - qr_3) - \frac{i}{8}(r_1q_{11} - q_1r_{11}) - \frac{3iqr}{8}(rq_1 - qr_1), \tag{2.41b}$$

$$\begin{aligned}
L_{23} =& \frac{i}{4}(q_2r_{11} - r_2q_{11}) + \frac{1}{2}(q_3r_1 + r_3q_1) + \frac{i}{8}(q_1r_{12} - r_1q_{12})\\
&+ \frac{3iqr}{8}(qr_2 - rq_2) + \frac{1}{8}q_{11}r_{11} - \frac{1}{4}qr(qr_{11} + rq_{11})\\
&+ \frac{1}{8}(qr_1 - rq_1)^2 + \frac{1}{2}q^3r^3\,.
\end{aligned} \tag{2.41c}$$

As proved in [SNC19a], the corresponding multiform Euler-Lagrange equations $\delta dL = 0$ are the familiar first two levels of the AKNS hierarchy

$$iq_2 + \frac{1}{2}q_{11} - q^2 r = 0, \qquad ir_2 - \frac{1}{2}r_{11} + qr^2 = 0\,, \tag{2.42a}$$

$$q_3 + \frac{1}{4}q_{111} - \frac{3}{2}qrq_1 = 0, \qquad r_3 + \frac{1}{4}r_{111} - \frac{3}{2}qrr_1 = 0\,. \tag{2.42b}$$

We have the equations

$$\begin{aligned}
\frac{\delta_{(12)}L_{12}}{\delta q} &= -qr^2 + \frac{1}{2}r_{11} - ir_2 = 0\,,\\
\frac{\delta_{(12)}L_{12}}{\delta r} &= -q^2 r + \frac{1}{2}q_{11} + iq_2 = 0\,,\\
\frac{\delta_{(13)}L_{13}}{\delta q} &= -ir_3 + \frac{3i}{2}qrr_1 - \frac{i}{4}r_{111}\,,\\
\frac{\delta_{(13)}L_{13}}{\delta r} &= iq_3 - \frac{3i}{2}qrq_1 + \frac{i}{4}q_{111}\,,
\end{aligned}$$

which are the Non-Linear Schrödinger and the modified Korteweg-de Vries equations, and

$$\begin{aligned}
\frac{\delta_{(23)}L_{23}}{\delta q} &= -\frac{1}{2}r_{13} + \frac{ir_{112}}{4} + \frac{3i}{2}qrr_2 - \frac{1}{2}qrr_{11} - \frac{1}{4}q_{11}r^2 + \frac{1}{4}qr_1^2 - \frac{1}{4}q_1rr_1 + \frac{3}{2}q^2r^3\,,\\
\frac{\delta_{(23)}L_{23}}{\delta r} &= -\frac{1}{2}q_{13} + \frac{iq_{112}}{4} - \frac{3i}{2}qrq_2 - \frac{1}{2}qrq_{11} - \frac{1}{4}q^2r_{11} + \frac{1}{4}q_1^2r - \frac{1}{4}qq_1r_1 + \frac{3}{2}q^3r^2\,,
\end{aligned}$$

that are differential consequences of the NLS and the mKdV equations. The corner equations

$$\begin{aligned}
\frac{\delta_{(12)}L_{12}}{\delta q_1} + \frac{\delta_{(23)}L_{23}}{\delta q_3} &= -\frac{1}{2}r_1 + \frac{1}{2}r_1 = 0\,,\\
\frac{\delta_{(12)}L_{12}}{\delta r_1} + \frac{\delta_{(23)}L_{23}}{\delta r_3} &= \frac{1}{2}q_1 - \frac{1}{2}q_1 = 0\,,\\
\frac{\delta_{(12)}L_{12}}{\delta q_2} - \frac{\delta_{(13)}L_{13}}{\delta q_3} &= \frac{ir}{2} - \frac{ir}{2} = 0\,,\\
\frac{\delta_{(12)}L_{12}}{\delta r_2} - \frac{\delta_{(13)}L_{13}}{\delta r_3} &= -\frac{iq}{2} + \frac{iq}{2} = 0\,,\\
\frac{\delta_{(13)}L_{12}}{\delta q_1} - \frac{\delta_{(23)}L_{23}}{\delta q_2} &= \frac{ir_{11}}{8} - \frac{3i}{8}qr^2 + \frac{ir_{11}}{8} - \left(\frac{ir_{11}}{4} - \frac{3i}{8}qr^2\right) = 0\,,\\
\frac{\delta_{(13)}L_{12}}{\delta r_1} - \frac{\delta_{(23)}L_{23}}{\delta r_2} &= -\frac{iq_{11}}{8} + \frac{3i}{8}q^2r - \frac{iq_{11}}{8} - \left(-\frac{iq_{11}}{4} + \frac{3i}{8}q^2r\right) = 0\,.
\end{aligned}$$

are all identically satisfied. In [SNC19b] this was extended to the first three levels of the hierarchy as

$$\mathscr{L} = L_{12}\,dx^{12} + L_{13}\,dx^{13} + L_{14}\,dx^{14} + L_{23}\,dx^{23} + L_{24}\,dx^{24} + L_{34}\,dx^{34}\,,$$

adding the coefficients

$$
\begin{aligned}
L_{14} =& \frac{i}{2}(rq_4 - qr_4) + \frac{5}{16}qr(qr_{11} + rq_{11}) \\
&+ \frac{3}{16}(q^2 r_1^2 + q_1^2 r^2) + \frac{1}{4}qrq_1r_1 - \frac{1}{8}q_{11}r_{11} - \frac{1}{4}q^3r^3\,,
\end{aligned} \tag{2.43a}
$$

$$
\begin{aligned}
L_{24} =& \frac{3i}{8}q^2r^2(rq_1 - qr_1) + \frac{1}{16}(q^2r_1r_2 + r^2q_1q_2) + \frac{5}{16}qr(qr_{12} + rq_{12}) \\
&- \frac{i}{8}qr(rq_{111} - qr_{111}) - \frac{i}{8}(q^2r_1r_{11} - ir^2q_1q_{11}) - \frac{i}{8}q_1r_1(rq_1 - qr_1) \\
&+ \frac{i}{4}qr(r_1q_{11} - q_1r_{11}) - \frac{3}{8}qr(q_1r_2 + r_1q_2) + \frac{1}{8}(q_{111}r_2 + r_{111}q_2) \\
&+ \frac{i}{16}(q_{111}r_{11} - r_{111}q_{11}) - \frac{1}{8}(q_{11}r_{12} + r_{11}q_{12}) + \frac{1}{2}(q_1r_4 + r_1q_4)\,,
\end{aligned} \tag{2.43b}
$$

$$
\begin{aligned}
L_{34} =& -\frac{1}{8}(q_{11}r_{13} + r_{11}q_{13}) + \frac{1}{8}(q_{111}r_3 + r_{111}q_3) + \frac{1}{32}q_{111}r_{111} \\
&- \frac{1}{32}(q^2r_{11}^2 + r^2q_{11}^2) - \frac{1}{32}q_1^2r_1^2 + \frac{3i}{8}qr(rq_4 - qr_4) - \frac{9}{32}q^4r^4 \\
&+ \frac{3}{16}q^2r^2(qr_{11} + rq_{11}) + \frac{1}{16}(q^2r_1r_3 + r^2q_1q_3) \\
&+ \frac{5}{16}qr(qr_{13} + rq_{13}) + \frac{i}{4}(q_{11}r_4 - r_{11}q_4) - \frac{3}{16}qr(q_1r_{111} + r_1q_{111}) \\
&- \frac{1}{16}qrq_{11}r_{11} + \frac{1}{16}q_1r_1(qr_{11} + rq_{11}) \\
&+ \frac{15}{16}q^2r^2q_1r_1 - \frac{3}{8}qr(q_1r_3 + r_1q_3) - \frac{i}{8}(q_1r_{14} - r_1q_{14})\,,
\end{aligned} \tag{2.43c}
$$

which in turn produce, besides the aforementioned NLS and mKdV equations,

$$
\begin{aligned}
iq_4 &= \frac{1}{8}q_{1111} + \frac{3}{4}q^3r^2 - \frac{1}{4}q^2r_{11} - \frac{1}{2}qq_1r_1 - qrq_{11} - \frac{3}{4}q_1^2r\,, \\
ir_4 &= -\frac{1}{8}r_{1111} - \frac{3}{4}q^2r^3 + \frac{1}{4}r^2q_{11} + \frac{1}{2}rq_1r_1 + qrr_{11} + \frac{3}{4}r_1^2q\,.
\end{aligned} \tag{2.44}
$$

The explicit proof of this fact is long and not very elegant, but it was originally shown using variational symmetry methods.

Chapter 3

The multisymplectic approach to a 1+1-dimensional field theory

In this Chapter we will describe what we take as the covariant Hamiltonian description of a $1 + 1$-dimension field theory. We work in the algebraic framework of the variational bi-complex as introduced in Section 2.2, which allows us to use two distinct differentials, the usual exterior one d (denoted horizontal) and a vertical one δ. Equipped with the above basic elements of the variational bi-complex, we now write how to describe a $1 + 1$-dimensional partial differential equation admitting a Lagrangian formulation into a covariant Hamiltonian formulation. As will be illustrated, covariant Hamiltonian field theory is still a topic of open discussion within the scientific community. We take more of a pragmatic approach, picking and choosing what suits best to our purpose from two main sources: the first one is the work of Dickey [D03], from where we take the definitions of multisymplectic form and of covariant Hamiltonian, and we take the idea behind the definition of covariant Poisson bracket and of admissible forms[1] from the work of Kanatchikov [K98].

3.1 Some context

The geometrisation of Hamiltonian dynamical systems led to a beautiful framework for classical mechanics, see e.g. [A78] for a modern exposition. The development of an analogous framework for classical field theories followed a less straightforward path and still is the object of current studies, see e.g. the recent book [LSV14]. One feature of field theories is that there are several independent (spacetime) coordinates on which the fields depend so that, starting from a Lagrangian description, one has to make a choice from the very beginning. Roughly speaking, one can distinguish two main avenues underlying the current state of the art.

[1] In [K98] they were called 'Hamiltonian forms'.

On the one hand, one can favour one particular coordinate (the time) to perform the Legendre transformation and develop the analogous geometrisation of Hamiltonian mechanics, resulting in an infinite dimensional Hamiltonian formalism. This point of view seems arbitrary, especially if one is interested in Lorentz invariant theories for instance. Nevertheless, it received a large amount of attention, with a boost coming in particular from the theory of classical integrable systems. The latter provided numerous examples of infinite dimensional Hamiltonian and Liouville integrable systems, since the early examples [ZF71, ZM74]. In that area, important developments such as the theory of Poisson-Lie groups [D83] and the classical r-matrix [S85] have led to an infinite-dimensional version of geometric Hamiltonian mechanics. In parallel, the 'algebraisation' of this framework, driven for instance by I.M. Gel'fand, L.A. Dickey and I. Dorfman, led to what is sometimes called formal (algebraic) variational calculus, see e.g. the books [D03, D93]. An important motivation for generalising the classical Hamiltonian theory to field theory in this way was the programme of canonical quantisation of integrable field theories into integrable quantum field theories. The classical r-matrix method proved to be fundamental to achieve this and it gave rise the notion of quantum R matrix and Quantum Inverse Scattering Method [S79, SF78, FST80, FT81].

On the other hand, the conceptual disadvantage of picking a special coordinate to perform the Legendre transformation emerged already in the early 1900's. The possibility to generalise the Legendre transformation to define conjugate momenta associated to each independent variable naturally leads to a generalisation of the standard Hamilton equations called for short covariant Hamiltonian field theory. This observation is at the basis of a theory discovered independently by De Donder and Weyl and now called De Donder-Weyl formalism [D30, W35]. Further developments followed and led to the Lepage-Dedecker theory, see [HK04] for a more recent exposition of this theory and a comparison with the de Donder-Weyl formalism. Despite being conceptually the same as the traditional Hamiltonian theory (Lagrangian and Hamiltonian pictures are related by a Legendre transformation), its geometrisation shows deep differences. In fact, there is not one established theory of what should play the role of the usual symplectic form and associated symplectic geometry, but instead a variety of related approaches (k-symplectic, polysymplectic or multisymplectic) as described in [LSV14]. Similarly, the familiar notion of phase space must be promoted to a covariant phase space whose definition and use come with certain difficulties. Such a successful framework is credited to Kijowski and Szczyrba [KS76] and later on Zuckerman [Z87]. The relation between multisymplectic formalism and the covariant phase space is investigated in [FR05] and also [H11] which contains an excellent review of the historical development of the many facets of this field and an account of covariant canonical quantization for free field theories. Alongside the problem of generalising symplectic geometry and the phase space comes the question of generalising to the field theoretic context the variational complex that one can associate to a (Lagrangian) system of (ordinary) equations in mechanics. The relevant structure is the variational bi-complex [A89], see e.g. [V08] for a review and a guide to the relevant literature and also [R04] for the relation between covariant phase space and variational bi-complex. A rigorous approach to the covariant phase space in the framework of jet spaces and Vinogradov secondary calculus was proposed in [V09].

To the best of our knowledge, these two avenues flourished rather independently, driven by motivations with little or no overlap, with the exception of one author, L.A. Dickey, who initiated the investigation of the second, covariant, point of view within the formalism of integrable systems in [D90]. This was further developed in the book [D03] where the aforementioned formal algebraic variational calculus was used to describe such objects as multisymplectic forms and the variational bi-complex. Dickey's goal was to study integrable hierarchies from the covariant Hamiltonian point of view, thus breaking the long tradition of the infinite dimensional Hamiltonian formalism that was used in that area, as already mentioned. This body of work does not seem to have been followed up, despite its importance as we now argue. One of the motivations for the endeavour in the aforementioned geometrisation of field theory is the programme of covariant canonical quantization as an alternative that would combine the advantages of manifest covariance (as in Feynman's path integral techniques) and 'simple' quantization rules (as in canonical quantization) without their disadvantages. Our point of view is that integrable field theories are the 'nicest' field theories one can work with, beyond free field theories, to test the framework.

3.2 Covariant Legendre transformation and covariant Hamiltonian equations

We focus on two-dimensional field theories: we set $M = \mathbb{R}^2$ and we start from a *Lagrangian volume* 2*-form*

$$\Lambda = L\, dx^1 \wedge dx^2.$$

$L \in \mathscr{A}$ is the Lagrangian density and depends on the fields u^k, $k = 1, \dots, N$ and their derivatives with respect to x^1 and x^2, up to some finite order. In most cases Λ will not depend explicitly on the space-time variables x^1 and x^2. Λ is the non-integrated version of the action $S = \int L\, dx\, dt$ of a $1+1$-dimensional field theory. The following results are taken from [D03] and specialised to a 2-dimensional space-time manifold, and will be illustrated with examples as we go along.

Proposition 3.1 *Let $F = f\, dx^1 \wedge dx^2 \in \mathscr{A}^{(0,2)}$ be a volume form, then δF can be represented as*

$$\delta F = \sum_{k=1}^{N} A_k\, \delta u_k \wedge dx^1 \wedge dx^2 + dG \tag{3.1}$$

where $G \in \mathscr{A}^{(1,1)}/d\mathscr{A}^{(1,0)}$.

Proof. The proof is obtained transforming the expression

$$\delta f \wedge dx^1 \wedge dx^2 = \sum_{k,(i)} \frac{\partial f}{\partial u_k^{(i)}} \delta u_k^{(i)} \wedge dx^1 \wedge dx^2$$

simply using integration by parts

$$\sum \frac{\partial f}{\partial u_k^{(i)}} \delta u_k^{(i)} = \sum (-1)^{|(i)|} \partial^{(i)} \left(\frac{\partial f}{\partial u_k^{(i)}} \right) \delta u_k^{(i)} + \sum \partial_\alpha B_\alpha$$

where B_α are vertical forms. One then just sets

$$A_k = \sum (-1)^{|(i)|} \partial^{(i)} \left(\frac{\partial f}{\partial u_k^{(i)}} \right) , \quad G = -B_1\, dx^2 + B_2\, dx^1 . \qquad \square$$

Thanks to the use of the Tulczyjev operator [D03], one can prove that the coefficients A_k are uniquely determined: they will be denoted $\frac{\delta f}{\delta u_k}$ and called *variational derivative* of f with respect to u_k. The form G is determined up to a horizontally closed form, and therefore lives in $\mathscr{A}^{(1,1)}/d\mathscr{A}^{(1,0)}$

In the case where the volume form is taken as a Lagrangian, we call $\Omega^{(1)} \equiv -G$:

$$\delta\Lambda = \sum_k \frac{\delta L}{\delta u_k} \delta u^k \wedge dx^1 \wedge dx^2 - d\Omega^{(1)} \tag{3.2}$$

where $\Omega^{(1)} \in \mathscr{A}^{(1,1)}/d\mathscr{A}^{(1,0)}$ is only determined up to a total d-differential. One then obtains the Euler-Lagrange equations by setting $\frac{\delta L}{\delta u^k} = 0$ for every k.

Remark 3.2: The content of this result is simply the local analog of the standard integration by parts procedure used when varying the action $\int \Lambda$, where the boundary term $\int d\Omega^{(1)}$ is usually discarded. The identification of $\Omega^{(1)}$ thus defined with the field-theoretic analog of the canonical 1-form can be found in [D03]. Despite not being well-known, this is a rather simple result that holds even in finite dimensional mechanics. Indeed let us consider an action of the type $S[q] = \int L(q, \dot{q})dt$. Its variation brings

$$\begin{aligned} \delta S &= \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \wedge dt \\ &= \int \left(\frac{\partial L}{\partial q} - \partial_t \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q \wedge dt + \int \partial_t \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \wedge dt \end{aligned}$$

where in the last term we recognise the canonical momentum $\frac{\partial L}{\partial \dot{q}}$. The minus sign in the definition of $\Omega^{(1)}$ is merely a convention.

Remark 3.3: From now on, we will only consider Lagrangians that do not depend explicitly on the space-time variables. Hence, neither $\Omega^{(1)}$ nor any object that will be derived from Λ and $\Omega^{(1)}$ will depend explicitly on the space-time variables.

We can reformulate the well-known fact that Lagrangians are equivalent up to a total differential (*i.e.* they bring the same equations of motion).

Proposition 3.4 (Equivalent Lagrangians) *The Lagrangian volume forms* Λ *and* $\Lambda' = \Lambda + d\varphi$, *where* $\varphi \in \mathscr{A}^{(0,1)}$ *produce the same Euler-Lagrange equations*

$$\delta\Lambda = E(\Lambda) - d\Omega^{(1)}\,, \qquad \delta\Lambda' = E(\Lambda) - d\Omega^{(1)'}\,,$$

with $\Omega^{(1)'} = \Omega^{(1)} + \delta\varphi$.

Proof. By direct calculation:

$$\delta\Lambda' = \delta\Lambda + \delta d\varphi = E(\Lambda) - d\Omega^{(1)} - d\delta\varphi \equiv E(\Lambda) - d\Omega^{(1)'}\,. \qquad \square$$

The next step is the following definition, which shows that for a field theory, $\Omega^{(1)}$ realises the Legendre transformation simultaneously with respect to all independent variables.

Definition 3.5 (Covariant Hamiltonian) *The* covariant Hamiltonian $H \in \mathscr{A}^{(0,2)}$ *related to the Lagrangian* $\Lambda \in \mathscr{A}^{(0,2)}$ *and* $\Omega^{(1)} \in \mathscr{A}^{(1,1)}/d\mathscr{A}^{(1,0)}$ *is*

$$H \coloneqq -\Lambda + \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega^{(1)}\,. \tag{3.3}$$

$h \in \mathscr{A}$ *such that* $H = h\, dx^1 \wedge dx^2$ *is called* covariant Hamiltonian density.

To understand the role played by $\Omega^{(1)}$, we remark the following facts. For a classical finite-dimensional Lagrangian system, the integration by parts provides (the pull-back to the tangent bundle of) the canonical one form $\frac{\partial L}{\partial \dot{q}}\delta q$, and one can obtain the symplectic form by taking its δ-differential. Similarly, in the case of field theories where Λ is taken to be a volume form, the form is $\Omega^{(1)} = \omega_1^{(1)} \wedge dx^1 + \omega_2^{(1)} \wedge dx^2$ where $\omega_1^{(1)}$ and $\omega_2^{(1)}$ each have a similar structure to the canonical one form of the finite dimensional case. It contains the *usual* symplectic structure $-\omega_1^{(1)}$ (if we consider x_2 as our 'time') but also the dual structure $\omega_2^{(1)}$ (which would correspond to performing the Legendre transformation when choosing x_1 as the time variable). In fact, the usual Hamiltonian formulation that is obtained with the choice $x^2 =$ 'time', is computed as

$$H_S = (-\widetilde{\partial}_2 \lrcorner \omega_1^{(1)} - L)\, dx^1 \wedge dx^2 \tag{3.4}$$

and integrating over the x^1 axis, whilst the dual one with the choice $x^1 =$'time' is the integral over the x^2 axis of

$$H_T = (\widetilde{\partial}_1 \lrcorner \omega_2^{(1)} - L)\, dx^1 \wedge dx^2 \,. \tag{3.5}$$

Equation (3.3) is in fact a *covariant Legendre transformation* of Λ: explicitly we have that

$$\begin{aligned} H =& - L\, dx^1 \wedge dx^2 + dx^1 \wedge \widetilde{\partial}_1 \lrcorner \Omega^{(1)} + dx^2 \wedge \widetilde{\partial}_2 \lrcorner \Omega^{(1)} \\ =& (\widetilde{\partial}_1 \lrcorner \omega_2^{(1)} - \widetilde{\partial}_2 \lrcorner \omega_1^{(1)} - L)\, dx^1 \wedge dx^2 \,. \end{aligned}$$

To make this definition clearer we will consider the following example:

Example: Let $\Lambda = L\, dx \wedge dt$ be

$$\Lambda = \left(\frac{u_t^2}{2} - \frac{u_x^2}{2} - V(u) \right) dx \wedge dt$$

where $V(u)$ is a smooth potential that only depends on the field u. The Euler-Lagrange equations are easily obtained:

$$\frac{\delta L}{\delta u} = -u_{tt} + u_{xx} - V'(u) = 0 \,.$$

The usual infinite-dimensional Hamiltonian formulation is brought by the following prescription: roughly, one considers as the phase space the set of space-dependent functions, and then defines the field momentum as

$$p(x) = \frac{\partial L}{\partial u_t}(x) = u_t(x) \,.$$

One then finds that the transformation $(u, u_t) \mapsto (u, p)$ is trivially invertible, and obtains the Hamiltonian as the integral

$$H_S = \int (p u_t - L) dx = \int (\frac{p^2}{2} + \frac{u_x^2}{2} + V(u)) dx$$

Alternatively, the 'dual' infinite-dimensional Hamiltonian formulation is brought by the different choice of phase space, now being the time-dependent functions, and the definition of another, time-dependent momentum as

$$\pi(t) = \frac{\partial L}{\partial u_x}(t) = -u_x(t)$$

and the definition of the dual Hamiltonian as the following integral

$$H_T = \int (\pi u_x - L) dt = \int (-\frac{\pi^2}{2} - \frac{u_t^2}{2} + V(u)) dt \,.$$

The *covariant Hamiltonian* is obtained by performing both Legendre transformations simultaneously in the following way. First of all, instead of choosing functions of just one of the space-time variables we take the differential algebra $\mathscr{A}$ as our phase space. Then, let us compute the δ-differential of Λ

$$\delta\Lambda = (u_t \delta u_t - u_x \delta u_x - V'(u)\delta u) \wedge dx \wedge dt$$

We want to express $\delta\Lambda$ as in (3.2), so we need to express $u_t \delta u_t \wedge dx \wedge dt$ and $u_x \delta u_x \wedge dx \wedge dt$ as total d-differential, which can be done as

$$\begin{aligned} u_t \delta u_t \wedge dx \wedge dt =& - u_{tt} \delta u \wedge dx \wedge dt + d(u_t \delta u \wedge dx) \\ -u_x \delta u_t \wedge dx \wedge dt =& u_{xx} \delta u \wedge dx \wedge dt + d(u_x \delta u \wedge dt) \end{aligned}$$

so that we have

$$\begin{aligned} \delta\Lambda &= (-u_{tt} + u_{xx} - V'(u))\delta u \wedge dx \wedge dt - d(-u_t \delta u \wedge dx - u_x \delta u \wedge dt) \\ &\equiv \frac{\delta L}{\delta u} \delta u \wedge dx \wedge dt - d\Omega^{(1)} . \end{aligned}$$

This defines $\Omega^{(1)}$ up to a d-differential. We notice that its coefficients are the single-time momenta, as in $\Omega^{(1)} = \pi \delta u \wedge dt - p \delta u \wedge dx$. The covariant Hamiltonian is then obtained as in Definition 3.5

$$\begin{aligned} H =& - L dx \wedge dt + dx \wedge \widetilde{\partial}_x \lrcorner \Omega^{(1)} + dt \wedge \widetilde{\partial}_t \lrcorner \Omega^{(1)} \\ =& (-\frac{u_t^2}{2} + \frac{u_x^2}{2} + V(u)) dx \wedge dt + dx \wedge (-u_x^2) dt + dt \wedge (-u_t^2) dx \\ =& (\frac{u_t^2}{2} - \frac{u_x^2}{2} + V(u)) dx \wedge dt . \end{aligned}$$

Proposition 3.6 (Covariant Hamilton equations and multisymplectic form) *The Euler-Lagrange equations* $\frac{\delta L}{\delta u^k} = 0$ *are equivalent to*

$$\delta H = \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega, \tag{3.6}$$

where $\Omega \in \mathscr{A}^{(2,1)}$ *is the* multisymplectic form

$$\Omega := \delta\Omega^{(1)} . \tag{3.7}$$

and H *is the covariant Hamiltonian related to* $\mathscr{L}$ *and* $\Omega^{(1)}$ *as in Definition 3.5.*

Proof. From the definition of H we get

$$\delta H = -\delta\Lambda - \sum_{j=1,2} dx^j \wedge \delta \widetilde{\partial}_j \lrcorner \Omega^{(1)}.$$

Thanks to the definition of $\Omega^{(1)}$ as in (3.2) the equations of motion are equivalent to

$$\begin{aligned}\delta H =& d\Omega^{(1)} - \sum_{j=1,2} dx^j \wedge \delta\widetilde{\partial}_j \lrcorner \Omega^{(1)} \\ =& \sum_{j=1,2} dx^j \wedge \partial_j \Omega^{(1)} - \sum_{j=1,2} dx^j \wedge \delta\widetilde{\partial}_j \lrcorner \Omega^{(1)}\end{aligned}$$

where we have also used $d\Omega^{(1)} = \sum_j dx^j \wedge \partial_j \Omega^{(1)}$. We now use $\partial_j = \partial'_j + \widetilde{\partial}_j$ to write

$$\begin{aligned}\delta H =& \sum_{j=1,2} dx^j \wedge (\partial'_j + \widetilde{\partial}_j)\Omega^{(1)} - \sum_{j=1,2} dx^j \wedge \delta\widetilde{\partial}_j \lrcorner \Omega^{(1)} \\ =& \sum_{j=1,2} dx^j \wedge (\partial'_j + \widetilde{\partial}_j \lrcorner \delta + \delta\widetilde{\partial}_j \lrcorner)\Omega^{(1)} - \sum_{j=1,2} dx^j \wedge \delta\widetilde{\partial}_j \lrcorner \Omega^{(1)}\end{aligned}$$

and we used the property $\widetilde{\partial}_j = \widetilde{\partial}_j \lrcorner \delta + \delta\widetilde{\partial}_j \lrcorner$. $\Omega^{(1)}$ does not depend explicitly on the space-time variables so $\partial'_j \Omega^{(1)} = 0$. We then have

$$\begin{aligned}\delta H =& \sum_{j=1,2} dx^j \wedge (\widetilde{\partial}_j \lrcorner \delta + \delta\widetilde{\partial}_j \lrcorner)\Omega^{(1)} - \sum_{j=1,2} dx^j \wedge \delta\widetilde{\partial}_j \lrcorner \Omega^{(1)} \\ =& \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \delta\Omega^{(1)} = \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega\end{aligned}$$

and the result is obtained by cancellation. □

Remark 3.7: In this thesis we are only dealing with $1+1$-dimensional field theories, where only 2 independent variables are considered. In general if a PDE involves k independent variables and admits a Lagrangian description, Λ and H are volume k-forms, $\Omega^{(1)} \in \mathscr{A}^{(1,k-1)}$ and $\Omega \in \mathscr{A}^{(2,k-1)}$, as explained in [D03]. We also remark that the multisymplectic form is vertically closed $\delta\Omega = 0$ (and more precisely, exact).

Proposition 3.8 *Equivalent Lagrangian volume forms define the same covariant Hamiltonian and multisymplectic form.*

Proof. We know that equivalent Lagrangians Λ and $\Lambda' = \Lambda + d\varphi$ bring the same Euler-Lagrange equations, but respectively $\Omega^{(1)}$ and $\Omega^{(1)'} = \Omega^{(1)} + \delta\varphi$. Since $\delta^2 = 0$ then $\Omega = \delta\Omega^{(1)}$ and $\Omega' = \delta\Omega^{(1)'} = \delta(\Omega^{(1)} + \delta\varphi) = \Omega + \delta^2\varphi$ coincide. It remains to check that also the covariant Hamiltonians

$$H = -\Lambda + \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega^{(1)} \quad \text{and} \quad H' = -\Lambda' + \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega^{(1)'}$$

coincide. In fact we have

$$\begin{aligned} H' =& -\Lambda - d\varphi + \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner (\Omega^{(1)} + \delta\varphi) \\ =& H - d\varphi + \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner \delta\varphi\,, \end{aligned}$$

and, using $\widetilde{\partial}_j = \widetilde{\partial}_j \lrcorner \delta + \delta \widetilde{\partial}_j \lrcorner$ we have

$$H' = H - d\varphi + \sum_j dx^j \wedge (\widetilde{\partial}_j - \delta\widetilde{\partial}_j \lrcorner)\varphi\,.$$

We now use the fact that $\varphi \in \mathscr{A}^{(0,1)}$, and so $\widetilde{\partial}_j \lrcorner \varphi = 0$, and that it does not depend explicitly on the space-time variables, so $\widetilde{\partial}_j \varphi = \partial_j \varphi$. Writing $d\varphi = \sum_j dx^j \wedge \partial_j \varphi$ we obtain the result. □

Example: Let us find the covariant Hamilton equations for the example above. We found

$$\begin{aligned} H &= (\frac{u_t^2}{2} - \frac{u_x^2}{2} + V(u))dx \wedge dt \\ \Omega^{(1)} &= -u_t \delta u \wedge dx - u_x \delta u \wedge dt\,, \end{aligned}$$

so the multisymplectic form is found as

$$\Omega = \delta\Omega^{(1)} = -\delta u_t \wedge \delta u \wedge dx - \delta u_x \wedge \delta u \wedge dt\,.$$

The covariant Hamilton equations are equivalent to $u_{tt} - u_{xx} + V'(u) = 0$, in fact $\delta H = dx \wedge \widetilde{\partial}_x \lrcorner \Omega + dt \wedge \widetilde{\partial}_t \lrcorner \Omega$ brings

$$\begin{aligned} &(u_t \delta u_t - u_x \delta u_x + V'(u)\delta u) \wedge dx \wedge dt \\ &\qquad = dx \wedge (-u_{xx}\delta u + u_x \delta u_x) \wedge dt + dt \wedge (-u_{tt}\delta u + u_t \delta u_t) \wedge dx \end{aligned}$$

and therefore

$$V'(u)\delta u \wedge dx \wedge dt = (u_{xx} - u_{tt})\delta u \wedge dx \wedge dt$$

which is equivalent to $\frac{\delta L}{\delta u} = 0$.

3.3 Covariant Poisson brackets

Equipped with a multisymplectic form we can consider a covariant Poisson bracket. We stress that the definition of a covariant Poisson bracket from a multisymplectic form, in a way that mimics the situation in classical mechanics, has been part of a rich activity since the early proposals. In particular, the Jacobi identity is a delicate issue, as well as the need to restrict to certain forms, called admissible, as we explain below. For our purpose, we will simply use Kanatchikov's ideas and adapt them to our needs. The results of [CS20a] show that, at least in our context, this leads to a satisfactory covariant Poisson bracket satisfying the Jacobi identity, thanks to the fact that the latter is satisfied by means of the classical Yang-Baxter equation for the classical r-matrix.

We need to restrict our attention to the a special class of forms called admissible.

Definition 3.9 (Admissible forms) *A horizontal form F is* admissible *with respect to Ω if there exists a (multi)vector field ξ_F such that*

$$\xi_F \lrcorner \Omega = \delta F\,. \tag{3.8}$$

Then ξ_F is called Hamiltonian vector field *related to the admissible form F.*

> **Remark 3.10:** In this thesis we only consider horizontal forms as candidates for being admissible, which is enough for our purposes. This also reflects the natural interpretation of admissible forms, *i.e.* a forms $F = F_1\,dx^1 + F_2\,dx^2$ that, when integrated over one of the space-time axes ($x^1 = 0$ or $x^2 = 0$), become the usual functionals $\int F^1\,dx^1$, in the latter case, or the dual $\int F^2\,dx^2$ in the former. Admissible forms with a vertical components have been proposed in [FPR03] with the terminology of *Poisson forms.*

Contrary to the usual symplectic case, the property of being an admissible form is quite restrictive. In the finite dimensional case, in fact, if ω is taken to be a symplectic form (and therefore non-degenerate), there is a one-to-one correspondence between vector fields and differentials of functions, so given a f, it's always possible to find a ξ_f such that $\xi_f \lrcorner \omega = df$. In the multisymplectic case, instead, Ω is often degenerate, and therefore this correspondence is missing. For this reason, from now on, we will always consider Hamiltonian vector fields modulo the kernel of Ω. On the other hand, thanks to the presence of two distinct differentials (a horizontal and a vertical one) and the fact that $\Omega \in \mathscr{A}^{(2,1)}$, we can allow a similar correspondence not only with scalar functions, but also to horizontal forms of any degree. However, as we soon find out, only 0- and 1-forms provide non-trivial admissible forms:

Proposition 3.11 *Let $G \in \mathscr{A}^{(0,2)}$. G is an admissible form with respect to $\Omega \in \mathscr{A}^{(2,1)}$ if and only if G is constant*[2]*, with $\xi_G = 0$.*

Proof. The proof is obtained by a simple counting argument: since $\Omega \in \mathscr{A}^{(2,1)}$, then there must exist ξ_G such that $\xi_G \lrcorner \Omega = \delta G \in \mathscr{A}^{(1,2)}$ which happens if and only if both $\delta G = 0$

[2] By constant we mean that $G = g\,dx^1 \wedge dx^2$, with $g \in \mathcal{K}$.

and $\xi_G = 0$. □

It is often the case that the multisymplectic form is fixed by the theory that we are considering. In this case, where it is not cause of confusion, we will refer to 'admissible forms', without specifying the multisymplectic form they are related to.

> **Remark 3.12:** We decided to change the terminology from *Hamiltonian forms* in [CS20a, CS20b, CS21, CSV21a] to *admissible forms*. The previous choice was motivated by its vast presence in the literature, e.g. in [FPR05], but it produces paradoxical statements such as 'The Hamiltonian is a form but it is not a Hamiltonian form'. The new choice, *admissible forms*, solves this problem and reflects (although with some changes) the terminology present for instance in the context of Dirac structures [CGM17, C90].

Let us denote by S_Ω the set of elements $\delta u_k^{(i)}$ that appear in the multisymplectic form. This is a finite set since we assume finite-jet dependence of Λ. We can therefore assume some ordering on S_Ω and label the $\delta u_k^{(i)}$'s as δv_j $j = 1, \dots, \#S_\Omega$. We then write

$$\Omega = \sum_{\substack{i<j \\ i,j\in I_1}} \omega_1^{ij} \delta v_i \wedge \delta v_j \wedge dx^1 + \sum_{\substack{i<j \\ i,j\in I_2}} \omega_2^{ij} \delta v_i \wedge \delta v_j \wedge dx^2 \,. \tag{3.9}$$

where $I_1, I_2 \subset \{ 1, \dots, \#S_\Omega \}$.

Proposition 3.13 (Necessary form of an admissible 1-form) *Suppose $F = F_1\, dx + F_2\, dt \in \mathscr{A}^{(0,1)}$ is an admissible form related to the multisymplectic form* (3.9)*. Then, F_1 can only depend (at most) on v_j, $j \in I_1$, and F_2 can only depend (at most) on v_i, $i \in I_2$.*

Proof. Assume F_1 depends on some $u_\ell^{(k)} \notin \{ v_j \,, \quad j \in I_1 \}$. On the one hand,

$$\delta F = \sum_{j\in I_1} \frac{\partial F_1}{\partial v_j} \delta v_j \wedge dx^1 + \frac{\partial F_1}{\partial u_\ell^{(k)}} \delta u_\ell^{(k)} \wedge dx^1 + \sum_{i\in I_2} \frac{\partial F_2}{\partial v_i} \delta v_i \wedge dx^2 \,.$$

On the other hand, since F is an admissible form, there exists a vector field ξ_F such that $\xi_F \lrcorner \Omega = \delta F$. This gives

$$\sum_{\substack{i<j \\ i,j\in I_1}} \omega_1^{ij} \xi_F \lrcorner (\delta v_i \wedge \delta v_j) \wedge dx^1 + \sum_{\substack{i<j \\ i,j\in I_2}} \omega_2^{ij} \xi_F \lrcorner (\delta v_i \wedge \delta v_j) \wedge dx^2$$

In particular, this requires

$$\sum_{j\in I_1} \frac{\partial F_1}{\partial v_j} \delta v_j \wedge dx^1 + \frac{\partial F_1}{\partial u_\ell^{(k)}} \delta u_\ell^{(k)} \wedge dx^1 = \sum_{\substack{i<j \\ i,j\in I_1}} \omega_1^{ij} \xi_F \lrcorner (\delta v_i \wedge \delta v_j) \wedge dx^1 \,,$$

so that necessarily $\frac{\partial F_1}{\partial v_j} = \sum_{i \in I_1} \omega_1^{ij} \xi_F \lrcorner \delta v_i$ and $\frac{\partial F_1}{\partial u_\ell^{(k)}} = 0$. The same argument holds for F_2. $\square$

Example: Let us characterise admissible forms for the multisymplectic form

$$\Omega = -\delta u_t \wedge \delta u \wedge dx - \delta u_x \wedge \delta u \wedge dt\,.$$

- **0-forms:** For a generic $K(u, u_x, u_t)$ let us assume that $\xi_K \lrcorner \Omega = \delta K$, where ξ_K is a generic vector field

$$\xi_K = a\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} + b\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial t} + c\frac{\partial}{\partial u_x} \wedge \frac{\partial}{\partial t} + d\frac{\partial}{\partial u_t} \wedge \frac{\partial}{\partial x}$$

(up to terms in $\ker \Omega$), where a, b, c and d are smooth functions of u, u_x and u_t to determine. We have started from a 2-vector field because we want to obtain $\delta K \in \mathscr{A}^{(1,0)}$ by insertion with $\Omega \in \mathscr{A}^{(2,1)}$, so ξ_K must have one vertical and one horizontal component. On the right hand-side we have

$$\delta K = \frac{\partial K}{\partial u}\delta u + \frac{\partial K}{\partial u_x}\delta u_x + \frac{\partial K}{\partial u_t}\delta u_t\,,$$

while on the left hand-side we have the following

$$\begin{aligned} a\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} \lrcorner \Omega &= a\delta u_t\,, \qquad & b\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial t} \lrcorner \Omega &= b\delta u_x \\ c\frac{\partial}{\partial u_x} \wedge \frac{\partial}{\partial t} \lrcorner \Omega &= -c\delta u\,, \qquad & d\frac{\partial}{\partial u_t} \wedge \frac{\partial}{\partial x} \lrcorner \Omega &= -d\delta u \end{aligned}$$

so $\xi_K \lrcorner \Omega = -(c+d)\delta u + b\delta u_x + a\delta u_t$. By comparison we have $b = \frac{\partial K}{\partial u_x}$ and $a = \frac{\partial K}{\partial u_t}$, whilst we see that we have a choice of both c and d, which is to be expected since $\frac{\partial}{\partial u_x} \wedge \frac{\partial}{\partial t} - \frac{\partial}{\partial u_t} \wedge \frac{\partial}{\partial x} \in \ker \Omega$. We can choose $d = 0$ and $c = -\frac{\partial K}{\partial u}$. Therefore we have that any $K(u, u_x, u_t) \in \mathscr{A}$ is an admissible form, with Hamiltonian vector field

$$\xi_K = \frac{\partial K}{\partial u_t}\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} + \frac{\partial K}{\partial u_x}\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial t} - \frac{\partial K}{\partial u}\frac{\partial}{\partial u_x} \wedge \frac{\partial}{\partial t}\,.$$

- **1-forms:** For a 1-form $F = F_1(u, u_t)dx + F_2(u, u_x)dt$, we proceed in a similar way starting from a vertical vector field

$$\xi_F = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial u_x} + c\frac{\partial}{\partial u_t}\,,$$

with coefficients to determine. After insertion with Ω we get

$$\xi_F \lrcorner \Omega = a\delta u_t \wedge dx + a\delta u_x \wedge dt - b\delta u \wedge dt - c\delta u \wedge dx$$

that we compare with

$$\begin{aligned}\delta F =&\frac{\partial F_1}{\partial u}(u,u_t)\delta u\wedge dx+\frac{\partial F_1}{\partial u_t}(u,u_t)\delta u_t\wedge dx\\ &+\frac{\partial F_2}{\partial u}(u,u_x)\delta u\wedge dt+\frac{\partial F_2}{\partial u_x}(u,u_x)\delta u_x\wedge dt\,,\end{aligned}$$

getting the following relations

$$a=\frac{\partial F_1}{\partial u_t}(u,u_t)=\frac{\partial F_2}{\partial u_x}(u,u_x)\,,$$
$$b=-\frac{\partial F_2}{\partial u}(u,u_x)\,,\qquad c=-\frac{\partial F_1}{\partial u}(u,u_t)\,.$$

We therefore see that $F\in\mathscr{A}^{(0,1)}$ is an admissible form if and only if F_1 and F_2 are respectively linear in u_t and u_x, such that $\frac{\partial F_1}{\partial u_t}=\frac{\partial F_2}{\partial u_x}$, with Hamiltonian vector field

$$\xi_F=\frac{\partial F_1}{\partial u_t}\frac{\partial}{\partial u}-\frac{\partial F_2}{\partial u}\frac{\partial}{\partial u_x}-\frac{\partial F_1}{\partial u}\frac{\partial}{\partial u_t}\,.$$

- **2-forms and beyond:** Any horizontal form G of degree greater or equal than two is an admissible form if and only if it is constant, *i.e.* $\delta G=0$.

Only for admissible forms can we define covariant Poisson brackets.

Definition 3.14 (Covariant Poisson brackets) *Given two admissible forms P and Q, of (horizontal) degree respectively r and s, we can define their* covariant Poisson bracket *as*

$$\{\!(P,Q)\!\}:=(-1)^r\xi_P\lrcorner\xi_Q\lrcorner\Omega\,. \tag{3.10}$$

The covariant Poisson brackets have the following properties:

- They are antisymmetric $\{\!(F,G)\!\}=-\{\!(G,F)\!\}$;
- They are bi-linear in the space of admissible forms.

We delay the discussion of the Jacobi identity to the end of this section.

We now prove the following theorem, which was only obtained explicitly on examples in [CS20a], but for which no general proof was given.

Theorem 3.15 *If the covariant Hamiltonian density $h\in\mathscr{A}$ is an admissible form, then we have for any admissible 1-form F that does not depend explicitly on the space-time variables*

$$dF=\{\!(h,F)\!\}\,dx^1\wedge dx^2. \tag{3.11}$$

Proof. Using (3.6) and the antisymmetry of Ω we have

$$\xi_F \lrcorner \delta H = \xi_F \lrcorner \left(\sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega \right) = - \sum_{j=1,2} dx^j \wedge \xi_F \lrcorner \widetilde{\partial}_j \lrcorner \Omega = \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \xi_F \lrcorner \Omega.$$

Since $\xi_F \lrcorner \Omega = \delta F$ we obtain

$$\xi_F \lrcorner \delta H = \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j \lrcorner \delta F.$$

Using the property $\widetilde{\partial}_j \lrcorner \delta = \widetilde{\partial}_j - \delta \widetilde{\partial}_j \lrcorner$

$$\xi_F \lrcorner \delta H = \sum_{j=1,2} dx^j \wedge \widetilde{\partial}_j F - \sum_{j=1,2} dx^j \wedge \delta \widetilde{\partial}_j \lrcorner F.$$

Since F is purely horizontal $\widetilde{\partial}_j \lrcorner F = 0$, and since it does not depend explicitly on the space-time variables $\widetilde{\partial}_j F = \partial_j F$, so that

$$\xi_F \lrcorner \delta H = \sum_{j=1,2} dx^j \wedge \partial_j F = dF.$$

Now we realise the covariant Poisson bracket:

$$dF = \xi_F \lrcorner \delta H = \xi_F \lrcorner \delta h \wedge dx^1 \wedge dx^2 = -\{\!\{F, h\}\!\} dx^1 \wedge dx^2 = \{\!\{h, F\}\!\} dx^1 \wedge dx^2 \,. \qquad \square$$

Remark 3.16: This is of course the multisymplectic analog of the well-known equation in Hamiltonian mechanics $\dot{f} = \{H, f\}$ giving the time evolution of a smooth real-valued function f on the phase space under the Hamiltonian flow of H.

The covariant Poisson brackets have an interesting property in terms of the single-time Poisson brackets. In particular, we know that

$$\Omega = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 \,, \quad \omega_{1,2} \in \mathscr{A}^{(2,0)}. \tag{3.12}$$

It may be that $\omega_{1,2}$ are traditional symplectic forms. In this case we can define the single-time Poisson brackets related to both ω_1 and ω_2 in the usual way: with respect to x^1

$$\{f, g\}_1 := -\gamma_f \lrcorner (\gamma_g \lrcorner \omega_1) = -\gamma_f \lrcorner \delta g \,, \quad \text{where } \gamma_f \lrcorner \omega_1 = \delta f \,, \quad \gamma_g \lrcorner \omega_1 = \delta g \,, \tag{3.13}$$

and, with respect to x^2

$$\{u, v\}_2 := -\eta_u \lrcorner (\eta_v \lrcorner \omega_2) = -\eta_u \lrcorner \delta v \,, \quad \text{where } \eta_u \lrcorner \omega_2 = \delta u \,, \quad \eta_v \lrcorner \omega_2 = \delta v \,. \tag{3.14}$$

These are traditional Poisson brackets, and in particular satisfy the Jacobi identity $\{a, \{b, c\}_k\}_k + \{b, \{c, a\}_k\}_k + \{c, \{a, b\}_k\}_k = 0$ for $k = 1, 2$.

Proposition 3.17 (Decomposition of the covariant Poisson brackets) *Let $F = F_1\, dx^1 + F_2\, dx^2$ and $G = G_1\, dx^1 + G_2\, dx^2$ be admissible 1-forms with respect to $\Omega = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2$. Then, if $\omega_{1,2}$ are symplectic forms,*

$$\{\!\!\{F, G\}\!\!\} = \{F_1, G_1\}_1\, dx^1 + \{F_2, G_2\}_2\, dx^2\,. \tag{3.15}$$

Proof. On the one hand, by definition

$$\delta F = \delta F_1 \wedge dx^1 + \delta F_2 \wedge dx^2\,,$$

and on the other hand, since F is an admissible form

$$\delta F = \xi_F \lrcorner (\omega_1 \wedge dx^1 + \omega_2 \wedge dx^2) = (\xi_F \lrcorner \omega_1) \wedge dx^1 + (\xi_F \lrcorner \omega_2) \wedge dx^2\,,$$

hence $\delta F_i = \xi_F \lrcorner \omega_i$. Next, consider the following chain of equalities

$$\begin{aligned}
\{\!\!\{F, G\}\!\!\} =& -\xi_F \lrcorner \delta G = -\xi_G \lrcorner (\delta G_1 \wedge dx^1 + \delta G_2 \wedge dx^2) \\
=& -\xi_F \lrcorner (\gamma_{G_1} \lrcorner \omega_1 \wedge dx^1 + \eta_{G_2} \lrcorner \omega_2 \wedge dx^2) \\
=& \gamma_{G_1} \lrcorner (\xi_F \lrcorner \omega_1) \wedge dx^1 + \eta_{G_2} \lrcorner (\xi_F \lrcorner \omega_2) \wedge dx^2 \\
=& \{F_1, G_1\}_1 dx^1 + \{F_2, G_2\}_2 dx^2
\end{aligned}$$

which concludes the proof. □

> **Remark 3.18:** In the case where $\omega_{1,2}$ are symplectic forms, then it is immediate to verify that the covariant Poisson bracket $\{\!\!\{\ ,\ \}\!\!\}$ also satisfies the Jacobi identity, as it satisfies it on the coefficients of dx^1 and dx^2.

The previous proposition provides not only an interpretation of the covariant Poisson brackets $\{\!\!\{\ ,\ \}\!\!\}$ between two 1-forms (it is a 1-form with coefficients being the usual and dual single-time Poisson brackets), but also a way to calculate the two brackets $\{\ ,\ \}_{1,2}$, which seems to be working even when the usual Legendre transformation is degenerate (e.g. the Non-Linear Schrödinger equation in Section 4.3) and one therefore should resort to the use of Dirac brackets [D50], as explained in Section A.3.

Example: We turn to our example, where

$$\begin{aligned}
h &= \frac{u_t^2}{2} - \frac{u_x^2}{2} + V(u)\,, \\
\Omega &= -\delta u_t \wedge \delta u \wedge dx - \delta u_x \wedge \delta u \wedge dt\,.
\end{aligned}$$

We first compute the covariant Poisson brackets between two admissible 1-forms $F = F_1 dx + F_2 dt$ and $G = G_1 dx + G_2 dt$, using the definition

$$\begin{aligned}
\{(F, G)\} &= -\,\xi_F \lrcorner \delta G = \frac{\partial F_1}{\partial u_t}\frac{\partial G_1}{\partial u}dx + \frac{\partial F_1}{\partial u_t}\frac{\partial G_2}{\partial u}dt - \frac{\partial F_2}{\partial u}\frac{\partial G_2}{\partial u_x}dt - \frac{\partial F_1}{\partial u}\frac{\partial G_1}{\partial u_t}dx \\
&= \left(\frac{\partial F_2}{\partial u_x}\frac{\partial G_1}{\partial u} - \frac{\partial F_1}{\partial u}\frac{\partial G_1}{\partial u_t}\right)dx + \left(\frac{\partial F_1}{\partial u_t}\frac{\partial G_2}{\partial u} - \frac{\partial F_2}{\partial u}\frac{\partial G_2}{\partial u_x}\right)dt \\
&= \left(\frac{\partial F_1}{\partial u_t}\frac{\partial G_1}{\partial u} - \frac{\partial F_1}{\partial u}\frac{\partial G_1}{\partial u_t}\right)dx + \left(\frac{\partial F_2}{\partial u_x}\frac{\partial G_2}{\partial u} - \frac{\partial F_2}{\partial u}\frac{\partial G_2}{\partial u_x}\right)dt
\end{aligned}$$

where in the last line we have used the admissible property $\frac{\partial F_1}{\partial u_t} = \frac{\partial F_2}{\partial u_x}$, $\frac{\partial G_1}{\partial u_t} = \frac{\partial G_2}{\partial u_x}$. We see that the coefficients of dx and dt are respectively the usual $\{\ ,\ \}_1$ and dual $\{\ ,\ \}_2$ single-time Poisson brackets obtained from the symplectic forms $\omega_1 = \delta u \wedge \delta u_t$ and $\omega_2 = \delta u \wedge \delta u_x$, as expected from Proposition 3.17. It is immediate to see that the covariant Poisson bracket between admissible 1-forms is anti-symmetric and bilinear in the space of admissible 1-forms. The Jacobi identity is a bit more cumbersome to verify, as we also need to show that the Poisson bracket of two admissible 1-forms is again an admissible 1-form, *i.e.*

$$\begin{aligned}
\frac{\partial}{\partial u_t}\left(\frac{\partial F_1}{\partial u_t}\frac{\partial G_1}{\partial u} - \frac{\partial F_1}{\partial u}\frac{\partial G_1}{\partial u_t}\right) &= \frac{\partial F_1}{\partial u_t}\frac{\partial^2 G_1}{\partial u_t \partial u} - \frac{\partial^2 F_1}{\partial u \partial u_t}\frac{\partial G_1}{\partial u_t} \\
&= \frac{\partial F_2}{\partial u_x}\frac{\partial^2 G_2}{\partial u_x \partial u} - \frac{\partial^2 F_2}{\partial u \partial u_x}\frac{\partial G_2}{\partial u_x} = \frac{\partial}{\partial u_x}\left(\frac{\partial F_2}{\partial u_x}\frac{\partial G_2}{\partial u} - \frac{\partial F_2}{\partial u}\frac{\partial G_2}{\partial u_x}\right)
\end{aligned}$$

where we have used the admissible properties of F and G. The Jacobi identity is then transferred from the Jacobi identities of $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$.

We now verify the validity of the covariant Hamilton equation in Poisson bracket form $dF = \{(h, F)\} dx \wedge dt$. The left hand-side is computed as

$$\begin{aligned}
dF &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial t}\right) dx \wedge dt \\
&= \left(\frac{\partial F_2}{\partial u}u_x + \frac{\partial F_2}{\partial u_x}u_{xx} - \frac{\partial F_1}{\partial u}u_t - \frac{\partial F_2}{\partial u_t}u_{tt}\right) dx \wedge dt \\
&= \left(\frac{\partial F_2}{\partial u}u_x - \frac{\partial F_1}{\partial u}u_t + \frac{\partial F_1}{\partial u_t}(u_{xx} - u_{tt})\right) dx \wedge dt
\end{aligned}$$

and the right hand-side is

$$\begin{aligned}
\xi_F \lrcorner \delta H &= \left(\frac{\partial F_1}{\partial u_t}\frac{\partial}{\partial u} - \frac{\partial F_2}{\partial u}\frac{\partial}{\partial u_x} - \frac{\partial F_1}{\partial u}\frac{\partial}{\partial u_t}\right) \lrcorner (u_t \delta u_t - u_x \delta u_x + V'(u)\delta u) \wedge dx \wedge dt \\
&= \left(\frac{\partial F_1}{\partial u_t}V'(u) + \frac{\partial F_2}{\partial u}u_x - \frac{\partial F_1}{\partial u}u_t\right) dx \wedge dt
\end{aligned}$$

and comparing the two we get $u_{xx} - u_{tt} = V'(u)$.

Chapter 4

Covariant Poisson brackets and classical r-matrix

In this chapter we illustrate applications of the theory explained in Chapter 3 to several integrable systems: the sine-Gordon equation (Section 4.1 and 4.2), the Non-Linear Schrödinger equation (Section 4.3), and the modified Korteweg-de Vries equation (Section 4.4), which is content from [CS20a], and the Zakharov-Mikhailov Lagrangian (Section 4.5), which is content from [CSV21b].

The starting point of the examples illustrated in this chapter is the Lagrangian form $\Lambda = L\, dx^1 \wedge dx^2$. We then use the procedure explained in Chapter 3 to introduce the following objects

- the multisymplectic form Ω,
- the covariant Hamiltonian $H = h\, dx^1 \wedge dx^2$,
- the covariant Poisson brackets $\{\!\!\{\ ,\ \}\!\!\}$,

to consistently obtain the classical r-matrix structure for the Lax connection $W(\lambda) = U(\lambda)\, dx^1 + V(\lambda)\, dx^2$ within the covariant Poisson brackets. In short, we provide the following result which is the covariant version of Sklyanin's fundamental discovery (2.12)

$$\{\!\!\{ W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu) \}\!\!\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]\,.$$

We also prove the covariant analog of the important fact that the zero-curvature condition for an integrable PDE can be cast in Hamiltonian form in the following way

$$dW(\lambda) = \{\!\!\{ H, W(\lambda) \}\!\!\}\, dx^1 \wedge dx^2 \quad \Longleftrightarrow \quad dW(\lambda) = W(\lambda) \wedge W(\lambda)\,.$$

Extension to $\mathfrak{gl}_N(\mathscr{A}^*)$ We naturally extend the formalism of the variational bi-complex to $\mathfrak{gl}_N(\mathscr{A}^*)$, *i.e.* $\mathscr{A}^*$-valued $\mathfrak{gl}_N$ matrices. Indeed, let $\{ E_{mn} \}$ be a basis for $\mathfrak{gl}_N$, then the

Lax connection (for a given λ) $W(\lambda) \in \mathfrak{gl}_N(\mathscr{A}^{(0,1)})$ can be written as[1] $\sum_{mn} W(\lambda)_{mn} E_{mn}$ where every $W_{mn}(\lambda) = \sum_i W^i_{mn}(\lambda) \in \mathscr{A}^{(0,1)}$ is a horizontal 1-form.
Note that the definition of an admissible form extends naturally to the case of matrix coefficients by requiring that each entry be an admissible form. Then, for each W_{mn} we can calculate its Hamiltonian vector field $\xi^{mn}_W \lrcorner \Omega = \delta W_{mn}$ and calculate the Poisson brackets between its coefficients. Moreover we define $\{\!\![W, F]\!\!\} := -\sum_{mn} (\xi^{mn}_W \lrcorner \delta F)\; E_{mn}$ for any admissible form F.
We also extend the tensor notation used in the Sklyanin bracket, as reviewed in Section 2.1, to the present situation as follows, denoting by $W(\lambda) = \sum_i W^i(\lambda) dx^i$

$$W_{\mathbf{1}}(\lambda) \equiv \sum_i W^i(\lambda) \otimes \mathbb{I}\, dx^i = \sum_i \sum_{mn} W^i_{mn}(\lambda)\, E_{mn} \otimes \mathbb{I}\, dx^i \equiv \sum_{mn} W_{mn}(\lambda) E_{mn} \otimes \mathbb{I}$$
$$W_{\mathbf{2}}(\lambda) \equiv \sum_i \mathbb{I} \otimes W^i(\lambda)\, dx^i = \sum_{i=0}^{\infty} \sum_{mn} W^i_{mn}(\lambda)\, \mathbb{I} \otimes E_{mn}\, dx^i \equiv \sum_{mn} W_{mn}(\lambda) \mathbb{I} \otimes E_{mn}\,.$$

We define the multi-time Poisson bracket between $W_{\mathbf{1}}(\lambda)$ and $W_2(\mu)$ by

$$\{\!\![W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\!\!\} = \sum_{m,n,k,\ell} \{\!\![W_{mn}(\lambda), W_{k\ell}(\mu)]\!\!\}\, E_{mn} \otimes E_{k\ell}\,. \tag{4.1}$$

Finally, we define the commutator of a matrix 0-form r and a matrix 1-form W by

$$[r, W] \equiv \sum_i [r, W^i]\, dx^i\,. \tag{4.2}$$

4.1 sine-Gordon equation in laboratory coordinates

The sine-Gordon model for the real scalar field $u(x,t)$ reads

$$u_{tt} - u_{xx} + \frac{m^2}{\beta} \sin \beta u = 0\,, \tag{4.3}$$

where m is the mass and β is the coupling constant. A Lagrangian form for it is given by

$$\Lambda = [\frac{1}{2}(u_t^2 - u_x^2) - \frac{m^2}{\beta^2}(1 - \cos \beta u)]\, dx \wedge dt\,. \tag{4.4}$$

Equation (4.3) is equivalent to the following zero-curvature equation which we set to hold as an identity in λ

$$\partial_t U(\lambda) - \partial_x V(\lambda) + [U(\lambda), V(\lambda)] = 0\,,$$

[1] The position of the indices indicating the coefficient of a matrix or of a differential form will not be important as it may change in the following depending of what makes the notation more understandable.

where the Lax pair (U,V) can be taken as

$$U(\lambda) = -ik_0(\lambda)\sin\frac{\beta u}{2}\sigma_1 - ik_1(\lambda)\cos\frac{\beta u}{2}\sigma_2 - \frac{i\beta}{4}u_t\sigma_3\,, \tag{4.5}$$

$$V(\lambda) = -ik_1(\lambda)\sin\frac{\beta u}{2}\sigma_1 - ik_0(\lambda)\cos\frac{\beta u}{2}\sigma_2 - \frac{i\beta}{4}u_x\sigma_3\,, \tag{4.6}$$

where $k_0(\lambda) = \frac{m}{4}(\lambda+\lambda^{-1})$ and $k_1(\lambda) = \frac{m}{4}(\lambda-\lambda^{-1})$. In the general notations of Section 2.2, here $N=1$ and the only field is $u_1 = u$. We will denote $u_k^{(i)}$, $(i) = (0,0)$, $(1,0)$, $(0,1)$, etc. as u, u_x, u_t, etc. for convenience. It is important to remember that u_x, u_t, etc. should be treated as coordinates in the differential algebra $\mathscr{A}$ when performing the calculations in the variational bi-complex.

Proposition 4.1 *The sine-Gordon equation* (4.3) *is the Euler-Lagrange equation for* Λ. *The form* $\Omega^{(1)}$ *is given by*

$$\Omega^{(1)} = -u_t\,\delta u\wedge dx - u_x\,\delta u\wedge dt\,. \tag{4.7}$$

and the multisymplectic form reads

$$\Omega = -\delta u_t\wedge\delta u\wedge dx - \delta u_x\wedge\delta u\wedge dt\,. \tag{4.8}$$

Proof. The δ-differential of Λ is

$$\delta\Lambda = [u_t\delta u_t - u_x\delta u_x - \frac{m^2}{\beta}\sin(\beta u)\delta u]\wedge dx\wedge dt.$$

Now, since $d(\delta u) = -\delta u_x\wedge dx - \delta u_t\wedge dt$, we get that $d(u_t\delta u\wedge dx) = u_{tt}dt\wedge\delta u\wedge dx + u_t d(\delta u)\wedge dx = u_{tt}\delta u\wedge dx\wedge dt + u_t\delta u_t\wedge dx\wedge dt$, and therefore

$$u_t\delta u_t\wedge dx\wedge dt = -u_{tt}\delta u\wedge dx\wedge dt + d(u_t\delta u\wedge dx),$$

and equivalently

$$-u_x\delta u_x\wedge dx\wedge dt = u_{xx}\delta u\wedge dx\wedge dt + d(u_x\delta u\wedge dt).$$

Therefore, the variation of Λ brings

$$\delta\Lambda = [-u_{tt} + u_{xx} - \frac{m^2}{\beta}\sin\beta u]\delta u\wedge dx\wedge dt + d(u_t\delta u\wedge dx + u_x\delta u\wedge dt).$$

By looking at $\frac{\delta\Lambda}{\delta u} = 0$ we obtain the Sine-Gordon equation. $\Omega^{(1)}$ then reads

$$\Omega^{(1)} = -u_t\delta u\wedge dx - u_x\delta u\wedge dt.$$

Its δ-differential $\delta\Omega^{(1)}$ is defined to be the multisymplectic form Ω

$$\Omega = \delta\Omega^{(1)} = -\delta u_t \wedge \delta u \wedge dx - \delta u_x \wedge \delta u \wedge dt\,. \tag{4.9}$$

$\square$

Equipped with the multisymplectic form Ω we can define the covariant Poisson bracket and also the two 'single-time' Poisson brackets as in Definition 3.14.

Proposition 4.2 *A 1-form $F = F_1(u, u_t)\, dx + F_2(u, u_x)\, dt$ is admissible for the multisymplectic form* (4.8) *if and only if*

$$\frac{\partial F_1}{\partial u_t} = \frac{\partial F_2}{\partial u_x}\,. \tag{4.10}$$

The respective Hamiltonian vector field is

$$\xi_F = \frac{\partial F_1}{\partial u_t}\frac{\partial}{\partial u} - \frac{\partial F_2}{\partial u}\frac{\partial}{\partial u_x} - \frac{\partial F_1}{\partial u}\frac{\partial}{\partial u_t}\,. \tag{4.11}$$

For any two admissible one-forms $F = A\, dx + B\, dt$ and $G = C\, dx + D\, dt$, we have following decomposition formula

$$\{\!(F, G)\!\} = \{A, C\}_1\, dx + \{B, D\}_2\, dt \tag{4.12}$$

where the single-time Poisson Brackets are given by

$$\{A, C\}_1 = \frac{\partial A}{\partial u_t}\frac{\partial C}{\partial u} - \frac{\partial A}{\partial u}\frac{\partial C}{\partial u_t}\,, \tag{4.13a}$$

$$\{B, D\}_2 = \frac{\partial D}{\partial u}\frac{\partial B}{\partial u_x} - \frac{\partial B}{\partial u}\frac{\partial D}{\partial u_x}. \tag{4.13b}$$

Proof. Let us consider the following (vertical) vector field

$$\xi_F = A\frac{\partial}{\partial u} + B\frac{\partial}{\partial u_x} + C\frac{\partial}{\partial u_t}$$

in the equation $\delta F = \xi_F \lrcorner \Omega$. The left hand-side reads

$$\delta F = \frac{\partial F_1}{\partial u}\delta u \wedge dx + \frac{\partial F_1}{\partial u_t}\delta u_t \wedge dx + \frac{\partial F_2}{\partial u}\delta u \wedge dt + \frac{\partial F_2}{\partial u_x}\delta u_x \wedge dt,$$

while the right hand-side is

$$\xi_F \lrcorner \Omega = A\delta u_t \wedge dx + A\delta u_x \wedge dt - B\delta u \wedge dt - C\delta u \wedge dx.$$

A direct comparison shows

$$A = \frac{\partial F_1}{\partial u_t} = \frac{\partial F_2}{\partial u_x}, \qquad B = -\frac{\partial F_2}{\partial u}, \qquad C = -\frac{\partial F_1}{\partial u}.$$

Then, (4.12) follows by a direct calculation from $\{\!(F,G)\!\} = -\xi_F \lrcorner \delta G$ and recognizing the single-time Poisson brackets as defined in the Proposition. □

Theorem 4.3 *The Lax form $W(\lambda) = U(\lambda)\,dx + V(\lambda)\,dt$ satisfies the following covariant Poisson bracket*

$$\{\!(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\!\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)] \tag{4.14}$$

where the classical r-matrix is that of the sine-Gordon model (see e.g. [FTR07])

$$r_{\mathbf{12}}(\lambda,\mu) = f(\lambda,\mu)(\mathbb{I}\otimes\mathbb{I} - \sigma_3\otimes\sigma_3) + g(\lambda,\mu)(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2), \tag{4.15}$$

with $f(\lambda,\mu) = -\frac{\beta^2}{16}\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}$ and $g(\lambda,\mu) = \frac{\beta^2}{8}\frac{\lambda\mu}{\lambda^2-\mu^2}$.

Proof. The proof is done by straightforward but long calculations. We give the details for this first example. We write $W(\lambda) = \sum_i W^i(\lambda)\sigma_i$, where $W^i(\lambda) = U^i(\lambda)\,dx + V^i(\lambda)\,dt$, so that

$$\begin{aligned}
W^1(\lambda) &= -ik_0(\lambda)\sin\frac{\beta u}{2}dx - ik_1(\lambda)\sin\frac{\beta u}{2}dt\,,\\
W^2(\lambda) &= -ik_1(\lambda)\cos\frac{\beta u}{2}dx - ik_0(\lambda)\cos\frac{\beta u}{2}dt\,,\\
W^3(\lambda) &= -\frac{i\beta}{4}u_t dx - \frac{i\beta}{4}u_x dt\,.
\end{aligned}$$

It can be checked that W^i, $i = 1,2,3$ are admissible forms. Therefore, using the decomposition property 4.12, we find that the only non-zero Poisson brackets are

$$\begin{aligned}
\{\!(W^1(\lambda), W^3(\mu))\!\} &= -\frac{\beta^2}{8}\cos\frac{\beta u}{2}(k_0(\lambda)dx + k_1(\lambda)dt),\\
\{\!(W^2(\lambda), W^3(\mu))\!\} &= \frac{\beta^2}{8}\sin\frac{\beta u}{2}(k_1(\lambda)dx + k_0(\lambda)dt),\\
\{\!(W^3(\lambda), W^1(\mu))\!\} &= \frac{\beta^2}{8}\cos\frac{\beta u}{2}(k_0(\mu)dx + k_1(\mu)dt),\\
\{\!(W^3(\lambda), W^2(\mu))\!\} &= -\frac{\beta^2}{8}\sin\frac{\beta u}{2}(k_1(\mu)dx + k_0(\mu)dt).
\end{aligned}$$

Thus we deduce, according to Definition 3.14, and using the auxiliary space notation as in Section 2.1,

$$\begin{aligned}
&\{\!(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\!\} \qquad (4.16)\\
&= \frac{\beta^2}{8}\Big(-\cos\frac{\beta u}{2}(k_0(\lambda)dx + k_1(\lambda)dt)\,\sigma_1\otimes\sigma_3 + \sin\frac{\beta u}{2}(k_1(\lambda)dx + k_0(\lambda)dt)\,\sigma_2\otimes\sigma_3\\
&\qquad + \cos\frac{\beta u}{2}(k_0(\mu)dx + k_1(\mu)dt)\,\sigma_3\otimes\sigma_1 - \sin\frac{\beta u}{2}(k_1(\mu)dx + k_0(\mu)dt)\,\sigma_3\otimes\sigma_2\Big)\,.
\end{aligned}$$

On the other hand, we can also compute $[r_{\mathbf{12}}(\lambda-\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]$ directly, using the commutation rules $[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and the property $[A\otimes\mathbb{I}, B\otimes C] = [A,B]\otimes C$. We find

$$
\begin{aligned}
&[r_{\mathbf{12}}(\lambda-\mu), W_{\mathbf{1}}(\lambda)+W_{\mathbf{2}}(\mu)] \\
&= [-f(\lambda,\mu)\sigma_3\otimes\sigma_3 + g(\lambda,\mu)\sigma_1\otimes\sigma_1 + g(\lambda,\mu)\sigma_2\otimes\sigma_2, W^1(\lambda)\sigma_1\otimes\mathbb{I} \\
&\qquad + W^2(\lambda)\sigma_2\otimes\mathbb{I} + W^3(\lambda)\sigma_3\otimes\mathbb{I} + W^1(\mu)\mathbb{I}\otimes\sigma_1 + W^2(\mu)\mathbb{I}\otimes\sigma_2 + W^3(\mu)\mathbb{I}\otimes\sigma_3] \\
&= -2i(f(\lambda,\mu)W^1(\lambda) + g(\lambda,\mu)W^1(\mu))\sigma_2\otimes\sigma_3 \\
&\quad + 2i(f(\lambda,\mu)W^2(\lambda) + g(\lambda,\mu)W^2(\mu))\sigma_1\otimes\sigma_3 \\
&\quad + 2i(f(\lambda,\mu)W^1(\mu) + g(\lambda,\mu)W^1(\lambda))\sigma_3\otimes\sigma_2 \\
&\quad - 2i(f(\lambda,\mu)W^2(\mu) + g(\lambda,\mu)W^2(\lambda))\sigma_3\otimes\sigma_1 \\
&\quad + 2i(g(\lambda,\mu)W^3(\mu) - g(\lambda,\mu)W^3(\lambda))\sigma_2\otimes\sigma_1 \\
&\quad + 2i(g(\lambda,\mu)W^3(\lambda) - g(\lambda,\mu)W^2(\mu))\sigma_1\otimes\sigma_2\,.
\end{aligned}
$$

Upon inserting the explicit expressions of W^i, f and g one recovers (4.16) and the claim is proved. □

We conclude this section on the sine-Gordon model with its covariant Hamiltonian formulation. The covariant Hamiltonian $H = h\,dx\wedge dt$ can be computed as $H = -\Lambda + dx\wedge\widetilde{\partial}_x\lrcorner\Omega^{(1)} + dt\wedge\widetilde{\partial}_t\lrcorner\Omega^{(1)}$ and its density is given by

$$h = \frac{1}{2}(u_t^2 - u_x^2) + \frac{m^2}{\beta^2}(1-\cos\beta u). \tag{4.17}$$

The corresponding Hamiltonian vector field ξ_h such that $\xi_h\lrcorner\Omega = \delta h$ can be taken as

$$\xi_h = u_t\frac{\partial}{\partial u}\wedge\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u}\wedge\frac{\partial}{\partial t} - \frac{m^2}{2\beta}\sin\beta u(\frac{\partial}{\partial u_t}\wedge\frac{\partial}{\partial x} + \frac{\partial}{\partial u_x}\wedge\frac{\partial}{\partial t}). \tag{4.18}$$

Let us now consider the Lax Form $W(\lambda) = U(\lambda)dx + V(\lambda)dt$. On the one hand, we have

$$
\begin{aligned}
dW(\lambda) =& ((-ik_1(\lambda)\cos\frac{\beta u}{2}u_x + ik_0(\lambda)\cos\frac{\beta u}{2}u_t)\sigma_1 \\
&+ (-ik_1(\lambda)\sin\frac{\beta u}{2}u_t + ik_0(\lambda)\sin\frac{\beta u}{2}u_x)\sigma_2 \\
&+ (\frac{i\beta}{4}u_{tt} - \frac{i\beta}{4}u_{xx})\sigma_3)dx\wedge dt
\end{aligned}
$$

and on the other hand,

$$\begin{aligned}\{(h, W(\lambda))\} =& \xi_H \lrcorner \delta W(\lambda) \\ =& \xi_H \lrcorner \Big(-i\frac{\beta k_0(\lambda)}{2}\cos\frac{\beta u}{2}\delta u\wedge dx - i\frac{k_1(\lambda)\beta}{2}\cos\frac{\beta u}{2}\delta u\wedge dt)\sigma_1 \\ & + (i\frac{k_1(\lambda)\beta}{2}\sin\frac{\beta u}{2}\delta u\wedge dx + i\frac{k_0(\lambda)\beta}{2}\sin\frac{\beta u}{2}\delta u\wedge dt)\sigma_2 \\ & -i\frac{\beta}{4}(\delta u_t\wedge dx + \delta u_x\wedge dt)\sigma_3\Big) \\ =& (\frac{i\beta}{2}(k_0(\lambda)u_t - k_1(\lambda)u_x)\cos\frac{\beta u}{2})\sigma_1 \\ & + (\frac{i\beta}{2}(k_0(\lambda)u_x - k_1(\lambda)u_t)\sin\frac{\beta u}{2})\sigma_2 - i\frac{m^2}{4}\sin\beta u\sigma_3.\end{aligned}$$

Therefore

$$dW(\lambda) = \{(h, W(\lambda))\}\, dx\wedge dt \quad \Leftrightarrow \quad u_{tt} - u_{xx} + \frac{m^2}{\beta}\sin\beta u = 0\,, \tag{4.19}$$

which is the desired covariant Hamiltonian form of the sine-Gordon equation. One can verify with a direct computation that $\{(h, W(\lambda))\} = [U(\lambda), V(\lambda)]$.

4.2 sine-Gordon equation in light-cone coordinates

We can also write the sine-Gordon equation (now we set $\beta = m = 1$ for simplicity) in light-cone coordinates $x^1 = \xi$ and $x^2 = \eta$ as

$$u_{\xi\eta} + \sin u = 0 \tag{4.20}$$

thanks to the change of coordinates $\xi = \frac{x+t}{2}$ and $\eta = \frac{t-x}{2}$. This equation is produced by the zero-curvature equation for the Lax form[2] $W(\lambda) = U(\lambda)\, d\xi + V(\lambda)\, d\eta$, where

$$U(\lambda) = -\frac{i}{4}\begin{pmatrix} u_\xi & \frac{2}{\lambda}e^{iu/2} \\ 2e^{-iu/2} & -u_\xi \end{pmatrix}, \qquad V(\lambda) = -\frac{i}{4}\begin{pmatrix} -u_\eta & 2e^{-iu/2} \\ 2\lambda e^{iu/2} & u_\eta \end{pmatrix}. \tag{4.21}$$

In fact we have

$$\begin{aligned}\partial_\xi V(\lambda) &= -\frac{i}{4}\begin{pmatrix} -u_{\eta\xi} & -iu_\xi e^{-iu/2} \\ i\lambda u_\xi e^{iu/2} & u_{\eta\xi} \end{pmatrix} \\ -\partial_\eta U(\lambda) &= \frac{i}{4}\begin{pmatrix} u_{\xi\eta} & \frac{iu_\eta}{\lambda}e^{iu/2} \\ -iu_\eta e^{-iu/2} & -u_{\xi\eta} \end{pmatrix} \\ [U(\lambda), V(\lambda)] &= \frac{1}{4}\begin{pmatrix} e^{-iu} - e^{iu} & -\frac{u_\eta}{\lambda}e^{iu/2} - u_\xi e^{-iu/2} \\ \lambda u_\xi e^{iu/2} + u_\eta e^{-iu/2} & e^{iu} - e^{-iu} \end{pmatrix}\end{aligned}$$

[2]We use the same symbols for the Lax matrices as the ones for the laboratory coordinates to avoid heavy notations.

and therefore $dW(\lambda) = W(\lambda) \wedge W(\lambda)$ is equivalent to (4.20). The sine-Gordon equation has also a Lagrangian formulation with

$$\Lambda = (\frac{1}{2}u_\xi u_\eta + \cos u) d\xi \wedge d\eta \,. \tag{4.22}$$

The following two propositions have proofs that are very similar to the laboratory coordinate case and therefore will be omitted.

Proposition 4.4 *The sine-Gordon equation in light-cone coordinates* (4.20) *is the Euler-Lagrange equation for* (4.22), *and with* $\Omega^{(1)} = -\frac{1}{2}u_\xi \delta u \wedge d\xi + \frac{1}{2}u_\eta \delta u \wedge d\eta$ *and*

$$\Omega = -\frac{1}{2}\delta u_\xi \wedge \delta u \wedge d\xi + \frac{1}{2}\delta u_\eta \wedge \delta u \wedge d\eta \,. \tag{4.23}$$

Proposition 4.5 *A 1-form* $F = F_1(u, u_\xi)\, d\xi + F_2(u, u_\eta)\, d\eta$ *is admissible for the multisymplectic form* (4.23) *if* $\frac{\partial F_1}{\partial u_\xi} = -\frac{\partial F_2}{\partial u_\eta}$ *with Hamiltonian vector field*

$$\xi_F = 2\frac{\partial F_1}{\partial u_\xi}\frac{\partial}{\partial u} - 2\frac{\partial F_1}{\partial u}\frac{\partial}{\partial u_\xi} + 2\frac{\partial F_2}{\partial u}\frac{\partial}{\partial u_\eta} \,. \tag{4.24}$$

The covariant Poisson bracket between two admissible 1-forms $F = F_1\, d\xi + F_2\, d\eta$ *and* $G = G_1\, d\xi + G_2\, d\eta$ *is* $\{\!(F, G)\!\} = \{F_1, G_1\}_\xi\, d\xi + \{F_2, G_2\}_\eta\, d\eta$ *where*

$$\{F_1, G_1\}_\xi = 2\left(\frac{\partial F_1}{\partial u}\frac{\partial G_1}{\partial u_\xi} - \frac{\partial F_1}{\partial u_\xi}\frac{\partial G_1}{\partial u}\right), \tag{4.25a}$$

$$\{F_2, G_2\}_\eta = 2\left(\frac{\partial F_2}{\partial u_\eta}\frac{\partial G_2}{\partial u} - \frac{\partial F_2}{\partial u}\frac{\partial G_2}{\partial u_\eta}\right). \tag{4.25b}$$

We are now ready to find the classical r-matrix within the covariant Poisson bracket $\{\!(\; , \;)\!\}$.

Theorem 4.6 *The Lax form* $W(\lambda) = U(\lambda)\, d\xi + V(\lambda)\, d\eta$ *satisfies the following covariant Poisson bracket*

$$\{\!(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\!\} = [r_{\mathbf{12}}(\lambda, \mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)] \,. \tag{4.26}$$

where the classical r-matrix is [S08, Eq. (4.22)]

$$r_{\mathbf{12}}(\lambda, \mu) = \frac{\mu + \lambda}{8(\mu - \lambda)}\left(\sigma_3 \otimes \sigma_3 + \frac{1}{2}\mathbb{I} \otimes \mathbb{I}\right) + \frac{\mu}{2(\mu - \lambda)}\sigma_+ \otimes \sigma_- + \frac{\lambda}{2(\mu - \lambda)}\sigma_- \otimes \sigma_+ \,. \tag{4.27}$$

Proof. We use the decomposition of the covariant Poisson bracket in $\{ \; , \; \}_\xi$ and $\{ \; , \; \}_\eta$. A direct calculation brings

$$
\begin{aligned}
\{U_{\mathbf{1}}(\lambda), U_{\mathbf{2}}(\mu)\}_\xi = & -\frac{1}{16}\Big(\{u_\xi, \frac{2}{\mu}e^{iu/2}\}_\xi \sigma_3 \otimes \sigma_+ + \{u_\xi, 2e^{-iu/2}\}_\xi \sigma_3 \otimes \sigma_- \\
& + \{\frac{2}{\lambda}e^{iu/2}, u_\xi\}_\xi \sigma_+ \otimes \sigma_3 + \{2e^{-iu/2}, u_\xi\}_\xi \sigma_- \otimes \sigma_3\Big) \\
= & \frac{ie^{iu/2}}{8}(\frac{\sigma_3 \otimes \sigma_+}{\mu} - \frac{\sigma_+ \otimes \sigma_3}{\lambda}) + \frac{ie^{-iu/2}}{8}(\sigma_- \otimes \sigma_3 - \sigma_3 \otimes \sigma_-)
\end{aligned}
$$

and an explicit calculation shows that this is equal to

$$
[\frac{\mu+\lambda}{8(\mu-\lambda)}\sigma_3 \otimes \sigma_3 + \frac{\mu}{2(\mu-\lambda)}\sigma_+ \otimes \sigma_- + \frac{\lambda}{2(\mu-\lambda)}\sigma_- \otimes \sigma_+, U(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes U(\mu)]\,.
$$

Similarly we obtain for the $d\eta$ coefficient

$$
\{V_{\mathbf{1}}(\lambda), V_{\mathbf{2}}(\mu)\}_\eta = [r_{\mathbf{12}}(\lambda,\mu), V_{\mathbf{1}}(\lambda) + V_{\mathbf{2}}(\mu)]\,. \qquad \square
$$

We can also show the covariant Hamiltonian nature of the zero-curvature equation. The covariant Hamiltonian form can be obtained as

$$
H = (\frac{1}{2}u_\xi u_\eta - \cos u)\, d\xi \wedge d\eta\,. \tag{4.28}
$$

The covariant Hamiltonian density $h = \frac{1}{2}u_\xi u_\eta - \cos u$ is an admissible 0-form with Hamiltonian vector field

$$
\xi_h = -u_\xi \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \eta} + u_\eta \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \xi} + 2\cos u \frac{\partial}{\partial u_\xi} \wedge \frac{\partial}{\partial \xi}\,. \tag{4.29}
$$

Applying Theorem 3.15 we know that, since $W(\lambda) = U(\lambda)\, d\xi + V(\lambda)\, d\eta$ is admissible,

$$
dW(\lambda) = \{\!\!\{h, W(\lambda)\}\!\!\}\, d\xi \wedge d\eta\,, \tag{4.30}
$$

on the sine-Gordon equation $dW = W \wedge W$. One can verify with a direct computation that $\{\!\!\{h, W(\lambda)\}\!\!\} = [U(\lambda), V(\lambda)]$, so that

$$
dW(\lambda) = \{\!\!\{h, W(\lambda)\}\!\!\}\, d\xi \wedge d\eta \quad \Longleftrightarrow \quad dW(\lambda) = W(\lambda) \wedge W(\lambda)\,. \tag{4.31}
$$

Remark 4.7: Unlike the other examples of this chapter, the sine-Gordon equation in light-cone coordinates is original of this thesis.

4.3 Non-Linear Schrödinger equation

By a slight abuse of language, we will call the following system of equations for two complex scalar fields q, r the nonlinear Schrödinger (NLS) equation

$$iq_t + \frac{1}{2}q_{xx} - q^2 r = 0\,, \qquad ir_t - \frac{1}{2}r_{xx} + r^2 q = 0\,. \tag{4.32}$$

Strictly speaking, the NLS appears under the reduction $r = \pm q^*$. We keep using as a Lagrangian volume form for (4.32)

$$\Lambda = (\frac{i}{2}(q_t r - q r_t) - \frac{1}{2}q_x r_x - \frac{1}{2}q^2 r^2)\, dx \wedge dt, \tag{4.33}$$

The system (4.32) is equivalent to the zero-curvature equation which must hold as an identity in λ

$$\partial_t U(\lambda) - \partial_x V(\lambda) + [U(\lambda), V(\lambda)] = 0\,.$$

where the Lax pair (U, V) can be taken as

$$U(\lambda) = -i\lambda\sigma_3 + q\sigma_+ + r\sigma_-\,, \tag{4.34}$$

$$V(\lambda) = \left(-i\lambda^2 - \frac{i}{2}qr\right)\sigma_3 + (\lambda q + \frac{i}{2}q_x)\sigma_+ + (\lambda r - \frac{i}{2}r_x)\sigma_-\,. \tag{4.35}$$

We will denote $u_k^{(i)}$, $k = 1, 2$, $(i) = (0, 0)$, $(1, 0)$, etc. as q, r, q_x, r_x, etc. for convenience.

Proposition 4.8 *The NLS equations* (4.32) *are the Euler-Lagrange equations for* Λ. *The form* $\Omega^{(1)}$ *is given by*

$$\Omega^{(1)} = \frac{i}{2}(q\delta r - r\delta q) \wedge dx - \frac{1}{2}(q_x \delta r + r_x \delta q) \wedge dt\,, \tag{4.36}$$

and the multisymplectic form reads

$$\Omega = i\delta q \wedge \delta r \wedge dx + (\frac{1}{2}\delta r \wedge \delta q_x + \frac{1}{2}\delta q \wedge \delta r_x) \wedge dt\,. \tag{4.37}$$

Proof. The δ-differential of the Lagrangian volume form is

$$\begin{aligned}\delta\Lambda = &\left(\frac{ir_t}{2} - qr^2\right)\delta q \wedge dx \wedge dt + \left(\frac{iq_t}{2} - q^2 r\right)\delta r \wedge dx \wedge dt \\ &+ \frac{ir}{2}\delta q_t \wedge dx \wedge dt - \frac{iq}{2}\delta r_t \wedge dx \wedge dt \\ &- \frac{1}{2}r_x \delta q_x \wedge dx \wedge dt - \frac{1}{2}q_x \delta r_x \wedge dx \wedge dt\,.\end{aligned}$$

Then, using

$$\begin{aligned}
\frac{ir}{2}\delta q_t \wedge dx \wedge dt &= d(\frac{ir}{2}\delta q \wedge dx) - \frac{ir_t}{2}\delta q \wedge dx \wedge dt\,, \\
-\frac{iq}{2}\delta r_t \wedge dx \wedge dt &= d(-\frac{iq}{2}\delta r \wedge dx) + \frac{iq_t}{2}\delta r \wedge dx \wedge dt\,, \\
-\frac{1}{2}r_x\delta q_x \wedge dx \wedge dt &= d(\frac{1}{2}r_x\delta q \wedge dt) + \frac{1}{2}r_{xx}\delta q \wedge dx \wedge dt\,, \\
-\frac{1}{2}q_x\delta r_x \wedge dx \wedge dt &= d(\frac{1}{2}q_x\delta r \wedge dt) + \frac{1}{2}q_{xx}\delta r \wedge dx \wedge dt\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
\delta\Lambda =&[(-ir_t + \frac{1}{2}r_{xx} - qr^2)\delta q + (iq_t + \frac{1}{2}q_{xx} - q^2 r)\delta r] \wedge dx \wedge dt \\
&- d((-\frac{ir}{2}\delta q + \frac{iq}{2}\delta r) \wedge dx + (-\frac{1}{2}r_x\delta q - \frac{1}{2}q_x\delta r) \wedge dt)
\end{aligned}$$

from which we can read off $\Omega^{(1)}$. We then compute $\Omega = \delta\Omega^{(1)}$ to get the stated result. $\square$

Proposition 4.9 *A 1-form $F = F_1(q,r)\,dx + F_2(q,r,q_x,r_x)\,dt$ is admissible for the multisymplectic form* (4.37) *if and only if*

$$\frac{\partial F_1}{\partial q} = -2i\frac{\partial F_2}{\partial q_x}, \qquad \frac{\partial F_1}{\partial r} = 2i\frac{\partial F_2}{\partial r_x}. \tag{4.38}$$

The respective Hamiltonian vector field is

$$\xi_F = -i\frac{\partial F_1}{\partial r}\frac{\partial}{\partial q} + i\frac{\partial F_1}{\partial q}\frac{\partial}{\partial r} - 2\frac{\partial F_2}{\partial r}\frac{\partial}{\partial q_x} - 2\frac{\partial F_2}{\partial q}\frac{\partial}{\partial r_x}. \tag{4.39}$$

Any two admissible 1-forms $F = A\,dx + B\,dt$ and $G = C\,dx + D\,dt$ satisfy the equation

$$\{\!(F,G)\!\} = \{A,C\}_1\,dx + \{B,D\}_2\,dt \tag{4.40}$$

where the single-time Poisson Brackets are given by

$$\{A,C\}_1 = -i\frac{\partial A}{\partial q}\frac{\partial C}{\partial r} + i\frac{\partial C}{\partial q}\frac{\partial A}{\partial r}\,, \tag{4.41a}$$

$$\{B,D\}_2 = 2\left(\frac{\partial B}{\partial q}\frac{\partial D}{\partial r_x} - \frac{\partial D}{\partial q}\frac{\partial B}{\partial r_x} + \frac{\partial B}{\partial r}\frac{\partial D}{\partial q_x} - \frac{\partial D}{\partial r}\frac{\partial B}{\partial q_x}\right)\,. \tag{4.41b}$$

Proof. We start from the Ansatz $\xi_F = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial r} + c\frac{\partial}{\partial q_x} + d\frac{\partial}{\partial r_x}$, and we want to find the coefficients by setting

$$\xi_F \lrcorner \Omega = \delta F. \tag{4.42}$$

The right hand-side reads

$$\delta F = \frac{\partial F_2}{\partial q}\delta q \wedge dt + \frac{\partial F_2}{\partial r}\delta r \wedge dt + \frac{\partial F_1}{\partial q_x}\delta q_x \wedge dt + \frac{\partial F_2}{\partial r_x}\delta r_x \wedge dt$$
$$+ \frac{\partial F_1}{\partial q}\delta q \wedge dx + \frac{\partial F_1}{\partial r}\delta r \wedge dx,$$

while the left hand-side is

$$\xi_F \lrcorner \Omega = ia\delta r \wedge dx + \frac{1}{2}a\delta r_x \wedge dt - ib\delta q \wedge dx + \frac{1}{2}b\delta q_x \wedge dt - \frac{1}{2}c\delta r \wedge dt - \frac{1}{2}d\delta q \wedge dt.$$

By matching the coefficients we get

$$a = -i\frac{\partial F_1}{\partial r} = 2\frac{\partial F_2}{\partial r_x}\,, \quad b = i\frac{\partial F_1}{\partial q} = 2\frac{\partial F_2}{\partial q_x}\,, \quad c = -2\frac{\partial F_2}{\partial r}\,, \quad d = -2\frac{\partial F_2}{\partial q}\,,$$

which is the first statement. The second statement then follows by a direct calculation from $\{(F,G)\} = -\xi_F \lrcorner \delta G$ and recognizing the single-time Poisson brackets as defined in the Proposition. □

Theorem 4.10 *The Lax form $W(\lambda) = U(\lambda)\,dx + V(\lambda)\,dt$ satisfies the following covariant Poisson bracket*

$$\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)] \tag{4.43}$$

where the classical r-matrix is that of the NLS equation (see e.g. [FTR07] and Section 2.1), the so-called rational r-matrix,

$$r_{\mathbf{12}}(\lambda,\mu) = \frac{P_{\mathbf{12}}}{2(\mu-\lambda)} = \frac{1}{2(\mu-\lambda)}(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ + \frac{\sigma_3 \otimes \sigma_3}{2} + \frac{\mathbb{I}\otimes\mathbb{I}}{2})\,. \tag{4.44}$$

Proof. Again, we give here the proof by direct computation. We write $W_{\mathbf{1}}(\lambda) = W^3(\lambda)\sigma_3 \otimes \mathbb{I} + W^+(\lambda)\sigma_+ \otimes \mathbb{I} + W^-(\lambda)\sigma_- \otimes \mathbb{I}$ and $W_{\mathbf{2}}(\mu) = W^3(\mu)\mathbb{I}\otimes\sigma_3 + W^+(\mu)\mathbb{I}\otimes\sigma_+ + W^-(\mu)\mathbb{I}\otimes\sigma_-$. For the right-hand side, we find

$$\begin{aligned}
&[r_{\mathbf{12}}(\lambda-\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)] \\
&= \frac{1}{2(\mu-\lambda)}[(2W^3(\mu) - 2W^3(\lambda))\sigma_+ \otimes \sigma_- + (W^-(\lambda) - W^-(\mu))\sigma_3 \otimes \sigma_- \\
&\qquad + (W^+(\lambda) - W^+(\mu))\sigma_+ \otimes \sigma_3 + (2W^3(\lambda) - 2W^3(\mu))\sigma_- \otimes \sigma_+ \\
&\qquad\quad + (W^+(\mu) - W^+(\lambda))\sigma_3 \otimes \sigma_+ + (W^-(\mu) - W^-(\lambda))\sigma_- \otimes \sigma_3] \\
&= -i(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)dx \\
&\quad + (-i(\mu+\lambda)(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) - \frac{r}{2}(\sigma_3 \otimes \sigma_- - \sigma_- \otimes \sigma_3) \\
&\qquad + \frac{q}{2}(\sigma_3 \otimes \sigma_+ - \sigma_+ \otimes \sigma_3))dt\,.
\end{aligned} \tag{4.45}$$

For the left-hand side, note that $W^3(\lambda)$, $W^+(\lambda)$ and $W^-(\lambda)$ are admissible forms. Thus, a direct calculation using the decomposition formula shows that the only nonzero covariant

Poisson bracket relations are the following

$$\begin{aligned}
\{\!\!\{ W^+(\lambda), W^-(\mu) \}\!\!\} =& - idx - i(\lambda+\mu)dt\,,\\
\{\!\!\{ W^+(\lambda), W^3(\mu) \}\!\!\} =& - \frac{q}{2}dt\,,\\
\{\!\!\{ W^-(\lambda), W^+(\mu) \}\!\!\} =& idx + i(\lambda+\mu)dt\,,\\
\{\!\!\{ W^-(\lambda), W^3(\mu) \}\!\!\} =& \frac{r}{2}dt\,,\\
\{\!\!\{ W^3(\lambda), W^+(\mu) \}\!\!\} =& \frac{q}{2}dt\,,\\
\{\!\!\{ W^3(\lambda), W^-(\mu) \}\!\!\} =& - \frac{r}{2}dt\,.
\end{aligned}$$

It remains to insert in the definition of $\{\!\!\{ W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu) \}\!\!\}$ to recognize that $\{\!\!\{ W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu) \}\!\!\}$ is precisely (4.45). □

We conclude the NLS example by a description of its covariant Hamiltonian formulation. The covariant Hamiltonian $H = h\, dx \wedge dt$ is given by

$$h = \frac{1}{2}(-q_x r_x + q^2 r^2). \tag{4.46}$$

Its Hamiltonian vector field ξ_h, such that $\xi_h \lrcorner \Omega = \delta h$ can be taken as

$$\xi_h = \left(-iq^2 r\frac{\partial}{\partial q} + iqr^2\frac{\partial}{\partial r}\right) \wedge \frac{\partial}{\partial x} + \left(-q_x\frac{\partial}{\partial q} - r_x\frac{\partial}{\partial r}\right) \wedge \frac{\partial}{\partial t}\,. \tag{4.47}$$

Equipped with this, we have the following result.

Proposition 4.11 *The covariant Hamiltonian formulation of the NLS equation is given by*

$$dW(\lambda) = \{\!\!\{ h, W(\lambda) \}\!\!\}\, dx \wedge dt\,, \tag{4.48}$$

where $W(\lambda)$ is the Lax Form.

Proof. On the one hand

$$dW(\lambda) = (-\frac{i}{2}(qr_x + rq_x)\sigma_3 + (-q_t + \lambda q_x + \frac{i}{2}q_{xx})\sigma_+ + (-r_t + \lambda r_x - \frac{i}{2}r_{xx})\sigma_-)dx \wedge dt\,,$$

while on the other hand,

$$\begin{aligned}
\{\!\!\{ h, W(\lambda) \}\!\!\} =& \xi_h \lrcorner \delta W(\lambda)\\
=& \xi_h \lrcorner (\sigma_+ \delta q \wedge dx + \sigma_- \delta r \wedge dx + (-\frac{i}{2} r\sigma_3 + \lambda\sigma_+)\delta q \wedge dt\\
& + (-\frac{i}{2} q\sigma_3 + \lambda\sigma_-)\delta r \wedge dt + \frac{i}{2}\sigma_+ \delta q_x \wedge dt - \frac{i}{2}\sigma_- \delta r_x \wedge dt\\
=& (iq^2 r + \lambda q_x)\sigma_+ + (-iqr^2 + \lambda r_x)\sigma_- - \frac{i}{2}(q_x r + qr_x)\sigma_3\,.
\end{aligned}$$

Therefore $dW(\lambda) = \{(h, W(\lambda))\}dx \wedge dt$ is equivalent to the NLS equation. □

One can verify with direct computation that $\{(h, W(\lambda))\} = [U(\lambda), V(\lambda)]$.

4.4 Modified Korteweg-de Vries equation

By a slight abuse of language, we call the following system of equations for two complex scalar fields q, r the modified Korteweg-de Vries (mKdV) equation,

$$q_t + \frac{1}{4}q_{xxx} - \frac{3}{2}qrq_x = 0\,, \qquad r_t + \frac{1}{4}r_{xxx} - \frac{3}{2}qrr_x = 0\,. \tag{4.49}$$

It is the next commuting flow in the so-called AKNS hierarchy [AKNS74] that also contains the NLS system (4.32). The original (real) modified KdV equation is obtained as the real reduction $r = q$ with q a real-valued field. A Lagrangian form for (4.49) is given by

$$\Lambda = (\frac{i}{2}(q_t r - qr_t) - \frac{i}{8}(q_{xx}r_x - r_{xx}q_x) - \frac{3i}{8}qr(q_x r_- qr_x))dx \wedge dt\,. \tag{4.50}$$

Remark 4.12: The reader may find the presence of an overall multiplicative constant i unnecessary or even confusing. This is only done at this stage for internal consistency with the rest of the thesis.

The system (4.49) is equivalent to the zero-curvature equation which must hold as an identity in λ

$$\partial_t U(\lambda) - \partial_x V(\lambda) + [U(\lambda), V(\lambda)] = 0 \tag{4.51}$$

where the Lax pair (U, V) can be taken as

$$U(\lambda) = -\,i\lambda\sigma_3 + q\sigma_+ + r\sigma_-\,, \tag{4.52}$$

$$\begin{aligned} V(\lambda) =& (-i\lambda^3 - \frac{i\lambda}{2}qr + \frac{1}{4}(q_x r - qr_x))\sigma_3 \\ &+ (\lambda^2 q + i\lambda q_x - \frac{1}{4}q_{xx} + \frac{1}{2}q^2 r)\sigma_+ + (\lambda^2 r - \frac{i\lambda}{2}r_x - \frac{1}{4}r_{xx} + \frac{1}{2}qr^2)\sigma_-\,. \end{aligned} \tag{4.53}$$

One reason for looking at this model, besides its physical relevance as a prototypical model related to the famous Korteweg-de Vries equation[3], is that it is degenerate both in the standard Legendre transformation and the dual one [ACDK16]. However, the method laid out by Dickey produces a multisymplectic form that is not sensitive to the degeneracy and both single-time forms are indeed symplectic (nondegenerate). In fact, they coincide with the ones obtained by the Dirac procedure in [ACDK16]. This feature is quite remarkable but its origin is not understood yet.

[3]This is obtained by a Miura transformation [M68].

Proposition 4.13 *The mKdV equations* (4.49) *are the Euler-Lagrange equations for* Λ. *The form* $\Omega^{(1)}$ *is given by*

$$\begin{aligned}\Omega^{(1)} =& \frac{i}{2}(q\delta r - r\delta q) \wedge dx \\ &+ \left(\left(\frac{i}{4} r_{xx} - \frac{3i}{8} qr^2 \right) \delta q + \left(-\frac{i}{4} q_{xx} + \frac{3i}{8} q^2 r \right) \delta r - \frac{i}{8} r_1 \delta q_1 + \frac{i}{8} q_1 \delta r_1 \right) \wedge dt \,,\end{aligned} \tag{4.54}$$

and the multisymplectic form reads

$$\Omega = i\delta q \wedge \delta r \wedge dx + \left(\frac{i}{4} \delta r_{xx} \wedge \delta q - \frac{i}{4} \delta q_{xx} \wedge \delta r + \frac{i}{4} \delta q_x \wedge \delta r_x + \frac{3iqr}{2} \delta q \wedge \delta r \right) \wedge dt \,. \tag{4.55}$$

Proof. By direct calculation as in the previous examples. □

Proposition 4.14 *A 1-form*

$$F = F_1(q, r)dx + F_2(q, r, q_x, r_x, q_{xx}, r_{xx})dt \,,$$

is admissible for the multisymplectic form (4.55) *if and only if*

$$\frac{\partial F_1}{\partial q} = -4 \frac{\partial F_2}{\partial q_{xx}} \,, \qquad \frac{\partial F_1}{\partial r} = -4 \frac{\partial F_2}{\partial r_{xx}} \,. \tag{4.56}$$

The corresponding Hamiltonian vector field is

$$\begin{aligned}\xi_F =& - i \frac{\partial F_1}{\partial r} \frac{\partial}{\partial q} + i \frac{\partial F_1}{\partial q} \frac{\partial}{\partial r} - 4i \frac{\partial F_2}{\partial r_x} \frac{\partial}{\partial q_x} + 4i \frac{\partial F_2}{\partial q_x} \frac{\partial}{\partial r_x} \\ &+ 4i (\frac{\partial F_2}{\partial r} + 6qr \frac{\partial F_2}{\partial r_{xx}}) \frac{\partial}{\partial q_{xx}} - 4i (\frac{\partial F_2}{\partial q} + 6qr \frac{\partial F_2}{\partial q_{xx}}) \frac{\partial}{\partial r_{xx}} \,.\end{aligned} \tag{4.57}$$

Any two admissible 1-forms $F = A\,dx + B\,dt$ *and* $G = C\,dx + D\,dt$ *satisfy the equation*

$$\{\!(F, G)\!\} = \{A, C\}_1 \,dx + \{B, D\}_2 \,dt \tag{4.58}$$

where the single-time Poisson Brackets are given by

$$\{A, C\}_1 = - i \left(\frac{\partial A}{\partial q} \frac{\partial C}{\partial r} - \frac{\partial C}{\partial q} \frac{\partial A}{\partial r} \right) \,, \tag{4.59a}$$

$$\begin{aligned}\{B, D\}_2 = - 4i \Bigg(& \frac{\partial B}{\partial r_{xx}} \frac{\partial D}{\partial q} - \frac{\partial B}{\partial q} \frac{\partial D}{\partial r_{xx}} - \frac{\partial B}{\partial q_{xx}} \frac{\partial D}{\partial r} + \frac{\partial B}{\partial r} \frac{\partial D}{\partial q_{xx}} \\ & \frac{\partial B}{\partial q_x} \frac{\partial D}{\partial r_x} - \frac{\partial B}{\partial r_x} \frac{\partial D}{\partial q_x} - 6qr \left(\frac{\partial B}{\partial q_{xx}} \frac{\partial D}{\partial r_{xx}} - \frac{\partial B}{\partial r_{xx}} \frac{\partial D}{\partial q_{xx}} \right) \Bigg) \,.\end{aligned} \tag{4.59b}$$

Proof. Inserting $\xi_F = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial r} + c\frac{\partial}{\partial q_x} + d\frac{\partial}{\partial r_x} + e\frac{\partial}{\partial q_{xx}} + f\frac{\partial}{\partial r_{xx}}$ into

$$\xi_F \lrcorner \Omega \,, \tag{4.60}$$

and matching the coefficients with δF. This gives the first statement. The second statement then follows by a direct calculation from $\{\!(F, G)\!\} = -\xi_F \lrcorner \delta G$ and recognizing the single-time Poisson brackets as defined in the Proposition. $\square$

Theorem 4.15 *The Lax form $W(\lambda) = U(\lambda)\,dx + V(\lambda)\,dt$ satisfies the following covariant Poisson bracket*

$$\{\!(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\!\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)] \tag{4.61}$$

where r is the rational classical r-matrix of the NLS equation.

$$r_{\mathbf{12}}(\lambda-\mu) = \frac{1}{2(\mu-\lambda)}(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ + \frac{\sigma_3 \otimes \sigma_3}{2} + \frac{\mathbb{I}\otimes\mathbb{I}}{2})\,. \tag{4.62}$$

Proof. The direct calculation follows exactly the same idea as before. $\square$

> **Remark 4.16:** A comment is in order regarding the fact that the same r-matrix as for the NLS appears here for the mKdV. In the standard Hamiltonian approach to the AKNS hierarchy, the only r-matrix structure is that given in (4.44) since all the higher flows share the same $U(\lambda)$ matrix $Q^{(1)}(\lambda)$. In our covariant context, since the same r-matrix appears for both the Lax matrices $U(\lambda)$ and $V(\lambda) = Q^{(2)}(\lambda)$ for the NLS and $V(\lambda) = Q^{(3)}(\lambda)$ for the mKdV and since both flows share the same $U(\lambda)$, we consistently find that the same r-matrix appears in the covariant Poisson structure for NLS and mKdV. We note however that this points to a deeper connection between our covariant approach and the notion of integrable hierarchies. Amazingly, this connection holds and was established in [CS21]. This will be presented in detail in Chapter 6.

We conclude the mKdV example by a description of its covariant Hamiltonian formulation. We find that the covariant Hamiltonian $H = h\,dx\wedge dt$ is given by

$$h = \frac{i}{4}(q_x r_{xx} - q_{xx} r_x) \tag{4.63}$$

Its Hamiltonian vector field ξ_H, such that $\xi_h \lrcorner \Omega = \delta h$ can be taken as

$$\begin{aligned}\xi_h =& \frac{3}{2}qr\left(q_x\frac{\partial}{\partial q} + r_x\frac{\partial}{\partial r}\right)\wedge\frac{\partial}{\partial x}\\ &-\left(q_x\frac{\partial}{\partial q} + r_x\frac{\partial}{\partial r} + q_{xx}\frac{\partial}{\partial q_x} + r_{xx}\frac{\partial}{\partial r_x}\right)\wedge\frac{\partial}{\partial t}\,.\end{aligned} \tag{4.64}$$

Equipped with this, we have the following result.

Proposition 4.17 *The covariant Hamiltonian formulation of the NLS equation is given by*

$$dW(\lambda) = \{\!(h, W(\lambda))\!\}\,dx\wedge dt\,, \tag{4.65}$$

where $W(\lambda)$ is the Lax form.

Proof. By direct computation as in the previous examples. $\square$

In the same way as in the two previous examples, one can show that $\{(h, W(\lambda))\} = [U(\lambda), V(\lambda)]$.

4.5 Zakharov-Mikhailov Lagrangian

The Zakharov-Mikhailov Lagrangian provides a variational principle for a class of integrable systems encapsulated by the Lax connection of Zakharov-Shabat type $W(z) = U(z)\, d\xi + V(z)\, d\eta$, where

$$U(z) = U_0 + \sum_{m=1}^{N_1} \frac{U_m}{z - a_m}\,, \qquad V(z) = V_0 + \sum_{n=1}^{N_2} \frac{V_n}{z - b_n}\,, \tag{4.66}$$

and each $U_m, V_n \in \mathfrak{gl}_N(\mathscr{A})$ are $\mathscr{A}$-valued $\mathfrak{gl}_N$ matrices. We also assume that $a_m \neq b_n$ $\forall m = 1, \dots, N_1$ $\forall n = 1, \dots, N_2$. By taking the residues in a_m and b_n, we can see that the zero-curvature condition $dW(z) = W(z) \wedge W(z)$, or equivalently $\partial_\xi V(z) - \partial_\eta U(z) = [U(z), V(z)]$ is also equivalent to the following equations, for $m = 1, \dots, N_1$ and $n = 1, \dots, N_2$

$$\begin{aligned} &\partial_\xi V_0 - \partial_\eta U_0 = [U_0, V_0]\,, \\ &\partial_\eta U_m = \left[V_0 + \sum_{n=1}^{N_2} \frac{V_n}{a_m - b_n}, U_m \right], \qquad \partial_\xi V_n = \left[U_0 + \sum_{m=1}^{N_1} \frac{U_m}{b_n - a_m}, V_n \right]. \end{aligned} \tag{4.67}$$

These are obtained by taking the regular part in z or the residues in $z = a_m$ or $z = b_n$ of $dW = W \wedge W$. In [ZM80], the authors proved that these equations have a variational origin, *i.e.* they are Euler-Lagrange equations of the Lagrangian form Λ. In the case where $U_0 = V_0 = 0$ this Lagrangian can be written as

$$\Lambda = \mathrm{Tr} \left(\sum_m \varphi_m^{-1} \partial_\eta \varphi_m U_m^{(0)} - \sum_n \psi_n^{-1} \partial_\xi \psi_n V_n^{(0)} - \sum_{m,n} \frac{U_m V_n}{a_m - b_n} \right) d\xi \wedge d\eta\,, \tag{4.68}$$

where in each sum, $m = 1, \dots, N_1$ and $n = 1, \dots, N_2$, and we have written each $U_m = \varphi_m U_m^{(0)} \varphi_m^{-1}$ and $V_n = \psi_n V_n^{(0)} \psi_n^{-1}$. The matrices $\varphi_m, \psi_n \in GL_N(\mathscr{A})$ (*i.e.* $\mathscr{A}$-valued non-singular $N \times N$ matrices) are dynamical, and they contain the fields of our theory and their derivatives. The matrices $U_m^{(0)}$ and $V_n^{(0)}$ are non-dynamical, meaning $\delta U_m^{(0)} = \delta V_n^{(0)} = 0$, but in general they may depend on the space time variables (ξ, η). However, to avoid some technical difficulties we will consider $U_m^{(0)}$ and $V_n^{(0)}$ to be constant, meaning that $U_m^{(0)}, V_n^{(0)} \in \mathfrak{gl}_N$.

Remark 4.18: We have set $U_0 = V_0 = 0$, which can be done thanks to the gauge freedom of U and V. In fact, from (4.67) there exists a non-singular matrix $g \in GL_N(\mathscr{A})$ such that

$$U_0 = \partial_\xi g g^{-1}\,, \qquad V_0 = \partial_\eta g g^{-1}\,. \tag{4.69}$$

In fact we have

$$\begin{aligned}\partial_\xi V_0 - \partial_\eta U_0 =& \partial_\xi(\partial_\eta g g^{-1}) - \partial_\eta(\partial_\xi g g^{-1})\\ =& \partial_\xi\partial_\eta g^{-1} - \partial_\eta g g^{-1}\partial_\xi g^{-1} - \partial_\eta\partial_\xi g g^{-1} + \partial_\xi g g^{-1}\partial_\eta g g^{-1}\\ =& [\partial_\xi g g^{-1}, \partial_\eta g g^{-1}]\\ =& [U_0, V_0]\,.\end{aligned}$$

It follows that the matrices

$$\begin{aligned}\bar{U} = g^{-1}Ug - g^{-1}(\partial_\xi g) = \sum_{k=1}^{N_1}\frac{g^{-1}U_k g}{z-a_k} \in \mathfrak{gl}_N(\mathscr{A})\,,\\ \bar{V} = g^{-1}Vg - g^{-1}(\partial_\eta g) = \sum_{k=1}^{N_2}\frac{g^{-1}V_k g}{z-b_k} \in \mathfrak{gl}_N(\mathscr{A})\end{aligned}$$

also satisfy $\partial_\xi\bar{V} - \partial_\eta\bar{U} = [\bar{U}, \bar{V}]$. Then one can take $\varphi_k \to g\varphi_k$ and $\psi_k \to g\psi_k$ and rename $\bar{U} \to U$ and $\bar{V} \to V$ to eliminate U_0 and V_0.

Remark 4.19: It was proved in [CSV21a] that the Lagrangian Λ can be obtained from a $4d$ Chern-Simons theory, see Appendix A.4. This is a result that follows the idea of [FSY20, CY19] with the introduction of minimally-coupled surface defects on the Riemann sphere that provides an additional family of models that can be derived from a $4d$ Chern-Simons theory.

Proposition 4.20 *The Euler-Lagrange equations of* Λ *are* (4.67) *with* $U_0 = V_0 = 0$ *i.e.*

$$\partial_\eta U_m + \left[U_m, \sum_{n=1}^{N_2}\frac{V_n}{a_m-b_n}\right] = 0 \qquad \partial_\xi V_n + \left[V_n, \sum_{m=1}^{N_1}\frac{U_m}{b_n-a_m}\right] = 0\,. \tag{4.70}$$

and the multisymplectic form is

$$\Omega = \mathrm{Tr}\left(\sum_{m=1}^{N_1}\varphi_m^{-1}\delta\varphi_m\wedge\varphi_m^{-1}\delta\varphi_m U_m^{(0)}\wedge d\xi + \sum_{n=1}^{N_2}\psi_n^{-1}\delta\psi_n\wedge\psi_n^{-1}\delta\psi_n V_n^{(0)}\wedge d\eta\right)\,. \tag{4.71}$$

Proof. We start by taking the δ-differential of Λ, which is

$$
\begin{aligned}
\delta\Lambda = &\operatorname{Tr}\sum_m \Big[-\partial_\eta\varphi_m U_m^{(0)}\varphi_m^{-1}\delta\varphi_m\varphi_m^{-1} + U_m^{(0)}\varphi_m^{-1}\delta(\partial_\eta\varphi_m)\Big]\wedge d\xi\wedge d\eta \\
&+\operatorname{Tr}\sum_n \Big[\partial_\xi\psi_n V_n^{(0)}\psi_n^{-1}\delta\psi_m\psi_n^{-1} - V_n^{(0)}\psi_m^{-1}\delta(\partial_\xi\psi_n)\Big]\wedge d\xi\wedge d\eta \\
&+\operatorname{Tr}\sum_{mn}\left[-\frac{[U_m,V_n]}{a_m-b_n}\delta\varphi_m\varphi_m^{-1} - \frac{[V_n,U_m]}{a_m-b_n}\delta\psi_n\psi_n^{-1}\right]\wedge d\xi\wedge d\eta \\
= &\operatorname{Tr}\sum_m\left[-\partial_\eta\varphi_m U_m^{(0)}\varphi_m^{-1} - \varphi_m\partial_\eta(U_m^{(0)}\varphi_m^{-1}) - \frac{[U_m,V_n]}{a_m-b_n}\right]\delta\varphi_m\varphi_m^{-1}\wedge d\xi\wedge d\eta \\
&+\operatorname{Tr}\sum_n\left[\partial_\xi\psi_n V_n^{(0)}\psi_n^{-1} + \psi_n\partial_\xi(V_n^{(0)}\psi_n^{-1}) - \frac{[V_n,U_m]}{a_m-b_n}\right]\delta\psi_n\psi_n^{-1}\wedge d\xi\wedge d\eta \\
&- d\operatorname{Tr}\left[-\sum_m U_m^{(0)}\varphi_m^{-1}\delta\varphi_m\wedge d\xi - \sum_n V_n^{(0)}\psi_n^{-1}\delta\psi_n\wedge d\eta\right]
\end{aligned}
$$

and we can read the equations (4.70) off the coefficients of $\delta\varphi_m$ and $\delta\psi_n$. We then take

$$
\Omega^{(1)} = -\operatorname{Tr}\left[\sum_m U_m^{(0)}\varphi_m^{-1}\delta\varphi_m\wedge d\xi + \sum_n V_n^{(0)}\psi_n^{-1}\delta\psi_n\wedge d\eta\right] \tag{4.72}
$$

and its δ-differential is $\Omega = \delta\Omega^{(1)}$ in (4.71). □

Our objective is to compute the covariant Poisson bracket à la Sklyanin for the Lax connection $W = U(z)d\xi + V(z)d\eta$ in the gauge where $U_0 = V_0 = 0$. Specifically, let $\{ E_{ij} \}$ be the canonical basis for $\mathfrak{gl}_N$ and write the Lax connection in this basis as

$$
W(z) = \sum_{i,j=1}^{N} W_{ij}(z)\, E_{ij}, \tag{4.73}
$$

where from now on we shall show the explicit dependence on the spectral parameter. To compute the covariant Poisson brackets between any two components of the Lax connection, we first need to show that these are admissible 1-forms.

For this we shall need the following useful identities. If $M \in GL_N(\mathscr{A})$ is any $\mathscr{A}$-valued matrix with components $M_{ij} \in \mathscr{A}$, $i,j = 1,\dots,N$ and C is any non-dynamical matrix (meaning $\delta C = 0$), then we have

$$
\sum_{k=1}^{N} M_{ik}\frac{\partial}{\partial M_{jk}} \lrcorner \operatorname{Tr}\big(M^{-1}\delta M\wedge M^{-1}\delta M C\big) = \delta(MCM^{-1})_{ij}\,, \tag{4.74a}
$$

$$
\sum_{k=1}^{N} M_{ik}\frac{\partial}{\partial M_{jk}} \lrcorner \delta(MCM^{-1})_{kl} = \delta_{jk}(MCM^{-1})_{il} - \delta_{il}(MCM^{-1})_{kj}\,. \tag{4.74b}
$$

In particular, we can use these with $M = \varphi_n$, $C = U_n^{(0)}$ and $M = \psi_n$, $C = V_n^{(0)}$. Then a direct calculation shows that

$$X_{ij}(z) = \sum_{m=1}^{N_1}\sum_{\beta=1}^{N} \frac{\varphi_{m,i\beta}}{z-a_m}\frac{\partial}{\partial\varphi_{m,j\beta}} + \sum_{n=1}^{N_2}\sum_{\beta=1}^{N} \frac{\psi_{n,i\beta}}{z-b_n}\frac{\partial}{\partial\psi_{n,j\beta}}\,, \tag{4.75}$$

satisfies $\delta W_{ij}(z) = X_{ij}(z) \lrcorner \Omega$. Therefore all the components $W_{ij}(z)$ for $i,j = 1,\dots,N$ of the Lax connection are admissible 1-forms, as required.

Theorem 4.21 *The covariant Poisson brackets of the Lax form satisfy the following relation*

$$\{\!(W_{\mathbf{1}}(z), W_{\mathbf{2}}(w))\!\} = \big[r_{\mathbf{12}}(z-w), W_{\mathbf{1}}(z) + W_{\mathbf{2}}(w)\big]\,, \tag{4.76}$$

where $r_{\mathbf{12}}(z) = -\frac{P_{\mathbf{12}}}{z}$ *is the rational r-matrix.*

We have used the permutation operator $P_{\mathbf{12}} = \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji}$ with the property

$$\sum_{i,j=1}^{N} (\delta_{jk}A_{il} - \delta_{il}A_{kj})\, E_{ij}\otimes E_{kl} = [A_{\mathbf{1}}, P_{\mathbf{12}}] = -[A_{\mathbf{2}}, P_{\mathbf{12}}]\,,$$

for any $A \in \mathfrak{gl}_N(\mathscr{A})$ with components $A_{ij}\in\mathscr{A}$ for $i,j = 1,\dots,N$.

Proof. We turn to the computation of the components on the left hand-side. We have

$$\begin{aligned}
\{\!(W_{ij}(z), W_{kl}(w))\!\} &= -\, X_{ij}(z)\lrcorner \delta W_{kl}(w)\\
&= -\sum_{m=1}^{N_1} \frac{\delta_{jk}(U_m)_{il} - \delta_{il}(U_m)_{kj}}{(z-a_m)(w-a_m)} d\xi\\
&\quad -\sum_{n=1}^{N_2} \frac{\delta_{jk}(V_n)_{il} - \delta_{il}(V_n)_{kj}}{(z-b_n)(w-b_n)} d\eta\,.
\end{aligned}$$

Noting that for any distinct $z, w, a \in \mathbb{C}$ we have the identity

$$\frac{1}{(z-a)(w-a)} = \frac{1}{w-z}\left(\frac{1}{z-a} - \frac{1}{w-a}\right)\,, \tag{4.77}$$

we may rewrite the covariant Poisson bracket as

$$\begin{aligned}
\{\!(W_{\mathbf{1}}(z), W_{\mathbf{2}}(w))\!\} &= \sum_{m=1}^{N_1} \frac{[P_{\mathbf{12}}, (U_m)_{\mathbf{1}}]}{(z-a_m)(w-a_m)} d\xi + \sum_{n=1}^{N_2} \frac{[P_{\mathbf{12}}, (V_n)_{\mathbf{1}}]}{(z-b_n)(w-b_n)} d\eta\\
&= \left[\frac{P_{\mathbf{12}}}{w-z}, W_{\mathbf{1}}(z) + W_{\mathbf{2}}(w)\right]\,.
\end{aligned}$$

□

Following our prescription, the covariant Hamiltonian related to Λ is found to be equal to

$$
\begin{aligned}
H = &-\Lambda + \sum_{m=1}^{N_1} \mathrm{Tr}\left(\varphi_m^{-1}\partial_\eta \varphi_m U_m^{(0)}\right) d\xi \wedge d\eta - \sum_{n=1}^{N_2} \mathrm{Tr}\left(\psi_n^{-1}\partial_\xi \psi_n V_n^{(0)}\right) d\xi \wedge d\eta \\
=&\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \mathrm{Tr}\, \frac{U_m V_n}{a_m - b_n}\, d\xi \wedge d\eta\,.
\end{aligned}
\tag{4.78}
$$

In the same way as the previous examples, we have shown that

$$
dW(z) = \{(h, W(z))\} d\eta \wedge d\xi\,, \quad \text{where} \quad H = h\, d\eta \wedge d\xi\,, \tag{4.79}
$$

in analogy to what one would do in the traditional Hamiltonian formalism, then since we have

$$
\{(h, W(z))\} d\eta \wedge d\xi = W(z) \wedge W(z)\,, \tag{4.80}
$$

we can conclude that $dW(z) = W(z) \wedge W(z)$. The main steps in the derivation of the crucial equality (4.80) are as follows. First, we have by definition

$$
\{(h, W(z))\} = \sum_{i,j=1}^{N} \left(X_{ij}(z) \lrcorner \delta h\right) E_{ij}\,. \tag{4.81}
$$

Second, we find

$$
\begin{aligned}
X_{ij}(z)\lrcorner\delta h =&\left(\sum_{m=1}^{N_1}\sum_{\beta=1}^{N} \frac{\varphi_{m,i\beta}}{z-a_m}\frac{\partial}{\partial \varphi_{m,j\beta}} + \sum_{n=1}^{N_2}\sum_{\beta=1}^{N} \frac{\psi_{n,i\beta}}{z-b_n}\frac{\partial}{\partial \psi_{n,j\beta}}\right)\lrcorner \\
&\left(\sum_{p=1}^{N_1}\sum_{q=1}^{N_2}\sum_{k,l=1}^{N} \frac{(\delta U_p)_{kl}(V_q)_{lk} + (U_p)_{lk}(\delta V_q)_{kl}}{a_p - b_q}\right) \\
=&\sum_{m=1}^{N_1}\sum_{q=1}^{N_2}\sum_{k,l=1}^{N} \frac{\left(\delta_{jk}(U_m)_{il} - \delta_{il}(U_m)_{kj}\right)(V_q)_{lk}}{(z-a_m)(a_m-b_q)} \\
&+\sum_{n=1}^{N_2}\sum_{p=1}^{N_1}\sum_{k,l=1}^{N} \frac{(U_p)_{lk}\left(\delta_{jk}(V_n)_{il} - \delta_{il}(V_n)_{kj}\right)}{(z-b_n)(a_p-b_n)} \\
=&\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \frac{([U_m, V_n])_{ij}}{(z-a_m)(z-b_n)}\,,
\end{aligned}
$$

where we have used the identity (4.74b) in the second equality and (4.77) in the last equality. Substituting the above into (4.81) we obtain (4.80).

4.6 The Korteweg-de Vries equation: an unsuccessful attempt

The above formalism seems to work really well to produce the equation $\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda)+W_{\mathbf{2}}(\mu)]$ for ultralocal field theories, *i.e.* theories for which the classical r-matrix is antisymmetric $r_{\mathbf{12}}(\lambda,\mu) = -r_{\mathbf{21}}(\mu,\lambda)$. We report here our unsuccessful attempt at applying the same construction to a non-ultralocal theory such as the Korteweg-de Vries equation (KdV). As the KdV is arguably among the most famous examples of integrable systems, we think the reader will be interested on the current state of understanding of this theory and what goes wrong and where. We must remark as well that we are not expecting a relation such as $\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda)+W_{\mathbf{2}}(\mu)]$, as the theory is non-ultralocal[4], but we will see that the problem arises well before, as the Poisson bracket $\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\}$ is already not defined.
We will treat the potential version of the KdV equation

$$v_{xt} = v_{xxxx} + 6v_x v_{xx} \tag{4.82}$$

where $u = v_x$ is the KdV field. As it is now costumary we start from the Lagrangian volume form[5]

$$\Lambda = (v_x v_t - 2(v_x)^3 + (v_{xx})^2)\, dx \wedge dt\,. \tag{4.83}$$

We compute the δ-differential of Λ as

$$\begin{aligned}\delta\Lambda =&(v_x\delta v_t + (v_t - 6(v_x)^2)\delta v_x + 2v_{xx}\delta v_{xx}) \wedge dx \wedge dt\\ =&(-2v_{xt} + 2v_{xxxx} + 12 v_x v_{xx})\delta v \wedge dx \wedge dt\\ &- d(-v_x\delta v \wedge dx + (v_t - 6(v_x)^2 - 2v_{xxx})\delta v \wedge dt + 2v_{xx}\delta v_x \wedge dt)\,,\end{aligned}$$

so that we have

$$\Omega^{(1)} = - v_x\delta v \wedge dx + (v_t - 6(v_x)^2 - 2v_{xxx})\delta v \wedge dt + 2v_{xx}\delta v_x \wedge dt\,, \tag{4.84}$$

$$\begin{aligned}\Omega =& - \delta v_x \wedge \delta v \wedge dx\\ &+ (\delta v_t \wedge \delta v - 12 v_x \delta v_x \wedge \delta v - 2\delta v_{xxx} \wedge \delta v + 2\delta v_{xx} \wedge \delta v_x) \wedge dt\,.\end{aligned} \tag{4.85}$$

If we use the familiar argument to investigate the presence of admissible 1-forms (*i.e.* starting from a generic vertical vector field and taking its interior product with the multisymplectic form) we realise that an admissible 1-form $F = F_1(v, v_x)\, dx + F_2(v, v_t, v_x, v_{xx}, v_{xxx})\, dt$

[4]We would expect an equation similar to [BBT03, Equation (2.10)], such as $\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda)] - [r_{\mathbf{21}}(\mu,\lambda), W_{\mathbf{2}}(\mu)]$.

[5]It differs from L_{13} of Section 2.3 by a total horizontal differential.

must satisfy the following requirements:

$$\frac{\partial F_1}{\partial v_x} = -\frac{\partial F_2}{\partial v_t} = \frac{1}{2}\frac{\partial F_2}{\partial v_{xxx}}\,, \tag{4.86a}$$
$$\frac{\partial F_1}{\partial v} = \frac{1}{2}\frac{\partial F_2}{\partial v_{xx}}\,. \tag{4.86b}$$

Next, we would need to find an admissible Lax connection for this equation, so that we can calculate the Poisson bracket $\{\!(W_{\mathbf{1}}, W_{\mathbf{2}})\!\}$. Unfortunately, we have not been able to do so. The only Lax pair we could find, *i.e.*

$$U(\lambda) = \begin{pmatrix} i\lambda & -v_x \\ 1 & -i\lambda \end{pmatrix}\,, \tag{4.87}$$

$$V(\lambda) = \begin{pmatrix} -4i\lambda^3 + 2iv_x\lambda + v_{xx} & 4v_x\lambda^2 - 2iv_{xx}\lambda - v_{xxx} - 2(v_x)^2 \\ -4\lambda^2 + 2v_x & 4i\lambda^3 - 2iv_x\lambda - v_{xx} \end{pmatrix}\,, \tag{4.88}$$

is not admissible. In fact, none of the above relations hold: it is not true that $\frac{\partial U}{\partial v_x} = -\sigma_+$ is equal to $-\frac{\partial V}{\partial v_t} = 0$ or to $\frac{1}{2}\frac{\partial V}{\partial v_{xxx}} = -\frac{1}{2}\sigma_+$, and $\frac{\partial U}{\partial v} = 0$ is not equal to $\frac{1}{2}\frac{\partial V}{\partial v_{xx}} = \sigma_3 - 2i\lambda\sigma_+$. There are two possible strategies that one could take at this stage. The first one is to extend our covariant Poisson brackets $\{\!(\;,\;)\!\}$ to non-admissible forms, allowing to keep the current Lax pair U, V as in (4.87)-(4.88). The extension of covariant Poisson bracket to non-admissible forms has been explored in literature, for instance in [FS15], but not in relation to integrable systems. The second strategy is to investigate other possibilities of Lax pairs that satisfy the equations (4.86) and are therefore admissible. This is currently still an open problem.

4.7 Concluding remarks

In this section we have shown for many archetypal examples of integrable systems the classical r-matrix structure of the Lax connection $W(\lambda)$ within the covariant Poisson bracket

$$\{\!(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\!\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]\,.$$

These Poisson brackets are only defined for a specific class of forms called *admissible*, *i.e.* forms F for which there exist a vector field ξ_F such that $\xi_F \lrcorner \Omega = \delta F$. The Poisson brackets $\{\!(\;,\;)\!\}$ were defined from the multisymplectic form Ω, which was obtained from the Lagrangian following the procedure explained in [D03]. Following [D03] we were also able to define the *covariant Hamiltonian* of the field theory in example $H = h\,dx \wedge dt$. We showed consistently that the zero-curvature equations $dW(\lambda) = W(\lambda) \wedge W(\lambda)$ can be recognised as a covariant Hamilton equation for the Lax connection as

$$dW(\lambda) = \{\!(h, W(\lambda))\!\}\,dx \wedge dt\,.$$

This opens up a series of questions. The Non-Linear Schrödinger and the modified Korteweg-de Vries equations belong to an integrable hierarchy, *i.e.* the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy. Ideally, it is interesting to see if the same covariant approach could be applied to more equations of the same hierarchy, and even to the hierarchy itself as a whole. This will be addressed in the following chapters, with the introduction of *Hamiltonian multiforms* in Chapter 5 and with its applications to the AKNS hierarchy in Chapter 6.

Moreover, as pointed out in Section 4.6, we have only been able to treat ultralocal field theories. This is because the non ultra-local theories that we tried to treat are expressed by a Lax connection that does not possess the right properties in order to calculate the covariant Poisson bracket $\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\}$ (*i.e.* the property of being *admissible*). These non-ultralocal field theories are extremely important to treat, as they include famous key systems such as the celebrated potential Korteweg-de Vries equation. This is a current issue of our approach, and it needs to be investigated further.

Finally, the consistency of these results points to a deeper generalisation, in terms of characterisation in terms of endomorphisms of a Lie algebra and Poisson-Lie groups, in the style of [RS88]. This will help with the generalisation of $\{(W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu))\} = [r_{\mathbf{12}}(\lambda, \mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]$ to other field theories, and to understand the theory in a deeper, non-phenomenological way.

Chapter 5

Hamiltonian multiform description of integrable hierarchies

In this chapter, which contains content from [CS20b], we aim to describe covariantly (*i.e.* treating space and time with equal footing) a whole integrable hierarchy of PDEs in a Hamiltonian fashion. This procedure generalises Dickey's construction of a covariant Hamiltonian (that has been reported and expanded upon in Chapter 3) to the case of a hierarchy, taking a *Lagrangian multiform* as a starting point as opposed to a Lagrangian volume form.

In Section 5.1, by means of what can be described as a 'covariant Legendre transformation with respect to all the times of the hierarchy' we produce the Hamiltonian counterpart of a Lagrangian multiform, that we call *Hamiltonian multiform*, and a new object which generalises the multisymplectic form to a whole hierarchy, that we call *symplectic multiform*. The multiform Euler-Lagrange equations are recovered as a natural extension of the covariant Hamilton equations, and the closure of the Lagrangian multiform is related to the closure of the Hamiltonian multiform, which resembles the usual conservation of the Hamiltonian function for finite dimensional mechanics. In Section 5.2 we introduce the *multi-time Poisson brackets*, which generalise the covariant Poisson bracket in the multiform framework. In Section 5.6 we relate our formalism to the results of [V20] regarding Lagrangian 1-forms (hierarchies of ODEs).

We use this new formalism to describe a few levels of the potential Korteweg-de Vries hierarchy in Section 5.3, the sine-Gordon hierarchy in light-cone coordinates in Section 5.4 and of the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy in Section 5.5. We anticipate that we will be able to describe the whole AKNS hierarchy in a closed form, but we delay its discussion to Chapter 6.

5.1 The Hamiltonian and symplectic multiform

The main observation at the basis of this chapter is that the objects and results illustrated in Chapter 4 can be extended to a Lagrangian multiform, *i.e.* a horizontal 2-form

$$\mathscr{L} = \sum_{i<j}^{n} L_{ij}\, dx^{ij}, \, , \tag{5.1}$$

for $L_{ij} \in \mathscr{A}$, required to satisfy a generalised variational principle associated to the action

$$S[\Gamma] = \int_\Gamma \mathscr{L} \, , \tag{5.2}$$

as explained in Section 2.3. Furthermore, we assume that the Lagrangian multiform $\mathscr{L}$ does not depend explicitly on the multi-time variables x^i. We can turn our attention to the generalisation of the form $\Omega^{(1)}$ in (3.2). We first use the following result from [V18, Proposition 6.3] and [V20], which we reproduce here with a little change of notation.

Proposition 5.1 *The field u is a critical point of $S[\Gamma] = \int_\Gamma \mathscr{L}$ for all (smooth) surfaces Γ in $\mathbb{R}^n$ if and only if there exists a (nonzero) form $\Omega^{(1)} \in \mathscr{A}^{(1,1)}$ such that*

$$\delta \mathscr{L} = -d\Omega^{(1)} \, . \tag{5.3}$$

We also recall that, as explained in the introduction we have that u is a critical point of S for all (smooth) surfaces Γ if and only if $\delta d\mathscr{L} = 0$. Equipped with this, let us write,

$$\mathcal{E}(\mathscr{L}) := \delta \mathscr{L} + d\Omega^{(1)} \, .$$

Then, a reformulation of the previous discussion is as follows:

$$\begin{aligned} \delta d\mathscr{L} = 0 \Leftrightarrow \\ & u \text{ is a critical point of } S[\Gamma] \text{ for all smooth surfaces } \Gamma \text{ in } \mathbb{R}^n \\ & \Leftrightarrow \mathcal{E}(\mathscr{L}) = 0 \, . \end{aligned} \tag{5.4}$$

Compared to the case of (3.2), in addition to the non-uniqueness of $\Omega^{(1)}$ induced by the freedom of adding a total differential $d\omega$ to $\mathscr{L}$ (as for a standard Lagrangian volume form), there is also some freedom in the integration by parts steps which lead to the expression

$$\delta \mathscr{L} = \mathcal{E}(\mathscr{L}) - d\Omega^{(1)} \, . \tag{5.5}$$

More precisely, in general we could also have another way of writing $\delta \mathscr{L}$,

$$\delta \mathscr{L} = \widetilde{\mathcal{E}}(\mathscr{L}) - d\widetilde{\Omega^{(1)}} \, , \tag{5.6}$$

with still $\widetilde{\mathcal{E}}(\mathscr{L}) = 0 \Leftrightarrow \delta d\mathscr{L} = 0$, following from Proposition 5.1 and reformulation (5.4). We will show that these two sources of freedom have no consequence on our constructions. Equipped with a pair $(\mathscr{L}, \Omega^{(1)})$, we define the Hamiltonian multiform associated to it.

Definition 5.2 (Hamiltonian multiform) *The* Hamiltonian multiform *associated to the pair* $(\mathscr{L}, \Omega^{(1)})$ *is defined by*

$$\mathcal{H} := -\mathscr{L} + \sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega^{(1)}. \tag{5.7}$$

As announced, this definition looks very similar to the definition of the covariant Hamiltonian in (3.5). However note that the sum involves $n+1$ terms here (the number of independent variables included in the Lagrangian multiform) and that $\mathcal{H}$ has the form $\mathcal{H} = \sum_{i<j} H_{ij} dx^{ij}$ and is in $\mathscr{A}^{(0,2)}$, like $\mathscr{L}$. $\mathcal{H}$ plays the role of the covariant Hamiltonian form in the multiform context.

Proposition 5.3 *The equivalent Lagrangian multiforms $\mathscr{L}$ and $\mathscr{L}' = \mathscr{L} + d\varphi$ for some $\varphi \in \mathscr{A}^{(0,1)}$ bring the same Hamiltonian multiform.*

Proof. Similar to the one of Proposition 3.8. In fact, let $\mathcal{H}$ be the Hamiltonian multiform associated to the pair $(\mathscr{L}, \Omega^{(1)})$. We have that $\mathcal{H}'$ is the one associated to the pair $(\mathscr{L}', \Omega^{(1)} + \delta\varphi)$. Then we prove that

$$\mathcal{H}' = \mathcal{H}\,. \tag{5.8}$$

□

The relevance of this lemma is related to the symplectic multiform defined below and the multiform Hamilton equations associated to it and $\mathcal{H}$.

We can easily see that there is a relation between the d-differential of $\mathcal{H}$ and the one of $\mathscr{L}$. The next result is important and connects the closure relation in the Lagrangian multiform to the Hamiltonian multiform formalism.

Theorem 5.4 *$d\mathcal{H} = -2d\mathscr{L}$ modulo the multiform Euler-Lagrange equations.*

Proof. We start from the definition of $\mathcal{H}$:

$$\begin{aligned} d\mathcal{H} =& -d\mathscr{L} + d\left(\sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega^{(1)}\right) = -d\mathscr{L} - \sum_{j=0}^{n} dx^j \wedge d\widetilde{\partial}_j \lrcorner \Omega^{(1)} \\ =& -d\mathscr{L} + \sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \lrcorner d\Omega^{(1)} \end{aligned}$$

where we used $d\widetilde{\partial}_j \lrcorner + \widetilde{\partial}_j \lrcorner d = 0$ (cf. [D03, Corollary 19.2.10]). Now we use the equation $\delta \mathscr{L} = -d\Omega^{(1)}$ to obtain

$$\begin{aligned} d\mathcal{H} =& - d\mathscr{L} - \sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \lrcorner \delta \mathscr{L} = -d\mathscr{L} - \sum_{j=0}^{n} dx^j \wedge (\widetilde{\partial}_j - \delta \widetilde{\partial}_j \lrcorner) \mathscr{L} \\ =& - d\mathscr{L} - \sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \mathscr{L} = -2d\mathscr{L}. \end{aligned}$$

In the last line we used the property $\widetilde{\partial}_j = \delta \widetilde{\partial}_j \lrcorner + \widetilde{\partial}_j \lrcorner \delta$, and the fact that $\mathscr{L}$ is purely horizontal and does not depend explicitly on the multi-time variables. □

Remark 5.5: In [SV16, V20] the closure of a pluri-Lagrangian form $\mathscr{L}$ was linked to the involution of the single-time Hamiltonians (that we will interpret in terms of Hamiltonian multiforms in Section 5.6), and in [V20] an analogue of Theorem 5.4 for the case of Lagrangian 1-forms was given. In the particular case where the Hamiltonian multiform is an admissible form in the sense defined below, we expect Theorem 5.4 to provide a general framework in which to recast these results (with appropriate modifications for the examples in $0 + 1$ dimensions presented in [SV16, V20]). This point is partially addressed in Section 5.6, but mainly left for future investigation.

Recalling that a Lagrangian multiform is defined to satisfy the closure relation $d\mathscr{L} = 0$ on the equations of motion, we obtain:

Corollary 5.6 (Closedness of $\mathcal{H}$) *The Hamiltonian multiform is horizontally closed on the multiform Euler-Lagrange equations $d\mathcal{H} = 0$. In other words, $\mathcal{H}$ satisfies the closure relation.*

We believe that these results justify our terminology *Hamiltonian multiform* since we have the closure relation for $\mathcal{H}$ if and only if $d\delta\mathscr{L} = 0$. This corollary is the multiform equivalent of the well known fact in finite-dimensional mechanics that the Hamiltonian is a conserved quantity $\frac{dH}{dt} = 0$ (recall that we do not include explicit dependence on the independent variables here).

We are now in a position to introduce the multiform analog of the multisymplectic form (3.7), again denoting it by Ω.

Definition 5.7 *The* symplectic multiform *associated to $\Omega^{(1)}$ is $\Omega := \delta\Omega^{(1)} \in \mathscr{A}^{(2,1)}$.*

Remark 5.8: Like the multisymplectic form, the symplectic multiform is vertically closed (more precisely, exact), has degree $(2, 1)$, in the case of 1+1-dimensional field theories considered here, and is of the form

$$\Omega = \sum_{j=0}^{n} \omega_j \wedge dx^j\,, \qquad \omega_j \in \mathscr{A}^{(2,0)}\,, \quad 0 \le j \le n\,. \tag{5.9}$$

If we were to consider a Lagrangian multiform for a hierarchy of k-dimensional field theories, $k < n$, the Lagrangian multiform would be a horizontal k-form, and the symplectic multiform (if it exists, and with the same definition) would be of degree $(2, k-1)$.

The symplectic multiform Ω achieves an important unification of the various (standard and dual) symplectic structures appearing in an integrable hierarchy, as originally observed in [ACDK16]. When x_1 is chosen to be the x variable and x_j, $2 \le j \le n$ to be the higher times t_j of the hierarchy then ω_1 represents (up to a sign) the usual symplectic form, while each ω_j, $j \neq 1$ represents the dual symplectic form related to the time t_j. For each $2 \le j \le n$, the multisymplectic form Ω_{1j} which would be obtained by considering the Lagrangian L_{1j} as a standalone Lagrangian, as in Chapter 3, is simply obtained by taking $\omega_1 \wedge dx^1 + \omega_j \wedge dx^j$.

Remark 5.9: The reader will hopefully forgive us for the choice of terminology, very similar to multisymplectic form. Another candidate, polysymplectic form, is already in use in the literature (see for instance [K98]). We could not simply keep multisymplectic form for our new object since, although both objects are derived in a similar fashion and play a similar role in the theory, they are quite different in concept. Indeed, the multisymplectic form is related to only a single field theory, while our symplectic multiform is related to a hierarchy. In the case of k-dimensional field theory, the multisymplectic form of degree $(2, k-1)$ is obtained considering a k-dimensional space-time manifold and a horizontal volume k-form as a Lagrangian. When we consider a hierarchy of such field theories, we extend the space-time manifold to a n-dimensional multi-time and therefore consider n independent variables. Moreover, we consider the k-form (previously taken as a Lagrangian) only as one of the terms of the Lagrangian multiform (which still is of degree k). Consequently, the multisymplectic form is extended to a symplectic multiform, which still has degree $(2, k-1)$, but contains other terms generated by the additional $n-k$ times.

Just like in the covariant case, it is clear from Proposition 5.3 that adding a total differential $d\varphi$ to $\mathscr{L}$, which amounts to adding $\delta\varphi$ to $\Omega^{(1)}$, has no consequence on Ω.

The following corollary gives support for our terminology as it is reminiscent of the fact that a symplectic form ω is closed in classical mechanics.

Corollary 5.10 *The symplectic multiform is horizontally closed on the multiform Euler-Lagrange equations:*

$$\delta d\mathscr{L} = 0 \qquad \Longrightarrow \qquad d\Omega = 0\,. \tag{5.10}$$

Proof. The equations are expressed as $\delta\mathscr{L} = -d\Omega^{(1)}$, so

$$0 = \delta^2\mathscr{L} = -\delta d\Omega^{(1)} = d\delta\Omega^{(1)} = d\Omega\,. \qquad \square$$

We now use the symplectic multiform to obtain the *multiform Hamilton equations*.

Proposition 5.11 (multiform Hamilton equations) *The multiform Euler-Lagrange equations for the Lagrangian multiform $\mathscr{L}$ are equivalent to*

$$\delta\mathcal{H} = \sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega. \tag{5.11}$$

Proof. The proof is a simple adaptation of the similar result obtained in [D03, Chapter 19] and in Proposition 3.6 to the multiform case. $\square$

> **Remark 5.12:** Lemma 5.3 ensures that the freedom of adding a total differential to $\mathscr{L}$ has no consequence on the multiform Hamilton equations as it should. The other source of freedom coming from (5.5)-(5.6) does not affect the result either. Indeed, suppose that $\widetilde{\mathcal{H}}$ is the Hamiltonian multiform associated to the pair $(\mathscr{L}, \widetilde{\Omega^{(1)}})$ of (5.6) and $\widetilde{\Omega}$ is associated to $\widetilde{\Omega^{(1)}}$ then exactly the same computation as above yields that the multiform Euler-Lagrange equations for the Lagrangian multiform $\mathscr{L}$ are equivalent to
> $$\delta\widetilde{\mathcal{H}} = \sum_{j=0}^{n} dx^j \wedge \widetilde{\partial}_j \lrcorner \widetilde{\Omega}\,.$$

5.2 The multi-time Poisson brackets

Continuing with the inspiration given by covariant Hamiltonian field theory, the next step is to construct a Poisson bracket related to our symplectic multiform and cast the multiform Hamilton equations into Poisson Bracket form. Similarly to the situation reviewed in Chapter 3, this can only be done for a restricted class of forms, called admissible forms. For convenience, we restrict again our attention to horizontal forms as this is sufficient for our purposes.

Definition 5.13 (Admissible forms) *We will say that a horizontal form P is* admissible *if there exists a (multi)vector field ξ_P such that $\xi_P \lrcorner \Omega = \delta P$. ξ_P is called the* Hamiltonian vector field *related to P.*

Proposition 5.14 *P can be a non-trivial admissible form only if either $P \in \mathscr{A}$ or $P \in \mathscr{A}^{(0,1)}$.*

Proof. The proof follows from a simple counting argument, and it is similar to the one of Proposition 3.11. Suppose $P \in \mathscr{A}^{(0,s)}$. Then, since $\Omega \in \mathscr{A}^{(2,1)}$, in order for a (p,q)-vector field[1] ξ_P to exist such that $\xi_P \lrcorner \Omega = \delta P$, then necessarily $2 - p = 1$ and $1 - q = s$. So $p = 1$ and $q = 1 - s \geq 0$, and therefore s can only be 0 or 1. $\square$

We now produce a statement that is similar to Proposition 3.13, but for the multiform case. The proof is easily obtained as an extension. We will use this result systematically without quoting it in our examples below.

Let us denote by S_Ω the set of basis elements $\delta u_l^{(i)}$ that appear explicitly the symplectic multiform. It is a finite set since Ω is derived from $\mathscr{L}$ which is assumed to depend on $u_l^{(i)}$ with $|i| \leq m$ for some m (finite jet dependence). Hence, we can assume some ordering on S_Ω such that we can label the $\delta u_l^{(i)}$'s as δv_j, $j = 1, \dots, \#S_\Omega$. We then write

$$\Omega = \sum_{k=0}^{n} \sum_{\substack{i<j \\ i,j \in I_k}} \omega_k^{ij} \delta v_i \wedge \delta v_j \wedge dx^k \tag{5.12}$$

where $I_k \subseteq \{1, \dots, \#S_\Omega\}$ for each k. Note that each $\omega_k^{ij} \in \mathscr{A}$ so has a dependence on the local coordinates $u_m^{(j)}$ which we do not show explicitly.

Proposition 5.15 (Necessary form of an admissible 1-form.) *Suppose $F = \sum_{k=0}^{n} F_k \, dx^k \in \mathscr{A}^{(0,1)}$ is an admissible 1-form for the symplectic multiform* (5.12). *Then, for each $0 \leq k \leq n$, F_k can only depend (at most) on v_j, $j \in I_k$.*

We can now define the multi-time Poisson brackets for admissible forms, in analogy with the covariant Poisson bracket.

Definition 5.16 (multi-time Poisson brackets) *For two admissible forms P and Q, of degree respectively r and s, we define their multi-time Poisson bracket as*

$$\{[P, Q]\} := (-1)^r \xi_P \lrcorner \delta Q. \tag{5.13}$$

> **Remark 5.17:** This definition is formally the same as the one of the covariant Poisson bracket (3.10). However, we stress that the symplectic multiform of the hierarchy is different from the the multisymplectic form of a singular field theory, as it includes additional terms. Therefore, the resulting Poisson bracket of two horizontal forms will be different. For this reason we have chosen to use two different notations,

[1] We mean that ξ_P is obtained with a wedge-product of p vertical vector fields and q horizontal vector fields.

i.e. $\{(\; , \;)\}$ for the covariant Poisson bracket, and $\{[\; , \;]\}$ for the multi-time Poisson bracket. The two brackets coincide, in the case of a $1+1$-dimensional field theory, when $n = 2$.

These Poisson brackets are graded antisymmetric and bilinear in the space of admissible forms. In particular

- $P, Q \in \mathscr{A}^{(0,1)}$, then $\{[P,Q]\} = -\xi_P \lrcorner \delta Q = -\{[Q,P]\} = \xi_Q \lrcorner \delta P$;
- $P \in \mathscr{A}^{(0,1)}$ and $H \in \mathscr{A}$, then $\{[H,P]\} = \xi_H \lrcorner \delta P = -\{[P,H]\} = \xi_P \lrcorner \delta H$.

As mentioned before for the covariant Poisson bracket, our definition may lead to issues regarding the Jacobi identity for instance. However, we investigate this further in connection with the r-matrix structure of the multi-time Poisson bracket whereby the Jacobi identity translates into the classical Yang-Baxter equation.

Theorem 5.18 *On the equations of motion*

$$dF = \xi_F \lrcorner \delta \mathcal{H} \tag{5.14}$$

for any admissible 1*-form that does not depend on the independent variables.*

Proof. The proof is easily obtained as an extension of the proof of Theorem 3.15 □

If the components H_{ij} of $\mathcal{H}$ are admissible 0-forms, then the previous proposition leads to:

Corollary 5.19 *On the equations of motion*

$$dF = \sum_{i<j=1}^{n} \{[H_{ij}, F]\} dx^{ij}. \tag{5.15}$$

for any admissible 1*-form that does not depend on the independent variables.*

Proof.

$$dF = \xi_F \lrcorner \delta \mathcal{H} = \sum_{i<j} \xi_F \lrcorner \delta H_{ij} \wedge dx^{ij} = -\sum_{i<j} \{[F, H_{ij}]\} dx^{ij} = \sum_{i<j} \{[H_{ij}, F]\} dx^{ij} \,. \qquad \square$$

This is a generalisation of the usual Hamilton equations in Poisson Bracket form for classical finite-dimensional mechanics $\dot{f} = \{H, f\}$. In our context, this result turns out to be useful in relation to conservation laws within an integrable hierarchy. Indeed, if F is a

1-form, we have

$$dF = \sum_{j=0}^{n} dx^j \wedge \partial_j F = \sum_{i,j=0}^{n} \partial_i F_j dx^i \wedge dx^j = \sum_{i<j} (\partial_i F_j - \partial_j F_i) dx^i \wedge dx^j$$

which means that, in fact if $dF = 0$ on the equations of motion, then

$$\partial_i F_j = \partial_j F_i, \qquad \forall i \neq j. \tag{5.16}$$

This suggests the following definition.

Definition 5.20 *We say that an admissible 1-form F is a* conservation law *if $dF = 0$ on the equations of motion.*

The next corollary then follows immediately from Proposition 5.18.

Corollary 5.21 *A admissible 1-form F is a conservation law if and only if on the equations of motion $\xi_F \lrcorner \delta\mathcal{H} = 0$ or, if each H_{ij} is admissible,*

$$\{\![H_{ij}, F]\!\} = 0 \qquad \forall ij\,. \tag{5.17}$$

This is clearly an extension of the concept of first integral in classical mechanics. As we will show on some examples below, a rather elegant byproduct of our approach is that the very definition of an admissible form being a conservation law can lead to its explicit form.

We now address the relationship between the multi-time Poisson bracket that we just defined and the single-time Poisson brackets that can be derived from the single Lagrangians L_{ij} using the usual construction. This generalises Proposition 3.17 to the case of Hamiltonian multiforms. Starting from the decomposition (5.9), for each $0 \leq i \leq n$, it is natural to want to define the i-th Poisson bracket of two 0-forms $f, g \in \mathscr{A}$ as

$$\{f, g\}_i := -\xi^i_f \lrcorner \delta g, \qquad \text{where} \qquad \xi^i_f \lrcorner \omega_i = \delta f. \tag{5.18}$$

We remark that there is no sum on the i index.

Theorem 5.22 (Decomposition of the multi-time Poisson Bracket) *Let $F = \sum_{i=0}^n F_i dx^i$ be an admissible* 1*-form, then for $i \geq 0$, F_i is admissible with respect to ω_i. Let $G = \sum_{i=0}^n G_i dx^i$ be another admissible* 1*-form, then the following decomposition of the multi-time Poisson bracket holds:*

$$\{\![F, G]\!\} = \sum_{i=0}^{n} \{F_i, G_i\}_i dx^i. \tag{5.19}$$

Proof. The proof is a generalisation of the one of Proposition 3.17 On the one hand, by definition

$$\delta F = \sum_{i=0}^{n} \delta F_i \wedge dx^i \,,$$

and on the other hand, since F is admissible

$$\delta F = \xi_F \lrcorner \sum_{i=0}^{n} \omega_i \wedge dx^i = \sum_{i=0}^{n} \xi_F \lrcorner \omega_i \wedge dx^i \,,$$

hence $\delta F_i = \xi_F \lrcorner \omega_i$ so F_i is admissible with respect to ω_i for each $0 \leq i \leq n$ and we can take $\xi^i_{F_i} = \xi_F$ for all $0 \leq i \leq n$ (modulo kernel of ω_i). Note that this gives an idea of how restrictive it is for F to be admissible. Next, consider the following chain of equalities

$$\begin{aligned}
\{[F, G]\} &= -\,\xi_F \lrcorner \delta G = -\xi_F \lrcorner (\sum_{i=0}^{n} \delta G_i \wedge dx^i) = -\xi_F \lrcorner (\sum_{i=0}^{n} \xi^i_{G_i} \lrcorner \omega_i \wedge dx^i) \\
&= \sum_{i=0}^{n} \xi^i_{G_i} \lrcorner \xi_F \lrcorner \omega_i \wedge dx^i = \sum_{i=0}^{n} \xi^i_{G_i} \lrcorner \delta F_i \wedge dx^i = \sum_{i=0}^{n} \{F_i, G_i\}_i dx^i
\end{aligned}$$

which concludes the proof. □

This is the generalization to an arbitrary number of flows in an integrable hierarchy of the decomposition theorem that was obtained in Proposition 3.17. This theorem describes the relationship between our multi-time Poisson bracket $\{[\ ,\ ]\}$, encapsulating an arbitrary number of flows in the hierarchy, and the usual and dual single-time Poisson brackets $\{\ ,\ \}_i$, which are related to each flow separately.

5.3 Potential Korteweg-de Vries hierarchy

In the following we will see the example of the Korteweg-de Vries (KdV) hierarchy with respect to its first two times, so in usual hierarchy notations, we would have $x_1 = x$, $x_2 = t_2$ and $x_3 = t_3$ (if one consider the KdV alone, t_3 is simply the time t). In fact, since the usual KdV equation does not admit a Lagrangian formulation, we consider its potential form instead. It is known that for KdV hierarchy the even flows are trivial $v_{2k} = 0\ \forall k$, so we will also treat the less trivial case of the first three odd times $x_1 = x$, $x_3 = t_3$ and $x_5 = t_5$. We use the Lagrangians multiforms presented in [V18].

5.3.1 Times 1,2 and 3

We formulate the first two levels of the (potential) KdV hierarchy, described by the Lagrangian multiform $\mathscr{L} = L_{12}\, dx^{12} + L_{23}\, dx^{23} + L_{13}\, dx^{13}$, where

$$L_{12} = v_1 v_2\,, \tag{5.20a}$$
$$L_{23} = -3v_1^2 v_2 - v_1 v_{112} + v_{11}v_{12} - v_{111}v_2\,, \tag{5.20b}$$
$$L_{13} = -2v_1^3 - v_1 v_{111} + v_1 v_3\,. \tag{5.20c}$$

In section 2.3 we have checked that the multiform Euler-Lagrange equations $\delta d\mathscr{L} = 0$ are equivalent to

$$v_2 = 0\,, \qquad v_3 = v_{111} + 3v_1^2\,. \tag{5.21}$$

and differential consequences. The potential KdV from $v_{13} = (v_3)_1 = v_{1111} + 6v_1 v_{11}$. We are now going to show the procedure to obtain the symplectic multiform from $\mathscr{L}$ and (5.21).

The symplectic multiform We start by computing the δ-differential of the Lagrangian multiform:

$$\begin{aligned}\delta\mathscr{L} =& v_1\delta v_2 \wedge dx^{12} + v_2\delta v_1 \wedge dx^{12}\\ &+ (-6v_1v_2 - v_{112})\delta v_1 \wedge dx^{23} + (-3v_1^2 - v_{111})\delta v_2 \wedge dx^{23} + v_{12}\delta v_{11} \wedge dx^{23}\\ &+ v_{11}\delta v_{12} \wedge dx^{23} - v_1\delta v_{112} \wedge dx^{23} - v_2\delta v_{111} \wedge dx^{23}\\ &+ (v_3 - v_{111} - 6v_1^2)\delta v_1 \wedge dx^{13} + v_1\delta v_3 \wedge dx^{13} - v_1\delta v_{111} \wedge dx^{13}.\end{aligned}$$

We now use the property $d\delta = -\delta d$ on some of the terms to obtain the desired expression $\delta\mathscr{L} = \mathcal{E}(\mathscr{L}) - d\Omega^{(1)}$, where $\mathcal{E}(\mathscr{L}) = 0$ is equivalent to (5.21). The reader can verify the following identities

$$\begin{aligned}
v_1\delta v_2 \wedge dx^{12} &= -v_{12}\delta v \wedge dx^{12} - v_{13}\delta v \wedge dx^{13} - v_1\delta v_3 \wedge dx^{13} - d(-v_1\delta v \wedge dx^1)\,,\\
v_2\delta v_1 \wedge dx^{12} &= -v_{12}\delta v \wedge dx^{12} + v_{23}\delta v \wedge dx^{23} + v_2\delta v_3 \wedge dx^{23} - d(v_2\delta v \wedge dx^2)\,,\\
(v_3 - v_{111} - 6v_1^2)\delta v_1 \wedge dx^{13} &= -(v_3 - v_{111} - 6v_1^2)_1\delta v \wedge dx^{13}\\
&\quad - (v_3 - v_{111} - 6v_1^2)_2\delta v \wedge dx^{23} - (v_3 - v_{111} - 6v_1^2)\delta v_2 \wedge dx^{23}\\
&\quad - d((v_3 - v_{111} - 6v_1^2)\delta v \wedge dx^3)\,,\\
-v_1\delta v_{111} \wedge dx^{13} &= v_{1111}\delta v \wedge dx^{13} + v_{1112}\delta v \wedge dx^{23} + v_{111}\delta v_2 \wedge dx^{23}\\
&\quad - v_{112}\delta v_1 \wedge dx^{23} - v_{11}\delta v_{12} \wedge dx^{23}\\
&\quad + v_{12}\delta v_{11} \wedge dx^{23} + v_1\delta v_{112} \wedge dx^{23}\\
&\quad - d(-v_1\delta v_{11} \wedge dx^3 + v_{11}\delta v_1 \wedge dx^3 - v_{111}\delta v \wedge dx^3).
\end{aligned}$$

Using these identities in $\delta\mathscr{L}$ we get

$$\begin{aligned}\delta\mathscr{L} =& -2v_{12}\delta v\wedge dx^{12} + (-2v_{13}+2v_{1111}+12v_1v_{11})\delta v\wedge dx^{13}\\
&+(2v_{1112}+12v_1v_{12})\delta v\wedge dx^{23} + (-6v_1v_2-2v_{112})\delta v_1\wedge dx^{23}\\
&+(-v_3+v_{111}+3v_1^2)\delta v_2\wedge dx^{23} + v_2\delta v_3 dx^{23} + 2v_{12}\delta v_{11}\wedge dx^{23}\\
&- v_2\delta v_{111}\wedge dx^{23}\\
&- d\big(-v_1\delta v\wedge dx^1 + v_2\delta v\wedge dx^2 + (v_3-2v_{111}-6v_1^2)\delta v\wedge dx^3\\
&\qquad + v_{11}\delta v_1\wedge dx^3 - v_1\delta v_{11}\wedge dx^3\big)\\
\equiv& \mathcal{E}(\mathscr{L}) - d\Omega^{(1)}\end{aligned}$$

where we define $\Omega^{(1)} = -v_1\delta v\wedge dx^1 + v_2\delta v\wedge dx^2 + (v_3-2v_{111}-6v_1^2)\delta v\wedge dx^3 + v_{11}\delta v_1\wedge dx^3 - v_1\delta v_{11}\wedge dx^3$. We see that $\mathcal{E}(\mathscr{L}) = \delta\mathscr{L} + d\Omega^{(1)} = 0$ is equivalent to the equations (5.21) and differential consequences. The symplectic multiform is then

$$\begin{aligned}\Omega =& -\delta v_1\wedge\delta v\wedge dx^1 + \delta v_2\wedge\delta v\wedge dx^2 + \delta v_3\wedge\delta v\wedge dx^3\\
&-2\delta v_{111}\wedge\delta v\wedge dx^3 - 12v_1\delta v_1\wedge\delta v\wedge dx^3 + 2\delta v_{11}\wedge\delta v_1\wedge dx^3.\end{aligned}$$

Multiform Hamilton equations The Hamiltonian multiform is computed as $\mathcal{H} = \sum_{i\leq j} H_{ij}\, dx^{ij}$, using $H_{ij} = \widetilde{\partial}_i\lrcorner\omega_j^{(1)} - \widetilde{\partial}_j\lrcorner\omega_i^{(1)} - L_{ij}$, and we find

$$H_{12} = v_1v_2, \tag{5.22a}$$
$$H_{23} = -3v_1^2v_2 - v_{111}v_2 \tag{5.22b}$$
$$H_{13} = v_1v_3 - 4v_1^3 + v_{11}^2 - 2v_1v_{111}. \tag{5.22c}$$

The multiform Hamiltonian equations are obtained as

- $\delta H_{12} = \widetilde{\partial}_2\lrcorner\omega_1 - \widetilde{\partial}_1\lrcorner\omega_2$:

$$v_1\delta v_2 + v_2\delta v_1 = -v_{12}\delta v + v_2\delta v_1 - v_{12}\delta v + v_1\delta v_2 \quad\Longrightarrow\quad v_{12}=0.$$

- $\delta H_{23} = \widetilde{\partial}_3\lrcorner\omega_2 - \widetilde{\partial}_2\lrcorner\omega_3$:

$$\begin{aligned}&-3v_1^2\delta v_2 - 6v_1v_2\delta v_1 - v_{111}\delta v_2 - v_2\delta v_{111}\\
&\quad= v_{23}\delta v - v_3\delta v_2 - v_{23}\delta v + v_2\delta v_3 + 2v_{1112}\delta v - 2v_2\delta v_{111}\\
&\qquad + 12v_1v_{12}\delta v - 12v_1v_2\delta v_1 - 2v_{112}\delta v_1 + 2v_{12}\delta v_{11}\end{aligned}$$

which implies the following system of equations

$$\begin{aligned}&v_2 = 0\,, && v_{12}=0\,,\\
&v_3 - 3v_1^2 - v_{111} = 0\,, && v_{112}+3v_1v_2 = 0\,,\\
&v_{1112}+6v_1v_{12} = 0\,.\end{aligned}$$

- $\delta H_{13} = \widetilde{\partial}_3 \lrcorner \omega_1 - \widetilde{\partial}_1 \lrcorner \omega_3$:

$$\begin{aligned} & v_1 \delta v_3 + v_3 \delta v_1 - 12 v_1^2 \delta v_1 + 2 v_{11} \delta v_{11} - 2 v_1 \delta v_{111} - 2 v_{111} \delta v_1 \\ & \quad = -v_{13} \delta v + v_3 \delta v_1 - v_{13} \delta v + v_1 \delta v_3 + 2 v_{1111} \delta v \\ & \qquad - 2 v_1 \delta v_{111} + 12 v_1 v_{11} \delta v - 12 v_1^2 \delta v_1 - 2 v_{111} \delta v_1 + 2 v_{11} \delta v_{11}, \end{aligned}$$

  which implies $v_{13} - v_{1111} - 6 v_1 v_{11} = 0$.

This system of equations is equivalent to (5.21) as expected.

Admissible forms and conservation laws We now describe admissible forms for this case. A 1-form $Q = Q_1(v, v_1)\, dx^1 + Q_2(v, v_2)\, dx^2 + Q_3(v, v_1, v_3, v_{11}, v_{111})\, dx^3$ for the symplectic multiform Ω is admissible if and only if

$$\frac{\partial Q_1}{\partial v_1} = -\frac{\partial Q_2}{\partial v_2} = -\frac{\partial Q_3}{\partial v_3} = \frac{1}{2} \frac{\partial Q_3}{\partial v_{111}}\,, \tag{5.23a}$$

$$\frac{\partial Q_1}{\partial v} = \frac{1}{2} \frac{\partial Q_3}{\partial v_{11}}\,. \tag{5.23b}$$

Its related Hamiltonian vector field is

$$\begin{aligned} \xi_Q = & \frac{\partial Q_1}{\partial v_1} \partial_v - \frac{\partial Q_1}{\partial v} \partial_{v_1} + \frac{\partial Q_2}{\partial v} \partial_{v_2} \\ & + \left(\frac{\partial Q_3}{\partial v} - 6 v_1 \frac{\partial Q_3}{\partial v_{11}} \right) \partial_{v_3} + \left(\frac{1}{2} \frac{\partial Q_3}{\partial v_1} - 3 v_1 \frac{\partial Q_3}{\partial v_{111}} \right) \partial_{v_{11}}. \end{aligned} \tag{5.24}$$

This can be proved as followed: one takes a generic vector field

$$\xi_Q = A \partial_v + B \partial_{v_1} + C \partial_{v_2} + D \partial_{v_3} + E \partial_{v_{11}} + D \partial_{v_{111}}$$

and determines the coefficients comparing the right and left hand-side of $\xi_Q \lrcorner \Omega = \delta Q$. This translates into constraints on the derivatives of Q_i with respect to the field and its derivatives, and determines the coefficients of the vector field.
We also verify that for any admissible 1-form Q and modulo the equations of motion $dQ = \xi_Q \lrcorner \delta \mathcal{H}$, or, more explicitly

- $\partial_1 Q_2 - \partial_2 Q_1 = \xi_Q \lrcorner \delta H_{12}$, which means

$$\begin{aligned} \frac{\partial Q_2}{\partial v} v_1 + \frac{\partial Q_2}{\partial v_2} v_{12} - \frac{\partial Q_1}{\partial v} v_2 - \frac{\partial Q_1}{\partial v_1} v_{12} &= -\frac{\partial Q_1}{\partial v} \frac{\partial H_{12}}{\partial v_1} + \frac{\partial Q_2}{\partial v} \frac{\partial H_{12}}{\partial v_2} \\ &= -\frac{\partial Q_1}{\partial v} v_2 + \frac{\partial Q_2}{\partial v} v_1 \\ \implies \quad & -2 v_{12} \frac{\partial Q_1}{\partial v_1} = 0. \end{aligned}$$

- $\partial_2 Q_3 - \partial_3 Q_2 = \xi_Q \lrcorner \delta H_{23}$, which means

$$\begin{aligned}
&\frac{\partial Q_3}{\partial v}v_2 + \frac{\partial Q_3}{\partial v_1}v_{12} + \frac{\partial Q_3}{\partial v_3}v_{23} + \frac{\partial Q_3}{\partial v_{11}}v_{112} + \frac{\partial Q_3}{\partial v_{111}}v_{1112} - \frac{\partial Q_2}{\partial v}v_3 - \frac{\partial Q_2}{\partial v_2}v_{23}\\
&\quad = -\frac{\partial Q_1}{\partial v}\frac{\partial H_{23}}{\partial v_1} + \frac{\partial Q_2}{\partial v}\frac{\partial H_{23}}{\partial v_2}\\
&\quad = 6v_1v_2\frac{\partial Q_1}{\partial v} - (3v_1^2 + v_{111})\frac{\partial Q_2}{\partial v},
\end{aligned}$$

which again is

$$v_2\frac{\partial Q_3}{\partial v} + v_{12}\frac{\partial Q_3}{\partial v_1} + 2v_{112}\frac{\partial Q_1}{\partial v_1} + (2v_{112} - 6v_1v_2)\frac{\partial Q_1}{\partial v} + (-v_3 + 3v_1^2 + v_{111})\frac{\partial Q_2}{\partial v} = 0.$$

- $\partial_1 Q_3 - \partial_3 Q_1 = \xi_Q \lrcorner \delta H_{13}$, which means

$$\begin{aligned}
&\frac{\partial Q_3}{\partial v}v_1 + \frac{\partial Q_3}{\partial v_1}v_{11} + \frac{\partial Q_3}{\partial v_3}v_{13} + \frac{\partial Q_3}{\partial v_{11}}v_{111} + \frac{\partial Q_3}{\partial v_{111}}v_{1111} - \frac{\partial Q_1}{\partial v}v_3 - \frac{\partial Q_1}{\partial v_1}v_{13}\\
&\quad = \; -\frac{\partial Q_1}{\partial v}\frac{\partial H_{13}}{\partial v_1} + \left(\frac{\partial Q_3}{\partial v} - 6v_1\frac{\partial Q_3}{\partial v_{111}}\right)\frac{\partial H_{13}}{\partial v_3} + \left(\frac{1}{2}\frac{\partial Q_3}{\partial v_1} - 3v_1\frac{\partial Q_3}{\partial v_{111}}\right)\frac{\partial H_{13}}{\partial v_{11}}\\
&\quad = (12v_1^2 + 2v_{111} - v_3)\frac{\partial Q_1}{\partial v} + v_1\frac{\partial Q_3}{\partial v} - 6v_1^2\frac{\partial Q_3}{\partial v_{11}} + v_{11}\frac{\partial Q_3}{\partial v_1} - 6v_1v_{11}\frac{\partial Q_3}{\partial v_{111}},
\end{aligned}$$

which again is $(2v_{13} - 2v_{1111} - 12v_1v_{11})\frac{\partial Q_3}{\partial v_3} = 0$.

We can find a conservation law for the Lagrangian multiform $\mathscr{L}$, i.e. a admissible 1-form $F = F_1(v,v_1)\,dx^1 + F_2(v,v_2)\,dx^2 + F_3(v,v_1,v_3,v_{11},v_{111})\,dx^3$ such that $\xi_F \lrcorner \delta\mathcal{H} = \xi_F\mathcal{H} = 0$:

- $\xi_F H_{12} = 0$ means that $-\frac{\partial F_1}{\partial v}v_2 + \frac{\partial F_2}{\partial v}v_1 = 0$. Since $\frac{\partial F_1}{\partial v_1} = -\frac{\partial F_2}{\partial v_2}$, necessarily $F_1 = a(v)v_1 + b(v)$ and $F_2 = -a(v)v_2 + c(v)$ for some a, b, c smooth functions of v. The condition above then translates to

$$-a'(v)v_1v_2 - b'(v)v_2 - a'(v)v_1v_2 + c'(v)v_1 = 0 \qquad \Longrightarrow \qquad a'(v) = b'(v) = c'(v) = 0.$$

  We will set $a = 1$, and $b = c = 0$, so we have $F_1 = v_1$ and $F_2 = -v_2$.

- $\xi_F H_{23} = 6v_1v_2\frac{\partial F_1}{\partial v} - (3v_1^2 + v_{111})\frac{\partial F_2}{\partial v} = 0$ automatically.

- Because of the admissibility constraint we have that $F_3 = -v_3 + 2v_{111} + d(v,v_1)$ where d is a smooth function of v, v_1. Now we solve for d the equation $\xi_F H_{13} = (12v_1^2 + 2v_{111} - v_3)\frac{\partial F_1}{\partial v} + v_1\frac{\partial F_3}{\partial v} - 6v_1^2\frac{\partial F_3}{\partial v_{11}} + v_{11}\frac{\partial F_3}{\partial v_1} - 6v_1v_{11}\frac{\partial F_3}{\partial v_{111}} = v_1\frac{\partial d(v,v_1)}{\partial v} + v_{11}\frac{\partial d(v,v_1)}{\partial v_1} - 12v_1v_{11} = 0$. This implies

$$\frac{\partial d}{\partial v} = 0, \quad \frac{\partial d}{\partial v_1} = 12v_1, \qquad \Longrightarrow \qquad d = 6v_1^2.$$

A conservation law is then

$$F = v_1 dx^1 - v_2 dx^2 + (-v_3 + 2v_{111} + 6v_1^2) dx^3. \tag{5.25}$$

In fact its differential dF is

$$\begin{aligned} & v_{12} dx^{21} + v_{13} dx^{31} - v_{12} dx^{12} - v_{23} dx^{32} \\ & \quad + (-v_{13} + 2v_{1111} + 12 v_1 v_{11}) dx^{13} + (-v_{23} + 2v_{1112} + 12 v_1 v_{12}) dx^{23} \\ & \quad\quad = -2v_{12} dx^{12} + (-2v_{13} + 2v_{1111} + 12 v_1 v_{11}) dx^{13} + (2v_{1112} + 12 v_1 v_{12}) dx^{23} \end{aligned}$$

which vanishes on the equations of motion.

Another Hamiltonian multiform formulation We now mention how to compute another symplectic multiform (and its related Hamiltonian multiform). One can perform an equivalent computation to the one above, making different choices as to what to apply $\delta d = -d\delta$ on, and obtain

$$\begin{aligned} \widetilde{\Omega^{(1)}} = & - v_1 \delta v \wedge dx^1 + \frac{v_2}{2} \delta v \wedge dx^2 \\ & + \frac{1}{2}(v_3 - 9v_1^2 - 3v_{111}) \delta v \wedge dx^3 + v_{11} \delta v_1 \wedge dx^3 - v_1 \delta v_{11} \wedge dx^3. \end{aligned} \tag{5.26}$$

Indeed it is easy to check that also $\delta\Lambda + d\widetilde{\Omega^{(1)}} = 0$ is equivalent to (5.21). Moreover, we notice that $d(\Omega^{(1)} - \widetilde{\Omega^{(1)}}) = 0$ is also equivalent to the equations of motion, as it should be since $d\Omega^{(1)} - d\widetilde{\Omega^{(1)}} = \delta\mathscr{L} - \delta\mathscr{L} = 0$. We then define

$$\begin{aligned} \widetilde{\Omega} = & - \delta v_1 \wedge \delta v \wedge dx^1 + \frac{1}{2} \delta v_2 \wedge \delta v \wedge dx^2 + \frac{1}{2} \delta v_3 \wedge \delta v \wedge dx^3 \\ & - 9 v_1 \delta v_1 \wedge \delta v \wedge dx^3 - \frac{3}{2} \delta v_{111} \wedge \delta v \wedge dx^3 + 2 \delta v_{11} \wedge \delta v_1 \wedge dx^3. \end{aligned} \tag{5.27}$$

The coefficients of Hamiltonian multiform $\widetilde{\mathcal{H}} = \widetilde{H}_{12}\, dx^{12} + \widetilde{H}_{23}\, dx^{23} + \widetilde{H}_{13}\, dx^{13}$ are

$$\widetilde{H}_{12} = \frac{1}{2} v_1 v_2\,, \tag{5.28a}$$

$$\widetilde{H}_{23} = - \frac{3}{2} v_1^2 v_2 - \frac{1}{2} v_2 v_{111}\,, \tag{5.28b}$$

$$\widetilde{H}_{13} = \frac{1}{2} v_1 v_3 + v_{11}^2 - \frac{5}{2} v_1^3 - \frac{5}{2} v_1 v_{111} \tag{5.28c}$$

and the multiform Hamilton equations for $\widetilde{\mathcal{H}}$ and $\widetilde{\Omega}$ bring the same set of equations as expected.

5.3.2 Times 1,3 and 5

In the previous section we considered the times 1 2 and 3 of (potential) KdV hierarchy. We can also describe the odd-time flows 1, 3 and 5, using the Lagrangian multiform

$\mathscr{L} = L_{13}\,dx^{13} + L_{15}\,dx^{15} + L_{35}\,dx^{35}$, where

$$\begin{aligned} L_{13} &= -\,2v_1^3 + v_1 v_3 - v_1 v_{111}\,, && (5.29a)\\ L_{15} &= -\,5v_1^4 + 10v_1 v_{11}^2 + v_1 v_5 - v_{111}^2\,, && (5.29b)\\ L_{35} &= 6v_1^5 - 10v_1^3 v_3 + 20v_1^3 v_{111} - 15v_1^2 v_{11}^2 + 3v_1^2 v_5 + 3v_1^2 v_{11111}\\ &\quad - 10v_1 v_3 v_{111} + 20v_1 v_{11} v_{13} - 12v_1 v_{11} v_{1111} + 6v_1 v_{111}^2\\ &\quad - 5v_3 v_{11}^2 + 7v_{11}^2 v_{111} + v_1 v_{115} - v_3 v_{11111} + v_5 v_{111} - v_{11} v_{15} && (5.29c)\\ &\quad + 2v_{13} v_{1111} - 2v_{111} v_{113} + v_{111} v_{11111} - v_{1111}^2\,. \end{aligned}$$

The multiform Euler-Lagrange equations are equivalent to

$$v_3 = v_{111} + 3v_1^2, \qquad v_5 = v_{11111} + 10v_1^3 + 5v_{11}^2 + 10v_1 v_{111} \tag{5.30}$$

and differential consequences. If we define the form $\Omega^{(1)}$ as

$$\begin{aligned} \Omega^{(1)} =& -v_1 \delta v \wedge dx^1 + (v_3 - 2v_{111} - 6v_1^2)\delta v \wedge dx^3 + v_{11}\delta v_1 \wedge dx^3 - v_1 \delta v_{11} \wedge dx^3\\ &+ (v_5 - 20v_1^3 - 20v_1 v_{111} - 10v_{11}^2 - 2v_{11111})\delta v \wedge dx^5\\ &+ (20v_1 v_{11} + 2v_{1111})\delta v_1 \wedge dx^5 - 2v_{111}\delta v_{11} \wedge dx^5, \end{aligned} \tag{5.31}$$

one can check that $\delta\Lambda + d\Omega^{(1)} = 0$ is equivalent to (5.30). The symplectic multiform is then $\Omega = \omega_1 \wedge dx^1 + \omega_3 \wedge dx^3 + \omega_5 \wedge dx^5$, where

$$\begin{aligned} \omega_1 &= \delta v \wedge \delta v_1\,, && (5.32a)\\ \omega_3 &= \delta v_3 \wedge \delta v - 2\delta v_{111} \wedge \delta v + 2\delta v_{11} \wedge \delta v_1 - 12v_1 \delta v_1 \wedge \delta v\,, && (5.32b)\\ \omega_5 &= \delta v_5 \wedge \delta v + (60v_1^2 + 20v_{111})\delta v \wedge \delta v_1 - 20v_1 \delta v_{111} \wedge \delta v\\ &\quad - 20v_{11}\delta v_{11} \wedge \delta v - 2\delta v_{11111} \wedge \delta v + 20v_1 \delta v_{11} \wedge \delta v_1 && (5.32c)\\ &\quad + 2\delta v_{1111} \wedge \delta v_1 - 2\delta v_{111} \wedge \delta v_{11}\,. \end{aligned}$$

The Hamiltonian multiform is obtained in the usual way and reads $\mathcal{H} = H_{13}\,dx^{13} + H_{35}\,dx^{35} + H_{15}\,dx^{15}$ where

$$\begin{aligned} H_{13} &= v_1 v_3 + v_{11}^2 - 2v_1 v_{111} - 4v_1^3\,, && (5.33a)\\ H_{15} &= v_1 v_5 - 15v_1^4 - 20v_1^2 v_{111} - 2v_{11111} v_1 + 2v_{1111} v_1 - v_{111}^2\,, && (5.33b)\\ H_{35} &= -\,10v_1^3 v_3 - 10v_1 v_{111} v_3 - 5v_{11}^2 v_3 - v_{11111} v_3 + v_{111} v_5\\ &\quad + 3v_1^2 v_5 - 6v_1^5 - 20v_1^3 v_{111} + 15v_1^2 v_{11}^2 - 3v_1^2 v_{11111} + 12v_1 v_{11} v_{1111} && (5.33c)\\ &\quad - 6v_1 v_{111}^2 - 7v_{11}^2 v_{111} - v_{111} v_{11111} + v_{1111}^2\,. \end{aligned}$$

One can then proceed in a similar way to the 123-times case and verify the validity of the multiform Hamilton equations:

$$\begin{aligned}\delta H_{13} =&\widetilde{\partial}_3 \lrcorner \omega_1 - \widetilde{\partial}_1 \lrcorner \omega_3\\ \delta H_{35} =&\widetilde{\partial}_5 \lrcorner \omega_3 - \widetilde{\partial}_3 \lrcorner \omega_5\\ \delta H_{15} =&\widetilde{\partial}_5 \lrcorner \omega_1 - \widetilde{\partial}_1 \lrcorner \omega_5 .\end{aligned}$$

We obtain that a 1-form

$$F = F_1(v, v_1)\, dx^1 + F_3(v, v_1, v_3, v_{11}, v_{111})\, dx^3 + F_5(v, v_1, v_5, v_{11}, v_{111}, v_{1111}, v_{11111})\, dx^5$$

is admissible if and only if

$$\frac{\partial F_1}{\partial v_1} = \frac{1}{2}\frac{\partial F_3}{\partial v_{111}} = \frac{1}{2}\frac{\partial F_5}{\partial v_{11111}} = -\frac{\partial F_3}{\partial v_3} = -\frac{\partial F_5}{\partial v_5}\,, \tag{5.34a}$$

$$\frac{\partial F_1}{\partial v} = \frac{1}{2}\frac{\partial F_3}{\partial v_{11}} = \frac{1}{2}\frac{\partial F_5}{\partial v_{1111}}\,, \tag{5.34b}$$

$$\frac{\partial F_5}{\partial v_{111}} = \frac{\partial F_3}{\partial v_1} + 4v_1\frac{\partial F_5}{\partial v_{11111}}\,. \tag{5.34c}$$

Its related Hamiltonian vector field is

$$\begin{aligned}\xi_F =&\frac{\partial F_1}{\partial v_1}\partial_v - \frac{\partial F_1}{\partial v}\partial_{v_1} + \left(\frac{\partial F_3}{\partial v} + +10v_{11}\frac{\partial F_3}{\partial v_{111}} + 4v_1\frac{\partial F_3}{\partial v_{11}} - \frac{\partial F_5}{\partial v_{11}}\right)\partial_{v_3}\\ &+ \left(\frac{\partial F_5}{\partial v} - 10v_1\frac{\partial F_5}{\partial v_{11}} + 10v_{11}\frac{\partial F_5}{\partial v_{111}} + (70v_1^2 - 10v_{111})\frac{\partial F_5}{\partial v_{1111}}\right)\partial_{v_5}\\ &+ \left(\frac{1}{2}\frac{\partial F_3}{\partial v_1} - 3v_1\frac{\partial F_3}{\partial v_{111}}\right)\partial_{v_{11}} + \left(-\frac{1}{2}\frac{\partial F_5}{\partial v_{11}} + 5v_{11}\frac{\partial F_5}{\partial v_{11111}} + 5v_1\frac{\partial F_5}{\partial v_{1111}}\right)\partial_{v_{111}}\\ &+ \left(\frac{1}{2}\frac{\partial F_5}{\partial v_1} - 5v_1\frac{\partial F_5}{\partial v_{111}} + (35v_1^2 - 5v_{111})\frac{\partial F_5}{\partial v_{11111}}\right)\partial_{v_{1111}}\,.\end{aligned} \tag{5.35}$$

From the equations (5.34) one can obtain an admissible conservation law:

$$\begin{aligned}F =&v_1\, dx^1 + (-v_3 + 2v_{111} + 6v_1^2)\, dx^3\\ &+ (-v_5 + 2v_{11111} + 20v_1v_{111} + 10v_{11}^2 + 20v_1^3)\, dx^5\,.\end{aligned} \tag{5.36}$$

In fact we have that $\partial_3 v_1 = \partial_1(-v_3 + 2v_{11} + 6v_1^2)$ implies

$$v_{13} = (v_{111} + 3v_1^2)_1; \tag{5.37}$$

the second equation $\partial_5 v_1 = \partial_1(-v_5 + 2v_{11111} + 20v_1v_{111} + 10v_{11}^2 + 20v_1^3)$ instead implies

$$v_{15} = (v_{11111} + 10v_1^3 + 5v_{11}^2 + 10v_1v_{111})_1; \tag{5.38}$$

and then $\partial_5(-v_3 + 2v_{11} + 6v_1^2) = \partial_3(-v_5 + 2v_{11111} + 20v_1v_{111} + 10v_{11}^2 + 20v_1^3)$ is satisfied using the previous equations.

5.4 sine-Gordon hierarchy

In this section we will show another example, i.e. the first two levels of the sine-Gordon hierarchy in light-cone coordinates. A Lagrangian multiform for this set of equations has been obtained for the first time in [S16] and is $\mathscr{L} = L_{12}\,dx^{12} + L_{13}\,dx^{13} + L_{23}\,dx^{23}$, where

$$L_{12} =\frac{1}{2}u_1u_2 + \cos u, \tag{5.39a}$$
$$L_{13} =\frac{1}{2}u_1u_3 + \frac{1}{2}u_{11}^2 - \frac{1}{8}u_1^4, \tag{5.39b}$$
$$L_{23} =-\frac{1}{2}u_2u_3 + u_{11}u_{12} + u_{11}\sin u - \frac{1}{2}u_1^2\cos u. \tag{5.39c}$$

The multiform Euler-Lagrange equations $d\delta\mathscr{L} = 0$ are equivalent to

$$u_{12} + \sin u = 0\,, \qquad u_3 - \frac{1}{2}u_1^3 - u_{111} = 0 \tag{5.40}$$

and differential consequences.

The symplectic and Hamiltonian multiform An similar computation to the ones above yields the form $\Omega^{(1)}$ as

$$\Omega^{(1)} = -\frac{1}{2}u_1\delta u\wedge dx^1 + \frac{1}{2}u_2\delta u\wedge dx^2 - (\frac{u_{111}}{2} + \frac{u_1^3}{4})\delta u\wedge dx^3 + u_{11}\delta u_1\wedge dx^3. \tag{5.41}$$

The δ-differential of $\Omega^{(1)}$ is the symplectic multiform $\Omega = \omega_1\wedge dx^1 + \omega_2\wedge dx^2 + \omega_3\wedge dx^3$, with

$$\omega_1 =\frac{1}{2}\delta u\wedge\delta u_1, \tag{5.42a}$$
$$\omega_2 =\frac{1}{2}\delta u_2\wedge\delta u, \tag{5.42b}$$
$$\omega_3 =-\frac{1}{2}\delta u_{111}\wedge\delta u - \frac{3u_1^2}{4}\delta u_1\wedge\delta u + \delta u_{11}\wedge\delta u_1. \tag{5.42c}$$

The Hamiltonian multiform $\mathcal{H} = H_{12}\,dx^{12} + H_{13}\,dx^{13} + H_{23}\,dx^{23}$ is computed as

$$H_{12} =\frac{1}{2}u_1u_2 + \cos u\,, \tag{5.43a}$$
$$H_{13} =-\frac{1}{2}u_1u_{111} + \frac{1}{2}u_{11}^2 - \frac{1}{8}u_1^4\,, \tag{5.43b}$$
$$H_{23} =-\frac{1}{2}u_2u_{111} - \frac{1}{4}u_1^3u_2 + u_{11}\sin u - \frac{1}{2}u_1^2\cos u\,. \tag{5.43c}$$

The multiform Hamilton equations are obtained as $\delta\mathcal{H} = \sum_{j=1}^{3} dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega$ and are equivalent to the multiform Euler-Lagrange equations, as required. In particular we have for $\delta H_{12} = \widetilde{\partial}_2 \lrcorner \omega_1 - \widetilde{\partial}_1 \lrcorner \omega_2$

$$\frac{u_1}{2}\delta u_2 + \frac{u_2}{2}\delta u_1 + \sin u\, \delta u = \frac{u_2}{2}\delta u_1 - \frac{u_{12}}{2}\delta u - \frac{u_{12}}{2}\delta u + \frac{u_1}{2}\delta u_2$$

which implies $u_{12} + \sin u = 0$, and for $\delta H_{23} = \widetilde{\partial}_3 \lrcorner \omega_2 - \widetilde{\partial}_2 \lrcorner \omega_3$

$$\begin{aligned}
&(u_{11}\cos u - \frac{u_1^2}{2}\sin u)\delta u + (-\frac{3}{4}u_1^2 u_2 + u_1 \cos u)\delta u_1 \\
&\quad + (-\frac{u_{111}}{2} - \frac{u_1^3}{4})\delta u_2 - \sin u \delta u_{11} - \frac{u_2}{2}\delta u_{111} \\
&\qquad\qquad = \frac{u_{23}}{2}\delta u - \frac{u_3}{2}\delta u_2 + \frac{u_{1112}}{2}\delta u - \frac{u_2}{2}\delta u_{111} \\
&\qquad\qquad\quad + \frac{3}{4}u_1^2 u_{12}\delta u - \frac{3}{4}u_1^2 u_2 \delta u_1 - u_{112}\delta u_1 + u_{12}\delta u_{11}
\end{aligned}$$

which implies

$$\begin{aligned}
&u_{12} + \sin u = 0\,, && u_3 = u_{111} + \frac{1}{2}u_1^3\,, \\
&u_{112} + u_1 \cos u = 0\,, && \frac{u_{23}}{2} + \frac{u_{1112}}{2} + \frac{3}{4}u_1^2 u_{12} + u_{11}\cos u + \frac{u_1^2}{2}\sin u = 0\,.
\end{aligned}$$

where the last two equations are differential consequences of the first two. Finally $\delta H_{13} = \widetilde{\partial}_3 \lrcorner \omega_1 - \widetilde{\partial}_1 \lrcorner \omega_3$:

$$\begin{aligned}
&-(\frac{u_{111}}{2} + \frac{u_1^3}{2})\delta u_1 + u_{11}\delta u_{11} - \frac{u_1}{2}\delta u_{111} \\
&\qquad\qquad = \frac{u_3}{2}\delta u_1 - \frac{u_{13}}{2}\delta u + \frac{u_{1111}}{2}\delta u - \frac{u_1}{2}\delta u_{111} + \frac{3}{4}u_1^2 u_{11}\delta u \\
&\qquad\qquad\quad - \frac{3}{4}u_1^3 \delta u_1 - u_{111}\delta u_1 + u_{11}\delta u_{11}
\end{aligned}$$

which implies again $u_3 = u_{111} + u_1^3/2$ and its differential consequence $u_{13} = u_{1111} + 3u_1^2 u_{11}/2$.

Admissible forms and multi-time Poisson brackets One can then investigate the presence of admissible forms:

- A 0-form $H(u, u_1, u_2, u_{11}, u_{111})$ is always admissible, with Hamiltonian vector field

$$\begin{aligned}
\xi_H = &\left(2\frac{\partial H}{\partial u_1} - 3u_1^2 \frac{\partial H}{\partial u_{111}}\right)\partial_u \wedge \partial_1 - 2\frac{\partial H}{\partial u_2}\partial_u \wedge \partial_2 \\
&+ 2\frac{\partial H}{\partial u_{111}}\partial_u \wedge \partial_3 + \left(\frac{3}{2}u_1^2 \frac{\partial H}{\partial u_{11}} - 2\frac{\partial H}{\partial u}\right)\partial_{u_1} \wedge \partial_1 - \frac{\partial H}{\partial u_{11}}\partial_{u_1} \wedge \partial_3.
\end{aligned}$$

  We remark that ξ_H is not unique;

- A 1-form $P = P_1\, dx^1 + P_2\, dx^2 + P_3\, dx^3$ is admissible if and only if $P_1 = P_1(u, u_1)$,

$P_2 = P_2(u, u_2)$, $P_3 = P_3(u, u_1, u_{11}, u_{111})$, and

$$\frac{\partial P_3}{\partial u_{11}} = 2\frac{\partial P_1}{\partial u}\,, \tag{5.44a}$$
$$\frac{\partial P_3}{\partial u_{111}} = -\frac{\partial P_2}{\partial u_2} = \frac{\partial P_1}{\partial u_1}\,, \tag{5.44b}$$

and its related vector field is

$$\begin{aligned}\xi_P =& 2\frac{\partial P_1}{\partial u_1}\partial_u - 2\frac{\partial P_1}{\partial u}\partial_{u_1} + 2\frac{\partial P_2}{\partial u}\partial_{u_2} \\ &+ \left(\frac{\partial P_3}{\partial u_1} - \frac{3}{2}u_1^2\frac{\partial P_3}{\partial u_{111}}\right)\partial_{u_{11}} + \left(\frac{3}{2}u_1^2\frac{\partial P_3}{\partial u_{11}} - 2\frac{\partial P_3}{\partial u}\right)\partial_{u_{111}}.\end{aligned} \tag{5.45}$$

- The only admissible 2-forms or 3-forms are the constant ones.

For such forms we can define the multi-time Poisson brackets. The Poisson bracket between an admissible 0-form H and an admissible 1-form $P = P_1\, dx^1 + P_2\, dx^2 + P_3\, dx^3$ is $\xi_P H$, therefore

$$\begin{aligned}\{\!\![H, P]\!\!\} =& 2\frac{\partial P_1}{\partial u_1}\frac{\partial H}{\partial u} - 2\frac{\partial P_1}{\partial u}\frac{\partial H}{\partial u_1} + 2\frac{\partial P_2}{\partial u}\frac{\partial H}{\partial u_2} - 2\frac{\partial P_3}{\partial u}\frac{\partial H}{\partial u_{111}} \\ &+ \frac{\partial P_3}{\partial u_1}\frac{\partial H}{\partial u_{11}} - \frac{3}{2}u_1^2\frac{\partial P_3}{\partial u_{111}}\frac{\partial H}{\partial u_{11}} + \frac{3}{2}u_1^2\frac{\partial P_3}{\partial u_{11}}\frac{\partial H}{\partial u_{111}}.\end{aligned} \tag{5.46}$$

If $P = \sum_{i=1}^{3} P_i dx^i$ and $Q = \sum_{i=1}^{3} Q_i dx^i$ are admissible 1-forms, then their Poisson bracket satisfies the decomposition

$$\{\!\![P, Q]\!\!\} = \{P_1, Q_1\}_1\, dx^1 + \{P_2, Q_2\}_2\, dx^2 + \{P_3, Q_3\}_3\, dx^3, \tag{5.47}$$

where

$$\{P_1, Q_1\}_1 = 2\frac{\partial P_1}{\partial u}\frac{\partial Q_1}{\partial u_1} - 2\frac{\partial P_1}{\partial u_1}\frac{\partial Q_1}{\partial u}\,, \tag{5.48a}$$
$$\{P_2, Q_2\}_2 = 2\frac{\partial P_2}{\partial u_2}\frac{\partial Q_2}{\partial u} - 2\frac{\partial P_2}{\partial u}\frac{\partial Q_2}{\partial u_2}\,, \tag{5.48b}$$
$$\begin{aligned}\{P_3, Q_3\}_3 =& 2\frac{\partial P_3}{\partial u}\frac{\partial Q_3}{\partial u_{111}} - 2\frac{\partial P_3}{\partial u_{111}}\frac{\partial Q_3}{\partial u} + \frac{\partial P_3}{\partial u_{11}}\frac{\partial Q_3}{\partial u_1} - \frac{\partial P_3}{\partial u_1}\frac{\partial Q_3}{\partial u_{11}} \\ &+ \frac{3}{2}u_1^2\frac{\partial P_3}{\partial u_{111}}\frac{\partial Q_3}{\partial u_{11}} - \frac{3}{2}u_1^2\frac{\partial P_3}{\partial u_{11}}\frac{\partial Q_3}{\partial u_{111}}\,.\end{aligned} \tag{5.48c}$$

Contrary to the potential KdV example (and AKNS example below), for the sine-Gordon hierarchy we were not able to find a admissible 1-form producing conservation laws in the sense of Definition 5.20. However, it is possible to find a non-admissible 1-form

$F = F_1\, dx^1 + F_2\, dx^2 + F_3\, dx^3$ that is closed on the equations of motion, as follows:

$$F_1 = \frac{1}{2}u_1^2\,, \tag{5.49a}$$

$$F_2 = \cos u\,, \tag{5.49b}$$

$$F_3 = \frac{3}{8}u_1^4 + u_1 u_{111} - \frac{1}{2}u_{11}^2\,. \tag{5.49c}$$

Then, on the equations of motion, one checks that

$$\partial_1 F_2 = \partial_2 F_1\,, \quad \partial_1 F_3 = \partial_3 F_1\,, \quad \partial_2 F_3 = \partial_3 F_2\,. \tag{5.50}$$

Thus, the sine-Gordon example points to a need to extend our approach to conservation laws beyond admissible forms.

5.5 The first four flows of Ablowitz-Kaup-Newell-Segur hierarchy

Our last example deals with the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy. For this example, we include one more time compared to previous example, to remind the reader that in principle we can keep adding more times in a multiform, corresponding to adding more and more flows in the hierarchy. However, as becomes clear in this example, the explicit expression soon becomes cumbersome. How to obtain a Lagrangian multiform for any number of flows of the AKNS hierarchy will be explained in Chapter 6.

Multiform Euler-Lagrange equations We start from the Lagrangian multiform adapted from the one in [SNC19b]

$$\mathscr{L} = L_{12}\, dx^{12} + L_{13}\, dx^{13} + L_{14}\, dx^{14} + L_{23}\, dx^{23} + L_{24}\, dx^{24} + L_{34}\, dx^{34}\,, \tag{5.51}$$

where

$$L_{12} = \frac{i}{2}(rq_2 - qr_2) - \frac{1}{2}q_1 r_1 - \frac{1}{2}q^2 r^2\,, \tag{5.52a}$$

$$L_{13} = \frac{i}{2}(rq_3 - qr_3) - \frac{i}{8}(r_1 q_{11} - q_1 r_{11}) - \frac{3iqr}{8}(rq_1 - qr_1)\,, \tag{5.52b}$$

$$\begin{aligned} L_{23} =& \frac{i}{4}(q_2 r_{11} - r_2 q_{11}) + \frac{1}{2}(q_3 r_1 + r_3 q_1) + \frac{i}{8}(q_1 r_{12} - r_1 q_{12}) \\ &+ \frac{3iqr}{8}(qr_2 - rq_2) + \frac{1}{8}q_{11} r_{11} - \frac{1}{4}qr(qr_{11} + rq_{11}) \\ &+ \frac{1}{8}(qr_1 - rq_1)^2 + \frac{1}{2}q^3 r^3\,, \end{aligned} \tag{5.52c}$$

and

$$\begin{aligned}
L_{14} =&\frac{i}{2}(rq_4 - qr_4) + \frac{5}{16}qr(qr_{11} + rq_{11})\\
&+ \frac{3}{16}(q^2r_1^2 + q_1^2r^2) + \frac{1}{4}qrq_1r_1 - \frac{1}{8}q_{11}r_{11} - \frac{1}{4}q^3r^3\,,
\end{aligned} \tag{5.53a}$$

$$\begin{aligned}
L_{24} =&\frac{3i}{8}q^2r^2(rq_1 - qr_1) + \frac{1}{16}(q^2r_1r_2 + r^2q_1q_2) + \frac{5}{16}qr(qr_{12} + rq_{12})\\
&- \frac{i}{8}qr(rq_{111} - qr_{111}) - \frac{i}{8}(q^2r_1r_{11} - ir^2q_1q_{11}) - \frac{i}{8}q_1r_1(rq_1 - qr_1)\\
&+ \frac{i}{4}qr(r_1q_{11} - q_1r_{11}) - \frac{3}{8}qr(q_1r_2 + r_1q_2) + \frac{1}{8}(q_{111}r_2 + r_{111}q_2)\\
&+ \frac{i}{16}(q_{111}r_{11} - r_{111}q_{11}) - \frac{1}{8}(q_{11}r_{12} + r_{11}q_{12}) + \frac{1}{2}(q_1r_4 + r_1q_4)\,,
\end{aligned} \tag{5.53b}$$

$$\begin{aligned}
L_{34} =& - \frac{1}{8}(q_{11}r_{13} + r_{11}q_{13}) + \frac{1}{8}(q_{111}r_3 + r_{111}q_3) + \frac{1}{32}q_{111}r_{111}\\
&- \frac{1}{32}(q^2r_{11}^2 + r^2q_{11}^2) - \frac{1}{32}q_1^2r_1^2 + \frac{3i}{8}qr(rq_4 - qr_4) - \frac{9}{32}q^4r^4\\
&+ \frac{3}{16}q^2r^2(qr_{11} + rq_{11}) + \frac{1}{16}(q^2r_1r_3 + r^2q_1q_3)\\
&+ \frac{5}{16}qr(qr_{13} + rq_{13}) + \frac{i}{4}(q_{11}r_4 - r_{11}q_4) - \frac{3}{16}qr(q_1r_{111} + r_1q_{111})\\
&- \frac{1}{16}qrq_{11}r_{11} + \frac{1}{16}q_1r_1(qr_{11} + rq_{11})\\
&+ \frac{15}{16}q^2r^2q_1r_1 - \frac{3}{8}qr(q_1r_3 + r_1q_3) - \frac{i}{8}(q_1r_{14} - r_1q_{14})\,,
\end{aligned} \tag{5.53c}$$

As proved in [SNC19b], the corresponding multiform Euler-Lagrange equations $\delta dL = 0$ produce the familiar first three levels of the AKNS hierarchy

$$iq_2 + \frac{1}{2}q_{11} - q^2r = 0\,, \qquad ir_2 - \frac{1}{2}r_{11} + qr^2 = 0\,, \tag{5.54a}$$

$$q_3 + \frac{1}{4}q_{111} - \frac{3}{2}qrq_1 = 0\,, \qquad r_3 + \frac{1}{4}r_{111} - \frac{3}{2}qrr_1 = 0\,, \tag{5.54b}$$

$$\begin{aligned}
iq_4 &= \frac{1}{8}q_{1111} + \frac{3}{4}q^3r^2 - \frac{1}{4}q^2r_{11} - \frac{1}{2}qq_1r_1 - qrq_{11} - \frac{3}{4}q_1^2r\,,\\
ir_4 &= -\frac{1}{8}r_{1111} - \frac{3}{4}q^2r^3 + \frac{1}{4}r^2q_{11} + \frac{1}{2}rq_1r_1 + qrr_{11} + \frac{3}{4}r_1^2q\,.
\end{aligned} \tag{5.54c}$$

The symplectic and Hamiltonian multiforms As done in the previous two examples, the computation of the form $\Omega^{(1)}$ from $\delta\mathscr{L}$ gives

$$\begin{aligned}
\Omega^{(1)} =& \left(-\frac{i}{2}r\delta q + \frac{i}{2}q\delta r\right) \wedge dx^1 + \left(-\frac{1}{2}q_1\delta r - \frac{1}{2}r_1\delta q\right) \wedge dx^2\\
&+ \left(\left(\frac{i}{4}r_{11} - \frac{3i}{8}qr^2\right)\delta q + \left(-\frac{i}{4}q_{11} + \frac{3i}{8}q^2r\right)\delta r - \frac{i}{8}r_1\delta q_1 + \frac{i}{8}q_1\delta r_1\right) \wedge dx^3\\
&+ \Bigg(\left(\frac{1}{8}r_{111} + \frac{1}{16}q_1r^2 - \frac{3}{8}qrr_1\right)\delta q + \left(\frac{1}{8}q_{111} + \frac{1}{16}q^2r_1 - \frac{3}{8}qrq_1\right)\delta r\\
&+ \left(-\frac{1}{8}r_{11} + \frac{5}{16}qr^2\right)\delta q_1 + \left(-\frac{1}{8}q_{11} + \frac{5}{16}q^2r\right)\delta r_1\Bigg) \wedge dx^4\,.
\end{aligned} \tag{5.55}$$

In fact, we have that $\delta\mathscr{L} = -d\Omega^{(1)}$ is equivalent to the equations (5.54). The δ-differential of $\Omega^{(1)}$ is the symplectic multiform

$$\Omega = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3 + \omega_4 \wedge dx^4, \tag{5.56}$$

where

$$\begin{aligned}
\omega_1 =& i\delta q \wedge \delta r\,, && (5.57a)\\
\omega_2 =& -\frac{1}{2}(\delta q_1 \wedge \delta r + \delta r_1 \wedge \delta q)\,, && (5.57b)\\
\omega_3 =& \frac{i}{4}(\delta r_{11} \wedge \delta q - \delta q_{11} \wedge \delta r) + \frac{i}{4}\delta q_1 \wedge \delta r_1 + \frac{3iqr}{2}\delta q \wedge \delta r\,, && (5.57c)\\
\omega_4 =& \frac{1}{8}\delta r_{111} \wedge \delta q + \frac{1}{8}\delta q_{111} \wedge \delta r - \frac{1}{8}\delta r_{11} \wedge \delta q_1\\
&- \frac{1}{8}\delta q_{11} \wedge \delta r_1 - \frac{1}{4}r^2 \delta q_1 \wedge \delta q - \frac{1}{4}q^2 \delta r_1 \wedge \delta r && (5.57d)\\
&- qr\delta q_1 \wedge \delta r - qr\delta r_1 \wedge \delta q - \frac{1}{2}(q_1 r - q r_1)\delta q \wedge \delta r\,.
\end{aligned}$$

The Hamiltonian multiform $\mathcal{H} = H_{12}\,dx^{12} + H_{13}\,dx^{13} + H_{14}\,dx^{14} + H_{23}\,dx^{23} + H_{24}\,dx^{24} + H_{34}\,dx^{34}$ can now be computed and brings

$$\begin{aligned}
H_{12} =& \frac{1}{2}(-q_1 r_1 + q^2 r^2)\,, && (5.58a)\\
H_{13} =& \frac{i}{4}(r_{11} q_1 - q_{11} r_1) && (5.58b)\\
H_{14} =& -\frac{1}{8}(q_1^2 r^2 + q^2 r_1^2) + \frac{1}{8}(q_{111} r_1 + q_1 r_{111})\\
&- \frac{1}{8}q_{11} r_{11} - qr q_1 r_1 + \frac{1}{4}q^3 r^3\,, && (5.58c)\\
H_{23} =& -\frac{1}{8}q_{11} r_{11} + \frac{qr}{4}(q r_{11} + q_{11} r) - \frac{1}{8}(q r_1 - q_1 r)^2 - \frac{1}{2}q^3 r^3\,, && (5.58d)\\
H_{24} =& \frac{iqr}{8}(r q_{111} - q r_{111}) + \frac{3i}{8}q^2 r^2 (q r_1 - r q_1) + \frac{i q_1 r_1}{8}(r q_1 - q r_1)\\
&+ \frac{iqr}{4}(q_1 r_{11} - r_1 q_{11}) + \frac{i}{16}(q_{11} r_{111} - q_{111} r_{11}) && (5.58e)\\
&+ \frac{i}{8}(q^2 r_1 r_{11} - r^2 q_1 q_{11})\,,\\
H_{34} =& -\frac{1}{16}(q^2 r_1 r_3 + r^2 q_1 q_3) - \frac{1}{32}q_{111} r_{111} + \frac{1}{32}(q^2 r_{11}^2 + q_{11}^2 r^2) + \frac{1}{32}q_1^2 r_1^2\\
&+ \frac{9}{32}q^4 r^4 - \frac{3}{16}q^2 r^2 (q r_{11} + q_{11} r) + \frac{3}{16}qr(q_1 r_{111} + r_1 q_{111}) && (5.58f)\\
&+ \frac{1}{16}qr q_{11} r_{11} - \frac{1}{16}q_1 r_1 (q r_{11} + q_{11} r) - \frac{15}{16}q^2 r^2 q_1 r_1\,.
\end{aligned}$$

The multiform Hamilton equations are obtained as $\delta\mathcal{H} = \sum_{j=1}^4 dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega$. One checks with a direct computation that they indeed reproduce the set of equations (5.54). We remark that H_{12} and H_{13} are the covariant Hamiltonian densities of respectively the NLS equations and the modified KdV equation already obtained for the first time in [CS20a] and reported in Chapter 4.

Admissible forms and multi-time Poisson brackets We have the following facts:

Proposition 5.23 • *Any 0-form H is admissible;*

- *A 1-form*

$$F = F_1(q,r)\,dx^1 + F_2(q,r,q_1,r_1)\,dx^2 + F_3(q,r,q_1,r_1,q_{11},r_{11})\,dx^3 + F_4(q,r,q_1,r_1,q_{11},r_{11},q_{111},r_{111})\,dx^4$$

is admissible if and only if the following relations hold

$$\frac{\partial F_1}{\partial r} = 2i\frac{\partial F_2}{\partial r_1} = -4\frac{\partial F_3}{\partial r_{11}} = -8i\frac{\partial F_4}{\partial r_{111}}\,, \tag{5.59a}$$

$$\frac{\partial F_1}{\partial q} = -2i\frac{\partial F_2}{\partial q_1} = -4\frac{\partial F_3}{\partial q_{11}} = 8i\frac{\partial F_4}{\partial q_{111}}\,, \tag{5.59b}$$

$$\frac{\partial F_2}{\partial r} = 2i\frac{\partial F_3}{\partial r_1} = -4\frac{\partial F_4}{\partial r_{11}}\,, \tag{5.59c}$$

$$\frac{\partial F_2}{\partial q} = -2i\frac{\partial F_3}{\partial q_1} = -4\frac{\partial F_4}{\partial q_{11}}\,, \tag{5.59d}$$

$$\frac{\partial F_4}{\partial r_1} = -\frac{i}{4}qr\frac{\partial F_1}{\partial r} - \frac{i}{2}\frac{\partial F_3}{\partial r} + \frac{i}{4}q^2\frac{\partial F_1}{\partial q}\,, \tag{5.59e}$$

$$\frac{\partial F_4}{\partial q_1} = \frac{i}{4}qr\frac{\partial F_1}{\partial q} + \frac{i}{2}\frac{\partial F_3}{\partial q} - \frac{i}{4}r^2\frac{\partial F_1}{\partial r}\,. \tag{5.59f}$$

and its Hamiltonian vector field is

$$\begin{aligned}\xi_F = &- i\frac{\partial F_1}{\partial r}\frac{\partial}{\partial q} + i\frac{\partial F_1}{\partial q}\frac{\partial}{\partial r} - 2\frac{\partial F_2}{\partial r}\frac{\partial}{\partial q_1} - 2\frac{\partial F_2}{\partial q}\frac{\partial}{\partial r_1}\\ &+ 4i\left(6qr\frac{\partial F_3}{\partial r_{11}} + \frac{\partial F_3}{\partial r}\right)\frac{\partial}{\partial q_{11}} - 4i\left(6qr\frac{\partial F_3}{\partial q_{11}} + \frac{\partial F_3}{\partial q}\right)\frac{\partial}{\partial r_{11}}\\ &+ 8\left(\frac{\partial F_4}{\partial r} + 2q^2\frac{\partial F_4}{\partial q_{11}} + 8qr\frac{\partial F_4}{\partial r_{11}} + 4(q_1 r - r_1 q)\frac{\partial F_4}{\partial r_{111}}\right)\frac{\partial}{\partial q_{111}}\\ &+ 8\left(\frac{\partial F_4}{\partial q} + 2r^2\frac{\partial F_4}{\partial r_{11}} + 8qr\frac{\partial F_4}{\partial q_{11}} - 4(q_1 r - r_1 q)\frac{\partial F_4}{\partial q_{111}}\right)\frac{\partial}{\partial r_{111}}\,.\end{aligned} \tag{5.60}$$

Proof. We start by proving that every 0-form $H(q,r,q_1,r_1,q_{11},r_{11},q_{111},r_{111})$ is admissible. This is achieved by starting from a generic multi-vector field (with the right degree) that up to elements of the kernel of Ω is

$$\begin{aligned}\xi_H =&\Big(a_1\frac{\partial}{\partial q}+b_1\frac{\partial}{\partial r}\Big)\wedge\frac{\partial}{\partial x^1}\\&+\Big(a_2\frac{\partial}{\partial q}+b_2\frac{\partial}{\partial r}+c_2\frac{\partial}{\partial q_1}+d_2\frac{\partial}{\partial r_1}\Big)\wedge\frac{\partial}{\partial x^2}\\&+\Big(a_3\frac{\partial}{\partial q}+b_3\frac{\partial}{\partial r}+c_3\frac{\partial}{\partial q_1}+d_3\frac{\partial}{\partial r_1}+e_3\frac{\partial}{\partial q_{11}}+f_3\frac{\partial}{\partial r_{11}}\Big)\wedge\frac{\partial}{\partial x^3}\\&+\Big(a_4\frac{\partial}{\partial q}+b_4\frac{\partial}{\partial r}+c_4\frac{\partial}{\partial q_1}+d_4\frac{\partial}{\partial r_1}+e_4\frac{\partial}{\partial q_{11}}+f_4\frac{\partial}{\partial r_{11}}+g_4\frac{\partial}{\partial q_{111}}+h_4\frac{\partial}{\partial r_{111}}\Big)\wedge\frac{\partial}{\partial x^4}\end{aligned}$$

and we look for its coefficients by imposing $\xi_H \lrcorner \Omega = \delta H$. By using the explicit expression of Ω we get

$$\begin{aligned}\xi_H\lrcorner\Omega =&ia_1\delta q-ib_1\delta r\\&+\frac{1}{2}a_2\delta r_1+\frac{1}{2}b_2\delta q_1-\frac{1}{2}c_2\delta r-\frac{1}{2}d_2\delta q\\&-\frac{i}{4}a_3\delta r_{11}+\frac{3i}{2}qra_3\delta r+\frac{i}{4}b_3\delta q_{11}-\frac{3i}{2}qrb_3\delta q+\frac{i}{4}c_3\delta r_1-\frac{i}{4}c_3\delta r_1-\frac{i}{4}d_3\delta q_1\\&-\frac{1}{8}a_4\delta r_{111}+\frac{1}{4}r^2a_4\delta q_1+qra_4\delta r_1-\frac{1}{2}(q_1r-qr_1)a_4\delta r\\&-\frac{1}{8}b_4\delta q_{111}+\frac{1}{4}q^2b_4\delta r_1+qrb_4\delta q_1+\frac{1}{2}(q_1r-qr_1)b_4\delta q\\&+\frac{1}{8}c_4\delta r_{11}-\frac{1}{4}r^2c_4\delta q-qrc_4\delta q+\frac{1}{8}d_4\delta q_{11}-\frac{1}{4}q^2d_4\delta r-qrd_4\delta r\\&-\frac{1}{8}e_4\delta r_1-\frac{1}{8}f_4\delta q_1+\frac{1}{8}g_4\delta q+\frac{1}{8}h_4\delta r\\=&\delta H\,.\end{aligned}$$

We must therefore have the following relations

$$\begin{aligned}\frac{\partial H}{\partial q}=&\frac{1}{8}g_4-qrd_4-\frac{1}{4}r^2c_4+\frac{1}{2}(q_1r-qr_1)b_4-\frac{3i}{2}qrb_3-\frac{1}{2}d_2+ia_1\\\frac{\partial H}{\partial r}=&\frac{1}{8}h_4-qrc_4-\frac{1}{4}q^2d_4-\frac{1}{2}(q_1r-qr_1)a_4+\frac{3i}{2}qra_3-\frac{1}{2}c_2-ib_1\\\frac{\partial H}{\partial q_1}=&-\frac{1}{8}f_4+qrb_4+\frac{1}{4}r^2a_4-\frac{i}{4}d_3+\frac{1}{2}b_2\\\frac{\partial H}{\partial r_1}=&-\frac{1}{8}e_4+qra_4+\frac{1}{4}q^2b_4+\frac{i}{4}c_3+\frac{1}{2}a_2\\\frac{\partial H}{\partial q_{11}}=&\frac{1}{8}d_4+\frac{i}{4}b_3\\\frac{\partial H}{\partial r_{11}}=&\frac{1}{8}c_4-\frac{i}{4}a_3\\\frac{\partial H}{\partial q_{111}}=&-\frac{1}{8}b_4\\\frac{\partial H}{\partial r_{111}}=&-\frac{1}{8}a_4\end{aligned}$$

that can be always be solved as there are more variables than equations. Therefore for every H we can find a vector field ξ_H such that $\xi_H \lrcorner \Omega = \delta H$.

We now treat the case of a 1-form F. A generic vertical vector field has the expression

$$\xi_F = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial r} + c\frac{\partial}{\partial q_1} + d\frac{\partial}{\partial r_1} + e\frac{\partial}{\partial q_{11}} + f\frac{\partial}{\partial r_{11}} + g\frac{\partial}{\partial q_{111}} + h\frac{\partial}{\partial r_{111}}\,,$$

The insertion with Ω is

$$\begin{aligned}
\xi_F \lrcorner \Omega =& - ib\delta q \wedge dx^1 + ia\delta r \wedge dx^1 \\
&- \frac{1}{2}d\delta q \wedge dx^2 - \frac{1}{2}c\delta r \wedge dx^2 + \frac{1}{2}b\delta q_1 \wedge dx^2 + \frac{1}{2}a\delta r_1 \wedge dx^2 \\
&+ i(\frac{f}{4} - \frac{3}{2}qrb)\delta q \wedge dx^3 + i(-\frac{e}{4} + \frac{3}{2}qra)\delta r \wedge dx^3 - \frac{id}{4}\delta q_1 \wedge dx^3 + \frac{ic}{4}\delta r_1 \wedge dx^3 \\
&+ \frac{ib}{4}\delta q_{11} \wedge dx^3 - \frac{ia}{4}\delta r_{11} \wedge dx^3 \\
&+ (\frac{1}{8}g - qrd - \frac{1}{4}r^2c + \frac{b}{2}(q_1r - qr_1))\delta q \wedge dx^4 \\
&+ (\frac{1}{8}h - qrc - \frac{1}{4}q^2d - \frac{a}{2}(q_1r - qr_1))\delta r \wedge dx^4 \\
&+ (-\frac{1}{8}f + qrb + \frac{1}{4}r^2a)\delta q_1 \wedge dx^4 + (-\frac{1}{8}e + qra + \frac{1}{4}q^2b)\delta r_1 \wedge dx^4 \\
&+ \frac{1}{8}d\delta q_{11} \wedge dx^4 + \frac{1}{8}c\delta r_{11} \wedge dx^4 - \frac{1}{8}b\delta q_{111} \wedge dx^4 - \frac{1}{8}a\delta r_{111} \wedge dx^4\,.
\end{aligned}$$

We have the following equations from $\xi_F \lrcorner \Omega = \delta F$

$$\begin{aligned}
\frac{\partial F_1}{\partial q} &= - ib & \frac{\partial F_1}{\partial r} &= ia \\
\frac{\partial F_2}{\partial q} &= - \frac{1}{2}d & \frac{\partial F_2}{\partial r} &= - \frac{1}{2}c \\
\frac{\partial F_2}{\partial q_1} &= \frac{1}{2}b & \frac{\partial F_2}{\partial r_1} &= \frac{1}{2}a \\
\frac{\partial F_3}{\partial q} &= \frac{if}{4} - \frac{3i}{2}qrb & \frac{\partial F_3}{\partial r} &= - \frac{ie}{4} + \frac{3i}{2}qra \\
\frac{\partial F_3}{\partial q_1} &= - \frac{id}{4} & \frac{\partial F_3}{\partial r_1} &= \frac{ic}{4} \\
\frac{\partial F_3}{\partial q_{11}} &= \frac{ib}{4} & \frac{\partial F_3}{\partial r_{11}} &= - \frac{ia}{4} \\
\frac{\partial F_4}{\partial q} &= \frac{1}{8}g - qrd - \frac{1}{4}r^2c + \frac{b}{2}(q_1r - qr_1) & \frac{\partial F_4}{\partial r} &= \frac{1}{8}h - qrc - \frac{1}{4}q^2d - \frac{a}{2}(q_1r - qr_1) \\
\frac{\partial F_4}{\partial q_1} &= - \frac{1}{8}f + qrb + \frac{1}{4}r^2a & \frac{\partial F_4}{\partial r_1} &= - \frac{1}{8}e + qra + \frac{1}{4}q^2b \\
\frac{\partial F_4}{\partial q_{11}} &= \frac{1}{8}d & \frac{\partial F_4}{\partial r_{11}} &= \frac{1}{8}c \\
\frac{\partial F_4}{\partial q_{111}} &= - \frac{1}{8}b & \frac{\partial F_4}{\partial r_{111}} &= - \frac{1}{8}a
\end{aligned}$$

for which one can find a solution if and only if the equations (5.59) hold. $\square$

Remark 5.24: Looking at the previous proof, the reader surely realises that the calculations, albeit still possible, become more and more cumbersome when we start including more times. As we discovered, this is due to the not ideal although traditional choice of coordinates q, r, and their derivatives with respect to x^1. It will be rather effortless to reobtain the same results with another choice of coordinates in Chapter 6, where we deal with the whole hierarchy. We still chose to report these calculations to show another traditional example of a Hamiltonian multiform.

We can derive the general expression of an admissible 1-form, given the first coefficient $F_1(q,r)$. This is important because it will allow us to find the coefficient of a conservation law (which is a special admissible 1-form) in a systematic way.

Proposition 5.25 *The general expressions of the coefficients of an admissible 1-form $F=\sum_{i=1}^4 F_i\,dx^i$ are, in terms the first F_1*

$$F_2 = \frac{i}{2}\left(\frac{\partial F_1}{\partial q}q_1 - \frac{\partial F_1}{\partial r}r_1\right) + a(q,r), \tag{5.61a}$$

$$\begin{aligned} F_3 = &-\frac{1}{4}\frac{\partial F_1}{\partial q}q_{11} - \frac{1}{4}\frac{\partial F_1}{\partial r}r_{11} - \frac{1}{8}\left(\frac{\partial^2 F_1}{\partial r^2}r_1^2 + \frac{\partial^2 F_1}{\partial q^2}q_1^2 - 2\frac{\partial^2 F_1}{\partial q\partial r}q_1r_1\right) \\ &+ \frac{i}{2}\frac{\partial a}{\partial q}q_1 - \frac{i}{2}\frac{\partial a}{\partial r}r_1 + b(q,r)\,, \end{aligned} \tag{5.61b}$$

$$\begin{aligned} F_4 = &\frac{i}{8}\left(\frac{\partial F_1}{\partial r}r_{111} - \frac{\partial F_1}{\partial q}q_{111}\right) \\ &+ \frac{i}{8}\left(\frac{\partial^2 F_1}{\partial r^2}r_1r_{11} - \frac{\partial^2 F_1}{\partial q^2}q_1q_{11} + \frac{\partial^2 F_1}{\partial q\partial r}(r_1q_{11} - q_1r_{11})\right) \\ &- \frac{1}{4}\left(\frac{\partial a}{\partial r}r_{11} + \frac{\partial a}{\partial q}q_{11}\right) + \frac{i}{4}\left(qrq_1 + q^2r_1\right)\frac{\partial F_1}{\partial q} - \frac{i}{4}\left(qrr_1 + r^2q_1\right)\frac{\partial F_1}{\partial r} \\ &+ \frac{i}{48}\left(\frac{\partial^3 F_1}{\partial r^3}r_1^3 - \frac{\partial^3 F_1}{\partial q^3}q_1^3\right) + \frac{i}{16}\left(\frac{\partial^3 F_1}{\partial q^2\partial r}q_1^2r_1 - \frac{\partial^3 F_1}{\partial q\partial r^2}q_1r_1^2\right) \\ &- \frac{1}{8}\left(\frac{\partial^2 a}{\partial q^2}q_1^2 + \frac{\partial^2 a}{\partial r^2}r_1^2 + 2\frac{\partial^2 a}{\partial q\partial r}q_1r_1\right) + \frac{i}{2}\left(\frac{\partial b}{\partial q}q_1 - \frac{\partial b}{\partial r}r_1\right) + c(q,r) \end{aligned} \tag{5.61c}$$

where $a(q,r)$, $b(q,r)$, and $c(q,r)$ are arbitrary smooth functions of q and r only.

Proof. Since $\frac{\partial F_1}{\partial r} = 2i\frac{\partial F_2}{\partial r_1}$ and $\frac{\partial F_1}{\partial q} = -2i\frac{\partial F_2}{\partial q_1}$ we find (5.61a). Then, since $\frac{\partial F_3}{\partial q_{11}} = -\frac{1}{4}\frac{\partial F_1}{\partial q}$ and $\frac{\partial F3}{\partial r_{11}} = -\frac{1}{4}\frac{\partial F_1}{\partial r}$ we have

$$F_3 = -\frac{1}{4}\frac{\partial F_1}{\partial q}q_{11} - \frac{1}{4}\frac{\partial F_1}{\partial r}r_{11} + (\dots)(q,r,q_1,r_1).$$

Then we use the fact that $\frac{\partial F_2}{\partial r} = 2i\frac{\partial F_3}{\partial r_1}$ and $\frac{\partial F_2}{\partial q} = -2i\frac{\partial F_3}{\partial q_1}$ to obtain

$$\frac{\partial F_3}{\partial q_1} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial q\partial r}r_1 - \frac{\partial^2 F_1}{\partial q^2}q_1\right) + \frac{i}{2}\frac{\partial a}{\partial q}, \qquad \frac{\partial F_3}{\partial r_1} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial q\partial r}q_1 - \frac{\partial^2 F_1}{\partial r^2}r_1\right) - \frac{i}{2}\frac{\partial a}{\partial r}.$$

we then use partial integration and find (5.61b). Similarly we can compute the fourth coefficient F_4, which results in (5.61c). □

For admissible forms we can define the multi-time Poisson brackets. The Poisson bracket between a 0-form $H(q,r,q_1,r_1,q_{11},r_{11},q_{111},r_{111})$ and an admissible 1-form $P = P_1\,dx^1 + P_2\,dx^2 + P_3\,dx^3 + P_4\,dx^4$ is $\xi_P H$

$$\begin{aligned}\{\![H,P]\!\} =& \frac{\partial H}{\partial q}\frac{\partial P_1}{\partial r} - \frac{\partial H}{\partial r}\frac{\partial P_1}{\partial q} - 2i\frac{\partial H}{\partial r_1}\frac{\partial P_2}{\partial q} - 2i\frac{\partial H}{\partial q_1}\frac{\partial P_2}{\partial r}\\ &+ 24qr\frac{\partial P_3}{\partial q_{11}}\frac{\partial H}{\partial r_{11}} - 24qr\frac{\partial P_3}{\partial r_{11}}\frac{\partial H}{\partial q_{11}} + 4\frac{\partial H}{\partial r_{11}}\frac{\partial P_3}{\partial q} - 4\frac{\partial H}{\partial q_{11}}\frac{\partial P_3}{\partial r}\\ &+ 8i\left(\frac{\partial P_4}{\partial r} + 2q^2\frac{\partial P_4}{\partial q_{11}} + 8qr\frac{\partial P_4}{\partial r_{11}} + 4(q_1r - r_1q)\frac{\partial P_4}{\partial r_{111}}\right)\frac{\partial H}{\partial q_{111}}\\ &+ 8i\left(\frac{\partial P_4}{\partial q} + 2r^2\frac{\partial P_4}{\partial r_{11}} + 8qr\frac{\partial P_4}{\partial q_{11}} - 4(q_1r - r_1q)\frac{\partial P_4}{\partial q_{111}}\right)\frac{\partial H}{\partial r_{111}}\,.\end{aligned} \tag{5.62}$$

If $P = \sum_{i=1}^4 P_i dx^i$ and $Q = \sum_{i=1}^4 Q_i dx^i$ are admissible 1-forms, then their Poisson bracket satisfies the decomposition

$$\{\![P,Q]\!\} = \{P_1,Q_1\}_1\,dx^1 + \{P_2,Q_2\}_2\,dx^2 + \{P_3,Q_3\}_3\,dx^3 + \{P_4,Q_4\}_4\,dx^4\,,$$

where

$$\{P_1,Q_1\}_1 = -i\left(\frac{\partial P_1}{\partial q}\frac{\partial Q_1}{\partial r} - \frac{\partial P_1}{\partial r}\frac{\partial Q_1}{\partial q}\right), \tag{5.63a}$$

$$\{P_2,Q_2\}_2 = 2\left(\frac{\partial P_2}{\partial q}\frac{\partial Q_2}{\partial r_1} - \frac{\partial P_2}{\partial r_1}\frac{\partial Q_2}{\partial q} + \frac{\partial P_2}{\partial r}\frac{\partial Q_2}{\partial q_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_2}{\partial r}\right), \tag{5.63b}$$

$$\begin{aligned}\{P_3,Q_3\}_3 =& -4i\Bigg(\frac{\partial P_3}{\partial r_{11}}\frac{\partial Q_3}{\partial q} - \frac{\partial P_3}{\partial q}\frac{\partial Q_3}{\partial r_{11}} - \frac{\partial P_3}{\partial q_{11}}\frac{\partial Q_3}{\partial r} + \frac{\partial P_3}{\partial r}\frac{\partial Q_3}{\partial q_{11}}\\ &+ \frac{\partial P_3}{\partial q_1}\frac{\partial Q_3}{\partial r_1} - \frac{\partial P_3}{\partial r_1}\frac{\partial Q_3}{\partial q_1} - 6qr\Big(\frac{\partial P_3}{\partial q_{11}}\frac{\partial Q_3}{\partial r_{11}} - \frac{\partial P_3}{\partial r_{11}}\frac{\partial Q_3}{\partial q_{11}}\Big)\Bigg),\end{aligned} \tag{5.63c}$$

$$\begin{aligned}\{P_4,Q_4\}_4 =& -8\left(\frac{\partial P_4}{\partial q_{111}}\frac{\partial Q_4}{\partial r} - \frac{\partial P_4}{\partial r}\frac{\partial Q_4}{\partial q_{111}}\right) + 8\left(\frac{\partial P_4}{\partial q}\frac{\partial Q_4}{\partial r_{111}} - \frac{\partial P_4}{\partial r_{111}}\frac{\partial Q_4}{\partial q}\right)\\ &- 8\left(\frac{\partial P_4}{\partial q_1}\frac{\partial Q_4}{\partial r_{11}} - \frac{\partial P_4}{\partial r_{11}}\frac{\partial Q_4}{\partial q_1}\right) + 8\left(\frac{\partial P_4}{\partial q_{11}}\frac{\partial Q_4}{\partial r_1} - \frac{\partial P_4}{\partial r_1}\frac{\partial Q_4}{\partial q_{11}}\right)\\ &+ 64qr\left(\frac{\partial P_4}{\partial q_{11}}\frac{\partial Q_4}{\partial r_{111}} - \frac{\partial P_4}{\partial r_{111}}\frac{\partial Q_4}{\partial q_{11}} + \frac{\partial P_4}{\partial r_{11}}\frac{\partial Q_4}{\partial q_{111}} - \frac{\partial P_4}{\partial q_{111}}\frac{\partial Q_4}{\partial r_{11}}\right)\\ &+ 16q^2\left(\frac{\partial P_4}{\partial q_{11}}\frac{\partial Q_4}{\partial q_{111}} - \frac{\partial P_4}{\partial q_{111}}\frac{\partial Q_4}{\partial q_{11}}\right)\\ &+ 16r^2\left(\frac{\partial P_4}{\partial r_{11}}\frac{\partial Q_4}{\partial r_{111}} - \frac{\partial P_4}{\partial r_{111}}\frac{\partial Q_4}{\partial r_{11}}\right)\\ &+ 32(q_1r - r_1q)\left(\frac{\partial P_4}{\partial r_{111}}\frac{\partial Q_4}{\partial q_{111}} - \frac{\partial P_4}{\partial q_{111}}\frac{\partial Q_4}{\partial r_{111}}\right)\end{aligned} \tag{5.63d}$$

Using this decomposition we can read the single-time Poisson brackets: $\{\ ,\ \}_1$ is (up to a sign) the usual equal-time Poisson bracket of the AKNS hierarchy, which in the traditional infinite dimensional setting provides the first structure (in the sense of bi-Hamiltonian theory) for the whole hierarchy, while $\{\ ,\ \}_{2,3}$ are the dual Poisson Bracket of respectively the NLS and mKdV which can be found in [ACDK16].

Remark 5.26: The multi-time Poisson brackets $\{\![\ ,\ ]\!\}$ satisfy a classical r-matrix structure, with the rational r-matrix $r_{12}(\lambda) = \frac{P_{12}}{2\lambda}$. This will be explained in Chapter 6 using a different set of coordinates that allows us to prove the r-matrix structure more elegantly and for the whole hierarchy.

Conservation laws Since the coefficients of the Hamiltonian multiform are admissible, the multiform Hamilton equations in a Poisson bracket form are

$$dF = \xi_F \lrcorner \delta\mathcal{H} = \sum_{i<j=1}^{4} \{\![H_{ij}, F]\!\}\, dx^{ij}$$

for any admissible 1-form $F = F_1\, dx^1 + F_2\, dx^2 + F_3\, dx^3 + F_4\, dx^4$. We can also find the first conservation laws for the AKNS hierarchy, i.e. F is a conservation law if

$$\{\![H_{ij}, F]\!\} = 0 \qquad \forall i < j\,. \tag{5.64}$$

We can solve the latter equation in the space of admissible forms (see Proposition 5.25 for the general expression of the coefficients) to find a conservation law. From $(i, j) = (1, 2)$ we get

$$\{\![H_{12}, F]\!\} = -iqr^2\frac{\partial F_1}{\partial r} + iq^2 r\frac{\partial F_1}{\partial q} + \frac{i}{2}\frac{\partial^2 F_1}{\partial q^2}q_1^2 - \frac{i}{2}\frac{\partial^2 F_1}{\partial r^2}r_1^2 + \frac{\partial a}{\partial q}q_1 + \frac{\partial a}{\partial r}r_1 = 0\,.$$

This translates into $r\frac{\partial F_1}{\partial r} = q\frac{\partial F_1}{\partial q}$ and $\frac{\partial^2 F_1}{\partial q^2} = \frac{\partial^2 F_1}{\partial r^2} = 0$, and therefore[2] $F_1 = qr$, and $\frac{\partial a}{\partial q} = \frac{\partial a}{\partial r} = 0$, so therefore a is constant, which we set to zero. The coefficients become then

$$F_1 = qr\,, \tag{5.65a}$$

$$F_2 = \frac{i}{2}\left(rq_1 - qr_1\right)\,, \tag{5.65b}$$

$$F_3 = -\frac{1}{4}rq_{11} - \frac{1}{4}qr_{11} + \frac{1}{4}q_1 r_1 + b(q, r)\,, \tag{5.65c}$$

$$F_4 = \frac{i}{8}\left(qr_{111} - rq_{111} + q_{11}r_1 - q_1 r_{11}\right) + \frac{i}{2}\left(\frac{\partial b}{\partial q}q_1 - \frac{\partial b}{\partial r}r_1\right) + c(q, r) \tag{5.65d}$$

with b and c left to determine. From $(i, j) = (1, 3)$ we get

$$\{\![H_{13}, F]\!\} = -\frac{3}{2}(qr^2 q_1 + q^2 r r_1) + q_1\frac{\partial b}{\partial q} + r_1\frac{\partial b}{\partial r} = 0\,,$$

[2] The solution $F_1 = 0$ would bring the trivial conservation law so it is rejected.

and therefore we choose $b = \frac{3}{4}q^2r^2$. The fourth coefficient becomes then $F_4 = \frac{i}{8}(qr_{111} - rq_{111}) + \frac{i}{8}(q_{11}r_1 - q_1r_{11}) + \frac{3i}{4}qr(q_1r - qr_1) + c(q, r)$. It can be verified by looking at the coefficient $(1, 4)$ that we have a conservation law when $c = 0$. A conservation law is then

$$\begin{aligned} F =& qr\, dx^1 + \frac{i}{2}(q_1r - r_1q)\, dx^2 + \frac{1}{4}(3q^2r^2 + q_1r_1 - q_{11}r - r_{11}q)\, dx^3 \\ &+ \left(\frac{i}{8}(qr_{111} - rq_{111}) + \frac{i}{8}(q_{11}r_1 - q_1r_{11}) + \frac{3i}{4}qr(q_1r - qr_1)\right) dx^4\,, \end{aligned} \tag{5.66}$$

which reproduces the known conservation laws and conserved quantities of the AKNS hierarchy: qr is interpreted as the mass, $q_1r - qr_1$ as the momentum, etc.

5.6 Hamiltonian 1-forms and involutivity of single-time Hamiltonians

We leave momentarily the realm of classical field theories to look at finite-dimensional Hamiltonian systems. In particular, we want to connect our results on Hamiltonian multiforms [CS20b] with the results of [V20], considering the case of a hierarchy of commuting ordinary differential equations, one for each time x^i, $0 \leq i \leq n$. We consider the configuration space to be $\mathbb{R}$ for simplicity, but this could be extended to other manifolds in general. In the Lagrangian multiform formalism, the dynamics are encapsulated by a Lagrangian 1-form $\mathscr{L} = \sum_{i=1}^n L_i dx^i$, where each of the $L_i \in \mathscr{A}$ is dependent on a field q and its derivatives with respect to the times x^1, x^i, and a generalised variational principle $\delta d\mathscr{L} = 0$. We consider the coefficients used in [V20]

$$L_1 = \frac{1}{2}q_1^2 - V_1(q)\,, \tag{5.67a}$$

$$L_i = q_1q_i - V_i(q, q_1)\,, \qquad i = 2, \ldots, n\,. \tag{5.67b}$$

This corresponds to the common case in which the first Lagrangian L_1 is 'Newtonian'[3] and the other Lagrangians are linear in the velocities. In [V20] for instance are listed some examples, including the Toda lattice and the Kepler problem. The multiform Euler-Lagrange equations for a generic Lagrangian 1-form $\mathscr{L}$ have been explicitly obtained in [SV16, Theorem 2.5], and in this case are

$$q_{11} = -\frac{\partial V_1}{\partial q}\,, \quad q_i = \frac{\partial V_i}{\partial q_1}\,, \quad q_{1i} = -\frac{\partial V_i}{\partial q}\,. \tag{5.68}$$

The work [V20] linked the closure relation $d\mathscr{L} = 0$ (modulo these equations) to the involutivity of the single-time Hamiltonians: if H_i is the Hamiltonian obtained in the usual way from the Lagrangian L_i with a Legendre transformation, and $\{\ ,\ \}$ are the

[3] *i.e.* of the form $L = K - V$, where K is quadratic in the velocities and V is a positional potential.

canonically constructed Poisson brackets, we have that

$$d\mathscr{L} = 0 \quad \Longleftrightarrow \quad \{H_j, H_k\} = 0\,. \tag{5.69}$$

This is a rather important results, as it links the closure relation $d\mathscr{L} = 0$ of a Lagrangian 1-form to Liouville integrability. We will re-interpret it as the closure of the Hamiltonian 1-form $\mathcal{H} = \sum_j H_j\, dx^j$.

Preliminarily, we remark that all the objects introduced in this section for the case of integrable hierarchies of classical field theories could be extended (or better, reduced) to the case of integrable hierarchies of ODEs: we keep the same definitions, just changing Lagrangian multiform from an object in $\mathscr{A}^{(0,2)}$ to an object in $\mathscr{A}^{(0,1)}$. As a consequence, we have that $\Omega^{(1)} \in \mathscr{A}^{(1,0)}$, and that $\delta\Omega^{(1)} = \Omega \in \mathscr{A}^{(2,0)}$ will represent the usual, single-time symplectic form. This is to be expected: as we do not have a $1+1$-dimensional field theory but only a ODE, we do not have to make any choice between *time* and *space*, but for each Lagrangian L_i we interpret x^i (the only possible coordinate) as time. To Ω we will associate the Poisson bracket

$$\{\ ,\ \}:\quad \mathscr{A} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

which in this case is at the same time a multi-time and a single-time Poisson bracket. Dickey's formula $\mathcal{H} = -\mathscr{L} + \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega^{(1)}$ produces a Hamiltonian multiform $\sum_j H_j\, dx^j \in \mathscr{A}^{(0,1)}$, whose coefficients H_j will be the single-time Hamiltonians.

The first step is to obtain the symplectic multiform $\Omega = \delta\Omega^{(1)}$, where (5.68) if and only if $\delta\mathscr{L} = -d\Omega^{(1)}$. We start by computing the δ-differential of $\mathscr{L}$, i.e.

$$\begin{aligned}\delta\mathscr{L} =& (q_1\delta q_1 - \frac{\partial V_1}{\partial q}\delta q) \wedge dx^1 \\ &+ \sum_{i>1}(q_1\delta q_i + q_i\delta q_1 - \frac{\partial V_i}{\partial q}\delta q - \frac{\partial V_i}{\partial q_1}\delta q_1) \wedge dx^i.\end{aligned}$$

We now want to turn the terms in $\delta q_1 \wedge dx^1$ into a total d-differential using the identity

$$q_1\delta q_1 \wedge dx^1 = -q_{11}\delta q \wedge dx^1 - \sum_{i>1}(q_{1i}\delta q + q_1\delta q_i) \wedge dx^i - d(q_1\delta q)\,,$$

and therefore obtaining

$$\begin{aligned}\delta\mathscr{L} =&(-q_{11} - \frac{\partial V_1}{\partial q})\delta q \wedge dx^1 \\ &+ \sum_{i>1}\left((q_1 - q_1)\delta q_i + (q_i - \frac{\partial V_i}{\partial q_1})\delta q_1 - (q_{1i} + \frac{\partial V_i}{\partial q})\delta q\right) \wedge dx^i - d(q_1\delta q) \\ =&(-q_{11} - \frac{\partial V_1}{\partial q})\delta q \wedge dx^1 \\ &+ \sum_{i>1}\left((q_i - \frac{\partial V_i}{\partial q_1})\delta q_1 - (q_{1i} + \frac{\partial V_i}{\partial q})\delta q\right) \wedge dx^i - d(q_1\delta q)\,,\end{aligned}$$

which implies $\Omega^{(1)} = q_1\delta q$, and therefore $\Omega = \delta q_1 \wedge \delta q$. The Hamiltonian 1-form (multiform of degree 1) $\mathcal{H} = \sum_i H_i\, dx^i$ is obtained as

$$\mathcal{H} = -\mathscr{L} + \sum_{i=1}^{n} dx^i \wedge \widetilde{\partial}_i \lrcorner \Omega^{(1)}\,. \tag{5.70}$$

Since $\widetilde{\partial}_i \lrcorner (q_1\delta q) = q_1 q_i$, we obtain $H_i = q_1 q_i - L_i$, and therefore we reobtain the Hamiltonians in [V20]:

$$H_1(q, q_q) = \frac{q_1^2}{2} + V_1(q)\,, \tag{5.71a}$$

$$H_i(q, q_1) = V_i(q, q_1)\,, \qquad i = 2, \dots, n\,. \tag{5.71b}$$

Any function $F(q, q_1)$ is admissible, i.e. there exists a vector field ξ_F such that $\xi_F \lrcorner \Omega = \delta F$, where

$$\xi_F = \frac{\partial F}{\partial q}\frac{\partial}{\partial q_1} - \frac{\partial F}{\partial q_1}\frac{\partial}{\partial q}\,. \tag{5.72}$$

We can therefore define the Poisson Brackets between two functions $F(q, q_1)$ and $G(q, q_1)$ as

$$\{F, G\} = -\xi_F G = \frac{\partial F}{\partial q_1}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial q_1}\,. \tag{5.73}$$

Moreover, for any function $F(q, q_1)$ we have that on the equation of motion (5.68) $dF = \sum_{i=1}^n \{H_i, F\}\, dx^i$, or equivalently

$$\partial_i F = \{H_i, F\}\,. \tag{5.74}$$

From the closure relation $d\mathcal{H} = 0$ we can easily recover the involution of the single-time Hamiltonians:

$$\begin{aligned} d\mathcal{H} =& \sum_{i<j}(\partial_i H_j - \partial_j H_i)\, dx^{ij} = \sum_{i<j}(\{H_i, H_j\} - \{H_j, H_i\})\, dx^{ij} \\ =& 2\sum_{i<j}\{H_i, H_j\}\, dx^{ij} = 0\,, \end{aligned}$$

and therefore $\{H_i, H_j\} = 0$, or $\partial_i H_j = 0 \ \forall i \neq j$. The equation $\partial_i H_i = 0$ is obtained from (5.74) by antisymmetry.

Chapter 6

Multi-time approach to the AKNS hierarchy and classical r-matrix

In Chapter 5 we showed how to describe integrable hierarchies in a Hamiltonian multiform fashion. This description could, in principle, be applied to any number of flows of the hierarchy, or even all the countably many flows. The problem is that, although theoretically possible, adding flows to a Lagrangian multiform (which is the starting point of our Hamiltonian description) becomes more and more computationally cumbersome the further up the hierarchy we go, if we resort to the newly developed techniques that appeared in [SNC19b, PV20]. In this chapter, including content from [CS21], we overcome this difficulty providing a Lagrangian and a Hamiltonian multiform for the complete AKNS hierarchy. We will also construct a multi-time Poisson bracket with a classical r-matrix structure that will generalise the results of Chapter 4. Our starting point will be the description of the AKNS hierarchy by Flashka, Newell and Ratiu [FNR83], but it will involve an equivalent but new approach to a hierarchy, that we call *multi-time approach*, as opposed to the traditional *field-theoretical approach* that has been used in the previous works.

The results in this chapter cast the results of [AC17] in a new light, realising the underlying goal of bypassing the need to specify an initial time in the AKNS hierarchy. Here we provide a multiform explanation for this behaviour, casting the single-time Poisson brackets in the greater structure of the multi-time Poisson brackets. In Section 6.1 we review the fundamental notions of the AKNS hierarchy, and we introduce the new multi-time approach. In Section 6.2 we introduce the generating Lagrangian multiform, and we use it to derive the equations of the hierarchy, and the symplectic and Hamiltonian multiforms in a compact form. In Section 6.3 we recover the classical r-matrix structure for the whole hierarchy, and we prove that the complete set of zero-curvature equations for each Lax pair of the hierarchy can be obtained as a multiform Hamilton equation. Finally, in Section 6.4 we recover the known results for the first three times of the hierarchy. We remark that the Lagrangian multiform for the AKNS hierarchy can be obtained from a *generating Lagrangian* in Chapter 7, together with other integrable models. Many of the long and not necessarily illuminating proofs are reported in Appendix B.

6.1 The Ablowitz-Kaup-Newell-Segur hierarchy

6.1.1 The traditional field-theoretical approach

In the 1983 paper [FNR83], Flashka, Newell and Ratiu introduced an algebraic formalism to cast the soliton equations associated with the AKNS hierarchy into what is known as the Adler-Kostant-Symes scheme [A79, K79, S80]. At the same time, the Russian school unraveled the structures underlying this type of construction which culminated in the classical r-matrix theory [S83], and the introduction of the notion of Poisson-Lie group [D83]. Here, we review some aspects of this topic, freely adapting and merging notations and notions coming from different sources. It had been known before [FNR83], since the work of [AKNS74], that the so-called AKNS hierarchy can be constructed by considering an auxiliary spectral problem of the form[1]

$$\partial_x \psi = \begin{pmatrix} -i\lambda & q(x,x^n) \\ r(x,x^n) & i\lambda \end{pmatrix} \psi \equiv P(x,x^n,\lambda)\psi \equiv (\lambda P_0 + P_1(x,x^n))\psi \,, \tag{6.1}$$

where

$$P_1(x,x^n) := \begin{pmatrix} 0 & q(x,x^n) \\ r(x,x^n) & 0 \end{pmatrix} \,, \qquad P_0 := -i\sigma_3 \,, \tag{6.2}$$

as well as another equation of the form

$$\partial_n \psi = Q^{(n)}(x,x^n,\lambda)\psi \,, \qquad \partial_n := \frac{\partial}{\partial x^n} \,, \tag{6.3}$$

with $Q^{(n)}(x,x^n,\lambda) = \lambda^n Q_0(x,x^n) + \lambda^{n-1}Q_1(x,x^n) + \cdots + Q_n(x,x^n)$ where each Q_i is a 2×2 traceless matrix. Then the compatibility condition $\partial_x\partial_n\psi = \partial_n\partial_x\psi$ translates into the well-known zero-curvature equation for the Lax pair $P(x,x^n,\lambda)$ and $Q^{(n)}(x,x^n,\lambda)$

$$\partial_n P(x,x^n,\lambda) - \partial_x Q^{(n)}(x,x^n,\lambda) + [P(x,x^n,\lambda), Q^{(n)}(x,x^n,\lambda)] = 0 \,. \tag{6.4}$$

The usual *field-theoretical* approach is described as follows. One 'forgets' the dependence on x^n (interpreted as the time variable) and considers the coefficients of $P_i(x)$ and $Q_i(x)$ to be fields in x (the space variable). By setting to zero every coefficient of λ one obtains a series of equations that allow to find $Q_0(x)$, …, $Q_n(x)$ recursively. This produces $Q_0(x) = P_0$, $Q_1(x) = P_1(x)$ (up to some normalisation constants) and the entries of $Q_j(x)$

[1] Traditionally, the flows thus defined are associated to 'time' variable t^n. However, one of the main points of [FNR83] is that they all play the same role as x which could be viewed as t^1 in this hierarchy. We simply denote them all by x^n since whether they play the role of a space or time variable is really up to interpretation.

with $j \geq 2$ are found to be polynomials in $q(x)$, $r(x)$ and their derivatives with respect to x. The last of these equations is the AKNS flow

$$\partial_n P_1(x) - \partial_x Q_n(x) + [P_1(x), Q_n(x)] = 0\,, \tag{6.5}$$

and produces a partial differential equation for q and r viewed as functions of x and x^n which is integrable (hence effectively 'injecting' the dependence on x^n at the last stage). Different values of n gives the successive equations of the AKNS hierarchy. We list them for $n = 0, 1, 2, 3$, giving the name of the corresponding famous example (which is usually obtained by a further reduction, e.g. $r = \pm q^*$ for $n = 2$ gives the (de)focusing nonlinear Schrödinger equation).

- **Scaling:** $q_0 = -2iq$, and $r_0 = 2ir$,
- **Translation:** $q_1 = q_x$, and $r_1 = r_x$,
- **NLS equation:** $iq_2 + \frac{1}{2}q_{xx} - q^2 r = 0$, and $ir_2 - \frac{1}{2}r_{xx} + qr^2 = 0$,
- **Modified KdV equation** $q_3 = -\frac{1}{4}q_{xxx} + \frac{3}{2}qrq_x$, and $r_3 = -\frac{1}{4}r_{xxx} + \frac{3}{2}qrr_x$.

We will show how to obtain the first three equations in detail.

- We start from the case $n = 0$: we set $Q^{(0)} = Q_0$ and study the equation $\partial_0 P - \partial_x Q^{(0)} + [P, Q^{(0)}] = 0$ which translates to

  $$-i\lambda\partial_0\sigma_3 + \partial_0 \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} - \partial_x Q_0 + [-i\lambda\sigma_3 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, Q_0] = 0\,.$$

  This is a polynomial in λ, and we set to zero each coefficient, starting from the highest power λ^1 and noticing that $\partial_0\sigma_3 = 0$

  $$-i[\sigma_3, Q_0] = 0$$

  which tells us that Q_0 is diagonal: we set $Q_0 = a\sigma_3$. The next equation is obtained setting to zero the coefficient of $\lambda^0 = 1$:

  $$\begin{pmatrix} 0 & q_0 \\ r_0 & 0 \end{pmatrix} + [\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, Q_0] = 0 \implies \begin{pmatrix} -a_0 & q_0 - 2aq \\ r_0 + 2ar & a_0 \end{pmatrix} = 0$$

  which in turns gives that a (and therefore Q_0) must be constant, and $q_0 = 2aq$, $r_0 = -2ar$. We obtain the desired equations by setting $a = -i$, and so $Q_0 = P_0$.

- We now treat the case $n = 1$:

  $$-i\lambda\partial_1\sigma_3 + \partial_1 \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} - \lambda\partial_x Q_0 - \partial_x Q_1 + [-i\lambda\sigma_3 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \lambda Q_0 + Q_1] = 0\,.$$

  The equation coming from the coefficient of the highest degree of λ (now λ^2) is still the same: $-i[\sigma_3, Q_0] = 0$, which means that we can parametrise $Q_0 = a\sigma_3$. The next equation in the list, the coefficient of λ^1 is

$$-\partial_x(a\sigma_3) + [-i\sigma_3, Q_1] + [\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, a\sigma_3] = 0$$

which implies that a is constant and we again set $Q_0 = -i\sigma_3$, and that the anti-diagonal part of Q_1 is the matrix $q\sigma_+ + r\sigma_-$. We denote the σ_3 component of Q_1 by $\bar{a}$. The last equation comes from the coefficient of λ^0 and is $-\partial_x Q_0 + [P_0, Q_1] + [P_1, Q_0] = 0$,

$$\begin{pmatrix} 0 & q_1 \\ r_1 & 0 \end{pmatrix} - \begin{pmatrix} \bar{a}_x & q_x \\ r_x & \bar{a}_x \end{pmatrix} + [\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \begin{pmatrix} \bar{a} & q \\ r & -\bar{a} \end{pmatrix}] = 0\,.$$

The diagonal part of this equation restricts $\bar{a}$ to be constant, which we set to zero $\bar{a} = 0$, while the non-diagonal part brings $q_1 = q_x$ ad $r_1 = r_x$ as desired.

- We report now the case $n = 2$, which generates the Non-Linear Schrödinger equation. The starting equation is $\partial_2 P(\lambda) - \partial_x Q^{(2)}(\lambda) + [P(\lambda), Q^{(2)}(\lambda)] = 0$. The equation coming from the coefficient of λ^3 is again $-i[\sigma_3, Q_0] = 0$, which means that we can parametrise $Q_0 = a\sigma_3$. The next is, again, $-\partial_x Q_0 + [P_0, Q_1] + [P_1, Q_0] = 0$ which means that we can set $Q_0 = P_0$ and $Q_1 = \bar{a}\sigma_3 + q\sigma_- + r\sigma_-$. The coefficient of λ^1 brings the equation
$$-\partial_x Q_1 + [P_0, Q_2] + [P_1, Q_1] = 0$$
which again implies that $\bar{a}$ is constant (we set it to zero so that $Q_1 = P_1$), and that the matrix Q_2 can be parametrised a $Q_2 = \widetilde{a}\sigma_3 + \frac{i}{2}q_x\sigma_+ - \frac{i}{2}r_x\sigma_-$. The final equation
$$\partial_2 P_1 - \partial_x Q_2 + [P_1, Q_2] = 0$$
generates the Non-Linear Schrödinger equation. The diagonal part brings $\widetilde{a}_x = -\frac{i}{2}(q_x r + q r_x)$ that we can solve by setting $\widetilde{a} = -\frac{i}{2}qr$, and the antidiagonal part then gives the desired system
$$iq_2 + \frac{1}{2}q_{xx} - q^2 r = 0 \qquad ir_2 - \frac{1}{2}r_{xx} + qr^2 = 0\,.$$

It is proved that all these equations can be interpreted as Hamiltonian flows which commute with each other and can therefore be imposed simultaneously on the variable q and r. This is ensured by that fact that the following zero-curvature equations hold for any $k, n \geq 1$ (by setting $x = x^1$ and $Q^{(1)} = P$),

$$\partial_n Q^{(k)}(\lambda) - \partial_k Q^{(n)}(\lambda) + [Q^{(k)}(\lambda), Q^{(n)}(\lambda)] = 0\,. \tag{6.6}$$

In [FNR83], these facts and several others were cast into the algebraic setup of the Adler-Kostant-Symes scheme whereby one can introduce integrable Hamiltonian systems based on the decomposition of a Lie algebra into two Lie subalgebras which are isotropic with respect to an ad-invariant nondegenerate symmetric bilinear form on the Lie algebra. For the AKNS hierarchy, [FNR83] use fields valued in the Lie algebra $\mathcal{L} \coloneqq s\ell(2,\mathbb{C}) \otimes \mathbb{C}(\!(\lambda^{-1})\!)$ of formal Laurent series in the variable $1/\lambda$ with coefficients in the Lie algebra $s\ell(2,\mathbb{C})$, *i.e.* the Lie algebra of elements of the form

$$X(\lambda) = \sum_{j=-N}^{\infty} X_j \lambda^{-j}\,, \quad X_j \in s\ell(2,\mathbb{C})\,; \quad \text{for some } N \in \mathbb{Z}\,, \tag{6.7}$$

with the bracket given by

$$[X,Y](\lambda) = \sum_k \sum_{i+j=k} [X_i, Y_j]\lambda^k\,. \tag{6.8}$$

There is a decomposition of $\mathcal{L}$ into Lie subalgebras $\mathcal{L} = \mathscr{K} \oplus \mathscr{N}$ where

$$\mathscr{K} = \{ \sum_{j=1}^{\infty} X_j \lambda^{-j} \}\,, \qquad \mathscr{N} = \{ \sum_{j=-N}^{0} X_j \lambda^{-j} \quad | \quad N \in \mathbb{Z}_{\geq 0} \}\,.$$

This yields two projectors P_+ on $\mathscr{N}$ and P_- on $\mathscr{K}$. The following ad-invariant nondegenerate symmetric bilinear form is used, for all $X(\lambda), Y(\lambda) \in \mathcal{L}$,

$$\langle X(\lambda), Y(\lambda)\rangle \coloneqq \sum_{i+j=0} \mathrm{Tr}(X_i Y_j)\,, \tag{6.9}$$

Without entering the details of the construction, we present the summarised results of interest for us. The entire ANKS hierarchy can be obtained by considering an element $Q(\lambda)$ of the annihilator of $\mathscr{K}$ as the following formal series

$$Q(\lambda) = \sum_{i=0}^{\infty} Q_i \lambda^{-i} = Q_0 + \frac{Q_1}{\lambda} + \frac{Q_2}{\lambda^2} + \frac{Q_3}{\lambda^3} + \dots\,, \tag{6.10}$$

$$Q_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \qquad a(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^{-i}\,, \quad b(\lambda) = \sum_{i=0}^{\infty} b_i \lambda^{-i}\,, \quad c(\lambda) = \sum_{i=0}^{\infty} c_i \lambda^{-i}\,,$$

and introducing the vector fields ∂_n by

$$\begin{aligned} \partial_n Q(\lambda) =& [P_+(\lambda^n Q(\lambda)), Q(\lambda)] \\ =& -[P_-(\lambda^n Q(\lambda)), Q(\lambda)] = [R(\lambda^n Q(\lambda)), Q(\lambda)]\,, \end{aligned} \tag{6.11}$$

where $R = \frac{1}{2}(P_+ - P_-)$ is the endomorphism form of the classical r-matrix and we used $\mathrm{Id} = P_+ + P_-$. It is well known that this operator satisfies the modified classical Yang-Baxter equation and allows one to define a second Lie bracket $[\ ,\ ]_R$ on $\mathcal{L}$ (see e.g. [S08])

$$[X,Y]_R = [RX, Y] + [X, RY]\,. \tag{6.12}$$

The significance of this reformulation is that the authors achieved several important results:

1. The equations (6.11) are commuting Hamiltonian flows associated to the Hamiltonian functions
$$g_k(X) = -\frac{1}{2}(S^k(X), X)\,, \quad k \in \mathbb{Z}\,, \quad (S^k X)(\lambda) = \lambda^k X(\lambda)\,. \tag{6.13}$$
which are Casimir functions with respect to the Lie-Poisson bracket associated to the Lie bracket (6.8). As a consequence, these functions are in involution with respect to the Lie-Poisson bracket associated to the second Lie bracket (6.12) on $\mathcal{L}$ and their Hamilton equations take the form of the Lax equation (6.11);

2. In this construction, one can get rid of the special role of the x variable, which is now the variable x^1, no different from any of the other x^n. They then propose to define a hierarchy of integrable PDEs as follows: use (6.11) for a fixed n as a starting point to determine all the Q_j. This yields that b_j, c_j for $j > n$ and a_j, $j > 1$ are polynomials in b_j, c_j, $j = 1, \ldots, n$, which are now viewed as functions of x^n, and in their derivatives with respect to x^n. Then, one can use any one of the other variables x^k to induce a Hamiltonian flow on the infinite dimensional phase space $b_j(x^n), c_j(x^n)$, $j = 1, \ldots, n$. The Hamilton equations take the form of a zero-curvature equation
$$\partial_k Q^{(n)}(x^n,\lambda) - \partial_n Q^{(k)}(x^n,\lambda) + [Q^{(n)}(x^n,\lambda), Q^{(k)}(x^n,\lambda)] = 0 \tag{6.14}$$
where $Q^{(n)}(x^n,\lambda)$ denotes $P_+(\lambda^n Q(\lambda))$ where the above substitution for a_j, b_j, c_j in terms of the finite number of fields $b_j(x^n), c_j(x^n)$, $j = 1, \ldots, n$ and their x^n derivatives has been performed. See [AC17] for more details about this.

3. There exist generalised conservation laws $\frac{\partial F_{jk}}{\partial x^\ell} = \frac{\partial F_{\ell k}}{\partial x^j}$ for all $j, k, \ell \geq 0$ where F_{kj} can be obtained efficiently from a generating function. For $j = 1$, they reproduce the usual AKNS conservation laws with F_{1k} being the conserved densities and $F_{\ell k}$ the corresponding fluxes.

Those results are reviewed in detail in [AC17] where the observation that one can start from an arbitrary flow x^n is used to prove the general result on the r-matrix structure of dual Lax pairs which was first observed in [CK15] and [C15b].

6.1.2 The multi-time approach

We want to stress that despite the deep observation that all independent variables x^j play the same role, both in [FNR83] and [AC17], the authors still implement the step of using (6.11) *first* for a fixed (but arbitrary) x^n in order to produce a phase space for a *field theory* consisting of a finite number of fields $b_j(x^n)$, $c_j(x^n)$ $j = 1, \ldots, n$. This leads to a rather complicated construction of the single-time Poisson brackets $\{\ ,\ \}_n$ and $\{\ ,\ \}_k$ in [AC17] whose common r-matrix structure is traced back to the original Lie-Poisson bracket associated to the second Lie bracket (6.12). In [CS20a] we achieved the goal of implementing a truly covariant Poisson bracket capable of accommodating any pair of independent variables x^n and x^k *simultaneously* and producing an r-matrix structure for the associated *Lax form* $W(\lambda) = Q^{(n)}(\lambda)\, dx^n + Q^{(k)}(\lambda)\, dx^k$. Another essential question was still pending, *i.e.* how to go beyond only a pair of times x^n and x^k, corresponding to a single zero-curvature equation, in order to include the entire hierarchy of flows.

In this chapter, we answer these questions by avoiding altogether the first step of fixing a given time x^n, and working with all the equations (6.11) at once using the formalism of Hamiltonian multiforms. The equations are interpreted as commuting Hamiltonian flows on a phase space with a countable (but infinite) number of coordinates b_j, c_j, $j \geq 1$. We claim that this interpretation, that we call *multi-time approach*, despite being less known than the standard field theory viewpoint provides a deeper insight into the structure of the hierarchy. In the author's opinion, this interpretation is also a true implementation of the original observation that all independent variables x^0, x^1, $x^2 \dots$ play a symmetric role, which is better captured by our use of a Lagrangian and Hamiltonian multiform that do not distinguish any particular independent variable as being special.

Our main objective is to construct a multi-time Poisson bracket $\{\![\ ,\ ]\!\}$ and a Hamiltonian multiform $H = \sum_{i<j=1}^{\infty} H_{ij}\, dx^i \wedge dx^j$ such that:

1. It is possible to compute $\{\![W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\!\}$ for the Lax form $W(\lambda) = \sum_{j=0}^{\infty} Q^{(i)}(\lambda)\, dx^j$ associated to the *entire* hierarchy, and to prove that it possesses the rational r-matrix structure;

2. The collection of all the equations $\partial_k Q(\lambda) = [Q^{(k)}(\lambda), Q(\lambda)]$, $k \geq 0$ or, equivalently[2], of all the zero-curvature equations

$$\partial_i Q^{(j)}(\lambda) - \partial_j Q^{(i)}(\lambda) + [Q^{(j)}(\lambda), Q^{(i)}(\lambda)] = 0\,, \quad i, j \geq 0\,, \tag{6.15}$$

can be written in Hamiltonian form as $dW(\lambda) = \sum_{i<j}^{\infty} \{\![H_{ij}, W(\lambda)]\!\}\, dx^i \wedge dx^j$.

In our exposition, the use of generating functions in the form of formal (Laurent) series will turn out to be extremely efficient. We use the the Lie algebra $\mathcal{L} \coloneqq s\ell(2, \mathscr{A}) \otimes \mathbb{C}(\!(\lambda^{-1})\!)$ of formal Laurent series in the variable $1/\lambda$ with coefficients being matrices in the Lie algebra $s\ell(2, \mathscr{A})$.

With this in mind, we collect the following set of compatible Lax equations for $Q(\lambda)$ as defined in (6.10) (that now is no longer a field, but valued in $\mathscr{A}$),

$$\partial_k Q(\lambda) = [Q^{(k)}(\lambda), Q(\lambda)]\,, \quad k = 0, 1, 2, \dots\,, \tag{6.16}$$

[2]This equivalence does not seem to be well-known but we use it all along and deal interchangeably with the FNR equations (6.11) and the zero-curvature equations (6.15). The implication (6.11)⇒(6.15) is shown for instance in [AC17, Lemma 3.13]. The converse is discussed in [N85, Chapter 5].

where $Q^{(k)}(\lambda) = P_+(\lambda^k Q(\lambda))$, into

$$D_\mu Q(\lambda) = \frac{[Q(\mu), Q(\lambda)]}{\mu - \lambda}\,, \tag{6.17}$$

where we introduced the derivation

$$D_\mu \coloneqq \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} \partial_k\,, \tag{6.18}$$

and used the formal series identity

$$\sum_{k=0}^{\infty} \frac{Q^{(k)}(\lambda)}{\mu^{k+1}} = \frac{Q(\mu)}{\mu - \lambda}\,. \tag{6.19}$$

It is important not to get confused by the notation D_μ which is not meant to be the partial derivative with respect to μ, but simply the generating expression (6.18). We remark that writing the AKNS hierarchy in the generating form (6.17) allows us to reproduce quickly known results. From the symmetry of the right-hand side in (6.17), we have $D_\mu Q(\lambda) = D_\lambda Q(\mu)$, which in component is

$$\partial_k Q_{j+1} = \partial_j Q_{k+1}\,, \quad j, k \geq 0\,. \tag{6.20}$$

Moreover, by means of the Jacobi identity we have

$$D_\lambda D_\mu Q(\nu) = D_\mu D_\lambda Q(\nu)\,, \tag{6.21}$$

which means that the flows ∂_j and ∂_k commute[3]. Finally, noting that the generating function of the Hamiltonian functions (6.13) is given by

$$g(\lambda) \coloneqq -\frac{1}{2} \operatorname{Tr} Q^2(\lambda) = -\frac{1}{2} \operatorname{Tr} Q_0^2 + \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} g_k\,, \tag{6.22}$$

we find

$$D_\mu g(\lambda) = 0\,. \tag{6.23}$$

This shows that the flows take place on the level surface $g(\lambda) = C(\lambda)$ where $C(\lambda)$ is a series in λ^{-1} with constant coefficients. Therefore, in line with [FNR83], we fix

$$\operatorname{Tr} Q^2(\lambda) = -2\,, \tag{6.24}$$

in the rest of this chapter.

[3]Of course, this had to be the case in the first place so as to allow us to consider those flows simultaneously and to define D_μ, but this is a good check of the generating function formalism and an argument in favour of its efficiency.

6.2 Lagrangian and Hamiltonian multiform description of the complete hierarchy

In this Section, we first introduce a Lagrangian multiform which allows us to implement the strategy reviewed in Section 5 to obtain the associated symplectic and Hamiltonian multiforms for the Ablowitz-Kaup-Newell-Segur hierarchy. In turn, this will allow us to show in the next section that Lax form of the entire hierarchy possesses the classical r-matrix structure with respect to our multi-time Poisson bracket.

6.2.1 Lagrangian multiform

Recall that the collection of flows in the Ablowitz-Kaup-Newell-Segur hierarchy is written in generating form as

$$D_\mu Q(\lambda) = \frac{[Q(\mu), Q(\lambda)]}{\mu - \lambda}\,, \tag{6.25}$$

where

$$D_\mu = \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} \partial_k\,, \qquad Q(\lambda) = \sum_{i=0}^{\infty} \frac{Q_i}{\lambda^i}\,,$$
$$Q(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{pmatrix}, \qquad Q_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix},$$
$$\frac{1}{2}\operatorname{Tr} Q(\lambda)^2 = a^2(\lambda) + b(\lambda)c(\lambda) = -1\,.$$

We remark that λ and μ are formal parameters. In order to find an appropriate Lagrangian multiform, it is convenient to note that we can write $Q(\lambda)$ as

$$Q(\lambda) = \varphi(\lambda) Q_0 \varphi(\lambda)^{-1} \tag{6.26}$$

with $Q_0 = -i\sigma_3$ being constant and

$$\varphi(\lambda) = \mathbb{I} + \sum_{j=1}^{\infty} \frac{\varphi_j}{\lambda^j}\,. \tag{6.27}$$

This has been established independently from various angles, in relation to the factorization theorem, see e.g. [S08] or in relation to vertex operators, see e.g. [N85, Chapter 5]. Contrary to the parametrisation used in the latter book, we find it useful to use the set of coordinates $e(\lambda), f(\lambda)$ found in [FNR83][4] and defined as

$$e(\lambda) := \frac{b(\lambda)}{\sqrt{i - a(\lambda)}} = \sum_{i=1}^{\infty} \frac{e_i}{\lambda^i}\,, \qquad f(\lambda) := \frac{c(\lambda)}{\sqrt{i - a(\lambda)}} = \sum_{i=1}^{\infty} \frac{f_i}{\lambda^i} \tag{6.28}$$

[4]The authors of [FNR83] use a different notation: the components of Q are called e, f, h (instead of our b, c, a) and the new coordinates are $\widetilde{e}, \widetilde{f}$ (instead of our e, f).

(note that $e_0 = f_0 = 0$) and set

$$\varphi(\lambda) = \frac{1}{\sqrt{2i}} \begin{pmatrix} \sqrt{2i - e(\lambda)f(\lambda)} & e(\lambda) \\ -f(\lambda) & \sqrt{2i - e(\lambda)f(\lambda)} \end{pmatrix} . \tag{6.29}$$

A direct calculation using $a^2(\lambda)+b(\lambda)c(\lambda) = -1$ shows that $\det \varphi(\lambda) = 1$ and $-i\varphi(\lambda)\sigma_3\varphi(\lambda)^{-1} = Q(\lambda)$ as required. The reader can find more about the coordinates $e(\lambda)$, $f(\lambda)$ in Appendix B.1. Their main property is that they provide Darboux coordinates for all the single-time Poisson brackets $\{\ ,\ \}_i$. We can now formulate the first main result of this section. We obtain the desired Lagrangian multiform $\mathscr{L} = \sum_{i<j=1}^{\infty} L_{ij}\, dx^{ij}$ using the generating function formalism and collecting the coefficients L_{ij} into a formal series in λ^{-1} and μ^{-1} as follows

$$\mathscr{L}(\lambda, \mu) = \sum_{i,j=0}^{\infty} \frac{L_{ij}}{\lambda^{i+1}\mu^{j+1}} . \tag{6.30}$$

By a slight abuse of language, we will also call $\mathscr{L}(\lambda, \mu)$ a Lagrangian multiform.

> **Remark 6.1:** As mentioned above, most results of this section are going to be generalised in Chapter 7. For this reason some of the proofs of this section will only be reported in the appendix.

Theorem 6.2 (Lagrangian multiform and multiform Euler-Lagrange equations) *Define* $\mathscr{L}(\lambda, \mu) = K(\lambda, \mu) - V(\lambda, \mu)$*, where*

$$K(\lambda, \mu) = \mathrm{Tr}\left(\varphi(\mu)^{-1} D_\lambda \varphi(\mu) Q_0 - \varphi(\lambda)^{-1} D_\mu \varphi(\lambda) Q_0\right) , \tag{6.31a}$$

$$V(\lambda, \mu) = -\frac{1}{2}\frac{\mathrm{Tr}(Q(\lambda) - Q(\mu))^2}{\lambda - \mu} . \tag{6.31b}$$

Then $\mathscr{L}(\lambda, \mu)$ is a Lagrangian multiform for the AKNS hierarchy equations (6.2.1).

Indeed, the multiform Euler-Lagrange equations $\delta d\mathscr{L} = 0$ are given by

$$D_\mu Q(\lambda) = \frac{[Q(\mu), Q(\lambda)]}{\mu - \lambda} , \tag{6.32}$$

and the closure relation $d\mathscr{L} = 0$ is satisfied on those equations. In generating form, the latter is equivalent to

$$D_\nu \mathscr{L}(\lambda, \mu) + D_\lambda \mathscr{L}(\mu, \nu) + D_\mu \mathscr{L}(\nu, \lambda) = 0. \tag{6.33}$$

In fact, we have that $d\mathscr{L} = \sum_{i<j<k}(\partial_k L_{ij} - \partial_j L_{ik} + \partial_i L_{jk})dx^{ijk}$ and

$$\begin{aligned} &D_\nu \mathscr{L}(\lambda,\mu) + D_\lambda \mathscr{L}(\mu,\nu) + D_\mu \mathscr{L}(\nu,\lambda) \\ &= \sum_{ijk} \frac{1}{\lambda^i \mu^j \nu^k}(\partial_k L_{ij} + \partial_i L_{jk} + \partial_j L_{ki}) = \sum_{ijk} \frac{1}{\lambda^i \mu^j \nu^k}(\partial_k L_{ij} + \partial_i L_{jk} - \partial_j L_{ik}). \end{aligned}$$

The proof is given in Appendix B.2.

Remark 6.3: Although we discovered it differently, we soon realised that the Lagrangian multiform $\mathscr{L}(\lambda,\mu)$ bears some striking resemblance with to the Zakharov-Mikhailov (ZM) Lagrangian appearing in [ZM80], despite the fact that the latter is a standard Lagrangian and not a multiform. The ZM Lagrangian was introduced to provide a variational description of the system of compatibility conditions (zero-curvature equations) corresponding to a Lax pair of matrices which are rational functions of the spectral parameter with distinct simple poles. We will see in Chapter 7 how the Zakharov-Mikhailov Lagrangian can be obtained from an extension of $\mathscr{L}(\lambda,\mu)$ by taking the appropriate residues in λ and μ.

Remark 6.4: A Lagrangian multiform constructed on the ZM Lagrangian was obtained in [SNC19a] and used to obtain a variational derivation of Lax pair equations themselves. In that same paper, the authors presented the first few coefficients of the Lagrangian multiform for the AKNS hierarchy but it was not clear how these derive directly from the ZM Lagrangian multiform. Our Lagrangian multiform and Theorem 6.2 fill in this gap and provides both the complete set of coefficients L_{ij} of the Lagrangian multiform for the AKNS hierarchy and the Zakharov-Mikhailov Lagrangian. We note that Lagrangians producing the zero-curvature equations (6.15) in potential form were obtained in [N86]. They involved a potential function denoted by H in that paper which produces the Lax matrices $Q^{(k)}$ we use here via the relation $Q^{(k)} = \partial_{k-1}H$. However, assembling all those Lagrangians into a two-form does not seem to provide a Lagrangian multiform for the set of AKNS equations. The closure relation does not hold for instance.

To help the reader recognize the most familiar models, we write some of the coefficients of the Lagrangian multiform explicitly using our formula. Using the expansion $\mathscr{L}(\lambda,\mu) = \sum_{i<j=1}^{\infty} L_{ij}\lambda^{-i-1}\mu^{-j-1}$ we have, for all $i,j \geq 0$

$$L_{ij} = \frac{1}{2}\sum_{k=1}^{j}(f_k\partial_i e_{j+1-k} - e_k\partial_i f_{j+1-k}) - \frac{1}{2}\sum_{k=1}^{i}(f_k\partial_j e_{i+1-k} - e_k\partial_j f_{i+1-k}) - V_{ij}\,.$$

The coefficients V_{ij} are gviven by the following proposition.

Proposition 6.5 *The coefficients of* $V(\lambda,\mu) = \sum_{i<j=1}^{\infty} V_{ij}\lambda^{-i-1}\mu^{-j-1}$ *are*

$$V_{ij} = \operatorname{Tr}\sum_{k=0}^{i} Q_k Q_{i+j-k+1}\,. \tag{6.34}$$

Proof. We being proving that the coefficients $V_{ij} \coloneqq \operatorname{Tr}\sum_{k=0}^{i} Q_k Q_{i+j-k+1}$ are antisymmetric $V_{ij} = -V_{ji}$: in fact we have

$$\begin{aligned} V_{ji} &= \operatorname{Tr}\sum_{k=0}^{j} Q_k Q_{i+j-k+1} \\ &= \operatorname{Tr}\sum_{k=0}^{i+j+1} Q_k Q_{i+j+1-k} - \operatorname{Tr}\sum_{k=j+1}^{i+j+1} Q_k Q_{i+j+1-k} \end{aligned}$$

The first term vanishes as it is the coefficient of λ^{i+j+1} in $\operatorname{Tr} Q^2(\lambda) = -2$. In the second term we make the substitution $m = i+j+1-k$ to obtain

$$V_{ji} = -\operatorname{Tr}\sum_{m=0}^{i} Q_m Q_{i+j+1-m} = -V_{ij}\,.$$

We now start from $V(\lambda,\mu) = -\frac{1}{2}\operatorname{Tr}\frac{(Q(\lambda)-Q(\mu))^2}{\lambda-\mu}$. Firstly, as $\operatorname{Tr}(Q(\lambda)-Q(\mu))^2 = 0$ when $\lambda = \mu$, then it is divisible by $\lambda-\mu$, and $V(\lambda,\mu) = \sum_{i<j=1}^{\infty} V_{ij}\lambda^{-i-1}\mu^{-j-1}$. Then, we formally use the identity[5]

$$\frac{1}{\lambda-\mu} = \frac{1}{2}\sum_{m=0}^{\infty}\frac{\mu^m}{\lambda^{m+1}} - \frac{1}{2}\sum_{m=0}^{\infty}\frac{\lambda^m}{\mu^{m+1}}$$

to look for the coefficient of $\lambda^{-i-1}\mu^{-j-1}$ of $V(\lambda,\mu)$

$$V(\lambda,\mu) = \frac{1}{2}\sum_{m.p,q=0}^{\infty}\frac{\operatorname{Tr} Q_p Q_q}{\lambda^{p+m+1}\mu^{q-m}} - \frac{1}{2}\sum_{m.p,q=0}^{\infty}\frac{\operatorname{Tr} Q_p Q_q}{\lambda^{p-m}\mu^{q+m+1}}\,,$$

where we used the fact that the terms proportional to $\operatorname{Tr} Q^2(\lambda) = -2$ do not contribute. The first contributes when $p+m = i$ and $q-m = j+1$, and the second when $p-m = i+1$ and $q+m = j$, and therefore we have that the i,j-th coefficient of V is

$$\frac{1}{2}\operatorname{Tr}\sum_{p=0}^{i} Q_p Q_{i+j+1-p} - \frac{1}{2}\operatorname{Tr}\sum_{q=0}^{j} Q_q Q_{i+j+1-q} = \frac{V_{ij}}{2} - \frac{V_{ji}}{2} = V_{ij}\,. \qquad \square$$

Recall that the elements a_j, b_j and c_j of Q can all be expressed in terms of the coordinates e_j and f_j (see Appendix B.1). At this stage, no particular choice of time has been made to

[5]This is easily obtainable as $\frac{1}{\lambda-\mu} = \frac{1}{2}\frac{1}{\lambda(1-\frac{\mu}{\lambda})} + \frac{1}{2}\frac{1}{\mu(\frac{\lambda}{\mu}-1)}$ and then using geometric sums. This will be reformulated in Chapter 7 as $\frac{1}{2}(\iota_{\lambda_\infty}\iota_{\mu_\infty} + \iota_{\mu_\infty}\iota_{\lambda_\infty})\frac{1}{\lambda-\mu}$, where ι_{λ_∞} denotes expansion in Laurent series in λ_∞

write these Lagrangians as field theory Lagrangian, in the spirit of [ACDK16] for instance. Hence, as an example, we simply have

$$L_{12} = \frac{1}{2}(f_1\partial_1 e_2 - e_1\partial_1 f_2 + f_2\partial_1 e_1 - e_2\partial_1 f_1) - \frac{1}{2}(f_1\partial_2 e_1 - e_1\partial_2 f_1) - V_{12}\,, \tag{6.35}$$

and

$$\begin{aligned} L_{13} =& \frac{1}{2}(f_1\partial_1 e_3 - e_1\partial_1 f_3 + f_2\partial_1 e_2 - e_2\partial_1 f_2 + f_3\partial_1 e_1 - e_3\partial_1 f_1) \\ &- \frac{1}{2}(f_1\partial_3 e_1 - e_1\partial_3 f_1) - V_{13}\,, \end{aligned} \tag{6.36}$$

which produce partial differential equations for the phase space coordinates e_j, f_j, $j = 1, 2, 3$.

> **Remark 6.6:** We use here the common choice of not including the time x^0 in our explicit multiforms, which would produce the scaling equations $q_0 = -2iq$ and $r_0 = 2ir$. Therefore, when we talk about the first m flows we will refer to the times $1, \dots, m$.

Now to make contact with the more familiar form of these Lagrangians and the corresponding equations of motion, we express the phase space coordinates in terms of $b_1 = q$, $c_1 = r$ and their x^1 derivatives[6]. Note that this amounts to choosing the x^1 equation in (6.11) and use it to *solve* for Q_j (standard field theory point of view). Doing so yields,

$$L_{12} = \frac{i}{4}(q_2 r - q r_2) + \frac{1}{8}(r q_{11} + q r_{11}) - \frac{1}{4}q^2 r^2\,, \tag{6.37}$$

and

$$L_{13} = \frac{i}{4}(r q_3 - q r_3) + \frac{i}{16}(q_{111} r - q r_{111}) + \frac{3i}{16} q r (q r_1 - r q_1)\,, \tag{6.38}$$

which are known Lagrangians whose Euler-Lagrange equations are

$$i q_2 + \frac{1}{2} q_{11} - q^2 r = 0\,, \qquad i r_2 - \frac{1}{2} r_{11} + q r^2\,, \tag{6.39a}$$

$$q_3 + \frac{1}{4} q_{111} - \frac{3}{2} q r q_1 = 0\,, \qquad r_3 + \frac{1}{4} r_{111} - \frac{3}{2} q r r_1 = 0\,. \tag{6.39b}$$

These are the (unreduced) NLS and mKdV systems respectively. We can just as easily produce the Lagrangian L_{23}, first in the e and f coordinates and then, if desired, in the q

[6] The reader can find the relations between the e_i's and f_i's and q and r and their derivative with respect to x^1 in Appendix B.1.

and r coordinates as before. It reads

$$\begin{aligned}L_{23} =&\frac{i}{16}(rq_{112}-qr_{112})+\frac{i}{16}(q_1r_{12}-q_{12}r_1)-\frac{i}{16}(q_{11}r_2-q_2r_{11})\\&-\frac{3i}{16}qr(rq_2-qr_2)-\frac{1}{8}(q_{13}r+qr_{13})+\frac{1}{8}(r_1q_3+q_1r_3)\\&+\frac{1}{16}q_{11}r_{11}-\frac{qr}{8}(qr_{11}+q_{11}r)+\frac{1}{16}(qr_1-q_1r)^2+\frac{1}{4}q^3r^3\,,\end{aligned}\tag{6.40}$$

and its Euler-Lagrange equations are just consequence of (6.39).

Remark 6.7: The partial Lagrangian multiform thus derived here for the first three times $L_{12}\,dx^{12}+L_{23}\,dx^{23}+L_{13}\,dx^{13}$ is equivalent to the one used in the previous sections, as it is the same up to an overall coefficient 2 and a total horizontal differential. This other normalisation is preferable in this case as it allows us to write a closed form for the coefficients of the Lagrangian multiform $\mathscr{L}$ in terms of the coordinates e, f used in [FNR83].

6.2.2 Symplectic multiform

Equipped with a Lagrangian multiform for the AKNS hierarchy, we now construct the associated symplectic multiform Ω. Again, it is very convenient to work with generating functions so we introduce

$$\Omega^{(1)}(\lambda)=\sum_{j=0}^{\infty}\frac{\omega_j^{(1)}}{\lambda^{j+1}}\,,\tag{6.41}$$

$$\Omega(\lambda)=\sum_{j=0}^{\infty}\frac{\omega_j}{\lambda^{j+1}}\,,\tag{6.42}$$

to represent respectively

$$\Omega^{(1)}=\sum_{j=0}^{\infty}\omega_j^{(1)}\wedge dx^j\,,\tag{6.43}$$

$$\Omega=\sum_{j=0}^{\infty}\omega_j\wedge dx^j\,.\tag{6.44}$$

As before, by a slight abuse of language, we also call $\Omega(\lambda)$ symplectic multiform.

Proposition 6.8 *The symplectic multiform associated to $\mathscr{L}(\lambda,\mu)$ is given by*

$$\Omega(\lambda)=-\operatorname{Tr}\left(Q_0\varphi(\lambda)^{-1}\delta\varphi(\lambda)\wedge\varphi(\lambda)^{-1}\delta\varphi(\lambda)\right)\,.\tag{6.45}$$

The proof is in Appendix B.3.

Remark 6.9: The expression for $\Omega(\lambda)$ is reminiscent of the well-known expression for the (pull-back to the group of the) Kostant-Kirillov symplectic form on a coadjoint orbit of the loop algebra $\mathcal{L}$ through the element Q_0. To make this more precise, let us use for instance the formulas in [BBT03, Section 3.3] giving the expression of the pull-back to the group of the Kostant-Kirillov form for the orbit through a diagonal matrix polynomial $A(\lambda)$,

$$\omega = \operatorname*{res}_{\lambda=0} \operatorname{Tr}\left(A(\lambda)g^{-1}(\lambda)\delta g(\lambda) \wedge g^{-1}(\lambda)\delta g(\lambda)\right) .$$

Here, choosing $A(\lambda) = -i\lambda^k\sigma_3$, $k \geq 0$, and $g(\lambda) = \varphi(\lambda)$, we get the connection between our symplectic multiform and the Kostant-Kirillov form

$$\omega = \operatorname*{res}_{\lambda=0} \lambda^k \Omega(\lambda) = \omega_k .$$

In particular, each single-time symplectic form ω_k corresponds to ω on the orbit of the element $-i\lambda^k\sigma_3$. Therefore, our symplectic multiform contains in a single object all those symplectic forms. This is the first time such an object is derived and, to our knowledge, it is the first time that a Kostant-Kirillov symplectic form is derived from a Lagrangian perspective.

As a consequence of the explicit formula for Ω, we get the following remarkable result that the e, f coordinates provide Darboux coordinates.

Corollary 6.10 *The symplectic multiform is written in Darboux form as*

$$\Omega(\lambda) = \delta f(\lambda) \wedge \delta e(\lambda) , \tag{6.46}$$

and hence, $\omega_0 = 0$ *and,*

$$\omega_k = \sum_{i=1}^{k} \delta f_i \wedge \delta e_{k+1-i} , \quad \forall k \geq 1 . \tag{6.47}$$

Proof. Direct calculation by inserting (6.29) into (6.45). We have

$$\sum_{j=0}^{\infty} \frac{\omega_j}{\lambda^{j+1}} = \delta \sum_{m=1}^{\infty} \frac{f_m}{\lambda^m} \wedge \delta \sum_{n=1}^{\infty} \frac{e_n}{\lambda^n} = \sum_{k=1}^{\infty} \sum_{m=1}^{k} \frac{1}{\lambda^{k+1}} \delta f_m \wedge \delta e_{k+1-m} . \qquad \square$$

6.2.3 Multiform Hamilton equations for the AKNS hierarchy

According to Definition (5.2), the coefficients of the Hamiltonian multiform $\mathcal{H} = \sum_{i<j=1}^{\infty} H_{ij}\, dx^{ij}$ associated to $\mathscr{L}$ and $\Omega^{(1)}$ are given by

$$H_{ij} = \widetilde{\partial}_i \lrcorner \omega_j^{(1)} - \widetilde{\partial}_j \lrcorner \omega_i^{(1)} - L_{ij} . \tag{6.48}$$

As is now customary, we rewrite this in generating form as

$$\mathcal{H}(\lambda,\mu) := \widetilde{D}_\lambda \lrcorner \Omega^{(1)}(\mu) - \widetilde{D}_\mu \lrcorner \Omega^{(1)}(\lambda) - \mathscr{L}(\lambda,\mu)\,, \tag{6.49}$$

where we introduce the notation $\widetilde{D}_\lambda = \sum_{i=0}^{\infty} \widetilde{\partial}_i / \lambda^{i+1}$ in line with (6.18).

Lemma 6.11 *The following holds*

$$\mathcal{H}(\lambda,\mu) = V(\lambda,\mu) = -\frac{1}{2}\frac{\mathrm{Tr}(Q(\lambda)-Q(\mu))^2}{\lambda-\mu}\,, \tag{6.50}$$

Hence, $\mathcal{H}(\lambda,\mu)$ satisfies the closure relation.

Proof. A direct calculation shows that $\widetilde{D}_\lambda \lrcorner \Omega^{(1)}(\mu) - \widetilde{D}_\mu \lrcorner \Omega^{(1)}(\lambda) = K(\lambda,\mu)$ hence $\mathcal{H}(\lambda,\mu) = V(\lambda,\mu)$. Finally, the closure relation of $\mathcal{H}$ is a general result that we reviewed in Corollary 5.6 but here, we get a direct confirmation from the structure of the proof of Theorem 6.2 which established that V is closed on the equations of motion, separately from K. □

For completeness, we now check the validity of the general result in Proposition 5.11 in our case.

Proposition 6.12 *The multiform Hamilton equations associated to $\mathcal{H}$ and Ω are*

$$D_\lambda Q(\mu) = \frac{[Q(\lambda),Q(\mu)]}{\lambda-\mu}\,. \tag{6.51}$$

Proof. The multiform Hamilton equations read $\delta\mathcal{H} = \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega$, or, in components,

$$\delta H_{ij} = \widetilde{\partial}_j \lrcorner \omega_i - \widetilde{\partial}_i \lrcorner \omega_j\,.$$

This is reformulated in generating form as,

$$\delta\mathcal{H}(\lambda,\mu) = \widetilde{D}_\mu \lrcorner \Omega(\lambda) - \widetilde{D}_\lambda \lrcorner \Omega(\mu)\,.$$

We have already computed $\delta\mathcal{H}(\lambda,\mu) = \delta V(\lambda,\mu)$ as

$$\delta\mathcal{H}(\lambda,\mu) = \mathrm{Tr}\left(\frac{1}{\mu-\lambda}\varphi(\lambda)^{-1}[Q(\mu),Q(\lambda)]\delta\varphi(\lambda) - \frac{1}{\lambda-\mu}\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\delta\varphi(\mu)\right).$$

We calculate the right hand-side, that reads

$$
\begin{aligned}
&\widetilde{D}_\mu \lrcorner \Omega(\lambda) - \widetilde{D}_\lambda \lrcorner \Omega(\mu) \\
&= \operatorname{Tr}\big(-Q_0\varphi(\lambda)^{-1}D_\mu\varphi(\lambda)\varphi(\lambda)^{-1}\delta\varphi(\lambda) + Q_0\varphi(\lambda)^{-1}\delta\varphi(\lambda)\varphi(\lambda)^{-1}D_\mu\varphi(\lambda) \\
&\qquad\qquad + Q_0\varphi(\mu)^{-1}D_\lambda\varphi(\mu)\varphi(\mu)^{-1}\delta\varphi(\mu) - Q_0\varphi(\mu)^{-1}\delta\varphi(\mu)\varphi(\mu)^{-1}D_\lambda\varphi(\mu)\big) \\
&= \operatorname{Tr}\big(\varphi^{-1}(\lambda)D_\mu Q(\lambda)\delta\varphi(\lambda) - \varphi^{-1}(\mu)D_\lambda Q(\mu)\delta\varphi(\mu)\big)\,.
\end{aligned}
$$

The result follows by reading the coefficient of $\delta\varphi(\mu)$ or equivalently $\delta\varphi(\lambda)$. □

6.2.4 The 0th time

In this section we remark that the 0th time x^0 can be included as well, by keeping the relations for $H_{pq} = \operatorname{Tr}\sum_{i=0}^{p} Q_i Q_{p+1+1-i}$ and $\omega_k = \sum_{i=1}^{k}\delta f_i \wedge \delta e_{k+1-i}$. Indeed we obtain

$$
H_{0q} = \operatorname{Tr} Q_0 Q_{q+1} = -2ia_{q+1} = -2i\sum_{i=1}^{q} f_i e_{q+1-i}\,, \tag{6.52}
$$

$$
\omega_0 = 0\,. \tag{6.53}
$$

The multiform Hamilton equations are $\delta\mathcal{H} = \sum_{i=0}^{\infty} dx^i \wedge \widetilde{\partial}_i \lrcorner \Omega$, where the left hand-side is $\sum_{p<q}^{\infty}\delta H_{pq}\wedge dx^{pq}$ and the right hand-side reads $\sum_{p<q=1}^{\infty}(\widetilde{\partial}_q\lrcorner\omega_p - \widetilde{\partial}_p\lrcorner\omega_q)\wedge dx^{pq}$. This becomes

$$
\begin{aligned}
&\sum_{q=1}^{\infty}\delta H_{0q}\wedge dx^{0q} + \sum_{1=p<q}^{\infty}\delta H_{pq}\wedge dx^{pq} \\
&\qquad = \sum_{q=1}^{\infty}(\widetilde{\partial}_q\lrcorner\omega_0 - \widetilde{\partial}_0\lrcorner\omega_q)\wedge dx^{0q} + \sum_{1=p<q}^{\infty}(\widetilde{\partial}_q\lrcorner\omega_p - \widetilde{\partial}_p\lrcorner\omega_q)\wedge dx^{pq}
\end{aligned}
$$

where we separated the equations involving the time x^0 from the other ones. Since $\omega_0 = 0$, the ones involving the time x^0 are then $\delta H_{0q} = -\widetilde{\partial}_0\lrcorner\omega_q$, where

$$
\delta H_{0q} = -2i\sum_{k=1}^{q}\left(\frac{\partial a_{q+1}}{\partial e_k}\delta e_k + \frac{\partial a_{q+1}}{\partial f_k}\delta f_k\right) = -2i\sum_{k=1}^{q}(f_{q+1-k}\delta e_k + e_{q+1-k}\delta f_k)
$$

and

$$
\begin{aligned}
-\widetilde{\partial}_0\lrcorner\omega_q &= -\,\widetilde{\partial}_0\lrcorner\sum_{k=1}^{q}\delta f_k\wedge\delta e_{q+1-k} \\
&= \sum_{k=1}^{q}(-\partial_0 f_k\delta e_{q+1-k} + \partial_0 e_{q+1-k}\delta f_k) = \sum_{k=1}^{q}(-\partial_0 f_{q+1-k}\delta e_k + \partial_0 e_{q+1-k}\delta f_k)
\end{aligned}
$$

so that the equations are

$$
\partial_0 f_k = 2if_k\,, \qquad \partial_0 e_k = -2ie_k\,, \tag{6.54}
$$

which correspond to the scaling equations in [FNR83] $q_0 = -2iq$ and $r_0 = 2ir$.

6.3 Classical r-matrix and zero-curvature equations

6.3.1 Admissible forms and multi-time Poisson bracket

Having the symplectic multiform Ω at our disposal, we can investigate in detail under which conditions a horizontal form is admissible and then compute the multi-time Poisson bracket for two such forms. Recall that in our case, only 0- and 1-forms can be non-trivial admissible forms. We have the following two propositions, the proofs of which are given in Appendix B.4 and B.5.

Proposition 6.13 *A 1-form $F = \sum_{k=0}^{\infty} F_k\, dx^k$ is admissible with respect to Ω if and only if F_0 is constant and, for all $k \geq 1$, F_k depends only on the coordinates $(e_1, \dots, e_k, f_1, \dots, f_k)$ and*

$$\frac{\partial F_k}{\partial e_j} = \frac{\partial F_{k+1}}{\partial e_{j+1}}\,, \qquad \frac{\partial F_k}{\partial f_j} = \frac{\partial F_{k+1}}{\partial f_{j+1}}\,, \quad j = 1, \dots, k\,. \tag{6.55}$$

Its Hamiltonian vector field is given by

$$\xi_F = \sum_{k=1}^{\infty} \left(-\frac{\partial F_k}{\partial f_1} \partial_{e_k} + \frac{\partial F_k}{\partial e_1} \partial_{f_k} \right)\,. \tag{6.56}$$

Proposition 6.14 *Every 0-form $H(e_1, \dots, f_1, \dots)$ is admissible with respect to Ω, with admissible vector field given by*

$$\xi_H = \sum_{i=1}^{\infty} \left(-\frac{\partial H}{\partial f_i} \partial_{e_1} \wedge \partial_i + \frac{\partial H}{\partial e_i} \partial_{f_1} \wedge \partial_i \right)\,. \tag{6.57}$$

Note that in practice, we will deal with 0-forms that depend only on a finite number of coordinates e_j, f_j in which case the sum in (6.57) truncates accordingly.

Remark 6.15: Proposition 6.13 gives an elegant reformulation of the rather complicated-looking conditions (5.59) in the variables q and r that the coefficients of an admissible 1-form F have to satisfy. They are of course equivalent. The first two lines are easily obtained respectively by taking

$$\frac{\partial F_1}{\partial f_1} = \frac{\partial F_2}{\partial f_2} = \frac{\partial F_3}{\partial f_3} = \frac{\partial F_4}{\partial f_4}\,, \qquad \text{and} \qquad \frac{\partial F_1}{\partial e_1} = \frac{\partial F_2}{\partial e_2} = \frac{\partial F_3}{\partial e_3} = \frac{\partial F_4}{\partial e_4}\,.$$

Let us also take for instance $\frac{\partial F_3}{\partial e_1} = \frac{\partial F_4}{\partial e_2}$ and let us write it in the old q and r

coordinates using the expression in Appendix B.1. We have

$$\begin{aligned}\frac{\partial F_3}{\partial e_1} =& \frac{\partial F_3}{\partial q}\frac{\partial q}{\partial e_1} + \frac{\partial F_3}{\partial q_{11}}\frac{\partial q_{11}}{\partial e_1} + \frac{\partial F_3}{\partial r_{11}}\frac{\partial r_{11}}{\partial e_1} \\ =& \sqrt{2i}\left(\frac{\partial F_3}{\partial q} + 6ie_1f_1\frac{\partial F_3}{\partial q_{11}} + 3if_1^2\frac{\partial F_3}{\partial r_{11}}\right) \\ =& \sqrt{2i}\left(\frac{\partial F_3}{\partial q} + 3qr\frac{\partial F_3}{\partial q_{11}} + \frac{3}{2}r^2\frac{\partial F_3}{\partial r_{11}}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F_4}{\partial e_2} =& \frac{\partial F_4}{\partial q_1}\frac{\partial q_1}{\partial e_2} + \frac{\partial F_4}{\partial q_{111}}\frac{\partial q_{111}}{\partial e_2} + \frac{\partial F_4}{\partial r_{111}}\frac{\partial r_{111}}{\partial e_2} \\ =& \sqrt{2i}\left(-2i\frac{\partial F_4}{\partial q_1} + 20e_1f_1\frac{\partial F_4}{\partial q_{111}} + 2f_1^2\frac{\partial F_4}{\partial r_{111}}\right) \\ =& \sqrt{2i}\left(-2i\frac{\partial F_4}{\partial q_1} - 10iqr\frac{\partial F_4}{\partial q_{111}} - ir^2\frac{\partial F_4}{\partial r_{111}}\right).\end{aligned}$$

Putting the two together and using the other equations (in particular $\frac{\partial F_4}{\partial q_{111}} = \frac{i}{2}\frac{\partial F_3}{\partial q_{11}} = -\frac{i}{8}\frac{\partial F_1}{\partial q}$, $\frac{\partial F_4}{\partial r_{111}} = -\frac{i}{2}\frac{\partial F_3}{\partial r_{11}} = \frac{i}{8}\frac{\partial F_1}{\partial r}$) we get the last of the equations (5.59)

$$\frac{\partial F_4}{\partial q_1} = \frac{i}{2}\frac{\partial F_3}{\partial q} + \frac{i}{4}qr\frac{\partial F_1}{\partial q} - \frac{i}{4}r^2\frac{\partial F_1}{\partial r}.$$

We can now define the multi-time Poisson bracket with respect to Ω between two admissible forms F and G as

$$\{\![F, G]\!\} = (-1)^r \xi_F \lrcorner \delta G \tag{6.58}$$

where r is the horizontal degree of F. We recall Proposition 5.22 which gives the decomposition of the multi-time Poisson brackets in terms of the single-time Poisson brackets $\{\ ,\ \}_k$. Given that we know the explicit form of the single-time symplectic forms ω_k, see (6.47), we obtain the following specialisation as a consequence.

Proposition 6.16 (Decomposition of the multi-time Poisson brackets) *The multi-time Poisson brackets with respect to Ω of two admissible 1-forms $F = \sum_{k=0}^{\infty} F_k\, dx^k$ and $G = \sum_{k=0}^{\infty} G_k\, dx^k$ satisfy the following decomposition:*

$$\begin{aligned}\{\![F, G]\!\} =& \sum_{k=0}^{\infty}\{F_k, G_k\}_k\, dx^k\,, \qquad \textit{where} \\ \{F_k, G_k\}_k =& \sum_{j=1}^{k}\left(\frac{\partial F_k}{\partial f_j}\frac{\partial G_k}{\partial e_{k-j+1}} - \frac{\partial F_k}{\partial e_j}\frac{\partial G_k}{\partial f_{k-j+1}}\right).\end{aligned} \tag{6.59}$$

Thanks to the the propositions above, we can prove by direct but long calculations that the multi-time Poisson bracket $\{\![\ ,\ ]\!\}$ satisfies the Jacobi identity.

Proposition 6.17 (Jacobi identity) *If* $F, G, K \in \mathscr{A}^{(0,1)}$ *and* $H \in \mathscr{A}$ *are admissible forms, we have that*

1. $\{\![F, G]\!\}$ *and* $\{\![F, H]\!\}$ *are respectively an admissible 1-form and an admissible 0-form,*
2. $\{\![\{\![F, G]\!\}, K]\!\} + \{\![\{\![K, F]\!\}, G]\!\} + \{\![\{\![G, K]\!\}, F]\!\} = 0,$
3. $\{\![\{\![F, G]\!\}, H]\!\} + \{\![\{\![H, F]\!\}, G]\!\} + \{\![\{\![G, H]\!\}, F]\!\} = 0.$

The proof can be found in Appendix B.6.

> **Remark 6.18:** It is known (see e.g. [FS15]) that the Jacobi identity is not necessarily satisfied by a covariant Poisson bracket. This problem could therefore be present in general for a multi-time Poisson bracket (which can be viewed as a generalisation of a covariant Poisson bracket). This is why the Jacobi identity was not discussed in [CS20b] and why we checked it here directly.

6.3.2 Classical r-matrix structure of the multi-time Poisson bracket

Definition 6.19 *We call* Lax form *the following horizontal* 1*-form with matrix coefficient*

$$W(\lambda) = \sum_{i=0}^{\infty} Q^{(i)}(\lambda)\, dx^i \tag{6.60}$$

where, for $i \geq 0$, $Q^{(i)}(\lambda) := P_+(\lambda^i Q(\lambda))$.

We are now ready to formulate the main result of this section, the proof of which is long but straightforward and is given in Appendix B.7.

Theorem 6.20 *The Lax form* $W(\lambda)$ *is admissible, with Hamiltonian vector field*

$$\xi_W(\lambda) = \sum_{k=1}^{\infty} \left(-\frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \partial_{e_k} + \frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \partial_{f_k} \right). \tag{6.61}$$

Its multi-time Poisson brackets possesses the linear Sklyanin bracket structure i.e.

$$\{\![W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\!\} = [r_{\mathbf{12}}(\lambda - \mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]\,, \tag{6.62}$$

where $r_{\mathbf{12}}(\lambda, \mu)$ *is the so-called rational classical r-matrix given by*

$$r_{\mathbf{12}}(\lambda) = -\frac{P_{\mathbf{12}}}{\lambda}\,. \tag{6.63}$$

> **Remark 6.21:** We have already shown directly that our multi-time Poisson bracket $\{\![\ ,\ ]\!\}$ satisfies the Jacobi identity for 0- and 1-forms. In the case of 1-forms, this is

also a corollary of Theorem 6.20 since $W(\lambda)$ contains all the coordinates of our phase space and it is known that the rational r-matrix satisfies the classical Yang-Baxter equation which implies the Jacobi identity.

6.3.3 Hamiltonian multiform nature of the zero-curvature equations

It is one of the most important results of the theory of integrable classical field theories that their zero-curvature representation admits a Hamiltonian formulation. In Chapter 4 we cast this result into a covariant framework, for the NLS and mKdV equations separately: the covariant Hamilton equations for the Lax form associated to each equation (thus containing only the two relevant $Q^{(j)}(x,\lambda)$) produce the respective zero-curvature condition. Here, we are in a position to prove the analogous result for the whole AKNS hierarchy at once, thanks to our Hamiltonian multiform and multi-time Poisson bracket. The following is the main result of this section

Theorem 6.22 *The multiform Hamilton equations for the Lax form $W(\lambda)=\sum_{k=0}^{\infty} Q^{(k)}(\lambda)\,dx^k$, i.e.*

$$dW(\lambda)=\sum_{i<j}\{\![H_{ij},W(\lambda)]\!\}\,dx^{ij}\,, \tag{6.64}$$

are equivalent to the complete set of zero-curvature equations of the AKNS hierarchy

$$\partial_i Q^{(j)}(\lambda)-\partial_j Q^{(i)}(\lambda)=[Q^{(i)}(\lambda),Q^{(j)}(\lambda)] \qquad \forall i<j\,. \tag{6.65}$$

The proof is given in Appendix B.8.

6.3.4 Conservation laws

We have introduced conservation laws in the context of Hamiltonian multiforms with Definition 5.20, and we have given an example of a conservation law for the first four flows of the AKNS hierarchy in Section 5.5. In this chapter we give the general expression for the coefficients of a conservation law for the whole hierarchy, which are obtained by considering the following 1-form.

Proposition 6.23 *The form $A=\sum_{k=0}^{\infty} A_k\,dx^k$, $A_k=a_{k+1}$ is a conservation law.*

Proof. From (B.2), we find $A_k=a_{k+1}=\sum_{i=1}^{k} e_i f_{k+1-i}$ so that $\frac{\partial A_i}{\partial f_j}=e_{i+1-j}=\frac{\partial A_{i+1}}{\partial f_{j+1}}$ and $\frac{\partial A_i}{\partial e_j}=f_{i+1-j}=\frac{\partial A_{i+1}}{\partial e_{j+1}}$. Hence A is Hamiltonian. Now,

$$\begin{aligned}dA=&\xi_A\lrcorner\delta\mathcal{H}=\sum_{m<n}\sum_{k=1}^{\infty}\left(-\frac{\partial A_k}{\partial f_1}\frac{\partial H_{mn}}{\partial e_k}+\frac{\partial A_k}{\partial e_1}\frac{\partial H_{mn}}{\partial f_k}\right)dx^{mn}\\ &=\sum_{m<n}\sum_{k=1}^{n}\left(-e_k\frac{\partial H_{mn}}{\partial e_k}+f_k\frac{\partial H_{mn}}{\partial f_k}\right)dx^{mn}\end{aligned}$$

where we have used $\frac{\partial A_k}{\partial e_1} = f_k$, $\frac{\partial A_k}{\partial f_1} = e_k$, and the fact that $\frac{\partial H_{mn}}{\partial e_k} = \frac{\partial H_{mn}}{\partial f_k} = 0$ if $k > n$ (without loss of generality, we consider $m < n$). From the explicit expression of $\mathcal{H}(\lambda, \mu)$, a direct argument shows that each H_{mn} is in fact a polynomial in $e_1, \dots, e_n, f_1, \dots, f_n$ of the form

$$H_{mn} = \sum_{(i),(j) \in \mathbb{N}^n} h_{(i)(j)} (e)^{(i)} (f)^{(j)} \,,$$

where the sum is finite (only a finite number of coefficients $h_{(i)(j)} \in \mathbb{C}$ are non zero) and we have used the notations $(e)^{(i)} = e_1^{i_1} e_2^{i_2} \dots e_m^{i_m}$, $(f)^{(j)} = f_1^{j_1} f_2^{j_2} \dots f_n^{j_n}$, and has the property that $\sum_{k=1}^n i_k = \sum_{k=1}^n j_k$. The result then follows since $\sum_{k=1}^n e_k \frac{\partial}{\partial e_k}$ and $\sum_{k=1}^n f_k \frac{\partial}{\partial f_k}$ are Euler operators with respect to the coordinates e_k and f_k respectively. □

This result provides a reinterpretation of the known fact the quantities $h_k = \frac{1}{k} \int a_{k+1}\, dx^1$, viewed as the traditional hierarchy of standard, single-time, Hamiltonians are indeed constant of the motion and in involution with respect to the traditional (single-time) Poisson bracket $\{\ ,\ \}_1$ (see e.g. [D03, Section 9.3]). We will recover the explicit expressions of the a_{k+1}'s in the next section.

6.4 Recovering previous results and the first three times

It is straightforward to recover our previous results of Chapter 4 by 'freezing' all times except a given pair. This singles out a single $1+1$-dimensional field theory within the hierarchy and our Lagrangian multiform, symplectic multiform, Hamiltonian multiform and multi-time Poisson bracket reduce respectively to a Lagrangian, multisymplectic form, covariant Hamiltonian and covariant Poisson bracket.

As the simplest example, let use freeze all times except $x^1 = x$ and $x^2 = t$: we specialise to NLS and recover all the results of Section 4.3 by direct calculation. The Lax form is simply

$$W(\lambda) = Q^{(1)}(\lambda)\, dx + Q^{(2)}(\lambda)\, dt \,, \tag{6.66}$$

which can be computed using again the coordinates q, r and derivatives with respect to x for instance to reproduce the well known NLS Lax pair. The Lagrangian multiform reduces to $\mathscr{L} = L_{12}\, dx \wedge dt$ where L_{12} is given in (6.37) while the Hamiltonian multiform only involves H_{12}. Using our general formula (which is just (6.34) written explicitly),

$$H_{ij} = \sum_{k=0}^{i} (2 a_k a_{i+j+1-k} + b_k c_{i+j+1-k} + c_k b_{i+j+1-k}) \,, \tag{6.67}$$

we find

$$\begin{aligned}
H_{12} =& 2a_0a_4 + b_0c_4 + c_0b_4 + 2a_1a_3 + b_1c_3 + c_1b_3 \\
=& -2i(e_1f_3 + e_2f_2 + e_3f_1) + 2ie_1(f_3 + \frac{i}{4}e_1f_1^2) + 2if_1(e_3 + \frac{i}{4}e_1^2f_1) \\
=& -2ie_2f_2 - e_1^2f_1^2 \\
=& -\frac{1}{4}(q_1r_1 - q^2r^2)\,.
\end{aligned}$$

This is the covariant Hamiltonian for NLS found in Equation (4.46) (up to an irrelevant factor). The symplectic multiform collapses into the following multisymplectic form

$$\Omega = \omega_1 \wedge dx + \omega_2 \wedge dt \qquad \text{where} \tag{6.68a}$$

$$\omega_1 = \frac{i}{2}\delta q \wedge \delta r\,, \quad \text{and} \quad \omega_2 = \frac{1}{4}\delta r \wedge \delta q_1 + \frac{1}{4}\delta q \wedge \delta r_1\,, \tag{6.68b}$$

also found first in [CS20a] and reported in Section 4.3 (up to irrelevant factors). It gives rise to a covariant Poisson bracket which is simply the reduction of our multi-time Poisson bracket to only two times and our main results, Theorems 6.20 and 6.22 restrict accordingly to the results of Section 4.3.

We stress however that we can instead choose any pair of times x^n and x^k and apply the same reasoning. Doing so provides a way to unify the results in [AC17] which established the r-matrix structure of dual Lax pairs for an arbitrary pair of times and the results in [CS20a] which provided a covariant formulation of this structure but only for the pair of times (x^1, x^2) and (x^1, x^3).

The salient features of the multiform theory appear when at least three times are combined together. In general, the coefficients L_{1n} (resp. H_{1n}) are not too difficult to construct but all the other ones are, and indeed up to now it was not known how to obtain them in general. For instance, freezing all times except x^1, x^2, x^3, the coefficient L_{23} was first obtained in [SNC19a] by complicated calculations. Here, we obtain it rather easily, see (6.40), as well as the associated coefficient H_{23} in the Hamiltonian multiform which reads

$$\begin{aligned}
H_{23} =& -2ie_3f_3 + \frac{1}{2}e_1f_1(f_1e_3 + e_1f_3) - (e_1f_2 + f_1e_2)^2 + \frac{i}{8}e_1^3f_1^3 \\
=& -\frac{1}{16}q_{11}r_{11} + \frac{qr}{8}(rq_{11} + qr_{11}) - \frac{1}{16}(rq_1 - qr_1)^2 - \frac{1}{4}q^3r^3\,.
\end{aligned} \tag{6.69}$$

For completeness, let us also give

$$\begin{aligned}
H_{13} =& -2i(e_2f_3 + e_3f_2) - \frac{3}{2}e_1f_1(f_1e_2 + e_1f_2)\,, \\
=& \frac{i}{8}\left(q_1r_{11} - r_1q_{11}\right)\,.
\end{aligned} \tag{6.70}$$

We remark again that these coefficients differ from those in the previous chapters by an expected factor $\frac{1}{2}$.

In the rest of this section, we illustrate in every detail the calculations involved in our general results when restricted to the first three times. This has only pedagogical value. We hope that this will help the reader familiarise themselves with some of the new formalism while dealing with the most familiar and easiest levels of the AKNS hierarchy. We now turn to the symplectic multiform $\Omega = \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3$, where

$$\omega_1 = \delta f_1 \wedge \delta e_1 \,, \tag{6.71a}$$

$$\omega_2 = \delta f_1 \wedge \delta e_2 + \delta f_2 \wedge \delta e_1 \,, \tag{6.71b}$$

$$\omega_3 = \delta f_1 \wedge \delta e_3 + \delta f_2 \wedge \delta e_2 + \delta f_3 \wedge \delta e_1 \,. \tag{6.71c}$$

As done above for ω_1 and ω_2, it is interesting to write ω_3 using $b_1 = q$, $c_1 = r$ and their derivatives with respect to x^1, denoted by q_1, r_1, q_{11}, r_{11}. We find

$$\omega_3 = \frac{i}{8}\delta r \wedge \delta q_{11} + \frac{i}{8}\delta r_{11} \wedge \delta q + \frac{i}{8}\delta q_1 \wedge \delta r_1 + \frac{3iqr}{4}\delta q \wedge \delta r \,, \tag{6.72}$$

and we remark that they also differ from the ones in [CS20b] by the same factor $\frac{1}{2}$, so that the multiform Hamilton equations $\delta\mathcal{H} = \sum_j dx^j \wedge \widetilde{\partial}_j \lrcorner \Omega$ are the same. Let us compute them, in the new e and f coordinates. In components we have

- $\delta H_{12} = \partial_2 \lrcorner \omega_1 - \partial_1 \lrcorner \omega_2$:

$$\partial_1 f_1 = 2if_2 \,, \qquad \partial_1 e_1 = -2ie_2 \,,$$
$$\partial_1 f_2 - \partial_2 f_1 = 2e_1 f_1^2 \,, \qquad \partial_2 e_1 - \partial_1 e_2 = 2e_1^2 f_1 \,.$$

  The top equations give the relations $b_2 = \frac{i}{2}\partial_1 b_1 = \frac{i}{2}q_1$ and $c_2 = -\frac{i}{2}\partial_1 c_1 = -\frac{i}{2}r_1$, and the bottom ones give the NLS equations.

- $\delta H_{13} = \partial_3 \lrcorner \omega_1 - \partial_1 \lrcorner \omega_3$:

$$\partial_1 f_1 = 2if_2 \,, \qquad \partial_1 e_1 = -2ie_2 \,,$$
$$\partial_1 f_2 = 2if_3 + \frac{3}{2}e_1 f_1^2 \,, \qquad \partial_1 e_2 = -2ie_3 - \frac{3}{2}e_1^2 f_1 \,,$$
$$\partial_1 f_3 - \partial_3 f_1 = \frac{3}{2}e_2 f_1^2 + 3e_1 f_1 f_2 \,, \qquad \partial_3 e_1 - \partial_1 e_3 = \frac{3}{2}e_1^2 f_2 + 3e_1 f_1 e_2 \,,$$

  where the top four equations give the relations $b_2 = \frac{i}{2}q_1$ and $c_2 = -\frac{i}{2}r_1$, and $b_3 = -\frac{1}{4}q_{11} + \frac{1}{2}q^2 r$ and $c_3 = -\frac{1}{4}r_{11} + \frac{1}{2}qr^2$, and the bottom ones are the mKdV equations.

- $\delta H_{23} = \partial_3 \lrcorner \omega_2 - \partial_2 \lrcorner \omega_3$:

$$\begin{aligned}
\partial_2 f_1 &= 2i f_3 - \frac{1}{2} e_1 f_1^2 \,, \\
\partial_2 e_1 &= -2i e_3 + \frac{1}{2} e_1^2 f_1 \,, \\
\partial_2 f_2 - \partial_3 f_1 &= 2 f_1^2 e_2 + 2 e_1 f_1 f_2 \,, \\
\partial_3 e_1 - \partial_2 e_2 &= 2 e_1^2 f_2 + 2 e_1 f_1 e_2 \,, \\
\partial_3 f_2 - \partial_2 f_3 &= \frac{1}{2} f_1^2 e_3 + e_1 f_1 f_3 + \frac{3i}{8} e_1^2 f_1^3 - 2 e_1 f_2^2 - 2 f_1 e_2 f_2 \,, \\
\partial_2 e_3 - \partial_3 e_2 &= \frac{1}{2} e_1^2 f_3 + e_1 f_1 e_3 + \frac{3i}{8} e_1^3 f_1^2 - 2 f_1 e_2^2 - 2 e_1 e_2 f_2 \,,
\end{aligned}$$

which reduce to differential consequences of the previous equations.

The single-time Poisson brackets $\{\ \ ,\ \ \}_k$ for $k = 1, 2, 3$

$$\{\ \ ,\ \ \}_k = \sum_{i=1}^{k} \left(\frac{\partial}{\partial f_i} \frac{\partial}{\partial e_{k+1-i}} - \frac{\partial}{\partial e_{k+1-i}} \frac{\partial}{\partial f_i} \right) , \tag{6.73}$$

can be re-expressed in the q and r coordinates as

$$\begin{aligned}
\{\ \ ,\ \ \}_1 =& 2i \left(\frac{\partial}{\partial r} \frac{\partial}{\partial q} - \frac{\partial}{\partial q} \frac{\partial}{\partial r} \right) , \\
\{\ \ ,\ \ \}_2 =& 4 \left(\frac{\partial}{\partial r} \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q} \frac{\partial}{\partial r_1} - \frac{\partial}{\partial q_1} \frac{\partial}{\partial r} - \frac{\partial}{\partial r_1} \frac{\partial}{\partial q} \right) , \\
\{\ \ ,\ \ \}_3 =& - 8i \Big(\frac{\partial}{\partial r} \frac{\partial}{\partial q_{11}} + \frac{\partial}{\partial r_{11}} \frac{\partial}{\partial q} + \frac{\partial}{\partial q_1} \frac{\partial}{\partial r_1} + 6qr \frac{\partial}{\partial q_{11}} \frac{\partial}{\partial r_{11}} \\
& - \frac{\partial}{\partial q_{11}} \frac{\partial}{\partial r} - \frac{\partial}{\partial q} \frac{\partial}{\partial r_{11}} - \frac{\partial}{\partial r_1} \frac{\partial}{\partial q_1} - 6qr \frac{\partial}{\partial r_{11}} \frac{\partial}{\partial q_{11}} \Big) .
\end{aligned}$$

These differ from the ones obtained in the previous chapters by an expected factor 2.

We will now show how to reobtain explicitly the classical r-matrix structure within the multi-time Poisson brackets for the first three times using these new coordinates. We will use the first three Lax matrices repackaged into the Lax form $W(\lambda) = Q^{(1)}(\lambda)\, dx^1 + Q^{(2)}(\lambda)\, dx^2 + Q^{(3)}(\lambda)\, dx^3$

$$\begin{aligned}
W^{+}(\lambda) &= b_1\, dx^1 + (\lambda b_1 + b_2)\, dx^2 + (\lambda^2 b_1 + \lambda b_2 + b_3)\, dx^3 \,, \\
W^{-}(\lambda) &= c_1\, dx^1 + (\lambda c_1 + c_2)\, dx^2 + (\lambda^2 c_1 + \lambda c_2 + c_3)\, dx^3 \,, \\
W^{3}(\lambda) &= -i\lambda dx^1 + (-i\lambda^2 - \frac{i}{2} b_1 c_1)\, dx^1 + (-i\lambda^3 - \frac{i\lambda}{2} b_1 c_1 - \frac{i}{2}(b_1 c_2 + c_1 b_2))\, dx^3 \,,
\end{aligned}$$

which we can also write in terms of the coordinates e and f as in

$$
\begin{aligned}
W^+(\lambda) &= \sqrt{2i}e_1\,dx^1 + \sqrt{2i}(\lambda e_1 + e_2)\,dx^2 + \sqrt{2i}(\lambda^2 e_1 + \lambda e_2 + e_3 + \frac{i}{4}e_1^2 f_1)\,dx^3\,,\\
W^-(\lambda) &= \sqrt{2i}f_1\,dx^1 + \sqrt{2i}(\lambda f_1 + f_2)\,dx^2 + \sqrt{2i}(\lambda^2 f_1 + \lambda f_2 + f_3 + \frac{i}{4}e_1 f_1^2)\,dx^3\,,\\
W^3(\lambda) &= -i\lambda dx^1 + (-i\lambda^2 + e_1 f_1)\,dx^1 + (-i\lambda^3 + \lambda e_1 f_1 + e_1 f_2 + e_2 f_1)\,dx^3\,.
\end{aligned}
$$

We can then compute the Hamiltonian vector field associated to each component of the Lax form:

$$
\begin{aligned}
\xi_{W^+}(\lambda) &= \sqrt{2i}\left(\partial_{f_1} + \lambda\partial_{f_2} + (\lambda^2 + \frac{i}{2}e_1 f_1)\partial_{f_3} - \frac{i}{4}e_1^2\partial_{e_3}\right)\,,\\
\xi_{W^-}(\lambda) &= \sqrt{2i}\left(-\partial_{e_1} - \lambda\partial_{e_2} + (-\lambda^2 - \frac{i}{2}e_1 f_1)\partial_{e_3} + \frac{i}{4}f_1^2\partial_{f_3}\right)\,,\\
\xi_{W^3}(\lambda) &= -e_1\partial_{e_2} + f_1\partial_{f_2} + (-\lambda e_1 - e_2)\partial_{e_3} + (\lambda f_1 + f_2)\partial_{f_3}\,.
\end{aligned}
$$

Let us now compute the multi-time Poisson bracket.

$$
\begin{aligned}
\{[W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\} &= \sum_{i,j=+,-,3}\{[W^i(\lambda), W^j(\mu)]\}\sigma_i\otimes\sigma_j\\
&= \{[W^+(\lambda), W^+(\mu)]\}\sigma_+\otimes\sigma_+ + \{[W^+(\lambda), W^-(\mu)]\}\sigma_+\otimes\sigma_- + \{[W^+(\lambda), W^3(\mu)]\}\sigma_+\otimes\sigma_3\\
&\quad + \{[W^-(\lambda), W^+(\mu)]\}\sigma_-\otimes\sigma_+ + \{[W^-(\lambda), W^-(\mu)]\}\sigma_-\otimes\sigma_- + \{[W^-(\lambda), W^3(\mu)]\}\sigma_-\otimes\sigma_3\\
&\quad + \{[W^3(\lambda), W^+(\mu)]\}\sigma_3\otimes\sigma_+ + \{[W^3(\lambda), W^-(\mu)]\}\sigma_3\otimes\sigma_- + \{[W^3(\lambda), W^3(\mu)]\}\sigma_3\otimes\sigma_3\,.
\end{aligned}
$$

The reader can check that $\{[W^+(\lambda), W^+(\mu)]\} = \{[W^-(\lambda), W^-(\mu)]\} = \{[W^3(\lambda), W^3(\mu)]\} = 0$, while the other non-zero Poisson brackets are

$$
\begin{aligned}
\{[W^+(\lambda), W^-(\mu)]\} &= -2i\,dx^1 - 2i(\lambda+\mu)\,dx^2 - 2i\,(\lambda^2 + \lambda\mu + \mu^2 + ie_1 f_1)\,dx^3\\
\{[W^-(\lambda), W^+(\mu)]\} &= 2i\,dx^1 + 2i(\lambda+\mu)\,dx^2 + 2i\,(\lambda^2 + \lambda\mu + \mu^2 + ie_1 f_1)\,dx^3\\
\{[W^+(\lambda), W^3(\mu)]\} &= -\sqrt{2i}e_1\,dx^2 - \sqrt{2i}((\lambda+\mu)e_1 + e_2)\,dx^3\,,\\
\{[W^3(\lambda), W^+(\mu)]\} &= \sqrt{2i}e_1\,dx^2 + \sqrt{2i}((\lambda+\mu)e_1 + e_2)\,dx^3\,,\\
\{[W^-(\lambda), W^3(\mu)]\} &= \sqrt{2i}f_1\,dx^2 + \sqrt{2i}((\lambda+\mu)f_1 + f_2)\,dx^3\,,\\
\{[W^3(\lambda), W^-(\mu)]\} &= -\sqrt{2i}f_1\,dx^2 - \sqrt{2i}((\lambda+\mu)f_1 + f_2)\,dx^3\,.
\end{aligned}
$$

Adding everything together one realises that $\{[W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\} = [\frac{P_{\mathbf{12}}}{\mu-\lambda}, W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]$, as desired.

Let us verify that for the first three times

$$
\sum_{i<j}\{[H_{ij}, W(\lambda)]\} = W(\lambda)\wedge W(\lambda) = \sum_{i<j}[Q^{(i)}(\lambda), Q^{(j)}(\lambda)]\,dx^{ij} \tag{6.74}
$$

or, in components,

$$\{\![H_{ij}, W(\lambda)]\!\} = [Q^{(i)}(\lambda), Q^{(j)}(\lambda)] \qquad i, j = 1, 2, 3\,. \tag{6.75}$$

We write explicitly the $(1,2)$ term. The coefficient of the Hamiltonian multiform $H_{12} = -2ie_2 f_2 - e_1^2 f_1^2$ has Hamiltonian vector field

$$\xi_{12} = 2e_1^2 f_1 \partial_{e_1} \wedge \partial_1 - 2e_1 f_1^2 \partial_{f_1} \wedge \partial_1 + 2ie_2 \partial_{e_1} \wedge \partial_2 - 2if_2 \partial_{f_1} \wedge \partial_2\,, \tag{6.76}$$

so that the left hand-side reads

$$\begin{aligned}
\{\![H_{12}, W(\lambda)]\!\} =& \xi_{12} \lrcorner \delta W(\lambda) \\
=& \xi_{12} \lrcorner (e_1 \delta f_1 \wedge dx^2 + f_1 \delta e_1 \wedge dx^2)\sigma_3 \\
&+ \xi_{12} \lrcorner (\sqrt{2i}\delta e_1 \wedge dx^1 + \sqrt{2i}\delta e_2 \wedge dx^2 + \sqrt{2i}\lambda \delta e_1 \wedge dx^2)\sigma_+ \\
&+ \xi_{12} \lrcorner (\sqrt{2i}\delta f_1 \wedge dx^1 + \sqrt{2i}\delta f_2 \wedge dx^2 + \sqrt{2i}\lambda \delta f_1 \wedge dx^2)\sigma_- \\
=& 2i(e_1 f_2 - f_1 e_2)\sigma_3 + \sqrt{2i}(-2e_1^2 f_1 - 2i\lambda e_2)\sigma_+ + \sqrt{2i}(2e_1 f_1^2 + 2i\lambda f_2)\sigma_- \\
=& [Q^{(1)}(\lambda), Q^{(2)}(\lambda)]\,.
\end{aligned}$$

Similarly one obtains $\{\![H_{13}, W(\lambda)]\!\} = [Q^{(1)}(\lambda), Q^{(3)}(\lambda)]$ and $\{\![H_{23}, W(\lambda)]\!\} = [Q^{(2)}(\lambda), Q^{(3)}(\lambda)]$.

Remark 6.24: As we pointed out before, the Lagrangian multiform of this chapter generates the Lagrangians that were previously used up to a total d-differential, and an overall multiplicative constant 2. This is the same constant that consistently turns the r-matrix $-\frac{P_{\mathbf{12}}}{\lambda}$ into the previously used one $-\frac{1}{2}\frac{P_{\mathbf{12}}}{\lambda}$ (4.44). In fact, one could define $\mathscr{L}' = 2\mathscr{L}$, and then $\mathcal{H}' = 2\mathcal{H}$ and $\Omega' = 2\Omega$ would follow. Then, the new Poisson bracket $\{\![\;,\;]\!\}' = \frac{1}{2}\{\![\;,\;]\!\}$ will be associated to Ω'. The relation $\{\![W_{\mathbf{1}}, W_{\mathbf{2}}]\!\} = [\frac{P_{\mathbf{12}}}{\mu-\lambda}, W_{\mathbf{1}} + W_{\mathbf{2}}]$ will then turn into $\{\![W_{\mathbf{1}}, W_{\mathbf{2}}]\!\}' = [\frac{1}{2}\frac{P_{\mathbf{12}}}{\mu-\lambda}, W_{\mathbf{1}} + W_{\mathbf{2}}]$. Moreover, $dW = \sum_{ij}\{\![H'_{ij}, W]\!\}' dx^{ij} = \sum_{ij}\{\![H_{ij}, W]\!\} dx^{ij}$.

We can also verify that $A = a_2 dx^1 + a_3 dx^2 + a_4 dx^3$ is indeed a conservation law in the usual coordinates q and r. In fact, we have that

$$a_2 = e_1 f_1 = -\frac{i}{2} qr\,, \tag{6.77a}$$

$$a_3 = e_1 f_2 + e_2 f_1 = \frac{1}{4}(q_1 r - q r_1)\,, \tag{6.77b}$$

$$a_4 = e_1 f_3 + e_2 f_2 + e_3 f_1 = \frac{i}{8} q r_{11} + \frac{i}{8} q_{11} r - \frac{3i}{8} q^2 r^2 - \frac{i}{8} q_1 r_1\,. \tag{6.77c}$$

Imposing $dA = 0$ is equivalent to the equations

$$\partial_1 a_3 = \partial_2 a_2\,, \quad \partial_1 a_4 = \partial_3 a_2\,, \quad \partial_2 a_4 = \partial_3 a_3\,, \tag{6.78}$$

which hold on the equations of motion.

Chapter 7

Generating Lagrangian multiform and classical Yang-Baxter equation

In Chapter 6 we provided a *generating Lagrangian multiform* for the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy. This was a formal series in $\lambda^{-1}\mu^{-1}$ and its coefficients could be identified with the coefficients of the AKNS Lagrangian multiform. In this chapter, containing content adapted from [CSV21b], we generalise those results by providing a *generating Lagrangian*, that provides the coefficients of a Lagrangian multiform for several integrable hierarchies other than the AKNS.

> **Remark 7.1:** It will be helpful to review some notation relative to formal power series.
>
> - $\mathbb{C}[\lambda]$ denotes the ring of complex polynomials in the variable λ, *i.e.* of the form $\sum_{j=0}^{N} p_j\lambda^j$, $N \in \mathbb{N}$.
> - $\mathbb{C}[\![\lambda]\!]$ denotes the ring of complex formal Taylor series in the variable λ, *i.e.* series of the form $\sum_{j=0}^{\infty} f_j\lambda^j$, $f_j \in \mathbb{C}\ \forall j$.
> - $\mathbb{C}(\!(\lambda)\!)$ denotes the ring of complex formal Laurent series in the variable λ, *i.e.* series of the form $\sum_{j=-N}^{\infty} f_j\lambda^j$, $N \in \mathbb{Z}$, $f_j \in \mathbb{C}\ \forall j$.

7.1 Algebraic setup

We will start from the Lie algebra $\mathcal{L}_a(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}(\!(\lambda_a)\!)$, $a \in \mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ and $\mathfrak{g} \subseteq \mathfrak{gl}_N$. Its elements are formal Laurent series

$$X^a(\lambda_a) = \sum_{j=-N}^{\infty} X_j^a \lambda_a^j\,, \qquad X_j^a \in \mathfrak{g}\,, \qquad N \in \mathbb{Z}\,, \tag{7.1}$$

where $\lambda_a \coloneqq \lambda - a$ if $a \in \mathbb{C}$ and $\lambda_\infty = 1/\lambda$. We use the Lie bracket naturally extended from the commutator of $\mathfrak{g}$

$$[X^a(\lambda_a), Y^a(\lambda_a)] \coloneqq \sum_r \sum_{i+j=r} [X_i^a, Y_j^a]\lambda_a^r .$$

For any $a \in \mathbb{C}\mathrm{P}^1$ the residue $\operatorname{res}_{\lambda=a} \colon \mathbb{C}(\!(\lambda_a)\!)d\lambda_a \to \mathbb{C}$ returns the coefficient of $\lambda_a^{-1}d\lambda_a$. If $a = \infty$ we note that $d\lambda = -\lambda_\infty^2 d\lambda_\infty$. In practice, the residue at ∞ returns the opposite of the coefficient of λ_∞.
Then, given a finite $S = \{ a_1, \dots, a_n \} \subset \mathbb{C}\mathrm{P}^1$ with $\#S = n$ we define the following Lie algebra

$$\begin{aligned}\mathcal{L}(\mathfrak{g}) \coloneqq & \bigoplus_{a\in S} \mathcal{L}_a(\mathfrak{g}) \\ =& \mathcal{L}_{a_1}(\mathfrak{g}) \oplus \mathcal{L}_{a_2}(\mathfrak{g}) \oplus \cdots \oplus \mathcal{L}_{a_n}(\mathfrak{g}) .\end{aligned} \tag{7.2}$$

We will denote an element of $\mathcal{L}(\mathfrak{g})$ as an n-tuple $X(\lambda) = (X^{a_1}(\lambda_{a_1}), \dots, X^{a_n}(\lambda_{a_n}))$. The Lie bracket of two elements $X(\lambda) = (X^a(\lambda_a))_{a\in S}$ and $Y(\lambda) = (Y^a(\lambda_a))_{a\in S}$ is defined component-wise as

$$[X(\lambda), Y(\lambda)] \coloneqq ([X^a(\lambda_a), Y^a(\lambda_a)])_{a\in S} .$$

For $\mathcal{L}(\mathfrak{g})$ we will choose a a pairing, *i.e.* a nondegenerate bilinear form

$$\langle X(\lambda), Y(\lambda)\rangle^{(k)} \coloneqq \sum_{a\in S} \operatorname*{res}_{\lambda=a} \operatorname{Tr} X^a(\lambda_a)Y^a(\lambda_a)\lambda_a^k d\lambda \tag{7.3}$$

for any $X(\lambda), Y(\lambda) \in \mathcal{L}(\mathfrak{g})$ and $k = 0, -1$. The identification of maximally isotropic subalgebras of $\mathcal{L}(\mathfrak{g})$ will allow us to identify an endomorphism $r \colon \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})$.

We will consider derivations ∂_k acting on the matrix elements of $\mathfrak{g}$, where t^k are the times of the hierarchy and the coordinates of our multi-time manifold M. We write $M = \bigoplus_{a\in S} M_a$ as a direct sum, where we use $(t_a^1, t_a^2, \dots)$ as coordinates on M_a. This can be done thanks to the fact that the number of coordinates of M are countable and that S is finite. We repackage each $\partial_{t_a^k}$ into the objects

$$D_{\lambda_a} \coloneqq \sum_{i=0}^{\infty} \lambda_a^i \partial_{t_a^i} , \quad a \in \mathbb{C} , \tag{7.4a}$$

$$D_{\lambda_\infty} \coloneqq \sum_{i=1}^{\infty} \lambda_\infty^{i+1} \partial_{t_\infty^i} \quad \text{for } k = 0 , \qquad D_{\lambda_\infty} \coloneqq \sum_{i=0}^{\infty} \lambda_\infty^{i} \partial_{t_\infty^i} \quad \text{for } k = -1. \tag{7.4b}$$

The D_{λ_a} are generalisations of the similar objects defined in Chapter 6. They also act like derivations, and are not to be confused with partial derivatives with respect to the spectral parameters ∂_{λ_a}. For instance, $\partial_\lambda X^b(\mu_b) = 0$, but $D_{\lambda_a} X^b(\mu_b) \neq 0$ in general. In principle, we could not consider elements D_{λ_a} of the form (7.4) if we were not sure that they would define commuting flows. This will be checked in Lemma 7.9.

Finally, it will be useful to define R_λ as the space of rational functions of the variable λ that are regular outside S. The map $\iota_{\lambda_a}\colon R_\lambda \to \mathbb{C}((\lambda_a))$ associates to each rational function in R_λ its Laurent series around the point $a \in S$. We will use extensively the following examples:

$$\iota_{\lambda_\infty} \frac{1}{\mu - \lambda} = -\sum_{r=0}^{\infty} \frac{\lambda_\infty^{r+1}}{\mu_\infty^r} = -\sum_{r=0}^{\infty} \frac{\mu^r}{\lambda^{r+1}}\,, \tag{7.5a}$$

$$\iota_{\lambda_a} \frac{1}{\mu - \lambda} = \sum_{r=0}^{\infty} \frac{\lambda_a^r}{\mu_a^{r+1}}\,. \tag{7.5b}$$

If we define $R_\lambda(\mathfrak{g}) \coloneqq \mathfrak{g} \otimes R_\lambda$ we have the embedding

$$\iota_\lambda\colon R_\lambda(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})\,, \qquad X \otimes f \mapsto (X \otimes \iota_{\lambda_a} f)_{a\in S}\,. \tag{7.6}$$

Proposition 7.2 $\iota_\lambda R_\lambda(\mathfrak{g}) \subset \mathcal{L}(\mathfrak{g})$ *is a maximally isotropic subalgebra with respect to* $\langle\ ,\ \rangle^{(k)}$ *for any* $k \in \mathbb{Z}$.

Proof. The proof is obtained using the residue theorem, the idea being as follows. Let $f, g \in R_\lambda$, and $X(\lambda) = \iota_\lambda f(\lambda)$ and $Y(\lambda) = \iota_\lambda g(\lambda)$. Then

$$\begin{aligned}\langle X(\lambda), Y(\lambda)\rangle^{(k)} &= \sum_{a\in S} \operatorname*{res}_{\lambda=S} \operatorname{Tr} X^a(\lambda_a) Y^a(\lambda_a) \lambda^k d\lambda = \sum_{a\in S} \operatorname*{res}_{\lambda=a} \operatorname{Tr}(\iota_{\lambda_a} f(\lambda))(\iota_{\lambda_a} g(\lambda))\lambda^k d\lambda \\ &= \sum_{a\in S} \operatorname*{res}_{\lambda=a} \operatorname{Tr} \iota_{\lambda_a} \left(f(\lambda) g(\lambda) \lambda^k \right) d\lambda = 0\,.\end{aligned}$$

Maximality is obtained by the strong residue theorem [T13, Corollary 1]. □

The subalgebra $\iota_\lambda R_\lambda(\mathfrak{g})$ will be complemented by another maximally isotropic subalgebra, denoted $\mathcal{L}^+(\mathfrak{g})$

$$\mathcal{L}(\mathfrak{g}) = \mathcal{L}^+(\mathfrak{g}) \oplus \iota_\lambda R_\lambda(\mathfrak{g})\,. \tag{7.7}$$

The definition of $\mathcal{L}^+(\mathfrak{g})$ will depend on the specific case. The classical r-matrix $r_{\mathbf{12}}(\lambda, \mu)$ is obtained from the endomorphism r in the following way

$$(rX)(\lambda) = \langle \frac{1}{2}(\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda) r_{\mathbf{12}}(\lambda, \mu), X_{\mathbf{2}}(\mu)\rangle_{\mathbf{2}}^{(k)} \tag{7.8}$$

where the subscripts $\mathbf{1}, \mathbf{2}$ denote the auxiliary spaces, and the bilinear form $\langle\ ,\ \rangle_{\mathbf{2}}^{(k)}$ is only taken on the space 2. It will satisfy the following relations:

- **Skew-symmetry:** $r_{\mathbf{12}}(\lambda, \mu) = -r_{\mathbf{21}}(\mu, \lambda)$,
- **Classical Yang-Baxter equation:** $[r_{\mathbf{12}}(\lambda, \mu), r_{\mathbf{13}}(\lambda, \nu)] + [r_{\mathbf{12}}(\lambda, \mu), r_{\mathbf{23}}(\mu, \nu)] + [r_{\mathbf{13}}(\lambda, \nu), r_{\mathbf{23}}(\mu, \nu)] = 0$.

Our hierarchy will be identified by choosing the algebra $\mathfrak{g}$, a collection S of points in

$\mathbb{C}\mathrm{P}^1$, which will turn out to be the poles of the Lax matrices, and the parameter k of the pairing (7.3).

- The choice $k = 0$ will bring the rational r-matrix, and either the Ablowitz-Kaup-Newell-Segur hierarchy, when we pick $S = \{\,\infty\,\}$ and $\mathfrak{g} = s\ell(2,\mathbb{C})$, or a hierarchy describing rational Lax matrices of Zakharov Shabat type [ZS79] for distinct and finite poles, containing the Lagrangian in [D03, Section 20.2] for

$$S = \{\, a_1, \ldots, a_{N_1}, b_1, \ldots, b_{N_2} \mid a_m, b_n \in \mathbb{C}, \quad \text{distinct} \,\}$$

  for $N_1, N_2 \in \mathbb{N}$, and $\mathfrak{g} = \mathfrak{gl}_N$ that we call *Dickey's Lagrangian*, of which the Zakharov-Mikhailov Lagrangian [ZM80] is a special case.

- The choice $k = -1$ and $\mathfrak{g} = s\ell(2,\mathbb{C})$ instead brings the trigonometric r-matrix and a hierarchy containing the sine-Gordon Lagrangian in light-cone coordinate with the choice $S = \{\, 0, \infty \,\}$.

Remark 7.3: Even if we only define the relevant objects for the specific cases of the integrable systems we aim to obtain, we anticipate that this formalism can be cast into the framework of *adéles* and will appear in [CSV21b]. This brings a more general definition of $\mathcal{L}(\mathfrak{g})$ and consequently of the generating Lagrangian, and will allow us to consider a wider class of systems. Here we prefer a more pragmatic and accessible, albeit less elegant approach.

7.1.1 Rational r-matrix

AKNS case In this section we fix $k = 0$, $\mathfrak{g} = s\ell(2,\mathbb{C})$. We also set $S = \{\,\infty\,\}$, so effectively $\mathcal{L}(\mathfrak{g}) = \mathcal{L}_\infty(\mathfrak{g})$. The maximally isotropic subalgebra playing the role of $\mathcal{L}^+(\lambda)$ will be obtained by considering the following subalgebra of $\mathcal{L}_\infty(\mathfrak{g})$

$$\mathcal{L}^{\mathrm{rat}}_\infty(\mathfrak{g}) \coloneqq \mathfrak{g} \otimes \lambda_\infty \mathbb{C}[\![\lambda_\infty]\!] \,. \tag{7.9}$$

Note that we have excluded the constant term from the Taylor series at infinity. We represent the pole part of $X^\infty(\lambda) \in \mathfrak{g} \otimes \mathbb{C}(\!(\lambda_\infty)\!)$ as

$$(X^\infty(\lambda_\infty))_- = \sum_{n=-N}^{0} X^\infty_n \lambda^n_\infty \in \mathfrak{g} \otimes \mathbb{C}[\lambda^{-1}_\infty] \,, \tag{7.10}$$

for any $X^\infty(\lambda_\infty) = \sum_{n=-N}^{\infty} X^\infty_n \lambda^n_\infty$. We can decompose $\mathcal{L}(\mathfrak{g})$ into the maximally isotropic Lie subalgebras

$$\mathcal{L}_\infty(\mathfrak{g}) = \mathcal{L}^{\mathrm{rat}}_\infty(\mathfrak{g}) \oplus \iota_{\lambda_\infty} R_\lambda(\mathfrak{g}) \,. \tag{7.11}$$

We have already proved that $\iota_{\lambda_\infty} R_\lambda(\mathfrak{g})$ is maximally isotropic. The exclusion from $\mathcal{L}^{\mathrm{rat}}$ of the constant term at infinity ensures the isotropy of the latter, and the maximality follows from the maximality of $\iota_{\lambda_\infty} R_\lambda(\mathfrak{g})$. We denote by π_+ and π_- the projections onto $\mathcal{L}^{\mathrm{rat}}_\infty(\mathfrak{g})$ and $\iota_{\lambda_\infty} R_\lambda(\mathfrak{g})$ respectively[1]. The projectors act as follows.

Proposition 7.4 (Projectors $\pi_\pm$ (AKNS case)) *For any* $X^\infty(\lambda_\infty) = \sum_{n=-N}^\infty X_n^\infty \lambda_\infty^n \in \mathcal{L}_\infty(\mathfrak{g})$ *we have*

$$(\pi_+ X)^\infty(\lambda_\infty) = \operatorname*{res}_{\mu=\infty} \mathrm{Tr}_{\mathbf{2}} \left(\iota_{\mu_\infty} \iota_{\lambda_\infty} \frac{P_{\mathbf{12}}}{\mu - \lambda} X^\infty(\mu_\infty)_{\mathbf{2}} \right) d\mu \tag{7.12a}$$

$$(\pi_- X)^\infty(\lambda_\infty) = - \operatorname*{res}_{\mu=\infty} \mathrm{Tr}_{\mathbf{2}} \left(\iota_{\lambda_\infty} \iota_{\mu_\infty} \frac{P_{\mathbf{12}}}{\mu - \lambda} X^\infty(\mu_\infty)_{\mathbf{2}} \right) d\mu \tag{7.12b}$$

Proof. We start by proving (7.12a). Let $X^\infty(\lambda_\infty) = \sum_{n=-N}^\infty X_n^\infty \lambda_\infty^n$. Explicitly we have

$$\begin{aligned} \operatorname*{res}_{\mu=\infty} \mathrm{Tr}_{\mathbf{2}} \left(\iota_{\mu_\infty} \iota_{\lambda_\infty} \frac{P_{\mathbf{12}}}{\mu - \lambda} X^\infty(\mu_\infty)_{\mathbf{2}} \right) d\mu &= \operatorname*{res}_{\mu=\infty} \iota_{\mu_\infty} \iota_{\lambda_\infty} \frac{X^\infty(\mu_\infty)}{\mu - \lambda} d\mu \\ = - \operatorname*{res}_{\mu=\infty} \iota_{\mu_\infty} \sum_{r=0}^\infty \frac{\lambda_\infty^{r+1}}{\mu_\infty^r} \sum_{n=-N}^\infty X_n^\infty \mu_\infty^n d\mu &= \sum_{n=1}^\infty X_n^\infty \lambda_\infty^n \in \mathcal{L}^{\mathrm{rat}}(\mathfrak{g}) \,, \end{aligned}$$

hence the result. We remark that if $X^\infty(\lambda_\infty) = \iota_{\lambda_\infty} f(\lambda)$ for some rational function f, then $(\pi_+ X)^\infty(\lambda_\infty) = 0$ by the residue theorem. The relation (7.12b) is obtained similarly:

$$\begin{aligned} - \operatorname*{res}_{\mu=\infty} \mathrm{Tr}_{\mathbf{2}} \left(\iota_{\lambda_\infty} \iota_{\mu_\infty} \frac{P_{\mathbf{12}}}{\mu - \lambda} X^\infty(\mu_\infty)_{\mathbf{2}} \right) d\mu &= - \operatorname*{res}_{\mu=\infty} \iota_{\lambda_\infty} \iota_{\mu_\infty} \frac{X^\infty(\mu_\infty)}{\mu - \lambda} d\mu \\ = - \operatorname*{res}_{\mu=\infty} \iota_{\lambda_\infty} \sum_{r=0}^\infty \sum_{n=-N}^\infty \frac{\mu_\infty^{n+r+1}}{\lambda_\infty^{r+1}} X^\infty(\lambda_\infty) d\mu &= \iota_{\lambda_\infty} \sum_{n=-N}^0 X_n^\infty \lambda_\infty^n \in \iota_{\lambda_\infty} R_\lambda(\mathfrak{g}) \,. \end{aligned} \tag{7.13}$$

Moreover, if $X^\infty(\lambda_\infty) \in \mathfrak{g} \otimes \lambda_\infty \mathbb{C}[\![\lambda_\infty]\!]$, we have $(\pi_- X)^\infty(\lambda_\infty) = 0$. □

Given the expressions of $\pi_\pm$ it follows that the kernel of the operator $r := \pi_+ - \pi_-$ is

$$(\iota_{\mu_\infty} \iota_{\lambda_\infty} + \iota_{\lambda_\infty} \iota_{\mu_\infty}) \frac{P_{\mathbf{12}}}{\mu - \lambda} \,. \tag{7.14}$$

Moreover, the kernel of the identity operator is given by

$$(\iota_{\mu_\infty} \iota_{\lambda_\infty} - \iota_{\lambda_\infty} \iota_{\mu_\infty}) \frac{P_{\mathbf{12}}}{\mu - \lambda} d\mu = \delta(\lambda_\infty, \mu_\infty) P_{\mathbf{12}} d\mu \,. \tag{7.15}$$

We have defined $\delta(\lambda_a, \mu_a) := \sum_{r \in \mathbb{Z}} \lambda_a^r \mu_a^{-r-1}$.

Zakharov-Shabat case In this section we fix $k = 0$, $\mathfrak{g} = \mathfrak{gl}_N$ and $S = \{ a_1, \dots, a_{N_1}, b_1, \dots, b_{N_2} \}$ where $a_i, b_j \in \mathbb{C}\ \forall i, j$. We consider the following algebras

$$\mathcal{L}^{\mathrm{rat}}_a(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\![\lambda_a]\!] \quad \forall\, a \in S \,. \tag{7.16}$$

[1]More properly they should be called $\pi_\pm^\infty$, but here $\mathcal{L}(\mathfrak{g}) = \mathcal{L}_\infty(\mathfrak{g})$.

We then define the subalgebra

$$\mathcal{L}^{\text{rat}}(\mathfrak{g}) \coloneqq \bigoplus_{a\in S} \mathcal{L}_a^{\text{rat}}(\mathfrak{g}) \subset \mathcal{L}(\mathfrak{g}) \tag{7.17}$$

that plays the role of $\mathcal{L}^+(\mathfrak{g})$. We represent the pole part of $X^a(\lambda_a) \in \mathcal{L}_a(\mathfrak{g})$, with $a \in S$ as

$$X^a(\lambda_a)_- = \sum_{n=-N_a}^{-1} X_n^a \lambda_a^n \in \mathfrak{g} \otimes \lambda_a^{-1}\mathbb{C}[\lambda_a^{-1}]\,, \tag{7.18}$$

for any $X^a(\lambda_a) = \sum_{n=-N_a}^{\infty} X_n^a \lambda_a^n$. As before we can decompose $\mathcal{L}(\mathfrak{g})$ into the maximally isotropic Lie subalgebras

$$\mathcal{L}(\mathfrak{g}) = \mathcal{L}^{\text{rat}}(\mathfrak{g}) \oplus \iota_\lambda R_\lambda(\mathfrak{g})\,. \tag{7.19}$$

We denote respectively by π_+ and π_- the projections onto $\mathcal{L}^{\text{rat}}(\mathfrak{g})$ and $\iota_\lambda R_\lambda(\mathfrak{g})$ respectively. The projectors are defined in the following way.

Proposition 7.5 (Projectors $\pi_\pm$ (Dickey case)) *Let $S = \{ a_1, \dots, a_{N_1}, b_1, \dots, b_{N_2} \}$. For any $X(\lambda) \in \mathcal{L}(\mathfrak{g})$ we have*

$$(\pi_+ X)^a(\lambda_a) = \sum_{b\in S} \operatorname*{res}_{\mu=b} \operatorname{Tr}_{\mathbf{2}} \left(\iota_{\mu_b}\iota_{\lambda_a} \frac{P_{\mathbf{12}}}{\mu-\lambda} X^b(\mu_b)_{\mathbf{2}} \right) d\mu \tag{7.20a}$$

$$(\pi_- X)^a(\lambda_a) = -\sum_{b\in S} \operatorname*{res}_{\mu=b} \operatorname{Tr}_{\mathbf{2}} \left(\iota_{\lambda_a}\iota_{\mu_b} \frac{P_{\mathbf{12}}}{\mu-\lambda} X^b(\mu_b)_{\mathbf{2}} \right) d\mu \tag{7.20b}$$

Proof. Similar to the AKNS case. □

Given the expressions of $\pi_\pm$ it follows that the kernel of the operator $r \coloneqq \pi^+ - \pi^-$ is

$$\left((\iota_{\mu_b}\iota_{\lambda_a} + \iota_{\lambda_a}\iota_{\mu_b}) \frac{P_{\mathbf{12}}}{\mu-\lambda} \right)_{a,b\in S}\,, \tag{7.21}$$

and that the kernel of the identity operator $\text{Id} = \pi^+ + \pi^-$ is given by

$$\left((\iota_{\mu_b}\iota_{\lambda_a} - \iota_{\lambda_a}\iota_{\mu_b}) \frac{P_{\mathbf{12}}}{\mu-\lambda} d\mu \right)_{a,b\in S} = (P_{\mathbf{12}}\delta_{ab}\delta(\lambda_a,\mu_a))_{a,b\in S}\,. \tag{7.22}$$

7.1.2 Trigonometric r-matrix

Let $\mathfrak{g} = s\ell(2,\mathbb{C})$. We preliminarily define

$$\rho_{\mathbf{12}}^+ \coloneqq \sigma_+ \otimes \sigma_-\,, \quad \rho_{\mathbf{12}}^0 \coloneqq \frac{1}{2}(\mathbb{I}\otimes\mathbb{I} + \sigma_3\otimes\sigma_3)\,, \quad \rho_{\mathbf{12}}^- \coloneqq \sigma_-\otimes\sigma_+\,, \tag{7.23}$$

so that $P_{\mathbf{12}} = \rho_{\mathbf{12}}^+ + \rho_{\mathbf{12}}^0 + \rho_{\mathbf{12}}^-$. We also define $P^\pm\colon \mathfrak{g} \to \mathfrak{n}_\pm$ and $P^0\colon \mathfrak{g}\to\mathfrak{h}$ where $\mathfrak{n}_\pm$ are the nilpotent subalgebras (spanned respectively by $\sigma_\pm$) and $\mathfrak{h}$ is the Cartan subalgebra (spanned by σ_3). They are given by

$$P^{\pm}X \coloneqq \mathrm{Tr}_{\mathbf{2}}\, \rho^{\pm}_{\mathbf{12}} X_{\mathbf{2}}\,, \quad P^0 X \coloneqq \mathrm{Tr}_{\mathbf{2}}\, \rho^0_{\mathbf{12}} X_{\mathbf{2}}\,, \tag{7.24}$$

for any $X \in \mathfrak{g}$, so that $\mathrm{Id}_{\mathfrak{g}} = P^+ + P^0 + P^-$.
To select the sine-Gordon and trigonometric case we choose $k = -1$ and $S = \{0, \infty\}$. The role played by $\mathcal{L}^{\mathrm{rat}}(\mathfrak{g})$ is now played by the following subalgebra

$$\mathcal{B}(\mathfrak{g}) \subset (\mathfrak{b}_+ \oplus \mathfrak{g} \otimes \lambda \mathbb{C}[\![\lambda]\!]) \times (\mathfrak{b}_- \oplus \mathfrak{g} \otimes \lambda^{-1}\mathbb{C}[[\lambda^{-1}]])\,. \tag{7.25}$$

Where we also define $\mathfrak{b}_\pm$ as the Borel subalgebras[2]. This is the Lie subalgebra consisting of pairs of Taylor series $X^0(\lambda) = \sum_{n=0}^\infty X^0_n \lambda^n$ and $X^\infty(\lambda) = \sum_{n=0}^\infty X^\infty_n \lambda^{-n}$, with $X^0_n, X^\infty_n \in \mathfrak{g}$ for all $n \geq 1$ but with $X^0_0 \in \mathfrak{b}_+$ and $X^\infty_0 \in \mathfrak{b}_-$, and with the constraint $P^0 X^0_0 = -P^0 X^\infty_0$. We also define the pole parts for this trigonometric case as

$$X^0(\lambda)_- \coloneqq (P^- + \frac{1}{2}P^0)X^0_0 + \sum_{n=-N_0}^{-1} X^0_n \lambda^n \in \mathfrak{b}_- \oplus \mathfrak{g} \otimes \lambda^{-1}\mathbb{C}[\lambda^{-1}]\,, \tag{7.26}$$

$$X^\infty(\lambda_\infty)_- \coloneqq (P^+ + \frac{1}{2}P^0)X^\infty_0 + \sum_{n=-N_\infty}^{-1} X^\infty_n \lambda^n_\infty \in \mathfrak{b}_+ \oplus \mathfrak{g} \otimes \lambda\mathbb{C}[\lambda]\,, \tag{7.27}$$

for any $X^0(\lambda) = \sum_{n=-N_0}^\infty X^0_n \lambda^n$ and $X^\infty(\lambda_\infty) = \sum_{n=-N_\infty}^\infty X^\infty_n \lambda^n_\infty$. We will use the following proposition.

Proposition 7.6 *We have a direct sum of vector spaces into maximal isotropic Lie subalgebras*

$$\mathcal{L}(\mathfrak{g}) = \mathcal{B}(\mathfrak{g}) \oplus \iota_\lambda R_\lambda(\mathfrak{g})\,. \tag{7.28}$$

Proof. To show that $\mathcal{B}(\mathfrak{g})$ is isotropic with respect to $\langle\ \ ,\ \ \rangle^{(-1)}$ let $X(\lambda), Y(\lambda) \in \mathcal{B}(\mathfrak{g})$ be arbitrary and calculate

$$\begin{aligned}&\operatorname*{res}_{\lambda=0} \mathrm{Tr}\, X^0(\lambda) Y^0(\lambda) \frac{d\lambda}{\lambda} + \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\, X^\infty(\lambda_\infty) Y^\infty(\lambda_\infty) \frac{d\lambda}{\lambda} = \mathrm{Tr}\, X^0_0 Y^0_0 - \mathrm{Tr}\, X^\infty_0 Y^\infty_0 \\ &= \mathrm{Tr}(P^0(X^0_0) P^0(Y^0_0))) - \mathrm{Tr}(P^0(X^\infty_0) P^0(Y^\infty_0)) = 0\,.\end{aligned}$$

To show that $\mathcal{B}(\mathfrak{g})$ is maximally isotropic it is sufficient to prove that it is in direct sum with $\iota_\lambda R_\lambda(\mathfrak{g})$. To any $X(\lambda) \in \mathcal{L}(\mathfrak{g})$ we associate

$$R_\lambda(\mathfrak{g}) \ni f_X(\lambda) = X^0(\lambda)_- + X^\infty(\lambda_\infty)_-\,. \tag{7.29}$$

Consider moreover the element $\widetilde{X}(\lambda) = (\widetilde{X}^a(\lambda_a))_{a=0,\infty}$ defined by

$$\widetilde{X}^a(\lambda_a) \coloneqq X^a(\lambda_a) - \iota_{\lambda_a} f_X(\lambda)\,, \qquad a = 0, \infty\,.$$

[2]In the $s\ell(2,\mathbb{C})$ case $\mathfrak{b}_+$ is spanned by σ_+ and σ_3, and $\mathfrak{b}_-$ is spanned by σ_- and σ_3.

As we are subtracting the pole parts around $\lambda = a$ we have that $\widetilde{X}^a(\lambda_a) \in \mathfrak{g} \otimes \mathbb{C}[\![\lambda_a]\!]$. At $a = 0$ we have

$$X^\infty(\lambda_\infty)_-|_{\lambda=0} = (P^+ + \frac{1}{2}P^0)X_0^\infty$$

so that $\widetilde{X}^0(\lambda_0) \in \mathfrak{b}_+ \oplus \mathfrak{g} \otimes \lambda\mathbb{C}[\![\lambda]\!]$ whose leading coefficient is given by

$$(P^+ + \frac{1}{2}P^0)(X_0^0 - X_0^\infty) \in \mathfrak{b}_+ \,.$$

Likewise at ∞ we have

$$X^0(\lambda_0)_-|_{\lambda=\infty} = (P^- + \frac{1}{2}P^0)X_0^0$$

so that $\widetilde{X}^\infty(\lambda_\infty) \in \mathfrak{b}_- \oplus \mathfrak{g} \otimes \lambda^{-1}\mathbb{C}[\![\lambda^{-1}]\!]$ with leading coefficient given by

$$(P^- + \frac{1}{2}P^0)(-X_0^0 + X_0^\infty) \in \mathfrak{b}_- \,.$$

We can conclude that $\widetilde{X}(\lambda) \in \mathcal{B}(\mathfrak{g})$, or in other words

$$X(\lambda) = \widetilde{X}(\lambda) + \iota_\lambda f_X(\lambda)$$

gives the desired decomposition of $X(\lambda) \in \mathcal{L}(\mathfrak{g})$ in terms of $\mathcal{B}(\mathfrak{g})$ and $\iota_\lambda R_\lambda(\mathfrak{g})$. This decomposition is unique since an element that belongs to both $\mathcal{B}(\mathfrak{g})$ and $\iota_\lambda R_\lambda(\mathfrak{g})$ must vanish. $\square$

Let $\pi_\pm$ the projections onto $\mathcal{B}(\mathfrak{g})$ and $\iota_\lambda R_\lambda$ respectively, and consider the following rational function given by

$$\begin{aligned} r_{\mathbf{12}}(\lambda,\mu) :=& \frac{1}{2}\left(\rho^+_{\mathbf{12}} - \rho^-_{\mathbf{12}} + \frac{\mu+\lambda}{\mu-\lambda}P_{\mathbf{12}}\right) \\ =& \frac{1}{4}\frac{\mu+\lambda}{\mu-\lambda}(\mathbb{I}\otimes\mathbb{I} + \sigma_3\otimes\sigma_3) + \frac{\mu}{\mu-\lambda}\sigma_+\otimes\sigma_- + \frac{\lambda}{\mu-\lambda}\sigma_-\otimes\sigma_+ \,. \end{aligned} \tag{7.30}$$

We have the following proposition.

Proposition 7.7 (Projector $\pi_\pm$ (sG case)) *For any $X(\lambda) \in \mathcal{L}(\mathfrak{g})$, its projections onto the subalgebras $\mathcal{B}$ and $\iota_\lambda R_\lambda(\mathfrak{g})$ are given respectively by $\pi_\pm X(\lambda) = ((\pi_\pm X)^a(\lambda_a))_{a=0,\infty}$ where*

$$(\pi_+ X)^a(\lambda_a) = \sum_{b=0,\infty} \operatorname*{res}_{\mu=b} \operatorname{Tr}_{\mathbf{2}}\, \iota_{\mu_b}\iota_{\lambda_a} r_{\mathbf{12}}(\lambda,\mu) X^b(\mu_b)_{\mathbf{2}} \frac{d\mu}{\mu} \tag{7.31a}$$

$$(\pi_- X)^a(\lambda_a) = -\sum_{b=0,\infty} \operatorname*{res}_{\mu=b} \operatorname{Tr}_{\mathbf{2}}\, \iota_{\lambda_a}\iota_{\mu_b} r_{\mathbf{12}}(\lambda,\mu) X^b(\mu_b)_{\mathbf{2}} \frac{d\mu}{\mu} \,. \tag{7.31b}$$

Proof. First of all we remark that $r_{\mathbf{12}}$ can be written as

$$r_{\mathbf{12}}(\lambda,\mu)=\frac{\mu P_{\mathbf{12}}}{\mu-\lambda}-\rho^{-}_{\mathbf{12}}-\frac{1}{2}\rho^{0}_{\mathbf{12}}\,. \tag{7.32}$$

Then, given any $X(\lambda)\in\mathcal{L}(\mathfrak{g})$ we consider

$$\begin{aligned}R_\lambda(\mathfrak{g})\ni f_X(\lambda)&=-\sum_{b=0,\infty}\operatorname*{res}_{\mu=b}\mathrm{Tr}_{\mathbf{2}}\,\iota_{\mu_b}r_{\mathbf{12}}(\lambda,\mu)X^b(\mu_b)_{\mathbf{2}}\frac{d\mu}{\mu}\\&=\sum_{b=0,\infty}\operatorname*{res}_{\mu=b}\left(\iota_{\mu_b}\mu^{-1}(P^-+\frac{1}{2}P^0)X^b(\mu_b)+\iota_{\mu_b}\frac{1}{\lambda-\mu}X^b(\mu_b)\right)d\mu\,.\end{aligned}$$

The residue at 0 is

$$(P^-+\frac{1}{2}P^0)X^0_0+\sum_{n=-N_0}^{-1}X^0_n\lambda^n=X^0(\lambda)_-\,,$$

while the residue at ∞ is

$$\begin{aligned}&-(P^-+\frac{1}{2}P^0)X^\infty_0+\sum_{n=-N_\infty}^{0}X^\infty_n\lambda^n_\infty\\&=-(P^-+\frac{1}{2}P^0)X^\infty_0+\sum_{n=-N_\infty}^{-1}X^\infty_n\lambda^n_\infty+(P^++P^0+P^-)X^\infty_0=X^\infty(\lambda_\infty)_-\,,\end{aligned}$$

where we have used $\mathrm{Id}=P^++P^0+P^-$. By construction then we have that $(\pi_-X)^a(\lambda_0)=\iota_{\lambda_a}f_X(\lambda)$ for every $a\in S$ as f_X is the function used in (7.29).
Then, if $X(\lambda)\in\mathcal{B}(\mathfrak{g})$ then $X^0(\lambda)_-=\frac{1}{2}P^0X^0_0$ and $X^\infty(\lambda_\infty)_-=\frac{1}{2}P^0_0X^\infty_0=-\frac{1}{2}P^0X^0_0$, so the two terms cancel in the sum, and $(\pi_-X)(\lambda)=0$ for any $X(\lambda)\in\mathcal{B}(\mathfrak{g})$.
Suppose now $X(\lambda)=\iota_\lambda f(\lambda)$ for some $f\in R_\lambda(\mathfrak{g})$. If it has a pole in the origin then its pole part in this point is given by

$$X^0(\lambda)_-+(P^-+\frac{1}{2}P^0)X^0_0=X^0(\lambda)_--(P^-+\frac{1}{2}P^0)X^\infty_0\,,$$

where $X^0_0=-X^0_\infty$ as f only has poles in 0 and ∞. The pole part at infinity is given by

$$X^\infty(\lambda_\infty)_-+(P^-+\frac{1}{2}P^0)X^\infty_0\,.$$

It follows then that this partial fraction decomposition of $f(\lambda)$ coincides with the right-hand side of (7.29), so $\pi_-X(\lambda)=X(\lambda)$, for any $X(\lambda)\in\iota_\lambda R_\lambda(\mathfrak{g})$.

Let us now consider π_+. For any $X(\lambda) \in \mathcal{L}(\mathfrak{g})$ we have at 0

$$\begin{aligned}(\pi_+ X)^0(\lambda_0) = &- \sum_{b=0,\infty} \operatorname*{res}_{\mu=b} \iota_{\mu_b}\mu^{-1}(P^- + \frac{1}{2}P^0)X^b(\mu_b)d\mu \\ &+ \sum_{b=0,\infty}\sum_{n=0}^{\infty} \operatorname*{res}_{\mu=b} \iota_{\mu_b}\frac{\lambda^n}{\mu^{n+1}}X^b(\mu_b)d\mu\,.\end{aligned} \tag{7.33}$$

If $X(\lambda) \in \iota_\lambda R_\lambda(\mathfrak{g})$ then both terms vanish by the residue theorem. The same happens at infinity with

$$\begin{aligned}(\pi_+ X)^\infty(\lambda_\infty) = &- \sum_{b=0,\infty} \operatorname*{res}_{\mu=b} \iota_{\mu_b}\mu^{-1}(P^- + \frac{1}{2}P^0)X^b(\mu_b)d\mu \\ &- \sum_{b=0,\infty}\sum_{n=0}^{\infty} \operatorname*{res}_{\mu=b} \iota_{\mu_b}\frac{\mu^n}{\lambda^{n+1}}X^b(\mu_b)d\mu\,,\end{aligned} \tag{7.34}$$

so $(\pi_+X)(\lambda) = 0$ for every $X(\lambda) \in \iota_\lambda R_\lambda(\mathfrak{g})$.
Suppose now $X(\lambda) \in \mathcal{B}(\mathfrak{g})$. The first term of π_+X in zero gets the contributions

$$\begin{aligned}&- \operatorname*{res}_{\mu=0} \mu^{-1}(P^- + \frac{1}{2}P^0)X^0(\mu_0)d\mu - \operatorname*{res}_{\mu=\infty}(P^- + \frac{1}{2}P^0)X^\infty(\mu_\infty)d\mu \\ &\quad = -(P^- + \frac{1}{2}P^0)X_0^0 + (P^- + \frac{1}{2}P^0)X_0^\infty = -(P^- + \frac{1}{2}P^0)X_0^0 + \frac{1}{2}P^0X_0^\infty \\ &\quad = -(P^- + \frac{1}{2}P^0)X_0^0 - \frac{1}{2}P^0X_0^0 \\ &\quad = (P^- + P^0)X_0^\infty = X_0^\infty\,.\end{aligned}$$

The contribution from the second term is $X^0(\lambda_0) - X_0^\infty$, so in total we get $(\pi_+X)^0(\lambda_0) = X^0(\lambda_0)$. Similarly one obtains $(\pi_+X)^\infty(\lambda_\infty) = X^\infty(\lambda_\infty)$. In conclusion

$$\pi_+X(\lambda) = X(\lambda) \qquad \forall X(\lambda) \in \mathcal{B}(\mathfrak{g})\,. \quad \square$$

We can now define the trigonometric r-matrix $r \coloneqq \pi_+ - \pi_-$, whose kernel reads

$$((\iota_{\mu_b}\iota_{\lambda_a} + \iota_{\lambda_a}\iota_{\mu_b})r_{\mathbf{12}}(\lambda,\mu))_{a,b=0,\infty}\,. \tag{7.35}$$

The kernel of the identity operator is instead given by

$$\big((\iota_{\mu_b}\iota_{\lambda_a} - \iota_{\lambda_a}\iota_{\mu_b})r_{\mathbf{12}}(\lambda,\mu)\mu^{-1}d\mu\big)_{a,b=0,\infty} = (P_{\mathbf{12}}\delta_{ab}\delta(\lambda_a,\lambda_b)d\mu_a)_{a,b=0,\infty}\,. \tag{7.36}$$

7.2 The generating Lagrangian multiform

In the rest of the section, we will keep k fixed to either -1 or 0. We will introduce the generating Lagrangian in terms of two ingredients:

1. $Q(\lambda) = (Q^a(\lambda_a))_{a\in S} \in \mathcal{L}^+(\mathfrak{g})$ is the collection of $\mathfrak{g}$-valued Taylor series in λ_a, $a \in S$,
$$Q^a(\lambda_a) = \sum_{k=N_a}^{\infty} Q_k^a\,\lambda_a^k\,, \qquad Q_k^a \in \mathfrak{g}\,. \tag{7.37}$$
The value of N_a depends on a and on the value we choose for k. The coefficients Q_k^a are independent, but there are conditions on the Q_0^a which depend on whether we are in the rational or trigonometric case. We consider elements $Q(\lambda)$ where each $Q^a(\lambda_a)$ can be factorised as
$$Q^a(\lambda_a) = \varphi^a(\lambda_a)X^a(\lambda_a)\varphi^a(\lambda_a)^{-1}\,. \tag{7.38}$$
The element $\varphi^a(\lambda_a) = \sum_{j=0}^{\infty}\varphi_k^a\,\lambda_a^k$ is a Taylor series in λ_a, such that φ_0^a is invertible. $\varphi^a(\lambda_a)$ is a holomorphic map in a neighbourhood of a with values in G (the Lie group corresponding to $\mathfrak{g}$). $\varphi^a(\lambda_a)^{-1}$ is its inverse, *i.e.* $\varphi^a(\lambda_a)\varphi^a(\lambda_a)^{-1} = \mathbb{I}$. For each $a \in S$ $X^a(\lambda_a) \in \mathcal{L}_a^+(\mathfrak{g})$ is a given constant element of the loop algebra. The matrices $\varphi(\lambda)$ contain the fields of our theory, while $X(\lambda)$ is non-dynamical and constant. The elements of S will become poles of the corresponding Lax matrix.

2. A skew-symmetric *classical r-matrix* $r_{\mathbf{12}}(\lambda,\mu) = -r_{\mathbf{21}}(\mu,\lambda)$, solution of the classical Yang-Baxter equation. The explicit expression of r will depend whether we are in the rational ($k=0$) or trigonometric ($k=-1$) case, but we require that, as was proved for the specific cases
$$(\iota_{\lambda_a}\iota_{\mu_b} - \iota_{\mu_b}\iota_{\lambda_a})r_{\mathbf{12}}(\lambda,\mu)\mu^k d\mu = P_{\mathbf{12}}\delta_{ab}\delta(\lambda_a,\mu_b)d\mu_a \tag{7.39}$$
where $\delta(\lambda_a,\mu_b) \coloneqq \sum_{n\in\mathbb{Z}}\lambda_a^n\mu_b^{-n-1}$.

We begin with the following two important lemmas.

Lemma 7.8 *We have*
$$\big[\mathrm{Tr}_{\mathbf{2}}\left(\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu)Q_{\mathbf{2}}(\mu)\right), Q_{\mathbf{1}}(\lambda)\big] = \big[\mathrm{Tr}_{\mathbf{2}}\left(\iota_\mu\iota_\lambda r_{\mathbf{12}}(\lambda,\mu)Q_{\mathbf{2}}(\mu)\right), Q_{\mathbf{1}}(\lambda)\big]\,.$$

Proof. Using the identity (7.39) we deduce that for any $a, b \in S$ we have
$$\big[\,\mathrm{Tr}_{\mathbf{2}}\left((\iota_{\lambda_a}\iota_{\mu_b} - \iota_{\mu_b}\iota_{\lambda_a})r_{\mathbf{12}}(\lambda,\mu)Q_{\mathbf{2}}^b(\mu_b)\right), Q_{\mathbf{1}}^a(\lambda_a)\big] \propto \delta(\lambda_a,\mu_a)[Q^a(\lambda_a), Q^a(\mu_a)]\,.$$
Since $[Q^a(\lambda_a), Q^a(\mu_a)]$ vanishes when $\lambda_a = \mu_a$, it is proportional to $\lambda_a - \mu_a$. Then, since[3] $\delta(\lambda_a,\mu_a)(\lambda_a - \mu_a) = 0$, it follows that the right hand side above vanishes, as required. □

[3]Indeed $\sum_{n\in\mathbb{Z}}\lambda_a^n\mu_a^{-n-1}(\lambda_a-\mu_a) = \sum_{n\in\mathbb{Z}}\lambda_a^{n+1}\mu_a^{-n-1} - \sum_{n\in\mathbb{Z}}\lambda_a^n\mu_a^{-n} = 0$.

Lemma 7.9 *Let* $D_\mu := (D_{\mu_a})_{a\in S}$. *The* generating Lax equations

$$D_\mu Q_{\mathbf{1}}(\lambda) = [\iota_\lambda \iota_\mu \operatorname{Tr}_{\mathbf{2}} r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu), Q_{\mathbf{1}}(\lambda)] \tag{7.40}$$

are compatible, in the sense that $D_\nu D_\mu Q_{\mathbf{1}}(\lambda) = D_\mu D_\nu Q_{\mathbf{1}}(\lambda)$.

Proof. The proof follows as a consequence of the classical Yang-Baxter equation for $r_{\mathbf{12}}$. We have

$$\begin{aligned} D_\nu D_\mu Q_{\mathbf{1}}(\lambda) &= \big[\operatorname{Tr}_{\mathbf{2}}\big(\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu) D_\nu Q_{\mathbf{2}}(\mu)\big), Q_{\mathbf{1}}(\lambda)\big] \\ &\quad + \big[\operatorname{Tr}_{\mathbf{2}}\big(\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu)\big), D_\nu Q_{\mathbf{1}}(\lambda)\big] \\ &= \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu)\big[\iota_\mu\iota_\nu r_{\mathbf{23}}(\mu,\nu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{2}}(\mu)\big], Q_{\mathbf{1}}(\lambda)\Big] \\ &\quad + \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu), \big[\iota_\lambda\iota_\nu r_{\mathbf{13}}(\lambda,\nu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{1}}(\lambda)\big]\Big]. \end{aligned} \tag{7.41}$$

By using the cyclicity of the trace over space 2 in the first term on the right hand side and the Jacobi identity on the last term, this can be rewritten as

$$\begin{aligned} D_\nu D_\mu Q_{\mathbf{1}}(\lambda) &= \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\mu\iota_\nu\big[r_{\mathbf{12}}(\lambda,\mu), r_{\mathbf{23}}(\mu,\nu)\big] Q_{\mathbf{2}}(\mu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{1}}(\lambda)\Big] \\ &\quad + \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\mu\iota_\nu\big[r_{\mathbf{12}}(\lambda,\mu), r_{\mathbf{13}}(\lambda,\nu)\big] Q_{\mathbf{2}}(\mu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{1}}(\lambda)\Big] \\ &\quad + \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\nu r_{\mathbf{13}}(\lambda,\nu) Q_{\mathbf{3}}(\nu), \big[\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu), Q_{\mathbf{1}}(\lambda)\big]\Big]. \end{aligned}$$

Likewise, exchanging $\mu \leftrightarrow \nu$ in (7.41) we obtain

$$\begin{aligned} D_\mu D_\nu Q_{\mathbf{1}}(\lambda) &= \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\nu r_{\mathbf{13}}(\lambda,\nu)\big[\iota_\nu\iota_\mu r_{\mathbf{32}}(\nu,\mu) Q_{\mathbf{2}}(\mu), Q_{\mathbf{3}}(\nu)\big], Q_{\mathbf{1}}(\lambda)\Big] \\ &\quad + \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\nu r_{\mathbf{13}}(\lambda,\nu) Q_{\mathbf{3}}(\nu), \big[\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu), Q_{\mathbf{1}}(\lambda)\big]\Big] \\ &= \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\mu\iota_\nu\big[r_{\mathbf{13}}(\lambda,\nu), r_{\mathbf{32}}(\nu,\mu)\big] Q_{\mathbf{2}}(\mu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{1}}(\lambda)\Big] \\ &\quad + \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\nu r_{\mathbf{13}}(\lambda,\nu) Q_{\mathbf{3}}(\nu), \big[\iota_\lambda\iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu), Q_{\mathbf{1}}(\lambda)\big]\Big], \end{aligned}$$

where the second equality we used Lemma 7.8 to swap the order of ι_ν and ι_μ in the first term, along with the cyclicity of the trace over space 3.

It now follows from combining the above that the difference $\big(D_\nu D_\mu - D_\mu D_\nu\big) Q_{\mathbf{1}}(\lambda)$ is given by

$$\begin{aligned} \operatorname{Tr}_{\mathbf{23}}\Big[\iota_\lambda\iota_\mu\iota_\nu\Big(&[r_{\mathbf{12}}(\lambda,\mu), r_{\mathbf{13}}(\lambda,\nu)] + [r_{\mathbf{12}}(\lambda,\mu), r_{\mathbf{23}}(\mu,\nu)] \\ &- [r_{\mathbf{13}}(\lambda,\nu), r_{\mathbf{32}}(\nu,\mu)]\Big) Q_{\mathbf{2}}(\mu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{1}}(\lambda)\Big] \end{aligned}$$

which vanishes using the classical Yang-Baxter equation, as required. □

This last result of commutativity of the flows D_λ's, when they act on our algebra as (7.40), and with the identification of the D_{λ_a}'s in terms of linear combinations of the $\partial_{t_a^k}$'s, allows us to consider the coordinates $\{\, t_a^k \mid \quad a \in S, k \geq 0 \,\}$ as times identifying commuting flows of an integrable hierarchy. As a result, we can effectively see our loop algebra $\mathcal{L}(\mathfrak{g})$ as valued in $\mathscr{A}$, the differential algebra underlying the variational bi-complex, and therefore we can take $\mathfrak{g} \subseteq \mathfrak{gl}_N(\mathscr{A})$.

Equations (7.40) will be shown to be variational, and they can be interpreted as multiform Euler-Lagrange equations for a collection of Lagrangian multiforms. These are introduced as a generating series in the same spirit of Equations (6.31), as

$$\mathscr{L}(\lambda, \mu) = K(\lambda, \mu) - V(\lambda, \mu)\,, \tag{7.42}$$

where we define the *kinetic part* as

$$K(\lambda, \mu) = \mathrm{Tr}\left(\varphi^{-1}(\lambda) D_\mu \varphi(\lambda) X(\lambda) - \varphi^{-1}(\mu) D_\lambda \varphi(\mu) X(\mu)\right) \tag{7.43}$$

and the *potential part* as

$$V(\lambda, \mu) = \frac{1}{2}\, \mathrm{Tr}_{\mathbf{12}}(\iota_\mu \iota_\lambda + \iota_\lambda \iota_\mu) r_{\mathbf{12}}(\lambda, \mu) Q_{\mathbf{1}}(\lambda) Q_{\mathbf{2}}(\mu)\,. \tag{7.44}$$

The above facts make the generating Lagrangian $\mathscr{L}(\lambda, \mu)$ a collection of double Laurent series in λ_a and μ_b, $a, b \in S$

$$\begin{aligned} \mathscr{L}(\lambda, \mu) =& \left(\sum_{m,n} L_{mn}^{ab}\, \lambda_a^m \mu_b^n\right)_{a,b\in S} \\ \equiv& (\mathscr{L}^{ab}(\lambda_a, \mu_b))_{a,b\in S}\,, \qquad\qquad \mathscr{L}^{ab}(\lambda_a, \mu_b) \in \mathfrak{g} \otimes \mathbb{C}(\!(\lambda_a, \mu_b)\!)\,. \end{aligned} \tag{7.45}$$

This will define a collection of Lagrangian multiforms, associated to the times $t_a^1, t_a^2, \ldots,$ and $t_b^1, t_b^2, \ldots$ for all $a, b \in S$ by using the following prescription

$$\mathscr{L}^{ab} := \sum_{m<n} L_{mn}^{ab}\, dt_a^m \wedge dt_b^n\,, \tag{7.46}$$

by the relation

$$L_{mn}^{ab} = \operatorname*{res}_{\lambda=a} \operatorname*{res}_{\mu=b} \frac{\mathscr{L}^{ab}(\lambda_a, \mu_b)}{\lambda_a^m \mu_b^n}\, \lambda^k d\lambda\, \mu^k d\mu\,. \tag{7.47}$$

In order to prove that this is the case, we need to calculate the corresponding multiform Euler-Lagrange equations $\delta d\mathscr{L} = 0$, and prove that on these equations $\mathscr{L}$ is horizontally closed (closure relation) $d\mathscr{L} = 0$. These computations will be carried over in *generating form*, *i.e.* by dealing with the formal series in λ and μ. For example, the coefficients of $d\mathscr{L}$ (as a form) are the coefficients of

$$D_\nu \mathscr{L}(\lambda, \mu) + D_\mu \mathscr{L}(\nu, \lambda) + D_\lambda \mathscr{L}(\mu, \nu) \tag{7.48}$$

as a triple series. We will use the same approach to calculate $\delta d\mathscr{L}$.

7.2.1 Multiform Euler-Lagrange equation

Proposition 7.10 *The multiform Euler-Lagrange equations for $\mathscr{L}$ are, in generating form*

$$D_\mu Q_{\mathbf{1}}(\lambda) = [\mathrm{Tr}_{\mathbf{2}}\, \iota_\lambda \iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu), Q_{\mathbf{1}}(\lambda)]\,, \tag{7.49}$$

Proof. We compute $\delta d\mathscr{L}$ in generating form as $\delta D_\nu \mathscr{L}(\lambda,\mu) + \circlearrowleft$, and separating between the kinetic and potential terms. We start with the kinetic

$$\begin{aligned} D_\nu K(\lambda,\mu) = \mathrm{Tr}(&-\varphi^{-1}(\lambda) D_\nu \varphi(\lambda) \varphi^{-1}(\lambda) D_\mu \varphi(\lambda) X(\lambda) \\ &+ \varphi^{-1}(\lambda) D_\nu D_\mu \varphi(\lambda) X(\lambda) + \varphi^{-1}(\lambda) D_\mu \varphi(\lambda) D_\nu X(\lambda) \\ &+ \varphi^{-1}(\mu) D_\nu \varphi(\mu) \varphi^{-1}(\mu) D_\lambda \varphi(\mu) X(\mu) \\ &- \varphi^{-1}(\mu) D_\nu D_\lambda \varphi(\mu) X(\mu) - \varphi^{-1}(\mu) D_\lambda \varphi(\mu) D_\nu X(\mu)) \end{aligned}$$

so that $D_\nu K(\lambda,\mu) + \circlearrowleft$ is

$$\begin{aligned} &\mathrm{Tr}(-\varphi^{-1}(\lambda) D_\nu \varphi(\lambda) \varphi^{-1}(\lambda) D_\mu \varphi(\lambda) X(\lambda) + \varphi^{-1}(\lambda) D_\mu \varphi(\lambda) D_\nu X(\lambda) \\ &\quad + \varphi^{-1}(\lambda) D_\mu \varphi^{-1}(\lambda) D_\nu \varphi(\lambda) X(\lambda) - \varphi(\lambda) D_\nu \varphi(\lambda) D_\mu X(\lambda)) + \circlearrowleft \end{aligned}$$

After we apply the δ-differential we get

$$\begin{aligned} \text{“}\delta dK\text{”} = \mathrm{Tr}((&\varphi^{-1}(\lambda) D_\nu \varphi(\lambda) \varphi^{-1}(\lambda) D_\mu Q(\lambda) - \varphi^{-1}(\lambda) D_\mu \varphi(\lambda) \varphi^{-1}(\lambda) D_\nu Q(\lambda)) \delta\varphi(\lambda) \\ &- \varphi^{-1}(\lambda) D_\mu Q(\lambda) \delta D_\nu \varphi(\lambda) + \varphi^{-1}(\lambda) D_\nu Q(\lambda) \delta D_\mu \varphi(\lambda)) + \circlearrowleft\ . \end{aligned}$$

Let us now compute δdV. Computing $D_\nu V(\lambda,\mu)$ we get, using $D_\nu r_{\mathbf{12}}(\lambda,\mu) = 0$

$$D_\nu V(\lambda,\mu) = \frac{1}{2}\,\mathrm{Tr}_{\mathbf{12}}(\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda)(r_{\mathbf{12}} D_\nu Q_{\mathbf{1}} Q_{\mathbf{2}} + r_{\mathbf{12}} Q_{\mathbf{1}} D_\nu Q_{\mathbf{2}})$$

where we dropped λ,μ for simplicity. After applying the δ-differential we get

$$\begin{aligned} \delta D_\nu V(\lambda,\mu) = \frac{1}{2}\,\mathrm{Tr}_{\mathbf{12}}(\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda)(&r_{\mathbf{12}} \delta D_\nu Q_{\mathbf{1}} Q_{\mathbf{2}} \\ &+ r_{\mathbf{12}} D_\nu Q_{\mathbf{1}} \delta Q_{\mathbf{2}} + r_{\mathbf{12}} \delta Q_{\mathbf{1}} D_\nu Q_{\mathbf{2}} + r_{\mathbf{12}} Q_{\mathbf{1}} \delta D_\nu Q_{\mathbf{2}}) \end{aligned}$$

We have the following identities:

$$\begin{aligned} \mathrm{Tr}_{\mathbf{12}}\, r_{\mathbf{12}} \delta D_\nu Q_{\mathbf{1}} Q_{\mathbf{2}} &= \mathrm{Tr}_{\mathbf{12}}(-Q_{\mathbf{2}} r_{\mathbf{12}} D_\nu Q_{\mathbf{1}} - Q_{\mathbf{1}} D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} Q_{\mathbf{2}} r_{\mathbf{12}} \\ &\quad + D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} Q_{\mathbf{2}} r_{\mathbf{12}} Q_{\mathbf{1}} + \varphi_{\mathbf{1}} D_\nu X_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} Q_{\mathbf{2}} r_{\mathbf{12}}) \delta\varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} \\ &\quad + \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}] \delta D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} \\ &= \mathrm{Tr}_{\mathbf{12}}([D_\nu Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}] - D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} [Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}]) \delta\varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} \\ &\quad + \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}] \delta D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1}\,, \end{aligned}$$

$$\begin{aligned}\mathrm{Tr}_{\mathbf{12}}\, r_{\mathbf{12}} Q_{\mathbf{1}} \delta D_\nu Q_{\mathbf{2}} =& \mathrm{Tr}_{\mathbf{12}}(-r_{\mathbf{12}} Q_{\mathbf{1}} D_\nu Q_{\mathbf{2}} - Q_{\mathbf{2}} D_\nu \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} r_{\mathbf{12}} Q_{\mathbf{1}} \\ &+ D_\nu \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} r_{\mathbf{12}} Q_{\mathbf{1}} Q_{\mathbf{2}} + \varphi_{\mathbf{2}} D_\nu X_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} r_{\mathbf{12}} Q_{\mathbf{1}}) \delta\varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} \\ &+ \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{2}}, r_{\mathbf{12}} Q_{\mathbf{1}}] \delta D_\nu \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} \\ =& \mathrm{Tr}_{\mathbf{12}}([D_\nu Q_{\mathbf{2}}, r_{\mathbf{12}} Q_{\mathbf{1}}] - D_\nu \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} [Q_{\mathbf{2}}, r_{\mathbf{12}} Q_{\mathbf{1}}]) \delta \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1} \\ &+ \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{2}}, r_{\mathbf{12}} Q_{\mathbf{1}}] \delta D_\nu \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1}\,,\end{aligned}$$

$$\mathrm{Tr}_{\mathbf{12}}\, r_{\mathbf{12}} \delta Q_{\mathbf{1}} D_\nu Q_{\mathbf{2}} = \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{1}}, r_{\mathbf{12}} D_\nu Q_{\mathbf{2}}] \delta \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1}\,,$$

$$\mathrm{Tr}_{\mathbf{12}}\, r_{\mathbf{12}} D_\nu Q_{\mathbf{1}} \delta Q_{\mathbf{2}} = \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{2}}, r_{\mathbf{12}} D_\nu Q_{\mathbf{1}}] \delta \varphi_{\mathbf{2}} \varphi_{\mathbf{2}}^{-1}\,.$$

We look at the terms coming from $\delta\varphi_{\mathbf{1}}$, $\delta D_\mu \varphi_{\mathbf{1}}$ and $\delta D_\nu \varphi_{\mathbf{1}}$. From δdK we have

$$\begin{aligned}&\mathrm{Tr}_{\mathbf{1}}(-D_\mu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} D_\nu Q_{\mathbf{1}} + D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} D_\mu Q_{\mathbf{1}}) \delta \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} \\ &+ \mathrm{Tr}_{\mathbf{1}}(-D_\mu \delta D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} + D_\nu Q_{\mathbf{1}} \delta D_\mu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1})\end{aligned}$$

and from δdV (from $\delta D_\nu V(\lambda,\mu)$ and $\delta D_\mu V(\nu,\lambda)$) and using $r_{ij} = -r_{ji}$

$$\begin{aligned}&\frac{1}{2}(\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda)\, \mathrm{Tr}_{\mathbf{12}}[Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}] \delta D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} - \frac{1}{2}(\iota_\lambda \iota_\nu + \iota_\nu \iota_\lambda)\, \mathrm{Tr}_{\mathbf{13}}[Q_{\mathbf{1}}, r_{\mathbf{13}} Q_{\mathbf{3}}] \delta D_\mu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} \\ &+ \frac{1}{2}(\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda)\, \mathrm{Tr}_{\mathbf{12}}([D_\nu Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}] - D_\nu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} [Q_{\mathbf{1}}, r_{\mathbf{12}} Q_{\mathbf{2}}] + [Q_{\mathbf{1}}, r_{\mathbf{12}} D_\nu Q_{\mathbf{2}}]) \delta \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} \\ &+ \frac{1}{2}(\iota_\lambda \iota_\nu + \iota_\nu \iota_\lambda)\, \mathrm{Tr}_{\mathbf{13}}(-[D_\mu Q_{\mathbf{1}}, r_{\mathbf{13}} Q_{\mathbf{3}}] + D_\mu \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1} [Q_{\mathbf{1}}, r_{\mathbf{13}} Q_{\mathbf{3}}] - [Q_{\mathbf{1}}, r_{\mathbf{13}} D_\mu Q_{\mathbf{3}}]) \delta \varphi_{\mathbf{1}} \varphi_{\mathbf{1}}^{-1}\,.\end{aligned}$$

By setting $\delta dK = \delta dV$, the coefficients of $\delta D_\mu \varphi_{\mathbf{1}}$ and $\delta D_\nu \varphi_{\mathbf{1}}$ bring the desired equations

$$D_\mu Q_{\mathbf{1}} = \frac{1}{2}(\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda)[\mathrm{Tr}_{\mathbf{2}}\, r_{\mathbf{12}} Q_{\mathbf{2}}, Q_{\mathbf{1}}]\,, \quad D_\nu Q_{\mathbf{1}} = \frac{1}{2}(\iota_\lambda \iota_\nu + \iota_\nu \iota_\lambda)[\mathrm{Tr}_{\mathbf{3}}\, r_{\mathbf{13}} Q_{\mathbf{3}}, Q_{\mathbf{1}}]\,.$$

The other coefficients are just differential consequences, and they follow from the commutativity of flows $D_\mu D_\nu Q_{\mathbf{1}} = D_\nu D_\mu Q_{\mathbf{1}}$. The coefficients of $\delta\varphi_{\mathbf{2}}$, $\delta\varphi_{\mathbf{3}}$ etc. give the equations with the other choices of spectral parameter and auxiliary space. As they are equivalent under the interchange of auxiliary space and formal variable, we will only keep the first one. We then use Lemma 7.8 to write the result. ☐

7.2.2 Closure relation and classical Yang-Baxter equation

In order to prove that $\mathscr{L}(\lambda,\mu)$ really generates a Lagrangian multiform we need to prove that it is horizontally closed under the equations generated by (7.49). We will see that, just like the commutativity of the flows, this translates in the classical Yang-Baxter equation. In a way, this result brings for the first time a variational origin of the classical Yang-Baxter equation, and provides another interesting feature of the Lagrangian multiform approach to an integrable system.

Proposition 7.11 (Closure relation) *On shell of the multiform Euler-Lagrange equations*

(7.49), *we have*

$$D_\nu \mathscr{L}(\lambda,\mu) + D_\mu \mathscr{L}(\nu,\lambda) + D_\lambda \mathscr{L}(\mu,\nu) = 0\,. \tag{7.50}$$

Proof. We start with contribution from the kinetic part. We have that

$$\begin{aligned}
&D_\nu K(\lambda,\mu)\\
&= D_\nu \operatorname{Tr}(\varphi^{-1}(\lambda)D_\mu\varphi(\lambda)X(\lambda) - \varphi^{-1}(\mu)D_\lambda\varphi(\mu)X(\mu))\\
&= \operatorname{Tr}(-\varphi^{-1}(\lambda)D_\nu\varphi(\lambda)\varphi^{-1}(\lambda)D_\mu\varphi(\lambda)X(\lambda) + \varphi^{-1}(\lambda)D_\nu D_\mu\varphi(\lambda)X(\lambda)\\
&\qquad + \varphi^{-1}(\mu)D_\nu\varphi(\mu)\varphi^{-1}(\mu)D_\lambda\varphi(\mu)X(\mu) - \varphi^{-1}(\mu)D_\nu D_\lambda\varphi(\mu)X(\mu))\,.
\end{aligned}$$

After we add $D_\mu K(\nu,\lambda) + D_\lambda K(\mu,\nu)$ we have that the terms with the double derivatives cancel out, while the others add up to

$$\operatorname{Tr}(D_\nu Q(\lambda)D_\mu\varphi(\lambda)\varphi^{-1}(\lambda) + D_\lambda Q(\mu)D_\nu\varphi(\mu)\varphi^{-1}(\mu) + D_\mu Q(\nu)D_\lambda\varphi(\nu)\varphi^{-1}(\nu))$$

Now we use the multiform Euler-Lagrange equations and get, associating the auxiliary spaces as usual and dropping the dependence on the spectral parameters

$$\begin{aligned}
&\operatorname{Tr}_{\mathbf{13}}([\iota_\lambda\iota_\nu r_{\mathbf{13}}Q_{\mathbf{3}}, Q_{\mathbf{1}}]D_\mu\varphi_{\mathbf{1}}\varphi_{\mathbf{1}}^{-1}) - \operatorname{Tr}_{\mathbf{12}}([\iota_\lambda\iota_\mu r_{\mathbf{12}}Q_{\mathbf{1}}, Q_{\mathbf{2}}]D_\nu\varphi_{\mathbf{2}}\varphi_{\mathbf{2}}^{-1})\\
&\quad - \operatorname{Tr}_{\mathbf{23}}([\iota_\mu\iota_\nu r_{\mathbf{23}}Q_{\mathbf{2}}, Q_{\mathbf{3}}]D_\lambda\varphi_{\mathbf{3}}\varphi_{\mathbf{3}}^{-1})\\
&= -\operatorname{Tr}_{\mathbf{13}}\,\iota_\lambda\iota_\nu r_{\mathbf{13}}Q_{\mathbf{3}}D_\mu Q_{\mathbf{1}} + \operatorname{Tr}_{\mathbf{12}}\,\iota_\lambda\iota_\mu r_{\mathbf{12}}Q_{\mathbf{1}}D_\nu Q_{\mathbf{2}} + \operatorname{Tr}_{\mathbf{23}}\,\iota_\mu\iota_\nu r_{\mathbf{23}}Q_{\mathbf{2}}D_\lambda Q_{\mathbf{3}}\\
&= \operatorname{Tr}_{\mathbf{123}}(-\iota_\lambda\iota_\nu r_{\mathbf{13}}Q_{\mathbf{3}}[\iota_\lambda\iota_\mu r_{\mathbf{12}}Q_{\mathbf{2}}, Q_{\mathbf{1}}] + \iota_\lambda\iota_\mu r_{\mathbf{12}}Q_{\mathbf{1}}[\iota_\mu\iota_\nu r_{\mathbf{23}}Q_{\mathbf{3}}, Q_{\mathbf{1}}]\\
&\qquad - \iota_\mu\iota_\nu r_{\mathbf{23}}Q_{\mathbf{2}}[\iota_\lambda\iota_\nu r_{\mathbf{13}}Q_{\mathbf{1}}, Q_{\mathbf{3}}])\\
&= \operatorname{Tr}_{\mathbf{123}}\,\iota_\lambda\iota_\mu\iota_\nu([r_{\mathbf{12}}, r_{\mathbf{13}}] + [r_{\mathbf{12}}, r_{\mathbf{23}}] + [r_{\mathbf{13}}, r_{\mathbf{23}}])Q_{\mathbf{1}}Q_{\mathbf{2}}Q_{\mathbf{3}}
\end{aligned}$$

which vanishes thanks to the classical Yang-Baxter equation. Let us treat the potential part $V(\lambda,\mu) = \operatorname{Tr}_{\mathbf{12}}\,\frac{1}{2}(\iota_\lambda\iota_\mu + \iota_\mu\iota_\lambda)r_{\mathbf{12}}Q_{\mathbf{1}}Q_{\mathbf{2}}$ (dropping the dependence on the spectral parameters). We have

$$\begin{aligned}
D_\nu V(\lambda,\mu) &= \frac{1}{2}\operatorname{Tr}_{\mathbf{12}}(\iota_\lambda\iota_\mu + \iota_\mu\iota_\lambda)(r_{\mathbf{12}}D_\nu Q_{\mathbf{1}}Q_{\mathbf{2}} + r_{\mathbf{12}}Q_{\mathbf{1}}D_\nu Q_{\mathbf{2}})\\
&= \frac{1}{2}\operatorname{Tr}_{\mathbf{123}}((\iota_\lambda\iota_\mu + \iota_\mu\iota_\lambda)r_{\mathbf{12}}[\iota_\lambda\iota_\nu r_{\mathbf{13}}Q_{\mathbf{3}}, Q_{\mathbf{1}}]Q_{\mathbf{2}} + r_{\mathbf{12}}Q_{\mathbf{1}}[\iota_\mu\iota_\nu r_{\mathbf{23}}Q_{\mathbf{3}}, Q_{\mathbf{2}}])\\
&= \operatorname{Tr}_{\mathbf{123}}\,\iota_\lambda\iota_\mu\iota_\nu([r_{\mathbf{12}}, r_{\mathbf{13}}] + [r_{\mathbf{12}}, r_{\mathbf{23}}])Q_{\mathbf{1}}Q_{\mathbf{2}}Q_{\mathbf{3}}
\end{aligned}$$

where in the last line we used Lemma 7.8. When we add the other terms $D_\mu V(\nu,\lambda) + D_\lambda V(\mu,\nu)$ we can see that they add up to

$$2\operatorname{Tr}_{\mathbf{123}}\,\iota_\lambda\iota_\mu\iota_\nu([r_{\mathbf{12}}, r_{\mathbf{13}}] + [r_{\mathbf{12}}, r_{\mathbf{23}}] + [r_{\mathbf{13}}, r_{\mathbf{23}}])Q_{\mathbf{1}}Q_{\mathbf{2}}Q_{\mathbf{3}}$$

which again vanishes thanks to the classical Yang-Baxter equation. □

7.2.3 Zero-curvature equations

The equations of motion (7.49) can be written succinctly as

$$D_\mu Q(\lambda) = [\iota_\lambda W(\lambda;\mu), Q(\lambda)] \tag{7.51}$$

where we have introduced the *generating Lax form*

$$W(\lambda;\mu) \coloneqq \mathrm{Tr}_{\mathbf{2}}\, \iota_\mu r_{\mathbf{12}}(\lambda,\mu) Q_{\mathbf{2}}(\mu)\,. \tag{7.52}$$

Note that we do *not* expand the right hand side in powers of λ_a for $a \in S$, *i.e.* we do not apply the homomorphism ι_λ. Instead, this expansion is taken explicitly in (7.51). In particular, the semi-colon in the notation $W(\lambda;\mu)$ is used to emphasise that λ is just a formal variable whereas μ is the usual notation used as a shorthand for a collection $\big(W^b(\lambda;\mu_b)\big)_{b\in S}$ where

$$W^b(\lambda;\mu_b) = \sum_{n=0}^{\infty} W^b_n(\lambda)\mu_b^n \in R_\lambda(\mathfrak{g}) \otimes \mathbb{C}[\![\mu_b]\!]. \tag{7.53}$$

Here $W^b_n(\lambda) \in R_\lambda(\mathfrak{g})$ are $\mathfrak{g}$-valued rational functions in λ with a pole at $\lambda = b$. By the following proposition, $W^b(\lambda;\mu_b)$ can be seen as a generating series in μ_b of a hierarchy of Lax matrices $W^b_n(\lambda)$ associated with the higher times t^n_b, and to every coefficient of μ_b is associated a coefficient of the Lax form $W^b(\lambda) = \sum_{n=0}^{\infty} W^b_n(\lambda)\, dt^n_b$.

Proposition 7.12 *We have the zero-curvature equation in generating form*

$$D_\nu W(\lambda;\mu) - D_\mu W(\lambda;\nu) + \big[W(\lambda;\mu), W(\lambda;\nu)\big] = 0.$$

Equivalently, in components we have

$$\partial_{t^b_n} W^a_m(\lambda) - \partial_{t^a_m} W^b_n(\lambda) + \big[W^a_m(\lambda), W^b_n(\lambda)\big] = 0$$

for every $a, b \in S$ and $m, n \geq 0$.

Proof. Using Proposition 7.49 we find

$$\begin{aligned} D_\nu W(\lambda;\mu) &= \mathrm{Tr}_{\mathbf{2}}\, \iota_\mu r_{\mathbf{12}}(\lambda,\mu) D_\nu Q_{\mathbf{2}}(\mu) \\ &= \mathrm{Tr}_{\mathbf{23}}\, \iota_\mu r_{\mathbf{12}}(\lambda,\mu)\big[\iota_\mu \iota_\nu r_{\mathbf{23}}(\mu,\nu) Q_{\mathbf{3}}(\nu), Q_{\mathbf{2}}(\mu)\big] \\ &= \mathrm{Tr}_{\mathbf{23}}\, \iota_\mu \iota_\nu \big[r_{\mathbf{12}}(\lambda,\mu), r_{\mathbf{23}}(\mu,\nu)\big] Q_{\mathbf{2}}(\mu) Q_{\mathbf{3}}(\nu), \end{aligned}$$

where in the last equality we used the cyclicity of the trace in space 2. Likewise, we also have

$$\begin{aligned}D_\mu W(\lambda;\nu) &= \mathrm{Tr}_{\mathbf{3}}\, \iota_\nu r_{\mathbf{13}}(\lambda,\nu) D_\mu Q_{\mathbf{3}}(\nu)\\ &= \mathrm{Tr}_{\mathbf{23}}\, \iota_\nu r_{\mathbf{13}}(\lambda,\nu)\big[\iota_\nu\iota_\mu r_{\mathbf{32}}(\nu,\mu)Q_{\mathbf{2}}(\mu), Q_{\mathbf{3}}(\nu)\big]\\ &= \mathrm{Tr}_{\mathbf{23}}\, \iota_\mu\iota_\nu \big[r_{\mathbf{13}}(\lambda,\nu), r_{\mathbf{32}}(\nu,\mu)\big] Q_{\mathbf{2}}(\mu)Q_{\mathbf{3}}(\nu).\end{aligned}$$

where in the final step we used Lemma 7.8 to swap the order of ι_ν and ι_μ, before using the cyclicity of the trace in space 3. Finally, we have

$$\begin{aligned}\big[W(\lambda;\mu), W(\lambda;\nu)\big] &= \mathrm{Tr}_{\mathbf{23}}\,\big[\iota_\mu r_{\mathbf{12}}(\lambda,\mu)Q_{\mathbf{2}}(\mu), \iota_\lambda\iota_\nu r_{\mathbf{13}}(\lambda,\nu)Q_{\mathbf{3}}(\nu)\big]\\ &= \mathrm{Tr}_{\mathbf{23}}\, \iota_\mu\iota_\nu\big[r_{\mathbf{12}}(\lambda,\mu), r_{\mathbf{13}}(\lambda,\nu)\big] Q_{\mathbf{2}}(\mu)Q_{\mathbf{3}}(\nu).\end{aligned}$$

The result now follows by the classical Yang-Baxter equation. □

7.3 The AKNS hierarchy

7.3.1 Lax matrices and zero-curvature equations

Lemma 7.13 *Let $X(\mu) = \sum_{n=-N}^{\infty} X_n^\infty \lambda_\infty^n \in \mathcal{L}_\infty^{rat}(\mathfrak{g})$. Then we have*

$$\iota_{\mu_\infty}\frac{X^\infty(\mu_\infty)}{\mu-\lambda} = \sum_{r=-N}^{\infty} \mu_\infty^{r+1}\left(\lambda_\infty^{-r}X^\infty(\lambda_\infty)\right)_- . \tag{7.54}$$

Proof. Obtained by direct calculation. We have

$$\begin{aligned}\iota_{\mu_\infty}\frac{X^\infty(\mu_\infty)}{\mu-\lambda} &= \sum_{r=0}^{\infty}\sum_{n=-N}^{\infty}\frac{\mu_\infty^{n+r+1}}{\lambda_\infty^r}X_n^\infty = \sum_{n=-N}^{\infty}\sum_{s=n}^{\infty} X_n^\infty \mu_\infty^{s+1}\lambda_\infty^{n-s}\\ &= \sum_{s=-N}^{\infty}\mu_\infty^{s+1}\sum_{n=-N}^{s}\lambda_\infty^{n-s}X_n^\infty = \sum_{r=-N}^{\infty}\mu_\infty^{r+1}\left(\lambda_\infty^{-r}X^\infty(\lambda_\infty)\right)_- .\end{aligned}$$ □

We can find the generating Lax form $W(\lambda;\mu)$ explicitly for the AKNS. Introducing $Q^\infty(\lambda_\infty) \in \mathcal{L}^{\mathrm{rat}}(\mathfrak{g})$ we have that

$$\begin{aligned}W^\infty(\lambda;\mu_\infty) &= \mathrm{Tr}_{\mathbf{2}}\, \iota_{\mu_\infty} r_{\mathbf{12}}(\lambda,\mu)Q^\infty(\mu_\infty)_{\mathbf{2}} = \mathrm{Tr}_{\mathbf{2}}\,\iota_{\mu_\infty}\frac{P_{\mathbf{12}}}{\mu-\lambda}Q^\infty(\mu_\infty)_{\mathbf{2}}\\ &= \sum_{r=1}^{\infty}\mu_\infty^{r+1}(\lambda_\infty^{-r}Q^\infty(\lambda_\infty))_- = Q_1^\infty\mu_\infty^2 + (Q_2^\infty + Q_1^\infty\lambda_\infty^{-1})\mu_\infty^3 + \dots\end{aligned} \tag{7.55}$$

As announced, the points in S (which only contains ∞ in this case), have become poles of the Lax matrices, *i.e.* the coefficients of W^∞. The zero-curvature equations in generating form become

$$\sum_{k=1}^{\infty}\mu_\infty^{k+1}\partial_{t_k^\infty}Q^\infty(\lambda_\infty) = \left[W^\infty(\lambda_\infty,\mu_\infty), Q^\infty(\lambda_\infty)\right]. \tag{7.56}$$

This means that for every $r \geq 1$ we have

$$\partial_{t_r^\infty} Q^\infty(\lambda_\infty) = \left[\sum_{n=1}^{r} Q_n^\infty \lambda_\infty^{n-r}, Q^\infty(\lambda_\infty) \right] . \tag{7.57}$$

Expanding both sides we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_\infty^n \partial_{t_r^\infty} Q_n^\infty &= \sum_{n=1}^{\infty} \sum_{s=1}^{r} \lambda_\infty^{s-r+n} \left[Q_s^\infty, Q_n^\infty \right] \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{r-1} \lambda_\infty^{n-k} \left[Q_{r-k}^\infty, Q_n^\infty \right] \\
&= \sum_{k=1}^{r-1} \sum_{n=1}^{k} \lambda_\infty^{n-k} \left[Q_{r-k}^\infty, Q_n^\infty \right] + \sum_{k=0}^{r-1} \sum_{n=k+1}^{\infty} \lambda_\infty^{n-k} \left[Q_{r-k}^\infty, Q_n^\infty \right] \\
&= \sum_{k=1}^{r-1} \sum_{p=0}^{k-1} \lambda_\infty^{-p} \left[Q_{r-k}^\infty, Q_{k-p}^\infty \right] + \sum_{k=0}^{r-1} \sum_{s=1}^{\infty} \lambda_\infty^{s} \left[Q_{r-k}^\infty, Q_{s+k}^\infty \right] \\
&= \sum_{p=0}^{r-1} \lambda_\infty^{-p} \sum_{k=p+1}^{r-1} \left[Q_{r-k}^\infty, Q_{k-p}^\infty \right] + \sum_{n=1}^{\infty} \lambda_\infty^{n} \sum_{k=0}^{r-1} \left[Q_{r-k}^\infty, Q_{n+k}^\infty \right]
\end{aligned}$$

Note that $\sum_{k=p+1}^{r-1} \left[Q_{r-k}^\infty, Q_{k-p}^\infty \right] = 0$ so we are left with the equations

$$\partial_{t_r^\infty} Q_n^\infty = \sum_{k=0}^{r-1} \left[Q_{r-k}^\infty, Q_{n+k}^\infty \right] .$$

We are now ready to connect with the results of [FNR83]: let us define $Q(\lambda) \coloneqq \lambda Q^\infty(\lambda^{-1})$, so that $Q_n = Q_{n+1}^\infty$, and redefine $x^n \coloneqq t_{n+1}^\infty$. Setting $x^1 = x$, we obtain the familiar

$$\partial_x Q_n = [Q_1, Q_n] + [Q_0, Q_{n+1}] , \tag{7.58}$$

which for each $n \geq 1$ we can solve recursively and obtain the AKNS hierarchy in the traditional fashion. Moreover, the generating Lax form becomes $\mu_\infty(\mu_\infty Q_0 + \mu_\infty^2(Q_1 + \lambda Q_0) + \mu_\infty^3(Q_2 + \lambda Q_1 + \lambda^2 Q_0) + \dots)$ that is

$$W^\infty(\lambda; \mu_\infty) = \mu_\infty \sum_{r=0}^{\infty} \frac{Q^{(r)}(\lambda)}{\mu^{r+1}} , \tag{7.59}$$

where $Q^{(r)}(\lambda)$ are the Lax matrices of the AKNS.

7.3.2 Extracting the AKNS multiform

The generating Lagrangian introduced in Chapter 6 is essentially the same as the one of this chapter, so we will only need to reformulate the procedure described there in terms of this more general language. Since we chose $S = \{ \infty \}$ we only need to factorise $Q(\lambda)$

and $Q(\mu)$ around infinity

$$Q^{\infty}(\lambda_\infty) = \varphi^{\infty}(\lambda_\infty) X^{\infty}(\lambda_\infty) \varphi^{\infty}(\lambda_\infty)^{-1} = \sum_{i=1}^{\infty} Q_i^{\infty} \lambda_i^{\infty} \tag{7.60}$$

where $\varphi^{\infty}(\lambda_\infty) = \sum_{k=0}^{\infty} \varphi_k^{\infty} \lambda_\infty^k$ and $\varphi^{\infty}(\lambda_\infty)^{-1} = \sum_{k=0}^{\infty} \widetilde{\varphi}_k^{\infty} \lambda_\infty^k$, and similarly with μ. The Lagrangian of Chapter 6 is obtained as follows (in terms of the old notation). We set $Q_n = Q_{n+1}^{\infty}$ and consistently

$$X^{\infty}(\lambda_\infty) = \frac{Q_0}{\lambda}\,, \quad \varphi^{\infty}(\lambda_\infty) = \varphi(\lambda)\,, \quad D_{\lambda_\infty} = \frac{D_\lambda}{\lambda}\,. \tag{7.61}$$

The kinetic term $K^{\infty\infty}(\lambda_\infty, \mu_\infty)$ then becomes, in terms of the old notation

$$\begin{aligned} K^{\infty}(\lambda_\infty, \mu_\infty) &= \operatorname{Tr}\left[\varphi^{-1}(\lambda)\left(\frac{D_\mu}{\mu}\right)\varphi(\lambda)\left(\frac{Q_0}{\lambda}\right) - \varphi^{-1}(\mu)\left(\frac{D_\lambda}{\lambda}\right)\varphi(\mu)\left(\frac{Q_0}{\mu}\right)\right] \\ &= -\sum_{m,n=0}^{\infty} \frac{K_{mn}}{\lambda^{m+1}\mu^{n+1}}\,, \end{aligned} \tag{7.62}$$

where the K_{mn}'s are the coefficient of the kinetic part of Chapter 6. The coefficients of the potential part are obtained as follows. We take the residues

$$\begin{aligned} &\operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{V^{\infty\infty}(\lambda_\infty, \mu_\infty)}{\lambda_\infty^{m+1}\mu_\infty^{n+1}} d\lambda d\mu \\ &= \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{1}{2} \operatorname{Tr}_{\mathbf{12}}(\iota_{\lambda_\infty}\iota_{\mu_\infty} + \iota_{\mu_\infty}\iota_{\lambda_\infty}) \frac{P_{\mathbf{12}} Q^{\infty}(\lambda_\infty)_{\mathbf{1}} Q^{\infty}(\mu_\infty)_{\mathbf{2}}}{(\mu-\lambda)\lambda_\infty^{m+1}\mu_\infty^{n+1}} d\lambda d\mu \\ &= -\frac{1}{2} \operatorname{Tr} \operatorname*{res}_{\lambda=\infty} \left(\frac{Q^{\infty}(\lambda_\infty)}{\lambda_\infty^{n+1}}\right)_- \frac{Q^{\infty}(\lambda_\infty)}{\lambda_\infty^{m+1}} d\lambda + \frac{1}{2} \operatorname{Tr} \operatorname*{res}_{\mu=\infty} \left(\frac{Q^{\infty}(\mu_\infty)}{\mu_\infty^{m+1}}\right)_- \frac{Q^{\infty}(\mu_\infty)}{\mu_\infty^{n+1}} d\mu \\ &= -\frac{1}{2} \operatorname{Tr} \operatorname*{res}_{\lambda=\infty} \sum_{i=1}^{n+1} Q_i^{\infty} \lambda_\infty^{i-n-1} \sum_{j=1}^{\infty} Q_j^{\infty} \lambda_\infty^{j-m-1} + \frac{1}{2} \operatorname{Tr} \operatorname*{res}_{\mu=\infty} \sum_{j=1}^{m+1} Q_j^{\infty} \mu_\infty^{j-m-1} \sum_{i=1}^{\infty} Q_i^{\infty} \mu_\infty^{i-n-1} \\ &= \frac{1}{2} \operatorname{Tr} \sum_{j=m+2}^{m+n+2} Q_{m+n-j+3}^{\infty} Q_j^{\infty} - \frac{1}{2} \operatorname{Tr} \sum_{j=n+2}^{m+n+2} Q_{m+n-j+3}^{\infty} Q_j^{\infty} \\ &= \frac{1}{2} \operatorname{Tr} \sum_{j=m+1}^{m+n+1} Q_{m+n-j+1} Q_j - \frac{1}{2} \operatorname{Tr} \sum_{j=n+1}^{m+n+1} Q_{m+n-j+1} Q_j \\ &= -\frac{1}{2} \operatorname{Tr} \sum_{j=0}^{m} Q_{m+n-j+1} Q_j + \frac{1}{2} \operatorname{Tr} \sum_{j=0}^{n} Q_{m+n-j+1} Q_j = -V_{mn}\,. \end{aligned}$$

Overall we get

$$\mathscr{L}^{\infty\infty}(\lambda_\infty, \mu_\infty) = -\sum_{m,n=0}^{\infty} \frac{K_{mn}}{\lambda^{m+1}\mu^{n+1}} + \sum_{m,n=0}^{\infty} \frac{V_{mn}}{\lambda^{m+1}\mu^{n+1}} \tag{7.63}$$

which is the generating Lagrangian of Chapter 6 up to an overall minus sign.

7.4 Zakharov-Shabat Lax pairs and Dickey's Lagrangian

7.4.1 Lax matrices and zero-curvature equations

Lemma 7.14 *Let $X(\mu) \in \mathcal{L}^{rat}(\mathfrak{g})$. Then we have*

$$\iota_{\mu_a} \frac{X(\mu)}{\mu - \lambda} = -\sum_{r=0}^{\infty} \mu_a^r \left(\frac{X^a(\lambda_a)}{\lambda_a^{r+1}} \right)_- , \qquad \forall a \in S \,. \tag{7.64}$$

Proof. Obtained by direct calculation. □

We can now find the generating Lax form explicitly for this case. Let $Q(\lambda) = (Q^a(\lambda_a))_{a \in S} \in \mathcal{L}^{\mathrm{rat}}(\mathfrak{g})$, where

$$Q^{a_i}(\lambda_{a_i}) = \sum_{k=0}^{\infty} Q_k^{a_i} (\lambda - a_i)^k \,, \quad i = 1, \ldots, N_1 \,, \tag{7.65a}$$

$$Q^{b_j}(\lambda_{b_j}) = \sum_{k=0}^{\infty} Q_k^{b_j} (\lambda - b_j)^k \,, \quad j = 1, \ldots, N_2 \,. \tag{7.65b}$$

We compute $W^{a_i}(\lambda; \mu - a_i) = \mathrm{Tr}\, \iota_{\mu_{a_i}} \frac{Q(\mu)}{\mu - \lambda}$ and we find

$$\begin{aligned}
\mathrm{Tr}\, \iota_{\mu_{a_i}} \frac{Q(\mu)}{\mu - \lambda} &= -\sum_{n_i=0}^{\infty} (\mu - a_i)^{n_i} \left(\frac{Q^{a_i}(\lambda - a_i)}{(\lambda - a_i)^{n_i+1}} \right)_- \\
&= -\sum_{n_i=0}^{\infty} (\mu - a_i) \sum_{k=0}^{n_i} \frac{Q_k^{a_i}}{(\lambda - a_i)^{n_i+1-k}} \,, \qquad i = 1, \ldots, N_1 \,, \\
\mathrm{Tr}\, \iota_{\mu_{b_j}} \frac{Q(\mu)}{\mu - \lambda} &= -\sum_{m_j=0}^{\infty} (\mu - b_j) \sum_{\ell=0}^{m_j} \frac{Q_\ell^{b_j}}{(\lambda - b_j)^{m_j+1-\ell}} \,, \qquad j = 1, \ldots, N_2 \,.
\end{aligned}$$

Therefore

$$W_{n_i}^{a_i}(\lambda) = -\sum_{k=0}^{n_i} \frac{Q_k^{a_i}}{(\lambda - a_i)^{n_i+1-k}} \,, \qquad W_{m_j}^{b_j}(\lambda) = -\sum_{\ell=0}^{m_j} \frac{Q_\ell^{b_j}}{(\lambda - b_j)^{m_j+1-\ell}} \,. \tag{7.66}$$

Now, introducing the times $t_{n_i}^{a_i}$ and $t_{m_j}^{b_j}$ we get that the zero-curvature condition in generating form is

$$
\begin{aligned}
&\partial_{t^{b_j}_{m_j}} W^{a_i}_{n_i}(\lambda) - \partial_{t^{a_i}_{n_i}} W^{b_j}_{m_j}(\lambda) + [W^{a_i}_{n_i}(\lambda), W^{b_j}_{m_j}(\lambda)] \\
&= \partial_{t^{a_i}_{n_i}} \sum_{\ell=0}^{m_j} \frac{Q^{b_j}_{\ell}}{(\lambda - b_j)^{m_j+1-\ell}} - \partial_{t^{b_j}_{m_j}} \sum_{k=0}^{n_i} \frac{Q^{a_i}_{k}}{(\lambda - a_i)^{n_i+1-k}} \\
&\qquad\qquad + \left[\sum_{k=0}^{n_i} \frac{Q^{a_i}_{k}}{(\lambda - a_i)^{n_i+1-k}}, \sum_{\ell=0}^{m_j} \frac{Q^{b_j}_{\ell}}{(\lambda - b_j)^{m_j+1-\ell}} \right] \\
&\equiv \partial_{t^{a_i}_{n_i}} V_j(\lambda) - \partial_{t^{b_j}_{m_j}} U_i(\lambda) + [U_i(\lambda), V_j(\lambda)] = 0 \,,
\end{aligned}
\tag{7.67}
$$

where we have defined

$$
U_i(\lambda) \coloneqq \sum_{k=0}^{n_i} \frac{Q^{a_i}_{k}}{(\lambda - a_i)^{n_i+1-k}} \,, \qquad V_j(\lambda) \coloneqq \sum_{\ell=0}^{m_j} \frac{Q^{b_j}_{\ell}}{(\lambda - b_j)^{m_j+1-\ell}} \,. \tag{7.68}
$$

Performing the sums $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2}$ we obtain (for given values of N_1, N_2, n_i $i = 1, \ldots, N_1$ and m_j $j = 1, \ldots, N_2$)

$$
\partial_\xi V(\lambda) - \partial_\eta U(\lambda) + [U(\lambda), V(\lambda)] = 0 \,, \tag{7.69}
$$

where

$$
U(\lambda) \coloneqq \sum_{i=1}^{N_1} U_i(\lambda) \,, \qquad V(\lambda) \coloneqq \sum_{j=1}^{N_2} V_j(\lambda) \tag{7.70}
$$

and $\partial_\xi \coloneqq \sum_{i=1}^{N_1} \partial_{t^{a_i}_{n_i}}$ and $\partial_\eta \coloneqq \sum_{j=1}^{N_2} \partial_{t^{b_j}_{m_j}}$. This corresponds to the auxiliary system of Zakharov-Shabat type studied by Dickey in [D03, Section 20.2], in the case where $U_0 = V_0 = 0$ and $a_i \neq b_j$ $\forall i, j$. The special case where $n_i = m_j = 0$ $\forall i, j$ corresponds to the ZM case [ZM80].

Remark 7.15: The case with coinciding poles $a_i = b_j$ is obtained by choosing some of the times $t^{a_i}_{n_i}$ and $t^{b_j}_{m_j}$ both in ∂_ξ and ∂_η. This, and the case with generic U_0, V_0 are still under current investigation and are objects of future research.

Remark 7.16: Equations (7.69) are the zero-curvature equations $dW(\lambda) + W(\lambda) \wedge W(\lambda) = 0$ for the Lax connection $W(\lambda) = U(\lambda)d\xi + V(\lambda)d\eta$, where we note the different sign in the definition of the curvature with respect to the rest of the thesis. This will bring a different sign in the ZM Lagrangian with respect to [ZM80] that uses the convention $F(W(\lambda)) = dW(\lambda) - W(\lambda) \wedge W(\lambda)$, that can be recovered by sending $W(\lambda) \mapsto -W(\lambda)$.

7.4.2 Extracting the Zakharov-Mikhailov Lagrangian

Our target is to obtain the Zakharov-Mikhailov (ZM) Lagrangian [ZM80] (see also Sections 4.5 and A.4) from the generating Lagrangian multiform $\mathscr{L}(\lambda,\mu)$. It will not be obtained as a coefficient a Lagrangian multiform, but a linear combination of coefficients

$$L_{ZM} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} L_{00}^{a_i b_j}\,.$$

As before we write $Q(\lambda) = (Q^a(\lambda_a))_{a\in S}$ as in (7.65), where $S = \{\, a_1,\dots,a_{N_1}, b_1,\dots,b_{N_2}\,\}$. Then, for each a_i we write

$$\begin{aligned}
&Q^{a_i}(\lambda_{a_i}) = \varphi^{a_i}(\lambda_{a_i})X^{a_i}(\lambda_{a_i})\,(\varphi^{a_i}(\lambda_{a_i}))^{-1}\,,\\
&\varphi^{a_i}(\lambda_{a_i}) = \sum_{k=0}^{\infty}\varphi_k^{a_i}(\lambda_{a_i})^k\,,\quad (\varphi^{a_i}(\lambda_{a_i}))^{-1} = \sum_{k=0}^{\infty}\widetilde{\varphi}_k^{a_i}(\lambda_{a_i})^k\,,\\
&X^{a_i}(\lambda_{a_i}) = \sum_{k=0}^{\infty} X_k^{a_i}\lambda_{a_i}^k\,,
\end{aligned} \tag{7.71}$$

and for each b_j

$$\begin{aligned}
&Q^{b_j}(\lambda_{b_j}) = \psi^{b_j}(\lambda_{b_j})X^{b_j}(\lambda_{b_j})(\varphi^{b_j}(\lambda_{b_j}))^{-1}\,,\\
&\psi^{b_j}(\lambda_{b_j}) = \sum_{k=0}^{\infty}\psi_k^{b_j}(\lambda_{b_j})^k\,,\quad (\psi^{b_j}(\lambda_{b_j}))^{-1} = \sum_{k=0}^{\infty}\widetilde{\psi}_k^{b_j}(\lambda_{b_j})^k\,,\\
&X^{b_j}(\lambda_{b_j}) = \sum_{k=0}^{\infty} X_k^{b_j}\lambda_{b_j}^k\,.
\end{aligned} \tag{7.72}$$

We remark that $\widetilde{\varphi}_0^{a_i} = (\varphi_0^{a_i})^{-1}$ and $\widetilde{\psi}_0^{b_j} = (\psi_0^{b_j})^{-1}$ for all i,j. Then, we obtain the Zakharov-Mikhailov Lagrangian by taking the following sums and residues

$$L_{ZM} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\frac{\mathscr{L}^{a_i b_j}(\lambda_{a_i},\mu_{b_j})}{\lambda_{a_i}\mu_{b_j}}\,. \tag{7.73}$$

Indeed we have

$$\begin{aligned}
&\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\frac{K^{a_ib_j}(\lambda_{a_i},\mu_{b_j})}{\lambda_{a_i}\mu_{b_j}}\\
&=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\frac{1}{\lambda_{a_i}\mu_{b_j}}\operatorname{Tr}\Bigg((\varphi^{a_i}(\lambda_{a_i}))^{-1}\sum_{k=0}^{\infty}\mu_{b_j}^k\partial_{t_k^{b_j}}\varphi^{a_i}(\lambda_{a_i})X^{a_i}(\lambda_{a_i})\\
&\qquad\qquad -(\psi^{b_j}(\mu_{b_j}))^{-1}\sum_{k=0}^{\infty}\lambda_{a_i}^k\partial_{t_k^{a_i}}\psi^{b_j}(\mu_{b_j})X^{b_j}(\mu_{b_j})\Bigg)\\
&=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname{Tr}\left((\varphi_0^{a_i})^{-1}\partial_{t_0^{b_j}}\varphi_0^{a_i}X_0^{a_i}-(\psi_0^{b_j})^{-1}\partial_{t_0^{a_i}}\psi_0^{b_j}X_0^{b_j}\right)\\
&\equiv\operatorname{Tr}\left(\sum_{i=1}^{N_1}\varphi_i^{-1}\partial_\eta\varphi_iU_i^{(0)}-\sum_{j=1}^{N_2}\psi_j^{-1}\partial_\xi\psi_jV_j^{(0)}\right)
\end{aligned}$$

and it corresponds to the kinetic part of the ZM Lagrangian under the identifications

$$\begin{aligned}
&\varphi_i\coloneqq\varphi^{a_i}\,,\qquad U_i^{(0)}\coloneqq X_0^{a_i}\,,\qquad U_i\coloneqq Q_0^{a_i}\,,\qquad \partial_\xi=\sum_{j=0}^{N_1}\partial_{t_0^{a_i}}\,,\\
&\psi_j\coloneqq\psi^{b_j}\,,\qquad V_j^{(0)}\coloneqq X_0^{b_j}\,,\qquad V_j\coloneqq Q_0^{b_j}\,,\qquad \partial_\eta=\sum_{j=0}^{N_2}\partial_{t_0^{b_j}}\,.
\end{aligned}\tag{7.74}$$

The potential part instead brings

$$\begin{aligned}
&\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\frac{V^{a_ib_j}(\lambda_{a_i},\mu_{b_j})}{\lambda_{a_i}\mu_{b_j}}\\
&=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\frac{1}{2}\operatorname{Tr}_{\mathbf{12}}(\iota_{\lambda_{a_i}}\iota_{\mu_{b_j}}+\iota_{\mu_{b_j}}\iota_{\lambda_{a_i}})\frac{P_{\mathbf{12}}Q^{a_i}(\lambda_{a_i})_{\mathbf{1}}Q^{b_j}(\mu_{b_j})_{\mathbf{2}}}{(\mu-\lambda)\lambda_{a_i}\mu_{b_j}}\\
&=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\operatorname{Tr}_{\mathbf{12}}\,\iota_{\lambda_{a_i}}\iota_{\mu_{b_j}}\frac{P_{\mathbf{12}}}{\mu-\lambda}\frac{Q^{a_i}(\lambda_{a_i})_{\mathbf{1}}}{\lambda_{a_i}}\frac{Q^{b_j}(\mu_{b_j})_{\mathbf{2}}}{\mu_{b_j}}
\end{aligned}$$

where we used the fact that $a_i\neq b_j$ and so $\iota_{\lambda_{a_i}}\iota_{\mu_{b_j}}=\iota_{\mu_{b_j}}\iota_{\lambda_{a_i}}$, and then, using $\operatorname{Tr}_{\mathbf{2}}P_{\mathbf{12}}A_{\mathbf{2}}=A_{\mathbf{1}}$ and $\iota_{\mu_{b_j}}\frac{1}{\mu-\lambda}=-\sum_{r=0}^{\infty}\mu_{b_j}^r\lambda_{b_j}^{r-1}$ we get

$$\begin{aligned}
&-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j}\iota_{\lambda_{a_i}}\operatorname{Tr}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}Q_k^{a_i}Q_\ell^{b_j}\lambda_{a_i}^{k-1}\lambda_{b_j}^{-r-1}\mu_{b_j}^{r+l-1}\\
&=-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname*{res}_{\lambda=a_i}\operatorname{Tr}\sum_{k=0}^{\infty}Q_k^{a_i}Q_0^{b_j}\lambda_{a_i}^{k-1}\lambda_{b_j}^{-1}\\
&=-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname{Tr}\frac{Q_0^{a_i}Q_0^{b_j}}{a_i-b_j}=-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\operatorname{Tr}\frac{U_iV_j}{a_i-b_j}
\end{aligned}$$

that is the ZM potential with the right sign according to the convention $F(W) = dW + W \wedge W$.

7.4.3 Extracting Dickey's Lagrangian

We now extract the more general Lagrangian of [D03, Section 20.2], which describes the zero-curvature condition $V(\lambda)_\xi - U(\lambda)_\eta + [U(\lambda), V(\lambda)] = 0$ where

$$U(\lambda) = \sum_{i=1}^{N_1} U_i(\lambda)\,, \qquad U_i(\lambda) = \sum_{r=0}^{n_i} \frac{U_r^i}{(\lambda - a_i)^{r+1}}\,, \tag{7.75a}$$

$$V(\lambda) = \sum_{j=1}^{N_2} V_j(\lambda)\,, \qquad V_j(\lambda) = \sum_{r=0}^{m_j} \frac{V_r^j}{(\lambda - b_j)^{r+1}}\,, \tag{7.75b}$$

which generalises the ZM system by allowing Lax matrices with poles of arbitrary degree. Similarly to the ZM case, it will not be obtained as a coefficient a Lagrangian multiform, but a linear combination of coefficients

$$L_D = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} L_{n_i m_j}^{a_i b_j}\,.$$

We take the following sums and residues

$$L_D = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \underset{\lambda=a_i}{\text{res}}\ \underset{\mu=b_j}{\text{res}}\ \frac{\mathscr{L}^{a_i,b_j}(\lambda_{a_i}, \mu_{b_j})}{\lambda_{a_i}^{n_i+1}\mu_{b_j}^{m_j+1}}\,, \tag{7.76}$$

proceeding similarly to the previous case (ZM): the kinetic part brings

$$\begin{aligned}
&\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \underset{\lambda=a_i}{\text{res}}\ \underset{\mu=b_j}{\text{res}}\ \frac{K^{a_i b_j}(\lambda_{a_i}^{n_i}, \mu_{b_j}^{m_j})}{\lambda_{a_i}^{n_i+1}\mu_{b_j}^{m_j+1}} \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \underset{\lambda=a_i}{\text{res}}\ \underset{\mu=b_j}{\text{res}}\ \frac{1}{\lambda_{a_i}^{n_i+1}\mu_{b_j}^{m_j+1}}\,\text{Tr}\Bigg((\varphi^{a_i}(\lambda_{a_i}))^{-1}\sum_{k=0}^{\infty}\mu_{b_j}^k \partial_{t_k^{b_j}}\varphi^{a_i}(\lambda_{a_i})X^{a_i}(\lambda_{a_i}) \\
&\qquad\qquad - (\psi^{b_j}(\mu_{b_j}))^{-1}\sum_{k=0}^{\infty}\lambda_{a_i}^k \partial_{t_k^{a_i}}\psi^{b_j}(\mu_{b_j})X^{b_j}(\mu_{b_j})\Bigg) \\
&= \sum_{i=1}^{N_1} \underset{\lambda=a_i}{\text{res}}\ \text{Tr}(\varphi^{a_i}(\lambda_{a_i}))^{-1}\sum_{j=1}^{N_2}\partial_{t_{m_j}^{b_j}}\varphi^{a_i}(\lambda_{a_i})\frac{X^{a_i}(\lambda_{a_i})}{\lambda_{a_i}^{n_j+1}} \\
&\qquad\qquad - \sum_{j=1}^{N_2}\text{Tr}\ \underset{\mu=b_j}{\text{res}}\,(\psi^{b_j}(\mu_{b_j}))^{-1}\sum_{i=1}^{N_1}\partial_{t_{n_i}^{a_i}}\psi^{b_j}(\mu_{b_j})\frac{X^{b_j}(\mu_{b_j})}{\mu_{b_j}^{m_j+1}}
\end{aligned}$$

We now call $\partial_\eta = \sum_{j=1}^{N_2} \partial_{t^{b_j}_{m_j}}$, and $\partial_\xi = \sum_{i=1}^{N_1} \partial_{t^{a_i}_{n_i}}$, and truncate $\varphi^{a_i}(\lambda_{a_i})$ and $\lambda^{-n_i-1} X^{a_i}(\lambda_{a_i})$ at n_i to obtain respectively

$$g_i(\lambda) = \sum_{r=0}^{n_i} g^i_r \lambda^r_{a_i}\,, \qquad A_i(\lambda) = \sum_{r=0}^{n_i} A^i_r \lambda^{-r-1}_{a_i} \tag{7.77}$$

and similarly $\psi^{b_j}(\lambda_{b_j})$ and $\lambda^{-m_j-1} X^{b_j}(\lambda_{b_j})$ at m_j to obtain respectively

$$h_j(\lambda) = \sum_{r=0}^{m_j} h^j_r \lambda^r_{b_j}\,, \qquad B_i(\lambda) = \sum_{r=0}^{m_j} B^i_r \lambda^{-r-1}_{b_j} \tag{7.78}$$

connecting our notation with the one of [D03]. The potential part then follows from the same identifications. In fact we have

$$\begin{aligned}
&\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j} \frac{V^{a_i b_j}(\lambda_{a_i},\mu_{b_j})}{\lambda_{a_i}^{n_i+1}\mu_{b_j}^{m_j+1}} \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j} \frac{1}{2}\operatorname{Tr}_{\mathbf{12}}(\iota_{\lambda_{a_i}}\iota_{\mu_{b_j}} + \iota_{\mu_{b_j}}\iota_{\lambda_{a_i}}) \frac{P_{\mathbf{12}} Q^{a_i}(\lambda_{a_i})_{\mathbf{1}} Q^{b_j}(\mu_{b_j})_{\mathbf{2}}}{(\mu-\lambda)\lambda_{a_i}^{n_i+1}\mu_{b_j}^{m_j+1}} \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j} \operatorname{Tr}_{\mathbf{12}}\, \iota_{\lambda_{a_i}}\iota_{\mu_{b_j}} \frac{P_{\mathbf{12}}}{\mu-\lambda} \frac{Q^{a_i}(\lambda_{a_i})_{\mathbf{1}}}{\lambda_{a_i}^{n_i+1}} \frac{Q^{b_j}(\mu_{b_j})_{\mathbf{2}}}{\mu_{b_j}^{m_j+1}} \\
&= -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \operatorname*{res}_{\lambda=a_i}\operatorname*{res}_{\mu=b_j} \iota_{\lambda_{a_i}} \operatorname{Tr} \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \left(\frac{Q^{a_i}_k}{\lambda_{a_i}^{n_i+1-k}}\right)\left(\frac{Q^{b_j}_\ell}{\mu_{b_j}^{m_j+1-\ell}}\right) \lambda_{b_j}^{-r-1}\mu_{b_j}^{r}\,.
\end{aligned}$$

We now take the residues in μ and take the pole parts in λ to obtain Dickey's potential identified as

$$= -\sum_{i=1}^{N_1}\sum_{j=2}^{N_2} \operatorname*{res}_{\lambda=a_i} (g_i A_i g_i^{-1})_-(h_j B_j h_j^{-1})_-\,. \tag{7.79}$$

7.5 sine-Gordon equation

7.5.1 Lax matrices and zero-curvature equations for the sine-Gordon case

We have the following lemma.

Lemma 7.17 *Let $X(\mu) \in \mathcal{L}(\mathfrak{g})$, with $X^0(\lambda) = \sum_{n=-N_0}^{\infty} X^0_n \lambda^n$ and $X^\infty(\lambda) = \sum_{n=-N_\infty}^{\infty} X^\infty_n \lambda^n_\infty$.*

We have

$$\iota_{\mu_0}\,\mathrm{Tr}_{\mathbf{2}}\, r_{\mathbf{12}}(\lambda,\mu_0)X^0(\mu_0)_{\mathbf{2}} = -\sum_{r=-N_0}^{\infty} \mu^r(\lambda^{-r}X^0(\lambda))_- \tag{7.80a}$$

$$\iota_{\mu_\infty}\,\mathrm{Tr}_{\mathbf{2}}\, r_{\mathbf{12}}(\lambda,\mu)X^\infty(\mu_\infty)_{\mathbf{2}} = \sum_{r=-N_\infty}^{\infty} \mu_\infty^r(\lambda_\infty^{-r}X^\infty(\lambda_\infty))_- \,. \tag{7.80b}$$

Proof. Obtained by direct calculation. □

We start with $Q(\lambda)\in\mathcal{B}(\mathfrak{g})$, where we parametrise the first few matrices of the expansion of $Q^0(\lambda)$ and $Q^\infty(\lambda_\infty)$ in the following way

$$Q_0^0 = \frac{i}{2}\begin{pmatrix} 0 & e^{iu/2} \\ 0 & 0\end{pmatrix} \in \mathfrak{b}_+\,, \qquad Q_1^0 = \frac{i}{2}\begin{pmatrix} v & e^{iu/2} \\ e^{-iu/2} & -v\end{pmatrix} \in \mathfrak{g}\,, \tag{7.81a}$$

$$Q_0^\infty = -\frac{i}{2}\begin{pmatrix} 0 & 0 \\ e^{iu/2} & 0\end{pmatrix} \in \mathfrak{b}_-\,, \qquad Q_1^\infty = -\frac{i}{2}\begin{pmatrix} -w & e^{-iu/2} \\ e^{iu/2} & w\end{pmatrix} \in \mathfrak{g}\,. \tag{7.81b}$$

The coordinate u will play the part of the sine-Gordon field, while v and w will be respectively u_ξ and u_η, ξ,η being the light-cone coordinates. We can then calculate the first terms of of the expansion of the generating Lax form. In zero we have

$$\begin{aligned} W^0(\lambda;\mu_0) &= \mathrm{Tr}_{\mathbf{2}}\,\iota_{\mu_0} r_{\mathbf{1}2}(\lambda,\mu_0)Q^0(\mu)_{\mathbf{2}} \\ &= -\sum_{r=0}^{\infty}\mu_0^r(\lambda^{-r}Q^0(\lambda))_- \\ &= -(P^- + \tfrac{1}{2}P^0)Q_0^0 - \mu(\lambda^{-1}Q_0^0 - (P^- + \tfrac{1}{2}P^0)Q_1^0) + \dots \end{aligned}$$

We then get $W_0^0(\lambda)=0$ and

$$U(\lambda) := W_1^0(\lambda) = -\frac{i}{4}\begin{pmatrix} v & \frac{2}{\lambda}e^{iu/2} \\ 2e^{-iu/2} & -v\end{pmatrix}\,. \tag{7.82}$$

At infinity on the other hand

$$\begin{aligned} W^\infty(\lambda;\mu_\infty) &= \mathrm{Tr}_{\mathbf{2}}\,\iota_{\mu_\infty} r_{\mathbf{12}}(\lambda,\mu_\infty)Q^\infty(\mu)_{\mathbf{2}} \\ &= \sum_{r=0}^{\infty}\mu_\infty^r(\lambda_\infty^{-r}Q^\infty(\lambda_\infty))_- \\ &= (P^+ + \tfrac{1}{2}P^0)Q_0^\infty + \mu(\lambda Q_0^\infty + (P^- + \tfrac{1}{2}P^0)Q_1^\infty + \dots \end{aligned}$$

and so $W_0^\infty(\lambda)=0$ and

$$V(\lambda) := W_1^\infty(\lambda) = -\frac{i}{4}\begin{pmatrix} -w & 2e^{-iu/2} \\ 2\lambda e^{iu/2} & w\end{pmatrix}\,. \tag{7.83}$$

$U(\lambda), V(\lambda)$ are, under the identification $v = u_\xi$ and $w = u_\eta$, precisely the Lax matrices (4.21) for the sine-Gordon equation in light-cone coordinates.

We now consider the zero-curvature equations for these coefficients, *i.e.*

$$\begin{aligned}&\partial_{t_1^\infty} W_1^0(\lambda) - \partial_{t_1^0} W_1^\infty(\lambda) + [W_1^a(\lambda), W_1^b(\lambda)] \\ &\equiv \partial_\eta U(\lambda) - \partial_\xi V(\lambda) + [U(\lambda), V(\lambda)] = 0\end{aligned} \tag{7.84}$$

under the identifications $\partial_\xi := \partial_{t_1^0}$ and $\partial_\eta := \partial_{t_1^\infty}$. This corresponds to the zero-curvature equation $dW(\lambda) = W(\lambda) \wedge W(\lambda)$ for the Lax form $W(\lambda) = U(\lambda)\, d\xi + V(\lambda)\, d\eta$ as calculated in Section 4.2, the only difference being that now this is equivalent to the system

$$\begin{cases} u_\eta = w \\ u_\xi = v \\ v_\eta + w_\xi + 2 \sin u = 0 \end{cases} \tag{7.85}$$

that implies $u_{\eta\xi} + \sin u = 0$.

7.5.2 Extracting the sine-Gordon Lagrangian

We parametrise $Q^0(\lambda) = \varphi^0(\lambda) X^0(\lambda) (\varphi^0(\lambda))^{-1}$ and therefore

$$(\varphi_0^0 + \varphi_1^0 \lambda + \dots) X^0(\lambda) (\widetilde{\varphi}_0^0 + \widetilde{\varphi}_1^0 \lambda + \dots) = Q_0^0 + Q_1^0 \lambda + \dots \tag{7.86}$$

where $\widetilde{\varphi}_0^0 = (\varphi_0^0)^{-1}$ and $\widetilde{\varphi}_1^0 = -\widetilde{\varphi}_0^0 \varphi_1^0 \widetilde{\varphi}_0^0$. We choose

$$X^0(\lambda) = \frac{i}{2}(\sigma_+ + \lambda \sigma_-) \in \mathfrak{b}_+ \oplus \mathfrak{g} \otimes \lambda \mathbb{C}[\![\lambda]\!]\,, \tag{7.87}$$

and we obtain, by setting $\varphi_0^0 (\frac{i}{2}\sigma_+) \widetilde{\varphi}_0^0 = Q_0^0$

$$\varphi_0^0 = \begin{pmatrix} e^{iu/4} & k \\ 0 & e^{-iu/4} \end{pmatrix}, \qquad \widetilde{\varphi}_0^0 = \begin{pmatrix} e^{-iu/4} & -k \\ 0 & e^{iu/4} \end{pmatrix} \tag{7.88}$$

for any k smooth function of the field u and its derivatives. At this stage k is arbitrary. Let us consider the next step, and set

$$\varphi_1^0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \widetilde{\varphi}_1^0 = \begin{pmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{c} & \widetilde{d} \end{pmatrix} = -\widetilde{\varphi}_0^0\, \varphi_1^0\, \widetilde{\varphi}_0^0\,. \tag{7.89}$$

We can partially fix the undetermined parameters using the following conditions.

1. $\det\varphi^0(\lambda)=1$:

$$\det\begin{pmatrix} e^{iu/4}+\lambda a+\dots & k+\lambda b+\dots \\ \lambda c+\dots & e^{-iu/4}+\lambda d+\dots \end{pmatrix}$$
$$=1+\lambda(de^{iu/4}+ae^{-iu/4}+ck)+O(\lambda^2)$$

   which implies $ck+de^{iu/4}+ae^{-iu/4}=0$.

2. $\varphi_0^0\sigma_-\widetilde{\varphi}_0^0+\varphi_1^0\sigma_+\widetilde{\varphi}_0^0+\varphi_0^0\sigma_+\widetilde{\varphi}_1^0=\left(\begin{smallmatrix} v & e^{iu/2} \\ e^{-iu/2} & -v\end{smallmatrix}\right)$:

$$\begin{pmatrix} ke^{-iu/4} & -k^2 \\ e^{-iu/2} & -ke^{-iu/4}\end{pmatrix}+\begin{pmatrix} 0 & ae^{iu/4} \\ 0 & ce^{iu/4}\end{pmatrix}+\begin{pmatrix} \widetilde{c}e^{iu/4} & \widetilde{d}e^{iu/4} \\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix} ke^{-iu/4}+\widetilde{c}e^{iu/4} & -k^2+ae^{iu/4}+\widetilde{d}e^{iu/4} \\ e^{-iu/2} & -ke^{-iu/4}+ce^{iu/4}\end{pmatrix}=\begin{pmatrix} u_0 & e^{iu/2} \\ e^{-iu/2} & -u_0\end{pmatrix},$$

   which implies the system

$$\begin{cases} ke^{-iu/4}+\widetilde{c}e^{iu/4}=v \\ -ke^{-iu/4}+ce^{iu/4}=-v \\ -k^2+ae^{iu/4}+\widetilde{d}e^{iu/4}=e^{iu/2} \end{cases}.$$

3. Finally, $\widetilde{\varphi}_1^0=-\widetilde{\varphi}_0^0\,\varphi_1^0\,\widetilde{\varphi}_0^0$:

$$\begin{cases} \widetilde{a}=kce^{-iu/4}-a^{-iu/2} \\ \widetilde{b}=k(ae^{-iu/4}+de^{iu/4}-kc)-b \\ \widetilde{c}=-c \\ \widetilde{d}=kce^{iu/4}-de^{iu/2} \end{cases}$$

   .

This would not lead to a complete determination: the first and second equation of point 2 imply the third equation of point 3, and therefore we have (at most) only seven equations for nine parameters. However, all we need to do to obtain the desired Lagrangian is to set

$$k=\frac{v}{2}e^{iu/4}$$

and as a consequence

$$c=-\frac{v}{2}e^{-iu/4}=-\widetilde{c}.$$

We fix all the other parameters but b consequently. We will see that we do not need the explicit expressions.

In a similar way we can construct the parametrisation around infinity $Q^\infty(\mu_\infty)=$

$\varphi^\infty(\mu_\infty)X^\infty(\mu_\infty)(\varphi^\infty(\mu_\infty))^{-1}$. We choose

$$X^\infty(\mu) = -\frac{i}{2}(\sigma_- + \frac{\sigma_+}{\mu}) \in \mathfrak{b}_- \oplus \mathfrak{g} \otimes \lambda^{-1}\mathbb{C}[\![\lambda^{-1}]\!]\,, \tag{7.90}$$

and obtain

$$\varphi_0^\infty = \begin{pmatrix} e^{-iu/4} & 0 \\ m & e^{iu/4} \end{pmatrix}, \qquad \widetilde{\varphi}_0^\infty = \begin{pmatrix} e^{iu/4} & 0 \\ -m & e^{-iu/4} \end{pmatrix} \tag{7.91}$$

$$\varphi_1^\infty = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad \widetilde{\varphi}_1^\infty = \begin{pmatrix} \widetilde{p} & \widetilde{q} \\ \widetilde{r} & \widetilde{s} \end{pmatrix}. \tag{7.92}$$

Similarly as before, we set

$$m = \frac{w}{2}e^{iu/4}\,, \qquad q = -\frac{w}{2}e^{-iu/4}\,,$$

and we leave r undetermined.

We can now obtain a Lagrangian for the sine-Gordon equation by taking the residues

$$L_{sG} := \operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty} \frac{\mathscr{L}^{0\infty}(\lambda_0,\mu_\infty)}{\lambda_0\mu_\infty}\frac{d\lambda}{\lambda}\frac{d\mu}{\mu} = \operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty} \frac{1}{\lambda^2}\mathscr{L}^{0\infty}(\lambda_0,\mu_\infty)d\lambda d\mu\,. \tag{7.93}$$

We start by calculating the kinetic part. We note that as we chose the pairing with $k=-1$, we take $D_{\mu_\infty} = \sum_{k=0}^\infty \mu_\infty^k \partial_{t_k^\infty}$. The first term brings

$$\begin{aligned} &\operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty} \frac{1}{\lambda^2}\operatorname{Tr}(\widetilde{\varphi}_0^0 + \widetilde{\varphi}_1^0\lambda + \dots)\sum_{k=0}^\infty \frac{1}{\mu^k}\partial_{t_k^\infty}(\varphi_0^0 + \varphi_1^0\lambda + \dots)X^0(\lambda) \\ &= -\frac{i}{2}\operatorname{Tr}(\widetilde{\varphi}_0^0\partial_\eta\varphi_0^0\sigma_- + \widetilde{\varphi}_0^0\partial_\eta\varphi_1^0\sigma_+ + \widetilde{\varphi}_1^0\partial_\eta\varphi_0^0\sigma_+) = \frac{u_\xi u_\eta}{4} \end{aligned} \tag{7.94}$$

where we have already imposed $v = u_\xi$. The other kinetic term brings, imposing $w = u_\eta$,

$$\begin{aligned} &\operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty} \frac{1}{\lambda^2}\operatorname{Tr}(\widetilde{\varphi}_0^\infty + \frac{\widetilde{\varphi}_1^\infty}{\mu} + \dots)\sum_{k=0}^\infty \lambda^k\partial_{t_k^0}(\varphi_0^\infty + \frac{\varphi_1^\infty}{\mu} + \dots)X^\infty(\mu) \\ &= \frac{i}{2}\operatorname{Tr}(\widetilde{\varphi}_0^\infty\partial_\xi\varphi_0^\infty\sigma_+ + \widetilde{\varphi}_0^\infty\partial_\xi\varphi_1^\infty\sigma_- + \widetilde{\varphi}_1^\infty\partial_\xi\varphi_0^\infty\sigma_-) = -\frac{u_\eta u_\xi}{4}\,. \end{aligned} \tag{7.95}$$

We now treat the potential part, starting from the contribution of the term $\rho^+_{\mathbf{12}} - \rho^-_{\mathbf{12}}$ of the r-matrix. We have

$$\begin{aligned} &\frac{1}{2}\operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty} \frac{1}{\lambda^2}\operatorname{Tr}_{12}(\rho^+_{\mathbf{12}} - \rho^-_{\mathbf{12}})Q^0_{\mathbf{1}}(\lambda)Q^\infty_{\mathbf{2}}(\mu_\infty) \\ &= -\frac{1}{2}\operatorname{Tr}_{\mathbf{12}}(\sigma_+Q_1^0)\otimes(\sigma_-Q_1^\infty) + \operatorname{Tr}_{\mathbf{12}}(\sigma_-Q_1^0)\otimes(\sigma_+Q_1^\infty) = -\frac{e^{-iu}}{8} + \frac{e^{iu}}{8}\,. \end{aligned} \tag{7.96}$$

The other contribution comes from

$$\begin{aligned}
&\frac{1}{2}\operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty}\frac{1}{\lambda^2}\operatorname{Tr}_{\mathbf{12}}\iota_\lambda\iota_\mu\frac{\mu+\lambda}{\mu-\lambda}P_{\mathbf{12}}Q^0_{\mathbf{1}}(\lambda)Q^\infty_{\mathbf{2}}(\mu_\infty)\\
&=\frac{1}{2}\operatorname*{res}_{\lambda=0}\operatorname*{res}_{\mu=\infty}\frac{1}{\lambda^2}\operatorname{Tr}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{\mu^k}+\sum_{k=1}^{\infty}\frac{\lambda^k}{\mu^k}\right)\sum_{i=0}^{\infty}Q^0_i\lambda^i\sum_{j=0}^{\infty}\frac{Q^\infty_j}{\mu^j}\\
&=-\operatorname{Tr}(Q^0_0Q^\infty_0+\frac{1}{2}Q^0_1Q^\infty_1)=\frac{u_\eta u_\xi}{4}-\frac{3e^{iu}}{8}-\frac{e^{-iu}}{8}\,.
\end{aligned}\tag{7.97}$$

Adding everything together we get

$$L_{sG}=\frac{u_\eta u_\xi}{4}+\frac{1}{2}\cos u \tag{7.98}$$

which is indeed the Lagrangian for the sine-Gordon equation in light-cone coordinates.

Remark 7.18: The Lagrangians overall multiplicative factor 2 between the two trigonometric r-matrices of this and Chapter 4, which creates (or is explained by, which at this stage is really up to interpretation), the factor $\frac{1}{2}$ between the Lagrangians (4.22) and (7.98).

Chapter 8

Conclusions and perspectives

> Few can foresee whither their road will lead them, till they come to its end.
>
> J. R. R. Tolkien

This thesis constitutes progress towards the understanding of the role played by multi-dimensional consistency and the application of covariant Hamiltonian field theory to integrable systems. We have developed a covariant formulation of integrable field theories in $1+1$ dimensions, and its generalisation to integrable hierarchies, called Hamiltonian multiforms, providing multiple examples. Moreover, we have proved that our formulation reproduces and generalises the classical r-matrix structure of the Poisson bracket via the formula

$$\{[W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]\,,$$

where $W(\lambda)$ is the Lax connection of the integrable field theory. Using the r-matrix structure and the classical Yang-Baxter equation we have also developed a technique that generates Lagrangian multiforms for several integrable hierarchies from a common object.

These results point to some interesting open questions, that will be object of future research.

A covariant $H = \operatorname{Tr} L^2$ formula Firstly, we remark that thanks to our formalism one can write the covariant equivalent for the famous $H = \operatorname{Tr} L^2$ relation that holds in classical finite-dimensional mechanics between the Hamiltonian function and the trace of the square of the Lax matrix. In fact, as it was first noticed in [CSV21a], we have that the covariant Hamiltonian for the ZM action can be written as

$$H_{ZM} = \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \operatorname*{res}_{\lambda=a_m} \operatorname*{res}_{w=b_n} \operatorname{Tr} \frac{W(\lambda) \wedge W(\mu)}{\lambda - \mu}\,.$$

In fact, we have that

$$
\begin{aligned}
&\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \operatorname*{res}_{\lambda=a_m} \operatorname*{res}_{w=b_n} \operatorname{Tr} \frac{W(\lambda)\wedge W(\mu)}{\lambda-\mu} \\
&= \sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \operatorname*{res}_{\lambda=a_m} \operatorname*{res}_{w=b_n} \frac{1}{\lambda-\mu} \operatorname{Tr} \left(\sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} \frac{U_k}{\lambda-a_k}\frac{V_\ell}{\mu-b_\ell} - \frac{V_\ell}{\lambda-b_\ell}\frac{U_k}{\mu-a_k} \right) d\xi\wedge d\eta \\
&= \sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \operatorname{Tr} \frac{U_m V_n}{a_m-b_n} d\xi\wedge d\eta = H_{ZM}\,.
\end{aligned}
$$

Surprisingly, a similar relation holds for the AKNS hierarchy as well, and we have that the AKNS Hamiltonian multiform can be obtained as

$$
\begin{aligned}
\mathcal{H} &= \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{1}{2} \frac{\operatorname{Tr} W(\lambda)\wedge W(\mu)}{\lambda-\mu} \\
&= \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{1}{2} \frac{1}{\lambda-\mu} \operatorname{Tr}\big(\sum_{m\geq 0} Q^{(m)}(\lambda)\,dx^m\big)\wedge\big(\sum_{n\geq 0} Q^{(n)}(\mu)\,dx^n\big) \\
&= \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{1}{2} \frac{1}{\lambda-\mu} \operatorname{Tr} \sum_{m<n} (Q^{(m)}(\lambda)Q^{(n)}(\mu) - Q^{(n)}(\lambda)Q^{(m)}(\mu)) dx^{mn} \\
&= \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{1}{2} \frac{1}{\lambda-\mu} \operatorname{Tr} \sum_{m<n}\sum_{i=0}^{m}\sum_{j=0}^{n} (\lambda^{m-i}\mu^{n-j}Q_iQ_j - \lambda^{n-j}\mu^{m-i}Q_jQ_i) dx^{mn} \\
&= \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \frac{1}{4} \left(\sum_{p\geq 0} \frac{\mu^p}{\lambda^{p+1}} - \sum_{p\geq 0}\frac{\lambda^p}{\mu^{p+1}} \right) \sum_{m<n}\sum_{i=0}^{m}\sum_{j=0}^{n} \operatorname{Tr}(\lambda^{m-i}\mu^{n-j}Q_iQ_j - \lambda^{n-j}\mu^{m-i}Q_jQ_i) dx^{mn} \\
&= \frac{1}{4} \operatorname*{res}_{\lambda=\infty} \operatorname*{res}_{\mu=\infty} \sum_{m<n}\sum_{p\geq 0}\sum_{i=0}^{m}\sum_{j=0}^{n} \operatorname{Tr}\Big(\frac{Q_iQ_j}{\lambda^{i+p+1-m}\mu^{j-n-p}} - \frac{Q_jQ_i}{\lambda^{j+p+1-n}\mu^{i-m-p}} \\
&\qquad\qquad - \frac{Q_iQ_j}{\lambda^{i-m-p}\mu^{j+p+1-n}} + \frac{Q_jQ_i}{\lambda^{j-n-p}\mu^{i+p+1-m}} \Big)\,.
\end{aligned}
$$

We apply the residue:

$$
\begin{aligned}
&= \frac{1}{4}\sum_{m<n} \left(\sum_{p=0}^{m} \operatorname{Tr}(Q_{m-p}Q_{p+n+1} + Q_{p+n+1}Q_{m-p}) - \sum_{p=0}^{n} \operatorname{Tr}(Q_{n-p}Q_{p+m+1} + Q_{p+m+1}Q_{n-p}) \right) dx^{mn} \\
&= \sum_{m<n} \left(\frac{1}{2}\sum_{k=0}^{m} Q_kQ_{m+n+1-k} - \frac{1}{2}\sum_{k=0}^{n} Q_kQ_{m+n+1-k} \right) dx^{mn} \\
&= \sum_{m<n} H_{mn}\,dx^{mn}
\end{aligned}
$$

as desired. This is definitely aesthetically pleasing, and it is interesting to see whether it holds for other hierarchies, possibly with other r-matrix structures besides the rational one, and its consequences on the integrable properties of the system. The conjecture is that the Hamiltonian multiform can be repackaged in a generating series (similarly to the

AKNS case in Chapter 6), in a formula such as

$$\mathcal{H}(\lambda,\mu)=\mathrm{Tr}_{\mathbf{12}}\, r_{\mathbf{12}}(\lambda,\mu)W_{\mathbf{1}}(\lambda)\wedge W_{\mathbf{2}}(\mu)\,.$$

Generating Hamiltonian multiform and multi-time Poisson brackets In the spirit of Chapter 7, one would like to define a generating Hamiltonian multiform, which we suspect could take the form

$$\mathcal{H}(\lambda,\mu)=\frac{1}{2}\,\mathrm{Tr}_{\mathbf{12}}(\iota_\lambda\iota_\mu+\iota_\mu\iota_\lambda)r_{\mathbf{12}}(\lambda,\mu)Q(\lambda)_{\mathbf{1}}Q(\mu)_{\mathbf{2}}\,,$$

and a Poisson bracket between the generating Lax forms

$$\{W(\lambda;\nu)_{\mathbf{1}},W(\mu;\sigma)_{\mathbf{2}}\}$$

as a double Laurent series in ν and σ, where λ and μ play the role of the spectral parameters. We would want this definition to reproduce $\{\![W(\lambda)_{\mathbf{1}},W(\mu)_{\mathbf{2}}]\!\}=[r_{\mathbf{12}}(\lambda,\mu),W(\lambda)_{\mathbf{1}}+W(\mu)_{\mathbf{2}}]$ and to be consistent with an equation as $dW(\lambda)=\{\![\mathcal{H},W(\lambda)]\!\}$. This is currently still under investigation.

Covariant (quantum) integrable systems The Hamiltonian multiform description of the Ablowitz-Kaup-Newell-Segur hierarchy cf. Chapter 6 has proved efficient in obtaining the infinite series of conservation laws, which are identified in a 1-form $A=\sum_k A_k dx^k$, with $dA=0$, and characterised by the familiar-looking requirement

$$dA=0\qquad\Longleftrightarrow\qquad \{\![H_{ij},A]\!\}=0\quad\forall i,j$$

where $\mathcal{H}=\sum_{ij}H_{ij}dx^{ij}$ is the Hamiltonian multiform and $\{\![\ ,\ ]\!\}$ are the multi-time Poisson brackets of the hierarchy. This points to a description of the hierarchy and its conservation laws that is more similar to the traditional approach to finite dimensional systems than to field theories, and opens a series of questions. Firstly, the conservation laws are obtained without resorting to the monodromy matrix [S82], and apparently without involving the r-matrix structure at the group level, which is the starting point of the traditional and well-known (quantum) Inverse Scattering Method. This is definitely remarkable, but it leaves us to understand if this is really the case, and if so, why.
Then, for 1-dimensional multiforms (*i.e.* hierarchies of ODEs), recent results ([V20] and Section 5.6) has linked the closure relation of the Lagrangian multiform $d\mathscr{L}=0$ to the involution of the Hamiltonians. It would be interesting to understand if we can relate the closure relation to the mutual involution of the single-time Hamiltonians in the case of field theories.
Moreover, we only managed to work with ultra-local field theories (*i.e.* where the classical r-matrix is skew-symmetric), since the non ultra-local theories that we tried to study are expressed by a Lax connection that is not admissible cf. Section 4.6. These non-ultralocal

field theories are extremely important to treat, as they include famous key systems such as the celebrated potential Korteweg-de Vries equation. This problem of extending the covariant Poisson brackets to non-Hamiltonian forms has been addressed in the literature (see for instance [FS15]), but not in relation to Integrable Systems.

This line of research is going towards the introduction of a notion of 'covariant integrability', which would relate covariant Hamiltonian field theory, multisymplectic geometry, Lagrangian multiforms and classical r-matrix theory. This offers the hope of carrying out a program of covariant canonical quantisation for integrable field theories, thus realizing the initial hope behind the effort of the Polish School for instance [K73] and attempted e.g. by Kanatchikov [K01]. We wish to remark that this thesis (and the works [CS20a, CS20b, CS21] and partly [CSV21a]) belongs to a programme whose overarching goal is a new approach to canonical covariant quantisation of an integrable system, and builds an important step towards this objective. We believe that the classical r-matrix structure within the covariant (and multi-time) Poisson bracket can provide a new outlook on how to perform this canonical quantisation in a covariant fashion.

Appendix A

Miscellanea

A.1 Matrix algebras

In this section we shall use Einstein's notation on repeated indices. Let $GL(2, \mathbb{C})$ the general linear group of invertible 2×2 (*i.e.* with non-zero determinant) matrices over $\mathbb{C}$. Its corresponding Lie algebra is the algebra of 2×2 matrices $M_{2\times 2} = \mathfrak{gl}_2$ over $\mathbb{C}$ with the usual commutator

$$A, B \in M_{2\times 2}\,, \qquad [A, B] \coloneqq AB - BA\,. \tag{A.1}$$

We use as a basis of the $\mathfrak{gl}_2$ algebra the set $\{\, E_{ij},\ i, j = 1, 2 \,\}$ where each E_{ij} is defined as the 2×2 matrix with the only non-zero entries being at the place (i, j), *i.e.*

$$(E_{ij})_{mn} = \delta_{im}\delta_{jn}\,. \tag{A.2}$$

They act on the canonical basis of $\mathbb{C}^2$ $\{\, e_i \,\}$ as $E_{ij}e_k = \delta_{jk}e_i$, and they have the following multiplication rule:

$$E_{ij}E_{mn} = \delta_{jm}E_{in}\,. \tag{A.3}$$

Let now $SL(2, \mathbb{C}) \subset GL(2, \mathbb{C})$ be the special linear group of invertible 2×2 matrices over $\mathbb{C}$ with determinant 1. It is a well-known fact that its corresponding Lie algebra is the one of traceless complex 2×2 matrices $s\ell(2, \mathbb{C})$

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in s\ell(2, \mathbb{C})\,, \qquad a, b, c \in \mathbb{C}\,, \tag{A.4}$$

with the Lie bracket given by the usual commutator. It is then easy to see that a matrix $A \in s\ell(2, \mathbb{C})$ can be expressed as

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{c-b}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.5}$$

The matrices

$$\sigma_1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{A.6}$$

are the famous *Pauli matrices*, and form a basis for $s\ell(2,\mathbb{C})$. They satisfy the following properties:

- $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{I}_2$.
- $\det \sigma_m = -1$ and $\operatorname{Tr} \sigma_m = 0$.
- The $s\ell(2,\mathbb{C})$ algebra rules

$$[\sigma_\ell, \sigma_m] = 2i\varepsilon_{\ell mn}\sigma_n \tag{A.7}$$

  where $\varepsilon_{\ell mn}$ is the Levi-Civita symbol.

We often prefer a different choice of basis, *i.e.*

$$\sigma_+ \coloneqq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad \sigma_- \coloneqq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_3 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.8}$$

These matrices satisfy the following properties:

- $\sigma_+^2 = \sigma_-^2 = 0$, $\sigma_3 = \mathbb{I}_2$.
- $\det \sigma_+ = \det \sigma_- = 0$, $\det \sigma_3 = -1$, and $\operatorname{Tr} \sigma_+ = \operatorname{Tr} \sigma_- = \operatorname{Tr} \sigma_3 = 0$.
- The $s\ell(2,\mathbb{C})$ algebra rules

$$[\sigma_+, \sigma_-] = \sigma_3\,, \quad [\sigma_3, \sigma_+] = 2\sigma_+\,, \quad [\sigma_3, \sigma_-] = -2\sigma_-\,. \tag{A.9}$$

Using the basis $\{\, \sigma_+,\, \sigma_-,\, \sigma_3 \,\}$ we can quickly write the commutator between two $s\ell(2,\mathbb{C})$ matrices $Q_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{smallmatrix}\right)$ and $Q_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{smallmatrix}\right)$

$$\begin{aligned} [Q_1, Q_2] =& a_1 b_2 [\sigma_3, \sigma_+] + a_1 c_2 [\sigma_3, \sigma_-] + b_1 a_2 [\sigma_+, \sigma_3] \\ & + b_1 c_2 [\sigma_+, \sigma_-] + c_1 a_2 [\sigma_-, \sigma_3] + c_1 b_2 [\sigma_-, \sigma_+] \\ =& 2a_1 b_2 \sigma_+ - 2a_1 c_2 \sigma_- - 2b_1 a_2 \sigma_+ + b_1 c_2 \sigma_3 + 2c_1 a_2 \sigma_- - c_1 b_2 \sigma_3 \\ =& \begin{pmatrix} b_1 c_2 - c_1 b_2 & 2(a_1 b_2 - b_1 a_2) \\ 2(c_1 a_2 - a_1 c_2) & c_1 b_2 - b_1 c_2 \end{pmatrix}\,. \end{aligned} \tag{A.10}$$

A.2 Auxiliary spaces notation and classical r-matrix

Let $\mathfrak{g} = \mathfrak{gl}_N$ with basis $\{\, E_{ij} \,\}$, so we can write its elements as $A = \sum_{ij} a_{ij} E_{ij}$. We shall the Lie algebra $\mathfrak{g} \otimes \mathfrak{g}$ with basis $\{\, E_{ij} \otimes E_{k\ell} \,\}$, with bracket extended from the one of $\mathfrak{g}$

(*i.e.* the usual commutator). If we consider $\mathfrak{g} = \mathfrak{gl}_2$, and in coordinates we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\,, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\,, \tag{A.11}$$

then we can view their tensor product as a 4×4 matrix, as

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \end{aligned} \tag{A.12}$$

In the same way, the tensor product of two $\mathbb{C}^2$ vectors $u = \sum_i u_i e_i$ and $v = \sum_k v_j e_j$ can be seen as a $\mathbb{C}^4$ vector as

$$u \otimes v = \begin{pmatrix} u_1 v \\ u_2 v \end{pmatrix} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \end{pmatrix}\,. \tag{A.13}$$

For each $A \in \mathfrak{g}$ define $A_{\mathbf{1}} = A \otimes \mathbb{I}$ and $A_{\mathbf{2}} = \mathbb{I} \otimes A$ where $\mathbb{I}$ is the 2×2 identity matrix. Note that for instance $A \otimes B = A_{\mathbf{1}} B_{\mathbf{2}}$. Matrices related to different auxiliary spaces commute: $A \otimes B = A_{\mathbf{1}} B_{\mathbf{2}} = B_{\mathbf{2}} A_{\mathbf{1}}$. We remark the well-known fact that not every element $C_{\mathbf{12}}$ of $\mathfrak{g} \otimes \mathfrak{g}$ can be written as $A \otimes B$, with some $A, B \in \mathfrak{g}$, but it is more generally $C_{\mathbf{12}} = \sum_{ij} c_{ij,k\ell} E_{ij} \otimes E_{k\ell}$.

The following definition of Sklyanin Poisson bracket [S82] is crucial in this thesis, since it allows us to identify classical r-matrix structures within various Poisson brackets.

Definition A.1 (Sklyanin Poisson bracket) *Given a Poisson bracket* $\{\ ,\ \}\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *we define the* Sklyanin Poisson bracket *[S82] between two matrices* $A = \sum_{ij} A_{ij} E_{ij}$ *and* $B = \sum_{k\ell} B_{k\ell} E_{k\ell}$ *as*

$$\{A_{\mathbf{1}}, B_{\mathbf{2}}\} := \sum_{ij,kl} \{A_{ij}, B_{kl}\} E_{ij} \otimes E_{kl} \in \mathfrak{g} \otimes \mathfrak{g}\,. \tag{A.14}$$

In other words, the Sklyanin Poisson bracket of two elements of $\mathfrak{g}$ allows us to calculate Poisson brackets of the different coefficients of these elements with respect to a given basis, casting them into an element of $\mathfrak{g} \otimes \mathfrak{g}$. Of course, if we are computing Sklyanin Poisson bracket of elements of $s\ell(2, \mathbb{C})$ and are using the basis $\{\,\sigma_3,\, \sigma_+,\, \sigma_-\,\}$ as

$$A = a_3\sigma_3 + a_+\sigma_+ + a_-\sigma_-\,, \qquad B = b_3\sigma_3 + b_+\sigma_+ + b_-\sigma_-\,, \tag{A.15}$$

we can compute it as

$$\{A_{\mathbf{1}}, B_{\mathbf{2}}\} = \sum_{i,j=3,+,-} \{a_i, b_j\}\, \sigma_i \otimes \sigma_i\,. \tag{A.16}$$

$P_{\mathbf{12}}$ is the so-called *permutation operator* on $\mathbb{C}^2 \otimes \mathbb{C}^2$: $P_{\mathbf{12}} u \otimes v = v \otimes u$. For $\mathfrak{gl}_N$ this can be written as $P_{\mathbf{12}} = \sum_{ij} E_{ij} \otimes E_{ji}$: in fact, we have that

$$\begin{aligned} P_{\mathbf{12}} u \otimes v =& (\sum_{ij} E_{ij} \otimes E_{ji})(\sum_{k\ell} u_k v_l\, e_k \otimes e_\ell) = \sum_{ij,k\ell} (u_k E_{ij} e_k) \otimes (v_\ell E_{ji} e_\ell) \\ =& \sum_{ij,k\ell} (u_k \delta_{jk} e_i) \otimes (v_\ell \delta_{i\ell} e_j) = \sum_{ij} (v_i e_i) \otimes (u_j e_j) = v \otimes u\,. \end{aligned}$$

In the case of $\mathfrak{gl}_2$ we have

$$\begin{aligned} P_{\mathbf{12}} =& E_{11} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + E_{22} \otimes E_{22} \\ =& \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ =& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{A.17}$$

For $\mathfrak{g} = s\ell(2, \mathbb{C})$ we have that the permutation operator can be written as $P_{\mathbf{12}} = (\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ + \frac{\mathbb{I} \otimes \mathbb{I}}{2} + \frac{\sigma_3 \otimes \sigma_3}{2})$.

Proposition A.2 *The permutation operator $P_{\mathbf{12}} = \sum_{ij} E_{ij} \otimes E_{ji}$ satisfies the following properties:*

1. $P_{\mathbf{12}}^2 = \mathrm{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2}$.
2. $P_{\mathbf{12}} A_{\mathbf{1}} B_{\mathbf{2}} P_{\mathbf{12}} = A_{\mathbf{2}} B_{\mathbf{1}}$*, where* $A, B \in \mathfrak{gl}_2$*. As a consequence we have that* $P_{\mathbf{12}} A_{\mathbf{1}} = A_{\mathbf{2}} P_{\mathbf{12}}$ *and* $P_{\mathbf{12}} B_{\mathbf{2}} = B_{\mathbf{1}} P_{\mathbf{12}}$*, and then*

$$[P_{\mathbf{12}}, A_{\mathbf{1}}] = -[P_{\mathbf{12}}, A_{\mathbf{2}}]\,. \tag{A.18}$$

Proof. 1. Directly from the definition, or

$$\begin{aligned} P_{\mathbf{12}} P_{\mathbf{12}} =& \sum_{ij} (E_{ij} \otimes E_{ji}) \sum_{mn} (E_{mn} \otimes E_{nm}) = \sum_{ijmn} E_{ij} E_{mn} \otimes E_{ji} E_{nm} \\ =& \sum_{ijmn} \delta_{jm} \delta_{in} E_{in} \otimes E_{jm} = \sum_{ij} E_{ii} \otimes E_{jj} = \mathbb{I} \otimes \mathbb{I} = \mathrm{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2}\,. \end{aligned}$$

2. We have that

$$P_{\mathbf{12}} A_{\mathbf{1}} B_{\mathbf{2}} P_{\mathbf{12}} = \sum_{\substack{ijmn \\ k\ell pq}} (E_{ij} \otimes E_{ji})(a_{mn} E_{mn} \otimes \mathbb{I})(b_{k\ell} \mathbb{I} \otimes E_{k\ell})(E_{pq} \otimes E_{qp})$$
$$= \sum_{\substack{ijmn \\ k\ell pq}} a_{mn} b_{k\ell}\, E_{ij} E_{mn} E_{pq} \otimes E_{ji} E_{k\ell} E_{qp} = \sum_{\substack{ijmn \\ k\ell pq}} \delta_{jm} \delta_{ik}\, a_{mn} b_{k\ell}\, E_{in} E_{pq} \otimes E_{j\ell} E_{qp}$$
$$= \sum_{\substack{ijmn \\ k\ell pq}} \delta_{jm} \delta_{ik} \delta_{np} \delta_{\ell q}\, a_{mn} b_{k\ell}\, E_{iq} \otimes E_{jp} = \sum_{ijln} e_{jn} b_{i\ell}\, E_{i\ell} \otimes E_{jn} = A_{\mathbf{2}} B_{\mathbf{1}}\,.$$

The relation $P_{\mathbf{12}} A_{\mathbf{1}} = A_{\mathbf{2}} P_{\mathbf{12}}$ is obtained by choosing $B = \mathbb{I}$ and multiplying by $P_{\mathbf{12}}$ on the right hand-side. In a similar way we have $P_{\mathbf{12}} B_{\mathbf{2}} = B_{\mathbf{1}} P_{\mathbf{12}}$. Finally,

$$[P_{\mathbf{12}}, A_{\mathbf{1}}] = P_{\mathbf{12}} A_{\mathbf{1}} - A_{\mathbf{1}} P_{\mathbf{12}} = A_{\mathbf{2}} P_{\mathbf{12}} - P_{\mathbf{12}} A_{\mathbf{2}} = -[P_{\mathbf{12}}, A_{\mathbf{2}}]\,. \qquad \square$$

We indicate with $\mathrm{Tr}_{\boldsymbol{k}}$ the usual trace taken on the space k, so that, for instance, $\mathrm{Tr}_{\mathbf{1}}\, A_{\mathbf{1}} = \mathrm{Tr}_{\mathbf{1}}(A \otimes \mathbb{I}) = (\mathrm{Tr}\, A)\mathbb{I}$. The following identities hold:

- **Symmetry of the auxiliary spaces:** $\mathrm{Tr}_{\mathbf{1}}\, A_{\mathbf{21}} = \mathrm{Tr}_{\mathbf{2}}\, A_{\mathbf{12}}$.
- **Cyclic property of the trace:** $\mathrm{Tr}_{\mathbf{1}}\, A_{\mathbf{1}} B_{\mathbf{12}} C_{\mathbf{1}} = \mathrm{Tr}_1\, C_{\mathbf{1}} A_{\mathbf{1}} B_{\mathbf{12}}$ and $\mathrm{Tr}_{\mathbf{2}}\, A_{\mathbf{2}} B_{\mathbf{21}} C_{\mathbf{2}} = \mathrm{Tr}_{\mathbf{2}}\, C_{\mathbf{2}} A_{\mathbf{2}} B_{\mathbf{21}}$.

Moreover, we have that $\mathrm{Tr}_{\mathbf{1}}\, P_{\mathbf{12}} A_{\mathbf{1}} = \mathrm{Tr}_{\mathbf{2}}\, P_{\mathbf{12}} A_{\mathbf{2}} = A$. In fact

$$\mathrm{Tr}_{\mathbf{1}}\, P_{\mathbf{12}} A_{\mathbf{2}} = \mathrm{Tr}_{\mathbf{1}} \sum_{ij} E_{ij} \otimes E_{ji} \sum_{k\ell} a_{k\ell} \mathbb{I} \otimes E_{k\ell} = \mathrm{Tr}_{\mathbf{1}} \sum_{ij} \sum_{k\ell} a_k l \delta_{ik} E_{ij} \otimes E_{k\ell}$$
$$= \mathrm{Tr}_{\mathbf{1}} \sum_{ij\ell} a_{i\ell} E_{ij} \otimes E_{j\ell} = \sum_{ij\ell} \delta_{ij} a_{i\ell} E_{j\ell} = \sum_{i\ell} a_{i\ell} E_{i\ell} = A$$

and similarly $\mathrm{Tr}_{\mathbf{2}}\, P_{\mathbf{12}} A_{\mathbf{2}} = A$.

Let $S_{\mathbf{12}}$ be an operator $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$.

$$S_{\mathbf{12}} = \sum_{ij,k\ell} s_{ij,k\ell} E_{ij} \otimes E_{k\ell}\,. \tag{A.19}$$

$S_{\mathbf{12}}$ will induce operators acting naturally on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$S_{\mathbf{12}} = \sum_{ij,k\ell} s_{ij,k\ell} E_{ij} \otimes E_{k\ell} \otimes \mathbb{I}\,, \tag{A.20a}$$
$$S_{\mathbf{23}} = \sum_{ij,k\ell} s_{ij,k\ell} \mathbb{I} \otimes E_{ij} \otimes E_{k\ell}\,, \tag{A.20b}$$
$$S_{\mathbf{13}} = \sum_{ij,k\ell} s_{ij,k\ell} E_{ij} \otimes \mathbb{I} \otimes E_{k\ell}\,. \tag{A.20c}$$

We use the same name for the operator $S_{\mathbf{12}}$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with a little abuse of notation. Such an operator is often non-constant, and it may depend on spectral parameters $\lambda, \mu \in \mathbb{C}$ in the following way

$$S_{\mathbf{12}}(\lambda, \mu) = \sum_{ij,k\ell} s_{ij,k\ell}(\lambda, \mu) E_{ij} \otimes E_{k\ell} \,. \tag{A.21}$$

An operator $r_{\mathbf{12}}(\lambda, \mu) : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ such that $r_{\mathbf{12}}(\lambda, \mu) = -r_{\mathbf{21}}(\mu, \lambda)$, that satisfies the *classical Yang-Baxter equation*

$$[r_{\mathbf{12}}(\lambda, \mu), r_{\mathbf{13}}(\lambda, \nu)] + [r_{\mathbf{12}}(\lambda, \mu), r_{\mathbf{23}}(\mu, \nu)] + [r_{\mathbf{13}}(\lambda, \nu), r_{\mathbf{23}}(\nu, \mu)] = 0 \tag{A.22}$$

is called an ultralocal *classical r-matrix.* Here are some examples:

- The rational r-matrix
$$r_{\mathbf{12}}(\lambda, \mu) = \frac{P_{\mathbf{12}}}{\lambda - \mu}, . \tag{A.23}$$

- The trigonometric r-matrix [FTR07]
$$r_{\mathbf{12}}(\lambda, \mu) = -\frac{1}{2}\frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}(\mathbb{I} \otimes \mathbb{I} - \sigma_3 \otimes \sigma_3) + \frac{\lambda\mu}{\lambda^2 - \mu^2}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \,. \tag{A.24}$$
The name *trigonometric* comes from the fact that when we perform the change of variables $\lambda = e^{i\alpha}$ and $\mu = e^{i\beta}$ we have that $r_{\mathbf{12}}$ becomes
$$r_{\mathbf{12}}(\alpha, \beta) = -\frac{i}{2}\frac{\cos(\alpha - \beta)}{\sin(\alpha - \beta)}(\mathbb{I} \otimes \mathbb{I} - \sigma_3 \otimes \sigma_3) - \frac{i}{2\sin(\alpha - \beta)}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \,. \tag{A.25}$$

- Another trigonometric r-matrix [S08, Section 4.5]
$$r_{\mathbf{12}}(\lambda, \mu) = \frac{1}{4}\frac{\mu + \lambda}{\mu - \lambda}(\sigma_3 \otimes \sigma_3 + \mathbb{I} \otimes \mathbb{I}) + \frac{\mu}{\mu - \lambda}\sigma_+ \otimes \sigma_- + \frac{\lambda}{\mu - \lambda}\sigma_- \otimes \sigma_+ \,. \tag{A.26}$$
These two trigonometric r-matrices are related by a 'gauge/twist' transformation, but we do not enter in further details here.

A.3 Dirac-Poisson brackets for the Non-Linear Schrödinger equation

This section is adapted from [ACDK16, Section 3.1]. We start with the Lagrangian for the (unreduced) Non-Linear Schrödinger (NLS) equation in the following form

$$L = \frac{i}{2}(q_t r - q r_t) - \frac{1}{2} q_x r_x - \frac{1}{2} q^2 r^2 \,. \tag{A.27}$$

which produces the Non-Linear Schrödinger equation in the form

$$iq_t + \frac{1}{2}q_{xx} - q^2 r = 0\,, \qquad ir_t - \frac{1}{2}r_{xx} + qr^2 = 0\,. \tag{A.28}$$

We compute the momenta conjugated to q and r in the usual way:

$$p^1 = \frac{\partial L}{\partial q_t} = \frac{ir}{2}\,, \qquad p^2 = \frac{\partial L}{\partial r_t} = -\frac{iq}{2}\,. \tag{A.29}$$

We see that these equations cannot be used to eliminate q_t and r_t in favour of the momenta $p^{1,2}$, but they relate variables that are supposed to be independent: they are therefore constraints

$$C_1 = p^1 - \frac{ir}{2}\,, \qquad C_2 = p^2 + \frac{iq}{2}\,, \tag{A.30}$$

and the 'constrained Hamiltonian', which takes these constraints into account, is

$$\begin{aligned} H_* =& H + \lambda_1 C_1 + \lambda_2 C_2\,, \\ =& p^1 q_t + p^2 r_t - L + \lambda_1 C_1 + \lambda_2 C_2\,. \end{aligned} \tag{A.31}$$

The canonical Poisson brackets are given by:

$$\{p^1, q\} = 1\,, \qquad \{p^2, r\} = 1\,. \tag{A.32}$$

At this stage, we have two possibilities: either we use the Poisson brackets $\{\ ,\ \}$ with the constrained Hamiltonian H_*, or we use the usual Hamiltonian H with the famous Dirac brackets $\{\ ,\ \}_D$ [D50]. Let us explore the second scenario. We compute

$$\{C_1, C_2\} = \{p^1 - \frac{ir}{2}, p^2 + \frac{iq}{2}\} = i\,, \tag{A.33}$$

which shows that these primary constraints are second class. The values of $\lambda_{1,2}$ are completely fixed by the requirement that the constraints are constant under the flow of H_*, in fact we have

$$\begin{aligned} \{H_*, C_1\} &= \{H, C_1\} + \lambda_2\{C_2, C_1\} = \{H, C_1\} - \lambda_2 = 0 \qquad \Longrightarrow \lambda_2 = \{H, C_1\}\,, \\ \{H_*, C_2\} &= \{H, C_2\} + \lambda_1\{C_1, C_2\} = \{H, C_2\} + \lambda_1 = 0 \qquad \Longrightarrow \lambda_1 = -\{H, C_2\}\,, \end{aligned}$$

which means that there are no more constraints besides $C_{1,2}$. We need the matrix M of the Poisson brackets between the constraints, such that $M_{ij} = \{C_i, C_j\}$, and its inverse

$$M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\,, \qquad M^{-1} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\,. \tag{A.34}$$

The Dirac-Poisson brackets $\{\ ,\ \}_D$ are defined as

$$\{f,g\}_D = \{f,g\} - \sum_{j,k=1}^{2} \{f,C_j\}(M^{-1})_{jk}\{C_k,g\} \tag{A.35}$$

for any f,g smooth functions of the dynamical variables. The Dirac-Poisson brackets between q and r therefore become

$$\begin{aligned} \{q,r\}_D =& \{q,r\} - \{q,C_1\}(M^{-1})_{12}\{C_2,r\} \\ =& 0 - i\{q,C_1\}\{C_2,r\} = i\,. \end{aligned} \tag{A.36}$$

This allows us to use the Poisson bracket $\{q,r\}_D = i$ (from now on renamed $\{\ ,\ \}$), and the Hamiltonian

$$H = p^1 q_t + p^2 r_t - L = \frac{1}{2}q_x r_x + \frac{1}{2}q^2 r^2\,, \tag{A.37}$$

to compute the Non-Linear Schrödinger equations as

$$\begin{aligned} q_t =& \{\int H dx, q\} = \int (\frac{1}{2}\{q_x r_x, q\} + \frac{1}{2}\{q^2 r^2, q\})dx = \frac{i}{2}q_{xx} - iq^2 r\,, \\ r_t =& \{\int H dx, r\} = \int (\frac{1}{2}\{q_x r_x, r\} + \frac{1}{2}\{q^2 r^2, r\})dx = -\frac{i}{2}r_{xx} + iqr^2\,. \end{aligned}$$

A.4 4d Chern-Simons origin of the Zakharov-Mikhailov Lagrangian

The setup Let $\Sigma := \mathbb{R}^2$ be the plane with light-cone coordinates ξ and η, and $X := \Sigma \times \mathbb{C}\mathrm{P}^1$. In $\mathbb{C}\mathrm{P}^1$ we use the coordinates $(z, \bar{z})$ We start from the *regularised 4d action*

$$S_{4d}(A) = -\frac{i}{4\pi} \int_X z \,\mathrm{Tr}(F(A) \wedge F(A))\,, \tag{A.38}$$

where $A = A_\xi \, d\xi + A_\eta \, d\eta + A_{\bar{z}} \, d\bar{z}$ is a $\mathfrak{gl}_N$-valued 1-form on X. The components of A are taken to be smooth functions anywhere on X but on a set of marked points $\{\, a_m \,\}_{m=1}^{N_1}$ and $\{\, b_n \,\}_{n=1}^{N_2}$. Specifically we require A_ξ and A_η to be singular at these points, and that they can be written locally as $A_\xi = (z - a_m)^{-1} B_{m,\xi}$ near each a_m and $A_\eta = (z - b_n)^{-1} B_{n,\eta}$ near each b_n. A will be often referred to as the *bulk field*, and S_{4d} as the *bulk action*. $F(A) = dA - A \wedge A$ is the curvature of A and has components

$$\begin{aligned} F(A) =&(\partial_\xi A_\eta - \partial_\eta A_\xi - [A_\xi, A_\eta])\, d\xi \wedge d\eta \\ &+ (\partial_\xi A_{\bar{z}} - \partial_{\bar{z}} A_\xi - [A_\xi, A_{\bar{z}}])\, d\xi \wedge d\bar{z} \\ &+ (\partial_\eta A_{\bar{z}} - \partial_{\bar{z}} A_\eta - [A_\eta, A_{\bar{z}}])\, d\eta \wedge d\bar{z} + dz \wedge \partial_z A\,. \end{aligned} \tag{A.39}$$

We used the notation

$$\mathrm{Tr}\left(\sum_{(I)} u_{(I)} dx^{(I)} \wedge \sum_{(J)} v_{(J)} dx^{(J)} \right) = \sum_{(I),(J)} \mathrm{Tr}(u_{(I)} v_{(J)}) dx^{(I)} \wedge dx^{(J)} \tag{A.40}$$

for $\mathfrak{gl}_N$ valued forms on X, where (I) and (J) are multi-indices.

Remark A.3: Note that we do not include a z-component in A, as the action is invariant with respect under local transformations $A \mapsto A + \chi dz$ for any $\chi \in C^\infty(X, \mathfrak{gl}_N)$. In fact we have that the curvature transform as

$$\begin{aligned} F(A + \chi dz) =& d(A + \chi dz) - (A + \chi dz) \wedge (A + \chi dz) \\ =& F(A) + d\chi \wedge dz - A \wedge \chi dz + \chi \wedge A dz \\ =& F(A) + (d\chi - [A, \chi]) \wedge dz\,. \end{aligned}$$

and therefore $z \,\mathrm{Tr}\, F(A) \wedge F(A)$ transforms as

$$\begin{aligned} & z \,\mathrm{Tr}\, F(A) \wedge F(A) + 2z \,\mathrm{Tr}\, F(A) \wedge (d\chi - [A, \chi]) \wedge dz \\ &= z \,\mathrm{Tr}\, F(A) \wedge F(A) + 2d(z dz \wedge \mathrm{Tr}\, F(A)\chi) \wedge dz\,, \end{aligned}$$

where we used $dF(A) = A \wedge F(A) - F(A) \wedge A$.

Remark A.4: The action S_{4d} is also invariant under gauge transformations

$$A \mapsto gAg^{-1} - dgg^{-1} \tag{A.41}$$

for any $g \in C^\infty(X, GL_N)$ thanks to the invariance of the trace.

We now couple the $4d$ bulk field A to a collection of $2d$ fields localised on the surface defects $\Sigma \times \{ a_m \}$ and $\Sigma \times \{ b_n \}$. We use the embeddings $\iota_{a_m} \colon \Sigma \times \{ a_m \} \hookrightarrow X$ and $\iota_{b_n} \colon \Sigma \times \{ b_n \} \hookrightarrow X$ for $m = 1, \ldots, N_1$ and $n = 1, \ldots, N_2$. To each point a_m we associate a Lie group valued field $\varphi_m \in C^\infty(\Sigma, GL_N)$ and to each b_n we associate $\psi_n \in C^\infty(\Sigma, GL_N)$. We also fix non-dynamical matrices $U_m^{(0)}, V_n^{(0)} \in \mathfrak{gl}_N$ for each $m = 1, \ldots, N_1$ and $n = 1, \ldots, N_2$. We remark that we take them to be constant for simplicity, but in principle they may be elements of $C^\infty(\Sigma, \mathfrak{gl}_N)$. In the next, we always mean $\sum_m = \sum_{m=1}^{N_1}$ and $\sum_n = \sum_{n=1}^{N_2}$, and $\varphi = \{ \varphi_m \}_{m=1}^{N_1}$, $\psi = \{ \psi_n \}_{n=1}^{N_2}$. We define

$$\begin{aligned} S_{def}(A, \varphi, \psi) = &- \sum_m \int_{\Sigma \times \{ a_m \}} \mathrm{Tr}(\varphi_m^{-1}(d_\Sigma - \iota_{a_m}^* A)\varphi_m U_m^{(0)}) \wedge d\xi \\ &- \sum_n \int_{\Sigma \times \{ b_n \}} \mathrm{Tr}(\psi_n^{-1}(d_\Sigma - \iota_{b_n}^* A)\psi_n V_n^{(0)}) \wedge d\eta \, , \end{aligned} \tag{A.42}$$

where d_Σ is the horizontal (de Rham) differential on Σ.

Remark A.5: In order to maintain gauge invariance we need to let the fields transform as $\varphi_m \mapsto g\varphi_m$ and $\psi_n \mapsto g\psi_n$.

We finally couple the bulk field A with the defects by considering the action

$$S(A, \varphi, \psi) = S_{4d}(A) + S_{def}(A, \varphi, \psi) \, . \tag{A.43}$$

From Chern-Simons to Zakharov-Mikhailov We can compute the bulk equations of motion and consider bulk variations $A \mapsto A + \varepsilon a$, where $a = a_\eta d\eta + a_\xi d\xi + a_{\bar z} d\bar z$ is a $\mathfrak{gl}_N$ valued 1-form on X of compact support

$$\begin{aligned} \delta_a S(A, \varphi, \psi) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon = 0} S(A + \varepsilon a, \varphi, \psi) \\ &= -\frac{i}{2\pi} \int_X dz \wedge \mathrm{Tr}(a \wedge F(A)) \\ &\quad - \sum_m \int_{\Sigma \times \{ a_m \}} \mathrm{Tr}(a_\eta U_m) d\xi \wedge d\eta + \sum_n \int_{\Sigma \times \{ b_n \}} \mathrm{Tr}(a_\xi V_n) d\xi \wedge d\eta \\ &= -\frac{i}{2\pi} \int_X dz \wedge \mathrm{Tr}(a \wedge F(A)) \\ &\quad - \sum_m \int_X \mathrm{Tr}(a_\eta U_m)\delta(z - a_m) d\xi \wedge d\eta \wedge dz \wedge d\bar z + \sum_n \int_X \mathrm{Tr}(a_\xi V_n)\delta(z - b_n) d\xi \wedge d\eta \wedge dz \wedge d\bar z \, . \end{aligned}$$

We have introduced $U_m := \varphi_m U^{(0)} \varphi_m^{-1}$ and $V_n := \psi_n V^{(0)} \psi_n^{-1}$. In the last line we used the δ-function, satisfying the property

$$\int_{\mathbb{C}\mathrm{P}^1} f(\xi, \eta, z)\delta(z - x)\, dz \wedge d\bar{z} = f(\xi, \eta, x) \tag{A.44}$$

for any $x \in \mathbb{C}$ and any smooth function f on X. The equations are the following:

$$\partial_\xi A_\eta - \partial_\eta A_\xi = [A_\xi, A_\eta]\,, \tag{A.45a}$$

$$\partial_\xi A_{\bar{z}} - \partial_{\bar{z}} A_\xi - [A_\xi, A_{\bar{z}}] = 2\pi i \sum_m U_m \delta(z - a_m)\,, \tag{A.45b}$$

$$\partial_\eta A_{\bar{z}} - \partial_{\bar{z}} A_\eta - [A_\eta, A_{\bar{z}}] = 2\pi i \sum_n V_n \delta(z - b_n)\,. \tag{A.45c}$$

We are now ready to turn A into the Lax connection. The first main issue is that the Lax connection has no $d\bar{z}$ component, which can be fixed by focusing on a field configuration of A where $A_{\bar{z}} = 0$.

> **Remark A.6:** This operation will break the gauge invariance, since now we must impose that $A \mapsto gAg^{-1} - dgg^{-1}$ does not recreate a $d\bar{z}$ component. In other words, we need a g such that $\partial_{\bar{z}} g g^{-1} = 0$. This can be achieved by picking a $g \in C^\infty(\Sigma, GL_N)$ only, *i.e.* that does not depend on $\mathbb{C}\mathrm{P}^1$. These residuals gauge transformations will correspond to the allowed gauge transformations of the Lax connection.

The second issue is that while $A = A_\xi\, d\xi + A_\eta\, d\eta$ is smooth on $\mathbb{C}\mathrm{P}^1$, with singularities on $\{\, a_m \,\}$ and $\{\, b_n \,\}$, the Lax connection is meromorphic on $\mathbb{C}\mathrm{P}^1$. This issue can be solved by focusing on a subset of the fields that satisfy the equations of motion. Using the identity $\partial_{\bar{z}} \frac{1}{z} = -2\pi i \delta(z)$ we can solve the a_ξ and a_η equations above as

$$A_\xi = U_0 + \sum_{m=1}^{N_1} \frac{U_m}{z - a_m} =: U\,, \tag{A.46a}$$

$$A_\eta = V_0 + \sum_{n=1}^{N_2} \frac{V_n}{z - b_n} =: V\,. \tag{A.46b}$$

We can also compute the defect equations of motion, and consider the variation of the action S with respect to the $2d$ fields φ_m and ψ_n. We consider variations $\varphi_m \mapsto e^{\varepsilon \alpha_m} \varphi_m$ and $\psi_n \mapsto e^{\varepsilon \beta_n} \psi_n$ for arbitrary $\alpha_m, \beta_n \in C^\infty(\Sigma, \mathfrak{gl}_N)$. This gives

$$\begin{aligned} \delta_{(\alpha,\beta)} S(A, \varphi, \psi) =& \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S(A, \{\, e^{\varepsilon\alpha_m} \varphi_m \,\}, \{\, e^{\varepsilon\beta_n} \psi_n \,\}) \\ =& -\sum_m \int_{\Sigma \times \{\, a_m \,\}} \mathrm{Tr}(-\alpha_m d_\Sigma U_m + \alpha_m [\iota_{a_m}^* A, U_m]) \wedge d\xi \\ & -\sum_n \int_{\Sigma \times \{\, b_n \,\}} \mathrm{Tr}(-\beta_n d_\Sigma V_n + \beta_n [\iota_{b_n}^* A, V_n]) \wedge d\eta\,. \end{aligned}$$

The defect equations of motions are then, once we use $A_\xi = U$, $A_\eta = V$ and $A_{\bar{z}} = 0$, the expected zero-curvature equations for the Lax connection $W = U\,d\xi + V\,d\eta$:

$$\partial_\eta U_m = [V_0 + \sum_n \frac{V_n}{a_m - b_n}, U_m]\,, \tag{A.47a}$$

$$\partial_\xi V_n = [U_0 + \sum_m \frac{U_m}{b_n - a_m}, V_n]\,. \tag{A.47b}$$

We now substitute the solution (A.46a)-(A.46b) for A into the action S to obtain the Zakharov-Mikhailov action. The bulk term becomes

$$\begin{aligned} S_{4d}(W) &= -\frac{i}{4\pi}\int_X z\,\mathrm{Tr}(F(W)\wedge F(W)) \\ &= \frac{1}{2\pi i}\int_X z(\partial_{\bar{z}}U\partial_z V - \partial_z U\partial_{\bar{z}}V)d\xi\wedge d\eta\wedge dz\wedge d\bar{z}\,. \end{aligned} \tag{A.48}$$

Using the explicit expressions of U, V and $\partial_{\bar{z}}\frac{1}{z} = -2\pi i\delta(z)$ we get

$$\begin{aligned} &S_{4d}(W) \\ &= \int_X z\sum_{mn}\mathrm{Tr}\,\frac{U_m V_n}{(a_m - b_n)^2}(\delta(z-a_m) - \delta(z-b_n))d\xi\wedge d\eta\wedge dz\wedge d\bar{z} \\ &= \sum_{mn}\int_\Sigma \mathrm{Tr}\,\frac{U_m V_n}{a_m - b_n}d\xi\wedge d\eta\,. \end{aligned} \tag{A.49}$$

On the other hand, the defects action becomes

$$\begin{aligned} &S_{def}(A,\varphi,\psi)\big|_{A=W} \\ &= -\sum_m \int_{\Sigma\times\{\,a_m\,\}} \mathrm{Tr}\,\varphi_m^{-1}(d_\Sigma - \iota^*_{a_m}V d\eta)\varphi_m U_m^{(0)}\wedge d\xi \\ &\qquad - \sum_n \int_{\Sigma\times\{\,b_n\,\}} \mathrm{Tr}\,\psi_n^{-1}(d_\Sigma - \iota^*_{b_n}U d\xi)\psi_n V_n^{(0)}\wedge d\eta \\ &= -\sum_m\int_\Sigma \mathrm{Tr}\,\varphi_m^{-1}(\partial_\eta - V_0 - \sum_n\frac{V_n}{a_m - b_n})\varphi_m U_m^{(0)} d\eta\wedge d\xi \\ &\qquad - \sum_n\int_\Sigma \mathrm{Tr}\,\psi_n^{-1}(\partial_\xi - U_0 - \sum_m\frac{U_m}{b_n - a_m})\psi_n V_n^{(0)} d\xi\wedge d\eta \\ &= \int_\Sigma \mathrm{Tr}\left(\sum_m \varphi_m^{-1}(\partial_\eta - V_0)\varphi_m U_m^{(0)}\right. \\ &\qquad\left. - \sum_n \psi_n^{-1}(\partial_\xi - U_0)\psi_n V_n^{(0)} - 2\sum_{mn}\frac{U_m V_n}{a_m - b_n}\right) d\xi\wedge d\eta\,. \end{aligned} \tag{A.50}$$

The Zakharov-Mikhailov action is obtained by adding the bulk and the defects action and we get the desired

$$
\begin{aligned}
S_{ZM}(\varphi,\psi) = \int_{\Sigma} \mathrm{Tr}\Bigg(& \sum_m \varphi_m^{-1}(\partial_\eta - V_0)\varphi_m U_m^{(0)} \\
& - \sum_n \psi_n^{-1}(\partial_\xi - U_0)\psi_n V_n^{(0)} - \sum_{mn} \frac{\varphi_m U_m^{(0)} \varphi_m^{-1} \psi_n V_n^{(0)} \psi_n^{-1}}{a_m - b_n} \Bigg) d\xi \wedge d\eta \,.
\end{aligned}
\tag{A.51}
$$

Appendix B

Proofs of Chapter 6

B.1 The e and f coordinates

In this section we discuss some of the properties of the coordinates e and f, which are defined as

$$e(\lambda) = \frac{b(\lambda)}{\sqrt{i - a(\lambda)}}\,, \qquad f(\lambda) = \frac{c(\lambda)}{\sqrt{i - a(\lambda)}}\,. \tag{B.1}$$

We remember that we are restricting to the subset where $a^2(\lambda) + b(\lambda)c(\lambda) = -1$, which means that $a(\lambda) = e(\lambda)f(\lambda) - i$, in fact

$$e(\lambda)f(\lambda) = \frac{b(\lambda)c(\lambda)}{i - a(\lambda)} = \frac{-1 - a^2(\lambda)}{i - a(\lambda)} = i + a(\lambda)\,. \tag{B.2}$$

The coefficients of e and f can be computed in the following way. First we write our series as Taylor series substituting $x = \lambda^{-1}$, as (with a slight abuse of notation)

$$b(x) = \sum_{j \geq 0} b_j x^j\,, \quad c(x) = \sum_{j \geq 0} c_j x^j\,, \quad a(x) = \sum_{j \geq 0} a_j x^j\,.$$

Then, we find

$$e_k = \frac{1}{k!}\frac{d^k}{dx^k}\bigg|_{x=0} \frac{b(x)}{\sqrt{i - a(x)}}\,, \qquad f_k = \frac{1}{k!}\frac{d^k}{dx^k}\bigg|_{x=0} \frac{c(x)}{\sqrt{i - a(x)}}\,. \tag{B.3}$$

We list the first few in the following, using in (B.3) that $b_0 = c_0 = a_1 = 0$:

$$e_0 = 0\,, \qquad f_0 = 0\,, \tag{B.4a}$$
$$e_1 = \frac{1}{\sqrt{2i}} b_1\,, \qquad f_1 = \frac{1}{\sqrt{2i}} c_1\,, \tag{B.4b}$$
$$e_2 = \frac{1}{\sqrt{2i}} b_2\,, \qquad f_2 = \frac{1}{\sqrt{2i}} c_2\,, \tag{B.4c}$$
$$e_3 = \frac{1}{\sqrt{2i}} (b_3 - \frac{1}{8} b_1^2 c_1)\,, \qquad f_3 = \frac{1}{\sqrt{2i}} (c_3 - \frac{1}{8} b_1 c_1^2)\,, \tag{B.4d}$$
$$e_4 = \frac{1}{\sqrt{2i}} (b_4 - \frac{1}{4} b_1 c_1 b_2 - \frac{1}{8} b_1^2 c_2)\,, \qquad f_4 = \frac{1}{\sqrt{2i}} (c_4 - \frac{1}{4} b_1 c_1 c_2 - \frac{1}{8} c_1^2 b_2)\,, \tag{B.4e}$$

Conversely, we have that

$$b(\lambda) = e(\lambda)\sqrt{2i - e(\lambda) f(\lambda)}\,, \qquad c(\lambda) = f(\lambda)\sqrt{2i - e(\lambda) f(\lambda)}\,, \tag{B.5}$$

and therefore

$$b_1 = \sqrt{2i} e_1\,, \qquad c_1 = \sqrt{2i} f_1\,, \tag{B.6a}$$
$$b_2 = \sqrt{2i} e_2\,, \qquad c_2 = \sqrt{2i} f_2\,, \tag{B.6b}$$
$$b_3 = \sqrt{2i}(e_3 + \frac{i}{4} e_1^2 f_1)\,, \qquad c_3 = \sqrt{2i}(f_3 + \frac{i}{4} e_1 f_1^2)\,, \tag{B.6c}$$
$$b_4 = \sqrt{2i}(e_4 + \frac{i}{2} e_1 f_1 e_2 + \frac{i}{4} e_1^2 f_2)\,, \qquad c_4 = \sqrt{2i}(f_4 + \frac{i}{2} e_1 f_1 f_2 + \frac{i}{4} f_1^2 e_2)\,. \tag{B.6d}$$

Also $a = ef - i$, so $a_k = \sum_{i=1}^{k-1} e_i f_{k-i}$:

$$a_0 = -i\,, \qquad a_1 = 0\,, \qquad a_2 = e_1 f_1\,,$$
$$a_3 = e_1 f_2 + e_2 f_1\,, \qquad a_4 = e_1 f_3 + e_2 f_2 + f_1 e_3\,, \qquad \ldots$$

and

$$a_2 = -\frac{i}{2} b_1 c_1\,, \qquad a_3 = -\frac{i}{2}(b_1 c_2 + b_2 c_1)\,,$$
$$a_4 = -\frac{i}{2}(b_1 c_4 + b_2 c_2 + b_4 c_1 - \frac{3}{8} b_1^2 c_1 c_2 - \frac{3}{8} b_1 c_1^2 b_2)\,, \qquad \ldots$$

It is also useful to express these relations in terms of the usual q and r coordinates (and their derivatives with respect to $x^1 = x$) we have the following identities

$$b_1 = q\,, \qquad c_1 = r\,, \tag{B.7a}$$
$$b_2 = \frac{i}{2} q_1\,, \qquad c_2 = -\frac{i}{2} r_1\,, \tag{B.7b}$$
$$b_3 = -\frac{1}{4} q_{11} + \frac{1}{2} q^2 r\,, \qquad c_3 = -\frac{1}{4} r_{11} + \frac{1}{2} q r^2\,, \tag{B.7c}$$
$$b_4 = -\frac{i}{8} q_{111} + \frac{3i}{4} q r q_1\,, \qquad c_4 = \frac{i}{8} r_{111} - \frac{3i}{4} q r r_1\,. \tag{B.7d}$$

$$e_1 = \frac{1}{\sqrt{2i}} q\,, \qquad f_1 = \frac{1}{\sqrt{2i}} r\,, \tag{B.8a}$$

$$e_2 = \frac{1}{\sqrt{2i}} \frac{i}{2} q_1\,, \qquad f_2 = -\frac{1}{\sqrt{2i}} \frac{i}{2} r_1\,, \tag{B.8b}$$

$$e_3 = \frac{1}{\sqrt{2i}} \left(-\frac{1}{4} q_{11} + \frac{3}{8} q^2 r \right)\,, \qquad f_3 = \frac{1}{\sqrt{2i}} \left(-\frac{1}{4} r_{11} + \frac{3}{8} q r^2 \right)\,, \tag{B.8c}$$

$$e_4 = \frac{1}{\sqrt{2i}} \left(-\frac{i}{8} q_{111} + \frac{5i}{8} q r q_1 + \frac{i}{16} q^2 r_1 \right)\,, \quad f_4 = \frac{1}{\sqrt{2i}} \left(\frac{i}{8} r_{111} - \frac{5i}{8} q r r_1 - \frac{i}{16} q_1 r^2 \right)\,. \tag{B.8d}$$

Conversely:

$$q = \sqrt{2i} e_1\,, \qquad r = \sqrt{2i} f_1\,, \tag{B.9a}$$

$$q_1 = -\sqrt{2i} 2i e_2\,, \qquad r_1 = \sqrt{2i} 2i f_2\,, \tag{B.9b}$$

$$q_{11} = \sqrt{2i} \left(-4e_3 + 3i e_1^2 f_1 \right)\,, \qquad r_{11} = \sqrt{2i} \left(-4f_3 + 3i e_1 f_1^2 \right)\,, \tag{B.9c}$$

$$q_{111} = \sqrt{2i} \left(8i e_4 + 20 e_1 f_1 e_2 - 2e_1^2 f_2 \right)\,, \quad r_{111} = \sqrt{2i} \left(-8i f_4 - 20 e_1 f_1 f_2 + 2 f_1^2 e_2 \right)\,. \tag{B.9d}$$

We can also write the expressions for the derivatives of Q with respect to the coordinates e and f:

$$\frac{\partial Q(\lambda)}{\partial e_k} = \frac{\lambda^{-k}}{\sqrt{i - a(\lambda)}} \begin{pmatrix} c(\lambda) & \frac{i - 3a(\lambda)}{2} \\ -\frac{c^2(\lambda)}{2(i - a(\lambda))} & -c(\lambda) \end{pmatrix}\,, \tag{B.10a}$$

$$\frac{\partial Q(\lambda)}{\partial f_k} = \frac{\lambda^{-k}}{\sqrt{i - a(\lambda)}} \begin{pmatrix} b(\lambda) & -\frac{b^2(\lambda)}{2(i - a(\lambda))} \\ \frac{i - 3a(\lambda)}{2} & -b(\lambda) \end{pmatrix}\,. \tag{B.10b}$$

Therefore we have that the derivatives of the coefficients of $Q(\lambda)$ with respect to e_j are

$$\frac{\partial a_i}{\partial e_j} = f_{i-j}\,, \tag{B.11a}$$

$$\frac{\partial b_i}{\partial e_j} = \left(\frac{i - 3a(\lambda)}{2\sqrt{i - a(\lambda)}} \right)_{i-j}\,, \tag{B.11b}$$

$$\frac{\partial c_i}{\partial e_j} = \left(\frac{-f^2(\lambda)}{2\sqrt{i - a(\lambda)}} \right)_{i-j} = \left(\frac{-c^2(\lambda)}{2(i - a(\lambda))^{3/2}} \right)_{i-j}\,, \tag{B.11c}$$

while the ones with respect to f_j are

$$\frac{\partial a_i}{\partial f_j} = e_{i-j}\,, \tag{B.12a}$$

$$\frac{\partial b_i}{\partial f_j} = \left(\frac{-e^2(\lambda)}{2\sqrt{i-a(\lambda)}}\right)_{i-j} = \left(\frac{-b^2(\lambda)}{2(i-a(\lambda))^{3/2}}\right)_{i-j}\,, \tag{B.12b}$$

$$\frac{\partial c_i}{\partial f_j} = \left(\frac{i-3a(\lambda)}{2\sqrt{i-a(\lambda)}}\right)_{i-j}\,. \tag{B.12c}$$

B.2 Proof of Theorem 6.2

This proof is generalised by the results of Section 7.2 that hold for a generic ultra-local r-matrix. However, we decided to keep this proof for completeness.

Proof. We need to calculate δdK and then δdV. We do so with the help of the generating functions as follows. Note that

$$dK = \sum_{i<j<k} \left(\partial_i K_{jk} + \partial_k K_{ij} + \partial_j K_{ki}\right)\, dx^{ijk}\,, \tag{B.13}$$

hence we associate to it the generating function[1] $D_\nu K(\lambda,\mu) + \circlearrowleft$. To obtain δdK, we simply calculate $D_\nu K(\lambda,\mu) + \circlearrowleft$. The same holds for δdV. We will need the following identities:

$$\begin{aligned}\operatorname{Tr} Q(\lambda)\delta(D_\nu Q(\mu)) = {}& \operatorname{Tr}\varphi(\mu)^{-1}\big([D_\nu Q(\mu), Q(\lambda)] \\ & + D_\nu\varphi(\mu)\varphi(\mu)^{-1}[Q(\lambda), Q(\mu)]\big)\,\delta\varphi(\mu) \\ & - \operatorname{Tr}\varphi(\mu)^{-1}[Q(\lambda), Q(\mu)]\delta(D_\nu\varphi(\mu))\,,\end{aligned} \tag{B.14}$$

$$\operatorname{Tr} D_\nu Q(\lambda)\delta Q(\mu) = \operatorname{Tr}\varphi(\mu)^{-1}[Q(\mu), D_\nu Q(\lambda)]\delta\varphi(\mu)\,. \tag{B.15}$$

We have that

$$\begin{aligned}D_\nu K(\lambda,\mu) = {}& \operatorname{Tr}(-\varphi(\mu)^{-1}D_\nu\varphi(\mu)\varphi(\mu)^{-1}D_\lambda\varphi(\mu)Q_0 + \varphi^{-1}(\mu)D_\nu D_\lambda\varphi(\mu)Q_0 \\ & + \varphi(\lambda)^{-1}D_\nu\varphi(\lambda)\varphi(\lambda)^{-1}D_\mu\varphi(\lambda)Q_0 - \varphi^{-1}(\lambda)D_\nu D_\mu\varphi(\lambda)Q_0)\,.\end{aligned}$$

[1]With $\circlearrowleft$ we mean the cyclic permutations of (ν,λ,μ).

We now apply the δ-differential.

$$\begin{aligned}
&\delta D_\nu K(\lambda,\mu)\\
&= \mathrm{Tr}(\varphi(\mu)^{-1}\delta\varphi(\mu)\varphi(\mu)^{-1}D_\nu\varphi(\mu)\varphi(\mu)^{-1}D_\lambda\varphi(\mu)Q_0 - \varphi(\mu)^{-1}\delta(D_\nu\varphi(\mu))\varphi(\mu)^{-1}D_\lambda\varphi(\mu)Q_0\\
&\quad + \varphi(\mu)^{-1}D_\nu\varphi(\mu)\varphi(\mu)^{-1}\delta\varphi(\mu)\varphi(\mu)^{-1}D_\lambda\varphi(\mu)Q_0 - \varphi(\mu)^{-1}D_\nu\varphi(\mu)\varphi(\mu)^{-1}\delta(D_\lambda\varphi(\mu))Q_0\\
&\quad - \varphi(\mu)^{-1}\delta\varphi(\mu)\varphi(\mu)^{-1}D_\nu D_\lambda\varphi(\mu)Q_0 + \varphi(\mu)^{-1}\delta(D_\nu D_\lambda\varphi(\mu))Q_0 - (\lambda\leftrightarrow\mu))\\
&= \mathrm{Tr}(\varphi(\mu)^{-1}\big(D_\nu\varphi(\mu)\varphi(\mu)^{-1}D_\lambda\varphi(\mu)\varphi(\mu)^{-1}Q(\mu)\\
&\quad + D_\lambda\varphi(\mu)\varphi(\mu)^{-1}Q(\mu)D_\nu\varphi(\mu)\varphi(\mu)^{-1} - D_\nu D_\lambda\varphi(\mu)\varphi(\mu)^{-1}Q(\mu)\big)\delta\varphi(\mu)\\
&\quad - \varphi(\mu)^{-1}D_\lambda\varphi(\mu)\varphi(\mu)^{-1}Q(\mu)\delta(D_\nu\varphi(\mu)) - \varphi(\mu)^{-1}Q(\mu)D_\nu\varphi(\mu)\varphi(\mu)^{-1}\delta(D_\lambda\varphi(\mu))\\
&\quad + \varphi(\mu)^{-1}Q(\mu)\delta(D_\nu D_\lambda\varphi(\mu)) - (\lambda\leftrightarrow\mu))\,.
\end{aligned}$$

We add the cyclic sum and we select the coefficients of $\delta\varphi(\mu)$, $\delta D_\nu\varphi(\mu)$, etc. adding the corresponding terms from $\delta D_\lambda K(\mu,\nu)$.

$$\begin{aligned}
\delta dK &= \mathrm{Tr}(\varphi(\mu)^{-1}\big(D_\mu\varphi(\mu)\varphi(\mu)^{-1}D_\lambda Q(\mu) - D_\lambda\varphi(\mu)\varphi(\mu)^{-1}D_\mu Q(\mu))\delta\varphi(\mu)\\
&\quad - \varphi(\mu)^{-1}D_\lambda Q(\mu)\delta(D_\mu\varphi(\mu)) + \varphi(\mu)^{-1}D_\mu Q(\mu)\delta(D_\lambda\varphi(\mu))\\
&\quad + \big(\varphi(\mu)^{-1}Q(\mu) - \varphi(\mu)^{-1}Q(\mu)\big)\delta(D_\mu D_\lambda\varphi(\mu)) + \circlearrowleft)\\
&= \mathrm{Tr}(\varphi(\mu)^{-1}\big(D_\mu\varphi(\mu)\varphi(\mu)^{-1}D_\lambda Q(\mu) - D_\lambda\varphi(\mu)\varphi(\mu)^{-1}D_\mu Q(\mu))\delta\varphi(\mu)\\
&\quad - \varphi(\mu)^{-1}D_\lambda Q(\mu)\delta(D_\mu\varphi(\mu)) + \varphi(\mu)^{-1}D_\mu Q(\mu)\delta(D_\lambda\varphi(\mu)) + \circlearrowleft)\,.
\end{aligned}$$

We do the same for $V(\lambda,\mu) = -\frac{1}{2}\frac{\mathrm{Tr}(Q(\lambda)-Q(\mu))^2}{\lambda-\mu} = \frac{2}{\lambda-\mu} + \frac{\mathrm{Tr}\,Q(\lambda)Q(\mu)}{\lambda-\mu}$.

$$D_\nu V(\lambda,\mu) = \frac{1}{\lambda-\mu}\,\mathrm{Tr}(D_\nu Q(\lambda)Q(\mu) + D_\nu Q(\mu)Q(\lambda))\,.$$

We now apply the δ-differential.

$$\delta D_\nu V(\lambda,\mu) = \frac{1}{\lambda-\mu}\,\mathrm{Tr}\,(D_\nu Q(\lambda)\delta Q(\mu) + Q(\lambda)\delta(D_\nu Q(\mu))) - (\lambda\leftrightarrow\mu)$$

and using the identities above we get

$$\begin{aligned}
&\delta D_\nu V(\lambda,\mu)\\
&= \frac{1}{\lambda-\mu}\,\mathrm{Tr}\Big(-\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\delta(D_\nu\varphi(\mu))\\
&\quad + \varphi(\mu)^{-1}\left([D_\nu Q(\mu),Q(\lambda)] + D_\nu\varphi(\mu)\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\right)\delta\varphi(\mu)\\
&\qquad + \varphi(\mu)^{-1}[Q(\mu),D_\nu Q(\lambda)]\delta\varphi(\mu)\Big) - (\lambda\leftrightarrow\mu)\\
&= \frac{1}{\lambda-\mu}\,\mathrm{Tr}\Big(-\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\delta(D_\nu\varphi(\mu))\\
&+ \varphi(\mu)^{-1}\left([D_\nu Q(\mu),Q(\lambda)] + [Q(\mu),D_\nu Q(\lambda)] + D_\nu\varphi(\mu)\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\right)\delta\varphi(\mu)\Big)\\
&\quad - (\lambda\leftrightarrow\mu)\,.
\end{aligned}$$

We add the cyclic sum and we select the coefficients of $\delta\varphi(\mu)$, $\delta D_\nu\varphi(\mu)$, etc. adding the corresponding terms from $\delta D_\lambda V(\mu,\nu)$.

$$
\begin{aligned}
&\delta dV \\
&= \frac{1}{\lambda-\mu}\operatorname{Tr}\Big(-\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\delta(D_\nu\varphi(\mu)) \\
&+\varphi(\mu)^{-1}\left([D_\nu Q(\mu),Q(\lambda)]+[Q(\mu),D_\nu Q(\lambda)]+D_\nu\varphi(\mu)\varphi(\mu)^{-1}[Q(\lambda),Q(\mu)]\right)\delta\varphi(\mu)\Big) \\
&\quad+\frac{1}{\nu-\mu}\operatorname{Tr}\Big(\varphi(\mu)^{-1}[Q(\nu),Q(\mu)]\delta(D_\lambda\varphi(\mu)) \\
&\quad-\varphi(\mu)^{-1}\left([D_\lambda Q(\mu),Q(\nu)]+[Q(\mu),D_\lambda Q(\nu)]+D_\lambda\varphi(\mu)\varphi(\mu)^{-1}[Q(\nu),Q(\mu)]\right)\delta\varphi(\mu)\Big) \\
&\qquad+\circlearrowleft\,.
\end{aligned}
$$

By comparing the coefficients of $\delta D_\nu\varphi(\mu)$ and of $\delta D_\lambda\varphi(\mu)$ we get the desired equations (6.2.1). The equations coming from the coefficients of $\delta\varphi(\mu)$ are differential consequences of them.

We turn to the closure relation. We are going to use the following identities:

$$\frac{1}{\mu-\nu}-\frac{1}{\lambda-\nu}=\frac{\lambda-\mu}{(\mu-\nu)(\lambda-\nu)}\,, \tag{B.16a}$$

$$\frac{1}{(\mu-\nu)(\lambda-\nu)}+\frac{1}{(\nu-\lambda)(\mu-\lambda)}+\frac{1}{(\nu-\mu)(\lambda-\mu)}=0\,, \tag{B.16b}$$

$$\operatorname{Tr}[Q(\lambda),Q(\mu)]Q(\lambda)=0\,, \tag{B.16c}$$

$$\operatorname{Tr}[Q(\lambda),Q(\nu)]Q(\mu)=\operatorname{Tr}[Q(\mu),Q(\lambda)]Q(\nu)\,. \tag{B.16d}$$

A direct computation shows that the kinetic term vanishes, in fact

$$
\begin{aligned}
&D_\nu K(\lambda,\mu)+D_\lambda K(\mu,\nu)+D_\mu K(\nu,\lambda)=\operatorname{Tr}(D_\nu\varphi(\lambda)\varphi(\lambda)^{-1}D_\mu Q(\lambda)+\circlearrowleft) \\
&=\operatorname{Tr}(\frac{1}{\lambda-\mu}D_\nu Q(\lambda)Q(\mu)+\circlearrowleft) \\
&=\operatorname{Tr}(\frac{1}{(\lambda-\mu)(\lambda-\mu)}[Q(\lambda),Q(\nu)]Q(\mu)+\circlearrowleft) \\
&=\operatorname{Tr}\left(\left(\frac{1}{(\lambda-\mu)(\lambda-\mu)}+\circlearrowleft\right)[Q(\lambda),Q(\nu)]Q(\mu)\right)=0\,.
\end{aligned}
$$

The potential term on the other hand brings

$$
\begin{aligned}
D_\nu V(\lambda,\mu)&=\frac{1}{\lambda-\mu}\operatorname{Tr}(D_\nu Q(\lambda)Q(\mu)+Q(\lambda)D_\nu Q(\mu)) \\
&=\frac{1}{\lambda-\mu}\frac{\operatorname{Tr}[Q(\nu),Q(\lambda)]Q(\mu)}{\nu-\lambda}+\frac{1}{\lambda-\mu}\frac{\operatorname{Tr}Q(\lambda)[Q(\nu),Q(\mu)]}{\nu-\mu} \\
&=\frac{1}{\lambda-\mu}\left(\frac{1}{\lambda-\nu}-\frac{1}{\mu-\nu}\right)\operatorname{Tr}[Q(\lambda),Q(\nu)]Q(\mu) \\
&=\frac{1}{(\lambda-\nu)(\nu-\mu)}\operatorname{Tr}[Q(\lambda),Q(\nu)]Q(\mu)\,,
\end{aligned}
$$

So that the cyclic sum then reads

$$\left(\frac{1}{(\lambda-\nu)(\nu-\mu)}+\frac{1}{(\nu-\mu)(\mu-\lambda)}+\frac{1}{(\mu-\lambda)(\lambda-\nu)}\right)\mathrm{Tr}[Q(\mu),Q(\lambda)]Q(\nu)=0$$

and we conclude that the Lagrangian multiform satisfies the closure relation $d\mathscr{L}=0$. □

B.3 Proof of Proposition 6.8

Proof. First, we claim that $\Omega^{(1)}$ is given by the generating function

$$\Omega^{(1)}(\lambda)=\mathrm{Tr}\left(Q_0\varphi(\lambda)^{-1}\delta\varphi(\lambda)\right)\,. \tag{B.17}$$

We need to show that $\delta\mathscr{L}+d\Omega^{(1)}=0$ on the multiform Euler-Lagrange equations $D_\mu Q(\lambda)=\frac{[Q(\mu),Q(\lambda)]}{\mu-\lambda}$. For convenience, let us denote $\psi(\lambda)\coloneqq\varphi^{-1}(\lambda)$. A direct computation shows that

$$\begin{aligned}\delta K(\lambda,\mu)=\mathrm{Tr}\Big(&D_\lambda\varphi(\mu)Q_0\delta\psi(\mu)+Q_0\psi(\mu)\delta(D_\lambda\varphi(\mu))\\&-D_\mu\varphi(\lambda)Q_0\delta\psi(\lambda)-Q_0\psi(\lambda)\delta(D_\mu\varphi(\lambda))\Big)\,,\end{aligned}$$

and

$$\delta V(\lambda,\mu)=\mathrm{Tr}\left(\frac{1}{\lambda-\mu}\psi(\lambda)[Q(\lambda),Q(\mu)]\delta\varphi(\lambda)-\frac{1}{\lambda-\mu}\psi(\mu)[Q(\lambda),Q(\mu)]\delta\varphi(\mu)\right)\,.$$

The coefficient of the generating function $\Omega^{(1)}(\lambda)=\sum_{k=0}^{\infty}\omega_k^{(1)}/\lambda^{k+1}$ are obtained as (note that $\omega_0^{(1)}=0$)

$$\omega_k^{(1)}=\mathrm{Tr}\sum_{i=1}^{k}Q_0\psi_i\delta\varphi_{k+1-i}\,. \tag{B.18}$$

Hence, for the corresponding form, we have using the variational bicomplex calculus,

$$\begin{aligned}&d\Omega^{(1)}\\&=d\left(\sum_{k=1}^{\infty}\omega_k^{(1)}\wedge dx^k\right)\\&=\mathrm{Tr}\sum_{k=0}^{\infty}\sum_{i=1}^{k}d(Q_0\psi_i\delta\varphi_{k+1-i})\wedge dx^k\\&=-\mathrm{Tr}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=1}^{k}(\partial_j\psi_i\delta\varphi_{k+1-i}+\psi_i\delta(\partial_j\varphi_{k+1-i}))\wedge dx^j\wedge dx^k\\&=\mathrm{Tr}\sum_{j<k=1}^{\infty}\sum_{i=1}^{k}(\partial_k\psi_i\delta\varphi_{j+1-i}+\psi_i\delta(\partial_k\varphi_{j+1-i})-\partial_j\psi_i\delta\varphi_{k+1-i}-\psi_i\delta(\partial_j\varphi_{k+1-i}))\wedge dx^{jk}\,.\end{aligned}$$

The associated generating function is given by

$$d\Omega^{(1)}(\lambda,\mu) = \operatorname{Tr}\Big(Q_0 D_\mu\psi(\lambda)\delta\varphi(\lambda) + Q_0\psi(\lambda)\delta(D_\mu\varphi(\lambda)) \\ - Q_0 D_\lambda\psi(\mu)\delta\varphi(\mu) - Q_0\psi(\mu)\delta(D_\lambda\varphi(\mu))\Big)\,.$$

So the sum $\delta K(\lambda,\mu) - \delta V(\lambda,\mu) + d\Omega^{(1)}(\lambda,\mu)$ reads

$$\begin{aligned}
&\operatorname{Tr}\Big(D_\lambda\varphi(\mu)Q_0\delta\psi(\mu) + Q_0\psi(\mu)\delta(D_\lambda\varphi(\mu)) - D_\mu\varphi(\lambda)Q_0\delta\psi(\lambda) - Q_0\psi(\lambda)\delta(D_\mu\varphi(\lambda)) \\
&\quad - \frac{1}{\lambda-\mu}\psi(\lambda)[Q(\lambda),Q(\mu)]\delta\varphi(\lambda) + \frac{1}{\lambda-\mu}\psi(\lambda)[Q(\lambda),Q(\mu)]\delta\varphi(\mu) \\
&\quad + Q_0 D_\mu\psi(\mu)\delta\varphi(\lambda) + Q_0\psi(\lambda)\delta(D_\mu\varphi(\lambda)) - Q_0 D_\lambda\psi(\mu)\delta\varphi(\mu) - Q_0\psi(\mu)\delta(D_\lambda\varphi(\mu))\Big) \\
&= \operatorname{Tr}\Big(\psi(\lambda)D_\mu Q(\lambda)\delta\varphi(\lambda) - \frac{1}{\lambda-\mu}\psi(\lambda)[Q(\lambda),Q(\mu)]\delta\varphi(\lambda) \\
&\qquad - \psi(\mu)D_\lambda Q(\mu) + \frac{1}{\lambda-\mu}\psi(\lambda)[Q(\lambda),Q(\mu)]\delta\varphi(\mu)\Big)\,.
\end{aligned}$$

This vanishes on the multiform Euler-Lagrange equations

$$D_\mu Q(\lambda) = \frac{[Q(\mu),Q(\lambda)]}{\mu-\lambda}\,,\qquad D_\lambda Q(\mu) = \frac{[Q(\lambda),Q(\mu)]}{\lambda-\mu}\,,$$

thus completing the argument. As a consequence,

$$\Omega(\lambda) = \delta\Omega^{(1)}(\lambda) = -\operatorname{Tr}\left(Q_0\varphi(\lambda)^{-1}\delta\varphi(\lambda)\wedge\varphi(\lambda)^{-1}\delta\varphi(\lambda)\right)\,, \tag{B.19}$$

as required. □

B.4 Proof of Proposition 6.13

Proof. We start with the general expression of the vertical vector field

$$\xi_F = \sum_{j=1}^{\infty}\left(A_j\partial_{f_j} + B_j\partial_{e_j}\right)\,,$$

and determine A_j, B_j such that $\xi_F \lrcorner \Omega = \delta F$ holds, or equivalently,

$$\xi_F \lrcorner \omega_k = \delta F_k\,,\qquad \forall k\geq 0\,.$$

Since $\omega_0 = 0$ we instantly get that F_0 has to be constant. The left-hand side reads

$$\begin{aligned} \xi_F \lrcorner \omega_k &= \sum_{i=1}^{k}\sum_{j=1}^{\infty}(A_j\delta_{i,j}\delta e_{k-i+1} - B_j\delta_{j,k-i+1}\delta f_i) = \sum_{i=1}^{k}(A_i\delta e_{k-i+1} - B_{k-i+1}\delta f_i) \\ &= \sum_{i=1}^{k}(A_{k-i+1}\delta e_i - B_{k-i+1}\delta f_i)\,, \end{aligned}$$

whilst the right hand-side is

$$\sum_{i=1}^{\infty}\left(\frac{\partial F_k}{\partial e_i}\delta e_i + \frac{\partial F_k}{\partial f_i}\delta f_i\right)\,.$$

Comparing the two we get

$$\begin{aligned} &\frac{\partial F_k}{\partial e_i} = \frac{\partial F_k}{\partial f_i} = 0\,, && \forall i > k\,, \\ &\frac{\partial F_k}{\partial e_i} = A_{k-i+1}\,, \qquad \frac{\partial F_k}{\partial f_i} = -B_{k-i+1}\,, && \forall i \leq k\,. \end{aligned}$$

The latter brings that

$$\frac{\partial F_k}{\partial e_i} = A_{k-i+1} = A_{(k+1)-(i+1)+1} = \frac{\partial F_{k+1}}{\partial e_{i+1}}\,,$$

and similarly for f_i. These conditions are necessary and sufficient. □

B.5 Proof of Proposition 6.14

Proof. We will show that

$$\xi_H \lrcorner \Omega = \delta H \quad \text{where} \quad \Omega = \sum_{k=1}^{\infty}\sum_{m=1}^{k}\delta f_m \wedge \delta e_{k+1-m} \wedge dx^k\,. \tag{B.20}$$

We start with the left hand-side

$$\begin{aligned} \xi_H \lrcorner \Omega =& \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{k}\left(-\frac{\partial H}{\partial f_i}\partial_{e_1} \wedge \partial_i + \frac{\partial H}{\partial e_i}\partial_{f_1} \wedge \partial_i\right) \lrcorner \left(\delta f_m \wedge \delta e_{k+1-m} \wedge \delta x^k\right) \\ =& \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{k}\left(\frac{\partial H}{\partial f_i}\delta_{ik}\delta_{k+1-m,1}\delta f_m\right) + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{k}\left(\frac{\partial H}{\partial e_i}\delta_{ik}\delta_{m,1}\delta e_{k+1-m}\right) \\ =& \sum_{i=1}^{\infty}\frac{\partial H}{\partial f_i}\delta f_i + \sum_{i=1}^{\infty}\frac{\partial H}{\partial e_i}\delta e_i = \delta H\,. \end{aligned}$$

□

B.6 Proof of Proposition 6.17

Proof. 1. The multi-time Poisson bracket $\{\![F,G]\!\}$ is an admissible 1-form. In fact we have that

$$\begin{aligned}
\frac{\partial}{\partial f_{m+1}}\left(\{\![F,G]\!\}\right)_{k+1} &= \frac{\partial}{\partial f_{m+1}}\{F_{k+1},G_{k+1}\}_{k+1} \\
&= \frac{\partial}{\partial f_{m+1}}\sum_{j=1}^{k+1}\left(\frac{\partial F_{k+1}}{\partial f_j}\frac{\partial G_{k+1}}{\partial e_{k+2-j}} - \frac{\partial F_{k+1}}{\partial e_j}\frac{\partial G_{k+1}}{\partial f_{k+2-j}}\right) \\
&= \sum_{j=1}^{k+1}\left(\frac{\partial^2 F_{k+1}}{\partial f_{m+1}\partial f_j}\frac{\partial G_{k+1}}{\partial e_{k+2-j}} + \frac{\partial F_{k+1}}{\partial f_j}\frac{\partial^2 G_{k+1}}{\partial f_{m+1}\partial e_{k+2-j}}\right. \\
&\qquad\left. - \frac{\partial^2 F_{k+1}}{\partial f_{m+1}\partial e_j}\frac{\partial G_{k+1}}{\partial f_{k+2-j}} - \frac{\partial F_{k+1}}{\partial e_j}\frac{\partial^2 G_{k+1}}{\partial f_{m+1}\partial f_{k+2-j}}\right) \\
&= \sum_{j=1}^{k+1}\left(\frac{\partial^2 F_{k}}{\partial f_{m}\partial f_j}\frac{\partial G_{k}}{\partial e_{k+1-j}} + \frac{\partial F_{k+1}}{\partial f_j}\frac{\partial^2 G_{k}}{\partial f_{m}\partial e_{k+2-j}}\right. \\
&\qquad\left. - \frac{\partial^2 F_{k}}{\partial f_{m}\partial e_j}\frac{\partial G_{k}}{\partial f_{k+1-j}} - \frac{\partial F_{k+1}}{\partial e_j}\frac{\partial^2 G_{k}}{\partial f_{m}\partial f_{k+2-j}}\right) \\
&= \sum_{j=1}^{k+1}\left(\frac{\partial^2 F_{k}}{\partial f_{m}\partial f_j}\frac{\partial G_{k}}{\partial e_{k+1-j}} - \frac{\partial^2 F_{k}}{\partial f_{m}\partial e_j}\frac{\partial G_{k}}{\partial f_{k+1-j}}\right) \\
&\quad + \sum_{j=1}^{k+1}\left(\frac{\partial F_{k+1}}{\partial f_j}\frac{\partial^2 G_{k}}{\partial f_{m}\partial e_{k+2-j}} - \frac{\partial F_{k+1}}{\partial e_j}\frac{\partial^2 G_{k}}{\partial f_{m}\partial f_{k+2-j}}\right).
\end{aligned}$$

Now we use the fact that the $(k+1)$-th term of the first sum vanishes as $\frac{\partial F_k}{\partial f_{k+1}} = \frac{\partial F_k}{\partial e_{k+1}} = 0$ and in the second sum we substitute $j \to j+1$

$$\begin{aligned}
\frac{\partial}{\partial f_{m+1}}\left(\{\![F,G]\!\}\right)_{k+1} &= \sum_{j=1}^{k}\left(\frac{\partial^2 F_{k}}{\partial f_{m}\partial f_j}\frac{\partial G_{k}}{\partial e_{k+1-j}} - \frac{\partial^2 F_{k}}{\partial f_{m}\partial e_j}\frac{\partial G_{k}}{\partial f_{k+1-j}}\right) \\
&\quad + \sum_{j=0}^{k}\left(\frac{\partial F_{k+1}}{\partial f_{j+1}}\frac{\partial^2 G_{k}}{\partial f_{m}\partial e_{k+1-j}} - \frac{\partial F_{k+1}}{\partial e_{j+1}}\frac{\partial^2 G_{k}}{\partial f_{m}\partial f_{k+1-j}}\right) \\
&= \sum_{j=1}^{k}\left(\frac{\partial^2 F_{k}}{\partial f_{m}\partial f_j}\frac{\partial G_{k}}{\partial e_{k+1-j}} - \frac{\partial^2 F_{k}}{\partial f_{m}\partial e_j}\frac{\partial G_{k}}{\partial f_{k+1-j}}\right) \\
&\quad + \sum_{j=0}^{k}\left(\frac{\partial F_{k}}{\partial f_{j}}\frac{\partial^2 G_{k}}{\partial f_{m}\partial e_{k+1-j}} - \frac{\partial F_{k}}{\partial e_{j}}\frac{\partial^2 G_{k}}{\partial f_{m}\partial f_{k+1-j}}\right) \\
&= \frac{\partial}{\partial f_m}\left(\{\![F,G]\!\}\right)_{k}
\end{aligned}$$

because the 0-th term of the second sum vanishes for the same reason. The proof of $\frac{\partial}{\partial e_m}\left(\{\![F,G]\!\}\right)_k = \frac{\partial}{\partial e_{m+1}}\left(\{\![F,G]\!\}\right)_{k+1}$ works in the same way.
The multi-time Poisson bracket $\{\![F,H]\!\}$ is admissible because it is a 0-form.

2. The proof follows from the Jacobi identity of the single-time Poisson bracket $\{\ ,\ \}_k$ (which is easy to prove as it is already written in Darboux form).

3. This part of the proof is the most laborious, and it is performed by computing the three terms separately and adding them together. We have

$$\begin{aligned}
\{\!\!\{\{\!\!\{F,G\}\!\!\},H\}\!\!\} =& -\xi_{\{\!\!\{F,G\}\!\!\}}\lrcorner\delta H \\
=& \sum_{k=1}^{\infty}\sum_{j=1}^{k}\Bigg(\frac{\partial^2 F_k}{\partial f_1\partial f_j}\frac{\partial G_k}{\partial e_{k-j+1}}\frac{\partial H}{\partial e_k}+\frac{\partial F_k}{\partial f_j}\frac{\partial^2 G_k}{\partial f_1\partial e_{k-j+1}}\frac{\partial H}{\partial e_k} \\
&-\frac{\partial^2 F_k}{\partial f_1\partial e_j}\frac{\partial G_k}{\partial f_{k-j+1}}\frac{\partial H}{\partial e_k}-\frac{\partial F_k}{\partial e_j}\frac{\partial^2 G_k}{\partial f_1\partial f_{k-j+1}}\frac{\partial H}{\partial e_k} \\
&-\frac{\partial^2 F_k}{\partial e_1\partial f_j}\frac{\partial G_k}{\partial e_{k-j+1}}\frac{\partial H}{\partial f_k}-\frac{\partial F_k}{\partial f_j}\frac{\partial^2 G_k}{\partial e_1\partial e_{k-j+1}}\frac{\partial H}{\partial f_k} \\
&+\frac{\partial^2 F_k}{\partial e_1\partial e_j}\frac{\partial G_k}{\partial f_{k-j+1}}\frac{\partial H}{\partial f_k}+\frac{\partial F_k}{\partial e_j}\frac{\partial^2 G_k}{\partial e_1\partial f_{k-j+1}}\frac{\partial H}{\partial f_k}\Bigg),
\end{aligned}$$

$$\begin{aligned}
\{\!\!\{\{\!\!\{H,F\}\!\!\},G\}\!\!\} =& \xi_G\lrcorner\xi_F\lrcorner\delta H \\
=& \sum_{k=1}^{\infty}\sum_{j=1}^{k}\Bigg(\frac{\partial^2 F_k}{\partial e_j\partial f_1}\frac{\partial G_j}{\partial f_1}\frac{\partial H}{\partial e_k}-\frac{\partial^2 F_k}{\partial e_j\partial e_1}\frac{\partial G_j}{\partial f_1}\frac{\partial H}{\partial f_k} \\
&-\frac{\partial^2 F_k}{\partial f_j\partial f_1}\frac{\partial G_j}{\partial e_1}\frac{\partial H}{\partial e_k}+\frac{\partial^2 F_k}{\partial f_j\partial e_1}\frac{\partial G_j}{\partial e_1}\frac{\partial H}{\partial f_k}\Big) \\
&+\sum_{j,k=1}^{\infty}\Big(\frac{\partial F_k}{\partial f_1}\frac{\partial G_j}{\partial f_1}\frac{\partial^2 H}{\partial e_j\partial e_k}-\frac{\partial F_k}{\partial e_1}\frac{\partial G_j}{\partial f_1}\frac{\partial^2 H}{\partial e_j\partial f_k} \\
&-\frac{\partial F_k}{\partial f_1}\frac{\partial G_j}{\partial e_1}\frac{\partial^2 H}{\partial f_j\partial e_k}+\frac{\partial F_k}{\partial e_1}\frac{\partial G_j}{\partial e_1}\frac{\partial^2 H}{\partial f_j\partial f_k}\Big),
\end{aligned}$$

$$\begin{aligned}
\{\!\!\{\{\!\!\{G,H\}\!\!\},F\}\!\!\} =& -\{\!\!\{F,\{\!\!\{G,H\}\!\!\}\}\!\!\} = \xi_F\lrcorner\{\!\!\{G,H\}\!\!\} \\
=& -\xi_F\lrcorner\xi_G\lrcorner\delta H \\
=& -\sum_{k=1}^{\infty}\sum_{j=1}^{k}\Big(\frac{\partial F_j}{\partial f_1}\frac{\partial^2 G_k}{\partial e_j\partial f_1}\frac{\partial H}{\partial e_k}-\frac{\partial F_j}{\partial f_1}\frac{\partial^2 G_k}{\partial e_j\partial e_1}\frac{\partial H}{\partial f_k} \\
&-\frac{\partial F_j}{\partial e_1}\frac{\partial^2 G_k}{\partial f_j\partial f_1}\frac{\partial H}{\partial e_k}+\frac{\partial F_j}{\partial e_1}\frac{\partial^2 G_k}{\partial f_j\partial e_1}\frac{\partial H}{\partial f_k}\Big) \\
&-\sum_{j,k=1}^{\infty}\Big(\frac{\partial F_j}{\partial f_1}\frac{\partial G_k}{\partial f_1}\frac{\partial^2 H}{\partial e_j\partial e_k}-\frac{\partial F_j}{\partial f_1}\frac{\partial G_k}{\partial e_1}\frac{\partial^2 H}{\partial e_j\partial f_k} \\
&-\frac{\partial F_j}{\partial e_1}\frac{\partial G_k}{\partial f_1}\frac{\partial^2 H}{\partial f_j\partial e_k}+\frac{\partial F_j}{\partial e_1}\frac{\partial G_k}{\partial e_1}\frac{\partial^2 H}{\partial f_j\partial f_k}\Big),
\end{aligned}$$

We add the last two together, simplifying the terms with the double derivative of

H, to obtain the expression for $\{[\{[H,F]\},G]\} + \{[\{[G,H]\},F]\}$

$$
\begin{aligned}
=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\Big(&\frac{\partial^2 F_k}{\partial e_j\partial f_1}\frac{\partial G_j}{\partial f_1}\frac{\partial H}{\partial e_k} - \frac{\partial^2 F_k}{\partial e_j\partial e_1}\frac{\partial G_j}{\partial f_1}\frac{\partial H}{\partial f_k} - \frac{\partial^2 F_k}{\partial f_j\partial f_1}\frac{\partial G_j}{\partial e_1}\frac{\partial H}{\partial e_k} + \frac{\partial^2 F_k}{\partial f_j\partial e_1}\frac{\partial G_j}{\partial e_1}\frac{\partial H}{\partial f_k}\\
&- \frac{\partial F_j}{\partial f_1}\frac{\partial^2 G_k}{\partial e_j\partial f_1}\frac{\partial H}{\partial e_k} + \frac{\partial F_j}{\partial f_1}\frac{\partial^2 G_k}{\partial e_j\partial e_1}\frac{\partial H}{\partial f_k} + \frac{\partial F_j}{\partial e_1}\frac{\partial^2 G_k}{\partial f_j\partial f_1}\frac{\partial H}{\partial e_k} - \frac{\partial F_j}{\partial e_1}\frac{\partial^2 G_k}{\partial f_j\partial e_1}\frac{\partial H}{\partial f_k}\Big)\,.
\end{aligned}
$$

We now use $\frac{\partial F_j}{\partial f_1} = \frac{\partial F_{j+k-j}}{\partial f_{1+k-j}} = \frac{\partial F_k}{\partial f_{k-j+1}}$, etc.

$$
\begin{aligned}
=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\Big(&\frac{\partial^2 F_k}{\partial e_j\partial f_1}\frac{\partial G_k}{\partial f_{k-j+1}}\frac{\partial H}{\partial e_k} - \frac{\partial^2 F_k}{\partial e_j\partial e_1}\frac{\partial G_k}{\partial f_{k-j+1}}\frac{\partial H}{\partial f_k} - \frac{\partial^2 F_k}{\partial f_j\partial f_1}\frac{\partial G_k}{\partial e_{k-j+1}}\frac{\partial H}{\partial e_k}\\
&+ \frac{\partial^2 F_k}{\partial f_j\partial e_1}\frac{\partial G_k}{\partial e_{k-j+1}}\frac{\partial H}{\partial f_k} - \frac{\partial F_k}{\partial f_{k-j+1}}\frac{\partial^2 G_k}{\partial e_j\partial f_1}\frac{\partial H}{\partial e_k} + \frac{\partial F_k}{\partial f_{k-j+1}}\frac{\partial^2 G_k}{\partial e_j\partial e_1}\frac{\partial H}{\partial f_k}\\
&+ \frac{\partial F_k}{\partial e_{k-j+1}}\frac{\partial^2 G_k}{\partial f_j\partial f_1}\frac{\partial H}{\partial e_k} - \frac{\partial F_k}{\partial e_{k-j+1}}\frac{\partial^2 G_k}{\partial f_j\partial e_1}\frac{\partial H}{\partial f_k}\Big)\,,
\end{aligned}
$$

that is also

$$
\begin{aligned}
=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\Big(&\frac{\partial^2 F_k}{\partial e_j\partial f_1}\frac{\partial G_k}{\partial f_{k-j+1}}\frac{\partial H}{\partial e_k} - \frac{\partial^2 F_k}{\partial e_j\partial e_1}\frac{\partial G_k}{\partial f_{k-j+1}}\frac{\partial H}{\partial f_k} - \frac{\partial^2 F_k}{\partial f_j\partial f_1}\frac{\partial G_k}{\partial e_{k-j+1}}\frac{\partial H}{\partial e_k}\\
&+ \frac{\partial^2 F_k}{\partial f_j\partial e_1}\frac{\partial G_k}{\partial e_{k-j+1}}\frac{\partial H}{\partial f_k} - \frac{\partial F_k}{\partial f_j}\frac{\partial^2 G_k}{\partial e_{k-j+1}\partial f_1}\frac{\partial H}{\partial e_k} + \frac{\partial F_k}{\partial f_j}\frac{\partial^2 G_k}{\partial e_{k-j+1}\partial e_1}\frac{\partial H}{\partial f_k}\\
&+ \frac{\partial F_k}{\partial e_j}\frac{\partial^2 G_k}{\partial f_{k-j+1}\partial f_1}\frac{\partial H}{\partial e_k} - \frac{\partial F_k}{\partial e_j}\frac{\partial^2 G_k}{\partial f_{k-j+1}\partial e_1}\frac{\partial H}{\partial f_k}\Big)\\
=-&\ \{[\{[F,G]\},H]\}
\end{aligned}
$$

as desired.

□

B.7 Proof of Theorem 6.20

Lemma B.1 *For each $k \geq 0$, the only non-zero single-time Poisson of a_i, b_i and c_i, $0 \leq i \leq k$, are given by*

$$\{a_i, b_j\}_k = b_{i+j-k-1}\,, \tag{B.21a}$$

$$\{a_i, c_j\}_k = -c_{i+j-k-1}\,, \tag{B.21b}$$

$$\{b_i, c_j\}_k = 2a_{i+j-k-1}\,. \tag{B.21c}$$

For convenience, we use the convention that a coefficient in a series vanishes when its index is negative. Hence, it is understood that $\{a_i, b_j\}_k = \{a_i, c_j\}_k = \{b_i, c_j\}_k = 0$ whenever $i + j < k + 1$.

Proof. We start with the fact that for any power series α and β we have

$$\sum_{\ell=1}^{k} \alpha_{i-\ell}\beta_{j+\ell-k-1} = (\alpha\beta)_{i+j-k-1} \,. \tag{B.22}$$

In fact, by limiting the sum only to the non-zero terms:

$$\sum_{\ell=1}^{k} \alpha_{i-l}\,\beta_{j+\ell-k-1} = \sum_{\ell=k+1-j}^{i} \alpha_{i-\ell}\,\beta_{j+\ell-k-1} = \sum_{m=0}^{i+j-k-1} \alpha_{i+j-k-1-m}\,\beta_m = (\alpha\beta)_{i+j-k-1} \,.$$

We study the case where $k+1-j \leq j$, namely $i+j \geq k+1$. If $i+j < k+1$ then the sum is empty, and the result is zero. We are now ready to compute the following Poisson brackets using the formulas in Appendix B.1.

$$\begin{aligned}
\{a_i, b_j\}_k =& \sum_{\ell=1}^{k} \left(\frac{\partial a_i}{\partial f_\ell}\frac{\partial b_j}{\partial e_{k+1-\ell}} - \frac{\partial b_j}{\partial f_\ell}\frac{\partial a_i}{\partial e_{k+1-\ell}} \right) \\
=& \sum_{\ell=1}^{k} \left(e_{i-\ell} \left(\frac{i-3a}{2\sqrt{i-a}} \right)_{j+\ell-k-1} - \left(\frac{-e^2}{2\sqrt{i-a}} \right)_{j-\ell} f_{i+\ell-k-1} \right) \\
=& \left(\frac{ie - 3ae + e^2 f}{2\sqrt{i-a}} \right)_{i+j-k-1} = \left(\frac{ie - 3ae + (i+a)e}{2\sqrt{i-a}} \right)_{i+j-k-1} \\
=& (e\sqrt{i-a})_{i+j-k-1} = b_{i+j-k-1} \,. \\
\{a_i, c_j\}_k =& \sum_{\ell=1}^{k} \left(\frac{\partial a_i}{\partial f_\ell}\frac{\partial c_j}{\partial e_{k+1-\ell}} - \frac{\partial c_j}{\partial f_\ell}\frac{\partial a_i}{\partial e_{k+1-\ell}} \right) \\
=& \sum_{\ell=1}^{k} \left(e_{i-\ell} \left(\frac{-f^2}{2\sqrt{i-a}} \right)_{j+\ell-k-1} - \left(\frac{i-3a}{2\sqrt{i-a}} \right)_{j-\ell} f_{i+\ell-k-1} \right) \\
=& \left(\frac{-ef^2 - f(i-3a)}{2\sqrt{i-a}} \right)_{i+j-k-1} = \left(\frac{-f(i+a) - if + 3af}{2\sqrt{i-a}} \right)_{i+j-k-1} \\
=& -(f\sqrt{i-a})_{i+j-k-1} = -c_{i+j-k-1} \,. \\
\{a_i, a_j\}_k =& \sum_{\ell=1}^{k} \left(\frac{\partial a_i}{\partial f_\ell}\frac{\partial a_j}{\partial e_{k+1-\ell}} - \frac{\partial a_j}{\partial f_\ell}\frac{\partial a_i}{\partial e_{k+1-\ell}} \right) \\
=& \sum_{\ell=1}^{k} \left(\left(\frac{-e^2}{2\sqrt{i-a}} \right)_{i-\ell} \left(\frac{-f^2}{2\sqrt{i-a}} \right)_{j+\ell-k-1} - \left(\frac{i-3a}{2\sqrt{i-a}} \right)_{j-\ell} \left(\frac{i-3a}{2\sqrt{i-a}} \right)_{i+\ell-k-1} \right) \\
=& \left(\frac{e^2 f^2 - (i-3a)^2}{4(i-a)} \right)_{i+j-k-1} = \left(\frac{(i+a)^2 - (i-3a)^2}{4(i-a)} \right)_{i+j-k-1} \\
=& 2a_{i+j-k-1} \,.
\end{aligned}$$

□

Remark B.2: These Poisson bracket coincides with the $\{\ ,\ \}_{-k}$ in [AC17]. In this instance we do not take the Poisson brackets of a_i, b_i, c_i for $i > k$ because they do not belong to the k-th single-time phase space.

Proof of Theorem 6.20. We start by proving that

$$\frac{\partial Q}{\partial e_k}(\lambda) = \lambda \frac{\partial Q}{\partial e_{k+1}}(\lambda)\,, \qquad \frac{\partial Q}{\partial f_k}(\lambda) = \lambda \frac{\partial Q}{\partial f_{k+1}}(\lambda)\,. \tag{B.23}$$

This is done for each matrix element. In fact

$$\begin{aligned}
\frac{\partial Q}{\partial e_k} &= \frac{\partial Q}{\partial e}\frac{\partial e}{\partial e_k} = \lambda^{-k}\frac{\partial Q}{\partial e} = \lambda\lambda^{-k-1}\frac{\partial Q}{\partial e} = \lambda\frac{\partial Q}{\partial e}\frac{\partial e}{\partial e_{k+1}} = \lambda\frac{\partial Q}{\partial e_{k+1}}\,,\\
\frac{\partial Q}{\partial f_k} &= \frac{\partial Q}{\partial f}\frac{\partial f}{\partial f_k} = \lambda^{-k}\frac{\partial Q}{\partial f} = \lambda\lambda^{-k-1}\frac{\partial Q}{\partial f} = \lambda\frac{\partial Q}{\partial f}\frac{\partial f}{\partial f_{k+1}} = \lambda\frac{\partial Q}{\partial f_{k+1}}\,.
\end{aligned}$$

By virtue of the previous result, and since Q_0 is constant, we have the following:

$$\frac{\partial Q}{\partial e_k} = \sum_{j=0}^{\infty}\frac{\partial Q_j}{\partial e_k}\lambda^{-j} \tag{B.24a}$$

$$\lambda\frac{\partial Q}{\partial e_{k+1}} = \lambda\sum_{i=0}^{\infty}\frac{\partial Q_i}{\partial e_{k+1}}\lambda^{-i} = \sum_{i=0}^{\infty}\frac{\partial Q_i}{\partial e_{k+1}}\lambda^{-i+1} = \sum_{j=0}^{\infty}\frac{\partial Q_{j+1}}{\partial e_{k+1}}\lambda^{-j}\,. \tag{B.24b}$$

If we look at the coefficients in λ we see that, for all j and k, $\frac{\partial Q_j}{\partial e_k} = \frac{\partial Q_{j+1}}{\partial e_{k+1}}$. Similarly one can obtain that $\frac{\partial Q_j}{\partial f_k} = \frac{\partial Q_{j+1}}{\partial f_{k+1}}$.
Finally, we check that the Lax form is admissible, using Proposition 6.13, i.e. that

$$\begin{aligned}
\frac{\partial Q^{(i)}}{\partial e_j} &= \sum_{k=0}^{i}\lambda^{i-k}\frac{\partial Q_k}{\partial e_j} = \sum_{k=0}^{i}\lambda^{i-k}\frac{\partial Q_{k+1}}{\partial e_{j+1}} = \sum_{k=0}^{i}\lambda^{(i+1)-(k+1)}\frac{\partial Q_{k+1}}{\partial e_{j+1}}\\
&= \sum_{k=1}^{i+1}\lambda^{i+1-k}\frac{\partial Q_k}{\partial e_{j+1}} = \frac{\partial Q^{(i+1)}}{\partial e_{j+1}} - \lambda^{i+1}\frac{\partial Q_0}{\partial e_{j+1}} = \frac{\partial Q^{(i+1)}}{\partial e_{j+1}}\,,
\end{aligned}$$

where we used that Q_0 is constant. Similarly $\frac{\partial Q^{(i)}}{\partial f_j} = \frac{\partial Q^{(i+1)}}{\partial f_{j+1}}$.

We now turn to the proof of (6.62). Thanks to the decomposition of the multi-time Poisson bracket into single-time Poisson brackets, we have that $\{[W_{\mathbf{1}}(\lambda), W_{\mathbf{2}}(\mu)]\} = [r_{\mathbf{12}}(\lambda,\mu), W_{\mathbf{1}}(\lambda) + W_{\mathbf{2}}(\mu)]$ if and only for all $k \geq 0$,

$$\{Q_{\mathbf{1}}^{(k)}(\lambda), Q_{\mathbf{2}}^{(k)}(\mu)\}_k = [r_{\mathbf{12}}(\lambda,\mu), Q_{\mathbf{1}}^{(k)}(\lambda) + Q_{\mathbf{2}}^{(k)}(\mu)]\,. \tag{B.25}$$

Writing $Q^{(k)}(\lambda) = Q_+^{(k)}(\lambda)\sigma_+ + Q_-^{(k)}(\lambda)\sigma_- + Q_3^{(k)}(\lambda)\sigma_3$, the right hand-side of (B.25) reads

$$\begin{aligned}
&[r_{\mathbf{12}}(\lambda,\mu), Q_{\mathbf{1}}^{(k)}(\lambda) + Q_{\mathbf{2}}^{(k)}(\mu)] \qquad \text{(B.26)}\\
&= \frac{2(Q_3^{(k)}(\mu) - Q_3^{(k)}(\lambda))}{\mu-\lambda}(\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+) + \frac{Q_+^{(k)}(\mu) - Q_+^{(k)}(\lambda)}{\mu-\lambda}(\sigma_3\otimes\sigma_+ - \sigma_+\otimes\sigma_3)\\
&\quad + \frac{Q_-^{(k)}(\lambda) - Q_-^{(k)}(\mu)}{\mu-\lambda}(\sigma_3\otimes\sigma_- - \sigma_-\otimes\sigma_3)\,,
\end{aligned}$$

while the left hand-side is given by

$$\begin{aligned}
&\{Q_1^{(k)}(\lambda), Q_2^{(k)}(\mu)\}_k\\
&= \sum_{i,j=0}^{k} \frac{(\lambda\mu)^k}{\lambda^i\mu^j}\left(\{a_i,a_j\}_k\sigma_3\otimes\sigma_3 + \{b_i,b_j\}_k\sigma_+\otimes\sigma_+ + \{c_i,c_j\}_k\sigma_-\otimes\sigma_-\right.\\
&\quad + \{b_i,c_j\}_k\sigma_+\otimes\sigma_- + \{c_i,b_j\}_k\sigma_-\otimes\sigma_+ + \{a_i,b_j\}_k\sigma_3\otimes\sigma_+\\
&\quad \left. + \{b_i,a_j\}_k\sigma_+\otimes\sigma_3 + \{a_i,c_j\}_k\sigma_3\otimes\sigma_- + \{c_i,a_j\}_k\sigma_-\otimes\sigma_3\right)\,.
\end{aligned} \qquad \text{(B.27)}$$

We now invoke Lemma B.1 which gives the necessary single-time Poisson brackets and allows us to check directly that (B.26) is equal to (B.27). We show it for the $\sigma_+\otimes\sigma_-$ component, as the others are obtained similarly. In the left hand-side we have

$$\frac{2}{\mu-\lambda}(Q_3^{(k)}(\mu) - Q_3^{(k)}(\lambda)) = \frac{2}{\mu-\lambda}\sum_{j=0}^{k}(\mu^{k-j} - \lambda^{k-j})a_j = 2\sum_{j=0}^{k}\sum_{i=1}^{k-j-1}\lambda^i\mu^{k-i-j-1}a_j\,,$$

while right hand-side is equal to

$$2\sum_{i,j=0}^{k}\frac{(\lambda\mu)^k}{\lambda^i\mu^j}a_{i+j-k-1} = 2\sum_{i=0}^{k}\sum_{m=0}^{i-1}\frac{a_m}{\lambda^{i-k}\mu^{m+1-i}} = 2\sum_{n=0}^{k}\sum_{m=0}^{k-n-1}\lambda^n\mu^{k-n-m-1}a_m\,.$$

This concludes the proof. □

B.8 Proof of Theorem 6.22

Proof. Note the set of zero-curvature equations can be written as

$$dW(\lambda) = W(\lambda)\wedge W(\lambda)\,, \qquad \text{(B.28)}$$

where the right-hand side is understood as

$$\begin{aligned}
&W(\lambda)\wedge W(\lambda)\\
&= \left(\sum_{i=0}^{\infty}Q^{(i)}(\lambda)dx^i\right)\wedge\left(\sum_{j=0}^{\infty}Q^{(j)}(\lambda)dx^j\right) = \sum_{i<j}[Q^{(i)}(\lambda), Q^{(j)}(\lambda)]\,dx^{ij}\,,
\end{aligned}$$

and the left-hand side is $dW(\lambda) = \sum_{i<j}(\partial_i Q^{(j)}(\lambda) - \partial_j Q^{(i)}(\lambda))\, dx^{ij}$. Thus, we will prove that

$$W(\lambda) \wedge W(\lambda) = \sum_{i<j} \{\![H_{ij}, W(\lambda)]\!\}\, dx^{ij}. \tag{B.29}$$

By definition

$$\sum_{i<j} \{\![H_{ij}, W(\lambda)]\!\} = \xi_W(\lambda) \lrcorner \delta\mathcal{H} = \sum_{i<j} (\xi_W(\lambda) \lrcorner \delta H_{ij})\, dx^{ij}\,,$$

where, using the expression (6.61) for $\xi_W(\lambda)$, we find

$$\xi_W(\lambda) \lrcorner \delta H_{ij} = \sum_{k=1}^{j} \left(\frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{\partial H_{ij}}{\partial f_k} - \frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \frac{\partial H_{ij}}{\partial e_k} \right) .$$

Hence (B.29) is equivalent to, for $i < j$,

$$[Q^{(i)}(\lambda), Q^{(j)}(\lambda)] = \sum_{k=1}^{j} \left(\frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{\partial H_{ij}}{\partial f_k} - \frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \frac{\partial H_{ij}}{\partial e_k} \right) . \tag{B.30}$$

We prove the latter in generating form as follows. We multiply both sides by $\mu^{-i-1}\nu^{-j-1}$ and form the following sums over i and j

$$\begin{aligned} &\sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{1}{\mu^{i+1}\nu^{j+1}} [Q^{(i)}(\lambda), Q^{(j)}(\lambda)] \\ &\qquad = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{1}{\mu^{i+1}\nu^{j+1}} \sum_{k=1}^{j} \left(\frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{\partial H_{ij}}{\partial f_k} - \frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \frac{\partial H_{ij}}{\partial e_k} \right) . \end{aligned}$$

We can rearrange the sums in the right-hand side to get

$$\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \sum_{i=0}^{j} \frac{1}{\mu^{i+1}\nu^{j+1}} (\cdots) = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{1}{\mu^{i+1}\nu^{j+1}} (\cdots)\,,$$

where we have used the fact that H_{ij} depends only on $e_1, \ldots, e_j$ and $f_1, \ldots, f_j$ in the second step to extend the sum over j from 0 instead of k. We can similarly form the sums with $\mu \leftrightarrow \nu$ and use the same trick to rearrange the sums in the right-hand side. Using the anti-symmetry of both left and right-hand side of (B.30), we come to the following generating form of (B.30)

$$\begin{aligned} &\sum_{i,j=0}^{\infty} \frac{1}{\mu^{i+1}\nu^{j+1}} [Q^{(i)}(\lambda), Q^{(j)}(\lambda)] \\ &= \sum_{k=1}^{\infty} \left(\frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{\partial}{\partial f_k} \sum_{i,j=0}^{\infty} \frac{H_{ij}}{\mu^{i+1}\nu^{j+1}} - \frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \frac{\partial}{\partial e_k} \sum_{i,j=0}^{\infty} \frac{H_{ij}}{\mu^{i+1}\nu^{j+1}} \right) , \end{aligned}$$

i.e. ,

$$\frac{[Q(\mu), Q(\nu)]}{(\mu-\lambda)(\nu-\lambda)} = \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{\partial \mathcal{H}(\mu,\nu)}{\partial f_k} - \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \frac{\partial \mathcal{H}(\mu,\nu)}{\partial e_k}, \tag{B.31}$$

where we have used

$$\sum_{i=0}^{\infty} \frac{Q^{(i)}(\lambda)}{\mu^{i+1}} = \frac{Q(\mu)}{\mu-\lambda}.$$

We now show that (B.31) holds by computing its right-hand side recalling that

$$\mathcal{H}(\mu,\nu) = -\frac{1}{2}\frac{\operatorname{Tr}(Q(\mu)-Q(\nu))^2}{\mu-\nu} = \frac{2}{\mu-\nu} + \frac{\operatorname{Tr} Q(\mu)Q(\nu)}{\mu-\nu}.$$

For convenience, denote $a(\mu), a(\nu), a(\lambda)$ by a, a', a'' respectively and similarly for b and c. We have

$$\frac{\partial \mathcal{H}(\mu,\nu)}{\partial f_k} = \frac{1}{(\mu-\nu)} \operatorname{Tr}\left(\frac{\partial Q(\mu)}{\partial f_k} Q(\nu) + \frac{\partial Q(\nu)}{\partial f_k} Q(\mu)\right) \tag{B.32}$$

and

$$\begin{aligned}
&\frac{1}{\mu-\nu} \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \operatorname{Tr} \frac{\partial Q(\mu)}{\partial f_k} Q(\mu) \\
&= \frac{1}{\mu-\nu} \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{1}{\mu^k \sqrt{i-a}} \operatorname{Tr} \begin{pmatrix} b & -\frac{b^2}{2(i-a)} \\ \frac{i-3a}{2} & -b \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \\
&= \frac{1}{\mu-\nu} \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \frac{1}{\mu^k \sqrt{i-a}} \left(2ba' - \frac{b^2 c'}{2(i-a)} + \frac{(i-3a)b'}{2}\right) \\
&= \frac{\mu}{(\mu-\nu)} \left(\frac{\partial}{\partial e_1} \frac{Q(\mu)}{\mu-\lambda}\right) \frac{1}{\sqrt{i-a}} \left(2ba' - \frac{b^2 c'}{2(i-a)} + \frac{(i-3a)b'}{2}\right) \\
&= \frac{1}{(\mu-\nu)(\mu-\lambda)} \frac{1}{i-a} \left(2ba' - \frac{b^2 c'}{2(i-a)} + \frac{(i-3a)b'}{2}\right) \begin{pmatrix} c & \frac{i-3a}{2} \\ -\frac{c^2}{2(i-a)} & -c \end{pmatrix},
\end{aligned}$$

where we have used that $\sum_{k=1}^{\infty} \frac{Q^{(k)}}{\mu^k} = \mu(\frac{Q(\mu)}{\mu-\lambda} - \frac{Q_0}{\mu})$ and that Q_0 is constant. Similarly, we have

$$\begin{aligned}
&\frac{1}{\mu-\nu} \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial e_1} \operatorname{Tr} \frac{\partial Q(\nu)}{\partial f_k} Q(\mu) = \\
&\frac{1}{(\mu-\nu)(\nu-\lambda)} \frac{1}{i-a'} (2b'a - \frac{b'^2 c}{2(i-a')} + \frac{(i-3a')b}{2}) \begin{pmatrix} c' & \frac{i-3a'}{2} \\ -\frac{c'^2}{2(i-a')} & -c' \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\nu-\mu} \sum_{k=1}^{\infty} \frac{\partial Q^{(k)}(\lambda)}{\partial f_1} \operatorname{Tr} \frac{\partial Q(\mu)}{\partial e_k} Q(\nu) = \\
&\frac{1}{(\nu-\mu)(\mu-\lambda)} \frac{1}{i-a} (2ca' - \frac{c^2 b'}{2(i-a)} + \frac{(i-3a)c'}{2}) \begin{pmatrix} b & -\frac{b^2}{2(i-a)} \\ \frac{i-3a}{2} & -b \end{pmatrix},
\end{aligned}$$

and

$$\frac{1}{\nu-\mu}\sum_{k=1}^{\infty}\frac{\partial Q^{(k)}(\lambda)}{\partial f_1}\operatorname{Tr}\frac{\partial Q(\nu)}{\partial e_k}(Q(\mu)) =$$
$$\frac{1}{(\nu-\mu)(\nu-\lambda)}\frac{1}{i-a'}(2c'a - \frac{c'^2 b}{2(i-a')} + \frac{(i-3a')c}{2})\begin{pmatrix} b' & -\frac{b'^2}{2(i-a')} \\ \frac{i-3a'}{2} & -b' \end{pmatrix}.$$

We collect all the contributions on the σ_3 component for instance (the other two are obtained similarly). The numerator of $\frac{N_1}{(\mu-\nu)(\mu-\lambda)(i-a)}$ is

$$\begin{aligned} N_1 =& 2bca' - \frac{b^2cc'}{2(i-a)} + b'c\frac{i-3a}{2} - 2bca' + \frac{c^2bb'}{2(i-a)} - bc'\frac{i-3a}{2} \\ =& \frac{1}{2(i-a)}(bc(b'c-bc') + (i-3a)(i-a)(b'c-bc')) \\ =& \frac{b'c-bc'}{2}(i+a+i-3a) \\ =& (b'c-bc')(i-a) \end{aligned}$$

where in the last equality, we have used that $bc = -1-a^2 = (i-a)(i+a)$. Similarly, the numerator of $\frac{N_2}{(\mu-\nu)(\nu-\lambda)(i-a')}$ is $-(i-a')(b'c-bc')$, by simply swapping μ and ν. So, in total the σ_3 component of the right-hand side of (B.31) is given by

$$\frac{bc'-b'c}{\nu-\mu}\left(\frac{1}{\mu-\lambda} - \frac{1}{\nu-\lambda}\right) = \frac{bc'-b'c}{(\mu-\lambda)(\nu-\lambda)}.$$

This is exactly the coefficient of the σ_3 component of $\frac{[Q(\mu),Q(\nu)]}{(\mu-\lambda)(\nu-\lambda)}$ as is readily seen. The other components are dealt with in the same way, and are omitted for brevity. □

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