Multidimensional Toeplitz and Truncated Toeplitz Operators

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The candidate confirms that the work submitted is his/her/their own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapter 4 of this thesis contains work based on the jointly authored publication

- M.C. Câmara, R. O'Loughlin, and J.R. Partington; Truncated Toeplitz operators on multiband spaces; Arxiv preprint 2012.14725; (2020).

The work in the above publication is directly attributable to all the authors (including the candidate). The largest contributions of the candidate were to Sections 3, 6 and 7 of the above publication.

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Abstract

In this thesis we extend previous studies of Toeplitz and truncated Toeplitz operators by studying both Toeplitz and truncated Toeplitz operators with matrix symbols.

We address the question of whether there is a smallest (matricial) Toeplitz kernel containing a given element or subspace of the Hardy space. This will in turn show how Toeplitz kernels can often be completely described by a fixed number of vectors, called maximal functions. We also discover an interesting and fundamental link between this topic and cyclic vectors for the backward shift.

We show that there is a link between the vector-valued nearly invariant subspaces and the scalar-valued nearly invariant subspaces with a finite defect. This powerful observation allows us to develop an all-encompassing approach to the study of the kernels of the Toeplitz operator, the truncated Toeplitz operator, the matrix-valued truncated Toeplitz operator and the dual truncated Toeplitz operator.

We study matrix-valued truncated Toeplitz operators with symbols having each entry in L^p for some $p \in (2, \infty]$. We develop an approach which bypasses the technical difficulties which arise when dealing with problems concerning matrix-valued truncated Toeplitz operators with unbounded symbols. Using this new approach we express the kernel of the matrix-valued truncated Toeplitz operator as an isometric image of an S^*-invariant subspace. Also, we construct a Toeplitz operator which is equivalent after extension to the matrix-valued truncated Toeplitz operator.

We characterise the dual, and in some cases the predual, of the backward shift

invariant subspaces of the Hardy space H^1. We then use our duality results to show that under certain conditions on the inner function I, every bounded truncated Toeplitz operator on the model space corresponding to I has a bounded symbol if and only if every compact truncated Toeplitz operator on the model space has a symbol which is of the form I multiplied by a continuous function.

Contents

1 Introduction

1.1 History of Toeplitz and truncated Toeplitz operators

Toeplitz operators are natural generalizations of so-called Toeplitz matrices. In the standard orthonormal basis of $\ell^2(\mathbb{Z}_+) = \left\{a = (a_n)_{n\geq 0} \,:\, \|a\|_2^2 := \sum_{n\geq 0} |a_n|^2 < \infty\right\}$, a Toeplitz operator is represented by the infinite matrix

$$T = \begin{pmatrix} u_0 & u_{-1} & u_{-2} & u_{-3} & \cdots \\ u_1 & u_0 & u_{-1} & u_{-2} & \ddots \\ u_2 & u_1 & u_0 & u_{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

where $(u_n)_{n\in\mathbb{Z}}$ is a given sequence. Although the Toeplitz operator is named after the German mathematician Otto Toeplitz (1881-1940), in his work [65, 66] Otto Toeplitz never actually studied the present day version of the Toeplitz operator. He studied Laurent Operators, which may be viewed as multiplication operators on $\ell^2(\mathbb{Z})$, and finite Toeplitz matrices. However, even without the present day formulation of the Toeplitz operator, one of the cornerstone theorems in the theory of Toeplitz operators was discovered by Toeplitz. Toeplitz showed that with T as above the upper bounds of the bilinear forms $\langle Tx, y\rangle$ and $\langle Lx, y\rangle$ (here $L : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denotes the Laurent operator with corresponding sequence $(u_n)_{n\in\mathbb{Z}}$) over the unit balls of the corresponding spaces are the same, and hence in modern notation, this is the well known formula

$$\|T\| = \left\| \sum_{j \in \mathbb{Z}} u_j z^j \right\|_\infty .$$

Although Toeplitz's interests were purely mathematical, Wiener and Hopf both independently came to study Toeplitz operators through applications. For Wiener the subject arose naturally during his studies of causal signals and the best quadratic predictions for random processes. Hopf came to study Toeplitz operators through his interest in integral equations and a problem related to radiative equilibrium (see [67]).

Onsanger (1903-1976), winner of the Nobel prize in Chemistry, showed that the problem of finding the thermodynamic limit of a system of particles lying in $\mathbb{Z} \times \mathbb{Z}$ may be reduced to an asymptotic question of Toeplitz determinants. In search of a mathematical colleague competent for this question (and able – as he wrote – to "fill out the holes in the mathematics and show the epsilons and deltas and all of that") Onsanger made contact with Szegö and this eventually led to the strong Szegö Theorem. This collaboration was the starting point for the vast field of study into the asymptotic properties of Toeplitz matrices and their diverse applications.

It is worth noting that the first appearance of a Toeplitz operator in its present day form $T : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$,

$$Tx = \left(\sum_{j \geqslant 0} c_{i-j} x_j \right)_{i \geqslant 0} ,$$

where $(c_i)_{i \in \mathbb{Z}}$ is some given sequence, took place in Odessa in 1948 [57, 56]. Since 1948 mathematicians have developed a rich theory surrounding the Toeplitz operator, which intertwines Riemann-Hilbert problems, Wiener-Hopf operators and more recently the invariant subspace problem. For a more detailed history on the Toeplitz operator we refer the reader to [54].

Truncated Toeplitz operators may be viewed as an operator theoretic generalisation of finite Toeplitz matrices (precise definitions will be given in the next section). Thus, from a historical perspective, it may seem appropriate to attribute the first mathematical study of truncated Toeplitz operators to Otto Toeplitz with his study of finite Toeplitz matrices in [66]. Although truncated Toeplitz operators were encountered naturally in the Sz.-Nagy-Foiaş model theory for Hilbert space contractions (see [53]) and Sarason's study of the Volterra operator [60], the first systematic study of truncated Toeplitz operators was initiated by Sarason in his seminal work of 2007 [63].

Sarason's work of 2007 has led to an explosion of research into truncated Toeplitz operators with far reaching applications. One notable reason operator theorists have taken a particular interest in truncated Toeplitz operators is because there seems to be a growing body of evidence to suggest that truncated Toeplitz operators might serve as some sort of model operator for various classes of complex symmetric operators. At this point, however, it is still too early to tell what exact form such a model theory should take. On the other hand, a surprising array of complex symmetric operators can be concretely realised in terms of truncated Toeplitz operators (or direct sums of such operators). We refer the reader to Section 9 of [37] for a detailed discussion of such results.

Other notable applications of the study of truncated Toeplitz operators are the Carathéodory and Pick problems [1], where truncated Toeplitz operators with an analytic symbol appear naturally, and extremal problems stemming from control theory and electrical engineering [33, 32] where one can compute the norm of a Hankel matrix by considering the norm of a truncated Toeplitz operator (see equation 2.9 in [55]).

In this thesis we build on the previous literature studying Toeplitz and truncated Toeplitz operators with a particular emphasis on extending the theory of these operators to a multidimensional setting. Although the study of multidimensional analogues of truncated Toeplitz operators is a fairly recent endeavour, these operators do find application in various problems. They appear naturally when one is considering the Sz.-Nagy and Foiaş model theory for Hilbert space contractions or when one wants to compute the norm of an associated (vectorial) Hankel operator. We refer the reader to Chapter 4 for a more detailed explanation of these links, with further applications to minimisation problems and Nehari's Theorem.

1.2 Background theory and notation

We let $\mathbb{T}$ denote the unit circle in the complex plane, let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and let m denote the normalised Lebesgue measure on $\mathbb{T}$. We denote $L^p = L^p(\mathbb{T}, dm)$. We now give two equivalent ways to view the Hardy space.

Definition 1.1. *For $0 < p < \infty$, we define the Hardy space, H^p, to be the class of holomorphic functions in the unit disc such that*

$$\sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\zeta}\right) \right|^p d\zeta \right)^{\frac{1}{p}} < \infty.$$

The class H^p is a vector space, and for $p \geqslant 1$ if we equip H^p with the norm given by

$$\|f\|_{H^p} := \sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\zeta}\right) \right|^p d\zeta \right)^{\frac{1}{p}} < \infty$$

then H^p becomes a Banach space.

Definition 1.2. *The space H^∞ is defined as the vector space of bounded holomorphic functions on the unit disk, with the norm*

$$\|f\|_{H^\infty} = \sup_{|z|<1} |f(z)|.$$

With this norm H^∞ is also a Banach space. Theorem 3.8 in Chapter 3 of [46] shows that given $f \in H^p$ with $0 < p \leqslant \infty$ the radial limit $\tilde{f}\left(e^{i\zeta}\right) := \lim_{r \to 1} f\left(re^{i\zeta}\right)$ exists for almost every $\zeta \in \mathbb{T}$ and $\|\tilde{f}\|_{L^p} = \|f\|_{H^p}$. We define $H^p(\mathbb{T})$ to be the vector subspace of L^p containing all the limit functions $\tilde{f}$ when $f \in H^p$. Then by Theorem 3.12 in Chapter 3 of [46] for $1 \leqslant p \leqslant \infty$ we have

$$g \in H^p(\mathbb{T}) \text{ if and only if } g \in L^p \text{ and } \hat{g}(n) = 0 \text{ for all } n < 0,$$

where $\hat{g}(n)$ are the Fourier coefficients of the function g,

$$\forall n \in \mathbb{Z}, \quad \hat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} g\left(e^{i\zeta}\right) e^{-in\zeta} d\zeta.$$

When $1 \leqslant p \leqslant \infty$ the space $H^p(\mathbb{T})$ is a closed subspace of L^p and thus is a Banach space.

With the above construction of $H^p(\mathbb{T})$, we start with the space H^p, defined on the disc, and obtain a closed subspace of L^p by taking radial limits. When $1 \leqslant p \leqslant \infty$, one can actually reverse this process and define the space H^p starting from the space $H^p(\mathbb{T})$. We define the Poisson kernel,

$$P_r(t) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \frac{1 - r^2}{1 - 2r\cos t + r^2} = \operatorname{Re}\left(\frac{1 + re^{it}}{1 - re^{it}}\right), \quad 0 \leq r < 1,$$

then we define

$$f\left(re^{it}\right) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t - \zeta) \tilde{f}\left(e^{i\zeta}\right) d\zeta, \quad r < 1.$$

Then Theorem 3.11 in Chapter 3 of [46] shows that $\tilde{f}$ belongs to $H^p(\mathbb{T})$ exactly when $f \in H^p$ and furthermore the proof of Theorem 3.12 in Chapter 3 of [46] shows the Fourier coefficients $(a_n)_{n\in\mathbb{N}}$ of $\tilde{f}$ are exactly the coefficients of the analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1.$$

Thus we have shown there are two equivalent ways to view the Hardy space, as either H^p or $H^p(\mathbb{T})$. We note that when $p < 1$ the Fourier coefficients of a function in $H^p(\mathbb{T})$ may not exist and we therefore can not view $H^p(\mathbb{T})$ as the subspace of L^p which has all negative Fourier coefficients having a value of 0. Following convention, we will not distinguish between $f \in H^p$ and $\tilde{f} \in H^p(\mathbb{T})$, and we will just use the notation $f \in H^p$. When we multiply a function $f \in H^p$ by any other function g defined almost everywhere on $\mathbb{T}$, this multiplication is to be understood as $\tilde{f}$ (defined as above) multiplied by g, i.e, the multiplication is understood on $\mathbb{T}$.

From the identification of H^p as a subset of L^p it is clear that for $p_1 < p_2$ we have $H^{p_2} \subseteq H^{p_1}$. Another key result in the theory of Hardy spaces, which can be found as Theorem 3.3 in [27], is the following.

Theorem 1.3. *For $0 < p < \infty$, H^p is the closure of the set of polynomials.*

We also note that by orthogonality of $\{z^n : n \in \mathbb{N}\}$ in L^2, the Hardy space H^2 consists of all functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

analytic in the unit disc $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Definition 1.4. *Let $0 < p \leqslant \infty$. We say a function $f^i \in H^p$ is inner if $|f| = 1$ a.e. on $\mathbb{T}$. We say an analytic function with radial boundary values defined almost everywhere, f^o, is outer if it is of the form*

$$f^o(z) = \alpha \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log \psi(e^{it}) dt \right\},$$

where α is a complex number of modulus one, $\psi(e^{it}) \geqslant 0$, $\log(\psi(e^{it})) \in L^1$.

As outlined after Definition 3.19 in [36], the significance of the above outer function lies in the fact that $|f^o| = |\psi| = \psi$ a.e. on $\mathbb{T}$.

Definition 1.5. *Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of points in $\mathbb{D}$ satisfying the property $\sum_k (1 - |a_k|) < \infty$. An inner function, B, of the form*

$$B(z) = \alpha z^m \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z},$$

where $\alpha \in \mathbb{C}$ has modulus one and $m \in \mathbb{Z}_+$, is called a Blaschke product.

We note that each a_k is a zero of B. We refer the reader to Theorem 2.4 in [27] for a proof that the specified function B is indeed inner.

Definition 1.6. *An inner function of the form*

$$S(z) = \alpha \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right\},$$

where α *is a constant of modulus one, and* μ *is a positive singular measure is called a singular inner function.*

We note that singular inner functions do not have any zeros in the disc. The most fundamental factorisation result within the theory of Hardy spaces is the following. (A proof of which may be found as Theorem 2.8 in [27].)

Theorem 1.7. *Let* $0 < p \leqslant \infty$. *Every non-zero* $f \in H^p$ *has a factorisation*

$$f(z) = f^i(z) f^o(z),$$

where f^i *is inner, and* f^o *is outer and lying in* H^p. *Furthermore this factorisation is unique up to multiplication by unimodular constants. We may further factorise* f^i *as*

$$f^i(z) = B(z)S(z),$$

where B *is a Blaschke product and* S *is a singular inner function. This factorisation of* f^i *is also unique up to multiplication by unimodular constants.*

Conversely, every such product $f(z) = f^i(z)f^o(z)$ *where* f^i *is inner and* f^o *is an outer function lying* L^p, *belongs to* H^p.

Throughout the thesis, for a function $f \in H^p$ we write $f = f^i f^o$, where f^i/f^o is an inner/outer factor of f respectively.

Theorem 3.5.6 in [24] gives us the following theorem.

Theorem 1.8. *Let $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Then*

1. $\ell \in (H^p)^*$ if and only if there is a $g \in H^q$ such that

$$\ell(f) = \ell_g(f) := \int_{\mathbb{T}} f(\zeta)\overline{g}(\zeta)dm(\zeta),$$

for all $f \in H^p$.

2. The norm of the above linear functional is equivalent to the H^q -norm of g.

We call the bounded map $S : H^p \to H^p$ given by $f \mapsto zf$ the (forward) shift. The shift invariant subspaces of H^p for $0 < p < \infty$ are characterised by Beurling's Theorem, which is the following.

Theorem 1.9. *Let $M \subseteq H^p$ be a nontrivial (closed) invariant subspace for S. Then there is an inner function $I \in H^\infty$ such that*

$$M = IH^p = \{If : f \in H^p\}.$$

Also, I is unique to within a constant of modulus 1.

Beurling's Theorem was originally proved by Beurling for the Hilbert space H^2 and then generalised by others to to the case when $0 < p < \infty$, see [35] page 132 and

[51] page 79.

The following theorem is a well known result, originally due to Riesz [58, 59]

Theorem 1.10. *Let $1 < p < \infty$. If $f \in L^p$ has the Fourier series*

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\zeta^n,$$

then the map $P_+ : L^p \to H^p$, defined by

$$(P_+ f)(z) := \sum_{n=0}^{\infty} \hat{f}(n) z^n = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\overline{\zeta} z} dm(\zeta), \quad z \in \mathbb{D},$$

is bounded.

We define $H_0^p := \{f \in H^p : f(0) = 0\}$ and we use the notation $\overline{f}$ to mean the conjugate of f (which is automatically in L^p whenever $f \in H^p$). We call the map P_+ the *Riesz projection*, and we note that when $p = 2$, P_+ is the orthogonal projection from L^2 to H^2. Similarly, we define $P_- := I_d - P_+$. When there is ambiguity over which space the projection is acting on, we will denote $P_{q,+}$ (respectively $P_{q,-}$) to mean the projection $L^q \to H^q$ (respectively $L^q \to \overline{H_0^q}$). We can observe that for $1 \leqslant p \leqslant \infty$ we have $H^p \cap \overline{H_0^p} = \{0\}$. Moreover, when $1 < p < \infty$ we have $L^p = P_+ L^p + P_- L^p$, which implies

$$L^p = H^p \oplus \overline{H_0^p},$$

where here we write $\overline{H_0^p}$ to mean the conjugate of H_0^p.

Definition 1.11. *Let $1 < p < \infty$. For $g \in L^\infty$ the Toeplitz operator, $T_g : H^p \to H^p$ is defined by*

$$T_g(f) = P_+(gf).$$

We call g the symbol for the Toeplitz operator.

Remark. Although the Toeplitz operator in the introduction is viewed on $\ell^2(\mathbb{Z}_+)$, viewing the Toeplitz operator to act instead on H^p will still give the same matrix representation of the operator (when one uses the canonical basis $\{z^n : n \in \mathbb{Z}_+\}$ for H^p) and we now have the added benefit that we can use function theoretic results developed for the Hardy space to study the Toeplitz operator.

Being the composition of two bounded maps, the Toeplitz operator is clearly bounded, and in fact by Theorem 2.1.5 in [54] we have the following.

Theorem 1.12. *For $g \in L^\infty$ the Toeplitz operator, $T_g : L^2 \to H^2$ satisfies $\|g\|_{L^\infty} = \|T_g\|$.*

For $1 < p < \infty$ the adjoint of $S : H^p \to H^p$, denoted S^*, is a continuous map given by

$$S^*(f)(z) = \frac{f(z) - f(0)}{z}.$$

We can also consider the above map S^* acting on other spaces of analytic functions in the disc, such as the Smirnov class. Using Beurling's Theorem one can show all closed S^*-invariant subspaces of H^p for $1 < p < \infty$ are of the form

$$K_I^p := I\overline{H_0^p} \cap H^p,$$

for some inner function I. Conversely for an inner function I, any set of the form $I\overline{H_0^p} \cap H^p$ is a closed S^*-invariant subspace of H^p. We refer the reader to Theorem 5.1.4 in [24] for a proof of this result.

Definition 1.13. *We call the set $K_I^p := I\overline{H_0^p} \cap H^p$ a model space.*

Theorem 5.10.1 in [24] is the following.

Theorem 1.14. *If $p \in (1, \infty)$, then $\ell \in \left(H^p \cap I\overline{H_0^p}\right)^*$ if and only if there is a $g \in H^q \cap I\overline{H_0^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that*

$$\ell(f) = \int f\overline{g}dm \quad f \in H^p \cap \overline{IH_0^p}.$$

Moreover the norm of ℓ is equivalent to the H^q norm of g.

For $1 < p < \infty$ and an inner function I, we define the surjective bounded projection $P_I : L^p \to K_I^p$ by $P_I := P_+ IP_-\overline{I}$. We observe that $K_I^p \cap IH^p = \{0\}$ and $L^p = P_- L^p + (P_I + (I_d - P_I))P_+ L^p$, which implies

$$L^p = \overline{H_0^p} \oplus K_I^p \oplus IH^p.$$

We again note that when $p = 2$ the projection P_I is orthogonal and the above decomposition is an orthogonal decomposition. When there is ambiguity on the index of which L^p space the projection is defined on we will use the notation $P_{I,q}$ to denote the projection from L^q to K_I^q.

Definition 1.15. *The truncated Toeplitz operator $A_g^I : K_I^2 \to K_I^2$ having symbol $g \in L^2$ is the densely defined operator*

$$A_g^I(f) = P_{I,2}(gf)$$

having domain

$$\{f \in K_I^2 : gf \in L^2\}.$$

We will use the abbreviation TTO for the truncated Toeplitz operator. In contrast to the Toeplitz operators on H^2 the truncated Toeplitz operator may be extended to a bounded operator on K_I^2 even for some unbounded symbols.

We now give a brief outline of some of the above results generalised to the multidimensional case.

For $1 \leqslant p < \infty$ the space $(L^p)^n$ is the space of column vectors of length $n \in \mathbb{N}$ with each coordinate taking values in L^p; $(L^p)^n$ is a Banach space when equipped with the norm

$$\left\| \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\| = (\|f_1\|_{L^p}^p + \ldots + \|f_n\|_{L^p}^p)^{\frac{1}{p}} .$$

The vector-valued Hardy space, denoted $(H^p)^n$, is the subspace of $(L^p)^n$ consisting of all vectors

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

such that $f_1, ..., f_n$ lie in H^p. We define

$$(\overline{H_0^p})^n := \{ \begin{pmatrix} \overline{f_1} \\ \vdots \\ \overline{f_n} \end{pmatrix} : \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in z(H^p)^n \}.$$

We can define (vectorial) projections, $P_+ : (L^p)^n \to (H^p)^n$, and $P_- : (L^p)^n \to (\overline{H_0^p})^n$

such that for $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in (L^p)^n$

$$P_+ \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} P_+(f_1) \\ \vdots \\ P_+(f_n) \end{pmatrix}, \quad P_- \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} P_-(f_1) \\ \vdots \\ P_-(f_n) \end{pmatrix}$$

where the projection maps on the right hand side of the above equalities are understood as in the scalar case. Like the scalar case we have the direct sum decomposition $(L^p)^n = (\overline{H_0^p})^n \oplus (H^p)^n$. As we have done in the scalar case, when there is ambiguity over which space the projection is acting on, we will denote $P_{q,+}$ (respectively $P_{q,-}$) to mean the projection $(L^q)^n \to (H^q)^n$ (respectively $(L^q)^n \to (\overline{H_0^q})^n$). The forward shift on the space $(H^p)^n$, S, is defined analogously to the scalar case, and so the adjoint of the forward shift, the backward shift, on the space $(H^p)^n$ is given by

$$S^* \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (z) = \frac{\begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} - \begin{pmatrix} f_1(0) \\ \vdots \\ f_n(0) \end{pmatrix}}{z}.$$

For $1 \leqslant p \leqslant \infty$, we denote $L^{(p,n\times n)}$ to be the space of n-by-n matrices with each entry taking values in L^p. We make an analogous definition for $H^{(p,n\times n)}$. For $G \in L^{(\infty,n\times n)}$ the *matricial Toeplitz operator* on the space $(H^p)^n$, with symbol G, is defined by

$$T_G(f) = P_+(Gf).$$

Remark. We note that in chapter 4 we use the same notation T_G for a natural generalisation of the matricial Toeplitz operator defined above. The full details of this generalisation are given in chapter 4.

Much like the scalar case, the matricial Toeplitz operator is bounded if and only if G is a bounded symbol. In other literature the above multidimensional generalisation of the Toeplitz operator is often called the vectorial Toeplitz operator or the block Toeplitz operator. When the context is clear we will also just refer to T_G as the Toeplitz operator.

The study of matricial generalisations of the truncated Toeplitz operator began as recently as 2018 [50]. Because this field of study is new we postpone the definition of the matrix-valued truncated Toeplitz operator until Chapter 4.

Layout of the thesis, notation and abbreviations

Here we list the notations and abbreviations which we will use consistently throughout the rest of the thesis. Note that we will make further definitions in each chapter as necessary.

- TTO is an abbreviation for truncated Toeplitz operator.
- We will use a.e. to abbreviate almost everywhere.
- Throughout we will use the notation I to denote an arbitrary inner function.
- Throughout we will fix the notation K_I^p to denote the model space.
- We write Toeplitz kernel to mean the kernel of a Toeplitz operator.
- We write m to denote the normalised Lebesgue measure on $\mathbb{T}$.
- All subspaces are assumed closed unless otherwise stated.
- In chapter 4 we use the abbreviation EAE for equivalent after extension.
- In chapter 4 we use the abbreviation MTTO for matrix-valued truncated Toeplitz operator.

This thesis is split into five chapters. Each of these chapters is split into sections and where necessary some sections may be split into subsections.

In Chapter 2 we show existence of a minimal kernel for any element of the vector-valued Hardy space and we determine a symbol for the corresponding Toeplitz operator. We show not all matricial Toeplitz kernels have a maximal function and in the case of $p = 2$ we find the exact conditions for when a Toeplitz kernel has a maximal function. We study the minimal Toeplitz kernel containing multiple elements of the Hardy space, which in turn allows us to deduce an equivalent condition for a function in the Smirnov class to be cyclic for the backward shift.

In Chapter 3 we study vector and scalar-valued nearly S^*-invariant subspaces of the Hardy space. We first produce some results on the structure of nearly S^*-invariant subspaces with a finite defect. In particular, we produce a powerful tool which allows us to relate the vector-valued nearly S^*-invariant subspaces to scalar-valued nearly S^*-invariant subspaces with a finite defect. These results then allow us to adopt a previously unknown universal approach to the study of the kernel of the Toeplitz operator, the truncated Toeplitz operator, the dual truncated Toeplitz operator and the matrix-valued truncated Toeplitz operator.

In Chapter 4 we study the matrix-valued truncated Toeplitz operator (abbreviated to MTTO). MTTOs are a vectorial generalisation of the truncated Toeplitz operator. We focus on studying the kernel of the MTTO and we also find a new form of Toeplitz operator which is equivalent after extension to the MTTO. We make a handy observation, that when studying a given property of the MTTO it is often convenient to initially modify the MTTO by changing its codomain (in a natural way), then one can deduce results about the MTTO from the modified MTTO. This approach allows us to tackle problems which were previously out of reach concerning MTTOs with unbounded symbols.

In Chapter 5 we provide two new overlapping results. We characterise the dual space of $K_I^1 = I\overline{H_0^1} \cap H^1$. Although the dual of K_I^p for $1 < p < \infty$ is easy to characterise, when we no longer have a reflexive Hardy space classical results break down and a complete description for when $p = 1$ is missing. In some cases we also characterise the predual of K_I^1. We then use our duality results to study the question, when does a bounded truncated Toeplitz operator have a bounded symbol? This question has generated much research interest and is one of the most fundamental problems concerning truncated Toeplitz operators. Surprisingly, we show that under certain assumptions on an inner function, I, every bounded truncated Toeplitz operator on K_I^2 has a bounded symbol if and only if every compact truncated Toeplitz operator on K_I^2 has a symbol which is of the form If where f is a continuous function on $\mathbb{T}$.

2 Minimal kernels and maximal functions

Throughout this chapter we will fix $1 < p < \infty$ and $n \in \mathbb{N}$. All Toeplitz operators are assumed bounded and hence have bounded symbols.

2.1 Minimal kernel of an element in $(H^p)^n$

It is easily shown that not all $\phi \in (H^p)^n$ lie in a one-dimensional Toeplitz kernel. In the scalar case (i.e when $n = 1$) Theorem 5.1 in [14] shows the existence of a Toeplitz kernel of smallest size containing $\phi \in H^p$ (formally known as the minimal kernel for ϕ and denoted $\kappa_{min}(\phi)$), furthermore a Toeplitz operator T_g is defined such that $\kappa_{min}(\phi) = \ker T_g$. This motivates our study for this section, where we address the question: is there a minimal Toeplitz kernel containing a given element $\phi \in (H^p)^n$?

Definition 2.1. *For G a bounded n-by-n matrix symbol we say $\ker T_G$ is the minimal kernel of $\phi := \begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix}^T$ if $\begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix}^T \in \ker T_G$, and if $\begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix}^T \in \ker T_H$ for any other bounded n-by-n matrix symbol H we have $\ker T_G \subseteq \ker T_H$. In this case we write $\kappa_{min}(\phi) = \ker T_G$.*

Although Section 5.1 in [14] addresses whether there always exists a minimal Toeplitz kernel containing a function in $(H^p)^n$, a complete answer to this question was not given. A partial result was given as Theorem 5.5 which shows the existence of a minimal Toeplitz kernel containing any rational ϕ in $(H^p)^n$. We will show existence of a minimal Toeplitz kernel containing any $\phi \in (H^p)^n$, and define an operator T_G such that $\kappa_{min}(\phi) = \ker T_G$.

Lemma 2.2. *For any $\phi_1 \ldots \phi_n \in H^p$ there exists an outer function u such that $|u|= |\phi_1|+ \ldots + |\phi_n|+1$.*

Proof. Outer functions have a representation

$$u(z) = \alpha \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log k(e^{it})dt\right),$$

where $|\alpha|= 1$, $\log k \in L^1(\mathbb{T})$ is real. Moreover $|k|= |u|$ a.e. on $\mathbb{T}$.

In the above representation, if we let $k = (|\phi_1|+\ldots+|\phi_n|+1)$ it then follows that $|u|= |\phi_1|+\ldots+|\phi_n|+1$. It can be seen that $\log k = \log(|\phi_1|+\ldots+|\phi_n|+1) \in L^1(\mathbb{T})$, as $0 < \log(1 + x) < x$ for all $x > 0$, and $\phi_1 \ldots \phi_n \in L^1$. □

Definition 2.3. *We say f belongs to the Smirnov class, denoted N^+, if f is holomorphic in the disc and*

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \log(1 + |f(rz)|)dm(z) = \int_{\mathbb{T}} \log(1 + |f(z)|)dm(z) < \infty.$$

On N^+ the metric is defined by

$$\rho(f, g) = \int_{\mathbb{T}} \log(1 + |f(z) - g(z)|)dm(z).$$

We let $\log L$ denote the class of complex measurable functions f on $\mathbb{T}$ for which

$\rho(f, 0) < \infty$. One can check $\log L$ is an algebra. Furthermore Section 3.6.3 of [24] along with the argument laid out on p. 122 of Gamelin's book [35] shows that when $\log L$ is equipped with ρ as a metric $\log L$ is a topological algebra ($f_n \to f$ and $g_n \to g$ in $\log L \implies f_n + g_n \to f + g$ and $f_n g_n \to fg$ in $\log L$). Proposition 3.6.10 in [24] further shows that N^+ is the closure of the analytic polynomials in $\log L$, and hence N^+ is a topological algebra.

Throughout various literature there have many equivalent ways to define the Smirnov class; for the sake of completeness we list these in the following proposition.

Proposition 2.4. *The following three statements are equivalent*

1. $f \in N^+$.

2. $f \in \{\frac{f_1}{f_2} : f_2 \text{ is outer }, f_1, f_2 \in H^\infty\}$.

3. $f \in \{\frac{f_1}{f_2} : f_2 \text{ is outer }, f_1, f_2 \in H^{1/2}\}$.

4. $f = b s_{\mu_1} f^o$, *where* b *is a Blaschke product,* s_{μ_1} *a singular inner function with respect to the measure* μ_1 *and* f^o *an outer function.*

Proof. Following the argument laid out in the proof of Theorem 2.10 in [27] shows the equivalence of 1 and 4. $2 \implies 3$ is immediate. $3 \implies 4$ follows from the fact that the reciprocal of an outer function is outer and so is the product of two outer functions. We now show 4 implies 2 to show all the statements are equivalent.

We construct two outer functions F_1, F_2 such that $|F_1|= min(1,|f|)$, and $|F_2|= min(1,|f|^{-1})$, as in Lemma 2.2 we only need to prove that $\log(min(1,|f|))$ and $\log(min(1,|f|^{-1}))$ are in L^1 in order to do this. We define $E := \{z \in \mathbb{T} : |f(z)|> 1\}$ and $F := \{z \in \mathbb{T} : |f(z)|\leqslant 1\}$. Then

$$\int_{\mathbb{T}} \log(min(1,|f|)) = \int_E \log(min(1,|f|)) + \int_F \log(min(1,|f|)) = 0 + \int_F \log|f|.$$

As $|f|$ is log integrable over the whole of $\mathbb{T}$ it is also log integrable over any subset of $\mathbb{T}$, so the expression above shows $\log min(1,|f|) \in L^1$. A similar computation shows $min(1,|f|^{-1})$ is log integrable and it then follows that $F_1, F_2 \in H^\infty$. As $|F_2||f|= |F_1|$ a.e. on $\mathbb{T}$, taking outer factors we can conclude $f^o = \frac{F_1}{F_2}$ so $f = \frac{bs_{\mu_1}F_1}{F_2}$. $\square$

Remark. From the equivalence of 2 and 3 in the above proposition one can also easily show that

$$N^+ = \{\frac{f_1}{f_2} : f_2 \text{ is outer } f_1, f_2 \in H^1\}.$$

Notice from the fourth characterisation of N^+ in the proposition above, that if $f \in N^+$ and the boundary function is in L^p, then $f \in H^p$, i.e., $N^+ \cap L^p = H^p$. This is a useful result we will freely use throughout this chapter.

We present the main theorem of this section.

Theorem 2.5. *Let u be an outer function such that $|u|= |\phi_1|+ \ldots + |\phi_n|+1$, where*

$\phi_1 \ldots \phi_n \in H^p$, *then*

$$\kappa_{min} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \ker T_{\begin{pmatrix} \overline{\phi_1}\overline{z}/\phi_1^o & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ -\phi_2/u & \phi_1/u & 0 & \ldots & \ldots & \ldots & 0 \\ -\phi_3/u & 0 & \phi_1/u & 0 & \ldots & \ldots & 0 \\ -\phi_4/u & 0 & 0 & \phi_1/u & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\phi_n/u & 0 & \ldots & \ldots & \ldots & 0 & \phi_1/u \end{pmatrix}}.$$

Proof. We denote the above symbol by G. It is clear that $\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix}^T \in \ker T_G$. It remains to show that if

$$\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix}^T \in \ker T_H,$$

for any bounded n-by-n matrix H, then every $\begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix}^T \in \ker T_G$ also lies in

$\ker T_H$. To this end let $\begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix}^T \in \ker T_H$, then if we write

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{pmatrix}$$

we have

$$\begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \overline{zp_1} \\ \vdots \\ \vdots \\ \overline{zp_n} \end{pmatrix},$$

for some $p_1 \dots p_n \in H^p$, so that $\phi_1 h_{i1} + \phi_2 h_{i2} + \dots + \phi_n h_{in} = \overline{zp_i}$ for each $i \in \{1 \dots n\}$. Let $\begin{pmatrix} f_1 & \dots & f_n \end{pmatrix}^T \in \ker T_G$, then $f_1 = \frac{\phi_1 \overline{p}}{\overline{\phi_1^o}}$ for some $p \in H^p$. Rows 2 to n of G take values in $N^+ \cap L^\infty = H^\infty$, so from row $i \in \{2 \dots n\}$ in $G \begin{pmatrix} f_1 & \dots & f_n \end{pmatrix}^T \in \overline{(H_0^p)^n}$, taking into account that $H^p \cap \overline{H_0^p} = \{0\}$, we deduce

$$f_1 \frac{\phi_i}{u} = f_i \frac{\phi_1}{u}.$$

Substituting our value for f_1 we can write f_i as,

$$f_i = \frac{\phi_i \overline{p}}{\overline{\phi_1^o}},$$

so

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \frac{\overline{p}}{\overline{\phi_1^o}} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix},$$

and hence

$$H \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = H \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \frac{\overline{p}}{\overline{\phi_1^o}} = \begin{pmatrix} \overline{zp_1} \\ \vdots \\ \overline{zp_n} \end{pmatrix} \frac{\overline{p}}{\overline{\phi_1^o}}.$$

Proposition 2.4 shows $\overline{zp_i}\frac{\overline{p}}{\overline{\phi_1^o}} \in \overline{zN^+} \cap L^p = \overline{H_0^p}$, so we conclude

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \ker T_H.$$

□

Remark. The above symbol for the minimal kernel is not unique. In fact we can show there are at least n different symbols (not including permuting the rows of the symbol) which represent the same kernel, each depending on the minimal kernel in the scalar case, of ϕ_j, where $j \in \{1 \dots n\}$. Consider the symbol

$$\begin{pmatrix} 0 & \dots & \dots & \dots & \dots & \dots & \overline{\phi_j}\overline{z}/\phi_j^o & 0 & \dots & 0 \\ 0 & \phi_j/u & 0 & \dots & \dots & \dots & -\phi_2/u & 0 & \dots & 0 \\ 0 & 0 & \phi_j/u & 0 & \dots & \dots & -\phi_3/u & 0 & \dots & 0 \\ 0 & 0 & 0 & \phi_j/u & 0 & \dots & -\phi_4/u & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_j/u & 0 & \dots & \dots & \dots & 0 & -\phi_1/u & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \phi_{j+1}/u & \phi_j/u & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & \phi_n/u & 0 & \dots & \phi_j/u \end{pmatrix},$$

where the first non-zero entry on the first row is in the j'th column, and the row where the first entry is non-zero is the j'th row. This can also be checked to be a symbol for the minimal kernel.

2.2 Maximal functions for $\ker T_G$

In this section we consider the following question: given any Toeplitz kernel K does there exist a ϕ such that $K = \kappa_{min}(\phi)$? We call such a ϕ a *maximal function* for K. It has been shown in [14] that in the scalar Toeplitz kernel case, whenever the kernel is non-trivial there does exist a maximal function. Theorem 3.17 in [18] shows that for $p = 2$ every matricial Toeplitz kernel which can be expressed as a fixed vector-valued function multiplied by a non-trivial scalar Toeplitz kernel also has a maximal function. The results of this section show not all non-trivial matricial Toeplitz kernels have a maximal function and for $p = 2$ we find the exact conditions for when a Toeplitz kernel has a maximal function. An interesting application of the study of maximal functions is given in [17], which fully characterises multipliers between Toeplitz kernels in terms of their maximal functions.

A simple explicit example to show not all matricial Toeplitz kernels have a maximal function is the following

$$\ker T_{\begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z} \end{pmatrix}} = \{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} : \lambda, \mu \in \mathbb{C} \}.$$

Suppose some fixed $\begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} \in \mathbb{C}^2$ give a maximal function, then

$$\begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} \in \ker T_{\begin{pmatrix} \mu_1 & -\lambda_1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} H^p,$$

but

$$\ker T_{\begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z} \end{pmatrix}} \not\subseteq \ker T_{\begin{pmatrix} \mu_1 & -\lambda_1 \\ 0 & 0 \end{pmatrix}},$$

so $\ker T_{\begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z} \end{pmatrix}}$ can not have a maximal function. We can build on this example to give a condition for when Toeplitz kernels do not have a maximal function.

We use the notation $\ker T_G(0) := \{f(0) : f \in \ker T_G\}$. For a matrix A with each entry of A being a holomorphic function in the disc we write $A(0)$ to mean A with each entry evaluated at 0.

Theorem 2.6. *If* $\ker T_G$ *is such that* $\dim \ker T_G(0) > 1$ *then* $\ker T_G$ *does not have a maximal function.*

Proof. Suppose for contradiction $\ker T_G$ is such that $\dim \ker T_G(0) > 1$ and $\ker T_G$ has a maximal function

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Then for any symbol H if $v \in \ker T_H$, we must have $\ker T_G \subseteq \ker T_H$. Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

be two linearly independent vectors in $\ker T_G(0)$. Pick $i, j \in \{1...n\}, i < j$ such that

$$\begin{pmatrix} 0 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \\ x_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ y_i \\ 0 \\ \vdots \\ 0 \\ y_j \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

span a two dimensional subspace of $\mathbb{C}^n$. Let n be the largest integer such that $\frac{v_i}{z^n}$

and $\frac{v_j}{z^n}$ lie in H^p, let u be an outer function such that $|u| = |v_i| + |v_j| + 1$, and let

$$H = \begin{pmatrix} 0 & \dots & 0 & \frac{v_j}{z^n u} & 0 & \dots & 0 & -\frac{v_i}{z^n u} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where the first non-zero entry is in the i'th column and the second is in the j'th column. As $\frac{v_i}{z^n u}, \frac{v_j}{z^n u} \in L^\infty \cap N^+ = H^\infty$, each entry of H takes values in H^∞. Furthermore $v \in \ker T_H$, so

$$\ker T_G \subseteq \ker T_H,$$

which means that

$$\ker T_G(0) \subseteq \ker T_H(0).$$

For $\begin{pmatrix} f_1 & \dots & f_n \end{pmatrix}^T \in \ker T_H$ we have $f_i \frac{v_j}{z^n u} = f_j \frac{v_i}{z^n u}$, and by dividing v_i, v_j by z^n, we have ensured there is a linear relation between $f_i(0)$ and $f_j(0)$. So the i'th and j'th coordinate of $\ker T_H(0)$ only span a one dimensional subspace of $\mathbb{C}^n$, but we have picked i, j so that the i'th and j'th coordinate of $\ker T_G(0)$ span a two dimensional subspace of $\mathbb{C}^n$, which is a contradiction. So we conclude that maximal functions do not exist whenever $\dim \ker T_G(0) > 1$. □

We now aim to generalise Dyakonov's decomposition of Toeplitz kernels, which is Theorem 1 in [29], to a matrix setting, we will then use this result to further study maximal functions. In the case of $p = 2$, Theorem 7.4 of [4] presents a similar formula to what we will obtain.

We define $N^{(+,n\times n)}$ to be the space of all $n \times n$ matrices taking values in N^+. An n-by-n matrix inner function Θ is an element of $H^{(\infty,n\times n)}$ such that for for almost every $z \in \mathbb{T}$, $\Theta(z)$ is unitary. We denote the adjoint of the matrix Θ by Θ^*. For a n-by-n matrix inner function Θ we denote the S^*-invariant subspace, $\ker T_{\Theta^*} = \Theta\overline{(H_0^p)^n} \cap (H^p)^n$ by K_Θ^p. K_Θ^p can easily be checked to be S^*-invariant by noting if $\Theta^* f \in \overline{(H_0^p)^n}$, then $\Theta^* f(0) \in \overline{(H^p)^n}$ and so $\Theta^*(f - f(0)) \in \overline{(H^p)^n}$, which implies $\Theta^* \frac{f-f(0)}{z} = \Theta^* S^*(f) \in \overline{(H_0^p)^n}$.

For a symbol G, if $\det G$ is an invertible element in L^∞ then

$$\int_{\mathbb{T}} \log \frac{1}{|\det G(z)|} dm(z) < \infty,$$

and so $\int_{\mathbb{T}} \log |\det G(z)| dm(z) = -\int_{\mathbb{T}} \log \frac{1}{|\det G(z)|} dm(z) > -\infty$. This means we can construct a scalar outer function q such that $|\det G| = |q|$.

Lemma 2.7. *Let $G \in L^{(\infty,n\times n)}$ be such that $\det G$ is invertible in L^∞ and let q be the outer function such that $|\det G| = |q|$. Then if we define $G' \in L^{(\infty,n\times n)}$ to be the matrix G with the first row divided by $\overline{q}$, we have $\ker T_G = \ker T_{G'}$. Furthermore $\det G'$ is unimodular.*

Proof. We only need to consider the first row of G'. Denote the first row of G (respectively G') by G_1 (respectively G_1'). As q is invertible in H^∞, for $f \in (H^p)^n$

we have $G_1 f \in \overline{H_0^p}$ if and only if $\frac{G_1}{\overline{q}} f \in \overline{H_0^p}$. The fact $\det G^{'}$ is unimodular is a result of linearity of the determinant in each row. $\square$

Under the assumption that $\det G$ is invertible in L^∞, by the argument laid out above we can assume without loss of generality that $\det G$ is actually unimodular. Theorem 4.2 of [7] states we can now write G as

$$G = {G_2}^* G_1, \tag{1}$$

with $G_1, G_2 \in H^{(\infty, n\times n)}$. Furthermore taking the determinant of our unimodular G shows us that $1 = |\det G_2^*||\det G_1|$ and so $\det G_2^*$ and $\det G_1$ are invertible in L^∞, which means G_2^* and G_1 are invertible in $L^{(\infty, n\times n)}$.

By (1) under the assumptions above we can write $f \in \ker T_G$ if and only if $f \in (H^p)^n$ and $G_2^* G_1 f \in \overline{(H_0^p)^n}$ i.e $G_1 f \in \ker T_{G_2^*}$. Furthermore the following proposition shows the kernel of $T_{G_2^*}$ can be simplified.

Proposition 2.8. *If* $G_2 \in H^{(\infty, n\times n)}$ *then* $\ker T_{G_2^*} = \ker T_{(G_2^i)^*}$.

Before we begin the proof we make a remark about inner-outer matrix factorisation. We follow definition 3.1 in [45] of outer functions in $N^{(+, n\times n)}$ and say that $E \in N^{(+, n\times n)}$ is outer if an only if $\det E$ is outer in N^+. Theorem 5.4 of [45] says that given a function $F \in N^{(+, n\times n)}$ such that $\det F$ is not equal to the 0 function, there exist matrix functions F^i inner and F^o outer (respectively $F^{i'}, F^{o'}$) such that we may write F as $F = F^i F^o$ (respectively $F = F^{o'} F^{i'}$).

Proof. Since $\det(G_2^o)$ is outer in H^∞ and invertible in L^∞, it is invertible in H^∞, so $(G_2^o)^*$ is invertible in $\overline{H^{(\infty,n\times n)}}$. Then, after writing G_2 as $G_2 = G_2^i G_2^o$, it immediately follows that $\ker T_{G_2^*} = \ker T_{(G_2^i)^*}$.

□

The following theorem is the generalisation of Dyakonov's decomposition of Toeplitz kernels to a matrix setting.

Theorem 2.9. *Using the decomposition for G given in (1),*

$$\ker T_G = (G_1^{i'})^* \left((G_1^{o'})^{-1} K^p_{G_2^i} \cap G_1^{i'} (H^p)^n \right).$$

Proof. Using the proposition above and (1) we may write $f \in \ker T_G$ if and only if $f \in (H^p)^n$ and $G_1 f \in K^p_{G_2^i}$. We write $G_1 = G_1^{o'} G_1^{i'}$. Since $\det G_1^{o'}$ is outer in H^∞ and invertible in L^∞, it is invertible in H^∞, which means $(G_1^{o'})^{-1} \in H^{(\infty,n\times n)}$. Hence we can write the condition $f \in (H^p)^n$ and $G_1 f \in K^p_{G_2^i}$ as $G_1^{i'} f \in (G_1^{o'})^{-1} K^p_{G_2^i} \cap G_1^{i'} (H^p)^n$ and so $f \in \ker T_G$ if and only if

$$f \in (G_1^{i'})^* \left((G_1^{o'})^{-1} K^p_{G_2^i} \cap G_1^{i'} (H^p)^n \right).$$

□

Proposition 2.10. *Let K be a S^*-invariant subspace of $(H^p)^n$ such that K evaluated at 0 is a one-dimensional subspace of $\mathbb{C}^n$. Then K is of scalar type i.e, K is a fixed* *vector multiplied by a scalar backward shift invariant subspace of H^p.*

Proof. Let K evaluated at 0 be equal to the span of $\begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T$, then by assumption for any $f \in K$ we must have $f(0) = x_0 \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T$ for some $x_0 \in \mathbb{C}$. Similarly $S^*(f)(0) = x_1 \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T$ for some $x_1 \in \mathbb{C}$, and we repeat this process recursively to obtain $S^{*i}(f)(0) = x_i \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T$ for each $i \in \mathbb{N}$. Noting that $S^{*i}(f)(0)$ is the coefficient of z^i for f and polynomials are dense in H^p, we deduce that $f = \sum_{i=0}^{\infty} x_i z^i \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T$. Furthermore

$$\left\{ \sum_{i=0}^{\infty} x_i z^i \in H^p : \sum_{i=0}^{\infty} x_i z^i \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T \in K \right\}$$

is S^*-invariant because K is. □

Corollary 2.11. *If $\ker T_G(0)$ is a one-dimensional subspace of $\mathbb{C}^n$ and in the decomposition of the kernel given in Theorem 2.9 we have $G_1^{i'} = I_d$ and $K^p_{G_2^i}$ is non-trivial, then $\ker T_G$ has a maximal function.*

Proof. If $G_1^{i'} = I_d$ then we have $G_1^{o'} \ker T_G = K^p_{G_2^i}$, so $G_1^{o'}(0) \ker T_G(0) = K^p_{G_2^i}(0)$. Which means either $K^p_{G_2^i}(0)$ is a one-dimensional subspace of $\mathbb{C}^n$ or is equal to 0, but as $K^p_{G_2^i}$ is S^*-invariant it can never be the case that $K^p_{G_2^i} \subseteq z(H^p)^n$. So we must have $K^p_{G_2^i}(0)$ is a one-dimensional subspace of $\mathbb{C}^n$. Then by the previous proposition $K^p_{G_2^i}$ must be equal to $\begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T K^p_{\mathcal{I}}$ for some $\begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T \in \mathbb{C}^n$, and some scalar inner function $\mathcal{I}$.

We now use Corollaries 2.20 and 2.21 which are proved later in this chapter but the proof is independent of any previous results. If we let $m^{'}$ be the maximal function of K_I^p (which exists by Corollary 2.20) then by Corollary 2.21 given any $f \in \ker T_G$ we can write $f = (G_1^{o'})^{-1}\begin{pmatrix}\lambda_1 & \dots & \lambda_n\end{pmatrix}^T m^{'}\overline{s}$ for some $s \in N^+$. So if $(G_1^{o'})^{-1}\begin{pmatrix}\lambda_1 & \dots & \lambda_n\end{pmatrix}^T m^{'} \in \ker T_H$ then

$$Hf = H(G_1^{o'})^{-1}\begin{pmatrix}\lambda_1 \\ \vdots \\ \lambda_n\end{pmatrix} m^{'}\overline{s} \in \overline{s(H_0^p)^n}.$$

Thus as N^+ is closed under multiplication we know each coordinate of Hf lies in $\overline{zN^+}$ and furthermore $f \in (H^p)^n$ and H is bounded so we must actually have $Hf \in \overline{(H_0^p)^n}$, and so $f \in \ker T_H$. As our f was arbitrary we have $\ker T_G \subseteq \ker T_H$. This shows $(G_1^{o'})^{-1}\begin{pmatrix}\lambda_1 & \dots & \lambda_n\end{pmatrix}^T m^{'}$ is a maximal vector for $\ker T_G$. □

For $1 < p < \infty$ and a Toeplitz operator $T_g : H^p \to H^p$, Theorem 2 in [41] shows existence of an extremal function $q \in \ker T_g$, and an inner function $\mathcal{I}$ vanishing at 0 such that:

1. If $p \leqslant 2$ then $qK_{\mathcal{I}}^2 \subseteq \ker T_g \subseteq qK_{\mathcal{I}}^p$.

2. If $p \geqslant 2$ then $qK_{\mathcal{I}}^p \subseteq \ker T_g \subseteq qK_{\mathcal{I}}^2$.

We now state a reformulation of this result, which may be viewed as a generalisation of the result given by Hayashi in [43] to $1 < p < \infty$.

Corollary 2.12. *1. If $p \leqslant 2$ then $\ker T_g = qK_{\mathcal{I}}^p \cap H^p$.*

2. If $p \geqslant 2$ then $\ker T_g = qK_{\mathcal{I}}^2 \cap H^p$.

Proof. We will prove statement (1). The $\subseteq$ inclusion is clear from the original result. To show the other inclusion we first observe that as $qK_{\mathcal{I}}^2 \subseteq \ker T_g$ we must have $q\mathcal{I}\overline{z} \in \ker T_g$. Then for all $p \in H^p$ we must then have

$$gq\mathcal{I}\overline{zp} \in \overline{zN^+},$$

and so if $q\mathcal{I}\overline{zp}$ also lies in H^p we must have $gq\mathcal{I}\overline{zp} \in \overline{H_0^p}$, and so consequently $q\mathcal{I}\overline{zp} \in \ker T_g$. The result now follows from the fact that any element of $qK_{\mathcal{I}}^p \cap H^p$ can be written as $q\mathcal{I}\overline{zp}$ for some $p \in H^p$. □

Although the existence of maximal functions in the scalar case has been established in [14], we can use the above corollary to give an alternate expression for a maximal function of a given scalar Toeplitz kernel.

Corollary 2.13. *If $\ker T_g$ is expressed as in Corollary 2.12, then $\kappa_{min}(q\mathcal{I}\overline{z}) = \ker T_g$.*

Proof. We will prove the statement in the case $p \leqslant 2$. It is clear $q\mathcal{I}\overline{z} \in \ker T_g$.

If $q\mathcal{I}\overline{z} \in \ker T_h$ for any other bounded symbol h, then for any $\overline{p} \in \overline{H^p}$ such that $q\mathcal{I}\overline{zp} \in H^p$, because $hq\mathcal{I}\overline{z} \in \overline{H_0^p}$, we must have $hq\mathcal{I}\overline{zp} \in \overline{H_0^p}$. □

2.2.1 Maximal functions when $p = 2$

For the remainder of Section 2.2 we set $p = 2$, and only consider Toeplitz operators

$$T_G : (H^2)^n \to (H^2)^n.$$

When considering whether a given Toeplitz kernel has a maximal function the space $W := \ker T_G \ominus (\ker T_G \cap \; z(H^2)^n)$ is central to this problem. We know from Corollary 4.5 in [20] that $\ker T_G$ can be written as

$$\ker T_G = [W_1, W_2, ...W_r]((H^2)^r \ominus \Phi(H^2)^{r'}), \tag{2}$$

where $W_1, ...W_r$ is an orthonormal basis for W, Φ is a r by r' matrix inner function vanishing at 0 (i.e Φ is such that multiplication by Φ is an isometry from $(H^2)^{r'}$ to $(H^2)^r$) and $r' \leqslant r$.

Lemma 2.14. $\dim \ker T_G(0) = \dim W$.

Proof. $W_1(0), ...W_r(0)$ are linearly independent, as if $W_k(0) = \sum_{i \neq k} \lambda_i W_i(0)$ this would mean $W_k - \sum_{i \neq k} \lambda_i W_i$ vanishes at 0 and therefore lies in $z(H^2)^n$. Next we show that $W_1(0), ...W_r(0)$ span $\ker T_G(0)$. Evaluating $\ker T_G$ at 0 gives

$$\ker T_G(0) = [W_1(0), W_2(0), ...W_r(0)]\mathbb{C}^r,$$

which is equal to the span of $W_1(0), ...W_r(0)$. So $W_1(0), ...W_r(0)$ are a basis for $\ker T_G(0)$. □

Taking into account Theorem 2.6 and the previous lemma we can conclude if $\ker T_G$ is such that $\dim W > 1$, then $\ker T_G$ does not have a maximal function. This leaves us with the following question: if $\ker T_G$ is such that $\dim W = 1$ does this Toeplitz kernel have a maximal function? When $\dim W = 1$, using the Sarason style decomposition (2) we can write

$$\ker T_G = W_1(H^2 \ominus \Phi H^2), \tag{3}$$

where Φ is a (scalar) inner function vanishing at 0 or $\Phi = 0$. So either:

1. $\ker T_G = W_1 K^2_\Phi$,

2. $\ker T_G = W_1 H^2$.

In case 1 K^2_Φ is a Toeplitz kernel so $\ker T_G$ has a maximal function given by $W_1 \Phi \overline{z}$ as shown in Theorem 3.17 in [18].

For case 2 we find that unlike the scalar Toeplitz kernel case there are non-trivial matricial Toeplitz kernels that are shift invariant, for example $\ker T_{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} H^2$. In case 2, $\ker T_G$ can not have a maximal function as if it did we would have $\kappa_{min}(\phi) = W_1 H^2$, but this can't be the case as Theorem 2.5 shows the minimal kernel of any element $\phi \in (H^2)^n$ is not shift invariant (in particular $\phi z \notin \kappa_{min}(\phi)$). We can summarise these results to conclude the following theorem.

Theorem 2.15. *A non-zero Toeplitz kernel, $\ker T_G$, has a maximal function if and only if both: $\dim W = 1$ (or equivalently $\dim \ker T_G(0) = 1$)), and when $\ker T_G$ is decomposed as in (3), $\ker T_G$ takes the form $\ker T_G = W_1 K_\Phi^2$.*

Remark. These two conditions can be concisely written as $\dim \ker T_G(0) = 1$ and $\ker T_G$ is not shift invariant.

Proof. Lemma 2.14 and Theorem 2.6 show that if $\dim W > 1$ then $\ker T_G$ does not have a maximal vector. Conversely if $\dim W = 1$ then the reasoning after (3) shows that when $\ker T_G$ is of the form $W_1 K_\Phi^2$, then it necessarily must have a maximal function and when $\ker T_G = W_1 H^2$, $\ker T_G$ can have no maximal function. □

Corollary 2.16. *If $\ker T_G$ is non-zero, then $\ker T_G$ is of scalar type if and only if* $\dim \ker T_G(0) = 1$.

Proof. If $\ker T_G$ is of scalar type it is clear that $\dim \ker T_G(0) = 1$. Conversely if

$\dim \ker T_G(0) = 1$, then Lemma 2.14 shows $\dim W = 1$, and then (3) shows $\ker T_G$ is of scalar type. □

We note that if $F \in \ker T_G \cap J(H^2)^n$, where J is a scalar inner function then $\frac{F}{J} \in \ker T_G$. This property is called near invariance and we will use this fact in the proof of multiple results in this chapter. We exploit this property further in the next chapter to study the kernels of truncated Toeplitz operators.

Theorem 2.17. *If $\Phi = 0$ in (3) i.e if $\ker T_G = W_1 H^2$, then for any $f \in H^2$ which is a cyclic vector for the backward shift on H^2, we have $\kappa_{min}(W_1, W_1 f) = W_1 H^2$.*

Proof. It is clear the two vectors are in the required kernel. Theorem 4.4 in [20] shows that multiplication by W_1 is an isometric mapping from H^2 to $(H^2)^2$, so W_1 is a 2-by-1 matrix inner function. If $W_1, W_1 f \in \ker T_H$ for any bounded H, then for any $\lambda \in \mathbb{C}$

$$W_1(f - \lambda) \in \ker T_H.$$

So setting $\lambda = f(0)$, and using the near invariance property of Toeplitz kernels we see that

$$W_1 \frac{f - f(0)}{z} = W_1 S^*(f) \in \ker T_H.$$

Repeating this inductively gives $W_1 S^{*n}(f) \in \ker T_H$ for all $n \in \mathbb{N}$, and as f is cyclic for the backward shift and W_1 is inner, we can deduce

$$W_1 H^2 \subseteq \ker T_H.$$

□

This demonstrates that the number of maximal functions needed to specify a matricial Toeplitz kernel is highly non-trivial and poses the question: for an arbitrary Toeplitz kernel $\ker T_G$, how large should k be such that we can find $\phi_1 \dots \phi_k$ where $\kappa_{min}(\phi_1 \dots \phi_k) = \ker T_G$? In this case we call $\phi_1 \dots \phi_k$ a *maximal k-tuple of functions* or when $k = 2$ a *maximal pair of functions* for $\ker T_G$.

We examine the case further for $n = 2$. We have seen if $\dim W = 2$ then $\ker T_G$ does not have a maximal function, however we will now show if $\dim W = 2$ under certain conditions $\ker T_G$ does have a maximal pair of functions. For a matrix A we denote $C_i(A)$ to be the i'th column of A.

Proposition 2.18. *If the decomposition of* $\ker T_G$ *in* (2) *is such that* Φ *is square i.e. if* $\ker T_G = [W_1, W_2]((H^2)^2 \ominus \Phi(H^2)^2)$*, then* $\ker T_G$ *has a maximal pair of functions given by* $[W_1, W_2]C_1(\Phi\overline{z})$*, and* $[W_1, W_2]C_2(\Phi\overline{z})$*.*

Proof. When Φ is square we have $\Phi\Phi^* = \Phi^*\Phi = I$, and so a computation shows $((H^2)^2 \ominus \Phi(H^2)^2) = \ker T_{\Phi^*}$. Then it is clear both vectors are in the required kernel.

Take any $x \in [W_1, W_2]((H^2)^2 \ominus \Phi(H^2)^2) = [W_1, W_2] \ker T_{\Phi^*}$, then

$$x = [W_1, W_2]\Phi \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix} = [W_1, W_2] \left(C_1(\Phi\overline{z})\overline{p_1} + C_2(\Phi\overline{z})\overline{p_2} \right),$$

for some $p_1, p_2 \in H^2$. If $[W_1, W_2]C_1(\Phi\overline{z}), [W_1, W_2]C_2(\Phi\overline{z}) \in \ker T_H$ for any bounded symbol H then

$$H[W_1, W_2]C_1(\Phi\overline{z})\overline{p_1} \in \overline{(zN^+)}^2 \text{ and } H[W_1, W_2]C_1(\Phi\overline{z})\overline{p_2} \in \overline{(zN^+)}^2.$$

Which means

$$Hx = H[W_1, W_2]\left(C_1(\Phi\overline{z})\overline{p_1} + C_2(\Phi\overline{z})\overline{p_2}\right) \in \overline{(zN^+)}^2,$$

but as $x \in (H^2)^n$ and H is bounded we can further conclude $Hx \in (\overline{H_0^2})^2$, and so $x \in \ker T_H$. Our $x \in \ker T_G$ was arbitrarily chosen so

$$\ker T_G \subseteq \ker T_H.$$

Thus $\ker T_G$ has a maximal pair of functions given by

$$\{[W_1, W_2]C_1(\Phi\overline{z}), [W_1, W_2]C_2(\Phi\overline{z})\}.$$

□

Remark. This result can be extended to show that if

$$\ker T_G = [W_1, W_2, ..., W_n]((H^2)^n \ominus \Phi(H^2)^n),$$

then $[W_1, W_2, ...W_n]C_i(\Phi\overline{z})$ for $i \in \{1 \dots n\}$ is a maximal n-tuple of functions for

$\ker T_G$.

2.3 Minimal kernel of multiple elements in H^p

Section 5 of [14] asks if there is a minimal Toeplitz kernel containing a closed subspace $E \subseteq (H^p)^n$, so in the final two sections of this chapter we turn our attention to finding the minimal kernel of multiple elements $f_1 \dots f_k \in (H^p)^n$. This in turn allows us to find the minimal Toeplitz kernel containing a finite-dimensional space E, as we can set $E =$span$\{f_1 \dots f_k\}$. When considering scalar Toeplitz kernels previous results considering the minimal kernel for multiple elements have been presented in [12]. In particular, Theorem 5.6 of [12] shows that when $\kappa_{min}(f_j) = K_{I_j}$ for some inner function I_j then $\kappa_{min}(f_1 \dots f_j) = K_{LCM(I_1,\dots I_j)}$. The corollaries of this section show a fundamental link between the minimal kernel of two elements in H^p and cyclic vectors for the backward shift. In fact, we deduce an equivalent condition for a function to be cyclic for the backward shift on N^+.

It has been shown in [14] that every $f \in H^p$ lies in a non-trivial Toeplitz kernel. If we try to consider the minimal kernel of two elements $f, g \in H^p$ we often find that $\kappa_{min}(f, g) = H^p$, and furthermore this seems to have a connection to cyclic vectors for the backward shift. This is demonstrated with the following example.

Example 2.1. Let f be a cyclic vector for the backward shift on H^p, then $\kappa_{min}(f, 1)$ is equal to H^p.

If for any symbol h, we have $f, 1 \in \ker T_h$, then $f - \lambda \in \ker T_h$ for any $\lambda \in \mathbb{C}$.

Hence $f - f(0) \in \ker T_h$, and by near invariance of Toeplitz kernels $\frac{f-f(0)}{z} = S^*(f) \in \ker T_h$. We can repeat this process inductively to give $S^{*n}(f) \in \ker T_h$, for all $n \in \mathbb{N}$ and as f is cyclic, we deduce $H^p \subseteq \ker T_h$.

One can show can that for any family $\mathcal{F}$ of inner functions, there is an inner function $I_{\mathcal{F}}$ with the property that (i) $I/I_{\mathcal{F}} \in H^\infty$ for all $I \in \mathcal{F}$; and (ii) if J is any inner function which divides every $I \in \mathcal{F}$, then J divides $I_{\mathcal{F}}$. The inner function $I_{\mathcal{F}}$ is called the greatest common divisor of $\mathcal{F}$. In this case we write $I_{\mathcal{F}} = \mathrm{GCD}(\{I : I \in \mathcal{F}\})$. See page 84 of [38] for a proof of the existence of a greatest common divisor.

The following theorem gives a sufficient condition for a given function g to be the symbol of a Toeplitz operator whose kernel is the minimal kernel of a given set of functions in H^p. This result may be viewed as a partial generalisation of Theorem 2.2 in [17].

Theorem 2.19. *If $f_1 \dots f_k \in H^p$ and $g \in L^\infty$ are such that for each $j \in \{1, ..., k\}$ we have $gf_j = \overline{zp_j}$ for some $p_j \in H^p$ and GCD$(p_1^i \dots p_k^i) = 1$, then $\kappa_{min}(f_1 \dots f_k) = \ker T_g$.*

Proof. It is clear that $f_j \in \ker T_g$ for all j. We can write g as $g = \frac{\overline{zp_j}}{f_j}$, and for all $x \in \ker T_g$ we have have $xg = \overline{zp}$ for some $p \in H^p$. Substituting our expression for g into $xg = \overline{zp}$ we may write $\frac{x\overline{zp_j}}{f_j} = \overline{zp}$, and so $x = \frac{f_j p_j^i \overline{p}}{\overline{p_j^o}}$ and then $hx = \frac{p_j^i (hf_j)\overline{p}}{\overline{p_j^o}} \in L^p$. Therefore if $f_j \in \ker T_h$, by Proposition 2.4 $\frac{(hf_j)\overline{p}}{\overline{p_j^o}} \in \overline{zN^+} \cap L^p = \overline{H_0^p}$, which means

$hx = \frac{p_j^i (hf_j)\overline{p}}{\overline{p_j^o}} \in p_j^i \overline{H_0^p}$, so by Proposition 5.5 in [36] $P_+(hx) \in K^p_{p_j^i}$ for all j. Now Corollary 5.9 in [36] shows us that $\bigcap_j K^p_{p_j^i} = K^p_1 = \{0\}$ and so $P_+(hx) = 0$. We conclude $x \in \ker T_h$ and then $\ker T_g \subseteq \ker T_h$. □

Although the following corollary can also be obtained from Corollary 5.1 in [14], we give an alternate proof.

Corollary 2.20. *Every non-trivial scalar Toeplitz kernel has a maximal function.*

Proof. Specialising the above theorem to $k = 1$, we see that if there exists an $f \in H^p$ such that $gf = \overline{zp}$ where $p \in H^p$ is outer then $\kappa_{min}(f) = \ker T_g$. If $\ker T_g$ is non-trivial then there exists a f' such that $gf' = \overline{zp'}$ for some $p' \in H^p$, multiplying both sides of this equality by $(p')^i$ we see that $f'(p')^i$ is a maximal function. □

Remark. Using the above corollary, we also obtain an explicit expression for a maximal function in a non-trivial Toeplitz kernel (when the symbol for the Toeplitz operator is known). This expression can also be derived from Theorem 2.2 in [17].

The following corollary can also be proved as a consequence of Theorem 2.2 in [17], but we provide an alternate proof here.

Corollary 2.21. *If m' is a maximal function for $\ker T_g$ then $\ker T_g = m'\overline{N^+} \cap H^p$.*

Proof. We first show the $\supseteq$ inclusion. As $m' \in \ker T_g$, we must have $m' g\overline{n} \in \overline{zN^+}$ for all $n \in N^+$, so consequently if $m'\overline{n} \in H^p$ we would have $gm'\overline{n} \in \overline{H_0^p}$. To show the $\subseteq$ inclusion we note that $gm' = \overline{zp_1^o}$ where p_1^o is an outer function in H^p, and if $f \in \ker T_g$ then $gf = \overline{zp_2}$ where $p_2 \in H^p$. Solving these expressions for f we see $f = m' \frac{\overline{zp_2}}{p_1^o} \in m' \overline{N^+} \cap H^p$. □

For clarity in the following theorem we will write $span^{N^+}$ to mean the closed linear span in N^+, and we will write *span* to mean the linear span.

Theorem 2.22. *Let $f, g \in H^p$. If $\frac{g}{f^o}$ is cyclic for the backward shift on N^+ then $\kappa_{min}(f, g) = H^p$.*

Proof. For any bounded h, if $f, g \in \ker T_h$ then near invariance shows $f^o \in \ker T_h$, and so for any $\lambda \in \mathbb{C}$,

$$g - \lambda f^o = f^o(\frac{g}{f^o} - \lambda) \in \ker T_h.$$

Letting $\lambda = \frac{g}{f^o}(0)$ we see that

$$f^o(\frac{g}{f^o} - \frac{g}{f^o}(0)) \in \ker T_h,$$

and near invariance gives

$$f^o \frac{(\frac{g}{f^o} - \frac{g}{f^o}(0))}{z} = f^o S^*(\frac{g}{f^o}) \in \ker T_h.$$

We can repeat this process inductively to give

$$span\{f^o S^{*n}(\frac{g}{f^o}) : n \in \mathbb{Z}_+\} \subseteq \ker T_h. \tag{4}$$

We now take the closure of both sides of this set inclusion in the H^p subspace topology of N^+. We first show $span^{N^+}\{f^o S^{*n}(\frac{g}{f^o}) : n \in \mathbb{Z}_+\} = N^+$.

We have $f^o \in N^+$ and for each n, $S^{*n}(\frac{g}{f^o}) \in N^+$, so as N^+ is closed under multiplication we have $\{f^o S^{*n}(\frac{g}{f^o}) : n \in \mathbb{Z}_+\} \subseteq N^+$ and hence $span^{N^+}\{f^o S^{*n}(\frac{g}{f^o}) : n \in \mathbb{Z}_+\} \subseteq N^+$, so one set inclusion is clear. We now show N^+ is contained in $span^{N^+}\{f^o S^{*n}(\frac{g}{f^o}) : n \in \mathbb{Z}_+\}$. Take any $x \in N^+$ then as $\frac{g}{f^o}$ is cyclic for N^+ and $\frac{x}{f^o} \in N^+$ there exists an $(x_k) \subseteq \text{span}\{S^{*n}(\frac{g}{f^o}) : n \in \mathbb{Z}_+\}$ such that $x_k \to \frac{x}{f^o}$ in N^+. Then as N^+ is a topological algebra we must have $f^o x_k \to x$ in N^+. So the closure of the left hand side of (4) in the H^p subspace topology of N^+ is equal to $N^+ \cap H^p = H^p$.

The closure of the right hand side of (4) in the H^p subspace topology of N^+ is the closure of $\ker T_h$ in N^+ intersected with H^p. This can be seen to equal $\ker T_h$ via the following observation. Let $x_k \in \ker T_h \subseteq N^+$ be such that $x_k \to x$ in $\log L$ (or equivalently N^+), then as $\log L$ is a topological algebra $\overline{zhx_k} \to \overline{zhx}$ in $\log L$. As $\overline{zhx_k} \in N^+$ and N^+ is closed in $\log L$ so we must have $\overline{zhx} \in N^+$. If $x \in H^p$ then $\overline{zhx} \in N^+ \cap L^p = H^p$ so $x \in \ker T_h$. We conclude

$$H^p \subseteq \ker T_h.$$

□

Corollary 2.23. *Let $f_1 \dots f_k \in H^p$. If for any pair f_j, f_l with $j, l \in \{1 \dots k\}$, we have that $\frac{f_j}{f_l^o}$ is a cyclic vector for the backward shift on N^+, then $\kappa_{min}(f_1 \dots f_k) = H^p$.*

We now find a minimal kernel for when $\frac{g}{f^o}$ is not a cyclic vector for the backward shift. It is immediate that if $\frac{g}{f^o}$ is not cyclic for N^+ then it lies inside some S^*-invariant subspace, and so to further understand $\kappa_{min}(f, g)$ we must discuss the S^*-invariant subspaces of N^+. As far as the author is aware the S^*-invariant subspaces of N^+ have not been described, however the following (unproved) conjecture is due to Aleksandrov and can be found in Section 11.15 of [42].

Conjecture 2.1.

The S^*-invariant subspaces of N^+ depend on three parameters:

1. An inner function $\mathcal{I}$.
2. A closed set $F \subseteq \mathbb{T}$ with $\sigma(\mathcal{I}) \cap \mathbb{T} \subseteq F$, where

   $$\sigma(\mathcal{I}) = \{z \in \mathbb{D}^- : \lim \inf_{\lambda \longrightarrow z} |\mathcal{I}(\lambda)| = 0\}$$

   is the spectrum of an inner function $\mathcal{I}$.
3. A function $k : F \to \mathbb{N} \cup \{\infty\}$ with the additional property $k(\eta) = \infty$ for all $\eta \in \sigma(\mathcal{I}) \cap \mathbb{T}$ and for all non-isolated points $\eta \in F$.

Define $\mathcal{E}(\mathcal{I}, F, k)$ to be the set of $f \in N^+$ with:

1. $\overline{z}\mathcal{I}\overline{f} \in N^+$.

2. f has an meromorphic continuation $\tilde{f}$ to a neighbourhood of $\hat{\mathbb{C}} \setminus F$.

3. η is a pole of $\tilde{f}$ of order at most $k(\eta)$ for all $\eta \in F$ with $k(\eta) \neq \infty$.

Then $\mathcal{E}(\mathcal{I}, F, k)$ is a proper S^*-invariant subspace of N^+ and for every non-trivial S^*-invariant subspace $\mathcal{E} \subseteq N^+$, there is a triple $(\mathcal{I}, F, k)$ such that $\mathcal{E} = \mathcal{E}(\mathcal{I}, F, k)$.

We will focus on S^*-invariant subspaces of N^+ of the form $\{f \in N^+ : \overline{z}\mathcal{I}\overline{f} \in N^+\} =: \mathcal{I}^*(N^+)$, where $\mathcal{I}$ is some fixed inner function and the above multiplication is understood on $\mathbb{T}$. We call S^*-invariant subspaces of this form *one component* S^*-invariant subspaces. We warn the reader that one component S^*-invariant subspaces are not related to one component inner functions. It seems this terminology has unfortunately been used twice independently to mean different things.

Proposition 2.24. *Let τ be a family of inner functions, then*

$$\bigcap_{\mathcal{I} \in \tau} \mathcal{I}^*(N^+) = GCD(\tau)^*(N^+).$$

Proof. The $\supseteq$ is clear. To prove the $\subseteq$ inclusion we start with the fact that the H^2 closure of $span\{\mathcal{I}H^2 : \mathcal{I} \in \tau\}$ is equal to $\text{GCD}(\tau)H^2$ (see Corollary 4.9 in [36]). This means we can find a sequence $h_n \in span\{\mathcal{I}H^2 : \mathcal{I} \in \tau\}$ such that $h_n \to \text{GCD}(\tau)$ in the H^2 norm. Using the fact $\log(1+x) < x$, this then implies $h_n \to \mathrm{GCD}(\tau)$ in the N^+ metric. So if $f \in \cap_{\mathcal{I}\in\tau}\mathcal{I}^*(N^+)$, then $\overline{z}\mathcal{I}\overline{f} \in N^+$ for all $\mathcal{I} \in \tau$, in particular as N^+ is an algebra $\overline{z}h_n\overline{f} \in N^+$. Taking the limit in the metric of $\log L$, noting $\log L$ is a topological algebra and N^+ is closed we see that $\overline{z}\mathrm{GCD}(\tau)\overline{f} \in N^+$. □

Although the S^*-invariant subspaces of N^+ have not been completely described, there is a partial result showing all S^*-invariant subspaces of N^+ are contained in a one component S^*-invariant subspace. The following can be found as Corollary 1, page 42 in [42].

Proposition 2.25. *Given a non-trivial S^*-invariant subspace of N^+, $\mathcal{E}$, there exists an inner function $\mathcal{J}$ such that $\mathcal{E} \subseteq \mathcal{J}^*(N^+)$.*

If $\frac{g}{f^o}$ is not cyclic, from the above proposition there exists a $\mathcal{J}$ such that $\frac{g}{f^o}$ lies in $\mathcal{J}^*(N^+)$. It then follows $f^i, \frac{g}{f^o}$ lie in a one component S^*-invariant subspace ($(\mathcal{J}f^i)^*(N^+)$ is one such example). Then Theorem 2.24 allows us to consider the smallest one component S^*-invariant subspace containing $f^i, \frac{g}{f^o}$.

Theorem 2.26. *Let $f, g \in H^p$. If $\frac{g}{f^o}$ is not cyclic for S^* then $\kappa_{min}(f,g) = \ker T_{\overline{f^o\mathcal{I}}/f^o}$, where $\mathcal{I}$ is such that $\mathcal{I}^*(N^+)$ is the smallest one component S^*-invariant subspace containing both $\frac{g}{f^o}$ and f^i.*

Proof. We first show $f, g \in \ker T_{\overline{f^o\mathcal{I}}/f^o}$. As $\frac{g}{f^o}, f^i \in \mathcal{I}^*(N^+)$,

$$\frac{g}{f^o} = \mathcal{I}\overline{zp_1},$$

and

$$f^i = \mathcal{I}\overline{zp_2},$$

for some $p_1, p_2 \in N^+$. So

$$g(\frac{\overline{f^o \mathcal{I}}}{f^o}) = \overline{f^o z p_1},$$

and

$$f(\frac{\overline{f^o \mathcal{I}}}{f^o}) = \overline{f^o z \ p_2},$$

both of which are in $\overline{zN^+} \cap L^p = \overline{H_0^p}$ (both can be seen to lie in L^p because the symbol for the operator is unimodular). Now by Theorem 2.19 all that remains to be proved is that $\mathrm{GCD}(p_1^i, p_2^i) = 1$.

Because f^i is inner this then forces p_2 to be inner. If $\mathrm{GCD}(p_2, p_1^i) = \alpha \neq 1$ then as $p_2 | \mathcal{I}$, this then forces $\alpha | \mathcal{I}$ and then this would imply $\frac{g}{f^o}, f^i \in (\mathcal{I}\overline{\alpha})^*(N^+) \subseteq \mathcal{I}(N^+)$. Which can not be the case by minimality of our choice of $\mathcal{I}$. □

Combining Theorem 2.22 and Theorem 2.26 we can now give a complete answer as to when $\kappa_{min}(f, g) = H^p$. This characterisation allows us to deduce an equivalent condition for a function to be cyclic for the backward shift on N^+.

Corollary 2.27. *Let $f, g \in H^p$. There are no non-trivial Toeplitz kernels containing both f and g if and only if $\frac{g}{f^o}$ is a cyclic vector for the backward shift on N^+.*

Due to symmetry of the above corollary and using the fact that the reciprocal of an outer function in N^+ is outer and in N^+ we can also deduce the following.

Corollary 2.28. *Let $f \in N^+$ be outer, then f is cyclic for the backward shift on N^+ if and only if $\frac{1}{f}$ is a cyclic vector for the backward shift on N^+.*

We know from Proposition 2.25 that an outer function $f \in N^+$ is not S^* cyclic for N^+ if and only if there exists an inner function α such that $\overline{\alpha} f \in \overline{zN^+}$. In the case that $f \in H^p$ we then clearly have f is not S^* cyclic for H^p if and only f is not S^* cyclic for N^+. Similarly when $f, \frac{1}{f}$ both lie in H^p from the above corollary we can deduce the following corollary.

Corollary 2.29. *Let f be an outer function in H^p and let $\frac{1}{f}$ also lie in H^p. The following statements are equivalent*

1. *f is not S^* cyclic for H^p,*
2. *f is not S^* cyclic for N^+,*
3. *$\frac{1}{f}$ is not S^* cyclic for N^+,*
4. *$\frac{1}{f}$ is not S^* cyclic for H^p.*

2.4 Minimal kernel of multiple elements in $(H^p)^2$

Keeping with earlier notation we will use Greek symbols for elements of $(H^p)^2$. When considering the minimal kernel of $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in (H^p)^2$, we find that the minimal kernel depends on the determinant of $M = \begin{pmatrix} \phi_1 & \psi_1 \\ \phi_2 & \psi_2 \end{pmatrix}$. We first consider the case when $\det M = \phi_1\psi_2 - \psi_1\phi_2$ is not identically equal to zero.

Theorem 2.30. *Let* $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in (H^p)^2$. *If* $\phi_1\psi_2 - \psi_1\phi_2$ *is not identically equal to zero then* $\kappa_{min}\left(\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right) = \ker T_{(\overline{u_1}/\overline{u_2})\overline{z}M^{-1}}$, *where* u_1 *is a scalar outer function with* $|u_1| = |\phi_1\psi_2 - \psi_1\phi_2|$, *and* u_2 *is a scalar outer function with* $|u_2| = |\phi_1| + |\phi_2| + |\psi_1| + |\psi_2| + 1$.

Proof. We first note that the specified symbol is in fact bounded. We have

$$(\overline{u_1}/\overline{u_2})\overline{z}M^{-1} = \overline{z}\frac{\overline{u_1}}{\phi_1\psi_2 - \psi_1\phi_2}\begin{pmatrix} \psi_2/\overline{u_2} & -\psi_1/\overline{u_2} \\ -\phi_2/\overline{u_2} & \phi_1/\overline{u_2} \end{pmatrix},$$

by construction $|\overline{z}\frac{\overline{u_1}}{\phi_1\psi_2 - \psi_1\phi_2}| = 1$ and each entry in $\begin{pmatrix} \psi_2/\overline{u_2} & -\psi_1/\overline{u_2} \\ -\phi_2/\overline{u_2} & \phi_1/\overline{u_2} \end{pmatrix}$ has modulus smaller than 1, hence $(\overline{u_1}/\overline{u_2})\overline{z}M^{-1}$ is a bounded matrix symbol.

For any $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ker T_{(\overline{u_1}/\overline{u_2})\overline{z}M^{-1}}$, we have

$$(\overline{u_1}/\overline{u_2})\overline{z}M^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{(H_0^p)^2}.$$

Dividing through by $\overline{u_1}$, then multiplying through by $\overline{u_2}$ we see that

$$\overline{z}M^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix},$$

for some $p_1, p_2 \in N^+$, so

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = M\begin{pmatrix} \overline{p_1} \\ \overline{p_2} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\overline{p_1} + \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\overline{p_2}.$$

Then for any other bounded matrix H we have

$$H\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = H\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\overline{p_1} + H\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\overline{p_2}.$$

So if $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \ker T_H$, then both coordinates of $H\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ lie in L^p and both $H\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\overline{p_1}$ and $H\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\overline{p_2}$ have both their coordinates lying in $\overline{zN^+}$, so therefore $H\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{(H_0^p)^2}$. We conclude

$$\ker T_{(\overline{u_1}/\overline{u_2})\overline{z}M^{-1}} \subseteq \ker T_H.$$

$\square$

We now consider the minimal kernel for when $\phi_1\psi_2 - \psi_1\phi_2 = 0$. In the following we let P_1' and P_2' denote the projections $(L^p)^2 \to L^p$ on to the first and second coordinate respectively.

Theorem 2.31. *Let* $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in (H^p)^2$ *and let u be an outer function such that* $|u| = |\phi_1| + |\phi_2| + 1$. *If* $\frac{\psi_2}{\phi_2^o}$ *is not a cyclic vector for the backward shift on* N^+ *and* $\phi_1\psi_2 - \psi_1\phi_2 = 0$, *then we have*

$$\kappa_{min}(\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}) = \ker T_{\begin{pmatrix}\phi_2/u & -\phi_1/u\\ 0 & \overline{\phi_2^o \mathcal{I}}/\phi_2^o\end{pmatrix}},$$

where $\mathcal{I}$ is such that $\mathcal{I}^*(N^+)$ is the smallest one component S^*-invariant subspace containing both $\frac{\psi_2}{\phi_2^o}$ *and* ϕ_2^i.

Remark. We note how $\mathcal{I}$ is the same inner function that appears in the symbol for the scalar minimal kernel of ϕ_2 and ψ_2.

Proof. Our choice of $\mathcal{I}$ guarantees both the vectors are in the required kernel.

Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & \overline{\phi_2^o \mathcal{I}}/\phi_2^o \end{pmatrix}}$, then we have

$$x_2 = \frac{\phi_2^o \mathcal{I} \overline{zh}}{\overline{\phi_2^o}},$$

for some $h \in H^p$. As in the scalar case for our choice of $\mathcal{I}$ we have $\frac{\psi_2}{\phi_2^o} = \mathcal{I}\overline{zp_1}$ and $\phi^i = \mathcal{I}\overline{zp_2}$, for some $p_1, p_2 \in N^+$, so $\mathcal{I}$ can be written as

$$\mathcal{I} = \frac{\psi_2 z p_1^i}{\phi_2^o \overline{p_1^o}},$$

and

$$\mathcal{I} = \phi_2^i z p_2,$$

where p_2 is inner. Substituting our two expressions for $\mathcal{I}$ into the above expression for x_2 gives

$$x_2 = \frac{\psi_2 p_1^i \overline{h}}{\phi_2^o \overline{p_1^o}}, \tag{5}$$

and

$$x_2 = \frac{\phi_2 p_2 \overline{h}}{\overline{\phi_2^o}}. \tag{6}$$

We also have that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfies

$$x_1\phi_2 - \phi_1 x_2 = 0,$$

so substituting $x_2 = \frac{\phi_2 p_2 \overline{h}}{\overline{\phi_2^o}}$ from (6) yields

$$x_1\phi_2 - \phi_1 \frac{\phi_2 p_2 \overline{h}}{\overline{\phi_2^o}} = 0,$$

and so

$$x_1 = \phi_1 \frac{p_2 \overline{h}}{\overline{\phi_2^o}}.$$

Consequently we may write all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & \overline{\phi_2^o \mathcal{I}}/\phi_2^o \end{pmatrix}}$ are of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \frac{p_2 \overline{h}}{\overline{\phi_2^o}}.$$

We will now find a similar expression relating $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Multiplying

$$x_1\phi_2 - \phi_1 x_2 = 0,$$

by $\frac{\psi_1}{\phi_1} = \frac{\psi_2}{\phi_2}$ gives

$$x_1\psi_2 - \psi_1 x_2 = 0,$$

and substituting $x_2 = \frac{\psi_2 p_1^i \overline{h}}{\overline{\phi_2^o}\overline{p_1^o}}$ from (5) into this expression yields

$$x_1\psi_2 - \psi_1 \frac{\psi_2 p_1^i \overline{h}}{\overline{\phi_2^o}\overline{p_1^o}} = 0,$$

so

$$x_1 = \psi_1 \frac{p_1^i \overline{h}}{\overline{\phi_2^o}\overline{p_1^o}}.$$

Consequently we can write

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \frac{p_1^i \overline{h}}{\overline{\phi_2^o}\overline{p_1^o}}.$$

Now we have two expressions for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & \overline{\phi_2^o \mathcal{I}}/\phi_2^o \end{pmatrix}}$,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \frac{p_1^i \overline{h}}{\overline{\phi_2^o p_1^o}},$$

and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \frac{p_2 \overline{h}}{\overline{\phi_2^o}}.$$

So if $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \ker T_H$, for any symbol H, then

$$H \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \frac{p_1^i \overline{h}}{\overline{\phi_2^o p_1^o}} = \Big(H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Big) \Big(\frac{p_1^i \overline{h}}{\overline{\phi_2^o p_1^o}} \Big).$$

By Proposition 2.4 $\frac{\overline{h}}{\overline{\phi_2^o o_1^o}} \in \overline{N^+}$ and $H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \overline{(H_0^p)^2}$, so both coordinates of $\Big(H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Big) \Big(\frac{\overline{h}}{\overline{\phi_2^o p_1^o}} \Big)$ are in $\overline{zN^+} \cap L^p = \overline{H_0^p}$, and so $H \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Big(H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Big) \Big(\frac{p_1^i \overline{h}}{\overline{\phi_2^o p_1^o}} \Big) \in$

$p_1^i\overline{(H_0^p)^2}$. Similarly

$$H\begin{pmatrix}x_1\\x_2\end{pmatrix} = H\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}\frac{p_2\overline{h}}{\overline{\phi_2^o}} = \Big(H\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}\Big)\Big(\frac{p_2\overline{h}}{\overline{\phi_2^o}}\Big) \in p_2\overline{(H_0^p)^2}.$$

So $P_+P_1'(H\begin{pmatrix}x_1\\x_2\end{pmatrix}) \in K_{p_2}^p \cap K_{p_1^i}^p = K_{\mathrm{GCD}(p_2,p_1^i)}^p$, but as in the scalar case we have chosen $\mathcal{I}$ such that $\mathrm{GCD}(p_2,p_1^i) = 1$, so $P_+P_1'(H\begin{pmatrix}x_1\\x_2\end{pmatrix}) \in K_1^p = \{0\}$. The same holds for $P_+P_2'(H\begin{pmatrix}x_1\\x_2\end{pmatrix})$ and so $P_+(H\begin{pmatrix}x_1\\x_2\end{pmatrix}) = 0$, and therefore

$$\ker T_{\begin{pmatrix}\phi_2/u & -\phi_1/u\\ 0 & \overline{\phi_2^o\mathcal{I}}/\phi_2^o\end{pmatrix}} \subseteq \ker T_H.$$

□

We now consider the case when $\frac{\psi_2}{\phi_2^o}$ is cyclic for S^*. In doing so we need to introduce some new theory. Let $(N^+)^2 := \left\{\begin{pmatrix}f_1\\f_2\end{pmatrix} : f_1, f_2 \in N^+\right\}$ with the metric on $(N^+)^2$ defined by

$$\rho^2\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right) = \rho(f_1, g_1) + \rho(f_2, g_2),$$

where ρ is the metric on N^+. It is easily checked that $(N^+)^2$ is also a metric space and a sequence in $(N^+)^2$ converges if and only if both of its coordinates converge in N^+. As outer functions are invertible in N^+ for a fixed $f \in N^+$, $fN^+ = f^i N^+$ is closed. For a fixed $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in (N^+)^2$, the following computation shows $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} N^+$ is closed in $(N^+)^2$. If $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} x_n \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then $f_1 x_n \to x_1$ so $x_1 = f_1 x_0$, for some $x_0 \in N^+$, then as $\log L$ is a topological algebra we can deduce $x_n \to x_0$. So then $f_2 x_n \to f_2 x_0$ and $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} x_n \to \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} x_0 \in \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} N^+$.

We can also let ρ^2 define a metric on $(\log L)^2 = \left\{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1, f_2 \in \log L\right\}$ and in this metric $(N^+)^2$ is a closed subspace of $(\log L)^2$.

Theorem 2.32. *Let* $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in (H^p)^2$, *let* $\beta = GCD(\phi_1^i, \phi_2^i)$ *and let* u *be an outer function such that* $|u| = |\phi_1| + |\phi_2| + 1$. *If* $\frac{\psi_2}{\beta\phi_2}$ *is a cyclic vector for the backward shift on* N^+ *and* $\phi_1\psi_2 - \psi_1\phi_2 = 0$, *then we have*

$$\kappa_{min}\left(\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right)=\ker T_{\begin{pmatrix}\phi_2/u & -\phi_1/u\\ 0 & 0\end{pmatrix}}.$$

The assumption $\phi_1\psi_2-\psi_1\phi_2=0$ ensures $\frac{\psi_2}{\overline{\beta}\phi_2}\in N^+$. Indeed, as $\overline{\beta}\phi_1\psi_2=\psi_1\overline{\beta}\phi_2$ and $\mathrm{GCD}(\overline{\beta}\phi_1^i,\overline{\beta}\phi_2^i)=1$, every inner factor of $\overline{\beta}\phi_2$ divides ψ_2.

In the following proof we will write $span^{N^+}$ to mean the closed linear span in $(N^+)^2$, and *span* to mean the linear span.

Proof. We split the proof up in to two stages. We first prove if for any bounded symbol H we have $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\in\ker T_H$, then $\overline{\beta}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}N^+\cap(H^p)^2\subseteq\ker T_H$. Then we prove $\ker T_{\begin{pmatrix}\phi_2/u & -\phi_1/u\\ 0 & 0\end{pmatrix}}=\overline{\beta}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}N^+\cap(H^p)^2$. If $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\in\ker T_H$ then near invariance of Toeplitz kernels guarantees $\overline{\beta}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\in\ker T_H$, and so for $\lambda\in\mathbb{C}$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}-\lambda\overline{\beta}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=\overline{\beta}\begin{pmatrix}\phi_1(\frac{\psi_1}{\overline{\beta}\phi_1}-\lambda)\\ \phi_2(\frac{\psi_2}{\overline{\beta}\phi_2}-\lambda)\end{pmatrix}\in\ker T_H.$$

Noting $\frac{\psi_1}{\overline{\beta}\phi_1} = \frac{\psi_2}{\overline{\beta}\phi_2}$, and letting $\lambda = \frac{\psi_1}{\overline{\beta}\phi_1}(0) = \frac{\psi_2}{\overline{\beta}\phi_2}(0)$ we see that,

$$\overline{\beta}\begin{pmatrix} \phi_1(\frac{\psi_2}{\overline{\beta}\phi_2} - \frac{\psi_2}{\overline{\beta}\phi_2}(0)) \\ \phi_2(\frac{\psi_2}{\overline{\beta}\phi_2} - \frac{\psi_2}{\overline{\beta}\phi_2}(0)) \end{pmatrix} \in \ker T_H,$$

and near invariance of Toeplitz kernels gives

$$\frac{\overline{\beta}\begin{pmatrix} \phi_1(\frac{\psi_2}{\overline{\beta}\phi_2} - \frac{\psi_2}{\overline{\beta}\phi_2}(0)) \\ \phi_2(\frac{\psi_2}{\overline{\beta}\phi_2} - \frac{\psi_2}{\overline{\beta}\phi_2}(0)) \end{pmatrix}}{z} = \overline{\beta}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} S^*(\frac{\psi_2}{\overline{\beta}\phi_2}) \in \ker T_H.$$

We can repeat this process inductively to get $\overline{\beta}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} S^{*n}(\frac{\psi_2}{\overline{\beta}\phi_2}) \in \ker T_H$ for each $n \in \mathbb{Z}_+$, and hence

$$span\{\overline{\beta}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} S^{*n}(\frac{\psi_2}{\overline{\beta}\phi_2}) : n \in \mathbb{Z}_+\} \subseteq \ker T_H. \tag{7}$$

We will now take the closure of both sides of this set inclusion in the $(H^p)^2$ subspace topology of $(N^+)^2$. The closure of the left hand side of (7) is equal to $span^{N^+}\{\overline{\beta}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} S^{*n}(\frac{\psi_2}{\overline{\beta}\phi_2})\}$ intersected with $(H^p)^2$. As $\overline{\beta}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} N^+$ is closed, $\frac{\psi_2}{\overline{\beta}\phi_2}$ is cyclic and N^+ is a topological algebra the closure of the left hand side of (7) equals

$$\overline{\beta}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} N^+ \cap (H^p)^2.$$

The closure of the right hand side of (7) is the closure of $\ker T_H$ in $(N^+)^2$ intersected with $(H^p)^2$. We now argue this is equal to $\ker T_H$. Let $\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix} \in \ker T_H$ be such that $\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix} \to \begin{pmatrix}x_1\\ x_2\end{pmatrix}$ in $(N^+)^2$, then $\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix} \to \begin{pmatrix}x_1\\ x_2\end{pmatrix}$ in $(\log L)^2$. As $\log L$ is a topological algebra and $H\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix} = \begin{pmatrix}h_{11}x_{1n}+h_{12}x_{2n}\\ h_{21}x_{2n}+h_{22}x_{2n}\end{pmatrix}$, we must have $\overline{zH\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix}} \to \overline{zH\begin{pmatrix}x_1\\ x_2\end{pmatrix}}$ in $(\log L)^2$. As $\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix} \in \ker T_H$ we have $\overline{zH\begin{pmatrix}x_{1n}\\ x_{2n}\end{pmatrix}} \in (N^+)^2$, and as $(N^+)^2$ is closed in $(\log L)^2$ we must have $\overline{zH\begin{pmatrix}x_1\\ x_2\end{pmatrix}} \in (N^+)^2$. So if $\begin{pmatrix}x_1\\ x_2\end{pmatrix} \in (H^p)^2$ then $\overline{zH\begin{pmatrix}x_1\\ x_2\end{pmatrix}} \in (N^+)^2 \cap (L^p)^2 = (H^p)^2$, so $\begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \ker T_H$. From this we deduce

$$\overline{\beta}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} N^+ \cap (H^p)^2 \subseteq \ker T_H.$$

It remains to prove that $\ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & 0 \end{pmatrix}} = \overline{\beta} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} N^+ \cap (H^p)^2$. The $\supseteq$ inclusion is clear. We will now show the $\subseteq$ inclusion. If we let $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ lie in $\ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & 0 \end{pmatrix}}$, then $F_1\phi_2 = F_2\phi_1$, and so $\overline{\beta}F_1\phi_2 = \overline{\beta}F_2\phi_1$ and F_1 can be written as $F_1 = \overline{\beta}\phi_1 \frac{F_2}{\overline{\beta}\phi_2}$. Furthermore $\frac{F_2}{\overline{\beta}\phi_2}$ is in the Smirnov class, because $\overline{\beta}F_1\phi_2 = \overline{\beta}F_2\phi_1$ and $\mathrm{GCD}(\overline{\beta}\phi_1, \overline{\beta}\phi_2) = 1$ so every inner factor of $\overline{\beta}\phi_2$ divides F_2. We can also write $F_2 = \overline{\beta}\phi_2 \frac{F_1}{\overline{\beta}\phi_1}$, and as $\overline{\beta}F_1\phi_2 = \overline{\beta}F_2\phi_1$, we have $\frac{F_1}{\overline{\beta}\phi_1} = \frac{F_2}{\overline{\beta}\phi_2}$, so $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in \overline{\beta} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} N^+ \cap (H^p)^2$.

Thus we have proved that if $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \ker T_H$ then

$$\ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & 0 \end{pmatrix}} \subseteq \ker T_H.$$

□

Proposition 2.33. *Let I be inner. Then $f \in N^+$ is cyclic for S^* if and only if If*

is cyclic for S^*.

Proof. If f is not cyclic then it lies in a non-trivial S^*-invariant subspace. Then by Proposition 2.25 $f \in \mathcal{I}^*(N^+)$ for some inner function $\mathcal{I}$, which then means $If \in (\mathcal{I}I)^*(N^+)$ and is therefore not cyclic for S^*. Conversely if If is not cyclic, If lies in some one component S^*-invariant subspace and hence so does f. So f can not be cyclic. $\square$

Combining the two previous theorems and the previous proposition we can deduce the following unifying theorem.

Theorem 2.34. *Let* $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in (H^p)^2$ *be such that* $\phi_1\psi_2 - \psi_1\phi_2 = 0$*. Then we have*

$$\kappa_{min}\left(\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right) = \ker T_{\begin{pmatrix} \phi_2/u & -\phi_1/u \\ 0 & \chi \end{pmatrix}},$$

where u is an outer function such that $|u| = \phi_1 + \phi_2 + 1$ and χ is our previously given symbol for the scalar Toeplitz kernel $\kappa_{min}(\phi_2, \psi_2)$. (Here if $\kappa_{min}(\phi_2, \psi_2) = H^p$ the symbol is formally defined to be 0.)

3 Nearly invariant subspaces

Throughout this chapter from Section 3.2 onward we assume the symbol of any truncated Toeplitz operator is bounded, and hence the truncated Toeplitz operator is bounded. Throughout we continue to let I be an arbitrary inner function.

Definition 3.1. *A closed subspace $M \subseteq (H^2)^n$ is said to be nearly S^*-invariant with defect d if and only if there exists a d-dimensional subspace of $(H^2)^n$, D, (which may be taken to be orthogonal to M) such that if $f \in M$ and $f(0)$ is the zero vector then $S^*f \in M \oplus D$.*

If M is nearly S^*-invariant with defect 0 then it is said to be nearly S^*-invariant.

The concept of (scalar) nearly backward shift invariant subspaces was first introduced by Hitt in [44] as a generalisation to Hayashi's results concerning Toeplitz kernels in [43]. These spaces were then studied further by Sarason [62]. The study of nearly backward shift invariant subspaces was then generalised to the vectorial case in [20], and generalised to include a finite defect in [22]. Kernels of Toeplitz operators are the prototypical example of nearly S^*- invariant subspaces.

3.1 Preliminary results

Although truncated Toeplitz operators share many properties with the classical Toeplitz operator, it is easily checked that the kernel of a truncated Toeplitz operator is not nearly S^*-invariant. For example the truncated Toeplitz operator $A_z^{z^3}$ has kernel given by span$\{z^2\}$ which is clearly not nearly S^*-invariant. This motivates our study for this section where we show under certain conditions the kernel of a truncated Toeplitz operator is in fact nearly S^*-invariant with defect 1. In many cases the study of Toeplitz operators becomes greatly simplified when the operator has an invertible symbol; in this section we also show that the symbol of a truncated Toeplitz operator, g, may be chosen such that $g^{-1} \in L^\infty$.

Theorem 3.2. *For any $g \in L^2$ we write $g = g^- + g^+$ where $g^- \in \overline{H_0^2}$ and $g^+ \in H^2$. If the outer function in H^2 with modulus equal to $2|g|+1$ is not cyclic for the backward shift then there exists a $\tilde{g} \in L^2$ such that $A_g^I = A_{\tilde{g}}^I$ and $\tilde{g}^{-1} \in H^\infty$.*

Proof. Theorem 3.1 of [63] shows that $A_{g_1}^I = A_{g_2}^I$ if and only if $g_1 - g_2 \in \overline{IH^2} + IH^2$, so we may initially assume without loss of generality that $g \in \overline{K_I^2} \oplus K_I^2$. Using Lemma 2.2 we can construct an outer function u such that $|u| = 2|g|+1$, furthermore $u \in L^2$ so $u \in H^2$. Then it follows that for any inner function α

$$g - \overline{\alpha u} \tag{8}$$

has the property that

$$|g - \overline{\alpha u}| \geqslant |u| - |g| > |g| + 1 > 0$$

almost everywhere on $\mathbb{T}$, and so $(g - \overline{\alpha u})^{-1} \in L^\infty$. Our construction of u shows $|\frac{1}{u}| \leqslant 1$ and as the reciprocal of an outer function in is outer, we have $\frac{1}{u}$ is outer and in L^∞, so $\frac{1}{u} \in H^\infty$. Furthermore by Corollary 2.29 we can say $\frac{1}{u} \in H^2$ is non-cyclic for S^* and hence must lie in a model space K^2_Φ. Define $\tilde{g} := (g - \overline{\Phi I u})$, then as previously stated $\tilde{g}^{-1} \in L^\infty$. We now show $\tilde{g}^{-1} = \sum_{k=0}^{\infty}(-1)g^k(\Phi I \frac{1}{\overline{u}})^{k+1}$ where the limit is taken in the sense of uniform convergence. We write $\tilde{g}_N^{-1}$ to be $\sum_{k=0}^{N}(-1)g^k(\Phi I \frac{1}{\overline{u}})^{k+1}$ then we have $\|\tilde{g}_N^{-1} - \tilde{g}^{-1}\|_\infty$ is equal to

$$\|\tilde{g}^{-1}\tilde{g}(\tilde{g}_N^{-1} - \tilde{g}^{-1})\|_\infty \leqslant \|\tilde{g}^{-1}\|_\infty \|\tilde{g}\tilde{g}_N^{-1} - 1\|_\infty \leqslant \|\tilde{g}^{-1}\|_\infty \|g^N(\Phi I \frac{1}{\overline{u}})^N\|_\infty.$$

By our construction of u this is less than $\|\tilde{g}^{-1}\|_\infty (\frac{1}{2})^N$, which clearly converges to 0. Now our choice of Φ ensures that $\Phi \frac{1}{\overline{u}} \in H^\infty$, we also have $Ig \in H^2$. This means $(-1)g^k(\Phi I \frac{1}{\overline{u}})^{k+1} \in H^2$ and is bounded by 1, so must actually lie in H^∞. So $\tilde{g}^{-1}$ being the uniform limit of a sequence in H^∞ must also be in H^∞. □

Examining the first part of the above proof we can also deduce the following proposition.

Proposition 3.3. *For any $g \in L^2$ there exists a $\tilde{g} \in L^2$ such that $A^I_g = A^I_{\tilde{g}}$ and $\tilde{g}^{-1} \in L^\infty$.*

Proof. In (8) if we set α to equal I, keep our construction of u the same and define $\tilde{g} = g - \overline{\alpha u}$ then $A^I_g = A^I_{\tilde{g}}$. Furthermore the computation immediately after (8) shows $\tilde{g}^{-1} \in L^\infty$. □

This has an interesting relation to Sarason's question posed in [63]; which is whether every bounded truncated Toeplitz operator has a bounded symbol. Although Sarason's question has been shown to be not true in general, the above proposition shows every bounded truncated Toeplitz operator has a symbol which has a bounded inverse.

These results suggest that under certain circumstances $\ker A_g^I$ may be a nearly invariant subspace with a finite defect. This is because $f \in \ker A_g^I$ if and only if $f \in K_I^2$ and

$$gf \in \overline{H_0^2} \oplus IH^2,$$

so if $f(0) = 0$ and $f \in \ker A_g^I$ then we must have

$$\frac{gf}{z} \in \overline{H_0^2} + \text{span}\{S^*(I)\} + IH^2.$$

This may lead us to believe that $\ker A_g^I$ is a nearly S^*-invariant subspace with a defect given by $g^{-1}\text{span}\{S^*(I)\}$, but the issue here is $g^{-1}S^*(I)$ need not necessarily lie in K_I^2 or even H^2. Theorem 3.2 shows us that under some weak restrictions we can choose our non-unique symbol g so that $g^{-1}S^*(I) \in H^2$, but to fully understand $\ker A_g^I$ as a nearly invariant subspace with a defect we must study vector-valued nearly invariant subspaces with a defect.

3.2 Vector-valued nearly invariant subspaces with a defect

In this section we prove a powerful result that shows for any $i \in \{1 \dots n\}$ the first i coordinates of a vector-valued nearly S^*-invariant subspace of $(H^2)^n$ is a nearly S^*-invariant subspace with a finite defect. We then generalise Theorem 3.2 in [20] and Corollary 4.5 in [22] to find a Hitt-style decomposition for the vector-valued nearly S^*-invariant subspaces with a finite defect.

Let $M \subseteq (H^2)^n$ be a nearly invariant subspace for the backward shift with a finite defect space D and let $\dim D = d$. If not all functions in M vanish at 0 then we define $W := M \ominus (M \cap z(H^2)^n)$ and Corollary 4.3 in [20] shows that $r := \dim W \leqslant n$, in this case we let $W_1 \dots W_r$ be an orthonormal basis of W. For $i = 1 \dots n$ we let $P_i : (H^2)^n \to (H^2)^i$ be the projection on to the first i coordinates.

Theorem 3.4. *For any $i \in \{1 \dots n\}$, $M_i := P_i(M)$ is a (not necessarily closed) nearly invariant subspace with a defect space* $\left(\frac{\text{span}\{P_i(W_1), \dots P_i(W_r)\}}{z} \cap (H^2)^i\right) + P_i(D)$.

Proof. We first consider the case when not all functions in M vanish at 0. Let $f_i \in M_i$, then f_i is the first i entries of some $F \in M$. We write F as

$$F = a_1 W_1 + \dots a_r W_r + F_1,$$

where $a_1 \dots a_r \in \mathbb{C}$ and $F_1 \in M \cap z(H^2)^n$. So if $f_i(0)$ is the zero vector, we then have $f_i(0)$ is zero and $F_1(0)$ is zero, which forces $P_i(a_1W_1 + \dots a_rW_r)$ to be zero. So

$$\frac{f_i}{z} - \frac{P_i(a_1W_1 + \dots a_rW_r)}{z} = P_i\left(\frac{F_1}{z}\right) \in M_i + P_i(D),$$

which means

$$\frac{f_i}{z} \in M_i + \left(\frac{\text{span}\{P_i(W_1), \dots P_i(W_r)\}}{z} \cap H^2\right) + P_i(D).$$

In the case when all functions in M vanish at 0 then $W = \{0\}$ and we would just have $\frac{F}{z} \in M + D$, so $\frac{f_i}{z} \in M_i + P_i(D)$. □

Remark. If $W = \{0\}$ we can interpret $\left(\frac{\text{span}\{P_i(W_1),\dots P_i(W_r)\}}{z} \cap (H^2)^2\right)$ to be the zero vector.

Corollary 3.5. *With the same assumptions as in Theorem 3.4, if $d = 0$ i.e. if M is a nearly S^*-invariant subspace, then M_i is a (not necessarily closed) nearly S^*-invariant subspace with a defect space $\left(\frac{\text{span}\{P_i(W_1),\dots P_i(W_r)\}}{z} \cap (H^2)^i\right)$.*

To further build on this result we will now give a Hitt style decomposition for a vector-valued nearly invariant subspace with a finite defect. This style of decomposition was first introduced by Hitt in [44] when he decomposed the nearly S^*-invariant subspaces. This was then generalised to the vectorial case as Corollary 4.5 in [20]. This style of proof was then adapted to produce a similar result for the (scalar) nearly invariant subspace with a defect, which is Theorem 3.2 in [22].

For a Hilbert space $\mathcal{H}$ and $x, y \in \mathcal{H}$ we define $x \otimes y(f) = \langle f, y\rangle x$. We say an operator T on $\mathcal{H}$ belongs to the class $C_{.0}$ if for all $x \in \mathcal{H}, \lim_{n\to\infty}\|(T^*)^n x\| = 0$. Consider a subspace M which is nearly S^*-invariant with defect 1, so that $D = \operatorname{span}\{e_1\}$, say, where $\|e_1\|_{(H^2)^n} = 1$. Suppose first that not all functions in M vanish at 0, then $1 \leqslant r = \dim W \leqslant n$. Let F_0 be the matrix with columns $W_1 \dots W_r$, and let P_W be the orthogonal projection on to W. For each $F \in M$ we may write

$$F = P_W(F) + F_1 = F_0 \begin{pmatrix} a_0^1 \\ \vdots \\ a_0^r \end{pmatrix} + F_1.$$

Now as $F_1(0) = 0$ we have $S^*(F_1) = G_1 + \beta_1 e_1$, where $G_1 \in M$ and $\beta_1 \in \mathbb{C}$. Thus

$$F(z) = F_0(z)A_0 + zG_1(z) + z\beta e_1(z),$$

where $A_0 = \begin{pmatrix} a_0^1 \\ \vdots \\ a_0^r \end{pmatrix}$. Moreover since the family $\{W_i\}_{i=1\dots r}$ forms an orthonormal basis of W, we obtain the following identity of norms:

$$\|F\|^2_{(H^2)^n} = \|F_0 A_0\|^2_{(H^2)^n} + \|F_1\|^2_{(H^2)^n} = \|A_0\|^2 + \|G_1\|^2_{(H^2)^n} + |\beta_1|^2.$$

We may now repeat this process on G_1 to obtain $G_1 = P_W(G_1) + F_2$, and $S^*(F_2) =$

$G_2 + \beta_2 e_1$, so $G_1 = F_0 A_1 + zG_2 + z\beta_2 e_1$. We iterate this process to obtain

$$F(z) = F_0(z)(A_0 + A_1 z + \ldots A_{n-1} z^{n-1}) + zG_n(z) + (\beta_1 z + \ldots + \beta_n z^n)\, e_1(z), \quad (9)$$

where

$$\|F\|^2_{(H^2)^n} = \sum_{k=0}^{n-1} \|A_k\|^2 + \|G_n\|^2_{(H^2)^n} + \sum_{k=1}^{n} |\beta_k|^2.$$

We now argue $\|G_n\|_{(H^2)^n} \to 0$ as $n \to \infty$. We can write $G_n = P_{e_1} S^* P_{W^\perp}(G_{n-1})$, where P_{e_1} is the projection with kernel span$\{e_1\}$ and $P_{W^\perp}$ is the projection with kernel span$\{W_1 \ldots W_r\}$. For all $n \geqslant 1$ we may write $G_{n+1} = P_{e_1} R^{n-1}(S^* P_{W^\perp}(G_1))$, where $R = S^* P_{W^\perp} P_{e_1}$ and so

$$\|G_{n+1}\|_{(H^2)^n} \leqslant \|P_{e_1}\| \|R^{n-1}(S^* P_{W^\perp}(G_1))\|_{(H^2)^n}. \quad (10)$$

As e_1 is orthogonal to W we have

$$P_{W^\perp} P_{e_1} = P_{e_1} P_{W^\perp} = I_d - e_1 \otimes e_1 - \sum_{j=1}^{r} W_j \otimes W_j,$$

and so the adjoint of R is

$$P_{e_1} P_{W^\perp} S = S - e_1 \otimes S^*(e_1) - \sum_{j=1}^{r} W_j \otimes S^*(W_j).$$

We now apply the second assertion of Proposition 2.1 from [20] to show the adjoint of R is of class $C_{.0}$, and so R^{n-1} applied to $S^* P_{W^\perp}(G_1)$ converges to 0; now from (10) we see $\|G_{n+1}\|_{(H^2)^n} \to 0$. As a consequence taking limits in (9) we may write

$$F(z) = \lim_{n\to\infty} \left(F_0(z)(A_0 + A_1 z + \ldots A_{n-1} z^{n-1}) + (\beta_1 z + \ldots + \beta_n z^n) e_1(z) \right).$$

We denote $a_n(z) = F_0(z)\left(A_0 + A_1 z + \ldots A_{n-1} z^{n-1}\right)$, and $a_0(z) = F_0\left(\sum_{k=0}^{\infty} A_k z^k\right)$, where $\left(\sum_{k=0}^{\infty} A_k z^k\right)$ is taken in the $(H^2)^n$ sense (this is defined by the equality of norms given immediately after (9)). Then in the $(H^1)^n$ norm we must have $\|a_n(z) - a_0(z)\|_{(H^1)^n}$ is equal to

$$\|F_0 \sum_{k=n}^{\infty} A_k z^k\|_{(H^1)^n} \leqslant \|W_1 \sum_{k=n}^{\infty} a_k^1 z^k\|_{(H^1)^n} + \ldots + \|W_r \sum_{k=n}^{\infty} a_k^r z^k\|_{(H^1)^n}.$$

For each $i \in \{1 \ldots r\}$ we define C_i to equal the maximum H^2 norm of each coordinate of W_i multiplied by n, then we apply Hölder's inequality on each coordinate to obtain

$$\|W_i \sum_{k=n}^{\infty} a_k^i z^k\|_{(H^1)^n} \leqslant C_i \|\sum_{k=n}^{\infty} a_k^i z^k\|_{(H^2)^n} \to 0.$$

Thus in the $(H^1)^n$ norm we have $a_n \to a_0$. A similar computation shows

$$(\beta_1 z + \ldots + \beta_n z^n)\, e_1(z)$$

converges to $(\sum_{k=1}^{\infty} \beta_k z^k)e_1$ in the $(H^1)^n$ norm so the $(H^1)^n$ limit of

$$F(z) = F_0(z)(A_0 + A_1 z + \ldots A_{n-1} z^{n-1}) + (\beta_1 z + \ldots + \beta_n z^n) e_1(z)$$

must be equal to

$$F(z) = F_0 \left(\sum_{k=0}^{\infty} A_k z^k \right) + \left(\sum_{k=1}^{\infty} \beta_k z^k \right) e_1.$$

Furthermore by taking limits in the equality of norms immediately after (9) we know

$$\|F\|^2_{(H^2)^n} = \sum_{k=0}^{\infty} \|A_k\|^2 + \sum_{k=1}^{\infty} |\beta_k|^2. \tag{11}$$

We may alternatively express this as saying $F \in M$ if and only if

$$F(z) = F_0 k_0 + z k_1 e_1, \tag{12}$$

where (k_0, k_1) lies in a subspace $K \subseteq (H^2)^r \times H^2$ which is identified with $(H^2)^{r+1}$.

By virtue of (11) we can see that K is the image of a isometric mapping, and hence closed. We now argue K is invariant under the backward shift on $(H^2)^{r+1}$. Since in the algorithm we have $k_0(0) = A_0$ and $k_1(0) = \beta_1$ we can write F as

$$F = F_0 A_0 + z F_0 S^*(k_0) + \beta_1 z e_1 + z^2 S^*(k_1) e_1,$$

consequently

$$F_0 S^*(k_0) + zS^*(k_1)e_1 = \frac{F - F_0 A_0 - \beta_1 z e_1}{z} = G_1 \in M. \tag{13}$$

Conversely if

$$M = \{F_0 k_0 + z k_1 e_1 : (k_0, k_1) \in K\},$$

is a closed subspace of $(H^2)^n$, where K is a S^*-invariant subspace of $(H^2)^{r+1}$, then M is nearly S^*-invariant with defect 1. To show this we first need a lemma, which has been shown to be true in the proof of Lemma 2.14.

Lemma 3.6. $W_1(0), ...W_r(0)$ *are linearly independent in* $\mathbb{C}^n$.

If $F \in M$ and $F(0) = 0$ then we must have $F_0(0)k_0(0)$ is equal to the zero vector. We now add $n - r$ vectors $X_1, \ldots, X_{n-r} \in \mathbb{C}$ which are linearly independent from $W_1(0), \ldots W_r(0)$ as extra columns to the matrix $F_0(0)$ to obtain a matrix

$$F_0^{'}(0) = [W_1(0), \ldots, W_r(0), X_1, \ldots, X_{n-r}].$$

We now add $n - r$ extra 0's to the end of the column vector $k_0(0)$ and label this $k_0^{'}(0)$. As $F_0(0)k_0(0)$ is equal to the zero vector, then $F_0^{'}(0)k_0^{'}(0)$ must also be equal to the zero vector. We can now invert $F_0^{'}(0)$ to obtain $k_0^{'}(0)$ is equal to the zero vector and hence $k_0(0)$ must be zero. This allows us to write

$$S^*(F) = F_0\frac{k_0}{z} + k_1e_1 = F_0\frac{k_0}{z} + zS^*k_1e_1 + k_1(0)e_1,$$

and as K is S^*-invariant this is clearly an element of $M \oplus \text{span}\{e_1\}$.

If all functions in M vanish at 0 then there is no non-trivial reproducing kernel at 0, but we may now write

$$F(z) = z\left(G_1(z) + \beta_1e_1(z)\right),$$

with $G_1 \in M$ and $\beta_1 \in \mathbb{C}$, and furthermore

$$\|F\|^2_{(H^2)^n} = \|G_1\|^2_{(H^2)^n} + |\beta_1|^2.$$

We can then iterate on G_1 as we have previously done to obtain

$$F(z) = \beta_1ze_1 + \beta_2z^2e_1 + \ldots.$$

For a general finite defect m the analogous calculations produce the following result.

Theorem 3.7. *Let M be nearly S^*-invariant with a finite defect d. Then:*

1. In the case where there are functions in M that do not vanish at 0,

$$M = \{F : F(z) = F_0(z)k_0(z) + z\sum_{j=1}^{d} k_j(z)e_j(z) : (k_0, \dots, k_d) \in K\},$$

where F_0 is the matrix with each column being an orthonormal element of W, $\{e_1, \dots e_d\}$ is any orthonormal basis for D, $k_0 \in (H^2)^r$ (where $r = \dim W$), $k_1, \dots k_d \in H^2$, and $K \subseteq (H^2)^{(r+d)}$ is a closed S^*-invariant subspace. Furthermore $\|F\|^2_{(H^2)^n} = \|k_0\|^2_{(H^2)^r} + \sum_{j=1}^{d} \|k_j\|^2_{H^2}$.

2. In the case where all functions in M vanish at 0,

$$M = \{F : F(z) = z\sum_{j=1}^{d} k_j(z)e_j(z) : (k_1, \dots, k_d) \in K\},$$

with the same notation as in 1, except that K is now a closed S^*-invariant subspace of $(H^2)^d$, and $\|F\|^2_{(H^2)^n} = \sum_{j=1}^{d} \|k_j\|^2_{H^2}$.

Conversely if a closed subspace $M \subseteq (H^2)^n$ has a representation as in 1 or 2, then it is a nearly S^*-invariant subspace with defect m.

Remark. The above theorem was also independently proved in [23].

3.3 Application to truncated Toeplitz operators

In this section we show that whenever a truncated Toeplitz operator has a bounded symbol, the kernel of the TTO is a nearly S^*-invariant subspace with defect 1; this then allows us to decompose the kernel into an isometric image of a model space. The approach of decomposing a kernel into an isometric image of a model space much resembles the works of Hayashi [43] and Hitt [44] for the classical Toeplitz operator. We also make the observation that we can decompose the kernel of a truncated Toeplitz operator into a nearly S^*-invariant subspace multiplied by a power of z (where $z \in \mathbb{D}$ is the independent variable). Then using the results of [44], this observation also gives us a second method to decompose the kernel into a isometric image of a model space. Furthermore we show that in general our two choices of decomposition of the kernel of a truncated Toeplitz operator yield different results. Finally we give a decomposition of a TTO when the inner function, I, corresponding to the model space K_I^2 satisfies extra assumptions.

Throughout this section (3.3) we assume g is bounded and so the truncated Toeplitz operator $A_g^I : K_I^2 \to K_I^2$ may be defined by

$$A_g^I(f) = P_{I,2}(gf).$$

It was originally observed in [15] that the kernel of A_g^I is the first coordinate of the kernel of the matricial Toeplitz operator with symbol

$$G = \begin{pmatrix} \overline{I} & 0 \\ g & I \end{pmatrix}.$$

Recall that scalar-type Toeplitz kernels (first introduced in [18]) are vector-valued Toeplitz kernels which can be expressed as the product of a space of scalar functions multiplied by a fixed vector function. Recall that a maximal function for $\ker T_G$ is an element $f \in \ker T_G$ such that if $f \in \ker T_H$ for any other bounded matricial symbol H, then $\ker T_G \subseteq \ker T_H$. By Corollary 3.9 in [18] $\ker T_G$ is of scalar type, and it is also easily checked that $\ker T_G$ is not shift invariant and so by Theorem 3.7 in [18] we must have that $\ker T_G$ has a maximal function. Now by Theorem 2.15 whenever the kernel is non-zero we can deduce that $W = \ker T_G \ominus (\ker T_G \cap z(H^2)^n)$ has dimension 1. If we denote $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ to be the normalised element of W then using Corollary 4.5 from [20] we can write

$$\ker T_G = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} K^2_{z\Phi},$$

where Φ is an inner function. We now can write

$$\ker A_g^I = w_1 K_{z\Phi}. \tag{14}$$

We describe Φ with the following proposition.

Proposition 3.8. *When* $\ker T_G = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} K^2_{z\Phi}$, Φ *is the unique (up to multiplication* *by a unimodular constant) inner function for which there exists* $p_1, p_2 \in H^2$ *such that*

$$G \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Phi = \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix},$$

and $GCD(p_1^i, p_2^i) = 1$.

We recall that GCD stands for greatest common divisor, and the reasoning following Theorem 2.19, shows the existence of a GCD of a family of inner functions.

Proof. We first show that up to multiplication by a unitary constant there can only be one inner function Φ satisfying

$$G \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Phi = \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix},$$

where $\mathrm{GCD}(p_1^i, p_2^i) = 1$. Suppose there are two inner functions Φ_1, Φ_2 such that

$$G \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Phi_1 = \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix},$$

and

$$G \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Phi_2 = \begin{pmatrix} \overline{zq_1} \\ \overline{zq_2} \end{pmatrix},$$

where both $\text{GCD}(p_1^i, p_2^i) = 1$ and $\text{GCD}(q_1^i, q_2^i) = 1$. This would then imply that

$$\overline{\Phi_1} \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix} = \overline{\Phi_2} \begin{pmatrix} \overline{zq_1} \\ \overline{zq_2} \end{pmatrix},$$

and so $(\Phi_1 p_1)^i = (\Phi_2 q_1)^i$ and $(\Phi_1 p_2)^i = (\Phi_2 q_2)^i$. By assumption we have $\text{GCD}(p_1^i, p_2^i)$ $= 1$ so $\text{GCD}((\Phi_1 p_2)^i, (\Phi_1 p_1)^i) = \Phi_1$, but substituting $(\Phi_1 p_1)^i$ for $(\Phi_2 q_1)^i$ we obtain

$$\text{GCD}((\Phi_1 p_2)^i, (\Phi_2 q_1)^i) = \Phi_1,$$

and so Φ_1 divides Φ_2. A similar computation shows Φ_2 divides Φ_1, and so we must have Φ_1 is a unitary constant multiple of Φ_2. We now show that Φ is such that

$$G \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Phi = \begin{pmatrix} \overline{zp_1} \\ \overline{zp_2} \end{pmatrix},$$

with $\text{GCD}(p_1^i, p_2^i) = 1$. If it is the case that $\alpha = \text{GCD}(p_1^i, p_2^i) \neq 1$ then it would follow that $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Phi\alpha \in \ker T_G$, which would be a contradiction as $\Phi\alpha \notin K_{z\Phi}^2$. $\square$

It is easily checked that $\ker T_G$ is nearly S^*-invariant, and in view of (14) we can use Corollary 3.5 to deduce the kernel of a truncated Toeplitz operator is nearly S^*-invariant with a defect given by $\text{span}\{\frac{w_1}{z}\} \cap H^2$. With this information we can use the following result given as Theorem 3.2 in [22] (or equivalently Theorem 3.7

with $n = 1$) to study $\ker A_g^I$.

Theorem 3.9. *Let $M \subseteq H^2$ be a closed subspace that is nearly S^*-invariant with a finite defect d. Then:*

1. *In the case where there are functions in M that do not vanish at 0,*

$$M = \{f : f(z) = f_0(z)k_0(z) + z\sum_{j=1}^{d} k_j(z)e_j(z) : (k_0, \dots, k_d) \in K\},$$

where f_0 is the normalised reproducing kernel for M at 0, $\{e_1, \dots e_d\}$ is any orthonormal basis for D, and K is a closed S^*-invariant subspace of $(H^2)^{(d+1)}$. Furthermore $\|f\|_{H^2}^2 = \sum_{j=0}^{d} \|k_j\|_{H^2}^2$.

2. *In the case where all functions in M vanish at 0,*

$$M = \{f : f(z) = z\sum_{j=1}^{d} k_j(z)e_j(z) : (k_1, \dots, k_d) \in K\},$$

with the same notation as in 1, except that K is now a closed S^*-invariant subspace of $(H^2)^d$, and $\|f\|_{H^2}^2 = \sum_{j=1}^{d} \|k_j\|_{H^2}^2$.

Conversely if a closed subspace $M \subseteq H^2$ has a representation as in 1 or 2, then it is a nearly S^*-invariant subspace with defect d.

To use Theorem 3.9 we have to assume that our defect space is orthogonal to $\ker A_g^I$; we consider two separate cases. We first assume that all functions in $\ker A_g^I$

vanish at 0. We set $O := \ker A_g^I + \operatorname{span}\{\frac{w_1}{z}\}$, $E := O \ominus \ker A_g^I$, we let e be $P_E(\frac{w_1}{z})$ and then e is orthogonal to $\ker A_g^I$. In this construction $e \neq 0$ as this would imply $\frac{w_1}{z} \in \ker A_g^I = w_1 K_{z\Phi}$ which is clearly a contradiction. Theorem 3.9 now yields

$$\ker A_g^I = ezK_\Psi,$$

where Ψ is some inner function and multiplication by ez is an isometry from K_Ψ to $\ker A_g^I$. This expression for $\ker A_g^I$ is more familiar than $w_1 K_{z\Phi}^2$ (which was obtained as equation (14)) as in this case the multiplication is an isometry as opposed to a contraction. We can also relate this expression to nearly S^*-invariant subspaces. If we let n be the greatest natural number such that $\frac{e}{z^n} \in H^2$ then $\frac{\ker A_g^I}{z^{n+1}} = \frac{e}{z^n} K_\Psi^2$, now $\frac{e}{z^n}(0) \neq 0$ so $\frac{\ker A_g^I}{z^{n+1}} = \frac{e}{z^n} K_\Psi^2$ is a nearly S^*-invariant subspace. We can conclude the following theorem in this case.

Theorem 3.10. *If n is the greatest natural number such that $\ker A_g^I \subseteq z^n H^2$, then $\frac{\ker A_g^I}{z^n}$ is a nearly S^*-invariant subspace.*

We now turn our attention to the case when not all functions in $\ker A_g^I$ vanish at 0. In this case it must also follow that $w_1(0) \neq 0$ as otherwise $w_1 K_{z\Phi}^2(0) = 0$, so using Corollary 3.5 we must have the defect space for $\ker A_g^I$ to be 0. So we can conclude the following theorem.

Theorem 3.11. *If $\ker A_g^I$ contains functions which do not vanish at 0 then it is nearly S^*-invariant.*

When $\ker A_g^I$ is nearly S^*-invariant we may proceed by using Proposition 3 of the paper of Hitt [44] to show $\ker A_g^I = uK^2_{z\psi}$ where $u \in \ker A_g^I \ominus (\ker A_g^I \cap zH^2)$ is an isometric multiplier and ψ is some inner function. As was noted in [41] we can call ψ the associated inner function to u, and it is easily checked (similar to the approach in Proposition 3.8) this is an inner function such that $gu\psi = \overline{zp_1} + Ip_2$ where p_1 is outer.

In fact using (14) we can view these two theorems as specialisations of the following theorem.

Theorem 3.12. *If $f \in H^2$, and $\mathcal{I}$ is an inner function such that $fK^2_{\mathcal{I}}$ is a closed subspace of H^2, then if $f(0) \neq 0$ then $fK^2_{\mathcal{I}}$ is a nearly invariant subspace. If $f(0) = 0$ then $fK^2_{\mathcal{I}}$ is both a nearly invariant subspace multiplied by a power of z and a nearly invariant subspace with a 1-dimensional defect space $\frac{f}{z}(K^2_{\mathcal{I}} \ominus (K^2_{\mathcal{I}} \cap zH^2))$.*

Proof. The only non-trivial statement to prove is if $f(0) = 0$ then $fK^2_{\mathcal{I}}$ is a nearly invariant subspace with a defect space $\frac{f}{z}(K^2_{\mathcal{I}} \ominus (K^2_{\mathcal{I}} \cap zH^2))$, but this follows from

$$\frac{fK^2_{\mathcal{I}}}{z} \in \frac{f}{z}(K^2_{\mathcal{I}} \ominus (K^2_{\mathcal{I}} \cap zH^2)) + f\left(\frac{K^2_{\mathcal{I}} \cap zH^2}{z}\right) \subseteq \frac{f}{z}(K^2_{\mathcal{I}} \ominus (K^2_{\mathcal{I}} \cap zH^2)) + fK^2_{\mathcal{I}}.$$

□

So under the assumptions $f \in H^2$ and $\mathcal{I}$ is an inner function such that $fK^2_{\mathcal{I}}$ is a closed subspace of H^2, if $f(0) = 0$ then Theorem 3.12 gives us two possible approaches to decomposing $fK^2_{\mathcal{I}}$.

1. Divide $fK_{\mathcal{I}}^2$ by z^n where $n \in \mathbb{N}$ is chosen such that $\frac{f}{z^n}(0) \neq 0$, then use the Hitt decomposition given in [44]. Then we could write $fK_{\mathcal{I}}^2$ as $z^n u$ multiplied by some model space, where $u \in \frac{fK_{\mathcal{I}}^2}{z^n} \ominus (\frac{fK_{\mathcal{I}}^2}{z^n} \cap zH^2)$.

2. Use Theorem 3.9 with $\frac{f}{z}(K_{\mathcal{I}}^2 \ominus (K_{\mathcal{I}}^2 \cap zH^2))$ as the defect space. Then we could write $fK_{\mathcal{I}}^2$ as ze multiplied by some model space , where e is chosen to be an element of $\frac{f}{z}(K_{\mathcal{I}}^2 \ominus (K_{\mathcal{I}}^2 \cap zH^2)) + fK_{\mathcal{I}}^2$ orthogonal to $fK_{\mathcal{I}}^2$.

In both of these cases we obtain a model space multiplied by an isometric multiplier.

Due to the similarities in the way these two decompositions are developed, one might expect that the two possible ways of decomposing $fK_{\mathcal{I}}^2$ might actually yield the same result. We show this is not the case and in general we have two different expressions with an example.

Example 3.1. Let $g = \frac{1}{1-\frac{z}{3}}(\overline{z}^3 + z^3)$ and let $\mathcal{I} = z^4$. We first find $\ker A_g^{\mathcal{I}}$ using linear algebra techniques. With respect to the basis $1, z, z^2, z^3$, $A_g^{\mathcal{I}}$ has the matrix representation

$$\begin{pmatrix} \frac{1}{3^3} & \frac{1}{3^2} & \frac{1}{3} & 1 \\ \frac{1}{3^4} & \frac{1}{3^3} & \frac{1}{3^2} & \frac{1}{3} \\ \frac{1}{3^5} & \frac{1}{3^4} & \frac{1}{3^3} & \frac{1}{3^2} \\ 1+\frac{1}{3^6} & \frac{1}{3^5} & \frac{1}{3^4} & \frac{1}{3^3} \end{pmatrix},$$

which has reduced row echelon form given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The kernel of this matrix has a basis given by

$$\begin{pmatrix} 0 \\ 1 \\ -\frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{3} \end{pmatrix},$$

and thus we can write $\ker A_g^{\mathcal{I}} = z(1 - \frac{z}{3})K_{z^2}^2$. We now will give two different decompositions of this kernel using Theorem 3.12. Let $f = z(1 - \frac{z}{3})$ and $K_{\mathcal{I}}^2 = K_{z^2}^2$, then $fK_{\mathcal{I}}^2 = z\mathrm{span}\{(1 - \frac{z}{3}), z(1 - \frac{z}{3})\}$. We first use approach 1. It can be checked that

$$1 - \frac{z}{3}K_{z^2}^2 \ominus 1 - \frac{z}{3}K_{z^2}^2 \cap zH^2$$

has a normalised basis element given by

$$u = \frac{3\sqrt{910}}{91}(1 - \frac{1}{30}z - \frac{1}{10}z^2),$$

and so $fK^2_{\mathcal{I}}$ can be written as zu multiplied by some model space, which we will denote $K^2_{\mathcal{I}_1}$. In order to find $\mathcal{I}_1$ we must solve

$$z(1-\frac{z}{3})K^2_{z^2} = zuK^2_{\mathcal{I}_1},$$

but $\frac{(1-\frac{z}{3})}{u}$ is a scalar multiple of $\frac{1}{1+\frac{3z}{10}}$, so $K^2_{\mathcal{I}_1}$ must be given by $\text{span}\{\frac{1}{1+\frac{3z}{10}}\frac{z}{1+\frac{3z}{10}}\}$, therefore $\mathcal{I}_1 = z\frac{z+\frac{3}{10}}{1+\frac{3z}{10}}$. So we conclude

$$z(1-\frac{z}{3})K^2_{z^2} = z\frac{3\sqrt{910}}{91}(1-\frac{1}{30}z-\frac{1}{10}z^2)K_{z(\frac{z+\frac{3}{10}}{1+\frac{3z}{10}})},$$

where multiplication by $z\frac{3\sqrt{910}}{91}(1-\frac{1}{30}z-\frac{1}{10}z^2)$ is an isometry on the model space. This can be simplified to

$$z(1-\frac{z}{3})K^2_{z^2} = z(30-z-3z^2)K^2_{z(\frac{z+\frac{1}{3}}{1+\frac{z}{3}})},$$

however in this case we no longer have the multiplication on the model space acting as an isometry. Now we use approach 2. We must find a normalised element $e \in z(1-\frac{z}{3})K^2_{z^2} + \text{span}\{(1-\frac{z}{3})\}$, which is orthogonal to $z(1-\frac{z}{3})K^2_{z^2}$. This can be checked to be

$$\sqrt{\frac{729}{74620}}(\frac{91}{9}-\frac{1}{27}z-\frac{1}{9}z^2-\frac{1}{3}z^3),$$

which means $fK^2_{\mathcal{I}}$ can also be written as ze multiplied by some model space, which we will denote $K^2_{\mathcal{I}_2}$. Now to find $\mathcal{I}_2$ we must solve

$$z(1-\frac{z}{3})K^2_{z^2} = zeK^2_{\mathcal{I}_2}.$$

we know e is a scalar multiple of

$$(273 - z - 3z^2 - 9z^3) = 3(1-\frac{z}{3})(9z^2+30z+91),$$

and so $K^2_{\mathcal{I}_2}$ must be $\text{span}\{\frac{1}{9z^2+30z+91}, \frac{z}{9z^2+30z+91}\}$. We now aim to find the inner function $\mathcal{I}_2$. We denote $A = \frac{1}{9z^2+30z+91}$ and $B = \frac{z}{9z^2+30z+91}$. $A(0) = \frac{1}{91}$, so

$$S^*(A)(z) = \frac{A(z) - \frac{1}{91}}{z} = \frac{-9z-30}{91(9z^2+30z+91)} = -\frac{30}{91}A - \frac{9}{91}B.$$

It is clear that $S^*(B) = A$. We now aim to find two eigenvectors of the backward shift operator (these are necessarily Cauchy kernels) which are in $\text{span}\{A, B\}$. If we use A, B as a basis for $\text{span}\{A, B\}$ then the matrix representation of the backward shift operator is given by

$$\begin{pmatrix} -\frac{30}{91} & 1 \\ -\frac{9}{91} & 0 \end{pmatrix},$$

which has eigenvalues given by $\frac{-15\pm 3i\sqrt{66}}{91}$. We denote $\lambda_1 = \frac{-15+3i\sqrt{66}}{91}$ and $\lambda_2 = \frac{-15-3i\sqrt{66}}{91}$, then the corresponding eigenvectors are given by $k_{\overline{\lambda_1}} = \frac{1}{1-\lambda_1 z}$ and $k_{\overline{\lambda_2}} = \frac{1}{1-\lambda_2 z}$. So $K^2_{\mathcal{I}_2} = \text{span}\{A, B\} = \text{span}\{k_{\overline{\lambda_1}}, k_{\overline{\lambda_2}}\}$, which means we must have $\mathcal{I}_2 =$

$(\frac{z-\overline{\lambda_1}}{1-\lambda_1 z})(\frac{z-\overline{\lambda_2}}{1-\lambda_2 z})$. We can conclude

$$z(1-\frac{z}{3})K^2_{z^2} = z\sqrt{\frac{729}{74620}}(\frac{91}{9}-\frac{1}{27}z-\frac{1}{9}z^2-\frac{1}{3}z^3)K^2_{(\frac{z-\overline{\lambda_1}}{1-\lambda_1 z})(\frac{z-\overline{\lambda_2}}{1-\lambda_2 z})},$$

where multiplication by $z\sqrt{\frac{729}{74620}}(\frac{91}{9}-\frac{1}{27}z-\frac{1}{9}z^2-\frac{1}{3}z^3)$ is an isometry on the model space. Again we can simplify this to

$$z(1-\frac{z}{3})K^2_{z^2} = z(273-z-3z^2-9z^3)K^2_{(\frac{z-\overline{\lambda_1}}{1-\lambda_1 z})(\frac{z-\overline{\lambda_2}}{1-\lambda_2 z})},$$

but in this expression we no longer have the multiplication on the model space acting as an isometry. Thus approach 1 and approach 2 give different decompositions.

In Chapter 4, we will build on the theory we have developed on nearly invariant subspaces to study the matrix-valued truncated Toeplitz operator. (In fact we will even consider matrix-valued truncated Toeplitz operators which do not possess a bounded symbol).

3.3.1 Separated symbols

We conclude this section by giving another decomposition of the kernel of a TTO under extra assumptions on the symbol of the TTO. In the following we use the convention that if $f \in \overline{H^2_0}$, we define the inner and outer factor of f to be the inner/outer factor of $\overline{f} \in H^2$ conjugated.

In this subsection (3.3.1) we continue to assume that g is bounded and we also assume the symbol g is separated. We call the symbol g, of A_g^I, separated when $g^{+i}\overline{g^{-i}} = I\alpha$, where $g = g^- + g^+$ with $g^+ \in H^2$ and $g^- \in \overline{H_0^2}$, and where α is an inner function.

This is a generalisation of the notion of separated introduced in [15] as we do not require that g^- and g^+ to be bounded. We also define the lowest common multiple (abbreviated to LCM) of two inner functions I_1, I_2 to be the inner function $I_* = \text{LCM}(I_1, I_2)$ such that I_1 and I_2 divide I_* (here I_1 dividing I_* means $\frac{I_1}{I_*} \in H^\infty$) and if I_1 and I_2 divide any other inner function β then I_* divides β. The lowest common multiple is unique up to multiplication by a unitary constant

Lemma 3.13. $\ker A_g^I = \ker A_{g^-+g^+}^I = \ker A_{g^{-i}}^I \cap \ker A_{g^{+i}}^I$.

Proof. By Theorem 3.1 of [63], we can assume throughout that $g^- \in \overline{K_I^2}$, and $g^+ \in K_I^2$. We first show the $\supseteq$ inclusion. If $a \in \ker A_{g^{-i}}^I \cap \ker A_{g^{+i}}^I$, then $ag^{+i} \in IH^2$, and multiplying by the outer factor of g we see $ag^+ \in IH^1$. Similarly we have $ag^{-i} \in \overline{H_0^2}$, so $ag^- \in \overline{H_0^1}$. Together these imply

$$ag = a(g^- + g^+) \in \overline{H_0^1} \oplus IH^1.$$

But $ag \in L^2$, so

$$ag \in \overline{H_0^2} \oplus IH^2$$

which means $a \in \ker A_g^I$.

We now show the $\subseteq$ inclusion. We first note that as $g \in L^\infty$ we can write g as $g = g^- + g^+$, where $g^- \in \overline{H_0^6}$, and $g^+ \in H^6$. The reason that we specify L^6 (and not L^2) is because then $ag \in L^{3/2}$ and $H^{3/2} = K_I^{3/2} \oplus IH^{3/2}$ whereas this decomposition does not hold for H^1. If $a \in \ker A^I_{g^- + g^+}$ then $ag^- + ag^+ \in \overline{H_0^2} \oplus IH^2 \subseteq \overline{H_0^{3/2}} \oplus IH^{3/2}$. We also have $a \in I\overline{H_0^2}$, so $ag^- \in I\overline{H_0^{3/2}}$, and similarly $ag^+ \in H^{3/2}$. Now, $H^{3/2} = K_I^{3/2} \oplus IH^{3/2}$, and $I\overline{H_0^{3/2}} = K_I^{3/2} \oplus \overline{H_0^{3/2}}$, so projecting from $L^{3/2}$ we have,

$$g^- a = P_{I,3/2}(g^- a) + P_{3/2,-}(g^- a)$$

and

$$g^+ a = P_{I,3/2}(g^+ a) + (P_{3/2,+} - P_{I,3/2})(g^+ a).$$

Adding these expressions together and noting $ga \in \overline{H_0^{\frac{3}{2}}} \oplus IH^{\frac{3}{2}}$ we obtain

$$g^- a + g^+ a = P_{3/2,-}(g^- a) + (P_{3/2,+} - P_{I,3/2})(g^+ a).$$

Multiplying by $\overline{g^{-i}}$ and rearranging we obtain

$$g^{-o} a - \overline{g^{-i}} P_{3/2,-}(g^- a) = -I\alpha g^{+o} a + \overline{g^{-i}}(P_{3/2,+} - P_{I,3/2})(g^+ a). \tag{15}$$

We know $a \in I\overline{H_0^2}$ and $g^{-o} \in \overline{H_0^6}$, so $ag^{-o} \in I\overline{H_0^{3/2}}$. Therefore the left hand side of the above equation is in $LCM(I, \overline{g^{-i}})\overline{H_0^{3/2}}$ whereas the right hand side of the equation is in $I\mathrm{GCD}(\alpha, \overline{g^{-i}})H^{3/2}$. We know $LCM(I, \overline{g^{-i}})$ is I multiplied by any factors of $\overline{g^{-i}}$ which aren't also a factor of I. But any factor of $\overline{g^{-i}}$ that isn't also a factor of I will be a factor of $\mathrm{GCD}(\alpha, \overline{g^{-i}})$, as $g^{+i}\overline{g^{-i}} = I\alpha$. So we have $LCM(I, \overline{g^{-i}})$ divides $I\mathrm{GCD}(\alpha, \overline{g^{-i}})$, therefore

$$LCM(I, \overline{g^{-i}})\overline{H_0^{3/2}} \cap I\mathrm{GCD}(\alpha, \overline{g^{-i}})H^{3/2} = \{0\},$$

which means both sides of (15) must be equal to 0. Multiplying the left hand side of (15) by g^{-i}, we have

$$g^{-}a = P_{3/2,-}(g^{-}a),$$

i.e

$$g^{-}a \in \overline{H_0^{3/2}}$$

which then implies

$$g^{+}a \in IH^{3/2}.$$

As $ag^{+} \in IH^{3/2}$, dividing through by g^{+o} and using Proposition 2.4 we see ag^{+i} is of the form I multiplied by an element of the Smirnov class as well as L^2, and therefore $ag^{+i} \in IH^2$. Similarly as $ag^{-} \in \overline{H_0^{3/2}}$, dividing through by the outer factor, again we see $ag^{-i} \in \overline{H_0^2}$. So we have

$$a \in \ker A^I_{g^{-i}} \cap \ker A^I_{g^{+i}}.$$

□

Theorem 3.14. *Let* $g^-, g^+ \neq 0$. *Then* $\ker A^I_{\overline{g^{-i}}} \cap \ker A^I_{g^{+i}} = K^2_{\overline{g^{-i}}} \cap I\overline{g^{+i}}H^2 \cap K^2_I$.

Proof. If $a \in \ker A^I_{\overline{g^{-i}}} \cap \ker A^I_{g^{+i}}$ then $ag^{-i} \in \overline{H^2_0}$ and $ag^{+i} \in IH^2$. So $a \in K^2_{\overline{g^{-i}}} \cap I\overline{g^{+i}}H^2 \cap K^2_I$. We now show the $\supseteq$ inclusion. If $a \in K^2_{\overline{g^{-i}}} \cap I\overline{g^{+i}}H^2 \cap K^2_I$, then $a = \overline{g^{-i}}\,\overline{zp_1} = I\overline{g^{+i}}p_2$ for some $p_1, p_2 \in H^2$, so

$$ag^{+i} = Ip_2,$$

and

$$ag^{-i} = \overline{zp_1},$$

so $a \in \ker A^I_{\overline{g^{-i}}} \cap \ker A^I_{g^{+i}}$. □

Lemma 3.15. *If* I_1, I_2 *are inner functions and* $GCD(I_1, I_2) = 1$ *then*

$$H^2 \cap \overline{I_2}H^2 I_1 = I_1 H^2.$$

Proof. To show the $\supseteq$ inclusion we trivially note that

$$I_1 x = \overline{I_2}(I_2 x) I_1.$$

We now show the $\subseteq$ inclusion. Corollary 5.9 in [36] shows us that

$$K_{I_1}^2 \cap K_{I_2}^2 = K_{\mathrm{GCD}(I_1,I_2)}^2 = \{0\}.$$

Which means we must have

$$I_1\overline{H_0^2} \cap I_2\overline{H_0^2} \subseteq \overline{H_0^2}, \tag{16}$$

because if there exists a non-zero $p_1 \in H^2$ and $p_2 \in H^2$ such that

$$\overline{zp_2} + p_1 \in I_1\overline{H_0^2} \cap I_2\overline{H_0^2},$$

then by Proposition 5.5 in [36] we would have $P_+(\overline{zp_2} + p_1) = p_1 \in K_{I_1}^2 \cap K_{I_2}^2 = K_{\mathrm{GCD}(I_1,I_2)}^2 = \{0\}$. Conjugating (16), we have

$$\overline{I_1}zH^2 \cap \overline{I_2}zH^2 \subseteq zH^2,$$

so

$$\overline{I_1}H^2 \cap \overline{I_2}H^2 \subseteq H^2,$$

which implies

$$H^2 \cap \overline{I_2}H^2 I_1 \subseteq I_1 H^2.$$

□

We can now conclude our main decomposition theorem for TTOs with a separated symbol g.

Theorem 3.16. *Let* $\Psi = GCD(I, g^{+i})$*, and* $\chi = GCD(\overline{g^{-i}}, I)$*. Then* $\ker A_g^I = I\overline{\Psi}K^2_{\chi\overline{I}\Psi}$.

Proof. We have $\ker A_g^I = \ker A^I_{g^{-i}} \cap \ker A^I_{g^{+i}} = K^2_{\overline{g^{-i}}} \cap I\overline{g^{+i}}H^2 \cap K_I^2$, which is equal to

$$K^2_\chi \cap I\overline{g^{+i}}H^2 = \chi\overline{H^2_0} \cap H^2 \cap I\overline{g^{+i}}H^2 = \chi\overline{H^2_0} \cap H^2 \cap I\overline{\Psi}H^2\overline{g^{+i}}\Psi.$$

But as a result of our previous lemma, this is equal to

$$\chi\overline{H^2_0} \cap I\overline{\Psi}H^2 = I\overline{\Psi}(\chi\overline{I}\Psi\overline{H^2_0} \cap H^2) = I\overline{\Psi}K^2_{\chi\overline{I}\Psi}.$$

□

By noting $A_g^I = A^I_{\overline{I}+g}$, when g is analytic the symbol $\overline{I} + g$ is separated and so we can deduce the following corollary

Corollary 3.17. *If* $g = g^+ \in H^\infty$*, then* $\ker A_g^I = I\overline{\Psi}K_\Psi$.

3.4 Application to dual truncated Toeplitz operators

In this section we study the kernel of dual truncated Toeplitz operator. Dual truncated Toeplitz operators have been studied in both [26, 19] as well as many other sources. The kernel of a dual truncated Toeplitz operator has been studied in [11]. Although the domain of the dual truncated Toeplitz operator is not a subspace of H^2, we still can use similar recursive techniques used in previous sections of this chapter to decompose the kernel into a fixed function multiplied by a S^*-invariant subspace of H^2.

It is easily checked that in L^2 we have $(K_I^2)^\perp = \overline{H_0^2} \oplus IH^2$. We denote R to be the orthogonal projection $R : L^2 \to (K_I^2)^\perp$.

Throughout this section (3.4) we continue to assume $g \in L^\infty$.

The dual truncated Toeplitz operator $D_g^I : (K_I^2)^\perp \to (K_I^2)^\perp$ is defined by

$$f \mapsto R(gf).$$

Theorem 6.6 in [11] shows that for a symbol g that is invertible in L^∞ we have $\ker D_g^I = g^{-1} \ker A_{g^{-1}}^I$, so given our observation (14) under the condition that g is invertible in L^∞ we can write $\ker D_g^I$ as an L^2 function multiplied by a model space. We now aim to use similar recursive methods that were used to prove Theorem 3.7 to obtain a decomposition theorem for $\ker D_g^I$.

Throughout this section (3.4) we assume that $\ker D_g^I$ is finite dimensional.

We define $A := \{f \in \ker D_g^I : gf \in K_I^2 \cap zH^2\}$ and $C := \ker D_g^I \cap (\overline{H_0^2} \oplus IzH^2) \cap A$, then using orthogonal decomposition we can write

$$\ker D_g^I = C \oplus (\ker D_g^I \ominus C).$$

Lemma 3.18. *If* $\ker D_g^I \subseteq C$ *then* $\ker D_g^I = \{0\}$.

Proof. Suppose we have a non-zero $f \in \ker D_g^I \subseteq C$, then by construction of C we must have $\frac{f}{z} \in \ker D_g^I \subseteq C$. Iterating this we can obtain $\frac{f}{z^n} \in \ker D_g^I$ for all $n \in \mathbb{N}$, which can't be true as given n sufficiently large $\frac{gf}{z^n} \notin H^2$. □

Corollary 3.19. *For any* $\ker D_g^I \neq \{0\}$ *we have* $1 \leqslant \dim(\ker D_g^I \ominus C) \leqslant 2$.

Proof. If $\ker D_g^I \neq \{0\}$ then Lemma 3.18 shows that $1 \leqslant \dim(\ker D_g^I \ominus C)$. Let F_1 be the orthogonal projection of $\overline{g}k_0$ on to $\ker D_g^I$ and F_2 be the orthogonal projection of Ik_0 on to $\ker D_g^I$, where $k_0 \in K_I^2$ is the reproducing kernel at 0, then $\ker D_g^I \ominus C$ is generated by F_1, F_2. Indeed if $f \in \ker D_g^I$ and f is orthogonal to F_1, F_2 then

$$\langle f, F_1 \rangle = \langle gf, k_0 \rangle = 0,$$

so $f \in A$, and

$$\langle f, F_2 \rangle = \langle \overline{I} f, k_0 \rangle = 0,$$

so we also have $P_+(\overline{I} f) \subseteq zH^2$, so $f \in C$. □

As we are working with a finite dimensional $\ker D_G^I$ we can consider $g \ker D_g^I = gC \oplus (g \ker D_g^I \ominus gC)$, and by Corollary 3.19 we must have $g \ker D_g^I \ominus gC$ is at most

2-dimensional. If $g \ker D_g^I \ominus gC$ is 2-dimensional then we denote its orthonormal basis elements by gf_0, gh_0. Then for all $f \in \ker D_g^I$ using orthogonal projections and the observation that $\frac{C}{z} \subseteq \ker D_g^I$ we can write

$$gf = \lambda_0 gf_0 + \mu_0 gh_0 + zgf_1,$$

where $gf_1 \in g \ker D_g^I$, and furthermore

$$\|gf\|_{H^2}^2 = |\lambda_0|^2 + |\mu_0|^2 + \|gf_1\|_{H^2}^2.$$

In a similar process to Theorem 3.7 we can iterate this process starting with gf_1 to obtain

$$gf = \sum_{i=0}^{N} gf_0 \lambda_i z^i + \sum_{j=0}^{N} gh_0 \mu_j z^j + z^{N+1} gf_{N+1},$$

with

$$\|gf\|_{H^2}^2 = \sum_{i=0}^{N} |\lambda_i|^2 + \sum_{j=0}^{N} |\mu_j|^2 + \|gf_{N+1}\|_{H^2}. \tag{17}$$

Following the argument laid out in Section 3.2 to deduce (10) we can deduce that in the H^2 norm $\|gf_{N+1}\| \to 0$ as $N \to \infty$. Then $\|gf_{N+1}\|$ must also converge to 0 in the L^1 norm, and so in the L^1 norm we must have

$$gf = \lim_{N \to \infty} \left(\sum_{i=0}^{N} gf_0 \lambda_i z^i + \sum_{j=0}^{N} gh_0 \mu_j z^j \right).$$

Now two applications of Hölder's inequality shows the L^1 limit of $\sum_{i=0}^{N} gf_0\lambda_i z^i + \sum_{j=0}^{N} gh_0\mu_j z^j$ is equal to $gf_0 \sum_{i=0}^{\infty} \lambda_i z^i + gh_0 \sum_{j=0}^{\infty} \mu_j z^j$, where $\sum_{i=0}^{\infty} \lambda_i z^i, \sum_{j=0}^{\infty} \mu_j z^j$ are limits in the H^2 sense . So we may write

$$gf = gf_0 \sum_{i=0}^{\infty} \lambda_i z^i + gh_0 \sum_{j=0}^{\infty} \mu_j z^j,$$

and furthermore by taking limits in (17) we can deduce

$$\|gf\|_{H^2}^2 = \sum_{i=0}^{\infty} |\lambda_i|^2 + \sum_{i=0}^{\infty} |\mu_i|^2.$$

Mimicking the argument from Section 3.2 between (12) and (13) we can say $f \in \ker D_g^I$ if and only if

$$gf = \begin{pmatrix} gf_0 & gh_0 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix},$$

where $\begin{pmatrix} k_0 \\ k_1 \end{pmatrix}$ lies in a closed S^*-invariant subspace of $(H^2)^2$. With obvious modifications for when $\dim \ker D_g^I \ominus C = 1$ we can deduce the following theorem.

Theorem 3.20. *1. If* $\dim(g \ker D_g^I \ominus gC) = 2$ *then*

$$g \ker D_g^I = \begin{pmatrix} gf_0 & gh_0 \end{pmatrix} K,$$

where K is a closed S^*-invariant subspace of $(H^2)^2$, gf_0, gh_0 are orthonormal basis elements of $(g\ker D_g^I \ominus gC)$ and for $f \in \ker D_g^I$ we have $\|gf\|_{H^2}^2 = \|k_0\|_{H^2}^2 + \|k_1\|_{H^2}^2$.

2. *If $\dim(g\ker D_g^I \ominus gC) = 1$ then*

$$g\ker D_g^I = gf_0 K_{\chi z},$$

where χ is some inner function, gf_0 is a normalised element of $(g\ker D_g^I \ominus gC)$ and for $f \in \ker D_g^I$ we have $\|gf\|_{H^2}^2 = \|k\|_{H^2}^2$.

Remark. In point 2 of the above theorem we know the inner function must be of the form χz (i.e. that $K_{\chi z}$ contains 1) because we know $f_0 \in \ker D_g^I$.

Cancelling the g and using the same notation as the previous theorem we obtain the following.

Corollary 3.21. 1. *If $\dim(\ker D_g^I \ominus C) = 2$ then*

$$\ker D_g^I = \begin{pmatrix} f_0 & h_0 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix}.$$

2. *If $\dim(\ker D_g^I \ominus C) = 1$ then*

$$\ker D_g^I = f_0 K_{\chi z}.$$

4 Matrix-valued truncated Toeplitz operators

In this chapter we will study the matrix-valued truncated Toeplitz operator (abbreviated to MTTO). MTTOs are a vectorial generalisation of the truncated Toeplitz operator. In particular, we focus on studying the kernel of the MTTO and we also find a new form of Toeplitz operator which is equivalent after extension to the MTTO. We make a powerful observation, that when studying a given property of the MTTO it is often convenient to initially modify the MTTO by changing its codomain (in a natural way), then one can deduce results about the MTTO from the modified MTTO. This approach allows us to tackle problems concerning MTTOs with unbounded symbols.

Recall that for a matrix $M \in L^{(\infty, n\times n)}$ the adjoint of $M \in L^{(\infty, n\times n)}$ is denoted M^* and an n-by-n matrix inner function Θ is an element of $H^{(\infty, n\times n)}$ such that for almost every $z \in \mathbb{T}$, we have $\Theta(z)$ is a unitary matrix. Throughout this chapter we use Θ to denote an arbitrary n-by-n inner function. We know from the Beurling-Lax Theorem that $\Theta(H^2)^n$ is a shift invariant subspace. Therefore using orthogonality one can see that the (matricial) model space, $K^2_\Theta := \Theta\overline{(H^2_0)^n} \cap (H^2)^n$ is S^*-invariant. Recall that for $k > n$ we write $P_n : (H^2)^k \to (H^2)^n$ to mean the projection onto the first n coordinates.

4.1 Basic properties, definition and motivation

As noted in the introduction the Riesz projections $P_{q+} : (L^q)^n \to (H^q)^n$ and $P_{q-} := I_d - P_{q+} : (L^q)^n \to (\overline{H_0^q})^n$ are bounded when $q \in (1, \infty)$. Furthermore for $n = 1$, P_{q+} can be expressed by

$$P_{q+}(f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} dm(\zeta),$$

which is independent of $q \in (1, \infty)$. Which means we can deduce the following.

Lemma 4.1. *For $q \in (1, 2)$ and $f \in (L^2)^n$, we have $P_{q+}(f) = P_{2+}(f)$ and $P_{q-}(f) = P_{2-}(f)$.*

Using the projection $P_{\Theta,q} := P_{q+}\Theta P_{q-}\Theta^*$ we can decompose, as we have done in the introduction for $n = 1$, $(H^q)^n$ as

$$(H^q)^n = K_\Theta^q \oplus \Theta(H^q)^n,$$

where $K_\Theta^q = \Theta\overline{(H_0^q)^n} \cap (H^q)^n$ and

$$(L^q)^n = (\overline{H_0^q})^n \oplus K_\Theta^q \oplus \Theta(H^q)^n.$$

We note that when $q = 2$ the above decompositions are orthogonal. As $P_{\Theta,2} = P_{2+}\Theta P_{2-}\Theta^*$ using Lemma 4.1 we can conclude the following.

Lemma 4.2. *For $q \in (1,2)$ and $f \in (L^2)^n$, we have*

$$P_{\Theta,2}(f) = P_{\Theta,q}(f).$$

We can also deduce that for $Q_{\Theta,q} := \Theta P_{q+}\Theta^* = P_{q+} - P_{\Theta,q} : (L^q)^n \to \Theta(H^q)^n$ we have the following.

Lemma 4.3. *For $q \in (1,2)$ and $f \in (L^2)^n$, we have*

$$Q_{\Theta,2}(f) = Q_{\Theta,q}(f).$$

We will use Lemmas 4.1, 4.2 and 4.3 freely throughout this chapter.

Matrix-valued truncated Toeplitz operators were first defined in [50] as a natural generalisation of truncated Toeplitz operators. They have further been studied in [49, 48]. We define the MTTO as follows. Let $G \in L^{(2,n\times n)}$, consider the map

$$f \mapsto P_{\Theta,2}(Gf), \tag{18}$$

defined on $K^2_\Theta \cap (H^\infty)^n$. It is shown in Section 4 of [50] that $K^2_\Theta \cap (H^\infty)^n$ is dense in K^2_Θ, so in the case when (18) is bounded this uniquely defines an operator $K^2_\Theta \to K^2_\Theta$, which we denote A^Θ_G and call a *matrix-valued truncated Toeplitz operator* (recall we abbreviate this to MTTO). We note that with this definition, all MTTOs are implicitly bounded. We call G the symbol of the MTTO, and we note that if we have the additional assumption that $G \in L^{(\infty,n\times n)}$ then (4) can always be extended to a bounded operator. In the case when $n = 1$, we recover the well known bounded truncated Toeplitz operator.

We say Θ is pure if $\|\Theta(0)\| < 1$. Matrix valued truncated Toeplitz operators with a pure inner function appear naturally in the Sz.-Nagy and Foiaş model theory for Hilbert space contractions. In particular, every bounded linear operator between two Hilbert spaces $T : H_1 \to H_2$ with defect indices (n, n) and with the property that for all $h \in H_1$, $T^{*n}(h) \to 0$ (S.O.T) is unitarily equivalent to A_z^{Θ} for some n-by-n inner function Θ. See Section 2, page 33, of [47] for a more detailed discussion. Although this is one of the main motivations for interest in the truncated Toeplitz operator (which is relevant when the defect indices are $(1, 1)$), there has been very little research done into the general case of the MTTO.

Let $I \in H^2$ be a scalar inner function and let $\phi \in H^\infty$. We denote the Hankel operator with symbol $g \in L^\infty$, by $H_g : H^2 \to \overline{H_0^2}$. This is defined by $H_\psi(p) = P_-(\psi p)$. It is well known that many questions about Hankel operators can be phrased in terms of truncated Toeplitz operators with an analytic symbol. In particular the relation

$$A_\phi^I = I H_{\bar{I}\phi}|_{K_I^2}$$

has long been exploited. Making natural generalisations so that $\Psi \in H^{(\infty,n\times n)}$, Θ is an n-by-n matrix inner function and $H : (H^2)^n \to (\overline{H_0^2})^n$ is a Hankel operator on the vector-valued Hardy space, we can also write the relation

$$A_\Psi^\Theta = \Theta H_{\Theta^*\Psi}|_{K_\Theta^2}.$$

So, just as is true in the scalar case, the matricial Hankel operator and MTTO are fundamentally linked. This has applications in minimisation problems and Nehari's Theorem, see Section 2.2 of [55].

4.2 The modified matrix-valued truncated Toeplitz operator

In this section we make some key observations which allow us to define the modified MTTO. The modified MTTO turns out to be a crucial tool in later sections of this chapter, when we are trying to understand properties of the MTTOs which do not possess a bounded symbol.

Definition 4.4. *Let $p \in (2, \infty]$, let $G \in L^{(p,n\times n)}$ and let $\frac{1}{2} + \frac{1}{p} = \frac{1}{q}$. Then the bounded operator $\tilde{A}_G^\Theta : K_\Theta^2 \to K_\Theta^q$ is defined by $\tilde{A}_G^\Theta(f) = P_{\Theta,q}(Gf)$. We call the operator $\tilde{A}_G^\Theta$ the modified matrix-valued truncated Toeplitz operator.*

Remark. Although $\tilde{A}_G^\Theta$ does have a specific p dependence depending on which space G lies in, we will omit this from our notation.

The following proposition shows that when $A_G^\Theta : K_\Theta^2 \to K_\Theta^2$ is a MTTO, up to a change in codomain, A_G^Θ and $\tilde{A}_G^\Theta$ are actually the same operator. In the next section of this chapter we will exploit this link to study the kernel of A_G^Θ.

Proposition 4.5. *Let the assumptions of Definition 4.4 hold and let $A_G^\Theta : K_\Theta^2 \to K_\Theta^2$ be a MTTO. Then for each $f \in K_\Theta^2$ we have $\tilde{A}_G^\Theta(f) = A_G^\Theta(f)$.*

Proof. For a given $f \in K_\Theta^2$, let $f_n \in K_\Theta^2 \cap (H^\infty)^n$ be such that $f_n \overset{(L^2)^n}{\to} f$. As $\tilde{A}_G^\Theta$ is bounded we have $P_{\Theta,q}(Gf_n) \overset{(L^q)^n}{\to} P_{\Theta,q}(Gf)$. By Lemma 4.2 this means

$$P_{\Theta,2}(Gf_n) \overset{(L^q)^n}{\to} P_{\Theta,q}(Gf) = \tilde{A}_G^\Theta(f). \tag{19}$$

Because $P_{\Theta,2}(Gf_n) \overset{(L^2)^n}{\to} P_{\Theta,2}(Gf) = A_G^\Theta(f)$ and convergence in $(L^2)^n$ is stronger than $(L^q)^n$ we must have

$$P_{\Theta,2}(Gf_n) \overset{(L^q)^n}{\to} P_{\Theta,2}(Gf) = A_G^\Theta(f). \tag{20}$$

Now by comparing (19) and (20), uniqueness of limits implies that $\tilde{A}_G^\Theta(f) = A_G^\Theta(f)$.

□

Corollary 4.6. *Let the assumptions of Definition 4.4 hold and let $A_G^\Theta : K_\Theta^2 \to K_\Theta^2$ be a MTTO. Then $Img\tilde{A}_G^\Theta \subseteq K_\Theta^2$.*

In fact we have the following;

Proposition 4.7. *Let the assumptions of Definition 4.4 hold. Then $Img\tilde{A}_G^\Theta \subseteq K_\Theta^2$ if and only if A_G^Θ is a MTTO (i.e the map (18) is bounded).*

Proof. The above corollary shows that when $A_G^\Theta : K_\Theta^2 \to K_\Theta^2$ is a MTTO, we have $\text{Img}\tilde{A_G^\Theta} \subseteq K_\Theta^2$. To show the other implication, we first change the codomain of $\tilde{A_G^\Theta}$, to view the map $\tilde{A_G^\Theta} : K_\Theta^2 \to K_\Theta^2$, which is well defined by the assumption $\text{Img}\tilde{A_G^\Theta} \subseteq K_\Theta^2$. We now use the Closed Graph Theorem to show $\tilde{A_G^\Theta} : K_\Theta^2 \to K_\Theta^2$ is continuous. Let $(f_n)_{n\in\mathbb{N}} \in K_\Theta^2$ and let

$$(f_n, \tilde{A_G^\Theta}(f_n)) \overset{K_\Theta^2 \times K_\Theta^2}{\to} (y_1, y_2),$$

then clearly $f_n \overset{K_\Theta^2}{\to} y_1$ and $\tilde{A_G^\Theta}(f_n) \overset{K_\Theta^2}{\to} y_2$. We also know that $\tilde{A_G^\Theta}(f_n) \overset{K_\Theta^q}{\to} \tilde{A_G^\Theta}(y_1)$, and as L^2 convergence is stronger than L^q convergence we can say that $\tilde{A_G^\Theta}(f_n) \overset{K_\Theta^q}{\to} y_2$. Uniqueness of limits now shows $(f_n, \tilde{A_G^\Theta}(f_n)) \overset{K_\Theta^2 \times K_\Theta^2}{\to} (y_1, \tilde{A_G^\Theta}(y_1))$, and hence the graph is closed. Now, again viewing $\tilde{A_G^\Theta} : K_\Theta^2 \to K_\Theta^2$, we have

$$\tilde{A_G^\Theta}(f) = A_G^\Theta(f)$$

for all $f \in K_\Theta^2 \cap (H^\infty)^n$. Thus boundedness of $\tilde{A_G^\Theta} : K_\Theta^2 \to K_\Theta^2$ ensures boundedness of (4). □

In a similar fashion to how we have changed the codomain of the MTTO to obtain the modified MTTO, we can also change the codomain of matricial Toeplitz operators. Let $p \in (2,\infty]$ and let $G \in L^{(p,n\times n)}$. Define

$$\mathcal{G} = \begin{pmatrix} \Theta^* & 0 \\ G & \Theta \end{pmatrix}, \tag{21}$$

where 0 denotes the n-by-n matrix with each entry being 0. Throughout the rest of this chapter, given two different Banach spaces X_1, X_2 we will equip the space

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1 \in X_1, f_2 \in X_2 \right\}$$

with the norm given by

$$\left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\| = \|f_1\|_{X_1} + \|f_2\|_{X_2}.$$

With this convention we can define $T_{\mathcal{G}} : \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix} \to \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$, where if $f_1 \in (H^2)^n$ and $f_2 \in (H^q)^n$ (where $\frac{1}{2} + \frac{1}{p} = \frac{1}{q}$),

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto \begin{pmatrix} P_{2+}(\Theta^* f_1) \\ P_{q+}(Gf_1 + \Theta f_2) \end{pmatrix}. \tag{22}$$

An application of Hölder's inequality shows $T_{\mathcal{G}}$ is bounded. In the following proposition recall that P_n denotes the projection on to the first n coordinates.

Proposition 4.8. *For the matrix $\mathcal{G}$ defined as (21) we have $P_n(\ker T_{\mathcal{G}}) = \ker \tilde{A}^{\Theta}_{G}$.*

Proof. Clearly, for $f_1 \in (H^2)^n$ and $f_2 \in (H^q)^n$, we have $(f_1, f_2) \in \ker T_{\mathcal{G}}$ if and only if $f_1 \in \ker T_{\Theta^*} = K^2_{\Theta}$ and $Gf_1 + \Theta f_2 \in (\overline{H^q_0})^n$. So $f_1 \in \ker A^{\Theta}_{G}$, and likewise given $f_1 \in \ker A^{\Theta}_{G}$ there exist $f_2 \in (H^q)^n$ with $(f_1, f_2) \in \ker T_{\mathcal{G}}$. □

4.3 The kernel

4.3.1 A decomposition of the kernel

In this subsection we aim to expand on the results in Chapter 3 to decompose the kernel of a MTTO into an isometric image of an S^*-invariant subspace. In Chapter 3 the kernel of a TTO with a bounded symbol is shown to be nearly invariant with defect 1. Following this result, we may suspect the kernel of a MTTO to be nearly S^*-invariant with defect n (where n is such that Θ and G are n-by-n matrices); in this subsection, under very mild assumptions, we show this is the case.

Recall from Chapter 3 that a closed subspace $M \subseteq (H^2)^n$ is said to be nearly S^*-invariant with defect d if and only if there exists a d-dimensional subspace D (which may be taken to be orthogonal to M) such that if $f \in M$ and $f(0)$ is the zero vector then $S^*f \in M \oplus D$. We call D the defect space. If M is nearly S^*-invariant with defect 0 then it is said to be nearly S^*-invariant. Similarly, we say a closed subspace $N \subseteq \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$ is nearly S^*-invariant if and only if all functions $f \in N$ with the property $f(0)$ is the zero vector satisfy $S^*(f) = \frac{f}{z} \in N$.

Define $\widetilde{W} := \ker T_{\mathcal{G}}(0) = \{F(0) : F \in \ker T_{\mathcal{G}}\} \subseteq \mathbb{C}^{2n}$. Let $\dim \widetilde{W} = r$, and pick $W_1, \ldots, W_r \in \ker T_{\mathcal{G}}$ such that $W_1(0), \ldots, W_r(0)$ are a basis for $\widetilde{W}$.

Proposition 4.9. *The space $P_n(\ker T_{\mathcal{G}})$ is nearly S^*-invariant with a defect space*

$$\left(\frac{span\{P_n(W_1), \ldots P_n(W_r)\}}{z} \cap (H^2)^n \right). \tag{23}$$

Remark. This may be viewed as a generalisation of Corollary 3.5, but the delicate issue here is that we are no longer working with a Hilbert space and so we can not use orthogonality.

Proof. Let $f_1 \in P_n(\ker T_{\mathcal{G}})$ with $f_1(0)$ equal to the zero vector. Pick $f_2 \in (H^q)^n$ such that $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ker T_{\mathcal{G}}$ and pick constants $\lambda_1 \ldots \lambda_r$ such that $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \lambda_1 W_1 - \ldots \lambda_r W_r$ evaluated at 0 is the zero vector, then

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \lambda_1 W_1 - \ldots \lambda_r W_r \in \ker T_{\mathcal{G}} \cap z \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}.$$

Near invariance of $\ker T_{\mathcal{G}}$ now ensures

$$\frac{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \lambda_1 W_1 - \dots \lambda_r W_r}{z} \in \ker T_{\mathcal{G}},$$

so

$$\frac{f_1}{z} - \frac{\lambda_1 P_n(W_1) - \dots \lambda_r P_n(W_r)}{z} \in P_n(\ker T_{\mathcal{G}}),$$

and therefore

$$\frac{f_1}{z} \in P_n(\ker T_{\mathcal{G}}) + \left(\frac{\operatorname{span}\{P_n(W_1), \dots P_n(W_r)\}}{z} \cap (H^2)^n \right).$$

□

Previous results on the kernel of the truncated Toeplitz operator (see Chapter 3, [18] and [16]) have been under the assumption that the symbol for the operator is bounded. Now using the operator $\tilde{A}_G^{\Theta}$ as an intermediate tool, this allows us to obtain a Hitt-style characterisation for the kernel of a MTTO and, unlike previous results, we do not require that the symbol of the MTTO is bounded for this characterisation to hold.

Theorem 4.10. *Let $p \in (2, \infty]$, and let $G \in L^{(p, n\times n)}$ be such that A_G^{Θ} is a MTTO. Then $\ker A_G^{\Theta}$ is nearly S^*-invariant with defect m, where $m \leqslant n$.*

Proof. From Proposition 4.5 it is clear that $\ker A_G^{\Theta} = \ker \tilde{A_G^{\Theta}}$, and Proposition 4.8 shows that $\ker \tilde{A_G^{\Theta}} = P_n(\ker T_{\mathcal{G}})$, so from Proposition 4.9 we can deduce that $\ker A_G^{\Theta}$ is a nearly invariant subspace with a defect space given by (23). If $r \leqslant n$ is it clear that the dimension of (23) is less than or equal to n, so it remains to prove that if $r = n + i$ for $i > 0$ then the dimension of (23) is at most n. Suppose $r = n + i$ for $i > 0$. We form a matrix

$$[W_1(0), \dots, W_{n+i}(0)],$$

then for $\begin{pmatrix} s_1 \\ \vdots \\ s_{n+i} \end{pmatrix} \in \mathbb{C}^{n+i}$ we have that $s_1 P_n(W_1) + \dots s_{n+i} P_n(W_{n+i}) \in z(H^2)^n$ if and only if

$$P_n \left([W_1(0), \dots, W_{n+i}(0)] \begin{pmatrix} s_1 \\ \vdots \\ s_{n+i} \end{pmatrix} \right)$$

is the zero vector. Hence the dimension of (23) is given by the dimension of

$$S = \left\{ \begin{pmatrix} s_1 \\ \vdots \\ s_{n+i} \end{pmatrix} \in \mathbb{C}^{n+i} : P_n \left([W_1(0), \dots, W_{n+i}(0)] \begin{pmatrix} s_1 \\ \vdots \\ s_{n+i} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

As $W_1(0), ...W_{n+i}(0) \in \mathbb{C}^{2n}$ are linearly independent, we may pick vectors

$$X_1, \dots X_{n-i} \in \mathbb{C}^{2n}$$

such that the vectors $W_1(0), \dots W_{n+i}(0), X_1, \dots, X_{n-i}$ are linearly independent. We then define $S^{'}$ as

$$\left\{ \begin{pmatrix} s_1 \\ \vdots \\ s_{n+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{2n} : P_n \left([W_1(0), \dots, W_{n+i}(0), X_1, \dots, X_{n-i}] \begin{pmatrix} s_1 \\ \vdots \\ s_{n+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

It is clear $\dim S = \dim S^{'}$, and moreover $S^{'}$ is contained in

$$\left\{ [W_1(0), \dots, W_{n+i}(0), X_1, \dots, X_{n-i}]^{-1} V : V \in \mathbb{C}^{2n} \text{and } P_n(V) = 0 \right\},$$

which has dimension n. Thus we can conclude that the dimension of (23) is equal to $\dim S = \dim S^{'} \leqslant n$. □

Theorem 3.7 (which was also independently proved in [23]) gives a decomposition for vector-valued nearly S^*-invariant subspaces with a defect. So combining the above theorem and Theorem 3.7 we obtain the following decomposition for the kernels of MTTOs in terms of S^*-invariant subspaces.

Theorem 4.11. *Let $p \in (2, \infty]$, and let $G \in L^{(p,n\times n)}$ be such that A_G^Θ is a MTTO. Let $\{e_1, \dots e_m\}$ be an orthonormal basis for the m-dimensional defect space (where $m \leqslant n$) for $\ker A_G^\Theta$ given by (23) and set $r = \dim(\ker A_G^\Theta \ominus (\ker A_G^\Theta \cap z(H^2)^n))$. Then*

1. *in the case where there are functions in $\ker A_G^\Theta$ that do not vanish at 0,*

$$\ker A_G^\Theta = \{F : F(z) = F_0(z)k_0(z) + z\sum_{j=1}^{m} k_j(z)e_j(z) : (k_0, \dots, k_m) \in K\},$$

   where F_0 is the matrix with each column being an orthonormal element of $\ker A_G^\Theta \ominus (\ker A_G^\Theta \cap z(H^2)^n)$, $k_0 \in (H^2)^r$, $k_1, \dots k_m \in H^2$, and $K \subseteq (H^2)^{(r+m)}$ is a closed S^*-invariant subspace. Furthermore $\|F\|^2_{(H^2)^n} = \|k_0\|^2_{(H^2)^r} + \sum_{j=1}^d \|k_j\|^2_{H^2}$.

2. *In the case where all functions in $\ker A_G^\Theta$ vanish at 0,*

$$\ker A_G^\Theta = \{F : F(z) = z\sum_{j=1}^{m} k_j(z)e_j(z) : (k_1, \dots, k_m) \in K\},$$

   with the same notation as in 1, except that K is now a closed S^*-invariant subspace of $(H^2)^m$, and $\|F\|^2_{(H^2)^n} = \sum_{j=1}^d \|k_j\|^2_{H^2}$.

We now give an example to show that under the conditions of Theorem 4.10, n is the smallest dimension of defect space for $\ker A_G^\Theta$, i.e. it is not true that for all inner functions Θ and symbols $G \in L^{(p,n\times n)}$, that $\ker A_G^{\Theta}$ has a j-dimensional defect where $j < n$.

Example 4.1. Let $\Theta = \begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix}$, and $G = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ then

$$\ker A_G^{\Theta} = \left\{ \begin{pmatrix} \lambda z \\ \mu z \end{pmatrix} : \lambda, \mu \in \mathbb{C} \right\},$$

which is clearly nearly S^*-invariant with defect 2.

The condition that we no longer require a bounded symbol to decompose $\ker A_G^{\Theta}$ is a significant extension to Chapter 3. This is because there are a wide class of MTTOs which do not have a bounded symbol but do have a symbol in $L^{(p,n\times n)}$, where $p \in (2, \infty)$. This can be shown in the case where $n = 1$ by using Theorem 5.3 in [6], which is the following;

Theorem 4.12. *Suppose $\mathcal{I}$ is a (scalar) inner function which has an ADC at $\zeta \in \mathbb{T}$ (i.e. the nontangential limits of I and the derivative of I exist at ζ and $|I(\zeta)|= 1$). Let $p \in (2, \infty)$. Then the following are equivalent:*

1. *the bounded truncated Toeplitz operator $k_\zeta^{\mathcal{I}} \otimes k_\zeta^{\mathcal{I}}$ has a symbol $\phi \in L^p$;*

2. *$k_\zeta^{\mathcal{I}} \in L^p$.*

Where in the above theorem $k_\zeta^{\mathcal{I}} = \frac{1-\overline{\mathcal{I}(\zeta)}\mathcal{I}(z)}{1-\bar{\zeta}z} \in K_{\mathcal{I}}^2$ is the reproducing kernel at ζ. In particular, the above theorem shows that if $2 < p_1 < p_2 < \infty$ and $k_\zeta^{\mathcal{I}} \in L^{p_1}$ but $k_\zeta^{\mathcal{I}} \notin L^{p_2}$, then $k_\zeta^{\mathcal{I}} \otimes k_\zeta^{\mathcal{I}}$ does not have a bounded symbol but does have a symbol in L^{p_1}.

The precise conditions for $k_\zeta^{\mathcal{I}}$ to lie in L^p for $p \in (1,\infty)$ are given in [2] and [25]. In particular, for a Blaschke product with zeros (a_k) we have $k_\zeta^{\mathcal{I}} \in L^p$ if and only if

$$\sum_k \frac{1-|a_k|^2}{|\zeta - a_k|^p} < \infty. \tag{24}$$

To obtain a bounded truncated Toeplitz operator which does not have a bounded symbol but does have a symbol in L^{p_1}, for some $p_1 \in (2,\infty)$, it is sufficient to have a point $\zeta \in \mathbb{T}$, and a Blaschke product which has an ADC at ζ such that (24) is true for some $p = p_1 \in (2,\infty)$ but not for some strictly larger value of p. An explicit example of this is a Blaschke product with zeros (a_k) accumulating to the point 1 such that

$$\sum_k \frac{1-|a_k|^2}{|1-a_k|^{p_1}} < \infty \quad \text{for some } 2 < p_1 < \infty,$$

but

$$\sum_k \frac{1-|a_k|^2}{|1-a_k|^{p_2}} = \infty \quad \text{for some } p_1 < p_2 < \infty.$$

Similarly, Theorem 5.1(b) in [63] states that if $\mathcal{I}$ has an ADC at $\zeta \in \mathbb{T}$, then $k_\zeta^{\mathcal{I}} \otimes k_\zeta^{\mathcal{I}}$ is a bounded truncated Toeplitz operator. Therefore by Theorem 5.1(b) in

[63] and the above theorem, we can construct an example of a bounded truncated Toeplitz operator which has a symbol in L^2, but does not have a symbol in L^p for any $p \in (2, \infty)$. Similar to our previous example, in order to do this it is sufficient to have a point $\zeta \in \mathbb{T}$ and a Blaschke product with an ADC at ζ such that (24) is true for $p = 2$ but not for any $p \in (2, \infty)$. A numerical example of such a point $\zeta \in \mathbb{T}$ and Blaschke product is the Blaschke product with zeros (accumulating to 1) given by $a_k = (1 - \epsilon_k)e^{i\delta_k}$ where $\epsilon_k = \frac{1}{k^2}$ and $\delta_k = \frac{log(k)}{k^{1/2}}$ for $k \in \mathbb{N}$. This observation shows that not every bounded truncated Toeplitz operator has a symbol in L^p for some $p \in (2, \infty)$.

We will consider the problem of determining when a bounded TTO has a bounded symbol in Chapter 5.

4.3.2 Analytic symbols and conjugations

In this subsection we continue to study the kernel of the MTTO, but we have a particular focus on when the symbol of the MTTO is analytic. We use a generalised notion of a conjugation map to deduce some elegant results about the MTTO when the symbol of the MTTO is analytic.

Recall Θ^T is the matrix transpose of Θ. We define the map $C : K^2_\Theta \to K^2_{\Theta^T}$ by $f \mapsto \Theta^T \overline{zf}$. One can check that C is a unitary map with adjoint given by $C_T : K^2_{\Theta^T} \to K^2_\Theta$, where $C_T(f) = \Theta \overline{zf}$. This pair of maps may be viewed a vectorial generalisation of the canonical conjugation map on the scalar model space. To the author's knowledge the map $C : K^2_\Theta \to K^2_{\Theta^T}$ was first introduced in [34]. It is worth noting these maps have been used before in [48] to deduce a spatial isomorphism theorem for MTTOs.

When considering the kernel of an analytic MTTO we have the following.

Proposition 4.13. *Suppose that* $G \in H^{\infty,n\times n}(\mathbb{C})$. *Then* $\ker A^\Theta_G = C_T(\ker T_{G'} \cap K^2_{\Theta^T})$, *where* $G^{'} = \Theta^T \overline{G\Theta}$.

Proof. We first show that $C(\ker A^\Theta_G) \subseteq \ker T_{G'} \cap K^2_{\Theta^T}$. It is clear that $C(\ker A^\Theta_G) \subseteq K^2_{\Theta^T}$, so we only require to prove that $C(\ker A^\Theta_G) \subseteq \ker T_{G'}$. Any element of $C(\ker A^\Theta_G)$ is of the form $\Theta^T\overline{zf}$ for some $f \in \ker A^\Theta_G$. By the definition of $G^{'}$ we have $G^{'}\Theta^T\overline{zf} = \Theta^T\overline{Gzf}$, and as $Gf \in \Theta(H^2)^n$, this means $\overline{Gzf} \in \overline{\Theta}(\overline{H^2_0})^n$, so

$$G^{'}\Theta^T\overline{zf} = \Theta^T\overline{Gzf} \in \Theta^T\overline{\Theta}(\overline{H^2_0})^n \in (\overline{H^2_0})^n.$$

Thus $C(\ker A^\Theta_G) \subseteq \ker T_{G'}$.

Next, we show $\ker A^\Theta_G \supseteq C_T(\ker T_{G'} \cap K^2_{\Theta^T})$. Any element of $C_T(\ker T_{G'} \cap K^2_{\Theta^T})$ is clearly contained in K^2_Θ and is of the form $\Theta\overline{zf}$, where $f \in \ker T_{G'} \cap K^2_{\Theta^T}$. So using our construction of $G^{'}$ and the fact $G^{'}f \in (\overline{H^2_0})^n$, we have

$$G\Theta\overline{zf} = \Theta\overline{G^{'}}\overline{zf} \in \Theta(H^2)^n.$$

Thus $\ker A^\Theta_G \supseteq C_T(\ker T_{G'} \cap K^2_{\Theta^T})$. □

As previously noted, kernels of Toeplitz operators are nearly S^*-invariant. It is also clear that the intersection of a nearly S^*-invariant subspace with a S^*-invariant subspace is a nearly S^*-invariant subspace. So we can make the following corollary.

Corollary 4.14. *Suppose that $G \in H^{\infty, n\times n}(\mathbb{C})$. Then $C(\ker A_G^\Theta)$ is a nearly S^*-invariant subspace of $K^2_{\Theta^T}$. Furthermore we can write*

$$\ker A_G^\Theta = \{\Theta \overline{z} \begin{pmatrix} \overline{k_1} \\ \vdots \\ \overline{k_n} \end{pmatrix} : \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in [V_1, \dots V_r] \left((H^2)^r \ominus z\Phi (H^2)^{r'} \right) \}$$

where $V_1, \dots V_r$ be an orthonormal basis of $C(\ker A_G^\Theta) \ominus (C(\ker A_G^\Theta) \cap z(H^2)^n)$, and Φ is a r-by-r' matrix inner function with $r' \leqslant r \leqslant n$.

Proof. The first statement is clear from the previous proposition. The final statement comes from the decomposition of vector-valued nearly S^*-invariant subspaces, which is Corollary 4.5 in [20]. $\square$

Remark. We note how in the above proposition, the lack of commutativity between G and Θ means we can only conclude $C(\ker A_G^\Theta)$ is nearly S^*-invariant and (unlike the scalar case) not S^*-invariant.

Below we provide a far reaching theorem for the case of analytic symbols. The following theorem may be specialised to give a decomposition of the kernel of A_z^Θ by setting $G = zI_d$.

Theorem 4.15. *Suppose that $G \in H^{\infty, n\times n}(\mathbb{C})$ and $\overline{G}\Theta^T = \Theta^T\overline{G}$. Then*

$$\ker A_G^\Theta = \{\Theta \begin{pmatrix} \overline{k_1} \\ \vdots \\ \overline{k_n} \end{pmatrix} \overline{z} : \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in K\} = \{C_T \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} : \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in K\}, \tag{25}$$

where K is the closed backward shift invariant subspace given by $K = \ker T_{\overline{G}} \bigcap K^2_{\Theta^T}$. Furthermore we can express K as $K = (H^2)^n \ominus \mathcal{I}(H^2)^n$, where $\mathcal{I}$ is the n-by-n matrix inner function that is the greatest common divisor of G^T and Θ^T, i.e. $\mathcal{I}$ is such that the closure of $G^T(H^2)^n + \Theta^T(H^2)^n$ is equal to $\mathcal{I}(H^2)^n$.

Remark. We note that the assumptions of the above theorem include the case when Θ is a diagonal matrix. Thus this theorem is particularly relevant when considering truncated Toeplitz operators on multiband spaces (see Section 4.5 and Theorem 4.27)

Proof. It is clear from the above proposition that

$$\ker A_G^\Theta = \{C_T \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} : \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in K\} = \{\Theta \begin{pmatrix} \overline{k_1} \\ \vdots \\ \overline{k_n} \end{pmatrix} \overline{z} : \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in K\}.$$

We now show that K is the backward shift invariant subspace given by $(H^2)^n \ominus \mathcal{I}(H^2)^n$. It is well known that $T_G^* = T_{G^*}$, and so if we denote $^\perp$ to be the orthogonal complement in $(H^2)^n$ then

$$\ker T_{\overline{G}} = \operatorname{Img}(T_{\overline{G}}^*)^\perp = \operatorname{Img}(T_{G^T})^\perp = (G^T(H^2)^n)^\perp. \tag{26}$$

This gives

$$K = \ker T_{\overline{G}} \bigcap K^2_{\Theta^T} = (G^T(H^2)^n)^\perp \bigcap \left(\Theta^T(H^2)^n\right)^\perp =$$

$$\left(G^T(H^2)^n + \Theta^T(H^2)^n\right)^\perp.$$

The Beurling-Lax Theorem now guarantees the closure of $G^T(H^2)^n + \Theta(H^2)^n$ is equal to $\mathcal{I}(H^2)^r$, where $r \leqslant n$ and $\mathcal{I}$ is a n-by-r matrix inner function (recall an n-by-r matrix inner function, Φ, is an element of $H^{\infty,n\times r}(\mathbb{C})$ such that $f \mapsto \Phi f$ is an isometry $(H^2)^r \to (H^2)^n$). After noting that the orthogonal complement of a set is equal to the orthogonal complement of its closure, we have

$$K = \left(G^T(H^2)^n + \Theta^T(H^2)^n\right)^\perp = \mathcal{I}(H^2)^{r\perp} = (H^2)^n \ominus \mathcal{I}(H^2)^r. \tag{27}$$

We now argue $r = n$, and so $\mathcal{I}$ is a square matrix inner function. Suppose for contradiction $r < n$. Let $\mathcal{I}_{ext}$ be the n-by-n matrix made by adding $n-r$ additional column vectors of length n with each entry being 0 as additional columns on the right hand side of $\mathcal{I}$. Then for any $F \in (H^2)^r$ define $F_{ext} \in (H^2)^n$ to be the column vector of length n with the first r coordinates of F_{ext} equal to F, and last $n-r$

coordinates arbitrarily chosen. Then for any choice of $F_1, \dots F_n \in (H^2)^r$, we form a matrix

$$[\mathcal{I}F_1, \mathcal{I}F_2, \dots, \mathcal{I}F_n] = [\mathcal{I}_{ext}F_{1,ext}, \mathcal{I}_{ext}F_{2,ext}, \dots, \mathcal{I}_{ext}F_{n,ext}] = \mathcal{I}_{ext}[F_{1,ext}, \dots, F_{n,ext}],$$

which has determinant zero because $\mathcal{I}_{ext}$ does.

However as $\Theta^T(H^2)^n \subseteq \mathcal{I}(H^2)^r$, there exists $F_1, \dots F_n \in (H^2)^r$ such that $\Theta^T = [\mathcal{I}F_1, \mathcal{I}F_2, \dots, \mathcal{I}F_n]$, but Θ^T does not have determinant equal to 0. So we conclude r must be equal to n. □

Corollary 4.16. *With the same assumptions as in Theorem 4.15, we have the following:*

1. $A_G^\Theta = 0$ *if and only if* $G^T \in \Theta^T H^{\infty, n\times n}(\mathbb{C})$ *;*
2. A_G^Θ *is injective if and only if* $\mathcal{I}$ *is the identity;*
3. $\dim \ker A_G^\Theta < \infty$ *if and only if* $\mathcal{I}$ *is a finite Blaschke–Potapov product.*

Proof. To prove (1), we first note that $A_G^\Theta = 0$ if and only if $\ker A_G^\Theta = K_\Theta^2$, which happens if and only if $\mathcal{I} = \Theta^T$. If $G^T \in \Theta^T H^{\infty, n\times n}(\mathbb{C})$, then clearly $\mathcal{I} = \Theta^T$. Conversely if $G^T \notin \Theta^T H^{\infty, n\times n}(\mathbb{C})$, then $G^T(H^2)^n \not\subseteq \Theta^T(H^2)^n$. Because if $G^T(H^2)^n \subseteq \Theta^T(H^2)^n$ then for e_i denoting the standard basis of $\mathbb{C}^n$ we would have

$$G^T e_i = \Theta^T F_i,$$

for some $F_i \in (H^\infty)^n$, which would mean

$$G^T = \Theta^T[F_1, \dots F_n] \in \Theta^T H^{\infty, n\times n}(\mathbb{C}).$$

So $G^T(H^2)^n \not\subseteq \Theta^T(H^2)^n$, and then trivially

$$G^T(H^2)^n + \Theta^T(H^2)^n \not\subseteq \Theta^T(H^2)^n,$$

which means $\mathcal{I} \neq \Theta^T$. Statement 2 is an immediate consequence of Theorem 4.15.

The proof of (3) follows from the fact that a vector-valued model space corresponding to an inner function is finite dimensional if and only if the inner function is a finite Blaschke–Potapov product. The proof of this can be found as Lemma 5.1 in Chapter 2 of [55].

□

With the same assumptions as in Theorem 3.4, in order to describe the point spectrum of A_G^Θ, we define B_λ to be the n-by-n matrix inner function such that the closure of $(G-\lambda I_d)^T(H^2)^n+\Theta^T(H^2)^n$ is equal to $B_\lambda(H^2)^n$. The existence of such an inner function is guaranteed by the Beurling-Lax Theorem, and the inner function can be seen to be n-by-n by mimicking the same argument laid out immediately after equation (27).

Corollary 4.17. *With the same assumptions as in Theorem 4.15, the point spectrum* *of* A_G^Θ *is the set*

$$\{\lambda : B_\lambda \neq I_d\}$$

and, for each λ *in the point spectrum, the corresponding eigenspace is given by*

$$E_\lambda = \{\Theta \begin{pmatrix} \overline{k_1} \\ \overline{k_2} \end{pmatrix} \overline{z} : \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in (H^2)^2 \ominus B_\lambda (H^2)^2\}.$$

Proof. If $\overline{G}\Theta^T = \Theta^T\overline{G}$, then $(\overline{G - \lambda I_d})\Theta^T = \Theta^T(\overline{G - \lambda I_d})$ and so, by Theorem 3.4, a necessary and sufficient condition for the kernel of the operator $A^\Theta_{G-\lambda I_d}$ to be non-zero is that B_λ is not the identity matrix; on the other hand, from the expression (25) we have $E_\lambda = \ker A^\Theta_{G-\lambda}$ given as above. □

4.4 Equivalence after extension

In this section we generalise the results of Section 6 in [16]. We first find a Toeplitz operator which is equivalent after extension (abbreviated to EAE) to the modified MTTO. As a corollary to this result, we can then easily find an operator which is EAE to the MTTO in the case when the symbol of the MTTO is bounded. We then also provide an EAE result for when the symbol of the MTTO is unbounded.

For Banach spaces $X, \tilde{X}, Y, \tilde{Y}$ the operators $T : X \to \tilde{X}$ and $S : Y \to \tilde{Y}$ are said to be (algebraically and topologically) equivalent if and only if $T = ESF$, where E and F are invertable operators. More generally T and S are equivalent after extension if and only if there exists (possibly trivial) Banach spaces X_0, Y_0, called extension spaces and invertable linear operators $E : \tilde{Y} \oplus Y_0 \to \tilde{X} \oplus X_0$ and $F : X \oplus X_0 \to Y \oplus Y_0$, such that

$$\begin{pmatrix} T & 0 \\ 0 & I_d \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_d \end{pmatrix} F,$$

where the I_d on the left hand side is the identity on X_0 and on the right hand side it is the identity on Y_0. In this case we write that $T \overset{*}{\backsim} S$.

The relation $\overset{*}{\backsim}$ is an equivalence relation. Operators that are equivalent after extension have many features in common. In particular, using the notation $X \simeq Y$ to say that two Banach spaces X and Y are isomorphic, i.e., that there exists an invertible operator from X onto Y , and the notation Img$A$ to denote the range of an operator A, we have the following.

Theorem 4.18 ([8]). *Let $T : X \to \tilde{X}, S : Y \to \tilde{Y}$ be operators and assume that $T \overset{*}{\backsim} S$. Then*

1. $\ker T \simeq \ker S$;

2. *Img T is closed if and only if Img$S$ is closed and, in that case, $\tilde{X}/\text{Img}T \simeq \tilde{Y}/\text{Img}S$;*

3. *if one of the operators T, S is generalised (left, right) invertible, then the other*

is generalised (left, right) invertible too;

4. *T is Fredholm if and only if S is Fredholm and in that case* $\dim\ker T = \dim\ker S$ *and codim Img$T$ = codim Img$S$.*

The above theorem highlights that when one wants to consider invertibility, Fredholmness and spectral properties, EAE extension results are very useful. Section 6 of [16] shows that a truncated Toeplitz operator with a bounded symbol is EAE to a matricial Toeplitz operator, and then consequently the spectral properties of the truncated Toeplitz operator were studied in [15]. Section 5 of [11] shows the dual truncated Toeplitz operator is EAE to a paired operator on $(L^2)^2$.

In the first part of this section we initially adapt the results in Section 6 of [16] to show that $T_{\mathcal{G}}$ is EAE to $\tilde{A}_G^{\Theta}$. Unlike the works of [16] we consider operators which only have unbounded symbols, and in order to overcome to problem of G not being bounded (and then necessarily the domain and codomain of $\tilde{A}_G^{\Theta}$ being different spaces) one must define a new normed space which mixes H^p and H^q spaces.

Throughout this section (4.4), unless otherwise stated, we assume that $G \in L^{(p,n\times n)}$ where $p \in (2,\infty]$. We let $q \in (1,2]$ be such that $\frac{1}{2}+\frac{1}{p}=\frac{1}{q}$. In this context, we write $T_G : (H^2)^n \to (H^q)^n$ to mean the map $f \mapsto P_{q+}(Gf)$.

Consider the operator

$$P_{\Theta,q}GP_{\Theta,2}+Q_{\Theta,2}:(H^2)^n\to K_\Theta^q+\Theta(H^2)^n,$$

where here the norm of $k+\Theta f\in K_\Theta^q+\Theta(H^2)^n$ is given by $\|k\|_{(L^q)^n}+\|\Theta f\|_{(L^2)^n}$. We first show that

$$\tilde{A}_G^\Theta \stackrel{*}{\backsim} P_{\Theta,q}GP_{\Theta,2}+Q_{\Theta,2}. \tag{28}$$

We have

$$\begin{pmatrix}\tilde{A}_G^\Theta & 0\\ 0 & I_{\Theta(H^2)^n}\end{pmatrix}=E_1\begin{pmatrix}P_{\Theta,q}GP_{\Theta,2}+Q_{\Theta,2} & 0\\ 0 & I_0\end{pmatrix}F_1,$$

where

$$F_1:K_\Theta^2\oplus\Theta(H^2)^n\to(H^2)^n\oplus\{0\}$$

is such that

$$\begin{pmatrix}k\\ \Theta f\end{pmatrix}\mapsto\begin{pmatrix}k+\Theta f\\ 0\end{pmatrix},$$

and

$$E_1:K_\Theta^q+\Theta(H^2)^n\oplus\{0\}\to K_\Theta^q\oplus\Theta(H^2)^n$$

is such that

$$\begin{pmatrix}k+\Theta f\\ 0\end{pmatrix}\mapsto\begin{pmatrix}k\\ \Theta f\end{pmatrix}.$$

On the other hand it is clear that

$$P_{\Theta,q}GP_{\Theta,2} + Q_{\Theta,2} \overset{*}{\backsim} \begin{pmatrix} P_{\Theta,q}GP_{\Theta,2} + Q_{\Theta,2} & 0 \\ 0 & P_{q+} \end{pmatrix}. \tag{29}$$

If we denote I_d to be the identity operator on $K^q_\Theta + \Theta(H^2)^n$, we also have

$$P_{\Theta,q}GP_{\Theta,2} + Q_{\Theta,2} = (I_d - P_{\Theta,q}T_GQ_{\Theta,q})(P_{\Theta,q}T_G + Q_{\Theta,2}).$$

We can see this by expanding the right hand side of the above expression to get

$$I_dP_{\Theta,q}T_G + I_dQ_{\Theta,2} - P_{\Theta,q}T_GQ_{\Theta,q}P_{\Theta,q}T_G - P_{\Theta,q}T_GQ_{\Theta,q}Q_{\Theta,2}, \tag{30}$$

but $Q_{\Theta,q}Q_{\Theta,2}(f) = Q_{\Theta,2}(f)$, $I_dP_{\Theta,q}T_G + I_dQ_{\Theta,2} = P_{\Theta,q}T_G + Q_{\Theta,2}$ and $Q_{\Theta,q}P_{\Theta,q} = 0$, so (30) is equal to $P_{\Theta,q}GP_{\Theta,2} + Q_{\Theta,2}$. Furthermore we also have:

Lemma 4.19. *The operator $I_d - P_{\Theta,q}T_GQ_{\Theta,q} : K^q_\Theta + \Theta(H^2)^n \to K^q_\Theta + \Theta(H^2)^n$ is invertible with inverse $I_d + P_{\Theta,q}T_GQ_{\Theta,q}$.*

Proof. As $Q_{\Theta,q}P_{\Theta,q} = 0$ we have

$$(I_d \pm P_{\Theta,q}T_GQ_{\Theta,q})(I_d \mp P_{\Theta,q}T_GQ_{\Theta,q}) = I_d \mp I_dP_{\Theta,q}T_GQ_{\Theta,q} \pm P_{\Theta,q}T_GQ_{\Theta,q}I_d = I_d.$$

□

In the following argument for ease of notation we write the domain and co-domain above the operator. For example, if the operator $A : X \to Y$, we will label this as $\overbrace{A}^{X\to Y}$. In the case when $A : X \to X$ we will denote this by $\overbrace{A}^{X}$. With this notation we will omit the specific q or 2 notation from the projections in the following matrices.

Thus with

$$T = \begin{pmatrix} \overbrace{I_d - P_\Theta T_G Q_\Theta}^{K_\Theta^q + \Theta(H^2)^n} & \overbrace{0}^{(H^q)^n \to \{0\}} \\ \overbrace{0}^{K_\Theta^q + \Theta(H^2)^n \to \{0\}} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix}, \tag{31}$$

we can write

$$\begin{pmatrix} \overbrace{P_\Theta G P_\Theta + Q_\Theta}^{(H^2)^n \to K_\Theta^q + \Theta(H^2)^n} & \overbrace{0}^{(H^q)^n \to K_\Theta^q + \Theta(H^2)^n} \\ \overbrace{0}^{(H^2)^n \to (H^q)^n} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix} =$$

$$T \begin{pmatrix} \overbrace{P_\Theta T_G + Q_\Theta}^{(H^2)^n \to K_\Theta^q + \Theta(H^2)^n} & \overbrace{0}^{(H^q)^n \to K_\Theta^q + \Theta(H^2)^n} \\ \overbrace{0}^{(H^2)^n \to (H^q)^n} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix} =$$

$$T \begin{pmatrix} \overbrace{T_\Theta}^{(H^2)^n \to \Theta(H^2)^n} & \overbrace{P_\Theta}^{(H^q)^n \to K_\Theta^q} \\ \overbrace{-P_+}^{(H^2)^n} & \overbrace{T_{\Theta^*}}^{(H^q)^n} \end{pmatrix} \begin{pmatrix} \overbrace{T_{\Theta^*}}^{(H^2)^n} & \overbrace{0}^{(H^q)^n \to (H^2)^n} \\ \overbrace{T_G - Q_\Theta T_G + Q_\Theta P_+}^{(H^2)^n \to (H^q)^n} & \overbrace{T_\Theta}^{(H^q)^n} \end{pmatrix},$$

where the last line follows by using the identity $P_+ - Q_\Theta = P_\Theta$ and $T_{\Theta^*} P_\Theta = 0$.

This can be factorised further to equal

$$T \begin{pmatrix} \overbrace{T_\Theta}^{(H^2)^n \to \Theta(H^2)^n} & \overbrace{P_\Theta}^{(H^q)^n \to K^q_\Theta} \\ \overbrace{-P_+}^{(H^2)^n} & \overbrace{T_{\Theta^*}}^{(H^q)^n} \end{pmatrix} T_{\mathcal{G}} \begin{pmatrix} \overbrace{P_+}^{(H^2)^n} & \overbrace{0}^{(H^q)^n \to (H^2)^n} \\ \overbrace{-T_{\Theta^*} T_G + T_{\Theta^*} P_+}^{(H^2)^n \to (H^q)^n} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix}, \tag{32}$$

where $T_{\mathcal{G}}$ is defined as in (22). In the above, we label the second factor as T_1 and the final factor as T_2.

1. The first factor, T, is invertible with inverse given by

$$\begin{pmatrix} \overbrace{I_d + P_\Theta T_G Q_\Theta}^{K^q_\Theta + \Theta(H^2)^n} & \overbrace{0}^{(H^q)^n \to \{0\}} \\ \overbrace{0}^{K^q_\Theta + \Theta(H^2)^n \to \{0\}} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix}.$$

   This is verified by Lemma 4.19.

2. The second factor, T_1, is invertible as a map $\begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix} \to \begin{pmatrix} K^q_\Theta + \Theta(H^2)^n \\ (H^q)^n \end{pmatrix}$ by Lemma 4.20 below.

3. The last factor, T_2, is invertible in $\begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$ by Lemma 4.21 below.

We note that Lemmas 4.19, 4.20 and 4.21 are generalisations of Lemmas 6.3, 6.4 and 6.5 respectively in [16].

Lemma 4.20. *The operator* $T_1 : \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix} \to \begin{pmatrix} K_\Theta^q + \Theta(H^2)^n \\ (H^q)^n \end{pmatrix}$ *defined by*

$$T_1 \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} \overbrace{T_\Theta}^{(H^2)^n \to \Theta(H^2)^n} & \overbrace{P_\Theta}^{(H^q)^n \to K_\Theta^q} \\ \overbrace{-P_+}^{(H^2)^n} & \overbrace{T_{\Theta^*}}^{(H^q)^n} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{2n} \end{pmatrix}$$

is invertible.

Proof. Given any $\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2n} \end{pmatrix} \in \begin{pmatrix} K_\Theta^q + \Theta(H^2)^n \\ (H^q)^n \end{pmatrix}$ we have $T_1 \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2n} \end{pmatrix}$ if and only if

$$\Theta \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} + P_\Theta \begin{pmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

and

$$- \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} + T_{\Theta^*} \begin{pmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{2n} \end{pmatrix}.$$

The first of these two equations implies that

$$\Theta \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = Q_\Theta \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \quad P_\Theta \begin{pmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{pmatrix} = P_\Theta \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \tag{33}$$

and from the second we have

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} + \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{2n} \end{pmatrix} = \Theta^* Q_\Theta \begin{pmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{pmatrix}; \tag{34}$$

therefore

$$Q_\Theta \begin{pmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{pmatrix} = \Theta \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} + \Theta \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{2n} \end{pmatrix} = Q_\Theta \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} + \Theta \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{2n} \end{pmatrix}. \tag{35}$$

So we must have

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \Theta^* Q_\Theta \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \tag{36}$$

and combining (33), (35) we can deduce

$$\begin{pmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} + T_\Theta \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{2n} \end{pmatrix}.$$

It follows that T_1 is injective (replacing $\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ and $\begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{2n} \end{pmatrix}$ by 0) and surjective as we can arbitrarily set $\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2n} \end{pmatrix} \in \begin{pmatrix} K_\Theta^q + \Theta(H^2)^n \\ (H^q)^n \end{pmatrix}$ in the above two formulae. Moreover T_1^{-1} is given by:

$$T_1^{-1} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2n} \end{pmatrix} = \begin{pmatrix} \overbrace{T_{\Theta^*}}^{K_\Theta^q + \Theta(H^2)^n \to (H^2)^n} & \overbrace{0}^{(H^q)^n \to (H^2)^n} \\ \overbrace{P_+}^{(H^q)^n} & \overbrace{T_\Theta}^{(H^q)^n} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2n} \end{pmatrix}.$$

□

Lemma 4.21. *The operator* $T_2 : \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix} \to \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$ *defined by*

$$T_2 = \begin{pmatrix} \overbrace{P_+}^{(H^2)^n} & \overbrace{0}^{(H^q)^n \to (H^2)^n} \\ \overbrace{-T_{\Theta^*}T_G + T_{\Theta^*}P_+}^{(H^2)^n \to (H^q)^n} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix}$$

is invertible with inverse given by

$$T_2^{-1} = \begin{pmatrix} \overbrace{P_+}^{(H^2)^n} & \overbrace{0}^{(H^q)^n \to (H^2)^n} \\ \overbrace{T_{\Theta^*}T_G - T_{\Theta^*}P_+}^{(H^2)^n \to (H^q)^n} & \overbrace{P_+}^{(H^q)^n} \end{pmatrix}.$$

Proof. This follows from the fact that T_2 is of the form

$$\begin{pmatrix} P_{2+} & 0 \\ A & P_{q+} \end{pmatrix}$$

where A is an operator such that $AP_{2+} = P_{q+}A$. □

We can now conclude the following;

Theorem 4.22. *$T_{\mathcal{G}}$ is equivalent after extension to $\tilde{A}_G^{\Theta}$.*

Proof. Using (28), (29) and the fact that $\overset{*}{\backsim}$ is transitive, we see that

$$\tilde{A_G^\Theta} \overset{*}{\backsim} \begin{pmatrix} P_{\Theta,q} G P_{\Theta,2} + Q_{\Theta,2} & 0 \\ 0 & P_{q+} \end{pmatrix}.$$

Now (32) and the reasoning immediately following (32) shows

$$\begin{pmatrix} P_{\Theta,q} G P_{\Theta,2} + Q_{\Theta,2} & 0 \\ 0 & P_{q+} \end{pmatrix} \overset{*}{\backsim} T_{\mathcal{G}}$$

and so transitivity of $\overset{*}{\backsim}$ gives us

$$\tilde{A_G^\Theta} \overset{*}{\backsim} T_{\mathcal{G}}.$$

□

Remark. In the case when $n = 1$ and $p = \infty$, Theorem 4.22 specialises to become (the symmetric case of) Theorem 6.6 in [16].

When G is bounded we have $\tilde{A_G^\Theta} = A_G^\Theta$, so we may specialise Theorem 4.22 to find an operator which is EAE to A_G^Θ when G is bounded.

Theorem 4.23. *Let $G \in L^{(\infty, n\times n)}$. Then $T_{\mathcal{G}} : (H^2)^{2n} \to (H^2)^{2n}$ is equivalent after extension to A_G^Θ.*

As operators which are EAE have isomorphic kernels and cokernels, Theorem 4.22 and Proposition 4.5 suggest that restricting the codomain of $T_{\mathcal{G}}$ may provide an operator which is EAE to A_G^Θ, where $G \in L^{(p,n\times n)}$, for $p \in (2,\infty)$. We now pursue this idea.

Throughout the remainder of this section (4.4) we now continue to assume that $G \in L^{(p,n\times n)}$ where $p \in (2,\infty]$, but we now we also make the extra assumption that A_G^Θ is a MTTO (and hence bounded).

The image of $T_{\mathcal{G}}$ is computed to be

$$\begin{pmatrix} 0 \\ \Theta(H^q)^n \end{pmatrix} + \begin{pmatrix} 0 \\ P_{q+}(GK_\Theta^2) \end{pmatrix} + \left\{ \begin{pmatrix} f \\ P_{q+}(G\Theta f) \end{pmatrix} : f \in (H^2)^n \right\},$$

where for $A \subseteq (L^q)^n$, $\begin{pmatrix} 0 \\ A \end{pmatrix}$ is the set of all vectors of length $2n$ with the last n coordinates taking a value $a \in A$. We now define the Banach space

$$\text{Co-d} := \begin{pmatrix} 0 \\ \Theta(H^q)^n \end{pmatrix} + \begin{pmatrix} 0 \\ K_\Theta^2 \end{pmatrix} + \left\{ \begin{pmatrix} f \\ P_{q+}(G\Theta f) \end{pmatrix} : f \in (H^2)^n \right\}, \tag{37}$$

where for $p_1 \in (H^q)^n, p_2 \in K_\Theta^2, p_3 \in (H^2)^n$ we have the well defined norm

$$\left\| \begin{pmatrix} 0 \\ \Theta p_1 \end{pmatrix} + \begin{pmatrix} 0 \\ p_2 \end{pmatrix} + \begin{pmatrix} p_3 \\ P_{q+}(G\Theta p_3) \end{pmatrix} \right\|_{\text{Co-d}} := \|\Theta p_1\|_{(H^q)^n} + \|p_2\|_{K_\Theta^2} + \|p_3\|_{(H^2)^n}.$$

We note that completeness of each of the spaces $(H^q)^n, K^2_\Theta$ and $(H^2)^n$ ensures completeness of Co-d. Corollary 4.6 ensures that $P_{q+}(GK^2_\Theta) \subseteq K^2_\Theta + \Theta(H^q)^n$ so this gives us a well defined bounded map

$$T^r_{\mathcal{G}} : \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix} \to \text{Co-d},$$

where for $f_1 \in (H^2)^n$ and $f_2 \in (H^q)^n$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto \begin{pmatrix} P_{2+}(\Theta^* f_1) \\ P_{q+}(Gf_1 + \Theta f_2) \end{pmatrix} = T_{\mathcal{G}} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Remark. In the case when $p = \infty$ and so $q = 2$, as sets we have Co-d $= \begin{pmatrix} (H^2)^n \\ (H^2)^n \end{pmatrix}$ and furthermore the Co-d norm is equivalent to the $\begin{pmatrix} (H^2)^n \\ (H^2)^n \end{pmatrix}$ norm.

Similar to the proof of Theorem 4.22, we can show that

$$A^\Theta_G \overset{*}{\backsim} \begin{pmatrix} \overbrace{P_{\Theta,q} G P_{\Theta,2} + Q_{\Theta,2}}^{(H^2)^n \to K^2_\Theta + \Theta(H^2)^n} & \overbrace{0}^{(H^q)^n \to K^q_\Theta + \Theta(H^2)^n} \\ \overbrace{0}^{(H^2)^n \to (H^q)^n} & \overbrace{P_{q+}}^{(H^q)^n} \end{pmatrix},$$

where we know by Corollary 4.6 that $P_{\Theta,q}(GK^2_\Theta) \subseteq K^2_\Theta$. It is also clear that for

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$ using (32) we still have

$$TT_1T^r_{\mathcal{G}}T_2 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \overbrace{P_{\Theta,q}GP_{\Theta,2} + Q_{\Theta,2}}^{(H^2)^n \to K^2_\Theta + \Theta(H^2)^n} & \overbrace{0}^{(H^q)^n \to K^q_\Theta + \Theta(H^2)^n} \\ \overbrace{0}^{(H^2)^n \to (H^q)^n} & \overbrace{P_{q+}}^{(H^q)^n} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

One can also check that the operator $TT_1 : \text{Co-d} \to \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$ is well defined, bounded and invertible. We know from Lemma 4.21 that $T_2 : \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix} \to \begin{pmatrix} (H^2)^n \\ (H^q)^n \end{pmatrix}$ is invertible. So we can conclude;

Theorem 4.24. *$A^\Theta_G \overset{*}{\backsim} T^r_{\mathcal{G}}$.*

4.5 Truncated Toeplitz operators on the multiband space

Definition 4.25. *Let I be an inner function and ϕ, ψ unimodular functions in L^∞ such that $\phi K^2_I \perp \psi K^2_I$. Define the space $M := \phi K^2_I \oplus \psi K^2_I$. Define the operator A^M_g by*

$$A^M_g u := P_M(gu),$$

where P_M is the orthogonal projection onto M. We refer to M as the multiband space, and we call A_g^M a truncated Toeplitz operator on the multiband space.

Remark. As with truncated Toeplitz operators these may initially be considered as densely defined operators when $g \in L^2$.

Truncated Toeplitz operators on multiband spaces were first introduced in [13] and are motivated by applications in speech processing and signal transmission. Theorem 2.2 in [13] shows how truncated Toeplitz operators on multiband spaces and MTTOs are fundamentally linked. We state this theorem below, and refer the reader to [13] for a proof.

Theorem 4.26. *Let A_g^M be a bounded truncated Toeplitz operator on the multiband space $M := \phi K_I^2 \oplus^{\perp} \psi K_I^2$, where I is inner and $\phi, \psi \in L^\infty$ are unimodular. Then A_g^M is unitarily equivalent to the block truncated Toeplitz operator*

$$W = \begin{pmatrix} A_g^I & A_{\overline{\phi}\psi g}^I \\ A_{\overline{\psi}\phi g}^I & A_g^I \end{pmatrix}, \tag{38}$$

on $K_I^2 \oplus K_I^2$. Hence $A_g^M = 0$ if and only if each of the four truncated Toeplitz operators composing W is 0.

Remark. We note that the candidate is not directly attributable to Theorem 4.26.

In the context of MTTOs we may rewrite the above theorem as the following;

Theorem 4.27. *With the assumptions of Theorem 4.26, A_g^M is unitarily equivalent to the MTTO A_Φ^Ω, where*

$$\Omega = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

and

$$\Phi = \begin{pmatrix} g & \overline{\phi}\psi g \\ \overline{\psi}\phi g & g \end{pmatrix}.$$

In the above theorem we denote the unitary map by U, and hence we can write $A_g^M = U^* A_\Phi^\Omega U$.

With the above theorem we may now specialise several results on MTTOs to produce results about TTOs on the multiband space. In particular, for a bounded TTO on a multiband space, A_g^M, we can deduce that $\ker A_g^M = U^* \ker A_\Phi^\Omega$. So from Theorem 4.10 we can deduce $\ker A_g^M$ is isometrically isomorphic to a nearly S^*-invariant subspace with defect less than or equal to 2. Due to the partly scalar nature of the TTO on the multiband space and the repetition in the matrix symbol appearing in Theorem 4.27 one may suspect that the defect of $\ker A_\Phi^\Omega$ (where Φ, Ω are defined as in Theorem 4.27) is actually strictly less than 2. However we will show this is not the case and in general $\ker A_\Phi^\Omega$ is nearly S^*-invariant with defect 2.

Example 4.2. With Θ and Ω defined as in Theorem 4.27, let $I = z^2$, $\phi = z$, $\psi = z^4$, $g = 2\overline{z}^2 + z + 2z^4$. We identify the basis of K_Θ^2 with a basis of $\mathbb{C}^4$ in the following way $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, then A^Ω_Φ has the following matrix representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}.$$

Thus $\ker A^\Omega_\Phi$ is given by the span of $\begin{pmatrix} z \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ z \end{pmatrix}$, which is clearly nearly S^*-invariant with defect 2.

In view of Theorem 4.27 we also can specialise the EAE result for MTTOs (which is Theorem 4.23) to TTOs on the multiband space.

Theorem 4.28. *For $g \in L^\infty$, one has $A^M_g \overset{*}{\sim} T_{\mathcal{G}}$ with*

$$\mathcal{G} = \begin{pmatrix} \overline{I} & 0 & 0 & 0 \\ 0 & \overline{I} & 0 & 0 \\ g & g\overline{\phi}\psi & I & 0 \\ g\phi\overline{\psi} & g & 0 & I \end{pmatrix}.$$

5 Symbols of truncated Toeplitz operators

Although there are an abundance of interesting questions concerning the symbols of MTTOs, there is still not a complete answer to several questions posed about the symbols of bounded TTOs. For this reason, in this chapter we only consider scalar truncated Toeplitz operators.

Throughout this chapter we continue to let I be an inner function. We let $C(\mathbb{T})$ be the space of continuous functions on the unit circle. We let BMOA denote the set of all analytic functions of bounded mean oscillation, i.e., $f \in$ BMOA means $f \in H^2$ and

$$\sup_{A} \frac{1}{|A|} \int_{A} |f - f_A| dm < \infty,$$

where the supremum is taken over all arcs $A \subseteq \mathbb{T}$ and

$$f_A := \frac{1}{|A|} \int_{A} f dm.$$

It can be checked that BMOA is a linear vector space and an easy adaptation of Proposition 2.5 in [39] shows that when equipped with the norm

$$\|f\|_{*} := \left| \int_{\mathbb{T}} g dm \right| + \sup_{A} \frac{1}{|A|} \int_{A} |f - f_A| dm,$$

BMOA becomes a Banach space. We let VMOA denote the set of all analytic functions of vanishing mean oscillation, i.e., $f \in$ VMOA means $f \in H^2$ and

$$\lim_{d\to 0}\left\{\sup_{|A|<d}\frac{1}{|A|}\int_A |f-f_A|dm\right\}=0.$$

Theorem 5.5 in [39] shows VMOA is a closed subspace of BMOA. We note that as BMOA $\subseteq H^2$ this allows to have a well defined map $P_I : BMOA \to P_I(BMOA)$.

We use the notation $\mathcal{T}(I)$ to denote the space of bounded truncated Toeplitz operators on K_I^2 and $\mathcal{T}_c(I)$ to denote the space of compact truncated Toeplitz operators on K_I^2.

In this chapter, before we can study the symbols of bounded truncated Toeplitz operators we must first obtain a description of both the dual and predual of K_I^1.

5.1 Duality results

5.1.1 Dual of K_I^1

Previous results in [9] identify the dual space of $K_I^1 \cap zH^1$ for a certain class of inner functions. The results in this subsection give an alternative description of the space dual to K_I^1, and furthermore this description is valid for all inner functions.

We first notice that we trivially have a surjective mapping P_I : BMOA $\to$ P_I(BMOA), and so for each $f \in P_I$(BMOA) this allows us to define the preimage of f, which we denote by $E_f :=\{g \in \text{BMOA such that } P_I(g) = f\}$. We define a norm on the space $P_I(\text{BMOA})$ given by

$$\|f\|_{\text{Img}} := \inf_{g \in E_f} \{\|g\|_*\},$$

for each $f \in P_I(\text{BMOA})$. We refer to this norm as the image norm. With this norm it is immediate that for each $g \in \text{BMOA}$

$$\frac{\|P_I(g)\|_{\text{Img}}}{\|g\|_*} \leqslant 1. \tag{39}$$

In order to show this norm is well defined the only non-trivial things to check are that it satisfies the triangle inequality and that $\|f\|_{\text{Img}} = 0$ implies that $f = 0$.

If $\|f\|_{\text{Img}} = 0$, then there exists a sequence (g_n) contained in E_f such that

$$g_n \overset{BMOA}{\to} \inf_{g \in E_f} \{\|g\|_*\} = 0.$$

As BMOA convergence implies convergence in H^2 (see the final statement of Theorem 2.1 in Chapter 9 of [3]) and $P_I : H^2 \to K_I^2$ is continuous, we must have $f = P_I(g_n) \overset{H^2}{\to} 0$, and so $f = 0$.

To show the triangle inequality holds, we let $f_1, f_2 \in P_I(\text{BMOA})$ and note that for any $g_1 \in E_{f_1}$ and $g_2 \in E_{f_2}$ there exists a $g := g_1 + g_2$ which lies in $E_{f_1+f_2}$ such that

$$\|g\|_* \leqslant \|g_1\|_* + \|g_2\|_*.$$

Thus taking the infimum of the above where $g \in E_{f_1+f_2}$ and $g_1 \in E_{f_1}$ and $g_2 \in E_{f_2}$ we obtain

$$\|f_1 + f_2\|_{\text{Img}} \leqslant \|f_1\|_{\text{Img}} + \|f_2\|_{\text{Img}}.$$

In order to show the normed space $(P_I(\text{BMOA}), \| \ \|_{\text{Img}})$ is complete we use the following well known result, a proof of this may be found as Proposition 2.2 in [52]. (Or alternatively one can make minor adaptations to the proof of Proposition 1.35 in [30].)

Lemma 5.1. *Let X be a Banach space, let Y be a normed vector space, and let $T : X \to Y$ be a surjective linear continuous map. Assume there exists some constant $C > 0$, such that for every $y \in Y$, there exists an $x \in X$ with $T(x) = y$ and $\|x\|_X \leqslant C\|y\|_Y$. Then Y is a Banach space.*

Proposition 5.2. *The space $(P_I(BMOA), \| \ \|_{Img})$ is a Banach space.*

Proof. The proof of this proposition is just an application of the previous lemma. In the notation of the previous lemma, we let $X = \text{BMOA}, Y = P_I(\text{BMOA})$ and $T = P_I : \text{BMOA} \to P_I(\text{BMOA})$. We know T is continuous by (39). For any non-zero function $f \in P_I(\text{BMOA})$, let us show that there exists a $g \in \text{BMOA}$ with $P_I(g) = f$ and $\|g\|_* \leqslant 2\|f\|_{\text{Img}}$. If f is non-zero then by definition there exists a sequence, (g_n), contained in E_f such that $g_n \overset{BMOA}{\to} \inf_{g \in E_f}\{\|g\|_*\} = \|f\|_{\text{Img}} > 0$ and $P_I(g_n) = f$ for all $n \in \mathbb{N}$. Now purely by means of the inertia principle there exists an $N \in \mathbb{N}$ such that $P_I(g_N) = f$ and such that $\|g_N\|_* \leqslant 2\|f\|_{\text{Img}}$. If f is zero then clearly $P_I(0) = 0$

and $\|0\|_*\leqslant 2\|0\|_{\text{Img}}$. □

When considering the dual space of K_I^1, one can deduce that $(K_I^1)^*$ is isometrically isomorphic to the quotient space $(H^1)^*/(K_I^1)^\perp$ (see Section 3.5 of [24] for details). Furthermore it is well known that $(H^1)^*$ is anti-linearly isomorphic to BMOA and a computation shows that $(K_I^1)^\perp = (IH^2 \cap \text{BMOA})$, so we can conclude that $(K_I^1)^*$ is anti-linearly isomorphic to BMOA$/(IH^2\cap\text{BMOA})$. However, as with the description for model spaces when $p\neq 1$, we can realise $(K_I^1)^*$ as a space of analytic functions on the unit disc.

Lemma 5.3 ([24] Lemma 5.8.14)**.** *K_I^2 is dense in K_I^1.*

Theorem 5.4. *$l\in(K_I^1)^*$ if and only if there is a $v\in P_I(BMOA)$ such that l is the continuous extension of the densely defined map*

$$l(f)=l_v(f):=\int_{\mathbb{T}} f(\zeta)\overline{v(\zeta)}dm(\zeta) \qquad\qquad f\in K_I^2$$

to K_I^1. Furthermore the norm of the above linear functional is equivalent to the $P_I(BMOA)$ norm of v.

Proof. First, we take $v\in P_I(\text{BMOA})$, then $v=P_I(g)$ for some $g\in$ BMOA. As BMOA is contained in H^2, we have that $l_v(f)$ agrees with the regular H^2 inner product of f and v whenever $f\in K_I^2$. Now as the projection $P_I : L^2\to K_I^2$ is self adjoint on the H^2 inner product, we can see that

$$\langle\ ,v\rangle_{H^2} = \langle\ ,P_I(g)\rangle_{H^2} = \langle\ ,g\rangle_{H^2},$$

when viewed as maps on $K_I^2 \subseteq K_I^1$. However, we know that $\langle\ ,g\rangle_{H^2} : K_I^2 \to \mathbb{C}$ extends continuously to K_I^1, because $K_I^1 \subseteq H^1$ and using the Hardy-BMO duality established by Fefferman-Stein [31] (or see Theorem 2.2 in Chapter 9 of [3]) we know BMOA is the dual of H^1. Thus, $l = l_v$ defined as above has a continuous extension to K_I^1 and defines an element of $(K_I^1)^*$.

Conversely if we take any $l \in (K_I^1)^*$, using the Hahn-Banach extension Theorem l can be extended to $l' \in (H^1)^*$. Using Theorem 2.2 in Chapter 9 of [3] we know there exists a $g \in$ BMOA such that l' is the continuous extension of

$$l'(f) = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}dm(\zeta) \qquad\qquad f \in H^2$$

to H^1. Furthermore as l' restricted to K_I^1 is equal to l, we know that l is the continuous extension of

$$l(f) = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}dm(\zeta) \qquad\qquad f \in K_I^2$$

to K_I^1. Now again, using the fact that $l(f) = \langle f, g\rangle_{H^2}$ when $f \in K_I^2$, we can see that

$$l(f) = l(P_I(f)) = \int_{\mathbb{T}} P_I(f)(\zeta)\overline{g(\zeta)}dm(\zeta) = \int_{\mathbb{T}} f(\zeta)\overline{P_I(g)(\zeta)}dm(\zeta) \tag{40}$$

when $f \in K_I^2$. So l must equally be the continuous extension of

$$f \mapsto \int_{\mathbb{T}} f(\zeta)\overline{v(\zeta)}dm(\zeta),$$

where $v = P_I(g)$, from K_I^2 to K_I^1.

We now prove the second statement of the theorem. We have shown there is a well defined surjective linear map

$$D : \overline{P_I(\text{BMOA})} \to (K_I^1)^*,$$

where $\overline{v} \mapsto l_v$. It is also clear this map is injective (as if $l_v = 0$, then $\|v\|_{H^2}^2 = l_v(v) = 0$). So if we equip $\overline{P_I(\text{BMOA})}$ with a norm given by $\|\overline{f}\| = \|f\|_{\text{Img}}$ and show D is bounded, then as a result of the Banach Isomorphism Theorem we will have shown D is an isomorphism and so the norm of l is equivalent to the $P_I(\text{BMOA})$ norm of v.

We write $v = P_I(g)$ where g is an element of E_v and by (40) it is clear that $l_v(k) = l_g(k)$ for each $g \in E_v$ and $k \in K_I^1$. As a result of Theorem 2.2 in Chapter 9 of [3] we know there exists a $C > 0$ such that

$$\frac{|l_v(k)|}{\|k\|_{K_I^1}} = \frac{|l_g(k)|}{\|k\|_{K_I^1}} \leqslant C\|g\|_*,$$

for each $g \in E_v$ and $k \in K_I^1$. If we take the infimum of the right hand side of the above expression over each $g \in E_v$, and the supremum of the left hand side over each $k \in K_I^1$, we obtain $\|l_v\| \leqslant C\|v\|_{Img}$. □

In light of the isomorphism D, as we do when $1 < p < \infty$, it is conventional to say $P_I(\text{BMOA})$ is the dual of K_I^1, and write $(K_I^1)^* = P_I(\text{BMOA})$.

Remark. One could also realise $(K_I^1)^*$ as $P_I(\text{BMOA})$ by defining the map

$$P_I : \text{BMOA} \to P_I(\text{BMOA}),$$

applying the First Isomorphism Theorem to deduce $\text{BMOA}/(IH^2 \cap \text{BMOA})$ is isomorphic to $P_I(\text{BMOA})$, and then noting $\text{BMOA}/(IH^2 \cap \text{BMOA})$ is anti-linearly isomorphic to $(K_I^1)^*$ by the reasoning laid out after Proposition 5.2. Although this description of $(K_I^1)^*$ is expressed as a space of analytic functions in the disc, this method only shows existence and we do not have an explicit description of the duality isomorphism.

Remark. One can also express $l_v(f)$ for $f \in K_I^1$ as

$$l(f) = l_v(f) := \lim_{r \to 1^-} \int_{\mathbb{T}} f_r(\zeta) v(\overline{\zeta}) dm(\zeta),$$

where $f_r(\zeta) = f(r\zeta)$.

We now seek to obtain a set theoretic description of $(K_I^1)^*$. The following result more so resembles classical duality result for model spaces, which is $(K_I^p)^* = K_I^q$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 5.5. $(K_I^1)^* = P_I(BMOA) = span(BMOA, I(BMOA)) \cap K_I^2$

Proof. The previous result shows the first equality, so we must only prove the second. As BMOA is contained in H^2, we can write P_I(BMOA) as

$$\{k \in K_I^2 : \text{ there exists a } h \in H^2 \text{ with } k + Ih \in \text{BMOA}\} := K.$$

Now because the space H^1 is invariant by multiplication by I, and BMOA is the dual space of H^1, we can deduce that BMOA is invariant under the Toeplitz operator $T_{\overline{I}}$. Thus, we can in fact write K as

$$\{k \in K_I^2 : \text{ there exists a } h \in \text{BMOA with } k + Ih \in \text{BMOA}\}.$$

The above line is clearly equal to $\text{span}(\text{BMOA}, I(\text{BMOA})) \cap K_I^2$, so we conclude

$$P_I(\text{BMOA}) = \text{span}(\text{BMOA}, I(\text{BMOA})) \cap K_I^2.$$

□

From the above proposition one can also show that the $\| \ \|_{Img}$ norm the on space $\text{span}(\text{BMOA}, I(\text{BMOA})) \cap K_I^2$ is given by

$$\|k\| = \inf_{h \in (\text{BMOA})} \|k + Ih\|_* = \inf_{h \in H^2} \|k + Ih\|_*,$$

for each $k \in \text{span}(\text{BMOA}, I(\text{BMOA})) \cap K_I^2$.

Remark. We note that in general $I\text{BMOA} \not\subseteq \text{BMOA}$. In fact, the conditions for when $I\text{BMOA} \subseteq \text{BMOA}$ can be found as Theorem 1 in [28].

Remark. In contrast to the case when $1 < p < \infty$, Theorem 3.8 of [64] shows that $P_I(\text{BMOA}) \subseteq \text{BMOA}$ if and only if I is finite Blaschke product. It is for this reason that we still must take an infimum in the above norm.

For ease of notation we denote the space $\text{span}(\text{BMOA}, I(\text{BMOA})) \cap K_I^2$ equipped with the $\| \ \|_{Img}$ norm by K_I^{BMOA}. We summarise the results of this subsection with a theorem;

Theorem 5.6. *The dual space of K_I^1 is K_I^{BMOA}.*

5.1.2 Pre-dual of K_I^1

As mentioned in the introduction, an inner function may be factorised into a Blaschke product multiplied by a singular inner function. The singular set of the inner function I, denoted sing(I), is defined to be the set of all $\zeta \in \mathbb{T}$ such that either ζ is an accumulation point of the zeros of I or ζ lies in the support of the singular measure associated to the singular factor of I.

As we have reserved the notation I for an arbitrary inner function, and in this subsection we require further assumptions on our inner function, in this subsection we will use the notation $\mathcal{I}$ to denote our inner function.

Throughout this subsection (5.1.2) we assume that $\mathcal{I}$ is a Blaschke product with a finite singular set. We note that as $\mathcal{I}$ a Blaschke product, sing$(\mathcal{I})$ is just the set of all $\zeta \in \mathbb{T}$, such that ζ is an accumulation point of the zeros of $\mathcal{I}$.

Just as we have done in the previous subsection, we can define a surjective map $P_{\mathcal{I}} : \text{VMOA} \to P_{\mathcal{I}}(\text{VMOA})$ and then equip $P_{\mathcal{I}}(\text{VMOA})$ with the image norm given by

$$\|f\|_{\text{Img}} = \inf\{\|v\|_* \text{ where } v \in \text{VMOA}, P_{\mathcal{I}}(v) = f\},$$

for $f \in P_{\mathcal{I}}(\text{VMOA})$. Mimicking the results of the previous subsection we can deduce that when $P_{\mathcal{I}}(\text{VMOA})$ is equipped with the image norm it is a Banach space and that $P_{\mathcal{I}} : \text{VMOA} \to P_{\mathcal{I}}(\text{VMOA})$ is continuous.

Throughout we will freely use the well known theorem, which can be found in [61], which states that $P_+(C(\mathbb{T})) = \text{VMOA}$. This result then immediately implies the disc algebra, $C(\mathbb{T}) \cap H^2$, is contained in VMOA.

In order to describe the predual of $K^1_{\mathcal{I}}$ we need the following lemma.

Lemma 5.7. *Let $g \in K^1_{\mathcal{I}}$ and let $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$. Then if*

$$\langle \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda, g\rangle = \int_{\mathbb{T}} \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda \overline{g} dm = 0$$

for every $\lambda \in \mathbb{D}$ such that $\mathcal{I}(\lambda) = 0$ and every non-negative integer i, where $i < j_\lambda$ and j_λ is the order of the zero $\mathcal{I}(\lambda)$, then $g = 0$.

Proof. Let the assumptions of the lemma hold and let $\lambda \in \mathbb{D}$ be such that $\mathcal{I}(\lambda) = 0$. We have

$$\langle k_\lambda, g\rangle = g(\lambda) = 0,$$

and so $g \in \frac{z-\lambda}{1-\overline{\lambda}z} H^1$. If $j_\lambda \geqslant 2$, then we may deduce $\overline{\frac{z-\lambda}{1-\overline{\lambda}z}} g \in H^1$ and $\overline{\frac{z-\lambda}{1-\overline{\lambda}z}} g(0) = 0$, and so $g \in \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^2 H^1$. We may then iterate this process to deduce $g \in \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^{j_\lambda} H^1$. As this argument holds for all $\lambda \in \mathbb{D}$ such that $\mathcal{I}(\lambda) = 0$ we must have $g \in \mathcal{I}H^1$, but $K^1_{\mathcal{I}} \cap \mathcal{I}H^1 = \{0\}$. □

Recall from Proposition 2.4 the Smirnov class, denoted N^+, can be expressed as

$$N^+ = \{\frac{f_1}{f_2} : f_2 \text{ is outer } f_1, f_2 \in H^1\}.$$

Theorem 5.8. *$l \in (P_{\mathcal{I}}(VMOA))^*$ if and only if there exists a $g \in K^1_{\mathcal{I}}$ such that*

$$l(f) = l_g(f) := \lim_{n\to\infty} \int_{\mathbb{T}} f\overline{g_n}dm, \tag{41}$$

where (g_n) is any sequence in $K^2_{\mathcal{I}}$ such that $g_n \to g$ in $K^1_{\mathcal{I}}$ (such a sequence always exists by Lemma 5.3). Furthermore the norm of l is equivalent to the $K^1_{\mathcal{I}}$ norm of g.

Proof. We first show l defined as above is well defined and bounded on $P_{\mathcal{I}}(\text{VMOA})$. We write $f = P_{\mathcal{I}}(v)$ for some $v \in \text{VMOA}$, then

$$\int_{\mathbb{T}} f\overline{g_n}dm = \int_{\mathbb{T}} P_{\mathcal{I}}(v)\overline{g_n}dm = \int_{\mathbb{T}} v\overline{P_{\mathcal{I}}(g_n)}dm = \int_{\mathbb{T}} v\overline{g_n}dm,$$

where the second equality holds because the above integrals may be expressed as a H^2 inner product and $P_{\mathcal{I}}$ is self adjoint. Now, by Fefferman's duality result given as Theorem 2.2 in Chapter 9 of [3] we know there exists a $C \geqslant 0$ such that

$$\left| \int_{\mathbb{T}} v\overline{g_n}dm \right| \leqslant C\|v\|_*\|g_n\|_{K^1_{\mathcal{I}}}. \tag{42}$$

Which shows $\int_{\mathbb{T}} v\overline{g_n}dm$ is a Cauchy sequence and hence converges to an element of $\mathbb{C}$. Similarly if $(g_n^{'})$ is another sequence in $K^2_{\mathcal{I}}$ which converges to g then (42) also shows that

$$\int_{\mathbb{T}} v\overline{g_n}dm - \int_{\mathbb{T}} v\overline{g_n^{'}}dm \to 0,$$

and thus $l(f)$ is independent of choice of sequence (g_n). So $l(f)$ is well defined. Finally if we take the limit as n tends to infinity in (42) and then then take the infimum over all $v \in \text{VMOA}$ such that $P_{\mathcal{I}}(v) = f$ we obtain

$$|l(f)| \leqslant C\|f\|_{\text{Img}}\|g\|_{K^1_{\mathcal{I}}}.$$

In order to show the forward implication part of the proof we need the following lemma, which gives a non standard description of VMOA*.

Lemma 5.9. *If $r \in (VMOA)^*$ then there exists a $h \in H^1$ such that for $w \in VMOA$*

$$r(w) = \lim_{n\to\infty} \int_{\mathbb{T}} w\overline{h_n} dm,$$

where h_n is any sequence in H^2 which converges to h in the H^1 norm.

Proof. By Theorem 3.5.27 in [24] we know there exists a $h \in H^1$ such that for $w \in \text{VMOA}$

$$r(w) = \lim_{r\to 1^-} \int_{\mathbb{T}} w_r \overline{h} dm,$$

where $w_r(\zeta) := w(r\zeta)$ for $r \in (0, 1)$. However, as with many spaces of analytic functions in the disc, by considering the duality on the corresponding sequence (obtained from the coefficients of w and h) we know that

$$\lim_{r\to 1^-} \int_{\mathbb{T}} w_r \overline{h} dm = \lim_{r\to 1^-} \int_{\mathbb{T}} w\overline{h_r} dm.$$

We now note

$$\lim_{r\to 1^-}\int_{\mathbb{T}} w\overline{h_r}dm = \lim_{n\to\infty}\int_{\mathbb{T}} w\overline{h_{r_n}}dm,$$

where r_n is any sequence which converges to 1 from below. Finally, by Theorem 3.2.3 in [24] we know that h_{r_n} converges to h in H^1, and so by a similar reasoning to that following (42) we can deduce

$$\lim_{n\to\infty}\int_{\mathbb{T}} w\overline{h_{r_n}}dm = \lim_{n\to\infty}\int_{\mathbb{T}} w\overline{h_n}dm,$$

where h_n is any sequence in H^2 such that $h_n \to h$ in H^1. □

With the above lemma we now proceed to show the forward implication part of the proof. Let $l \in (P_{\mathcal{I}}(\text{VMOA}))^*$, then

$$v \mapsto l(P_{\mathcal{I}}(v))$$

is continuous on VMOA. So by the above lemma there exists a $g \in H^1$ such that

$$l(P_{\mathcal{I}}(v)) = \lim_{n\to\infty}\int_{\mathbb{T}} v\overline{g_n}dm, \tag{43}$$

where (g_n) is any sequence in H^2 converging to g in the H^1 norm. Now, we denote $\text{sing}(\mathcal{I}) = \{x_1, ..., x_N\}$, and we set $J := \mathcal{I}(z - x_1)...(z - x_N)$. As $\mathcal{I}$ has an analytic continuation to any point in $\mathbb{T}\backslash\text{sing}(\mathcal{I})$, it is clear that J is continuous on $\mathbb{T}\backslash\text{sing}(\mathcal{I})$.

Furthermore if we define $J(x) = 0$ for all $x \in \text{sing}(\mathcal{I})$, then for each $k \geqslant 0$, Jz^k is analytic in $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$, and thus lies in VMOA. So using (43) and the fact $Jz^k \in \mathcal{I}H^2$ we deduce

$$0 = l(P_{\mathcal{I}}(Jz^k)) = \lim_{n\to\infty} \int_{\mathbb{T}} Jz^k \overline{g_n} dm = \int_{\mathbb{T}} Jz^k \overline{g} dm,$$

for each $k \geqslant 0$. Thus, by our construction of H^1 viewed as a subspace of L^1 we know $J\overline{g} \in zH^1$ and as $(z - x_1)...(z - x_N)$ is outer we have $g \in \mathcal{I}\overline{zN^+} \cap H^1 = \mathcal{I}\overline{H_0^1} \cap H^1 = K_{\mathcal{I}}^1$. Now by Lemma 5.3, we may choose our sequence (g_n) to be any sequence $K_{\mathcal{I}}^2$. Thus, we have shown for each $l \in (P_{\mathcal{I}}(\text{VMOA}))^*$ there exists a $g \in K_{\mathcal{I}}^1$ such that

$$l(P_{\mathcal{I}}(v)) = \lim_{n\to\infty} \int_{\mathbb{T}} v\overline{g_n} dm,$$

where (g_n) is any sequence in $K_{\mathcal{I}}^2$ converging to g in $K_{\mathcal{I}}^1$. So, as we have done previously, if we write $f = P_{\mathcal{I}}(v)$ then we have

$$l(f) = \lim_{n\to\infty} \int_{\mathbb{T}} v\overline{g_n} dm = \lim_{n\to\infty} \int_{\mathbb{T}} v\overline{P_{\mathcal{I}}(g_n)} dm = \lim_{n\to\infty} \int_{\mathbb{T}} f\overline{g_n} dm. \tag{44}$$

This proves the first statement of the theorem.

In order to show the norm of l is equivalent to the $K_{\mathcal{I}}^1$ norm of g, one may directly adapt the proof the second statement in Theorem 5.4 to show there is a well defined bounded surjective linear map

$$\overline{K^1_{\mathcal{I}}} \to (P_{\mathcal{I}}(\text{VMOA}))^*,$$

where $\overline{g} \mapsto l_g$ (defined as in (41)). We now argue that this map is also injective and so consequently we can apply Banach's Isomorphism Theorem to deduce the norm of l is equivalent to the $K^1_{\mathcal{I}}$ norm of g. For each $\lambda \in \mathbb{D}$ such that $\mathcal{I}(\lambda) = 0$ and every $i < j_\lambda$, where j_λ is the order of the zero $\mathcal{I}(\lambda)$ we have $\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda \in C(\mathbb{T}) \cap K^2_{\mathcal{I}}$. Now as $P_+(C(\mathbb{T})) = \text{VMOA}$, this means the specified $\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda$ lie in VMOA and trivially $P_{\mathcal{I}}(\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda) = \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda$, so $\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda \in P_{\mathcal{I}}(\text{VMOA})$. Now if $l_g = 0$ then for each of the specified $\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda$ we have

$$l_g(\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda) = \int_{\mathbb{T}} \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda \overline{g} dm = 0,$$

(note we can omit taking the limit in the above expression as $\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda$ is continuous). Now applying Lemma 5.7 we see that $g = 0$. $\square$

Just as we have done with the analogous BMOA space we can deduce the following.

Proposition 5.10. *$P_{\mathcal{I}}(VMOA) = span(VMOA, \mathcal{I}(H^2)) \cap K^2_{\mathcal{I}}$.*

Furthermore, as we have done with BMOA previously, one may equip

$$K_{\mathcal{I}}^{\mathrm{VMOA}} := \mathrm{span}(\mathrm{VMOA}, \mathcal{I}(H^2)) \cap K_{\mathcal{I}}^2$$

with the VMOA image norm, and in light of Theorem 5.8 we may write $(K_{\mathcal{I}}^{\mathrm{VMOA}})^* = K_{\mathcal{I}}^1$. However, in the next section it is more convenient to actually continue to realise the predual of $K_{\mathcal{I}}^1$ as the image of $P_{\mathcal{I}} : \mathrm{VMOA} \to P_{\mathcal{I}}(\mathrm{VMOA})$.

5.2 Application to truncated Toeplitz operators

The question of whether every bounded TTO has a bounded symbol is an interesting one. This question has led to much research activity within the community with many questions being answered and many new questions being posed. Here we give the reader a brief background on this topic. In Sarason's seminal work of 2007 [63] he initiated a systematic study of TTOs with symbols in L^2. In this paper one of the most natural questions posed was whether every bounded TTO has a bounded symbol. This question was then shown to be negative in 2009 (see [6]). In fact, the authors actually constructed a bounded rank one TTO which was shown to have no bounded symbol. To build on this work, in [5] the authors gave a condition on an inner function, I, which is equivalent to every bounded TTO on K_I^2 having a bounded symbol. (See Theorem 5.11 below.)

Motivated by these findings, a similar study into the symbols of compact TTOs was initiated. Section 5 of [21] gives an overview of many results in this area. In particular, the role played by bounded symbols in the case of bounded TTOs on K_I^2 seems to be replaced by symbols of the form $IC(\mathbb{T})$ when we are considering compact TTOs. Specifically, Proposition 5.4 of [21] shows that if $\phi \in IC(\mathbb{T})$ then A_ϕ^I is compact, however much like the case for bounded TTOs, Corollary 5.13 shows the converse of this statement does not hold in general. One question posed in [21] was whether there was a compact TTO on K_I^2 with a symbol in $IC(\mathbb{T}) + IH^\infty$ that has no continuous symbol. This question was then answered affirmative in [40] when a compact TTO with this property was then constructed.

Following the results in [5] one may suspect that there are conditions on the inner function I which are equivalent to every compact TTO on K_I^2 having a symbol in $IC(\mathbb{T})$. We may further suspect that these conditions may be similar in nature to the condition on the inner function I which is equivalent to every bounded TTO on K_I^2 having a bounded symbol. In fact, in this section we will prove that under the assumptions on I given later, every compact TTO has a symbol in $IC(\mathbb{T})$ if and only if every bounded TTO has a bounded symbol. We show this with Theorem 5.12 below.

In the following we define $\mathcal{C}_p(I)$ to be the set of all finite complex Borel measures μ on the unit circle such that the embedding $K_I^p \to L^p(|\mu|)$ is continuous.

Theorem 5.11. *[5]*

The following are equivalent:

1. *any bounded TTO on* K_I^2 *admits a bounded symbol;*

2. $\mathcal{C}_1(I^2) = \mathcal{C}_2(I^2)$*;*

3. *for any* $f \in K^1_{I^2/z}$ *there exists* $x_i, y_i \in K_I^2$ *with* $\sum_i \|x_i\|_{K_I^2} \|y_i\|_{K_I^2} < \infty$ *such that* $f = \sum_i x_i y_i$.

The inner function I is said to be one-component if and only if there exists an η such that

$$\{z \in \mathbb{D} : |I(z)| < \eta\}$$

is connected. We remark that by Corollary 2.5 in [5] the equivalent conditions of the theorem below are fulfilled when I is a one component inner function.

Throughout the remainder of this section (5.2), unless otherwise stated, we suppose that the inner function $\mathcal{I}$ is such that $\mathcal{I}(0) = 0$ and in order to use the previous results concerning the predual of a model space we also impose the condition that $\mathcal{I}$ is a Blaschke product with a finite singular set. All TTOs are assumed to be defined on the space $K_{\mathcal{I}}^2$.

In this section we will see that our previous duality results allow us to retrieve information about the symbols of bounded TTOs.

The following is the main theorem that we shall prove:

Theorem 5.12. *The equivalent conditions of Theorem 5.11 (with the inner function* I *now replaced by* $\mathcal{I}$*) are satisfied if and only if any compact TTO on* $K_{\mathcal{I}}^2$ *has a symbol* *in* $\mathcal{I}C(\mathbb{T})$.

We postpone the proof of Theorem 5.12.

Corollary 5.13. *There are compact TTOs on* $K_{\mathcal{I}}^2$ *without a symbol in* $\mathcal{I}C(\mathbb{T})$.

Proof. This follows from Theorem 5.12 and the existence of bounded TTOs with no bounded symbol shown in [6]. We note that the examples of the bounded TTOs with no bounded symbol in [6] can be defined on $K_{\mathcal{I}}^2$ where $\mathcal{I}$ satisfies the assumptions given above. □

Following the results of [5], we define the Banach spaces

$$X = \{\sum x_i\overline{y_i} : x_i, y_i \in K_{\mathcal{I}}^2, \sum \|x_i\|_{K_{\mathcal{I}}^2}\|y_i\|_{K_{\mathcal{I}}^2} < \infty\},$$

and

$$X_a = \{\sum x_i y_i : x_i, y_i \in K_{\mathcal{I}}^2, \sum \|x_i\|_{K_{\mathcal{I}}^2}\|y_i\|_{K_{\mathcal{I}}^2} < \infty\}.$$

The norm in the space of X and X_a is defined as the infimum of $\sum \|x_i\|_{K_{\mathcal{I}}^2}\|y_i\|_{K_{\mathcal{I}}^2}$ over all possible representations. We note there is an isometric isomorphism from X to X_a given by

$$f \mapsto \overline{z}\mathcal{I}f, \tag{45}$$

and one can also show that the inclusion $X_a \to K^1_{\mathcal{I}^2/z}$ is bounded. One key result we will use, which is given as Theorem 2.3 in [5], is the following.

Theorem 5.14. *The dual space of X can be naturally identified with $\mathcal{T}(\mathcal{I})$. Namely, continuous linear functionals over X are of the form*

$$\Phi_A(f) = \sum_i \langle Ax_i, y_i \rangle, \qquad f = \sum_i x_i \overline{y_i} \in X,$$

with $A \in \mathcal{T}(\mathcal{I})$, and the correspondence between X and $\mathcal{T}(\mathcal{I})$ is one to one and isometric.

We can define a bounded linear map

$$L : X \to K^1_{\mathcal{I}^2/z},$$

given by

$$f \mapsto \overline{z}\mathcal{I}f.$$

Now taking into account Theorem 5.6 and Theorem 5.14, when considering the adjoint, L^*, of L we obtain a bounded map

$$L^* : K^{\mathrm{BMOA}}_{\mathcal{I}^2/z} \to \mathcal{T}(\mathcal{I}).$$

Explicitly, L^* is the unique map satisfying

$$\langle L(k), g \rangle = \langle k, L^*(g) \rangle, \tag{46}$$

for each $k \in X$ and $g \in K^{\mathrm{BMOA}}_{\mathcal{I}^2/z}$ (here the duality pairings, denoted $\langle\ ,\rangle$, are given by Theorem 5.4 and 5.14 respectively). If we denote $A^{\mathcal{I}}$ to be the TTO $L^*(g)$, then for $k = x\overline{y}$ with $x \in K^\infty_{\mathcal{I}}, y \in K^2_{\mathcal{I}}$ equating both sides of equation (46) gives

$$\int_{\mathbb{T}} x\overline{y}\mathcal{I}\overline{z}\overline{g}dm = \int_{\mathbb{T}} A^{\mathcal{I}}(x)\overline{y}dm.$$

Now by density of $K^\infty_{\mathcal{I}}$ in $K^2_{\mathcal{I}}$ and non-degeneracy of the integral we can deduce

$$A^{\mathcal{I}} = A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}.$$

We conclude the following result;

Theorem 5.15. *There is a bounded (anti-linear) map* $L^* : K^{BMOA}_{\mathcal{I}^2/z} \to \mathcal{T}(\mathcal{I})$*, given by*

$$g \mapsto A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}. \tag{47}$$

Remark. The reason we have an anti-linear map as opposed to a linear map is because the identification of the dual space of X with $\mathcal{T}(\mathcal{I})$ given in Theorem 5.14 is a linear map, whereas the duality given in 5.4 is defined by an antilinear map.

It is well known that the symbol of a TTO is not unique. As previously pointed out, $A^{\mathcal{I}}_\phi = 0$ if and only if $\phi \in \overline{\mathcal{I}H^2} + \mathcal{I}H^2$, which means every TTO has unique symbol in $\overline{K^2_{\mathcal{I}}}+K^2_{\mathcal{I}}$. This unique symbol is called the *standard symbol*, and we remark that the standard symbol is easily obtained from any symbol, ψ, by projecting ψ on to the space $\overline{K_{\mathcal{I}}^2} + K_{\mathcal{I}}^2$. We recognise in the above theorem that $\mathcal{I}\overline{gz}$ is the standard symbol for $A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}$.

As $P_+(L^\infty) = \text{BMOA}$ (see Theorem 3.5.1 from [24] and references thereafter) and $P_{\mathcal{I}^2/z}P_+ = P_{\mathcal{I}^2/z}$, we can deduce $K^{\text{BMOA}}_{\mathcal{I}^2/z} = P_{\mathcal{I}^2/z}(L^\infty) = P_{\mathcal{I}^2/z}(\text{BMOA})$. This observation allows us to make the following corollary.

Corollary 5.16. *The image of L^* is exactly all elements of $\mathcal{T}(\mathcal{I})$ which possess a bounded symbol.*

Proof. We have $g \in K^{\text{BMOA}}_{\mathcal{I}^2/z} = P_{\mathcal{I}^2/z}(L^\infty)$ if and only if there exists $h_1, h_2 \in H^2$ such that

$$\overline{zh_2} + g + \frac{\mathcal{I}^2}{z}h_1 \in L^\infty,$$

which happens if and only if

$$\mathcal{I}\overline{z}(\overline{\frac{\mathcal{I}^2}{z}h_1 + \overline{g} + zh_2}) = \overline{\mathcal{I}h_1} + \mathcal{I}\overline{gz} + \mathcal{I}h_2 \in L^\infty.$$

Which is clearly equivalent to $A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}$ possessing a bounded symbol. □

We now consider the pre-adjoint of L. Making a minor adaptation to the second part of Theorem 2.3 in [5], we can write the following.

Theorem 5.17. *The dual space of $\mathcal{T}_c(\mathcal{I})$ can be identified with X, via the duality* *pairing* $\sum_i x_i\overline{y_i} \mapsto L_{\sum_i x_i\overline{y_i}}$*, where*

$$L_{\sum_i x_i\overline{y_i}}(A^{\mathcal{I}}) = \sum_i \langle A^{\mathcal{I}} y_i, x_i\rangle,$$

for each compact TTO $A^{\mathcal{I}}$*. Furthermore the duality pairing* $\sum_i x_i\overline{y_i} \mapsto L_{\sum_i x_i\overline{y_i}}$ *is one-to-one and isometric map between* X *and* $(\mathcal{T}_c)^*$*.*

In the general case the pre-adjoint of a bounded linear map may not exist. Nonetheless we may define the map ${}^*L : P_{\mathcal{I}^2/z}(\text{VMOA}) \to \mathcal{T}_c(\mathcal{I})$, where

$$g \mapsto A^{\mathcal{I}}_{\overline{\mathcal{I}}gz}.$$

Proposition 5.18. *The map* ${}^*L$ *is a well defined bounded, linear, injective map and* $({}^*L)^* = L$ *(i.e.* ${}^*L$ *is the pre-adjoint of* L*).*

Proof. The map ${}^*L$ is clearly linear. If $A^{\mathcal{I}}_{\overline{\mathcal{I}}gz} = 0$, then $\overline{\mathcal{I}}gz \in \overline{\mathcal{I}H^2} + \mathcal{I}H^2$, and so $g \in \overline{H^2_0} + \frac{\mathcal{I}^2}{z}H^2$, but as $g \in K^2_{\mathcal{I}^2/z}$, this means $g = 0$ and hence ${}^*L$ is injective. To show the map is well defined we note that $A^{\mathcal{I}}_{\overline{\mathcal{I}}gz}$ is compact if and only if $(A^{\mathcal{I}}_{\overline{\mathcal{I}}gz})^* = A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}$ is compact, so it suffices to show $A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}$ is compact. Recall by [61] we know $P_+(C(\mathbb{T}) = \text{VMOA}$ and trivially we have $P_{\mathcal{I}^2/z}P_+ = P_{\mathcal{I}^2/z}$ so this must mean that $P_{\mathcal{I}^2/z}(\text{VMOA}) = P_{\mathcal{I}^2/z}(C(\mathbb{T})))$, and hence $g \in P_{\mathcal{I}^2/z}(C(\mathbb{T}))$. Thus we know there exists $p_1, p_2 \in H^2$ such that $\overline{zp_1} + g + \frac{\mathcal{I}^2}{z}p_2 := g' \in C(\mathbb{T})$. Now it is easy to see that $A^{\mathcal{I}}_{\mathcal{I}\overline{gz}} = A^{\mathcal{I}}_{\mathcal{I}\overline{g'z}}$, and by Proposition 5.4 in [21] we know that $A^{\mathcal{I}}_{\mathcal{I}\overline{g'z}}$ is compact.

To show boundedness, we again use the fact that $\|A^{\mathcal{I}}_{\overline{\mathcal{I}}gz}\| = \|(A^{\mathcal{I}}_{\overline{\mathcal{I}}gz})^*\| = \|A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}\|$, and we observe that

$$\sup_{g\in P_{\mathcal{I}^2/z}(\text{VMOA})} \frac{\|A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}\|}{\|g\|_{\text{Img}}} \leqslant \sup_{g\in P_{\mathcal{I}^2/z}(\text{BMOA})} \frac{\|A^{\mathcal{I}}_{\mathcal{I}\overline{gz}}\|}{\|g\|_{\text{Img}}}$$

which is finite due to (47).

We now argue $({}^*L)^* = L$. We know $({}^*L)^*$ is a linear map satisfying

$$\langle {}^*L(g), x\overline{y}\rangle = \langle g, ({}^*L)^*(x\overline{y})\rangle,$$

for every $g \in P_{\mathcal{I}^2/z}(\text{VMOA})$ and every $x\overline{y} \in X$, where $x, y \in K^2_{\mathcal{I}}$. Here the duality pairing on the left hand side is understood by the duality described in Theorem 5.17 and on the right hand side the duality is described by Theorem 5.8. Explicitly, this means $({}^*L)^*$ is a linear map satisfying

$$\langle \overline{\mathcal{I}}gzy, x\rangle_{L^2} = \lim_{n\to\infty} \langle g, ({}^*L)^*(x\overline{y})_n\rangle_{L^2}, \tag{48}$$

where $({}^*L)^*(x\overline{y})_n$ is any sequence in $K^2_{\mathcal{I}^2/z}$ converging to $({}^*L)^*(x\overline{y})$ in the $K^1_{\mathcal{I}^2/z}$ norm. Continuous functions in $K^2_{\mathcal{I}^2/z}$ also lie in $P_{\mathcal{I}^2/z}(\text{VMOA})$ and so in (48) we can set g to equal $\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda$ for every $\lambda \in \mathbb{D}$ such that $\frac{\mathcal{I}^2}{z}(\lambda) = 0$ and every $i < j_\lambda$ where j_λ is the order of the zero $\frac{\mathcal{I}^2}{z}(\lambda)$. Then as each $g = \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^i k_\lambda$ is bounded we can omit taking the limit in our duality pairing and we obtain

$$\int_{\mathbb{T}} g\overline{\mathcal{I}\overline{z}x\overline{y} - ({}^*L)^*(x\overline{y})}dm = 0.$$

Now an application of Lemma 5.7 gives us that $\mathcal{I}\overline{z}x\overline{y} - ({}^*L)^*(x\overline{y}) = 0$, and so $({}^*L)^*(x\overline{y}) = L(x\overline{y})$. Now, Proposition 4.1 in [5] states that every element of X can be expressed as a sum of four elements of the form $x\overline{y}$ for $x, y \in K^2_{\mathcal{I}}$, and furthermore L and $({}^*L)^*$ are linear so we must indeed have $({}^*L)^* = L$. □

We can make a result which is analogous to Proposition 5.16 but in the case of continuous symbols.

Proposition 5.19. *The image of* ${}^*L$ *is all TTOs of the form* $A^{\mathcal{I}}_{\phi}$ *where* $\phi \in \overline{\mathcal{I}}C(\mathbb{T})$

Proof. Let $A^{\mathcal{I}}_{\phi}$ lie in the image of ${}^*L$. Then $A^{\mathcal{I}}_{\phi} = A^{\mathcal{I}}_{\overline{\mathcal{I}}gz}$ for some $g \in P_{\mathcal{I}^2/z}(\text{VMOA}) = P_{\mathcal{I}^2/z}(C(\mathbb{T}))$. As g lies in $P_{\mathcal{I}^2/z}(C(\mathbb{T}))$ there exists $p_1, p_2 \in H^2$ such that $\overline{zp_1} + g + \frac{\mathcal{I}^2}{z}p_2 := g' \in C(\mathbb{T})$. Now it is easy to see that $A^{\mathcal{I}}_{\phi} = A^{\mathcal{I}}_{\overline{\mathcal{I}}gz} = A^{\mathcal{I}}_{\overline{\mathcal{I}}g'z}$, and clearly $g'z \in C(\mathbb{T})$.

Conversely if $A^{\mathcal{I}}_{\phi}$ is such that $\phi \in \overline{\mathcal{I}}C(\mathbb{T})$ then as pointed out in the proof of the previous proposition $A^{\mathcal{I}}_{\phi}$ is compact. As division by z is continuous we can write $\phi = \overline{\mathcal{I}}\phi' z$ where ϕ' is continuous. Now it is easy to see that $A^{\mathcal{I}}_{\phi} = A^{\mathcal{I}}_{\overline{\mathcal{I}}\phi' z} = A^{\mathcal{I}}_{\overline{\mathcal{I}}P_{\mathcal{I}^2/z}(\phi')z} = {}^*L(P_{\mathcal{I}^2/z}(\phi') \in {}^*L(P_{\mathcal{I}^2/z}(C(\mathbb{T}))) = {}^*L(P_{\mathcal{I}^2/z}(\text{VMOA}))$. □

As the map L is clearly injective, ${}^*L$ must have dense range and we can make the following corollary which has been previously noticed in the proof of Lemma 3.5 in [10].

Corollary 5.20. *Truncated Toeplitz operators of the form* $A^{\mathcal{I}}_{\phi}$ *where* $\phi \in \overline{\mathcal{I}}C(\mathbb{T})$ *are dense in* $\mathcal{T}_c(\mathcal{I})$.

Lemma 5.21. *Every compact TTO on* $K^2_{\mathcal{I}}$ *is of the form* $A^{\mathcal{I}}_{\mathcal{I}\phi}$ *where* $\phi \in C(\mathbb{T})$ *if and only if every compact TTO on* $K^2_{\mathcal{I}}$ *is of the form* $A^{\mathcal{I}}_{\overline{\mathcal{I}\psi}}$*, where* $\psi \in C(\mathbb{T})$.

Proof. This follows from the fact that for all $g \in L^\infty$, $A^{\mathcal{I}}_g$ is compact if and only if $(A^{\mathcal{I}}_g)^* = A^{\mathcal{I}}_{\overline{g}}$ is compact and that $\overline{C(\mathbb{T})} = C(\mathbb{T})$. □

We now can prove a one way implication of Theorem 5.12.

Theorem 5.22. *If every compact TTO on* $K^2_{\mathcal{I}}$ *is of the form* $A^{\mathcal{I}}_{\mathcal{I}\phi}$ *where* $\phi \in C(\mathbb{T})$ *then every bounded TTO has a bounded symbol.*

Proof. If every compact TTO on $K^2_{\mathcal{I}}$ is of the form $A^{\mathcal{I}}_{\mathcal{I}\phi}$ where $\phi \in C(\mathbb{T})$ then by Lemma 5.21 and Proposition 5.19 we know that ${}^*L$ is surjective (and hence isomorphic). Now by Proposition 5.18 we must also have that $({}^*L)^* = L$ is isomorphic, and hence L^* is (anti-linear) isomorphic. Now by Proposition 5.16 this must mean every bounded TTO has a bounded symbol. □

In order to prove the converse of the above theorem, we need the following lemma. As the following lemma holds for any inner function, we prove the lemma in the context of an arbitrary inner function I.

Lemma 5.23. *If I is written as a product $I = \theta_1 ... \theta_N$, where each θ_i is inner, then*

$$\bigcap_{i=1,\ldots,N} \mathcal{C}_2(\theta_i) = \mathcal{C}_2(I).$$

Proof. For each i we have $K^2_{\theta_i} \subseteq K^2_I$, so the $\supseteq$ inclusion is immediate. We now assume $\mu \in \bigcap_{i=1,\ldots,N} \mathcal{C}_2(\theta_i)$. For each $f \in K^2_I$, we can use an orthogonal decomposition to write f as

$$f = k_1 + \theta_1 k_2 + \ldots + \theta_1 \ldots \theta_{N-1} k_N,$$

where $k_i \in K^2_{\theta_i}$ and $\|f\|^2_{K_I} = \|k_1\|^2_{K^2_{\theta_1}} + \|\theta_1 k_2\|^2_{K^2_{\theta_2}} + \ldots + \|\theta_1 \ldots \theta_{N-1} k_N\|^2_{K^2_{\theta_N}}$. By the triangle inequality we have

$$\|f\|_{L^2(|\mu|)} \leqslant \|k_1\|_{L^2(|\mu|)} + \|\theta_1 k_2\|_{L^2(|\mu|)} + \ldots + \|\theta_1 \ldots \theta_{N-1} k_N\|_{L^2(|\mu|)},$$

and as each θ_i is inner this is equal to

$$\|k_1\|_{L^2(|\mu|)} + \|k_2\|_{L^2(|\mu|)} + \ldots + \|k_N\|_{L^2(|\mu|)}. \tag{49}$$

Now if we denote C_i to be the least bound such that $\|\tilde{k}\|_{L^2(|\mu|)} \leqslant C_i \|\tilde{k}\|_{K^2_{\theta_i}}$ for all $\tilde{k} \in K^2_{\theta_i}$, and $C := \max\{C_1, \ldots C_N\}$ then equation (49) is less than or equal to

$$C \left(\|k_1\|_{K^2_{\theta_1}} + \|k_2\|_{K^2_{\theta_2}} + \ldots + \|k_N\|_{K^2_{\theta_N}} \right)$$

which is equal to

$$C\left(\|k_1\|_{K^2_{\theta_1}} + \|\theta_1 k_2\|_{K^2_{\theta_2}} + \ldots + \|\theta_1 \ldots \theta_{N-1} k_N\|_{K^2_{\theta_N}}\right). \tag{50}$$

Now it is easily checked that K^2_I with the conventional norm is equivalent to K^2_I equipped with the norm where $\|f\| = \|k_1\|_{K^2_{\theta_1}} + \|\theta_1 k_2\|_{K^2_{\theta_2}} + \ldots + \|\theta_1 \ldots \theta_{N-1} k_N\|_{K^2_{\theta_N}}$. This means there exist a $B \geqslant 0$ such that (50) is less than or equal to $CB\|f\|_{K^2_I}$. □

Corollary 5.24. *$\mathcal{C}_2(I^2) = \mathcal{C}_2(I)$.*

Theorem 5.25. *If every bounded TTO on $K^2_{\mathcal{I}}$ has a bounded symbol then every compact TTO on $K^2_{\mathcal{I}}$ is of the form $A^{\mathcal{I}}_{\mathcal{I}\phi}$ where $\phi \in C(\mathbb{T})$.*

Proof. If we assume every bounded TTO on $K^2_{\mathcal{I}}$ has a bounded symbol then by Theorem 5.11 we must also have $\mathcal{C}_2(\mathcal{I}^2) = \mathcal{C}_1(\mathcal{I}^2)$, and then consequently by the above corollary we must also have $\mathcal{C}_2(\mathcal{I}) = \mathcal{C}_1(\mathcal{I}^2)$. Now under this condition Theorem 5.2 in [21] states that every compact TTO on $K^2_{\mathcal{I}}$ is of the form $A^{\mathcal{I}}_{\mathcal{I}\phi}$ where $\phi \in C(\mathbb{T})$. □

We now easily state the proof of our main result.

Proof of Theorem 5.12. The forward implication is Theorem 5.25 and the backward implication is Theorem 5.22.

A long standing open conjecture regarding symbols of bounded TTOs is the following.

Conjecture 5.1. Let I be an inner function. Every bounded TTO on K_I^2 has a bounded symbol if and only if I is one-component.

Under the conditions given on our inner function $\mathcal{I}$ the results of this section show one may approach the above conjecture from a different viewpoint. When one considers the inner function $\mathcal{I}$ an alternative formulation of the above conjecture is the following.

Conjecture 5.2. Every compact TTO on $K_{\mathcal{I}}^2$ has a symbol in $\mathcal{I}C(\mathbb{T})$ if and only if $\mathcal{I}$ is one-component.

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