A THEORETICAL AND EXPERIMENTAL STUDY OF CERTAIN CYLINDRICALLY SYMMETRICAL ELECTRIC AND MAGNETIC FIELDS

THESIS SUBMITTED FOR THE DEGREE OF Ph.D.

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The electric field between two spheres

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The exact expression of the self-inductance of circular turn of wire by means of Toroidal Functions

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List of Symbols

The following list contains those symbols that have been used consistently throughout the thesis. Because of the large number of quantities to be represented and the undesirability of using alphabets other than English and Greek, it has been necessary to use some symbols to represent different quantities at different times and places. In every instance the symbol has been defined where introduced to avoid misinterpretation of its meaning.

$\underline{A}$	Vector potential of the magnetic field
$\underline{B}$	Magnetic flux density
E	Voltage gradient or electric intensity
E(k), F(k)	Elliptic integrals
i,j,k	Unit vectors, Cartesian coordinates
I	Total current
H	Magnetic intensity
n	Unit vector normal to the surface
P_n, Q_n	Legendre functions of order n
p_n, q_n	Toroidal functions of order n
s	Distance
V	Electric potential
L	Self-inductance

r, θ, ϕ	Spherical coordinates
ρ, z, χ	Cylindrical coordinates
x, y, z	Cartesian Coordinates
ϕ	Potential function
η_0	Primary magnetic constant($=1.257 \times 10^{-6}$)
Φ	Flux linkage
u, v, w	Toroidal coordinates
W	Work, energy
Δ, δ	Small increment of a quantity

SUMMARY

Cylindrically symmetrical electric or magnetic fields are often present in electrical apparatus. It is usually considered to be prudent to study these fields experimentally rather than mathematically, since analytic methods require more advanced mathematics than is normally possessed by practical engineers.

The author has derived in the present paper some analytic solutions for a few common field problems of this kind.

The paper is essentially in three parts: The first two parts are concerned with electric fields; the last being devoted to a magnetic field.

The first part deals with the general theory of zonal toroidal functions and in particular with the application of these functions to the problem of the electric field of an electrode system consisting of a rod and a concentric ring. The experimental investigations undertaken by the author to justify the usefulness of these functions were carried out with the aid of an electrolytic tank.

The second part is devoted to the allied problem of the sphere gap. Legendre's functions and dispherical coordinates are used.

The third part deals with the exact calculation of the self-inductance of a circular turn of wire by means of toroidal functions.

Finally, a number of appendices have been included, dealing with a variety of interesting details which it has been considered advisible to separate from the main text of the thesis.

PART 1

THE APPROXIMATE CALCULATION OF THE ELECTRIC FIELD BETWEEN A ROD AND A CONCENTRIC RING BY MEANS OF TOROIDAL FUNCTIONS

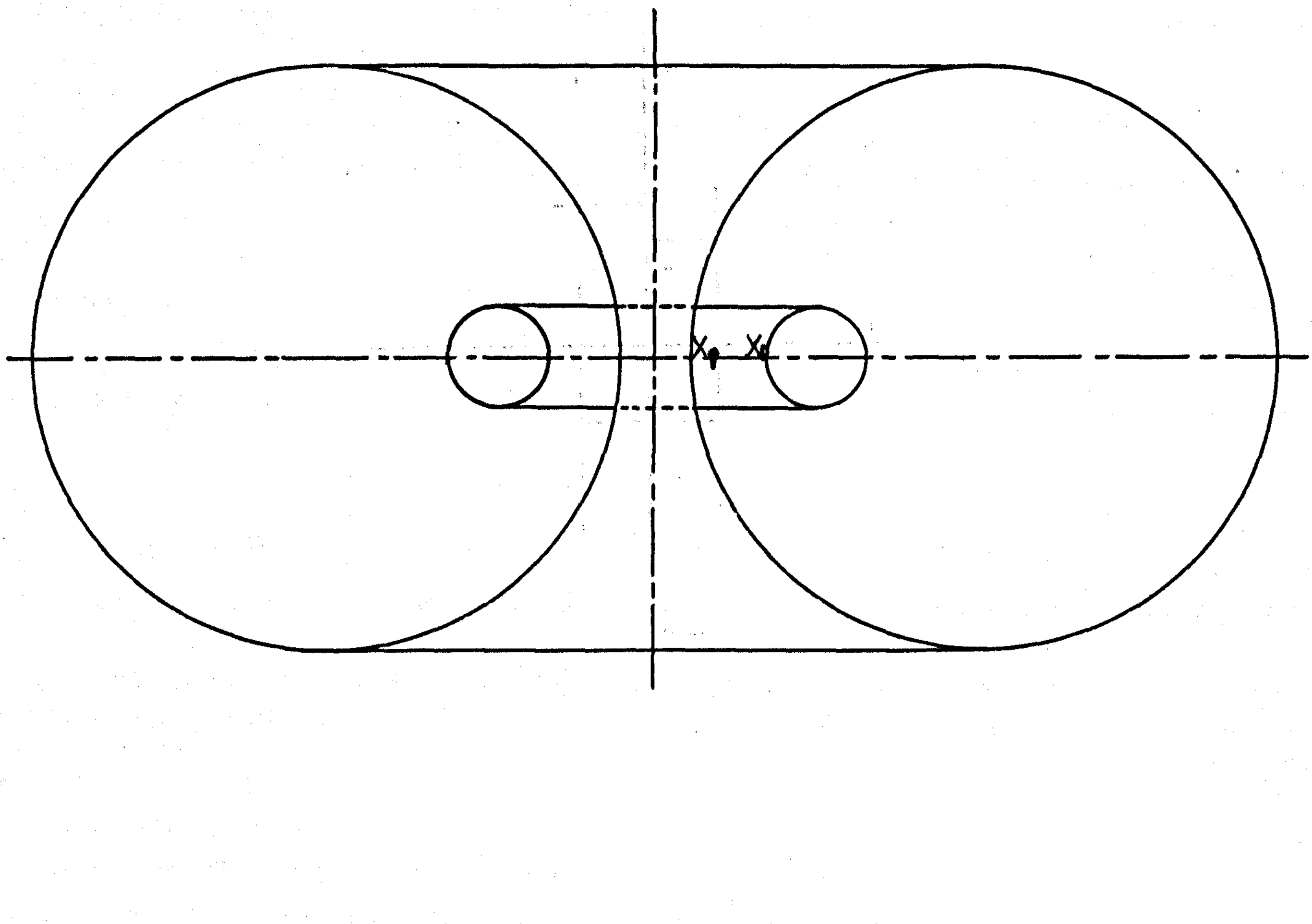

FIG. 1.2. EQUIVALENT SYSTEM WITH TOROIDAL ELECTRODES

1.1. INTRODUCTION.

The electric field in an electrode system consisting of a cylindrical rod surrounded by a concentric ring of circular section (Fig. 1.1) is not exactly calculable by any known analytical method; but it has an important bearing on the design of bushings, which may be regarded as practical embodiments of this ideal arrangement. The problem is discussed by Schwaiger[1] in terms of experimentally determined breakdown voltages, mostly obtained by measuring the sparkover voltage between a pair of crossed rods. On the theoretical side the allied problem of a wire passing through a circular hole in a plate was attacked by Bolliger[2] by a method which would furnish an approximate solution to our problem, namely by regarding the electrodes as two members of the system of hyperboloids formed by rotating a set of confocal hyperbolas about their conjugate axis.

The theoretical part of the present paper also proceeds by attacking an allied problem, that of the field between electrodes which are tores of the system formed when a set of coaxial circles is rotated about its radical axis (Fig. 1.2). Thus the ring electrode is correctly represented, but the rod is replaced by an hour-glass-shaped solid having the same radius on the central plane.

The " Toroidal Functions " necessary for the solution of this problem were first introduced by Neumann[3] in 1864, in connection with the problem of the distribution of heat in a solid anchor-ring and were discussed in detail by other investigators, in particular, by Hicks[4] [5]; but the author has only been able to trace one numerical table of these functions, that published by Fouquet[6] in 1937. As Fouquet's table does not cover a sufficient range for our purpose, an extended table has been computed and included in this paper. The values have been used to calculate the potential gradients on the electrodes at the points of nearest approach. The question of the error introduced in replacing the rod by an hour-glass is investigated with the aid of an electrolytic tank, and conclusions are drawn about the most efficient radii for the conductors.

The word " Tore " used later in this paper means an anchor-ring which has a circular cross-section. It is also to be understood that the " Toroidal Functions " satisfy Laplace's equation and are suitable for conditions given over the surfaces of the tore.

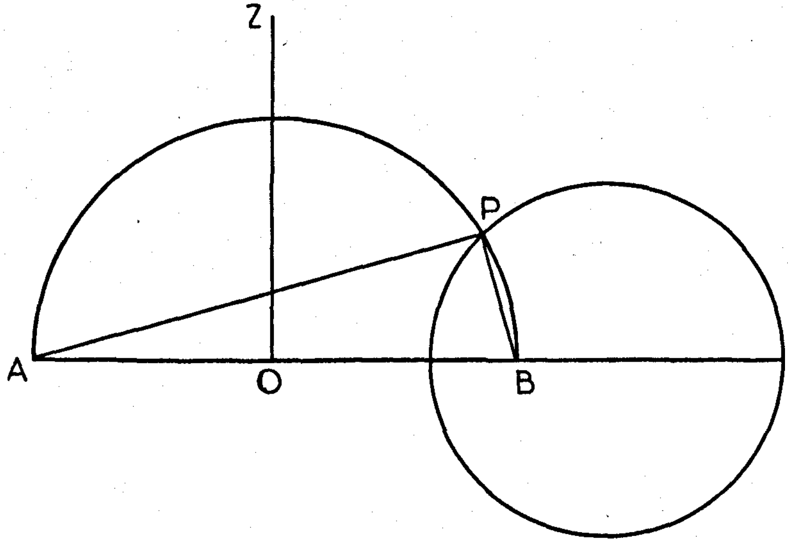

FIG. 1.3.

1.2. TOROIDAL CO-ORDINATES.

We shall now consider the " toroidal functions " which arise when Laplace's equation is transformed so that the three co-ordinates, which are then taken as independent variables, are the parameters 'u' of a family of anchor-rings or tores, the parameters 'v' of a family of spherical bowls orthogonal to the anchor-rings and the parameters 'w' of a family of half-planes orthogonal to the tores and bowls.

If A, B, are points on a straight line through the origin, O, perpendicular to the z-axis, and making an angle 'w' with the x-axis, we take as the co-ordinates of a point P, in the plane w= constant, the value of log(AP/BP) which may be denoted by 'u', the angle APB, denoted by 'v', and the azimuthal angle 'w'. The distance, 2a, between A and B is taken to be constant. (see Fig. 1.3)

It is clear that as 'w' increases from 0 to 2π, the surfaces for which 'u' has constant values will be the family of tores generated by the revolution round the z-axis of the circles of the family of coaxial circles of which A and B are the limiting points. Also the surfaces for which 'v' has constant values will be the family of spherical bowls.

(a) RECTANGULAR CARTESIAN CO-ORDINATES.

The relation between the rectangular Cartesian coordinates (x,y,z) and the toroidal coordinates (u,v,w) is

$$x = \frac{a \sinh u \cos w}{\cosh u - \cos v}$$

$$y = \frac{a \sinh u \sin w}{\cosh u - \cos v} \qquad \text{---------------} \quad (1.2.1)$$

$$z = \frac{a \sin v}{\cosh u - \cos v}$$

$$\rho = (x^2 + y^2)^{1/2}$$

$$= \frac{a \sinh u}{\cosh u - \cos v}$$

From (1.2.1), the line u = constant gives the equation,

$$\rho^2 + z^2 - 2a \coth u \, \rho + a^2 = 0 \qquad \text{------} \quad (1.2.2)$$

This gives a family of coaxial circles with limiting points at $\pm a$.

Similarly, the line v=constant has the equation,

$$\rho^2 + z^2 - 2a \cdot \cot v \cdot z - a^2 = 0 \qquad \text{-------} \quad (1.2.3)$$

For the element of length ds, we have in the new co-ordinates (u,v,w,)

$$(ds)^2 = (d\rho)^2 + (dz)^2 + \rho^2(dw)^2$$

which is easily found to reduce to

$$(ds)^2 = \frac{a^2}{(\cosh u-\cos v)^2}\left\{(du)^2+(dv)^2+\sinh^2 u(dw)^2\right\}$$

(b) CYLINDRICAL CO-ORDINATES.

The cylindrical co-ordinates (ρ,z,χ) and the toroidal co-ordinates (u,v,w) are related by a well-known mathematical equation given by

$$u + jv = \log_e \frac{\rho+a+jz}{\rho-a+jz}$$

so that,

$$u = \frac{1}{2}\log_e \frac{z^2+(\rho+a)^2}{z^2+(\rho-a)^2}$$

$$v = -\tan^{-1}\frac{2az}{\rho^2+z^2-a^2}$$

$$\rho+jz = \frac{a\,(e^{u+jv}+1)}{e^{u+jv}-1}$$

$$\frac{du}{dn} = \frac{\cosh u-\cos v}{a}$$

$$= (\sinh u/\rho)$$

1.3. TOROIDAL FUNCTIONS.

It is well known that if u,v,w be any system of orthogonal curvilinear co-ordinates, Laplace's equation may be expressed in the form

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial w}\right)\right\}$$

$$= 0 \qquad \text{-------------(1.3.1)}$$

where

$$h_1^2 = \left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial u}\right)^2$$

$$h_2^2 = \left(\frac{\partial X}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial v}\right)^2 + \left(\frac{\partial Z}{\partial v}\right)^2$$

$$h_3^2 = \left(\frac{\partial X}{\partial w}\right)^2 + \left(\frac{\partial Y}{\partial w}\right)^2 + \left(\frac{\partial Z}{\partial w}\right)^2$$

and

$$x = X(u,v,w)$$

$$y = Y(u,v,w)$$

$$z = Z(u,v,w)$$

In the present case, u,v, being conjugate functions of ρ and z

$$h_1^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$= \left(\frac{\partial v}{\partial \rho}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = h_2^2$$

and since

$$w = \tan^{-1}(y/x)$$

Thus,

$$h_3^2 = \rho^2$$

so that equation (1.3.1) becomes

$$\frac{\partial}{\partial u}\left(\rho\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\rho\frac{\partial\phi}{\partial v}\right) + \frac{1}{\rho}\left\{\left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\}^{-1}\frac{\partial^2\psi}{\partial w^2} = 0 \quad \text{--------(1.3.2)}$$

We are concerned solely with problems having cylindrical symmetry, so that the variable "w" becomes irrelevant. For such problems, Laplace's equation, when expressed in terms of u and v, becomes

$$\frac{\partial}{\partial u}\left(\frac{\sinh u}{\cosh u - \cos v}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\sinh u}{\cosh u - \cos v}\frac{\partial\phi}{\partial v}\right) = 0$$

$$\text{-------------(1.3.3)}$$

where ϕ is the potential in the field.

The substitution

$$\phi = \psi(\cosh u - \cos v)^{1/2} \quad \text{-----(1.3.4)}$$

reduces equation (1.3.3) to

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} + \coth u\,\frac{\partial\psi}{\partial u} + \frac{1}{4}\psi = 0$$

$$\text{-------------(1.3.5)}$$

Now we will seek a solution of the type $\psi = U\,V$, where U, V, are functions of u, v respectively.

We get

$$\frac{1}{U}\frac{d^2U}{du^2} + \frac{1}{U}\coth u\frac{dU}{du} + \frac{1}{4} = -\frac{1}{V}\frac{d^2V}{dv^2} = +n^2, \text{ say}$$

---------------(1.3.6)

Hence

$$V = C\cos nv + D\sin nv$$

and

$$\frac{d^2U}{du^2} + \coth u\frac{dU}{du} + \left(\frac{1}{4} - n^2\right)U = 0$$

---------------(1.3.7)

The solution of equation (1.3.7) may be written

$$U = a_n p_n(u) + b_n q_n(u)$$

---------------(1.3.8)

where a_n, b_n, C, D are arbitrary constants, while the functions $p_n(u)$ and $q_n(u)$ are defined by the integrals.

$$p_n(u) = \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(\cosh u - \sinh u\cos\theta)^{n+\frac{1}{2}}}$$

---------------(1.3.9)

$$q_n(u) = \int_0^{\infty}\frac{d\theta}{(\cosh u + \sinh u\cosh\theta)^{n+\frac{1}{2}}}$$

---------------(1.3.10)

$p_n(u)$, $q_n(u)$ are <u>toroidal functions</u>*

The properties of these functions are found to have analogies with those of the ordinary spherical harmonics, but with essential differences.

The general solution of equation (1.3.3) may be written (see Appendix 1)

$$\phi = (\cosh u - \cos v)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left\{ a_n p_n(u) + b_n q_n(u) \right\} \cos nv \qquad \text{------------------} \quad (1.3.11)$$

Terms in 'sinnv' are formally possible, but they cannot occur in a problem such as this, wherein the central plane is a plane of symmetry.

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* $p_n(u)$, $q_n(u)$ are Legendre functions of order $(n-\frac{1}{2})$, the relation being $p_n(u) = P_{n-\frac{1}{2}}(ju)$, $q_n(u) = Q_{n-\frac{1}{2}}(ju)$; but we have thought it desirable to use a new notation. Hicks uses $P_n(u)$ for $\pi p_n(u)$, Q_n for $q_n(u)$; but this borrowing of the customary symbols for Legendre functions may lead to confusion.

(a) TABLE OF TOROIDAL FUNCTIONS.

As Fouquet's table does not cover a sufficient range for our purpose and also undoubtedly contains a number of errors, it is considered advisable that an extended table should be computed and included in this thesis.

It will be shown in the latter sections that the toroidal functions can be expressed in terms of elliptic integrals. Using the necessary formulae given, the required table of values for the first four functions of each kind has been worked out.

The table is computed in three stages as follows:-

(1) A difference table is constructed for the 10-figure table of the complete elliptic integrals published by Milne-Thomson.[12]

(2) By means of the difference table and the appropriate equations, we calculate the values for the first two functions of each kind.

(3) The other two functions are obtained by using the recurrence formulae provided.

Although a 7-figure table was initially calculated, only a 4-figure table is included in this thesis; this is due to the fact that the 4-figure table will cover our computation sufficiently. The tabular intervals are chosen so that where possible linear interpolation or at worst the inclusion of second difference suffice. An ordinary difference table and a logarithmic difference table are computed to ensure the accuracy of the table and are found to be smooth and satisfactory. The tables are the result of original calculations, no undue steps have been taken to secure the rigorous accuracy of half a unit in the last decimal. In no case should any error greater than ± 0.52 units of the last decimal be found.

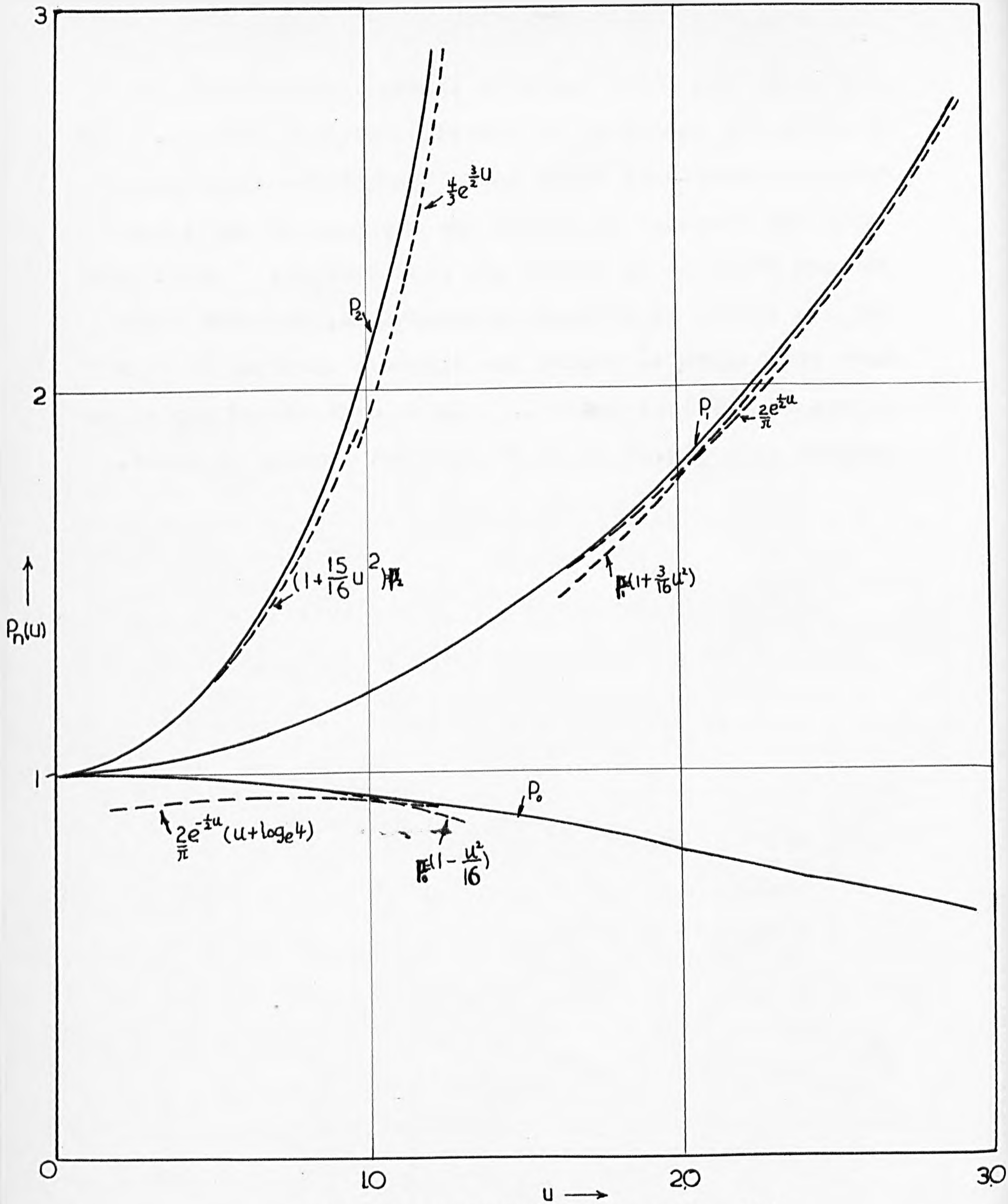


FIG. 1.4. TOROIDAL FUNCTIONS $P_n(u)$

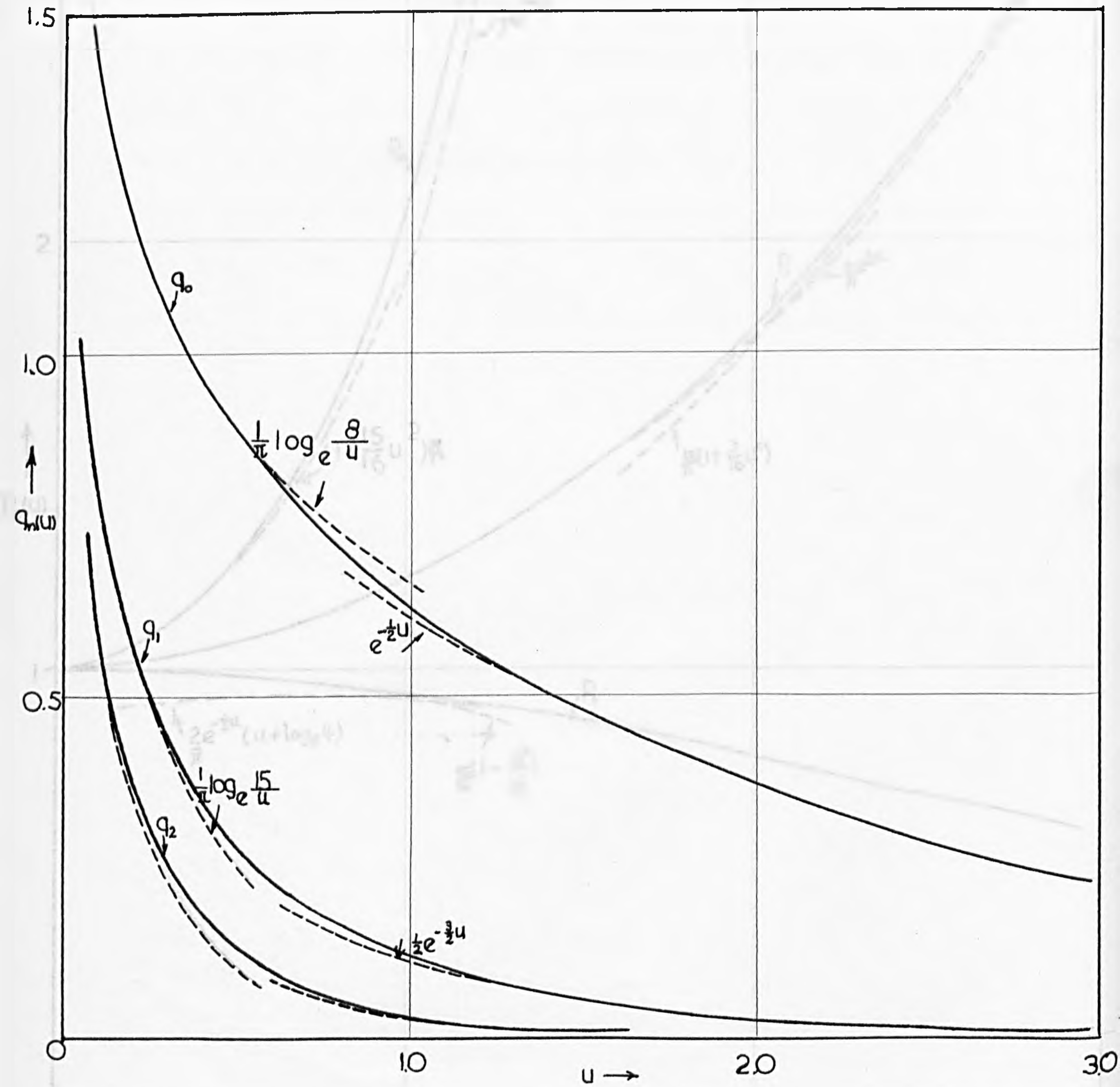


FIG. 1.5. TOROIDAL FUNCTIONS $q_n(u)$

TABLE 1

TOROIDAL FUNCTIONS (FIRST KIND)

u	$p_0(u)$	$p_1(u)$	$p_2(u)$	$p_3(u)$
0.0	1.000	1.000	1.000	1.000
.1	.9994	1.002	1.009	1.022
.2	.9975	1.008	1.038	1.089
.3	.9944	1.017	1.086	1.206
.4	.9901	1.030	1.155	1.379
0.5	.9846	1.047	1.246	1.620
.6	.9780	1.068	1.362	1.942
.7	.9702	1.092	1.504	2.366
.8	.9614	1.120	1.678	2.918
.9	.9516	1.153	1.885	3.631
1.0	.9409	1.189	2.132	4.551
.1	.9292	1.229	2.424	5.733
.2	.9168	1.273	2.767	7.252
.3	.9035	1.321	3.169	9.202
.4	.8897	1.373	3.640	(1) 1.170
1.5	.8752	1.429	4.190	1.491
.6	.8601	1.489	4.832	1.903
.7	.8446	1.554	5.580	2.432
.8.	.8286	1.624	6.452	3.110
.9	.8123	1.698	7.466	3.981
2.0	.7957	1.777	8.646	5.098

TABLE 1 (CONTINUED)-------TOROIDAL FUNCTIONS

u	$p_o(u)$	$p_1(u)$	$p_2(u)$	$p_3(u)$
2.1	.7788	1.860	(1) 1.002	(1) 6.532
.2	.7617	1.949	1.162	8.373
.3	.7444	2.043	1.347	(2) 1.074
.4	.7271	2.142	1.563	1.377
2.5	.7097	2.247	1.814	1.766
.6	.6923	2.358	2.105	2.266
.7	.6750	2.476	2.444	2.908
.8	.6577	2.599	2.838	3.732
.9	.6405	2.730	3.296	4.790
3.0	.6234	2.867	3.828	6.148
.1	.6064	3.012	4.446	7.892
.2	.5897	3.164	5.164	(3) 1.013
.3	.5731	3.324	5.998	1.300
.4	.5568	3.495	6.967	1.670
3.5	.5407	3.671	8.093	2.144
.6	.5248	3.858	9.402	2.753
.7	.5092	4.054	(2) 1.092	3.534
.8	.4939	4.261	1.269	4.537
.9	.4788	4.479	1.474	5.826
4.0	.4641	4.708	1.713	7.480

TABLE 2

TOROIDAL FUNCTIONS (SECOND KIND)

u	$q_0(u)$	$q_1(u)$	$q_2(u)$	$q_3(u)$
0.0				
.1	4.380	2.389	1.742	1.367
.2	3.681	1.712	1.102	(-1) 7.714
.3	3.267	1.329	(-1) 7.640	4.802
.4	2.969	1.069	5.512	3.120
0.5	2.735	(-1) 8.768	4.067	2.076
.6	2.540	7.283	3.043	1.407
.7	2.373	6.101	2.300	(-2) 9.567
.8	2.226	5.143	1.750	6.582
.9	2.095	4.354	1.338	4.549
1.0	1.975	3.700	1.027	3.158
.1	1.866	3.152	(-2) 7.907	2.200
.2	1.765	2.690	6.102	1.536
.3	1.672	2.300	4.717	1.073
.4	1.585	1.969	3.651	(-3) 7.514
1.5	1.503	1.687	2.830	5.268
.6	1.426	1.447	2.195	3.697
.7	1.354	1.242	1.704	2.596
.8	1.286	1.067	1.324	1.825
.9	1.222	(-2) 9.164	1.029	1.283
2.0	1.161	7.875	(-3) 7.999	(-4) 9.024

TABLE 2 (CONTINUED) ------- TOROIDAL FUNCTIONS

u	$q_o(u)$	$q_1(u)$	$q_2(u)$	$q_3(u)$
2.1	1.104	(-2) 6.769	(-3) 6.221	(-4) 6.350
.2	1.049	5.821	4.839	4.469
.3	(-1) 9.973	5.006	3.765	3.147
.4	9.482	4.305	2.930	2.216
2.5	9.016	3.703	2.281	1.560
.6	8.574	3.186	1.775	1.099
.7	8.154	2.741	1.382	(-5) 7.741
.8	7.754	2.359	1.076	5.453
.9	7.375	2.030	(-4) 8.377	3.841
3.0	7.014	1.747	6.523	2.706
.1	6.671	1.503	5.079	1.907
.2	6.346	1.294	3.955	1.343
.3	6.036	1.113	3.080	(-6) 9.466
.4	5.741	(-3) 9.581	2.398	6.670
3.5	5.460	8.246	1.868	4.700
.6	5.194	7.097	1.454	3.312
.7	4.941	6.108	1.133	2.333
.8	4.700	5.257	(-5) 8.820	1.644
.9	4.470	4.524	6.869	1.159
4.0	4.252	3.894	5.349	(-7) 8.165

The numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied

e.g. $q_3(3.0) = 0.00002706$

1.4. DISCUSSION ON $p_n(u)$.

We shall first discuss the integral already obtained.

$$p_n(u) = \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(\cosh u-\sinh u\cdot\cos\theta)^{n+\frac{1}{2}}}$$

---------------(1.3.9)

It can be shown that the above integral is the same as

$$\frac{1}{\pi}\int_0^{\pi}(\cosh u-\sinh u\cdot\cos\theta)^{n-\frac{1}{2}}\,d\theta$$

---------------(1.4.1)

If we differentiate $p_n(u)$ in (1.3.9) w.r.t. "u", we get

$$\frac{dp_n(u)}{du} = -(n+\tfrac{1}{2})\,\frac{1}{\pi}\int_0^{\pi}\frac{(\sinh u-\cosh u\cdot\cos\theta)d\theta}{(\cosh u-\sinh u\cdot\cos\theta)^{n+\frac{3}{2}}}$$

whence

$$\frac{2\sinh u}{2n+1}\,\frac{dp_n(u)}{du} = p_{n+1}(u)-\cosh u\,p_n(u)$$

---------------(1.4.2)

Similarly from (1.4.1)

$$\frac{dp_n(u)}{du} = (n-\tfrac{1}{2})\int_0^{\pi}\Big\{(\cosh u-\sinh u\cdot\cos\theta)^{n-\frac{3}{2}}$$

$$\times(\sinh u-\cosh u\cdot\cos\theta)\Big\}d\theta$$

Thus

$$\frac{2\sinh u}{2n-1}\,\frac{dp_n(u)}{du} = \cosh u\cdot p_n(u) - p_{n-1}(u)$$

--------------(1.4.3)

Combining (1.4.2) and (1.4.3) we get

$$(2n+1)p_{n+1}(u) - 4n\cosh u\, p_n(u) + (2n-1)p_{n-1}(u) = 0$$

----------------(.4.4)

This sequence equation may also be deduced at once from (1.3.9) or (1.4.1)

The function of order zero, $p_o(u)$, may be expressed in terms of an elliptic integral as follows:

$$p_o(u) = \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(\cosh u - \sinh u\,\cos\theta)^{1/2}}$$

$$= \frac{1}{\pi}\,2k'^{1/2}\int_0^{\pi/2}\frac{d\theta}{(1 - k^2\sin^2\theta\,)^{1/2}}$$

$$= \frac{1}{\pi}\,2k'^{1/2}\,F(k)$$ -----------(1.4.5)

where

$$k^2 = \frac{2\sinh u}{\cosh u + \sinh u} = 1 - e^{-2u}$$

and

$$k'^2 = \frac{1}{(\cosh u + \sinh u)^2} = e^{-2u}$$

$F(k)$ is known as the complete elliptic integral of the first kind.

$$p_1(u) = \frac{1}{\pi}\int_0^\pi (\cosh u - \sinh u \cdot \cos\theta)^{1/2}\, d\theta$$
$$= \frac{2}{\pi\sqrt{k'}}\int_0^\pi (1-k^2\cdot\sin^2\theta)^{\frac{1}{2}}\, d\theta$$
$$= \frac{2}{\pi\sqrt{k}} E(k)$$

from the recurrence formulas, we find

$$p_2(u) = \frac{4}{3}\cosh u\; p_1(u) - \frac{1}{3}\, p_o(u)$$

$$p_3(u) = \frac{8}{5}\cosh u\; p_2(u) - \frac{3}{5}\, p_1(u)$$

The value of $p_n(u)$ when $u=0$ is 1

The value of $p_n(u)$ when $u=\infty$ is ∞

These statements are at once seen to be true. Since 'u' becomes infinite along the critical circle it follows that the functions $p_n(u)$ are not suitable functions to use by which to express functions which are finite in spaces containing the critical circle, i.e. within any tore. But it is finite and continuous for all space outside any tore.

1.5. BEHAVIOUR OF $p_n(u)$ FOR SMALL AND LARGE VALUES OF "u".

We know

$$p_n(u) = \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(\cosh u-\sinh u\cdot\cos\theta)^{n+\frac{1}{2}}}$$

$$=\frac{1}{\pi}(\cosh u)^{-n-\frac{1}{2}}\int_0^{\pi}(1-\tanh u\cdot\cos\theta)^{-n-\frac{1}{2}}d\theta$$

$$=\frac{1}{\pi}(\cosh u)^{-n-\frac{1}{2}}\int_0^{\pi}\Big\{1+(n+\tfrac{1}{2})\tanh u\cdot\cos\theta$$

$$+\frac{(n+\frac{1}{2})(n+\frac{3}{2})}{2!}\tanh^2 u\cdot\cos^2\theta+\dots\Big\}\,d\theta$$

$$=\frac{1}{\pi}(\cosh u)^{-n-\frac{1}{2}}\Big\{\pi+\frac{\pi}{4}(n+\tfrac{1}{2})(n+\tfrac{3}{2})\tanh^2 u+\dots\Big\}$$

------------------(1.5.1)

(a) when "u" is small and the terms higher than second may be neglected, we have

$$\cosh u \doteq (1+\tfrac{1}{2}u^2+\dots\dots)$$

---------------(1.5.2)

and

$$\tanh u \doteq u$$ ------------------(1.5.3)

Substituting (1.5.2) and (1.5.3) into equation (1.5.1), we get

$$p_n(u) = \frac{1}{\pi}(1+\tfrac{1}{2}u^2+\dots)^{-n-\frac{1}{2}}\Big(\pi+\frac{1}{4}\pi\frac{(2n+1)}{2}\frac{(2n+3)}{2}u^2\dots\Big)$$

$$\doteq \Big(1+\frac{4n^2-1}{16}u^2\Big)$$ ---------(1.5.4)

This approximate form obtained for $p_n(u)$ for small values of "u" is surprisingly accurate, even beyond the expected range of validity. However for higher values of "n", the range of validity will be slightly reduced.

(b) with large values of "u", we have

$$\cosh u = \tfrac{1}{2}(e^{u} + e^{-u})$$

and

$$\sinh u = \tfrac{1}{2}(e^{u} - e^{-u})$$

Substituting these relations into $p_n(u)$, we get

$$p_n(u) = \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{\left\{\tfrac{1}{2}(e^{u}+e^{-u})-\cos\theta\tfrac{1}{2}(e^{u}-e^{-u})\right\}^{n+\frac{1}{2}}}$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(e^{u}\sin^2\tfrac{1}{2}\theta + e^{-u}\cos^2\tfrac{1}{2}\theta)^{n+\frac{1}{2}}}$$

$$= \frac{2}{\pi}\ e^{-(n+\frac{1}{2})u}\int_0^{\pi/2}\frac{d\theta'}{(\cos^2\theta' + e^{-2u}\sin^2\theta')^{n+\frac{1}{2}}} \qquad \text{------------------(1.5.7)}$$

where $\theta' = \tfrac{1}{2}(\pi - \theta)$

Now, $$\int_0^{\pi/2}\frac{d\theta'}{(\cos^2\theta' + e^{-2u}\sin^2\theta')^{\frac{1}{2}}}$$

$$\doteq \log(4e^{u}),\ \text{if } e^{-2u} \text{ is small}$$

(approximation for complete elliptic integral) when n=1

$$\int_0^{\pi/2} \frac{d\theta'}{(\cos^2\theta' + e^{-2u}\sin^2\theta')^{3/2}}$$

$$= 2\frac{d}{d\lambda}(\log 4 - \tfrac{1}{2}\log\lambda) \qquad \text{where } \lambda = e^{-2u}$$

$$= 1/\lambda = e^{2u}$$

when n=2

$$\int_0^{\pi/2} \frac{d\theta'}{(\cos^2\theta' + e^{-2u}\sin^2\theta')^{5/2}}$$

$$= \frac{2}{3}\, e^{4u}$$

similarly

$$\int_0^{\pi/2} \frac{d\theta'}{(\cos^2\theta' + e^{-2u}\sin^2\theta')^{n+\frac{1}{2}}}$$

$$= 2\,\frac{2\cdot 4\cdot 6\ \ldots(2n-2)}{3\cdot 5\cdot 7\ldots.(2n-1)}\, e^{4u}$$

Thus

$$p_o(u) \doteq \frac{2}{\pi}e^{-\frac{1}{2}u}(\log_e 4+u) \longrightarrow \frac{2}{\pi}ue^{-\frac{1}{2}u}$$

$$p_1(u) \doteq \frac{2}{\pi}\, e^{\frac{1}{2}u}$$

$$p_n(u) \doteq \frac{2}{\pi}\,\frac{2\cdot 4\ \ldots(2n-2)}{3\cdot 5\ \ldots(2n-1)}\, e^{(n-\frac{1}{2})u}$$

1.6. DISCUSSION ON $q_n(u)$.

We know

$$q_n(u) = \int_0^\infty \frac{d\theta}{(\cosh u + \sinh u \cdot \cosh\theta)^{n-\frac{1}{2}}}$$

--------------------(1.3.10)

and may be expressed in terms of $p_n(u)$

$$q_n(u) = p_n(u) \int_u^\infty \frac{du}{p_n^2(u) \cdot \sinh u}$$

--------------------(1.6.1)

Similarly, for $q_n(u)$ as in the case of $p_n(u)$ can also be easily shown that

$$\frac{2\sinh u}{2n+1} \frac{dq_n(u)}{du} = q_{n+1}(u) - \cosh u \cdot q_n(u)$$

$$\frac{2\sinh u}{2n-1} \frac{dq_n(u)}{du} = \cosh u \cdot q_n(u) - q_{n-1}(u)$$

------------------(1.6.2)

and

$$(2n+1)q_{n+1}(u) - 4n \cosh u \; q_n(u) + (2n-1)q_{n-1}(u) = 0$$

------------------(1.6.3)

Again, if we wish to express $q_n(u)$ in terms of Elliptic integrals, we have to change θ into 2θ, write $\cosh\theta = \sec\phi$; $\sinh u = \tan\phi$; $d\theta = \sec\phi d\phi$ and when $\theta = 0$ or ∞,

$\phi=0$ or $\pi/2$.

Then

$$q_o(u)=2\int_0^{\pi/2}\frac{d\phi}{\{\cosh u+\sinh u-(\cosh u-\sinh u)\sin^2\phi\}^{1/2}}$$

$$= 2k'^{1/2}\,F'(k) \qquad \text{------------------} \quad (1.6.4)$$

Similarly,

$$q_1(u) = 2k'^{1/2}\{F'(k) - E'(k)\}$$

$$\text{------------------} \quad (1.6.5)$$

The value of $q_n(u)$ for $u=0$ is ∞, and for $u=\infty$ is zero. Hence $q_n(u)$ is suitable for the space within a tore, and not for space including the axis.

A useful integral $\int_0^{\pi}\frac{\cos nv\,dv}{(\cosh u-\cos v)^{1/2}} = \sqrt{2}\,q_n(u)$

$$\text{------------------} \quad (1.6.6)$$

which often appears in connection with the toroidal functions.

It has been observed, during the calculation for toroidal functions of $q_2(u)$ and $q_3(u)$, that for high values of u, $q_n(u)$ becomes extremely difficult to calculate and the accuracy of the value obtained is doubtful. The recurrence formulae derived by Hicks seem to be unsuitable for our purpose and a new approach, independent of the elliptic integrals had to be found in order that the required accuracy of the table might be preserved. For this, the author has derived a new formula for the calculation of $q_n(u)$ for any values of ' u '.

1.7. CALCULATION OF $q_n(u)$.

We know,

$$q_n(u)=\int_0^\infty \frac{d\theta}{(\cosh u+\sinh u\cdot\cosh\theta)^{n+\frac{1}{2}}}$$

$$=\int_0^\infty \frac{d\theta}{\left\{\frac{1}{2}e^u(1+\cosh\theta)+\frac{1}{2}e^u(1-\cosh\theta)\right\}^{n+\frac{1}{2}}}$$

$$= e^{-(n+\frac{1}{2})u}\int_0^\infty \frac{d\theta}{(\cosh^2\tfrac{1}{2}\theta - e^{-2u}\sinh^2\tfrac{1}{2}\theta)^{n+\frac{1}{2}}}$$

$$= e^{-(n+\frac{1}{2})u}\int_0^\infty \cosh^{-(2n+1)}\tfrac{1}{2}\theta \left\{1+(n+\tfrac{1}{2})e^{-2u}\tanh^2\tfrac{1}{2}\theta + \tfrac{1}{2}(n+\tfrac{1}{2})(n+\tfrac{3}{2})e^{-4u}\tanh^4\tfrac{1}{2}\theta + \ldots\ldots\right\} d\theta$$

$$= e^{-(n+\frac{1}{2})u}\left\{\int_0^\infty \frac{\cosh\tfrac{1}{2}\theta\cdot d\theta}{(1+\sinh^2\tfrac{1}{2}\theta)^{n+1}} + \frac{2n+1}{2}e^{-2u}\int_0^\infty \frac{\cosh\tfrac{1}{2}\theta\,\sinh^2\tfrac{1}{2}\theta\,d\theta}{(1+\sinh^2\tfrac{1}{2}\theta)^{n+2}} + \ldots\ldots\ldots\ldots\right\}$$

Let $\sinh\frac{1}{2}\theta=x$, $\frac{1}{2}\cosh\frac{1}{2}\theta d\theta=dx$, we get

$$2e^{-(n+\frac{1}{2})u}\left\{\int_0^\infty \frac{dx}{(1+x^2)^{n+1}} + e^{-2u}\left(\frac{2n+1}{2}\right)\int_0^\infty \frac{x^2\,dx}{(1+x^2)^{n+2}} + \ldots\ldots\right\}$$

Let $x=\tan\phi$, $dx=\sec^2\phi d\phi$; we now get

$$2e^{-(n+\frac{1}{2})u}\left\{\int_0^{\pi/2}\cos^{2n}\phi\,d\phi + e^{-2u}(n+\tfrac{1}{2})\int_0^{\pi/2}\sin^2\phi\cdot\cos^{2n}\phi\,d\phi\right.$$

$$\left. + e^{-4u}\ \frac{(2n+1)\ (2n+3)}{8}\int_0^{\pi/2}\sin^4\phi\cdot\cos^{2n}\phi\,d\phi + \cdots\cdots\right\}$$

$$= \pi e^{-(n+\frac{1}{2})u}\left\{\frac{(2n-1)(2n-3)\ldots 1}{2n\ (2n-2)\ldots\ldots 2} + \left(\frac{2n+1}{2}\right)e^{-2u}\ \frac{(2n-1)(2n-3)\ldots 1}{(2n+2)\ 2n\ldots\ldots 2}\right.$$

$$\left. + \frac{(2n+1)(2n+3)}{8}\ e^{-4u}\ \frac{(2n-1)(2n-3)\ldots\ldots 1}{(2n+4)(2n+2)\ldots\ldots 2} + \cdots\right\}$$

$$= \pi e^{-(n+\frac{1}{2})u}\ \frac{(2n-1)(2n-3)\ldots\ldots 1}{2n(2n-2)\ \ldots\ldots 2}\left\{1 + e^{-2u}\ \frac{2n+1}{2}\ \frac{1}{2n+2}\right.$$

$$\left. + e^{-4u}\ \frac{(2n+1)(2n+3).1.3}{(2n+2)(2n+4).2.4} + \cdots\cdots\cdots\right\}$$

if $n = 0$

$$q_0(u) = \pi e^{-\frac{1}{2}u}\ \left(1 + \tfrac{1}{4}e^{-2u} + \frac{9}{64}\ e^{-4u} + \frac{225}{2304}\ e^{-6u}\ \ldots\right)$$

if $n = 1$

$$q_1(u) = \frac{\pi}{2}\ e^{-u}\left(1 + \frac{3}{8}e^{-2u} + \frac{15}{64}\ e^{-4u} + \cdots\cdots\right)$$

if $n = 2$

$$q_2(u) = \frac{3\pi}{8}e^{-u}\ \left(1 + \frac{5}{12}\ e^{-2u} + \frac{35}{128}\ e^{-4u} + \cdots\right)$$

if $n = 3$

$$q_3(u) = \frac{5}{16}\pi\, e^{-\frac{1}{2}u}\left(1+\frac{7}{16}e^{-2u}+\frac{189}{640}e^{-4u}+\frac{1155}{5120}e^{-6u}+\ldots\right)$$

1.8. SOLUTION OF THE PROBLEM OF COAXIAL TORES.

The potential of the field between charged coaxial tores is given by

$$\phi = (\cosh u-\cos v)^{1/2}\sum_{n=0}^{\infty}\left\{a_n p_n(u)+b_n q_n(u)\right\}\cos nv \qquad \text{-----------------------(1.8.1)}$$

If $\phi = 0$ when $u = u_1$, for all values of v, we get

$$a_n p_n(u_1) + b_n q_n(u_1) = 0$$

Therefore,

$$b_n = -\frac{p_n(u_1)}{q_n(u_1)}\,a_n \qquad \text{-----------------------(1.8.2)}$$

Substituting (1.8.2) into (1.8.1), we have

$$\phi = (\cosh u-\cos v)^{1/2}\sum_{n=0}^{\infty}a_n\left\{p_n(u)-\frac{p_n(u_1)}{q_n(u_1)}q_n(u)\right\}\cos nv \qquad \text{-----------------------(.1.8.3)}$$

If $\phi = V$, when $u = u_2$ for all values of v, we have

$$\phi = (\cosh u_2-\cos v)^{1/2}\sum_{n=0}^{\infty}a_n\left\{p_n(u_2)-\frac{p_n(u_1)}{q_n(u_1)}q_n(u_2)\right\}\cos nv$$

and

$$\frac{V}{(\cosh u_2 - \cos v)^{1/2}} = \sum_{n=0}^{\infty} a_n \left\{ p_n(u_2) - \frac{p_n(u_1)}{q_n(u_1)} q_n(u_2) \right\} \cos nv$$

---------------------- (1.8.4)

Multiplying both sides by 'cosnv dv' and integrating them from 0 to π, we will have

$$\delta_n V \int_0^{\pi} \frac{\cos nv \, dv}{(\cosh u_2 - \cos v)^{1/2}} = \frac{\pi}{2} a_n \left\{ p_n(u_2) - \frac{p_n(u_1)}{q_n(u_1)} q_n(u_2) \right\}$$

---------------------- (1.8.5)

where $\delta_n = 1$ when $n \geqslant 1$, but $\delta_0 = 1/2$.

But equation (1.6.6) gives

$$\int_0^{\pi} \frac{\cos nv \, dv}{(\cosh u - \cos v)^{1/2}} = \sqrt{2} \; q_n(u)$$

Hence

$$a_n = \frac{2\sqrt{2}}{\pi} V \frac{\delta_n q_n(u_1) \cdot q_n(u_2)}{p_n(u_2) q_n(u_1) - p_n(u_1) q_n(u_2)}$$

$$b_n = -\frac{2\sqrt{2}}{\pi} V \frac{\delta_n q_n(u_2) \cdot p_n(u_1)}{p_n(u_2) q_n(u_1) - p_n(u_1) q_n(u_2)}$$

-------------------- (1.8.6)

Therefore the general equation of potential of the field between two charged tores, u_1 and u_2 is given by

$$\phi = \frac{2\sqrt{2}}{\pi} V\,(\cosh u - \cos v)^{1/2} \sum_{n=0}^{\infty} \frac{\delta_n\, q_n(u_1) q_n(u_2)}{p_n(u_2) q_n(u_1) - p_n(u_1) q_n(u_2)} \left\{ p_n(u) - \frac{p_n(u_1)}{q_n(u_1)}\, q_n(u) \right\} \cos nv$$

---------------(1.8.7)

The potential on the line $v = \pi$, is given by

$$\phi = \frac{2\sqrt{2}}{\pi} V(1+\cosh u)^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{\delta_n\, q_n(u_1) q_n(u_2)}{p_n(u_2) q_n(u_1) - p_n(u_1) q_n(u_2)} \left\{ p_n(u) - \frac{p_n(u_1)}{q_n(u_1)}\, q_n(u) \right\}$$

---------------(1.8.8)

We are chiefly interested in the potential gradients on the surfaces of the two conductors; since u is constant on each of these surfaces, the direction of the field is the direction of variation of u. The gradient is $(-\partial\phi/\partial s_u)$, thus

$$E_u = -\frac{\partial \phi}{\partial s_u} = -\frac{(\cosh u - \cos v)}{a} \frac{\partial \phi}{\partial u}$$

---------------(1.8.9)

The points of especial interest are X_o, X_1 (see Fig.1.2), where the two conductors are closest together, and where the coordinates (u, v) are (u_1, π) and (u_2, π), u_1 being greater than u_2 if u_2 denotes the central conductor.

At any point on the plane $v = \pi$

$$E_u = - \frac{(1 + \cosh u)}{a} \frac{\partial \phi}{\partial u}$$

Therefore, the potential gradient

$$E_u = \sqrt{2}\, \frac{(\cosh u+1)}{a} \sum_{n=0}^{\infty} (-1)^n \quad a_n \left\{ \cosh \tfrac{1}{2}u \left[p_n'(u) - \frac{p_n(u_1)}{q_n(u_1)} q_n'(u) \right] \right.$$

$$\left. + \tfrac{1}{2}\sinh \tfrac{1}{2}u \left[p_n(u) - \frac{p_n(u_1)}{q_n(u_1)} \; q_n(u) \right] \right\}$$

------------------(1.8.10)

Evaluating this with the help of the identities (1.6.2), we obtain

$$E_u = \frac{\sqrt{2}}{a} \frac{\cosh^2 \tfrac{1}{2}u}{\sinh \tfrac{1}{2}u} \sum_{n=0}^{\infty} (-1)^n \left\{ (n\cosh u + \tfrac{1}{2}) \left[a_n p_n(u) + b_n q_n(u) \right] \right.$$

$$\left. -(n+\tfrac{1}{2}) \left[a_n p_{n+1}(u) + b_n q_{n+1}(u) \right] \right\}$$

------------------(1.8.11)

On the earthed conductor $u = u_1$, equation (1.8.11) reduces to

$$E_1 = -\frac{\sqrt{2}}{a}\,\frac{\cosh^2\frac{1}{2}u}{\sinh\frac{1}{2}u}\sum_{n=0}^{\infty}(-1)^n\left\{a_n p_{n+1}(u_1) + b_n q_{n+1}(u_1)\right\} \qquad \text{(1.8.12)}$$

while on the live conductor $u = u_2$,

$$E_2 = \frac{\sqrt{2}}{a}\,\frac{\cosh^2\frac{1}{2}u}{\sinh\frac{1}{2}u}\sum_{n=0}^{\infty}(-1)^n\left\{\frac{2\sqrt{2}}{\pi}V(n\cosh u_2+\tfrac{1}{2})\delta_n q_n(u_2) - (n+\tfrac{1}{2})\left[a_n p_{n+1}(u_2) + b_n q_{n+1}(u_2)\right]\right\} \qquad \text{(1.8.13)}$$

with the aid of the identity, given by Hicks,

$$p_n'(u)q_n(u) - p_n(u)q_n'(u) = 1 / \sinh u \qquad \text{(1.8.14)}$$

we obtain

$$q_n'(u) = \frac{1}{p_n(u)}\left\{p_n'(u)q_n(u) - \frac{1}{\sinh u}\right\} \qquad \text{(1.8.15)}$$

substituting (1.8.15) into (1.8.10), we get

$$E_u = \frac{\sqrt{2}\,(1+\cosh u)}{a}\sum_{n=0}^{\infty}(-1)^n a_n\left\{\cosh\tfrac{1}{2}u\left[p_n'(u) - \frac{p_n(u_1)}{q_n(u_1)}\right]\times\left(\frac{p_n'(u)q_n(u)}{p_n(u)} - \frac{1}{\sinh u\cdot p_n(u)}\right)\right] + \tfrac{1}{2}\sinh\tfrac{1}{2}u\left[p_n(u) - \frac{p_n(u_1)}{q_n(u_1)}q_n(u)\right]\right\} \qquad \text{(1.8.16)}$$

On the earthed conductor $u = u_1$, equation (1.8.16), reduces to

$$E_1 = \frac{\sqrt{2}(1+\cosh u_1)}{a}\sum_{n=0}^{\infty}(-1)^n \, a_n\left\{\frac{\pi}{\sinh u_1 \cdot q_n(u_1)}\right\}\cosh\tfrac{1}{2}u_1$$

$$= \frac{2\sqrt{2}\cdot\cosh^2\tfrac{1}{2}u_1}{a}\sum_{n=0}^{\infty}(-1)^n \frac{a_n\cdot\cosh\tfrac{1}{2}u_1}{2\sinh\tfrac{1}{2}u_1\cdot\cosh\tfrac{1}{2}u_1\cdot q_n(u_1)}$$

$$= \frac{\sqrt{2}\cdot\cosh^2\tfrac{1}{2}u_1}{a\cdot\sinh\tfrac{1}{2}u_1}\sum_{n=0}^{\infty}(-1)^n \, a_n\left\{\frac{1}{q_n(u_1)}\right\}$$

---------------------(1.8.17)

The potential gradient on the surfaces of the conductor can now be easily calculated by substituting the values of u_1 and u_2 into the appropriate equations.

It is considered to be convenient and practical, to employ the dimension ratios r_1/R, r_2/R (see Fig. 1.1), instead of using u_1 and u_2 as independent variables.

The relations between r_1, r_2, R and a, u_1, u_2 are

$$a^2 = R^2 - r_1^2$$

$$u_2 = \operatorname{sech}^{-1}(r_1/R)$$

$$u_1 = 2\tanh^{-1}\frac{r_2}{\sqrt{(R^2 - r_1^2)}}$$

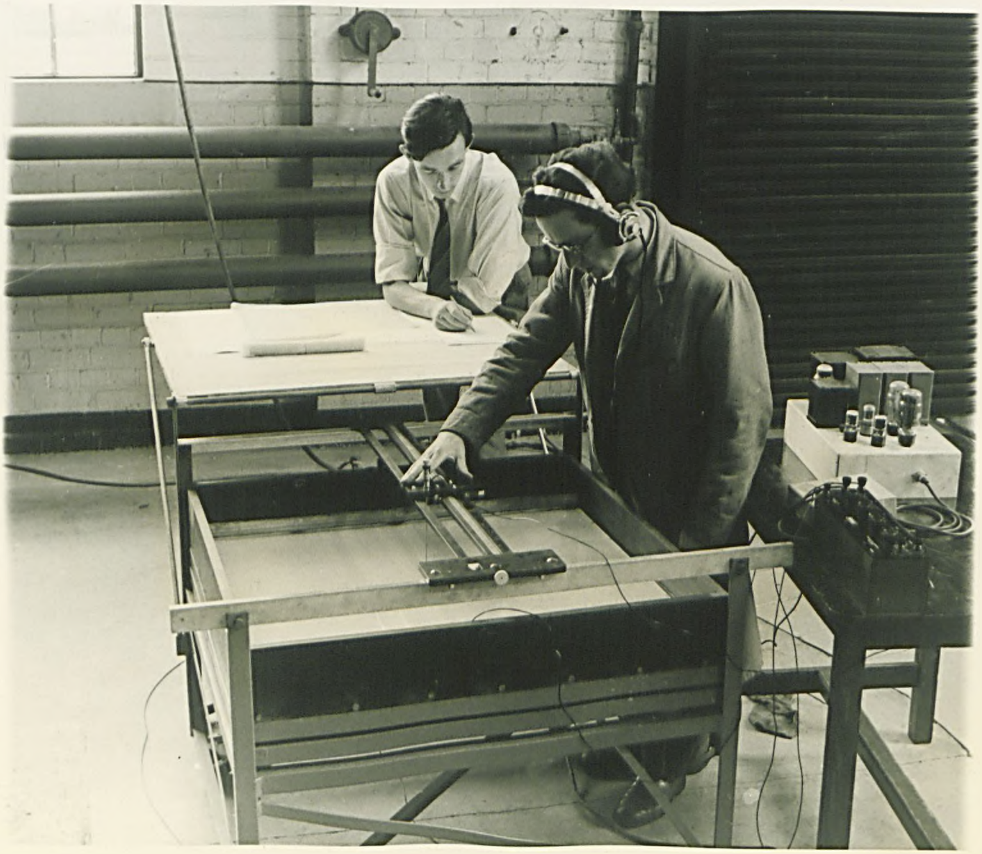
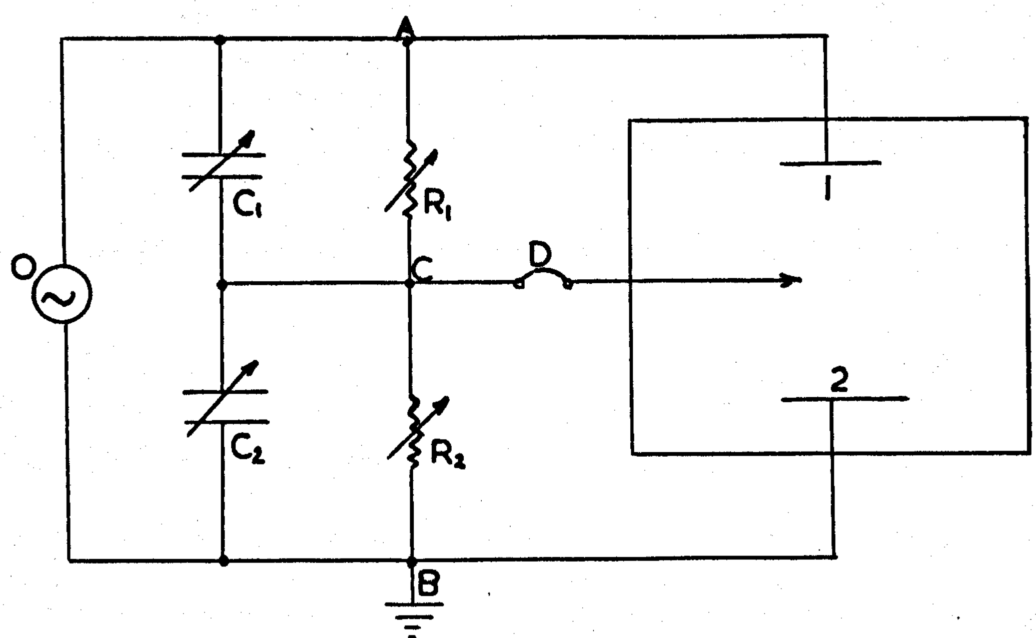


FIG. 1.6. ELECTROLYTIC TANK UNIT



O. OSCILLATOR OUTPUT 50 V AT 1KC/S

D. DETECTOR

FIG. 1.7. CIRCUIT DIAGRAM OF ELECTROLYTIC TANK

1.9. ELECTROLYTIC TANK INVESTIGATION.

To ascertain how closely the field between coaxial tores approximates to that between a coaxial rod and ring, the latter was investigated in a wedge-shaped electrolytic tank. (see Fig. 1.6)

The principle, from which the field plotting in an electrolytic tank is derived, is based entirely on Maxwell's equations. In a homogeneous isotropic dielectric which is bounded by two electrodes and in which there are no internal charges, Maxwell's equations reduce to

$$\nabla^2\phi = 0 \qquad \text{(1.9.1)}$$

This is well known as Laplace's equation which must be satisfied at every point in the particular field in question.

The electric circuit shown in Fig. 1.7. is essentially a Wheatstone bridge, with two arms formed by the probe and the electrodes 1 and 2, and the other two arms AC and BC on the calibrated potentiometer. The probe is moved until its potential is equal to the selected value on the tap C of the potentiometer as indicated by the detector.

Detector.

The headphones are chosen as the detector for the purpose of convenience. Of course, other devices such as a cathode-ray oscilloscope may be used as detector. In order to reduce the noise in the headphones to a minimum, a matching transformer is inserted between the phones and the probe.

Electrolyte.

Ordinary tap water has been used throughout the test and found to be satisfactory. Other solutions such as N/1000 of sodium hydroxide and N/2000 of sulphuric acid have been tried by various investigators and used with satisfaction.

Supply.

The supply is obtained from an oscillator of a two stage Wien bridge type with thermistor stabilization. The frequency used is approximately 1,000 c/s. It represents a compromise between error introduced by polarisation and stray capacitance, the former decreases quickly with increasing frequency up to 1,000 - 1,500 c/s; but thereafter reductions are outweighted by errors resulting from stray capacitance.

Electrode.

Different combinations of electrode materials and electrolytes have been tried in order to reduce the surface impedance to a minimum, to avoid oxidation yet to keep within the confines of simplicity and availability.

It has been found, with satisfaction, by the author that electrodes made of copper coated with " Aquadag" (a solution of graphite) have a very low contact drop when the ordinary tap water is used as electrolyte.

From a preliminary test with electrodes representing concentric cylinders it was concluded that the curve of potential variation could be drawn with an accuracy of 1 per cent.

By tilting the tank, a wedge-shaped bath is formed to represent the axially symmetrical fields which are under investigation. It is very important that the wetting line should coincide with the axis of symmetry. Unfortunately, owing to the surface tension between the electrolyte and the floor of the tank (i.e. glass), it is hardly possible to obtain a straight wetting line as desired. This difficulty has however been overcome by placing a piece of adhesive tape along the wetting line to cause a ' forced axis of symmetry'.

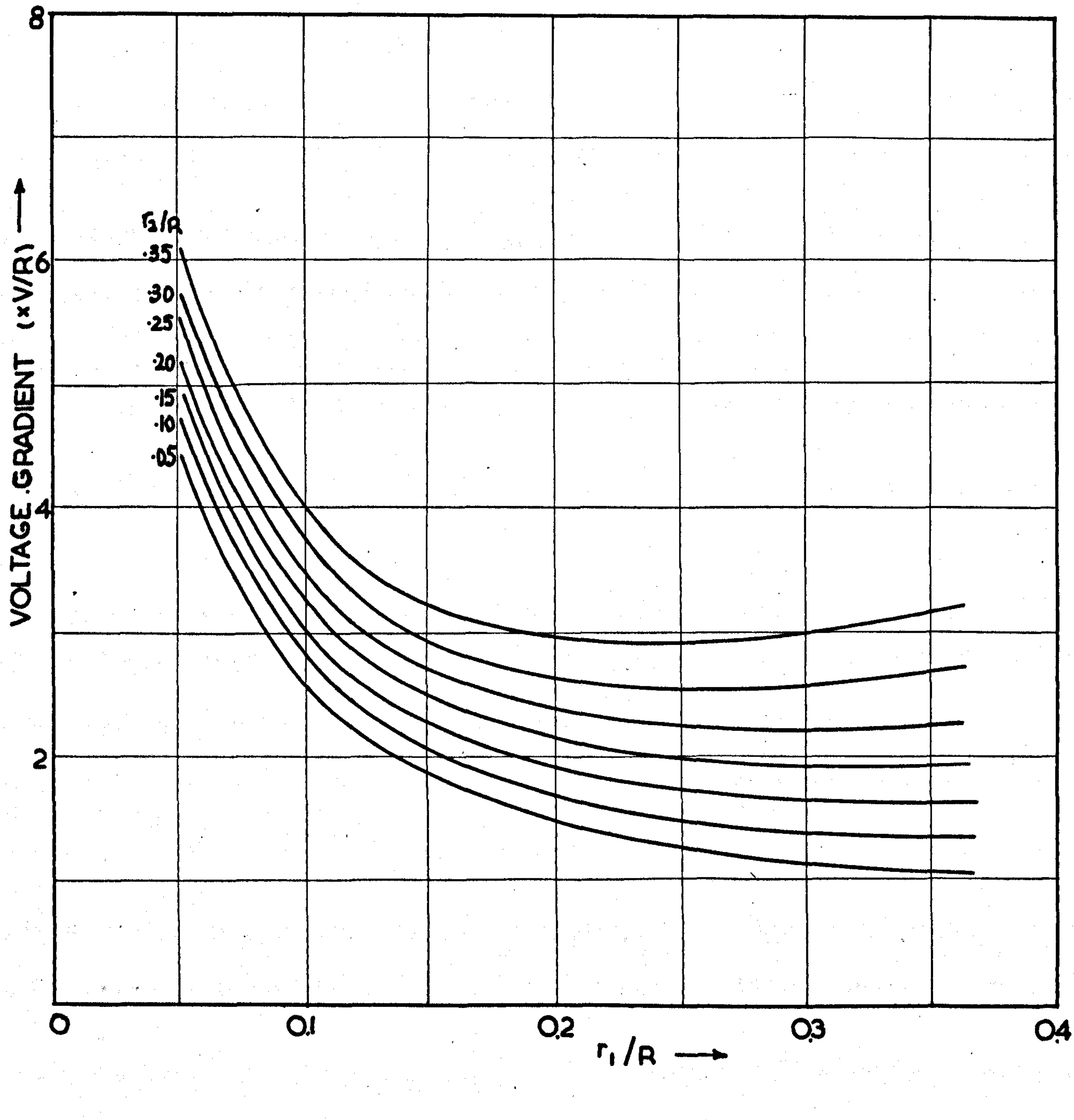


FIG. 1.8. VOLTAGE GRADIENT ON TORE — CALCULATED

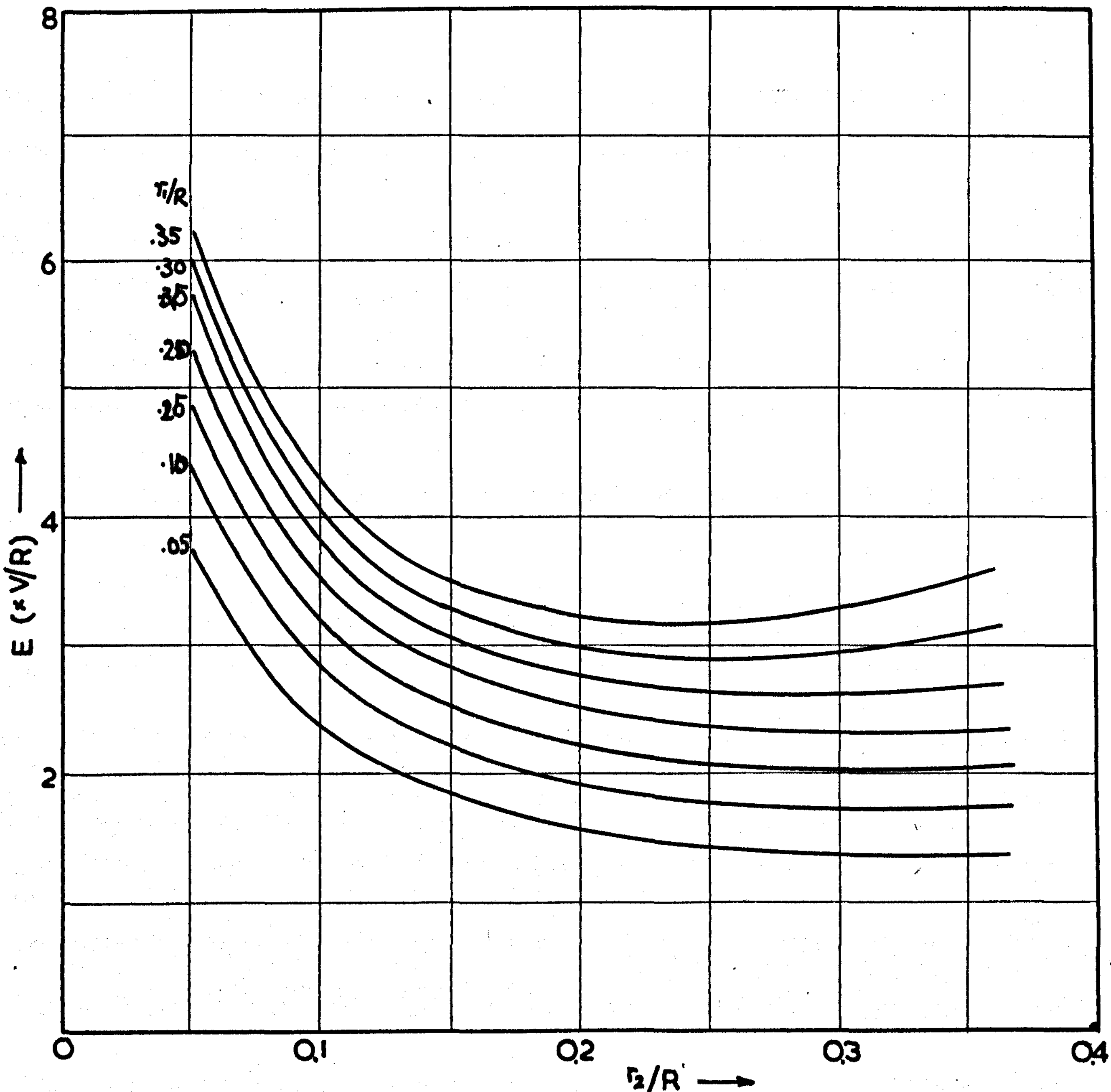


FIG. 1.9. VOLTAGE GRADIENT ON CENTRAL CONDUCTOR — CALCULATED

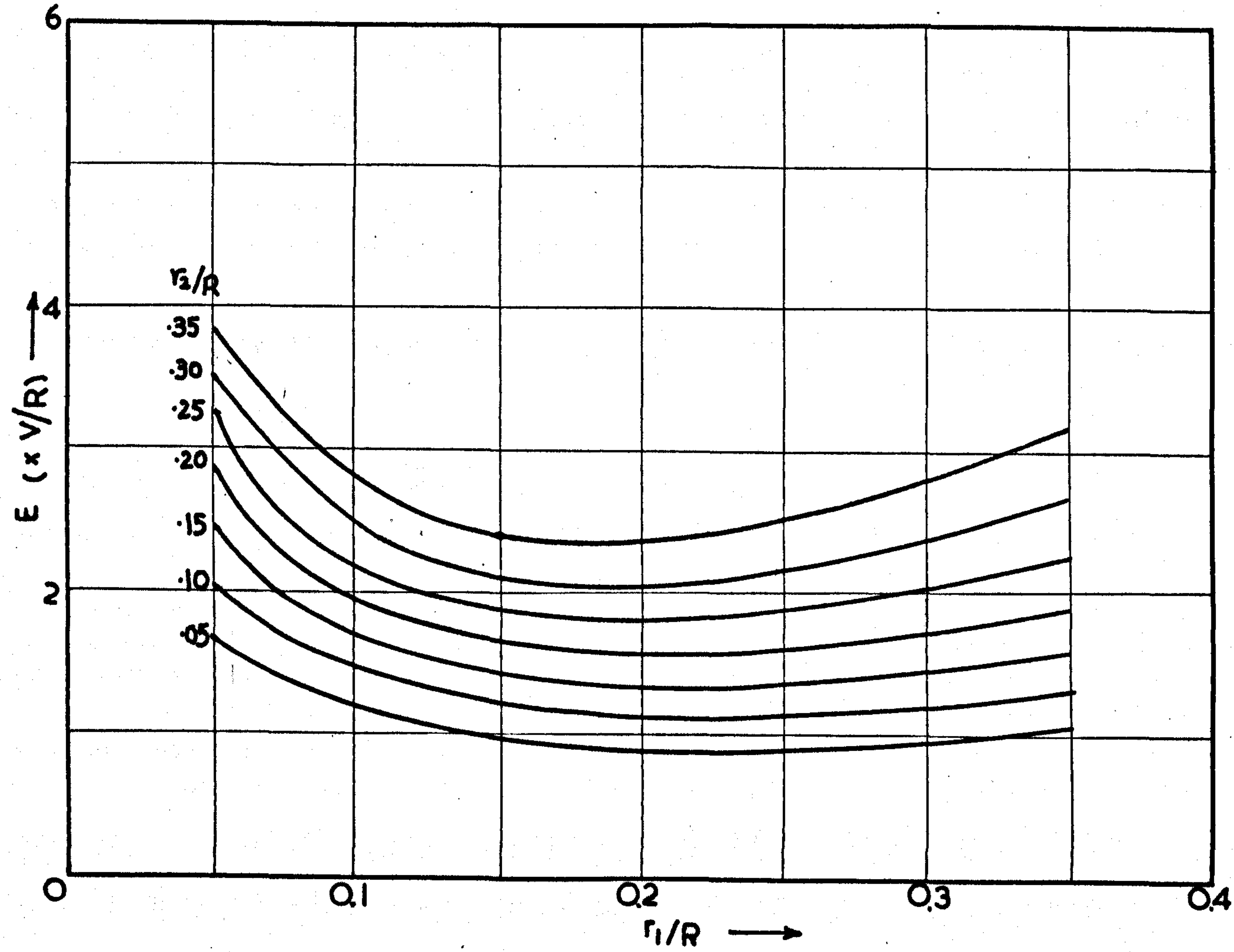


FIG. 1.10. VOLTAGE GRADIENT ON TORE — MEASURED

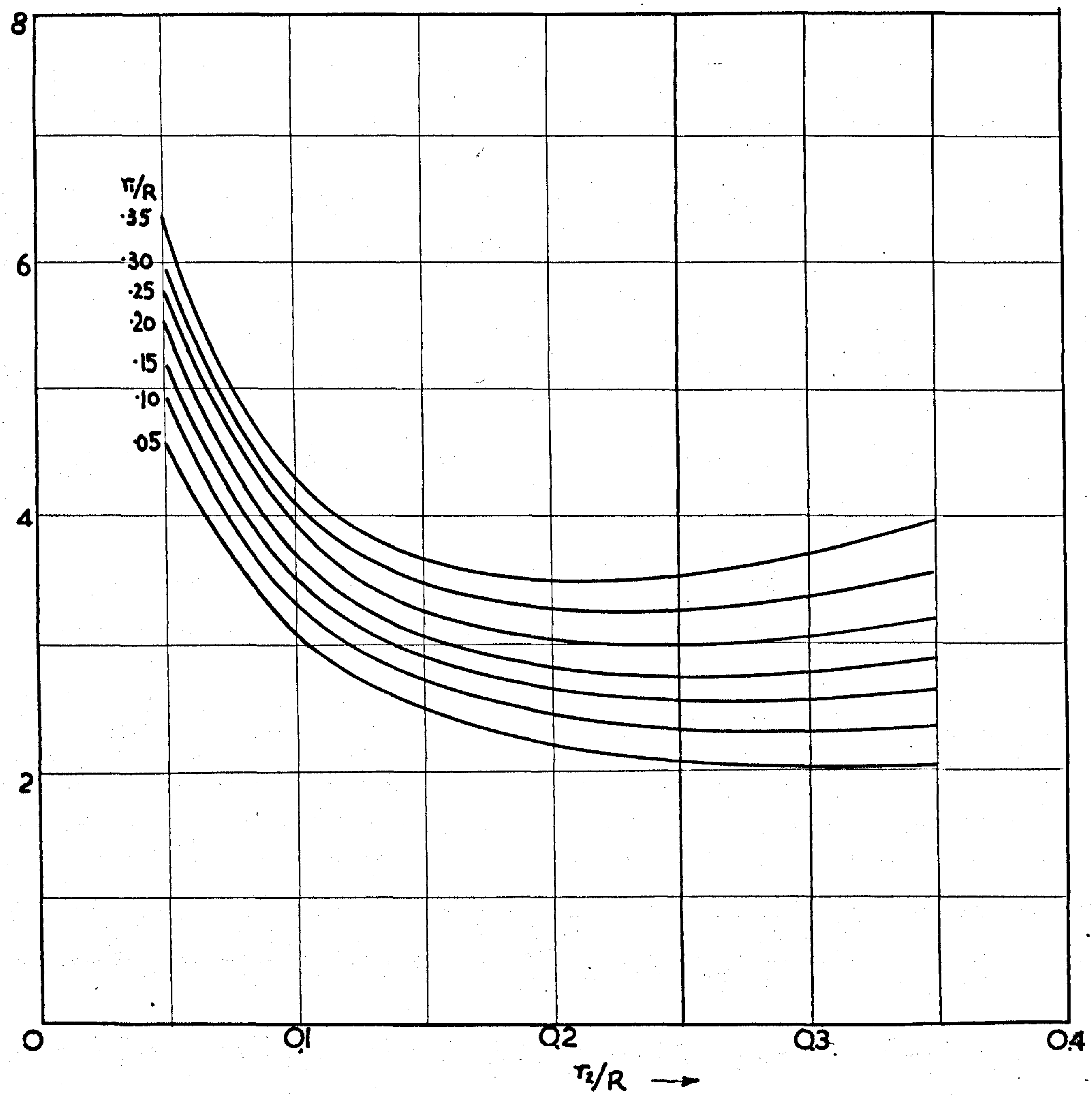


fig 1.11. voltage gradient on central conductor – measured

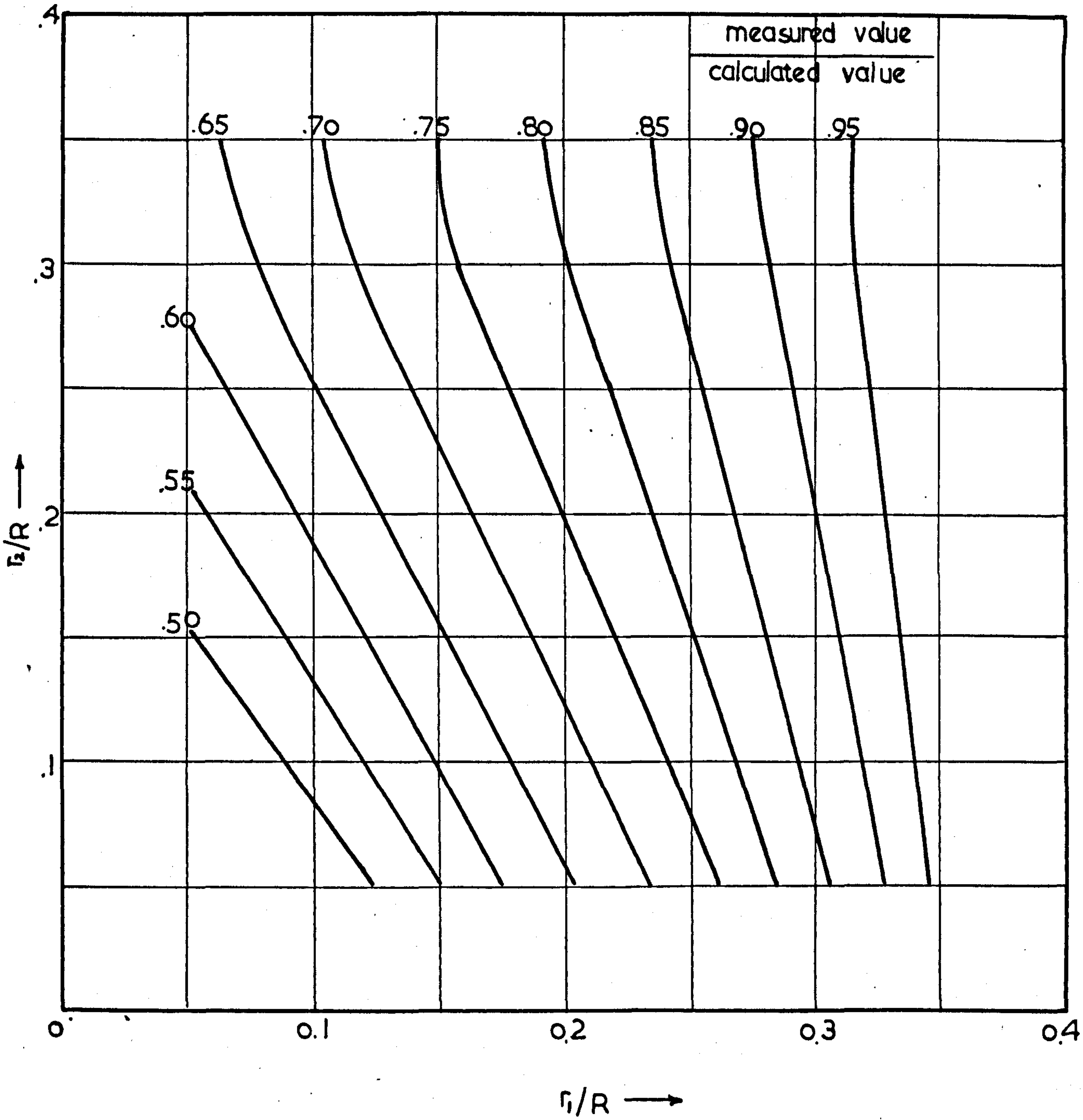


FIG.1.12. APPROXIMATE CORRECTION CHART FOR VOLTAGE GRADIENT ON TORE

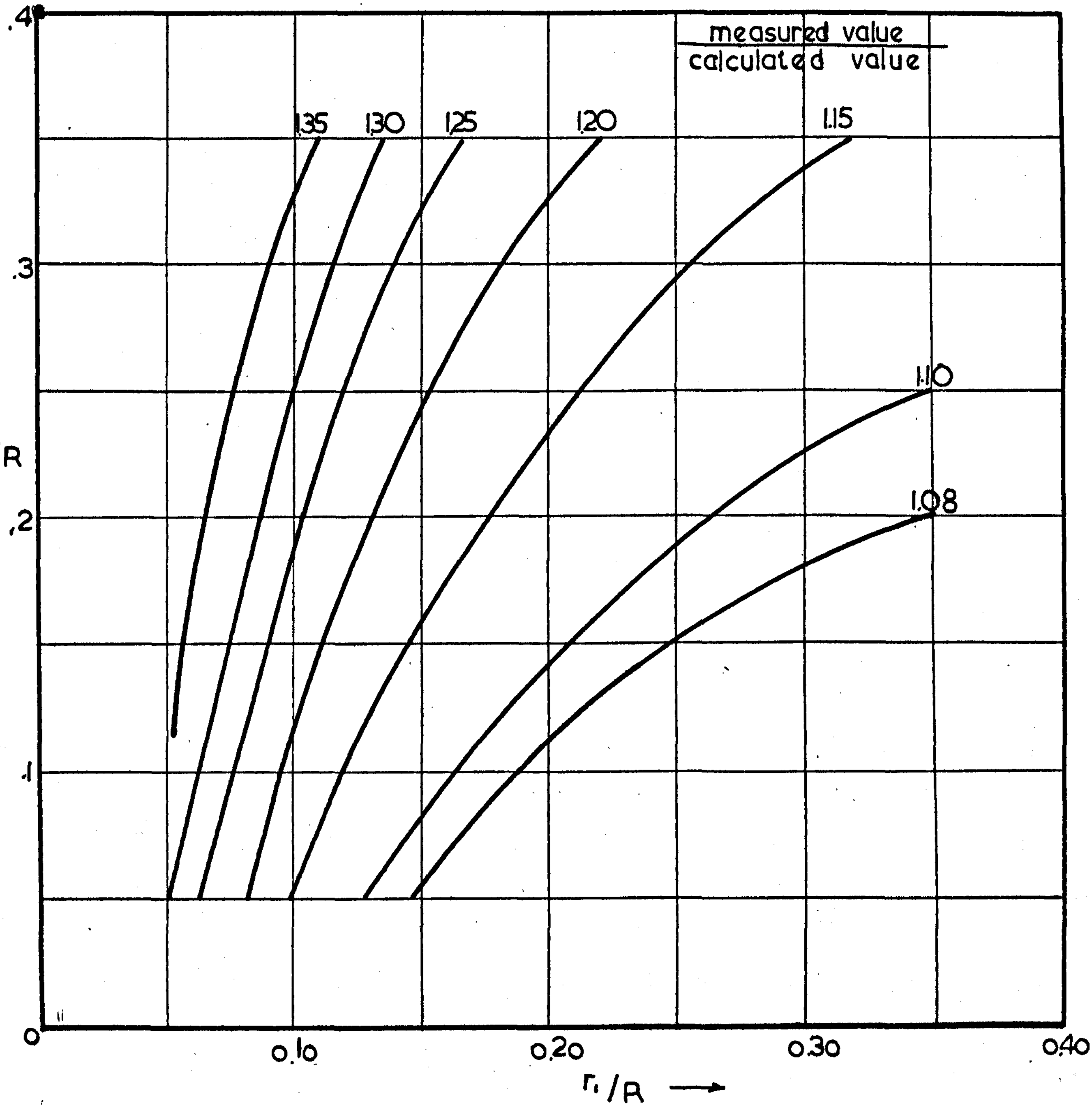


FIG 1.13. APPROXIMATE CORRECTION CHART FOR VOLTAGE GRDIENT ON CENTRAL CONDUCTOR

For exploration of the field close to the axis as needed for calculating the electric stress on the surfaces of the electrodes, a large-scale model is desirable in order to avoid the capillary rise of the electrolyte on the probe, which may cause considerable error in very shallow water.

The radial variation in potential in the mid-plane was plotted for different combinations of electrode radii, and the voltage gradients on the electrodes were deduced; the results are set forth in Figs.1.10 and 1.11. Fig. 1.10, showing the voltage gradient on the tore, is to be compared with Fig. 1.8.; Fig.1.11. showing that on the central conductor, with Fig. 1.9.

The relation between the two sets of curves is summed up in Figs. 1.12 and 1.13., in which the ratio of measured to calculated voltage gradient is plotted. On the tore, the ratio approaches unity for large values of r_2 (that is, for thick rings); but in general the measured value is less than the calculated. On the central conductor the reverse is true, the measured value exceeding the calculated, and again the ratio approaches unity for large values of r_2. These tendencies can be deduced from a general consideration of the field, the hour-glass shape of the central conductor tending to produce a reduced gradient on itself and an increased gradient on the surrounding ring as compared with a central conductor of cylindrical shape. It may therefore be said that the calculations confirm the general correctness of the measurements.

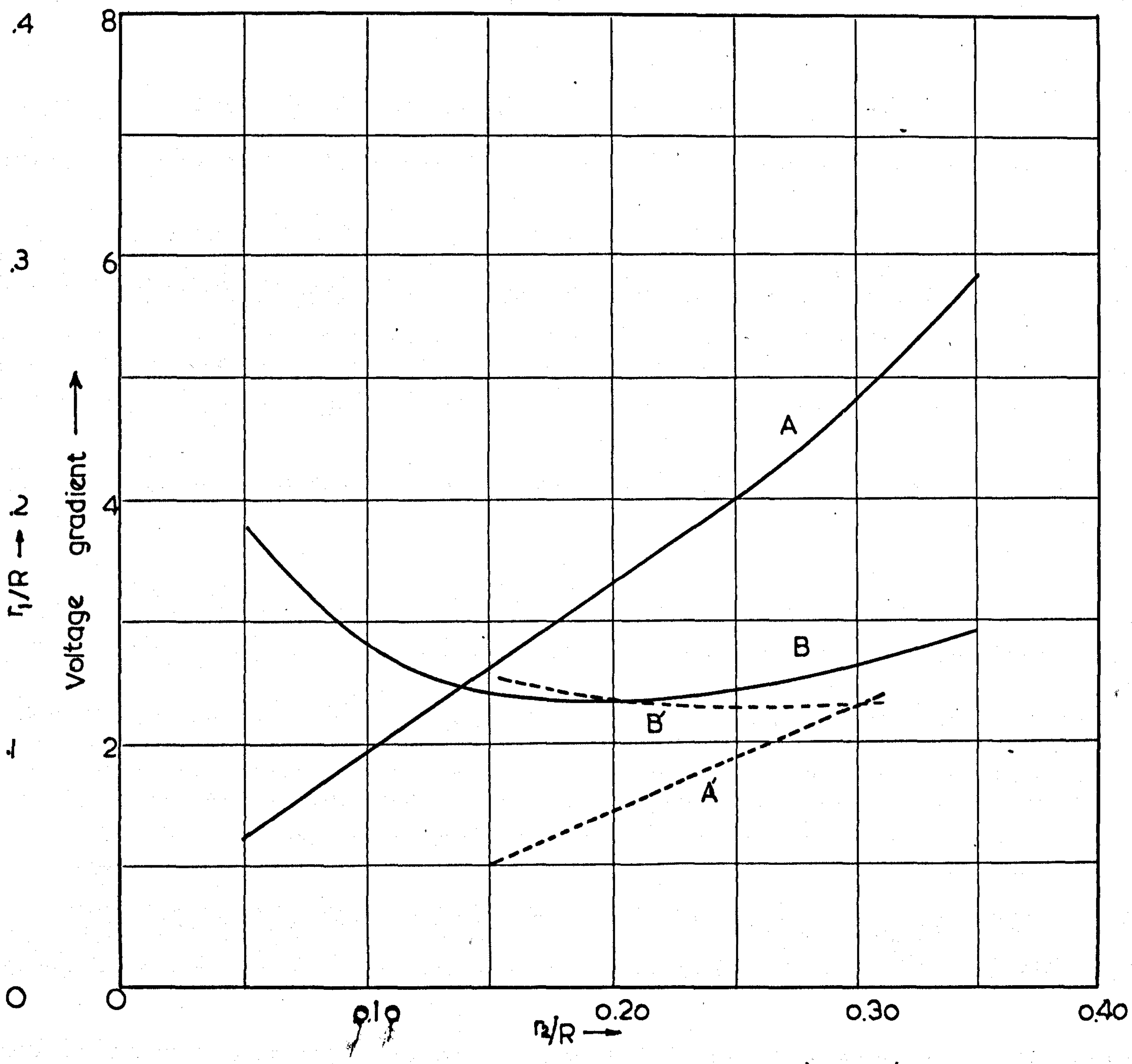


FIG. 1.14. A, A' RELATION BETWEEN r_1/R & r_2/R FOR EQUAL VOLTAGE GRADIENTS.

B', B VALUE OF THE GRADIENT ON EITHER ELECTRODE WHEN RELATION A OR A' IS SATISFIED.

A, B BY TOROIDAL FUNCTIONS.

A', B' BY ELECTROLYTIC TANK.

1.10. CONCLUSIONS.

The electric field in an electrode system consisting of a rod surrounded by a concentric ring of circular section is approximately calculated by replacing the rod by an hour-glass-shaped conductor of the same minimum radius. The resulting field is then calculated in terms of toroidal functions, numerical tables of which are given. In order to find how closely the calculated results correspond with the true values for a rod and a concentric ring, a systematic electrolytic tank study is undertaken and charts of the differences between theory and experiment are given.

For any given value of (r_1/R), there must be a value of (r_2/R) which will make the maximum voltage gradients on the two electrodes equal. This relation is given by curves A and A' in Fig. 1.14. , curve A referring to the calculated values of voltage gradient and curve A' to the measured values. The considerable discrepancy between these two curves is due to the fact that the measured voltage gradients on the two electrodes diverge from the calculated ones in opposite directions.

The relation given by these curves represents an optimum design condition. Assuming it to be satisfied, the voltage gradient on either electrode is given by curve B or B', curve B referring to the calculated values and curve B' to the measured ones. Either curve shows a minimum attainable gradient of about (2.3V/R); theory suggests that this will occur when r_1=0.20R, r_2=0.165R, while measurement corrects these values to r_1=0.27R, r_2=0.10R. So long as the relation between r_1 and r_2 is correctly maintained, their values can depart quite a long way from those cited without greatly increasing the voltage gradient.

PART 2

THE ELECTRIC FIELD BETWEEN TWO SPHERES

2.1. INTRODUCTION.

Sphere gaps are used extensively nowadays in high voltage engineering to measure impulse or surge voltages, and at normal frequency to calibrate other measuring devices. The problem of calculating the electric field in a sphere gap is a very old one and has intrigued some of the great minds in electrical theory. Kirchhoff, Kelvin, Alexander Russell and others have made contributions to it. The method has been that of approximation by successive images, and it appears that no general analytical method of solution has been developed as in the case of the analogous problems for the ellipsoid and anchor-ring. In this paper a general solution of Laplace's equation is obtained in a form suitable for problems in which the boundary conditions are given over two spherical surfaces and the electric field in a sphere gap is calculated.

2.2. DIPOLAR CO-ORDINATES.

The co-ordinates system known as ' Dipolar or Dispherical co-ordinates ' are defined by rotating about the z axis the system of circles, in any plane, through two fixed points on the axis and the orthogonal system of circles. Thus, if x,y,z, are the Cartesian co-ordinates and $\rho = (x^2+y^2)^{1/2}$, and the distance between the fixed points 2a, we have a system of orthogonal curvilinear co-ordinates, u,v,w, where

$$u+jv = \log_e \frac{\rho + j(z+a)}{\rho + j(z-a)}$$

$$w = \tan^{-1}(y/x) \qquad \text{-----------(2.2.1)}$$

The surfaces u=constant will then be a series of non-intersecting coaxial spheres having a common diametral plane u=0. It is obvious that the origin and the value of 'a' can be so chosen that any two given non-intersecting spheres will be included in the system. These co-ordinates are similar to those employed by Hicks in his memoir on ' Toroidal Functions ', the difference being that in the present case the circles are rotated about the line through the limiting points instead of about their common radical axis. Further, in one case the surface conditions are given over spheres, while in the other case they are given over 'tores' or 'anchor-rings'. It will be useful to set down here in a compact form, formulas relating to these functions, which will be required later on. Most of them are easily proved and are set down without proof.

$$\rho = \frac{a \sin v}{\cosh u - \cos v}$$

$$z = \frac{a \sinh u}{\cosh u - \cos v} \qquad \text{------------------(2.2.2)}$$

$$\text{or} \quad z + j\rho = ja \cdot \cot \tfrac{1}{2}(u + jv) \qquad \text{--------------(2.2.3)}$$

$$\left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left\{\left(\frac{\partial \rho}{\partial u}\right)^2 + \left(\frac{\partial \rho}{\partial v}\right)^2\right\}^{-1}$$

$$= \frac{(\cosh u - \cos v)^2}{a^2} \qquad \text{-------(2.2.4)}$$

If 'r' be the radius of any sphere of the system, and 'd' the distance of its centre from the origin,

$$r = a/|\sinh u|$$

$$d = a \cdot \coth u \qquad \text{----------------(2.2.5)}$$

The detailed account of the dipolar coordinates may be found in Jeffery's paper[1].

2.3. SOLUTION OF LAPLACE'S EQUATION, $\nabla^2\phi=0$.

It is well known that if u,v,w, be any system of the orthogonal curvilinear co-ordinates, Laplace's equation may be written in the form,

$$\nabla^2\phi = h_1h_2h_3\left\{\frac{\partial}{\partial u}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_2}{h_3h_1}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_3}{h_1h_2}\frac{\partial\phi}{\partial w}\right)\right\} = 0 \qquad \text{------------------(2.3.1)}$$

where

$$h_1^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$h_2^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2$$

$$h_3^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$

In the present case, u,v being conjugate functions of ρ and z,

$$h_1^2=\left(\frac{\partial u}{\partial \rho}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{\partial v}{\partial \rho}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2=h_2^2$$

and since

$$w= \tan^{-1}(y/x)$$

therefore,

$$h_3^2 = 1/\rho^2$$

Substituting these values into (2.3.1),

Laplace's equation then becomes,

$$\frac{\partial}{\partial u}\left(\rho\frac{\partial\phi}{\partial u}\right)+\frac{\partial}{\partial v}\left(\rho\frac{\partial\phi}{\partial v}\right)+\rho^{-1}\left\{\left(\frac{\partial u}{\partial \rho}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right\}^{-1}\frac{\partial^2\phi}{\partial\omega^2}=0 \qquad \text{------------------(2.3.2)}$$

Write $\phi=\Psi/\sqrt{\rho}$; we obtain,

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{\psi}{4\rho^2}\left\{\left(\frac{\partial\rho}{\partial u}\right)^2+\left(\frac{\partial\rho}{\partial v}\right)^2\right\}+\rho^{-2}\left\{\left(\frac{\partial u}{\partial \rho}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right\}^{-1}\frac{\partial^2\psi}{\partial\omega^2}=0 \qquad \text{------------------(2.3.3)}$$

thus,

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{1}{\sin^2 v}\left(\frac{\psi}{4}+\frac{\partial^2\psi}{\partial\omega^2}\right)=0 \qquad \text{-----------(2.3.4)}$$

Since the problem we are interested in is symmetrical about z axis,

i.e. $$\frac{\partial^2\psi}{\partial\omega^2}=0$$

equation (2.3.4) then becomes,

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{\psi}{4}\frac{1}{\sin^2 v}=0 \qquad \text{-----------(2.3.4a)}$$

After the usual manner we will seek a solution of the type $\psi=U\,V$, where U, V are functions of u, v respectively. Substituting this in (2.3.4a) we at once obtain,

$$-\frac{1}{U}\frac{d^2U}{du^2}=\frac{1}{V}\frac{d^2V}{dv^2}+\frac{1}{4\sin^2 v}=\text{constant}=-(n+\tfrac{1}{2})^2,\ \text{say}$$

Hence,

$$U = A\cosh(n+\tfrac{1}{2})u + B\sinh(n+\tfrac{1}{2})u$$

and

$$\frac{d^2V}{dv^2} + \left\{(n+\tfrac{1}{2})^2 + \frac{1}{4\sin^2 v}\right\} V = 0$$

Put

$V = (\sin v)^{\frac{1}{2}}\chi$, we have

$$\frac{d^2\chi}{dv^2} + \cot v\,\frac{d\chi}{dv} + (n^2+n)\chi = 0;$$

finally writing $\cos v = \mu$, we have

$$(1-\mu^2)\frac{d^2\chi}{d\mu^2} - 2\mu\frac{d\chi}{d\mu} + n(n+1)\chi = 0$$

The solution of which is well known to be

$$\chi = aP_n(\mu) + bQ_n(\mu)$$

$P_n(\mu)$ being the Legendre functions of order n, $Q_n(\mu)$ is the corresponding function of the ' second kind '. Hence,

$$V = (\sin v)^{\frac{1}{2}}\left\{aP_n(\mu) + bQ_n(\mu)\right\}$$

and a particular solution of equation (2.3.4a) may be expressed in the form,

$$\phi=(\cosh u-\cos v)^{\frac{1}{2}}\left\{A\cosh(n+\tfrac{1}{2})u+B\sinh(n+\tfrac{1}{2})u\right\}\left\{aP_n(\mu)+bQ_n(\mu)\right\}$$

------------------- (2.3.5)

We shall find that for physical applications it is sufficient to confine our attention to integral values of n. Moreover, the solution corresponding to $n=-(m+1)$ is identical in form with that corresponding to $n=+m$. It will, therefore, be sufficient to consider only positive integral values of 'n' and we may write the general solution for the potential.

$$(\cosh u-\cos v)^{\frac{1}{2}}\sum_{n=0}^{\infty}\left\{A_n\cosh(n+\tfrac{1}{2})u+B_n\sinh(n+\tfrac{1}{2})u\right\}\left\{a_nP_n(\mu)+b_nQ_n(\mu)\right\}$$

------------------- (2.3.5a)

The function ϕ and its first differential coefficients must be finite and continuous at all points of the field except those which correspond to some special physical conditions such as a source or a charge. $P_n(\mu)$ is finite and continuous for all real values of v, but $Q_n(\mu)$ becomes infinite when $v = 0$ or π, hence cannot occur in the expression for ϕ which is to hold throughout a region including any point of axis of z. It is to such cases that we confine our attention in the present paper, so that

$$\phi=(\cosh u-\cos v)^{1/2}\sum_{n=0}^{\infty}\left\{A_n\cosh(n+\tfrac{1}{2})u + B_n\sinh(n+\tfrac{1}{2})u\right\} P_n(\mu) \tag{2.3.6}$$

2.4. SOLUTION OF THE PROBLEM OF THE SPHERE GAP.

Equation (2.3.6) shows that the potential of the field between two charged spheres is given by

$$\phi = (\cosh u - \cos v)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left\{ A_n \cosh(n+\tfrac{1}{2})u + B_n \sinh(n+\tfrac{1}{2})u \right\} P_n(\mu)$$

Let u_1, u_2 be any two spheres such that $u_1 > 0$ but u_2 is unrestricted. The potential ϕ is constant over each of these spheres and we can without loss of generality suppose it to be zero over the surface u_2 and V over surface u_1. It is obvious that ϕ will be of the form

$$\phi = (\cosh u - \cos v)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left\{ A_n \sinh(n+\tfrac{1}{2})(u-u_2) \right\} P_n(\cos v) \qquad \text{------------------- (2.411)}$$

from which we have

$$V(\cosh u_1 - \cos v)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A_n \sinh(n+\tfrac{1}{2})(u_1-u_2) \times P_n(\cos v) \qquad \text{------------------- (2.4.2)}$$

The left-hand side of the above equation may be written

$$\sqrt{2}\, V\, e^{-\frac{1}{2}u_1} (1 - 2e^{-u_1}\cos v + e^{-2u_1})^{-\frac{1}{2}}$$

$$= \sqrt{2}\, V \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})u_1} P_n(\cos v) \qquad \text{------- (2.4.3)}$$

since $u_1 > 0$

Equating the coefficients of $P_n(\cos v)$, we obtain the value of A_n and the potential function then becomes,

$$\phi = \sqrt{2}\ V(\cosh u - \cos v)^{1/2} \sum_{n=0}^{\infty} \frac{\sinh(n+\frac{1}{2})(u-u_2)}{\sinh(n+\frac{1}{2})(u_1-u_2)}\ e^{-(n+\frac{1}{2})u_1}\, P_n(\mu)$$

-------------------- (2.4.4)

We are actually interested in the voltage gradient of the field, especially the voltage gradient on the surface of two equal spheres (i.e. $u=+u_1$ and $u=-u_1$). Since it is symmetrical about the plane $u=0$, the voltage gradient will be obtained by differentiating equation (2.4.4) and substituting $u_2=-u_1$.

The potential gradient $\dfrac{\partial\phi}{\partial s_n} = \dfrac{\cosh u - \cos v}{a}\dfrac{\partial\phi}{\partial u}$

---------------------(2.4.5)

The breakdown usually occurs at the points where the gradient is maximum, i.e. on the line v= .
Thus,

$$\frac{\partial\phi}{\partial s_n} = \frac{2}{a} V(1+\cosh u)\sum_{n=0}^{\infty}(-1)^n \frac{e^{-(n+\frac{1}{2})u_1}}{\sinh(n+\frac{1}{2})u_1}\left\{\frac{n+1}{2}\cosh(n+1)u_1 + \frac{n}{2}\cosh nu\right\}$$

-------------------- (2.4.6)

The voltage gradient at $u=u_1$ is

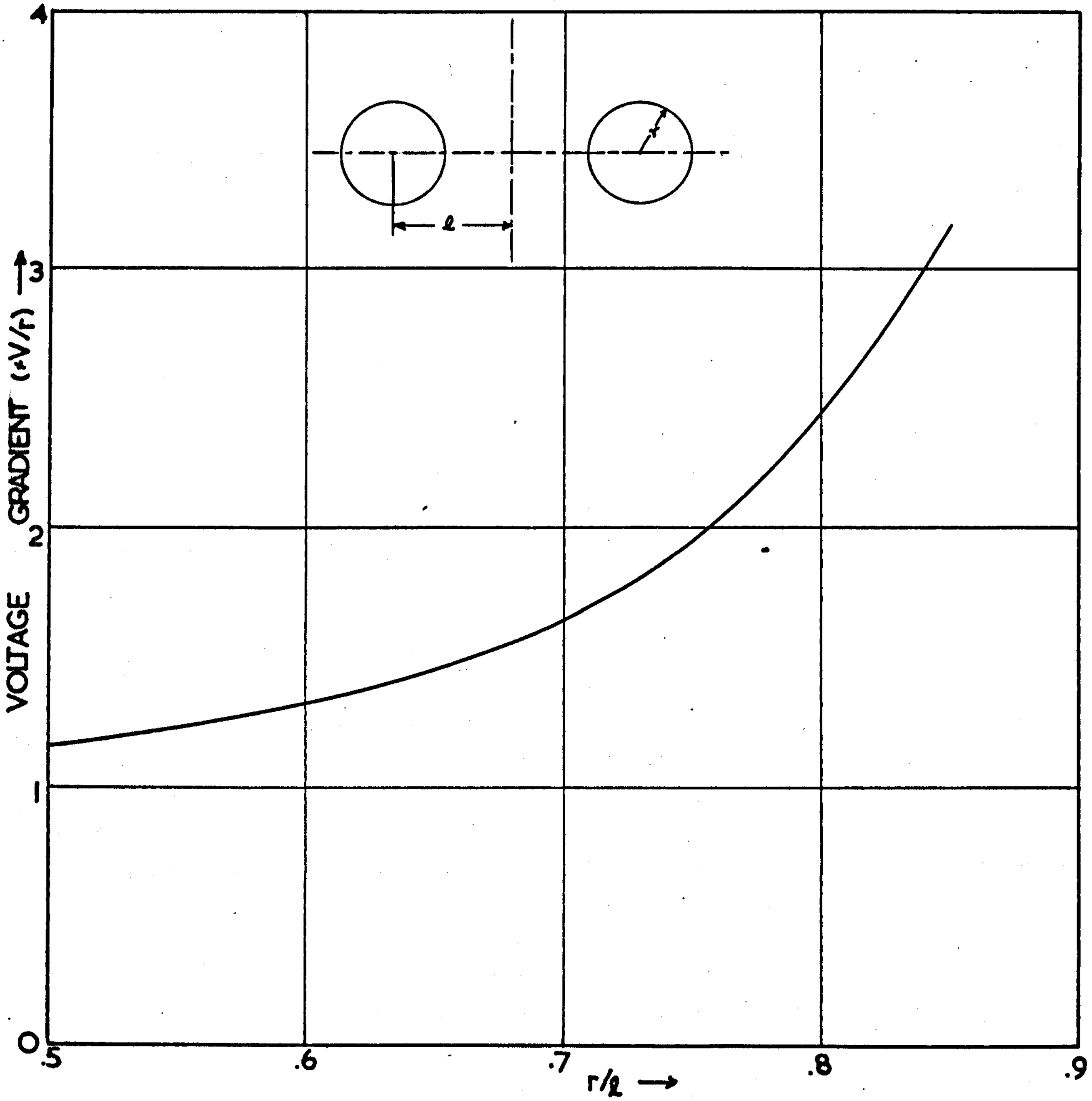


FIG. 2.1. VOLTAGE GRADIENT ON SPHERE

BY DIPOLAR COORDINATES

$$E_{u_1} = \frac{4}{a} V \cdot \cosh^2 \frac{u_1}{2} \sum_{n=0}^{\infty} (-1)^n e^{-(n+\frac{1}{2})u_1} \left\{ \frac{1}{2} \sinh \frac{u_1}{2} + (n+\tfrac{1}{2}) \cosh \frac{u_1}{2} \coth \left(\frac{2n+1}{2}\right) u_1 \right\}$$

-----------------------(2.4.7)

The voltage gradient given by equation (2.4.7) is plotted in Fig.(2.1). Instead of using u_1 as independent variable the dimension ratio r/ℓ is employed. The relations between r, ℓ, and a, u, are

$$a^2 = (\ell^2 + r^2)$$

$$\ell = r \cdot \cosh u_1$$

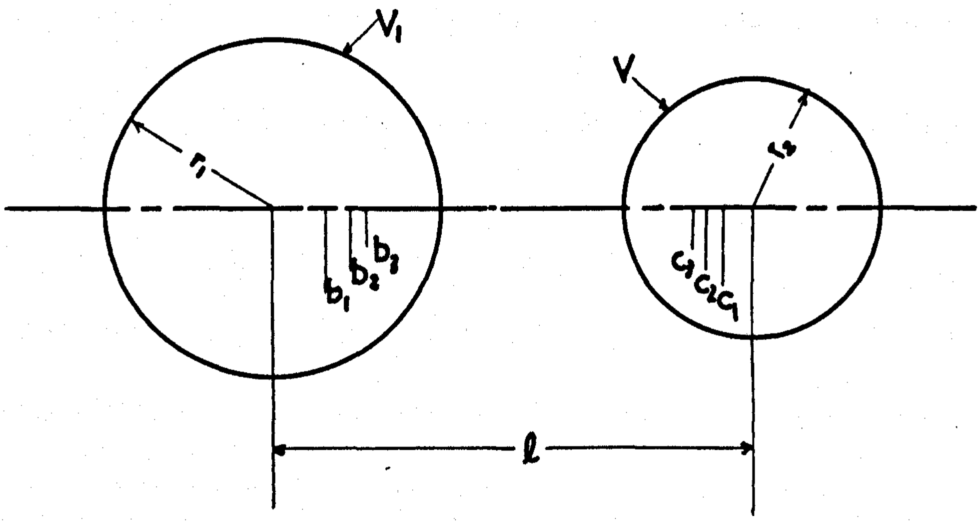


FIG. 2.2.

2.5. METHOD OF IMAGES.

It is not necessary to give more than a brief outline of the method of images, since the details are always given in general textbooks of electromagnetism.

It is well known that a sphere of radius r, distance ℓ from a charge q_1 may be made a zero equipotential surface by placing with it a charge $q_2=-q_1 \cdot r/\ell$ at a distance $c=r^2/\ell$ from the centre of r. The point at a is called an ' inverse point ' and the charge q_2 an ' inverse charge '.

We are now to determine a system of charges such as will preserve two spheres of radii r_1 and r_2 with a distance ℓ between centres, at potential V_1 and V_2 respectively, (Fig. 2.2). We shall employ the methods of superposition and successive approximations in four stages as follows:-

(a) Let r_2 be an isolated sphere at potential V_2. Its charge may then be considered as concentrated at its centre and equal to $q_2 = V_2 r_2$.

(b) Now bring up the sphere r_1 of zero potential. To hold it at zero potential in the presence of q_2 we must place a charge $q_{b1} = -q_2 r_1/\ell$ $= -V_2 r_1 r_2/\ell$ at a distance $b_1 = r_1^2/\ell$. But the pre-

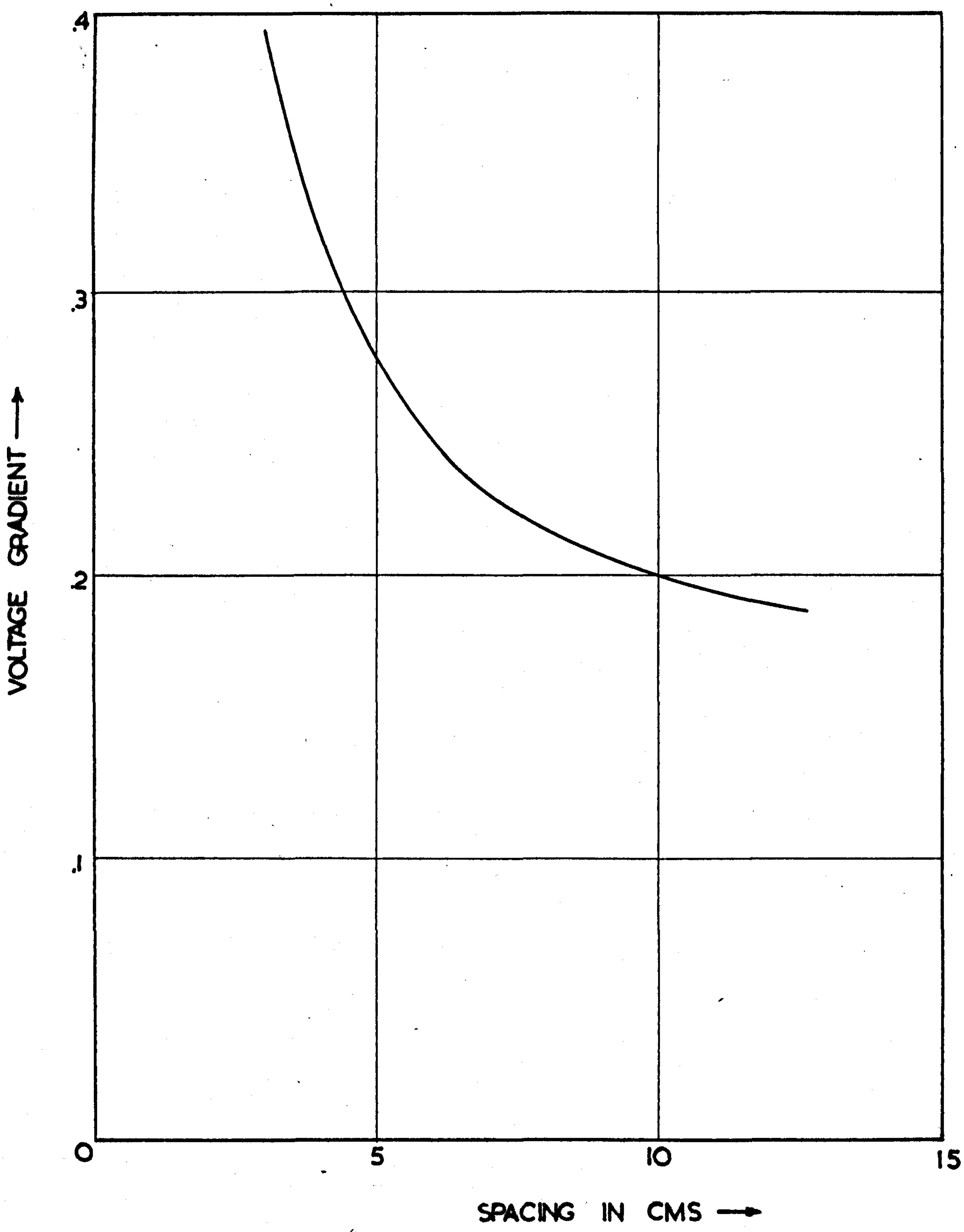


FIG. 2.3. VOLTAGE GRADIENT ON SPHERE

sence of this charge would change the potential of r_2, so we must place a charge $q_{c1} = -q_{b1} \cdot r_2 / (\ell - b_1) = V_2 r_2^2 r_1 / \ell(\ell - b_1)$ at $c_1 = r_2^2 / (\ell - b_1)$ to return r_2 to its proper potential V_2. But this in turn necessitates a charge q_{b2} at b_2, and so on. The process is continued each new charge becoming smaller than the one before it, until the effect of additional charge becomes quite negligible. We thus arrive at a series of images which maintain r_2 at V_2 and r_1 at $V_1=0$.

(c) In exactly the same way a series of charges can be found which will maintain r_2 at $V_2=0$ and r_1 at V_1.

(d) Finally, by superposition of the two systems of charges arrived at in (b) and (c) above, we obtain the necessary charges to hold r_2 at V_2 and r_1 at V_1.

The potential gradients for a particular system of two spheres having equal radii of 6.25cm. are plotted in(Fig.2.3).

The new method derived can be checked against the method of images simply by substituting the dimension ratio into (2.4.7). It was found with satisfaction that the two values agreed.

2.6. CONCLUSIONS.

The method derived in this paper for the calculation of voltage gradient of the electric field in a sphere gap involves more advanced mathematics than is used in successive images, but the time and labour saving is great.

In practice, we are chiefly interested in the breakdown voltage of the sphere gap whose tables of numerical value for different radii and spacings are calculated experimentally and given in the British Standard No.358 of 1939. Unfortunately, as the electric stress is raised, the behaviour of the dielectric in the sphere gap becomes very complicated and changes, depending on a number of factors such as surface of spheres, conditioning of spheres, correction for air density etc.. Until the behaviour of the dielectric before breakdown is thoroughly known, the theoretical method derived in this paper gives only a rough estimation of the breakdown voltage. However, it has been shown that the new method is far more powerful in attacking this kind of problem than the method of images.

PART 3

THE EXACT CALCULATION OF THE SELF-INDUCTANCE OF A CIRCULAR TURN OF WIRE BY MEANS OF TOROIDAL FUNCTIONS

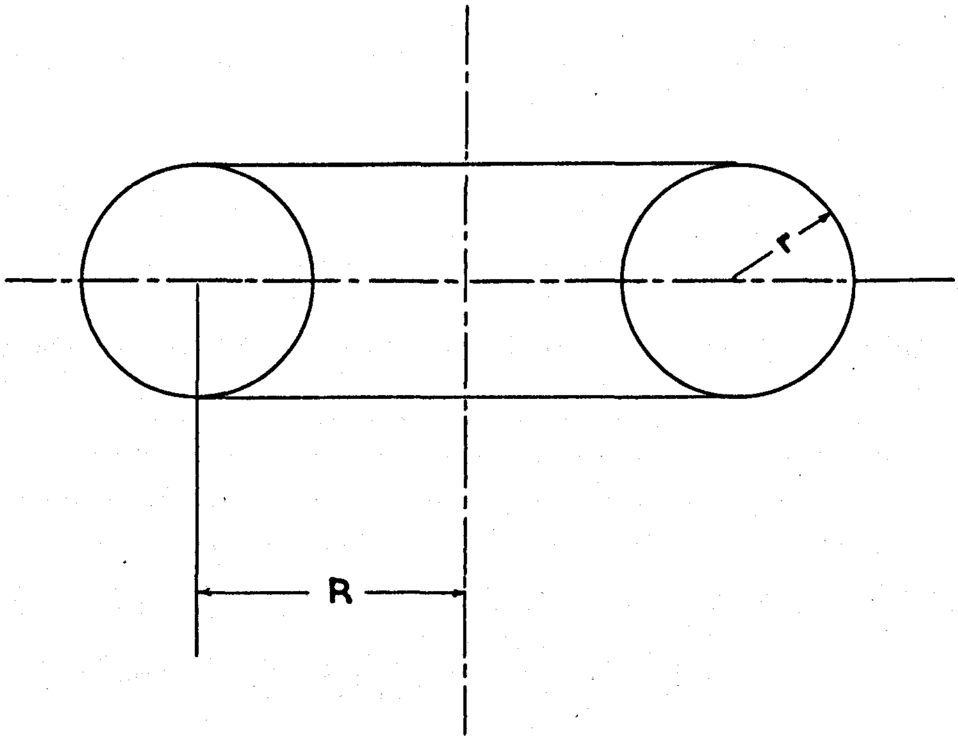


FIG. 3.1.

3.1. INTRODUCTION.

The calculation of the self-inductance of a circular turn of wire (see Fig. 3.1) has a number of applications in the design of inductors and is a problem of importance in electrical engineering and especially in the field of communication. Although this quantity has been approximately calculated by various investigators[1] [2] [3] whose formulae are to be found in various electrical engineering handbooks and notably in the publications of the National Bureau of Standards, the exact calculation which is the object of the present investigation cannot be found. These approximate formulae are only valid when the dimension ratio r/R is very small (that is, for thin wire).

In this investigation, the toroidal coordinates are so chosen that the form of the wire is truly represented by the surface $u = u_0$, and it is also assumed that the current flows in the wire in such a way that no magnetic flux cuts the surface of the wire. The assumed condition is, in fact, approached with very high-frequency currents. The functions necessary for the solution of this problem are similar to those used in part 1; in fact they are the first derivatives of the toroidal functions $p_n(u)$, $q_n(u)$. A numerical table of these functions has been computed and included in the thesis. An accuracy of a part in a thousand is aimed at in general, but better precision is obtainable over most of the table.

3.2. VECTOR POTENTIAL.

It is not necessary to give more than a brief outline of the theory of the vector potential, since the details are to be found in general textbooks of electromagnetism.

We know that the vector field of current density is derived from the vector field of magnetic force by the process of taking its curl; this fact is represented by the vector equation

$$\mathrm{curl}\,\underline{H} = \underline{J} \qquad \text{------------------} \quad (3.2.1)$$

In the same way, we may take a vector $\underline{A}$ such that we write for the magnetic induction in vector form.

$$\mathrm{curl}\,\underline{A} = \underline{B} \qquad \text{------------------} \quad (3.2.2)$$

Then '$\underline{A}$' is called the 'vector potential' due to the current density $\underline{J}$; its mathematical and physical properties are easily explained. In words, this equation means that the magnetic induction is to be found from the vector potential by taking the curl; i.e. by finding the direction of the axis around which the line integral of the vector potential is greatest and the amount of the line integral; the first is the direction of the magnetic induction, the second is proportional to its magnitude.

The components of curl$\underline{A}$ expressed in toroidal coordinates are

$$\frac{1}{\delta s_v\ \delta s_w}\left\{\delta s_v \frac{\partial}{\partial s_v}(A_w\delta s_w)-\delta s_w\frac{\partial}{\partial s_w}(A_v\delta s_v)\right\}=\frac{1}{\lambda\rho}\left\{\frac{\partial}{\partial v}(\rho A_w)-\frac{\partial}{\partial w}(\lambda A_v)\right\}$$

$$\frac{1}{\delta s_u\ \delta s_w}\left\{\delta s_w \frac{\partial}{\partial s_w}(A_u\cdot\delta s_u)-\delta s_u\frac{\partial}{\partial s_u}(A_w\delta s_w)\right\}=\frac{1}{\lambda\rho}\left\{\frac{\partial}{\partial w}(\lambda A_u)-\frac{\partial}{\partial u}(\rho A_w)\right\}$$

$$\frac{1}{\delta s_u\cdot\delta s_v}\left\{\delta s_u \frac{\partial}{\partial s_u}(A_v\cdot\delta s_v)-\delta s_v\cdot\frac{\partial}{\partial s_v}(A_u\delta s_u)\right\}=\frac{1}{\lambda^2}\left\{\frac{\partial}{\partial u}(\lambda A_v)-\frac{\partial}{\partial v}(\lambda A_u)\right\}$$

------------------ (3.2.3)

We are chiefly interested in cases where the vector potential $\underline{A}$ has only a w-component i.e. $A=(0,0,A_W)$; so A_W is henceforth called A; and

$$\text{curlA} = \frac{1}{\lambda\rho}\frac{\partial}{\partial v}(\rho A), \quad -\frac{1}{\lambda\rho}\frac{\partial}{\partial u}(\rho A), \quad 0$$

------------------ (3.2.4)

If the current flows in the surface of the wire (along circumferential paths) with the current density so distributed that $u = u_o$ is a line of force: that is to say that there is no line of force penetrating the surface: then

$$B_u = 0 \quad \text{on surface}$$

or

$$(\text{curl}\underline{A})_u = 0 \qquad \text{---------------} \quad (3.2.5)$$

This gives

$$\frac{\partial}{\partial u}(\rho A) = 0$$

or

$$\rho A = \text{constant} \qquad \text{---------------} \quad (3.2.6)$$

In space, curl$\underline{B}$ = 0, this gives

$$\text{curl curl}\underline{A} = 0 \qquad \text{---------------} \quad (3.2.7)$$

$\underline{A}$ is presumed to be a function of u and v only; so the equation (3.2.7) becomes

$$\text{curl curl}\underline{A} = \frac{1}{\lambda^2}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\rho}\frac{\partial}{\partial u}(\rho A)\right] + \frac{\partial}{\partial v}\left[\frac{1}{\rho}\frac{\partial}{\partial v}(\rho A)\right]\right\}$$

$$= \frac{1}{\lambda^2}\left\{\frac{\partial}{\partial u}\left[\frac{\partial A}{\partial u} + \frac{A}{\rho}\frac{\partial \rho}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{\partial A}{\partial v} + \frac{A}{\rho}\frac{\partial \rho}{\partial v}\right]\right\}$$

$$= \frac{1}{\lambda^2}\left\{\frac{\partial^2 A}{\partial u^2} + \frac{\partial^2 A}{\partial v^2} + \frac{1}{\rho}\left[\frac{\partial \rho}{\partial u}\frac{\partial A}{\partial u} + \frac{\partial \rho}{\partial v}\cdot\frac{\partial A}{\partial v}\right] + \frac{A}{\rho}\left(\frac{\partial^2 \rho}{\partial u^2} + \frac{\partial^2 \rho}{\partial v^2}\right) - \frac{A}{\rho^2}\left[\left(\frac{\partial \rho}{\partial u}\right)^2 + \left(\frac{\partial \rho}{\partial v}\right)^2\right]\right\}$$

$$= 0 \qquad \text{-----------------} \quad (3.2.8)$$

Since ρ, z are conjugate functions of u, v,

$$\frac{\partial^2 \rho}{\partial u^2} + \frac{\partial^2 \rho}{\partial v^2} = 0$$

and

$$\left(\frac{\partial \rho}{\partial u}\right)^2 + \left(\frac{\partial \rho}{\partial v}\right)^2 = \lambda^2$$

Therefore the equation (3.2.7) becomes

$$\frac{1}{\lambda^2}\left\{\frac{\partial^2 A}{\partial u^2}+\frac{\partial^2 A}{\partial v^2}+\frac{1}{\rho}\left(\frac{\partial \rho}{\partial u}\frac{\partial A}{\partial u}+\frac{\partial \rho}{\partial v}\frac{\partial A}{\partial v}\right)-\frac{A}{\rho^2}\lambda^2\right\}=0$$

------------------ (3.2.9)

Laplace's equation in toroidal coordinates is

$$\frac{1}{\lambda^2}\left\{\frac{\partial^2 \phi}{\partial u^2}+\frac{\partial^2 \phi}{\partial v^2}+\frac{1}{\rho}\left(\frac{\partial \rho}{\partial u}\frac{\partial \phi}{\partial u}+\frac{\partial \rho}{\partial v}\frac{\partial \phi}{\partial v}\right)\right\}=0$$

------------------ (3.2.10)

where ϕ is independent of w.

Thus the equation (3.2.9) obeyed by $\underline{A}$ is

$$\nabla^2 \underline{A} = \frac{A}{\rho^2}$$ ---------------- (3.2.11)

This is known as 'Poisson's equation'.

3.3. THE GENERAL SOLUTIONS OF THE EQUATION curl curl A = 0.

We have already obtained that when the vector potential A has only a w-component, the equation curl curl A = 0 becomes

$$\frac{1}{\lambda^2}\left\{\frac{\partial^2 A}{\partial u^2}+\frac{\partial^2 A}{\partial v^2}+\frac{1}{\rho}\left(\frac{\partial \rho}{\partial u}\frac{\partial A}{\partial u}+\frac{\partial \rho}{\partial v}\frac{\partial A}{\partial v}\right)-\frac{A}{\rho^2}\lambda^2\right\}=0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.2.9)$$

The substitution,

$$A = \psi(\cosh u - \cos v)^{1/2} \quad \ldots\ldots\ldots \quad (3.3.1)$$

$$\frac{\partial A}{\partial u}=\rho^{-\frac{1}{2}}\frac{\partial \psi}{\partial u}-\frac{1}{2}\rho^{-\frac{3}{2}}\psi\frac{\partial \rho}{\partial u}$$

and

$$\frac{\partial^2 A}{\partial u^2}=\rho^{-\frac{1}{2}}\frac{\partial^2 \psi}{\partial u^2}-\rho^{-\frac{3}{2}}\frac{\partial \rho}{\partial u}\frac{\partial \psi}{\partial u}+\frac{3}{4}\rho^{-\frac{5}{2}}\psi\left(\frac{\partial \rho}{\partial u}\right)^2-\frac{1}{2}\rho^{-\frac{3}{2}}\psi\frac{\partial^2 \rho}{\partial u^2}$$

will reduce equation (3.2.9) to

$$\frac{\partial^2 \psi}{\partial u^2}+\frac{\partial^2 \psi}{\partial v^2}-\frac{3}{4}\frac{\psi}{\rho^2}\lambda^2=0 \quad \ldots\ldots\ldots \quad (3.3.2)$$

Substituting $\rho=\frac{a\sinh u}{\cosh u-\cos v}$ and $\lambda=\frac{a}{\cosh u-\cos v}$ into equation (3.3.2), we then get

$$\frac{\partial^2 \psi}{\partial u^2}+\frac{\partial^2 \psi}{\partial v^2}-\frac{3}{4}\frac{1}{\sinh^2 u}\psi=0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.3.3)$$

This equation is solved in the usual manner by assuming $\psi = U\,V$, where U is a function of u only and V a function of v only. By this substitution, equation (3.3.3) is then transformed into

$$\frac{1}{U}\left(\frac{d^2U}{du^2} + \frac{3}{4}\frac{U}{\sinh^2 u}\right) + \frac{1}{V}\frac{d^2V}{dv^2} = 0$$

Since the first term cannot contain v and the second term cannot contain u, both must be constant, and are written + n^2 and $-n^2$ respectively. Thus the separate differential equations for U and V are

$$\frac{d^2U}{du^2} - \frac{3}{4}(\sinh u)^{-2}\,U = n^2 U \qquad \text{------------ (3.3.4)}$$

and

$$\frac{d^2V}{dv^2} = -n^2 V \qquad \text{------------ (3.3.5)}$$

The independent solutions of the equation (3.3.5) are cos nv and sin nv, and it is known from the theory of Fourier series that a sum of terms in which n takes all integral values from zero to infinity will be capable of describing any distribution of current on the surface of a tore. By direct substitution, it may be proved that the independent solutions of equation (3.3.4) are $p_n'(u)$ and $q_n'(u)$. (The proof is to be found in Appendix 4) $p_n'(u)$ and $q_n'(u)$ are the first derivatives of the toroidal functions $p_n(u)$ and $q_n(u)$ respectively.

$p_n'(u)$ and $q_n'(u)$ can be expressed in terms of $p_n(u)$ and $q_n(u)$; the necessary formulae are given in (1.6.2). Using these, the following table of values for the first four functions of each kind has been worked out. There are differences between these figures and those contained in Fouquet's table, which contains a number of errors.

The general solution of the equation (3.3.3) is the product of the solutions of the equations (3.3.4) and (3.3.5), and may be written

$$\psi = \sum_{n=0}^{\infty} \left\{ a_n P_n'(u) + b_n Q_n'(u) \right\} \cos(nv + \alpha_n) \qquad (3.3.6)$$

where a_n, b_n, α_n are arbitrary constants. When the central plane is a plane of symmetry (as here), $\alpha_n = 0$; therefore, the vector potential of the field, set up by the current flowing circumferentially in such a way that the surface of the tore is a line of force, is given by

$$A = \sqrt{(\cosh u - \cos v)} \sum_{n=0}^{\infty} \left\{ a_n P_n'(u) + b_n Q_n'(u) \right\} \cos nv \qquad (3.3.7)$$

TABLE 3

TOROIDAL FUNCTIONS

u	$p_0'(u)$	$p_1'(u)$	$p_2'(u)$	$p_3'(u)$
0.0	0	0	0	0
.1	(-2)-1.246	(-2)3.747	(-1)1.881	(-1)4.417
.2	-2.486	7.502	3.809	9.107
.3	-3.713	(-1)1.126	5.822	*1.435
.4	-4.909	1.502	7.970	2.047
0.5	-6.077	1.880	*1.030	2.785
.6	-7.201	2.259	1.287	3.693
.7	-8.281	2.640	1.574	4.829
.8	-9.310	3.023	1.897	6.262
.9	(-1)-1.028	3.410	2.264	8.082
1.0	-1.120	3.801	2.682	(1)1.040
.1	-1.205	4.196	3.161	1.337
.2	-1.283	4.596	3.713	1.716
.3	-1.355	5.003	4.350	2.203
.4	-1.420	5.416	5.086	2.827
1.5	-1.479	5.838	5.937	3.627
.6	-1.531	6.270	6.924	4.655
.7	-1.577	6.712	8.067	5.972
.8	-1.616	7.166	9.393	7.664
.9	-1.650	7.634	(1)1.093	9.836
2.0	-1.678	8.117	1.272	(2)1.262

TABLE 3 (CONTINUED) ---------- TOROIDAL FUNCTIONS

u	$p'_0(u)$	$p'_1(u)$	$p'_2(u)$	$p'_3(u)$
2.1	(-1)-1.700	(-1)8.616	(1)1.479	(2)1.620
.2	-1.717	9.132	1.720	2.080
.3	-1.729	9.669	2.000	2.670
.4	-1.736	*1.022	2.325	3.420
2.5	-1.739	1.080	2.702	4.401
.6	-1.738	1.140	3.141	5.650
.7	-1.734	1.203	3.650	7.253
.8	-1.726	1.269	4.241	9.313
.9	-1.715	1.338	4.928	(3)1.196
3.0	-1.701	1.410	5.727	1.535
.1	-1.685	1.485	6.655	1.971
.2	-1.666	1.563	7.732	2.531
.3	-1.646	1.645	8.984	3.249
.4	-1.623	1.731	(2)1.044	4.172
3.5	-1.599	1.822	1.213	5.357
.6	-1.573	1.917	1.409	6.879
.7	-1.546	2.017	1.637	8.833
.8	-1.518	2.122	1.902	(4)1.134
.9	-1.489	2.232	2.210	1.456
4.0	-1.460	2.347	2.568	1.870

TABLE 4

TOROIDAL FUNCTIONS

u	$q_0'(u)$	$q_1'(u)$	$q_2'(u)$	$q_3'(u)$
0.0	0	0	0	0
.1	(1) -1.004	-9.876	-9.566	-9.178
.2	*5.071	-4.802	-4.381	-3.915
.3	-3.424	-3.082	-2.615	-2.151
.4	-2.606	-2.208	-1.728	-1.302
0.5	-2.118	-1.676	-1.204	(-1)-8.278
.6	-1.793	-1.317	(-1)-8.639	-5.398
.7	-1.561	-1.060	-6.630	-3.621
.8	-1.387	(-1)-8.662	-4.734	-2.447
.9	-1.250	-7.163	-3.561	-1.670
1.0	-1.138	-5.976	-2.699	-1.148
.1	-1.048	-5.017	-2.058	(-2)-7.935
.2	(-1)-9.697	-4.234	-1.575	-5.499
.3	-9.023	-3.587	-1.210	-3.829
.4	-8.432	-3.049	(-2)-9.323	-2.671
1.5	-7.906	-2.597	-7.198	-1.868
.6	-7.433	-2.217	-5.566	-1.308
.7	-7.004	-1.895	-4.310	(-3)-9.165
.8	-6.611	-1.622	-3.341	-6.431
.9	-6.249	-1.390	-2.592	-4.516
2.0	-5.913	-1.192	-2.012	-3.174

TABLE 4 (CONTINUED) ----------TOROIDAL FUNCTIONS

u	$q_0'(u)$	$q_1'(u)$	$q_2'(u)$	$q_3'(u)$
2.1	(-1)-5.601	(-1)-1.023	(-2)-1.563	(-3)-2.231
.2	-5.310	(-2)-8.785	-1.215	-1.569
.3	-5.037	-7.546	(-3)-9.445	-1.104
.4	-4.780	-6.485	-7.346	(-4)-7.771
2.5	-4.539	-5.574	-5.715	-5.470
.6	-4.311	-4.792	-4.446	-3.852
.7	-4.095	-4.121	-3.460	-2.712
.8	-3.892	-3.545	-2.693	-1.910
.9	-3.699	-3.049	-2.096	-1.345
3.0	-3.516	-2.623	-1.632	(-5)-9.478
.1	-3.342	-2.257	-1.271	-6.677
.2	-3.178	-1.942	(-4)-9.892	-4.704
.3	-3.022	-1.671	-7.702	-3.314
.4	-2.874	-1.438	-5.998	-2.335
3.5	-2.733	-1.238	-4.670	-1.645
.6	-2.599	-1.065	-3.637	-1.159
.7	-2.472	(-3)-9.164	-2.832	(-6)-8.169
.8	-2.351	-7.887	-2.205	-5.756
.9	-2.236	-6.788	-1.717	-4.056
4.0	-2.127	-5.842	-1.337	-2.858

3.4. DETERMINATION OF THE COEFFICIENT "a_n".

We have already obtained

$$A=\sqrt{(\cosh u-\cos v)}\sum_{n=0}^{\infty}\left\{a_n p_n'(u)+b_n q_n'(u)\right\}\cos nv$$

—————————— (3.3.7)

As the vector potential contains the central axis, i.e. $u = 0$, the term $b_n q_n'(u)$ must disappear. The equation (3.3.7) then becomes

$$A=\sqrt{(\cosh u-\cos v)}\sum_{n=0}^{\infty} a_n p_n'(u)\cos nv$$

—————————— (3.4.1)

We have assumed that the u-component of the magnetic flux B on the surface of the wire is zero, thus

$$\begin{aligned} B_u &= (\text{curl } A)_u \\ &= \frac{1}{\lambda\rho}\frac{\partial}{\partial v}(\rho A) \\ &= 0 \end{aligned}$$

or $\left(\dfrac{a \sinh u}{\cosh u - \cos v} A\right) =$ constant, say c.

—————————— (3.4.2)

On the tore $u = u_0$, the vector potential

$$A=\sqrt{(\cosh u_0-\cos v)}\sum_{n=0}^{\infty} a_n p_n'(u_0)\cos nv$$

—————————— (3.4.3)

Substituting (3.4.3) into (3.4.2), we get

$$\frac{a\sinh u_0}{\sqrt{(\cosh u_0-\cos v)}}\sum_{n=0}^{\infty} a_n p_n'(u_0)\cos nv = c$$

This gives

$$\sum_{n=0}^{\infty} a_n p_n'(u_o)\cos nv = \frac{c(\cosh u_o - \cos v)^{1/2}}{a \sinh u_o} \cos nv\, dv$$

$$\text{------------------} \quad (3.4.4)$$

Multiplying both sides by cosnv and integrating from 0 to π, we get

$$\delta_n \pi a_n p_n'(u_o) = \int_0^{\pi} \frac{c(\cosh u_o - \cos v)}{a \sinh u_o} \cos nv\, dv$$

$$\text{------------------} \quad (3.4.5)$$

where $\delta_n = \frac{1}{2}$ when $n \geq 1$, but $\delta_o =$ [redacted].

Hence

$$a_n = \frac{c}{\delta_n \pi p_n'(u_o) a \sinh u_o} \int_0^{\pi} (\cosh u_o - \cos v)^{1/2} \cos nv\, dv$$

$$\text{------------------} \quad (3.4.6)$$

But

$$\int_0^{\pi} (\cosh u_o - \cos v)^{1/2} \cos nv\, dv$$

$$= \int_0^{\pi} \frac{(\cosh u_o - \cos v)}{(\cosh u_o - \cos v)} \cos nv\, dv \quad \text{-----} \quad (3.4.7)$$

$$= \int_0^{\pi} \frac{(\cosh u_o \cos nv)dv}{(\cosh u_o - \cos v)^{1/2}} - \int_0^{\pi} \frac{\cos v \cos nv\, dv}{(\cosh u_o - \cos v)^{1/2}}$$

Evaluating this with the help of the identities of Hicks' equation (23), we obtain that the first term is

$$\sqrt{2}\ \cosh u_o\ q_n(u_o) \qquad \text{----------------} \quad (3.4.8)$$

and the second term may be split into

$$-\int_0^\pi \frac{\tfrac{1}{2}\cos(n+1)v\ dv}{(\cosh u_o - \cos v)^{1/2}} \quad -\int_0^\pi \frac{\tfrac{1}{2}\cos(n-1)v\ dv}{(\cosh u_o - \cos v)^{1/2}}$$

$$= -\sqrt{2}\left\{ \tfrac{1}{2}q_{n+1}(u_o) + \tfrac{1}{2}q_{n-1}(u_o)\right\}$$

$$\text{----------------} \quad (3.4.9)$$

Thus the equation (3.4.7) becomes

$$\sqrt{2}\left\{\cosh u_o q_n(u_o) - \tfrac{1}{2}q_{n+1}(u_o) - \tfrac{1}{2}q_{n-1}(u_o)\right\}$$

$$\text{----------------} \quad (3.4.10)$$

By use of the relation (1.6.3), equation (3.4.10) is further simplified to

$$\frac{2\sqrt{2}}{4n^2-1}\sinh u_o\ q_n'(u_o)$$

Thus the expression for the coefficient 'a_n' may be written as

$$a_n = \frac{2\sqrt{2}}{4n^2-1}\ \frac{c}{\delta_n \pi a}\ \frac{q_n'(u_o)}{p_n'(u_o)}$$

$$\text{----------------} (3.4.11)$$

3.5. TOTAL CURRENT.

Ampére's law atated in the integral form is

$$\int B\cdot ds = \eta_o \Sigma i \quad \text{------------} \quad (3.5.1)$$

where $\sum i$ indicates the total current in the circuit.

On the surface $u=u_o$, we know $B_u=B_w=0$,

$$B_v = -\frac{1}{\lambda\rho}\frac{\partial}{\partial u}(\rho A)$$

$$= \frac{(\cosh u_o-\cos v)}{a}\,\frac{(\cosh u_o-\cos v)}{a\sinh u_o}\frac{\partial}{\partial u}\left\{\frac{a\sinh u_o}{(\cosh u_o-\cos s)}\;A\right\}$$

$$= -\frac{(\cosh u_o-\cos v)^2}{a^2\sinh u_o}\frac{d}{du}\left\{\frac{a\sinh u_o}{(\cosh u_o-\cos v)^{1/2}}\sum_{n=0}^{\infty}a_n p_n'(u_o)\cos nv\right\}$$

$$= -\frac{(\cosh u_o-\cos v)^2}{a^2\sinh u_o}\left\{\left[\frac{a\sinh u_o}{(\cosh u_o-\cos v)^{1/2}}-\frac{a\sinh^2 u_o}{a^{3/2}(\cosh u_o-\cos v)}\right]\right.$$

$$\left.\sum a_n p_n'(u_o)\cos nv + \left[\frac{a\sinh u_o}{(\cosh u_o-\cos v)^{1/2}}\sum a_n p_n''(u_o)\cos nv\right\}$$

$$= \frac{(\cosh u_o-\cos v)^{1/2}}{a\sinh u_o}\left\{\left[\tfrac{1}{2}\sinh^2 u_o-\cosh u_o(\cosh u_o-\cos v)\right]\right.$$

$$\left.\sum a_n p_n'\cos nv-\left[\sinh u_o(\cosh u_o-\cos v)\sum a_n p_n''\cos nv\right]\right\}$$

$$\text{----------------} \quad (3.5.2)$$

Substituting (3.5.2) into (3.5.1), we get the expression for the total current,

$$I = \frac{1}{\eta_o}\int_{-\pi}^{\pi} B\cdot ds$$

$$= \frac{2}{\eta_o}\int_{0}^{\pi} B_v \cdot \frac{a}{(\cosh u_o - \cos v)}\, dv$$

$$= \frac{2a}{\eta_o}\int_{0}^{\pi} \frac{(\cosh u_o - \cos v)^{1/2}}{a \sinh u_o} \Big\{ \Big[\tfrac{1}{2}\sinh^2 u_o - \cosh u_o(\cosh u_o - \cos v)$$

$$\Sigma a_n p_n' \cos nv\Big] - \Big[\sinh u_o(\cosh u_o - \cos v)\Sigma a_n p_n'' \cos nv\Big]$$

$$\frac{a}{(\cosh u_o - \cos v)}\, dv \Big\}$$

------------------ (3.5.3)

Evaluating this with the help of the identities of Hicks' equation (23), it becomes

$$I = \frac{\sqrt{2}}{\eta_o} \Sigma a_n \Big\{ \sinh u_o p_n' q_n - \frac{4q_n'}{4n^2-1}\big[\cosh u_o p_n' + \sinh u_o p_n''\big]\Big\}$$

------------------ (3.5.4)

This expression is further simplified with the aid of the identity (1.6.3). The final result is

$$I = \frac{\sqrt{2}}{\eta_o}\Sigma a_n \qquad \text{------------------ (3.5.5)}$$

3.6. TOTAL FLUX.

The total flux through the circuit is defined by the integral form as

$$\int B_n \cdot da \qquad \text{------------------} \quad (3.6.1)$$

But

$$\underline{B} = \text{curl}\underline{A}$$

We get therefore

$$\Phi = \int (\text{curl}\underline{A})_n \, da \qquad \text{------------} \quad (3.6.2)$$

By Stokes's theorem, equation (3.6.2) becomes

$$\Phi = \int A \cdot ds$$

In the present case, $A_u = A_v = 0$ then

$$\Phi = \int_{-\pi}^{\pi} A \, ds_w$$

$$= \int_{-\pi}^{\pi} \frac{a \sinh u}{\cosh u - \cos v} A \cdot dw$$

As the vector potential A is a function of u and v only, we get

The total flux, $\Phi = \dfrac{2\pi a}{\cosh u - \cos v} \sinh u (\cosh u - \cos v)^{1/2} \sum_{n=0}^{\infty} a_n p_n' \cos nv$

$$= \frac{2\pi a \sinh u}{(\cosh u - \cos v)^{1/2}} \sum_{n=0}^{\infty} a_n p_n' (u) \cos nv$$

$$= 2\pi c \qquad \text{----------------} \quad (3.6.3)$$

where 'c' is defined in equation (3.4.2)

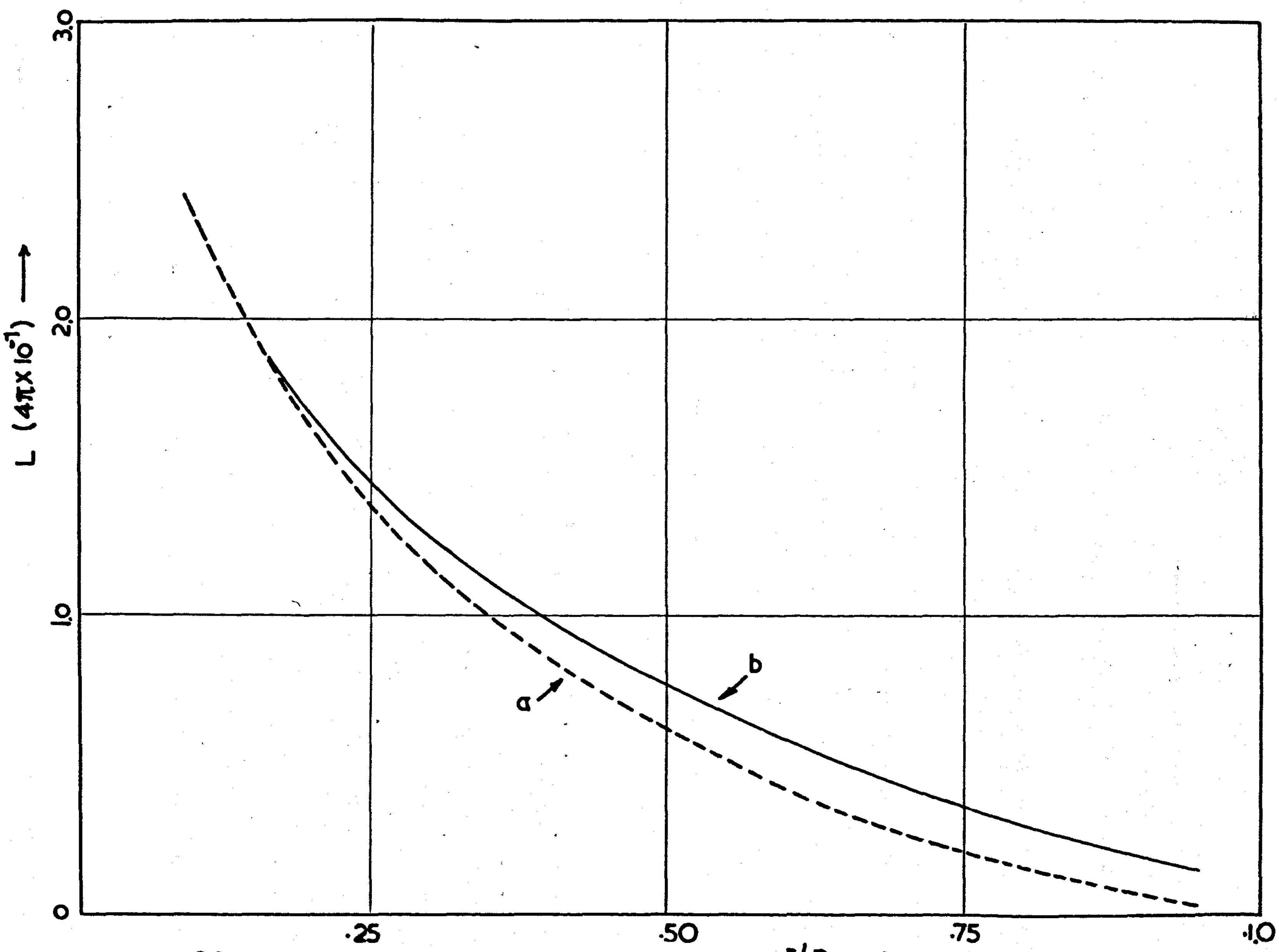


FIG. 3.2. CALCULATION OF SELF-INDUCTANCE

a. TOROIDAL FUNCTIONS,
b. APPROX. FORMULA.

3.7. EXACT EXPRESSION FOR THE SELF-INDUCTANCE OF A CIRCULAR TURN OF WIRE.

The self-inductance of a circuit is defined as the total flux linkage per unit current and may be written as

$$L = \frac{\Phi \text{ (total flux linkage)}}{I \text{ (total current)}} \text{ in Henry} \qquad \text{-----------------} (3.7.1)$$

Substituting (3.5.5) and (3.6.3) into the above expression for self-inductance, we get

$$L = \frac{2\pi c}{\frac{\sqrt{2}}{\eta_0}\Sigma a_n} \qquad \text{-----------------} (3.7.2)$$

This exact expression for the self-inductance of a circular turn of wire may be expressed in terms of the toroidal functions simply by substituting (3.4.11) into the above equation.

Thus

$$L = \frac{2\eta_o\pi^2 a}{\sum\frac{1}{\delta_n(n^2-\frac{1}{4})}\left\{\frac{q_n'(u_o)}{p_n'(u_o)}\right\}} \qquad \text{-----------------} (3.7.3)$$

The self-inductances given by the equation (3.7.3) are calculated and plotted in Fig (3.2_a).

Instead of using u_o and a as independent variables, the dimension ratio (r/R) has been used. The relations between r, R and a, u_o are

$$a = R\left\{1-(r/R)^2\right\}^{1/2}$$
$$u_o = \text{arc sech}(r/R) \qquad \text{———————— (3.7.4)}$$

3.8. APPROXIMATE FORMULA FOR THE SELF-INDUCTANCE OF A CIRCULAR TURN OF WIRE.

In a magnetic field, energy is required to produce the field, but no energy is required to maintain it. If the magnetic flux density is $\underline{B}$ in a given volume element dV in the field (the field being ' in vacuo ' or in a material not containing magnetizable material), then it can be shown that

$$W = \frac{1}{2\eta_0}\iiint B^2 dV \qquad \text{-------------} \quad (3.8.1)$$

This stored energy may also be expressed in terms of the current and the self-inductance,

$$W = \frac{1}{2} LI^2 \qquad \text{-----------------} \quad (3.8.2)$$

Combining equations (3.8.1) and (3.8.2), we get

$$\frac{1}{2} LI^2 = \frac{1}{2\eta_0}\iiint B^2 dV \qquad \text{-----------} \quad (3.8.3)$$

For our present purpose, assuming that the dimension ratio (r/R) is very small, we obtain

$$L = \eta_0 R \left(\log \frac{8R}{r} - 2 \right)$$

$$\text{-----------------} \quad (3.8.4)$$

The detailed account of the integration is to be found in Abraham's book [4].

The self-inductance given by the equation (3.8.4) is plotted in Fig. (3.2_b) to be compared with Fig. (3.2_a).

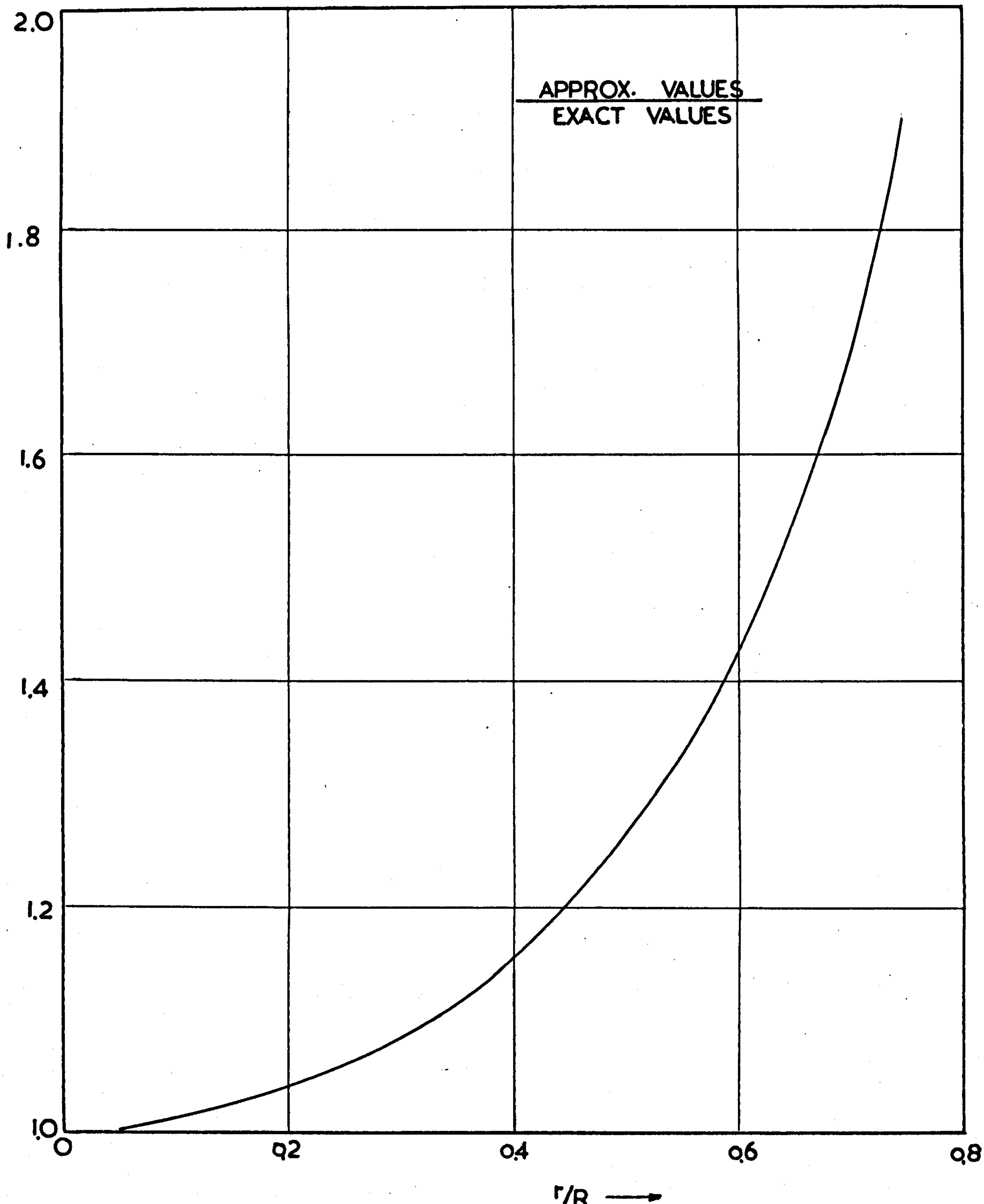


FIG. 3.3.

3.9. CONCLUSIONS

The relation between the two curves is summed up in Fig (3.3) in which the ratio of the approximate value of self-inductance of a circular turn of wire to the exact is plotted. The ratio approaches unity for small values of r/R (that is, for thin wire); but in general the approximate value is greater than the exact. The tendency for an increased self-inductance, when calculated from the approximate formula, can be easily deduced from a general consideration of the field; the current which was assumed to be concentrated at the axis in the calculation produces a stronger magnetic flux-linkage.

The error introduced by the approximate formula is found to be significant (that is, more than 10%), when the dimension ratio r/R has a value of more than 0.35. Thus the genéral correctness of the exact calculation is confirmed by the approximate one.

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Appendix 1.

$p_n(u)$ and $q_n(u)$ are the general solution of the equation

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\coth u\frac{\partial\psi}{\partial u}+\frac{1}{4}\psi=0$$

Laplace's equation expressed in toroidal coördinates (u, v, w) has the form

$$\frac{\partial}{\partial u}\left(\frac{\sinh u}{\cosh u-\cos v}\frac{\partial\psi}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{\sinh u}{\cosh u-\cos v}\frac{\partial\psi}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{1}{\sinh u(\cosh u-\cos v)}\frac{\partial\psi}{\partial w}\right)=0$$

------------------------(1)

Where there is radial symmetry, the last term disappears. In such case the above equation will reduce to

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\coth u\frac{\partial\psi}{\partial u}+\frac{1}{4}\psi=0 \quad \text{------------(2)}$$

The general solution of equation (2) may be written

$$\psi=\sum_{n=0}^{\infty}\left\{a_np_n(u)+b_nq_n(u)\right\}\cos nv$$

where

$$p_n(u)=\frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(\cosh u-\sinh u\cdot\cos\theta)^{n+\frac{1}{2}}}$$

-------(3)

$$q_n(u)=\int_0^{\infty}\frac{d\theta}{(\cosh u+\sinh u\cdot\cosh\theta)^{n+\frac{1}{2}}}$$

The proof that $p_n(u)$ and $q_n(u)$ are the solutions of the equation is as follows:

Since

$$p_n(u) = \frac{1}{\pi}\int_0^\pi \frac{d\theta}{(\cosh u - \sinh u\cdot\cos\theta)^{n+\frac{1}{2}}}$$

$$\frac{dp_n(u)}{du} = -\frac{1}{\pi}\,\frac{2n+1}{2}\int_0^\pi \frac{(\sinh u - \cosh u\cdot\cos\theta)\,d\theta}{(\cosh u - \sinh u\cdot\cos\theta)^{n+\frac{3}{2}}}$$

------------------(4)

Also

$$\frac{-\pi 2}{2n+1}\cdot\frac{d^2p_n}{du^2} = \frac{d}{du}\left\{\int_0^\pi \frac{(\sinh u - \cosh u\cdot\cos\theta)}{(\cosh u - \sinh u\cdot\cos\theta)^{n+\frac{3}{2}}}\,d\theta\right\}$$

$$= -\frac{2n+3}{2}\int_0^\pi \frac{(\sinh u - \cosh u\cdot\cos\theta)^2}{(\cosh u - \sinh u\cdot\cos\theta)^{n+\frac{5}{2}}}\cdot d\theta$$

$$+\int_0^\pi \frac{d\theta}{(\cosh u - \sinh u\cdot\cos\theta)^{n+\frac{1}{2}}}$$

------------------(5)

We know

$$(\sinh u - \cosh u\cdot\cos\theta)^2 - (\cosh u - \sinh u\cdot\cos\theta)^2$$

$$= \cos^2\theta - 1$$

$$= -\sin^2\theta$$ ------------------(6)

Hence, the first term of equation (5) becomes,

$$-\frac{2n+3}{2}\int_0^{\pi}\frac{-\sin^2\theta + (\cosh u-\sinh u\cdot\cos\theta)^2}{(\cosh u-\sinh u\cdot\cos\theta)^{n+5/2}}\,d\theta$$

$$=\frac{2n+3}{2}\int_0^{\pi}\frac{\sin^2\theta\ \ d\theta}{(\cosh u-\sinh u\cdot\cos\theta)^{n+5/2}} - \frac{2n+3}{2}\pi\cdot p_{n(u)}$$

Therefore,

$$\frac{d^2p_n}{du^2} = \frac{(2n+1)^2}{2^2}\cdot p_n - \frac{2n+1}{2}\cdot\frac{1}{\pi\sinh u}\int_0^{\pi}\frac{\cos\theta\cdot d\theta}{(\cosh u-\sinh u\cdot\cos\theta)^{n+3/2}} \quad \text{---------(7)}$$

Let the potential $\psi = U\cdot V$, where U and V are functions of u and v respectively, equation (2) then reduces to

$$\frac{d^2V}{dv^2} + n^2V = 0 \quad \text{---------(8)}$$

and

$$\frac{d^2U}{du^2} + \coth u\frac{dU}{du} - \frac{(4n^2-1)}{4}U = 0 \quad \text{---------(9)}$$

Equation (8) has the general solution given by

$$V = a'\cos nv + b'\sin nv$$

If equation (9) has the general solution given by

$$U = a_np_n(u) + b_n\cdot q_n(u)$$

then p_n and q_n must satisfy the equation (9) separately. With p_n defined by equation (3),

$$\frac{d^2p_n}{du^2} + \coth u \frac{dp_n}{du}$$

$$= \frac{(2n+1)^2}{4} p_n - \frac{2n+1}{2} \cdot \frac{1}{\pi \sinh u} \int_0^{\pi} \frac{\cos\theta \, d\theta}{(\cosh u - \sinh u \cdot \cos\theta)^{n+3/2}}$$

$$+ \frac{\cosh u}{\sinh u} \cdot \frac{2n+1}{2} \cdot \frac{-1}{\pi} \int_0^{\pi} \frac{\sinh u - \cosh u \cdot \cos\theta}{(\cosh u - \sinh u \cdot \cos\theta)^{n+3/2}} \, d\theta$$

$$= \frac{(2n+1)^2}{4} p_n - \frac{2n+1}{2} \frac{1}{\pi \sinh u} \int_0^{\pi} \frac{\cosh u \sinh u - \sinh^2 u \cdot \cos\theta}{(\cosh u - \sinh u \cdot \cos\theta)^{n+3/2}} \, d\theta$$

$$= \frac{(2n+1)^2}{4} p_n - \frac{2n+1}{2} \frac{1}{\pi} \int_0^{\pi} \frac{d\theta}{(\cosh u - \sinh u \cdot \cos\theta)^{n+1/2}}$$

$$= \frac{4n^2-1}{4} p_n$$

Hence p_n is a solution of the equation (2). Similarly, it can also be proved in the same manner that q_n is a solution of the equation.

Appendix 2.

$q_n(u)$ expressed in terms of $p_n(u)$.

Let $\psi = c_n p_n(u) \cdot F_n(u)$ ------------------(10)

substituting this into equation (9), we get

$$p_n \frac{d^2 F_n}{du^2} + 2 \frac{dp_n}{du} \frac{dF_n}{du} + \coth u \, p_n \frac{dF_n}{du} = 0$$

which may be written

$$\frac{\frac{d^2 F_n}{du^2}}{\frac{dF_n}{du}} + 2 \frac{\frac{dp_n}{du}}{p_n} + \coth u = 0$$

------------------(11)

The integral of this equation is

$$\log_e \frac{dF_n}{du} + 2 \log_e \cdot p_n + \log_e \cdot \sinh u = 0$$

------------------(12)

Thus

$$\frac{dF_n}{du} = \frac{A}{\sinh u \; p_n^2}$$

where A is the constant of integration.

Hence

$$F_n = A \int_u^\infty \frac{du}{\sinh u \cdot p_n^2}$$ --------------(13)

comparing equation (13) with the given solution, we have

$$q_n(u) = A p_n(u) \int_u^{\infty} \frac{du}{\sinh u \, (p_n)^2}$$

---------------------(14)

The following relations between p_n and q_n will be useful in application;

(a) $p_{n+1} \cdot q_n - p_n \cdot q_{n+1} = \frac{2}{2n+1}$

(b) $p_n' \cdot q_n - p_n \cdot q_n' = \frac{1}{\sinh u}$

(c) $p_n' \cdot q_{n+1}' - p_{n+1}' \cdot q_n' = \frac{(2n+1)}{2}$

They are easily proved, for substituting for p_{n+1}, q_{n+1} from their sequence equation.

Appendix 3

VECTOR OPERATIONS IN TOROIDAL COORDINATES.

Let us assume three orthogonal coordinates u, v, w, so that the three sets of coordinate surfaces, u = constant, v = constant, w = constant, intersect at right angles, though in general the surfaces will be curved. Now let us move a distance ds_u normal to a surface u = constant. When we do so, v and w do not change, but we reach another surface on which u has increased by du, which in general is different from ds_u. In general, we have

$$\begin{aligned} ds_u &= h_1 du \\ ds_v &= h_2 dv \qquad \text{---------------} \quad (15) \\ ds_w &= h_3 dw \end{aligned}$$

where

$$h_1^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

similarly for h_2 and h_3.

The first step in setting up vector operations in any set of coordinates is to derive these h's, which can be done by elementary geometrical methods. Thus in toroidal coordinates where the coordinates are u, v, w, we have

$$\delta s_u = \lambda \delta u$$

$$\delta s_v = \lambda \delta v \qquad \text{-----------------} \quad (16)$$

$$\delta s_w = \rho \delta w$$

where

$$\lambda = \left| \frac{d(\rho + jz)}{d(u+jv)} \right|$$

$$= \frac{a}{(\cosh u - \cos v)}$$

which corresponds to " $1/\xi$ " in Hicks' paper.

GRADIENT.

The component of the gradient of a scalar S' in any direction is its rate of change in that direction. Thus the component in the direction 'u' (normal to the surface u=constant) is

$$\frac{dS'}{ds_u} = \frac{1}{h_1} \frac{\partial S'}{\partial u}$$

with similar formulas for the other components. Thus in toroidal coordinates, we have

$$\text{grad}_u S' = \frac{1}{\lambda} \frac{\partial S'}{\partial u}$$

$$\text{grad}_v S' = \frac{1}{\lambda} \frac{\partial S'}{\partial v}$$

$$\text{grad}_w S' = \frac{1}{\rho} \frac{\partial S'}{\partial w}$$

DIVERGENCE.

Let us apply the divergence theorem to a small volume element $dV = ds_u\ ds_v\ ds_w$, bounded by co-ordinate surfaces at u_1, u_1+du_1 etc.. If we have a vector $\underline{A}$, with components A_u, A_v, A_w along the three curvilinear axes, the flux into the volume over the face at u, whose area is $ds_v\ ds_w$, is $(A_u\ ds_v\ ds_w)_u$, and the corresponding flux out over the opposite face is $(A_u\ ds_v\ ds_w)_{u+du}$, where we note that the area $ds_v\ ds_w$ changes with u as well as the flux density A_u. Thus the flux out over these two faces is

$$\frac{\partial}{\partial u}(A_u \cdot ds_v \cdot ds_w)\ du$$

$$= \frac{\partial}{\partial u}(A_u \cdot h_2 \cdot h_3) du\ dv\ dw$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u}\ (A_u \cdot h_2 \cdot h_3)\ dv$$

Proceeding similarly with the other pairs of faces, and setting the whole outward flux equal to $\text{div}\underline{A}\ dv$, we have

$$\text{div}\underline{A} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u}(A_u h_2 h_3) + \frac{\partial}{\partial v}(A_v h_3 h_1) + \frac{\partial}{\partial w}(A_w h_1 h_2)\right\} \qquad (17)$$

Thus in toroidal coordinates

$$\text{div } \underline{A} = \frac{1}{\lambda^2 \rho}\left\{ \frac{\partial}{\partial u}(A_u \cdot \rho \cdot \lambda) + \frac{\partial}{\partial v}(A_v \cdot \rho \cdot \lambda) + \frac{\partial}{\partial w}(A_w \cdot \lambda^2) \right\}$$

--------------------(18)

LAPLACIAN.

Writing the Laplacian of a scalar S as divgradS, and placing $A_u = \text{grad}_u S$, etc., in the expression for divA, we have

$$\nabla^2 S = \frac{1}{h_1 h_2 h_3}\left\{ \frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial S}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial S}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial S}{\partial w}\right) \right\} = 0$$

--------------------(19)

Thus in toroidal coordinates, we will have

$$\nabla^2 S = \frac{1}{\lambda^2 \rho}\left\{ \frac{\partial}{\partial u}\left(\rho \frac{\partial S}{\partial u}\right) + \frac{\partial}{\partial v}\left(\rho \frac{\partial S}{\partial v}\right) + \frac{\lambda^2}{\rho}\frac{\partial^2 S}{\partial w^2} \right\}$$

--------------------(20)

CURL.

We apply Stokes's theorem to an approximately rectangular area bounded by u, u+du, v, v+dv. The line integral of a vector $\underline{A}$ about the circuit is

$$A_u(u,v)dS_u + A_v(u+du,v)dS_v - A_u(u,v+dv)dS_u - A_v(u,v)dS_v$$

This is approximately equal to

$$\left\{\frac{\partial}{\partial u}(h_2 \cdot A_v) - \frac{\partial}{\partial v}(h_1 \cdot A_u)\right\} dudv$$

---------------------------(21)

Since this must be equal to $curl_w A \cdot dS_u \cdot dS_v$, we ahve

$$curl_w A = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial u}(h_2 A_v) - \frac{\partial}{\partial v}(A_u h_1)\right\}$$

---------------------------(21)

With the similar expressions for the other components.

Thus in toroidal coordinates, we have

$$curl_u A = \frac{1}{\lambda\rho}\left\{\frac{\partial}{\partial v}(A_w \cdot \rho) - \frac{\partial}{\partial w}(A_v \cdot \lambda)\right\}$$

$$curl_v A = \frac{1}{\lambda\rho}\left\{\frac{\partial}{\partial w}(Au \cdot \lambda) - \frac{\partial}{\partial u}(A_w \cdot \rho)\right\}$$

$$curl_w A = \frac{1}{\lambda\rho}\left\{\frac{\partial}{\partial u}(A_v \cdot \lambda) - \frac{\partial}{\partial v}(A_u \cdot \lambda)\right\}$$

Appendix 4

$p_n'(u)$ and $q_n'(u)$ are the general solutions of the equation

$$\frac{d^2U}{du^2} n^2U - \frac{3U}{4\sinh^2 u} = 0.$$

If we put

$$U = (\sinh u)^{1/2} \phi \quad \text{----------} \quad (22)$$

then

$$\frac{dU}{du} = (\sinh u)^2 \frac{d\phi}{du} + \tfrac{1}{2}(\sinh u)^{-1/2} \cosh u \, \phi$$

$$\frac{d^2U}{du^2} = (\sinh u)^{1/2} \frac{d^2\phi}{du^2} + (\sinh u)^{-1/2} \cosh u \frac{d\phi}{du}$$

$$+ \tfrac{1}{2}(\sinh u)^{1/2} \phi - \tfrac{1}{4}(\sinh u)^{-\frac{3}{2}} \cosh^2 u \, \phi$$

Thus

$$\frac{d^2U}{du^2} + n^2U - \frac{3U}{4\sinh^2 u}$$

$$= (\sinh u)^{1/2} \left\{ \frac{d^2\phi}{du^2} + \coth u \frac{d\phi}{du} - (n^2 - \tfrac{1}{4})\phi - (\sinh u)^{-2} \phi \right\}$$

or

$$\frac{d^2\phi}{du^2} + \coth u \frac{d\phi}{du} - (n^2 - \tfrac{1}{4})\phi - (\sinh u)^{-2} \phi = 0$$

$$\text{----------------} \quad (23)$$

The toroidal functions $p_n(u)$ and $q_n(u)$ satisfy

$$\frac{d^2p}{du^2} + \coth u \frac{dp}{du} - (n^2-\tfrac{1}{4})p = 0 \quad \text{(Hicks' equ. 9)}$$

Differentiate this with respect to 'u' and write p' for $\frac{dp}{du}$, we get

$$\frac{d^2p'}{du^2} + \coth u \frac{dp'}{du} - \operatorname{cosech}^2 u \; p' - (n^2-\tfrac{1}{4})p' = 0$$

which is equation (23). Thus the solutions of equations (23) are $p_n'(u)$ and $q_n'(u)$.