On Galois representations associated to low weight Hilbert–Siegel modular forms

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In loving memory of Rivka Lehmann ז"ל

Abstract

Under the Langlands correspondence, where automorphic representations of GL_n should correspond to n-dimensional Galois representations, *cuspidal* automorphic representations should correspond to *irreducible* Galois representations. More generally, heuristically, one expects that the image of an automorphic Galois representation should be as large as possible, unless there is an automorphic reason for it to be small. This thesis addresses the consequence of this heuristic for low weight, genus 2 Hilbert–Siegel modular forms.

Let F be a totally real field and $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, whose archimedean components lie in the holomorphic (limit of) discrete series. If π is not CAP or endoscopic, then we show that its associated ℓ-adic Galois representation $\rho_{\pi,\ell}$ is irreducible and crystalline for 100% of primes. If, moreover, π is neither an automorphic induction nor a symmetric cube lift, then we show that, for 100% of primes ℓ, the image of its mod ℓ Galois representation contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$.

Contents

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Introduction

Under the Langlands correspondence, where automorphic representations of GL_n should correspond to n-dimensional Galois representations, *cuspidal* automorphic representations should correspond to *irreducible* Galois representations. More generally, heuristically, one expects that the image of an automorphic Galois representation should be as large as possible, unless there is an automorphic reason for it to be small.

In this thesis, we address the consequence of this heuristic for low weight, genus 2 Hilbert–Siegel modular forms. These automorphic forms, which are the genus 2 analogue of weight 1 Hilbert modular forms, are of particular interest due to their conjectural relationship with abelian surfaces. Our main result is the following theorem:

Theorem A. *Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ such that, for each archimedean place v, the local component π_v lies in the holomorphic (limit of) discrete series. Assume that π is not CAP (cuspidal associated to a parabolic) or endoscopic. For each prime number ℓ, let*

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{Q}}_\ell)$$

be the ℓ-adic Galois representation associated to π.

1. *Suppose that $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$ and that $\ell \gg 0$. Then $\rho_{\pi,\ell}$ is irreducible.*
2. *If π is not an automorphic induction, then $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$ for a set of primes ℓ of Dirichlet density 1.*

In particular, $\rho_{\pi,\ell}$ is irreducible for 100% of primes.

The corresponding result for elliptic modular forms was proven by Ribet [Rib77] using class field theory and the Ramanujan bounds for the Hecke eigenvalues of modular forms. For high weight Hilbert–Siegel modular forms, irreducibility for all but finitely many primes follows from the work of Ramakrishnan [Ram13]. Here, potential modularity is used in place of class field theory, and results from p-adic Hodge theory—in particular, the fact that the Galois representations are Hodge–Tate regular—are used in place of the Ramanujan bounds. All other recent results proving the irreducibility of automorphic Galois representations rely crucially on inputs that are only available under the hypothesis of regularity (see, for example, [BLGGT14, Xia19]).

The novelty of this thesis is to prove an irreducibility theorem in a situation where these key inputs are not available. In the case of low weight Hilbert–Siegel modular forms, the Hodge–Tate–Sen weights of $\rho_{\pi,\ell}$ are irregular, purity is an open problem and, a priori, crystallinity is not known. Indeed, a priori, we do not even know that $\rho_{\pi,\ell}$ is Hodge–Tate. In place of these inputs, we exploit the fact that $\rho_{\pi,\ell}$ is symplectic with odd similitude character in combination with partial results towards the generalised Ramanujan conjecture and a criterion of Jorza [Jor12], which provides a sufficient condition for $\rho_{\pi,\ell}$ to be crystalline.

We also analyse the images of the mod ℓ Galois representations attached to π and prove the following big image theorem:

Theorem B. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, whose archimedean components lie in the holomorphic (limit of) discrete series. Suppose that π is not CAP or endoscopic. For each prime ℓ, let*

$$\overline{\rho}_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{F}}_\ell)$$

be the mod ℓ Galois representation associated to π. Let S_{crys} be the set of primes ℓ such that $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$. Then:

1. *$\overline{\rho}_{\pi,\ell}$ is irreducible for 100% of primes ℓ.*
2. *If π is neither a symmetric cube lift nor an automorphic induction and if the weights of π are not all of the form $(2k_v - 1, k_v + 1)_{v|\infty}$ for $k_v \geq 2$,[1] then the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for 100% of primes ℓ.*

Moreover, assuming Serre's conjecture for F, the following stronger result holds:

1. *For all but finitely many primes $\ell \in S_{crys}$, $\overline{\rho}_{\pi,\ell}$ is irreducible.*
2. *If π is neither a symmetric cube lift nor an automorphic induction, then, for all but finitely many primes $\ell \in S_{crys}$, the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$.*

This theorem generalises the work of Ribet and Momose [Rib85, Mom81] for elliptic modular forms. For high weight Hilbert–Siegel modular forms, residual irreducibility for 100% of primes follows from irreducibility in characteristic 0 by applying [BLGGT14, Proposition 5.3.2]. If π is a totally generic regular algebraic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_\mathbf{Q})$ that is not CAP or endoscopic and is neither an automorphic induction nor a symmetric cube lift, Dieulefait–Zenteno [DZ] prove that the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for 100% of primes. This result does not apply directly to Siegel modular forms, whose automorphic representations are not totally generic. However, applying Arthur's classification (see Section 2.3) gives a result for high weight Siegel modular forms. In the high weight case, the results of this paper strengthen previous results: we generalise Dieulefait–Zenteno's results to automorphic representations over totally real fields and, when $F = \mathbf{Q}$, we prove that the image is large for all but finitely many primes.

0.1 A history of the problem

0.1.1 Congruences between modular forms

Images of modular Galois representations were first studied by Serre [Ser73] and Swinnerton-Dyer [SD73] in order to analyse congruences between modular forms. For

[1] This condition is automatic if π is non-cohomological.

an example of how the image of a Galois representation can encode these congruences, consider the Ramanujan delta function

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n \in S_{12}(\mathrm{SL}_2(\mathbf{Z}))$$

and the Eisenstein series of weight 12 and level 1

$$G_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty}\sigma_{11}(n)q^n \in M_{12}(\mathrm{SL}_2(\mathbf{Z})).$$

It was first observed by Ramanujan that, for all primes p,

$$\tau(p) \equiv 1 + p^{11} \pmod{691},$$

and hence that Δ and G_{12} are congruent modulo 691. Serre [Ser69] observed that this congruence can be recast into the language of Galois representations: the mod 691 Galois representation $\overline{\rho}_{\Delta,691}$ attached to Δ is reducible and

$$\overline{\rho}_{\Delta,691} \simeq \mathbf{1} \oplus \overline{\varepsilon}_{691}^{-11} \simeq \overline{\rho}_{G_{12},691}.$$

In particular, the image of the 691-adic Galois representation attached to Δ is small: up to conjugation, its image consists of elements that are upper-triangular modulo 691.

More generally, if f is a non-CM cuspidal eigenform, then a mod ℓ congruence between f and a CM form or an Eisenstein series is encoded in the image of the ℓ-adic Galois representation of f. In particular, in showing that the image of the ℓ-adic Galois representation of f is as large as possible for all but finitely many primes, Ribet [Rib75, Rib77, Rib85] and Momose [Mom81] proved that f can be congruent modulo ℓ to a CM form or to an Eisenstein series for only finitely many primes ℓ.

0.1.2 Selmer groups and Iwasawa theory

While the authors of first papers studying the images of automorphic Galois representations [Ser73, SD73, Rib75] were certainly aware of the irreducibility of the ℓ-adic Galois representations,[2] these papers primarily focus on the images of the mod ℓ Galois representations for all but finitely many primes. The question of ℓ-adic irreducibility was first considered explicitly by Ribet [Rib76] in the context of a different problem.

Let A be the class group of $\mathbf{Q}(\zeta_\ell)$ and let $C = A/A^\ell$. Letting $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ act on C through its quotient $\mathrm{Gal}(\mathbf{Q}(\zeta_\ell)/\mathbf{Q})$, C decomposes as

$$C = \bigoplus_{i \pmod{\ell-1}} C(\overline{\varepsilon}_\ell^i),$$

where $\overline{\varepsilon}$ is the mod ℓ cyclotomic character and $C(\overline{\varepsilon}_\ell^i)$ is the subset of elements $c \in C$ upon which $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts as $g(c) = \overline{\varepsilon}_\ell^i(g)c$. Ribet proved that the numerator of the Bernoulli number B_k is divisible by ℓ only if $C(\overline{\varepsilon}_\ell^{1-k}) \neq 0$.[3]

[2]Indeed, in [Rib75], on the way to proving an open image theorem, Ribet proves that the Galois representations attached to level 1 modular forms are irreducible. However, the proof is indirect: ℓ-adic irreducibility is deduced, in Theorem 5.5, as a corollary of residual irreducibility for all but finitely many primes.

[3]The converse had been proven by Herbrand.

The irreducibility of the ℓ-adic Galois representations associated to cuspidal modular eigenforms is a crucial ingredient of Ribet's proof. If $\ell \mid B_k$, then Ribet showed that the weight k Eisenstein series, whose leading term is B_k, is congruent modulo ℓ to a cusp form f. If $\rho_{f,\ell}$ is the ℓ-adic Galois representation attached to f, then Ribet showed that $\rho_{f,\ell}$ is irreducible and that, as a result, there is a choice of lattice for $\rho_{\pi,\ell}$ such that $\rho_{f,\ell} \pmod{\ell}$ is not semisimple: i.e. $\rho_{f,\ell} \pmod{\ell}$ is of the form

$$\begin{pmatrix} \overline{\varepsilon}_\ell^{1-k} & * \\ 0 & \mathbf{1} \end{pmatrix}.$$

Using the extension class of $\rho_{f,\ell} \pmod{\ell}$, Ribet constructed an unramified ℓ-extension of $\mathbf{Q}(\zeta_\ell)$ and deduced that $C(\overline{\varepsilon}_\ell^{1-k}) \neq 0$.

This technique of using congruences to produce irreducible ℓ-adic Galois representations with non-semisimple residual representations is still one of the key techniques for producing elements in Selmer groups of Galois representations. For example, the technique was applied to Hida families by Wiles [Wil90] to prove the Iwasawa main conjecture for GL_1, and to automorphic representations of $\mathrm{U}(2,2)$ by [SU14] to prove the Iwasawa main conjecture for GL_2.

We conclude with an application of our results to theory of Euler systems when $F = \mathbf{Q}$. To use the existence of an Euler system to bound the Selmer group of a Galois representation, one needs to know that the image of the Galois representation is sufficiently large. The following corollary, which is immediate from Theorem B (c.f. [LSZ17, Remark 11.1.3]), says that any stable lattice T of $\rho_{\pi,\ell}$ satisfies the condition $\mathrm{Hyp}(\mathbf{Q}_\infty, T)$ of [Rub00].

Corollary 0.1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_\mathbf{Q})$ such that π_∞ lies in the holomorphic (limit of) discrete series. Suppose that π is not CAP, endoscopic, an automorphic induction or a symmetric cube lift. Let $E/\mathbf{Q}_\ell$ be a finite extension, with residue field $\mathbf{F}_E$, over which $\rho_{\pi,\ell}$ is defined, and let T be a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable lattice for $\rho_{\pi,\ell}$. Then, for all but finitely many $\ell \in S_{crys}$,*

1. *$T^\vee \otimes \mathbf{F}_E$ is irreducible as an $\mathbf{F}_E[\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_{\ell^\infty}))]$-module.*
2. *There exists $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_{\ell^\infty}))$ such that $T^\vee/(g-1)T^\vee$ is free of rank 1 over $\mathcal{O}_E$.*

In particular, if the Euler system constructed by [LSZ17, LPSZ19] is non-zero, then it can be used to bound the Selmer group of $\rho_{\pi,\ell}$ (see [LSZ17, Section 11]).

0.2 Dependence on Arthur's classification

The results of this thesis rely crucially on Arthur's classification of the discrete automorphic spectrum of GSp_4, which was announced in [Art04]. In particular, we require the local-global compatibility results proven by Mok [Mok14], which rely on the existence of a transfer map between automorphic representations of GSp_4 and of GL_4.

A proof of Arthur's classification has been given by Gee–Taïbi [GT18]. However, this work is itself dependent on [Art13] and on the twisted weighted fundamental lemma, which was announced in [CL10], but whose proof is yet to appear.

0.3 The structure of this thesis

The first four chapters of this thesis cover background material that is needed to prove our results. In Chapter 1, we give an expository account of the Irreducibility Conjecture and of the techniques have been used to prove cases of it. For example, in Section 1.2, we show how to use general potential automorphy theorems to prove general irreducibility theorems. The results of this section, in particular Corollary 1.2.5, form the basis of the proof of Theorem A.

In Chapter 2, we review the theory of automorphic representations of GSp_4. Using the local Langlands correspondence, we describe the (limit of) discrete series representations of $\mathrm{GSp}_4(\mathbf{R})$. We also discuss the transfer map from automorphic representations of GSp_4 to automorphic representations of GL_4, and we describe Arthur's classification of the discrete spectrum of GSp_4.

In Chapter 3, we discuss Lafforgue's G-pseudorepresentations, which we use in Chapter 4 to show that the Galois representations attached to low weight Hilbert–Siegel modular forms are valued in GSp_4.

Finally, in Chapter 4, we conclude the background section of this thesis with a discussion of the Galois representations attached to Hilbert–Siegel modular forms. These Galois representations are the main objects of study in this thesis.

In Chapter 5, we prove the first part of Theorem A. Our proof follows a similar structure to the proof of [Ram13, Theorem B]. In Theorem 5.2.1, we prove that, if $\rho_{\pi,\ell}$ is reducible, then it decomposes as a direct sum of irreducible subrepresentations that are two-dimensional, Hodge–Tate regular and totally odd. If $\rho_{\pi,\ell}$ is crystalline and ℓ is sufficiently large, then these representations are potentially modular, and the arguments of Section 1.2 lead to a contradiction. In the cohomological setting, the fact that the subrepresentations of $\rho_{\pi,\ell}$ are regular comes for free and the proof that they are odd uses the regularity of $\rho_{\pi,\ell}$. In place of these inputs, we use partial results towards the generalised Ramanujan conjecture in combination with the facts that $\rho_{\pi,\ell}$ is essentially self-dual and that its similitude character is odd.

In Chapter 6, we complete the proof of Theorem A. Our key tool is [Jor12, Theorem 3.1], which states that $\rho_{\pi,\ell}$ is crystalline for a place $v \mid \ell$ if the Satake parameters of π_v are distinct. Hence, if we fix a prime ℓ_0, we can show that $\rho_{\pi,\ell}$ is crystalline at v for a place $v \nmid \ell_0$ by considering the characteristic polynomial of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$. If $\rho_{\pi,\ell}$ is not crystalline at all places $v \mid \ell$ for a positive density of primes ℓ, then the characteristic polynomials of $\rho_{\pi,\ell_0}(\mathrm{Frob}_v)$ have repeated roots for those primes. This severe restriction on the image of ρ_{π,ℓ_0} implies that ρ_{π,ℓ_0} cannot be irreducible. In Section 6.1, we prove that $\rho_{\pi,\ell}$ is crystalline for 100% of primes *assuming* that $\rho_{\pi,\ell}$ is irreducible for at least one prime. In Section 6.2, we remove this assumption by proving that $\rho_{\pi,\ell}$ is crystalline for a positive density of primes and invoking the first part of Theorem A.

Finally, in Chapter 7, we prove Theorem B. As with Theorem A, we reduce to the case that, for infinitely many primes ℓ, $\overline{\rho}_{\pi,\ell}$ splits as a direct sum of subrepresentations that are two-dimensional, odd and have Serre weights bounded independently of ℓ. The fact that we can reduce to this case (in particular, that the subrepresentations are odd) is new even in the high weight case, and uses the fact that the similitude character of $\rho_{\pi,\ell}$ is odd in combination with the results of Section 6.1.

0.4 Background, notation and conventions

0.4.1 Basic objects

If F is a number field and v is a place of F, then we let $\mathcal{O}_F$ be the ring of integers of F and F_v be the completion of F at v, with ring of integers $\mathcal{O}_{F_v}$ and residue field $\mathbf{F}_v$. Let $\mathbf{A}_F$ denote the adèle ring of F.

We say that F is a totally real (resp. totally complex) field if all its archimedean places $v \mid \infty$ are real (resp. complex). We say that F is a CM field if it is a totally complex quadratic extension of a totally real field.

For a perfect field k, we let $\overline{k}$ denote its algebraic closure.

Denote by ζ_m a primitive m^{th} root of unity.

We fix once and for all an isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$ for each prime ℓ. Usually, in order to simplify notation, we use this isomorphism implicitly.

0.4.2 The group GSp_4

We denote by GSp_4 the split reductive group over $\mathbf{Z}$ defined by

$$\mathrm{GSp}_4(R) = \left\{ g \in \mathrm{GL}_4(R) : gJg^t = \mathrm{sim}(g)J \right\}$$

for a ring R, where

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

and $\mathrm{sim}(g) \in R^\times$. We denote by

$$\mathrm{sim} : \mathrm{GSp}_4 \to \mathbf{G}_m$$

the similitude character given by mapping $g \mapsto \mathrm{sim}(g)$ and we let Sp_4 denote its kernel.

0.4.3 The local Langlands correspondence

We let W_{F_v} denote the Weil group of F_v and

$$W'_{F_v} = \begin{cases} W_{F_v} & v \text{ archimedean} \\ W_{F_v} \times \mathrm{SL}_2(\mathbf{C}) & v \text{ non-archimedean} \end{cases}$$

denote the Weil–Deligne group of F_v. We define a map

$$|\cdot| : W_{F_v} \to \mathbf{R}_{>0}$$

via

$$W_{F_v} \twoheadrightarrow W_{F_v}^{ab} \xrightarrow{\sim} F_v^\times \xrightarrow{|\cdot|_v} \mathbf{R}_{>0},$$

where the middle map is the isomorphism of local class field theory. When v is non-archimedean, we let $\mathrm{Frob}_v \in W_{F_v}$ be a geometric Frobenius element and normalise this isomorphism so that Frob_v maps to a uniformiser of F_v. When $F_v \cong \mathbf{R}^\times$, the isomorphism $W_{F_v}^{ab} \to \mathbf{R}^\times$ is given by $z \in \mathbf{C}^\times \mapsto z\bar{z}$ and $j \mapsto -1$ (see Section 2.1).

If (r, N) is a Weil–Deligne representation, we denote by $(r, N)^{F\text{-}ss} = (r^{ss}, N)$ its Frobenius semisimplification and by $(r, N)^{ss} = (r^{ss}, 0)$ its semisimplification.

If Π_v is an irreducible admissible representation of $\mathrm{GL}_n(F_v)$, then we denote by

$$\mathrm{rec}(\Pi_v) : W'_{F_v} \to \mathrm{GL}_n(\mathbf{C})$$

the Weil–Deligne representation associated to it by the local Langlands correspondence for GL_n [Lan89, Hen00, HT01]. If π_v is an irreducible admissible representation of $\mathrm{GSp}_4(F_v)$, then we denote by

$$\mathrm{rec}_{\mathrm{GT}}(\pi_v) : W'_{F_v} \to \mathrm{GSp}_4(\mathbf{C})$$

the Weil–Deligne representation associated to it by the local Langlands correspondence for GSp_4 [GT11]. Note that rec is injective, while $\mathrm{rec}_{\mathrm{GT}}$ is finite-to-one.

0.4.4 Galois representations

We denote by $\mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. By an ℓ-adic Galois representation, we always mean a continuous, semisimple representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbf{Q}}_\ell)$$

for some reductive group G. We only consider continuous representations of $\mathrm{Gal}(\overline{F}/F)$.

Representation theory. We say that two G-valued representations $\rho_1 : \Gamma \to G(k)$ and $\rho_2 : \Gamma \to G(k)$ of a group Γ are equivalent if they are conjugate by an element of $G(k)$, and write $\rho_1 \simeq \rho_2$.

When $G = \mathrm{GL}_n$, we define

$$(\rho_1, \rho_2) := \dim \mathrm{Hom}_\Gamma(\rho_1, \rho_2)$$

to be the dimension of the space of Γ-invariant homomorphisms from ρ_1 to ρ_2.

For a representation ρ, we let $\rho^\vee$ denote its dual. Then

$$(\rho_1, \rho_2) = \dim(\rho_1^\vee \otimes \rho_2)^\Gamma = (\rho_1^\vee \otimes \rho_2, \mathbf{1}),$$

where $\mathbf{1}$ is the trivial representation.

Induction and restriction. Let E/F be a finite extension. If ρ is a representation of $\mathrm{Gal}(\overline{F}/F)$, then we denote the restriction of ρ to $\mathrm{Gal}(\overline{F}/E)$ by $\rho|_E$ or, occasionally, by $\mathrm{Res}_E^F(\rho)$. If r is a representation of $\mathrm{Gal}(\overline{F}/E)$, then we let

$$\mathrm{Ind}_E^F(r) := \mathrm{Ind}_{\mathrm{Gal}(\overline{F}/E)}^{\mathrm{Gal}(\overline{F}/F)}(r)$$

be its induction to $\mathrm{Gal}(\overline{F}/F)$.

The conjugation action. Let E^{gal} denote the Galois closure of E/F. If $\sigma \in \mathrm{Gal}(\overline{F}/F)$ has image $\overline{\sigma} \in \mathrm{Gal}(E^{gal}/F)$, then we denote by r^σ the representation of $\sigma^{-1}\,\mathrm{Gal}(\overline{F}/E)\sigma = \mathrm{Gal}(\overline{F}/\overline{\sigma}^{-1}E)$ given by

$$r^\sigma(g) = r(\sigma g \sigma^{-1}).$$

Since r^σ is independent of the choice of lift of $\overline{\sigma}$, we usually use σ to denote both the element of $\mathrm{Gal}(E^{gal}/F)$ and its lift to $\mathrm{Gal}(\overline{F}/F)$.

Galois representations attached to automorphic representations. If π is an automorphic representation, we denote by $\rho_{\pi,\ell}$ its associated ℓ-adic Galois representation and by $\overline{\rho}_{\pi,\ell}$ the semisimplification of the mod ℓ reduction of $\rho_{\pi,\ell}$. We refer to Chapter 4 for details on how we normalise this association.

Compatible systems. If ℓ_1, ℓ_2 are primes, we say that two Galois representations

$$\rho_{\ell_1} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_{\ell_1})$$

$$\rho_{\ell_2} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_{\ell_2})$$

are *compatible* if there exists a finite set S of places of F and a number field E such that, for all places $v \notin S \cup \{v : v \mid \ell_1\ell_2\}$, ρ_{ℓ_1} and ρ_{ℓ_2} are unramified at v, and the characteristic polynomials of $\rho_{\ell_1}(\mathrm{Frob}_v)$ and $\rho_{\ell_2}(\mathrm{Frob}_v)$ are contained in $E[X]$ and are equal.

We say that a collection of Galois representations $(\rho_\ell)_\ell$ with

$$\rho_\ell : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

is a *compatible system* if it is strictly compatible in the sense of [Ser98]: i.e. if there is a fixed set S of places of F and a fixed number field E such that for all ℓ and for all $v \notin S \cup \{v : v \mid \ell\}$, ρ_ℓ is unramified at v, and the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ is contained in $E[X]$ and is independent of ℓ. In this definition, we do not assume that ρ_ℓ satisfies any conditions at the places of F dividing ℓ.

Twists by the cyclotomic character. Let ε_ℓ denote the ℓ-adic cyclotomic character: if $F(\zeta_{\ell^\infty})$ denotes the field obtained by adjoining to F all ℓ-power roots of unity, then ε_ℓ is defined by

$$\begin{aligned}\varepsilon_\ell : \mathrm{Gal}(\overline{F}/F) &\twoheadrightarrow \mathrm{Gal}(F(\zeta_{\ell^\infty})/F)\\ &\cong \varprojlim_n \mathrm{Gal}(F(\zeta_{\ell^n})/F)\\ &\to \varprojlim_n (\mathbf{Z}/\ell^n\mathbf{Z})^\times\\ &\cong \mathbf{Z}_\ell^\times,\end{aligned}$$

where the map

$$\mathrm{Gal}(F(\zeta_{\ell^n}) \to (\mathbf{Z}/\ell^n\mathbf{Z})^\times$$

is defined by

$$(\zeta \mapsto \zeta^a) \mapsto a.$$

Hence, if $g \in \mathrm{Gal}(\overline{F}/F)$ and ζ is an ℓ-power root of unity, we have

$$g(\zeta) = \zeta^{\varepsilon_\ell(g)}.$$

We let $\overline{\varepsilon}_\ell$ denote the mod ℓ cyclotomic character, obtained from the action of $\mathrm{Gal}(\overline{F}/F)$ on the ℓ^{th} roots of unity.

If ρ is an ℓ-adic Galois representation and $n \in \mathbf{Z}$, then we define

$$\rho(n) := \rho \otimes \varepsilon_\ell^n.$$

Weil–Deligne representations. Given a local Galois representation

$$\rho_v : \mathrm{Gal}(\overline{F_v}/F_v) \to G(\overline{\mathbf{Q}}_\ell),$$

we denote by

$$\mathrm{WD}(\rho_v) : W'_{F_v} \to G(\mathbf{C})$$

its associated Weil-Deligne representation.[4]

0.4.5 ℓ-adic Hodge theory

We refer the reader to [Pat19, Section 2.2] for a concise summary of the aspects of ℓ-adic Hodge theory that we require in this thesis. We refer the reader to [BC09b] for a detailed introduction to ℓ-adic Hodge theory and for a precise definition of de Rham and crystalline representations.

Hodge–Tate representations. Let L be a finite extension of $\mathbf{Q}_\ell$ and let $\mathbf{C}_L$ be the completion of the algebraic closure of L. Let $B_{HT} = \bigoplus_{n\in\mathbf{Z}} \mathbf{C}_L(n)$, a graded ring with a natural action of $\mathrm{Gal}(\overline{L}/L)$.

If V is a finite-dimensional representation of $\mathrm{Gal}(\overline{L}/L)$ over $\mathbf{Q}_\ell$, then we define

$$D_{HT}(V) = (B_{HT} \otimes_{\mathbf{Q}_\ell} V)^{\mathrm{Gal}(\overline{L}/L)},$$

an L-vector space of dimension at most $\dim_{\mathbf{Q}_\ell}(V)$. We say that V is *Hodge–Tate* if

$$\dim_{\mathbf{Q}_\ell}(V) = \dim_L(D_{HT}(V)).$$

More generally, if E is an extension of $\mathbf{Q}_\ell$ and V is a finite-dimensional representation of $\mathrm{Gal}(\overline{L}/L)$ over E, then D_{HT} is naturally an $L \otimes_{\mathbf{Q}_\ell} E$-module of rank at most $\dim_E(V)$, and we say that V is *Hodge–Tate* if

$$\dim_E(V) = \mathrm{rk}_{L\otimes_{\mathbf{Q}_\ell} E}(D_{HT}(V)).$$

Typically, we assume that $E = \overline{\mathbf{Q}}_\ell$. In this case, for each embedding $\tau : L \hookrightarrow \overline{\mathbf{Q}}_\ell$, we define the τ-Hodge–Tate weights of V to be the multi-set of integers n for which $(\mathbf{C}_L(n) \otimes_{\overline{\mathbf{Q}}_\ell,\tau} V)^{\mathrm{Gal}(\overline{L}/L)}$ is non-zero, where the multiplicity of n is the dimension of $(\mathbf{C}_L(n) \otimes_{\overline{\mathbf{Q}}_\ell,\tau} V)^{\mathrm{Gal}(\overline{L}/L)}$ as a $\overline{\mathbf{Q}}_\ell$-vector space.

[4] Note that we are implicitly using the fixed isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$.

If F is a number field and if

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

is a Galois representation, then we say that ρ is Hodge–Tate if $\rho|_{F_v}$ is Hodge–Tate for every place $v \mid \ell$ of F and we index the Hodge–Tate weights via

$$\{v : v \mid \infty\} = \mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}_\ell) = \prod_{v|\ell} \mathrm{Hom}_{\mathbf{Q}_\ell}(F_v, \overline{\mathbf{Q}}_\ell).$$

Note that, with this convention, the Hodge–Tate weight of the cyclotomic character ε_ℓ is -1 at every embedding.

We say that the Galois representation ρ is *regular* or *Hodge–Tate regular* if it is Hodge–Tate and if, for each embedding $\tau : F \hookrightarrow \overline{\mathbf{Q}}_\ell$, each τ-Hodge–Tate weight of ρ has multiplicity 1.

Hodge–Tate–Sen weights. In general, the Galois representations $\rho_{\pi,\ell}$ that we consider in this thesis are not known to be Hodge–Tate, which prevents us from using the Hodge–Tate weights of $\rho_{\pi,\ell}$ to understand its subrepresentations. Sen's theory [Sen81] circumvents this problem by allowing us to attach a set of weights to any Galois representation V, which, when V is Hodge–Tate, agree with the Hodge–Tate weights of V.

If L is a finite extension of $\mathbf{Q}_\ell$ and V is a $\mathbf{C}_L$-semilinear representation of $\mathrm{Gal}(\overline{L}/L)$, then there is a $\mathbf{C}_L$-semilinear endomorphism Θ_V, the *Sen operator*, which is an invariant of V. V is Hodge–Tate if and only if Θ_V is semisimple with integer eigenvalues. In general, we define the *Hodge–Tate–Sen* weights of V to be the eigenvalues of Θ_V. In particular, if these eigenvalues are distinct integers, then Θ_V is diagonalisable and hence V is Hodge–Tate.

We note that Hodge–Tate–Sen weights behave well under direct sums and tensor products. For example (see [Pat19, Theorem 2.2.4]), if V_1, V_2 are representations, we have

$$\Theta_{V_1 \oplus V_2} = \Theta_{V_1} \oplus \Theta_{V_2}$$

and

$$\Theta_{V_1 \otimes V_2} = \Theta_{V_1} \otimes \mathrm{id}_{V_2} \oplus \Theta_{V_2} \otimes \mathrm{id}_{V_1}.$$

0.4.6 Automorphic representations

For a reductive group G, we let $\Pi(G)$ denote the set of automorphic representations of G.

Following the convention of [BCGP18], we usually use π to denote automorphic representations of GSp_4, Π to denote automorphic representations of GL_4 and $\boldsymbol{\pi}$ to denote automorphic representations of GL_2. Usually, Π is the transfer of π to GL_4.

By a Hecke character, we mean an automorphic representation of GL_1, namely a continuous character

$$\eta : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times.$$

If $x \in F^\times \otimes_{\mathbf{Q}} \mathbf{R} \subseteq \mathbf{A}_F^\times$, then we can write

$$\eta(x) = \prod_{v \text{ real } \leftrightarrow\ \tau : F \hookrightarrow \mathbf{R}} \tau(x)^{n_\tau} \prod_{v \text{ complex } \leftrightarrow\ \tau, c\tau : F \hookrightarrow \mathbf{C}} \tau(x)^{n_\tau} c\tau(x)^{n_{c\tau}}$$

and we say that η is *algebraic* if $n_\tau, n_{c\tau} \in \mathbf{Z}$ for all τ. If F is totally real, then the existence of global units ensures that the n_τ are independent of τ.[5] Hence, every algebraic Hecke character of F is of the form $\eta_0 |\cdot|^a$, where $a \in \mathbf{Z}$, η_0 is a finite order character and

$$|\cdot| : (x_v)_v \mapsto \prod_v |x_v|_v$$

is the norm character.

If π is an automorphic representation of GL_n, then we let $L(\pi, s)$ denote its L-function. If S is a finite set of places of F, then we denote by $L^S(\pi, s)$ the partial L-function of π, where we have removed the Euler factors at the places contained in S. If we do not need to make S explicit, then we denote this partial L-function by $L^*(\pi, s)$. If π is an automorphic representation of GSp_4, then we let $L(\pi, s)$ denote its spin L-function, i.e. the L-function of its transfer to GL_4.

Conductors. If $\pi = \otimes'_v \pi_v$ is an automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, then, abusing notation somewhat, we define its conductor via its transfer to GL_4. If $\iota : \mathrm{GSp}_4 \to \mathrm{GL}_4$ is the usual embedding, we define the conductor of π to be the ideal $\mathfrak{N}_\pi \subseteq \mathcal{O}_F$ given by

$$\mathfrak{N}_\pi = \prod_{v \text{ finite}} \mathrm{cond}(\iota \circ \mathrm{rec}_{\mathrm{GT}}(\pi_v)),$$

where $\mathrm{cond}(\iota \circ \mathrm{rec}_{\mathrm{GT}}(\pi_v))$ denotes the conductor of the GL_4-valued Weil–Deligne representation $\iota \circ \mathrm{rec}_{\mathrm{GT}}(\pi_v)$.

Satake parameters. Let $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ and let v be a place of F such that $\pi_v = (\pi_v, V)$ is unramified. The representation π_v is completely determined by a tuple of complex numbers—its Satake parameters—in a manner that we now explain.

Let $K \subseteq G(F_v)$ be a hyperspecial subgroup and let $\mathcal{H}_v$ be the corresponding Hecke algebra. Then V^K is an $\mathcal{H}_v$-module. Since π_v is unramified, $\dim(V^K) = 1$ and thus, by Schur's lemma, $\mathcal{H}_v$ acts via a character. It follows that π_v is determined by an element of $\mathrm{Hom}_{\mathbf{C}\text{-alg}}(\mathcal{H}_v, \mathbf{C})$. If $\widehat{T} \subseteq \widehat{\mathrm{GSp}_4} = \mathrm{GSp}_4(\mathbf{C})$ is a maximal torus and W is the corresponding Weyl group, then the *Satake isomorphism* is an isomorphism

$$H_v \xrightarrow{\sim} \mathbf{C}[\widehat{T}]^W.$$

Via this isomorphism, π_v corresponds to an element of $\mathrm{Hom}_{\mathbf{C}\text{-alg}}(\mathcal{H}_v, \mathbf{C}) \cong \widehat{T}(\mathbf{C})/W$. One often identifies $\widehat{T}(\mathbf{C})/W$ with $(\mathbf{C}^\times)^3/W$. Instead, following an alternative convention (see, for example, [Tay93, pp. 294]), we identify $\widehat{T}(\mathbf{C})/W$ with the diagonal torus in $\mathrm{GSp}_4 \subseteq \mathrm{GL}_4$. Thus, by the *Satake parameters* of π_v, we mean a tuple

[5] We follow [Loe]. If $u \in \mathcal{O}_F^\times$, then $\eta(u) = 1$ and, replacing u by some power, we can assume that $\eta(u) = \eta|_{F^\times \otimes_{\mathbf{Q}} \mathbf{R}}(u)$. Hence, the vector $(n_\tau)_{\tau : F \hookrightarrow \mathbf{R}}$ is orthogonal to a finite index subgroup of the lattice spanned by the vectors $(\log|\tau(u)|)_{\tau : F \hookrightarrow \mathbf{R}}$ for $u \in \mathcal{O}_F^\times$. By Dirichlet's unit theorem, this lattice has rank $[F : \mathbf{Q}] - 1$. Hence, its orthogonal complement has rank at most 1, and it is spanned by $(1, \ldots, 1)$.

$(\alpha, \beta, \gamma, \delta) \in (\mathbf{C}^\times)^4/S_4$ such that, after reordering, we have $\alpha\delta = \beta\gamma$. This convention has the advantage that the Satake parameters correspond to the Frobenius eigenvalues of the associated Galois representation.

The strong multiplicity one theorem. We make frequent use of the following theorem, due to Piatetski-Shapiro [PS79] and Jacquet–Shalika [JS81b]:

Theorem 0.4.1 (Strong multiplicity one)**.** *Let Π_1, Π_2 be isobaric, unitary automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ and let S be a finite set of places of F. Suppose that, for almost all places $v \notin S$, the local components of Π_1 and Π_2 at v are isomorphic. Then $\Pi_1 \cong \Pi_2$.*

We deduce the following corollary:

Corollary 0.4.2. *Let Π_1, Π_2 be isobaric, unitary automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ and let S be a finite set of places of F that contains the ramified places of Π_1 and Π_2. Suppose that Π_1 and Π_2 have associated ℓ-adic Galois representations $\rho_{\Pi_1,\ell}$ and $\rho_{\Pi_2,\ell}$. Then*

$$\Pi_1 \cong \Pi_2 \iff \rho_{\Pi_1,\ell} \simeq \rho_{\Pi_2,\ell}.$$

Proof. By definition, for any place $v \notin S \cup \{v \mid \ell\}$, the Satake parameters of $\Pi_{i,v}$ correspond to the eigenvalues of $\rho_{\Pi_i,\ell}(\mathrm{Frob}_v)$. If $\rho_{\Pi_1,\ell} \simeq \rho_{\Pi_2,\ell}$, then $\Pi_{1,v}$ and $\Pi_{2,v}$ have the same Satake parameters for all $v \notin S \cup \{v \mid \ell\}$ and are thus isomorphic. Hence, by the strong multiplicity one theorem, $\Pi_1 \cong \Pi_2$. The converse follows from the Chebotarev density theorem and the fact that a semisimple representation is determined by its trace. □

Chapter 1
The Irreducibility Conjecture

The Irreducibility Conjecture *asserts that cuspidal automorphic representations should have irreducible Galois representations. In this chapter, we discuss the history of this conjecture and the efforts that have been made to prove it.*

All progress towards proving the Irreducibility Conjecture depends on a single strategy: if $\rho_{\pi,\ell}$ is reducible and its subrepresentations are modular, then, using the modularity of these subrepresentations, we can apply automorphic techniques to contradict the cuspidality of π. Therefore, given general modularity theorems, we can prove general irreducibility theorems.

It is this dependence on modularity theorems that makes proving the Irreducibility Conjecture so difficult. Given the rarity of modularity theorems, proofs of cases of the Irreducibility Conjecture must make the most of the few modularity theorems that we have. In particular, in Section 1.2, following ideas of Dieulefait [Die07] and Ramakrishnan [Ram13], we demonstrate how to use potential modularity *to prove irreducibility theorems.*

1.1 The Irreducibility Conjecture

Assuming the Langlands conjectures, there is a correspondence between (algebraic) automorphic representations and (geometric) Galois representations. It is natural to ask what kinds of Galois representations should correspond to the cuspidal automorphic representations, which are, in a sense, the "building blocks" of automorphic representations.

Conjecture 1.1.1 (Irreducibility Conjecture)**.** *Let π be a cuspidal automorphic representation of GL_n. Suppose that there exists an ℓ-adic Galois representation $\rho_{\pi,\ell}$ attached to π. Then $\rho_{\pi,\ell}$ is irreducible.*

Aesthetically, this conjecture makes sense: the "building blocks" of automorphic representations should correspond to the "building blocks" of Galois representations. Moreover, the conjecture can be thought of as an automorphic version of the Tate conjecture for algebraic varieties.

1.1.1 The Irreducibility Conjecture for Hilbert modular forms

The Irreducibility Conjecture was proven for classical and for Hilbert modular forms by Ribet [Rib77, Rib84], building on previous results of Serre [Ser98], Swinnerton-Dyer [SD73] and Deligne–Serre [Del71, Section 8.7]. The strategy used by these authors is essentially the only one that has been successful in proving cases of the conjecture.

Theorem 1.1.2 (Irreducibility Conjecture for Hilbert modular forms)**.** *Let F be a totally real field and let f be a cuspidal Hilbert modular form over F. For each prime ℓ, let*

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

be the ℓ-adic Galois representation associated to f. Then $\rho_{f,\ell}$ is irreducible.

Proof. We follow the proof given in [Rib84]. Suppose, for contradiction, that $\rho_{f,\ell}$ is reducible. We derive a contradiction in three steps:

1. Since $\rho_{f,\ell}$ is reducible, it decomposes as a direct sum of one-dimensional Galois representations. Moreover, since $\rho_{f,\ell}$ is Hodge–Tate, so are these subrepresentations. When F is totally real, the Hodge–Tate characters of $\mathrm{Gal}(\overline{F}/F)$ are products of finite order characters and powers of the cyclotomic character.[1] Hence, we can write
$$\rho_{f,\ell} = \chi_1\varepsilon_\ell^{n_1} \oplus \chi_2\varepsilon_\ell^{n_2},$$
where χ_1, χ_2 are finite order characters, ε_ℓ is the ℓ-adic cyclotomic character and $n_1, n_2 \in \mathbf{Z}$ with $n_1 \geq n_2$.

2. By class field theory, the finite order Galois characters χ_1, χ_2 are the Galois representations associated to finite order Hecke characters η_1, η_2.

3. Hence, away from a finite set of places of F, we have an equality of partial L-functions
$$\begin{aligned} L^*(\rho_{f,\ell} \otimes \chi_1^{-1}, s) &= L^*(\varepsilon_\ell^{n_1} \oplus \chi_2\chi_1^{-1}\varepsilon_\ell^{n_2}, s) \\ &= L^*(\varepsilon_\ell^{n_1}, s)L^*(\chi_2\chi_1^{-1}\varepsilon_\ell^{n_2}, s) \\ &= \zeta_F^*(s - n_1)L^*(\eta_2\eta_1^{-1}, s - n_2). \end{aligned}$$
When $s = 1 + n_1$, the Dedekind zeta function $\zeta_F^*(s - n_1)$ has a simple pole and $L^*(\eta_2\eta_1^{-1}, s-n_2)$ is non-zero. Hence, the right hand side has a pole at $s = 1+n_1$. On the other hand, since $f \otimes \eta_1^{-1}$ is cuspidal, its L-function is entire. Hence, the left hand side is finite when $s = 1 + n_1$, which is a contradiction.

□

1.1.2 A general strategy

The proof of Theorem 1.1.2 translates into a general strategy for proving the irreducibility of automorphic Galois representations. Suppose that π is a cuspidal automorphic representation of GL_n and that we can associate an ℓ-adic Galois representation $\rho_{\pi,\ell}$ to π. The strategy is to derive a contradiction in three steps:

[1] By class field theory, this fact follows from the corresponding fact that any algebraic Hecke character over a totally real field is a product of a finite order character and a power of the norm character [Wei56].

1. Suppose, for contradiction, that $\rho_{\pi,\ell}$ is reducible, and apply Galois theoretic arguments (perhaps with some automorphic input) to show that $\rho_{\pi,\ell}$ decomposes as
$$\rho_{\pi,\ell} = \bigoplus_{i=1}^{m} \rho_i,$$
where the ρ_i are Galois representations with good properties (e.g. Hodge–Tate, crystalline, of a certain form).

2. Apply an automorphy theorem—in Ribet's case, class field theory—to attach to each ρ_i an automorphic representation π_i.

3. Apply automorphic arguments, such as the analytic properties of automorphic L-functions or the generalised Ramanujan conjecture, to contradict the cuspidality of π.

It is the second step that makes the Irreducibility Conjecture so difficult to prove: automorphy theorems are extremely rare and often rely on strong hypotheses. Indeed, the only case for which automorphy is completely known is for one-dimensional representations, where it follows from class field theory.

1.1.3 A weaker form of the Irreducibility Conjecture

Only one other case of the Irreducibility Conjecture has been completely proven: when F is totally real and π is an essentially self-dual automorphic representation of $\mathrm{GL}_3(\mathbf{A}_F)$ [BR92]. This case can still rely on class field theory, while higher-dimensional cases must use higher-dimensional modularity results. However, progress has been made towards the following weaker conjecture:

Conjecture 1.1.3. *Let π be a cuspidal automorphic representation of GL_n over a number field F. Suppose that there is a compatible system of ℓ-adic Galois representations $(\rho_{\pi,\ell})_\ell$,*
$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$
attached to π. Then $\rho_{\pi,\ell}$ is irreducible for all primes $\ell \in \mathcal{L}$ for a set of primes $\mathcal{L}$ of Dirichlet density 1.

If F is a CM field, $n \leq 6$ and π is a regular, algebraic, cuspidal, polarisable automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, then Conjecture 1.1.3 has been proven by [Ram13, CG13, Xia19]. More generally, assuming the additional, highly restrictive hypothesis that π is *extremely regular*, Barnet-Lamb–Gee–Geraghty–Taylor [BLGGT14] have proven Conjecture 1.1.3 for all n. Their proofs all follow the strategy outlined in Section 1.1.2. Indeed, the main result of [BLGGT14] is a potential automorphy theorem for polarisable automorphic representations of GL_n, and the assumption of extreme regularity is used to ensure that any subrepresentations of ρ_ℓ satisfy the hypotheses of this automorphy theorem.

The following table summarises the current state of the field:

Table 1.1: Progress towards proving the Irreduciblity Conjecture

Dimension	Case	Modularity theorem used	Reference
2	F totally real	Class field theory	[Rib84]
	F CM	Class field theory	[Tay94]
3	F totally real π polarisable	Class field theory	[BR92]
$n \leq 4$	$F = \mathbf{Q}$[2] π regular Irreducibility for all but finitely many primes ℓ for which $\rho_{\pi,\ell}$ is crystalline.[3]	Potential modularity for GL_2 [Tay06]	[Ram13]
$n \leq 5$	F totally real π regular and polarisable Irreducibility for 100% of primes	Potential automorphy for GL_n [BLGGT14]	[CG13]
$n \leq 6$	F CM π regular and polarisable Irreducibility for 100% of primes	Potential automorphy for GL_n [BLGGT14, PT15]	[Xia19]
All n	F CM π extremely regular and polarisable Irreducibility for 100% of primes	Potential automorphy for GL_n [BLGGT14]	[BLGGT14]
All n	F CM π regular and polarisable Irreducibility for a positive density of primes	Potential automorphy for GL_n [BLGGT14, PT15]	[PT15]

In the case that π is irregular, to the best of our knowledge, this thesis and [Wei18b] give the first proof of a case of the Irreducibility Conjecture when $n \geq 4$.

1.2 Potential automorphy and irreducibility

Let $\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ be the ℓ-adic Galois representation associated to a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$. The strategy outlined in Section 1.1.2 relies on the existence of automorphy theorems that would apply to any subrepresentations of $\rho_{\pi,\ell}$. However, in general, it is too optimistic to expect to prove that a subrepresentation of $\rho_{\pi,\ell}$ is automorphic. In this section, we demonstrate how

[2] See Remark 5.3.2.

[3] In particular, if π is polarisable, then we obtain irreducibility for all but finitely many primes.

to prove that $\rho_{\pi,\ell}$ is irreducible knowing only that its subrepresentations are *potentially automorphic*.

We outline two similar approaches for using potential automorphy to understand how $\rho_{\pi,\ell}$ decomposes into irreducible subrepresentations. Both approaches originate with Brauer's proof that any Artin representation is isomorphic to a virtual sum of representations induced from finite order characters. More generally, a potentially automorphic Galois representation is isomorphic to a virtual sum of representations induced from automorphic Galois representations. We use this fact in two different ways:

1. Following [Ram13, Theorem C] and [BLGGT14, Theorem 5.5.2], we use the potential automorphy of a Galois representation to understand the analytic properties of its L-function. If $\rho_{\pi,\ell}$ is reducible and its subrepresentations are potentially automorphic, then these analytic properties contradict the cuspidality of π.
2. Following an argument of Dieulefait [Die07] and its generalisation in [BLGGT14, Theorem 5.5.1], we use the potential automorphy of a Galois representation to put it into a compatible system of Galois representations. In particular, if ρ_{π,ℓ_0} is reducible for a single prime ℓ_0 and its subrepresentations are potentially automorphic, then $\rho_{\pi,\ell}$ is reducible for all primes.

The key representation theoretic input is Brauer's Induction Theorem.

Theorem 1.2.1 (Brauer's Induction Theorem)**.** *Let G be a finite group and let ρ be a representation of G. Then there exist nilpotent subgroups $H_i \leq G$, one-dimensional representations ψ_i of H_i and integers n_i such that, as virtual representations, we have*

$$\rho = \bigoplus_i n_i \operatorname{Ind}_{H_i}^G(\psi_i).$$

1.2.1 Potential automorphy and L-functions

In this subsection, we prove the following theorem, which is a generalisation of [Ram13, Theorem C].

Theorem 1.2.2. *Let F be a number field and let*

$$\rho_1 : \operatorname{Gal}(\overline{F}/F) \to \operatorname{GL}_{n_1}(\overline{\mathbf{Q}}_\ell)$$
$$\rho_2 : \operatorname{Gal}(\overline{F}/F) \to \operatorname{GL}_{n_2}(\overline{\mathbf{Q}}_\ell)$$

be irreducible Galois representations. Suppose that there exists a finite, Galois extension E/F and that, for each i, there is a unitary cuspidal automorphic representation π_i of $\operatorname{GL}_{n_i}(\mathbf{A}_E)$, such that $\rho_i|_E$ is irreducible and

$$L^*(\pi_i, s - \frac{w}{2}) = L^*(\rho_i|_E, s)$$

for some fixed integer w.

Then $L^*(\rho_1 \otimes \rho_2^\vee, s)$ has meromorphic continuation to the whole of $\mathbf{C}$ and

$$-\operatorname{ord}_{s=1} L^*(\rho_1 \otimes \rho_2^\vee, s) = (\rho_1, \rho_2).$$

In words, $L^*(\rho_1 \otimes \rho_2^\vee, s)$ is non-zero at $s = 1$, and has a simple pole at $s = 1$ if and only if $\rho_1 \simeq \rho_2$.

Remark 1.2.3. In [Ram13, Theorem C], Ramakrishnan proves a similar result when ρ_1, ρ_2 are two-dimensional under slightly different assumptions: Ramakrishnan does not require ρ_1, ρ_2 to become automorphic over the same finite extension E, but instead assumes that $\rho_1 \oplus \rho_2$ is associated to an isobaric automorphic representation of GL_4. In practice, potential automorphy results can usually be carried out simultaneously for a finite number of representations.

Our key tool is the following theorem of Jacquet–Shalika and Shahidi:

Theorem 1.2.4 ([JS81a, (3.3), (3.6), (3.7)] [Sha81, Theorem 5.2])**.** *Let F be a number field and let π_1, π_2 be unitary, cuspidal automorphic representations of $\mathrm{GL}_{n_1}(\mathbf{A}_F)$ and $\mathrm{GL}_{n_2}(\mathbf{A}_F)$ respectively. Let S be a finite set of places of F, containing all the archimedean places and all the places at which either π_1 or π_2 is ramified. Then*

1. *$L^S(\pi_1 \otimes \pi_2^\vee, s)$ has meromorphic continuation to the whole of $\mathbf{C}$.*
2. *$L^S(\pi_1 \otimes \pi_2^\vee, s)$ is non-zero when $\mathrm{Re}(s) > 1$.*
3. *We have*
$$-\operatorname{ord}_{s=1} L^S(\pi_1 \otimes \pi_2^\vee, s) = \begin{cases} 1 & \pi_1 \cong \pi_2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 1.2.2. The proof is similar to the proof of [BLGGT14, Theorem 5.5.2], except that we do not assume that ρ_1, ρ_2 are pure and regular.

By Theorem 1.2.1, we can write the trivial character $\mathbf{1}$ of $\mathrm{Gal}(\overline{F}/F)$ as a virtual representation
$$\mathbf{1} = \bigoplus_j n_j \operatorname{Ind}_{E_j}^F(\psi_j),$$
where, for each j, $n_j \in \mathbf{Z}$, $F \subseteq E_j \subseteq E$, E/E_j is solvable (and hence nilpotent) and ψ_j is a character of $\mathrm{Gal}(\overline{F}/E_j)$ that is trivial on $\mathrm{Gal}(\overline{F}/E)$. Hence, as virtual representations, we have
$$\rho_1 \simeq \left(\bigoplus_j n_j \operatorname{Ind}_{E_j}^F(\psi_j) \right) \otimes \rho_1 = \bigoplus_j n_j \operatorname{Ind}_{E_j}^F(\rho_1|_{E_j} \otimes \psi_j)$$
and
$$\rho_1 \otimes \rho_2^\vee = \bigoplus_j n_j \operatorname{Ind}_{E_j}^F(\rho_1|_{E_j} \otimes \rho_2|_{E_j}^\vee \otimes \psi_j).$$

Now, since E/E_j is solvable, each $\rho_i|_{E_j}$ is associated to a cuspidal automorphic representation $\pi_i^{E_j}$ of $\mathrm{GL}_{n_i}(\mathbf{A}_{E_j})$, which base changes to π_i over E. Indeed, by induction, we may assume that E/E_j is cyclic of prime order p and write $\mathrm{Gal}(E/E_j) = \langle \sigma \rangle$. For each $i = 1, 2$, because $\rho_i|_E$ is the restriction of a representation of $\mathrm{Gal}(\overline{F}/F)$, we have $\rho_i|_E \simeq \rho_i|_E^\sigma$. Hence, by the strong multiplicity one theorem for GL_n (see Corollary 0.4.2), $\pi_i \cong \pi_i^\sigma$. By cyclic base change [AC89, Theorem 4.2], π_i extends to p distinct cuspidal automorphic representations $\pi_i^{E_j} \otimes \eta^m$ of $\mathrm{GL}_{n_i}(\mathbf{A}_{E_j})$, where $m = 0, \ldots, p-1$, and η is the Dirichlet character corresponding to the extension E/E_j. Since $\rho_i|_{E_j}$ is irreducible, it is the Galois representation associated to $\pi_i^{E_j} \otimes \eta^m$ for exactly one m. Without loss of generality, we may take $m = 0$.

Hence, for each j, writing η_j for the Hecke character attached to ψ_j, we have

$$\begin{aligned}
-\operatorname{ord}_{s=1} L^*(\rho_1|_{E_j} \otimes \rho_2|_{E_j}^\vee \otimes \psi_j, s) &= -\operatorname{ord}_{s=1} L^*(\pi_1^{E_j} \otimes \eta_j \otimes (\pi_2^{E_j})^\vee, s) \\
&= \begin{cases} 1 & \pi_1^{E_j} \otimes \eta_j \cong \pi_2^{E_j} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \rho_1|_{E_j} \otimes \psi_j \cong \rho_2|_{E_j} \\ 0 & \text{otherwise} \end{cases} \\
&= (\rho_1|_{E_j} \otimes \psi_j, \rho_2|_{E_j}).
\end{aligned}$$

The second equality follows from Theorem 1.2.4, the third equality follows from the Chebotarev density theorem and the fourth equality follows from Schur's lemma and uses the irreducibility of $\rho_i|_E$.

Hence,

$$\begin{aligned}
-\operatorname{ord}_{s=1} L^*(\rho_1 \otimes \rho_2^\vee, s) &= -\operatorname{ord}_{s=1} L^*\left(\bigoplus_j n_j \operatorname{Ind}_{E_j}^F(\rho_1|_{E_j} \otimes \rho_2|_{E_j}^\vee \otimes \psi_j), s\right) \\
&= -\sum_j n_j \operatorname{ord}_{s=1} L^*(\operatorname{Ind}_{E_j}^F(\rho_1|_{E_j} \otimes \rho_2|_{E_j}^\vee \otimes \psi_j), s) \\
&= -\sum_j n_j \operatorname{ord}_{s=1} L^*(\rho_1|_{E_j} \otimes \rho_2|_{E_j}^\vee \otimes \psi_j, s) \\
&= \sum_j n_j (\rho_1|_{E_j} \otimes \psi_j, \rho_2|_{E_j}) \\
&= \sum_j n_j (\operatorname{Ind}_{E_j}^F(\rho_1|_{E_j} \otimes \psi_j), \rho_2) \\
&= \left(\bigoplus_j n_j \operatorname{Ind}_{E_j}^F(\rho_1|_{E_j} \otimes \psi_j), \rho_2\right) \\
&= (\rho_1, \rho_2)
\end{aligned}$$

□

Corollary 1.2.5. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Suppose that π has an associated ℓ-adic Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$, and write*

$$\rho = \rho_1 \oplus \cdots \oplus \rho_m,$$

where each ρ_i is irreducible. Suppose that there exists a finite, Galois extension E/F and an integer $w \in \mathbf{Z}$ such that, for each i, $\rho_i|_E$ is irreducible and automorphic: there is a unitary cuspidal automorphic representation π_i with $L^*(\pi_i, s - \frac{w}{2}) = L^*(\rho_i|_E, s)$. Then $m = 1$, i.e. ρ is irreducible.

Proof. By Theorem 1.2.2 we have

$$\begin{aligned}
-\operatorname{ord}_{s=1} L^*(\rho \otimes \rho^\vee, s) &= \sum_{1 \le i,j \le m} -\operatorname{ord}_{s=1} L^*(\rho_i \otimes \rho_j^\vee, s) \\
&= \sum_{1 \le i,j \le m} (\rho_i, \rho_j)
\end{aligned}$$

$$= m + 2\left(\sum_{1\le i<j\le m} (\rho_i, \rho_j)\right)$$
$$\ge m.$$

Since π is cuspidal, the partial L-function $L^*(\pi \otimes \pi^\vee, s) = L^*(\rho \otimes \rho^\vee, s)$ has a simple pole at $s = 1$. It follows that $m = 1$. □

Finally, we record a proof of the Irreducibility Conjecture for automorphic representations $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_F)$ for any number field F, assuming the existence of Galois representations. This result is presumably well-known, but we do not know of a reference in the literature. When F is totally real, this result is due to Ribet [Rib84]; when F is imaginary quadratic, Taylor [Tay94, Section 3] proved irreducibility in many cases.

Theorem 1.2.6. *Let F be a number field and let $\boldsymbol{\pi}$ be a unitary*[4] *cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbf{A}_F)$*. Suppose that there is a compatible system of ℓ-adic Galois representations*[5]

$$\rho_{\boldsymbol{\pi},\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

such that, for each ℓ, $L^*(\boldsymbol{\pi}, s - \frac{w}{2}) = L^*(\rho_{\boldsymbol{\pi},\ell}, s)$ for some integer w. Then $\rho_{\boldsymbol{\pi},\ell}$ is irreducible.

Proof. Suppose that $\rho_{\boldsymbol{\pi},\ell} = \chi \oplus \det(\rho_{\boldsymbol{\pi},\ell})\chi^{-1}$ is reducible. Then $\rho_{\boldsymbol{\pi},\ell}$ is abelian and is rational over a number field E by assumption. Hence, by [Hen82, Theorem 2], $\rho_{\boldsymbol{\pi},\ell}$ is Hodge–Tate. Therefore, by class field theory, there is an algebraic Hecke character η associated to χ.

By definition, there are integers $n_\tau, n_{c\tau}$, such that, for any $x \in F^\times \otimes_{\mathbf{Q}} \mathbf{R} \subseteq \mathbf{A}_F^\times$, we have

$$\eta(x) = \prod_{v \text{ real} \leftrightarrow \tau:F\hookrightarrow\mathbf{R}} \tau(x)^{n_\tau} \prod_{v \text{ complex} \leftrightarrow \tau,c\tau:F\hookrightarrow\mathbf{C}} \tau(x)^{n_\tau} c\tau(x)^{n_{c\tau}}.$$

By Weil's classification of Hecke characters [Wei56], there is a CM or totally real field $F' \subseteq F$ such that n_τ depends only on $\tau|_{F'}$, and there is a fixed integer w' such that $w' = n_\tau + n_{c\tau}$ is independent of τ, where c is any choice of complex conjugation. In particular, χ is pure of weight w': for almost all places v of F, we have

$$|\chi(\mathrm{Frob}_v)| = N(v)^{w'/2}.$$

Now, by the assumption that $\boldsymbol{\pi}$ is unitary, for almost all places v of F, we have

$$|\det(\rho_{\boldsymbol{\pi},\ell})(\mathrm{Frob}_v)| = N(v)^w.$$

Moreover, by [JS81b, Corollary 2.5], since $\chi(\mathrm{Frob}_v)$ is a Satake parameter of $\boldsymbol{\pi}$, we must have

$$N(v)^{w-1/2} < |\chi(\mathrm{Frob}_v)| < N(v)^{w+1/2}.$$

Hence, $w' = w$. If $\chi_{\boldsymbol{\pi}}$ denotes the central character of $\boldsymbol{\pi}$, it follows that

$$L^*(\eta, s - \frac{w}{2}) = L^*(\chi, s) \text{ and } L^*(\chi_{\boldsymbol{\pi}}\eta^{-1}, s - \frac{w}{2}) = L^*(\det(\rho_{\boldsymbol{\pi},\ell})\chi^{-1}, s).$$

The result follows from Corollary 1.2.5. □

[4] We can always twist an automorphic representation by a power of $|\det|$ to ensure that it is unitary.

[5] In particular, we assume that there is a number field E, such that $\mathrm{Tr}(\rho_{\boldsymbol{\pi},\ell}(\mathrm{Frob}_v)) \in E$ for almost all places v of F.

1.2.2 Potential automorphy and compatible systems

In this subsection, we show how to use potential automorphy to put Galois representations into compatible systems. The following result is based on [BLGGT14, Theorem 5.5.1], which is itself based on [Die07].

Theorem 1.2.7. *Let F be a number field, let ℓ_0 be a prime and let*

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_{\ell_0})$$

be an irreducible Galois representation. Suppose that there exists a finite, Galois extension E/F such that $\rho|_E$ is irreducible and is contained in a compatible system $(r_\ell)_\ell$ of Galois representations. Suppose moreover that:

1. *For each prime ℓ, r_ℓ decomposes as a direct sum of distinct irreducible subrepresentations.*
2. *For any intermediate extension $F \subseteq E' \subseteq E$ with $\mathrm{Gal}(E/E')$ solvable, $(r_\ell)_\ell$ extends to a compatible system $(r_\ell^{(E')})_\ell$ of $\mathrm{Gal}(\overline{F}/E')$, such that*
   (a) $\rho|_{E'} = r_{\ell_0}^{(E')}$.
   (b) *For each ℓ, r_ℓ and $r_\ell^{(E')}$ have the same number of irreducible subrepresentations.*

Then r_ℓ extends to a compatible system of Galois representations of $\mathrm{Gal}(\overline{F}/F)$ that contains ρ.

While this theorem is purely Galois theoretic, in practice, one uses potential automorphy to produce the compatible system $(r_\ell)_\ell$.

Corollary 1.2.8. *Let F be a CM (or totally real) field, let ℓ_0 be a prime and let*

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_{\ell_0})$$

be an irreducible Galois representation. Suppose that there exists a finite, Galois, CM (or totally real) extension E/F, such that:

1. *$\rho|_E$ is the Galois representation associated to a regular algebraic, polarisable cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_E)$.*
2. *$\rho|_E$ is irreducible.*
3. *If G is the Zariski closure of $\rho(\mathrm{Gal}(\overline{F}/F))$, G° is its identity connected component and $F^\circ = F^{\rho^{-1}(G^\circ)}$, then E is linearly disjoint from F°.*

Then ρ is contained in a compatible system of Galois representations.

Proof. Let $(r_\ell)_\ell$ be the compatible system of Galois representations attached to π by [HT01, Shi11] et al. We check that $(r_\ell)_\ell$ satisfies the hypothesis of Theorem 1.2.7.

Since π is regular, $(r_\ell)_\ell$ has distinct Hodge–Tate weights. Therefore, for each ℓ, r_ℓ decomposes as a direct sum of distinct irreducible subrepresentations.

Suppose that $F \subseteq E' \subseteq E$ is an intermediate extension with $\mathrm{Gal}(E/E')$ solvable. As in Theorem 1.2.2, by base change in cyclic layers [AC89], we may extend π to an automorphic representation $\pi^{E'}$ of $\mathrm{GL}_n(\mathbf{A}_{E'})$, whose associated ℓ_0-adic Galois representation is $\rho|_{E'}$. If we define $(r_\ell^{(E')})_\ell$ to be the compatible system of Galois representations attached to $\pi^{E'}$, then these representations give the required extensions of r_ℓ.

It remains to check that, for each prime ℓ, r_ℓ and $r_\ell^{(E')}$ have the same number of subrepresentations. Define $G_{E,\ell}$ to be the Zariski closure of the image of $r_\ell(\mathrm{Gal}(\overline{F}/E))$, $G_{E',\ell}$ to be the Zariski closure of the image of $r_\ell^{(E')}(\mathrm{Gal}(\overline{F}/E'))$ and $G^\circ_{E,\ell}$, $G^\circ_{E',\ell}$ to be their identity connected components.

By a theorem of Serre (c.f. [LP92, Theorem 6.14]), the groups $G_{E,\ell}/G^\circ_{E,\ell}$ and $G_{E',\ell}/G^\circ_{E',\ell}$ are independent of ℓ.

By the third hypothesis, we have $G = G_{E,\ell_0} = G_{E',\ell_0}$. Hence, for any ℓ,

$$\begin{aligned} G_{E,\ell}/G^\circ_{E,\ell} &\cong G_{E,\ell_0}/G^\circ_{E,\ell_0} \\ &\cong G_{E',\ell_0}/G^\circ_{E',\ell_0} \\ &\cong G_{E',\ell}/G^\circ_{E',\ell}. \end{aligned}$$

Since $G_{E,\ell}$ is a union of connected components of $G_{E',\ell}$, it follows that $G_{E,\ell} = G_{E',\ell}$. Hence, r_ℓ and $r_\ell^{(E')}$ have the same number of subrepresentations. The result follows from Theorem 1.2.7. □

Remark 1.2.9. Conjecturally, any compatible system of Galois representations should satisfy condition 2 of Theorem 1.2.7: as we have seen in the proof of Corollary 1.2.8, condition 2 is a corollary of automorphy and solvable base change. However, without knowing that the compatible system is automorphic, we do not know of any way to verify this condition. This gap in our knowledge lies at the heart of why automorphic input is required to prove irreducibility theorems.

If ℓ_1, ℓ_2 are primes and

$$r_{\ell_1} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_{\ell_1}), \qquad r_{\ell_2} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_{\ell_2})$$

are compatible Galois representations, then, up to reordering, their subrepresentations should be compatible as well. If we were able to prove this compatibility, then we could prove that the decomposition type of a compatible system (r_ℓ) is independent of ℓ, which would reduce the problem of proving the irreducibility of all members of a compatible system to verifying the irreducibility of a single member. However, even in simple cases, it does not seem possible to prove this compatibility without automorphic input.

To see how the compatibility of subrepresentations relates to condition 2 of Theorem 1.2.7, consider the case that E/F is quadratic in Theorem 1.2.7. Then $\rho|_E$ is contained in a compatible system (r_ℓ) of representations of $\mathrm{Gal}(\overline{F}/E)$. Suppose that r_ℓ extends to representations $s_\ell, s_\ell \otimes \chi_{E/F}$ of $\mathrm{Gal}(\overline{F}/F)$, where $\chi_{E/F}$ is the quadratic character that cuts out the extension E/F. With condition 2, it follows that, without loss of generality, s_ℓ and ρ are compatible. Without condition 2, even though the representations

$$\mathrm{Ind}_E^F(r_\ell) = s_\ell \oplus (s_\ell \otimes \chi_{E/F})$$

and

$$\mathrm{Ind}_E^F(r_{\ell_0}) = \rho \oplus (\rho \otimes \chi_{E/F})$$

are compatible, it does not seem possible to deduce that ρ and s_ℓ are compatible. In particular, condition 2 of Theorem 1.2.7 is needed even if E/F is a quadratic extension!

Proof of Theorem 1.2.7

We now prove Theorem 1.2.7, following the proof of [BLGGT14, Theorem 5.5.1]. We begin with some auxiliary lemmas. Fix a prime ℓ and an intermediate extension $F \subseteq E' \subseteq E$, such that E/E' is solvable.

Write
$$r_\ell = r_{1,\ell} \oplus \cdots \oplus r_{t,\ell}$$
for the decomposition of r_ℓ into t distinct irreducible subrepresentations. Then, by assumption,
$$r_\ell^{(E')} = r_{1,\ell}^{(E')} \oplus \cdots \oplus r_{t,\ell}^{(E')},$$
where each $r_{i,\ell}^{(E')}$ is irreducible and extends $r_{i,\ell}$.

Let $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Since r_{ℓ_0} extends to a representation of $\mathrm{Gal}(\overline{F}/F)$, it follows that $r_{\ell_0}^\sigma \simeq r_{\ell_0}$. In particular, for all but finitely many places v of F, we have $\mathrm{Tr}(r_{\ell_0}(\mathrm{Frob}_v)) = \mathrm{Tr}(r_{\ell_0}^\sigma(\mathrm{Frob}_v))$. Since r_ℓ and r_{ℓ_0} are compatible, it follows that $\mathrm{Tr}(r_\ell(\mathrm{Frob}_v)) = \mathrm{Tr}(r_\ell^\sigma(\mathrm{Frob}_v))$. Hence, by the Chebotarev density theorem and the fact that a semisimple representation is determined by its trace, we see that $r_\ell^\sigma \simeq r_\ell$. It follows that the representation
$$(r_\ell^{(E')})^\sigma = (r_{1,\ell}^{(E')})^\sigma \oplus \cdots \oplus (r_{t,\ell}^{(E')})^\sigma$$
is an extension of r_ℓ to $\mathrm{Gal}(\overline{F}/\sigma^{-1}E')$. We show that these representations are isomorphic to the representations $r_{i,\ell}^{(\sigma^{-1}E')}$ given by the hypotheses of Theorem 1.2.7.

Lemma 1.2.10. *Let $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Then for each $i = 1, \ldots, t$,*
$$(r_{i,\ell}^{(E')})^\sigma \simeq r_{i,\ell}^{(\sigma^{-1}E')}.$$

Proof. By assumption, $\rho|_{\sigma^{-1}E'} = r_{\ell_0}^{(\sigma^{-1}E')}$ and $(\rho|_{E'})^\sigma \simeq (r_{\ell_0}^{(E')})^\sigma$. Since $\rho^\sigma \simeq \rho$, it follows that
$$(r_{\ell_0}^{(E')})^\sigma \simeq r_{\ell_0}^{(\sigma^{-1}E')}.$$
By the Chebotarev density theorem, we have
$$(r_\ell^{(E')})^\sigma \simeq r_\ell^{(\sigma^{-1}E')} \tag{1.1}$$
as well.

We first show that, for each i, $r_{i,\ell} \simeq r_{i,\ell}^\sigma$. If not, then, for each integer j, $r_{i,\ell}^{\sigma^j}$ is a subrepresentation of $r_\ell^{\sigma^j} \simeq r_\ell$. Since σ has finite order in $\mathrm{Gal}(E/F)$, there is a smallest integer m_0 such that $r_{i,\ell}^{\sigma^{m_0}} \simeq r_{i,\ell}$. If m is the order of σ in $\mathrm{Gal}(E/F)$, then $m_0 \mid m$ and, by Mackey theory, the representation
$$r_{i,\ell} \oplus r_{i,\ell}^\sigma \oplus \cdots \oplus r_{i,\ell}^{\sigma^{m-1}}$$
of r_ℓ extends to a direct sum of $\frac{m}{m_0}$ irreducible representations of $\mathrm{Gal}(\overline{F}/E^{\langle\sigma\rangle})$.[6] But, by assumption, r_ℓ and $r_\ell^{(E^{\langle\sigma\rangle})}$ have the same number of subrepresentations, a contradiction. Hence, $r_{i,\ell} \simeq r_{i,\ell}^\sigma$.

[6]Indeed, by Frobenius reciprocity, $(\mathrm{Ind}_E^{E^{\langle\sigma\rangle}}(r_{i,\ell}), \mathrm{Ind}_E^{E^{\langle\sigma\rangle}}(r_{i,\ell})) = (r_{i,\ell}, \mathrm{Res}_E^{E^{\langle\sigma\rangle}} \mathrm{Ind}_E^{E^{\langle\sigma\rangle}}(r_{i,\ell}))$ and Mackey theory tells us that
$$\mathrm{Res}_E^{E^{\langle\sigma\rangle}} \mathrm{Ind}_E^{E^{\langle\sigma\rangle}}(r_{i,\ell}) \simeq r_{i,\ell} \oplus r_{i,\ell}^\sigma \oplus \cdots \oplus r_{i,\ell}^{\sigma^{m-1}}.$$
Hence, $(\mathrm{Ind}_E^{E^{\langle\sigma\rangle}}(r_{i,\ell}), \mathrm{Ind}_E^{E^{\langle\sigma\rangle}}(r_{i,\ell})) = \frac{m}{m_0}$.

Similarly,

$$(r_{i,\ell}^{(E')})^\sigma \cong r_{i,\ell}^{(\sigma^{-1}E')}$$

for each i. Indeed, by Equation (1.1), both $r_{i,\ell}^{(\sigma^{-1}E')}$ and $(r_{i,\ell}^{(E')})^\sigma$ are subrepresentations of $r_\ell^{(\sigma^{-1}E')}$ and, by definition,

$$r_{i,\ell}^{(\sigma^{-1}E')}|_E = r_{i,\ell}$$

and

$$(r_{i,\ell}^{(E')})^\sigma|_E = r_{i,\ell}^\sigma \simeq r_{i,\ell}.$$

Hence, $r_{i,\ell}^{(\sigma^{-1}E')} \simeq (r_{i,\ell}^{(E')})^\sigma$. □

Corollary 1.2.11. *Let $F \subseteq E_1, E_2 \subseteq E$ be intermediate extensions with E/E_1 and E/E_2 solvable. Let ψ_1 and ψ_2 be characters of $\mathrm{Gal}(\overline{F}/E_1)$ and $\mathrm{Gal}(\overline{F}/E_2)$ that are trivial on $\mathrm{Gal}(\overline{F}/E)$. Then, for each $i = 1, \ldots, t$,*

$$\left(\mathrm{Ind}_{E_1}^F(\psi_1 \otimes r_{i,\ell}^{(E_1)}), \mathrm{Ind}_{E_2}^F(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right) = \left(\mathrm{Ind}_{E_1}^F(\psi_1), \mathrm{Ind}_{E_2}^F(\psi_2)\right).$$

Proof. By Frobenius reciprocity

$$\left(\mathrm{Ind}_{E_1}^F(\psi_1 \otimes r_{i,\ell}^{(E_1)}), \mathrm{Ind}_{E_2}^F(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right) = \left(\psi_1 \otimes r_{i,\ell}^{(E_1)}, \mathrm{Res}_{E_1}^F \mathrm{Ind}_{E_2}^F(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right).$$

By Mackey theory,

$$\mathrm{Res}_{E_1}^F \mathrm{Ind}_{E_2}^F(\psi_2 \otimes r_{i,\ell}^{(E_2)}) = \bigoplus_\sigma \mathrm{Ind}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1} \left(\mathrm{Res}_{E_2\cdot(\sigma E_1)}^{E_2}(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right)^\sigma, \tag{1.2}$$

where the sum runs over $\sigma \in \mathrm{Gal}(\overline{F}/E_2)\backslash \mathrm{Gal}(\overline{F}/F)/\mathrm{Gal}(\overline{F}/E_1)$. Hence,

$$\begin{aligned}&\left(\psi_1 \otimes r_{i,\ell}^{(E_1)}, \mathrm{Res}_{E_1}^F \mathrm{Ind}_{E_2}^F(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right)\\ &= \sum_\sigma \left(\psi_1 \otimes r_{i,\ell}^{(E_1)}, \mathrm{Ind}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}\left(\mathrm{Res}_{E_2\cdot(\sigma E_1)}^{E_2}(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right)^\sigma\right)\\ &= \sum_\sigma \left(\mathrm{Res}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}(\psi_1 \otimes r_{i,\ell}^{(E_1)}), \left(\mathrm{Res}_{E_2\cdot(\sigma E_1)}^{E_2}(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right)^\sigma\right),\end{aligned}$$

where the second equality follows from Frobenius reciprocity. For each σ and for each $i = 1, \ldots, t$, by Lemma 1.2.10 with $E' = E_2 \cdot (\sigma E_1)$, we have

$$\begin{aligned}\mathrm{Res}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}(r_{i,\ell}^{(E_1)}) &\simeq r_{i,\ell}^{(\sigma^{-1}E_2)\cdot E_1}\\ &\simeq (r_{i,\ell}^{(E_2\cdot(\sigma E_1))})^\sigma\\ &\simeq \left(\mathrm{Res}_{E_2\cdot(\sigma E_1)}^{E_2}(r_{i,\ell}^{(E_2)})\right)^\sigma.\end{aligned}$$

Since $\mathrm{Res}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}(r_{i,\ell}^{(E_1)})$ is irreducible and ψ_1, ψ_2 are characters, we see that

$$\left(\mathrm{Res}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}(\psi_1 \otimes r_{i,\ell}^{(E_1)}), \left(\mathrm{Res}_{E_2\cdot(\sigma E_1)}^{E_2}(\psi_2 \otimes r_{i,\ell}^{(E_2)})\right)^\sigma\right)$$

is either 1 or 0, depending on whether or not the two representations are isomorphic. Moreover, since $r_{i,\ell}$ is irreducible, if χ is any non-trivial character of $\mathrm{Gal}(\overline{F}/(\sigma^{-1}E_2)\cdot E_1)$ that is trivial on E, we must have

$$\mathrm{Res}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}(r_{i,\ell}^{(E_1)}) \not\simeq \mathrm{Res}_{(\sigma^{-1}E_2)\cdot E_1}^{E_1}(r_{i,\ell}^{(E_1)}) \otimes \chi.$$

It follows that

$$\left(\mathrm{Res}^{E_1}_{(\sigma^{-1}E_2)\cdot E_1}(\psi_1 \otimes r^{(E_1)}_{i,\ell}), \left(\mathrm{Res}^{E_2}_{E_2\cdot(\sigma E_1)}(\psi_2 \otimes r^{(E_2)}_{i,\ell}\right)^{\sigma}\right)$$
$$= \left(\mathrm{Res}^{E_1}_{(\sigma^{-1}E_2)\cdot E_1}(\psi_1), \mathrm{Res}^{E_2}_{E_2\cdot(\sigma E_1)}(\psi_2)^{\sigma}\right).$$

Hence,

$$\begin{aligned}
\left(\mathrm{Ind}^F_{E_1}(\psi_1 \otimes r^{(E_1)}_{i,\ell}), \mathrm{Ind}^F_{E_2}(\psi_2 \otimes r^{(E_2)}_{i,\ell})\right) &= \sum_{\sigma} \left(\mathrm{Res}^{E_1}_{(\sigma^{-1}E_2)\cdot E_1}(\psi_1), \left(\mathrm{Res}^{E_2}_{E_2\cdot(\sigma E_1)}(\psi_2)\right)^{\sigma}\right) \\
&= \sum_{\sigma} \left(\psi_1, \mathrm{Ind}^{E_1}_{(\sigma^{-1}E_2)\cdot E_1}\left(\mathrm{Res}^{E_2}_{E_2\cdot(\sigma E_1)}(\psi_2)\right)^{\sigma}\right) \\
&\overset{(1.2)}{=} (\psi_1, \mathrm{Res}^F_{E_1}\mathrm{Ind}^F_{E_2}(\psi_2)) \\
&= (\mathrm{Ind}^F_{E_1}(\psi_1), \mathrm{Ind}^F_{E_2}(\psi_2))
\end{aligned}$$

as required. □

We are now ready to complete the proof of Theorem 1.2.7

Proof of Theorem 1.2.7. We use Brauer induction to extend r_ℓ to a representation of $\mathrm{Gal}(\overline{F}/F)$ that is compatible with ρ.

Let $\mathbf{1} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_1(\overline{\mathbf{Q}})$ be the trivial representation. By Theorem 1.2.1 we can write

$$\mathbf{1} = \bigoplus_j n_j \,\mathrm{Ind}^F_{E_j}(\psi_j),$$

where $n_j \in \mathbf{Z}$, $F \subseteq E_j \subseteq E$, E/E_j is solvable and ψ_j is a character of $\mathrm{Gal}(\overline{F}/E_j)$ that is trivial on $\mathrm{Gal}(\overline{F}/E)$. Since each ψ_j is a finite order character, it takes values in $\mathrm{GL}_1(\overline{\mathbf{Q}}) \subset \mathrm{GL}_1(\overline{\mathbf{Q}}_\ell)$ for any prime ℓ.

For each $i = 1, \dots, t$, write $\rho_{i,\ell}$ for the virtual representation of $\mathrm{Gal}(\overline{F}/F)$ given by

$$\rho_{i,\ell} = \bigoplus_j n_j \,\mathrm{Ind}^F_{E_j}(\psi_j \otimes r^{(E_j)}_{i,\ell}).$$

Then, by Corollary 1.2.11,

$$\begin{aligned}
(\rho_{i,\ell}, \rho_{i,\ell}) &= \sum_{j,k} n_j n_k \left(\mathrm{Ind}^F_{E_j}(\psi_j \otimes r^{(E_j)}_{i,\ell}), \mathrm{Ind}^F_{E_k}(\psi_k \otimes r^{(E_k)}_{i,\ell})\right) \\
&= \sum_{j,k} n_j n_k \left(\mathrm{Ind}^F_{E_j}(\psi_j), \mathrm{Ind}^F_{E_k}(\psi_k)\right) \\
&= \left(\sum_j n_j \,\mathrm{Ind}^F_{E_j}(\psi_j), \sum_k n_k \,\mathrm{Ind}^F_{E_k}(\psi_k)\right) \\
&= (\mathbf{1}, \mathbf{1}) \\
&= 1.
\end{aligned}$$

It follows that $\rho_{i,\ell}$ is an irreducible representation of $\mathrm{Gal}(\overline{F}/F)$ and not just a virtual representation. Now, set

$$\rho_\ell = \rho_{1,\ell} \oplus \cdots \oplus \rho_{t,\ell}.$$

It remains to show that the ρ_ℓ form a compatible system. It is sufficient to show that ρ_ℓ and ρ are compatible at all unramified places of F. We have

$$\rho_{\ell_0} = \bigoplus_j n_j \operatorname{Ind}_{E_j}^F (\psi_j \otimes \rho|_{E_j}) = \rho$$

and, for any unramified place v of F,

$$\begin{aligned}
\operatorname{Tr} \rho_\ell(\operatorname{Frob}_v) &= \sum_i \operatorname{Tr} \rho_{i,\ell}(\operatorname{Frob}_v) \\
&= \sum_j \sum_{\substack{\sigma \in \operatorname{Gal}(\overline{F}/F)/\operatorname{Gal}(\overline{F}/E_j), \\ \sigma \operatorname{Frob}_v \sigma^{-1} \in \operatorname{Gal}(\overline{F}/E_j)}} \psi_j^\sigma(\operatorname{Frob}_v) \operatorname{Tr} r_\ell^{(E_j)}(\sigma \operatorname{Frob}_v \sigma^{-1}) \\
&= \sum_j \sum_{\substack{\sigma \in \operatorname{Gal}(\overline{F}/F)/\operatorname{Gal}(\overline{F}/E_j), \\ \sigma \operatorname{Frob}_v \sigma^{-1} \in \operatorname{Gal}(\overline{F}/E_j)}} \psi_j^\sigma(\operatorname{Frob}_v) \operatorname{Tr} r_\ell^{(E_j)}(\operatorname{Frob}_v) \\
&= \sum_j \sum_{\substack{\sigma \in \operatorname{Gal}(\overline{F}/F)/\operatorname{Gal}(\overline{F}/E_j), \\ \sigma \operatorname{Frob}_v \sigma^{-1} \in \operatorname{Gal}(\overline{F}/E_j)}} \psi_j^\sigma(\operatorname{Frob}_v) \operatorname{Tr} r_{\ell_0}^{(E_j)}(\operatorname{Frob}_v) \\
&= \operatorname{Tr} \rho(\operatorname{Frob}_v).
\end{aligned}$$

Hence, the ρ_ℓ form a compatible system. □

Chapter 2

Automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_F)$

The primary objects of this thesis, Hilbert–Siegel modular forms, give rise to cuspidal automorphic representations of GSp_4 *over totally real fields. These automorphic representations are more convenient to work with than the Hilbert–Siegel modular forms from which they arise. This chapter summarises their key properties. In particular:*

- *We describe the holomorphic (limit of) discrete series representations of* $\mathrm{GSp}_4(\mathbf{R})$*, which are the archimedean components of automorphic representations that arise from Hilbert–Siegel modular forms.*
- *We review the relation between automorphic representations of* GSp_4 *and automorphic representations of* GL_4*, a relation that will be crucial for attaching Galois representations to Hilbert–Siegel modular forms.*
- *We discuss Arthur's classification, which categorises automorphic representations in terms of how—and whether—they are constructed from automorphic representations of smaller groups. On the Galois side, these categories should determine how the associated Galois representations split into irreducible subrepresentations.*

In this chapter, we review aspects of the local and global theory of automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_F)$ for totally real fields F.

In Section 2.1, we define the (limit of) discrete series representations of $\mathrm{GSp}_4(\mathbf{R})$ via the archimedean local Langlands correspondence. In particular, we define what we mean by the "weights" of an automorphic representation π, and what it means for π to have low or high weight.

In Section 2.2, we study the functorial transfer from GSp_4 to GL_4 and its consequences.

Finally, in Section 2.3, we review Arthur's classification of the discrete spectrum of GSp_4, which is crucial for establishing the transfer from GSp_4 to GL_4 in general.

2.1 Archimedean L-parameters

If F is a totally real field and π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, then π can be decomposed as a restricted tensor product $\pi = \otimes'_v \pi_v$, with each π_v an admissible representation of $\mathrm{GSp}_4(F_v)$. In this thesis, we are concerned with automorphic representations π with components π_v at archimedean places v lying in the holomorphic (limit of) discrete series. The aim of this chapter is to define these representations.

Rather than studying the representation theory of $\mathrm{GSp}_4(\mathbf{R})$ directly—which would be tangential to the goals of this thesis—we instead describe the L-parameters of the holomorphic (limits of) discrete series. Our exposition follows [Mok14, §3.1].

We begin by recording some notation. Let $W_{\mathbf{C}} = \mathbf{C}^\times$ denote the Weil group of $\mathbf{C}$ and let $W_{\mathbf{R}}$ be the Weil group of $\mathbf{R}$. Then

$$W_{\mathbf{R}} = \mathbf{C}^\times \sqcup \mathbf{C}^\times j,$$

where $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbf{C}^\times$. We define a map

$$|\cdot| : W_{\mathbf{R}} \to \mathbf{R}^\times$$

by sending $z \in \mathbf{C}^\times$ to $z\overline{z} = |z|^2$ and $j \mapsto -1$. This map induces an isomorphism $W_{\mathbf{R}}^{ab} \cong \mathbf{R}^\times$. Note that, under this isomorphism, the usual norm map

$$\begin{aligned} |\cdot| : \mathbf{R}^\times &\to \mathbf{R}^\times \\ x &\mapsto |x| \end{aligned}$$

corresponds to the map $|\cdot|^2 : W_{\mathbf{R}} \to \mathbf{R}^\times$.

If $z = re^{i\theta} \in \mathbf{C}^\times$, we let $(z/\overline{z})^{n/2} = e^{in\theta}$.

2.1.1 L-parameters for $\mathrm{GL}_2(\mathbf{R})$

We begin by defining the L-parameters that correspond to the (limit of) discrete series representations of $\mathrm{GL}_2(\mathbf{R})$. Let w, n be integers, with $n \geq 0$ and $n \equiv w + 1 \pmod 2$. Define

$$\begin{aligned} \phi_{(w;n)} : W_{\mathbf{R}} &\to \mathrm{GL}_2(\mathbf{C}) \\ z &\mapsto |z|^{-w} \begin{pmatrix} (z/\overline{z})^{n/2} & \\ & (z/\overline{z})^{-n/2} \end{pmatrix} \qquad \text{for } z \in \mathbf{C}^\times \\ j &\mapsto \begin{pmatrix} & 1 \\ (-1)^n & \end{pmatrix}. \end{aligned}$$

When $n \geq 1$, $\phi_{(w;n)}$ corresponds to the weight $n+1$ discrete series representation $\mathcal{D}(n+1)$ of $\mathrm{GL}_2(\mathbf{R})$, with central character $a \mapsto a^{-w}$ for $a \in \mathbf{R}^\times$. When $n = 0$, $\phi_{(w;n)}$ corresponds to the limit of discrete series representation $\mathcal{D}(1)$, with central character $a \mapsto a^{-w}$. The parity condition on w ensures that, in both cases, $\phi_{(w;n)}$ is the archimedean L-parameter attached to a classical modular form of weight $n+1$.

2.1.2 L-parameters for $\mathrm{GSp}_4(\mathbf{R})$

Next, we define the L-parameters whose L-packets contain the (limit of) discrete series representations of $\mathrm{GSp}_4(\mathbf{R})$. For integers w, m_1, m_2, with $m_1 > m_2 \geq 0$ and $m_1 + m_2 \equiv w + 1 \pmod 2$, we define an L-parameter

$$\phi_{(w;m_1,m_2)} : W_{\mathbf{R}} \to \mathrm{GSp}_4(\mathbf{C})$$

by

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\overline{z})^{(m_1+m_2)/2} & & & \\ & (z/\overline{z})^{(m_1-m_2)/2} & & \\ & & (z/\overline{z})^{-(m_1-m_2)/2} & \\ & & & (z/\overline{z})^{-(m_1+m_2)/2} \end{pmatrix}$$

for $z \in \mathbf{C}^\times$ and

$$j \mapsto \begin{pmatrix} & & & 1 \\ & & 1 & \\ & (-1)^{m_1+m_2} & & \\ (-1)^{m_1+m_2} & & & \end{pmatrix}.$$

The image of $\phi_{(w;m_1,m_2)}$ lies in $\mathrm{GSp}_4(\mathbf{C})$ and has similitude character given by

$$\begin{aligned} z &\mapsto |z|^{-2w} \\ j &\mapsto (-1)^w = (-1)^{m_1+m_2+1}. \end{aligned}$$

If we compose $\phi_{(w;m_1,m_2)}$ with the inclusion $\mathrm{GSp}_4(\mathbf{C}) \hookrightarrow \mathrm{GL}_4(\mathbf{C})$, the resulting representation of $W_{\mathbf{R}}$ is isomorphic to a direct sum of parameters of discrete series representations $\phi_{(w;m_1+m_2)} \oplus \phi_{(w;m_1-m_2)}$.

The L-packet corresponding to $\phi_{(w;m_1,m_2)}$ has two elements:

$$\left\{\pi^H_{(w;m_1,m_2)}, \pi^W_{(w;m_1,m_2)}\right\}.$$

Both $\pi^H_{(w;m_1,m_2)}$ and $\pi^W_{(w;m_1,m_2)}$ have central character given by $a \mapsto a^{-w}$ for $a \in \mathbf{R}^\times$. When $m_2 \geq 1$ they are (up to twist) discrete series representations: $\pi^H_{(w;m_1,m_2)}$ is in the holomorphic discrete series and $\pi^W_{(w;m_1,m_2)}$ is in the generic discrete series. When $m_2 = 0$, $\pi^H_{(w;m_1,0)}$ is a holomorphic limit of discrete series and $\pi^W_{(w;m_1,0)}$ is a generic limit of discrete series.

2.1.3 The "weight" of a holomorphic (limit of) discrete series

The Blattner parameter (k_1, k_2) of the holomorphic (limit of) discrete series representation $\pi^H_{(w;m_1,m_2)}$ is defined to be

$$k_1 = m_1 + 1, \quad k_2 = m_2 + 2.$$

If π is the automorphic representation corresponding to a classical Hilbert–Siegel modular form of weights $(k_{1,v}, k_{2,v})_{v|\infty}$,[1] then, for each archimedean place v, the component

[1] i.e. a vector valued Hilbert–Siegel modular form with weight $\prod_{v|\infty} \mathrm{Sym}^{k_{1,v}-k_{2,v}} \det^{k_{2,v}}$.

π_v is a holomorphic (limit of) discrete series representation with Blattner parameter $(k_{1,v}, k_{2,v})$. Hence, if π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ such that, for each archimedean place v of F, π_v lies in the holomorphic (limit of) discrete series, then we refer to the Blattner parameters $(k_{1,v}, k_{2,v})_{v|\infty}$ of π_v as the weights of π.

We say that π is *cohomological* or has *high weight* if $k_{2,v} \geq 3$ for all places v. If $k_{2,v} = 2$ for some place v, then we say that π is *non-cohomological* or has *low weight*. Cohomological automorphic representations can be realised in the étale cohomology of a local system of a suitable Shimura variety, while non-cohomological automorphic representations can only be realised in coherent cohomology.

2.1.4 Other non-degenerate limits of discrete series for $\mathrm{GSp}_4(\mathbf{R})$

Finally, for completeness, we note that there is another L-packet of non-degenerate limits of discrete series. Following [Sch17a], if m is a positive integer, then this L-packet corresponds, up to twist, to the L-parameter $W_{\mathbf{R}} \to \mathrm{GSp}_4(\mathbf{C})$ that sends

$$z \mapsto \begin{pmatrix} (z/\overline{z})^{m/2} & & & \\ & 1 & & \\ & & 1 & \\ & & & (z/\overline{z})^{-m/2} \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

If π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, whose archimedean components are all either discrete series or non-degenerate limits of discrete series, then, by [GK19, Theorem 10.5.1], we can attach Galois representations to π. Assuming local-global compatibility results for these Galois representations as in Theorem 4.2.1, it is likely that the arguments of this thesis could be extended to prove a big image theorem for these Galois representations.

2.2 The transfer map

The principle of Langlands functoriality predicts that there is a global transfer map from automorphic representations of GSp_4 to automorphic representations of GL_4. Indeed, ${}^L\mathrm{GSp}_4 = \mathrm{GSp}_4(\mathbf{C})$ and ${}^L\mathrm{GL}_4 = \mathrm{GL}_4(\mathbf{C})$, and the existence of the embedding $\mathrm{GSp}_4(\mathbf{C}) \hookrightarrow \mathrm{GL}_4(\mathbf{C})$ of L-groups means that there should be a corresponding lifting

$$\Pi(\mathrm{GSp}_4) \to \Pi(\mathrm{GL}_4)$$

of automorphic representations. This lifting should be compatible with the local Langlands correspondence: if $\pi \in \Pi(\mathrm{GSp}_4)$ lifts to $\Pi \in \Pi(\mathrm{GL}_4)$, then, for each place v of F, the Weil–Deligne representation

$$W'_{F_v} \to \mathrm{GL}_4(\mathbf{C})$$

corresponding to Π_v via the local Langlands correspondence for GL_4 should be isomorphic to the Weil-Deligne representation

$$W'_{F_v} \to \mathrm{GSp}_4(\mathbf{C}) \hookrightarrow \mathrm{GL}_4(\mathbf{C})$$

corresponding to π_v via the local Langlands correspondence for GSp_4. Moreover, an automorphic representation $\Pi \in \Pi(\mathrm{GL}_4)$ should be in the image of this lifting if and only if it is of *symplectic type*: i.e. if there is an automorphic representation $\chi \in \Pi(\mathrm{GL}_1)$ such that $\Pi \cong \Pi^\vee \otimes \chi$ and the partial L-function $L^*(\bigwedge^2(\Pi) \otimes \chi^{-1}, s)$[2] has a pole at $s = 1$.

This lifting has been achieved for *globally generic* cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_F)$ by Asgari–Shahidi [AS06]. However, the automorphic representations that correspond to Hilbert–Siegel modular forms are not globally generic: their archimedean components are holomorphic (limit of) discrete series representations, which are not generic. The existence of this lifting for all automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_F)$ follows from Arthur's classification for GSp_4, which is the subject of Section 2.3.

If F is a totally real field, then we will see in Chapter 4 that the transfer map is essential to attaching Galois representations to the automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_F)$ that arise from Hilbert–Siegel modular forms. Moreover, if $\pi \in \Pi(\mathrm{GSp}_4)$ is not generic, then the usual techniques for proving the analytic properties of the L-functions of π do not apply directly, but can be applied to the transfer of π to GL_4. It is for these reasons that this thesis is dependent on Arthur's classification.

2.3 Arthur's classification for GSp_4

In this subsection, we recall Arthur's classification of the discrete spectrum of automorphic representations of GSp_4.

Let F be a number field and let π be a discrete automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$. Arthur's classification places π into a *global A-packet*, a multi-set of (not necessarily automorphic) representations of $\mathrm{GSp}_4(\mathbf{A}_F)$. This A-packet is parametrised by an *Arthur parameter*, which can be one of six types. When π has an associated ℓ-adic Galois representation $\rho_{\pi,\ell}$, this type will determine how $\rho_{\pi,\ell}$ decomposes into subrepresentations.

We begin by giving a brief overview of Arthur parameters and A-packets for GSp_4, before describing the six types of global A-packets. For a more general introduction to Arthur's conjecture, we refer the reader to Arthur's papers, in particular [Art13], as well as the expository paper [BC09a, Appendix A] and the notes and lectures from the conference, *On the Langlands Program: Endoscopy and Beyond*, which took place in Singapore in 2018 [Sin]. Our main references for this subsection are [Art04] and [Sch17b, Sch18].

As mentioned in Section 0.2, Arthur's classification for GSp_4 is still dependent on several unpublished papers.

[2]The automorphic representation $\bigwedge^2(\Pi) \otimes \chi^{-1} \in \Pi(\mathrm{GL}_6)$ exists by [Kim03].

2.3.1 Arthur parameters and A-packets

Definition 2.3.1 ([Sch17b, pp. 4])**.** Fix a character χ of $\mathrm{GL}_1(\mathbf{A}_F)$. An *Arthur parameter* ψ for $\mathrm{GSp}_4(\mathbf{A}_F)$ with central character χ is a formal, unordered expression

$$\psi = (\mu_1 \boxtimes \nu(n_1)) \boxplus \cdots \boxplus (\mu_r \boxtimes \nu(n_r))$$

such that:

- For each i, μ_i is a unitary, cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbf{A}_F)$ for some integer m_i, and $\mu_i \cong \mu_i^\vee \otimes \chi$.
- For each i, $\nu(n_i)$ is the irreducible representation of $\mathrm{SL}_2(\mathbf{C})$ of dimension n_i.
- $\sum_{i=1}^r m_i n_i = 4$.
- $\mu_i \boxtimes \nu(n_i) \neq \mu_j \boxtimes \nu(n_j)$ for $i \neq j$.
- If n_i is odd, then μ_i is symplectic (i.e. $L(\bigwedge^2(\mu_i) \otimes \chi^{-1}, s)$ has a pole at $s = 1$). If n_i is even, then μ_i is orthogonal (i.e. $L(\mathrm{Sym}^2(\mu_i) \otimes \chi^{-1}, s)$ has a pole at $s = 1$).

Given an Arthur parameter ψ and a place v of F, using the local Langlands correspondence for GL_n, we can associate to ψ a *local Arthur parameter*

$$\psi_v : W'_{F_v} \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GSp}_4(\mathbf{C}),$$

where W'_{F_v} is the Weil–Deligne group of F_v. Indeed, for each i, let $\mathrm{WD}(\mu_i)$ be the Weil–Deligne representation attached to μ_i by the local Langlands correspondence for GL_n, and define

$$\psi_v := \sum_{i=1}^r \mathrm{WD}(\mu_i) \boxtimes \nu(n_i).$$

While ψ is a formal object, the local parameter ψ_v is a bona fide representation, and it is valued in $\mathrm{GSp}_4(\mathbf{C})$ by our assumptions on μ_i, n_i. Moreover, we can obtain from ψ_v a Weil–Deligne representation

$$\phi_{\psi_v} : W'_{F_v} \to \mathrm{GSp}_4(\mathbf{C}),$$

by setting

$$\phi_{\psi_v}(w) = \psi_v \left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix} \right)$$

for each $w \in W'_{F_v}$.

To each local Arthur parameter, Arthur associates a *local A-packet* $A(\psi_v)$, consisting of admissible representations of $\mathrm{GSp}_4(F_v)$. This local A-packet contains the L-packet $L(\phi_{\psi_v})$ of ϕ_{ψ_v}, as well as additional representations to ensure that the packet satisfies certain stability conditions and endoscopic character identities coming from the trace formula. If $\psi_v|_{\mathrm{SL}_2(\mathbf{C})}$ is trivial, then $A(\psi_v) = L(\phi_{\psi_v})$.

For each place v, we choose a *base point* π_{ψ_v} in the local A-packet as follows:

- If $\psi_v|_{\mathrm{SL}_2(\mathbf{C})}$ is trivial, we let π_{ψ_v} be the unique generic representation in the L-packet of ϕ_{ψ_v}.

- If $\psi_v|_{\mathrm{SL}_2(\mathbf{C})}$ is non-trivial, then it turns out that the L-packet of ϕ_{ψ_v} contains a unique irreducible representation of $\mathrm{GSp}_4(F_v)$, and we let π_{ψ_v} be this representation.

Definition 2.3.2. The *global A-packet* $A(\psi)$ of an Arthur parameter ψ is

$$A(\psi) := \left\{\pi = \otimes_v' \pi_v : \pi_v \in A(\psi_v), \quad \pi_v = \pi_{\psi_v} \text{ for almost all } v\right\}.$$

The global A-packet $A(\psi)$ consists of representations that may or may not be automorphic. For each representation $\pi \in A(\psi)$, the number of times that π appears in the discrete automorphic spectrum of GSp_4 is determined by *Arthur's multiplicity formula.*

2.3.2 Arthur's classification

Let π be a discrete automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with central character χ. Arthur's classification attaches an A-packet to π, which is associated to an Arthur parameter ψ of one of the following types:

(G) General type:

$$\psi = \Pi \boxtimes \nu(1),$$

where Π is a unitary, cuspidal automorphic representation of $\mathrm{GL}(4)$ of symplectic type with $\Pi \cong \Pi^\vee \otimes \chi$.

(Y) Yoshida type:

$$\psi = \psi_1 \boxplus \psi_2 = (\boldsymbol{\pi}_1 \boxtimes \nu(1)) \boxplus (\boldsymbol{\pi}_2 \boxtimes \nu(1)),$$

where $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ are distinct, unitary, cuspidal automorphic representations of GL_2, each with central character χ.

(Q) Soudry type:

$$\psi = \boldsymbol{\pi} \boxtimes \nu(2),$$

where $\boldsymbol{\pi}$ is a unitary cuspidal automorphic representation of GL_2 of orthogonal type,[3] with central character $\chi_{\boldsymbol{\pi}}$ such that $\chi_{\boldsymbol{\pi}}\chi^{-1}$ is a quadratic character.[4]

(P) Saito–Kurokawa type:

$$\psi = \psi_1 \boxplus \psi_2 = (\chi_1 \boxtimes \nu(2)) \boxplus (\boldsymbol{\pi} \boxtimes \nu(1)),$$

where χ_1 is an automorphic representation of GL_1 and $\boldsymbol{\pi}$ is a unitary cuspidal automorphic representation of GL_2, with central character $\chi_{\boldsymbol{\pi}}$ such that $\chi_1^2 = \chi_{\boldsymbol{\pi}} = \chi$.

(B) Howe–Piatetski-Shapiro type:

$$\psi = \psi_1 \boxplus \psi_2 = (\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2)),$$

where χ_1, χ_2 are distinct automorphic representations of GL_1 with $\chi_1^2 = \chi_2^2 = \chi$.

[3] In other words, $\boldsymbol{\pi}$ is an automorphic induction from a character of a quadratic extension K/F.
[4] Note that there is a typo on [Art04, pp.79]: the second line should read $\chi_\mu^2 = \chi^2$.

(F) One-dimensional type:

$$\psi = \chi_1 \boxtimes \nu(4),$$

where χ_1 is an automorphic representation of GL_1 with $\chi_1^4 = \chi$.

Remark 2.3.3. If π is not of general or Yoshida type and if v is a real place of F, then the base point π_{ψ_v} is never a (limit of) discrete series representation. Indeed, it is straightforward to check that the archimedean L-parameter ϕ_{ψ_v} associated to ψ_v is not of the form of any of the representations given in Section 2.1. However, the A-packet of ψ_v can still contain a (limit of) discrete series representation, and will often contain only one of the holomorphic or generic (limit of) discrete series representations.[5]

As shown in [Sch18, Sch17b], high weight representations π can only be of types **(G)**, **(Y)**, **(P)**. If π has low weight, then π can also be of type **(Q)** or **(B)**. Representations of type **(F)** are never cuspidal.

In this thesis, we focus almost exclusively on automorphic representations π of type **(G)**. If π is an algebraic automorphic representation of any other type, then its Galois representations are reducible, decomposing as a direct sum of one- and two-dimensional subrepresentations, corresponding to the automorphic representations of GL_1 and GL_2 that are the components of π. These Galois representations are well understood. We discuss their construction in Section 4.3.

[5] Note that a local A-packet need not be a union of local L-packets.

Chapter 3
Lafforgue pseudorepresentations

Let Γ be a group and let R be a ring. In [Tay91], Taylor introduced the notion of a pseudorepresentation, a function

$$T : \Gamma \to R,$$

whose properties make it look like the trace of a representation $\rho : \Gamma \to \mathrm{GL}_n(R)$. When R is an algebraically closed field of characteristic 0, Taylor proved that every pseudorepresentation is the trace of a true representation. Due to their comparative versatility, pseudorepresentations can be used to construct Galois representations in situations where direct constructions are not possible.

In his work on the geometric Langlands correspondence [Laf18], V. Lafforgue introduced the notion of a G-pseudorepresentation for a general reductive group G, whose properties make it look like the "trace" of a representation $\rho : \Gamma \to G(R)$. When R is an algebraically closed field, Lafforgue proved that every G-pseudorepresentation arises from a true representation.

In a precise sense, the trace function of GL_n generates its ring of invariants $\mathbf{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$, and a classical pseudorepresentation can be thought of as a map which assigns to each $\chi \in \mathbf{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ a map $T_\chi : \Gamma \to R$ in a compatible way. More generally, a G-pseudorepresentation is a collection of maps, which compatibly assigns to each $\chi \in \mathbf{Z}[G^m]^G$ a map $T_\chi : \Gamma \to R$. Following [Wei18a], these compatibilities can be neatly encoded by defining a G-pseudorepresentation as a natural transformation between two specific functors.

Despite their complexity, G-pseudorepresentations are more versatile than G-valued representations. We take advantage of this versatility in the next chapter to prove that the ℓ-adic Galois representations attached to Siegel modular forms are valued in GSp_4.

Let π be an automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, with F a totally real field. When π is cohomological, its associated ℓ-adic Galois representation can be extracted from the étale cohomology of a suitable Shimura variety. When π is non-cohomological its ℓ-adic Galois representation is constructed, via its corresponding pseudorepresentation, as a limit of Galois representations attached to cohomological automorphic representations. After taking this limit, it is no longer clear that the Galois representation attached to π is symplectic. In this chapter, we define Lafforgue pseudorepresentations, which we will use in place of classical pseudorepresentations to circumvent this problem.

Lafforgue pseudorepresentations were introduced by Vincent Lafforgue as part of his proof of the automorphic-to-Galois direction of the geometric Langlands correspondence for general reductive groups [Laf18]. Rather than following Lafforgue's original approach [Laf18, Section 11], we use a categorical approach due to Weidner [Wei18a].

3.1 FFS-algebras

Let **FFS** be the category of free, finitely-generated semigroups. If I is a finite set, let $\mathrm{FS}(I)$ denote the free semigroup generated by I.

If $I \to J$ is a morphism of sets, then there is a corresponding semigroup homomorphism $\mathrm{FS}(I) \to \mathrm{FS}(J)$. However, not all morphisms in **FFS** are of this form.

Lemma 3.1.1 ([Wei18a, Lemma 2.1])**.** *Any morphism in* **FFS** *is a composition of morphisms of the following types:*

- *morphisms* $\mathrm{FS}(I) \to \mathrm{FS}(J)$ *that send generators to generators, i.e. those induced by morphisms* $I \to J$ *of finite sets;*
- *morphisms*

$$\mathrm{FS}(\{x_1, \ldots, x_n\}) \to \mathrm{FS}(\{y_1, \ldots, y_{n+1}\})$$
$$x_i \mapsto \begin{cases} y_i & i < n \\ y_n y_{n+1} & i = n. \end{cases}$$

Definition 3.1.2. Let R be a ring. An **FFS**-*algebra* is a functor from **FFS** to the category R-**alg** of R-algebras. A morphism of **FFS**-algebras is a natural transformation of functors.

We will be interested in the following two examples:

Example 3.1.3. Let Γ be a group and let A be an R-algebra. We define a functor

$$\mathrm{Map}(\Gamma^\bullet, A) : \mathbf{FFS} \to R\text{-}\mathbf{alg}$$

as follows. For each finite set I, let $\mathrm{Map}(\Gamma^I, A)$ denote the R-algebra of set maps $\Gamma^I \to A$. If I is a finite set, then there is a natural isomorphism

$$\Gamma^I = \mathrm{Hom}_{\mathbf{Set}}(I, \Gamma) \cong \mathrm{Hom}_{\mathbf{FFS}}(\mathrm{FS}(I), \Gamma).$$

Hence, the association

$$\mathrm{FS}(I) \mapsto \mathrm{Map}(\mathrm{Hom}_{\mathbf{FFS}}(\mathrm{FS}(I), \Gamma), A) \cong \mathrm{Map}(\Gamma^I, A)$$

is well-defined. Moreover, a morphism $\phi : \mathrm{FS}(I) \to \mathrm{FS}(J)$ in **FFS** induces a morphism of sets

$$\Gamma^J \cong \mathrm{Hom}_{\mathbf{FFS}}(\mathrm{FS}(J), \Gamma) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathbf{FFS}}(\mathrm{FS}(I), \Gamma) \cong \Gamma^I,$$

and therefore a morphism of R-algebras

$$\mathrm{Map}(\Gamma^I, A) \to \mathrm{Map}(\Gamma^J, A).$$

Hence, the functor

$$\mathrm{Map}(\Gamma^\bullet, A) : \mathrm{FS}(I) \mapsto \mathrm{Map}(\Gamma^I, A)$$

is an **FFS**-algebra.

If Γ is a topological group and A is a topological R-algebra, then this construction works with continuous maps in place of set maps.

Example 3.1.4. Let G, X be affine group schemes over R and let G act on X. For any finite set I, G acts diagonally on X^I, and hence G acts on the coordinate ring $R[X^I]$ of X^I.

For each finite set I, let $R[X^I]^G$ be the R-algebra of fixed points of $R[X^I]$ under the action of G. A morphism $\phi : \mathrm{FS}(I) \to \mathrm{FS}(J)$ in **FFS** induces a morphism of R-schemes $X^J \to X^I$, and thus an R-algebra morphism $R[X^I]^G \to R[X^J]^G$. The corresponding functor

$$R[X^\bullet]^G : \mathrm{FS}(I) \mapsto R[X^I]^G$$

is an **FFS**-algebra.

3.2 Lafforgue pseudorepresentations

Let R be a ring and let G be a reductive group over R. Let G° denote the identity connected component of G, which we assume is split. Then G° acts on G by conjugation and we can form the **FFS**-algebra $R[G^\bullet]^{G^\circ}$.

Definition 3.2.1. Let Γ be a group and let A be a R-algebra. A *G-pseudorepresentation* of Γ over A is an **FFS**-algebra morphism

$$\Theta^\bullet : R[G^\bullet]^{G^\circ} \to \mathrm{Map}(\Gamma^\bullet, A).$$

Remarks 3.2.2.

1. Unwinding this definition recovers Lafforgue's original definition [Laf18, Définition-Proposition 11.3]. Indeed, Lafforgue defines a pseudorepresentation as a collection $(\Theta_n)_{n\geq 1}$ of algebra maps

$$\Theta_n : R[G^n]^{G^\circ} \to \mathrm{Map}(\Gamma^n, A)$$

that are compatible in the following sense:

(a) If $n, m \geq 1$ are integers and $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, then for every $f \in R[G^m]^{G^\circ}$ and $\gamma_1, \dots \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}),$$

where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.

(b) For every integer $n \geq 1$, $f \in R[G^n]^{G^\circ}$ and $\gamma_1, \dots \gamma_{n+1} \in \Gamma$, we have

$$\Theta_{n+1}(\hat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1}),$$

where $\hat{f}(g_1, \dots, g_{n+1}) = f(g_1, \dots, g_{n-1}, g_n g_{n+1})$.

By definition, an **FFS**-algebra morphism $R[G^\bullet]^{G^\circ} \to \mathrm{Map}(\Gamma^\bullet, A)$ consists of a collection of R-algebra morphisms $\Theta^I : R[G^I]^{G^\circ} \to \mathrm{Map}(\Gamma^I, A)$ such that, for any semigroup homomorphism $\phi : \mathrm{FS}(I) \to \mathrm{FS}(J)$, the following diagram commutes:

$$\begin{array}{ccc} R[G^I]^{G^\circ} & \xrightarrow{\Theta^I} & \mathrm{Map}(\Gamma^I, A) \\ \downarrow & & \downarrow \\ R[G^J]^{G^\circ} & \xrightarrow{\Theta^J} & \mathrm{Map}(\Gamma^J, A). \end{array}$$

Here, the vertical arrows are those induced by ϕ. By Lemma 3.1.1, checking that this diagram commutes for all morphisms ϕ is equivalent to verifying conditions (a) and (b) above.

2. Suppose that G is a connected linear algebraic group with a fixed embedding $G \hookrightarrow \mathrm{GL}_r$ for some r. Let χ denote the composition of this embedding with the usual trace function. Then $\chi \in \mathbf{Z}[G]^G$ and, if $\Theta^\bullet$ is a G-pseudorepresentation of Γ over A, then
$$\Theta^1(\chi) \in \mathrm{Map}(\Gamma, A)$$
is a classical pseudorepresentation. In fact, we will see in Section 3.4 that when $G = \mathrm{GL}_n$, $\Theta^\bullet$ is completely determined by $\Theta^1(\chi)$ [Laf18, Remarque 11.8]. In particular, the notion of a G-pseudorepresentation is a generalisation of the notion of a classical pseudorepresentation.

We finish this subsection by recording a generalisation of [Tay91, Lemma 1], which notes that we are free to change the R-algebra A. Part 1 is part of [BHKT, Lemma 4.4].

Lemma 3.2.3. *Let A be an R-algebra and let Γ be a group.*

1. *Let $h : A \to A'$ be a morphism of R-algebras and let $\Theta^\bullet$ be a G-pseudorepresentation of Γ over A. Then $h_*(\Theta) = h \circ \Theta^\bullet$ is a G-pseudorepresentation of Γ over A'.*

2. *Let $h : A \hookrightarrow A'$ be an injective morphism of R-algebras. Define a collection of maps $\Theta^\bullet$, where, for each finite set I, $\Theta^I : R[G^I]^{G^\circ} \to \mathrm{Map}(\Gamma^I, A)$ is a map of sets.*

   Suppose that $h \circ \Theta^\bullet$ is a G-pseudorepresentation of Γ over A'. Then $\Theta^\bullet$ is a G-pseudorepresentation over A.

Proof. We prove 2. We need to show that the maps $\Theta^I : R[G^I]^{G^\circ} \to \mathrm{Map}(\Gamma^I, A)$ form **FFS**-algebra morphisms $R[G^\bullet]^{G^\circ} \to \mathrm{Map}(\Gamma^\bullet, A)$. Hence, we need to show that the maps Θ^I are R-algebra maps and that, for every semigroup homomorphism $\phi : \mathrm{FS}(I) \to \mathrm{FS}(J)$, the diagram

$$\begin{array}{ccc} R[G^I]^{G^\circ} & \xrightarrow{\Theta^I} & \mathrm{Map}(\Gamma^I, A) \\ \downarrow & & \downarrow \\ R[G^J]^{G^\circ} & \xrightarrow{\Theta^J} & \mathrm{Map}(\Gamma^J, A) \end{array}$$

commutes.

By assumption, $h \circ \Theta^I$ is an R-algebra morphism. Hence, since h is injective, it follows that Θ^I is an R-algebra morphism as well. Moreover, in the diagram

$$\begin{array}{ccccc} R[G^I]^{G^\circ} & \xrightarrow{\Theta^I} & \mathrm{Map}(\Gamma^I, A) & \xrightarrow{h} & \mathrm{Map}(\Gamma^I, A') \\ \downarrow & & \downarrow & & \downarrow \\ R[G^J]^{G^\circ} & \xrightarrow{\Theta^J} & \mathrm{Map}(\Gamma^J, A) & \xrightarrow{h} & \mathrm{Map}(\Gamma^J, A'), \end{array}$$

the right-hand square clearly commutes, and the outermost square

$$\begin{array}{ccc} R[G^I]^{G^\circ} & \xrightarrow{h\circ\Theta^I} & \mathrm{Map}(\Gamma^I, A') \\ \downarrow & & \downarrow \\ R[G^J]^{G^\circ} & \xrightarrow{h\circ\Theta^J} & \mathrm{Map}(\Gamma^J, A') \end{array}$$

commutes by the assumption that $h \circ \Theta^\bullet$ is a pseudorepresentation. Since h is injective, it follows that the left-hand square commutes as well. The result follows. □

3.3 Lafforgue pseudorepresentations and G-valued representations

The key motivation for introducing Lafforgue pseudorepresentations is their connection to G-valued representations. From now on, we assume that $R = \mathbf{Z}$. Then G is a reductive group over $\mathbf{Z}$ with G° split, and A is a ring.

Lemma 3.3.1 ([Laf18, Définition-Proposition 11.3])**.** *Let $\rho : \Gamma \to G(A)$ be a representation of Γ. Define*

$$(\mathrm{Tr}\,\rho)^\bullet : \mathbf{Z}[G^\bullet]^{G^\circ} \to \mathrm{Map}(\Gamma^\bullet, A)$$

by

$$(\mathrm{Tr}\,\rho)^I(f)\big((\gamma_i)_{i\in I}\big) = f\big((\rho(\gamma_i))_{i\in I}\big)$$

for each finite set I and for each $f \in \mathbf{Z}[G^I]^{G^\circ}$.

Then $(\mathrm{Tr}\,\rho)^\bullet$ is a G-pseudorepresentation of Γ over A. Moreover, $(\mathrm{Tr}\,\rho)^\bullet$ depends only on the $G^\circ(A)$-conjugacy class of ρ.

In fact, in many cases, the converse of Lemma 3.3.1 is also true. Let k be an algebraically closed field and let $\rho : \Gamma \to G(k)$ be a representation of Γ. If $G = \mathrm{GL}_n$ and ρ is semisimple, then Taylor [Tay91] proved that ρ can be recovered from its classical pseudorepresentation. To state the generalisation of this fact for G-pseudorepresentations, we first define what it means for ρ to be semisimple in general.

Definition 3.3.2 ([BHKT, Definitions 3.3, 3.5])**.** Let H denote the Zariski closure of $\rho(\Gamma)$.

- We say that ρ is *G-irreducible* if there is no proper parabolic subgroup of G containing H.
- We say that ρ is *semisimple* or *G-completely reducible* if, for any parabolic subgroup $P \subseteq G$ containing H, there exists a Levi subgroup of P containing H.

Theorem 3.3.3 ([Laf18, Proposition 11.7], [BHKT, Theorem 4.5])**.** *Let k be an algebraically closed field. The assignment $\rho \mapsto (\mathrm{Tr}\,\rho)^\bullet$ defines a bijection between the following two sets:*

1. *The set of $G^\circ(k)$-conjugacy classes of G-completely reducible homomorphisms $\rho : \Gamma \to G(k)$;*

2. *The set of G-pseudorepresentations $\Theta^\bullet : \mathbf{Z}[G^\bullet]^{G^\circ} \to \mathrm{Map}(\Gamma^\bullet, k)$ of Γ over k.*

3.4 GSp_4-pseudorepresentations

Definition 3.4.1 ([Wei18a, Definition 2.6])**.** Let $A^\bullet$ be an **FFS**-algebra. Given a subset $\Sigma \subseteq \bigcup_I A^I$, define the *span* of Σ in $A^\bullet$ to be the smallest **FFS**-subalgebra $B^\bullet$ of $A^\bullet$ such that $\Sigma \subseteq \bigcup_I B^I$. We say that Σ generates $A^\bullet$ if the span of Σ in $A^\bullet$ is the whole of $A^\bullet$.

Examples 3.4.2.

1. Suppose that k is a field of characteristic 0. By results of Procesi [Pro76], the **FFS**-algebra $k[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is generated by the elements $\mathrm{Tr}, \det^{-1} \in k[\mathrm{GL}_n]^{\mathrm{GL}_n}$. If **FFG** denotes the category of free, finitely-generated groups, then we can define **FFG**-algebras analogously to **FFS**-algebras and, as an **FFG**-algebra, $k[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is generated by Tr.[1] Hence, if $\Theta^\bullet$ is any GL_n-pseudorepresentation, by [Wei18a, Corollary 2.13], $\Theta^\bullet$ is completely determined by its classical pseudorepresentation $\Theta^1(\mathrm{Tr}) \in \mathrm{Map}(\Gamma, k)$.

2. Under the assumption that k is a field of characteristic 0, for many reductive groups G, Weidner [Wei18a, Section 3] has computed generating sets for $k[G^\bullet]^G$ as **FFG**-algebras. For example, if $G = \mathrm{GSp}_{2n}$, then $k[G^\bullet]^G$ is generated as an **FFG**-algebra by $\mathrm{Tr}, \mathrm{sim} \in k[G]^G$.

The above examples all assume that k is a field of characteristic 0. In our examples, k will often be a finite field or a ring of integers. Hence, we will require the following lemma.

Lemma 3.4.3. *For an element $X \in \mathrm{GSp}_4$, let $t^4 + \sum_{i=1}^4 (-1)^i s_i(X) t^{4-i}$ be its characteristic polynomial. The **FFS**-algebra $\mathbf{Z}[\mathrm{GSp}_4^\bullet]^{\mathrm{GSp}_4}$ is generated by the elements $s_1, s_2, \mathrm{sim}^{\pm 1} \in \mathbf{Z}[\mathrm{GSp}_4]^{\mathrm{GSp}_4}$. In particular, a GSp_4-pseudorepresentation of Γ over A is completely determined by the image of these elements in $\mathrm{Map}(\Gamma, A)$.*

Proof. Let M_4 denote the additive algebraic group of 4×4 matrices. The embedding

$$\begin{aligned} \mathrm{GSp}_4 &\to \mathbf{G}_m \times \mathrm{M}_4 \\ X &\mapsto (\mathrm{sim}(X)^{-1}, X) \end{aligned}$$

gives GSp_4 the structure of a closed subscheme of $\mathbf{G}_m \times \mathrm{M}_4$, which is stable under the conjugation action of GSp_4. By [Ses77, Theorem 3(iii)], $\mathrm{Spec}(\mathbf{Z}[\mathrm{GSp}_4^n]^{\mathrm{GSp}_4})$ is therefore

[1] If $X \in \mathrm{GL}_n$, then $\det(X)$ can be expressed as a polynomial in $\mathrm{Tr}(X^i)$. Hence, as an **FFG**-algebra, $\det^{-1}$ is in the **FFG**-subalgebra generated by Tr.

a closed subscheme of $\mathrm{Spec}(\mathbf{Z}[(\mathbf{G}_m \times \mathrm{M}_4)^n]^{\mathrm{GSp}_4}$. It follows that the corresponding map

$$\mathbf{Z}[(\mathbf{G}_m \times \mathrm{M}_4)^n]^{\mathrm{GSp}_4} \to \mathbf{Z}[\mathrm{GSp}_4^n]^{\mathrm{GSp}_4}$$

is surjective.

Moreover, since GSp_4 acts trivially on $\mathbf{G}_m^n$, we find that

$$(\mathbf{Z}[\mathbf{G}_m^n] \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{M}_4^n])^{\mathrm{GSp}_4} = \mathbf{Z}[\mathbf{G}_m^n]^{\mathrm{GSp}_4} \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{M}_4^n]^{\mathrm{GSp}_4} = \mathbf{Z}[\mathbf{G}_m^n] \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{M}_4^n]^{\mathrm{Sp}_4}.$$

Indeed, if $f \otimes g \in (\mathbf{Z}[\mathbf{G}_m^n] \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{M}_4^n])^{\mathrm{GSp}_4}$, then, automatically, $f \in \mathbf{Z}[\mathbf{G}_m^n]^{\mathrm{GSp}_4}$ and hence $g \in \mathbf{Z}[\mathrm{M}_4^n]^{\mathrm{GSp}_4}$.

We deduce that $\mathbf{Z}[\mathbf{G}_m^n]^{\mathrm{Sp}_4} \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{M}_4^n]^{\mathrm{Sp}_4}$ surjects onto $\mathbf{Z}[\mathrm{GSp}_4^n]^{\mathrm{GSp}_4}$.

Now, by [Pro76, Theorem 10.1], $\mathbf{Z}[\mathrm{M}_4^n]^{\mathrm{Sp}_4}$ is generated by maps of the form

$$(\gamma_1, \dots \gamma_n) \mapsto s_i(\mu_{\zeta(1)}^{a_1} \cdots \mu_{\zeta(m)}^{a_m}),$$

as ζ runs over all functions $\{1 \dots, m\} \to \{1, \dots, n\}$, $i = 1, 2$, $a_j \in \mathbf{N}$ and where μ_j is either γ_j or its dual (with respect to the symplectic pairing induced by the matrix J) γ_j^*. We also have $\mathbf{Z}[\mathbf{G}_m^n] \cong \mathbf{Z}[x_i, x_i^{-1} : 1 \leq i \leq n]$.

We deduce that $\mathbf{Z}[\mathrm{GSp}_4^n]^{\mathrm{GSp}_4}$ is generated by the image of these maps. It follows that, for each integer $n \geq 1$, $\mathbf{Z}[\mathrm{GSp}_4^n]^{\mathrm{GSp}_4}$ is generated by functions of the form

$$(g_1, \dots, g_n) \mapsto f(h_{\zeta(1)}^{a_1} h_{\zeta(2)}^{a_2} \cdots h_{\zeta(m)}^{a_m}),$$

where $f \in \{s_i, \mathrm{sim}^{\pm 1}\}$, $1 \leq j \leq r$, $m \geq 1$, $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$ and $a_j \in \mathbf{N}$ and $h_{\zeta(i)}$ is either $g_{\zeta(i)}$ or its dual.

However, since $g_{\zeta(i)} \in \mathrm{GSp}_4$, by definition, $g_{\zeta(i)}^* = \mathrm{sim}(g_{\zeta(i)}) g_{\zeta(i)}^{-1}$. It follows that, for each integer $n \geq 1$, $\mathbf{Z}[\mathrm{GSp}_4^n]^{\mathrm{GSp}_4}$ is generated by functions of the form

$$(g_1, \dots, g_n) \mapsto f(g_{\zeta(1)}^{a_1} g_{\zeta(2)}^{a_2} \cdots g_{\zeta(m)}^{a_m}),$$

where $f \in \{s_i, \mathrm{sim}^{\pm 1}\}$, $1 \leq j \leq r$, $m \geq 1$, $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$ and $a_j \in \mathbf{Z}$. Rephrasing this fact in terms of **FFS**-algebras, we deduce that $\mathbf{Z}[\mathrm{GSp}_4^\bullet]^{\mathrm{GSp}_4}$ is generated by $\{s_i, \mathrm{sim}^{\pm 1}\}$. □

Chapter 4

Galois representations associated to Hilbert–Siegel modular forms

If π is a Hilbert–Siegel modular form and if ℓ is a prime, then we can associate an ℓ-adic Galois representation $\rho_{\pi,\ell}$ to π. While the isomorphism class of $\rho_{\pi,\ell}$ is independent of the way that it is constructed, different constructions yield different information about $\rho_{\pi,\ell}$. As a result, many of the properties of $\rho_{\pi,\ell}$ that are known when π is cohomological—for example, geometricity and local-global compatibility—are not known when π is non-cohomological.

In this chapter, we review the constructions and properties of $\rho_{\pi,\ell}$ in both cases. When π is non-cohomological, we use the methods of Chapter 3 to extend Taylor's construction of $\rho_{\pi,\ell}$ [Tay91], in order to show that its image is valued in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$.

In this chapter, we review the construction of the Galois representations associated to Hilbert–Siegel modular forms. In Section 4.1, we record the properties of Galois representations attached to cohomological automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_F)$, before doing the same for non-cohomological automorphic representations in Section 4.2. In Section 4.3, we recall the construction of Galois representations for cuspidal automorphic representations that are not of type **(G)**. Finally, in Section 4.4, we prove that the image of the ℓ-adic Galois representation attached to a low weight automorphic representation is valued in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$.

For the remainder of this thesis, π will be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$, and central character χ_π. We assume that there exists an integer w such that

$$\pi^\circ := \pi \otimes |\,\mathrm{sim}\,|^{w/2}$$

is unitary, and such that $w+1 \equiv k_{1,v} + k_{2,v} - 3 \pmod 2$ for all archimedean places v.[1] In particular, we can write $\chi_\pi = \chi_{\pi^\circ}|\,\mathrm{sim}\,|^{-w}$, where χ_{π° is the central character of π°.

[1]This condition ensures that π is either C-algebraic or L-algebraic in the sense of [BG14].

4.1 The cohomological case

We review the construction of Galois representations attached to high weight Hilbert–Siegel modular forms.

Theorem 4.1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 3$, and central character $\chi_\pi = \chi_{\pi^\circ} |\operatorname{sim}|^{-w}$. Let S denote the set of places of F at which π is ramified. Then, for each prime number ℓ, there exists a continuous, semisimple, symplectic Galois representation*

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$$

that satisfies the following properties:

1. *The similitude character $\operatorname{sim}(\rho_{\pi,\ell})$ is equal to $\rho_{\chi_{\pi^\circ},\ell}\varepsilon_\ell^{-w-3}$, where $\rho_{\chi_{\pi^\circ},\ell}$ is the ℓ-adic Galois representation associated to χ_{π°. Moreover, $\operatorname{sim}(\rho_{\pi,\ell})$ is totally odd*[2], *and*
$$\rho_{\pi,\ell} \simeq \rho_{\pi,\ell}^\vee \otimes \operatorname{sim}(\rho_{\pi,\ell}).$$

2. *$\rho_{\pi,\ell}$ is unramified at all finite places v of F such that $v \notin S, v \nmid \ell$.*

3. *Local-global compatibility is satisfied up to semisimplification: for any place v of F,*
$$\mathrm{WD}(\rho_{\pi,\ell}|_{F_v})^{ss} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\operatorname{sim}|_v^{-3/2})^{ss}.$$

4. *$\rho_{\pi,\ell}$ is de Rham at all places $v \mid \ell$ and crystalline if $v \notin S$. Identifying the embeddings $F \hookrightarrow \overline{\mathbf{Q}}_\ell$ with the archimedean places v via our fixed isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$, the Hodge–Tate weights at the embedding corresponding to $v \mid \infty$ are given by*
$$\delta_v + \{0, k_{2,v} - 2, k_{1,v} - 1, k_{1,v} + k_{2,v} - 3\},$$
where $\delta_v = \frac{1}{2}\big(w + 3 - (k_{1,v} + k_{2,v} - 3)\big)$.

Moreover, if π is of type ***(G)*** *or* ***(Y)*** *in Arthur's classification (see Section 2.3)—i.e. if π is not CAP—then $\rho_{\pi,\ell}$ satisfies the following stronger properties:*

5. *Local-global compatibility is satisfied: for any place v,*
$$\mathrm{WD}(\rho_{\pi,\ell}|_{F_v})^{F\text{-}ss} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\operatorname{sim}|_v^{-3/2}).$$

6. *$\rho_{\pi,\ell}$ is pure of weight $w+3$: if $\rho_{\pi,\ell}$ is unramified at a place v and if $\alpha \in \mathbf{C}$ is a root of the characteristic polynomial of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$, then $|\alpha| = N(v)^{\frac{1}{2}(w+3)}$.*

Finally, suppose that π is of type ***(G)*** *(i.e. that π is neither CAP nor endoscopic). Then:*

7. *$\rho_{\pi,\ell}$ is irreducible for 100% of primes ℓ. When $F = \mathbf{Q}$, if $\ell > 2(k_1 + k_2 - 3) - 1$ and $\rho_{\pi,\ell}$ is crystalline at ℓ, then $\rho_{\pi,\ell}$ is irreducible.*

Proof. When π is CAP or endoscopic, the construction of $\rho_{\pi,\ell}$ follows from the construction of Galois representations for Hilbert modular forms. A discussion of these

[2] We say that a character is ω is totally odd if $\omega(c_v) = -1$ for any choice of complex conjugation c_v.

cases is given in [Mok14, pp. 537–538] (see also Section 4.3). In these cases, $\rho_{\pi,\ell}$ is reducible. Henceforth, we assume that π is of type **(G)**.

When $F = \mathbf{Q}$, the Galois representations were constructed by Laumon [Lau05] and Weissauer [Wei05], building on previous work of Taylor [Tay93]. Their construction works directly with a symplectic Shimura variety: the Galois representations are constructed from the étale cohomology of Siegel threefolds. The fact that the Galois representations are valued in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$ was proven by Weissauer [Wei08]. This construction does not rely on Arthur's classification. However, the arguments break down when $F \neq \mathbf{Q}$, and are not sufficient to prove local-global compatibility at ramified primes.

When $F \neq \mathbf{Q}$, the Galois representations were constructed by Sorensen [Sor10]. This construction uses the transfer map from GSp_4 to GL_4 in combination with Harris–Taylor's construction of Galois representations for automorphic representations of GL_4, which uses unitary Shimura varieties [HT01]. Sorensen's construction assumes that π has a weak lift to GL_4 and, therefore, initially only applies to generic automorphic representations (see Section 2.2). Applying his construction to automorphic representations coming from Hilbert–Siegel modular forms requires Arthur's classification (see Sections 2.2 and 2.3). Using this construction, Mok [Mok14, Theorem 3.5] proves local-global compatibility at ramified primes. The fact that the Galois representations are valued in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$ was proven by Bellaïche–Chenevier [BC11].

Irreducibility for 100% of primes is due to [CG13]. When $F = \mathbf{Q}$, irreducibility when ℓ is sufficiently large follows by applying [Ram13, Theorem B] to the transfer of π to GL_4 (see Remark 5.3.2 for the limitations of applying Ramakrishnan's result when $F \neq \mathbf{Q}$). □

4.2 The non-cohomological case

The situation for low weight automorphic representations is far less comprehensive. Since the automorphic representations are non-cohomological, their Galois representations cannot be constructed directly from the étale cohomology of symplectic or unitary Shimura varieties. Instead, they are constructed as limits of cohomological Galois representations. The process of taking a limit of Galois representations loses information, in particular information about local-global compatibility and geometricity at ℓ.

Theorem 4.2.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$, and central character $\chi_\pi = \chi_{\pi^\circ}|\,\mathrm{sim}\,|^{-w}$. Let S denote the set of places of F at which π is ramified. Suppose that π is of type **(G)**. Then for each prime ℓ, there exists a continuous, semisimple Galois representation*

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{Q}}_\ell)$$

that satisfies the following properties:

1. *There is a character, $\mathrm{sim}(\rho_{\pi,\ell})$ such that*

$$\rho_{\pi,\ell} \simeq \rho_{\pi,\ell}^\vee \otimes \mathrm{sim}(\rho_{\pi,\ell}).$$

The character $\mathrm{sim}(\rho_{\pi,\ell})$ *is equal to* $\rho_{\chi_{\pi^\circ},\ell}\varepsilon_\ell^{-w-3}$*, where* $\rho_{\chi_{\pi^\circ},\ell}$ *is the* ℓ*-adic Galois representation associated to* χ_{π°*. Moreover,* $\mathrm{sim}(\rho_{\pi,\ell})$ *is totally odd.*

2. $\rho_{\pi,\ell}$ *is unramified at all finite places* v *of* F *such that* $v \notin S, v \nmid \ell$*.*

3. *Local-global compatibility is satisfied up to semisimplification: for any place* v *of* F*,*
$$\mathrm{WD}(\rho_{\pi,\ell}|_{F_v})^{ss} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\,\mathrm{sim}\,|_v^{-3/2})^{ss}.$$

4. *Identifying the embeddings* $F \hookrightarrow \overline{\mathbf{Q}}_\ell$ *with the archimedean places* v *via our fixed isomorphism* $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$*, the Hodge–Tate–Sen weights at the embedding corresponding to* $v \mid \infty$ *are given by*
$$\delta_v + \{0, k_{2,v} - 2, k_{1,v} - 1, k_{1,v} + k_{2,v} - 3\},$$
where $\delta_v = \frac{1}{2}\big(w + 3 - (k_{1,v} + k_{2,v} - 3)\big)$*. If* $k_{2,v} > 2$*, then* $\rho_{\pi,\ell}$ *is Hodge–Tate at* v*.*

5. *If* $v \mid \ell$*,* $v \notin S$ *and the roots of the Satake parameters of* π_v *are pairwise distinct, then* $\rho_{\pi,\ell}$ *is crystalline at* v*.*

Moreover, if $F = \mathbf{Q}$ *or if* π *is unramified at all places of* F *above* ℓ*, then* $\rho_{\pi,\ell}$ *is isomorphic to a representation that is valued in* $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$*, with similitude character* $\mathrm{sim}(\rho_{\pi,\ell})$*.*

Proof. There are two different constructions of the ℓ-adic Galois representation attached to π. In both cases, $\rho_{\pi,\ell}$ is constructed, via its pseudorepresentation, as a limit of cohomological Galois representations.

- When $F = \mathbf{Q}$, the original construction, due to Taylor [Tay91], uses the Hasse invariant to find congruences between the Hecke eigenvalue system of π and mod ℓ^n cohomological eigenforms π_n. The associated Galois pseudorepresentation is constructed as the limit of the Galois pseudorepresentations attached to the π_n. This approach has been generalised to when $F \neq \mathbf{Q}$ by Goldring–Koskivirta [GK19] to construct ℓ-adic Galois representations when π is unramified at all primes dividing ℓ. This construction is sufficient to prove the existence of the of Galois representation and parts 1-2 of the theorem. When $F = \mathbf{Q}$, this construction does not use Arthur's classification.

- A second construction, due to Mok [Mok14], extends the work of Sorensen [Sor10], and constructs an eigenvariety for GSp_4. This construction does require Arthur's classification. Using this construction, Mok [Mok14, Theorem 3.5] proves local-global compatibility at ramified primes up to semisimplification. Part (5) is due to Jorza [Jor12, Theorem 4.1] (c.f [Mok14, Proposition 4.16]), and also uses this construction.

Finally, the fact that the Galois representation is valued in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$ is Theorem 4.4.1 of this thesis; this fact does not rely on Arthur's classification when $F = \mathbf{Q}$. ∎

Remark 4.2.2. If π is not an automorphic induction, then the Satake parameters of π_v should always be pairwise distinct. However, without this condition, in general, we do not even know that $\rho_{\pi,\ell}$ is Hodge–Tate at v. A key result of this thesis (Proposition 6.1.11) is that either π is an automorphic induction, or this condition holds for 100% of places v.

Corollary 4.2.3. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, where $k_{1,v} \geq k_{2,v} \geq 2$. Suppose that π is of type **(G)**. Then there exists an ideal $\mathfrak{N} \subseteq \mathcal{O}_F$, such that the Serre conductor of $\rho_{\pi,\ell}$ divides $\mathfrak{N}$ for all ℓ.*

Proof. Fix a prime ℓ and let $\mathfrak{N}_\rho$ be the Serre conductor of $\rho_{\pi,\ell}$. Let S be the set of places of F at which π is ramified, and let

$$\mathfrak{N}_\pi = \prod_{v\in S} \mathrm{cond}(\mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\,\mathrm{sim}\,|_v^{-3/2})),$$

via the local Langlands correspondence [GT11]. Since S is finite, we can assume that ℓ is large enough that $v \notin S$ for any $v \mid \ell$. By definition,

$$\mathfrak{N}_\rho = \prod_{v\in S} \mathrm{cond}(\mathrm{WD}(\rho_{\pi,\ell}|_{F_v})^{F\text{-}ss}).$$

A Weil-Deligne representation (V, ρ, N) of W_{F_v} has conductor

$$\mathrm{cond}(\rho)\mathfrak{p}_v^{\dim(V^I)-\dim(V_N^I)},$$

where $\mathfrak{p}_v$ is the prime of F corresponding to v, V^I is the subspace of V fixed by the inertia group, and $V_N^I = \ker(N)^I$. We have $(V, \rho, N)^{ss} = (V, \rho^{ss}, 0)$. If (V, ρ, N) is Frobenius semisimple, then $\rho^{ss} = \rho$, and it follows that

$$\mathrm{cond}(V, \rho, N) \mid \mathrm{cond}(\rho)\mathfrak{p}_v^{\dim(V)}.$$

Hence, $\mathfrak{N}_\rho$ divides $\prod_{v\in S} \mathrm{cond}(\mathrm{WD}(\rho_{\pi,\ell}|_{F_v})^{ss})\mathfrak{p}_v^4$. By part (3) of Theorem 4.2.1,

$$\mathrm{cond}(\mathrm{WD}(\rho_{\pi,\ell}|_{F_v})^{ss})\mathfrak{p}_v^4 = \prod_{v\in S} \mathrm{cond}(\mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\,\mathrm{sim}\,|_v^{-3/2})^{ss})\mathfrak{p}_v^4,$$

which divides

$$\prod_{v\in S} \mathrm{cond}(\mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\,\mathrm{sim}\,|_v^{-3/2}))\mathfrak{p}_v^4.$$

Since

$$\prod_{v\in S} \mathrm{cond}(\mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\,\mathrm{sim}\,|_v^{-3/2}))\mathfrak{p}_v^4 = N_\pi \prod_{v\in S} \mathfrak{p}_v^4,$$

we deduce that $\mathfrak{N}_\rho \mid \mathfrak{N}_\pi \prod_{v\in S} \mathfrak{p}_v^4$, and the result follows. □

4.3 Galois representations for CAP and endoscopic automorphic representations

For completeness, we review the construction of Galois representations for automorphic representations π that are not of type **(G)**. This analysis has been carried out in [Mok14] when π is cohomological and in [BCGP18] in general. The other possible types of π were defined in Section 2.3.

Suppose that π has weights $(k_{1,v}, k_{2,v})_{v|\infty}$ and central character $\chi_\pi = \chi_{\pi^\circ}|\,\mathrm{sim}\,|^{-w}$.

- If π is of type **(Y)**, then the A-packet of π corresponds to the Arthur parameter $(\boldsymbol{\pi}_1 \boxtimes \nu(1)) \boxplus (\boldsymbol{\pi}_2 \boxtimes \nu(1))$, where $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ are distinct, unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$, each with central character χ_π. In this case, we can follow the arguments of [Mok14, pp. 537].

  Computing the archimedean L-parameters using Section 2.1, we can arrange that, at each real place v, the L-parameter of $\boldsymbol{\pi}_1$ is $\phi_{(w;k_{1,v}+k_{2,v}-3)}$ and the L-parameter of $\boldsymbol{\pi}_2$ is $\phi_{(w;k_{1,v}-k_{2,v}+1)}$. Both parameters correspond to discrete series representations of $\mathrm{GL}_2(\mathbf{R})$. Hence, for each $i = 1, 2$, we can associate an ℓ-adic Galois representation

$$\rho_{\boldsymbol{\pi}_i,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

  to $\boldsymbol{\pi}_i$ such that

$$\mathrm{WD}(\rho_{\boldsymbol{\pi}_i,\ell})^{F\text{-}ss} \cong \mathrm{rec}((\boldsymbol{\pi}_i)_v \otimes |\det|_v^{-1/2}).$$

  We can then define[3]

$$\rho_{\pi,\ell} = \rho_{\boldsymbol{\pi}_1,\ell}(-1) \oplus \rho_{\boldsymbol{\pi}_2,\ell}(-1).$$

- If π is of type **(Q)**, then the A-packet of π corresponds to the Arthur parameter $\boldsymbol{\pi} \boxtimes \nu(2)$, where $\boldsymbol{\pi}$ is a unitary cuspidal automorphic representation of GL_2 of orthogonal type with central character $\chi_{\boldsymbol{\pi}}$ such that $\chi_{\boldsymbol{\pi}}^2 = \chi_\pi^2$. By [Sch18, Table 3], this case can only occur if $(k_{1,v}, k_{2,v}) = (2, 2)$ for all archimedean places v. Hence, at each real place v, the archimedean L-parameter of π is $\phi_{(w;1,0)}$ and the L-parameter of $\boldsymbol{\pi}$ at v is $\phi_{(w;0)}$. Since w is even, $\phi_{(w;0)}$ does not satisfy the parity condition in Section 2.1 (it is C-algebraic, and not L-algebraic in the sense of [BG14]). However, it follows that the representations $\boldsymbol{\pi}|\cdot|^{\pm\frac{1}{2}}$ are L-algebraic and have associated ℓ-adic Galois representations

$$\rho_{\boldsymbol{\pi}|\cdot|^{\pm\frac{1}{2}},\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

  whose Hodge–Tate weights at v are $\left\{\frac{w+1}{2} \pm \frac{1}{2}, \frac{w+1}{2} \pm \frac{1}{2}\right\}$. We can then define

$$\rho_{\pi,\ell} = \rho_{\boldsymbol{\pi}|\cdot|^{\frac{1}{2}},\ell}(-1) \oplus \rho_{\boldsymbol{\pi}|\cdot|^{-\frac{1}{2}},\ell}(-1) = \rho_{\boldsymbol{\pi}|\cdot|^{\frac{1}{2}},\ell}(-1) \oplus \rho_{\boldsymbol{\pi}|\cdot|^{\frac{1}{2}},\ell}(-2).$$

- If π is of type **(P)**, then the A packet of π corresponds to the Arthur parameter $(\chi_1 \boxtimes \nu(2)) \boxplus (\boldsymbol{\pi} \boxtimes \nu(1))$, where χ_1 is an automorphic representation of GL_1 and $\boldsymbol{\pi}$ is a unitary cuspidal automorphic representation of GL_2 with central character $\chi_{\boldsymbol{\pi}}$ such that $\chi_1^2 = \chi_{\boldsymbol{\pi}} = \chi_\pi$. In this case, we can follow the arguments of [Mok14, pp. 538].

  By [Sch18, Table 2], we must have $k_{1,v} = k_{2,v}$ for all archimedean places v. In particular, since $w + 1 \equiv k_{1,v} + k_{2,v} - 3 \pmod 2$, w is even and, therefore, χ_1 is an algebraic Hecke character. Writing $k_v = k_{1,v} = k_{2,v}$ and computing the archimedean L-parameters using [Sch18] and Section 2.1, it follows that, at each archimedean place v, the L-parameter of $\boldsymbol{\pi}$ at v is $\phi_{(w;2k_v-3)}$, corresponding to a discrete series representation of $\mathrm{GL}_2(\mathbf{R})$. Hence, we can associate an ℓ-adic Galois representation

$$\rho_{\boldsymbol{\pi},\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

  to $\boldsymbol{\pi}$, and an ℓ-adic character $\rho_{\chi_1,\ell}$ to χ_1. We then define

$$\rho_{\pi,\ell} = \rho_{\boldsymbol{\pi},\ell}(-1) \oplus \rho_{\chi_1,\ell}(-1) \oplus \rho_{\chi_1,\ell}(-2).$$

[3] The additional twists by the inverse of the cyclotomic character come from our normalisations.

- If π is of type **(B)**, then the A-packet of π corresponds to the Arthur parameter $(\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2))$, where χ_1, χ_2 are distinct automorphic representations of GL_1 with $\chi_1^2 = \chi_2^2 = \chi_\pi$. By [Sch18, Table 1], this case can only occur if $(k_{1,v}, k_{2,v}) = (2,2)$ for all v. It follows that w is an even integer, and hence that χ_1, χ_2 are algebraic Hecke characters. Writing $\rho_{\chi_i,\ell}$ for the ℓ-adic Galois representation attached to χ_i, $i = 1, 2$, we can define

$$\rho_{\pi,\ell} = \rho_{\chi_1,\ell}(-1) \oplus \rho_{\chi_1,\ell}(-2) \oplus \rho_{\chi_2,\ell}(-1) \oplus \rho_{\chi_2,\ell}(-2).$$

In each of these cases, the Galois representations are direct sums of twists of Galois representations associated to Hilbert modular forms and Hecke characters.

4.4 Galois representations valued in GSp_4

In this section, we show that, when π is non-cohomological, the image of $\rho_{\pi,\ell}$ is contained in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$, as predicted by the Langlands conjectures.

Theorem 4.4.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})$, $k_{1,v} \geq k_{2,v} \geq 2$, and let*

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{Q}}_\ell)$$

be its ℓ-adic Galois representation. Suppose either that $F = \mathbf{Q}$ or that π is unramified at all places $v \mid \ell$. Then $\rho_{\pi,\ell}$ is isomorphic to a representation that factors through $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$.

Remark 4.4.2. This theorem is apparently known to experts, however, prior to [Wei18b, Section 1.3], a full proof has not appeared in the literature. We are grateful to B. Stroh for providing an outline of the proof.

The idea is to reformulate Taylor's original construction of $\rho_{\pi,\ell}$, using V. Lafforgue's G-pseudorepresentations [Laf18] in place of Taylor's pseudorepresentations [Tay91]. Lafforgue's G-pseudorepresentations were introduced in Chapter 3.

4.4.1 Taylor's construction

In [Tay91, Example 1], Taylor gives a blueprint for attaching Galois representations to low weight Siegel modular forms, by utilising congruences with Siegel modular forms of cohomological weight. Taylor's approach to generating these congruences, using Hasse invariants, has been generalised to general totally real fields by [GK19]. In this subsection, we give an overview of the construction of these Galois representations.

Recall that π is the cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ corresponding to a cuspidal Hilbert–Siegel modular eigenform F_π of weights $(k_{1,v}, k_{2,v})_{v|\infty}$ and level Γ for some congruence subgroup $\Gamma \subseteq \mathrm{Sp}_4(\mathbf{Z})$. Fix a prime ℓ, and let E be the finite extension of $\mathbf{Q}_\ell$ spanned by the Hecke eigenvalues of π. Let $\mathbf{T}$ denote the abstract Hecke algebra generated by the Hecke operators of π, and for each tuple $\vec{k} = (k_{1,v}, k_{2,v})_{v|\infty}$

of weights, let $\mathbf{T}_{\vec{k}}$ denote the Hecke algebra acting on forms of weight $\vec{k}$ and level Γ. Then $\mathbf{T}_{\vec{k}}$ is a finite $\mathbf{Z}$-algebra and, for each $\vec{k}$, there is a map $\theta_{\vec{k}} : \mathbf{T} \to \mathbf{T}_{\vec{k}}$.

Associated to π is a map $\theta : \mathbf{T} \to \mathbf{T}_{(k_{1,v},k_{2,v})_{v|\infty}} \to \mathcal{O}_E$. In [Tay91, Example 1], Taylor shows how to attach a Galois representation to π given the following data:

- A set of cohomological weights $\vec{k}_i$ such that, for each place $v \mid \infty$, $\vec{k}_{i,v} \equiv (k_{1,v}, k_{2,v}) \pmod{\ell^i}$.[4]
- For each i, a cuspidal mod ℓ^i Hilbert–Siegel modular eigenform of weight $\vec{k}_i$ whose Hecke eigenvalues are equal to the Hecke eigenvalues of F_π modulo ℓ^i. In the language of Hecke algebras, the existence of this mod ℓ^i eigenform is encoded in the existence of an algebra map

$$r_i : \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_E \to \mathcal{O}_E/\ell^i, \tag{4.1}$$

  and the congruence with F_π is equivalent to asking that the diagram

$$\begin{CD} \mathbf{T} @>{\theta}>> \mathcal{O}_E \\ @V{\theta_{\vec{k}_i}}VV @VVV \\ \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_E @>{r_i}>> \mathcal{O}_E/\ell^i \end{CD}$$

  commutes (c.f. [Gol14, Corollary 6.3.1] for this diagram).

For every i, $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_E$ is a finite product of local rings, each corresponding to a cohomological Hilbert–Siegel modular form of weight $\vec{k}_i$. It follows that there is a finite extension E_i/E and a Galois representation

$$\rho_i : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}_4(\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i)$$

such that $\mathrm{Tr}(\rho_i(\mathrm{Frob}_v)) = \theta_{\vec{k}_i}(T_v)$ whenever $v \notin S_i$, for some finite set of places S_i.

If we could compose ρ_i with r_i to construct a representation

$$\overline{\rho}_i : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}_4(\mathcal{O}_E/\ell^i),$$

then we would be able to construct $\rho_{\pi,\ell}$ as the limit $\varprojlim_i \overline{\rho}_i$. The problem is that, while $\mathrm{Tr}(\rho_i(\mathrm{Frob}_v)) \in \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$ for all places $v \notin S_i$, it is not necessarily true that ρ_i can be chosen to be valued in $\mathrm{GSp}_4(\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i})$. The solution to this problem is to work with pseudorepresentations.

Associated to ρ_i is a pseudorepresentation

$$T_i = \mathrm{Tr}(\rho_i) : \mathrm{Gal}(\overline{F}/F) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i,$$

and, at this level, since $T_i(\mathrm{Frob}_v) \in \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$ for all $v \notin S_i$, it is clear that

$$T_i : \mathrm{Gal}(\overline{F}/F) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$$

is valued in $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$. Composing with r_i, we obtain a pseudorepresentation

$$\overline{T}_i : \mathrm{Gal}(\overline{F}/F) \to \mathcal{O}_{E_i}/\ell^i.$$

[4] In other words, in $\mathbf{Z}_\ell$, $\vec{k}_{i,v} \to (k_{1,v}, k_{2,v})$ as $i \to \infty$.

A computation [Tay91, pp. 291] shows that each $\overline{T}_i$ is in fact valued in $\mathcal{O}_E/\ell^i$, and that for $i \geq m$, $\overline{T}_m \equiv \overline{T}_i \pmod{\ell^m}$. Hence, there is a pseudorepresentation

$$T = \varprojlim_i \overline{T}_i : \mathrm{Gal}(\overline{F}/F) \to \mathcal{O}_E \subset \overline{\mathbf{Q}}_\ell$$

It follows from the theory of pseudorepresentations [Tay91, Theorem 1] that there is a semisimple Galois representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{Q}}_\ell)$$

associated to T, which is, by construction, the Galois representation associated to π.

Hence, to apply Taylor's strategy, it remains to show the existence of the maps in Equation (4.1). When $F = \mathbf{Q}$, these maps were constructed by Taylor [Tay04, Proposition 3] using powers of the Hasse invariant. If $F/\mathbf{Q}$ is totally real and π is unramified at all places $v \mid \ell$, then Taylor's approach using Hasse invariants was generalised by Goldring–Koskivirta [GK19, Theorem 10.5.1].

Taylor's construction via pseudorepresentations is sufficient to show that $\rho_{\pi,\ell}$ is valued in $\mathrm{GL}_4(\overline{\mathbf{Q}}_\ell)$, but is insufficient to show that the representation is isomorphic to one which is valued in $\mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell)$: taking the trace of ρ_i 'forgets' the fact that ρ_i is symplectic. The proof of Theorem 4.4.1 follows the same structure as Taylor's proof, replacing pseudorepresentations with Lafforgue's G-pseudorepresentations.

4.4.2 A reformulation of Taylor's construction

We now prove Theorem 4.4.1. We keep the notation and assumptions of the previous section.

Proof of Theorem 4.4.1. Recall from Chapter 3 that there is a GSp_4-pseudorepresentation $(\mathrm{Tr}\,\rho_i)^\bullet$ associated to

$$\rho_i : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}_4(\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i).$$

By Lemma 3.4.3, $\mathrm{Tr}(\rho_i)^\bullet$ is completely determined by the elements

$$(\mathrm{Tr}\,\rho_i)^1(s_1) = T_i : \mathrm{Gal}(\overline{F}/F) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i,$$

$$(\mathrm{Tr}\,\rho_i)^1(s_2) : \mathrm{Gal}(\overline{F}/F) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i,$$

and

$$(\mathrm{Tr}\,\rho_i)^1(\mathrm{sim}^{\pm 1}) : \mathrm{Gal}(\overline{F}/F) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i$$

of $\mathrm{Map}(\mathrm{Gal}(\overline{F}/F), (\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i))$. Since each of these maps factors through $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$, by part 2 of Lemma 3.2.3, $(\mathrm{Tr}\,\rho_i)^\bullet$ can be viewed is a GSp_4-pseudorepresentation of $\mathrm{Gal}(\overline{F}/F)$ over $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$. Then, by part 1 of Lemma 3.2.3, we can compose $(\mathrm{Tr}\,\rho_i)^\bullet$ with the map $r_i : \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i} \to \mathcal{O}_{E_i}/\ell^i$ to produce a new GSp_4-pseudorepresentation $(r_i \circ \mathrm{Tr}\,\rho_i)^\bullet$ of $\mathrm{Gal}(\overline{F}/F)$ over $\mathcal{O}_{E_i}/\ell^i$. This GSp_4-pseudorepresentation is determined by the images of s_1, s_2 and $\mathrm{sim}^{\pm 1}$, all of which take values in $\mathcal{O}_E/\ell^i$. Hence, by part 2 of Lemma 3.2.3, $(r_i \circ \mathrm{Tr}\,\rho_i)^\bullet$ can be viewed as a GSp_4-pseudorepresentation over $\mathcal{O}_E/\ell^i$. Moreover, if $i \geq m$, then

$$(r_m \circ \mathrm{Tr}\,\rho_m)^\bullet = (r_i \circ \mathrm{Tr}\,\rho_i)^\bullet \pmod{\ell^m}.$$

We can thus form a GSp_4-pseudorepresentation

$$\Theta^\bullet = \varprojlim_i (r_i \circ \operatorname{Tr} \rho_i)^\bullet$$

of $\mathrm{Gal}(\overline{F}/F)$ over $\mathcal{O}_E$. Finally, viewing $\mathcal{O}_E$ as a subalgebra of $\overline{\mathbf{Q}}_\ell$, we may view $\Theta^\bullet$ as a GSp_4-pseudorepresentation over $\overline{\mathbf{Q}}_\ell$.

By Theorem 3.3.3, there is a representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_\ell),$$

such that $\Theta^\bullet = (\operatorname{Tr} \rho)^\bullet$. This is the Galois representation associated to π. Indeed, by construction,

$$\Theta^1(s_1) = T : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}_\ell$$

is exactly the classical pseudorepresentation constructed by Taylor. □

Chapter 5
Irreducibility

In this chapter, we prove that $\rho_{\pi,\ell}$ is irreducible if it is crystalline and if ℓ is sufficiently large.

Our proof uses the strategy outlined in Section 1.1.2.

1. *In Theorem 5.2.1, we show that, if $\rho_{\pi,\ell}$ is reducible, then it splits as a direct sum of irreducible subrepresentations ρ_1, ρ_2 that are distinct, two-dimensional, Hodge–Tate regular and totally odd. This theorem is the key technical result of this chapter, and makes no assumptions on the prime ℓ.*

2. *If ℓ is sufficiently large and if $\rho_{\pi,\ell}$ is crystalline, then ρ_1 and ρ_2 are potentially modular.*

3. *By appealing to the techniques outlined in Section 1.2, we deduce that, if $\rho_{\pi,\ell}$ is reducible, then π cannot be cuspidal, a contradiction.*

The results of this chapter, especially Theorem 5.2.1, lay the groundwork for Chapter 6, where we prove that $\rho_{\pi,\ell}$ is crystalline—and therefore irreducible—for 100% of primes.

Recall that F is a totally real field and π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$, and central character $\chi_\pi = \chi_{\pi^\circ}|\operatorname{sim}|^{-w}$. Let ℓ be a prime. In Chapter 4, we constructed the ℓ-adic Galois representation

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{Q}}_\ell)$$

attached to π. If π is not of type **(G)**, then $\rho_{\pi,\ell}$ is reducible, and its subrepresentations are well understood. Hence, from now on, we assume that π is of general type (see Section 2.3).

In Section 5.1, we address the case that π is of type **(G)**, but is still a lift from a smaller group. In this situation, we use standard techniques to prove that $\rho_{\pi,\ell}$ is irreducible.

In Section 5.2, we prove the main technical result of this chapter, which restricts the ways in which $\rho_{\pi,\ell}$ can decompose, without any assumptions on ℓ. Using this result, in Section 5.3, we prove Theorem 5.3.1, that $\rho_{\pi,\ell}$ is irreducible if it is crystalline and if ℓ is sufficiently large.

5.1 Lifts from smaller groups

5.1.1 Automorphic inductions

Let Π be the transfer of π to GL_4 (see Section 2.2). Since π is of type **(G)**, Π is cuspidal.

Definition 5.1.1. We say that π is an *automorphic induction* if there is a quadratic extension K/F and a cuspidal automorphic representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_K)$ such that $\Pi = \mathrm{AI}_K^F(\boldsymbol{\pi})$ is automorphically induced from $\boldsymbol{\pi}$ in the sense of [AC89].

If

$$\rho_{\boldsymbol{\pi},\ell} : \mathrm{Gal}(\overline{F}/K) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

is the ℓ-adic Galois representation attached to $\boldsymbol{\pi}$, then, comparing the Hecke parameters of Π with the Frobenius traces of $\mathrm{Ind}_K^F(\rho_{\boldsymbol{\pi}})$ and applying the Chebotarev density theorem, we see that

$$\rho_{\pi,\ell} \simeq \mathrm{Ind}_K^F(\rho_{\boldsymbol{\pi},\ell}).$$

Remark 5.1.2. If K is totally real, then $\boldsymbol{\pi}$ corresponds to a Hilbert modular form, and we can directly attach ℓ-adic Galois representations to $\boldsymbol{\pi}$. On the other hand, if K is not totally real, then the ℓ-adic Galois representations attached to $\boldsymbol{\pi}$ are constructed via the ℓ-adic Galois representations attached to π [Mok14], [BCGP18, Theorem 2.7.3]. This latter case can only occur if π has low weight. In either case, by Theorem 1.2.6, $\rho_{\boldsymbol{\pi},\ell}$ is irreducible.

We deduce the following, well-known proposition:

Proposition 5.1.3. *Suppose that π is an automorphic induction. Then $\rho_{\pi,\ell}$ is irreducible.*

Proof. By assumption,

$$\rho_{\pi,\ell} \simeq \mathrm{Ind}_K^F(\rho_{\boldsymbol{\pi},\ell}),$$

and $\rho_{\boldsymbol{\pi},\ell}$ is irreducible.

Since Π is cuspidal, for any $\sigma \in \mathrm{Gal}(K/F)$, $\boldsymbol{\pi}^\sigma \not\cong \boldsymbol{\pi}$ [AC89, Theorem 4.2]. It follows that $\rho_{\boldsymbol{\pi},\ell}^\sigma \not\simeq \rho_{\boldsymbol{\pi},\ell}$ and, hence, by Mackey theory, that $\rho_{\pi,\ell}$ is irreducible. □

5.1.2 Symmetric cube lifts

A case of Langlands functoriality, proven by Kim–Shahidi [KS02], associates to the map $\mathrm{Sym}^3 : \mathrm{GL}_2(\mathbf{C}) \to \mathrm{GL}_4(\mathbf{C})$ a functorial lift

$$\Pi(\mathrm{GL}_2) \to \Pi(\mathrm{GL}_4).$$

In fact, the image of Sym^3 lies in $\mathrm{GSp}_4(\mathbf{C})$.

Definition 5.1.4. Let Π be the transfer of π to GL_4. We say that π is a *symmetric cube lift* if there is a cuspidal automorphic representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_F)$ such that

$$\Pi \cong \mathrm{Sym}^3(\boldsymbol{\pi}).$$

Since F is totally real, we can directly attach an ℓ-adic representation

$$\rho_{\boldsymbol{\pi},\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

to $\boldsymbol{\pi}$. This representation is irreducible by Theorem 1.1.2. If $\Pi \cong \mathrm{Sym}^3(\boldsymbol{\pi})$, then

$$\rho_{\pi,\ell} \simeq \mathrm{Sym}^3(\rho_{\boldsymbol{\pi},\ell})$$

is irreducible.

We note that if π is a symmetric cube lift, then the weights of π must be of the form $(2k_v - 1, k_v + 1)_{v|\infty}$ for $k_v \geq 2$. In particular, π must be cohomological.

5.2 Restrictions on the decomposition of $\rho_{\pi,\ell}$

In Section 5.1, we proved that, if π is either an automorphic induction or a symmetric cube lift, then $\rho_{\pi,\ell}$ is irreducible. Therefore, it remains to address the case that π is neither a symmetric cube lift nor an automorphic induction.

For each prime ℓ, the ℓ-adic Galois representation $\rho_{\pi,\ell}$ associated to π is either irreducible, or it decomposes as a direct sum of irreducible subrepresentations. The goal of this section is to prove the following theorem, which strongly restricts the ways in which $\rho_{\pi,\ell}$ can decompose.

Theorem 5.2.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$. Suppose, moreover, that π is of type* (**G**).

Then, either $\rho_{\pi,\ell}$ is irreducible, or it decomposes as a direct sum $\rho_1 \oplus \rho_2$ of distinct, irreducible, two-dimensional, Hodge–Tate representations, both with determinant $\mathrm{sim}(\rho_{\pi,\ell})$.

In particular, for each $i = 1, 2$, ρ_i is Hodge–Tate regular, and the Hodge–Tate weights of ρ_1, ρ_2 at the embedding corresponding to $v \mid \infty$ are $\{\delta_v + k_{1,v} - 2, \delta_v + k_{2,v} - 1\}$ and $\{\delta_v, \delta_v + k_{1,v} + k_{2,v} - 3\}$.

We will apply this theorem in two different settings:

1. If $\rho_{\pi,\ell}$ is crystalline and if ℓ is sufficiently large, then Theorem 5.2.1 allows us to apply potential modularity results to any subrepresentations of $\rho_{\pi,\ell}$. Applying the arguments of Section 1.2 will allow us to deduce that $\rho_{\pi,\ell}$ is irreducible.
2. Theorem 5.2.1 gives a restriction on the image of $\rho_{\pi,\ell}$, which we will use in Chapter 6 to prove that $\rho_{\pi,\ell}$ is crystalline for a positive density of primes ℓ.

If π is cohomological, then Theorem 5.2.1 is essentially known. Indeed, since π is regular, $\rho_{\pi,\ell}$ has distinct Hodge–Tate weights and, hence, its subrepresentations are distinct and have regular Hodge–Tate weights. Moreover, the works of Weissauer [Wei05, Theorem II] and Ramakrishnan [Ram13, Theorem A] show that these subrepresentations are two-dimensional[1] and odd. The fact that $\det(\rho_1) = \det(\rho_2) = \mathrm{sim}(\rho_{\pi,\ell})$ follows from

[1] While Weissauer's result is stated when $F = \mathbf{Q}$, his proof applies more generally.

combining [Ram13, Theorem A] with the arguments in the proof of [SU06, Théorème 3.2.1].[2] These results make extensive use of results about $\rho_{\pi,\ell}$ that are only known when π is cohomological. In the remainder of this section, we prove Theorem 5.2.1 in the case that π is non-cohomological.

Remark 5.2.2. Theorem 5.2.1 states that, if π is of type **(G)** in Arthur's classification, then $\rho_{\pi,\ell}$ is either irreducible or else it looks like a Galois representation associated to an automorphic representation of type **(Y)**. In Lemmas 5.2.3 and 5.2.4, we use a weak form of the Ramanujan conjecture to show that $\rho_{\pi,\ell}$ can never look like the Galois representation associated to an automorphic representation of type **(Q)**, **(P)**, **(B)** or **(F)**. The key technical component of the proof of Theorem 5.2.1 is Lemma 5.2.5, where we show that, if $\rho_{\pi,\ell}$ is reducible, then its subrepresentations are odd.

5.2.1 Subrepresentations are two-dimensional

Suppose, for contradiction, that $\rho_{\pi,\ell} = \rho_1 \oplus \rho_2$ is reducible. To prove that ρ_1 and ρ_2 are two-dimensional, we show that $\rho_{\pi,\ell}$ cannot have a one-dimensional subrepresentation. In the cohomological case, Weissauer [Wei05, Theorem II] establishes this fact by appealing to the Ramanujan conjecture and the fact that $\rho_{\pi,\ell}$ is Hodge–Tate. Our key observation is that we can establish this fact by appealing to a very weak form of the Ramanujan conjecture combined with knowledge of the Hodge–Tate–Sen weights of $\rho_{\pi,\ell}$.

Lemma 5.2.3. *Any irreducible subrepresentation of $\rho_{\pi,\ell}$ is two-dimensional.*

Proof. We prove that $\rho_{\pi,\ell}$ cannot have a one-dimensional subrepresentation. Suppose, for contradiction, that ρ_1 is a one-dimensional subrepresentation of $\rho_{\pi,\ell}$. Then ρ_1 is a Hodge–Tate character of $\mathrm{Gal}(\overline{F}/F)$ and, since F is totally real, $\rho_1 = \varepsilon^{-i}\chi$ for some $i \in \mathbf{Z}$ and for some finite order character χ. By Theorem 4.2.1 $\rho_{\pi,\ell}$ has Hodge–Tate–Sen weights $\{\delta_v, \delta_v + k_{2,v} - 2, \delta_v + k_{1,v} - 1, \delta_v + k_{1,v} + k_{2,v} - 3\}$, where $\delta_v = \frac{1}{2}(w + 3 - (k_{1,v} + k_{2,v} - 3)))$. It follows that

$$\begin{aligned} i \in \{&\delta_v, \delta_v + k_{2,v} - 2, \delta_v + k_{1,v} - 1, \delta_v + k_{1,v} + k_{2,v} - 3\} \\ = \bigg\{&\frac{1}{2}(w+3+(3-k_1-k_2)), \frac{1}{2}(w+3+(-1-k_1+k_2)), \\ &\frac{1}{2}(w+3+(1+k_1-k_2)), \frac{1}{2}(w+3+(k_1+k_2-3))\bigg\}. \end{aligned}$$

Note that, since $k_1 \geq k_2 \geq 2$, i cannot equal $\frac{1}{2}(w+3)$.

Choose any place v of F at which $\rho_{\pi,\ell}$ is unramified. Then

$$\alpha_v := \rho_1(\mathrm{Frob}_v) = \varepsilon^{-i}(\mathrm{Frob}_v)\chi(\mathrm{Frob}_v) = N(v)^i\chi(\mathrm{Frob}_v)$$

is an eigenvalue of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$. The complex absolute value of α_v is $|\alpha_v| = N(v)^i$. The generalised Ramanujan conjecture predicts that $|\alpha_v| = N(v)^{\frac{1}{2}(w+3)}$. Hence, if $\rho_{\pi,\ell}$ has a one-dimensional subrepresentation, then the generalised Ramanujan conjecture fails

[2][Ram13, Theorem A] shows that ρ_1 and ρ_2 are odd, and [SU06] prove that $\det(\rho_1) = \det(\rho_2)$, assuming that ρ_1, ρ_2 are odd. Since $\det(\rho_{\pi,\ell}) = \mathrm{sim}(\rho_{\pi,\ell})^2 = \det(\rho_1)^2$ and $\mathrm{sim}(\rho_{\pi,\ell})$ and $\det(\rho_1)$ are both odd, it follows that they are equal.

for all places v at which $\rho_{\pi,\ell}$ is unramified. Although the Ramanujan conjecture is not known when π has non-cohomological weight, by [JS81b, Corollary 2.5], we have

$$N(v)^{-\frac{1}{2}} < |\alpha_v| N(v)^{-\frac{12}{(}w+3)} < N(v)^{\frac{1}{2}}.$$

Since $i \in \mathbf{Z}$, it follows that $i = \frac{1}{2}(w+3)$, a contradiction. □

5.2.2 Subrepresentations are Hodge–Tate

As a result of Lemma 5.2.3, if $\rho_{\pi,\ell}$ is reducible, then we can write

$$\rho_{\pi,\ell} = \rho_1 \oplus \rho_2,$$

where ρ_1, ρ_2 are two-dimensional. We now show that both ρ_1 and ρ_2 are Hodge-Tate, and that their determinants have the same Hodge–Tate weight as $\mathrm{sim}(\rho_{\pi,\ell})$.

Lemma 5.2.4. *The representations ρ_1 and ρ_2 are Hodge–Tate, and $\det(\rho_1), \det(\rho_2)$ both have Hodge–Tate weight $w+3$ for all places $v \mid \infty$.*

Proof. Fix an archimedean place v and let h be the Hodge–Tate weight of $\det(\rho_1)$ at v. Suppose that $h \neq w+3$. Recall that the Hodge–Tate–Sen weights of $\rho_{\pi,\ell}$ are

$$\{\delta_v, \delta_v + k_{2,v} - 2, \delta_v + k_{1,v} - 1, \delta_v + k_{1,v} + k_{2,v} - 3\},$$

where $\delta_v = \frac{1}{2}(w + 3 - (k_{1,v} + k_{2,v} - 3))$. Switching ρ_1 and ρ_2 if necessary, it follows that $h > w+3$. If v is any place of F at which $\rho_{\pi,\ell}$ is unramified and if α_v, β_v are the roots of the characteristic polynomial of $\rho_1(\mathrm{Frob}_v)$, then

$$|\alpha_v \beta_v| = |\det(\rho_1(\mathrm{Frob}_v))| = N(v)^h > N(v)^{(w+3)}.$$

After relabelling, it follows that $|\alpha_v| \geq N(v)^{\frac{h}{2}}$. Since v was arbitrary, we deduce that, for almost all places v of F, $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ has an eigenvalue $\alpha_v \in \mathbf{C}$ with

$$|\alpha_v| N(v)^{-\frac{w+3}{2}} \geq N(v)^{\frac{h-(w+3)}{2}} \geq N(v)^{\frac{1}{2}},$$

which contradicts [JS81b, Corollary 2.5], as in Lemma 5.2.3.

Hence, $\det(\rho_1)$ and $\det(\rho_2)$ both have Hodge–Tate weight $w+3$ at v. Therefore, up to reordering, the Hodge–Tate–Sen weights of ρ_1 and ρ_2 are

$$\{\delta_v + k_{2,v} - 2, \delta_v + k_{1,v} - 1\} \text{ and } \{\delta_v, \delta_v + k_{1,v} + k_{2,v} - 3\}.$$

Since the Hodge–Tate–Sen weights are distinct integers, it follows that the Sen operator is semisimple (c.f. Section 0.4.5) and, hence, that both ρ_1 and ρ_2 are Hodge–Tate at v. □

5.2.3 Subrepresentations are odd

Lemma 5.2.5. *Both ρ_1 and ρ_2 have determinant $\mathrm{sim}(\rho_{\pi,\ell})$. In particular, both ρ_1 and ρ_2 are odd.*

Proof. Let $\chi = \det(\rho_1)^{-1}\,\mathrm{sim}(\rho_{\pi,\ell})$. By Lemma 5.2.4, χ has Hodge–Tate weight 0 for all places $v \mid \infty$, so χ is a finite order character. We show that χ is the trivial character.

First, since

$$\mathrm{sim}(\rho_{\pi,\ell})^2 \simeq \det(\rho_{\pi,\ell}) \simeq \det(\rho_1)\det(\rho_2),$$

it follows that

$$\chi \simeq \det(\rho_2)\,\mathrm{sim}(\rho_{\pi,\ell})^{-1}.$$

Moreover, since $\rho_{\pi,\ell}^\vee \otimes \mathrm{sim}(\rho_{\pi,\ell}) \simeq \rho_{\pi,\ell}$, we have

$$\begin{aligned}
\rho_1 \oplus \rho_2 \simeq \rho_{\pi,\ell} &\simeq \rho_{\pi,\ell}^\vee \otimes \mathrm{sim}(\rho_{\pi,\ell}) \\
&\simeq \left(\rho_1^\vee \otimes \mathrm{sim}(\rho_{\pi,\ell})\right) \oplus \left(\rho_2^\vee \otimes \mathrm{sim}(\rho_{\pi,\ell})\right) \\
&\simeq \left(\rho_1 \otimes \det(\rho_1)^{-1} \otimes \mathrm{sim}(\rho_{\pi,\ell})\right) \oplus \left(\rho_2 \otimes \det(\rho_2)^{-1} \otimes \mathrm{sim}(\rho_{\pi,\ell})\right) \\
&\simeq (\rho_1 \otimes \chi) \oplus \left(\rho_2 \otimes \chi^{-1}\right).
\end{aligned}$$

By Schur's lemma, it follows that either:

1. $\rho_1 \otimes \chi \simeq \rho_1$ and $\rho_2 \otimes \chi \simeq \rho_2$;
2. Or $\rho_1 \otimes \chi \simeq \rho_2$.

In the first case, we see that

$$\rho_{\pi,\ell} \otimes \chi \simeq \rho_{\pi,\ell}.$$

Let Π be the transfer of π to GL_4, and let η be the Hecke character attached to χ by class field theory. By local-global compatibility at unramified primes, we see that the automorphic representations Π and $\Pi \otimes \eta$ have the same Hecke parameters at almost all primes. The assumption that π is of type **(G)** ensures that Π is cuspidal. Hence, by the strong multiplicity one theorem for GL_4 (see Corollary 0.4.2), we see that $\Pi \cong \Pi \otimes \eta$. Therefore, by [AC89, Lemma 3.6.6], Π is an automorphic induction. However, if Π is an automorphic induction, then, by Proposition 5.1.3, $\rho_{\pi,\ell}$ is irreducible, so this case cannot occur.

In the second case,[3] we find that

$$\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_1 \otimes \chi.$$

Let n be the order of χ. We want to show that $n = 1$. Note that if $n = 2$, then $\chi = \chi^{-1}$ and $\rho_{\pi,\ell} \otimes \chi \simeq \rho_{\pi,\ell}$ as in the first case. Hence, we can assume that $n \geq 3$.

Consider the representations

$$r_i := \rho_1 \otimes \rho_1^\vee \otimes \chi^i$$

for each $i \in \mathbf{Z}$, and let

$$a_i = \mathrm{ord}_{s=1} L(r_i, s).$$

We prove the following facts to reach a contradiction:

1. The partial L-functions $L^*(r_i, s)$ have meromorphic continuation to the entire complex plane. In particular, $a_i \in \mathbf{Z}$ for all i.

[3] In this case, ρ_1 and $\rho_2 \simeq \rho_1 \otimes \chi$ have the same Hodge–Tate weights. In particular, this case can only occur when $\rho_{\pi,\ell}$ has irregular Hodge–Tate weights.

2. For all $i \not\equiv 0 \pmod n$, we have

$$a_{i+1} = -2a_i - a_{i-1}$$

and $a_0 = -1$.

3. On the other hand, $a_{n-1} = 1 - a_1$.

The combination of these facts contradicts the fact that $a_1 \in \mathbf{Z}$.

Claim 1. *The partial L-functions $L^*(r_i, s)$ have meromorphic continuation to the entire complex plane.*

Proof. Consider the exterior square $\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1}$. Using the fact that, for general two-dimensional representations σ_1, σ_2, we have

$$\bigwedge^2(\sigma_1 \oplus \sigma_2) \simeq (\sigma_1 \otimes \sigma_2) \oplus \det(\sigma_1) \oplus \det(\sigma_2),$$

it follows that

$$\begin{aligned}
\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1} &\simeq \bigwedge^2(\rho_1 \oplus \rho_1 \otimes \chi) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1} \\
&\simeq \big(\rho_1 \otimes \rho_1 \otimes \chi \oplus \det(\rho_1) \oplus \det(\rho_1)\chi^2\big) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1} \\
&\simeq (\rho_1 \otimes \rho_1 \otimes \det(\rho)^{-1}) \oplus \chi^{-1} \oplus \chi \\
&\simeq (\rho_1 \otimes \rho_1^\vee) \oplus \chi \oplus \chi^{-1}.
\end{aligned}$$

Let Π be the transfer of π to GL_4. There is an automorphic representation $\bigwedge^2(\Pi)$ of GL_6 with the property that $L^*(\bigwedge^2(\Pi), s) = L^*(\bigwedge^2(\rho_{\pi,\ell}), s)$ outside a finite set of places [Kim03]. In particular, for any Hecke character η, $L^*(\bigwedge^2(\Pi) \otimes \eta, s)$ has meromorphic continuation to the entire complex plane. Moreover, we have an equality of partial L-functions

$$L^*(\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1}, s) = L^*(\rho_1 \otimes \rho_1^\vee, s) L^*(\chi, s) L^*(\chi^{-1}, s), \tag{5.1}$$

from which it follows that $L^*(\rho_1 \otimes \rho_1^\vee, s)$ has meromorphic continuation to the entire complex plane. Similarly, by considering $L^*(\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1} \otimes \chi^i, s)$, we deduce that, for any i, $L^*(r_i, s)$ has meromorphic continuation to the entire complex plane. □

In particular, we can define integers

$$a_i := \mathrm{ord}_{s=1} L^*(\rho_1 \otimes \rho_1^\vee \otimes \chi^i, s)$$

for all $i \in \mathbf{Z}$. Since, by definition, χ has order n, it follows that $a_i = a_{i+n}$ for all i. We use the fact that $a_i \in \mathbf{Z}$ for all i to reach a contradiction.

Claim 2. *We have $a_0 = -1$ and $a_{i+1} = -2a_i - a_{i-1}$ for all $i \not\equiv 0 \pmod n$.*

Proof. By [Sch17b, Lemma 1.2], since π is cuspidal and is not CAP or endoscopic, the exterior square L-function $L^*(\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1}, s)$ has a simple pole at $s = 1$. Since χ is a finite order character that is not the trivial character, $L^*(\chi, s)$ and $L^*(\chi^{-1}, s)$ are defined and non-zero at $s = 1$. Hence, by Equation (5.1), $a_0 = -1$.

Moreover, for all i, we have

$$\begin{aligned}\rho_{\pi,\ell} \otimes \rho_{\pi,\ell}^\vee \otimes \chi^i &\simeq (\rho_1 \oplus \rho_1 \otimes \chi) \otimes (\rho_1^\vee \oplus \rho_1^\vee \otimes \chi^{-1}) \otimes \chi^i \\ &\simeq \left(\rho_1 \otimes \rho_1^\vee \otimes \chi^i\right)^{\oplus 2} \oplus \left(\rho_1 \otimes \rho_1^\vee \otimes \chi^{i+1}\right) \oplus \left(\rho_1 \otimes \rho_1^\vee \otimes \chi^{i-1}\right).\end{aligned}$$

Taking L-functions of both sides, outside a finite set of places, we find that

$$L^*(r_{i+1}, s) = \frac{L^*(\rho_{\pi,\ell} \otimes \rho_{\pi,\ell}^\vee \otimes \chi^i)}{L^*(r_i, s)^2 L^*(r_{i-1}, s)}. \tag{5.2}$$

Let Π be the transfer of π to GL_4 and write η for the Hecke character associated to χ by class field theory. If Π is an automorphic induction, then, by Proposition 5.1.3, $\rho_{\pi,\ell}$ is irreducible. Hence, we may assume that Π is not an automorphic induction and, by [AC89, Theorem 4.2], that

$$\Pi \not\cong \Pi \otimes \eta^i$$

for all $i \not\equiv 0 \pmod{n}$. Hence, away from a finite set of places, by Theorem 1.2.4, we have

$$\operatorname{ord}_{s=1} L^*(\rho_{\pi,\ell} \otimes \rho_{\pi,\ell}^\vee \otimes \chi^i, s) = \operatorname{ord}_{s=1} L^*(\Pi \otimes \Pi^\vee \otimes \eta^i, s) = 0. \tag{5.3}$$

Combining (5.2) and (5.3), it follows that

$$a_{i+1} = -2a_i - a_{i-1}$$

for all $i \not\equiv 0 \pmod{n}$, as required. □

Solving this difference equation, we find that, for all $i \not\equiv 0 \pmod{n}$,

$$a_{i+1} = (-1)^i + (i+1)(1-a_1)(-1)^{i+1}. \tag{5.4}$$

Let Π be the transfer of π to GL_4. Since Π is cuspidal, away from a finite set of places, by Theorem 1.2.4, we have

$$\operatorname{ord}_{s=1} L^*(\rho_{\pi,\ell} \otimes \rho_{\pi,\ell}^\vee, s) = \operatorname{ord}_{s=1} L^*(\Pi \otimes \Pi^\vee, s) = -1. \tag{5.5}$$

By Equation (5.2) with $i = 0$, we have

$$a_1 = \operatorname{ord}_{s=1} L^*(\rho_{\pi,\ell} \otimes \rho_{\pi,\ell}^\vee, s) - 2a_0 - a_{n-1}$$

Since we know that $a_0 = -1$, by Equation (5.5), it follows that

$$a_1 = -1 - 2a_0 - a_{n-1} = 1 - a_{n-1}.$$

Comparing this equality with (5.4) when $i = n-2$ gives

$$1 - a_1 = a_{n-1} = (-1)^n + (n-1)(1-a_1)(-1)^{n-1}$$

from which it follows that

$$a_1 = 1 + \frac{(-1)^{n+1}}{1 + (-1)^n(n-1)}.$$

Hence, a_1 is an integer only if $n = 1$ or $n = 3$. When $n = 3$, we deduce that $a_1 = 0$, and hence that $a_2 = 1 - a_1 = 1$. But using the fact that $a_3 = a_0 = -1$, we have

$$-1 = a_3 = -2a_2 - a_1 = -2,$$

which is a contradiction. The result follows. □

5.2.4 Subrepresentations are distinct

Finally, we show that ρ_1 and ρ_2 are distinct, which completes the proof of Theorem 5.2.1. This fact is clear in the case that π_v is a holomorphic discrete series for some place $v \mid \infty$ of F, but requires proof in the case that π_v is a holomorphic limit of discrete series at all archimedean places.

Lemma 5.2.6. *If $\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are irreducible and two-dimensional, then $\rho_1 \not\simeq \rho_2$.*

Proof. Suppose that $\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_1$ where ρ_1 is irreducible and two-dimensional. Then on the one hand,

$$\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1} \simeq \mathrm{ad}^0(\rho_1) \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}.$$

As before, $L^*(\bigwedge^2(\rho_{\pi,\ell}) \otimes \mathrm{sim}(\rho_{\pi,\ell})^{-1}, s)$ has meromorphic continuation to the entire complex plane [Kim03], from which it follows that $L^*(\mathrm{ad}^0(\rho_1), s)$ has meromorphic continuation to the entire complex plane. On the other hand, we have

$$\rho_{\pi,\ell} \otimes \rho_{\pi,\ell}^\vee \simeq \mathrm{ad}^0(\rho_1)^{\oplus 4} \oplus \mathbf{1}^{\oplus 4}.$$

It follows from (5.5) that

$$4(\mathrm{ord}_{s=1} L^*(\mathrm{ad}^0(\rho_1), s)) = 3,$$

which is impossible, since $L^*(\mathrm{ad}^0(\rho_1), s)$ is meromorphic. □

5.3 Irreducibility when ℓ is large and $\rho_{\pi,\ell}$ is crystalline.

We have now shown that either $\rho_{\pi,\ell}$ is irreducible or $\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are distinct, totally odd, irreducible, Hodge–Tate regular, two-dimensional representations. We now prove that if ℓ is sufficiently large and if $\rho_{\pi,\ell}$ is crystalline, then $\rho_{\pi,\ell}$ is irreducible.

Theorem 5.3.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$. Suppose, moreover, that π is of type $(\mathbf{G})$.*

Then, there exists an integer M such that, if $\ell > M$ and if $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$, then $\rho_{\pi,\ell}$ is irreducible.

If $F = \mathbf{Q}$, then we can take $M = 2(k_1 + k_2 - 3) + 1$.

Proof. If $\rho_{\pi,\ell}$ is reducible, then, by Theorem 5.2.1, $\rho_{\pi,\ell} = \rho_1 \oplus \rho_2$ decomposes as a direct sum of two-dimensional representations that are crystalline, odd, irreducible, Hodge–Tate regular and have determinant $\mathrm{sim}(\rho_{\pi,\ell})$.

By Corollary 1.2.5, it is enough to show that there is a totally real extension E/F such that $\rho_1|_E, \rho_2|_E$ are modular. Indeed, suppose that there are cuspidal automorphic representations $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ of $\mathrm{GL}_2(\mathbf{A}_E)$ such that, for each i, $\rho_i|_E$ is the Galois representation associated to $\boldsymbol{\pi}_i$. By Theorem 1.1.2, $\rho_i|_E$ is necessarily irreducible. Since $\det(\rho_1) = \det(\rho_2)$, we can choose $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$ to be unitary and to satisfy $L^*(\boldsymbol{\pi}_i, s + \frac{w+3}{2}) = L^*(\rho_i, s)$. In this case, the result follows from Corollary 1.2.5.

If $F = \mathbf{Q}$ and $\ell > 2(k_1 + k_2 - 3) + 1$, then, by the main theorem of [Tay06], ρ_1, ρ_2 are potentially modular, and the result follows.[4]

If $F \neq \mathbf{Q}$ and the mod ℓ reductions $\overline{\rho}_1$ and $\overline{\rho}_2$ of ρ_1 and ρ_2 are irreducible, then, when ℓ is sufficiently large, we can apply the potential automorphy theorem of [BLGGT14]. Indeed, by [CG13, Lemma 2.6], if $\ell > 2k+1$, then $\overline{\rho}_1|_{F(\zeta_\ell)}$ and $\overline{\rho}_2|_{F(\zeta_\ell)}$ are irreducible. Moreover, when ℓ is unramified in F and $\ell > k+1$, by [BLGGT14, Lemma 1.4.3], ρ_1 and ρ_2 are potentially diagonalisable[5] at all primes above ℓ. Moreover, when ℓ is large enough, we can assume that $\zeta_\ell \notin F$. Hence, by [BLGGT14, Theorem 4.5.1] and the arguments of the proof of [BLGGT14, Corollary 4.5.2], there exists a totally real field E/F and automorphic representations $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ of $\mathrm{GL}_2(\mathbf{A}_E)$ such that for each i, $\rho_i|_E$ is the Galois representation associated to $\boldsymbol{\pi}_i$. The result follows in this case.

It remains to address the case where $F \neq \mathbf{Q}$ and $\overline{\rho}_1$ and $\overline{\rho}_2$ are reducible. We show that this case does not occur when ℓ is sufficiently large. Let S_{crys} be the set of primes ℓ such that $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$. In Section 7.2.1, we show that, for all but finitely many primes $\ell \in S_{crys}$, the residual representation $\overline{\rho}_{\pi,\ell}$ does not contain a one-dimensional subrepresentation.[6] Hence, when ℓ is sufficiently large, $\overline{\rho}_1$ and $\overline{\rho}_2$ are irreducible. The result follows. □

Remark 5.3.2. In the proofs of [Ram13, Theorems B and C], the author applies the potential modularity theorem of [Tay06] to irreducible, two-dimensional representations $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$. However, as stated, the results of [Tay06] only apply to representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The results of [Tay06] have been generalised to totally real fields F, assuming the Taylor–Wiles hypothesis that $\overline{\rho}|_{F(\zeta_\ell)}$ is irreducible, however, the representations considered in [Ram13] could be residually reducible. To the best of our knowledge, when $F \neq \mathbf{Q}$, potential modularity for GL_2 in the residually reducible case is an open problem.

[4] Taylor shows that ρ_1, ρ_2 are potentially modular, but not necessarily simultaneously potentially modular. However, Taylor's proof can be modified to ensure simultaneous potential modularity (see, for example, [BLGHT11, pp. 93]. Alternatively, we could apply [Ram13, Theorem C], which does not require simultaneous potential modularity, in place of Corollary 1.2.5.

[5] See [BLGGT14, pp. 3] for the definition.

[6] The results of Section 7.2.1 do not need to assume that $\rho_{\pi,\ell}$ is irreducible.

Chapter 6
Crystallinity

In the previous chapter, we proved that $\rho_{\pi,\ell}$ is irreducible if it is crystalline at all places $v \mid \ell$ and if ℓ is sufficiently large. In this chapter, we prove that $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$ for 100% of primes ℓ. By a criterion of Jorza (Theorem 6.1.3), it is sufficient to show that π_v has distinct Satake parameters for all places $v \mid \ell$ for 100% of primes ℓ.

If π_v has repeated Satake parameters for an unramified place v of F, then, for any *prime ℓ with $v \nmid \ell$, $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ has repeated eigenvalues. If this repetition occurs for a positive density of primes, then we obtain a severe restriction on the image of $\rho_{\pi,\ell}$ for all primes ℓ. In particular, $\rho_{\pi,\ell}$ can never be irreducible.*

In this chapter, we prove Theorem A as follows:

1. *Use Theorem 5.2.1 to show that π_v has distinct Satake parameters at all places $v \mid \ell$ for a positive density of primes ℓ.*
2. *Apply Theorem 5.3.1 to show that $\rho_{\pi,\ell}$ is irreducible for such primes ℓ.*
3. *Deduce that π_v has distinct Satake parameters at all places $v \mid \ell$ for 100% of primes ℓ.*
4. *Apply Theorem 5.3.1 again to deduce Theorem A.*

6.1 Crystallinity and the image of $\rho_{\pi,\ell}$

In this section, we prove the following lemma:

Lemma 6.1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v\mid\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$. Suppose that π is of type $(\mathbf{G})$ and that π is not an automorphic induction.*

Suppose that there exists a prime ℓ_0 at which π is unramified, such that ρ_{π,ℓ_0} is irreducible. Then for 100% of primes ℓ, the representation $\rho_{\pi,\ell}$ is crystalline at v for all places $v \mid \ell$.

In Theorem 5.3.1, we showed that $\rho_{\pi,\ell}$ is irreducible if it is crystalline and if ℓ is sufficiently large. Hence, using Lemma 6.1.1, we can prove Theorem A by showing that there is single prime ℓ_0 such that ρ_{π,ℓ_0} is irreducible.

Remarks 6.1.2.

1. If π is of cohomological weight, then, by Theorem 4.1.1, $\rho_{\pi,\ell}$ is crystalline at v whenever π_v is unramified. The content of Lemma 6.1.1 is in the case that π is of non-cohomological weight.

2. If π is not of type (**G**), then, in Section 4.3, we showed that $\rho_{\pi,\ell}$ is built up from Galois representations attached to Hilbert modular forms and to Hecke characters. If π is unramified at all places above ℓ, then these Galois representations are crystalline and it follows that $\rho_{\pi,\ell}$ is crystalline as well.

3. If π is an automorphic induction, then we do not prove that $\rho_{\pi,\ell}$ is crystalline. However, in this case, by Proposition 5.1.3, $\rho_{\pi,\ell}$ is irreducible for all primes ℓ.

Our key tool for proving Lemma 6.1.1 is the following theorem of Jorza, which reduces the problem of showing that $\rho_{\pi,\ell}$ is crystalline at v to showing that the Satake parameters of π_v are distinct.

Theorem 6.1.3 ([Jor12, Theorem 4.1], [Mok14, Proposition 4.16])**.** *Let v be a place of F which divides ℓ. Suppose that π is unramified at v and that the Satake parameters of π_v are pairwise distinct. Then $\rho_{\pi,\ell}$ is crystalline at v.*

If $\ell \neq \ell_0$ is a prime and $v \mid \ell$ is a place of F, then, by Theorem 4.2.1, the Satake parameters of π_v are exactly the eigenvalues of $\rho_{\pi,\ell_0}(\mathrm{Frob}_v)$. In particular, Theorem 6.1.3 allows us to examine the crystallinity of $\rho_{\pi,\ell}$ at places $v \mid \ell$ for all ℓ by analysing the image of the single representation ρ_{π,ℓ_0}.

6.1.1 Big image and distinctness of Satake parameters

In this section, we relate the distinctness of the eigenvalues of $\rho_{\pi,\ell_0}(\mathrm{Frob}_v)$ to the image of ρ_{π,ℓ_0}. Our key tool is the following theorem, due to Rajan:

Theorem 6.1.4 ([Raj98, Theorem 3])**.** *Let E be a finite extension of $\mathbf{Q}_\ell$ and let G be an algebraic group over E. Let X be a subscheme of G, defined over E, that is stable under the adjoint action of G. Suppose that*

$$R : \mathrm{Gal}(\overline{F}/F) \to G(E)$$

is a Galois representation and let

$$C = X(E) \cap \rho(\mathrm{Gal}(\overline{F}/F)).$$

Let H denote the Zariski closure of $R(\mathrm{Gal}(\overline{F}/F))$ in $G_{/E}$, with identity connected component H° and component group $\Phi = H/H^\circ$. For each $\phi \in \Phi$, let H^ϕ denote its corresponding connected component. Let

$$\Psi = \left\{ \phi \in \Phi : H^\phi \subseteq X \right\}.$$

Then the set of places v of F with $R(\mathrm{Frob}_v) \in C$ has density $\frac{|\Psi|}{|\Phi|}$.

Using Theorem 6.1.4, we now prove Lemma 6.1.1 under the additional assumption that the Zariski closure of the image of ρ_{π,ℓ_0} is GSp_4. Later, in Proposition 6.1.11, we will show that, if π is of type **(G)** and is neither an automorphic induction nor a symmetric cube lift, then this assumption always holds.

Lemma 6.1.5. *Let F be a number field, let E be a finite extension of $\mathbf{Q}_\ell$ and let*

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(E)$$

be a Galois representation. Suppose that the Zariski closure of $\rho(\mathrm{Gal}(\overline{F}/F))$ in $\mathrm{GL}_{4/E}$ is GSp_4. Then, for 100% of primes p of $\mathbf{Q}$ and for all places $v \mid p$ of F, the eigenvalues of $\rho(\mathrm{Frob}_v)$ are distinct.

Proof. Let F^{gal} be the Galois closure of F and let p be a prime that is unramified in F^{gal} such that ρ is unramified at v for all places $v \mid p$ of F. If $\rho|_{F^{gal}}(\mathrm{Frob}_w)$ has distinct eigenvalues for all primes $w \mid p$ of F^{gal}, then $\rho(\mathrm{Frob}_v)$ has distinct eigenvalues for all places $v \mid p$ of F. Indeed, if w is a place of F^{gal} above a place v of F with residue degree f, then the characteristic polynomial of $\rho_{\pi,\ell}|_{F^{gal}}(\mathrm{Frob}_w)$ is the same as the characteristic polynomial of $\rho_{\pi,\ell}(\mathrm{Frob}_v^f)$. Hence, if the eigenvalues $\rho_{\pi,\ell}|_{F^{gal}}(\mathrm{Frob}_w)$ are distinct, then so are the eigenvalues of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$. Therefore, without loss of generality, we may assume that F is a Galois extension of $\mathbf{Q}$.

To prove Lemma 6.1.5, we apply Theorem 6.1.4 to the representation $\mathrm{Ind}_F^{\mathbf{Q}}(\rho)$.

First, observe that $\mathrm{Gal}(F/\mathbf{Q})$ acts on the set of representations

$$\{\rho^\sigma : \sigma \in \mathrm{Gal}(F/\mathbf{Q})\}\,.$$

Let $\mathrm{Gal}(F/L)$ be the stabiliser of this action, for an intermediate field $\mathbf{Q} \subseteq L \subseteq F$. Let $d = [L : \mathbf{Q}]$ and let $m = [F : \mathbf{Q}]$.

Write

$$R = \mathrm{Ind}_F^{\mathbf{Q}}(\rho) : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_{4m}(E)$$

for the induction of ρ to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and let p be a prime that is unramified in F and at which R is unramified. If v is a place of F above p, then let $\alpha_v, \beta_v, \gamma_v$ and δ_v be the eigenvalues of $\rho(\mathrm{Frob}_v)$. Let f be the residue degree of p in F, which is independent of v since $F/\mathbf{Q}$ is Galois. Denote by Ev_p the set of eigenvalues of $R(\mathrm{Frob}_p)$. Explicitly,

$$\mathrm{Ev}_p = \left\{\zeta \sqrt[f]{\alpha_v}, \zeta \sqrt[f]{\beta_v}, \zeta \sqrt[f]{\gamma_v}, \zeta \sqrt[f]{\delta_v} : \zeta^f = 1, v \mid p\right\}.$$

If $\sigma \in \mathrm{Gal}(F/L)$, then

$$\{\alpha_v, \beta_v, \gamma_v, \delta_v\} = \{\alpha_{v^\sigma}, \beta_{v^\sigma}, \gamma_{v^\sigma}, \delta_{v^\sigma}\}\,.$$

Hence, $\#\mathrm{Ev}_p \le 4d$. We prove that $\#\mathrm{Ev}_p = 4d$, from which the result follows.

Let H be the Zariski closure of $R(\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}))$ in $\mathrm{GL}_{4m/E}$, let H° be its identity connected component and let $\Psi = H/H^\circ$. Let X be the subscheme of GL_{4m} consisting of elements $g \in \mathrm{GL}_{4m}$ such that a root of the characteristic polynomial of g has multiplicity strictly greater than $\frac{m}{d}$.[1] If $\#\mathrm{Ev}_p < 4d$, then $R(\mathrm{Frob}_p) \in X$. Hence, if $\#\mathrm{Ev}_p < 4d$ for a positive density of primes p, then, by Theorem 6.1.4, there is a connected component H^ϕ of H such that $H^\phi \subseteq X$.

We now argue as in [Pat19, Proposition 3.4.9]. Since the Zariski closure of $\rho(\mathrm{Gal}(\overline{F}/F))$ is GSp_4, which is connected, and since $\rho^\sigma \not\simeq \rho$ for all $\sigma \in \mathrm{Gal}(F/\mathbf{Q}) \setminus \mathrm{Gal}(F/L)$, we

[1] So X is the vanishing set of the discriminant of the $(\frac{m}{d} - 1)^{\mathrm{th}}$ derivative of the characteristic polynomial.

see that $H^\circ = \mathrm{GSp}_4^d$. If V is the vector space corresponding to ρ, we see that R can be represented in block-matrix form via the decomposition

$$R|_{R^{-1}(H^\circ)} = \bigoplus_{\sigma \in \mathrm{Gal}(L/\mathbf{Q})} (\sigma V)^{m/d}.$$

Writing $H^\phi = TH^\circ$ for some matrix T, we see that the elements

$$T \cdot \mathrm{diag}\,\{M_1, \cdots M_1, M_2, \cdots M_2, \cdots, M_d, \cdots, M_d\}$$

are all contained in the subscheme X, where the M_i are arbitrary elements of $\mathrm{GSp}_4(E)$.

However, no non-zero block-matrix T can have this property. Indeed, we can write

$$T = (T_{ij})_{1 \leq i,j \leq m}$$

in block-matrix form, with each T_{ij} either 0 or an element of $\mathrm{GL}_4(E)$. Suppose that the entries $T_{\frac{m}{d},j_1}, T_{\frac{2m}{d},j_2}, \ldots, T_{\frac{dm}{d},j_d}$ are non-zero, for distinct integers $j_1, \ldots, j_d$. Then the $(\frac{im}{d}, j_i)^{\text{th}}$ entry of

$$T \cdot \mathrm{diag}\,\{M_1, \cdots M_1, M_2, \cdots M_2, \cdots, M_d, \cdots, M_d\}$$

is $T_{\frac{im}{d},j_i} \cdot M_i$. Since each M_i is an arbitrary element of $\mathrm{GSp}_4(E)$, we can easily choose the M_i to ensure that $H^\phi = TH^\circ \not\subseteq X$, contradicting our previous assumption. The result follows. □

Hence, to prove Lemma 6.1.1, we need to show that, if ρ_{π,ℓ_0} is irreducible and π is neither an automorphic induction nor a symmetric cube lift,[2] then the Zariski closure G of its image is GSp_4. In the next subsection, we prove that, in these cases, ρ_{π,ℓ_0} is Lie-irreducible. Then, by analysing the Lie algebra of G, we prove that $G = \mathrm{GSp}_4$.

6.1.2 Lie-irreducibility

Recall that, in the statement of Lemma 6.1.1, we have assumed π is not an automorphic induction and that ρ_{π,ℓ_0} is irreducible for some prime ℓ_0. Using these assumptions, we now show that ρ_{π,ℓ_0} is Lie-irreducible.

Definition 6.1.6. Let G be a group and let k be a field. We say that a representation

$$\rho : G \to \mathrm{GL}_n(k)$$

is *Lie-irreducible* if $\rho|_H$ is irreducible for all finite index subgroups $H \leq G$.

Remark 6.1.7. If G is a connected algebraic group with associated Lie algebra $\mathfrak{g}$ and

$$d\rho : \mathfrak{g} \to \mathfrak{gl}_n(k)$$

is the Lie algebra representation associated to ρ, then $d\rho$ is irreducible if and only if ρ is irreducible. More generally, if G is a algebraic group with identity connected component G°, then G° is of finite index in G and any finite index subgroup of G is a

[2] The case of automorphic inductions is addressed in Remarks 6.1.2. If π is a symmetric cube lift and is unramified at all places above ℓ, then $\rho_{\pi,\ell}$ is the symmetric cube lift of a crystalline representation, so is itself crystalline; hence, Lemma 6.1.1 is clear in this case.

union of connected components. In particular, ρ is Lie-irreducible if and only if $\rho|_{G^\circ}$ is irreducible if and only if the Lie algebra representation

$$d\rho : \mathfrak{g} \to \mathfrak{gl}_n(k)$$

associated to ρ is irreducible.

Definition 6.1.8. Let G be a group and let k be a field. We say that an absolutely irreducible representation

$$\rho : G \to \mathrm{GL}_n(k)$$

is *imprimitive* if there is a finite index subgroup $H < G$ and a $\overline{k}$-representation σ of H such that $\rho \otimes_k \overline{k} \simeq \mathrm{Ind}_H^G \sigma$. Otherwise, we say that ρ is *primitive*.

Lemma 6.1.9. *Suppose that ρ_{π,ℓ_0} is absolutely irreducible. Then ρ_{π,ℓ_0} is imprimitive if and only if π is an automorphic induction.*

Proof. Let Π be the transfer of π to GL_4 (see Section 2.2). We show that ρ_{π,ℓ_0} is imprimitive if and only if there is a quadratic extension K/F and an automorphic representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_K)$ such that Π is induced from $\boldsymbol{\pi}$. Note that $\boldsymbol{\pi}$ itself could also be an automorphic induction.

By [AC89, Lemmas 6.4, 6.6], Π is automorphically induced from an automorphic representation $\boldsymbol{\pi}$ as above if and only if

$$\Pi \cong \Pi \otimes \eta_{K/F},$$

where $\eta_{K/F}$ is the finite order Hecke character corresponding to the quadratic extension K/F. By the Chebotarev density theorem and the strong multiplicity one theorem for GL_4 (see Corollary 0.4.2), this isomorphism of automorphic representations is equivalent to to the isomorphism

$$\rho_{\pi,\ell_0} \simeq \rho_{\pi,\ell_0} \otimes \omega_{K/F}$$

of Galois representations, where $\omega_{K/F}$ is the Galois character corresponding to $\eta_{K/F}$. Since ρ_{π,ℓ_0} is irreducible, it follows that ρ_{π,ℓ_0} is induced from a representation of $\mathrm{Gal}(\overline{F}/K)$.

It remains to show that, if $\rho_{\pi,\ell_0} = \mathrm{Ind}_K^F \sigma$ is imprimitive, then K can be chosen to be a quadratic extension. Since ρ_{π,ℓ_0} is four-dimensional, by counting dimensions, we see that $[K : F] = 2$ or 4. If K contains a quadratic subextension K', then $\rho_{\pi,\ell_0} = \mathrm{Ind}_{K'}^F \left(\mathrm{Ind}_K^{K'} \sigma\right)$. Finally, if K does not contain a quadratic subextension, then the proof of [GT10, Lemma 5.3] shows that ρ_{π,ℓ_0} is induced from a different quadratic extension. □

Proposition 6.1.10. *Suppose that $F = \mathbf{Q}$ or that π is unramified at all places above ℓ_0. If ρ_{π,ℓ_0} is irreducible, but not Lie-irreducible, then π is an automorphic induction.*

Proof. By Lemma 6.1.9, we may assume that ρ_{π,ℓ_0} is primitive. Then, by [Pat19, Propositon 3.4.1], we can write

$$\rho_{\pi,\ell_0} = \sigma \otimes \omega,$$

where σ is a Lie-irreducible representation of dimension d with $d \mid 4$ and ω is an Artin representation of dimension $\frac{4}{d}$.

If ρ_{π,ℓ_0} is not Lie-irreducible, then the fact that ρ_{π,ℓ_0} is not a twist of an Artin representation ensures that σ is two-dimensional. If ω is imprimitive—say $\omega \simeq \mathrm{Ind}_K^F(\chi)$ for some quadratic extension K/F and character χ of $\mathrm{Gal}(\overline{F}/K)$—then

$$\rho_{\pi,\ell_0} \simeq \mathrm{Ind}_K^F(\sigma|_K \otimes \chi)$$

is also imprimitive. Hence, we may assume that both σ and ω are primitive. It follows that $\mathrm{Sym}^2(\sigma)$ and $\mathrm{Sym}^2(\omega)$ are both irreducible. Taking exterior squares, we find that

$$\bigwedge^2(\rho_{\pi,\ell_0}) \simeq \bigwedge^2(\sigma \otimes \omega) \simeq \left(\bigwedge^2(\sigma) \otimes \mathrm{Sym}^2(\omega)\right) \oplus \left(\bigwedge^2(\omega) \otimes \mathrm{Sym}^2(\sigma)\right)$$

does not contain a one-dimensional subrepresentation, contradicting Theorem 4.4.1, that ρ_{π,ℓ_0} is symplectic. □

6.1.3 Distinctness of Satake parameters

We are now ready to prove Lemma 6.1.1, which follows immediately from Theorem 6.1.3 and the following proposition:

Proposition 6.1.11. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$ that are not of the form $(2k_v - 1, k_v + 1)_{v|\infty}$, $k_v \geq 2$ for all $v \mid \infty$.*[3] *Assume that π is of type $(\mathbf{G})$ and that π is not an automorphic induction.*

Suppose that there exists a prime ℓ_0 such that π is unramified at all places above ℓ_0 and such that ρ_{π,ℓ_0} is irreducible. Then π_v has distinct Satake parameters for all $v \mid \ell$ for 100% of primes ℓ.

Proof. Choose a finite extension E of $\mathbf{Q}_\ell$ over which ρ_{π,ℓ_0} is defined. By Theorem 4.4.1, we may assume that the image of ρ_{π,ℓ_0} is contained in $\mathrm{GSp}_4(E)$.

Let G be the Zariski closure of $\rho_{\pi,\ell_0}(\mathrm{Gal}(\overline{F}/F))$ in $\mathrm{GSp}_{4/E}$, let $\mathfrak{g}$ be its Lie algebra and let $\mathfrak{g}'$ be the derived subalgebra of $\mathfrak{g}$. Since ρ_{π,ℓ_0} is a semisimple representation, G is a reductive group. Hence, $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{g}'$ is a semisimple Lie algebra.

Since $\mathfrak{g}$ is reductive, it decomposes as the direct sum of $\mathfrak{g}'$ and an abelian Lie algebra $\mathfrak{a}$. Moreover, since the similitude of ρ_{π,ℓ_0} has infinite image, we must have $\mathfrak{a} = \mathfrak{gl}_1(E)$.

The result will follow if we can show that $\mathfrak{g}' = \mathfrak{sp}_4(E)$. Indeed, if $\mathfrak{g}' = \mathfrak{sp}_4(E)$, then

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{gl}_1(E) = \mathfrak{sp}_4(E) \oplus \mathfrak{gl}_1(E) \cong \mathfrak{gsp}_4(E).$$

If so, then G is a subgroup of $\mathrm{GSp}_{4/E}$ whose Lie algebra is $\mathfrak{gsp}_4(E)$ and therefore $G = \mathrm{GSp}_{4/E}$. The result then follows from Lemma 6.1.5.

Hence, we show that $\mathfrak{g}' = \mathfrak{sp}_4(E)$. Using the classification of semisimple Lie algebras and the fact that $\mathfrak{g}' \subseteq \mathfrak{sp}_4(E)$, we deduce that $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell$ (for a fixed embedding $E \hookrightarrow \overline{\mathbf{Q}}_\ell$) is one of the following Lie algebras [HT15, 9.3.1]:

1. $\mathfrak{sp}_4(\overline{\mathbf{Q}}_\ell)$

[3] If the weights of π are of this form, then π is necessarily cohomological. In this case, crystallinity is already known (c.f Theorem 4.1.1) and Lemma 6.1.1 is vacuously true.

2. $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell) \times \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$
3. $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded in a Klingen parabolic subalgebra
4. $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded in a Siegel parabolic subalgebra
5. $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded via the symmetric cube representation $\mathrm{SL}_2 \to \mathrm{Sp}_4$
6. $\{1\}$.

Let

$$d\rho_{\pi,\ell_0} : \mathfrak{g} \to \mathfrak{gsp}_4(E)$$

be the Lie algebra representation associated to ρ_{π,ℓ_0}. Since π is not an automorphic induction and ρ_{π,ℓ_0} is irreducible, by Proposition 6.1.10, ρ_{π,ℓ_0} is Lie-irreducible. Hence, by Remark 6.1.7, $d\rho_{\pi,\ell_0}$ is irreducible.

Since $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{gl}_1(E)$ and $d\rho_{\pi,\ell_0}$ is irreducible, we can view $d\rho_{\pi,\ell_0}$ as the tensor product of an irreducible representation of $\mathfrak{g}'$ with an irreducible representation of $\mathfrak{gl}_1(E)$. Since every irreducible representation of $\mathfrak{gl}_1(E)$ is one-dimensional, it follows that the restriction of $d\rho_{\pi,\ell_0}$ to $\mathfrak{g}'$ is irreducible. In particular, $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell$ cannot be as in cases (2), (3), (4) or (6).

To eliminate case (5), suppose that $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell \cong \mathrm{Sym}^3 \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$. Let G° be the identity connected component of G. Then $(G^\circ)'$ is a connected semisimple subgroup of $\mathrm{Sp}_{4/E}$. The above classification of the semisimple Lie subalgebras of $\mathfrak{sp}_4(\overline{\mathbf{Q}}_\ell)$ gives rise to a corresponding classification of the connected semisimple subgroups of $\mathrm{Sp}_{4/\overline{\mathbf{Q}}_\ell}$ and we deduce that

$$(G^\circ)' \times_E \overline{\mathbf{Q}}_\ell \cong \mathrm{Sym}^3 \mathrm{SL}_2 \,.$$

Therefore, since the similitude of $\rho_{\pi,\ell}$ does not have finite image, we have

$$G^\circ \times_E \overline{\mathbf{Q}}_\ell \cong \mathrm{Sym}^3 \mathrm{GL}_2 \,.$$

There is a finite extension K/F such that $\rho_{\pi,\ell}(\mathrm{Gal}(\overline{F}/K)) \subseteq G^\circ(\overline{\mathbf{Q}}_\ell)$. The fact that $G^\circ(\overline{\mathbf{Q}}_\ell) \cong \mathrm{Sym}^3 \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ ensures that $\rho_{\pi,\ell}|_K \simeq \mathrm{Sym}^3 \rho_1'$ for some two-dimensional representation ρ_1'. Since, for some place $v \mid \infty$, the weights of π_v are not of the form $(2k_v - 1, k_v + 1)_{v|\infty}, k_v \geq 2$, we see that $\rho_{\pi,\ell_0}|_K$ being a symmetric cube lift is incompatible with the Hodge–Tate–Sen weights of ρ_{π,ℓ_0}, giving the required contradiction.

Hence, by exhaustion, we are left with case (1), that $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell \cong \mathfrak{sp}_4(\overline{\mathbf{Q}}_\ell)$. Now, $\mathfrak{g}'$ is a vector subspace of $\mathfrak{sp}_4(E)$; the fact that $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell \cong \mathfrak{sp}_4(E) \otimes_E \overline{\mathbf{Q}}_\ell$ shows that the two vector spaces have the same dimension and, hence, are equal, as required. □

Finally, in Chapter 7, we will require the following lemma, which applies specifically when π is cohomological and has the same weights as a symmetric cube lift.

Lemma 6.1.12. *Suppose that ρ_{π,ℓ_0} is irreducible for some prime ℓ_0 at which π is unramified. Moreover, suppose that π is not an automorphic induction and that the weights of π are of the form $(2k_v - 1, k_v + 1)_{v|\infty}$, $k_v \geq 2$ for all $v \mid \infty$. Then π_v has distinct Satake parameters for 100% of places of F.*

Proof. By the proof of Proposition 6.1.11, it is enough to consider the case that $\rho_{\pi,\ell_0} \simeq \mathrm{Sym}^3(\rho')$ for some two-dimensional representation ρ' of $\mathrm{Gal}(\overline{F}/F)$.

By Proposition 6.1.10, we see that ρ_{π,ℓ_0} is Lie-irreducible and, hence, so is ρ'. It follows from Lemma 6.2.2 below that $\rho'(\mathrm{Frob}_v)$ has distinct eigenvalues α_v, β_v for 100% of places v. Moreover, by [Pat19, Proposition 3.4.9], $\alpha_v \neq -\beta_v$ for 100% of places v. Lastly, if $\alpha_v = \beta_v e^{2\pi i/3}$ for a positive density of places v, then $\mathrm{Tr}(\mathrm{Sym}^2(\rho')(\mathrm{Frob}_v)) = 0$ for all such v, and it follows from [Pat19, Proposition 3.4.9] that $\mathrm{Sym}^2(\rho')$ is irreducible, but not Lie-irreducible, a contradiction. We deduce that the eigenvalues α_v^3, $\alpha_v^2\beta_v$, $\alpha_v\beta_v^2$, β_v^3 of $\rho_{\pi,\ell_0}(\mathrm{Frob}_v)$ are distinct for 100% of places v, as required. □

6.2 The main result

In the previous section, we proved that $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$ for 100% of primes ℓ if it is irreducible for at least one prime ℓ_0. In this section, we use the following lemma to prove Theorem A:

Lemma 6.2.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$. Suppose that π is of type* (**G**) *and that π is not an automorphic induction.*

Then $\rho_{\pi,\ell}$ is crystalline at v for all places $v \mid \ell$ for a positive density of primes ℓ.

Theorem A follows immediately:

Proof of Theorem A. If π is an automorphic induction, then the result follows from Proposition 5.1.3.

If not, then, by Lemma 6.2.1 combined with Theorem 5.3.1, we deduce that $\rho_{\pi,\ell}$ is irreducible for at least one prime. Hence, by Lemma 6.1.1, $\rho_{\pi,\ell}$ is crystalline for 100% of primes. Applying Theorem 5.3.1 again, we deduce that $\rho_{\pi,\ell}$ is irreducible for 100% of primes. □

6.2.1 Proof of Lemma 6.2.1

Fix a prime ℓ such that π_v is unramified at all places $v \mid \ell$. If $\rho_{\pi,\ell}$ is irreducible, then Lemma 6.2.1 follows from Lemma 6.1.1. Hence, we may assume that $\rho_{\pi,\ell}$ is reducible. By Theorem 5.2.1, we know that $\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$ decomposes as a direct sum of distinct, odd, Hodge–Tate regular, two-dimensional representations, both with determinant $\mathrm{sim}(\rho_{\pi,\ell})$.

We may suppose that $\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$ is defined and reducible over a finite extension E of $\mathbf{Q}_\ell$. For each $i = 1, 2$, let H_i be the Zariski closure of the image of ρ_i in $\mathrm{GL}_{2/E}$, let $\mathfrak{h}_i$ be its Lie algebra and let $\mathfrak{h}_i'$ be the derived subalgebra of $\mathfrak{h}_i$.

Lemma 6.2.2. *Suppose that*

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

is a Lie-irreducible Galois representation, whose determinant has infinite image. Then, for 100% of places v of F, the eigenvalues of $\rho(\mathrm{Frob}_v)$ are distinct.

Proof. Let $E/\mathbf{Q}_\ell$ be the field of definition of ρ. Let G be the Zariski closure of $\rho(\mathrm{Gal}(\overline{F}/F))$ in $\mathrm{GL}_{2/E}$, let G' be its derived subgroup and let $\mathfrak{g}'$ be the Lie algebra of $\mathfrak{g}'$.

By well-known arguments [Rib77], $\mathfrak{g} = \mathfrak{gl}_2(E)$. Indeed, $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell$ is a semisimple Lie subalgebra of $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$, so is either $\{1\}$ or $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$. Since ρ is Lie-irreducible, we conclude that $\mathfrak{g}' \otimes_E \overline{\mathbf{Q}}_\ell = \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$. Hence,

$$\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell) \subseteq \mathfrak{g} \otimes_E \overline{\mathbf{Q}}_\ell \subseteq \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$$

and, since the determinant of ρ has infinite image, $\mathfrak{g} \otimes_E \overline{\mathbf{Q}}_\ell = \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$. Finally, since $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_2(E)$, it follows that $\mathfrak{g} = \mathfrak{gl}_2(E)$.

Since G is a subgroup of GL_2 whose dimension is equal to the dimension of $\mathfrak{g}$, it follows that $G = \mathrm{GL}_2$.

Let $X \subseteq \mathrm{GL}_2$ be the subscheme of elements of GL_2 with indistinct eigenvalues. Since $G = \mathrm{GL}_2$ is connected and $G \nsubseteq X$, it follows from Theorem 6.1.4 that $\rho(\mathrm{Frob}_v) \in X$ for a set of places of F of density 0. The result follows. □

Lemma 6.2.3. *At least one of ρ_1 and ρ_2 is Lie-irreducible.*

Proof. Note that if ρ_1, say, is not Lie-irreducible, then it is automorphic. Indeed, if $\rho_1|_L$ is reducible for some finite extension L/F, then, since ρ_1 is Hodge–Tate regular, the irreducible components of $\rho_1|_K$ are distinct. It follows that

$$\rho_1 = \mathrm{Ind}_K^F(\chi),$$

where $F \subseteq K \subseteq L$ is an intermediate extension with K/F quadratic, and χ is a character of $\mathrm{Gal}(\overline{F}/K)$. Since ρ_1 is Hodge–Tate, so is χ and hence χ is the Galois character attached to a Hecke character η of $\mathrm{GL}_1(\mathbf{A}_K)$. It follows that ρ_1 is the Galois representation attached to the automorphic induction $\mathrm{AI}_K^F(\eta)$ of η.

In particular, at least one of the ρ_i must be Lie-irreducible. Indeed, if both ρ_1 and ρ_2 are not Lie-irreducible, then, for each $i = 1, 2$, there is a Hecke character η_i of $\mathrm{GL}_1(\mathbf{A}_{K_i})$ such that ρ_i is the Galois representation attached to $\boldsymbol{\pi}_i := \mathrm{AI}_{K_i}^F(\eta_i)$. If Π is the transfer of π to GL_4, it follows that Π and $\boldsymbol{\pi}_1 \boxplus \boldsymbol{\pi}_2$ have isomorphic Galois representations and are therefore isomorphic by the strong multiplicity one theorem for GL_4. In particular, Π is not cuspidal, a contradiction. □

Hence, without loss of generality, we may assume that ρ_1 is Lie-irreducible. Lemma 6.2.1 follows immediately from the following lemma:

Lemma 6.2.4. *Suppose that $\rho_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$ is reducible, with ρ_1, ρ_2 irreducible and ρ_1 Lie-irreducible. Then, for 100% of places v of F, $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ has distinct eigenvalues.*

Proof. We first note that, by Lemma 6.2.2, $\rho_1(\mathrm{Frob}_v)$ has distinct eigenvalues for 100% of places v of F.

If ρ_2 is Lie-irreducible, then, by the same reasoning, $\rho_2(\mathrm{Frob}_v)$ has distinct eigenvalues for 100% of places v of F. If not, then we can write

$$\rho_2 \simeq \mathrm{Ind}_K^F(\chi),$$

where K/F is a quadratic extension and χ is a character of $\mathrm{Gal}(\overline{F}/K)$. If v is a place of F that is inert in K, then $\mathrm{Tr}(\rho_2(\mathrm{Frob}_v)) = 0$, from which it follows that the eigenvalues of $\rho_2(\mathrm{Frob}_v)$ are distinct. And if v is a place of F that splits as $v = ww^c$ in K, then the eigenvalues of $\rho_2(\mathrm{Frob}_v)$ are $\chi(\mathrm{Frob}_w)$ and $\chi(\mathrm{Frob}_{w^c}) = \chi^c(\mathrm{Frob}_w)$, where c is the non-trivial element of $\mathrm{Gal}(K/F)$. Since ρ_2 is irreducible, $\chi \not\simeq \chi^c$. Hence, if $\chi(\mathrm{Frob}_w) = \chi(\mathrm{Frob}_{w^c})$ for a positive density of places v of F, then, by [Raj98, Theorem 3], there must be a finite order character ω of $\mathrm{Gal}(\overline{F}/K)$ such that $\chi \simeq \chi^c \otimes \omega$, which is a contradiction, since χ and χ^c have different Hodge–Tate weights.[4]

For each $i = 1, 2$ and for each place $v \nmid \ell$ of F at which $\rho_{\pi,\ell}$ is unramified, let $\alpha_{v,i}, \beta_{v,i}$ be the roots of the characteristic polynomial of $\rho_i(\mathrm{Frob}_v)$. We have shown that, for a set of places of density 1, $\alpha_{v,i} \neq \beta_{v,i}$. Hence, if the eigenvalues of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ are indistinct for a positive density of primes, then, without loss of generality, we must have $\alpha_{v,1} = \beta_{v,1}$ and, hence, since ρ_1, ρ_2 have the same determinant, $\alpha_{v,2} = \beta_{v,2}$.

It follows that, for a set of primes of positive density, $\mathrm{Tr}(\rho_1(\mathrm{Frob}_v)) = \mathrm{Tr}(\rho_2(\mathrm{Frob}_v))$. We can derive a contradiction as in [Raj98, Corollary 1]. Since ρ_1 is Lie-irreducible, by the proof of Lemma 6.2.2, the Zariski closure of its image in $\mathrm{GL}_{2/E}$ is connected. Hence, by [Raj98, Theorem 3], there is a finite order character ω of $\mathrm{Gal}(\overline{F}/F)$ such that

$$\rho_1 \simeq \rho_2 \otimes \omega.$$

Since $\det(\rho_1) \simeq \det(\rho_2)$, it follows that ω is a quadratic character. Hence

$$\rho_{\pi,\ell} \otimes \omega \simeq \rho_{\pi,\ell},$$

contradicting the fact that π is not an automorphic induction. The result follows. □

[4]Since ρ_2 is Hodge–Tate regular, K/F must be a CM extension.

Chapter 7
Residual irreducibility and the image of Galois

Now that we have shown that $\rho_{\pi,\ell}$ is irreducible for 100% of primes ℓ, we next study the images of the residual representations $\overline{\rho}_{\pi,\ell}$. Let S_{crys} be the set of primes ℓ at which $\rho_{\pi,\ell}$ is crystalline. In this chapter, we prove:

- *If the weights of π are such that π cannot be a symmetric cube lift,*[1] *then the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for 100% of primes.*
- *If Serre's conjecture holds for F, then the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for all but finitely many $\ell \in S_{crys}$.*

In particular, when $F = \mathbf{Q}$ and π is cohomological, we prove that the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for all but finitely many primes, strengthening the results of [DZ].

Our methods resemble those of Chapter 5, except that, in place of the Hodge–Tate weights of $\rho_{\pi,\ell}$, we use Fontaine–Laffaille theory and the inertial weights of $\overline{\rho}_{\pi,\ell}$.

In this chapter, we prove Theorem B.

Theorem B. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, whose archimedean components lie in the holomorphic (limit of) discrete series. Suppose that π is not CAP or endoscopic. For each prime ℓ, let*

$$\overline{\rho}_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{\mathbf{F}}_\ell)$$

be the mod ℓ Galois representation associated to π. Let S_{crys} be the set of primes ℓ such that $\rho_{\pi,\ell}$ is crystalline at all places $v \mid \ell$. Then:

1. *$\overline{\rho}_{\pi,\ell}$ is irreducible for 100% of primes ℓ.*
2. *If π is neither a symmetric cube lift nor an automorphic induction and if the weights of π are not all of the form $(2k_v - 1, k_v + 1)_{v|\infty}$ for $k_v \geq 2$, then the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for 100% of primes ℓ.*

Moreover, assuming Serre's conjecture for F, the following stronger result holds:

1. *For all but finitely many primes $\ell \in S_{crys}$, $\overline{\rho}_{\pi,\ell}$ is irreducible.*

[1] This condition is automatic if π is non-cohomological

2. *If π is neither a symmetric cube lift nor an automorphic induction, then, for all but finitely many primes $\ell \in S_{crys}$, the image of $\overline{\rho}_{\pi,\ell}$ contains* $\mathrm{Sp}_4(\mathbf{F}_\ell)$.

When π is non-cohomological, the results of this chapter are new. When π is cohomological and $F = \mathbf{Q}$, Dieulefait–Zenteno [DZ] have proven that the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for 100% of primes ℓ and our result extends theirs.

We begin, in Section 7.1, with an introduction to Fontaine–Laffaille theory, which is one of our key tools for proving Theorem B.

Then, in the remainder of the chapter, we prove Theorem B. In Section 7.2, we prove, assuming Serre's conjecture, that $\overline{\rho}_{\pi,\ell}$ is residually irreducible for almost all $\ell \in S_{crys}$. Of particular note is Section 7.2.3, which shows that $\overline{\rho}_{\pi,\ell}$ cannot split as a sum of two-dimensional even representations for infinitely many primes ℓ; this result is new even when π is cohomological (c.f. [DZ, Remark 3.4]).

In Section 7.3, we prove unconditionally that $\overline{\rho}_{\pi,\ell}$ is irreducible for 100% of primes ℓ. The proof is a direct application of [PSW18] in combination with our results about the ℓ-adic Galois representations $\rho_{\pi,\ell}$.

Finally, in Section 7.4, assuming Serre's conjecture, we prove that the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for all but finitely many $\ell \in S_{crys}$. Our argument follows the structure of [Die02, Die07, DZ], in that our key tools are the classification of the maximal subgroups of $\mathrm{Sp}_4(\mathbf{F}_{\ell^n})$ and Fontaine–Laffaille theory.

7.1 Fontaine–Laffaille theory

In Chapter 5, we used knowledge about the Hodge–Tate–Sen weights of $\rho_{\pi,\ell}$ to deduce results about its irreducibility. In this chapter, we apply a similar analysis to $\overline{\rho}_{\pi,\ell}$ using knowledge about its inertial weights. If $\rho_{\pi,\ell}$ is crystalline and if ℓ is sufficiently large, then Fontaine–Laffaille theory provides a recipe for computing the inertial weights of $\overline{\rho}_{\pi,\ell}$ from the Hodge–Tate weights of $\rho_{\pi,\ell}$. In this section, we give an overview of this theory. Our main reference is [Bar18].

Let L be a finite, unramified extension of $\mathbf{Q}_\ell$ with residue field $\mathbf{F}_L$, and let

$$\overline{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_n(\overline{\mathbf{F}}_\ell)$$

be a Galois representation. Let $\mathbf{Z}^n_+$ denote the set of n-tuples of integers $(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$ such that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

7.1.1 Inertial weights

In this subsection, we attach a subset

$$W(\overline{\rho})^{inert} \subset (\mathbf{Z}^n_+)^{\mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)}$$

to $\overline{\rho}$, called the *inertial weights* of $\overline{\rho}$. These weights, which are an analogue of Hodge–Tate weights for mod ℓ representations, depend only on the restriction of $\overline{\rho}$ to the inertia subgroup of $\mathrm{Gal}(\overline{L}/L)$.

We first attach inertial weights to $\overline{\rho}$ is the case that $\overline{\rho}$ is irreducible. If $\overline{\rho}$ is irreducible, then we can write

$$\overline{\rho} = \mathrm{Ind}_K^L(\chi),$$

where K/L is an unramified extension and χ is a character of $\mathrm{Gal}(\overline{L}/K)$. Indeed, we first observe that $\overline{\rho}$ must be tamely ramified: if not, then the wild inertia group, which is pro-ℓ, acts non-trivially on $\mathbf{F}^n$ for some characteristic ℓ field $\mathbf{F}$. However, since the orbits of this action must all have ℓ-power order and the orbit of $\{0\}$ is itself, a counting argument leads to a contradiction [Ser72, Proposition 4]. Hence, the restriction of $\overline{\rho}$ to the inertia subgroup $I_L \subseteq \mathrm{Gal}(\overline{L}/L)$ factors through the tame inertia group I_L^t, which is abelian. Therefore, by Schur's lemma, it follows that $\overline{\rho}|_{I_L}$ is a sum of one-dimensional representations. Finally, since $\overline{\rho}$ is irreducible and has finite image and since $\mathrm{Gal}(\overline{L}/L)/I_L$ is pro-cyclic, by Frobenius reciprocity, it follows that ρ is induced from a character of some field K, with $L^{\ker(I_L)} \subseteq K \subseteq L$.

Therefore, to classify the irreducible representations $\overline{\rho}$ of $\mathrm{Gal}(\overline{L}/L)$, it is sufficient to classify the one-dimensional representations of $\mathrm{Gal}(\overline{L}/K)$ for unramified extensions K/L. Let K be an unramified extension of L with residue field $\mathbf{F}_K$ and let $f_K = [\mathbf{F}_K : \mathbf{F}_\ell]$. Let $I_K \subseteq \mathrm{Gal}(\overline{L}/K)$ be the inertia group and let I_K^t be the tame inertia group. We have [Ser72, Proposition 2] a $\mathrm{Gal}(\overline{L}/K)$-equivariant isomorphism

$$I_K^t \cong \varprojlim_{k/\mathbf{F}_K \text{ finite}} k^\times.$$

A representation of I_K^t extends to I_K if and only if it is stable under the conjugation action of I_K on I_K^t. Therefore, if $\chi : \mathrm{Gal}(\overline{L}/K) \to \overline{\mathbf{F}}_\ell$ is a character, then χ is tamely ramified and $\chi|_{I_K}^{p^{f_K}} = \chi|_{I_K}$. Hence,

$$\chi|_{I_K} = \prod_\sigma \psi_\sigma^{-r_\sigma},$$

where the product runs over the embeddings $\sigma : \mathbf{F}_K \hookrightarrow \overline{\mathbf{F}}_\ell$, $r_\sigma \in \mathbf{Z}$ and ψ_σ, defined as

$$\begin{array}{ccccc} I_K & & \xrightarrow{\quad\psi_\sigma\quad} & & \overline{\mathbf{F}}_\ell^\times \\ & \searrow\!\!\!\!\!\searrow & & & \nearrow \sigma \\ & I_K^t \cong \varprojlim_{k/\mathbf{F}_K \text{ finite}} k^\times & \twoheadrightarrow & \mathbf{F}_K^\times & \end{array}$$

is the σ^{th} fundamental character (the bottom surjective arrow is the natural projection map). Characters $\chi : \mathrm{Gal}(\overline{L}/K) \to \overline{\mathbf{F}}_\ell$ are classified by the integers r_σ up to unramified twist.

For any representation $\overline{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_n(\overline{\mathbf{F}}_\ell)$, there exist unramified extensions K_χ/L and characters $\chi : \mathrm{Gal}(\overline{L}/K_\chi) \to \overline{\mathbf{F}}_\ell$ such that

$$\overline{\rho}^{ss} = \bigoplus_\chi \mathrm{Ind}_{K_\chi}^L(\chi),$$

with each summand irreducible. This decomposition is not unique in general.

Definition 7.1.1. We say that $\Lambda = (\lambda_\tau)_\tau \in (\mathbf{Z}_+^n)^{\mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)}$ is an *inertial weight* of $\overline{\rho}$ if we can write

$$\overline{\rho}^{ss} = \bigoplus_\chi \mathrm{Ind}_{K_\chi}^L(\chi)$$

as above, with

$$\chi|_{I_{K_\chi}} = \prod_\sigma \psi_\sigma^{-r_{\sigma,\chi}},$$

such that

$$\{\lambda_{\tau,1}, \ldots, \lambda_{\tau,n}\} = \{r_{\sigma,\chi} : \sigma|_{\mathbf{F}} = \tau\}.$$

We denote the set of inertial weights by $W(\overline{\rho})^{inert}$.

Example 7.1.2. The mod ℓ cyclotomic character

$$\overline{\varepsilon}_\ell : \mathrm{Gal}(\overline{L}/L) \to \overline{\mathbf{Q}}_\ell^\times$$

is equal to

$$\prod_{\tau \in \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)} \psi_\tau.$$

Hence, $(-1)_{\tau \in \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)}$ is an inertial weight of $\overline{\varepsilon}_\ell$.

7.1.2 Fontaine–Laffaille theory

Let

$$\rho : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_n(\overline{\mathbf{Z}}_\ell)$$

be a Galois representation and let

$$\overline{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_n(\overline{\mathbf{F}}_\ell)$$

be its reduction modulo ℓ. If $\rho \otimes \overline{\mathbf{Q}}_\ell$ is Hodge–Tate, then let

$$\mathrm{HT}(\rho) = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n})_\tau \in (\mathbf{Z}_+^n)^{\mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)}$$

denote the Hodge–Tate weights of ρ. Note that, since $L/\mathbf{Q}_\ell$ is unramified, we can index the Hodge–Tate weights by embeddings $\mathbf{F}_L \hookrightarrow \overline{\mathbf{F}}_\ell$ rather than embeddings $L \hookrightarrow \overline{\mathbf{Q}}_\ell$.

In Section 7.1.1, we attached a set

$$W(\overline{\rho})^{inert} \subset (\mathbf{Z}_+^n)^{\mathrm{Hom}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)}$$

of *inertial weights* to $\overline{\rho}$. Fontaine–Laffaille theory [FL82] gives a connection between the Hodge–Tate weights of ρ and the inertial weights of $\overline{\rho}$.

Theorem 7.1.3 (Fontaine–Laffaille). *Suppose that ρ is crystalline and that, for each $\tau \in \mathrm{Hom}(\mathbf{F}_L, \overline{\mathbf{F}}_\ell)$, we have $\lambda_{\tau,n} - \lambda_{\tau,1} \leq \ell - 1$. Then*

$$\mathrm{HT}(\rho) \in W(\overline{\rho})^{inert}.$$

Now, suppose that F is a number field that is unramified at ℓ. For each place v of F, let F_v denote the completion of F at v and let $\mathbf{F}_v$ denote its residue field. Let

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

be an ℓ-adic Galois representation, and let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_\ell)$$

be the semisimplification of its mod ℓ reduction. Recall that we have fixed an isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$. Hence, we can identify

$$\begin{aligned}\{v : v \mid \infty\} &\cong \mathrm{Hom}_{\mathbf{Q}}(F, \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}_\ell) \\ &\cong \prod_{v \mid \ell} \mathrm{Hom}_{\mathbf{Q}_\ell}(F_v, \overline{\mathbf{Q}}_\ell) \cong \prod_{v \mid \ell} \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell).\end{aligned} \tag{7.1}$$

If ρ is crystalline at all places $v \mid \ell$, then let

$$\mathrm{HT}(\rho) = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n})_\tau \in (\mathbf{Z}_+^n)^{\mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}_\ell)} = (\mathbf{Z}_+^n)^{\prod_{v \mid \ell} \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell)}$$

be the Hodge–Tate weights of ρ. If $\lambda_{\tau,n} - \lambda_{\tau,1} \leq \ell - 1$ for all $\tau \in \mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}_\ell)$, then, by Theorem 7.1.3, $(\lambda_\tau)_{\tau \in \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell))}$ is an inertial weight for $\overline{\rho}|_{F_v}$.

7.2 Residual irreducibility assuming Serre's conjecture

We now return to the notation from the beginning of the chapter. In this section, we prove part 1 of the second part of Theorem B, namely, we prove results about the irreducibility of $\overline{\rho}_{\pi,\ell}$, assuming that Serre's conjecture holds for F (c.f. [BDJ10, Conjecture 1.1]).

Conjecture 7.2.1 (Serre's Conjecture for F)**.** *Let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ be an irreducible, totally odd Galois representation. Then there exists a cuspidal automorphic representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\overline{\rho} \simeq \overline{\rho}_{\boldsymbol{\pi},\ell}$.*

When $F = \mathbf{Q}$, Conjecture 7.2.1 is a theorem of Khare–Wintenberger [KW09]. Assuming Serre's conjecture, we can deduce information about the possible weights and conductors of $\boldsymbol{\pi}$. The weight part of the following theorem follows from [GLS15, Theorem A] and the conductor part follows from [Raj01, Fuj06] and [BDJ10, Proposition 2.5].

Theorem 7.2.2. *Let $\ell > 5$ be a prime that is unramified in F and let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ be an irreducible, totally odd Galois representation. Suppose that:*

- *$\overline{\rho}|_{F(\zeta_\ell)}$ is irreducible.*
- *$\det(\overline{\rho}) = \chi\overline{\varepsilon}_\ell^{-w-1}$ for some integer w and some character χ that is unramified at ℓ.*
- *$\overline{\rho}$ has Serre conductor $\mathfrak{N}$.*
- *$\overline{\rho}$ has an inertial weight of the form $(\gamma_v, \gamma_v + k_v - 1)_{v \mid \infty}$, where, for all places $v \mid \infty$, $k_v > 1$,*
$$\gamma_v = \frac{1}{2}(w + 1 - (k_v - 1))$$
and the integers $((k_v)_{v\mid\infty}, w + 2)$ all have the same parity.
- *$\overline{\rho}$ is modular.*

Then there exists a cuspidal automorphic representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $((k_v)_{v\mid\infty}, w + 2)$ and conductor $\mathfrak{N}$ such that $\overline{\rho} \simeq \overline{\rho}_{\boldsymbol{\pi},\ell}$.

Proof. Since $\overline{\rho}|_{F(\zeta_\ell)}$ is irreducible, $(\gamma_v, \gamma_v + k_v - 1)_{v|\infty}$ is an inertial weight of $\overline{\rho}$ and $\overline{\rho}$ is modular, it follows from [GLS15, Theorem A] that $\overline{\rho}$ is modular[2] of weight σ, where, identifying $\{v : v \mid \infty\}$ with $\prod_{v|\ell} \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell)$,

$$\sigma \cong \bigotimes_{v|\ell} \bigotimes_{\tau \in \mathrm{Hom}_{\mathbf{F}_\ell}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell)} \det{}^{\gamma_\tau} \mathrm{Sym}^{k_\tau - 2} \mathbf{F}_v^2 \otimes_\tau \overline{\mathbf{F}}_\ell.$$

By [BDJ10, Propsition 2.5], it follows that $\overline{\rho} = \overline{\rho}_{\boldsymbol{\pi}',\ell}$ for an automorphic representation $\boldsymbol{\pi}'$ of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $((k_v)_{v|\infty}, 2\gamma_v + k_v)) = ((k_v)_{v|\infty}, w + 2)$ and level prime to ℓ. Finally, by the level lowering results of [Raj01, Fuj06], we can find another automorphic representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_F)$ of the same weight as $\boldsymbol{\pi}'$, with $\overline{\rho} \simeq \overline{\rho}_{\boldsymbol{\pi},\ell}$. The result follows. □

Using Serre's conjecture, we prove the following theorem, which is part 1 of the second part of Theorem B.

Theorem 7.2.3. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$ with weights $(k_{1,v}, k_{2,v})_{v|\infty}$, $k_{1,v} \geq k_{2,v} \geq 2$, and central character $\chi_\pi = \chi_{\pi^\circ}|\mathrm{sim}|^{-w}$. Let S_{crys} denote the set of primes ℓ such that $\rho_{\pi,\ell}$ is crystalline at v for all places $v \mid \ell$.*

Assume Conjecture 7.2.1. Then $\overline{\rho}_{\pi,\ell}$ is irreducible for all but finitely many primes $\ell \in S_{crys}$.

Fix a prime ℓ. Since our goal is to prove that $\overline{\rho}_{\pi,\ell}$ is irreducible for all but finitely many $\ell \in S_{crys}$, we may assume that F is unramified at ℓ, that $\ell \geq 5$ and that $\ell - 1 \geq k$, where $k = \max_{v|\infty}(k_{1,v} + k_{2,v} - 3)$. Hence, we can use Theorem 7.1.3 to understand the inertial weights of $\overline{\rho}_{\pi,\ell}$. Suppose that π has conductor $\mathfrak{N}$.

If $\overline{\rho}_{\pi,\ell}$ is reducible, then $\overline{\rho}_{\pi,\ell}$ decomposes in one of the following ways:

1. $\overline{\rho}_{\pi,\ell}$ has a one-dimensional subrepresentation.
2. $\overline{\rho}_{\pi,\ell}$ decomposes as $\overline{\rho}_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$, where: ρ_1, ρ_2 are irreducible, two-dimensional representations; $\det(\rho_1) = \chi\overline{\varepsilon}_\ell^{-w-3}$, where χ is a character that is unramified at all places $v \mid \ell$ and has conductor dividing $\mathfrak{N}$; and $\det(\rho_1)$ is totally odd.
3. $\overline{\rho}_{\pi,\ell}$ decomposes as $\overline{\rho}_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$, where: ρ_1, ρ_2 are irreducible, two-dimensional representations; $\det(\rho_1) = \chi\overline{\varepsilon}_\ell^{-w-3}$, where χ is a character that is unramified at all places $v \mid \ell$ and has conductor dividing $\mathfrak{N}$; and $\det(\rho_1)$ is not totally odd.
4. $\overline{\rho}_{\pi,\ell}$ decomposes as $\overline{\rho}_{\pi,\ell} \simeq \rho_1 \oplus \rho_2$, where ρ_1, ρ_2 are irreducible, two-dimensional representations, and $\det(\rho_1)$ and $\overline{\varepsilon}_\ell^{-w-3}$ have different inertial types.

In the remainder of this section, we prove that $\overline{\rho}_{\pi,\ell}$ cannot decompose in any of these ways for infinitely many $\ell \in S_{crys}$, thereby proving Theorem 7.2.3.

7.2.1 One-dimensional subrepresentations

We first rule out that $\overline{\rho}_{\pi,\ell}$ has a one-dimensional subrepresentation for infinitely many $\ell \in S_{crys}$.

[2] Here, we mean modular in the sense of [BDJ10, Definition 2.1]

Lemma 7.2.4. *For all but finitely many primes $\ell \in S_{crys}$, $\overline{\rho}_{\pi,\ell}$ does not contain a one-dimensional subrepresentation.*

To prove this lemma, we use class field theory in combination with the following result:

Lemma 7.2.5 ([CG13, Lemma 5.1])**.** *Let $(\lambda_\tau)_{\tau\in\mathrm{Hom}(F,\mathbf{C})} \in \mathbf{Z}^{\mathrm{Hom}(F,\mathbf{C})}$ be a fixed weight. Suppose that there exist infinitely many primes ℓ such that $\mathrm{Gal}(\overline{F}/F)$ admits a character*

$$\chi_\ell : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{F}}_\ell^\times$$

with the following properties:

1. *(λ_τ) is an inertial weight of χ_ℓ.*[3]
2. *The (Serre) conductor of χ_ℓ is bounded independently of ℓ.*

Then there exists a Hecke character η with infinity type (λ_τ) such that, for infinitely many ℓ,

$$\chi_\ell \simeq \overline{\rho}_{\eta,\ell},$$

where $\overline{\rho}_{\eta,\ell}$ is the mod ℓ Galois representation attached to η. In particular, since F is totally real, λ_τ is independent of τ.

Proof. Suppose that there exists an algebraic Hecke character $\eta : F^\times\backslash\mathbf{A}_F^\times \to \mathbf{C}^\times$ with infinity type (λ_τ). Then, for each prime ℓ, the mod ℓ Galois representation $\overline{\rho}_{\eta,\ell}$ attached to η has inertial weight (λ_τ). Hence, we can write

$$\chi_\ell \simeq \phi_\ell \otimes \overline{\rho}_{\eta,\ell}$$

where ϕ_ℓ is a character $\mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{F}}_\ell^\times$ that is unramified at all places $v \mid \ell$. By assumption, the conductor of ϕ_ℓ is bounded independently of ℓ. Since there are only finitely many characters $\mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^\times$ with bounded conductor, we see that, for infinitely many ℓ, ϕ_ℓ is the reduction modulo ℓ of a fixed Galois character $\phi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^\times$. Writing ϕ again for the Hecke character attached to ϕ, we see that χ_ℓ is the mod ℓ Galois representation attached to the Hecke character $\eta\phi$ for infinitely many ℓ, as required.

Hence, it remains to show that such an algebraic Hecke character exists. By Weil's classification of the Hecke characters [Wei56], there exists a Hecke character with infinity type (λ_τ) if and only if there is a finite index subgroup $U \subseteq \mathcal{O}_F^\times$ such that

$$\prod_{\tau\in\mathrm{Hom}(F,\mathbf{C})} \tau(x)^{-\lambda_\tau} = 1 \tag{7.2}$$

for all $x \in U$. By class field theory, there is a surjective homomorphism

$$r : F^\times\backslash\mathbf{A}_F^\times \twoheadrightarrow \mathrm{Gal}(\overline{F}/F)^{ab}$$

such that, for each finite place v of F, $r(\mathcal{O}_{F_v}^\times) = I_{K_v}^{ab}$. For each prime ℓ, pulling back χ_ℓ along r, we obtain a map

$$\widetilde{\chi}_\ell : F^\times\backslash\mathbf{A}_F^\times \to \overline{\mathbf{F}}_\ell.$$

[3] We are implicitly using Equation (7.1) to view (λ_τ) as an inertial weight.

Since the conductor of χ_ℓ is bounded independently of ℓ, there exists a fixed finite index subgroup $U \subseteq \mathcal{O}_F^\times$ such that, for all primes ℓ and for all $x \in U$,

$$1 = \widetilde{\chi}_\ell(x) = \prod_{v|\ell} \widetilde{\chi}_\ell(x_v) \in \overline{\mathbf{F}}_\ell.$$

Since χ_ℓ has inertial type (λ_τ), identifying $\mathrm{Hom}(F, \mathbf{C})$ with $\prod_{v|\ell} \mathrm{Hom}(\mathbf{F}, \overline{\mathbf{F}}_\ell)$, we have

$$1 = \widetilde{\chi}_\ell(x) = \prod_{v|\ell} \widetilde{\chi}_\ell(x_v) = \prod_{v|\ell} \prod_{\tau \in \mathrm{Hom}(\mathbf{F}, \overline{\mathbf{F}}_\ell)} \tau(\overline{x}_v)^{-\lambda_\tau}$$

for all $x \in U$, where $\overline{x}_v$ is the reduction of x modulo v. Since this equality holds for infinitely many ℓ, we see that Equation (7.2) must be satisfied. Hence, there exists a Hecke character with infinity type (λ_τ) and the result follows.

□

Proof of Lemma 7.2.4. Suppose that $\ell \in S_{crys}$ is sufficiently large and that $\overline{\rho}_{\pi,\ell}$ contains a one-dimensional subrepresentation χ_ℓ. By Fontaine–Laffaille theory, χ_ℓ has an inertial weight $(\lambda_v)_{v|\infty}$, where, for each place $v \mid \infty$,

$$\lambda_v \in \delta_v + \{0, k_{2,v} - 2, k_{1,v} - 1, k_{1,v} + k_{2,v} - 3\}.$$

If $\overline{\rho}_{\pi,\ell}$ contains a one-dimensional subrepresentation χ_ℓ for infinitely many $\ell \in S_{crys}$, then some fixed weight $(\lambda_v)_{v|\infty}$ as above is an inertial weight for χ_ℓ for infinitely many ℓ. Hence, by Lemma 7.2.5, there exists a Hecke character η such that

$$\chi_\ell \simeq \overline{\rho}_{\eta,\ell}$$

for infinitely many ℓ. Moreover, η is of the form $|\cdot|^{-i}\phi$ with $i \in \mathbf{Z}$ and ϕ a finite order character. Hence, $\chi_\ell = \overline{\varepsilon}_\ell^{-i}\overline{\rho}_{\phi,\ell}$. Certainly, $i \neq \frac{1}{2}(w+3)$, where w is the weight appearing in the central character of π.

We give a mod ℓ analogue of the argument in Lemma 5.2.3. Choose any finite place $v \nmid \ell$ of F at which π is unramified. If ϖ_v is a uniformiser of F_v, then

$$\overline{\alpha}_{v,\ell} := \chi_\ell(\mathrm{Frob}_v) \equiv N(v)^i \phi(\varpi_v) \pmod{\ell}$$

is the reduction mod ℓ of an eigenvalue α_v of $\rho_{\pi,l}(\mathrm{Frob}_v)$. The eigenvalues of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ are independent of ℓ. Hence, for infinitely many ℓ, $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ has an eigenvalue α_v with

$$\alpha_v \equiv N(v)^i \phi(\varpi_v) \pmod{\ell}$$

and, therefore, $|\alpha_v| = N(v)^i$. However, by [JS81b, Corollary 2.5], we have

$$N(v)^{-\frac{1}{2}} < |\alpha_v| N(v)^{-\frac{1}{2}(w+3)} < N(v)^{\frac{1}{2}}.$$

Since $i \in \mathbf{Z}$ it follows that $i = \frac{1}{2}(w+3)$, a contradiction.

□

Remark 7.2.6. Suppose that π is automorphically induced from a representation $\boldsymbol{\pi}$ of $\mathrm{GL}_2(\mathbf{A}_K)$ for some quadratic extension K/F. If $\overline{\rho}_{\boldsymbol{\pi},\ell}$, the mod ℓ Galois representation attached to $\boldsymbol{\pi}$, is reducible, then $\overline{\rho}_{\boldsymbol{\pi},\ell}$ is a sum of characters and, hence,

$\overline{\rho}_{\pi,\ell} = \mathrm{Ind}_K^F(\overline{\rho}_{\boldsymbol{\pi},\ell})$ is a sum of characters. Therefore, by the arguments above, $\overline{\rho}_{\boldsymbol{\pi},\ell}$ is irreducible for all but finitely many primes in S_{crys}.[4] If $\mathrm{Gal}(K/F) = \langle c \rangle$ and $\overline{\rho}_{\boldsymbol{\pi},\ell} \simeq \overline{\rho}^c_{\boldsymbol{\pi},\ell}$ for infinitely many primes ℓ, then $\rho_{\boldsymbol{\pi},\ell} \simeq \rho^c_{\boldsymbol{\pi},\ell}$, contradicting the cuspidality of π. It follows that if π is an automorphic induction, then $\overline{\rho}_{\pi,\ell}$ is irreducible for all but finitely many primes $\ell \in S_{crys}$. Therefore, when π is an automorphic induction, Theorem 7.2.3 is true.

7.2.2 Two-dimensional constituents with $\det(\rho_1) = \chi\overline{\varepsilon}_\ell^{-w-3}$ odd

Hence, for all but finitely many primes $\ell \in S_{crys}$, either $\overline{\rho}_{\pi,\ell}$ is irreducible or it decomposes as a direct sum $\rho_1 \oplus \rho_2$ where ρ_1, ρ_2 are irreducible and two-dimensional.

Lemma 7.2.7. *Assume that Serre's conjecture (Conjecture 7.2.1) holds for F. Then, for at most finitely many primes $\ell \in S_{crys}$, there is a character χ such that $\det(\rho_1) = \chi\overline{\varepsilon}_\ell^{-w-3}$ and $\det(\rho_1)$ is totally odd.*

Proof. Suppose that this case occurs for infinitely many primes ℓ. Then, taking ℓ sufficiently large, we may assume that χ is unramified at all places $v \mid \ell$. Observe that, when ℓ is sufficiently large, ρ_1 and ρ_2 have inertial weights $(\lambda_v^{(1)})_{v|\infty}$ and $(\lambda_v^{(2)})_{v|\infty}$ such that, for each place $v \mid \infty$

$$\big\{\lambda_v^{(1)}, \lambda_v^{(2)}\big\} = \big\{(\delta_v, \delta_v + k_{1,v} + k_{2,v} - 3), (\delta_v + k_{2,v} - 2, \delta_v + k_{1,v} - 1)\big\}.$$

Indeed, by Theorem 7.1.3, $\overline{\rho}_{\pi,\ell}$ has an inertial weight equal to the Hodge–Tate weights of $\rho_{\pi,\ell}$, and the assumption that $\det(\rho_1) = \chi\overline{\varepsilon}^{-w-3}$ ensures that the weights split up as above.

In order to apply Theorem 7.2.2, we must verify that $\rho_1|_{F(\zeta_\ell)}$ and $\rho_2|_{F(\zeta_\ell)}$ are irreducible. If, for example, $\rho_1|_{F(\zeta_\ell)}$ is reducible, then ρ_1 is induced from the unique quadratic subextension of $F(\zeta_\ell)$ and it follows that $\rho_1 \otimes \psi \simeq \rho_1$, where ψ is a character whose inertial weight at some v is $\frac{\ell-1}{2}$. When ℓ is large enough, this equivalence is incompatible with the possible inertial weights of ρ_1 (c.f [CG13, Lemma 2.6]).

Hence, by Serre's conjecture and Theorem 7.2.2, there are cuspidal automorphic representations $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ of $\mathrm{GL}_2(\mathbf{A}_F)$ associated to ρ_1 and ρ_2 such that:

- At each place $v \mid \infty$, each of $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ has weight either $k_{1,v}+k_{2,v}-2$ or $k_{1,v}-k_{2,v}+1$.
- $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ have conductors bounded independently of ℓ.
- $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ have central characters $\chi_{\boldsymbol{\pi}_1^\circ}|\det|^{-w-1}$ and $\chi_{\boldsymbol{\pi}_2^\circ}|\det|^{-w-1}$.

The set of automorphic representations with these properties are finite: there are only finitely many Hilbert modular eigenforms with bounded weight and level. Hence, if $\overline{\rho}_{\pi,\ell}$ splits in this way infinitely often, then there exist fixed automorphic representations $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ as above such that

$$\overline{\rho}_{\pi,\ell} \simeq \overline{\rho}_{\boldsymbol{\pi}_1,\ell} \oplus \overline{\rho}_{\boldsymbol{\pi}_2,\ell}$$

for infinitely many ℓ. We see that $\mathrm{Tr}(\rho_{\pi,\ell}(\mathrm{Frob}_v)) \in \overline{\mathbf{Z}}$ and

$$\mathrm{Tr}(\rho_{\boldsymbol{\pi}_1,\ell}(\mathrm{Frob}_v)) + \mathrm{Tr}(\rho_{\boldsymbol{\pi}_2,\ell}(\mathrm{Frob}_v)) \in \overline{\mathbf{Z}}$$

[4] It should be possible to show this fact directly. However, in general, K need not be a CM field, in which case, $\rho_{\boldsymbol{\pi},\ell}$ is not known to be crystalline.

are equal in $\overline{\mathbf{F}}_\ell$ for infinitely many ℓ and, hence, are equal in $\overline{\mathbf{Z}}$. Therefore, by the Chebotarev density theorem, it follows that

$$\rho_{\pi,\ell} \simeq \rho_{\boldsymbol{\pi}_1,\ell} \oplus \rho_{\boldsymbol{\pi}_2,\ell}$$

for all ℓ, contradicting Theorem A. □

7.2.3 Two-dimensional constituents with $\det(\rho_1) = \chi\overline{\varepsilon}_\ell^{-w-3}$ not totally odd

Lemma 7.2.8. *For at most finitely many primes $\ell \in S_{crys}$, there is a character χ such that $\det(\rho_1) = \chi\overline{\varepsilon}_\ell^{-w-3}$ and $\det(\rho_1)$ is not totally odd.*

Proof. Suppose that this case holds for infinitely many ℓ. Then, taking ℓ sufficiently large, we may assume that χ is unramified at all places $v \mid \ell$. We show that π does not have distinct Satake parameters at 100% of primes, contradicting Proposition 6.1.11 and Lemma 6.1.12.

For each $i = 1, 2$, let $\omega_{i,\ell} = \det(\rho_i)^{-1}\operatorname{sim}(\overline{\rho}_{\pi,\ell})$. By assumption, $\omega_{i,\ell}$ is unramified at all places $v \mid \ell$, has conductor dividing $\mathfrak{N}$ and is non-trivial, since $\operatorname{sim}(\overline{\rho}_{\pi,\ell})$ is totally odd. Since $\overline{\rho}_{\pi,\ell}^\vee \otimes \operatorname{sim}(\overline{\rho}_{\pi,\ell}) \simeq \overline{\rho}_{\pi,\ell}$, we deduce that either:

1. $\rho_i \simeq \rho_i^\vee \otimes \operatorname{sim}(\overline{\rho}_{\pi,\ell}) \simeq \rho_i \otimes \omega_{i,\ell}$ for each i;
2. $\rho_1 \simeq \rho_2^\vee \otimes \operatorname{sim}(\overline{\rho}_{\pi,\ell}) \simeq \rho_2 \otimes \omega_{2,\ell}$.

In the first case, we see that the $\omega_{i,\ell}$ are quadratic characters and, since $\omega_{1,\ell}\omega_{2,\ell} = \det(\rho_{\pi,\ell})^{-1}\operatorname{sim}(\overline{\rho}_{\pi,\ell})^2 = \mathbf{1}$, it follows that $\omega_{1,\ell} = \omega_{2,\ell}$. Hence,

$$\overline{\rho}_{\pi,\ell} \otimes \omega_{1,\ell} \simeq \overline{\rho}_{\pi,\ell}.$$

If this case occurs for infinitely many ℓ, then, by the pigeonhole principle, there is a finite order character $\omega_1 : \operatorname{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^\times$ such that $\omega_{1,\ell} \equiv \omega_1 \pmod{\ell}$ for infinitely many ℓ. Thus, $\rho_{\pi,\ell}\otimes\omega_1 \simeq \rho_{\pi,\ell}$, from which it follows that $\rho_{\pi,\ell}$ is imprimitive. Hence, by Lemma 6.1.9, π is an automorphic induction. However, in this case, by Remark 7.2.6, $\overline{\rho}_{\pi,\ell}$ is irreducible for all but finitely many $\ell \in S_{crys}$, and the result follows.

Suppose that the second case occurs for infinitely many primes. Then, since $\omega_{2,\ell}$ has conductor dividing $\mathfrak{N}$ and is unramified at all places $v \mid \ell$, there exists a finite order character $\omega_2 : \operatorname{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^\times$ such that $\omega_{2,\ell} \equiv \omega_2 \pmod{\ell}$ for infinitely many ℓ. Hence, for any place v of F at which π is unramified, we have (up to conjugation)

$$\rho_1(\operatorname{Frob}_v) = \rho_2(\operatorname{Frob}_v)\omega_2(\operatorname{Frob}_v),$$

and for the set of places v of positive density at which $\omega_2(\operatorname{Frob}_v) = 1$, we have

$$\rho_1(\operatorname{Frob}_v) = \rho_2(\operatorname{Frob}_v).$$

For each place v at which π is unramified, consider the Satake parameters of π_v. Then for all places v as above and for infinitely many ℓ, the parameters are not distinct in $\overline{\mathbf{F}}_\ell$. Hence they are not distinct in $\overline{\mathbf{Q}}$. We find that for a set of places v of F of positive density, the Satake parameters of π_v are not distinct, which contradicts Proposition 6.1.11 and Lemma 6.1.12. □

7.2.4 Other two-dimensional constituents

It remains to address the case that $\overline{\rho}_{\pi,\ell} \simeq \rho_{1,\ell} \oplus \rho_{2,\ell}$, where $\rho_{1,\ell}, \rho_{2,\ell}$ are two-dimensional, irreducible representations such that $\det(\rho_{1,\ell})$ and $\overline{\varepsilon}^{-w-3}$ have different inertial types.

Lemma 7.2.9. *For at most finitely many primes $\ell \in S_{crys}$, $\det(\rho_1)$ and $\overline{\varepsilon}_\ell^{-w-3}$ have different inertial types.*

Proof. Suppose that this case occurs for infinitely many primes $\ell \in S_{crys}$. If ℓ is large enough, then $\det(\rho_{1,\ell})$ has an inertial weight $(\lambda_v)_{v|\infty}$, where, for each archimedean place v of F, λ_v is a sum of two elements of

$$\delta_v + \{0, k_{2,v} - 2, k_{1,v} - 1, k_{1,v} + k_{2,v} - 3\}\,.$$

Moreover, by assumption, λ_v cannot equal $w + 3$ for all v.

Since the number of possible inertial weights is finite, some fixed weight (λ_v) is an inertial weight of $\det(\rho_{1,\ell})$ for infinitely many ℓ. Hence, by Lemma 7.2.5, there is a Hecke character $\eta = |\cdot|^{-i}\phi$ with $i \in \mathbf{Z}$, and ϕ a finite order character, such that

$$\det(\rho_{1,\ell}) \simeq \overline{\rho}_{\eta,\ell}$$

for infinitely many ℓ. Hence, $\det(\rho_{1,\ell}) = \overline{\varepsilon}_\ell^{-i}\overline{\rho}_{\phi,\ell}$. By assumption, $i \neq w + 3$. Thus, for any place $v \nmid \ell$ of F at which π is unramified, we see that

$$\overline{\alpha}_{v,\ell}\overline{\beta}_{v,\ell} := \det(\rho_{1,\ell}) \equiv N(v)^i\phi(\varpi_v) \pmod{\ell}$$

is the reduction mod ℓ of a product of two eigenvalues α_v, β_v of $\rho_{\pi,l}(\mathrm{Frob}_v)$. Hence, for infinitely many ℓ, $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ has eigenvalues α_v, β_v with

$$\alpha_v\beta_v \equiv N(v)^i\phi(\varpi_v) \pmod{\ell}$$

and, hence, $|\alpha_v\beta_v| = N(v)^i$. Since $i \neq w + 3$, we obtain a contradiction with [JS81b, Corollary 2.5], as in Lemma 5.2.4. □

This lemma completes the proof of Theorem 7.2.3.

7.3 Residual irreducibility unconditionally

In the previous section, we proved, assuming Serre's conjecture, that $\overline{\rho}_{\pi,\ell}$ is irreducible for all but finitely many primes $\ell \in S_{crys}$. In this section, we prove unconditionally a weaker result that $\overline{\rho}_{\pi,\ell}$ is irreducible for 100% of primes ℓ. The result follows from [PSW18] once we have established that $\rho_{\pi,\ell}$ is rational over a number field E.

Lemma 7.3.1. *There exists a number field E such for all ℓ, $\rho_{\pi,\ell}$ is conjugate to a representation*

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(E_\lambda)$$

for some prime $\lambda \mid \ell$.

Proof. Let E' be the finite extension of $\mathbf{Q}$ generated by the Hecke parameters of π.[5] By Proposition 6.1.11, for a positive density of places v, π_v is unramified and has distinct Satake parameters. Let E be the extension of E' obtained by adjoining these parameters for two places v, v' with distinct residue characteristics.

For each ℓ, we can view $\rho_{\pi,\ell}$ as a representation

$$\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_4(\overline{E}_\lambda).$$

Moreover, by assumption, $\mathrm{Tr}(\rho_{\pi,\ell}(\gamma)) \in E_\lambda$ for all $\gamma \in \mathrm{Gal}(\overline{F}/F)$. If our first chosen place v is prime to ℓ, then the characteristic polynomial of $\rho_{\pi,\ell}(\mathrm{Frob}_v)$ is well-defined and has distinct, E_λ-rational roots. If not, then $\rho_{\pi,\ell}(\mathrm{Frob}_{v'})$ has distinct, E_λ-rational roots. The result follows from [BLGGT14, Lemma A.1.5]. □

Theorem 7.3.2. *The residual representation $\overline{\rho}_{\pi,\ell}$ is irreducible for all primes ℓ in a set of Dirichlet density* 1*.*

Proof. By Theorem A, $\rho_{\pi,\ell}$ is irreducible and crystalline for all primes ℓ in a set of Dirichlet density 1. Moreover, by Lemma 7.3.1, $\rho_{\pi,\ell}$ is rational over a number field E.

The result follows immediately from [PSW18, Theorem 1.2]. □

7.4 The image of Galois

Suppose that π is not an automorphic induction or a symmetric cube lift. In this section, assuming Serre's conjecture, we prove part 2 of the second part of Theorem B, namely, that the image of $\overline{\rho}_{\pi,\ell}$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for almost all primes $\ell \in S_{crys}$. If the weights of π are not all of the form $(2k_v - 1, k_v + 1)_{v|\infty}$ for $k_v \geq 2$—i.e. if the weights of π are incompatible with symmetric cube lifts—we do not need to assume Serre's conjecture.

Our proof mirrors that of [DZ, 3.2-3.5]. By the classification of the maximal subgroups of $\mathrm{PSp}_4(\mathbf{F}_{\ell^n})$, [Mit14][DZ, Theorem 3.2], if $\overline{\rho}_{\pi,\ell}$ is irreducible and does not contain $\mathrm{Sp}_4(\mathbf{F}_\ell)$, then one of the following cases must hold:

1. The image of $\overline{\rho}_{\pi,\ell}$ contains a reducible index two subgroup. Hence, $\overline{\rho}_{\pi,\ell}$ is induced from a quadratic extension.

2. $\overline{\rho}_{\pi,\ell}$ is isomorphic to the symmetric cube of a two-dimensional representation.

3. The image $\overline{\rho}_{\pi,\ell}$ is a small exceptional group.

We show that each of these cases can only happen for finitely many primes $\ell \in S_{crys}$.

[5] The finiteness of $E'/\mathbf{Q}$ follows from the fact the Hecke algebra $\mathbf{T}$ acting on Hilbert–Siegel modular forms of fixed weight and level is a finitely generated $\mathbf{Q}$-algebra. Hence, the map $\mathbf{T} \to \mathbf{C}$ corresponding to π must have image in a number field. See [Tay91, Theorem 2] for a complete argument in the case that $F = \mathbf{Q}$.

7.4.1 $\mathrm{Im}(\overline{\rho}_{\pi,\ell})$ contains a reducible index two subgroup

Suppose that the image of $\overline{\rho}_{\pi,\ell}$ contains a reducible index two subgroup. Then $\overline{\rho}_{\pi,\ell}$ is irreducible, but becomes reducible over a quadratic extension L/F. It follows that $\overline{\rho}_{\pi,\ell}$ is induced from a character of $\mathrm{Gal}(\overline{F}/L)$ and, hence, that

$$\overline{\rho}_{\pi,\ell} \simeq \overline{\rho}_{\pi,\ell} \otimes \chi_\ell,$$

where χ_ℓ is the quadratic character that cuts out the extension L/F. If χ_ℓ is ramified at ℓ, then, since χ_ℓ is a quadratic character, χ_ℓ must have an inertial weight of the form (λ_τ), where $\lambda_\tau \in \{0, \frac{\ell-1}{2}\}$ for all τ with not all the λ_τ's equal to 0. Comparing this inertial weight with the inertial weights of $\overline{\rho}_{\pi,\ell}$, we see that χ_ℓ must be unramified at ℓ when ℓ is sufficiently large.

Suppose that there exists a quadratic character χ_ℓ such that

$$\overline{\rho}_{\pi,\ell} \simeq \overline{\rho}_{\pi,\ell} \otimes \chi_\ell$$

for infinitely many primes ℓ. Then we may assume that χ_ℓ is unramified at ℓ and hence that its conductor is bounded independently of ℓ. Therefore, there is a finite order character $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^\times$ such that χ_ℓ is the reduction of χ modulo ℓ for infinitely many primes ℓ. It follows that

$$\rho_{\pi,\ell} \simeq \rho_{\pi,\ell} \otimes \chi.$$

Thus, $\rho_{\pi,\ell}$ is imprimitive, contradicting Lemma 6.1.9.

7.4.2 $\overline{\rho}_{\pi,\ell}$ is a symmetric cube lift

We follow the argument of [Die02, Section 4.5]. Suppose that

$$\overline{\rho}_{\pi,\ell} \simeq \mathrm{Sym}^3(\rho_\ell)$$

for some two-dimensional representation ρ_ℓ. Computing the matrix $\mathrm{Sym}^3\begin{pmatrix} a & * \\ * & d \end{pmatrix}$ directly, we see that the image of $\overline{\rho}_{\pi,\ell}$ consists of matrices of the form

$$\begin{pmatrix} a^3 & * & * & * \\ * & a^2d & * & * \\ * & * & ad^2 & * \\ * & * & * & d^3 \end{pmatrix}$$

for $a, d \in \overline{\mathbf{F}}_\ell$. Observe that

$$(a^3)^2 d^3 = (a^2d)^3 \quad \text{and} \quad a^3(d^3)^2 = (ad^2)^3.$$

In particular, if $(\lambda_\tau) \in (\mathbf{Z}_+^4)^{\tau \in \mathrm{Hom}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell)}$ is an inertial weight of $\overline{\rho}_{\pi,\ell}$ and if ℓ is large enough, then for every τ, we must have

$$2\lambda_{1,\tau} + \lambda_{4,\tau} = 3\lambda_{2,\tau} \quad \text{and} \quad \lambda_{1,\tau} + 2\lambda_{4,\tau} = 3\lambda_{3,\tau}.$$

The Hodge–Tate weights of $\rho_{\pi,\ell}$ satisfy these relations if and only if π has weights $(2k_v + 1, k_v + 2)_{v|\infty}$ and w is a multiple of 3. In particular, if the weights of π are not of this form, then $\overline{\rho}_{\pi,\ell}$ can only be a symmetric cube lift for finitely many primes $\ell \in S_{crys}$.

Suppose now that the weights of π are $(2k_v+1, k_v+2)_{v|\infty}$, where $k_v \geq 2$. Identifying $\{v : v \mid \infty\}$ with $\prod_{v|\ell} \mathrm{Hom}(\mathbf{F}_v, \overline{\mathbf{F}}_\ell)$ and using Fontaine–Laffaille theory, we see that ρ_ℓ has an inertial weight of the form

$$(\lambda_{1,v}, \lambda_{2,v}) = \delta'_v + (0, k_v - 1),$$

where

$$\delta'_v = \frac{1}{2}(w' + 1 - (k_v - 1))$$

with $w' = \frac{1}{3}(w-3)$. Observe that $\det(\rho_\ell)^3 = \mathrm{sim}(\overline{\rho}_{\pi,\ell})$ and hence that $\det(\rho_\ell)$ is totally odd. If this case occurs for infinitely many primes ℓ, arguing as in Section 7.2.2, by Serre's conjecture, we deduce that $\rho_{\pi,\ell}$ is itself a symmetric cube lift, contradicting our assumptions.

7.4.3 The remaining images

In the remaining cases, the projective image of $\overline{\rho}_{\pi,\ell}$ is bounded independently of ℓ. Since $\rho_{\pi,\ell}$ is not an Artin representation, by Fontaine–Laffaille theory, it is clear that the image of $\overline{\rho}_{\pi,\ell}$ grows at least linearly with ℓ (c.f. [CG13, Lemma 5.3]). Hence, these cases can occur for only finitely many ℓ.

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