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Modularity of abelian surfaces over imaginary quadratic fields

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Modular forms for GL(2) over an imaginary quadratic field K are known as Bianchi modular forms. Standard modularity conjectures assert that every weight 2 rational Bianchi newform has either an associated elliptic curve over K or an associated abelian surface with quaternionic multiplication over K. We give explicit evidence in the way of examples to support this conjecture in the latter case. Furthermore, the quaternionic surfaces given correspond to *genuine* Bianchi newforms, which answers a question posed by J. Cremona as to whether this phenomenon can happen.

Contents

Introduction

One of the crowning achievements of mathematics in the 20th century was the proof of Fermat's last theorem. Whilst the result itself is of huge importance in the mathematical community for historical reasons, perhaps of equal importance are the striking techniques that were used in order to achieve the proof. Broadly speaking, Wiles, Taylor-Wiles et al. [Wil95, TW95, BCDT01] proved a deep conjecture asserting a correspondence between two of the central objects in number theory: elliptic curves and modular forms over the rationals. One direction of this correspondence was established by Eichler and Shimura. The converse direction, from elliptic curves to modular forms, was the content of the so-called the Taniyama-Shimura-Weil conjecture and is now called the modularity theorem. Fermat's last theorem followed from this result.

The modularity theorem is in fact part of a large network of conjectures and correspondences known as the Langlands Programme. This framework connects objects such as abelian varieties, modular forms, Galois representations and L-functions and it is fair to say that a huge part of modern number theory sits within this framework. The methods initiated by Wiles and Taylor-Wiles sparked much activity and progress in the Langlands programme. It is expected that some version of the modularity theorem generalises to varieties over different fields and arbitrary dimensions. To date there has been progress in this direction, for example: elliptic curves over totally real fields [FLHS15]; GL_2-type varieties over $\mathbb{Q}$ [KW09]; $\mathbb{Q}$-curves [ES01] Calabi-Yau threefolds [GY11] and abelian surfaces over $\mathbb{Q}$ [BCGP18].

Whereas there has been a lot of work done in the totally real case (cf. [BDJ10]), there has been comparatively little done beyond such fields. This is because there is a dichotomy between the two cases which make them qualitatively different to study. For example, for fields which are not totally real the associated locally symmetric spaces does not have a complex structure and so the techniques of algebraic geometry cannot be applied.

This thesis is primarily concerned with the objects involved for modularity in the imaginary quadratic field setting. Note that a modular form over an imaginary quadratic field is called a *Bianchi modular form*.

It was first noted by J.-P Serre [Ser70] that there could be an analogous connection between elliptic curves over imaginary quadratic fields and Bianchi modular forms. There is computational evidence for this conjecture starting with [GM78] and including [Cre84, CW94, Byg98, Lin05]. Very recently, there has been a major breakthrough in proving modularity lifting theorems over imaginary quadratic fields [ACC+18].

As opposed to totally real fields, in the case of imaginary quadratic fields it is not necessarily the case that a weight 2 rational Bianchi newform has to correspond to an elliptic curve. It is possible that it might instead correspond to an abelian surface with quaternionic endomorphisms, i.e. a *QM-surface*. Note that these objects are often referred to *fake/false elliptic curves* based on the observation that such a surface is isogenous to the square of an elliptic curve modulo every prime of good reduction. This makes for the following conjectural diagram:

$$\left\{\begin{array}{c}\text{weight 2 rational}\\ \text{Bianchi newforms}/K\end{array}\right\} \overset{1:1}{\dashleftarrow\dashrightarrow} \begin{array}{c}\left\{\begin{array}{c}\text{non-CM by } K\\ \text{elliptic curves}/K\\ \text{up to isogeny}\end{array}\right\}\\ \bigsqcup \\ \left\{\begin{array}{c}\text{QM surfaces}/K\\ \text{up to isogeny}\end{array}\right\}\end{array}$$

The conductor of the elliptic curve or false elliptic curve should be equal to the level or the square of the level of the Bianchi newform respectively.

This connection between QM surfaces and Bianchi newforms was first explicated by P. Deligne in a letter to J. Mennicke in 1979 [Del79]. It was further studied by J. Cremona [Cre92] who gave explicit examples of QM-surfaces which are modular by using the following construction: let f be a classical newform of weight 2 with a real quadratic Hecke eigenvalue field $K_f = \mathbb{Q}(\{a_i\})$ and denote $\langle\sigma\rangle = \mathrm{Gal}(K_f/\mathbb{Q})$. Suppose that f has an *inner twist*, i.e. $f^\sigma = f \otimes \chi_K$ where χ_K is the quadratic Dirichlet character associated to some imaginary quadratic field K. It follows that f and f^σ must base change to the same Bianchi newform over K.

The abelian surface $A_f/\mathbb{Q}$ associated to f is of GL_2-type in the sense that $\mathrm{End}^0_{\mathbb{Q}}(A_f) \simeq K_f$. Moreover, $L(A_f/\mathbb{Q}, s) = L(f, s)L(f^\sigma, s)$ and the base-change surface $A \otimes_{\mathbb{Q}} K$ is a QM surface such that $L(A/K, s) = L(F, s)^2$, where F is the induction from $\mathbb{Q}$ to K of f.

We use the term *genuine* for Bianchi newforms that are not (the twist of) base-change of a classical newform and similarly for abelian surfaces. The above construction motivates the following question (cf. [Cre92, Question 1']):

Question. *Is it possible for a QM surface over an imaginary quadratic field to be* ***genuine****?*

The results of this thesis are twofold:

- Genuine QM-surfaces over imaginary quadratic fields exist and explicit examples are given.
- These examples are verified to be modular.

The thesis will be laid out in the following way. In Chapter 1 background material is presented on abelian varieties, Bianchi modular forms and Quaternion algebras. Chapter 2 is devoted to aspects of QM-surfaces, including the family of QM-surfaces used to find the genuine examples mentioned above. Furthermore, arithmetic aspects of Galois representations attached to QM-surfaces will be discussed. The third chapter is where the main result of the thesis is presented: that genuine QM-surfaces exist. A detailed overview on how the explicit examples were found is included, as is a brief discussion on the connection the Paramodularity Conjecture. Finally, in the last chapter the method of Faltings-Serre is applied to show that the genuine QM-surfaces presented in the thesis are modular. This is achieved by proving that the attached Galois representations are isomorphic via the tools of class field theory and comparing the traces of Frobenius at a carefully chosen finite set of primes.

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Chapter 1

Background

§ 1.1 Abelian varieties

We begin this chapter by reviewing the basic notions around abelian varieties. A classical reference for the theory when the base field is not algebraically closed is [Mil86].

1.1.1 Abelian varieties

Let K be a field, not necessarily of characteristic zero. A *group variety* G is an algebraic variety that is also a group such that the multiplication and inversion operations

$$\begin{aligned} m : G \times G &\longrightarrow G; \\ \iota : G &\longrightarrow G, \end{aligned}$$

are defined by regular maps on G.

There is an important disctinction between algebaic groups which are projective – these encompass abelian varieties – and those that are affine: linear algebraic groups. The theories differ somewhat considerably and we will not pursue the affine case, which one can think of as subgroups of GL_n.

Definition 1.1.1. An *abelian variety* is a complete connected group variety.

Thankfully, the terminology is consistent:

Proposition 1.1.2. *Any abelian variety is commutative.*

Proof. For an abelian variety A consider the map

$$\begin{aligned} A(\overline{K}) \times A(\overline{K}) &\longrightarrow A(\overline{K}) \\ (x, y) &\longmapsto y \cdot x \cdot y^{-1} \cdot x^{-1}. \end{aligned}$$

The restriction of this operation to $A(\overline{K}) \times e_A$ or $e_A \times A(\overline{K})$ is just the identity. We can then apply the Rigidity Theorem (cf. [Mil86, Theorem 1.1]). □

Proposition 1.1.3. *A group variety is an abelian variety if and only if it is projective.*

1.1.2 Isogenies and the Tate module

Recall that if $f : X \to Y$ is a surjective morphism of algebraic varieties over K then the *degree* of f is the degree of the finite field extension of the function field $K(X)$ over $f^*K(Y)$.

Definition 1.1.4. A K*–isogeny* $A \to B$ of abelian varieties over K is a homomorphism such that the kernel is finite. The *degree* of an isogeny is equal to the order of the kernel.

For a homomorphism $f : A \to B$ the following are equivalent:

1 f is an isogeny;

2 $\dim(A) = \dim(B)$ and f is surjective.

Definition 1.1.5. The K*–rational endomorphism ring* of A is defined as

$$\mathrm{End}_K(A) = \{f : A \to A \mid f \text{ is a homomorphism defined over } K\}.$$

There is an injection of the integers into the ring of endomorphisms $\mathbb{Z} \hookrightarrow \mathrm{End}_K(A)$ since for a positive integer n one naturally defines

$$\begin{aligned} [n] : A &\longrightarrow A \\ x &\longmapsto x + \cdots + x. \end{aligned}$$

The map $[-1]$ is just defined to be the inverse and by composition we deduce that any integer can be viewed as an endomorphism of A. We will follow the convention that A has *trivial endomorphisms* over K if $\mathrm{End}_K(A) \simeq \mathbb{Z}$. Given an isogeny $f : A \to B$ of degree d there exists an isogeny $g : B \to A$ such that $g \circ f = f \circ g = [d]$.

Definition 1.1.6. For a positive integer n we define the *n–torsion points* of A to be $A[n] = \mathrm{Ker}([n] : A(\overline{K}) \to A(\overline{K}))$.

If K has characteristic prime to n then as a group $A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Definition 1.1.7. Let ℓ be a prime number different from the characteristic of K. The *ℓ–adic Tate modules* of A are defined to be

$$T_\ell A = \lim_{\leftarrow n} A[\ell^n] \quad \text{and} \quad V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The Tate module is isomorphic to $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$.

There is a natural action of the absolute Galois group G_K on $A(\overline{K})$

$$\begin{aligned} G_K \times A(\overline{K}) &\longrightarrow A(\overline{K}) \\ (\sigma, P) &\longmapsto \sigma(P). \end{aligned}$$

The action restricts to the ℓ–torsion points and hence we can define the residual Galois representation

$$\overline{\rho}_{A,\ell} : G_K \longrightarrow \mathrm{Aut}(A[\ell]) \simeq \mathrm{GL}_{2g}(\mathbb{F}_\ell)$$

and the full ℓ–adic continous representation

$$\rho_{A,\ell} : G_K \longrightarrow \mathrm{Aut}(T_\ell(A)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell).$$

The representations are compatible in the sense that if $\pi : \mathbb{Z}_\ell \to \mathbb{F}_\ell$ is the natural projection map then $\overline{\rho}_{A,\ell} = \pi \circ \rho_{A,\ell}$.

1.1.3 Jacobians

An important source of abelian varieties is to take the Jacobian of a curve. Let us explain this construction.

By a *curve* over K we shall mean a smooth, projective, geometrically integral K–algebraic variety of dimension one. For the remainder C will be used to denote a curve over a perfect field K.

Given a set of points $P_i \in C(\overline{K})$ for $i = 1 \ldots k$, the formal linear combination $\sum_{i=1}^{k} n_i P_i$ where $n_i \in \mathbb{Z}$, is called a *divisor*. These form a group Div_C called the *group of divisors*. The group G_K acts on Div_C in a natural way. The fixed points of this action are called *K–rational divisors* and form a subgroup $\mathrm{Div}_C(K)$.

The *degree* of a divisor $D = \sum_{i=1}^{k} n_i P_i$ is simply the integer

$$\deg(D) = \sum_{i=1}^{k} n_i$$

and the degree map is a homomorphism $\mathrm{Div}_C \to \mathbb{Z}$. The kernel of this map is denoted by Div_C^0.

For $f \in \overline{K}(C)\backslash\{0\}$ a rational function on C, the *divisor* of f is defined to be

$$\mathrm{div}(f) = \sum_{P \in C(\overline{K})} v_P(f) P,$$

where $v_P(f)$ is the order of vanishing of f at P. A divisor is said to be *principal* if it is of the form $\mathrm{div}(f)$ for some nonzero rational function $f \in \overline{K}(C)$. It turns out that principal divisors have degree 0 so we denote by Princ_C the subgroup of Div_C^0 of principal divisors.

Definition 1.1.8. The *Picard groups* are the quotients

$$\mathrm{Pic}_C = \mathrm{Div}_C / \mathrm{Princ}_C$$

and

$$\mathrm{Pic}_C^0 = \mathrm{Div}_C^0 / \mathrm{Princ}_C.$$

Theorem 1.1.9. *As a G_K–module Pic_C^0 is isomorphic to an abelian variety defined over K. This is called the Jacobian of C, denoted by $\mathrm{Jac}(C)$. The dimension of $\mathrm{Jac}(C)$ is equal to the genus of C.*

Broadly speaking, the *dual abelian variety* $A^\vee$ of A parameterises particular line bundles on A. It is an abelian variety over K which is isogenous to A. See [Mil86, Chapter 1.8] for a precise definition. In general, an abelian variety and its dual are not necessarily isomorphic.

A *K-polarisation* of the abelian variety A/K is a K–isogeny $\lambda : A \longrightarrow A^\vee$ such that over $\overline{K}$ it is of the form $\lambda_{\mathcal{L}}$ for some ample line bundle $\mathcal{L}$. A *principal polarisation* is one which induces an isomorphism $A \simeq A^\vee$ and we say that A is principally polarisable if such an isomorphism exists. It is not true that principal polarisations necessarily exist, however, the Jacobian of a curve is principally polarisable.

Definition 1.1.10. For a K–polarisation $\lambda : A \to A^\vee$ of degree d with dual isogeny $\hat{\lambda} : A^\vee \to A$ such that $\hat{\lambda} \circ \lambda = [d]$, the *Rosati involution* is defined on the endomorphism algebra by

$$\begin{aligned} \dagger : \mathrm{End}^0_K(A) &\longrightarrow \mathrm{End}^0_K(A) \\ \varphi &\longmapsto \frac{1}{d}\hat{\lambda} \circ \hat{\varphi} \circ \lambda. \end{aligned}$$

Let us finally note that Torelli's theorem states that the Jacobi map from the moduli space of curves of genus g to the moduli space of principally polarised abelian varieties of dimension g is injective. For $g = 2$ the dimension of each moduli spaces is 3.

§ 1.2 Bianchi modular forms

We give a brief overview on *Bianchi modular forms*. These are modular forms for imaginary quadratic fields; we shall focus on Bianchi newforms of weight 2 as this is what will be needed for the purposes of the thesis. An excellent and concise account is given in [CW94, §2].

Let K be an imaginary quadratic field of class number 1 with ring of integers $\mathcal{O}_K$. The notation $\mathfrak{n}$ will be used for a non-zero ideal of $\mathcal{O}_K$. Many of the notions in the theory of classical newforms have an analogue here.

Denote by $\mathbb{H}_3$ the model of hyperbolic 3-space

$$\mathbb{H}_3 = \{ \ (z, t) \mid z \in \mathbb{C}, \ t \in \mathbb{R}, \ t > 0 \ \}.$$

There is a transitive isometric action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}_3$. Given the quaternion algebra presentation of the Hamiltonians $\left(\frac{-1,-1}{\mathbb{R}}\right)$, we can consider $\mathbb{H}_3$ to be contained inside of the Hamiltonians via $(z, t) \mapsto z + tj$. Then the action takes the familiar form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1}.$$

Let

$$\beta = \left(-\frac{dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t}\right)$$

be a basis for the left invariant differential forms on $\mathbb{H}_3$.

Define the congruence subgroup

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) \mid c \equiv 0 \bmod \mathfrak{n} \right\}.$$

Definition 1.2.1. A *Bianchi cusp form* $F \in S_2(\Gamma_0(\mathfrak{n}))$ of weight 2 and level $\mathfrak{n}$ is a function

$$F : \mathbb{H}_3 \longrightarrow \mathbb{C}^3$$

such that

- $F \cdot \beta$ is a harmonic differential one-form on $\mathbb{H}_3$ which is invariant under $\Gamma_0(\mathfrak{n})$.
- $\int_{\mathbb{C}/\mathcal{O}_K} (F|\sigma)(z,t)dz = 0$ for all $\sigma \in \mathrm{SL}_2(\mathcal{O}_K)$.

Bianchi cusp forms have Fourier expansions of the form

$$F = (F_1, F_2, F_3) = \sum_{\mathfrak{p}} c(\mathfrak{p}) t^2 K\left(\frac{4\pi|\mathfrak{p}|t}{\sqrt{|D|}}\right) \psi\left(\frac{\mathfrak{p}z}{\sqrt{D}}\right),$$

where $\psi(z) = exp(2\pi i(z + \bar{z}))$ for $z \in \mathbb{C}$ and

$$K(t) = \left(-\frac{i}{2}K_1(t), K_0(t), \frac{i}{2}K_1(t)\right)$$

for $t \in \mathbb{R}_{>0}$. The two functions K_1 and K_2 are hyperbolic Bessel functions.

As in the classical theory, the space of cusp forms $S_2(\Gamma_0(\mathfrak{n}))$ is a finite dimensional complex vector space. There is a commutative algebra of Hecke operators that act on $S_2(\Gamma_0(\mathfrak{n}))$ and for prime ideals $\mathfrak{p} \nmid \mathfrak{n}$, the Hecke operator takes a form with coefficients $c(\alpha)$ to one with coefficients $N(\mathfrak{p})c(\alpha\mathfrak{p}) + c(\alpha/\mathfrak{p})$.

Definition 1.2.2. A *Bianchi newform* for $S_2(\Gamma_0(\mathfrak{n}))$ is a cusp form which is an eigenform for all of the Hecke operators $\{T_{\mathfrak{p}}\}_{\mathfrak{p}\nmid\mathfrak{n}}$ and is not induced from a form of strictly smaller level $\mathfrak{m}$, with $\mathfrak{m}|\mathfrak{n}$. It is assumed to be normalised in the sense that $c(1) = 1$ and hence $T_{\mathfrak{p}}F = c(\mathfrak{p})F$ for all $\mathfrak{p} \nmid \mathfrak{n}$.

Attached to F is the formal Dirichlet series

$$L(F,s) = \sum_{\mathfrak{p}} c(\mathfrak{p}) N(\mathfrak{p})^{-s}.$$

It has an Euler product expansion

$$L(F,s) = \prod_{\mathfrak{p}} (1 - c(\mathfrak{p})N(\mathfrak{p})^{-s} + \chi(\mathfrak{p})N(\mathfrak{p})^{1-2s})^{-1}$$

where

$$\chi(\mathfrak{p}) = \begin{cases} 0 & \text{if } \mathfrak{p} | \mathfrak{n}; \\ 1 & \text{if } \mathfrak{p} \nmid \mathfrak{n}. \end{cases}$$

Remark 1.2.3. The account of Bianchi newforms above as vector valued functions is perhaps the most intuitive given a knowledge of classical modular forms. However, it is possible to interpret these objects in a variety of ways: see [Tay94] for a treatment of these forms as regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_K)$ and see [Har75] to view them as certain classes in the cohomology of quotients of hyperbolic three space by congruence subgroups of $\mathrm{GL}_2(\mathcal{O}_K)$.

§ 1.3 Quaternion algebras

In this section we present a self-contained introduction to the theory of quaternion algebras. We will follow the concise treatment given by [MR03]. Two excellent classical references are [Vig80, Piz76] and a more recent compendium of all things quaternion algebra related is [Voi19].

1.3.1 Overview

Let F be a field with algebraic closure $\overline{F}$. Recall that an *algebra* over F is a ring $\mathcal{B}$ with an embedding $F \hookrightarrow \mathcal{B}$ such that the image is in the center of $\mathcal{B}$.

Definition 1.3.1. A quaternion algebra $\mathcal{B}$ over a field F is an algebra of dimension 4 over F which is:

- *Central*: the centre of $\mathcal{B}$ is exactly F, and

- *Simple*: $\mathcal{B}$ contains no non-trivial two-sided ideals.

From now on we suppose that F is a field of characteristic different from 2. The above definition is equivalent to saying that $\mathcal{B}$ has an F-basis denoted by $1, i, j$ and k subject to

$$i^2 = a,\ j^2 = b \text{ and } ij = -ji = k, \tag{1.1}$$

with $a, b \in F^\times$. Such a quaternion algebra is denoted by the Hilbert symbol

$$\mathcal{B} = \left(\frac{a, b}{F}\right).$$

One readily sees that the presentation is unique up to squares, i.e $\left(\frac{a,b}{F}\right) \simeq \left(\frac{ax^2,by^2}{F}\right)$ for $x, y \in F^\times$, that a and b can be swapped, or one of a or b replaced by $-ab$ since $k^2 = -ab$ and i, j and k anti-commute.

Example 1.3.2. The set of two by two matrices $M_2(F)$ is a quaternion algebra isomorphic to $\left(\frac{1,1}{F}\right)$. An explicit map is given by

$$i \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and one can easily check the identity (1.1). More generally, $\left(\frac{1,a}{F}\right) \simeq \left(\frac{a,-a}{F}\right) \simeq M_2(F)$.

Recall that Wedderburn's Structure Theorem states that a simple algebra of finite dimension over a field F is isomorphic to the matrix algebra $M_n(D)$ where D is a division algebra over F. The fact that the quaternion algebra $\mathcal{B}$ has dimension 4 over F and $\dim_F(M_n(D)) = n^2 \cdot \dim_F(D)$ forces $n = 1$ or 2 in this case. Hence we distinguish between the two possibilities.

Definition 1.3.3. A quaternion algebra $\mathcal{B}$ over F is said to be *split* if it is isomorphic to $M_2(F)$ and *non-split* otherwise.

It is clear that if F is algebraically closed then $\mathcal{B}$ is split. However, it is possible to extend scalars by a quadratic extension and get the matrix algebra. In this regard note that if K is a field extension of F then

$$\left(\frac{a, b}{F}\right) \otimes_F K \simeq \left(\frac{a, b}{K}\right)$$

and we say that K *splits* $\mathcal{B}$ if $\mathcal{B} \otimes_F K \simeq M_2(K)$.

An important aspect of quaternion algebras is that they have a notion of trace and norm.

Definition 1.3.4. For an element $w \in \mathcal{B}$ write

$$w = a_0 + a_1 i + a_2 j + a_3 k, \quad a_0, \ldots, a_3 \in F \tag{1.2}$$

and define the *conjugate* of w to be

$$\overline{w} = a_0 - a_1 i - a_2 j - a_3 k. \tag{1.3}$$

Then we define the (reduced) *trace* and (reduced) *norm* functions respectively as

$$tr : \mathcal{B} \longrightarrow F \tag{1.4}$$
$$w \longmapsto w + \overline{w}; \tag{1.5}$$
$$nm : \mathcal{B}^* \longrightarrow F^\times \tag{1.6}$$
$$w \longmapsto w \cdot \overline{w}. \tag{1.7}$$

These agree with the usual trace and norm maps on a matrix algebra. Furthermore, they are in fact homomorphisms so the set of invertible elements $\mathcal{B}^*$ are precisely those with non-zero norm. We will denote

$$\mathcal{B}^1 = \{\ w \in \mathcal{B} \mid nm(w) = 1\ \} \subseteq \mathcal{B}^*.$$

The element $w \in \mathcal{B}$ satisfies the quadratic equation

$$x^2 - tr(w)x + nm(w) = 0.$$

Proposition 1.3.5. *Let $\mathcal{B}$ be a division quaternion algebra over F. Then for every element $w \in \mathcal{B}\backslash F$, the quadratic extension $F(w)$ splits the quaternion algebra as $\mathcal{B} \otimes_F F(w) \simeq M_2(F(w))$.*

Proof. $F(w)$ is a commutative subring of $\mathcal{B}$ and also a division ring so is hence a field. As $\mathcal{B}$ is central there are strict inclusions $\mathcal{B} \supset F(w) \supset F$ and w satisfies the quadratic equation above so $F(w)$ is a quadratic extension of F.

To show that $F(w)$ splits $\mathcal{B}$, let us pick an element $y \in F(w)\backslash F$ such that $y^2 \in F$. Now $-y = zyz^{-1}$ for some element $z \in \mathcal{B}$ by the Skolem Noether Theorem. Since $F(w)$ is commutative it is clear that $z \notin F(w)$. Thus $\{1, y, z, yz\}$ is an F-basis of $\mathcal{B}$. Finally, the fact that y is square in $F(w)$ means that $B \otimes_F F(w) \simeq \left(\frac{y,z}{F(w)}\right) \simeq \left(\frac{1,z}{F(w)}\right) \simeq M_2(F(w))$. □

1.3.2 Orders in quaternion algebras

In this subsection we briefly introduce the theory of orders in quaternion algebras, which are an important aspect when looking at quaternion algebras over a global field. Throughout let R denote a Dedekind domain whose field of quotients k is either a number field or a local field.

Definition 1.3.6. An *R-lattice* L of a vector space V/k is a finitely generated R-module contained in V, which we say is *complete* if $L \otimes_R k \simeq V$.

Definition 1.3.7. An element w in the quaternion algebra $\mathcal{B}/k$ is an *integer* if $R[w]$ is an R-lattice. This is equivalent to having $tr(w), nm(w) \in R$.

Definition 1.3.8. For the quaternion algebra $\mathcal{B}/k$:

1 An *ideal* of $\mathcal{B}$ is a complete R-lattice.

2 An *order* of $\mathcal{B}$ is an ideal which is also a ring.

3 An order is *maximal* if it is maximal with respect to inclusion.

For an ideal $I \subset \mathcal{B}$, the *order on the left* and *right* of I are defined respectively as

$$\mathcal{O}_l(I) = \{w \in \mathcal{B} | wI \subset I\}, \quad \mathcal{O}_r(I) = \{w \in \mathcal{B} | Iw \subset I\}$$

which are indeed both orders as one would expect. The ideal I is said to be *two-sided* if $\mathcal{O}_l(I) = \mathcal{O}_r(I)$. The *norm* of an ideal is the fractional ideal of R generated by $\{ \ nm(w) \mid w \in I \ \}$.

Orders are characterised as rings of integers $\mathcal{O}$ in $\mathcal{B}$ which contain R and are such that $k\mathcal{O} = \mathcal{B}$. It is a fact that maximal orders exist and every order is contained in a maximal order. In the case that $\mathcal{B} = M_2(k)$ then one maximal order is $M_2(R)$ and if R is a PID then all maximal orders are conjugate. Let us define a special class of orders.

Definition 1.3.9. An *Eichler order* is an order that is the intersection of two distinct maximal orders.

Example 1.3.10. $M_2(R)$ is a maximal order of $M_2(k)$ as one might hope. For any ideal J of R, an Eichler order is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, d \in R, c \in J \right\}.$$

As above we set $\mathcal{O}^1 = \{ \ w \in \mathcal{O} \mid nm(w) = 1 \ \}$.

1.3.3 Quaternion algebras over local fields

Quaternion algebras over local fields are in many ways simpler than those over global fields. In fact, they are used to classify quaternion algebras over global fields up to isomorphism by looking at their structure under completion.

Throughout this subsection let k be a non-archimedean local field, R its ring of integers, π a uniformiser such that $\mathcal{P} = \pi R$ is the unique maximal ideal. Let $\mathcal{B}$ be a quaternion algebra over k. Write $\nu : k^* \to \mathbb{Z}$ for the non-archimedean valuation and define

$$\upsilon : \mathcal{B}^* \longrightarrow \mathbb{Z} \tag{1.8}$$
$$w \longmapsto \nu(nm(w)) \tag{1.9}$$

where nm is the norm function on $\mathcal{B}$.

Lemma 1.3.11. *The map $\upsilon : \mathcal{B}^* \longrightarrow \mathbb{Z}$ defines a valuation on $\mathcal{B}^*$.*

The valuation gives us two important features:

$$\mathcal{O} = \{ \ w \in \mathcal{B} \mid \upsilon(w) \geq 0 \ \},$$

which is the unique maximal order of $\mathcal{B}$ and

$$\mathcal{J} = \{ \ w \in \mathcal{B} \mid \upsilon(w) > 0 \ \},$$

a two-sided ideal. It is a principal ideal given by $\mathcal{J} = \mathcal{O}j$ and any two-sided ideal of $\mathcal{O}$ is a power of $\mathcal{J}$.

The main theorem about quaternion algebras over non-archimedean local fields is the following.

Theorem 1.3.12. *There is a unique division quaternion algebra over k which is isomorphic to $\left(\frac{\pi,u}{k}\right)$, where $k(\sqrt{u})$ is the unique unramified quadratic extension of k.*

An immediate consequence of this theorem is that a quaternion algebra over k is either isomorphic to $M_2(K)$ or the unique division quaternion algebra over k. When $\mathcal{B}$ is the unique division quaternion algebra it is often useful to embed it into the matrix algebra. This will be especially useful when comparing Galois representations in Chapter 4. To this end, we make use of the fact that $L = k(\sqrt{u})$ splits $\mathcal{B}$. There is an explicit isomorphism of k-algebras

$$\mathcal{B} \simeq \left\{ \begin{pmatrix} \alpha & \beta \\ \pi\beta' & \alpha' \end{pmatrix} \mid \alpha, \beta \in L, \ ' : L \to L \text{ is conjugation in } L/k \right\} \subseteq M_2(L); \quad (1.10)$$

$$i \mapsto \begin{pmatrix} \sqrt{u} & 0 \\ 0 & -\sqrt{u} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}. \quad (1.11)$$

1.3.4 Quaternion algebras over global fields

We can use the results of the previous section to classify quaternion algebras over global fields. Let K denote a number field throughout this subsection and ν a valuation of K. The completion of K at ν will be denoted by k_ν. The behaviour of the quaternion algebra $\mathcal{B}/K$ over completions of K is an important notion. Define the quaternion algebra $\mathcal{B}_\nu = \mathcal{B} \otimes_K k_\nu$.

Definition 1.3.13. The quaternion algebra $\mathcal{B}/K$ is said to be *ramified* at ν if $\mathcal{B}_\nu$ is the unique division quaternion algebra over k_ν.

The following local-to-global result tells us that the splitting behaviour of $\mathcal{B}$ can be interpreted in terms completions.

Theorem 1.3.14. *The quaternion algebra $\mathcal{B}/K$ is split if and only if $\mathcal{B}_\nu$ is split for every place ν.*

The set of places at which $\mathcal{B}$ ramifies is finite and of even cardinality. Let us make the following important definition.

Definition 1.3.15. The set of places at which $\mathcal{B}$ is ramified is denoted by $\mathrm{Ram}(\mathcal{B})$ and the *discriminant* of $\mathcal{B}$ is the ideal

$$\prod_{\nu \text{ non-archimedean}} \mathfrak{p}_\nu$$

where $\mathfrak{p}_\nu$ is the prime ideal of K corresponding to ν.

Theorem 1.3.16. *The quaternion algebras $\mathcal{B}$ and $\mathcal{B}'$ over K are isomorphic if and only if $Ram(\mathcal{B}) = Ram(\mathcal{B}')$.*

Conversely, for any finite set of an even number of places of K there is a quaternion algebra ramified at exactly these places.

Chapter 2

Surfaces with quaternionic multiplication

In this chapter we shall discuss various aspects of abelian surfaces with quaternionic endomorphism algebras. We will give an overview of these objects and in the last section present a detailed analysis of the arithmetic of the associated Galois representations.

Furthermore, a thorough description of the family used to obtain the main result in the next chapter will be presented here. For the interested reader a collection of other families of QM-surfaces will be reviewed in brief.

Let K be an imaginary quadratic field. A simple abelian surface over K whose algebra of K-endomorphisms is an indefinite quaternion algebra over $\mathbb{Q}$ is commonly known as a *QM-abelian surface*, or just *QM-surface*. In 1970, in his Master's thesis, Y. Morita [Mor70] proved that modulo every prime of good reduction a QM-surface splits as the square of an elliptic curve over $\overline{\mathbb{F}}_p$. This was then improved by H. Yoshida [Yos73, Lemma 6] to show that in fact a QM-surface defined over a finite field is isogenous to the square of an elliptic curve over this finite field. The proof of this fact is shown in the next section and makes good use of J. Tate's work on abelian varieties over finite fields [Tat66].

This important fact means that whilst a QM-surface might not globally be a product of elliptic curves, locally it looks like one in the sense that it has similar arithmetic. In particular, the Euler polynomial of a QM-surface is the square of a quadratic polynomial with rational coefficients. This observation led J.-P. Serre to coin the name *fausses courbes elliptiques* [DR73, §0.7]. Hence these objects are commonly referred to as *false/fake elliptic curves* in the literature.

Whilst it was becoming clear what type of automorphic object a GL_2-type variety should correspond to, it was not until a few years later that this question would be considered for QM-surfaces. The novelty in this situation is compounded by the fact that QM-surfaces only exist over totally complex fields. Bianchi newforms - the automorphic objects for imaginary quadratic fields - were not well understood at the time.

The correspondence between QM-surfaces and Bianchi newforms was first observed by P. Deligne in a letter to J. Mennicke in 1979 [Del79]. The letter was written in response to the article [GM78], in which they assume the conjectural existence of a bijection between elliptic curves over imaginary quadratic fields and Bianchi newforms. The original manuscript remains unpublished (cf. [GHM78, §8] where this correspondence is discussed by the authors).

P. Deligne explains that he is ‘sceptical’ about being able to assert bijectivity since there should be a similar automorphic representation for abelian surfaces ‘with multiplication by a quaternion algebra D, split at ∞’. He also remarks that whilst it is not clear how to construct such a correspondence, D must be split by K because the associated Lie algebra is a K-linear 2-dimensional representation of D.

The first time this suggestion appears in the literature is in [EGM82, p.267], credited to P. Deligne (note the similar sounding article [EGMa82] by the authors from a similar time period). A concise version of the expected correspondence for QM-surfaces (and their generalisations to arbitrary dimension) is presented in [Tay95, Conjecture 3].

§ 2.1 Surfaces with quaternionic multiplication

In this brief subsection we introduce QM-surfaces and reproduce perhaps the most fundamental result relating to these objects.

For an abelian variety A defined over a field K, we will use the notation that the endomorphism rings $\mathrm{End}_K(A)$ and $\mathrm{End}_{\overline{K}}(A)$ will denote the endomorphisms of A which are defined over K and the algebraic closure $\overline{K}$ respectively. Throughout this chapter we shall assume that $\mathcal{O}$ is a maximal order in a rational indefinite quaternion algebra B.

As opposed to the name 'false/fake elliptic curve', we shall stick to the terminology: A/K is a QM-surface if $\mathrm{End}_K(A)$ contains $\mathcal{O}$ and A has *potential QM* if the action of $\mathcal{O}$ is defined over some extension of K. A more precise definition will be given below.

Remark 2.1.1. For reasons of parsimony we restrict throughout to working with maximal orders. For the most part this is because the work of B. Jordan [Jor86] on Galois representations attached to QM-surfaces is for maximal orders. Nonetheless, there are many results relating to non-maximal orders. For example [DR04] studies the field of definition of the quaternion action for so-called *hereditary orders*.

Let D denote the discriminant on B and recall $x \mapsto \overline{x}$ is the canonical anti-involution on B as in §1.3. For any positive anti-involution $^* : B \to B$, the Noether-Skolem theorem tells us that * is conjugate to the canonical involution, which means that there exists $\mu \in B^\times$ such that

$$x^* = \mu^{-1}\overline{x}\mu$$

for all $x \in B$. The trace of μ is zero so it satisfies a quadratic equation $\mu^2 + \delta = 0$ for some $\delta > 0$. In fact we can pick $\mu \in \mathcal{O}$ such that $\mu^2 + D = 0$ (see [Rot03] for more details).

Hence fix $^* : B \to B$ to be the positive anti-involution defined by $x \mapsto \mu^{-1}\overline{x}\mu$, where $\mu^2 + D = 0$. This element μ is often referred to as a principal polarization of $\mathcal{O}$. Recall that $\dagger : \mathrm{End}^0(A) \to \mathrm{End}^0(A)$ is a canonical anti-involution called the Rosati involution defined in Section 1.1.

Definition 2.1.2 (Cf. [Voi19], §43.6)**.** We say that A/K has quaternionic multiplication (QM) by $(\mathcal{O}, \mu)$ if there is an embedding $\iota : \mathcal{O} \hookrightarrow \mathrm{End}_K(A)$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\iota} & \mathrm{End}^0_K(A) \\ \downarrow{*} & & \downarrow{\dagger} \\ B & \xrightarrow{\iota} & \mathrm{End}^0_K(A) \end{array}$$

commutes.

The remainder of this subsection will be devoted to the proof of the striking fact that QM-surfaces split as the square of an elliptic curve modulo every prime of good reduction. The proof makes use of the following result.

Theorem 2.1.3 ([Tat66, Theorem 2])**.** *Let A be an abelian variety of dimension n over a field k of characteristic $p > 0$ and π the Frobenius endomorphism of A relative to k.*

1 *$\mathbb{Q}(\pi)$ is the centre of the semisimple algebra $\mathrm{End}_k^0(A)$. The algebra is commutative if and only if $\dim_{\mathbb{Q}}(\mathrm{End}_k^0(A)) = 2n$ and*

$$2n \leq \dim_{\mathbb{Q}}(\mathrm{End}_k^0(A)) \leq (2n)^2.$$

2 *The following are equivalent:*

- *The centre of $\mathrm{End}_k^0(A)$ is $\mathbb{Q}$;*
- *$\mathrm{End}_k^0(A) \simeq M_n(B_p)$, where B_p is the rational quaternion algebra of discriminant p;*
- *A is k-isogenous to the power of a super-singular elliptic curve whose endomorphisms are all defined over k.*

These properties can be used to show that the reduction of a QM-surface defined over a number field will split as the square of an elliptic curve for almost all primes because there is an injection of endomorphism rings under the natural reduction map.

Theorem 2.1.4 ([Yos73, Lemma 6])**.** *Let A be a QM-surface defined over a finite field $\mathbb{F}_q$ of characteristic p with $\mathcal{O} \subset \mathrm{End}_{\mathbb{F}_q}(A)$ and $p \nmid \mathrm{Disc}(\mathcal{O})$. Then A is isogenous to $E \times E$, where E is an elliptic curve over $\mathbb{F}_q$.*

Proof. It was shown by Y. Morita [Mor70] that A is isogenous to the square of an elliptic curve E' over $\overline{\mathbb{F}}_q$. Let $\mathcal{D} = \mathrm{End}_{\overline{\mathbb{F}}_q}^0(E')$. Then $\mathcal{D}$ is either isomorphic to a quadratic field F or to B_p, the rational quaternion algebra of discriminant p. We have that

$$B \subseteq \mathrm{End}_{\mathbb{F}_q}^0(A) \subseteq M_2(\mathcal{D}).$$

If $\mathbb{Q}(\pi) \simeq \mathbb{Q}$ then by the above theorem A is the square of a super-singular elliptic curve and we are done.

Suppose that $\mathbb{Q}(\pi) \neq \mathbb{Q}$. We know that $4 \leq \dim_{\mathbb{Q}}(\mathrm{End}^0_{\mathbb{F}_q}(A)) \leq 16$ but it cannot be the case that $\dim_{\mathbb{Q}}(\mathrm{End}^0_{\mathbb{F}_q}(A)) = 4$ or 16 because the centre is not $\mathbb{Q}$ so $\dim_{\mathbb{Q}}(\mathrm{End}^0_{\mathbb{F}_q}(A)) = 8$. Also since $\dim_{\mathbb{Q}}(\mathrm{End}^0_{\mathbb{F}_q}(A)) \neq 4$ we know that $\mathrm{End}^0_{\mathbb{F}_q}(A)$ is not commutative.

If A is $\mathbb{F}_q$-simple then $\mathrm{End}^0_{\mathbb{F}_q}(A)$ is a quaternion division algebra over $\mathbb{Q}(\pi)$ ramified at the places above p. But $B \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \simeq \mathrm{End}^0_{\mathbb{F}_q}(A)$ which contradicts the fact that $p \nmid \mathrm{Disc}(B)$. □

§ 2.2 Shimura curves

In analogy to modular curves, Shimura curves are moduli spaces for abelian surfaces with quaternionic multiplication.

We keep the notation that B is an indefinite rational division quaternion algebra. Fix an embedding

$$\iota_\infty : B \longrightarrow B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$$

and for a point $z \in \mathbb{H}$, set $v_z = (z \;\; 1)^t$ and construct the lattice $\Lambda_z = \iota_\infty(\mathcal{O})v_z$. Then there is an abelian surface $A_z = \mathbb{C}^2/\Lambda_z$. On Λ_z there is the Riemann form

$$E_z(\iota_\infty(\lambda_1)v_z, \iota_\infty(\lambda_2)v_z) = tr(\lambda_1^* \mu \lambda_2).$$

This defines a principal polarization ρ_z on A_z such that the Rosati involution on $\mathrm{End}(A_z)$ corresponds to $x \mapsto x^*$. Hence we have defined a triple

$$[A_z, \rho_z, \iota_z]$$

from a point $z \in \mathbb{H}$, where $\iota_z : \mathcal{O} \hookrightarrow \mathrm{End}(A_z)$.

There is an equivalence relation on triples governed by $[A_{z_1}, \rho_{z_1}, \iota_{z_1}] \sim [A_{z_2}, \rho_{z_2}, \iota_{z_2}]$ if there is an isomorphism $\phi : A_1 \to A_2$ such that $\phi^*(\rho_2) = \rho_1$ and the isomorphism commutes with the action of $\iota_{z_1}(\mathcal{O})$ and $\iota_{z_2}(\mathcal{O})$.

Let

$$\Gamma^1(\mathcal{O}) = \iota_\infty(\mathcal{O}^1)/\{\pm 1\} \leq \mathrm{PSL}_2(\mathbb{R})$$

and define the quotient

$$X^1 = \Gamma^1(\mathcal{O})\backslash\mathbb{H}.$$

It is compact since B is non-split.

Theorem 2.2.1. *There is a bijection*

$$X^1 \longleftrightarrow \left\{ \begin{array}{c} \textit{Principally polarized abelian} \\ \textit{surfaces with QM by } \mathcal{O} \\ \textit{up to isomorphism} \end{array} \right\}$$
$$\Gamma^1(\mathcal{O})z \longmapsto \{[A_z, \rho_z, \iota_z]\}.$$

Proof. See [Voi19, Theorem 43.6.14] for a detailed exposition of this fact. □

It has been mentioned already that QM-surfaces only occur over totally complex fields. This result is due to Shimura, encapsulated in the following theorem.

Theorem 2.2.2. *If B is a division algebra then*

$$X^1(\mathbb{R}) = \emptyset.$$

Proof. This is a special case of a more general question considered in [Shi75].

There is an element $\eta \in \mathcal{O}^\times$ such that $\mathrm{nm}(\eta) = -1$. This means that $\eta^2 \in \mathcal{O}^1$ and η naturally defines an anti-holomorphic involution on X^1.

Suppose $X^1(\mathbb{R}) \neq \emptyset$. Then there is $z \in \mathbb{H}$ such that

$$z = \iota_\infty(\eta) \cdot z = \frac{a\overline{z} + b}{c\overline{z} + d}$$

where $\iota_\infty(\eta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Hence $a\overline{z} + b = c|z|^2 + dz$. Equating imaginary parts we deduce that $\mathrm{tr}(\eta) = a + d = 0$ and hence η satisfies the polynomial $x^2 - 1 = 0$. Since B is indefinite we conclude that $\eta = \pm 1$, which is a contradiction. □

One of the striking facts about Shimura curves is that, despite the fact that they have no real points, the quotient X^1 is indeed a curve with a model defined over the rationals.

Theorem 2.2.3 ([Shi67])**.** *There exists a projective non-singular curve $X^1_{\mathbb{Q}}$ defined over the rationals and an isomorphism*

$$\phi : \Gamma^1(\mathcal{O})\backslash\mathbb{H} \longrightarrow X^1_{\mathbb{Q}}(\mathbb{C}).$$

§ 2.3 Families of surfaces with quaternionic multiplication

Here we present a list of families that specialise to give hyperelliptic curves whose Jacobians have quaternionic multiplication. Whilst it transpired that only one of the families was needed for the main results in Chapter 3, we display others found in the literature in the hope that the interested reader might find it useful. Since the Baba-Granath family played an important part in finding explicit models for the *genuine* QM-surfaces in this thesis, a more detailed overview is given.

2.3.1 Baba-Granath

In [BG08] the authors construct genus 2 curves whose Jacobians are surfaces with quaternionic multiplication in the case of discriminant 6 and 10. An overview of some of the computational aspects used in this thesis is discussed in §3.4. In this subsection we describe how the families were derived. We restrict to the case of discriminant 6 since the discriminant 10 case is similar.

The distinct advantage of this family is that the genus 2 curves are directly related to the moduli space via a parameterisation. This parameterisation is encapsulated using a parameter j which has arithmetic properties that allow us to search for the QM-surfaces we wish to find.

Let $\mathcal{O}$ be the maximal order of the rational quaternion algebra B_6 of discriminant 6. As explained in the previous section, the quotient $V_6 = \mathbb{H}/\mathcal{O}^1$ is a Shimura curve which is the moduli space for surfaces with QM by $\mathcal{O}$. The strategy of the authors was to make a correspondence between points $z \in \mathbb{H}$ and genus 2 curves whose Jacobians have QM.

Let $\mathcal{M}_2$ denote the moduli space of genus 2 curves and $\mathcal{A}_2$ denote the moduli space of principally polarized abelian surfaces. The Torelli map is an injective map $\mathcal{M}_2 \to \mathcal{A}_2$ whose image is Zariski open. If $\tilde{E}$ is the image of the natural map $V_6 \to \mathcal{A}_2$ and E is the intersection with $\mathcal{M}_2$ then there is the following picture

$$\begin{array}{ccccc} & & \mathcal{A}_2 & \longleftarrow & \mathcal{M}_2 \\ & & \uparrow & & \uparrow \\ V_6 & \longrightarrow & \tilde{E} & \longleftarrow & E. \end{array}$$

The starting point is to take the QM-family derived in [HM95] (cf. QM-family 1). It is a hyperelliptic curve $Y^2 = f(X; t, s) \in \overline{\mathbb{Q}}(t, s)[X]$ subject to the parameters satisfying a quadratic equation $g(s, t) = 4s^2t^2 - s^2 + t^2 + 2 = 0$. The authors write down two simultaneous equations in the Igusa invariants $[J_2 : J_4 : J_6 : J_{10}]$ which are satisfied by the above curve. In particular, this allows them to get an embedding $E \to \mathbb{P}^1$ and in this way define an arithmetic j-function $j : E \to \mathbb{P}^1 \backslash \{0, 1\}$.

Now we wish to define an analytic j-function that coincides with this one. Let $\tilde{\Gamma}$ denote the normalizer group $N_{B+}(\mathcal{O})$ and Γ be the subgroup of this generated by elements of norm 1. The space of holomorphic weight k forms for Γ is denoted by $S_k(\Gamma)$. Then $S_4(\Gamma)$ is generated by a form $h_4(z)$ and likewise $S_6(\Gamma)$ is generated by $h_6(z)$. $S_{12}(\Gamma)$ is generated by h_4^3, h_6^2 and h_{12}. These satisfy $h_{12}^2 + 3h_6^4 + h_4^6 = 0$.

The authors prove that as a graded ring

$$\bigoplus_{k=0}^{\infty} S_{2k}(\Gamma) \simeq \mathbb{C}[h_4, h_6, h_{12}]/(h_{12}^2 + 3h_6^4 + h_4^6).$$

They define the weight 0 modular form

$$j_m = \frac{4h_6^2}{3h_4^3}.$$

It is an isomorphism $j_m : V_6 \to \mathbb{P}^1$.

The authors then use knowledge of two abelian surfaces with CM that correspond to CM points on the Shimura curve and by computing the value of j at these points it is deduced that $j = j_m^2$ as functions on $\tilde{E}$ [BG08, Proposition 3.9]. Hence the map $f : V_6 \to X_6 = \{X^2 + 3Y^2 + Z^2 = 0\}$ is an isomorphism given by

$$f(z) = [h_4(z)^3 : h_6(z)^2 : h_{12}(z)] = [4 : 3\sqrt{j} : \sqrt{-27j - 16}].$$

Define the genus 2 curve

$$\begin{aligned} C_j : y^2 = &(-4 + 3s)x^6 + 6tx^5 + 3t(28 + 9s)x^4 - 4t^2x^3 \\ &+ 3t^2(28 - 9s)x^2 + 6t^3x - t^3(4 + 3s), \end{aligned}$$

where $t = -2(27j + 16)$ and $s = \sqrt{-6j}$. Then the Jacobian of C_j has QM of discriminant 6.

2.3.2 QM-families

Here were present a few families of QM-surfaces found in the literature.

QM-family 1 ([HM95]). *The first family of QM-surfaces was derived by K. Hashimoto and N. Murabayashi in the form of a hyperelliptic curve*

$$Y^2 = f(X;t,s) \in \overline{\mathbb{Q}}(t,s)[X].$$

Let B be a rational indefinite quaternion algebra. The strategy was to embed the Shimura curve S_B into the moduli space of principally polarized abelian surfaces

$$S_B \longrightarrow \mathcal{A}_2(\mathbb{C}) \simeq Sp(4,\mathbb{Z})\backslash\mathbb{H} \simeq \mathcal{M}_2(\mathbb{C})$$

and describe the the image in the moduli space $\mathcal{M}_2(\mathbb{C})$ of genus 2 curves via the Torelli map.

This relates to classical work of Humbert who approached this for surfaces with real multiplication. The authors then show that if two real multiplications generate the maximal order $\mathcal{O} \subset B$ then the desired fibre space will be a component of the intersection of two Humbert surfaces. This was done explicitly for the the quaternion algebras of discriminant 6 and 10.

For discriminant 6 this is given by

$$\mathcal{S}_6(t,s) : Y^2 = X(X^4 + (A-B)X^3 + QX^2 + (A+B)X + 1);$$
$$\textit{where} \quad A = \frac{s}{2t}, \quad B = \frac{1+3t^2}{1-3t^2},$$
$$Q = -\frac{(1-2t^2+9t^4)(1-28t^2+166t^4-252t^6+81t^8)}{4t^2(1-3t^2)^2(1-t^2)(1-9t^2)}$$

and

$$s^2 + 3 - 14t^2 + 27t^4 = 0.$$

This is in fact a variant [HT99] of the original family, which has the advantage that the quaternionic multiplication is defined over the field of definition $\mathbb{Q}(\sqrt{-3+14t^2-27t^4})$.

This allowed the authors to produce the first known examples of geometrically simple QM-surfaces.

QM-family 2 ([PS11]). *Define the curve*

$$C(\lambda) : w^3 = z(z-1)(z-\lambda_1)(z-\lambda_2)$$

where

$$\lambda = (\lambda_1, \lambda_2) \in \Lambda = \{\ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2) \neq 0\ \}.$$

Every period of $C(\lambda)$ *is a* $\mathbb{Z}[\zeta_3]$*-linear combination of three periods* $\{\eta_1, \eta_2, \eta_3\}$ *given explicitly. One defines the matrix*

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and chooses a vector $c \in \mathbb{Z}[\zeta_3]^3$ *such* $cH^t\overline{c} > 0$*. Then the authors prove that*

$$Jac(C(\lambda)) \simeq E_0 \times A(\lambda)$$

where $E_0 \simeq \mathbb{C}^2/(\mathbb{Z} + \eta_3\mathbb{Z})$ *is an elliptic curve and* $A(\lambda)$ *is a QM-surface with quaternionic multiplication by*

$$\left(\frac{-3, \langle c, c\rangle_H}{\mathbb{Q}}\right).$$

QM-family 3 ([DFL$^+$16]). *Let* i, j, k, N *be integers such that* $1 \leq i, j, k \leq N$*. For a fixed* λ *the generalized Legendre curve is defined by*

$$C(\lambda)^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^k.$$

The authors let $J(\lambda)^{new}$ *denote the abelian variety of dimension* $\varphi(N)$ *which is the primitive part of* $\mathrm{Jac}(C(\lambda)^{[N;i,j,k]})$*.*

Fix the notation

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

where $\Gamma(s)$ *is the the standard gamma function. Then the main result that the authors prove is that for* $N = 3, 4, 6$ *and* $N \nmid i + j + k$*, the endomorphism algebra of* $J(\lambda)^{new}$ *contains a quaternion algebra if and only if*

$$B\Big(\frac{N-i}{N}, \frac{N-j}{N}\Big)/B\Big(\frac{k}{N}, \frac{2N-(i+j+k)}{N}\Big) \in \overline{\mathbb{Q}}.$$

§ 2.4 Galois representations attached to abelian surfaces with quaternionic multiplication

In this section we describe the Galois representation attached to a QM surface. In particular, we shall investigate the arithmetic properties of the representations in the case that the prime ℓ divides the discriminant of the quaternion algebra. This will be necessary for Chapter 4, where we prove modularity of the QM-surfaces using the Faltings-Serre-Livné method at the prime $\ell = 2$.

Let K be an imaginary quadratic field and A/K be a QM surface with $\mathcal{O} \hookrightarrow \mathrm{End}_K(A)$ a maximal order in the quaternion algebra $B/\mathbb{Q}$. As in §1.1.2 the Tate module is denoted

$$T_\ell A = \lim_{\leftarrow n} A[\ell^n] \quad \text{and} \quad V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

We will write $\sigma_\ell : G_K \longrightarrow GL_4(\mathbb{Z}_\ell)$ for the representation coming from the action of G_K on $T_\ell A$.

Denote by $\mathcal{O}_\ell = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $B_\ell = B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Proposition 2.4.1. *For each prime ℓ the Tate module $T_\ell A$ is free of rank 1 as a left $\mathcal{O}_\ell$-module.*

Proof. See [Oht74]. It was also found independently by [Jak74]. □

Recall that for the ring $R \subset \mathrm{End}_R(A)$ and the natural map $\iota : R \hookrightarrow \mathrm{End}(T_\ell A)$ we define

$$\mathrm{End}_R(T_\ell A) = \{ \ \phi \in \mathrm{End}(T_\ell A) \mid \phi \circ \iota(r) = \iota(r) \circ \phi \ \forall r \in R \ \}.$$

It follows from the above proposition that $\mathrm{Aut}_{\mathcal{O}}(T_\ell A) \simeq \mathcal{O}_\ell^\times$ where $\mathcal{O}_\ell^\times$ acts on $T_\ell A$ by right multiplication.

The action of G_K commutes with $\iota(R)$ and so there is an associated ℓ-adic representation

$$\rho_\ell : \ G_K \longrightarrow \mathrm{Aut}_{\mathcal{O}}(T_\ell A) \simeq \mathcal{O}_\ell^\times \subseteq B_\ell^\times.$$

Furthermore, the ρ_ℓ form a strictly compatible system of ℓ-adic representations [Jor86, §5]. The image $\rho_\ell(G_K)$ is an open subgroup of $\mathrm{Aut}_{\mathcal{O}}(T_\ell A)$ and is surjective for almost all ℓ [Oht74].

Theorem 2.4.2. *If $\ell \nmid Disc(B)$ this precisely means that $\mathcal{O}_\ell^\times \simeq GL_2(\mathbb{Z}_\ell)$ and in this case there is a decomposition*

$$\sigma_\ell \simeq \rho_\ell \oplus \rho_\ell.$$

Proof. This is a specialisation of [Chi90, Theorem A] to abelian surfaces, which states that as a G_K-module the Tate module of an abelian variety of 'Type II' is the sum of two isomorphic submodules.

The prime ℓ does not divide the discriminant of B, so fix an isomorphism $B_\ell \longrightarrow M_2(\mathbb{Q}_\ell)$. Under this identification, $M_2(\mathbb{Q}_\ell)$ acts on $V_\ell(A)$ and commutes with the action of G_K.

Let t and v be two elements of $M_2(\mathbb{Q}_\ell)$ such that

$$t^2 = v^2 = 1 \quad \text{and} \quad tv = -vt.$$

For example

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Write $e = 1/2(I + t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and put $U = eV_\ell(A)$ and $W = (1 - e)V_\ell(A)$. From the identity $(t - 1)(t + 1) = t^2 - 1 = 0$ one sees that U is the eigenspace for the eigenvalue 1 of t. Similarly, W is the eigenspace for -1 of t. Furthermore,

$$v : U \longrightarrow W$$

is an isomorphism due to the fact that $(t+1)(v)(1+t) = tv+tvt+v+vt = v(1-t^2) = 0$ and the inverse is $v^{-1} = v$. Hence we can conclude that $V_\ell(A) \simeq U \oplus W$ as G_K-modules and $\text{Dim}(U) = \text{Dim}(W) = 1/2\,\text{Dim}(V_\ell(A)) = 2$. □

The reason we can pick v and t with the desired properties in the proof above is because of the description of B_ℓ as a quaternion algebra: $M_2(\mathbb{Q}_\ell) \simeq \left(\frac{1,1}{\mathbb{Q}_\ell}\right)$. This would not be possible if $\ell | \text{Disc}(B)$, although if we extend scalars by a field that splits B_ℓ we would obtain the same result.

For the remainder of the section let ℓ be a prime that divides $\text{Disc}(B)$. This means that ℓ is ramified in B and so B_ℓ is isomorphic to the unique division quaternion algebra over $\mathbb{Q}_\ell$. It can be represented as

$$\left(\frac{\pi, u}{\mathbb{Q}_\ell}\right) \simeq \mathbb{Q}_\ell \cdot 1 + \mathbb{Q}_\ell \cdot i + \mathbb{Q}_\ell \cdot j + \mathbb{Q}_\ell \cdot ij;\ i^2 = u,\ j^2 = \pi;$$

where π is the uniformiser of $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell(\sqrt{u})$ is the unique unramified quadratic extension of $\mathbb{Q}_\ell$.

Any quadratic extension of $\mathbb{Q}_\ell$ splits the ramified quaternion algebra. So let us denote $L = \mathbb{Q}_\ell(\sqrt{u})$ and R_L as its ring of integers. Then $B \otimes_{\mathbb{Q}_\ell} L \simeq M_2(L)$ and there is an explicit isomorphism of $\mathbb{Q}_\ell$-algebras

$$B_\ell \simeq \left\{ \begin{pmatrix} \alpha & \beta \\ \pi\beta' & \alpha' \end{pmatrix} \mid \alpha, \beta \in L,\ ' : L \to L \text{ is conjugation in } L/\mathbb{Q}_\ell \right\} \subseteq M_2(L); \quad (2.1)$$

$$i \mapsto \begin{pmatrix} \sqrt{u} & 0 \\ 0 & -\sqrt{u} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}. \quad (2.2)$$

It will often be useful to consider the representation ρ_ℓ as having its image in $M_2(L)$.

Define $\lambda_\ell \subseteq \mathcal{O}$ to be the unique two-sided ideal of reduced norm ℓ such that $\lambda_\ell^2 = (\ell)$.

Proposition 2.4.3. *If $\ell | \operatorname{Disc}(B)$ then $A[\lambda_\ell]$ is the unique proper $\mathcal{O}$-submodule of $A[\ell]$ and it has order ℓ^2.*

Proof. As explained in [Jor86, §4], $A[\ell]$ is free of rank 1 over the algebra $\mathcal{O}/\ell$ and there is a bijection between non-zero proper $\mathcal{O}$-submodules of $A[\ell]$ and non-zero proper left ideals of $\mathcal{O}/\ell$. Since $\mathcal{O}/\ell$ has exactly one non-zero proper left ideal the result follows. □

In the terminology of B. Jordan, $A[\lambda_\ell]$ is called the *canonical torsion subgroup.*

Proposition 2.4.4. *The torsion subgroups $A[\ell]$ and $A[\lambda_\ell]$ are free of rank 1 over the $\mathbb{F}_\ell$-algebras $\mathcal{O}/\ell$ and $\mathcal{O}/\lambda_\ell$ respectively. Explicitly, these have the structure*

$$\mathcal{O}/\ell \simeq \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^\ell \end{smallmatrix} \right) \mid \alpha, \beta \in \mathbb{F}_{\ell^2} \right\} \subseteq M_2(\mathbb{F}_{\ell^2}),$$
$$\mathcal{O}/\lambda_\ell \simeq \mathbb{F}_{\ell^2}.$$

Proof. This is because $T_\ell A$ is free of rank 1 as a left $\mathcal{O}_\ell$-module and the structure is deduced from (2.1). □

Denote the residual representations by

$$\begin{aligned} \overline{\tau}_\ell &: G_K \longrightarrow \mathrm{Aut}_{\mathcal{O}}(A[\ell]) \leq \mathrm{GL}_2(\mathbb{F}_{\ell^2}), \\ \overline{\rho}_\ell &: G_K \longrightarrow \mathrm{Aut}_{\mathcal{O}}(A[\lambda_\ell]) \simeq \mathbb{F}_{\ell^2}^\times. \end{aligned}$$

One can think of $\overline{\rho}_\ell$ as a character with the following property:

Proposition 2.4.5 ([Oht74]). *Let K^{ab} denote the abelian closure of K in $\overline{K}$. Then there is a commutative diagram*

$$\begin{array}{ccc} G_K & \xrightarrow{\overline{\rho}_\ell} & \mathbb{F}_{\ell^2}^\times \\ & \searrow^{\chi_\ell} & \downarrow{\scriptstyle N_{\mathbb{F}_{\ell^2}/\mathbb{F}_\ell}} \\ & & \mathbb{F}_\ell^\times \end{array}$$

where $\chi_\ell : \mathrm{Gal}(K^{ab}/K) \to \mathbb{F}_\ell^\times$ is the ℓ-cyclotomic character.

As noted above, the representations $\{\rho_\ell\}$ form a strictly compatible system of representations with values in the algebraic group H defined such that $H(\mathbb{Q}) = B^\times$. The set of primes S of bad reduction for A is the smallest set such that ρ_ℓ is unramified at every prime not in S and any prime above ℓ.

For an element $\sigma \in G_K$ we define

$$P_\ell(\sigma) = N_{B_\ell/\mathbb{Q}_\ell}(1 - \rho_\ell(\sigma)t) \in \mathbb{Q}_\ell[t].$$

At Frobenius elements F_v with $v \notin S$ the Hecke polynomial is given by

$$P_\ell(F_v) = N_{B_\ell/\mathbb{Q}_\ell}(1 - \rho_\ell(F_v)t) = 1 - a_v t + N_v t^2.$$

The polynomial $P_\ell(F_v)$ is independent of ℓ and has integer coefficients.

We can attach an L-series by setting

$$L_\rho(s) = \prod_{v \notin S} P_\ell(F_v)(N_v^{-s})^{-1}$$

and $L(A/K, s) = L_\rho(s)^2$ [Jor86, Proposition 5.1].

There is a commutative diagram:

$$\begin{array}{ccc} G_K & \xrightarrow{\rho_\ell} & \mathrm{Aut}_{\mathcal{O}}(T_\ell A) \\ & \overset{\overline{\rho}_\ell}{\searrow} & \downarrow \\ & & \mathrm{Aut}_{\mathcal{O}}(A[\lambda_\ell]). \end{array}$$

Hence we see that

$$P_{\rho_\ell}(F_v) \bmod \ell = (1 - a_v t + N_v t^2) \bmod \ell = (1 - \overline{\rho}_\ell(F_v)t)(1 - \overline{\rho}_\ell(F_v)^\ell t).$$

Under the quaternion algebra identification (2.1) and projecting as in the commutative diagram, the image of $\overline{\rho}_\ell$ can be thought of to lie in $\mathrm{GL}_2(\mathbb{F}_{\ell^2})$. We describe its image under this identification. First though, we note that the image can be assumed to lie in $\mathrm{GL}_2(\mathbb{F}_\ell)$ due to the following lemma.

Lemma 2.4.6 ([Jon16, Lemma 3.1])**.** *Let $\rho : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a Galois representation with rational traces of Frobenius and $\overline{\rho} : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be the residual representation. Then there exists an element $t \in \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ such that $t\overline{\rho}(g)t^{-1} \in \mathrm{GL}_2(\mathbb{F}_\ell)$ for all $g \in G_K$.*

We give a precise description of the image of the residual representation.

Proposition 2.4.7. *As a representation $\overline{\rho}_\ell : G_K \longrightarrow \mathrm{GL}_2(\mathbb{F}_\ell)$ the image is contained in the non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$ up to conjugation, i.e. the unique cyclic subgroup of order $\ell^2 - 1$.*

Proof. The traces of Frobenius are all integral hence we can assume that the image lies in $\mathrm{GL}_2(\mathbb{F}_\ell)$ rather than $\mathrm{GL}_2(\mathbb{F}_\ell^2)$ by the previous lemma. Then note that $\mathbb{F}_{\ell^2}$ is a 2-dimensional $\mathbb{F}_\ell$ vector space and there is a natural map $\mathbb{F}_{\ell^2}^\times \to \mathrm{Aut}_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^2}) \simeq \mathrm{GL}_2(\mathbb{F}_\ell)$ given by left multiplication. $\square$

It may be desirable to be able to find the image of $\overline{\rho}_\ell$ from prior knowledge of $\overline{\tau}_\ell$, which is often more amenable to computation. The images of the two representations are related in the following way.

Theorem 2.4.8. *Let A be a geometrically simple abelian surface defined over an imaginary quadratic field K such that $\mathrm{End}_K(A)$ is isomorphic to a maximal order $\mathcal{O}$ in an indefinite quaternion algebra $B/\mathbb{Q}$. Suppose that the prime ℓ divides $\mathrm{Disc}(B)$ and $\overline{\tau}_\ell, \overline{\rho}_\ell$ are the residual Galois representations on the torsion* *subgroups* $A[\ell]$ *and* $A[\lambda_\ell]$ *respectively. Then there is a short exact sequence of groups*

$$1 \longrightarrow \overline{\epsilon} \longrightarrow Im(\overline{\tau}_\ell) \longrightarrow Im(\overline{\rho}_\ell) \longrightarrow 1,$$

where $\overline{\epsilon} \leq \mathbb{F}^+_{\ell^2}$.

Proof. Since $\mathrm{Aut}_{\mathcal{O}}(A[\ell]) \simeq (\mathcal{O}/\ell)^\times$ and $\mathrm{Aut}_{\mathcal{O}}(A[\lambda_\ell]) \simeq (\mathcal{O}/\lambda_\ell)^\times \simeq \mathbb{F}^\times_{\ell^2}$ it is enough to show that there is a short exact sequence

$$1 \longrightarrow \mathbb{F}^+_{\ell^2} \longrightarrow (\mathcal{O}/\ell)^\times \longrightarrow \mathbb{F}^\times_{\ell^2} \longrightarrow 1.$$

Let r be the projection $r : (\mathcal{O}/\ell)^\times \to (\mathcal{O}/\lambda_\ell)^\times$. Then $\ker(r)$ consists of the cosets $\phi + \ell$ such that $\phi \in \lambda_\ell + 1$. It follows that $\ker(r) \simeq (1 + \lambda_\ell)/(1 + \ell)$ which is isomorphic to $\mathbb{F}^+_{\ell^2}$. □

Remark 2.4.9. If F is a weight 2 Bianchi newform with rational coefficients which corresponds to a QM surface, then by definition the attached Galois representations are isomorphic. It is worth noting that this means for primes ℓ dividing the discriminant of the acting quaternion algebra, the residual representation $\overline{\rho}_{F,\ell}$ will have cyclic image and $\rho_{F,\ell}$ cannot have its coefficient field conjugated into $\mathbb{Q}_\ell$.

Given a weight 2 Bianchi newform with rational coefficients, it would be desirable to have a criterion which determines whether f should correspond to an elliptic curve or a QM surface which can be determined from computing the trace of Frobenius for a finite set of primes. More generally, given an automorphic object how can we determine whether it should correspond to a variety with quaternionic multiplication?

Chapter 3

Genuine aspects

This chapter is concerned with presenting the most novel aspect of this thesis, namely that *genuine* QM-surfaces exist. The term genuine is used to indicate that they do not arise via base-change.

The motivation for this is that if we take a non-CM newform f with quadratic coefficient field which has a non-trivial inner twist (σ, χ_K), then the base-change to K of the abelian surface A_f has a quaternionic multiplication over K. This provides us with a source of QM-surfaces and was first studied in [Cre92]. Hence it is natural to ask whether this is the only way QM-surfaces arise over imaginary quadratic fields (see Question 3.2.3). In this chapter we show that it is not necessarily the case that QM-surfaces only arise in this way.

Included are four explicit examples of QM-surfaces which are genuine. Also discussed will be the computational methods used to find these examples and their relation to the Paramodularity Conjecture.

§ 3.1 Base change

Base-changing is a procedure that one can take on geometric objects, but this will naturally have a counterpoint on the automorphic side, both of which we need to consider. Whilst the geometric viewpoint is straightforward, deep work of Langlands is necessary to establish the automorphic behaviour under base-change.

3.1.1 Base change

Let A be an abelian variety defined over the field K. For a field extension L/K we say that $A \otimes_K L$ is the base-change of A from K to L. If A is the base-change of an abelian variety from a smaller field we simply say that A is *base-change*.

Slightly more generally, we will have to consider abelian varieties which are a *twist of base-change*. By a twist of A we mean an abelian variety A_ξ/K defined by the element $\xi \in H^1(\mathrm{Gal}(L/K), \mathrm{Aut}_K(A))$, which comes with an isomorphism $\theta : A_\xi \to A$ defined over L such that

$$\xi_\sigma = \theta^\sigma \circ \theta^{-1} \quad \text{for all } \sigma \in \mathrm{Gal}(L/K).$$

So we say that A is a twist of base-change if there is a twist A_ξ of A which is base-change. For a discussion on twists of abelian varieties see [Kid95].

Now we give a precise account of lifting modular forms to imaginary quadratic fields [Asa78, GL79]. This can be described in the classical language of automorphic forms (cf. [Lan80, AC89]), however, we wish to be as explicit as possible.

Let f be a cusp form of weight 2 for $\Gamma_0(N)$ and K an imaginary quadratic field. Then f lifts to a cusp form F of weight 2 for $\Gamma_0(\mathfrak{n})$ over K as in §1.2. We call such a form F *base-change* and if a form G is the twist of the form $F \otimes \psi$ for some character ψ, we say that G is a *twist of base-change*.

We can describe the lifting in terms of the coefficients. Suppose that f is a newform. Then the coefficients of the lift of f to K are indexed by prime ideals $\mathfrak{p}$ and for $\mathfrak{p} \nmid \mathfrak{n}$ these are given by

$$c_\mathfrak{p} = \begin{cases} a_p & \text{if } \chi_K(p) = 1, \text{ i.e. if } p \text{ is split in } \mathcal{O}_K; \\ a_p^2 - 2p & \text{if } \chi_K(p) = -1, \text{ i.e. if } p \text{ is inert in } \mathcal{O}_K; \\ a_p & \text{if } \chi_K(p) = 0, \text{ i.e. if } p \text{ is ramified in } \mathcal{O}_K, \end{cases}$$

where $\mathfrak{p}$ is the prime ideal over p and χ_K is the quadratic character of K. A cusp form of weight 2 for $\Gamma_0(\mathfrak{n})$ is base-change if and only if $c_\mathfrak{p} = c_{\bar{\mathfrak{p}}}$ for all primes $\mathfrak{p}$.

The Hecke polynomial of f at p can be written as

$$X^2 - a_p X + p = (X - \alpha)(X - \beta).$$

If p is inert in $\mathcal{O}_K$, then considering the trace of Frobenius we can explain the above relation by the fact that $a_\mathfrak{p} = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = a_p^2 - 2p$.

Recall that f has CM by K if $f \otimes \chi_K = f$. If this were the case then the lift of f to K would be an Eisenstein series. However, if f does not have CM by K then the lift of f is also a newform. Note that f and its twist $f \otimes \chi_K$ lift to the same form.

We can describe the level of the base-change form exactly (see [Ţur18, Lemma 5.2]). Suppose that the classical newform f of weight 2 has level N_1 and its twist $f \otimes \chi_K$ has level N_2. Then f and $f \otimes \chi_K$ lift to the same newform over K of level $\mathfrak{n}$ with

$$N_{K/\mathbb{Q}}(\mathfrak{n}) = \frac{N_1 N_2}{\Delta_K^2}.$$

3.1.2 Inner twists

Let $f = \sum a_n q^n$ be a newform of weight 2 for $\Gamma_0(N)$ with nebentypus ϵ. Denote by

$$K_f = \mathbb{Q}(\{a_n\})$$

the coefficient field of f. It is well known that K_f is a number field. For every automorphism σ of K_f there is a newform of the same level defined as

$$f^\sigma = \sum \sigma(a_n) q^n.$$

The collection of newforms obtained from all automorphisms of K_f is called the *Hecke orbit* of f.

Definition 3.1.1. Suppose that $f = \sum a_n q^n \in S_2(\Gamma_0(N), \epsilon)$ is a newform and there is a Dirichlet character χ such that

$$\sigma(a_p) = \chi(p) a_p$$

for almost all primes p. Then f has an *inner twist* by (σ, χ).

This is of course the same requirement that f^σ and $f \otimes \chi$ are the same newform. We will assume from now on that f does not have complex multiplication.

The 'inner twist' phenomenon was first studied by K. Ribet [Rib80] with an emphasis on the effect on endomorphism algebras and also independently by F. Momose [Mom81]; many of the facts we recall here are first seen in these papers.

It is clear from comparing nebentypus that we have the relation $\epsilon\chi^2 = \sigma(\epsilon)$. The conductor of χ is only divisible by the primes dividing N and if ϵ is trivial then χ must be a quadratic character.

The field K_f is either totally real or a CM field depending on whether ϵ is trivial or non-trivial respectively. Now let Γ be the group of automorphisms of K_f which give rise to an inner twist. Then Γ is an abelian group. The fixed field $F = K_f^\Gamma$ is a totally real field generated by

$$\{a_p^2/\epsilon(p)\}_{p\nmid N}.$$

If the level N is square-free and ϵ is trivial then f has no inner twists. In fact we can be even more precise: the primes p dividing the level of a newform with trivial nebentypus and an inner twist have exponent [Has96, Proposition 2]

$$\begin{cases} 2 \leq \nu_p(N) \leq 10 & \text{if } p = 2; \\ 2 \leq \nu_p(N) \leq 5 & \text{if } p = 3; \\ \nu_p(N) = 2 & \text{if } p \geq 5. \end{cases}$$

Furthermore, N is divisible by 2^5 or a prime p such that $p \equiv 3 \bmod 4$.

For the newform $f \in S_2(\Gamma_0(N), \epsilon)$, Shimura attached to f an abelian variety A_f that is of GL_2-type [Shi71]. It has dimension equal to $[K_f : \mathbb{Q}]$ and $\mathrm{End}^0_{\mathbb{Q}}(A_f) \simeq K_f$. The variety A_f factors up to isogeny over $\overline{\mathbb{Q}}$ as a product of r copies of a simple abelian variety B_f. These B_f are called building blocks as introduced by [Rib94, Pyl04]. The group of inner twists measures how A_f factors into its building blocks and the field extensions over which this splitting happens.

The endomorphism algebra $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B_f)$ is a division algebra which is either isomorphic to the totally real field F and $\dim(B_f) = [F : \mathbb{Q}]$ or a quaternion algebra over F and $\dim(B_f) = 2[F : \mathbb{Q}]$. For a discussion on the field of definition of the building blocks and how these are computed see [Que09].

We will be interested in the case where $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B_f)$ is a quaternion algebra over $\mathbb{Q}$ and the quaternionic action descends to an imaginary quadratic field. Hence, suppose that ϵ is trivial, $K_f = \mathbb{Q}(\sqrt{m})$ is a (real) quadratic field and f has an inner twist by (σ, χ) where $\langle\sigma\rangle = \mathrm{Gal}(K_f/\mathbb{Q})$.

Denote by $K = \mathbb{Q}(\sqrt{d})$ the quadratic field associated to χ. Now A_f is an abelian surface over $\mathbb{Q}$. Its full endomorphism algebra is described in [Cre92] as the quaternion algebra

$$\mathrm{End}^0_{\mathbb{Q}}(A_f) \simeq \mathrm{End}^0_K(A_f) \simeq \left(\frac{m, d}{\mathbb{Q}}\right).$$

Hence A_f is geometrically a QM-surface if and only if $\left(\frac{m,d}{\mathbb{Q}}\right)$ is non-split, otherwise it is a product of two elliptic curves.

Proposition 3.1.2. *If A_f is geometrically a QM-surface then K is imaginary quadratic.*

Proof. This is [Cre92, Theorem 2]. □

Remark 3.1.3. There is a conjecture attributed to R. Coleman that states that up to bounded dimension g and degree d there should be only finitely many rings that occur as the endomorphism ring of an abelian variety of dimension g over a number field of degree d. For the case of abelian surfaces with quaternionic multiplication this is investigated in [BFGR06, Rot08]. More recently it has been shown how this relates to other conjectures in arithmetic geometry [OSZ18].

§ 3.2 Genuine QM-surfaces and newforms

Let K be an imaginary quadratic field.

Definition 3.2.1. We say that a Bianchi newform is *genuine* over K if it is not (the twist of) base-change of a classical newform over $\mathbb{Q}$. Similarly, an abelian variety is defined to be genuine over K if it is not a simple factor of a twist of base-change.

As predicted by the Langlands programme, there is a conjectural connection between Bianchi newforms and QM-varieties [Tay95, Conjecture 3].

Conjecture 3.2.2. *Let F be a weight 2 Bianchi newform over K of level $\mathfrak{n}$ with coefficient field E_F and $[E_F : \mathbb{Q}] = n$. Then there is a $2n$-dimensional abelian variety A_F defined over K with* $\mathrm{Cond}(A) = \mathfrak{n}^{2n}$ *and quaternionic multiplication by a (possibly split) quaternion algebra over E_F such that*

$$L(A_F, s) = \prod_{\sigma : E_F \hookrightarrow \mathbb{C}} L(F^\sigma, s)^2.$$

The case where the quaternion algebra is split is the most frequently occuring and this happens precisely when A_F is the square of a GL_2-type variety. We will be mostly interested in Bianchi newforms that have rational coefficients since conjecturally attached to these are QM-surfaces.

As explained in the previous subsection, one source of QM-surfaces over imaginary quadratic fields is the base-change of GL_2-type surfaces which are attached to classical weight two newforms with quadratic coefficients and a non-trivial inner twist.

It is natural to ask whether all QM-surfaces over imaginary quadratic fields arise this way. Motivated by the conjectural connection to Bianchi newforms, J. Cremona asked whether this is the case [Cre92, Question 1'], phrased in the following way (see also [DGP10, Conjecture 1]).

Question 3.2.3. *If f is a rational weight 2 Bianchi newform over K for $\Gamma_0(\mathfrak{n})$, is there either an elliptic curve E/K with $L(f,s) = L(E/K,s)$, or a quadratic character χ of G_K such that $f \otimes \chi$ is base-change?*

The answer, as the examples in the next section show, is that a genuine Bianchi newform does *not* necessarily correspond an elliptic curve.

Theorem 3.2.4. *There exists QM-surfaces over imaginary quadratic fields that are not twists of base-change.*

Remark 3.2.5. One can interpret this statement in terms of rational points on Shimura curves. For $m|D$ there is a quotient curve defined by the Atkin-Lehner involution $X_D^{(m)} = X_D/\langle\omega_m\rangle$ and a natural projection map $\pi_m : X_D \to X_D^{(m)}$. A rational point on $X_D^{(m)}$ represents a GL_2-type surface over $\mathbb{Q}$ with real multiplication by $\mathbb{Q}(\sqrt{m})$ and quaternionic multiplication of discriminant D over K [BFGR06]. Hence the above theorem implies that $\bigcup_{m|D} X_D(K)\backslash\pi_m^{-1}(\mathbb{Q}) \neq \emptyset$ for some imaginary quadratic field K and discriminant D.

Question 3.2.6. *For which pairs (K, D) is the set $\bigcup_{m|D} X_D(K)\backslash\pi_m^{-1}(\mathbb{Q})$ non-empty?*

Remark 3.2.7. As it turns out, it is possible for a QM-surface not to be a quadratic twist of base-change. Rather, it could be a simple factor of the twist of a GL_2-type variety of dimension at least 4. This is the contents of [CDP+19].

§ 3.3 Explicit examples

Theorem 3.3.1. *The Jacobians of the following genus 2 curves are QM surfaces which correspond to a genuine Bianchi newform as in Conjecture 3.2.2.*

1. $C_1 : y^2 = x^6 + 4ix^5 + (-2i-6)x^4 + (-i+7)x^3 + (8i-9)x^2 - 10ix + 4i + 3,$
   Bianchi newform: 2.0.4.1-34225.3-a;

2. $C_2 : y^2 = x^6 + (-2\sqrt{-3}-10)x^5 + (10\sqrt{-3}+30)x^4 + (-8\sqrt{-3}-32)x^3 + (-4\sqrt{-3}+16)x^2 + (-16\sqrt{-3}-12)x - 4\sqrt{-3} + 16,$
   Bianchi newform: 2.0.3.1-61009.1-a;

3. $C_3 : y^2 = (104\sqrt{-3}-75)x^6 + (528\sqrt{-3}+456)x^4 + (500\sqrt{-3}+1044)x^3 + (-1038\sqrt{-3}+2706)x^2 + (-1158\sqrt{-3}+342)x - 612\sqrt{-3} - 1800,$
   Bianchi newform: 2.0.3.1-67081.3-a;

4. $C_4 : y^2 = x^6 - 2\sqrt{-3}x^5 + (2\sqrt{-3}-3)x^4 + 1/3(-2\sqrt{-3}+54)x^3 + (-20\sqrt{-3}+3)x^2 + (-8\sqrt{-3}-30)x + 4\sqrt{-3} - 11,$
   Bianchi newform: 2.0.3.1-123201.1-b.

Proof. See Chapter 4. □

At the time of writing there are 161343 rational Bianchi newforms of weight 2 in the LMFDB [LMF13] and these are for the quadratic fields $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11$. Among these, up to conjugation and twist there are only four genuine newforms for which no corresponding elliptic curve has been found. These are all accounted for by Theorem 3.3.1.

Curve 1. *Let C_1 be the genus 2 curve as in Theorem 3.3.1:*

$$C_1 : y^2 = x^6 + 4ix^5 + (-2i-6)x^4 + (-i+7)x^3 + (8i-9)x^2 - 10ix + 4i + 3.$$

- *The surface $A = Jac(C_1)$ has conductor $\mathfrak{p}_{5,1}^4 \cdot \mathfrak{p}_{37,2}^4$ with norm 34225^2.*
- *$\mathcal{O} \hookrightarrow End_{\mathbb{Q}(i)}(A)$ where $\mathcal{O}$ is the maximal order of the rational quaternion algebra of discriminant 6.*
- *There is a genuine Bianchi newform $f \in S_2(\Gamma_0(\mathfrak{p}_{5,1}^2 \cdot \mathfrak{p}_{37,2}^2))$ which is modular to A and is listed on the LMFDB database with label 2.0.4.1-34225.3-a.*

Curve 2. *Let C_2 be the genus 2 curve as in Theorem 3.3.1:*

$$C_2 : y^2 = x^6 + (-2\sqrt{-3} - 10)x^5 + (10\sqrt{-3} + 30)x^4 + (-8\sqrt{-3} - 32)x^3 + (-4\sqrt{-3} + 16)x^2 + (-16\sqrt{-3} - 12)x - 4\sqrt{-3} + 16.$$

- *The surface $A = Jac(C_2)$ has conductor $\mathfrak{p}_{13,1}^4 \cdot \mathfrak{p}_{19,1}^4$ with norm 61009^2.*
- *$\mathcal{O} \hookrightarrow End_{\mathbb{Q}(\sqrt{-3})}(A)$ where $\mathcal{O}$ is the maximal order of the rational quaternion algebra of discriminant 10.*
- *There is a genuine Bianchi newform $f \in S_2(\Gamma_0(\mathfrak{p}_{13,1}^2 \cdot \mathfrak{p}_{19,1}^2))$ which is modular to A and is listed on the LMFDB database with label 2.0.3.1-61009.1-a.*

Curve 3. *Let C_3 be the genus 2 curve as in Theorem 3.3.1:*

$$C_3 : y^2 = (104\sqrt{-3} - 75)x^6 + (528\sqrt{-3} + 456)x^4 + (500\sqrt{-3} + 1044)x^3 + (-1038\sqrt{-3} + 2706)x^2 + (-1158\sqrt{-3} + 342)x - 612\sqrt{-3} - 1800.$$

- *The surface $A = Jac(C_3)$ has conductor $\mathfrak{p}_{7,1}^4 \cdot \mathfrak{p}_{37,2}^4$ with norm 67081^2.*
- *$\mathcal{O} \hookrightarrow End_{\mathbb{Q}(\sqrt{-3})}(A)$ where $\mathcal{O}$ is the maximal order of the rational quaternion algebra of discriminant 10.*
- *There is a genuine Bianchi newform $f \in S_2(\Gamma_0(\mathfrak{p}_{7,1}^2 \cdot \mathfrak{p}_{37,2}^2))$ which is modular to A and is listed on the LMFDB database with label 2.0.3.1-67081.3-a.*

Curve 4. *Let C_4 be the genus 2 curve as in Theorem 3.3.1:*

$$C_4 : y^2 = x^6 - 2\sqrt{-3}x^5 + (2\sqrt{-3} - 3)x^4 + 1/3(-2\sqrt{-3} + 54)x^3 + (-20\sqrt{-3} + 3)x^2 + (-8\sqrt{-3} - 30)x + 4\sqrt{-3} - 11.$$

- *The surface $A = Jac(C_4)$ has conductor $\mathfrak{p}_3^{12} \cdot \mathfrak{p}_{13,1}^4$ with norm 123201^2.*
- *$\mathcal{O} \hookrightarrow End_{\mathbb{Q}(\sqrt{-3})}(A)$ where $\mathcal{O}$ is the maximal order of the rational quaternion algebra of discriminant 6.*
- *There is a genuine Bianchi newform $f \in S_2(\Gamma_0(\mathfrak{p}_3^6 \cdot \mathfrak{p}_{13,1}^2))$ which is modular to A and is listed on the LMFDB database with label 2.0.3.1-123201.1-b.*

§ 3.4 Computational aspects

Here we explain in detail how the examples in the previous section were found using computational tools. The majority of the work was carried out in MAGMA [BCP97].

3.4.1 Overview

The aim is to find examples of QM-surfaces which are

1 *genuine*;

2 of *small* conductor.

The second condition is to ensure that the corresponding Bianchi newform will be in the realms of what it is possible to compute. This is quite a strict limitation because the typical QM-surface will have a very large conductor due to the following:

Proposition 3.4.1. *Let A/K be a QM-surface. Then any prime ideal dividing the conductor of A appears to at least exponent 4.*

Proof. This is [GMŞ16, Proposition 2.4]. □

The strategy will be to use a family of QM-surfaces and try to run a large search by varying the parameters and catching those with the desired properties.

To this effect, the family given by [HT99] (see §2.3.2) would seem ideal: it is given as a two-parameter family $\mathcal{S}_6(t, s)$ of genus two curves such that the Jacobian of each curve is a QM-surface. However, it turns out that this family is not useful for our purposes since it would seem by checking numerically that all specialisations of $\mathcal{S}_6(t, s)$ are (a twist of) base-change.

Hence we instead make use of two families of discriminant 6 and 10 found in [BG08], which in turn were derived from the family above. The big advantage with these two families is that they are derived from the moduli space and there is a parameterisation in which we can control the arithmetic properties of the resulting genus 2 curves.

For a detailed overview of the two families of QM-surfaces worked out by S. Baba and H. Granath, including how they were constructed, see §2.3. Let us quote the main result here for the quaternion algebra of discriminant 6, the case of discriminant 10 is similar.

Let the maximal order of $\mathcal{B}_6$ be denoted by $\mathcal{O}$ and the set of norm 1 elements denoted by $\mathcal{O}^1$. These act properly discontinuously as isometries on the upper half plane $\mathcal{H}_2$ via an embedding $\mathcal{B}_6 \hookrightarrow M_2(\mathbb{R})$ and the resulting quotient $X_6 = \mathcal{H}_2/\mathcal{O}^1$ is the Shimura curve which is a moduli space for abelian surfaces with quaternionic multiplication by $\mathcal{O}$. An explicit model for this curve is $X_6 : X^2 + 3Y^2 + Z^2 = 0$.

Theorem 3.4.2. *Let*

$$P_j = (4 : 3\sqrt{j} : \sqrt{-27j - 16}) \in X_6$$

be a point on the conic X_6 which is parameterised by $\mathbb{P}^1$. Define the genus 2 curve

$$\begin{aligned} C_j : y^2 = (-4 + 3s)x^6 + 6tx^5 + 3t(28 + 9s)x^4 - 4t^2x^3 \\ + 3t^2(28 - 9s)x^2 + 6t^3x - t^3(4 + 3s), \end{aligned}$$

where $t = -2(27j+16)$ and $s = \sqrt{-6j}$. Then the Jacobian of C_j is a QM surface. The curve is defined over the field $\mathbb{Q}(\sqrt{j}, \sqrt{-6})$ and the field of moduli for C_j is $\mathbb{Q}(j)$.

Proof. See [BG08, §3.5]. □

In this way we can generate numerous QM surfaces over various number fields by, for example, taking any $j \in \mathbb{Q}$. However, for the purposes of modularity we need to fix an imaginary quadratic field K and try to define a QM-surface over this field.

3.4.2 Computation

In order to find a genuine QM-surface with QM of discriminant 6 (the case of discriminant 10 is very similar) one can take the following steps:

1 Fix a field K such that $X_6(K) \neq \emptyset$ and $K \hookrightarrow \mathcal{B}_6$.

2 Find a point $P_0 \in X_6(K)$

3 Fix a parameterisation $X_6(K)$ from the base point P_0.

4 Find a point $P = (a : b : c)$ on the conic $X_6(K)$ using the parameterisation.

5 Define the quantity $j = (\frac{4b}{3a})^2$.

6 Create the hyperelliptic curve C_j in Theorem 3.4.2 which is defined over $\mathbb{Q}(\sqrt{j}, \sqrt{-6})$.

7 Use the commands `IgusaClebschInvariants()` and `HyperellipticCurveFromIgusaClebsch()` to find a model C of C_j which is defined over K.

8 For a set S consisting of split primes $\mathfrak{p}, \overline{\mathfrak{p}}$ of K which do not divide the discriminant of C, reduce the curve C modulo $\mathfrak{p}$ and $\overline{\mathfrak{p}}$.

9 Use the commands `Jacobian()` and `EulerPolynomial()` to get Euler polynomials of the form $(X^2 - a_{\mathfrak{p}}X + N(\mathfrak{p}))^2$.

10 If $a_{\mathfrak{p}} \neq \pm a_{\overline{\mathfrak{p}}}$ for any prime $\mathfrak{p} \in S$ then the surface $\mathrm{Jac}(C)$ is genuine over K.

If $\mathrm{Jac}(C)$ is genuine then the set of primes S needed to verify this should not be too large in general. By searching for points $P \in X_6(K)$ using a parameterisation and following the above procedure it is reasonably straight forward to find a QM-surface which is genuine. However, if we are trying to find a QM-surface with small or prescribed conductor the process is more involved.

In the steps above we require that $K \hookrightarrow \mathcal{B}_6$ to ensure the C_j has a model defined over K. Let us explain this point.

For a non-singular genus 2 curve C, Igusa defined invariants $\{J_i(C)\}_{i=2,4,6,10}$ which are homogeneous polynomials in the coefficients of C. Two curves are isomorphic if and only if they define the same point $[J_2, J_4, J_6, J_{10}]$ in weighted projective space.

Definition 3.4.3. The minimal field over which the point $[J_2, J_4, J_6, J_{10}]$ is defined is called the *field of moduli* of C. The minimal field over which a model for C is defined is called the *field of definition* of C.

J.-F. Mestre [Mes91] solved the problem of how to construct a model for the genus 2 curve C from its Igusa invariants. If the field of moduli of C is K, then assuming that C does not have any automorphisms other than the hyperelliptic involution,

C cannot be defined over K in general. There is a quaternion algebra H_C, called the *Mestre obstruction*, such that for a field extension L/K, it is possible to define C over L if and only if L splits H_C. In our specific case there is a simple criterion.

Proposition 3.4.4. *The Mestre obstruction of C_j is the quaternion algebra*

$$\left(\frac{-6j, -2(27j+16)}{\mathbb{Q}(j)}\right).$$

In particular, C_j has a model defined over K if and only if K splits $\mathcal{B}_6$.

Proof. The Mestre obstruction is derived in [BG08, Proposition 3.13]. The second statement just follows from the fact that

$$\left(\frac{-6j, -2(27j+16)}{\mathbb{Q}(j)}\right) \simeq \left(\frac{-6, 2}{K}\right) \simeq \mathcal{B}_6 \otimes_{\mathbb{Q}} K.$$

□

In order to find explicit examples of genuine QM-surfaces which are amenable to computation we wish to find curves with *nice* models and whose Jacobians have *small* conductors. This is done by searching over values of j by using a parameterisation as above.

One would hope that values of j that are of small height correspond to curves C_j with small arithmetic invariants. This is indeed the case as we can see from the Igusa invariants, which in terms of j are ([BG08, Proposition 3.6])

$$[12(j+1), 6(j^2+j+1), 4(j^3-2j^2+1), j^3].$$

We can also use the following to control ramification properties.

Proposition 3.4.5. *Let C_j/K be a genus 2 curve as above. Then C_j has potentially good reduction at a prime $\mathfrak{p} \nmid 6$ if and only if $\nu_{\mathfrak{p}}(j) = 0$.*

Proof. See [BG08, Proposition 3.19]. □

The models which the function `HyperellipticCurveFromIgusaClebsch()` produce are typically very large, often many pages long. This is of practical importance because of the discrepancy between the discriminant of the curve C and the conductor of the surface $\mathrm{Jac}(C)$.

Lemma 3.4.6. *Let C be a genus 2 curve over a number field K with integral coefficients. Then there is an invariant $\Delta_{min} \trianglelefteq \mathcal{O}_K$ called the minimal discriminant of C defined in [BSS+16, §2] and we say that C is a global minimal model if $(\Delta(C)) = \Delta_{min}(C)$. It has the property that Cond(Jac(C))$|\Delta_{min}$ [Liu94].*

The converse is not necessarily true and if $\mathfrak{p}$ divides $\Delta_{\min}$ but not the conductor of the Jacobian, $\mathfrak{p}$ is typically large [BSS+16, Remark 5.3.3]. If K has class number 1 then C has a global minimal model, but not necessarily otherwise.

It is necessary to know the conductor exactly since we wish to find the conjecturally associated Bianchi newform. The odd part of the conductor can be found using MAGMA. Computing the even part has recently been made possible using machinery developed in [DD19]. The support of the ideal generated by the discriminant of a genus 2 hyperelliptic curve contains the support of the conductor of its Jacobian and the inclusion can in fact be strict. This phenomenon arises especially when one works with curves that have very large coefficients.

To find small models for the curves in §3.3 it was required to use as yet unpublished code by L. Dembélé. The input is:

1 A list of sufficiently many Euler polynomials;

2 The field cut out by the 2-torsion of the surface Jac(C).

The Euler polynomials are computed from the conjecturally corresponding Bianchi newform. To compute the 2-torsion we use the following fact.

Proposition 3.4.7. *The field of 2-torsion of the surface Jac(C) is the same as the splitting field of the hyperelliptic polynomial of C.*

Proof. See [Wil98, Lemma 4.4.2]. □

Example 3.4.8. Let us illustrate how this is done with the field $K = \mathbb{Q}(i)$ and the curve C_1 in §3.3.

Fix the base point

$$P = (-i, 0, 1)$$

from which we will make the parameterisation. Then there is a map

$$\begin{aligned} \varphi : \mathbb{P}^1(K) &\longrightarrow X_6(K) \\ (g : h) &\longmapsto (ig^2 - 3ih^2, 2igh, -g^2 - 3h^2). \end{aligned}$$

with inverse $(X,Y,Z)\mapsto(X-iZ,Y)$. The base point corresponds to $(1:0)$. Hence we can set $h=1$ and search over values of g. The point $(g:1)$ corresponds to the value $j=(\frac{8ig}{ig^2-3i})^2$. One can then do a search over values of g of bounded height.

Let $g=\frac{-1636i+248}{4107}$ which corresponds to the point

$$P_j=(4:\frac{48i+66}{37}:\frac{-176i+54}{37})$$

with

$$j=\frac{704i+228}{1369}.$$

As an ideal j factorises as $(j)=\mathfrak{p}_2^4\cdot\mathfrak{p}_{5,1}^2\cdot\mathfrak{p}_{37,2}^{-2}$. This allows us to write down the hyperelliptic curve C_j which is defined over $K(\sqrt{-6})$. By taking the Igusa-Clebsch invariants and using the function above we can define a hyperelliptic curve C which is defined over K and is isomorphic to C_j over some extension of K.

Now that we have a model C we can compute the field cut out by the 2-torsion of $\mathrm{Jac}(C)$ by computing the splitting field. This then allows us to define a smaller model for C using the code of L. Dembélé. Hence we produce the curve

$$C_1: y^2=x^6+4ix^5+(-2i-6)x^4+(-i+7)x^3+(8i-9)x^2-10ix+4i+3$$

and $\mathrm{Jac}(C_1)$ is a genuine QM-surface.

§ 3.5 Connection with the Paramodularity Conjecture

The genuine QM-surfaces we present also have an interesting connection to the Paramodularity Conjecture. Recall that the original Paramodularity Conjecture posits a correspondence between abelian surfaces $A/\mathbb{Q}$ with trivial endomorphisms and genus 2 paramodular rational Siegel newforms of weight 2 that are not Gritsenko lifts [BK14, Conjecture 1.1]. Let us briefly state the details.

Under the general Langlands framework, H. Yoshida conjectured that for every abelian surface over $\mathbb{Q}$ with trivial endomorphisms there should be a discrete group $\Gamma\leq\mathrm{Sp}_4(\mathbb{Q})$ and a certain Siegel modular form of weight 2 for Γ with the same L-function as the abelian surface [Yos84].

Definition 3.5.1. For a natural number N define the *paramodular group*

$$K(N) = \mathrm{Sp}_4(\mathbb{Q}) \cap \begin{pmatrix} * & * & */N & * \\ *N & * & * & * \\ *N & *N & * & *N \\ *N & * & * & * \end{pmatrix}.$$

Let $S_2^k(K(N))$ denote the $\mathbb{C}$-vector space of Siegel cusp forms of weight k and degree 2 for $K(N)$ called *paramodular forms.*

One can lift Jacobi forms of index N to paramodular forms for $K(N)$ of weight 2 via the Gritsenko lift Grit: $J_{2,N}^{cusp} \longrightarrow S_2^2(K(N))$. These lifts are not interesting for applications to arithmetic geometry because the eigenvalues are too large.

It was conjectured that there is a bijection between lines of paramodular newforms $f \in S_2^2(K(N))$ with rational eigenvalues that are not Gritsenko lifts and isogeny classes of abelian surfaces A with trivial endomorphisms and conductor N. In this correspondence

$$L(A, s, \text{Hasse-Weil}) = L(f, s, \text{spin}).$$

Evidence for the conjecture is given in [PY15] by computing spaces of cusp forms for primes N less than 600.

It has been recently pointed out by F. Calegari et al. [BCGP18, §10] that this conjectural correspondence needs to be amended. In the bijection the geometric side needs to take into account abelian 4-folds $B/\mathbb{Q}$ with $\mathrm{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q}$ an indefinite quaternion algebra over $\mathbb{Q}$. This can be illustrated using our genuine QM-surfaces.

Let C/K be any of the four curves given in Theorem 3.3.1. Define A/K to be the QM surface $\mathrm{Jac}(C)$ with $\mathrm{End}_K(A) \otimes \mathbb{Q} \simeq D/\mathbb{Q}$ an indefinite quaternion algebra. Then the Weil restriction $B = \mathrm{Res}_{K/\mathbb{Q}}(A)$ of A from K to $\mathbb{Q}$ is a simple abelian 4-fold such that $\mathrm{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q} \simeq D/\mathbb{Q}$.

We prove in Chapter 4 that there is a genuine rational weight 2 Bianchi newform f over K such that $L(A/K, s) = L(f, s)^2$. Now let F be the paramodular rational Siegel newform of weight 2 that is the theta lift of f. It follows from the properties of Weil restriction [Mil72] and theta lifting [BDPŞ15] that $L(B/\mathbb{Q}, s) = L(A/K, s) = L(f, s)^2 = L(F, s)^2$.

In analogy to the case of QM surfaces, at any prime ℓ unramified in D, the 8-dimensional ℓ-adic Tate module of $B/\mathbb{Q}$ splits as the square of a 4-dimensional submodule [Chi92, §7]. Then the 4-dimensional ℓ-adic Galois representation has similar arithmetic to one that arises from an abelian surface over $\mathbb{Q}$ with trivial endomorphisms. Indeed, our example above shows that via the representation afforded by the submodule, $B/\mathbb{Q}$ corresponds to a Siegel newform of the type considered in the Paramodularity Conjecture.

Let us then state the modified paramodularity conjecture [BK19].

Conjecture 3.5.2. *Let $\mathcal{A}_N$ denote the set of isogeny classes of abelian surfaces of conductor N with trivial endomorphisms and $\mathcal{B}_N$ denote the set of isogeny classes of QM-fourfolds of conductor N^2. Then there is a bijection between paramodular newforms of level N up to scaling that are non-lifts and $\mathcal{A}_N \cup \mathcal{B}_N$. In this correspondence*

$$L(A, s, \textit{Hasse-Weil}) = L(f, s, \textit{spin}) \quad \textit{if } A \in \mathcal{A}_N$$

and

$$L(A, s, \textit{Hasse-Weil}) = L(f, s, \textit{spin})^2 \quad \textit{if } A \in \mathcal{B}_N.$$

Chapter 4

The Faltings-Serre-Livné method

In this chapter we shall prove that the genuine QM-surfaces in Theorem 3.3.1 are modular in the sense that the attached Galois representations are isomorphic to ones coming from Bianchi newforms. This will be achieved by using what is often called the *Faltings-Serre method*, although we shall actually use a criterion of R. Livné because the residual image of the Galois representations will be absolutely reducible in this case.

The method is used for determining the equivalence of two Galois representations

$$\rho_1, \rho_2 : G \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$

This comes out of Faltings' seminal work [Fal83] and is elaborated upon by J.-P. Serre in a letter to J. Tate dated 26th October 1984 [Ser15, pp.699-705]. It gives an effective method to check whether two representations are isomorphic by computing the trace of Frobenius for only finitely many primes. As an example of its utility Serre shows that the elliptic curve of conductor 11 is modular using these relatively simple techniques.

The contribution of Livné [Liv87] to this topic was to give an analogous treatment for representations which are absolutely reducible at $\ell = 2$.

This overall method for verifying whether two representations are isomorphic was extended to imaginary quadratic fields by [DGP10], which we shall follow. Note that many properties of Galois representations can be computed from a finite number of traces of Frobenius using the algorithms developed in [AGC18].

For the remainder of the chapter, unless stated otherwise let K be an imaginary quadratic field and

$$\rho_1, \rho_2 : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$$

be two 2-adic Galois representations with residual representations

$$\overline{\rho}_1, \overline{\rho}_2 : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_2).$$

§ 4.1 Galois representations attached to Bianchi newforms

Let π be any regular cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ with unitary central character ω (cf. [Tay94] for definitions). By the Langlands philosophy, one expects that attached to π is a compatible family of continuous irreducible Galois representations $\{\rho_\ell\}$ such that the associated L-functions agree. That is, the Frobenius polynomials of ρ agree with the Hecke polynomials of π at each place.

Under the assumption that ω is equal to its complex conjugate, it is possible to relate π to a holomorphic Siegel modular form via the theta lift and get a version of the predicted correspondence. In [HST93, Tay94], with some technical assumptions the authors succeeded in attaching to π a compatible family of 2-dimensional Galois representations with the Frobenius and Hecke polynomials agreeing outside of a density zero set of places.

This result was then strengthened by [BH07] in which the authors removed the technical assumptions and proved the equivalence of polynomials outside of a finite explicit set. This was then improved by [Mok14] who also extended the result to CM fields.

We quote the result here.

Theorem 4.1.1. *Let S denote the set of places of K which divide ℓ or where π is ramified and assume that $\omega = \omega^c$. Then there is a compatible system $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\}$ of continuous irreducible representations such that if the prime $v \notin S$ then the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ agrees with the Hecke polynomial of π at v. In other words, $L(\rho_\ell, s) = L(\pi, s)$ away from S.*

It will be important in the following sections to control the field of definition of the image of these representations, for which we can use the result below due to R. Taylor. Let the notation be as in the previous theorem. Then for $v \notin S$,

$\{\alpha_v, \beta_v\} \subset \mathbb{C}^\times$ are the Satake parameters of π. Define the field F_π as the subfield of $\mathbb{C}$ generated by $\alpha_v + \beta_v$ and $\alpha_v\beta_v$ for $v \notin S$. In fact, F_π is a number field.

Theorem 4.1.2. *Let f be a Bianchi newform over the imaginary quadratic field K. If for $i = 1, 2$, the $v_i \notin S$ are places of K with $\alpha_{v_i} \neq \beta_{v_i}$ and if v_i is split then $\alpha_{\overline{v_i}} \neq -\beta_{\overline{v_i}}$, we let $E = F_\pi(\alpha_{v_1}, \alpha_{v_2})$. Then E/F_π is an extension of degree at most four and the representation is defined over*

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E_\lambda)$$

where E_λ is the completion of E at a prime λ above ℓ.

Proof. See [Tay94, Corollary 1]. □

§ 4.2 Comparing residual representations

The strategy of the Faltings-Serre method is to first establish that the two residual representations are isomorphic and given this, show that the full representations are isomorphic. Much of the work is in this first step.

As usual let $\rho_1, \rho_2 : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ be two Galois representations. We briefly outline how to define the residual representations $\overline{\rho}_1, \overline{\rho}_2$, borrowing from [Jon15, §6.1].

It is well known that such a continuous representation $\rho_\ell : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ has its image contained in $\mathrm{GL}_2(E)$, where E is a finite extension of $\mathbb{Q}_\ell$. Then one can show that the representation $\rho_\ell : G_K \longrightarrow \mathrm{GL}_2(E)$ fixes a lattice in E, i.e. an $\mathcal{O}_E$ module Λ such that $\Lambda \otimes_{\mathcal{O}_E} E \simeq E^2$. This means that up to isomorphism we can realise it as a representation

$$\rho_\ell : G_K \longrightarrow \mathrm{GL}_2(\mathcal{O}_E).$$

Hence we can define the residual representation $\overline{\rho}_\ell$ by composing with the natural reduction map $\mathrm{GL}_2(\mathcal{O}_E) \to \mathrm{GL}_2(k)$, where k is the residue field of $\mathcal{O}_E$.

However, the projection induced by $\mathcal{O}_E \to k$ introduces the subtlety that the residual representation is dependent on the choice of lattice Λ. Two lattices differ by an element of $\mathrm{GL}_2(E)$, but as representations over k the two residual representations need not be isomorphic.

To remedy this, let $A = k[G_K]$ and $\mathcal{V}, \mathcal{W}$ be the vector space k^2 with $k[G_K]$-action defined by two distinct lattices of E^2. Whilst the lattices are distinct, the full representations associated to them are equivalent up to conjugation which means that they have equal characteristic polynomials with integral coefficients in $\mathcal{O}_E$. In particular, the reduced characteristic polynomials over k associated to $\mathcal{V}$ and $\mathcal{W}$ are equal.

The well known Brauer-Nesbitt theorem tells us that since the characteristic polynomials of $\mathcal{V}$ and $\mathcal{W}$ are the same, their composition factors must be equal. That is to say, in the composition series

$$\mathcal{V} = \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_n = 0;$$
$$\mathcal{W} = \mathcal{W}_0 \supsetneq \mathcal{W}_1 \supsetneq \cdots \supsetneq \mathcal{W}_n = 0$$

with simple composition factors $\mathcal{V}_i/\mathcal{V}_{i+1}$ and $\mathcal{W}_i/\mathcal{W}_{i+1}$, that $\mathcal{V}_i/\mathcal{V}_{i+1} \simeq \mathcal{W}_i/\mathcal{W}_{i+1}$.

Recall that the *semisimplification* of $\mathcal{V}$ is defined as

$$\mathcal{V}^{ss} = \bigoplus_{0}^{n-1} \mathcal{V}_i/\mathcal{V}_{i+1}$$

and $\mathcal{V}^{ss}$ is unique up to isomorphism by a theorem due to Jordan-Hölder.

Definition 4.2.1. The *residual representation* $\overline{\rho}_\ell : G_k \longrightarrow \mathrm{GL}_2(k)$ is defined to be the semisimplification of the representation obtained by composing $\rho_\ell : G_K \longrightarrow \mathrm{GL}_2(\mathcal{O}_E)$ with the reduction map $\mathrm{GL}_2(\mathcal{O}_E) \to \mathrm{GL}_2(k)$.

The residual representation is independent of the choice of lattice Λ.

From now on let us suppose that the representations have rational trace of Frobenius. We can simplify the situation by assuming that the residual representations have image in $\mathrm{GL}_2(\mathbb{F}_2)$ by Lemma 2.4.6.

Let L_i denote the fixed field of $\mathrm{Ker}(\overline{\rho}_i)$ and note that $\mathrm{Gal}(L_i/K) \simeq G_K/\mathrm{Ker}(\overline{\rho}_i) \simeq \mathrm{Im}(\overline{\rho}_i)$.

By applying Brauer-Nesbitt it is enough to show that $L_1 \simeq L_2$ in order to prove that the two residual representations are isomorphic. Hence we use the tools of class field theory to test whether the two fields are isomorphic which is possible because $\mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$ is solvable.

The *ray class group* $Cl(\mathcal{O}_K, \mathfrak{m})$ is defined as the group of fractional ideals of K coprime to the ideal $\mathfrak{m}$, modulo the group of principal ideals coprime to $\mathfrak{m}$. It is a finite abelian group and significantly there is an abelian extension $K(\mathfrak{m})/K$ called the *ray class field* which is unramified away from the primes dividing $\mathfrak{m}$ and $\mathrm{Gal}(K(\mathfrak{m})/K) \simeq Cl(\mathcal{O}_K, \mathfrak{m})$. The isomorphism sends a prime $\mathfrak{p}$ to the Frobenius element $\mathrm{Frob}_{K(\mathfrak{m})/K}(\mathfrak{p})$ in $\mathrm{Gal}(K(\mathfrak{m})/K)$. This is called the *Artin map* and establishes a correspondence between subgroups of $Cl(\mathcal{O}_K, \mathfrak{m})$ and subfields of $K(\mathfrak{m})/K$.

Theorem 4.2.2. *Let K be an imaginary quadratic field and L/K an abelian extension of prime degree p unramified outside of the finite set of places S. Define the modulus*

$$\mathfrak{m}_K = \prod_{\mathfrak{p} \in S} \mathfrak{p}^{e(p)},$$

where

$$\begin{cases} e(\mathfrak{p}) = 1 & \text{if } \ \mathfrak{p} \nmid p; \\ e(\mathfrak{p}) = \lfloor \frac{pe(\mathfrak{p}/p)}{p-1} \rfloor + 1 & \text{if } \ \mathfrak{p} | p. \end{cases}$$

Then $\mathrm{Gal}(L/K)$ corresponds to a subgroup of the ray class group $Cl(\mathcal{O}_K, \mathfrak{m}_K)$.

Proof. This is [Coh00, Propositions 3.3.21-22]. □

Thus to test the isomorphism of field extensions of K we have to look at the ray class group $Cl(\mathcal{O}_K, \mathfrak{m}_K)$. The following will be used regularly to enumerate and compare field extensions.

Proposition 4.2.3. *Let K be an imaginary quadratic field and $\mathfrak{m}_K$ a modulus. Then there is a $(\mathbb{Z}/n\mathbb{Z})$-basis $\{\psi_1, \dots, \psi_s, \chi_1, \dots, \chi_t\}$ of the characters of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$ of order exactly n and a set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ of prime ideals of $\mathcal{O}_K$ such that $\psi_i(\mathfrak{p}_j) = 0$ for all i, j and the vectors $\{(\chi_1(\mathfrak{p}_j), \dots, \chi_t(\mathfrak{p}_j))\}_{1 \le j \le r}$ span $(\mathbb{Z}/n\mathbb{Z})^t$.*

For any such set of primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$, if χ is a non-trivial order n character of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$ not lying in the span of $\{\psi_i\}_{1 \le i \le s}$, then $\chi(\mathfrak{p}_j) \neq 0$ for some prime $\mathfrak{p}_j$.

Proof. See [Jon15, Proposition 6.3.2]. □

There are four distinct possibilities for the image: $\mathrm{Im}(\overline{\rho}_1) \simeq \mathrm{id}, C_2, C_3$ or S_3. We will restrict ourselves to looking at the third case since this is the situation for the QM-surfaces in §3.3 by Proposition 2.4.7.

Assumption: $\mathrm{Im}(\overline{\rho}_1) \simeq C_3$.

Then there are two steps to show that $\overline{\rho}_1 \simeq \overline{\rho}_2$:

1 Prove that $\mathrm{Im}(\overline{\rho}_2) \simeq C_3$;

2 Show that the image of $\overline{\rho}_2$ factors through L_1.

We will make frequent use of the fact that the elements of order 3 in $\mathrm{GL}_2(\mathbb{F}_2)$ are precisely those of odd trace and the elements of order 2 all have even trace as can be seen from the following table.

Element	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$
Order	1	2	2	2	3	3
Trace	0	0	0	0	1	1

Table 4.1: Elements of $\mathrm{GL}_2(\mathbb{F}_2)$.

As in Theorem 4.2.2 define the modulus

$$\mathfrak{m}_K = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{e(p)},$$

where

$$e(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \nmid 6; \\ 2e(\mathfrak{p}/p)+1 & \text{if } \mathfrak{p}|2; \\ 3\lfloor \frac{pe(\mathfrak{p}/p)}{2} \rfloor + 1 & \text{if } \mathfrak{p}|3 \end{cases}$$

and S is the set of primes at which ρ_2 is unramified away from.

Step 1. First we establish whether L_2 contains any quadratic subfields. If $\mathrm{Im}(\overline{\rho}_2)$ contains an element of order 2 then there must be a quadratic character of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$ corresponding to some quadratic subfield F_2 of L_2. Then for an inert prime $\mathfrak{p}$ of F_2, $\overline{\rho}_2(\mathrm{Frob}_{\mathfrak{p}})$ must have order 2 and hence even trace.

Let $\{\chi_1, \ldots, \chi_t\}$ be a $(\mathbb{Z}/2\mathbb{Z})$-basis for the quadratic characters of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$. Suppose that the vectors $\{(\chi_1(\mathfrak{p}_i), \ldots, \chi_t(\mathfrak{p}_i))\}_{1\le i\le r}$ span $(\mathbb{Z}/2\mathbb{Z})^t$ for some chosen set of primes $\mathcal{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of $\mathcal{O}_K$ not dividing $\mathfrak{m}_K$. From Proposition 4.2.3 we can infer that for any quadratic extension of K unramified outside of S with associated character χ, there is a prime $\mathfrak{p} \in \mathcal{P}$ such that $\chi(\mathfrak{p}) \neq 0$. In particular there is a prime $\mathfrak{p} \in \mathcal{P}$ which is inert in this quadratic extension.

It follows from the above that if $\mathrm{Tr}(\overline{\rho}_2(\mathrm{Frob}_\mathfrak{p}))$ is odd for all primes $\mathfrak{p} \in \mathcal{P}$ then L_2 cannot contain any quadratic subfields and hence $\mathrm{Im}(\overline{\rho}_2) \simeq C_3$.

Step 2. Let ψ_1 be the character associated to the extension L_1 which we have assumed to be cubic and extend it to a $(\mathbb{Z}/3\mathbb{Z})$-basis $\{\psi_1, \chi_1, \ldots, \chi_t\}$ of the cubic characters of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$. As in Proposition 4.2.3, choose a set of primes $\mathcal{P}_2 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of $\mathcal{O}_K$ not dividing $\mathfrak{m}_K$ such that $\psi_1(\mathfrak{p}_i) = 0$ for all i and the vectors $\{(\chi_1(\mathfrak{p}_i), \ldots, \chi_t(\mathfrak{p}_i))\}_{1\le i\le r}$ span $(\mathbb{Z}/3\mathbb{Z})^t$.

Now let ψ_2 be the cubic character associated to L_2. We can express the character as

$$\psi_2 = \epsilon\psi_1 + \sum_{i=1}^{t} \epsilon_i \chi_i, \quad \epsilon, \epsilon_i \in \mathbb{Z}/3\mathbb{Z}.$$

If $\psi_2 \neq \epsilon\psi_1$ then by Proposition 4.2.3 there is a prime $\mathfrak{p} \in \mathcal{P}_2$ such that $\psi_2(\mathfrak{p}) \neq 0$. This would mean that $\overline{\rho}_2(\mathrm{Frob}_\mathfrak{p})$ has order 3, whilst $\overline{\rho}_2(\mathrm{Frob}_\mathfrak{p})$ is the trivial element. In particular we would have $\mathrm{Tr}(\overline{\rho}_1(\mathrm{Frob}_\mathfrak{p})) \neq \mathrm{Tr}(\overline{\rho}_2(\mathrm{Frob}_\mathfrak{p}))$.

Hence if $\mathrm{Tr}(\overline{\rho}_1(\mathrm{Frob}_\mathfrak{p})) = \mathrm{Tr}(\overline{\rho}_2(\mathrm{Frob}_\mathfrak{p}))$ for all primes $\mathfrak{p} \in \mathcal{P}_2$ we can conclude that $L_1 \simeq L_2$ and hence $\overline{\rho}_1 \simeq \overline{\rho}_2$. Otherwise the two representations are not isomorphic.

§ 4.3 Livné's criterion

Once we have shown that the residual representations $\overline{\rho}_1$ and $\overline{\rho}_2$ are isomorphic, we are now in a position to ascertain whether the full representations are isomorphic up to semisimplification. We use the following result due to Livné in the case that the image is cyclic.

Theorem 4.3.1. *Let K be a number field, E a finite extension of $\mathbb{Q}_2$ and $\mathcal{O}_E$ its ring of integers with maximal ideal $\mathcal{M}$. Let*

$$\rho_1, \rho_2 : \mathrm{Gal}(\overline{K}/K) \longrightarrow GL_2(E)$$

be two continous representations unramified outside of a finite set of primes S and $K_{2,S}$ the compositum of all quadratic extensions of K unramified outside of S.

Suppose that

1. $Tr(\rho_1) \equiv Tr(\rho_2) \equiv 0 \pmod{\mathcal{M}}$ *and* $Det(\rho_1) \equiv Det(\rho_2) \equiv 1 \pmod{\mathcal{M}}$*;*
2. *There is a finite set of primes T disjoint from S such that* $\{\mathrm{Frob}_{\mathfrak{p}} | \mathfrak{p} \in T\}$ *surjects onto* $\mathrm{Gal}(K_{2,S}/K)$.

If the characteristic polynomials of ρ_1 and ρ_2 are equal on $\{\mathrm{Frob}_{\mathfrak{p}} | \mathfrak{p} \in T\}$ *then ρ_1 and ρ_2 have isomorphic semisimplifications.*

Proof. See [Liv87, Theorem 4.3] and also [Che08, Theorem 5.4.9] for this precise statement. □

As in the previous section, suppose that $\mathrm{Im}(\overline{\rho}_1) \simeq \mathrm{Im}(\overline{\rho}_1) \simeq C_3$, which is the case we are interested in. Denote by $L \simeq L_1 \simeq L_2$ the cubic extension of K cut out by the residual representations. Then to apply the above theorem we have to prove isomorphism (up to semisimplication) of $\rho_1|_{G_L} \simeq \rho_2|_{G_L}$ at the level of $G_L = \mathrm{Gal}(\overline{K}/L)$.

If the representations are isomorphic (up to semisimplification) over a cyclic extension, the original representations can differ by a character. To prove that the full representations are isomorphic (up to semisimplification) we have to show that this character is trivial.

Proposition 4.3.2. *Suppose that L is a cyclic extension of K and denote by $G_L = \mathrm{Gal}(\overline{K}/L)$. If $\rho_1|_{G_L} \simeq \rho_2|_{G_L}$, then $\rho_1 \simeq \rho_2 \otimes \chi$ for some character χ of G_K whose restriction to G_L is trivial. If* $\mathrm{Tr}(\overline{\rho}_1(\mathrm{Frob}_{\mathfrak{p}})) = \mathrm{Tr}(\overline{\rho}_2(\mathrm{Frob}_{\mathfrak{p}}))$ *for some prime $\mathfrak{p}$ of K which is inert in L, then ρ_1 and ρ_2 are isomorphic up to semisimplification.*

Proof. This is essentially what is shown by [SW05, pp. 362], who found themselves in this situation whilst trying to apply Livné's criterion to an elliptic curve over $\mathbb{Q}(\sqrt{509})$.

As explained in [Jon16, p. 161], by Frobenius reciprocity we know that $\rho_1 \simeq \rho_2 \otimes \chi$ with the character χ being trivial on G_L.

Now suppose that $\mathfrak{p}$ is inert in L and $\mathrm{Tr}(\overline{\rho}_1(\mathrm{Frob}_\mathfrak{p})) = \mathrm{Tr}(\overline{\rho}_2(\mathrm{Frob}_\mathfrak{p}))$. Then clearly $\chi(\mathfrak{p}) = 1$. However, the fact that $\mathfrak{p}$ is inert in L means the only possibility is that χ is trivial and hence we can conclude that $\rho_1 \simeq \rho_2$. □

§ 4.4 Proof of modularity

In this final section we shall apply the methods outlined in this chapter to prove that the genuine QM-surfaces in §3.3 are modular. In particular we shall use Livné's criterion because the residual image of our representations are cyclic of order 3. For computational reasons it is necessary to consider the ℓ-adic representations for the prime $\ell = 2$. This is possible because 2 does not divide the conductors of any of the QM-surfaces presented.

To compute coefficients of Bianchi newforms we use the implementation of D. Yasaki's work in MAGMA [Yas10].

For the sake of clarity and exposition we shall follow the proof for the second QM-surface in §3.3. However, the list of primes needed to prove modularity of the other three QM-surfaces will be included at the end.

So let $K = \mathbb{Q}(\sqrt{-3})$, C_2 be the genus 2 curve

$$\begin{aligned} C_2 : y^2 =& x^6 + (-2\sqrt{-3} - 10)x^5 + (10\sqrt{-3} + 30)x^4 + (-8\sqrt{-3} - 32)x^3 \\ &+ (-4\sqrt{-3} + 16)x^2 + (-16\sqrt{-3} - 12)x - 4\sqrt{-3} + 16. \end{aligned}$$

and A the Jacobian of C_2. The surface $A = Jac(C_2)$ has conductor $\mathfrak{p}_{13,1}^4 \cdot \mathfrak{p}_{19,1}^4$ with norm 61009^2 and $\mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{Q}(\sqrt{-3})}(A)$ where $\mathcal{O}$ is the maximal order of the rational quaternion algebra of discriminant 10. The endomorphism algebra can be independently verified using the machinery developed in [CMSV19].

Let $f \in S_2(\Gamma_0(\mathfrak{p}_{13,1}^2 \cdot \mathfrak{p}_{19,1}^2))$ be the genuine Bianchi newform which is listed on the LMFDB database with label 2.0.3.1-61009.1-a. We will show that f corresponds to A. As explained in §4.1, we can associate an ℓ-adic Galois representation $\rho_{f,\ell} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ to f such that $L(f, s) = L(\rho_{f,\ell}, s)$.

As in §2.4, attached to the QM-surface A is a Galois representation

$$\rho_{A,2} : \mathrm{Gal}(\overline{K}/K) \longrightarrow (B \otimes_\mathbb{Q} \mathbb{Q}_2)^\times,$$

where B is the rational quaternion algebra of discriminant 10. First we have to control the image of the two attached representations.

Lemma 4.4.1. *The representations*

$$\rho_{A,2}, \rho_{f,2} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$$

have image contained in $\mathrm{GL}_2(E)$*, where* E *is the unique unramified quadratic extension of* $\mathbb{Q}_2$*.*

Proof. For $\rho_{A,2}$ it follows from the fact that, whilst $B \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is a division algebra, any quadratic extension E of $\mathbb{Q}_2$ splits it as $B \otimes_{\mathbb{Q}} E \simeq M_2(E)$.

Let us now consider $\rho_{f,2}$. It is possible to directly apply the result 4.1.2 of Taylor. So take the prime 31, which is split in $\mathbb{Q}(\sqrt{-3})$ and whose Hecke eigenvalues at the primes above it are distinct. We get the field $\mathbb{Q}(\sqrt{-43}, \sqrt{-123})$ by adjoining the roots of the Hecke polynomials. The completion at either of the primes above 2 in this field gives the unique unramified quadratic extension of $\mathbb{Q}_2$ and so we can take this as the coefficient field E. □

Now we can show that the residual representations are isomorphic.

Lemma 4.4.2. *The residual representations* $\overline{\rho}_{A,2}, \overline{\rho}_{f,2}$ *are isomorphic and have image* $C_3 \subset \mathrm{GL}_2(\mathbb{F}_2)$*.*

Proof. Denote by L_A and L_f the fields cut out by $\overline{\rho}_{A,2}$ and $\overline{\rho}_{f,2}$ respectively. It is clear already from Proposition 2.4.7 that the image of $\overline{\rho}_{A,2}$ is a subgroup of C_3. We compute that the field cut out by the 2-torsion of A is isomorphic to A_4 which has only one proper normal subgroup. This subgroup has order 4 and so applying the short exact sequence of Theorem 2.4.8, the image of $\overline{\rho}_{A,2}$ must be C_3.

We first note that it can be assumed $\mathrm{Im}(\overline{\rho}_{f,2}) \subset GL_2(\mathbb{F}_2)$ by Lemma 2.4.6, due to the fact that the traces of Frobenius are all rational. To show that $\mathrm{Im}(\overline{\rho}_{f,2}) \simeq C_3$, following §4.2 let $\mathfrak{m}$ denote the modulus

$$\mathfrak{m} = \mathfrak{p}_2^3 \cdot \mathfrak{p}_{13,1} \cdot \mathfrak{p}_{19,1}.$$

If $\mathrm{Im}(\overline{\rho}_{f,2})$ is not equal to C_3 there must be a quadratic extension of K contained in L_f which corresponds to a quadratic character of $Cl(\mathcal{O}_K, \mathfrak{m})$. We compute the ray class group to be

$$Cl(\mathcal{O}_K, \mathfrak{m}) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/36\mathbb{Z}).$$

Let us choose $\{\chi_1, \dots, \chi_4\}$ as an $\mathbb{F}_2$-basis for the quadratic characters of $Cl(\mathcal{O}_K, \mathfrak{m})$. Then the vectors $\{(\chi_1(\mathfrak{p}), \dots, \chi_4(\mathfrak{p}))\}_{\mathfrak{p}\in\mathcal{P}}$ span $\mathbb{F}_2^4$, where $\mathcal{P} = \{\mathfrak{p}_{7,1}, \mathfrak{p}_{7,2}, \mathfrak{p}_{13,2}, \mathfrak{p}_{19,2}, \mathfrak{p}_5\}$. If L_f contains a quadratic subfield then by Proposition 4.2.3 the associated quadratic character must be non-zero for one of the primes in $\mathcal{P}$. Hence there must be a prime $\mathfrak{p} \in \mathcal{P}$ that is inert in this subfield and so $\overline{\rho}_{f,2}(\mathrm{Frob}_{\mathfrak{p}})$ must have order 2. However, we compute that the trace of Frobenius is odd for all primes in $\mathcal{P}$ and therefore L_f is a cubic extension of K.

To show that the representations are isomorphic let ψ_A denote the cubic character associated to L_A. Extend this to an $\mathbb{F}_3$-basis $\{\psi_A, \chi_1\}$ of the cubic characters of $Cl(\mathcal{O}_K, \mathfrak{m})$. We find that the prime $\mathfrak{p}_{37,1}$ is such that $\psi_A(\mathfrak{p}_{37,1}) = 0$ and $\chi_1(\mathfrak{p}_{37,1}) \neq 0$. So if χ_f is the cubic character associated to L_f and χ_f is not in the span of χ_A then $\psi_f(\mathfrak{p}_{37,1})$ must be non-zero. In particular, $\overline{\rho}_{f,2}(\mathrm{Frob}_{\mathfrak{p}})$ must have order 3 but we find that $\mathrm{Tr}(\overline{\rho}_{f,2}(\mathrm{Frob}_{\mathfrak{p}_{37,1}})) = \mathrm{Tr}(\overline{\rho}_{A,2}(\mathrm{Frob}_{\mathfrak{p}_{37,1}}))$ and so we can conclude that the residual representations are isomorphic. □

Now that we have shown that the residual representations are isomorphic it remains to show that the full representations are isomorphic up to semisimplifcation. The residual images are cyclic and note that this will always be the case for $\overline{\rho}_{A,2}$ when the prime ℓ divides the discriminant of the acting quaternion algebra. Since the images are absolutely reducible we can apply Theorem 4.3.1.

Theorem 4.4.3. *Up to semisimplification, there is an isomorphism of Galois representations*

$$\rho_{A,2} \simeq \rho_{f,2}.$$

Proof. Restricting the representations to the absolute Galois group of the cubic extension L/K cut out by the residual representation, the residual image becomes trivial. We are now in a position to apply Livné's criterion.

Over L_A we define the modulus

$$\mathfrak{m}_{L_A} = \mathfrak{p}_2^7 \cdot \mathfrak{p}_{13,1} \cdot \mathfrak{p}_{19,1}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A}) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/216\mathbb{Z}).$$

Let $\{\chi_1, \dots, \chi_6\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A})$. Any set of primes $\{\mathfrak{p}_i\}$ for which the vectors $\{(\chi_1(\mathfrak{p}_i), \dots, \chi_6(\mathfrak{p}_i))\}$ cover $\mathbb{F}_2^6 \backslash \{0\}$ will satisfy the criterion. Following the algorithm [DGP10, §2.3, step (7)] we compute the set

$$T(C_2) = \{3, 37, 43, 61, 67, 73, 97, 103, 127, 151, 157, 193, 211, 307, 313, 343, 373, 433, 463, 499, 523, 631, 661, 823, 1321, 2197, 2557, 2917\}.$$

The traces of Frobenius agree on this set and so the G_L representations are isomorphic at the level of the cubic extension.

As explained in Proposition 4.3.2 this means that the full G_K representations could differ by a character. To show that the character is trivial we find that the prime above 5 is inert in the cubic extension and that the traces of Frobenius agree on this prime. Hence we can conclude that the two representations are isomorphic up to semisimplification. □

We finish with the main result:

Theorem 4.4.4. *The Jacobians of the following genus 2 curves are QM surfaces which are modular by a genuine Bianchi newform as in Conjecture 3.2.2.*

1 $C_1 : y^2 = x^6 + 4ix^5 + (-2i-6)x^4 + (-i+7)x^3 + (8i-9)x^2 - 10ix + 4i + 3$,
Bianchi newform: 2.0.4.1-34225.3-a;

2 $C_2 : y^2 = x^6 + (-2\sqrt{-3} - 10)x^5 + (10\sqrt{-3} + 30)x^4 + (-8\sqrt{-3} - 32)x^3 + (-4\sqrt{-3} + 16)x^2 + (-16\sqrt{-3} - 12)x - 4\sqrt{-3} + 16$,
Bianchi newform: 2.0.3.1-61009.1-a;

3 $C_3 : y^2 = (104\sqrt{-3} - 75)x^6 + (528\sqrt{-3} + 456)x^4 + (500\sqrt{-3} + 1044)x^3 + (-1038\sqrt{-3} + 2706)x^2 + (-1158\sqrt{-3} + 342)x - 612\sqrt{-3} - 1800$,
Bianchi newform: 2.0.3.1-67081.3-a;

4 $C_4 : y^2 = x^6 - 2\sqrt{-3}x^5 + (2\sqrt{-3} - 3)x^4 + 1/3(-2\sqrt{-3} + 54)x^3 + (-20\sqrt{-3} + 3)x^2 + (-8\sqrt{-3} - 30)x + 4\sqrt{-3} - 11$,
Bianchi newform: 2.0.3.1-123201.1-b.

Proof. We have proved this for C_2. The same techniques are applied below to show modularity of the other three QM-surfaces.

- **C1**: We define the modulus

$$\mathfrak{m}_K = \mathfrak{p}_2^5 \cdot \mathfrak{p}_{5,1} \cdot \mathfrak{p}_{37,2}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_K, \mathfrak{m}_K) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/36\mathbb{Z}).$$

Let $\{\psi_1, \ldots, \psi_4\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$. The set $\mathcal{P}$ of primes above $\{3, 5, 13\}$ ensures that $\{(\psi_1(\mathfrak{p}), \ldots, \psi_4(\mathfrak{p}))\}_{\mathfrak{p}\in\mathcal{P}}$ spans $\mathbb{F}_2^4$. The traces of Frobenius are all odd for the primes in $\mathcal{P}$ and hence $\mathrm{Im}(\overline{\rho}_{f,2}) \simeq C_3$.

There is only one cubic character so we can immediately deduce that $L_A \simeq L_f$.

Over L_A we define the modulus

$$\mathfrak{m}_{L_A} = \mathfrak{p}_2^3 \cdot \mathfrak{p}_{5,1} \cdot \mathfrak{p}_{37,1}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A}) = (\mathbb{Z}/2\mathbb{Z})^4.$$

Let $\{\chi_1, \ldots, \chi_4\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A})$. The primes needed to show that the full representations are isomorphic are the ones above

$$T(C_1) = \{5, 17, 61, 73, 121, 125, 157\}$$

since the vectors $\{(\chi_1(\mathfrak{p}), \ldots, \chi_4(\mathfrak{p}))\}_{\mathfrak{p}\in T(C_1)}$ cover $\mathbb{F}_2^4\backslash\{0\}$. The traces of Frobenius for the two representations agree on $T(C_1)$ so we can conlcude that $\rho_{A,2} \simeq \rho_{f,2}$ up to semisimplification.

- **C3**: We define the modulus

$$\mathfrak{m}_K = \mathfrak{p}_2^3 \cdot \mathfrak{p}_{7,1} \cdot \mathfrak{p}_{37,2}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_K, \mathfrak{m}_K) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/36\mathbb{Z}).$$

Let $\{\psi_1, \ldots, \psi_4\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$. The set $\mathcal{P}$ of primes above $\{13, 19\}$ ensures that $\{(\psi_1(\mathfrak{p}), \ldots, \psi_4(\mathfrak{p}))\}_{\mathfrak{p}\in\mathcal{P}}$ spans $\mathbb{F}_2^4$. The traces of Frobenius are all odd for the primes in $\mathcal{P}$ and hence $\mathrm{Im}(\overline{\rho}_{f,2}) \simeq C_3$.

Let φ_A denote the cubic character associated to L_A and set $\{\varphi_A, \chi\}$ to be a basis for the cubic characters. The prime $\mathfrak{p}_{43,1}$ is such that $\varphi_A(\mathfrak{p}_{43,1}) = 0$

and $\chi(\mathfrak{p}_{43,1}) \neq 0$. At this prime $\mathrm{Tr}(\overline{\rho}_{f,2}(\mathrm{Frob}_{\mathfrak{p}_{43,1}})) = \mathrm{Tr}(\overline{\rho}_{A,2}(\mathrm{Frob}_{\mathfrak{p}_{43,1}}))$ and so the cubic character associated to L_f must be in the span of φ_A. Hence we can conclude that the residual representations are isomorphic.

Over L_A we define the modulus

$$\mathfrak{m}_{L_A} = \mathfrak{p}_2^7 \cdot \mathfrak{p}_{7,1} \cdot \mathfrak{p}_{37,2}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A}) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/36\mathbb{Z}).$$

Let $\{\chi_1, \ldots, \chi_6\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A})$. The primes needed to show that the full representations are isomorphic are the ones above

$$T(C_3) = \{3, 13, 19, 31, 43, 73, 79, 103, 157, 163, 181, 199, 307, 313, 397, 409, 457, 487, 643, 661, 673, 691, 823, 829, 997, 1063, 1447, 1621, 2377, 2689\}$$

since the vectors $\{(\chi_1(\mathfrak{p}), \ldots, \chi_6(\mathfrak{p}))\}_{\mathfrak{p} \in T(C_3)}$ cover $\mathbb{F}_2^6 \backslash \{0\}$. The traces of Frobenius for the two representations agree on $T(C_3)$ so we can conlcude that $\rho_{A,2} \simeq \rho_{f,2}$ up to semisimplification.

- **C4**: We define the modulus

$$\mathfrak{m}_K = \mathfrak{p}_2^3 \cdot \mathfrak{p}_3 \cdot \mathfrak{p}_{13,1}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_K, \mathfrak{m}_K) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/12\mathbb{Z}).$$

Let $\{\psi_1, \ldots, \psi_4\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_K, \mathfrak{m}_K)$. The set $\mathcal{P}$ of primes above $\{7, 13, 19\}$ ensures that $\{(\psi_1(\mathfrak{p}), \ldots, \psi_4(\mathfrak{p}))\}_{\mathfrak{p} \in \mathcal{P}}$ spans $\mathbb{F}_2^4$. The traces of Frobenius are all odd for the primes in $\mathcal{P}$ and hence $\mathrm{Im}(\overline{\rho}_{f,2}) \simeq C_3$.

There is only one cubic character so we can immediately deduce that $L_A \simeq L_f$.

Over L_A we define the modulus

$$\mathfrak{m}_{L_A} = \mathfrak{p}_2^4 \cdot \mathfrak{p}_{7,1} \cdot \mathfrak{p}_{37,2}$$

and compute the ray class group to be

$$Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A}) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z}).$$

Let $\{\chi_1, \ldots, \chi_3\}$ be a dual basis of quadratic characters of $Cl(\mathcal{O}_{L_A}, \mathfrak{m}_{L_A})$. The primes needed to show that the full representations are isomorphic are the ones above

$$T(C_4) = \{7, 13, 61, 79, 97\}$$

since the vectors $\{(\chi_1(\mathfrak{p}), \ldots, \chi_3(\mathfrak{p}))\}_{\mathfrak{p}\in T(C_4)}$ cover $\mathbb{F}_2^3 \backslash \{0\}$. The traces of Frobenius for the two representations agree on $T(C_4)$ so we can conlcude that $\rho_{A,2} \simeq \rho_{f,2}$ up to semisimplification.

$\square$

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