On univalence, Rezk completeness and presentable quasi-categories

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The initial idea to consider the nerve of the internal category object associated to a fibration p and to compare completeness of the Reedy fibrant replacement of this Segal object with univalence of p, as done in Chapter 6, was suggested by Richard Garner and arose in a discussion with him and Nicola Gambino at the Logic Colloquium 2016 at the University of Leeds.

Abstract

This thesis is concerned with constructions in fibration categories and model categories motivated by Homotopy Type Theory and the relationship between homotopical algebra and higher category theory in the sense of Joyal and Lurie.

We present some general results on univalence in type theoretic fibration categories and type theoretic model categories, extending results of Shulman and generalizing results of Lumsdaine and Kapulkin. We then study the model structure for Bousfield-Segal spaces introduced by Bergner and relate the associated model structure for complete Bousfield-Segal spaces to the work of Rezk, Schwede and Shipley and of Cisinski, showing that it yields a model of Homotopy Type Theory. We further formulate and prove a strong relationship between Rezk's completeness condition of Segal objects and the univalence condition of fibrations in a large class of type theoretic model categories.

We give a definition of combinatorial model categories with universal homotopy colimits and semi-left exact left Bousfield localizations. Building on results of Dugger, Rezk, Lurie and Gepner and Kock, we show that these notions relate to locally cartesian closed presentable quasi-categories and semi-left exact localizations in the sense of Gepner and Kock in the same way as model toposes and left exact Bousfield localizations in the sense of Rezk relate to Grothendieck ∞-toposes and left exact localizations in the sense of Lurie. We further relate semi-left exactness to right properness.

We show that relative compact maps in presentable quasi-categories are exactly those maps presented by small fibrations between fibrant objects in Dugger's model categorical "small presentation" and discuss generalizations of this comparison to simplicial presheaf categories over small simplicial categories.

Contents

Introduction

Context

This thesis concerns higher categorical structures and properties related to the homotopical semantics of intensional type theory. The treatment of its topics therefore ranges between the areas of Homotopy Type Theory in the sense of [**41**], homotopical algebra à la Quillen [**42**] and Brown [**11**], and higher category theory in the sense of Joyal and Lurie [**36**]. There is a natural path to move from one area to another by mapping a type theory $\mathcal{T}$ to its syntactic category $\mathcal{C}(\mathcal{T})$ and a category $\mathbb{C}$ with weak equivalences to its underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{C})$.

$$
\begin{tikzcd}
 & \text{Homotopical algebra} \arrow[dr, "\mathrm{Ho}_\infty"] & \\
\text{Homotopy Type Theory} \arrow[ur, "\mathcal{C}"] & & (\infty,1)\text{-Category theory}
\end{tikzcd} \tag{1}
$$

These two assignments serve as compilers between different languages designed to express homotopy theory. They do not only allow us to transfer a plethora of constructions and statements from one setting to the other along the direction of their translation, but also often allow us to lift constructions and statements in the converse direction.

In homotopical algebra, the concept of homotopy is realized by structure built on top of ordinary categorical structure. Therefore, discourse about model categories (or fibration categories etc.) has two levels by nature, since to discuss functors and constructions which respect this structure requires us to discuss potentially non homotopy invariant constructions and functors on underlying categories in the first place.

Both $(\infty,1)$-category theory and Homotopy Type Theory express mathematics which is homotopy invariant by design. On the one hand, in $(\infty,1)$-category theory this is incarnated in the definition of higher categorical equivalences and the general principle that equivalent $(\infty,1)$-categories are indistinguishable. On the other hand, in Homotopy Type Theory this is achieved axiomatically via the

Univalence Axiom [**41**, Axiom 2.10.3], stipulating that for every two types the type of equivalences between them is equivalent to the type of propositional equalities between them. This is to say that equivalences satisfy Leibniz' Law – that is Indiscernibility of Identicals and Identity of Indiscernibles – as assumed implicitly in all branches of category theory as very nicely put in Barry Mazur's exposé [**38**].

To date, every interpretation of intensional type theory in $(\infty, 1)$-category theory factors through homotopical algebra, mainly because the syntactic rules as given by the calculus require on-the-nose constructions to interpret judgemental equality. Such interpretations have been given using ordinary category theoretic tools finding cleavages for certain Grothendieck fibrations. These cleavages correspond to pullback stable representatives of associated isomorphism classes, a level of "equality" between identity and homotopy equivalence not existent in $(\infty, 1)$-category theory.[1]

Restricting to suitable subclasses of objects in each realm we also obtain inverse assignments to (1). For example, every fibration category $\mathbb{C}$ yields an internal intensional type theory $\mathcal{T}_{\mathbb{C}}$ with dependent sum types. Denoting the collection of intensional type theories with dependent sum types by $\mathrm{ITT}_{\sum}$, Kapulkin and Szumiło in fact have shown in [**34**] that the edges of the following triangle are equivalences of homotopy theories in a suitable sense.

$$\begin{tikzcd} & \text{Fibration categories} \arrow[dr, "\mathrm{Ho}_\infty"] & \\ \mathrm{ITT}_{\sum} \arrow[ur, "\mathcal{C}"] & & \text{Left exact } (\infty,1)\text{-categories} \end{tikzcd} \tag{2}$$

Extending the three languages by further structure such as dependent products and universes renders the compiling procedure increasingly difficult and invertibility of the assignments $\mathcal{C}$ and Ho_∞ an open question. This question leads to the famous hypothesis that Homotopy Type Theory with all type formers from [**41**, A.2] and the theory of elementary ∞-toposes (thought of as locally cartesian closed $(\infty, 1)$-categories with enough impredicative power) are equivalent in a way analogous to (2).

The translation of such additional categorical structure becomes more tangible when restricting attention to presentable models in the following sense. For every

[1] Recent efforts to skip this intermediate level and interpret Homotopy Type Theory in $(\infty, 1)$-category theory via finding cleavages for corresponding cartesian fibrations and representatives of families of equivalences directly have been expressed for instance by Denis-Charles Cisinski.

presentable quasi-category $\mathcal{C}$ there is a combinatorial model category $\mathrm{Pres}(\mathcal{C})$ whose underlying quasi-category is equivalent to $\mathcal{C}$, such that also $\mathrm{Pres}(\mathrm{Ho}_\infty(\mathbb{M}))$ is Quillen equivalent to $\mathbb{M}$ for every combinatorial model category $\mathbb{M}$. The presentation $\mathrm{Pres}(\mathcal{C})$ can be chosen in such a way that the model category is type theoretic – i.e. its internal type theory also exhibits dependent function types as described in [**51**] – whenever $\mathcal{C}$ is locally cartesian closed by [**23**, Section 7]. Certainly every type theoretic model category $\mathbb{M}$ gives rise to its category $\mathbb{M}^f$ of fibrant objects which is type theoretic, too, so the transition forth and back may be depicted as follows.

$$
\begin{tikzcd}
\begin{array}{c}\text{Type theoretic fibration}\\ \text{categories}\\ \text{(with additional structure?)}\end{array} \arrow[d, dashed, bend left, "\mathcal{T}"] & \begin{array}{c}\text{Type theoretic model}\\ \text{categories}\\ \text{(with additional structure?)}\end{array} \arrow[l, "(\_)^f"'] \\
\begin{array}{c}\mathrm{ITT}_{\Sigma,\Pi}\\ \text{(with additional structure?)}\end{array} \arrow[u, bend left, "\mathcal{C}"] & \begin{array}{c}\text{Type theoretic combinatorial model}\\ \text{categories}\\ \text{(with additional structure?)}\end{array} \arrow[u, hook] \arrow[d, bend left, "\mathrm{Ho}_\infty"] \\
 & \begin{array}{c}\text{Locally cartesian closed presentable}\\ (\infty,1)\text{-categories}\\ \text{(with additional structure?)}\end{array} \arrow[u, dashed, bend left, "\mathrm{Pres}"]
\end{tikzcd}
\tag{3}
$$

Diagram (3) is the landscape this thesis dwells about, discussing transitions of additional structures between the vertices of the diagram with a focus on homotopical algebra and $(\infty, 1)$-category theory. In the following, we give an overview of the main contributions of this thesis.

Outline and main results

Chapter 1 recalls some fundamentals of the semantics of intensional type theories from [**51**] and discusses the interpretations of (-1)-truncatedness, function extensionality and univalence therein. Formalizing the meaning of univalence as a homotopy-uniqueness property, we construct the object $P_{q,p}$ of pullback representations of fibrations q along fibrations p and characterize univalence of p by means of (-1)-truncatedness of these objects, generalizing [**33**, Theorem 3.5.3] as follows.

Theorem 1.5.2. Let $\mathbb{C}$ be a type theoretic fibration category with function extensionality. Then a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$ is univalent if and only if for every fibration $q\colon X \twoheadrightarrow Y$ in $\mathbb{C}$, the object $P_{q,p}$ is (-1)-truncated.

This characterization will be used to show that univalence between two fibrations is invariant under homotopy equivalence (Corollary 1.6.4).

In Chapter 2 we discuss univalence of fibrations $p\colon E \twoheadrightarrow B$ and related notions in type theoretic model categories by means of diagrammatic properties of their associated canonical local class F_p. The *weak equivalence extension property* and the *fibration extension property* of F_p have been useful tools to verify univalence (and fibrancy of the base B) of p in the literature, so here we want to give a precise analysis of how these properties relate. While most proofs are quite straightforward, it seems that one of these relations – given in Lemma 2.4.3 – has not been mentioned in any published work to date, although it shortens proofs on the existence of universes in the literature considerably. It follows for example that the use of minimal fibrations in the proofs of fibrancy of the universe (cf. [**33**] and consequently [**52**] for example) is not necessary. Indeed, from Lemma 2.4.3 and [**52**] we can derive the following theorem.

Theorem 2.5.10. Let $\mathbb{D}$ be a small category and consider $\mathrm{sPsh}(\mathbb{D})$ as equipped with the injective model structure. Then $\mathrm{sPsh}(\mathbb{D})$ is a type theoretic model category. Let $\kappa > \mathfrak{c}(\mathrm{sPsh}(\mathbb{D}))$ be an inaccessible cardinal. Then the class S_κ of κ-small maps has the weak equivalence extension property and the fibration extension property with respect to a set of generating acyclic cofibrations. Hence, if the codomains of the generating acyclic cofibrations are representable, $\mathrm{sPsh}(\mathbb{D})$ supports a univalent universal fibration for S_κ with fibrant base.

In order to move forward in the landscape (3) of combinatorial model categories and presentable quasi-categories, in Chapter 3 we recall the essential basics of the theory of combinatorial model categories and left Bousfield localizations from Hirschhorn's standard reference [**25**]. Here, we set the basic foundation for the rest of the thesis and briefly discuss in this specific context some of the notions we have considered in the first two chapters. More precisely, we note that left Bousfield localization preserves the weak equivalence extension property for every class $\mathcal{S}$ of maps, and we further note that Cisinski's strictification of univalent universes in the locally constant model structure over elegant Reedy categories from [**14**, Proposition 1.1] can be applied to every left Bousfield localization of a right proper model category which comes equipped with a strict univalent universe itself.

In Chapters 4 and 5 we discuss a concrete type theoretic combinatorial model category $(s\mathbf{S}, \mathrm{CB})$ which gives an alternative presentation of the quasi-category $\mathcal{S}$ of spaces. The model category $(s\mathbf{S}, \mathrm{CB})$ is constructed via the model structure $(s\mathbf{S}, \mathrm{B})$ for *Bousfield-Segal spaces* on bisimplicial sets as introduced by Bergner in [**7**, Section 6]. In Chapter 4 we give the definition of $(s\mathbf{S}, \mathrm{B})$, present basic constructions and properties of Bousfield-Segal spaces and show in Theorem 4.4.7 that the fibrant objects in this model category are exactly the Segal spaces with invertible edges.

Theorem 4.4.7. Every Bousfield-Segal space is a Segal space. In particular, the model structures $(s\mathbf{S}, B)$ and $\mathcal{L}_B(s\mathbf{S}, S)$ coincide.

Chapter 5 discusses the model category $(s\mathbf{S}, \mathrm{CB})$ of complete Bousfield-Segal spaces and relates it to both Rezk, Schwede and Shipley's canonical model structure from [**46**] and the work of Cisinski on locally constant model structures over elegant Reedy categories (since these two coincide here). This shows that $(s\mathbf{S}, \mathrm{CB})$ is Quillen equivalent to the model category $(\mathbf{S}, \mathrm{Kan})$ for Kan complexes in Theorem 5.1.14 and yields a model of Homotopy Type Theory with all standard type formers including an infinite sequence of univalent universes in Section 5.3.

Having discussed univalence and completeness independently of each other so far, in Chapter 6 we go back to type theoretic model categories and make precise an analogy between univalence and completeness that has been subject to informal discussions in the research community (see e.g. [**50**]). More precisely, we combine notions treated in Chapter 1 and Chapter 5 to generalize the univalence property from fibrations to Segal objects, and give another characterization of univalence (of fibrations) via completeness of associated Reedy fibrant Segal objects, as stated below.

Theorem 6.4.4. Let X be a Segal object in a type theoretic model category $\mathbb{M}$ such that all fibrant objects in $\mathbb{M}$ are cofibrant. Then X is univalent if and only if for any Reedy fibrant replacement $\mathbb{R}X$ of X, the Segal object $\mathbb{R}X$ is complete.

Given a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}^f$, as a special case we obtain that p is a univalent fibration in the type theoretic fibration category $\mathbb{M}^f$ if and only if for any Reedy fibrant replacement $\mathbb{R}Np$ of its associated internal nerve Np, the Segal object $\mathbb{R}Np$ is complete.

In Chapter 7 we shift our focus entirely to combinatorial model categories and presentable $(\infty, 1)$-categories. We recall the presentation results for presentable quasi-categories and Grothendieck ∞-toposes due to Dugger, Rezk and Lurie, but focus on a class of quasi-categories which lies inbetween these two classes. This is the class of presentable locally cartesian closed quasi-categories treated in [**23**]. We introduce and discuss combinatorial model categories with *universal homotopy colimits* and their presentation as *semi-left exact* left Bousfield localizations of simplicial presheaf categories to give a systematic treatment of notions considered in [**23**] for $(\infty, 1)$-categories in the realm of combinatorial model categories.

Theorem 7.2.4. A combinatorial model category $\mathbb{M}$ has universal homotopy colimits if and only if its associated quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ has universal colimits.

The notion of semi-left exactness for reflective localizations of ordinary presentable categories has been studied for example in [**22**], its $(\infty, 1)$-categorical generalization was introduced in [**23**]. Here, we define semi-left exactness for left Bousfield localizations consistent with Gepner and Kock's definition of semi-left exactness on underlying quasi-categories. While, on the one hand, we therefore will see that semi-left exactness and universality of homotopy-colimits are strongly related, in Lemma 7.3.9 we show that semi-left exactness also characterizes right properness.

This is interesting, since, although we will see that right properness and universality of homotopy colimits are related in special cases, they are generally independent of each other. Combining these observations, in Corollary 7.3.17 we obtain a presentation result à la Dugger and Rezk for combinatorial model categories $\mathbb{M}$ such that the presentation of $\mathbb{M}$ is right proper if and only if $\mathbb{M}$ has universal homotopy colimits.

In Chapter 8 we broach two issues associated with Diagram (3) in the context of the question whether every ∞-Grothendieck topos $\mathcal{C}$ can be presented by a fibration category $\mathbb{C}$ which models Homotopy Type Theory in the sense of [**51**]. This is motivated by the above mentioned hypothesis that HoTT is the internal language of elementary ∞-toposes. The work of [**23**, 7] shows that every presentable locally cartesian closed quasi-category is presented by a type theoretic model category which models intensional type theory with all standard type formers but potentially univalent universes. But it is open to date whether the existence of small object classifiers in an ∞-topos $\mathcal{C}$ induces the existence of universal fibrations classifying small fibrations in a presentation $\mathbb{M}$ which still models all of the rest of HoTT. A weak form of this problem is given by the question whether small object classifiers in $\mathcal{C}$ correspond to weakly universal small fibrations in $\mathbb{M}$. In other words, do (weak or strict) Tarski universes as commonly interpreted in homotopical algebra "compile correctly" to universes in ∞-toposes? Necessary for such a correspondence – and in fact the only non-trivial obstruction – is the comparison of the two potentially different smallness notions as will be explained in Section 8.1. Therefore, on the one hand, although with various limitations, we will provide such a comparison between κ-small fibrations in $\mathbb{M}$ and relative κ-compact maps in $\mathcal{C}$ for sufficiently large cardinals κ. Such a comparison is given in Corollary 8.3.13 for Dugger's presentations of combinatorial model categories and in Theorem 8.3.14 more generally as follows.

Theorem 8.3.14. Let $\mathbf{C}$ be a small simplicial category and T a subset of arrows in $\mathrm{sPsh}(\mathbf{C})$. Let $\mathbb{M}$ be the left Bousfield localization $\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}$. Then every κ-small fibration $p \in \mathbb{M}$ between fibrant objects is relative κ-compact in the underlying quasi-category. Vice versa, if a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact, then there is a κ-small map $g \in \mathrm{sPsh}(\mathbf{C})$ such that $g \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.

On the other hand, since interpreting intensional type theory in $(\infty,1)$-category theory factors through homotopical algebra as depicted in Diagram (1), a potential converse process of generating an internal language $\mathcal{T}$ associated to a given quasi-category $\mathcal{C}$ has to factor through the choice of a fibration category $\mathbb{C}$ presenting $\mathcal{C}$. That means we have to choose a fibration category $\mathbb{C}$ whose underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{C})$ is equivalent to $\mathcal{C}$, such that $\mathbb{C}$ comes with enough structure to yield an adequately expressive internal type theory $\mathcal{T}_{\mathbb{C}}$ in the sense of [**34**] or [**51**]. While we do not aim to make this precise, the point we want to make here is that this presentation $\mathbb{C}$ of $\mathcal{C}$ has to be chosen carefully with respect to the categorical structure in $\mathcal{C}$ that we want to express type theoretically in order for this to work. Indeed, in Section 8.4 we will provide a large class of presheaf ∞-categories $\mathcal{C}$ and for each such a presentation $\mathbb{M}$ which does come equipped with an internal type theory $\mathcal{T}_{\mathbb{M}^f}$ supporting all type constructors including univalent Tarski universes in the sense of [**51**] (Theorem 8.4.8). But $\mathcal{T}_{\mathbb{M}^f}$ will fail to be an internal language of $\mathcal{C}$ in the sense that the type constructions in $\mathcal{T}_{\mathbb{M}^f}$ do not translate to the corresponding quasi-categorical constructions in $\mathcal{C}$ (Theorem 8.4.9).

Overall, the results of this thesis are intended to provide progress in the understanding of categorical concepts motivated by the type theoretical semantics in homotopy theory and higher category theory. As such, it contributes to the long-term objective of creating an interplay between intensional and univalent type theories and higher-dimensional categories that is as close, precise and fruitful as the one existing between extensional type theories and one-dimensional categories. Since our work does make as little direct reference to the syntax of type theory as possible, we hope that it is accessible also to readers who are not experts in the type theory.

Notation

Throughout this thesis, we adopt the following notational conventions. The symbol $\mathbf{S}$ denotes the category of simplicial sets and $\mathcal{S}$ its associated quasi-category of spaces. Whenever $\mathbf{C}$ and $\mathbf{D}$ are simplicial categories (i.e. $\mathbf{C}$ and $\mathbf{D}$ are $\mathbf{S}$-enriched), their associated simplicial category of $\mathbf{S}$-enriched functors is denoted by $[\mathbf{C},\mathbf{D}]_{\mathbf{S}}$. The underlying category of the simplicial category $[\mathbf{C}^{op},\mathbf{S}]_{\mathbf{S}}$ of simplicial presheaves over $\mathbf{C}$ is denoted by sPsh($\mathbf{C}$). The category $[\Delta^{op},\mathbb{C}]$ of simplicial objects in an ordinary category $\mathbb{C}$ will be denoted by $s\mathbb{C}$. In particular, $s\mathbf{S}$ denotes the category of bisimplicial sets. Following standard conventions, given a functor $f\colon \mathbf{C}\to\mathbf{D}$, the associated restriction functor between simplicial presheaf categories is denoted by f^* with left adjoint $f_!$ and right adjoint f_*.

In an ordinary category $\mathbb{C}$, the slice category over $C\in\mathbb{C}$ is denoted by $\mathbb{C}/C$. Given an arrow $f\colon C\to D$ in $\mathbb{C}$, the associated pullback functor is denoted by $f^*\colon \mathbb{C}/D\to\mathbb{C}/C$ with left adjoint $\sum_f$ (the "dependent sum" along f) and right adjoint $\prod_f$ (the "dependent product" along f) whenever they exist.

The "freely walking arrow" category, generated by two distinct objects and one morphism between them, is denoted by [1]. So for every category $\mathbb{C}$, we obtain the arrow category $\mathbb{C}^{[1]}$ whose objects are morphisms in $\mathbb{C}$ and whose morphisms are squares in $\mathbb{C}$. The free groupoid generated by the walking arrow [1] will be denoted by I and, again following standard conventions, referred to as the "freely walking isomorphism".

Model categories will be denoted by tuples $(\mathbb{M}, M)$ where $\mathbb{M}$ is a bicomplete category and M is a name for the model structure $(\mathcal{C}_M, \mathcal{W}_M, \mathcal{F}_M)$ on $\mathbb{M}$. Here, the classes $\mathcal{C}_M$, $\mathcal{W}_M$ and $\mathcal{F}_M$ denote the classes of cofibrations, weak equivalences and fibrations respectively. The arrows in these will sometimes be called M-(co)fibrations and weak M-equivalences, maps in $\mathcal{C}_M \cap \mathcal{W}_M$ are M-acyclic cofibrations. The category of cofibrant objects will be denoted by $\mathbb{M}^c$ and the category of fibrant objects by $\mathbb{M}^f$. As it is standard, diagrammatically cofibrations are depicted by arrows of the form "$\hookrightarrow$", weak equivalences by "$\xrightarrow{\sim}$" and fibrations by "$\twoheadrightarrow$". Variables freely ranging over some class of model categories will be denoted by $(\mathbb{M}, M)$ or $(\mathbb{N}, N)$ and short handedly referred to by their underlying categories $\mathbb{M}$ and $\mathbb{N}$. A Quillen pair consisting of a left Quillen functor $F\colon \mathbb{M} \to \mathbb{N}$ and its right adjoint G will be abbreviated by $(F, G)\colon \mathbb{M} \to \mathbb{N}$.

The standard examples of model categories relevant for this thesis are the Joyal model structure $(\mathbf{S}, \mathrm{Qcat})$ for quasi-categories and the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$ for Kan complexes on simplicial sets.

CHAPTER 1

Univalence and homotopy pullbacks

1.1. A short foreword on univalence

The development of Homotopy Type Theory was sparked by the understanding of intensional type theory as a possible internal language of suitable homotopy theories. Here, such a suitable homotopy theory is given by a fibration category or a model category (giving rise to the fibration category of its fibrant objects) satisfying a few additional axioms. The crucial observation linking intensional type theories to fibration categories was first presented in [**2**]. The authors noted that the *J-eliminator* for identity types corresponds to a lift in certain squares proving that the class of maps $r_A\colon A \to \mathrm{Id}_A$ associated to the introduction rule of identity types has the left lifting property with respect to all display maps. It was shown in [**40**] that the rules for identity types in fact generate a weak factorization system on the syntactical category of any given intensional type theory, whose elements are thought of as acyclic cofibrations and fibrations respectively. The classical approach of modelling extensional dependent type theories in locally cartesian closed categories is adapted to the idea of modelling type families as fibrations – in order to respect the syntactical rules of identity types – by requiring the class of fibrations to satisfy certain closure properties. In [**51**, Sections 2 and 3], Shulman introduced the notions of type theoretic fibration categories and type theoretic model categories isolating sufficient criteria for such categories to support a model of intensional type theory with dependent sums and dependent functions. These notions will be the basic notions underlying all type theoretic considerations in this thesis.

Following the insight that the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$ on the category of simplicial sets supports a model of intensional type theory ([**33**]), many homotopical models of intensional type theory have been constructed. In all these models identity types are interpreted as path objects. The fact that identity on a type A is an equivalence relation on A is interpreted by the homotopically coherent concatenation, inversion and unitality rules on paths.

Given a universe type U as a special case, the type family of equivalences is another equivalence relation on U and it was a fundamental observation of Voevodsky's that in $(\mathbf{S}, \mathrm{Kan})$ the type family of equivalences is a path object for U. This idea is captured syntactically by the Univalence Axiom which states that every universal type family $\pi\colon \tilde{U} \twoheadrightarrow U$ is *univalent* in a sense to be recalled below, stating that propositional equality $A =_U B$ of types $A, B : U$ and equivalence $A \simeq B$ of such types in U are equivalent relations within the type theory. Identity types were originally introduced by Martin-Löf, while the type of equivalences was introduced by Voevodsky in [**56**]. The definitions can be found in [**41**, A.2.10] and [**51**, Section 5] respectively.

The property of univalence is very central to Homotopy Type Theory and, therefore, this first chapter is a discussion of type theoretic fibration categories and univalent fibrations therein.

Section 1.2 recalls the corresponding fundamentals. In Section 1.3 we give a definition of function extensionality which will be a property assumed throughout most of this chapter and briefly discuss the type of equivalences between two fibrations. In Section 1.4 we recall a few characterizations of (-1)-truncated fibrations and show that for such fibrations the paths in the total space are (up to homotopy) exactly paths in the base space between endpoints lying in the total space. This will prove useful in later sections. In Section 1.5 we give the definition of univalence in type theoretic fibration categories and characterize univalent fibrations by means of (-1)-truncatedness and their pullback representations, generalizing [**33**, Theorem 3.5.3]. Using results from Section 1.4, Section 1.6 assures that the notions of the previous section are homotopy invariant, also generalizing the corresponding results from [**33**, Section 3].

1.2. Preliminaries

In the following we recall the definitions of type theoretic fibration categories and type theoretic model categories from [**51**] and cite a few basic facts about the homotopy theory within such. For various related notions of fibration categories, see [**11**] for categories of fibrant objects and [**29**] for tribes.

Definition 1.2.1 ([**51**, Definition 2.1])**.** A *type theoretic fibration category* is a pair $(\mathbb{C}, \mathcal{F})$ where $\mathbb{C}$ is a category and $\mathcal{F} \subseteq \mathbb{C}$ is a class of arrows, called *fibrations*, such that the following hold.

(1) $\mathbb{C}$ has a terminal object 1.

(2) $\mathcal{F} \subseteq \mathbb{C}$ is closed under composition and contains all isomorphisms and all morphisms with codomain 1.
(3) All pullbacks of fibrations exist and are fibrations.
(4) Let $\mathcal{AC}$ be the class of morphisms with the left lifting property with respect to all fibrations. Elements of $\mathcal{AC}$ are called acyclic cofibrations and depicted by arrows of the form $A \overset{\sim}{\hookrightarrow} B$. Then every morphism factors as an acyclic cofibration followed by a fibration.
(5) Given an object X in $\mathbb{C}$, let $\mathcal{F}/X$ denote the full subcategory of $\mathbb{C}/X$ whose objects are the fibrations with codomain X. Then for every fibration $p\colon X \twoheadrightarrow Y$, the pullback functor $p^*\colon \mathcal{F}/Y \to \mathcal{F}/X$ has a right adjoint $\prod_p$.
(6) In the following commutative diagram

$$\begin{tikzcd}
X_1 \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & V_1 \arrow[dr, hook, "\sim"] \arrow[ddr, bend right] & \\
& X_2 \arrow[rr] \arrow[d] & & V_2 \arrow[d] \\
& Y \arrow[rr] & & W,
\end{tikzcd}$$

if the maps $V_i \to W$ are fibrations, $V_1 \to V_2$ is an acyclic cofibration, and all squares are pullbacks (hence $X_i \to Y$ are fibrations), then $X_1 \to X_2$ is also an acyclic cofibration.

Note that condition (5) in Definition 1.2.1 implies that acyclic cofibrations are stable under pullback along fibrations. The class of fibrations in a type theoretic fibration category $\mathbb{C}$ generates a weak factorization system $(\mathcal{AC}, \mathcal{AC}^{\pitchfork})$ on $\mathbb{C}$ where $\mathcal{AC}^{\pitchfork}$ denotes the class of morphisms with the right lifting property with respect to all acyclic cofibrations. In particular we have $\mathcal{F} \subseteq \mathcal{AC}^{\pitchfork}$. The existence of pullbacks along fibrations and factorizations as required in Definition 1.2.1.(5) induces a notion of path objects in type theoretic fibration categories $\mathbb{C}$. Recall that a *path object* for a fibration $X \twoheadrightarrow Y$ is a factorization $X \overset{\sim}{\hookrightarrow} P_Y X \twoheadrightarrow X \times_Y X$ of the diagonal $X \to X \times_Y X$ over Y into an acyclic cofibration followed by a fibration. We will sometimes denote the left map by $r_X\colon X \overset{\sim}{\hookrightarrow} P_Y X$ and the right map by $\partial_X\colon P_Y X \twoheadrightarrow X \times_Y X$, omitting the index when appropriate. We hence obtain a notion of *right homotopy* and *homotopy equivalence* in $\mathbb{C}$ as explained in [**51**, Section 2]. An *acyclic fibration* is a fibration which also is a homotopy equivalence.

Lemma 1.2.2 ([**51**, Section 3])**.** *Let $\mathbb{C}$ be a type theoretic fibration category.*

(1) A map $f\colon A \to B$ in $\mathbb{C}$ is an acyclic cofibration if and only if there is a retract $r\colon B \to A$ with a homotopy $fr \sim id_B$ under A.
(2) A fibration $p\colon X \twoheadrightarrow Y$ in $\mathbb{C}$ is an acyclic fibration if and only if there is a section $s\colon Y \to X$ with a homotopy $sp \sim id_X$ over Y.
(3) $\mathbb{C}$ with its class $\mathcal{W}$ of homotopy equivalences is a category of fibrant objects à la Brown (see [**11**, Part 1.1] *for the definition), so in particular $\mathcal{W}$ satisfies 2-for-3 and pullbacks of acyclic fibrations are acyclic fibrations.*
(4) Given an object $C \in \mathbb{C}$, let $(\mathbb{C}/C)_f$ denote the full subcategory of $\mathbb{C}/C$ whose objects are the fibrations over C in $\mathbb{C}$. This is a type theoretic fibration category. Then a morphism $f\colon X_1 \to X_2$ in $\mathbb{C}/Y$ between fibrations $X_1 \twoheadrightarrow Y$ and $X_2 \twoheadrightarrow Y$ is a homotopy equivalence in $(\mathbb{C}/Y)_f$ if and only if it is a homotopy equivalence in $\mathbb{C}$.

□

Shulman has shown in [**51**, Section 4.2] that every type theoretic fibration category $\mathbb{C}$ has an internal type theory $\mathcal{T}_{\mathbb{C}}$ with dependent sums and dependent function types. In that sense, we sometimes will refer to a fibration $X \twoheadrightarrow Y$ in a type theoretic fibration category $\mathbb{C}$ as a type X in context Y.

Following [**51**, Definition 2.12], we further give a definition of type theoretic model categories. For the definition of a model category see [**42**] or [**27**].

Definition 1.2.3. A *type theoretic model category* is a model category $\mathbb{M}$ with the following additional properties.

(1) Limits preserve cofibrations, i.e. any natural transformation that is a levelwise cofibration induces a cofibration between the limits.
(2) $\mathbb{M}$ has the *Frobenius property*, i.e. acylic cofibrations are stable under pullback along fibrations.
(3) Pullback p^* along any fibration p has a right adjoint $\prod_p$.

Note that condition (1) in Definition 1.2.3 implies that cofibrations in type theoretic model categories are stable under pullback. Under condition (1), condition (2) is equivalent to right properness. It is easy to see that every type theoretic model category yields a type theoretic fibration category as follows.

Proposition 1.2.4 ([**51**, Proposition 2.13])**.** *Let $\mathbb{M}$ be a model category. If $\mathbb{M}$ is type theoretic, then its full subcategory $\mathbb{M}^f$ of fibrant objects is a type theoretic fibration category.*

□

However, the converse does not hold in general. We will encounter a family of examples in Section 8.4.

1.3. Function extensionality

In the following sections, we adopt the notation for the standard type constructors from [**41**]. In particular, given a type Y, the type $y =_Y y'$ denotes the identity type of y and y' in Y, and given a type family $y : Y \vdash X$ type, the type $\prod_{y:Y} X(y)$ denotes the type of dependent functions from Y to X. The latter is abbreviated by $Y \to X$ whenever X is a constant type family.

Recall that in an intensional type theory, for every type family $y : Y \vdash X$ type and every two dependent functions $f, g : \prod_{y:Y} X(y)$, there is a canonical map

$$\mathsf{happly} : (f = g) \to \prod_{y:y} f(y) =_{X(y)} g(y)$$

obtained by path induction. Reading types as propositions, the term happly witnesses that equality of two functions implies pointwise equality of their values.

A type theory satisfies *function extensionality* if for every type family $y : Y \vdash X$ type and every two dependent functions $f, g : \prod_{y:Y} X(y)$, the function happly is an equivalence. In homotopical algebraic terms, the principle of function extensionality holds in the internal language of a type theoretic fibration category $\mathbb{C}$ if and only if for every pair of fibrations $p \colon X \twoheadrightarrow Y$ and $q \colon Y \twoheadrightarrow Z$, the canonical map

$$\mathsf{happly} \colon P_Z(\prod_q X) \to \prod_q (P_Y X),$$

which is obtained as a diagonal filler in the square

$$\begin{tikzcd}
\prod_q X \arrow[r, "\prod_q r_X"] \arrow[d, hook, "r_{\prod_q X}"', "\wr"] & \prod_q P_Y X \arrow[d, two heads, "\prod_q \partial_X"] \\
P_Z(\prod_q X) \arrow[r, two heads, "\partial_{\prod_q X}"'] & \prod_q X \times_Z \prod_q X,
\end{tikzcd}$$

is a homotopy equivalence. Or in other words, if and only if $\prod_q (P_Y X)$ is (up to homotopy equivalence) a path object for $\prod_q X$ over Z. In fact, this is equivalent to assume the existence of a map over $\prod_q X \times_Z \prod_q X$ in the converse direction (which is a priori not necessarily a homotopy equivalence) as shown for instance in [**51**, Theorem 5.6].

Towards another characterization of function extensionality, let $\mathbb{C}$ be a type theoretic fibration category. Say that dependent products along fibrations preserve homotopy equivalences between fibrations in $\mathbb{C}$ if for all fibrations $X_1 \twoheadrightarrow Y$, $X_2 \twoheadrightarrow Y$ and $q\colon Y \twoheadrightarrow Z$ together with a homotopy equivalence $X_1 \xrightarrow{\simeq} X_2$ over Y,

$$
\begin{tikzcd}
X_1 \arrow[rr] \arrow[dr, "\wr" sloped] \arrow[ddr, bend right] & & \prod_q X_2 \arrow[dr] \arrow[ddr, bend right] & \\
& X_2 \arrow[rr] \arrow[d] & & \prod_q X_2 \arrow[d] \\
& Y \arrow[rr, two heads, "q"'] & & Z,
\end{tikzcd}
\tag{1.3.1}
$$

the natural map $\prod_q X_1 \to \prod_q X_2$ is a homotopy equivalence, too.

Lemma 1.3.1. *Let $\mathbb{C}$ be a type theoretic fibration category. Then the following conditions are equivalent.*

(1) The internal language $\mathcal{T}_{\mathbb{C}}$ satisfies function extensionality.
(2) Dependent products along fibrations in $\mathbb{C}$ preserve path objects (up to homotopy).
(3) Dependent products along fibrations preserve acyclicity of fibrations.
(4) Dependent products along fibrations preserve homotopy equivalences.

Proof. The equivalence of (1) and (2) was shown above. The equivalence of (1) and (3) was shown in [**51**, Lemma 5.9] using a formulation of function extensionality in terms of contractibility of certain dependent function types due to Voevodsky. The equivalence of (3) and (4) is easy to verify, below we give a direct proof of the equivalence (2) and (4).

Assume the internal language of $\mathbb{C}$ satisfies function extensionality. Given a diagram of the form (1.3.1), we want to show that the map $\prod_q X_1 \to \prod_q X_2$ is a homotopy equivalence. Therefore, let $q\colon Y \twoheadrightarrow Z$ be a fibration in $\mathbb{C}$ and $v\colon X_1 \to X_2$ be a homotopy equivalence between fibrations $X_1 \twoheadrightarrow Y$ and $X_2 \twoheadrightarrow Y$ with homotopy inverse v^{-1}. By functoriality of the dependent product $\prod_q$, we obtain maps

$$\begin{tikzcd} \prod_q X_1 \arrow[r, shift left, "\prod_q v"] & \prod_q X_2 \arrow[l, shift left, "\prod_q v^{-1}"] \end{tikzcd}$$

over Z and we have to show that they are mutually inverse. But a homotopy $\prod_q v \prod_q v^{-1} \sim \mathrm{id}$ corresponds to a lift

$$\begin{tikzcd} & P_Z \prod_q X_1 \arrow[d, two heads] \\ \prod_q X_1 \arrow[ur, dashed] \arrow[r, "(\prod_q (vv^{-1}), \mathrm{id})"'] & \prod_q X_1 \times_Z \prod_q X_1. \end{tikzcd}$$

Denoting the counit of the adjoint pair $(q^*, \prod_q)$ by $\varepsilon\colon q^* \prod_q X \to X$, this in turn corresponds to a lift

$$\begin{tikzcd} & P_Z X \arrow[d, two heads] \\ q^* \prod_q X \arrow[ur, dashed] \arrow[r, "(\varepsilon \circ q^* \prod_q (vv^{-1}), \varepsilon)"'] & X \times_Z X \end{tikzcd}$$

by function extensionality. But the square

$$\begin{tikzcd} q^* \prod_q X \arrow[r, "\varepsilon"] \arrow[d, "q^* \prod_q (vv^{-1})"'] & X \arrow[d, "vv^{-1}"] \\ q^* \prod_q X \arrow[r, "\varepsilon"'] & X \end{tikzcd}$$

commutes by naturality of the counit, giving the desired lift by a homotopy $vv^{-1} \sim \mathrm{id}$ precomposed with ϵ. Constructing a homotopy $\prod_q v^{-1} \prod_q v \sim \mathrm{id}$ works analogously. The other direction is immediate. □

Definition 1.3.2. We say that $\mathbb{C}$ satisfies *function extensionality* if any of the conditions of Lemma 1.3.1 are satisfied.

Remark 1.3.3. Applying [**51**, Lemma 5.9], Shulman remarks in [**51**, Section 5] that, if the acyclic fibrations are the right class in a weak factorization system, then function extensionality is equivalent to pullback stability of the corresponding left class (the “cofibrations”) along fibrations. This is the case in any type theoretic model category in which every weak equivalence between fibrant objects is a homotopy equivalence.

Lemma 1.3.4. *Let $\mathbb{M}$ be a type theoretic model category. Then every weak equivalence between fibrant objects is a homotopy equivalence if and only if all fibrant objects in $\mathbb{M}$ are cofibrant.*

Proof. If all fibrant objects are cofibrant, then weak equivalences between such are homotopy equivalences by [**27**, Proposition 1.2.8]. On the other hand, if weak equivalences between fibrant objects are homotopy equivalences, let X be fibrant and X' be a cofibrant replacement of X in $\mathbb{M}$. Thus we have an acyclic fibration $p\colon X' \twoheadrightarrow X$ in $\mathbb{M}$ (that is a fibration which is a weak equivalence) between the fibrant objects X' and X. This is a homotopy equivalence by assumption and hence an acyclic fibration in the type theoretic fibration category $\mathbb{M}^f$. By Lemma 1.2.2.(2) p has a section, so that X is a retract of the cofibrant object X'. Therefore X is cofibrant itself. □

Cofibrancy of all fibrant objects will be a recurring assumption on type theoretic model categories in most of the following sections as well as in Chapters 2 and 6.

Remark 1.3.5. A morphism in $\mathbb{C}$ between two fibrations with a common codomain C is a homotopy equivalence in $\mathbb{C}$ if and only if it is so in $\mathbb{C}/C$ by [**51**, Corollary 3.14]. It follows immediately by Lemma 1.3.1 that function extensionality is a local property of $\mathbb{C}$, i.e. dependent products along fibrations preserve homotopy equivalences between fibrations in $\mathbb{C}$ if and only if the same is true in all its slices $\mathbb{C}/C$.

An analogous statement holds for every type theoretic model category $\mathbb{M}$ in which all fibrant objects are cofibrant. For such a model category $\mathbb{M}$ even the slice fibration categories $(\mathbb{M}/C)^f$ for not necessarily fibrant objects $C \in \mathbb{M}$ satisfy function extensionality.

For the following sections, recall that for every two fibrations $p\colon X_1 \twoheadrightarrow Y$ and $q\colon X_2 \twoheadrightarrow Y$ in a type theoretic fibration category $\mathbb{C}$, the internal hom-object $[X_1, X_2]_Y \cong \prod_{X_1}(X_1 \times_Y X_2)$ models the type of functions $X_1 \to X_2$ in context Y. The type of equivalences

$$\mathrm{Equiv}_Y(X_1, X_2) \twoheadrightarrow [X_1, X_2]_Y,$$

is defined in [**51**, 5.5.4]. It is constructed in such a way that, in the words of the author of [**51**], to give a global element of $\mathrm{Equiv}_Y(X_1, X_2)$ over a global element $f\colon 1 \to [X_1, X_2]_Y$ is to give a homotopy section and a homotopy retraction of the morphism $X_1 \to X_2$ over Y named by f. More precisely, we have fibrations

$\mathrm{Linv}_Y(X_1,X_2) \twoheadrightarrow [X_1,X_2]_Y$ and $\mathrm{Rinv}_Y(X_1,X_2) \twoheadrightarrow [X_1,X_2]_Y$ given by the type families

$$\mathrm{Linv}_Y(X_1,X_2) :\equiv \sum_{g:[X_2,X_1]_Y} g\circ f = \mathrm{id}_{X_1}$$
$$\mathrm{Rinv}_Y(X_1,X_2) :\equiv \sum_{h:[X_2,X_1]_Y} f\circ h = \mathrm{id}_{X_2}$$

such that $\mathrm{Equiv}_Y(X_1,X_2) :\equiv \mathrm{Linv}_Y(X_2,X_1)\times_{[X_1,X_2]_Y}\mathrm{Rinv}_Y(X_1,X_2)$ over $[X_1,X_2]_Y$.

1.4. (-1)-truncated fibrations

In the following, let $\mathbb{C}$ be a type theoretic fibration category.

Definition 1.4.1. A fibration $p\colon X \twoheadrightarrow Y$ in $\mathbb{C}$ is (-1)-*truncated* if the path object fibration $\partial\colon P_YX \twoheadrightarrow X\times_Y X$ has a section.

In syntactical terms, it is immediate that $p\colon X \twoheadrightarrow Y$ is (-1)-truncated if and only if the type family X is a mere proposition in context Y, i.e. if the type $\mathsf{isProp}(X) :\equiv \prod_{x,x':X} x =_X x'$ is inhabited in context Y. Note that this definition of (-1)-truncatedness also coincides with the corresponding conventional homotopy theoretical definition whenever the latter can be formulated. We show this in Lemma 3.2.3, and so in particular we obtain a coincidence of notions whenever $\mathbb{C}$ is the fibration category associated to a combinatorial model category $\mathbb{M}$.

Lemma 1.4.2. *Let $p\colon X \twoheadrightarrow Y$ be a fibration in $\mathbb{C}$.*

(1) p is (-1)-truncated if and only if any two maps $f,g\colon A\to X$ over Y are homotopic over Y;
(2) p is acyclic if and only if it is (-1)-truncated and has a section.

Proof. We show part (1). If p is (-1)-truncated, let s be a section to $\partial\colon P_YX \twoheadrightarrow X\times_Y X$. Then s lifts any two maps $(f,g)\colon A\to X\times_Y X$ to a homotopy between f and g over Y. Vice versa, if any two maps into X over Y are homotopic over Y, then so are the projections $X\times_Y X \twoheadrightarrow X$ over Y. The corresponding homotopy is the desired section.

We show part (2). From left to right, assume p is an acyclic fibration. Then so is the projection $X\times_Y X\to X$ as a pullback of p by Lemma 1.2.2.(3). But the composition $P_YX \twoheadrightarrow X\times_Y X \twoheadrightarrow X$ is a homotopy equivalence, too, so $P_YX \twoheadrightarrow X\times_Y X$ is an acyclic fibration by 2-for-3. Then the existence of the section follows from Lemma 1.2.2.(2). Conversely, given a section t to p, we have to show that $tp \sim \mathrm{id}_X$ over Y. But this follows from part (1). □

There are various ways to verify (-1)-truncatedness of a given fibration, some are gathered in the following lemma.

Lemma 1.4.3. *For any fibration $p\colon X \twoheadrightarrow Y$ in $\mathbb{C}$, the following conditions are equivalent.*

(1) p is (-1)-truncated;
(2) the path object fibration $\partial\colon P_Y X \twoheadrightarrow X \times_Y X$ is an acyclic fibration;
(3) the projections $\pi_1, \pi_2\colon X \times_Y X \twoheadrightarrow X$ are acyclic fibrations;
(4) the projections $\pi_1, \pi_2\colon X \times_Y X \twoheadrightarrow X$ are homotopic to each other;
(5) there is a section $X \to \mathsf{isContr}(X)$.

Here, recall the type family $\mathsf{isContr}$ given by $\mathsf{isContr}(X) := \sum_{x:X} \prod_{x':X} x =_X x'$.

Proof. We use Lemma 1.2.2 throughout.

We prove (1) $\Rightarrow$ (2). Given a section t to ∂, we only have to show that $t\partial \sim \mathrm{id}_{P_Y X}$. But both are maps $P_Y X \to P_Y X$ over X, hence are homotopic over X by both parts of Lemma 1.4.2 as $P_Y X \twoheadrightarrow X$ is an acyclic fibration.

The implication (2) $\Rightarrow$ (1) follows by Lemma 1.2.2.(2).

The equivalence (2) $\Leftrightarrow$ (3) follows by the 2-for-3 property for homotopy equivalences. Part (5) is equivalent to Definition 1.4.1 since sections $X \to \mathsf{isContr}(X)$ are in 1-1 correspondence to sections $X \times_Y X \to P_Y X$ by construction.

The equivalence (1) $\Leftrightarrow$ (4) is immediate, because $\mathrm{id} = (\pi_1, \pi_2)\colon X \times_Y X \to X \times_Y X$. □

Corollary 1.4.4. (-1)*-truncated fibrations are closed under pullback.*

Proof. Let $p\colon X \twoheadrightarrow Y$ be a (-1)-truncated fibration and $f\colon A \to Y$ be a map in $\mathbb{C}$. Then f^*p is a fibration. By Lemma 1.4.3.(3) it suffices to show that the projections $\pi_i\colon f^*X \times_A f^*X \twoheadrightarrow f^*X$ are acyclic fibrations. But

$$\begin{tikzcd}
f^*X \times_A f^*X \arrow[r] \arrow[d, two heads, "\pi_i"] & X \times_Y X \arrow[d, two heads, "\pi_i"] \\
f^*X \arrow[r, "p^*f"'] & X
\end{tikzcd}$$

is a pullback square, too. Again by Lemma 1.4.3.(3), the projections $\pi\colon X \times_Y X \twoheadrightarrow X$ are acyclic and acyclic fibrations are closed under pullback by Lemma 1.2.2.(3). This finishes the proof. □

The central example of a (-1)-truncated fibration in this chapter is the following.

Lemma 1.4.5. *Let $\mathbb{C}$ be a type theoretic fibration category with function extensionality. Then for any two fibrations $X_1 \twoheadrightarrow Y$ and $X_2 \twoheadrightarrow Y$ in $\mathbb{C}$, the fibration*

$$\mathrm{Equiv}_Y(X_1, X_2) \twoheadrightarrow [X_1, X_2]_Y$$

is (-1)-truncated. If $\mathbb{M}$ is a type theoretic model category in which all fibrant objects are cofibrant, then the same holds for any two fibrations (with not necessarily fibrant base Y) in $\mathbb{M}$.

Proof. The first part is [**51**, Lemma 5.12] using the equivalences shown in Lemma 1.4.3. The second part follows as the objects $[X_1, X_2]_Y$ and $\mathrm{Equiv}_Y(X_1, X_2)$ are constructions in the slice $\mathbb{M}/Y$ and (-1)-truncatedness is invariant under taking slices. So the proof can be reduced to show that the fibration $\mathrm{Equiv}_Y(X_1, X_2) \twoheadrightarrow [X_1, X_2]_Y$ is (-1)-truncated in the type theoretic fibration category $(\mathbb{M}/Y)^f$. But $(\mathbb{M}/Y)^f$ satisfies function extensionality as noted in Remark 1.3.5, so the statement follows from part (1). □

By Lemma 1.4.3.(2) we see that (-1)-truncatedness of a fibration $p\colon X \twoheadrightarrow Y$ is characterized by acyclicity of its path object fibration $P_X Y \twoheadrightarrow X \times_Y X$ over Y, which can be thought of as consisting of paths *in the fibres* of p. In the following we note on the other hand that general paths in X – this is paths in $\sum_Y X$ – are characterized by paths in Y whose endpoints lie in X. Therefore, given fibrations $X_1 \twoheadrightarrow Y$ and $X_2 \twoheadrightarrow Y$, observe that every map $f\colon X_1 \to X_2$ over Y induces a *transport of paths* operation $P_Y f$ by lifting the acyclic cofibration $X_1 \overset{\sim}{\hookrightarrow} P_Y X_1$ against the fibration $P_Y X_2 \twoheadrightarrow X_2 \times_Y X_2$ as in the diagram below.

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, hook, "r_{X_1}"', "\wr"] & X_2 \arrow[d, hook, "r_{X_2}", "\wr"'] \\
P_Y X_1 \arrow[r, dashed, "P_Y f"] \arrow[d, two heads, "\partial_{X_1}"'] & P_Y X_2 \arrow[d, two heads, "\partial_{X_2}"] \\
X_1 \times_Y X_1 \arrow[r, "{(f,f)}"] & X_2 \times_Y X_2
\end{tikzcd}$$

This induces an associated natural map $P_Y X_1 \to (X_1 \times_Y X_1) \times_{(X_2 \times_Y X_2)} P_Y X_2$ over $X_1 \times_Y X_1$.

Proposition 1.4.6. *Let $X \overset{p}{\twoheadrightarrow} Y \overset{q}{\twoheadrightarrow} Z$ be fibrations and p be (-1)-truncated. Then the map*

$$(\partial, P_Z p) \colon P_Z X \to (X \times_Z X) \times_{(Y \times_Z Y)} P_Z Y$$

over $X \times_Z X$ is a homotopy equivalence.

Proof. We give a syntactic proof of the proposition just to avoid overly convoluted notation. Recall that for a type family $Y \vdash X\mathsf{type}$, the identity type of any two terms w, w' in the dependent sum $\sum_{y:Y} X(y)$ is characterized in [**41**, Theorem 2.7.2] via an equivalence of type

$$\left(w =_{\sum_{y:Y} X(y)} w'\right) \simeq \left(\sum_{t:\mathrm{pr}_1(w) =_Y \mathrm{pr}_1(w')} t_* \mathrm{pr}_2(w) =_{X(\mathrm{pr}_1(w))} \mathrm{pr}_2(w')\right). \tag{1.4.1}$$

Given fibrations $X \overset{p}{\twoheadrightarrow} Y \overset{q}{\twoheadrightarrow} Z$ in a type theoretic fibration category $\mathbb{C}$ such that p is (-1)-truncated, we obtain an equivalence of the form (1.4.1) for the associated type family $Y \vdash X\mathsf{type}$ of mere propositions in context Z. By [**41**, Lemma 3.11.10], the identity types $x =_{X(y)} x'$ are contractible for all $y \in Y$ and $x, x' \in X(y)$, and hence so are in particular the identity types $t_* \mathrm{pr}_2(w) =_{X(\mathrm{pr}_1(w))} \mathrm{pr}_2(w')$ for all $t : \mathrm{pr}_1(w) =_Y \mathrm{pr}_1(w')$ and $w, w' \in \sum_{y:Y} X(y)$. But this gives an equivalence

$$\left(\sum_{t:\mathrm{pr}_1(w) =_Y \mathrm{pr}_1(w')} t_* \mathrm{pr}_2(w) =_{X(\mathrm{pr}_1(w))} \mathrm{pr}_2(w')\right) \simeq \left(\mathrm{pr}_1(w) =_Y \mathrm{pr}_1(w')\right)$$

by [**41**, Lemma 3.11.9.(i)] . This induces an equivalence on total spaces

$$\sum_{w,w':\sum_{y:Y} X(y)} \left(w =_{\sum_{y:Y} X(y)} w'\right) \simeq \sum_{w,w':\sum_{y:Y} X(y)} \left(\mathrm{pr}_1(w) =_Y \mathrm{pr}_1(w')\right)$$

by [**41**, Theorem 4.7.6]. But, by the induction principle for $\sum$-types, we have

$$\sum_{w,w':\sum_{y:Y} X(y)} \left(\mathrm{pr}_1(w) =_Y \mathrm{pr}_1(w')\right) \simeq \sum_{y,y':Y} X(y) \times X(y') \times (y =_Y y').$$

We obtain a composite equivalence

$$(w =_{\sum_{y:Y} X(y)} w') \simeq \sum_{y,y':Y} X(y) \times X(y') \times (y =_Y y')$$

and a formal but straightforward translation shows that this equivalence corresponds to the map

$$(\partial, P_Z p)\colon P_Z X \to (X \times_Z X) \times_{(Y\times_Z Y)} P_Z Y$$

as stated. □

Remark 1.4.7. For fibrations $X_1 \twoheadrightarrow Y$ and $X_2 \twoheadrightarrow Y$ in $\mathbb{C}$, applying Proposition 1.4.6 to the pair of fibrations $\mathrm{Equiv}_Y(X_1, X_2) \twoheadrightarrow [X_1, X_2]_Y \twoheadrightarrow Y$ generalizes the observation made in [**33**, Lemma 3.2.9], stating that, in the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$, homotopies in $\mathrm{Equiv}_Y(X_1, X_2)$ are exactly homotopies in $[X_1, X_2]_Y$ whose endpoints lie in $\mathrm{Equiv}_Y(X_1, X_2)$.

From Lemma 1.4.3.(1) it follows immediately that two (-1)-truncated fibrations $X_1 \twoheadrightarrow Y$ and $X_2 \twoheadrightarrow Y$ are homotopy equivalent whenever there are maps between X_1 and X_2 over Y in both directions. In type theoretical terms this means that any two mere propositions which imply each other are equivalent as types. This is shown in [**41**, Lemma 3.3.3] directly. Via Proposition 1.4.6 we can show that this statement still holds when the two (-1)-truncated fibrations have not necessarily the same, but homotopy equivalent bases.

Corollary 1.4.8. *Let $Y_1 \twoheadrightarrow Z$ and $Y_2 \twoheadrightarrow Z$ be fibrations in $\mathbb{C}$. Suppose $p\colon X_1 \twoheadrightarrow Y_1$ and $q\colon X_2 \twoheadrightarrow Y_2$ are fibrations in $\mathbb{C}$ together with commutative squares*

$$\begin{tikzcd}
X_1 \arrow[r, "\bar{f}"] \arrow[d, two heads, "p"] & X_2 \arrow[r, "\bar{g}"] \arrow[d, two heads, "q"] & X_1 \arrow[d, two heads, "p"] \\
Y_1 \arrow[r, "f"] & Y_2 \arrow[r, "g"] & Y_1
\end{tikzcd}$$

over Z such that f and g are mutual homotopy inverses. If both p and q are (-1)-truncated, then $\bar{f}$ and $\bar{g}$ are mutually homotopy inverse, too.

Proof. We construct a homotopy $\bar{g}\bar{f} \sim \mathrm{id}_{X_1}$, the other case works analogously. By assumption there is a homotopy $H\colon gf \sim \mathrm{id}_{Y_1}$, so we obtain a square

$$\begin{tikzcd}
X_1 \arrow[r, "(\mathrm{id}_{X_1}, \bar{g}\bar{f})"] \arrow[d, "Hp"'] & X_1 \times_Z X_1 \arrow[d, "p\times p"] \\
P_Z Y_1 \arrow[r, "\partial"'] & Y_1 \times_Z Y_1.
\end{tikzcd}$$

This induces a map $X_1 \to P_Z Y_1 \times_{(Y_1 \times_Z Y_1)} (X_1 \times_Z X_1)$ over $X_1 \times_Z X_1$ and hence, by Proposition 1.4.6, we obtain a map

$$\begin{tikzcd} & P_Z X_1 \arrow[d, "\partial"] \\ X_1 \arrow[ur] \arrow[r, "(\mathrm{id},\bar{g}\bar{f})"'] & X_1 \times_Z X_1. \end{tikzcd}$$

□

1.5. Univalence and homotopy-uniqueness of pullback presentations

Let $\mathbb{C}$ be a type theoretic fibration category. For a fibration $p\colon E \twoheadrightarrow B$, thought of as a type family E over B, consider (in the language of [**33**, Section 1]) the *generic* function type family

$$\mathrm{Fun}p := [\pi_1^* p, \pi_2^* p]_{B\times B} \overset{(s,t)}{\twoheadrightarrow} B \times B.$$

To give a map $A \xrightarrow{(f,g)} B \times B$ together with a lift

$$\begin{tikzcd} & \mathrm{Fun}p(s,t) \arrow[d, two heads] \\ A \arrow[ur, dashed, "l"] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd}$$

is the same as to give a map

$$\begin{tikzcd} f^*E \arrow[rr, dashed, "\bar{l}"] \arrow[dr, two heads, "f^*p"'] & & g^*E \arrow[dl, two heads, "g^*p"] \\ & A & \end{tikzcd}$$

over A. The type $\mathrm{Fun}p$ naturally comes with a unit

$$\begin{tikzcd} B \arrow[rr, "\eta"] \arrow[dr, "\Delta"'] & & \mathrm{Fun}p \arrow[dl, "(s,t)"] \\ & B \times B & \end{tikzcd}$$

associated to the identity map 1_E and a composition

$$\mu\colon \mathrm{Fun}p \times_B \mathrm{Fun}p \to \mathrm{Fun}p$$

which together yield an internal category object in $\mathbb{C}$. Further, consider the *generic* type family $\mathrm{Eq}p := \mathrm{Equiv}_{B\times B}(\pi_1^* p, \pi_2^* p) \twoheadrightarrow B \times B$ of homotopy equivalences associated to p. For terms $x, y : B$ and a function $f : \mathrm{Fun}p(x, y)$ in the internal type theory $\mathcal{T}_{\mathbb{C}}$ of $\mathbb{C}$, it is given by

$$\mathrm{Eq}p(x, y, f) :\equiv \mathrm{Linv}_p(x, y, f) \times \mathrm{Rinv}_p(x, y, f)$$

for the types

$$\mathrm{Linv}_p(x, y, f) :\equiv \sum_{g:\mathrm{Fun}p[\frac{y}{x}, \frac{x}{y}]} \prod_{e:E(x)} \mu(g, f)(e) =_{E(x)} e$$
$$\mathrm{Rinv}_p(x, y, f) :\equiv \sum_{h:\mathrm{Fun}p[\frac{y}{x}, \frac{x}{y}]} \prod_{e:E(y)} \mu(f, h)(e) =_{E(y)} e$$

of left- and right-inverses of f respectively.

Translating these into notions in $\mathbb{C}$, for $i \in \{0, 1\}$ we obtain the generic type families of left- and right-invertible maps Linv_p and Rinv_p over $B \times B$ respectively, whose exact diagrammatical formulas are not essential at this point, but will be given in Section 6.2. By construction, we have fibrations

$$\mathrm{Eq}p := \mathrm{Equiv}_{B\times B}(\pi_1^*p, \pi_2^*p) \twoheadrightarrow [\pi_1^*p, \pi_2^*p]_{B\times B} \twoheadrightarrow B \times B$$

and hence obtain a canonical composite source and target pair $(s, t)\colon \mathrm{Eq}p \twoheadrightarrow B \times B$. To give a map $A \xrightarrow{(f,g)} B \times B$ together with a lift

$$\begin{tikzcd} & \mathrm{Eq}p \arrow[d, two heads, "(s,t)"] \\ A \arrow[r, "(f,g)"'] \arrow[ur, dashed, "l"] & B \times B \end{tikzcd}$$

is the same as to give a homotopy equivalence

$$\begin{tikzcd} f^*E \arrow[rr, dashed, "\bar{l}", "\simeq"'] \arrow[dr, two heads, "f^*p"'] & & g^*E \arrow[dl, two heads, "g^*p"] \\ & A & \end{tikzcd}$$

over A.

The unit $\eta\colon B \to \mathrm{Fun}p$ lifts to a retract

$$r\colon B \to \mathrm{Eq}p$$

of the compositions $\pi_i(s, t)\colon \mathrm{Eq}p \to B$, internally assigning to each $b : B$ the identity function $\mathrm{id}_b\colon E_b \to E_b$ together with the canonical proof that the function id_b is an involution.

Definition 1.5.1. A fibration p is *univalent* if the unit map $r\colon B \to \mathrm{Eq}p$ is a homotopy equivalence. In other words, p is univalent if and only if the type $\mathrm{Eq}p$ of equivalences over $B \times B$ is a path object for B.

Thus, a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$ is univalent if the natural map

$$\begin{tikzcd} PB \arrow[rr, "\mathsf{idtoequiv}_p"] \arrow[dr, two heads] & & \mathrm{Eq}p \arrow[dl, two heads] \\ & B \times B & \end{tikzcd}$$

induced by the weak factorization system in $\mathbb{C}$ is a homotopy equivalence. By the 2-for-3 property, this holds if and only if the target or source fibration

$$\mathrm{Eq}p \to B \times B \to B$$

is acyclic.

Syntactically, recall that a type family $p\colon (b : B, e : E) \twoheadrightarrow (b : B)$ corresponding to the judgement

$$b : B \vdash E\mathsf{type}$$

is *univalent* if and only if the canonical map

$$\mathsf{idtoequiv}_p : \prod_{b,b':B} \big(b =_B b' \to E(b) \simeq E(b')\big)$$

is an equivalence, i.e., if the type $\prod_{b,b':B} \mathrm{isEquiv}(\mathsf{idtoequiv}_p(b, b'))$ is inhabited.

In [**33**, Theorem 3.5.3] it is shown that a fibration $p\colon E \twoheadrightarrow B$ in the category of simplicial sets (equipped with the Quillen model structure) is univalent if and only if for every other fibration $q\colon Y \twoheadrightarrow X$ the object $P_{q,p}$ representing homotopy-cartesian squares from q to p is either empty or contractible. The object $P_{q,p}$ is chosen such that maps from a simplicial set A into $P_{q,p}$ correspond to maps $f\colon A \times X \to B$ and a weak equivalence $w\colon A \times Y \to f^*E$ over $A \times X$.

In the following paragraph we show that this correspondence of univalence and uniqueness of homotopy pullback representations is not a peculiarity of simplicial sets but a purely syntactical fact.

Namely, take an intensional type theory with the usual type formers – in particular it comes with $\sum$-types (satisfying the extensional η-rule) and so without loss of generality all contexts are of length 1 – satisfying function extensionality.

For any two type families

$$y : Y \vdash X \ \mathsf{type}$$

$$b : B \vdash E \text{ type}$$

with associated fibrations $q\colon (y : Y, x : X) \to (y : Y)$ and $p\colon (b : B, e : E) \to (b : B)$, consider the type

$$P_{q,p} :\equiv \sum_{f:Y\to B} \prod_{y:Y} \mathrm{Equiv}(X(y), E(fy))\text{type}$$

of "homotopy-cartesian squares" from q to p.

Given fibrations $q\colon X \twoheadrightarrow Y$ and $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$ with internal hom-object $B^Y := [Y, B]_1$ and terminal map $!_{B^Y} : B^Y \to 1$ we have

$$P_{q,p} \cong \sum_{!_{B^Y}} \prod_{\pi_1} \mathrm{Equiv}_{B^Y\times Y}(\pi_2^* X, \mathrm{ev}^* p) \tag{1.5.1}$$

such that for every $A \in \mathbb{C}$,

$$\mathbb{C}(A, P_{q,p}) \cong \left\{ (f, w) \mid f\colon A \to B^Y, \quad \begin{tikzcd} \pi_1^* A \arrow[rr, "w"] \arrow[dr, "\pi^* f"'] & & \mathrm{Equiv}_{B^Y\times Y}(\pi_2^* X, \mathrm{ev}^* E) \arrow[dl] \\ & B^Y \times Y & \end{tikzcd} \right\}$$

$$\cong \left\{ (f, w) \mid f\colon A \to B^Y, \quad \begin{tikzcd} A \times Y \arrow[rr, "w"] \arrow[dr, "f\times \mathrm{id}_Y"'] & & \mathrm{Equiv}_{B^Y\times Y}(\pi_2^* X, \mathrm{ev}^* E)) \arrow[dl] \\ & B^Y \times Y & \end{tikzcd} \right\}$$

$$\cong \left\{ (f, w) \mid \quad \begin{tikzcd} & (f^{ad})^* E \arrow[rr] \arrow[ddl, bend left] \arrow[drr, phantom, "\lrcorner"] & & \mathrm{ev}^* E \arrow[r] \arrow[ddl, bend left] & E \arrow[dd, two heads, "p"] \\ A \times X \arrow[ur, dashed, "w", "\simeq"'] \arrow[rr] \arrow[d, "A\times q"'] \arrow[dr, phantom, "\lrcorner"] & & \pi_2^* X \arrow[d, two heads] & & \\ A \times Y \arrow[rr, "f\times \mathrm{id}_Y"] \arrow[rrrr, dashed, bend right, "f^{ad}"'] & & B^Y \times Y \arrow[rr, "\mathrm{ev}"] & & B \end{tikzcd} \right\}$$

$$\cong \left\{ (f, w) \mid f\colon A \times Y \to B, w\colon A \times X \xrightarrow{\simeq} f^* p \text{ over } A \times Y \right\}$$

Theorem 1.5.2. *Let $\mathbb{C}$ be a type theoretic fibration category with function extensionality. Then a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$ is univalent if and only if for every fibration $q\colon X \twoheadrightarrow Y$ in $\mathbb{C}$, the object $P_{q,p}$ is (-1)-truncated.*

We give a proof of Theorem 1.5.2 in the internal language of $\mathbb{C}$. Although it can be translated into purely semantical terms, the corresponding diagrammatical proof is notationally very convoluted. However in the case $\mathbb{C}$ is the fibration category associated to a type theoretic model category with all fibrant objects cofibrant, the diagrammatical proof is comparatively appealing and is also given below for the interested reader.

For the following proof we freely use notation from [**41**] and suppress non-essential subscripts.

Proof. We have to show that a type family p in $\mathcal{T}_{\mathbb{C}}$ is univalent if and only if for every type family q the type $P_{q,p}$ is a mere proposition.

Suppose p is univalent and denote $\mathsf{idtoequiv}_p$'s (point-wise) inverse by ua_p. Let $q\colon (y : Y, x : X) \twoheadrightarrow (y : Y)$ be a type family. We have to construct a witness of type

$$\mathsf{isProp}(P_{q,p}) :\equiv \prod_{s,t:P_{q,p}} s =_{P_{q,p}} t.$$

Therefore, by the induction principle for $\sum$-types, let $(f, w), (g, v) : P_{q,p}$. Towards the construction of a path of type $(f, w) =_{P_{q,p}} (g, v)$, recall that there is an equivalence

$$w_\sigma : \big((f, w) =_{P_{q,p}} (g, v)\big) \simeq \sum_{u:f=g} u_* w = v \tag{1.5.2}$$

by [**41**, Theorem 2.7.2]. Thus, first we note that the components

$$w : \prod_{y:Y} \mathrm{Equiv}(X(y), E(fy)),$$

$$v : \prod_{y:Y} \mathrm{Equiv}(X(y), E(gy))$$

give rise to the composition

$$v \circ w^{-1} : \prod_{y:Y} \mathrm{Equiv}(E(fy), E(gy)).$$

This yields $\mathsf{ua}_p(v \circ w^{-1}) : \prod_{y:Y} fy = gy$ and by function extensionality we obtain

$$\varphi :\equiv \mathrm{funext} \circ \mathsf{ua}_p(v \circ w^{-1}) : f = g.$$

So it remains to construct a witness of the identity $\varphi_* w = v$ between terms of type $\prod_{y:Y} \mathrm{Equiv}(X(y), E(gy))$. But there is

$$\theta :\equiv \lambda f.\lambda g.\lambda u.\lambda y.\mathsf{idtoequiv}_u(fy, gy, \mathsf{happly}(u, y))$$

of type

$$\prod_{f,g:Y\to B} \Big(f = g \to \prod_{y:Y} \mathrm{Equiv}(E(fy), E(gy))\Big)$$

and a simple path induction on the type $Y \to B$ constructs a witness of type

$$\prod_{f,g:Y\to B}\prod_{u:f=g} u_* w = \theta(u)\circ w.$$

Clearly

$$\begin{aligned}\theta(\varphi) &\equiv \lambda y.\mathsf{idtoequiv}_p(fy, gy, \mathsf{happly}(\varphi, y))\\ &\equiv \lambda y.\mathsf{idtoequiv}_p(fy, gy, \mathsf{happly}(\mathrm{funext}(\mathsf{ua}_p(fy, gy, v\circ w^{-1})))(y))\\ &= \lambda y.\mathsf{idtoequiv}_p(fy, gy)(\mathsf{ua}_p(fy, gy)(v\circ w^{-1})(y))\\ &= \lambda y.(v\circ w^{-1})(y)\\ &\equiv v\circ w^{-1}\end{aligned}$$

eventually giving a path of type $\varphi_* w = v$ by composition.

Vice versa, assume the types $P_{q,p}$ are mere propositions. We have to construct an inverse to

$$\mathsf{idtoequiv}_p : \prod_{b,b':B}\big(b =_B b'\big) \to E(b) \simeq E(b').$$

Therefore, let $b, b' : B$ and $w\colon E(b) \simeq E(b')$. We obtain the map $b\colon \mathbf{1} \to B$ and terms $(b, \mathrm{id}_{E(b)}), (b', w) \in P_{q_b,p}$ for $q_b\colon (z : \mathbf{1}, e : E(b)) \twoheadrightarrow (z : \mathbf{1})$. By assumption, there is a witness

$$\omega : \prod_{s,t:P_{q_b,p}} s = t$$

and by the equivalence corresponding to (1.5.2) we obtain

$$(\varphi_1(w), \varphi_2(w)) :\equiv (\mathrm{pr}_1(v(b, b', w)), \mathrm{pr}_2(v(b, b', w))) : \sum_{p:b=b'} p_*\mathrm{id} = w$$

for $v(b, b', w) :\equiv w_\sigma(\omega((b, \mathrm{id}_{E(b)}), (b', w))$. This constructs a map

$$\varphi_1 :\equiv \lambda w.\varphi_1(w) : \big(E(b) \simeq E(b')\big) \to (b =_B b').$$

It remains to show that φ_1 is an inverse to $\mathsf{idtoequiv}_p$. On one hand, again by path induction over B we see that $(\varphi_1(w))_*\mathrm{id} = \mathsf{idtoequiv}_p(b, b')(\varphi_1(w)) \circ \mathrm{id} \equiv \mathsf{idtoequiv}_p(b, b')(\varphi_1(w))$, which yields a proof of

$$\prod_{w:E(b)\simeq E(b')} \mathsf{idtoequiv}_p(b, b')(\varphi_1(w)) = w.$$

On the other hand, in order to show that also $\varphi_1 \circ \mathsf{idtoequiv}_p = \mathrm{id}$, by function extensionality and path induction it suffices to verify $\varphi_1(\mathsf{idtoequiv}_p(b, b)(\mathrm{refl}_b)) = \mathrm{refl}_b$. But

$$\varphi_1(\mathsf{idtoequiv}_p(b, b)(\mathrm{refl}_b)) \equiv \varphi_1(\mathrm{id})$$

holds judgementally and we recall that $\varphi_1(\mathrm{id}) \equiv \mathrm{pr}_1(v(b, b, \mathrm{id}_{E(b)}))$. Now every mere proposition is an (h-)set, so there is a term of type

$$\omega((b, \mathrm{id}_{E(b)}), (b, \mathrm{id}_{E(b)})) = \mathrm{refl}_{(b,\mathrm{id}_{E(b)})}$$

which induces a term of type $\mathrm{pr}_1(v(b, b, \mathrm{id}_{E(b)})) = \mathrm{pr}_1(w_\sigma(\mathrm{refl}_{(b,\mathrm{id}_{E(b)})})) \equiv \mathrm{refl}_b$ by function application. □

While a direct transcription of the syntactical proof to the categorical framework is basically unintelligible, it turns out comparatively short if $\mathbb{C}$ allows to replace right homotopies by left homotopies. Therefore, we give another proof Theorem 1.5.2 in the case when $\mathbb{C}$ is the category of fibrant objects associated to a type theoretic model category $\mathbb{M}$ with all fibrant objects cofibrant. The cofibrancy condition is assumed so that both left and right homotopies as well as homotopy equivalences and weak equivalences between fibrations in $\mathbb{C}$ coincide respectively (by [**27**, Corollary 1.2.6] and [**27**, Proposition 1.2.8] applied to the slices of $\mathbb{M}$). Note that in particular function extensionality holds in $\mathbb{C}$ by Remark 1.3.3.

Denoting for a given object $D \in \mathbb{C}$ its associated cylinder by $Z(D)$, pullback stability of cofibrations along fibrations in $\mathbb{C}$ assures that $D \coprod D \to D \times Z(1) \to D$ is a cylinder object for D, so we can pick the functorial choice $Z(D) := D \times Z(1)$ of cylinder objects.

Proposition 1.5.3 (Special case of Theorem 1.5.2)**.** *Let $\mathbb{M}$ be a type theoretic model category with all fibrant object cofibrant. Then a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}^f$ is univalent if and only if for every fibration $q\colon X \twoheadrightarrow Y$ in $\mathbb{M}^f$, the object $P_{q,p}$ is (-1)-truncated.*

The set-up of this proposition anticipates the notions treated in Chapter 2. In fact the following proof will make use of Lemma 2.3.2, the reference will be given when applied.

Proof. Let $p\colon E \twoheadrightarrow B$ and $q\colon X \twoheadrightarrow Y$ be fibrations with fibrant base in $\mathbb{M}$. First, suppose all $P_{q,p}$ are (-1)-truncated. Let

$$\begin{tikzcd}
\mathrm{Eq}p \arrow[rr, "f"] \arrow[dr, two heads, "s"'] & & \mathrm{Eq}p \arrow[dl, two heads, "s"] \\
& B &
\end{tikzcd}$$

be any endomorphism in $\mathbb{M}/B$. By definition, f corresponds to a homotopy equivalence $v\colon s^*E \to (tf)^*E$ over $\mathrm{Eq}p$. The same holds for the identity $\mathrm{id}_{\mathrm{Eq}p}$ corresponding to a homotopy equivalence $w\colon s^*E \to t^*E$ over $\mathrm{Eq}p$. Therefore, both (tf, v) and (t, w) represent global elements of $P_{s^*p,p}$. By assumption, there is a homotopy $H\colon Z(1) \to P_{s^*p,p}$ between (tf, v) and (t, w) which amounts to a diagram of the form

$$\begin{tikzcd}
 & s^*E \times Z(1) \arrow[r, two heads, "e_H", "\sim"'] \arrow[dr, two heads] & f_H^*E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
s^*E \sqcup s^*E \arrow[ur] \arrow[r, "w \sqcup v"] \arrow[dr, two heads] & t^*E \sqcup (tf)^*E \arrow[ur] \arrow[d, two heads] & \mathrm{Eq}p \times Z(1) \arrow[r, "f_H"] & B \\
 & \mathrm{Eq}p \sqcup \mathrm{Eq}p \arrow[ur, hook] \arrow[urr, bend right, "(s,t)"] \arrow[urr, bend right=40, "(s,tf)"'] & &
\end{tikzcd}$$

with pullbacks $s^*E \xrightarrow{w} t^*E$ and $s^*E \xrightarrow{v} (tf)^*E$ over $\mathrm{Eq}p$ along the respective coproduct inclusions. But this yields the data of a map

$$\begin{tikzcd}
\mathrm{Eq}p \times Z(1) \arrow[rr, "\bar{H}"] \arrow[dr, "s\pi_1"'] & & \mathrm{Eq}p \arrow[dl, "s"] \\
 & B &
\end{tikzcd}$$

being a homotopy from $\mathrm{id}_{\mathrm{Eq}p}$ to f.

So we have seen that every map $f\colon \mathrm{Eq}p \to \mathrm{Eq}p$ over B is homotopic to the identity. This in particular holds for the composition $\mathrm{Eq}p \xrightarrow{s} B \xrightarrow{r} \mathrm{Eq}p$ which gives a homotopy of the maps $s, t\colon \mathrm{Eq}p \to B$ by post-composition with the target fibration. In other words, there is a lift

$$\begin{tikzcd}
 & PB \arrow[d, two heads] \\
\mathrm{Eq}p \arrow[ur, dashed, "\mathsf{ua}_p"] \arrow[r, two heads, "(s,t)"'] & B \times B.
\end{tikzcd}$$

Vice versa there is the map $\mathsf{idtoequiv}_p\colon PB \to \mathrm{Eq}p$ over $B \times B$. The composition $\mathsf{ua}_p \circ \mathsf{idtoequiv}_p\colon PB \to PB$ is homotopic to the identity over B, because PB is contractible over B. On the other hand, $\mathsf{idtoequiv}_p \circ \mathsf{ua}_p\colon \mathrm{Eq}p \to \mathrm{Eq}p$ is a map over B and we have seen that such maps are homotopic to the identity. So we have shown that $\mathsf{idtoequiv}_p$ is a homotopy equivalence.

Conversely, assume p is univalent and pick a fibration $q\colon X \twoheadrightarrow Y$. By Lemma 1.4.3, we have to show that the projections $P_{q,p} \times P_{q,p} \twoheadrightarrow P_{q,p}$ are acyclic. Therefore, due to pullback stability of the type of equivalences according to the diagram

$$\begin{tikzcd}
\mathrm{Equiv}_{P_{q,p}\times(B^Y\times Y)}(\pi_{\hat{1}}^*\pi_2^*X, \pi_{\hat{1}}^*\mathrm{ev}^*E) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Equiv}_{B^Y\times Y}(\pi_2^*X, \mathrm{ev}^*E) \arrow[d, two heads] \\
P_{q,p}\times(B^Y\times Y) \arrow[r, two heads, "\pi_{\hat{1}}"] \arrow[d, two heads, "\pi_{\hat{3}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^Y\times Y \arrow[d, two heads] \\
P_{q,p}\times B^Y \arrow[r, two heads] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^Y \arrow[d, two heads] \\
P_{q,p} \arrow[r, two heads] & 1
\end{tikzcd}$$

note that

$$\begin{aligned}
P_{q,p}\times P_{q,p} &\cong P_{q,p}\times \sum_{!_{B^Y}}\prod_{\pi_1}\mathrm{Equiv}_{B^Y\times Y}(\pi_2^*X, \mathrm{ev}^*E)\\
&\cong \sum_{\pi_1}\prod_{\pi_{\hat{3}}}\pi_{\hat{1}}^*\mathrm{Equiv}_{B^Y\times Y}(\pi_2^*X, \mathrm{ev}^*E)\\
&\cong \sum_{\pi_1}\prod_{\pi_{\hat{3}}}\mathrm{Equiv}_{P_{q,p}\times(B^Y\times Y)}(\pi_3^*X, (\mathrm{ev}\pi_{\hat{1}})^*E)
\end{aligned}$$

holds. Further, the identity on $P_{q,p}$ induces a homotopy equivalence $\omega_{\mathrm{id}}\colon P_{q,p}\times X\to f_{\mathrm{id}}^*E$ over $P_{q,p}\times Y$ which in turn can be pulled back along the projection $\pi_{\hat{2}}\colon P_{q,p}\times(B^Y\times Y)\to P_{q,p}\times Y$ to give a homotopy equivalence

$$\begin{tikzcd}
\pi_3^*X\cong P_{q,p}\times B^Y\times X \arrow[rr, "\pi_{\hat{2}}^*w_{\mathrm{id}}", "\sim"'] \arrow[dr, two heads] & & \pi_{\hat{2}}^*f_{\mathbf{id}}^*E. \arrow[dl, two heads, "\pi_{\hat{2}}^*f_{\mathrm{id}}^*p"] \\
& P_{q,p}\times B^Y\times Y &
\end{tikzcd}$$

By preservation of homotopy equivalences between cofibrant-fibrant objects, this induces a homotopy equivalence

$$\sum_{\pi_1}\prod_{\pi_{\hat{3}}}\mathrm{Equiv}_{P_{q,p}\times(B^Y\times Y)}(\pi_3^*X, (\mathrm{ev}\pi_{\hat{1}})^*E)\simeq\sum_{\pi_1}\prod_{\pi_{\hat{3}}}\mathrm{Equiv}_{P_{q,p}\times(B^Y\times Y)}(\pi_{\hat{2}}^*f_{\mathbf{id}}^*E, (\mathrm{ev}\pi_{\hat{1}})^*E)$$

over $P_{q,p}$. So it suffices to show that the map

$$\varphi\colon \sum_{\pi_1}\prod_{\pi_{\hat{3}}}\mathrm{Equiv}_{P_{q,p}\times(B^Y\times Y)}((f_{\mathbf{id}}\pi_{\hat{2}})^*E, (\mathrm{ev}\pi_{\hat{1}})^*E)\twoheadrightarrow P_{q,p}$$

is an acylic fibration. As the projection $P_{q,p}\times P_{q,p}\to P_{q,p}$ always comes with the diagonal section, the fibration φ has a section, too, and so it suffices to show that any two maps into φ over $P_{q,p}$ are homotopic to each other (so in particular

φ composed with this section is homotopic to the identity). So given an object $D \xrightarrow{d} P_{q,p}$ in $\mathbb{M}/P_{q,p}$, we have

$$(\mathbb{M}/P_{q,p})(d,\varphi) \cong \left\{ g\colon D\times Y \to B, w\colon (f_{\mathrm{id}}(d,\mathrm{id}))^*E \xrightarrow{\sim} g^*E \text{ over } D\times Y \right\}$$

by a diagram chase similar to the ones performed above. So let $\gamma, \gamma'\colon d \to \varphi$ be two arbitrary maps. We are left to construct a homotopy between them.
The two maps γ, γ' correspond to maps $g, g'\colon D\times Y \to B$ and homotopy equivalences $w\colon (f_{\mathrm{id}}(d,\mathrm{id}))^*E \xrightarrow{\sim} g^*E$ and $w'\colon (f_{\mathrm{id}}(d,\mathrm{id}))^*E \xrightarrow{\sim} (g')^*E$ over $D\times Y$ respectively. We obtain a lift

$$\begin{tikzcd}
 & \mathrm{Eq}p \arrow[d, two heads] \\
D\times Y \arrow[ur, "{(g,g',w'\circ w^{-1})}"] \arrow[r, "{(g,g')}"'] & B\times B
\end{tikzcd}$$

and hence due to $\mathsf{ua}_p\colon \mathrm{Eq}p \to PB$ a homotopy $H\colon g \sim g'$. This gives rise to a diagram of the form

$$\begin{tikzcd}
 & & H^*E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, two heads, "p"] \\
(f_{\mathrm{id}}(d,\mathrm{id}))^*E \sqcup (f_{\mathrm{id}}(d,\mathrm{id}))^*E \arrow[r, "w\sqcup v", "\sim"'] \arrow[dr, two heads] & g^*E \sqcup (g')^*E \arrow[ur] \arrow[d, two heads] & (D\times Y)\times Z(1) \arrow[r, "H"] & B \\
 & (D\times Y)\sqcup(D\times Y) \arrow[ur, hook] \arrow[urr, "{(g,g')}"'] & &
\end{tikzcd}$$

which, by univalence of p (respectively by the *weak equivalence extension property* of F_p, see Definition 2.3.1 and Lemma 2.3.2) can be completed to a diagram of the form

$$\begin{tikzcd}
 & & J^*E \arrow[r, "\sim"] \arrow[dr, two heads] & H^*E \arrow[d, two heads] \\
(f_{\mathrm{id}}(d,\mathrm{id}))^*E \sqcup (f_{\mathrm{id}}(d,\mathrm{id}))^*E \arrow[urr] \arrow[r, "w\sqcup v", "\sim"'] \arrow[dr, two heads] & g^*E \sqcup (g')^*E \arrow[urr] \arrow[d, two heads] & & (D\times Z(1))\times Y \\
 & (D\times Y)\sqcup(D\times Y) \arrow[urr, hook] & &
\end{tikzcd}$$

for some map $J\colon (D\times Y)\times Z(1) \to B$ such that all faces are pullback squares. We see that J^*E is a cylinder object for $(f_{\mathrm{id}}(d,\mathrm{id}))^*E$ and hence we obtain a homotopy equivalence

$$\begin{tikzcd}
(f_{\mathrm{id}}(d,\mathrm{id}))^*E \times Z(1) \arrow[r, "\sim"] \arrow[dr, two heads] & J^*E \arrow[d, two heads] \\
& (D \times Z(1)) \times Y
\end{tikzcd}$$

whose composition with $J^*E \xrightarrow{\sim} H^*E$ completes the diagram

$$\begin{tikzcd}
& (f_{\mathrm{id}}(d,\mathrm{id}))^*E \times Z(1) \arrow[r, "\sim"] \arrow[dr, two heads] & H^*E \arrow[d, two heads] \\
(f_{\mathrm{id}}(d,\mathrm{id}))^*E \sqcup (f_{\mathrm{id}}(d,\mathrm{id}))^*E \arrow[ur] \arrow[r, "w \sqcup v", "\sim"'] \arrow[dr, two heads] & g^*E \sqcup (g')^*E \arrow[ur] \arrow[d, two heads] & (D \times Z(1)) \times Y \\
& (D \times Y) \sqcup (D \times Y) \arrow[ur, hook] &
\end{tikzcd}$$

where the faces still are pullback squares. Eventually, noticing that $(f_{\mathrm{id}}(d,\mathrm{id}))^*E \times Z(1)$ and $(f_{\mathrm{id}}(d\pi_1,\mathrm{id}))^*E$ are isomorphic, we have gathered the data necessary to construct a homotopy $D \times Z(1) \to \mathrm{dom}\varphi$ between γ and γ'. This finishes the proof. $\square$

1.6. Invariance under homotopy equivalence

We conclude this chapter by showing stability of univalence and of the objects $P_{q,p}$ under homotopy equivalence[1]. The statements follow from the following generalization of [**33**, Proposition 3.2.9].

Proposition 1.6.1. *Let $\mathbb{C}$ be a type theoretic fibration category with function extensionality. Then for every object $Z \in \mathbb{C}$ and all fibrations X_1, X_2, Y_1, Y_2 over Z together with homotopy equivalences*

$$\begin{tikzcd}
X_1 \arrow[dr, "v"] \arrow[ddr, two heads] & & Y_1 \arrow[dr, "w"] \arrow[ddl, two heads] & \\
& X_2 \arrow[d, two heads] & & Y_2 \arrow[dll, two heads] \\
& Z & &
\end{tikzcd}$$

over Z, there is a homotopy equivalence

$$\mathrm{Equiv}_Z(X_1, Y_1) \to \mathrm{Equiv}_Z(X_2, Y_2)$$

[1]This is proved without assuming the existence of univalent universes. Indeed, that assumption makes stability under homotopy equivalence trivially true for all type theoretic properties.

of the associated types of equivalences.

Proof. Consider the maps

$$[X_1, Y_1]_Z \xrightarrow{[v^{-1}, Y_1]_Z} [X_2, Y_1]_Z \xrightarrow{[X_2, w]_Z} [X_2, Y_2]_Z \tag{1.6.1}$$

over Z induced by pre- and postcomposition of the given homotopy equivalences respectively. By Lemma 1.3.1, postcomposition $[X_2, w]_Z$ is a homotopy equivalence, but note that function extensionality implies that precomposition $[v^{-1}, Y_1]_Z$ is a homotopy equivalence, too. Indeed, the proof is almost identical to the proof of Lemma 1.3.1.

Claim 1.6.2. Let $v\colon X_1 \to X_2$ be a homotopy equivalence between fibrations $X_1 \twoheadrightarrow Z$ and $X_2 \twoheadrightarrow Z$ and let $Y_1 \twoheadrightarrow Z$ be a fibration. Then the map $[v, Y_1]_Z\colon [X_2, Y_1]_Z \to [X_1, Y_1]_Z$ is a homotopy equivalence, too.

In order to show that $[v, Y_1]_Z$ is a homotopy inverse of $[v^{-1}, Y_1]_Z$, we have to construct a homotopy between $[vv^{-1}, Y_1]_Z$ and the identity on $[X_1, Y_1]_Z$ and vice versa. This is a lift

$$\begin{tikzcd}
 & P_Z[X_1, Y_1]_Z \arrow[d] \\
{[X_1, Y_1]_Z} \arrow[ur, dashed] \arrow[r, "\left([vv^{-1}, Y_1]_Z, \mathrm{id}\right)"'] & {[X_1, Y_1]_Z \times_Z [X_1, Y_1]_Z}
\end{tikzcd}$$

which, by function extensionality, corresponds to a lift

$$\begin{tikzcd}
 & P_{X_1}(X_1 \times_Z Y_1) \arrow[d] \\
X_1 \times_Z [X_1, Y_1]_Z \arrow[ur, dashed] \arrow[r, "\left(\mathrm{ev}\circ(X_1 \times_Z [vv^{-1}, Y_1]_Z, \mathrm{ev}\right)"'] & (X_1 \times_Z Y_1) \times_{X_1} (X_1 \times_Z Y_1).
\end{tikzcd}$$

But the square

$$\begin{tikzcd}
X_1 \times_Z [X_1, Y_1]_Z \arrow[r, "X_1 \times_Z [vv^{-1}, Y_1]_Z"] \arrow[d, "vv^{-1} \times_Z [X_1, Y_1]_Z"'] & X_1 \times_Z [X_1, Y_1]_Z \arrow[d, "\mathrm{ev}"] \\
X_1 \times_Z [X_1, Y_1]_Z \arrow[r, "\mathrm{ev}"'] & X_1 \times_Z Y_1
\end{tikzcd}$$

commutes. Therefore, a homotopy $H\colon vv^{-1} \sim \mathrm{id}_{Y_1}$ yields the homotopy

$$(X_1 \times_Z H) \circ \mathrm{ev}\colon X_1 \times_Z (vv^{-1}) \circ \mathrm{ev} \sim \mathrm{ev}$$

and hence the desired lift. The converse direction is proven analogously, so this shows that precomposition $[v, Y_1]_Z$ is a homotopy equivalence whenever v is such. This proves the claim.

In particular, the composition (1.6.1) is a homotopy equivalence. Denoting the natural fibration $\mathrm{Equiv}_Z(X_1, Y_1) \twoheadrightarrow Z$ by e_1, we obtain a square

$$\begin{tikzcd}
e_1^*X_1 \arrow[rr] \arrow[ddr, two heads] & & e_1^*Y_1 \arrow[dr, "e_1^*w"] \arrow[ddl, two heads] & \\
 & e_1^*X_2 \arrow[ul, "e_1^*v^{-1}"'] \arrow[rr] \arrow[d, two heads] & & e_1^*Y_2 \arrow[dll, two heads] \\
 & \mathrm{Equiv}_Z(X_1, Y_1) & &
\end{tikzcd}$$

where the top and bottom maps are induced by the universal property of the internal hom-objects $[X_i, Y_i]_Z$. The top map $e_1^*X_1 \to e_1^*Y_1$ is a homotopy equivalence by the universal property of $\mathrm{Equiv}_Z(X_1, Y_1)$ while the maps $e_1^*v^{-1}$ and e_1^*w are homotopy equivalences by pullback stability of homotopy equivalences along fibrations (which holds by [**51**, Corollary 3.15]). Hence, by 2-for-3, the bottom map $e_1^*X_1 \to e_1^*Y_1$ is a homotopy equivalence, too. This means, we obtain a lift

$$\begin{tikzcd}
\mathrm{Equiv}_Z(X_1, Y_1) \arrow[r, dashed] \arrow[d, two heads] & \mathrm{Equiv}_Z(X_2, Y_2) \arrow[d, two heads] \\
{[X_1, Y_1]_Z} \arrow[r, "{[X_2,w]_Z \circ [v^{-1}, Y_1]_Z}"'] & {[X_2, Y_2]_Z}
\end{tikzcd}$$

between the respective types of equivalences $\mathrm{Equiv}_Z(X_1, Y_1)$ and $\mathrm{Equiv}_Z(X_2, Y_2)$. The same argument lifts the map $[v, Y_1]_Z \circ [X_2, w^{-1}]_Z$ between the respective types of equivalences in converse direction. So we obtain conversely directed maps

$$\begin{tikzcd}
\mathrm{Equiv}_Z(X_1, Y_1) \arrow[r, shift left] \arrow[d, two heads] & \mathrm{Equiv}_Z(X_2, Y_2) \arrow[l, shift left] \arrow[d, two heads] \\
{[X_1, Y_1]_Z} \arrow[r, shift left, "{[X_2,w]_Z \circ [Y_1, v^{-1}]}"] & {[X_2, Y_2]_Z} \arrow[l, shift left, "{[v,Y_1]_Z \circ [X_2, w^{-1}]_Z}"]
\end{tikzcd}$$

over the internal hom objects $[X_i, Y_i]_Z$. But the fibrations $\mathrm{Equiv}_Z(X_i, Y_i) \twoheadrightarrow [X_i, Y_i]_Z$ are (-1)-truncated by Lemma 1.4.5, so that these two maps are homotopy inverses to one another by Corollary 1.4.8. $\square$

Remark 1.6.3. Let $\mathbb{M}$ be a type theoretic model category with all fibrant objects cofibrant. Then, given a (not necessarily fibrant) object $Z \in \mathbb{M}$ and fibrant objects

X_1, X_2, Y_1, Y_2 over Z together with homotopy equivalences as in Lemma 1.6.1 over Z, there is a homotopy equivalence

$$\mathrm{Equiv}_Z(X_1, Y_1) \to \mathrm{Equiv}_Z(X_2, Y_2)$$

of the associated types of equivalences by Proposition 1.6.1 applied to the slice fibration category $(\mathbb{M}/Z)_f$.

Corollary 1.6.4. *Let $p\colon E \twoheadrightarrow B$ and $p'\colon E' \twoheadrightarrow B'$ be fibrations in a type theoretic fibration category $\mathbb{C}$ with function extensionality. Then*

(1) if p and p' are homotopy equivalent, then $P_{q,p}$ and $P_{q,p'}$ are homotopy equivalent for all fibrations q in $\mathbb{C}$;

(2) univalence is stable under homotopy equivalence, i.e. if p and p' are homotopy equivalent, then p is univalent if and only if p' is;

(3) if $P_{q,p}$ and $P_{q,p'}$ are homotopy equivalent for all fibrations q in $\mathbb{M}$, then p is univalent if and only if p' is univalent;

(4) if $P_{q,p}$ and $P_{q,p'}$ are homotopy equivalent for all fibrations q in $\mathbb{M}$ and p is univalent, then p and p' are homotopy equivalent.

Proof. We prove part (1). Recalling the formula in (1.5.1) and using Proposition 1.6.1, it is easy to see that $\mathrm{Equiv}_{B^X \times X}(\pi_2^* Y, \mathrm{ev}^* p)$ and $\mathrm{Equiv}_{(B')^X \times X}(\pi_2^* Y, \mathrm{ev}^* p')$ are homotopy equivalent whenever p and p' are. By function extensionality and Beck-Chevalley, this implies homotopy equivalence of $\prod_{\pi_1} \mathrm{Equiv}_{B^X \times X}(\pi_2^* Y, \mathrm{ev}^* p)$ and $\prod_{\pi_1} \mathrm{Equiv}_{(B')^X \times X}(\pi_2^* Y, \mathrm{ev}^* p')$. Clearly, this gives a homotopy equivalence between $P_{q,p}$ and $P_{q,p'}$.

Part (2) follows immediately from part (1).

Part (3) follows directly from Theorem 1.5.2 and Lemma 1.4.3.

We prove part (4). By assumption, the identities id_E and $\mathrm{id}_{E'}$ yield global sections of $P_{p,p'}$ and $P_{p',p}$ respectively. Recalling from (1.5.1) the functors that these objects represent, we obtain homotopy cartesian squares

$$\begin{tikzcd}
E' \arrow[dr, "\sim", "w"'] \arrow[ddr, bend right, "p'"'] & & \\
& \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\cdot\lrcorner"] & E \arrow[d, two heads, "p"] \\
& B' \arrow[r] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[dr, "\sim", "v"'] \arrow[ddr, bend right, "p"'] & & \\
& \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\cdot\lrcorner"] & E' \arrow[d, two heads, "p'"] \\
& B \arrow[r] & B'
\end{tikzcd}$$

Pasting these two squares horizontally yields global sections in $P_{p,p}$ and $P_{p',p'}$. Due to univalence of p (and hence univalence of p'), the objects $P_{p,p}$ and $P_{p',p'}$ are (-1)-truncated, so that in both cases we obtain a homotopy to the global section induced by the respective identity. These homotopies induce homotopies between the compositions $B \to B' \to B$, $B' \to B \to B'$ and the respective identities. Hence, $B' \to B$ is a homotopy equivalence and also

$$\begin{tikzcd} E' \arrow[r] \arrow[d, two heads, "p'"'] & E \arrow[d, two heads, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$

is a homotopy equivalence. $\square$

CHAPTER 2

On univalent fibrations in model categories

This chapter is a categorical analysis of univalence by means of related properties such as the *weak equivalence extension property* and the *fibration extension property* in well behaved type theoretic model categories.

2.1. Motivation and preliminaries

We know that any given type theoretic model category $\mathbb{M}$ yields a model of intensional type theory via its associated type theoretic fibration category $\mathbb{M}^f$ ([**51**, 4]). While this model has identity types, dependent sums, dependent function types and a unit type, it is not guaranteed to have any other standard type formers as listed for example in [**41**, A]. In particular – in the context of HoTT most crucially – it is not guaranteed to have univalent type universes $\mathcal{U}$. In order to obtain the latter in the internal type theory of $\mathbb{M}^f$ one has to exhibit univalent universal fibrations $\pi\colon \tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ (with fibrant bases $\mathcal{U}$ in $\mathbb{M}$). Such fibrations in some classes of Cisinski model structures were constructed in [**52**] (both generalizing and revising constructions of [**33**] and [**14**, Section 1]).

In fact, all the constructions of universal fibrations in the literature ultimately rely on essentially the same procedure, either directly or by use of already established universal fibrations constructed in the following way. First, given an inaccessible cardinal κ, a map $\pi_\kappa\colon \tilde{\mathcal{U}}_\kappa \to \mathcal{U}_\kappa$ is constructed by representability arguments. Second, one sees that π_κ is a κ-small fibration which, by design, classifies the class of κ-small fibrations up to *isomorphism*. Third, one verifies the *weak equivalence extension property* for κ-small fibrations which implies univalence of π_κ and the *fibration extension property* of κ-small fibrations which implies fibrancy of $\mathcal{U}_\kappa$ independently of each other. Some of the steps can be performed in a general class of type theoretic model categories while others are special to the authors' chosen set-ups. The aim of Sections 2.2, 2.3 and 2.4 is to give an overview of all these properties in a general setting and to analyse the conditions needed to prove implications between each other. Section 2.5 applies the results to the special case when $\mathbb{M}$ is a cofibrantly generated model structure on a presheaf category. A direct benefit of this analytic presentation is Theorem 2.5.4 which shows that the universal fibrations π_κ in all of the referred approaches automatically have fibrant base $\mathcal{U}_\kappa$ whenever the corresponding weak equivalence extension property can be verified. This observation in fact is no surprise, giving a counterpart to Coquand et al.'s result that "glueing" implies "composition" for the universe in cubical type theory (see [**15**]). But it seems that it has been overlooked as the literature (such as [**33**], [**51**], [**52**], [**14**]) makes no mention of it, while the authors go through considerable computations to prove fibrancy of $\mathcal{U}_\kappa$ although the weak equivalence extension property is verified in each case independently. Notably, all these computations make use of minimal fibrations either directly, or by reference to the base case treated in [**33**, Lemma 2.2.5, Theorem 2.2.1]. It turns out that this is not necessary.

Notation. For this chapter, we fix a model category $\mathbb{M}$ as described in Remark 1.3.3, that is, a type theoretic model category such that all fibrant objects are cofibrant.

For a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}$, let F_p denote the class of fibrations obtained by pullback of p, i.e.

$$F_p := \{q\colon X \twoheadrightarrow Y \mid \exists(\gamma\colon Y \to B) : q \cong \gamma^* p\}.$$

Definition 2.1.1. Given a class of fibrations $S \subseteq \mathbb{M}$, say $p \in S$ is *(strictly) universal* for S if $S = F_p$.

As $\mathbb{M}$ is right proper, the class F_p is a class of representatives of the homotopy pullback classes of p for every fibration p in $\mathbb{M}$.

Although in this chapter our ultimate goal is to compare properties of a fibration p and properties of its associated class F_p of maps in $\mathbb{M}$, the presentation of the corresponding material appears the most clear when discussing the latter properties for arbitrary classes S of maps in $\mathbb{M}$ generally without assuming the existence of an S-universal map p.

2.2. The fibration extension property

Definition 2.2.1. Let S be a class of maps in $\mathbb{M}$. Say S has the *fibration extension property* if every solid span

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner\bullet"] & W \arrow[d, dashed, "\bar{q}"] \\ Y \arrow[r, hook, "\sim", "j"'] & Z \end{tikzcd}$$

where $q \in S$ is a fibration and j is an acyclic cofibration can be complemented to a cartesian square such that $\bar{q} \in S$ is a fibration, too.

Lemma 2.2.2. *Let $p\colon E \to B$ be a fibration in $\mathbb{M}$. If B is fibrant, then F_p satisfies the fibration extension property.*

Proof. Immediate by the right lifting property of the fibration $B \twoheadrightarrow 1$ against all acyclic cofibrations. $\square$

In the following we introduce conditions on a fibration p such that the converse direction of Lemma 2.2.2 holds.

Definition 2.2.3. Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{M}$. For a pair of fibrations $q_1\colon X_1 \twoheadrightarrow Y_1$, $q_2\colon X_2 \twoheadrightarrow Y_2$ consider diagrams of the form

(2.2.1)
$$\begin{tikzcd}
X_1 \arrow[rr, "x_1"] \arrow[dr] \arrow[dd, two heads, "q_1"'] & & E \arrow[dd, two heads, "p"] \\
& X_2 \arrow[dd, two heads, "q_2"] & \\
Y_1 \arrow[rr, "y_1"] \arrow[dr, "j"'] & & B. \\
& Y_2 &
\end{tikzcd}$$

The fibration p satisfies the *acyclic stratification property* if for every pair of fibrations q_1, q_2 as above, every diagram of the form (2.2.1) with $j\colon Y_1 \to Y_2$ an acyclic cofibration, and every cartesian square

$$\begin{tikzcd}
X_2 \arrow[r, "x"] \arrow[d, two heads, "q_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
Y_2 \arrow[r, "y"'] & B,
\end{tikzcd}$$

there is a cartesian square (x_2, y_2) between the same fibrations which makes the diagram

$$\begin{tikzcd}
X_1 \arrow[rr, "x_1"] \arrow[dr] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, two heads, "q_1"'] \arrow[ddr, phantom, near start, "\lrcorner"] & & E \arrow[dd, two heads, "p"] \\
& X_2 \arrow[ur, dashed, "x_2"'] \arrow[dd, two heads, near start, "q_2"] & \\
Y_1 \arrow[rr, crossing over, "y_1"] \arrow[dr, "j"'] & & B. \\
& Y_2 \arrow[ur, dashed, "y_2"'] &
\end{tikzcd}$$

commute.

In other words, a fibration p satisfies the acyclic stratification property if, given a fibration q_1 together with a representation $y_1 \colon Y_1 \to B$ and an "extension" q_2 of q_1 along an acyclic cofibration $Y_1 \to Y_2$, the fibration q_2 has a representation $y_2 \colon Y_2 \to B$ compatible with y_1 whenever it has a representation in p at all.

Proposition 2.2.4. *Suppose $p \colon E \twoheadrightarrow B$ has the acyclic stratification property. Then B is fibrant if and only if F_p has the fibration extension property.*

Proof. One direction is given by Lemma 2.2.2, while the other is obtained by checking the right lifting property of the object B against acyclic cofibrations. Given an acylic cofibration $j \colon Y_1 \overset{\sim}{\hookrightarrow} Y_2$ and a map $y_1 \colon Y_1 \to B$, consider the cartesian square

$$\begin{tikzcd}
y_1^*E \arrow[r] \arrow[d, two heads, "y_1^*p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, two heads, "p"] \\
Y_1 \arrow[r, "y_1"'] & B.
\end{tikzcd}$$

which in turn yields a cartesian square

$$\begin{tikzcd}
y_1^*E \arrow[r] \arrow[d, two heads, "y_1^*p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X_2 \arrow[d, two heads, "p"] \\
Y_1 \arrow[r, hook, "\sim", "j"'] & Y_2.
\end{tikzcd}$$

by the fibration extension property. Then the acyclic stratification property in particular generates a lift

$$\begin{tikzcd} Y_1 \arrow[r, "y_1"] \arrow[d, hook, "j"', "\wr"] & B \\ Y_2 \arrow[ur, dashed] & \end{tikzcd}$$

as desired. □

2.3. The weak equivalence extension property

Definition 2.3.1. Let S be a class of maps in $\mathbb{M}$. Say S has the *(acyclic) weak equivalence extension property* if every solid diagram of the form

$$\begin{tikzcd} X_1 \arrow[rr, dashed] \arrow[dr, "w"] \arrow[ddr, bend right, "q_1"'] & & W_1 \arrow[dr, dashed, "v"] \arrow[ddr, dashed, "\bar{q}_1"', pos=0.4] & \\ & X_2 \arrow[rr] \arrow[d, "q_2"] \arrow[drr, phantom, "\lrcorner", pos=0.1] & & W_2 \arrow[d, "\bar{q}_2"] \\ & Y \arrow[rr, hook, "\iota"'] & & Z \end{tikzcd}$$

where w is a weak equivalence between fibrations $q_1, q_2 \in S$, ι is an (acyclic) cofibration and $\bar{q}_2 \in S$ is a fibration, has a dashed extension as above such that v is a weak equivalence between fibrations $\bar{q}_1 \in S$ and $\bar{q}_2 \in S$ and the back square is cartesian, too.

For a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}^f$, recall the associated fibration $\mathrm{Eq}p \twoheadrightarrow B \times B$ and the definition of univalence from Section 1.5. Note that the construction of $\mathrm{Eq}p \twoheadrightarrow B \times B$ exists for all fibrations $p\colon E \twoheadrightarrow B$ in $\mathbb{M}$ with not necessarily fibrant base B, too. Therefore, Definition 1.5.1 can be formulated for all fibrations in $\mathbb{M}$.

Lemma 2.3.2. *Let $p\colon E \to B$ be a fibration in $\mathbb{M}$. If p is univalent and B is fibrant, then F_p satisfies the weak equivalence extension property.*

Proof. If B is fibrant, the projections $B \times B \to B$ are fibrations and so the composite map $\mathrm{Eq}p \twoheadrightarrow B \times B \twoheadrightarrow B$ is a fibration, too. If p is univalent, the fibration $\mathrm{Eq}p \twoheadrightarrow B$ is an acylic fibration by the 2-for-3 property and so the weak equivalence extension property of F_p follows by the right lifting property of $\mathrm{Eq}p \twoheadrightarrow B$ against cofibrations. □

In analogy to Definition 2.2.3, in the following we introduce conditions on a fibration p such that the converse direction of Lemma 2.3.2 holds.

Definition 2.3.3. A fibration p in $\mathbb{M}$ satisfies the *stratification property* if diagrams of the form (2.2.1) with $C_1 \to C_2$ a cofibration can be extended just as in Definition 2.2.3.

Just as in the acyclic case, the stratification property aligns lifts to p in such a way that a "discrete" statement about mere existence in F_p yields a "continuous" version of the statement about p directly.

Proposition 2.3.4. *Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{M}$.*

(1) Suppose p has the acyclic stratification property. Then the target map

$$t\colon \mathrm{Eq}p \xrightarrow{(s,t)} B \times B \xrightarrow{\pi_2} B$$

is a fibration if and only if F_p satisfies the acyclic weak equivalence extension property.

(2) Suppose p has the stratification property and B is fibrant. Then, p is univalent if and only if F_p has the weak equivalence extension property.

Proof. One direction of part (2) is given by Lemma 2.3.2, the corresponding direction of part (1) is shown in the same way. The other direction of both parts is obtained by checking the right lifting property of the target map $t\colon \mathrm{Eq}p \to B$ against (acyclic) cofibrations; given a diagram as obtained by the (acyclic) weak equivalence extension property of F_p, the respective stratification property generates the desired lift. $\square$

Remark 2.3.5. Let $p\colon E \twoheadrightarrow B$, $p'\colon E' \twoheadrightarrow B'$ be fibrations in $\mathbb{M}^f$ such that p is univalent. Recall from Corollary 1.6.4 that p and p' are homotopy equivalent (and in particular p' is univalent) if and only if their associated objects of homotopy pullback representations $P_{q,p}$ and $P_{q,p'}$ coincide for all fibrations $q \in \mathbb{C}$.

Note that the class F_p is the union of global sections of the objects $P_{q,p}$ for all fibrations q in $\mathbb{M}$. Obviously if p and p' are isomorphic fibrations, then the classes F_p and F_q coincide, but note that the converse is not true in general. In fact, they need not be even homotopy equivalent. Easy counter examples are coproducts $p \sqcup p$ for fibrations $p\colon E \twoheadrightarrow B$ in model structures $\mathbb{M}$ satisfying the Frobenius Property and very mild further conditions; that is, assume that the object $B \in \mathbb{M}$ is such that the coproduct $B \sqcup B$ is disjoint with non-homotopical inclusions $\iota_i\colon B \to B \sqcup B$ which preserve coproducts along pullback, the coproduct $p \sqcup p\colon E \sqcup E \to B \sqcup B$ is a fibration and the initial object $0 \in \mathbb{M}$ has no non-trivial elements. Then, the squares

$$\begin{tikzcd}
E \sqcup E \arrow[r, "\nabla"] \arrow[d, two heads, "p \sqcup p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r, "\iota_i"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \sqcup E \arrow[d, two heads, "p \sqcup p"] \\
B \sqcup B \arrow[r, "\nabla"'] & B \arrow[r, "\iota_i"'] & B \sqcup B
\end{tikzcd}$$

are pullbacks, and therefore we have $F_p = F_{p \sqcup p}$. But for every fibration $q \in F_p$ at least two non-homotopical maps into $B \sqcup B$ classifying q exist by postcomposition with the two inclusions. If, for example, p is univalent, then $F_{p \sqcup p} = F_p$ satisfies the weak equivalence extension property, but the inclusions $\iota_i \colon B \to B \sqcup B$ admit a lift of the form

$$\begin{tikzcd}
 & \mathrm{Eq}(p \sqcup p) \arrow[d, two heads, "(s,t)"] \\
B \arrow[ur, "(\iota_1, \iota_2, 1_E)"] \arrow[r, "(\iota_1, \iota_2)"'] & (B \sqcup B) \times (B \sqcup B)
\end{tikzcd}$$

even though they are not homotopic. Hence, $p \sqcup p$ cannot be univalent.

Another way to look at this is to understand why $p \sqcup p$ does not satisfy the stratification property. Therefore, take a cylinder object $B \sqcup B \overset{j}{\hookrightarrow} Z(B) \underset{\sim}{\overset{z}{\twoheadrightarrow}} B$ of B and consider the diagram

$$\begin{tikzcd}
E \sqcup E \arrow[rr, equal] \arrow[dd, two heads, "p \sqcup p"'] \arrow[dr] & & E \sqcup E \arrow[dd, two heads, "p \sqcup p"] \\
 & z^*E \arrow[dd, two heads, "z^*p", near start] & \\
B \sqcup B \arrow[rr, equal, crossing over] \arrow[dr, "j"'] & & B \sqcup B. \\
 & ZB &
\end{tikzcd}$$

Note that the square from $p \sqcup p$ to z^*p is cartesian, because p is a fibration such that the pullback functor $(z^*p)^*$ exists, has a right adjoint and hence preserves colimits. Then dotted arrows extending the diagram as required in Definition 2.3.3.(1) cannot exist if the inclusions ι_i are not left homotopic.

2.4. Interplay of the properties

Given a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}$, we add another condition on the class F_p in order to show the following diagram of implications.

(2.4.1)

$$\begin{tikzcd}
\text{w.e.e.p.}(F_p) \arrow[r, phantom, "\Rightarrow"] \arrow[d, Leftrightarrow] & \text{acyclic w.e.e.p.}(F_p) \arrow[r, phantom, "\Leftrightarrow"] \arrow[d, Leftrightarrow] & \text{f.e.p.}(F_p) \\
t\colon \mathrm{Eq}p \to B \text{ is am acyclic fibration} \arrow[r, phantom, "\Rightarrow"] \arrow[d, Leftrightarrow] & t\colon \mathrm{Eq}p \to B \text{ is a fibration} \arrow[d, Leftrightarrow] & \\
p \text{ is univalent \& } B \text{ is fibrant} \arrow[r, phantom, "\Rightarrow"] & B \text{ is fibrant} \arrow[uur, Leftrightarrow, bend right] &
\end{tikzcd}$$

Here, w.e.e.p. is short for the weak equivalence extension property and f.e.p. is short for the fibration extension property. Note that some implications are trivial while some others do not require any conditions on $\mathbb{M}$ (not even the Frobenius property). The first row of vertical arrows was shown in the previous section and, in the downwards direction, requires p to satisfy the (acyclic) stratification property. All other implications together with their respective necessary conditions will be shown in this section.

The following statement is the only remaining one we can prove without further assumptions.

Lemma 2.4.1. *Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{M}$. If F_p satisfies the fibration extension property, then it satisfies the acyclic weak equivalence extension property.*

Proof. Let

$$\begin{tikzcd}
X_1 \arrow[dr, "\sim", "w"'] \arrow[ddr, two heads, bend right, "q_1"'] & & \\
& X_2 \arrow[r, hook, "\sim"] \arrow[d, two heads, "q_2"'] \arrow[dr, phantom, "\lrcorner"] & W_2 \arrow[d, two heads, "\bar{q}_2"] \\
& Y \arrow[r, hook, "\sim", "j"'] & Z
\end{tikzcd}$$

be a diagram as in Definition 2.3.1. By assumption, the diagram

$$\begin{tikzcd}
X_1 \arrow[d, two heads, "q_1"'] & \\
Y \arrow[r, hook, "\sim", "j"'] & Z
\end{tikzcd}$$

has an extension of the form

$$\begin{tikzcd} X_1 \arrow[r, hook, "\sim", "k"'] \arrow[d, two heads, "q_1"'] \arrow[dr, phantom, "\lrcorner"] & W_1, \arrow[d, two heads, "\bar{q}_1 \in F_p"] \\ Y \arrow[r, hook, "\sim", "j"'] & Z \end{tikzcd}$$

and the square $\begin{tikzcd} X_1 \arrow[r, "\bar{q}_2^* j \circ w"] \arrow[d, hook, "k"', "\wr"] & W_2 \arrow[d, two heads, "\bar{q}_2"] \\ W_1 \arrow[r, "\bar{q}_1"'] & Z \end{tikzcd}$ exhibits a lift $v\colon W_1 \to W_2$ completing the diagram

$$\begin{tikzcd} X_1 \arrow[rr, "k"] \arrow[dr, "w"] \arrow[ddr, bend right, "q_1"'] & & W_1 \arrow[dr, "v"] \arrow[ddr, dashed, two heads, "\bar{q}_1"] & \\ & X_2 \arrow[rr, hook, "\sim"] \arrow[d, two heads, "q_2"] \arrow[drr, phantom, "\lrcorner"] & & W_2 \arrow[d, two heads, "\bar{q}_2"] \\ & Y \arrow[rr, hook, "\sim", "j"'] & & Z \end{tikzcd}$$

as required by Definition 2.3.1. The map v is a weak equivalence by 2-for-3, since the maps k, $\bar{q}_2^* j$ and w are all weak equivalences. ☐

To prove the converse direction as well as all other remaining directions, we introduce the following condition.

Definition 2.4.2. Say a class $S \subseteq \mathbb{M}$ of morphisms is *closed under fibrant replacement* if it is closed under composition, $\mathcal{C} \cap \mathcal{W} \subseteq S$ and every map in S can be factored into an acyclic cofibration followed by a fibration in S.

Lemma 2.4.3. *Let $S \subseteq \mathbb{M}$ be a class of maps closed under fibrant replacement and suppose S (or equivalently $S \cap \mathcal{F}_{\mathbb{M}}$) has the acyclic weak equivalence extension property. Then S (or equivalently $S \cap \mathcal{F}_{\mathbb{M}}$) has the fibration extension property.*

Proof. Let $q\colon X \twoheadrightarrow Y$ be a fibration in S followed by an acyclic cofibration $j\colon Y \overset{\sim}{\hookrightarrow} Z$. Factor the composition $jq \in S$ into an acyclic cofibration k followed by a fibration $\bar{q} \in S$ to obtain a square

$$\begin{tikzcd}
X \arrow[r, hook, "\sim", "k"'] \arrow[d, two heads, "q"'] & W \arrow[d, two heads, "\bar{q}"] \\
Y \arrow[r, hook, "\sim", "j"'] & Z.
\end{tikzcd}$$

This square is homotopy cartesian due to the Frobenius property (Definition 1.2.3.(2)), i.e. the natural map $X \to X_2$ in the diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, two heads, "q"'] & & \\
& X_2 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, two heads, "\bar{q}"] \\
& Y \arrow[r, hook, "\sim", "j"'] & Z
\end{tikzcd}$$

is a weak equivalence. All three vertical fibrations are elements of S and hence, assuming S has the acyclic weak equivalence extension property, there is an extension

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[dr] \arrow[ddr, bend right, two heads, "q"'] & W_1 \arrow[dr, dashed] \arrow[ddr, dashed, two heads] & \\
& X_2 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, two heads, "\bar{q}"] \\
& Y \arrow[r, hook, "\sim", "j"'] & Z
\end{tikzcd}$$

with $W_1 \twoheadrightarrow Z$ in S such that all three squares are cartesian. Thus, the back square verifies the fibration extension property of S. □

Remark 2.4.4. Let $S \subseteq \mathbb{M}$ be a class of maps closed under fibrant replacement and $p\colon E \twoheadrightarrow B$ a fibration which is universal for $S \cap \mathcal{F}_{\mathbb{M}}$. Then we can show by the same line of reasoning as in the proof of Lemma 2.4.3 that the base B is fibrant and contractible if we can extend all maps (instead of only weak equivalences) between fibrations in F_p along cofibrations to some map (instead of a weak equivalence) between fibrations in F_p as depicted in Definition 2.3.1.

To complete Diagram 2.4.1, we are left to show the following lemma.

Lemma 2.4.5. *Let $p\colon E \twoheadrightarrow B$ be a fibration and $S \subseteq \mathbb{M}$ be a class of morphisms closed under fibrant replacement with $S \cap \mathcal{F} = F_p$. Then the target map $t\colon \mathrm{Eq}p \to B$*

is a fibration if and only if B is fibrant. In particular, the target map t is an acyclic fibration if and only if p is univalent and B is fibrant.

Proof. If B is fibrant, the projection $\pi_2 \colon B \times B \to B$ is a fibration and hence the composition $t = \pi_2(s,t)$ is a fibration, too. Vice versa, suppose the target map is a fibration and let $j \colon Y \overset{\sim}{\hookrightarrow} Z$ be an acyclic cofibration and $y \colon Y \to B$ be a map. We have to construct a lift of y to Z. Following the proof of Lemma 2.4.3, we recall that by our assumptions there is a diagram

$$\begin{tikzcd}
y^*E \arrow[dr, "w"] \arrow[drr, hook, bend left, "k", "\sim"'] \arrow[ddr, two heads, bend right, "y^*p"'] & & \\
& X \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & W \arrow[d, two heads, "q"] \\
& Y \arrow[r, hook, "\sim", "j"'] & Z
\end{tikzcd}$$

where $q = z^*p$ for some map $z \colon Z \to B$ (which does not have to be compatible with y). This gives a square together with a lift

$$\begin{tikzcd}
Y \arrow[r, "{(y,jz,w)}"] \arrow[dd, hook, "j"', "\wr"] & \mathrm{Eq}p \arrow[d, "{(s,t)}"] \\
& B \times B \arrow[d, "\pi_2"] \\
Z \arrow[r, "z"'] \arrow[uur, dashed, "l"] & B
\end{tikzcd}$$

so we in particular get a lift of y along j via

$$\begin{tikzcd}
Y \arrow[rrr, bend left, "y"] \arrow[d, hook, "j", "\wr"'] & \mathrm{Eq}p \arrow[r, "{(s,t)}"'] & B \times B \arrow[r, "\pi_1"'] & B. \\
Z \arrow[ur, "l"'] & & &
\end{tikzcd}$$

The second part of the statement is an immediate consequence of the 2-for-3 property. $\square$

Remark 2.4.6. Lemma 2.4.5 also follows directly from the priorly proven instances of Diagram 2.4.1 if p satisfies the stratification property, since the right half of the diagram is a chain of equivalences.

2.5. The class of small maps in presheaf categories

Due to the closure properties of type theoretical universes under dependent type formers, any interpretation of such universes as universal fibrations in a set theoretical background theory will embody some notion of smallness. More precisely, such universes are modelled as *small* fibrations classifying all *small* fibrations. An analysis of this restriction shows that such a class of small maps may be axiomatized quite generally in a style similar to the presentation of small maps in [**31**]. We omit giving such an axiomatization here and restrict our attention to the class of κ-small maps for a cardinal κ on a presheaf category $\mathbb{M} = [\mathbb{D}^{\mathrm{op}}, \mathrm{Set}]$. The reason being that the author is simply not aware of any examples of such a class of small maps which is not directly derived from the set theoretical notion of κ-smallness for some cardinal κ.

In this section we want to apply the observations from Section 2.4 to the class of κ-small maps in presheaf categories. As a result, in Theorem 2.5.4 we will see that the common constructions of univalent universal κ-small fibrations presented in the literature always have fibrancy of the base built in.

Throughout this section, $\mathbb{M}$ is assumed to be a cofibrantly generated model structure on a presheaf category $\widehat{\mathbb{D}} := [\mathbb{D}^{\mathrm{op}}, \mathrm{Set}]$ satisfying the Frobenius property (Definition 1.2.3.(2)). For the definition of cofibrantly generated model categories, see [**25**, Chapter 11].

We fix generating sets I and J of cofibrations and acyclic cofibrations for $\mathbb{M}$, respectively. For a small category $\mathbb{D}$ and a cofibrantly generated model structure $\mathbb{M}$ on $\widehat{\mathbb{D}}$, we denote the cardinality of $\mathbb{D}$ by

$$|\mathbb{D}| := \sum_{D,D' \in \mathbb{D}} |\mathrm{Hom}_{\mathbb{D}}(D, D')|$$

and, given a presheaf $X \in \widehat{\mathbb{D}}$, the size of X is given by

$$|X| := \sum_{D \in \mathbb{D}} |X(D)| = \sum_{D \in \mathbb{D}} |\widehat{\mathbb{D}}(yD, X)|.$$

Then let

$$\mathfrak{c}(\mathbb{M}) := \sum_{i \in I} (|\mathrm{dom} i| + |\mathrm{codom} i|)^{+} + \sum_{j \in J} (|\mathrm{dom} j| + |\mathrm{codom} j|)^{+} + |\mathbb{D}|$$

be an upper bound of the size of arrows in I and J. So $\mathfrak{c}(\mathbb{M}) \geq |\mathbb{D}|$ and for every $j \in J$, the codomain of j is $\mathfrak{c}(\mathbb{M})$-compact.

Given a cardinal κ, we consider the class

$$\{X \in \widehat{\mathbb{D}} \mid |X| < \kappa\}$$

of κ-small objects in $\widehat{\mathbb{D}}$ and its corresponding full subcategory $\widehat{\mathbb{D}}_\kappa \subset \widehat{\mathbb{D}}$. We say that a morphism $f\colon X \to Y$ in $\mathbb{M}$ is *κ-small* if for all elements $c\colon yC \to Y$, the pullback c^*X is contained in $\widehat{\mathbb{D}}_\kappa$. We denote the class of all κ-small maps in $\widehat{\mathbb{D}}$ by S_κ.

Remark 2.5.1. Note that if $\kappa > \mathfrak{c}(\mathbb{M})$ is regular, then $J \subset \widehat{\mathbb{D}}_\kappa$ and every acyclic cofibration in $\mathbb{M}$ is κ-small. This is because the class of acyclic cofibrations is the smallest saturated class containing J, and $\widehat{\mathbb{D}}_\kappa$ certainly is closed under retracts and $< \mathfrak{c}(\mathbb{M})$-sequential colimits. The fact that it is also closed under pushouts along arbitrary maps follows from compactness of the representables, hence the class of κ-small maps indeed is saturated.

Remark 2.5.2. In [**33**] and [**52**], the authors construct universal fibrations $\pi_\kappa\colon \tilde{\mathcal{U}}_\kappa \twoheadrightarrow \mathcal{U}_\kappa$ for S_κ in specific *Cisinski model structures* $\mathbb{M}$ on presheaf categories for suitably large cardinals κ. While Shulman's construction in [**52**, Theorem 3.2] satisfies the stratification property by design – it is incorporated in his Condition "(2′)" that he checks to prove universality for S_κ – the authors of [**33**] use representability arguments to construct π_κ. By their definition of $\mathcal{U}_\kappa$, for every object $Y \in \mathbb{M}$ where $\mathbb{M}$ specifically is the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$ on simplicial sets, the canonical map

$$\mathbf{S}(Y, \mathcal{U}_\kappa) \to \{\text{well ordered fibrations into } Y\}$$

is an isomorphism. The fibrations π_κ, which correspond under this isomorphism to the identity map on $\mathcal{U}_\kappa$, automatically satisfy the stratification property, because the diagram in Definition 2.3.3 commutes for every dotted extension by uniqueness of such extensions (up to the choice of a well order).

In order to apply the results of Section 2.4 to the class S_κ directly, we would have to prove that S_κ is closed under fibrant replacement as defined in Definition 2.4.2. Unfortunately, this is not true in general, there are counter examples already in the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$.[1] Therefore, we have to work with the following simple refinement of the fibration extension property instead.

[1] Thanks to Mike Shulman for pointing them out to me.

Definition 2.5.3. Le S be a class of maps in $\widehat{\mathbb{D}}$. Say S has the *fibration extension property with respect to* J if every solid span

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & W \arrow[d, dashed, "\bar{q}"] \\ Y \arrow[r, hook, "\sim", "j"'] & Z \end{tikzcd}$$

where $q \in S$ is a fibration and $j \in J$ is an acyclic cofibration can be complemented to a cartesian square such that $\bar{q} \in S$ is a fibration, too.

Theorem 2.5.4. *Let $\mathbb{M}$ be a cofibrantly generated model structure on a presheaf category $\widehat{\mathbb{D}}$. Let $\kappa > \mathfrak{c}(\mathbb{M})$ be a regular cardinal such that the class S_κ of κ-small maps in $\mathbb{M}$ satisfies the weak equivalence extension property.*

(1) Then S_κ satisfies the fibration extension property with respect to J.

(2) If further there is an S_κ-universal fibration $\pi_\kappa \colon \tilde{U}_\kappa \twoheadrightarrow U_\kappa$ in $\mathbb{M}$ which satisfies the acyclic stratification property, then the base U_κ is fibrant. In particular, S_κ satisfies the fibration extension property (with respect to all acylic cofibrations).

Proof. Let $\mathbb{M}$ and κ be as stated. For part (1), given a solid span as in Definition 2.5.3, the composition $qj \colon X \to Z$ is contained in $\widehat{\mathbb{D}}_\kappa$ and hence, by [**16**, Proposition 2.3.(iii)], we can factor qj into an acyclic cofibration $k \colon X \overset{\sim}{\hookrightarrow} W$ and a κ-small fibration $\bar{q} \colon W \twoheadrightarrow Z$. Then the construction of the dashed arrows in the diagram of Definition 2.5.3 follows exactly along the lines of the proof of Lemma 2.4.3.

Part (2) is proven exactly like Lemma 2.2.4, since the base U_κ is fibrant if and only if it has the right lifting property against all generating acylic cofibrations. □

Theorem 2.5.4.(2) shows that under the given assumptions the effort to prove univalence and fibrancy of the given universal fibration independently from one another is redundant. As an example, recall the main results from [**52**], rephrased in the language used throughout this chapter. The following amounts to [**52**, Theorem 3.1] with a minor correction on the cardinal bound of κ.

Theorem 2.5.5. *Let $\mathbb{M} = \widehat{\mathbb{D}}$ be a simplicial model category such that the cofibrations are exactly the monomorphisms. If κ is an infinite cardinal such that $\mu^{|\mathbb{D}|} < \kappa$ for all $\mu < \kappa$, then S_κ has the weak equivalence extension property in $\mathbb{M}$.*

□

Remark 2.5.6. The condition on κ to satisfy $\mu^{|\mathbb{D}|} < \kappa$ for all $\mu < \kappa$ is unnecessary in the case $\mathbb{D} = \Delta$ due to finiteness of the representables in this case which is the reason it does not appear as a condition in [**33**, Theorem 3.4.1].

Corollary 2.5.7. *If further $\kappa > \mathfrak{c}(\mathbb{M})$ is regular, then the class S_κ of κ-small maps has the fibration extension property with respect to J.*

Proof. Follows immediately from Theorem 2.5.4 and Theorem 2.5.5. □

The following amounts to [**52**, Theorem 3.2].

Theorem 2.5.8. *Suppose $\mathbb{M} = \widehat{\mathbb{D}}$ is cofibrantly generated, all cofibrations in $\mathbb{M}$ are monomorphisms and the codomains of generating acyclic cofibrations in J are representable. Then, for every regular cardinal $\kappa > \mathfrak{c}(\mathbb{M})$, there is a fibration $\pi_\kappa \colon \tilde{U}_\kappa \twoheadrightarrow U_\kappa$ universal for S_κ with the stratification property.* □

Although the author of [**52**] did not mention the stratification property in his paper, in his case it holds trivially by construction as mentioned in Remark 2.5.2.

Corollary 2.5.9. *Suppose all cofibrations in $\mathbb{M}$ are monomorphisms and the codomains of generating acyclic cofibrations in J are representable. Let $\kappa > \mathfrak{c}(\mathbb{M})$ be inaccessible. Then the universal fibration $\pi_\kappa \colon \tilde{U}_\kappa \twoheadrightarrow U_\kappa$ for S_κ from Theorem 2.5.8 is univalent and the codomain U_κ is fibrant.*

Proof. By Corollary 2.5.7 and Theorem 2.5.8. □

This captures the essential content of [**52**, Section 6] on elegant Reedy structures and the Reedy assumption turns out unnecessary as far as it goes beyond assuring a cofibrantly generated model structure with a set of generating acyclic cofibrations with representable domain whose cofibrations are exactly the monomorphisms. Indeed, generalizing [**52**, Theorem 5.1, Theorem 6.4] we can summarize the situation in the following theorem. Therefore, for a category $\mathbb{D}$, recall the category of simplicial presheaves $\mathrm{sPsh}(\mathbb{D}) := [\mathbb{D}^{\mathrm{op}}, \mathbf{S}]$ over $\mathbb{D}$ isomorphic to the presheaf category $\widehat{\mathbb{D} \times \Delta}$. We consider it equipped with the injective model structure, i.e. the cofibrations are exactly the monomorphisms in $\mathrm{sPsh}(\mathbb{D})$ and the weak equivalences are the point-wise weak equivalences in $(\mathbf{S}, \mathrm{Kan})$.

Theorem 2.5.10. *Let $\mathbb{D}$ be a small category and consider $\mathrm{sPsh}(\mathbb{D})$ as equipped with the injective model structure. Then $\mathrm{sPsh}(\mathbb{D})$ is a type theoretic model category. Let $\kappa > \mathfrak{c}(\mathrm{sPsh}(\mathbb{D}))$ be an inaccessible cardinal.*

(1) The class S_κ of κ-small maps has the weak equivalence extension property and the fibration extension property with respect to J.

(2) Hence, if the codomains of the generating acyclic cofibrations are representable, sPsh($\mathbb{D}$) *supports a univalent universal fibration for S_κ with fibrant base.*

Proof. The fact that sPsh($\mathbb{D}$) equipped with the injective model structure is a type theoretic model category is well known and easy to verify. Part (1) follows directly from Theorem 2.5.5 and Corollary 2.5.7. Part (2) follows directly from Theorem 2.5.4. $\square$

The class of examples of categories $\mathbb{D}$ that Theorem 2.5.10 applies to is rather specific unfortunately, since in general there is no canonical set of generating acyclic cofibrations for the injective model structure on sPsh($\mathbb{D}$).

We see that the only missing ingredient for the injective model structure on sPsh($\mathbb{D}$) to fully support intensional type theory with univalent universes is the existence of S_κ-universal fibrations which satisfy the stratification property.

CHAPTER 3

An interlude for left Bousfield localizations

The remaining chapters of this thesis discuss ideas and constructions based on Bousfield localizations of combinatorial model categories. Therefore, in this chapter we briefly discuss the relevant technical underlying material and review some of the notions discussed in the two prior chapters in this specific framework. In Section 3.1, we gather basic definitions, examples and statements about left Bousfield localizations of combinatorial model categories. In Section 3.2, we re-introduce the notion of (-1)-truncated maps from Section 1.4 in the context of left proper simplicial combinatorial model categories, and in Section 3.3 we have a look at the behaviour of univalent universal fibrations and the weak equivalence extension property as treated in Chapter 2 under left Bousfield localization.

Throughout this chapter, "Bousfield localization" always refers to "left Bousfield localization".

3.1. General theory

Definition 3.1.1. Given a bicomplete category $\mathbb{M}$ with two model structures M_i for $i \in \{0, 1\}$, M_2 is said to be a Bousfield localization of M_1 if $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. If this holds, the identity $\mathrm{id}\colon (\mathbb{M}, M_1) \to (\mathbb{M}, M_2)$ is a left Quillen functor.

For a model category $\mathbb{M}$ and objects $A, B \in \mathbb{M}$, let $[A, B]_h$ denote the *homotopy function complex* (or, in some literature, the *derived mapping space*) from A to B as defined for example in [**27**, Chapter 5].

Definition 3.1.2. Given a class T of arrows in a model category $\mathbb{M}$, an object $X \in \mathbb{M}$ is said to be T-*local* if

$$f^*\colon [B, X]_h \to [A, X]_h$$

is a weak equivalence for all $f\colon A \to B$ in T. In turn, an arrow $g\colon A \to B$ is said to be a T-*local equivalence* if

$$g^*\colon [B, X]_h \to [A, X]_h$$

is a weak equivalence for all T-local objects X in $\mathbb{M}$. We say that a model structure M_T on the same underlying category is the Bousfield localization of M at T if

(1) $\mathcal{C}_T = \mathcal{C}_{\mathbb{M}}$ and
(2) $\mathcal{W}_T = \{T\text{-local equivalences}\}$

Clearly, if the Bousfield localization of $\mathbb{M}$ at T exists it is unique, because model structures are uniquely determined by their cofibrations and weak equivalences. In that case we denote the Bousfield localization of $\mathbb{M}$ at T by $\mathcal{L}_T(\mathbb{M})$. Also note that $\mathcal{W}_M \subseteq \mathcal{W}_T$ holds whenever $\mathcal{L}_T\mathbb{M}$ exists.

Remark 3.1.3. Let λ be the left part of the functorial weak factorization system in $\mathbb{M}$ which assigns to any map $f\colon A \to B$ a cofibration $\lambda f\colon A \hookrightarrow \bar{B}$ and let $\mathbb{L}$ be the associated cofibrant replacement functor. Replacing the maps in a class T by cofibrations with cofibrant domain by setting

$$\mathbb{L}T := \{\lambda \mathbb{L} f \mid f \in T\}$$

yields equal Bousfield localizations $\mathcal{L}_T\mathbb{M}$ and $\mathcal{L}_{\mathbb{L}T}\mathbb{M}$, because the homotopy function-complexes $[\,\_\,,\,\_\,]_h$ preserve weak equivalences in both variables. Furthermore, working with $\mathbb{L}T$, we might as well require $\mathbb{L}T$-local objects to be fibrant. Then, if $\mathbb{M}$ is a simplicial model category, we can replace the homotopy function complexes $[\,\_\,,\,\_\,]_h$ in the definition by the regular function complexes $[\,\_\,,\,\_\,]_{\mathbb{M}}$. This cofibrant replacement of T obviously becomes redundant if all objects of $\mathbb{M}$ are cofibrant, so that in this case we can consider regular function complexes $[\,\_\,,\,\_\,]_{\mathbb{M}}$ in Definition 8.4.1 for any class T whenever $\mathbb{M}$ is simplicial. Both the simplicial enrichment of $\mathbb{M}$ and cofibrancy of all objects of $\mathbb{M}$ will be satisfied in all the examples $\mathbb{M}$ we consider.

Recall that a model category $(\mathbb{M}, M)$ is said to be combinatorial if the model structure M is cofibrantly generated and the underlying category $\mathbb{M}$ is locally presentable. The following classical existence result is generally attributed to Jeff Smith.

Theorem 3.1.4 ([**48**, Theorem 4.1])**.** *Given a left proper combinatorial simplicial model category $\mathbb{M}$ and a set T of arrows in $\mathbb{M}$, the Bousfield localization $\mathcal{L}_T\mathbb{M}$ of $\mathbb{M}$ at T exists and is again a left proper combinatorial simplicial model category (with the same function complexes). The fibrant objects in $\mathbb{M}_T := \mathcal{L}_T\mathbb{M}$ are exactly the T-local objects which are fibrant in $\mathbb{M}$.*

Hirschhorn gives a proof of a corresponding existence result for left proper *cellular* model categories in [**25**, Theorem 4.1.1].

Observe that by the uniqueness property of Bousfield localizations in terms of Properties (1) and (2) in Definition 3.1.2, the order of successive Bousfield localizations does not matter, i.e. for any simplicial and combinatorial model category $\mathbb{M}$ and classes T_1, T_2 of arrows in $\mathbb{M}$, the model structures $\mathcal{L}_{T_2}\mathcal{L}_{T_1}\mathbb{M}$, $\mathcal{L}_{T_1\cup T_2}\mathbb{M}$ and $\mathcal{L}_{T_1}\mathcal{L}_{T_2}\mathbb{M}$ coincide whenever they exist.

Example 3.1.5. Recall that the Joyal model structure Qcat on the category $\mathbf{S}$ of simplicial sets forms a cellular model category (although it is not simplicial). The inner horn inclusions $\{h_i^n\colon \Lambda_i^n \hookrightarrow \Delta^n \mid 0 < i < n\}$ are acyclic cofibrations in $(\mathbf{S}, \mathrm{Qcat})$ and all weak categorical equivalences – these are the weak equivalences in $(\mathbf{S}, \mathrm{Qcat})$ – are weak homotopy equivalences. This can easily be seen from Joyal's definition of the truncation functors $\tau_0\colon \mathbf{S} \to \mathbf{Set}$ and $\tau_1\colon \mathbf{S} \to \mathbf{Cat}$ as for example defined in [**32**, Section 1]. The acyclic cofibrations in $(\mathbf{S}, \mathrm{Kan})$ are generated by all horn inclusions $\{h_i^n\colon \Lambda_i^n \hookrightarrow \Delta^n \mid 0 \leq i \leq n\}$ while the cofibrations in both $(\mathbf{S}, \mathrm{Qcat})$ and $(\mathbf{S}, \mathrm{Kan})$ are generated by the boundary inclusions $\{\delta_n\colon \partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$. It follows that $(\mathbf{S}, \mathrm{Kan})$ is the Bousfield localization of $(\mathbf{S}, \mathrm{Qcat})$ at the outer horn inclusions

$$l \cup r := \{h_0^n\colon \Lambda_0^n \to \Delta^n \mid n \geq 2\} \cup \{h_n^n\colon \Lambda_n^n \to \Delta^n \mid n \geq 2\}.$$

In fact, the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$ is the Bousfield localization of $(\mathbf{S}, \mathrm{Qcat})$ already at the left (respectively right) horn inclusions. Indeed, one can show that

$$\begin{aligned} X \text{ is fibrant in } (\mathbf{S}, \mathrm{Qcat}) \text{ and } l\text{-local} \quad &\Leftrightarrow \quad X \text{ is left-fibrant} \\ &\Leftrightarrow \quad X \text{ is a Kan complex.} \end{aligned}$$

The last equivalence follows immediately from Joyal's Theorem as proven for example in [**36**, Prop. 1.2.4.3]. Hence, given a map $f : A \to B$ in $\mathbf{S}$, once can show that

$$\begin{aligned} f \text{ is an } l\text{-equivalence} \quad &\Leftrightarrow \quad f^*\colon [B, X]_h \xrightarrow{\sim} [A, X]_h \text{ for all } l\text{-local } X \in \mathrm{Qcat} \\ &\Leftrightarrow \quad f^*\colon [B, X]_h \xrightarrow{\sim} [A, X]_h \text{ for all } X \in \mathrm{Kan} \\ &\Leftrightarrow \quad f \text{ is a weak equivalence in } (\mathbf{S}, \mathrm{Kan}). \end{aligned}$$

Dually, the arguments hold for right horn inclusions.

This construction of a homotopy theory for ∞-groupoids from a homotopy theory for $(\infty, 1)$-categories on $\mathbf{S}$ works out analogously on the category $s\mathbf{S}$ of bisimplicial sets. Indeed, we will see in Chapter 5 that localizing the model structure $(s\mathbf{S}, \mathrm{CS})$ for complete Segal spaces (in fact even the Reedy model structure on bisimplicial sets) at a bisimplicial version of the left horn inclusions generates a model for the homotopy theory of Kan complexes.

We will also make use of the following lemma.

Lemma 3.1.6. *Let* $(F,G)\colon \mathbb{M} \xrightarrow{\simeq} \mathbb{N}$ *be a Quillen equivalence between model categories* $\mathbb{M}$ *and* $\mathbb{N}$.

(1) *Let* $T \subseteq \mathbb{M}$ *be a class of arrows such that the left Bousfield localizations* $\mathcal{L}_T\mathbb{M}$ *and* $\mathcal{L}_{\mathbb{L}F[T]}\mathbb{N}$ *exist. Then the Quillen equivalence* (F,G) *descends to a Quillen equivalence*

$$(F,G)\colon \mathcal{L}_T\mathbb{M} \xrightarrow{\simeq} \mathcal{L}_{\mathbb{L}F[T]}\mathbb{N}.$$

(2) *Let* $T \subseteq \mathbb{N}$ *be a class of arrows such that the left Bousfield localizations* $\mathcal{L}_T\mathbb{N}$ *and* $\mathcal{L}_{\mathbb{R}G[T]}\mathbb{M}$ *exist. Then the Quillen equivalence* (F,G) *descends to a Quillen equivalence*

$$(F,G)\colon \mathcal{L}_{\mathbb{R}G[T]}\mathbb{M} \xrightarrow{\simeq} \mathcal{L}_T\mathbb{N}.$$

Proof. Part (1) is [**25**, 3.3.20.(i)]. To show part (2), let $T \subseteq \mathbb{N}$ and $U := \mathbb{R}G[T]$. Then by part (1), (F,G) descends to a Quillen equivalence

$$(F,G)\colon \mathcal{L}_U\mathbb{M} \xrightarrow{\simeq} \mathcal{L}_{\mathbb{L}F[U]}\mathbb{N}.$$

But for every arrow $f\colon A \to B$ in T, we have a span of weak equivalences $f \xrightarrow{\rho_f} \mathbb{R}f \xleftarrow{\varepsilon} \mathbb{L}F\mathbb{R}Gf$, so $X \in \mathbb{N}$ is T-local if and only if X is $\mathbb{L}F\mathbb{R}Gf$-local for every $f \in T$. This in turn holds if and only if X is $\mathbb{L}F\mathbb{R}G[T]$-local. Therefore,

$$\mathcal{L}_{\mathbb{L}F[U]}\mathbb{N} = \mathcal{L}_{\mathbb{L}F\circ\mathbb{R}G[T]}\mathbb{N} = \mathcal{L}_T\mathbb{N}.$$

$\square$

Lemma 3.1.7 ([**25**, Proposition 3.3.15.(1)])**.** *Let* $\mathbb{M}$ *be a model category,* S *be a class of maps in* $\mathbb{M}$ *such that the Bousfield localization* $\mathcal{L}_S\mathbb{M}$ *exists and*

$$\begin{tikzcd} A \arrow[rr, "e", "\sim"'] \arrow[dr, two heads, "q"'] & & X \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

be a diagram in $\mathbb{M}$ *such that* e *is a weak equivalence and both* p *and* q *are fibrations in* $\mathbb{M}$. *Suppose that* p *is further a fibration in* $\mathcal{L}_S\mathbb{M}$. *Then* q *is a fibration in* $\mathcal{L}_S\mathbb{M}$, *too.*

Recall that the slices $\mathbb{M}/B$ of a left proper combinatorial simplicial model category $\mathbb{M}$ are again left proper, combinatorial and simplicial. Hence, we can localize the slices of $\mathbb{M}$ at a set of maps whenever we can do so with $\mathbb{M}$. It can be shown in that case that for every set T of maps in $\mathbb{M}$ there is a set T_B of maps in $\mathbb{M}/B$ such that the model categories $\mathcal{L}_T\mathbb{M}/B$ and $\mathcal{L}_{T_B}(\mathbb{M}/B)$ coincide.

Just as ordinary locally presentable categories are equivalent to the localization of the presheaf category over their subcategory of λ-compact objects for some cardinal λ, Dugger has shown that combinatorial model categories can be presented in an analogous fashion.

Proposition 3.1.8 ([**16**])**.** *Let $\mathbb{M}$ be a combinatorial model category. Then there is a a regular cardinal λ such that*

(1) there is a set $\mathcal{A} \subset \mathbb{M}$ of λ-small objects which generates $\mathbb{M}$ under λ-filtered colimits;

(2) there are cofibrant and fibrant replacement functors which preserve λ-filtered colimits;

(3) λ-filtered colimits of weak equivalences are again weak equivalences;

(4) there are functorial factorizations of maps $X \to Y$ in $\mathbb{M}$ of the form $X \hookrightarrow \tilde{Y} \overset{\sim}{\twoheadrightarrow} Y$ and $X \overset{\sim}{\hookrightarrow} \tilde{X} \twoheadrightarrow Y$ such that whenever X and Y are λ-small, then so are $\tilde{X}$ and $\tilde{Y}$.

Let the set M_λ denote a skeleton of all λ-compact objects in $\mathbb{M}$. Then $(\mathbb{M}_\lambda)^c$ generates $\mathbb{M}$ under homotopy colimits, i.e. there is a set $T \subset \mathrm{sPsh}(\mathbb{M}_\lambda^c)$ of maps such that the natural Quillen pair

$$\mathrm{Re}\colon\ \mathcal{L}_T\mathrm{sPsh}(\mathbb{M}_\lambda^c) \rightleftarrows \mathbb{M}\ :\mathrm{Sing}$$

is a Quillen equivalence.

Various exactness conditions on this localization characterizing exactness properties of $\mathbb{M}$ are subject to Chapter 7. Note that this presentation $\mathcal{L}_T\mathrm{sPsh}(\mathbb{M}_\lambda^c)$ is a left proper combinatorial simplicial model category.

3.2. (-1)-truncated and (-1)-connected maps

Recalling [**45**, Proposition 7.5], consider the following definition.

Definition 3.2.1. Let $\mathbb{M}$ be a simplicial combinatorial model category and λ a regular cardinal as in Theorem 3.1.8. For $\delta_k\colon \partial\Delta^k \hookrightarrow \Delta^k$ the k-th boundary inclusion and a fixed integer $n \geq -2$, consider

$$T_n := \{\delta_{n+2} \otimes Z \colon \partial\Delta^k \otimes Z \to \Delta^k \otimes Z \mid Z \in \mathbb{M}^c_\lambda\},$$

so we can construct the Bousfield localization $\mathcal{L}_{T_n}\mathbb{M}$. Say an object $X \in \mathbb{M}$ is *n-truncated* if it is T_n-local, while X is *n-connected* if it is contractible in $\mathcal{L}_{T_n}\mathbb{M}$. Analogously, a map $(f\colon X \to Y) \in \mathbb{M}$ is *n-truncated* in $\mathbb{M}$ if it is a n-truncated object in $\mathbb{M}/Y$, while f is *n-connected* if it is a n-connected object in $\mathbb{M}/Y$.

Remark 3.2.2. Observe the object-wise definition of truncation and connectivity, it is not claimed that the fibrations in $\mathcal{L}_{T_n}\mathbb{M}$ are the n-truncated fibrations. Still, a fibration $f\colon X \twoheadrightarrow Y$ is n-truncated in $\mathbb{M}$ if and only if for all maps $g \in \mathbb{M}/Y$, the homotopy function complex $[g, f]_h$ is a n-truncated Kan complex. By the fibre-wise nature of n-truncatedness in $(\mathbf{S}, \mathrm{Kan})$, if Y is fibrant, this in turn holds if and only if for all cofibrant objects $Z \in \mathbb{M}$, the map $f_*\colon [Z, X]_{\mathbb{M}} \to [Z, Y]_{\mathbb{M}}$ is a n-truncated map of simplicial sets. This last formulation does not involve any instance of a homotopy theory on the slice categories anymore. Indeed, the collection of n-truncated fibrations is (homotopy) pullback stable in $\mathbb{M}$, because $[Z, \_]_h$ preserves homotopy pullbacks. By reducing commutative squares to commutative triangles via pullback, it is easy to see that the pair (n-connected cofibrations, n-truncated fibrations) is a weak factorization system on $\mathbb{M}$.

Lemma 3.2.3. *Let $\mathbb{M}$ be a type theoretic model category which is combinatorial, left proper and simplicial, such that all fibrant objects in $\mathbb{M}$ are cofibrant. Let $p\colon X \twoheadrightarrow Y$ be a fibration in $\mathbb{M}^f$. Then the following are equivalent.*

(1) The fibration p is (-1)-truncated in the sense of Definition 1.4.1.
(2) The fibration p is (-1)-truncated in the sense of Definition 3.2.1.

Proof. Since $\mathbb{M}$ is simplicially enriched, we can choose the path object $P_Y X$ to be given by the cotensor X^{Δ^1} and $\partial\colon P_Y X \to X \times_Y X$ to be X^{δ_1}.

We show (1)⇒(2). If (1) holds, by Lemma 1.4.3 we know that the path object fibration $\partial\colon P_Y X \twoheadrightarrow X \times_Y X$ is acylic. It follows that for every cofibrant object $Z \in \mathbb{M}/Y$, the induced map $[Z, P_Y X]_{\mathbb{M}/Y} \twoheadrightarrow [Z, X \times_Y X]_{\mathbb{M}/Y}$ is a trivial fibration in $\mathbf{S}$. But in virtue of the simplicial enrichment, this map is isomorphic to

$$[\Delta^1 \otimes_Y Z, X]_{\mathbb{M}/Y} \twoheadrightarrow [\partial\Delta^1 \otimes_Y Z, X]_{\mathbb{M}/Y}.$$

In particular, the fibration p is $\{\delta_1 \otimes_Y Z \mid Z \in (\mathbb{M}/Y)^c\}$-local.

For the converse direction, assume (2) holds. Then, as X is cofibrant, the fibration p is $\delta_1 \otimes_Y X$-local and so, by cotensoring, the natural map

$$[X, P_Y X]_{\mathbb{M}/Y} \twoheadrightarrow [X, X \times_Y X]_{\mathbb{M}/Y}$$

is an acyclic fibration. In particular, we obtain a section of $\partial\colon P_Y X \twoheadrightarrow X \times_Y X$ as required by Definition 1.4.1. $\square$

3.3. Univalence in left Bousfield localizations

Notation 3.3.1. Given a model category $\mathbb{M}$ and a class T of maps in $\mathbb{M}$, the (co)fibrations and weak equivalences in the Bousfield localization $\mathcal{L}_T\mathbb{M}$ are referred to simply by T-(co)fibrations and T-weak equivalences.

Recall the weak equivalence extension property from Definition 2.3.1.

Lemma 3.3.2. *Let S and T be classes of maps in $\mathbb{M}$. If S has the weak equivalence extension property in $\mathbb{M}$, then it has the weak equivalence extension property in $\mathcal{L}_T\mathbb{M}$.*

Proof. Suppose we are given a solid span

$$\begin{tikzcd}
A_1 \arrow[rr, dashed] \arrow[dr, "w"] \arrow[ddr, bend right, "q_1"'] & & D_1 \arrow[dr, dashed, "v"] \arrow[ddr, dashed, "\bar{q}_1"] & \\
& A_2 \arrow[rr] \arrow[d, "q_2"] \arrow[dr, phantom, "\lrcorner"] & & D_2 \arrow[d, "\bar{q}_2"] \\
& B \arrow[rr, hook, "\iota"'] & & C.
\end{tikzcd}$$

in $\mathcal{L}_T\mathbb{M}$ as in Definition 2.3.1. Thus, it follows that q_1, q_2 and $\bar{q}_2$ are T-fibrations in S, ι is a cofibration and the map w, being a T-weak equivalence between T-fibrations, is a weak equivalence in $\mathbb{M}$. By the weak equivalence extension property of S in $\mathbb{M}$ we obtain the dashed arrows such that v is a weak equivalence in $\mathbb{M}$, $\bar{q}_1$ is an $\mathbb{M}$-fibration in S and the back square is cartesian. By Lemma 3.1.7, the map $\bar{q}_1$ turns out to be a fibration in $\mathcal{L}_T\mathbb{M}$, and v is a weak equivalence in $\mathcal{L}_T\mathbb{M}$ anyway. $\square$

It follows that Theorem 2.5.10 applies to all right proper left Bousfield localizations of simplicial presheaf categories with the injective model structure.

Another useful transfer result in the passage to Bousfield localizations is given by a straightforward generalization of Cisinski's work in [**14**, Theorem 1.1]. In the following, we say that a fibration $p\colon E \twoheadrightarrow B$ is weakly universal for a class S of maps if for every fibration $q\colon X \twoheadrightarrow Y$ in S there is a homotopy cartesian square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r] & B. \end{tikzcd}$$

Theorem 3.3.3. *Let $\mathbb{M}$ be a right proper model category, S be a class of maps in $\mathbb{M}$ and $p\colon \tilde{U} \twoheadrightarrow U$ a univalent $(S \cap \mathcal{F}_{\mathbb{M}})$-universal fibration with fibrant base U. Let $\mathcal{L}_T\mathbb{M}$ be a Bousfield localization of $\mathbb{M}$. Then every univalent weakly $(S \cap \mathcal{F}_T)$-universal T-fibration $p_T\colon \tilde{U}_T \to U_T$ in $\mathcal{L}_T\mathbb{M}$ with T-fibrant base U_T can be rectified to a univalent and strictly $(S \cap \mathcal{F}_T)$-universal T-fibration with T-fibrant base.*

Proof. This proof is a straightforward transcription of Cisinski's construction of fibrant universes in the locally constant model structure, see [**14**, Theorem 1.1]. Let $p_T\colon \tilde{U}_T \twoheadrightarrow U_T$ be a weakly $(S \cap \mathcal{F}_T)$-universal T-fibration with T-fibrant codomain and let $p\colon \tilde{U} \twoheadrightarrow U$ be a univalent $(S \cap \mathcal{F}_T)$-universal fibration with fibrant codomain in $\mathbb{M}$. I.e. the latter is a fibration between fibrant objects in $\mathbb{M}$ such that every $\mathbb{M}$-fibration in S arises as a strict pullback of p. Hence, choose a strict pullback square

$$\begin{tikzcd} \tilde{U}_T \arrow[r, "\alpha"] \arrow[d, "p_T"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ U_T \arrow[r, "\beta"'] & U \end{tikzcd}$$

and denote this square by S_U. The proof consists essentially of four steps.

(1) Given a T-fibration $q\colon A \to B$ in S, pick a strict pullback square S_1 from q to p, and separately a homotopy cartesian square S_2 from q to p_T in $\mathbb{M}_T$.

(2) Argue that S_2 is also homotopy cartesian in $\mathbb{M}$. Thus, $S_U \circ S_2\colon q \to p$ is a homotopy cartesian square from q to p, and the classifying maps $\zeta_1\colon B \to U$ and $\beta \circ \zeta_2\colon B \to U$ associated to S_1 and $S_U \circ S_2$ respectively have to be homotopic by univalence of p.

(3) If β was a fibration, we could lift the corresponding homotopy to a homotopy in U_T between ζ_2 and some map $\bar{\zeta}$ in the fibre of ζ_1. But ζ_1 and β are both strictly classifying maps (yielding associated strictly cartesian

squares), hence $\bar{\zeta}$ is also strictly classifying the map $\mathrm{dom}S_1 = q$. Therefore, first

(0) fibrantly replace the weakly $(S \cap \mathcal{F}_T)$-universal T-fibration p_T by another such universe π over U, such that the map β is a fibration and we can perform Step 3. Replace p_T with π in Steps 1,2 and 3.

More precisely, pick a factorization $\begin{tikzcd} U_T \arrow[r, "u"'] \arrow[rr, bend left, two heads, "\beta"] & V \arrow[r, two heads, "w"'] & U \end{tikzcd}$ into an acyclic cofibration u and a fibration w in $\mathbb{M}$. Factoring the pullback square S_U correspondingly through V yields a diagram of the form

$$\begin{tikzcd}
& \tilde{V} \arrow[dd, "\pi"] \arrow[dr, "\tilde{v}"] & \\
\tilde{U}_T \arrow[ur, dashed] \arrow[rr, "\alpha"] \arrow[dd, "p_T"'] & & \tilde{U} \arrow[dd, "p"] \\
& V \arrow[dr, "v"] & \\
U_T \arrow[ur, "u"] \arrow[rr, "\beta"'] & & U,
\end{tikzcd}$$

thus so far we replace p_T by a fibration π whose classifying map v into p is a fibration, too. But the square from p_T to π is a pullback square, too, and $\mathbb{M}$ is right proper, therefore the natural map $\tilde{U}_T \to \tilde{V}$ is a weak equivalence in $\mathbb{M}$. By construction, the objects $\tilde{U}_T$ and U_T are T-local. Hence, by Lemma 3.1.7, V and $\tilde{V}$ are T-local, too. So the objects $\tilde{V}$ and V are fibrant in $\mathbb{M}_T$. Further, π, being a fibration between T-fibrant objects, also turns out to be a T-fibration. We now show that π is the fibrant universe for $\mathbb{M}_T$ we are looking for.

Let $q\colon A \to B$ be a T-fibration in S, and pick both a pullback square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "q"'] \arrow[dr, phantom, "I"] & \tilde{U} \arrow[d, "p"] \\
B \arrow[r, "\beta"'] & U,
\end{tikzcd}$$

and a homotopy cartesian square

$$\begin{tikzcd}
A \arrow[r, "\mu"] \arrow[d, "q"'] \arrow[dr, phantom, "II"] & \tilde{V} \arrow[d, "\pi"] \\
B \arrow[r, "\lambda"'] & V
\end{tikzcd}$$

in $\mathbb{M}_T$. Square II comes from the composition of a homotopy cartesian square from q to p_T (in $\mathbb{M}_T$) and the square from p_T into π which is homotopy cartesian in $\mathbb{M}_T$, too.

Claim. Square II also is homotopy cartesian in $\mathbb{M}$.

Proof. Indeed, π is a T-fibration (and hence an $\mathbb{M}$-fibration), and factoring μ via

$$\begin{tikzcd}
A \arrow[r, dashed, "\eta"] \arrow[rr, bend left, "\mu"] \arrow[dr, bend right, "q"'] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{V} \arrow[d, "\pi"] \\
& B \arrow[r, "\lambda"'] & V
\end{tikzcd}$$

through the pullback along λ, we want to show that η is a weak equivalence in $\mathbb{M}$. But we know that η is a weak T-equivalence, and that both q and $\lambda^*\pi$ are T-fibrations. The model structure $\mathbb{M}_T/B$ is still a Bousfield localization of $\mathbb{M}/B$, and we just have spelled out that η is a weak equivalence over fibrant objects in $\mathbb{M}_T/B$. Therefore, η is already a weak equivalence in $\mathbb{M}/B$, in particular a weak equivalence in $\mathbb{M}$. $\square$

Thus, the composition

$$\begin{tikzcd}
A \arrow[r, "\mu"] \arrow[d, "q"'] \arrow[dr, phantom, "II"] & \tilde{V} \arrow[r, "\tilde{v}"] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & \tilde{U} \arrow[d, "p"] \\
B \arrow[r, "\lambda"'] & V \arrow[r, "v"'] & U
\end{tikzcd}$$

is homotopy cartesian in $\mathbb{M}$. But Square I is already homotopy cartesian in $\mathbb{M}$, too, and hence by univalence of p, the map

$$\begin{tikzcd}
B \arrow[r, "{(v\lambda,\beta,(A \xrightarrow{\sim} \beta^*\tilde{U})\circ(A \xrightarrow{\sim} (v\lambda)^*\tilde{U})^{-1})}"] \arrow[dr, "{(v\lambda,\beta)}"'] & \mathrm{Eq}p \arrow[d] \\
& U \times U
\end{tikzcd}$$

yields a homotopy between the classifying maps β and $v\lambda$. Thus, there is a homotopy $h\colon B \times p_2^*\Delta^1 \to U_v$ with endpoints $h|_0 = \beta$ and $h|_1 = v\lambda$. By fibrancy of v, we can lift h to V,

$$\begin{tikzcd}
B \times \{1\} \arrow[r, "\lambda"] \arrow[d, "\mathrm{id}\times d_0"'] & V \arrow[d, "v"] \\
B \times p_2^*\Delta^1 \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & U,
\end{tikzcd}$$

with $\bar{h}|_0 =: \bar{\beta}$, $\bar{h}|_1 = \lambda$ and $v\bar{\beta} = h|_0 = \beta$. By naturality, the map $\bar{\beta}$ yields the cartesian square

$$\begin{tikzcd}
& \tilde{V} \arrow[dr, "\tilde{v}"] \arrow[dd, "\pi", near end] \arrow[d, phantom, "\lrcorner", near start, shift left=2] & \\
A \arrow[ur, dashed, "\gamma"] \arrow[rr, "\alpha", near start] \arrow[ddd, "q"] \arrow[dr, phantom, "\lrcorner", near start] & & \tilde{U} \arrow[ddd, "p"] \\
& V \arrow[ddr, "v"] & \\
& & \\
B \arrow[uur, "\bar{\beta}"] \arrow[rr, "\beta"'] & & U,
\end{tikzcd}$$

so eventually turns out to be a classifying map for q. $\square$

CHAPTER 4

Bousfield-Segal spaces

4.1. Introduction

In [**7**, 6], Julie Bergner introduced a model structure for (complete) Bousfield-Segal spaces, while the notion of Bousfield-Segal spaces itself originated in [**9**]. The structure was meant to support a theory of groupoidal Segal spaces. Her main approach working with simplicial presheaves on an "invertible simplex"-category $I\Delta$ turned out to model the homotopy theory of $(\infty,1)$-categories with an involution rather than the homotopy theory of ∞-groupoids ([**8**]). The approach towards a model for ∞-groupoids via complete Bousfield-Segal spaces was presented briefly as an alternative, and includes some parallel statements to those of her approach via $I\Delta$. The primary topic of this chapter and the next is to study the model structure for (complete) Bousfield-Segal spaces in the style of [**32**] with a view towards homotopy type theoretical semantics.

In order to explain the approach pursued here and in Chapter 5, let us recall that the category **Gpd** of groupoids arises as a localization of the category **Cat** of (small) categories. If by I we denote the free groupoid generated by the walking arrow $[1]$ (that is the "walking isomorphism"), then **Gpd** is the localization of **Cat** at the inclusion $e_1\colon [1] \to I$. Likewise, the category of simplicial groupoids is a localization of the category of simplicial categories. The model structure for Kan complexes can be obtained similarly as the left Bousfield localisation of the model structure for quasi-categories as explained in Example 3.1.5, such that Kan complexes are understood as quasi-categories with invertible edges. Modelling higher category theory in the category $s\mathbf{S}$ of bisimplicial sets, Charles Rezk introduced model structures $(s\mathbf{S}, \mathrm{S})$ and $(s\mathbf{S}, \mathrm{CS})$ for Segal spaces and complete Segal spaces, respectively, in [**44**]. The homotopy theory associated to the latter is a model for $(\infty,1)$-category theory equivalent to the one associated to the model category for quasi-categories. Correspondingly, we will see that the model structure $(s\mathbf{S}, \mathrm{CB})$ for complete Bousfield-Segal spaces is a model for ∞-groupoids equivalent to the one associated to Kan complexes, as stated without proof in [**7**, Theorem 6.12]. We will further see that $(s\mathbf{S}, \mathrm{CB})$ also supports a model of homotopy type theory with univalent universes. In fact, we will do this by showing that the model structure for Bousfield-Segal spaces – as introduced by Bergner in [**7**] – is a left Bousfield localization of the model structure for Segal spaces at a canonical map induced by the inclusion $e_1\colon [1] \to I$.

Therefore, Section 4.2 recalls the Reedy model structure $(s\mathbf{S}, R_v)$ on bisimplicial sets and Section 4.3 introduces Bousfield-Segal spaces in the sense of [**7**]. We present how every Bousfield-Segal space X comes equipped with a fraction operation (unique up to homotopy) which induces an associated homotopy groupoid $\mathrm{Ho}_B(X)$. In Section 4.4 we will show that such a fraction operation on a Bousfield-Segal space X induces an invertible composition operation on X, proving that every Bousfield-Segal space is in fact a Segal space and the associated model structure $(s\mathbf{S}, \mathrm{B})$ for Bousfield-Segal spaces as introduced by Bergner is a left Bousfield localization of $(s\mathbf{S}, \mathrm{S})$. We will also see that the homotopy category $\mathrm{Ho}(X)$ of a Bousfield-Segal space X associated to it *as a Segal space* (following [**44**, 5.5]) is a groupoid and coincides with the construction $\mathrm{Ho}_B(X)$. Hence, many of Rezk's results in [**44**] and Joyal and Tierney's results in [**32**] carry over to the model structure for (complete) Bousfield-Segal spaces. In Section 4.5 we use this to describe Bousfield-Segal spaces as the Segal spaces with invertible edges in a precise way.

4.2. Preliminaries on bisimplicial sets

A bisimplicial set $X \in s\mathbf{S}$ can be understood as a functor $X\colon \Delta^{op} \times \Delta^{op} \to \mathrm{Set}$, and whenever done so, will be denoted by $X_{\bullet\bullet}$ to highlight its two components. Currying to the left and to the right yields a simplicial object in $\mathbf{S}$, whose evaluation at an object $[n] \in \Delta^{op}$ is the n-th row $X_{\bullet n}$ and the n-th column $X_n := X_{n\bullet}$ respectively.

The box product and its adjoints. To recall some constructions which are very convenient in describing the generating sets for the model structures on bisimplicial sets we are interested in, we briefly summarise some constructions from [**32**, Section 2].

By left Kan extension of the Yoneda embedding $y\colon \Delta \times \Delta \to s\mathbf{S}$ along the product of Yoneda embeddings $y \times y\colon \Delta \times \Delta \to \mathbf{S} \times \mathbf{S}$ one obtains a bicontinuous functor $\_ \,\Box\, \_ \colon \mathbf{S} \times \mathbf{S} \to s\mathbf{S}$, often called the *box product*. The box product is divisible on both sides, i.e. gives rise to adjoint pairs

$$A \,\square\, \_ : \mathbf{S} \longleftrightarrow s\mathbf{S} \colon A \setminus \_$$

and

$$\_ \,\square\, B \colon \mathbf{S} \longleftrightarrow s\mathbf{S} \colon \_/B$$

for all simplicial sets A and B. In particular, for any bisimplicial set X the simplicial set $\Delta^n \setminus X \cong X_n$ is the n-th column and $X/\Delta^n \cong X_{\bullet n}$ is the n-th row of X. Vice versa, for a given $X \in s\mathbf{S}$, the induced functors

$$\_ \setminus X \colon \mathbf{S}^{op} \longleftrightarrow \mathbf{S} \colon X/\_$$

are mutually right adjoint, i.e. both pairs $(\_ \setminus X, X/\_)$ and $(X/\_, \_ \setminus X)$ are adjoint pairs. Considering the Leibniz construction (see e.g. [**47**, Definition 4.4]) for the box product and its dual, we get a functor

$$\_ \,\square'\, \_ : \mathbf{S}^{[1]} \times \mathbf{S}^{[1]} \to (s\mathbf{S})^{[1]}$$

on the arrow-categories, taking a pair of arrows $u \colon A \to B$, $v \colon A' \to B'$ in $\mathbf{S}$ to the natural map

$$\begin{tikzcd}
A \,\square\, A' \arrow[r, "A\square u"] \arrow[d, "v\square A'"'] & A \,\square\, B' \arrow[d] \arrow[ddr, bend left, "v\square B'"] & \\
B \,\square\, A' \arrow[r] \arrow[drr, bend right, "B\square u"'] & Q \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, "u\square' v"] & \\
& & B \,\square\, B'
\end{tikzcd}$$

in $s\mathbf{S}$. The functor $\_ \,\square'\, \_$ is divisible on both sides, too, the respective right adjoints for a given map $f \in s\mathbf{S}$ are denoted by

$$\langle f \setminus \_ \rangle, \langle \_/f \rangle \colon (s\mathbf{S})^{[1]} \to \mathbf{S}^{[1]}.$$

Proposition 4.2.1 ([**32**, Proposition 2.1])**.** *For any two maps $u, v \in \mathbf{S}$ and $f \in s\mathbf{S}$, we have*

$$(u \,\square'\, v) \pitchfork f \iff u \pitchfork \langle f/v \rangle \iff v \pitchfork \langle u \setminus f \rangle.$$

$\square$

In analogy to [**32**, Lemma 2.11] we have the following lemma.

Lemma 4.2.2. *For every triple $A, B, C \in \mathbf{S}$ the diagonal d^* yields*

$$d^*(A \,\square\, B) = A \times B \quad \textit{and} \quad A \setminus d_* C \cong C^A \cong d_* C/A.$$

More precisely, these equations also hold for morphisms, such that we obtain isomorphisms of bifunctors.

Proof. The first equation is clear. The other two are easily derived from the adjunctions associated to the three left adjoints d^*, $A \times \_$ and $A \square \_$. $\square$

The vertical and horizontal Reedy model structures. It is well known that the Reedy and injective model structures on $s\mathbf{S}$ coincide since the simplex-category Δ is an elegant Reedy category (in fact it is the archetype of such a Reedy category). The reason why elegance is a desirable property is that the associated model structure on simplicial presheaves over such Reedy categories combines the best of both worlds. While the (acyclic) cofibrations in the injective model structure have a level-wise description and hence inherit properties like pullback stability from the codomain model category (in this case that is from $(\mathbf{S}, \mathrm{Kan})$), their generating sets have no practical description in general. Vice versa, the (acyclic) cofibrations in the Reedy model structure are generated by well understood combinations of boundary inclusions of representables, while the sets of (acyclic) cofibrations generally do not inherit much from the homotopical algebra of the codomain. We loosely follow the language and structure of [**32**] and call this model structure the *vertical Reedy model* structure, denoted by R_v. Its cofibrations are the (point-wise) monomorphisms, its weak equivalences the point-wise weak equivalences and its fibrations the maps with the right lifting property to acyclic cofibrations.

For $n \geq 0$ we denote by $\delta_n \colon \partial\Delta^n \hookrightarrow \Delta^n$ the n-th boundary inclusion of the n-simplex $\Delta^n \in \mathbf{S}$ and, for $0 \leq i \leq n$, by $h_i^n \colon \Lambda_i^n \hookrightarrow \Delta^n$ the corresponding i-th horn inclusion. Recall that the set $\{\delta_n \mid n \geq 0\}$ of boundary inclusions generates the class of cofibrations and the set $\{h_i^n \mid 0 \leq i \leq n\}$ of horn inclusions generates the class of acyclic cofibrations in the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$. In terms of the general calculus of Reedy structures as presented for example in [**27**, Section 5.2], the object $\partial\Delta^n \backslash X$ is the n-th matching object of X. Hence, by [**27**, Theorem 5.2.5], a map $f \colon X \to Y$ in $(s\mathbf{S}, R_v)$ is an (acyclic) v-fibration if and only if the associated maps $\langle \delta_m \backslash f \rangle \colon X_n \to Y_n \times_{(\partial\Delta^n \backslash Y)} (\partial\Delta^n \backslash X)$ are (acyclic) Kan fibrations in $\mathbf{S}$. Then it is easy to see that the class of cofibrations $\mathcal{C}_v$ of $(s\mathbf{S}, R_v)$ is generated by the set

(4.2.1)
$$\mathcal{I}_v := \{\delta_n \,\square'\, \delta_m \colon (\Delta^n \,\square\, \partial\Delta^m) \cup_{\partial\Delta^n \square \partial\Delta^m} (\partial\Delta^n \,\square\, \Delta^m) \to (\Delta^n \,\square\, \Delta^m) \mid 0 \leq m, n\},$$

and the class $\mathcal{W}_v \cap \mathcal{C}_v$ of acyclic cofibrations is generated by the set

(4.2.2)
$$\mathcal{J}_v := \{\delta_n \,\square'\, h_i^m \colon (\Delta^n \,\square\, \Lambda_i^m) \cup_{\partial\Delta^n \square \Lambda_i^m} (\partial\Delta^n \,\square\, \Delta^m) \to (\Delta^n \,\square\, \Delta^m) \mid 0 \leq i \leq m, n\}.$$

Proposition 4.2.3 ([**32**, Proposition 2.5])**.** *A map $f \in s\mathbf{S}$ is a fibration in $(s\mathbf{S}, R_v)$, say a* v-fibration, *if and only if it satisfies one of the following equivalent conditions:*

(1) $\langle \delta_m \setminus f \rangle$ is a Kan fibration for all $m \geq 0$,
(2) $\langle u \setminus f \rangle$ is a Kan fibration for all monomorphims $u \in \mathbf{S}$,
(3) $\langle f / h_i^n \rangle$ is a trivial Kan fibration for all $0 \leq i \leq n$,
(4) $\langle f / v \rangle$ is a trivial Kan fibration for all anodyne maps $v \in \mathbf{S}$.

□

The projection $p_2 \colon \Delta \times \Delta \to \Delta$ onto the second component and the corresponding inclusion $\iota_2 = \langle [0], \mathrm{id} \rangle \colon \Delta \to \Delta \times \Delta$ constitute an adjoint pair $p_2 \dashv \iota_2$, and hence give rise to an adjoint pair

$$p_2^* \colon \mathbf{S} \longleftrightarrow s\mathbf{S} \colon \iota_2^*,$$

with $(p_2^* A)_n = A$ for all $n \geq 0$, and $\iota_2^* X = X_0$ the 0th column of X. We obtain a simplicial enrichment of $s\mathbf{S}$ via

$$\mathrm{Hom}_2(X, Y) := \iota_2^*(Y^X)$$

for bisimplicial sets X and Y.

Proposition 4.2.4 ([**32**, Propositions 2.4 and 2.6])**.** *The simplicial enrichment $Hom_2(X, Y)$ on $s\mathbf{S}$ turns $(s\mathbf{S}, R_v)$ into a simplicial model category.*

□

It is immediate that properness of $(\mathbf{S}, \mathrm{Kan})$ implies properness of $(s\mathbf{S}, R_v)$. The permutation $\sigma := \langle p_2, p_1 \rangle : \Delta \times \Delta \to \Delta \times \Delta$ induces an isomorphism $\sigma^* : s\mathbf{S} \to s\mathbf{S}$ which transports the vertical Reedy model structure into the *horizontal Reedy model structure* R_h with

$$\mathcal{C}_h = \{\text{monomorphims in } s\mathbf{S}\}$$

and

$$\mathcal{W}_h = \{f \colon X \to Y \mid f_{\bullet n} \colon X_{\bullet n} \to Y_{\bullet n} \text{ is a weak homotopy equivalence for all } n \geq 0\}.$$

Its cofibrations and acyclic cofibrations are generated by the sets

$$\mathcal{I}_h = \mathcal{I}_v$$

and

$$\mathcal{J}_h = \{h_i^n \,\Box'\, \delta_m \colon (\Delta^n \,\Box\, \partial\Delta^m) \cup_{\Lambda_i^n \Box \partial\Delta^m} (\Lambda_i^n \,\Box\, \Delta^m) \to (\Delta^n \,\Box\, \Delta^m) \mid 0 \leq i \leq m, n\}$$

respectively. A map is a weak equivalence in $(s\mathbf{S}, R_h)$ if and only if it is a row-wise weak homotopy equivalence in $\mathbf{S}$.

In analogy to the pair $p_2^* \dashv \iota_2^*$, we have an adjunction between the projection to the first component and the corresponding inclusion

$$p_1^* \colon \mathbf{S} \longleftrightarrow s\mathbf{S} \colon \iota_1^* \tag{4.2.3}$$

with $(p_1^* A)_{\bullet n} = A$ for all $n \geq 0$, and $\iota_1^* X = X_{\bullet 0}$ the 0th row of X.

In the following we cite several propositions proved in [**32**] which the authors use to investigate the model structure for (complete) Segal spaces. They show that $(s\mathbf{S}, R_v)$ naturally comes equipped with two orthogonal projections, a Quillen adjunction $p_1^* \colon (s\mathbf{S}, R_v) \to (\mathbf{S}, \mathrm{Kan})$ on the one hand, and a mere adjunction $p_2^* \colon s\mathbf{S} \to \mathbf{S}$ on the other. In order to construct a homotopy theory of $(\infty, 1)$-categories in $s\mathbf{S}$, the authors localize $(s\mathbf{S}, R_v)$ at a suitable set of maps such that the horizontal projection $p_2^* \colon s\mathbf{S} \to \mathbf{S}$ becomes a Quillen adjunction (and in fact a Quillen equivalence) to the Joyal model structure $(\mathbf{S}, \mathrm{Qcat})$. In this spirit, the authors are interested in the row-wise "categorical" homotopy theory in $s\mathbf{S}$. In order to construct a homotopy theory of ∞-*groupoids*, we localize $(s\mathbf{S}, R_v)$ at a larger class of maps such that the horizontal projection $p_2^* \colon s\mathbf{S} \to \mathbf{S}$ becomes a Quillen adjunction (and in fact a Quillen equivalence) to the model structure for *Kan complexes* $(\mathbf{S}, \mathrm{Kan})$. Therefore, we are interested in the row-wise "homotopical" homotopy theory, while discussing the categorical statements in [**32**] only so much as they help us to establish their groupoidal counterparts. Thus, some statements of [**32**] are cited as stated originally, while others are "homotopical" versions of results in [**32**] often with mostly identical proofs.

Now, for every $X \in s\mathbf{S}$ the unique map $!_{[n]} \colon [n] \to [0]$ in the category Δ induces a map $X/!_{[n]} \colon X_{\bullet 0} \to X_{\bullet n}$ from the 0-th to the n-th row of X.

In analogy to [**32**, 2.7-2.9,Theorem 2.12], we have the following.

Definition 4.2.5. A bisimplicial set X is *categorically (respectively groupoidally) constant* if the map $X/!_{\Delta^n} : X_{\bullet 0} \to X_{\bullet n}$ is a weak equivalence in $(\mathbf{S}, \mathrm{Qcat})$ (respectively in $(\mathbf{S}, \mathrm{Kan})$) for all $n \geq 0$.

Clearly every categorically constant object is also groupoidally constant since every categorical equivalence is a weak homotopy equivalence in $\mathbf{S}$.

Proposition 4.2.6. *Every v-fibrant $X \in s\mathbf{S}$ is categorically constant. In particular, v-fibrant bisimplicial sets are groupoidally constant.*

Proof. By [**32**, Proposition 2.8]. $\square$

Corollary 4.2.7. *A map $f \in s\mathbf{S}$ between v-fibrant X and Y is a row-wise weak homotopy equivalence if and only if $f_{\bullet 0} : X_{\bullet 0} \to Y_{\bullet 0}$ is a weak homotopy equivalence.*

$\square$

Theorem 4.2.8. *The diagonal d^* is part of Quillen pairs $(s\mathbf{S}, R_v) \xrightarrow{(d^*, d_*)} (\mathbf{S}, \mathrm{Kan})$ and $(s\mathbf{S}, R_h) \xrightarrow{(d^*, d_*)} (\mathbf{S}, \mathrm{Kan})$.*

Proof. The functor d^* preserves monomorphims, and hence cofibrations in both cases, being the right adjoint to the left Kan extension $d_!$. Further, it preserves both row-wise and column-wise weak homotopy equivalences by the Realization Lemma, see e.g. [**24**, IV, Proposition 1.7] for the columnise statement. $\square$

4.3. Bousfield-Segal spaces

Let $i_n : I_n \hookrightarrow \Delta^n$ be the nth spine-inclusion, i.e.

$$I_n = \bigcup_{i<n} j_i[\Delta^1]$$

for $j_i : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto i+1$. Localizing $(s\mathbf{S}, R_v)$ at the set of horizontally constant diagrams

$$\mathrm{S} := \{p_1^*(i_n) : p_1^*(I_n) \hookrightarrow p_1^*(\Delta^n) \mid 2 \leq n\}$$

yields the left-proper combinatorial simplicial model structure $(s\mathbf{S}, S) := \mathcal{L}_S(s\mathbf{S}, R_v)$ whose fibrant objects are the *Segal spaces* as defined in [**44**, Section 4.1] and [**32**, Definition 3.1]. By construction, these are v-fibrant bisimplicial sets X such that the maps

$$(p_1^*(i_n))^* : \mathrm{Hom}_2(p_1^*(\Delta^n), X) \to \mathrm{Hom}_2(p_1^*(I_n), X)$$

are weak homotopy equivalences for all $n \geq 2$. In other words, these are v-fibrant bisimplicial sets X such that the maps $i_n \backslash X : \Delta^n \backslash X \to I_n \backslash X$ are weak homotopy equivalences for all $n \geq 2$. Recall that we have $\Delta^n \setminus X \cong X_n$ and note that $I_n \setminus X \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the pullback taken along the boundaries $d_0 \setminus X$ and $d_1 \setminus X$ successively. In the following, we denote this pullback by $X_1 \times^S_{X_0} \cdots \times^S_{X_0} X_1$ or $(X_{1/X_0})^n_S$, where n is the number of components. Then we define the *Segal maps*

$$\xi_n \colon X_n \to X_1 \times^S_{X_0} \cdots \times^S_{X_0} X_1 \tag{4.3.1}$$

via $\xi_n := i_n \setminus X$ for $n \geq 2$, such that Segal spaces are the v-fibrant bisimplicial sets whose associated Segal maps are acyclic fibrations. One can think of Segal spaces X as horizontal simplicial collections of Kan complexes

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right, "d_1"'] \arrow[l, bend left, "d_0"] \arrow[r, bend left, "s_0"] \arrow[r, bend right, "s_1"'] & X_2 \arrow[l, "d_0"'] \arrow[l, bend right=60, "d_1"'] \arrow[l, bend left=60, "d_2"] \arrow[r, bend left, "s"] & \cdots \arrow[l, bend left, "d"] \arrow[r, bend left, "s"] & X_n \arrow[l, bend left, "d"] \arrow[r, bend left, "s"] & \cdots, \arrow[l, bend left, "d"] \end{tikzcd}$$

where X_0 is the space of objects and X_1 is the space of morphisms. It comes equipped with a horizontal weak composition via the Segal maps just like quasi-categories are simplicial collections of sets which come equipped with a weak composition via the lifts of inner horn inclusions. In analogy to Kan complexes, which are quasi-categories with lifts for left horn inclusions as described in Example 3.1.5, we consider Segal spaces with the corresponding lifting property. Namely, for the map $\gamma_i \colon [1] \to [n]$ with $0 \mapsto 0$, $1 \mapsto i$ let

$$C_{0,n} := \bigcup_{0<i} \gamma_i[\Delta^1]$$

be the 1-skeletal cone whose pinnacle is the initial vertex $0 \in \Delta^n$. We will refer to its edges as the *initial edges* of Δ^n and let $\iota_{0,n} \colon C_{0,n} \hookrightarrow \Delta^n$ denote the canonical inclusion. Localizing $(s\mathbf{S}, R_v)$ at the set of horizontally constant diagrams

$$B := \{p_1^*(\iota_{0,n}) \colon p_1^*(C_{0,n}) \to p_1^*(\Delta^n) \mid n \geq 2\}$$

yields a model structure $(s\mathbf{S}, B) := \mathcal{L}_B(s\mathbf{S}, R_v)$. This model structure was considered in [**7**, 6], Bergner calls its fibrant objects *Bousfield-Segal spaces.* They were first introduced in [**9**] and referred to as "very special bisimplicial sets". Note that a v-fibrant bisimplicial set X is B-local if and only if the fibrations $\iota_{0,n} \setminus X \colon \Delta^n \setminus X \twoheadrightarrow C_{0,n} \setminus X$ are weak homotopy equivalences. Here, $C_{0,n} \setminus X \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the n-fold fibre product of X_1 over X_0 along d_1 everywhere. We distinguish this pullback notationally by $X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$ or $(X_{1/X_0})^n_B$. We define the *Bousfield maps*

$$\beta_n \colon X_n \to X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$$

of X via $\beta_n := \iota_{0,n} \setminus X$.

Definition 4.3.2. Let X be a v-fibrant bisimplicial set X. We say that X is a *Bousfield-Segal space* (B-space for short) if the Bousfield maps

$$\beta_n \colon X_n \to X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1 \tag{4.3.2}$$

are weak homotopy equivalences for all $n \geq 0$.

Given a B-space X, the Bousfield maps $\beta_n : X_n \to X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$ are acyclic fibrations between Kan complexes, in particular the map β_2 exhibits a section $\mu_2 \colon X_1 \times^B_{X_0} X_1 \to X_2$ and thus the composite map

$$\_/\_ : X_1 \times^B_{X_0} X_1 \xrightarrow{\mu_2} X_2 \xrightarrow{d_0} X_1.$$

From now on we refer to this map as the *fraction* operation associated to X.

Notation 4.3.4. For vertices $x \in X_{00}$ we write $1_x := s_0 x$ and for $v, w \in X_{n0}$ we write $v \sim w$ if $[v] = [w] \in \pi_0 X_n$. Given a bisimplicial set W and points $x, y \in W$, the *hom-space* $W(x, y)$ denotes the pullback of $\langle d_1, d_0 \rangle \colon W_1 \to W_0 \times W_0$ along $(x, y) \in W_{00} \times W_{00}$.

Lemma 4.3.5. *For any B-space X and $x, y, z \in X_{00}$, the fraction operation restricts to a map*

$$\_/\_ : X_1(x, y) \times X_1(x, z) \to X_1(z, y).$$

On the horizontal Kan complexes $X_{\bullet m}$ it maps edges as follows,

$$\begin{tikzcd} & z & \\ x \arrow[ur, bend left, "g"] \arrow[rr, bend right, "f"'] & & y \end{tikzcd} \mapsto \begin{tikzcd} & z \arrow[dr, bend left, "f/g"] \arrow[d, phantom, "\mu_2(f,g)"] & \\ x \arrow[ur, bend left, "g"] \arrow[rr, bend right, "f"'] & {} & y \end{tikzcd} \mapsto \begin{tikzcd} z \arrow[dr, bend left, "f/g"] & \\ & y. \end{tikzcd}$$

Then

(1) $f/f \sim 1_y$ for all vertices $f \colon x \to y$ in X_1,
(2) $f/1_x \sim f$ for all vertices $f \colon x \to y$ in X_1,
(3) $f/g \sim (f/h)/(g/h)$ for all vertices $(f, g, h) \in X_1 \times^B_{X_0} X_1 \times^B_{X_0} X_1$.

Proof. Straightforward calculation. $\square$

The maps μ_2 and d_0 are natural transformations of simplicial sets, hence $\_/\_$ descends to homotopy classes. Therefore, for the family of sets

$$\mathrm{Ho}_B(X) := \langle \pi_0 X_1(x,y) \mid x, y \in X_{00} \rangle$$

indexed over the set of vertices X_{00} we obtain the following corollary.

Corollary 4.3.6. *The family of sets* $\mathrm{Ho}_B(X)$ *comes equipped with an operation*

$$\_/\_ \colon \mathrm{Ho}_B(X)(x,y) \times \mathrm{Ho}_B(X)(x,z) \to \mathrm{Ho}_B(z,y)$$

satisfying

(1) $[f]/[f] = [1_y]$ *for all* $f \in X_1(x,y)$,
(2) $[f]/[1_x] = f$ *for all* $f \in X_1(x,y)$,
(3) $[f]/[g] = ([f]/[h])/([g]/[h])$ *for all* $(f,g,h) \in X_1(x,y) \times X_1(x,z) \times X_1(x,w)$.

□

Proposition 4.3.7. *The family* $\mathrm{Ho}_B(X)$ *together with the operation* $\_ \circ \_$, *defined as the postcomposite*

$$\mathrm{Ho}_B(X)(y,z) \times \mathrm{Ho}_B(X)(x,y) \to \mathrm{Ho}_B(X)(y,z) \times \mathrm{Ho}_B(X)(y,x) \to \mathrm{Ho}_B(X)(x,z)$$

$$([g],[f]) \mapsto [g]/([1_x]/[f]) = [g/(1_x/f)],$$

of $\mathrm{id} \times (1_x/\_)$ *with* $\_/\_$, *is a groupoid.*

Proof. Straightforward calculation. □

This fraction operation on B-spaces is referred to in [**7**, Section 6] and in its essence also already used in [**9**].

4.4. Bousfield-Segal spaces are B-local Segal spaces

Despite the suggestive name it is not clear a priori that Bousfield-Segal spaces as defined in the previous section are in fact Segal spaces. In this section we dispose of this potential ambiguity in notation and show that Bousfield-Segal spaces and B-local Segal spaces are the exact same thing.

Therefore, we start with the following combinatorial lemma which is essential to later calculations. Let

$$k_n \colon C_{0,n} \to \Lambda^n_0 \tag{4.4.1}$$

be the canonical inclusion of simplicial sets, such that $\iota_{0,n} = h^n_0 \circ k_n$.

The proof of the next lemma is a variation of [**32**, Lemma 3.5] which is a similar statement for essential edges.

Lemma 4.4.1. *Let $A \subseteq \mathbf{S}$ be a saturated class of morphisms. Suppose further that A has the right cancellation property for monomorphims, i.e. $vu \in A$ and $u \in A$ imply $v \in A$ for all monomorphims $u, v \in \mathbf{S}$. Then $(h_0^n)_{n\geq 2} \subseteq A$ if and only if $(\iota_{0,n})_{n\geq 2} \subseteq A$.*

Proof. The inclusion $\iota_{0,n} \colon C_{0,n} \hookrightarrow \Delta^n$ factors through the inclusions

$$C_{0,n} \xrightarrow{k_n} \Lambda_0^n \xrightarrow{h_0^n} \Delta^n,$$

so it suffices to show that $k_n \in A$ for all $n \geq 2$ for both directions.

Suppose $(\iota_{0,n})_{n\geq 2} \subseteq A$. Then clearly $k_2 = \mathrm{id}_{C_{0,2}}$ is contained in A. For $n \geq 2$, we construct k_{n+1} from the inclusions $\iota_{0,m}$ for $m \leq n$ by a recursive pasting procedure. Therefore, let $n \geq 2$ and assume that the inclusion

$$C_{0,n} \hookrightarrow C_{0,n} \cup \bigcup_{0<j\leq i} d^j[\Delta^{n-1}]$$

is contained in A for every $0 < i \leq n$. Note that for $n = 2$ this is trivial and for $i = n$ this inclusion is k_n. We now show that the inclusion

$$C_{0,n+1} \hookrightarrow C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n]$$

is contained in A for every $0 < i \leq n+1$. There is a pushout square

$$\begin{tikzcd} C_{0,n} \arrow[r, "\cong", "d^1"'] \arrow[d, hook, "\iota_{0,n}"'] & d^1[\Delta^n] \cap C_{0,n+1} \arrow[r, hook] \arrow[d, hook] & C_{0,n+1} \arrow[d, hook] \\ \Delta^n \arrow[r, "\cong", "d^1"'] & d^1[\Delta^n] \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & C_{0,n+1} \cup d^1[\Delta^n] \end{tikzcd} \tag{4.4.2}$$

where the boundaries d^1 in the left square are isomorphisms, because the coboundary $d^1 \colon [n] \to [n+1]$ is a monomorphism. This implies that the inclusion

$$\iota_{(0,1,n+1)} \colon C_{0,n+1} \hookrightarrow C_{0,n+1} \cup d^1[\Delta^n]$$

is contained in A. Similarly, note that for $0 < i \leq n$ the boundaries

$$\begin{tikzcd} C_{0,n} \cup \bigcup_{0<j\leq i} d^j[\Delta^{n-1}] \arrow[r, "\cong", "d^{i+1}"'] \arrow[d, hook] & d^{i+1}[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n]) \arrow[d, hook] \\ \Delta^n \arrow[r, "\cong", "d^{i+1}"'] & d^{i+1}[\Delta^n] \end{tikzcd}$$

are isomorphisms. Indeed, the upper boundary d^{i+1} is an isomorphism, because

$$\begin{aligned}
d^{i+1}[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n]) &= (d^{i+1}[\Delta^n] \cap C_{0,n+1}) \cup \bigcup_{0<j\leq i} (d^{i+1}[\Delta^n] \cap d^j[\Delta^n]) \\
&= (d^{i+1}[\Delta^n] \cap C_{0,n+1}) \cup \bigcup_{0<j\leq i} d^{i+1}d^j[\Delta^{n-1}] \\
&\cong_{d_{i+1}} C_{0,n} \cup \bigcup_{0<j\leq i} d^j[\Delta^{n-1}]).
\end{aligned}$$

By assumption, the inclusion $C_{0,n} \hookrightarrow C_{0,n} \cup \bigcup_{0<j\leq i} d^j[\Delta^n]$ is contained in A. But then, by the right cancellation property of A, the inclusion $C_{0,n} \cup \bigcup_{0<j\leq i} d^j[\Delta^n] \hookrightarrow \Delta^n$ is contained in A, too. Therefore, since the square

$$\begin{tikzcd}
d^{i+1}[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n]) \arrow[r, hook] \arrow[d, hook] & C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n] \arrow[d, hook] \\
d^{i+1}[\Delta^n] \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" near end] & C_{0,n+1} \cup \bigcup_{0<j\leq i+1} d^j[\Delta^n]
\end{tikzcd} \tag{4.4.3}$$

is a pushout, the inclusion

$$\iota_{(0,i+1,n+1)} \colon C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n] \hookrightarrow C_{0,n+1} \cup \bigcup_{0<j\leq i+1} d^j[\Delta^n]$$

is contained in A for every $0 < i \leq n+1$. But then the composition

$$\iota_{(0,i+1,n+1)} \circ \dots \iota_{(0,2,n+1)} \circ \iota_{(0,1,n+1)} \colon C_{0,n+1} \hookrightarrow \bigcup_{0<j\leq i+1} d^j[\Delta^n]$$

is contained in A which finishes the induction. In particular, k_{n+1} as the composition of all $\iota_{(0,i,n+1)}$ for $0 < i \leq n+1$ is contained in A.

For the other direction, assume that $(h_0^n)_{n\geq 2} \subseteq A$. For $n = 2$, we have $C_{0,2} = \Lambda_0^2$ and $h_0^2 = \iota_{0,2}$, hence $\iota_{0,2}$ is contained in A. Suppose $n \geq 2$ and $\iota_{0,m} \in A$ for all $2 \leq m \leq n$. As we have seen above, by Diagrams (4.4.2) and (4.4.3), this implies $k_{n+1} \in A$. This in turn implies $\iota_{0,n+1} \in A$, because $\iota_{0,n+1} = h_0^{n+1} \circ k_{n+1}$. □

Corollary 4.4.2. *Let $X \in s\mathbf{S}$ be v-fibrant. Then the following two statements are equivalent.*

(1) $\iota_{0,n} \setminus X$ is an acyclic fibration for all $n \geq 2$.
(2) $h_0^n \setminus X$ is an acyclic fibration for all $n \geq 2$.

Both conditions imply that $k_n \setminus X$ is an acyclic fibration for all $n \geq 2$.

Proof. Let X be v-fibrant. The class

$$A := \{f \in \mathbf{S} \mid f \text{ is a monomorphim and } f \setminus X \text{ is an acyclic fibration}\}$$

has the right cancellation property for monomorphisms and is saturated by Proposition 4.2.1 and the fact that the class of monomorphims in $\mathbf{S}$ is saturated. Therefore, (1) and (2) are equivalent by Lemma 4.4.1. Further, in the proof of Lemma 4.4.1 we have seen that A contains $(k_n)_{n\geq 2}$ whenever it contains $(\iota_{0,n})_{n\geq 2}$ or $(h_0^n)_{n\geq 2}$, so the last part follows immediately. $\square$

Now, let X be a Bousfield-Segal space and recall the notation from (4.3.1) and (4.3.2) for its associated Segal and Bousfield maps respectively. Then its Bousfield maps $\beta_n\colon X_n \twoheadrightarrow (X_{1/X_0})^n_B$ are acyclic fibrations and in order to show that X is a Segal space, we have to infer that its Segal maps $\xi_n\colon X_n \twoheadrightarrow (X_{1/X_0})^n_S$ are acyclic, too. We have seen in the previous section that X comes equipped with a fraction operation $\_/\_\colon (X_{1/X_0})^2_B \to X_1$ and hence, for $n \geq 2$, with induced maps $\kappa_n := \langle \pi_1, \pi_2/\pi_1, \ldots, \pi_n/\pi_{n-1}\rangle$ as follows.

$$\begin{aligned} \kappa_n\colon (X_{1/X_0})^n_B &\to (X_{1/X_0})^n_S \\ (f_1, \ldots, f_n) &\mapsto (f_1, f_2/f_1, \ldots, f_n/f_{n-1}) \end{aligned}$$

We want to use these κ_n as a comparison between the Bousfield maps and the Segal maps of X, therefore note that there are maps

$$\gamma_n\colon (X_{1/X_0})^n_S \to (X_{1/X_0})^n_B$$

$$(f_1, \ldots, f_n) \mapsto (f_1, f_2/(1_{d_1 f_1}/f_1), \ldots, f_i/(1_{d_1 f_{i-1}}/\gamma_n(f_1, \ldots, f_n)_{i-1}), \ldots)_{i\geq 0}$$

in the converse direction constructed by recursion on $n \geq 2$.

Lemma 4.4.3. *Let X be a Bousfield-Segal space. Then there are homotopies*

(1) $H_2^{\gamma\kappa}\colon \mathrm{id} \sim \gamma_2 \circ \kappa_2$,

(2) $H_2^{\kappa\gamma}\colon \mathrm{id} \sim \kappa_2 \circ \gamma_2$

which are constant on vertices (i.e. the homotopies are constant after applying the boundaries $X_1 \times_{X_0} X_1 \to X_0 \times X_0 \times X_0$).

To distinguish the various projections present, given a simplicial set W, we distinctly denote the first projection $W \times \Delta^1 \to W$ by pr_1 and thus the constant homotopy $W \times \Delta^1 \to Z$ from a map $g\colon W \to Z$ to itself simply by $g\mathrm{pr}_1$.

Proof. For part (1) we have to prove that there is a homotopy $H_2^{\gamma\kappa} = (H_2^1, H_2^2)$ between the identity and

$$\begin{aligned} \gamma_2 \circ \kappa_2\colon X_1 \times^B_{X_0} X_1 &\to X_1 \times^B_{X_0} X_1 \\ (f_1, f_2) &\mapsto (f_1, (f_2/f_1)/(1_{d_1 f_1}/f_1)). \end{aligned}$$

That means we have to construct homotopies

* $H_2^1\colon (X_1 \times_{X_0}^B X_1) \times \Delta^1 \to X_1$ between π_1 and $\pi_1\gamma_2\kappa_2 = \pi_1$,
* $H_2^2\colon (X_1 \times_{X_0}^B X_1) \times \Delta^1 \to X_1$ between π_2 and $\pi_2\gamma_2\kappa_2 = (\pi_2/\pi_1)/(1_{d_1\pi_1}/\pi_1)$

whose "whiskering" with d_1 coincide on the base X_0. Since $X_1 \times_{X_0}^B X_1$ is a homotopy pullback, in quasi-categorical terms this is exactly the necessary construction in order to show that the map $\gamma_2\kappa_2$ is a vertex in the contractible space

$$\mathrm{Hom}_{\mathcal{S}/(d_0,d_1)}(X_1 \times_{X_0}^B X_1, X_1 \times_{X_0}^B X_1)$$

for $\mathcal{S}$ the quasi-category of spaces and the diagram $(d_0, d_1)\colon \Lambda_1^2 \to \mathcal{S}$ given by the boundaries $d_0, d_1\colon X_1 \to X_0$.

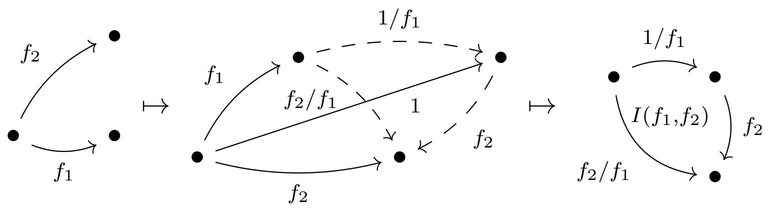
Clearly, the constant homotopy $H_2^1 = \pi_1\mathrm{pr}_1$ does half the deal. We construct H_2^2 via the section and right-homotopy inverse μ_2 of β_2 and a section μ_0^3 to the map

$$h_0^3 \setminus X\colon X_3 \overset{\sim}{\twoheadrightarrow} X_2 \times_{X_1} X_2 \times_{X_1} X_2$$

which is an acyclic fibration by Corollary 4.4.2. Namely, the two sections induce a map $I\colon X_1 \times_{X_0}^B X_1 \to X_2$ as the composite of

$$\begin{aligned} X_1 \times_{X_0}^B X_1 \to X_2 \times_{X_1} X_2 \times_{X_1} X_2 &\xrightarrow{\mu_0^3} X_3 \xrightarrow{d_0} X_2 \\ (f_1, f_2) \mapsto (\mu_2(1, f_1), \mu_2(f_2, f_1), s_0 f_2) &\mapsto I(f_1, f_2). \end{aligned}$$

On the horizontal simplicial sets $X_{\bullet m}$, the composite I assigns pairs of edges (f_1, f_2) to 2-simplices in $X_{\bullet m}$ in the following way.



By construction, we have $I(f_1, f_2) \in \beta_2^{-1}(f_2/f_1, 1/f_1,)_m$ for every tuple $(f_1, f_2) \in X_{1m} \times_{X_{0m}}^B X_{1m}$. Since μ_2 is also a homotopy right-inverse to β_2, there is a homotopy

$$H\colon X_2 \times \Delta_1 \to X_2$$

from the identity to $\mu_2\beta_2$ over $X_1 \times^B_{X_0} X_1$. This induces a homotopy $H^2_2\colon (X_1 \times^B_{X_0} X_1) \times \Delta^1 \to X_1$ as the composite of the top maps in the following diagram.

$$
\begin{tikzcd}
(X_1 \times^B_{X_0} X_1) \times \Delta^1 \arrow[r, "\langle I,\mathrm{id}\rangle"] & X_2 \times \Delta^1 \arrow[r, "H"] & X_2 \arrow[r, "d_0"] & X_1 \\
X_1 \times^B_{X_0} X_1 \arrow[u, shift left, "\iota_1"'] \arrow[u, shift right, "\iota_0"] \arrow[r, "I"] & X_2 \arrow[u, shift left, "\iota_1"'] \arrow[u, shift right, "\iota_0"] \arrow[ur, shift left, "\mathrm{id}"] \arrow[ur, shift right, "\mu_2\beta_2"'] & &
\end{tikzcd}
$$

H^2_2 is a homotopy between $H^2_2|_{\{0\}} = d_0 I = \pi_2$ and $H^2_2|_{\{1\}} = d_0\mu_2\beta_2 I = \pi_2\gamma_2\kappa_2$ (as can be checked by a straightforward element-wise calculation).

We have to show that H^1_2 and H^2_2 coincide on d_1. Certainly $d_1 H^2_1 = d_1\pi_1\mathrm{pr}_1$, and

$$
\begin{aligned}
d_1 H^2_2(f_1, f_2, \sigma) &= d_1 d_0 H\langle I, \mathrm{id}\rangle(f_1, f_2, \sigma) \\
&= d_0 d_2 H(I(f_1, f_2), \sigma) \\
&= d_0 \pi_2\beta_2 H(I(f_1, f_2), \sigma) \\
&= d_0 \pi_2 \beta_2 I(f_1, f_2) \\
&= d_0 d_2 I(f_1, f_2) \\
&= d_1 d_0 I(f_1, f_2) = d_1 f_2,
\end{aligned}
$$

so $d_1 H^2_2 = d_1\pi_2\mathrm{pr}_1$. But $d_1\pi_1$ and $d_1\pi_2$ coincide on $X_1 \times^B_{X_0} X_1$. Similarly we get $d_0 H^2_1 = d_0\pi_1\mathrm{pr}_1$ and $d_0 H^2_2 = d_0\pi_2\mathrm{pr}_1$.

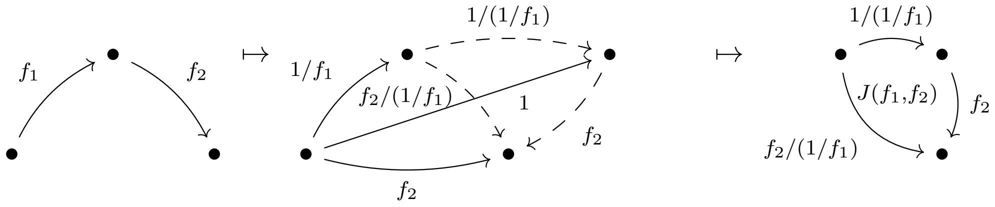
For part (2), again, we have to construct homotopies

* $L^1_2\colon (X_1 \times^S_{X_0} X_1) \times \Delta^1 \to X_1$ between π_1 and $\pi_1\kappa_2\gamma_2 = \pi_1$,
* $L^2_2\colon (X_1 \times^S_{X_0} X_1) \times \Delta^1 \to X_1$ between π_2 and $\pi_2\kappa_2\gamma_2 = [\pi_2/(1_{d_1\pi_1}/\pi_1)]/\pi_1$

such that the boundary conditions are satisfied. Just as in the first case, the constant homotopy $L^1_2 = \pi_1\mathrm{pr}_1$ will do. Towards a formula for the homotopy L^2_2, consider the map $J\colon X_1 \times^S_{X_0} X_1 \to X_2$ defined as the composite

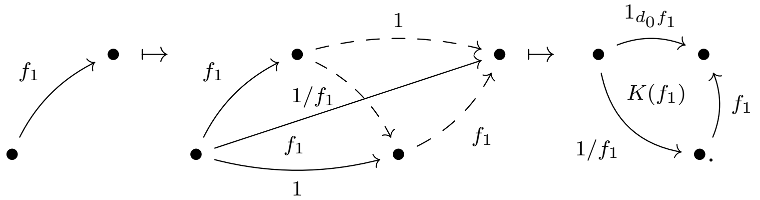
$$
\begin{aligned}
X_1 \times^S_{X_0} X_1 \to X_2 \times_{X_1} X_2 \times_{X_1} X_2 &\xrightarrow{\mu^3_0} X_3 \xrightarrow{d_0} X_2 \\
(f_1, f_2) \mapsto (\mu_2(1, 1/f_1), \mu_2(f_2, 1/f_1), s_0 f_2)) &\mapsto J(f_1, f_2).
\end{aligned}
$$

On the horizontal simplicial sets $X_{\bullet m}$, it assigns pairs of edges (f_1, f_2) to 2-simplices in $X_{\bullet m}$ in the following way.



Further, as a witness of the relation $(f^{-1})^{-1} \sim f$, consider the map $K\colon X_1 \to X_2$ defined as the composite

$$\begin{aligned} X_1 \to X_2 \times_{X_1} X_2 \times_{X_1} X_2 &\xrightarrow{\mu_0^3} X_3 \xrightarrow{d_0} X_2 \\ f_1 \mapsto (s_1 f_1, \mu_2(1, f_1), s_0 f_1)) &\mapsto K(f_1), \end{aligned}$$



These two maps yield homotopies $H_J := H \circ \langle J, \mathrm{id}\rangle$ and $H_K := \bar{H} \circ \langle K, \mathrm{id}\rangle$, where $H\colon \mathrm{id} \sim \mu_2\beta_2$ is the homotopy introduced above and $\bar{H}$ denotes the flipped homotopy from $\mu_2\beta_2$ to the identity over $X_1 \times^B_{X_0} X_1$.

$$\begin{tikzcd}
(X_1 \times^S_{X_0} X_1) \times \Delta^1 \arrow[r, "\langle J,\mathrm{id}\rangle"] \arrow[rr, bend left, "H_J"] & X_2 \times \Delta^1 \arrow[r, "H"] & X_2 \\
X_1 \times^S_{X_0} X_1 \arrow[r, "J"] \arrow[u, shift left, "\iota_0"] \arrow[u, shift right, "\iota_1"'] & X_2 \arrow[u, shift left, "\iota_0"] \arrow[u, shift right, "\iota_1"'] \arrow[ur, shift left, "\mathrm{id}"] \arrow[ur, shift right, "\mu_2\beta_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \times \Delta^1 \arrow[r, "\langle K,\mathrm{id}\rangle"] \arrow[rr, bend left, "H_K"] & X_2 \times \Delta^1 \arrow[r, "\bar{H}"] & X_2 \\
X_1 \arrow[r, "K"] \arrow[u, shift left, "\iota_1"] \arrow[u, shift right, "\iota_0"'] & X_2 \arrow[u, shift left, "\iota_1"] \arrow[u, shift right, "\iota_0"'] \arrow[ur, shift left, "\mathrm{id}"] \arrow[ur, shift right, "\mu_2\beta_2"'] &
\end{tikzcd}$$

Via H_K we obtain a new homotopy $H'_K\colon (X_1 \times^S_{X_0} X_1) \times \Delta^1 \to X_1 \times^B_{X_0} X_1$ as follows.

$$\begin{tikzcd}
(X_1 \times^S_{X_0} X_1) \times \Delta^1 \arrow[r, "\langle \pi_1, \mathrm{id}\rangle"] \arrow[dr, dashed, "\exists ! H'_K"'] \arrow[dd, "\mathrm{pr}_1"'] & X_1 \times \Delta^1 \arrow[r, "\langle K, \mathrm{id}\rangle"] \arrow[rr, bend left, "H_K"] & X_2 \times \Delta^1 \arrow[r, "\bar{H}"] & X_2 \arrow[d, "d_0"] \\
 & X_1 \times^B_{X_0} X_1 \arrow[rr, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & & X_1 \arrow[d, "d_1"] \\
X_1 \times^S_{X_0} X_1 \arrow[r, "\pi_2/(1/\pi_1)"'] & X_1 \arrow[rr, "d_1"] & & X_0
\end{tikzcd}$$

The outer rectangle commutes, because $\bar{H}$ is a homotopy over β_2. By construction we have $H'_K|_{\{i\}} = \langle \pi_2/(1/\pi_1), d_0 H_K|_{\{i\}}\rangle$, and thus the composition

$$(X_1 \times^S_{X_0} X_0) \times \Delta^1 \xrightarrow{H'_K} X_1 \times^B_{X_0} X_1 \xrightarrow{\mu_2} X_2$$

is a homotopy beginning at

$$\begin{aligned}
\mu_2 H'_K|_{\{0\}} &= \mu_2 \langle \pi_2/(1/\pi_1), d_0 \mu_2 \beta_2 K \pi_1 \rangle \\
&= \mu_2 \langle \pi_2/(1/\pi_1), 1/(1/\pi_1) \rangle && \text{since } d_0 \mu_2 \beta_2 K \pi_1 = 1/(1/\pi_1), \\
&= \mu_2 \beta_2 J && \text{by the definition of } J, \\
&= H_J|_{\{1\}}.
\end{aligned}$$

Therefore, the pushforward of the concatenation of H_J with $\mu_2 H'_K$ along d_0, that is $d_0(H_J * \mu_2 H'_K)$, is a homotopy between

$$d_0 H_J|_{\{0\}} = d_0 J = \pi_2$$

and

$$d_0 \mu_2 H'_K|_{\{1\}} = d_0 \mu_2 \langle \pi_2/(1/\pi_1), \pi_1 \rangle = [\pi_2/(1/\pi_1)]/\pi_1 = \pi_2 \kappa_2 \gamma_2.$$

To ensure that L^1_2 and $d_0(H_J * \mu_2 H'_K)$ yield a homotopy $L = (L^1_2, d_0(H_J * \mu_2 H'_K))$ into the pullback $X_1 \times^S_{X_0} X_1$, we have to choose the concatenation $H_J * \mu_2 H'_K$ constant over d_1 and d_0. Namely, the fact that $d_0 L^2_1 = d_0 \pi_1 \mathrm{pr}_1$ requires us to check that $d_1 L^2_2(f_1, f_2, \sigma) = d_0 f_1$ holds for all triples $(f_1, f_2, \sigma) \in (X_1 \times^S_{X_0} X_1) \times \Delta^1$. This is satisfied indeed by the homotopies $d_0 H_J$ and $d_0 \mu_2 H'_K$; indeed

$$\begin{aligned}
d_1 d_0 H_J(f_1, f_2, \sigma) &= d_0 d_2 H(J(f_1, f_2), \sigma) && \text{since } d_1 d_0 = d_0 d_2, \\
&= d_0 \pi_2 \beta_2 H(J(f_1, f_2), \sigma) \\
&= d_0 \pi_2 \beta_2 J(f_1, f_2) && \text{because } H \text{ is constant over } \beta_2,
\end{aligned}$$

$$
\begin{aligned}
&= d_0 1/(1/f_1) && \text{by definition of } J, \\
&= d_0 f_1
\end{aligned}
$$

and

$$
\begin{aligned}
d_1 d_0 \mu_2 H'_K(f_1, f_2, \sigma) &= d_0 \pi_2 H'_K(f_1, f_2, \sigma) \\
&= d_0 d_0 \bar{H}(K f_1 \sigma) \\
&= d_0 d_1 \bar{H}(K f_1, \sigma) \\
&= d_0 \pi_1 \beta_2 \bar{H}(K f_1, \sigma) \\
&= d_0 \pi_1 \beta_2 K f_1 && \text{because } \bar{H} \text{ is constant over } \beta_2, \\
&= d_0 1_{d_0 f_1} && \text{by the definition of } K, \\
&= d_0 f_1.
\end{aligned}
$$

Also $d_0 d_0 H_J = d_0 \pi_2 \mathrm{pr}_1$ and $d_0 d_0 H'_K = d_0 \pi_2 H'_K = d_0 \pi_2 \mathrm{pr}_1$ hold by the same line of equations. Defining $Q := \Big(\big((X_1 \times^S_{X_0} X_1) \times \Delta^1 \big) \times \{0\} \Big) \cup \Big(\big((X_1 \times^S_{X_0} X_1) \times \{1\} \big) \times \Delta^1 \Big)$, these computations render the diagram

$$
\begin{tikzcd}
Q \arrow[rr, "\langle H_J \mathrm{pr}_{\hat{3}}, \mu_2 H'_K \mathrm{pr}_{\hat{2}} \rangle"] \arrow[d, hook, "\wr"'] \arrow[ddrr, dashed, "\langle d_0\pi_1\mathrm{pr}^2_1, d_0\pi_1\mathrm{pr}^2_1 \rangle \times \langle d_0\pi_2\mathrm{pr}^2_1, d_0\pi_2\mathrm{pr}^2_1 \rangle"] & & X_2 \arrow[dd, dashed, two heads, "d_1 d_0 \times d_0 d_0"] \\
\big((X_1 \times^S_{X_0} X_1) \times \Delta^1 \big) \times \Delta^1 \arrow[drr, dashed, bend right, "d_0\pi_1\mathrm{pr}^2_1 \times d_0\pi_2\mathrm{pr}^2_1"'] & & \\
& & X_0 \times X_0
\end{tikzcd}
$$

commutative. Thus, we have a diagram

$$
\begin{tikzcd}
Q \arrow[r, "\langle H_J \mathrm{pr}_{\hat{3}}, \mu_2 H'_K \mathrm{pr}_{\hat{2}} \rangle"] \arrow[d, hook, "\wr"'] & X_2 \\
\big((X_1 \times^S_{X_0} X_1) \times \Delta^1 \big) \times \Delta^1 &
\end{tikzcd}
$$

in the slice $\mathbf{S}/(X_0 \times X_0)$. Observe that $d_1 d_0 \times d_0 d_0 \colon X_2 \to X_0 \times X_0$ is a Kan fibration by Lemma 4.2.3, since $d^0 d^1 \sqcup d^0 d^0 \colon \Delta^0 \sqcup \Delta^0 \to \Delta^2$ is a cofibration and X

is v-fibrant. Therefore, we obtain a lift

$$\begin{tikzcd} Q \arrow[r, "\langle H_J \mathrm{pr}_{\hat{3}}, \mu_2 H'_K \mathrm{pr}_{\hat{2}} \rangle"] \arrow[d, hook, "\wr"'] & X_2 \\ \big((X_1 \times^S_{X_0} X_1) \times \Delta^1\big) \times \Delta^1 \arrow[ur, dashed, "H_{JK}"'] & \end{tikzcd}$$

in $\mathbf{S}/(X_0 \times X_0)$. For the diagonal $\Delta\colon \Delta^1 \to \Delta^1 \times \Delta^1$, set

$$L_2^2 := d_0 H_{JK} \langle \mathrm{id}, \Delta \rangle \colon (X_1 \times^S_{X_0} X_1) \times \Delta^1 \to X_2.$$

Then $d_1 L_2^2 = d_1 d_0 H_{JK} \langle \mathrm{id}, \Delta \rangle = d_0 \pi_1 \mathrm{pr}_1^2 \langle \mathrm{id}, \Delta \rangle = \pi_1 \mathrm{pr}_1$ holds by construction. Note that

$$d_0 L_2^2 = d_0 \pi_2 \mathrm{pr}_1 \tag{4.4.4}$$

holds, too, and so $H_2^{\gamma\kappa} := (L_2^1, L_2^2)$ is a homotopy as required. □

Lemma 4.4.5. *Let X be a Bousfield-Segal space. Then the maps κ_n and γ_n are mutually homotopy inverse for all $n \geq 2$, i.e. for all $n \geq 2$ there are homotopies*

(1) $H_n^{\gamma\kappa} \colon \mathrm{id} \sim \gamma_n \circ \kappa_n$,

(2) $H_n^{\kappa\gamma} \colon \mathrm{id} \sim \kappa_n \circ \gamma_n$.

In the following, given a product $A_1 \times \cdots \times A_n$ and a sequence of numbers $\{i_1, \ldots, i_k\}$ between 1 and n, the map $\pi_{\{i_1,\ldots,i_k\}} \colon A_1 \times \cdots \times A_n \to A_{i_1} \times \cdots \times A_{i_k}$ denotes the projection into the components specified by the sequence. Given a number $m \leq n$, $\hat{m}$ denotes the sequence of all numbers $1 \leq k \leq n$ with $k \neq m$ and $\pi_{\hat{m}}$ denotes the corresponding projection.

Proof. By Lemma 4.4.3 there are homotopies $H_2^{\kappa\gamma} \colon \mathrm{id} \sim \kappa_2 \gamma_2$ and $H_2^{\gamma\kappa} \colon \mathrm{id} \sim \gamma_2 \kappa_2$ which are constant on vertices. Suppose further for all $2 \leq m \leq n$ there are homotopies $H_m^{\kappa\gamma} \colon \mathrm{id} \sim \kappa_m \gamma_m$ and $H_m^{\gamma\kappa} \colon \mathrm{id} \sim \gamma_m \kappa_m$ such that

(i) $\pi_{\{1,\ldots,m\}} H_n^{\gamma\kappa} = H_m^{\gamma\kappa} \langle \pi_{\{1,\ldots,m\}}, \mathrm{id}_{\Delta^1} \rangle$ and $\pi_{\{1,\ldots,m\}} H_n^{\kappa\gamma} = H_m^{\kappa\gamma} \langle \pi_{\{1,\ldots,m\}}, \mathrm{id}_{\Delta^1} \rangle$ for all $m \leq n$,

(ii) $d_i \pi_m H_n^{\gamma\kappa} = d_i \pi_m \mathrm{pr}_1$ for all $i \in \{0,1\}$ and $m \leq n$, i.e. the homotopy is constant on vertices,

(iii) $d_0 \pi_n H_n^{\kappa\gamma} = d_0 \pi_n \mathrm{pr}_1$, i.e. the homotopy is constant on the last vertex.

We then construct homotopies $H_{n+1}^{\gamma\kappa}$ and $H_{n+1}^{\kappa\gamma}$ satisfying conditions (i), (ii) and (iii).

Towards a formula for $H_{n+1}^{\gamma\kappa} \colon \mathrm{id} \sim \gamma_{n+1} \kappa_{n+1}$ for part (1), construct homotopies

* $H^1_{n+1}\colon (X_{1/X_0})^{n+1}_B \times \Delta^1 \to (X_{1/X_0})^n_B$ between $\pi_{(\widehat{n+1})}$ and $\pi_{(\widehat{n+1})}\gamma_{n+1}\kappa_{n+1}$,
* $H^2_{n+1}\colon (X_{1/X_0})^{n+1}_B \times \Delta^1 \to X_1$ between π_{n+1} and $\pi_{n+1}\gamma_{n+1}\kappa_{n+1}$

such that the homotopies

$$d_1\pi_1 H^1_{n+1}, d_1 H^2_{n+1}\colon (X_{1/X_0})^{n+1}_B \times \Delta^1 \to X_0$$

coincide. Recalling the definitions of γ_{n+1} and κ_{n+1}, we note that on the first n-many components we have $\pi_{(\widehat{n+1})}\gamma_{n+1}\kappa_{n+1} = \gamma_n\kappa_n\pi_{(\widehat{n+1})}$. So H^1_{n+1} defined as

$$H^{\gamma\kappa}_n \pi_{(\widehat{n+1})}\colon \pi_{(\widehat{n+1})} \sim \pi_{(\widehat{n+1})}\gamma_{n+1}\kappa_{n+1}$$

gives us the first homotopy.

Towards a formula for H^2_{n+1}, note that on the $(n+1)$-st component we have

$$\pi_{n+1}\gamma_{n+1}\kappa_{n+1} = (\pi_{n+1}/\pi_n)/(1_{d_1\pi_1}/\pi_n\gamma_n\kappa_n\pi_{(\widehat{n+1})}).$$

By assumption, $\gamma_n\kappa_n$ is homotopic to the identity, so we only have to construct a homotopy between $\pi_{n+1}\gamma_{n+1}\kappa_{n+1}$ and $(\pi_{n+1}/\pi_n)/(1/\pi_n)$, and make sure that the homotopies concatenate well. Therefore, consider the following diagram.

$$\begin{tikzcd}
(X_{1/X_0})^{n+1}_B \times \Delta^1 \arrow[r, "{\langle \pi_{(\widehat{n+1})}, \mathrm{id}\rangle}"] \arrow[dd, "\mathrm{pr}_1"'] \arrow[dr, dashed, "\exists!(H^2_{n+1})'"'] & (X_{1/X_0})^n_B \times \Delta^1 \arrow[r, "H^{\gamma\kappa}_n"] & (X_{1/X_0})^n_B \arrow[d, "\pi_n"] \\
 & X_1 \times^B_{X_0} X_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & X_1 \arrow[d, "d_1"] \\
(X_{1/X_0})^{n+1}_B \arrow[r, "s_0 d_1 \pi_1"'] & X_1 \arrow[r, "d_1"'] & X_0
\end{tikzcd}$$

The outer square commutes by Condition (ii) for n, and the natural map $(H^2_{n+1})'$ is a homotopy from $\langle s_0 d_1\pi_1, \pi_n\rangle$ to $\langle s_0 d_1 \pi_1, \pi_n\gamma_n\kappa_n\pi_{(\widehat{n+1})}\rangle$. Further, we have a diagram

$$\begin{tikzcd}
(X_{1/X_0})^{n+1}_B \times \Delta^1 \arrow[r, "(H^2_{n+1})'"] \arrow[dd, "\mathrm{pr}_1"'] \arrow[dr, dashed, "\exists!(H^2_{n+1})''"'] & X_1 \times^B_{X_0} X_1 \arrow[r, "\mu_2"] & X_2 \arrow[d, "d_0"] \\
 & X_1 \times^B_{X_0} X_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & X_1 \arrow[d, "d_1"] \\
(X_{1/X_0})^{n+1}_B \arrow[r, "\pi_{n+1}/\pi_n"'] & X_1 \arrow[r, "d_1"'] & X_0,
\end{tikzcd}$$

where the outer rectangle commutes again by Condition (ii), such that the resulting map $(H^2_{n+1})''$ yields a homotopy of the form

$$d_0\mu_2(H^2_{n+1})''\colon (\pi_{n+1}/\pi_n)/(1/\pi_n) \sim \pi_{n+1}\gamma_{n+1}\kappa_{n+1}.$$

We also have

$$\pi_2 H_2^{\gamma\kappa}\pi_{\{n,n+1\}}\colon \pi_2\pi_{\{n,n+1\}} \sim \pi_2\gamma_2\kappa_2\pi_{\{n,n+1\}},$$

with $\pi_2\pi_{\{n,n+1\}} = \pi_{n+1}$ and $\pi_2\gamma_2\kappa_2\pi_{\{n,n+1\}} = (\pi_{n+1}/\pi_n)/(1/\pi_n)$. A small calculation using both Conditions (i) and (ii) shows that the two homotopies $d_0\mu_2(H^2_{n+1})''$ and $\pi_2 H_2^{\gamma\kappa}\pi_{\{n,n+1\}}$ coincide over the boundaries $(d_0, d_1)\colon X_1 \to X_0 \times X_0$. Hence, as for the construction of L^2_2 in the proof of Lemma 4.4.3, we can choose a concatenation

$$\big(\pi_2 H_2^{\gamma\kappa}\pi_{\{n,n+1\}}\big) * \big(d_0\mu_2(H^2_{n+1})''\big)\colon (X_{1/X_0})^{n+1}_B \times \Delta^1 \to X_1$$

over (d_0, d_1) which we denote by H^2_{n+1}. In particular, $H^2_{n+1}|_{\{0\}} = \pi_{n+1}$ and $H^2_{n+1}|_{\{1\}} = \pi_{n+1}\gamma_{n+1}\kappa_{n+1}$, while $d_1 H^2_{n+1} = d_1\pi_{n+1}\mathrm{pr}_1$ and $d_0 H^2_{n+1} = d_0\pi_{n+1}\mathrm{pr}_1$ are constant. Set

$$H^{\gamma\kappa}_{n+1} := (H^1_{n+1}, H^2_{n+1})$$

which is a homotopy from the identity to $\gamma_{n+1}\kappa_{n+1}$ by construction. Then the first half of Condition (i) holds by

$$\begin{aligned}
\pi_{\{1,\dots,m\}} H^{\gamma\kappa}_{n+1} &= \pi_{\{1,\dots,m\}}\pi_{(\widehat{n+1})} H^{\gamma\kappa}_{n+1} \\
&= \pi_{\{1,\dots,m\}} H^1_{n+1} && \text{by definition,} \\
&= \pi_{\{1,\dots,m\}} H^{\gamma\kappa}_n \pi_{(\widehat{n+1})} && \text{by definition,} \\
&= H^{\gamma\kappa}_m \pi_{(\widehat{n+1})} && \text{since Condition (i) holds for } n \text{ by assumption,} \\
&= H^{\gamma\kappa}_m \pi_{\{1,\dots,m\}}
\end{aligned}$$

for all $m \leq n$. Towards Condition (ii), whenever $m \leq n$, it follows that

$$d_i\pi_m H^{\gamma\kappa}_{n+1} = d_i\pi_m H^{\gamma\kappa}_m \langle \pi_{\{1,\dots,m\}}, \mathrm{id}\rangle = d_i\pi_m\pi_{\{1,\dots,m\}}\mathrm{pr}_1 = d_i\pi_m\mathrm{pr}_1,$$

and for $m = n+1$, we have $d_1\pi_{n+1}H^{\gamma\kappa}_{n+1} = d_1 H^2_{n+1}$ and a few lines above we have seen that $d_1 H^2_{n+1} = d_1\pi_{n+1}\mathrm{pr}_1$ is constant. Analogously,

$$d_0\pi_{n+1}H^{\gamma\kappa}_{n+1} = d_0 H^2_{n+1} = d_0\pi_{n+1}\mathrm{pr}_1.$$

Thus, the construction of the $H^{\gamma\kappa}_n$ for $n \geq 2$ succeeds.

For part (2), towards a formula for the homotopy $H^{\kappa\gamma}_{n+1}$, once again we have to construct homotopies

* $L^1_{n+1}\colon (X_{1/X_0})^{n+1}_S \times \Delta^1 \to (X_{1/X_0})^n_S$ between $\pi_{(\hat{n+1})}$ and $\pi_{(\hat{n+1})}\kappa_{n+1}\gamma_{n+1}$,
* $L^2_{n+1}\colon (X_{1/X_0})^{n+1}_S \times \Delta^1 \to X_1$ between π_{n+1} and $\pi_{n+1}\kappa_{n+1}\gamma_{n+1}$,

such that

$$d_0\pi_n L^1_{n+1}\colon (X_{1/X_0})^{n+1}_S \times \Delta^1 \to X_0$$
$$d_1 L^2_{n+1}\colon (X_{1/X_0})^{n+1}_S \times \Delta^1 \to X_0$$

coincide and Conditions (i) and (iii) are satisfied. As in the prior case, because $\pi_{(\hat{n+1})}\kappa_{n+1}\gamma_{n+1} = \kappa_n\gamma_n\pi_{(\hat{n+1})}$ holds, we can define the homotopy L^1_{n+1} simply to be

$$H^{\kappa\gamma}_n\langle\pi_{(\hat{n+1})}, \mathrm{id}\rangle\colon \pi_{(\hat{n+1})} \sim \pi_{(\hat{n+1})}\kappa_{n+1}\gamma_{n+1}.$$

Towards a formula for the homotopy L^2_{n+1} on the $(n+1)$-st component, note that

$$\begin{aligned}\pi_{n+1}\kappa_{n+1}\gamma_{n+1} &= \big(\pi_{n+1}/(1/\pi_n\gamma_{n+1})\big)/\pi_n\gamma_{n+1}\\ &= \big(\pi_{n+1}/(1/\pi_n\gamma_n\pi_{(\hat{n+1})})\big)/\pi_n\gamma_n\pi_{(\hat{n+1})}\\ &= \pi_2\kappa_2\gamma_2\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle.\end{aligned}$$

Therefore, simply set L^2_{n+1} to be

$$\pi_2 H^{\kappa\gamma}_2\langle\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle, \mathrm{id}_{\Delta^1}\rangle\colon (X_{1/X_0})^{n+1}_S \times \Delta^1 \to X_1,$$

such that $L^2_{n+1}|_{\{0\}} = \pi_2\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle = \pi_n$ and $L^2_{n+1}|_{\{1\}} = \pi_{n+1}\kappa_{n+1}\gamma_{n+1}$. By Condition (iii), we have

$$d_0\pi_n L^1_{n+1} = d_0\pi_n H^{\kappa\gamma}_n\langle\pi_{(\hat{n+1})}, \mathrm{id}\rangle = d_0\pi_n\mathrm{pr}_1$$

and

$$\begin{aligned}d_1 L^2_{n+1} &= d_1\pi_2 H^{\kappa\gamma}_2\langle\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle, \mathrm{id}\rangle\\ &= d_1 L^2_2\langle\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle, \mathrm{id}\rangle\\ &= d_0\pi_1\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle\mathrm{pr}_1 \qquad \text{by Lemma 4.4.3,}\\ &= d_0\pi_n\gamma_n\pi_{(\hat{n+1})}\mathrm{pr}_1\\ &= d_0\pi_n\mathrm{pr}_1.\end{aligned}$$

So the boundary condition is satisfied and we are left to verify Conditions (i) and (iii) for $H^{\kappa\gamma}_{n+1} := (L^1_{n+1}, L^2_{n+1})$. But Condition (i) is straightforward by the definition of L^1_{n+1} and the inductive hypothesis, and Condition (iii) holds by

$$\begin{aligned}d_0\pi_{n+1}H^{\kappa\gamma}_{n+1} &= d_0 L^2_{n+1}\\ &= d_0\pi_2 H^{\kappa\gamma}_2\langle\langle\pi_n\gamma_n\pi_{(\hat{n+1})}, \pi_{n+1}\rangle, \mathrm{id}\rangle\end{aligned}$$

$$
\begin{aligned}
&= d_0 L_2^2 \langle\langle \pi_n \gamma_n \pi_{\widehat{(n+1)}}, \pi_{n+1}\rangle, \mathrm{id}\rangle \\
&= d_0 \pi_2 \mathrm{pr}_1 \langle\langle \pi_n \gamma_n \pi_{\widehat{(n+1)}}, \pi_{n+1}\rangle, \mathrm{id}\rangle \\
&= d_0 \pi_{n+1}.
\end{aligned}
$$

So the induction succeeds. This finishes the proof. □

So we have seen that, if X is a B-space, the maps

$$\kappa_n \colon (X_{1/X_0})^n_B \to (X_{1/X_0})^n_S$$

are homotopy equivalences. The following lemma shows that this comparison of pullbacks in fact yields a comparison between the Bousfield maps and the Segal maps of X.

Lemma 4.4.6. *Let X be a Bousfield-Segal space. Then for every $n \geq 2$ there is a section and homotopy right-inverse μ_n of β_n such that the square*

$$
\begin{tikzcd}
X_n \arrow[r, "\mu_n\beta_n"] \arrow[d, two heads, "\beta_n"'] & X_n \arrow[d, two heads, "\zeta_n"] \\
(X_{1/X_0})^n_B \arrow[r, "\kappa_n"'] & (X_{1/X_0})^n_S
\end{tikzcd}
\tag{4.4.5}
$$

commutes.

Proof. Recall that acyclic fibrations $p\colon X \twoheadrightarrow Y$ between cofibrant objects X, Y always exhibit a section s together with a homotopy $H\colon ps \sim \mathrm{id}$ over p as for example shown in [**25**, Proposition 7.6.11.(2)]. First, in order to find such μ_n such that (4.4.5) commutes, we construct a distinguished factorization $\beta_n = \beta_n^{2,1} \circ \beta_n^{n,2}$ such that we control the essential edges under the resulting homotopy inverses $\mu_n^{n,2}$ and $\mu_n^{2,1}$. Note that in order to render the square in (4.4.5) commutative, we do not need to care about the output of $\mu_n\beta_n$ at any edges but the initial and essential ones. Hence it suffices to control the specific 2-simplices which, given adjacent initial edges f_i and f_{i+1}, generate the essential edges f_{i+1}/f_i. Therefore, we consider the factorization

$$C_{0,n} \overset{a_n}{\hookrightarrow} \bigcup_{0<i<n} \Delta^2_i \hookrightarrow \Delta^n$$

of $\iota_{0,n}\colon C_{0,n} \hookrightarrow \Delta^n$, where $\Delta^2_i \subseteq \Delta^n$ is given by the 2-simplex $\sigma_i \in \Delta^n_2$ with $d_1\sigma_i = \Delta^{\{0,i+1\}}$ the edge from 0 to $i+1$ and $d_2\sigma_i = \Delta^{\{0,i\}}$ the edge from 0 to i.

For $0 < j \leq n$, let

$$t_{n,j} \colon C_{0,2} \to C_{0,n} \cup \bigcup_{0<i<j} \Delta^2_i \subset \Delta^n$$

be the inclusion given by $\Delta^{\{0,i\}} \mapsto \Delta^{\{0,j+i-1\}}$. For any such $j \leq n$, we have

$$\begin{tikzcd} C_{0,2} \arrow[r, "t_{n,j}"] \arrow[d, "\iota_{0,2}"'] & C_{0,n} \cup \bigcup_{0<i<j} \Delta^2_i \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C_{0,n} \cup \bigcup_{0<i\leq j} \Delta^2_i \end{tikzcd}$$

and $C_{0,n} \cup \bigcup_{0<i<n} \Delta^2_i = \bigcup_{0<i<n} \Delta^2_i$. Since a_n is a finite composition of cobase changes of $\iota_{0,2}$, it induces an acyclic fibration

$$a_n \setminus X \colon (X_{2/X_1})^{n-1}_B \xrightarrow{\sim} (X_{1/X_0})^n_B,$$

where $(X_{2/X_1})^{n-1}_B := X_2 \times_{X_1} \cdots \times_{X_1} X_2 \cong \bigcup_{0<i<n} \Delta^2_i \setminus X$ is the pullback consecutively taken along adjacent initial edges. We denote this fibration by $\beta^{2,1}_n$. Further, the spine inclusion $\iota_n \colon I_n \hookrightarrow \Delta^n$ also factors via

$$I_n \overset{l_n}{\hookrightarrow} \bigcup_{0<i<n} \Delta^2_i \hookrightarrow \Delta^n,$$

so in order to show that the Segal maps $\zeta_n = \iota_n \setminus X$ are weak equivalences, by 2-for-3 it suffices to show that the inclusion l_n yields an acyclic fibration $l_n \setminus X$. Note that $l_n \setminus X$ is the map

$$\langle d_2\pi_1, d_0\pi_1, d_0\pi_2, \ldots, d_0\pi_{n-1}\rangle \colon (X_{2/X_1})^{n-1}_B \to (X_{1/X_0})^n_B. \tag{4.4.6}$$

We will show acyclicity of the fibration $l_n \setminus X$ by constructing a weak homotopy equivalence $\mu^{2,1}_n \colon (X_{1/X_0})^n_B \xrightarrow{\sim} (X_{2/X_1})^{n-1}_B$ for every $n \geq 2$ such that the triangle

$$\begin{tikzcd} (X_{1/X_0})^n_B \arrow[r, "\mu^{2,1}_n"] \arrow[dr, "\kappa_n"'] & (X_{2/X_1})^{n-1}_B \arrow[d, "l_n\setminus X"] \\ & (X_{1/X_0})^n_S \end{tikzcd} \tag{4.4.7}$$

commutes. Since the κ_n are homotopy equivalences by Lemma 4.4.5, the statement follows again by 2-for-3.

Therefore, denote the projection away from the n-th component by $\pi_{\hat{n}}$, and let $\sigma_n \colon (X_{1/X_0})^n_B \to (X_{1/X_0})^n_B$ be the permutation which reverses the components' order, that is $(f_1, \ldots, f_n) \mapsto (f_n, \ldots, f_1)$. We construct $\mu^{2,1}_n$ by recursion on n first and prove commutativity of (4.4.7) by induction afterwards. For the case $n = 2$, we take $\mu_n^{2,1} := \mu_2$. For $n = 3$, we get a cube of the following form.

$$\begin{tikzcd}
& X_2 \times^B_{X_1} X_2 \arrow[rr, "\pi_1"] \arrow[d, "\pi_2"] \arrow[drr, phantom, "\lrcorner", very near start] & & X_2 \arrow[d, "d_1"] \\
& X_2 \arrow[rr, "d_2"'] & & X_1 \\
(X_{1/X_0})^3_B \arrow[rr, "\sigma\pi_{\hat{3}}"] \arrow[d, "\sigma\pi_{\hat{1}}"'] \arrow[uur, dashed, "\mu_3^{2,1}"] \arrow[drr, phantom, "\lrcorner", very near start] & & X_1 \times^B_{X_0} X_1 \arrow[d, "\pi_1"'] \arrow[uur, "\mu_2"] & \\
X_1 \times^B_{X_0} X_1 \arrow[rr, "\pi_2"'] \arrow[uur, "\mu_2"'] & & X_1 \arrow[uur, equal] &
\end{tikzcd} .$$

All involved simplicial sets are Kan complexes and the maps in the pullback squares are fibrations. $(\mathbf{S}, \mathrm{Kan})$ is right proper, so the natural map $\mu_3^{2,1}$ is a weak homotopy equivalence. The permutations σ are inserted to obtain the correct orientation of the resulting 2-simplices after application of $\mu_3^{2,1}$ which is necessary for the triangle (4.4.7) to commute. Analogously, for $n = 4$, the diagram

$$\begin{tikzcd}
& (X_{2/X_1})^3_B \arrow[rr, "\pi_{\hat{3}}"] \arrow[d, "\pi_{\hat{1}}"] \arrow[drr, phantom, "\lrcorner", very near start] & & X_2 \times^B_{X_1} X_2 \arrow[d, "\pi_{\hat{1}}"] \\
& X_2 \times^B_{X_1} X_2 \arrow[rr, "\pi_{\hat{2}}"'] & & X_2 \\
(X_{1/X_0})^4_B \arrow[rr, "\pi_{\hat{4}}"] \arrow[d, "\pi_{\hat{1}}"'] \arrow[uur, dashed, "\mu_4^{2,1}"] \arrow[drr, phantom, "\lrcorner", very near start] & & (X_{1/X_0})^3_B \arrow[d, "\sigma\pi_{\hat{1}}"'] \arrow[uur, "\mu_3^{2,1}"] & \\
(X_{1/X_0})^3_B \arrow[rr, "\sigma\pi_{\hat{3}}"'] \arrow[uur, "\mu_3^{2,1}"'] & & X_1 \times^B_{X_0} X_1 \arrow[uur, "\mu_2"'] &
\end{tikzcd} ,$$

yields a weak equivalence $\mu_4^{2,1}$ for the same reasons. In virtue of the explicit choice of μ_2 and κ_n this case has to be considered separately since the permutations σ have to be inserted to make the diagram commute. This is not necessary for the following diagrams relating $\mu_n^{2,1}$ and $\mu_{n-1}^{2,1}$ when $n \geq 4$.

Let $n \geq 4$ and suppose we have constructed weak homotopy equivalences $\mu_m^{2,1}$ for all $4 \leq m \leq n$ such that the squares

$$\begin{tikzcd}
(X_{1/X_0})_B^n \arrow[r, "\mu_n^{2,1}", "\sim"'] \arrow[d, "\pi_{\hat{1}}"'] & (X_{2/X_1})_B^{n-1} \arrow[d, "\pi_{\hat{1}}"] \\
(X_{1/X_0})_B^{n-1} \arrow[r, "\mu_{n-1}^{2,1}", "\sim"'] & (X_{2/X_1})_B^{n-2}
\end{tikzcd}
\qquad
\begin{tikzcd}
(X_{1/X_0})_B^n \arrow[r, "\mu_n^{2,1}", "\sim"'] \arrow[d, "\pi_{\hat{n}}"'] & (X_{2/X_1})_B^{n-1} \arrow[d, "\pi_{(\widehat{n-1})}"] \\
(X_{1/X_0})_B^{n-1} \arrow[r, "\mu_{n-1}^{2,1}", "\sim"'] & (X_{2/X_1})_B^{n-2}
\end{tikzcd}
\tag{4.4.8}$$

commute. Then we get a natural map

$$\begin{tikzcd}
 & (X_{2/X_1})_B^n \arrow[rr, "\pi_{\hat{n}}"] \arrow[d, "\pi_{\hat{1}}"] \arrow[drr, phantom, "\lrcorner", very near start] & & (X_{2/X_1})_B^{n-1} \arrow[d, "\pi_{\hat{1}}"] \\
 & (X_{2/X_1})_B^{n-1} \arrow[rr, "\pi_{(\widehat{n-1})}"'] & & (X_{2/X_1})_B^{n-2} \\
(X_{1/X_0})_B^{n+1} \arrow[uur, dashed, "\mu_{n+1}^{2,1}"] \arrow[rr, "\pi_{(\widehat{n+1})}"] \arrow[d, "\pi_{\hat{1}}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & (X_{1/X_0})_B^n \arrow[uur, "\mu_n^{2,1}"] \arrow[d, "\pi_{\hat{1}}"'] & \\
(X_{1/X_0})_B^n \arrow[uur, "\mu_n^{2,1}"'] \arrow[rr, "\pi_{\hat{n}}"'] & & (X_{1/X_0})_B^{n-1} \arrow[uur, "\mu_{n-1}^{2,1}"'] &
\end{tikzcd},$$

because the squares on the bottom and the right face commute by assumption. The map $\mu_{n+1}^{2,1}$ is again a weak homotopy equivalence by right properness and the assumption that $\mu_n^{2,1}$ and $\mu_{n-1}^{2,1}$ are weak homotopy equivalences.

We are left to prove that the composition $l_n \setminus X \circ \mu_n^{2,1}$ is a factorization of κ_n by induction on $n \geq 2$. Clearly $l_2 \setminus X \circ \mu_2^{2,1} = \kappa_2 \setminus X$ holds by definition of κ_2. Further, by (4.4.6) for $n = 3$ we have

$$\begin{aligned}
l_3 \setminus X \circ \mu_3^{2,1} &= \langle d_2\pi_1\mu_3^{2,1}, d_0\pi_1\mu_3^{2,1}, d_0\pi_2\mu_3^{2,1}\rangle \\
&= \langle d_2\mu_2\sigma\pi_{\hat{3}}, d_0\mu_2\sigma\pi_{\hat{3}}, d_0\mu_2\sigma\pi_{\hat{1}}\rangle \\
&= \langle \pi_1, \pi_2/\pi_1, \pi_3/\pi_2\rangle \\
&= \kappa_3.
\end{aligned}$$

For $n \geq 3$, assume $l_n \setminus X \circ \mu_n^{2,1} = \kappa_n$ holds. Then by (4.4.6) and (4.4.8), we obtain

$$\begin{aligned}
l_{n+1} \setminus X \circ \mu_{n+1}^{2,1} &= \langle d_2\pi_1\mu_{n+1}^{2,1}, d_0\pi_1\mu_{n+1}^{2,1}, d_0\pi_2\mu_{n+1}^{2,1}, \ldots, d_0\pi_{n-1}\mu_{n+1}^{2,1}\rangle \\
&= \langle d_2\pi_1\pi_{\hat{n}}\mu_{n+1}^{2,1}, d_0\pi_1\pi_{\hat{n}}\mu_{n+1}^{2,1}, \ldots, d_0\pi_{n-1}\pi_{\hat{n}}\mu_{n+1}^{2,1}, d_0\pi_{n-1}\pi_{\hat{1}}\mu_{n+1}^{2,1}\rangle \\
&= \langle d_2\pi_1\mu_n^{2,1}\pi_{n\hat{+}1}, d_0\pi_1\mu_n^{2,1}\pi_{n\hat{+}1}, \ldots, d_0\pi_{n-1}\mu_n^{2,1}\pi_{n\hat{+}1}, d_0\pi_{n-1}\mu_n^{2,1}\pi_{\hat{1}}\rangle
\end{aligned}$$

$$= \langle l_n \setminus X \circ \mu_n^{2,1} \circ \pi_{(\hat{n+1})}, \pi_n \circ l_n \setminus X \circ \mu_n^{2,1} \circ \pi_{\hat{1}} \rangle$$
$$= \langle \kappa_n \circ \pi_{(\hat{n+1})}, \pi_n \circ \kappa_n \circ \pi_{\hat{1}} \rangle$$
$$= \kappa_{n+1}.$$

This finishes the proof. □

Theorem 4.4.7. *Every Bousfield-Segal space is a Segal space. In particular, the model structures* $(s\mathbf{S}, B)$ *and* $\mathcal{L}_B(s\mathbf{S}, S)$ *coincide.*

Proof. Let X be a Bousfield-Segal space. By Lemma 4.4.6, there is a section μ_n of β_n such that the square

$$\begin{tikzcd} X_n \arrow[r, "\mu_n\beta_n"] \arrow[d, two heads, "\beta_n"] & X_n \arrow[d, two heads, "\zeta_n"] \\ (X_{1/X_0})^n_B \arrow[r, "\kappa_n"'] & (X_{1/X_0})^n_S \end{tikzcd}$$

commutes. But β_n and μ_n are weak homotopy equivalences, and so is κ_n by Lemma 4.4.5. Hence, the Segal maps ζ_n are weak homotopy equivalences by 2-for-3 and X is a Segal space. This means that every fibrant object in $(s\mathbf{S}, B)$ is also fibrant in $\mathcal{L}_B(s\mathbf{S}, S)$. But fibrant objects in $\mathcal{L}_B(s\mathbf{S}, S)$ are v-fibrant and B-local by construction, so the left Bousfield localizations $(s\mathbf{S}, B)$ and $\mathcal{L}_B(s\mathbf{S}, S)$ have the same class of fibrant objects and hence coincide. □

Theorem 4.4.7 implies that all constructions from [**44**] apply to the class of B-spaces. In particular every B-space X comes equipped with a homotopy category $\mathrm{Ho}(X)$ as constructed in [**44**, 5.5]. Recall the groupoid $\mathrm{Ho}_B(X)$ associated to X in Proposition 4.3.7.

Corollary 4.4.8. *For any B-space* X*, the categories* $\mathrm{Ho}(X)$ *and* $\mathrm{Ho}_B(X)$ *coincide. In particular,* $\mathrm{Ho}(X)$ *is a groupoid.*

Proof. Let X be a B-space. Clearly the families $\mathrm{Ho}_B(X)$ and $\mathrm{Ho}(X)$ of sets coincide and have the same identity, so we have to show that the corresponding compositions $\circ_B$ and $\circ_S$ coincide, too. By Theorem 4.4.7, let η_2 be a section to $\xi_2 \colon X_2 \xrightarrow{\sim} X_1 \times^S_{X_0} X_1$, such that $\_ \circ_S \_ := d_1\eta_2$ is a composition for the Segal space X. For any two morphisms $f \in X(x, y)$ and $g \in X(y, z)$, the inner 3-horn map $\eta_2(f,g) \cup \mu_2(1_x, f) \cup \mu_2(g, 1_x/f)\colon \Lambda^3_1 \to X_{\bullet 0}$ of the form

$$\begin{tikzcd}
& y \arrow[rr, bend left, "1_x/f"] \arrow[dr, "g"] & & x \arrow[dl, dashed, "g\circ_B f = g/(1_x/f)"] \\
x \arrow[ur, bend left, "f"] \arrow[urrr, dashed, "1_x"'] \arrow[rr, dashed, bend right, "g\circ_S f"'] & & z &
\end{tikzcd}$$

has a lift $L(f,g)\colon \Delta^3 \to X_{\bullet 0}$. Both the simplex

$$\begin{tikzcd}
x \arrow[r, bend left, "1_x"] \arrow[dr, bend right, "g\circ_S f"'] \arrow[dr, phantom, "d_1 L(f,g)"] & x \arrow[d, bend left, "g\circ_B f"] \\
& z
\end{tikzcd}$$

and $s_0(g \circ_S f)$ lie in the fibre $\beta_2^{-1}(g \circ_S f, 1_x)_0$. But β_2 is a trivial fibration, hence $d_1 L(f,g)$ and $s_0(g \circ_S f)$ lie in the same connected component of X_2. Therefore, by naturality of d_0, we have

$$[g \circ_B f] = [d_0 d_1 L(f,g)] = [d_0 s_0(g \circ_S f)] = [g \circ_S f]$$

in $\pi_0 X_1(x,z) = \mathrm{Ho}(X)(x,z) = \mathrm{Ho}_B(X)(x,z)$. $\square$

4.5. Further characterizations

In this section we prove a few basic properties of B-spaces and characterize B-spaces as those Segal spaces with invertible edges.

Proposition 4.5.1. *A Segal space X is a Bousfield-Segal space if and only if the Bousfield map*

$$\beta_2 = \iota_{0,2} \setminus X \colon X_2 \to X_1 \times^B_{X_0} X_1$$

is an acyclic fibration. In particular, the model structures $\mathcal{L}_{\iota_{0,2}}(s\mathbf{S}, S)$ and $(s\mathbf{S}, B)$ coincide.

Proof. As both model structures are left Bousfield localizations of the same Reedy structure, we only have to compare their fibrant objects. Clearly, every B-space is fibrant in $\mathcal{L}_{\iota_{0,2}}(s\mathbf{S}, S)$. Vice versa, we have to show that fibrant objects in $\mathcal{L}_{\iota_{0,2}}(s\mathbf{S}, S)$ are $p_1^*\iota_{0,n}$-local for all $n \geq 2$. Consider the class

$$A := \{f \in \mathbf{S} \mid f \text{ is a monomorphism and } p_1^* f \text{ is an acyclic cofibration in } \mathcal{L}_{\iota_{0,2}}(s\mathbf{S}, S)\}.$$

A is saturated and has the right cancellation property for monomorphims, by construction $S \cup \{\iota_{0,2}\}$ is a subset of A. In a similar fashion to the proof of Lemma 4.4.1, we show $\iota_{0,n} \in A$ by induction on n. Suppose $\iota_{0,m} \in A$ for all $2 \le m \le n$. In the proof of Lemma 4.4.1 we have seen that under these assumptions we have

$$\big(C_{0,n+1} \hookrightarrow (C_{0,n+1} \cup \bigcup_{0<j\le n+1} d^j[\Delta^n])\big) \in A.$$

The same proof, just replacing the boundary $d^1\colon [n] \to n+1$ with the boundary d^2 in Diagram (4.4.2) and continuing the line of reasoning accordingly, shows that the inclusion

$$C_{0,n+1} \hookrightarrow (C_{0,n+1} \cup \bigcup_{1<j\le n+1} d^j[\Delta^n]) \tag{4.5.1}$$

is contained in A, too. We observe that the 0-face of the codomain of the map (4.5.1) is

$$\begin{aligned}
d^0[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{1<j\le n+1} d^j[\Delta^n]) &= (d^0[\Delta^n] \cap C_{0,n+1}) \cup \bigcup_{1<j\le n+1} (d^0[\Delta^n] \cap d^j[\Delta^n]) \\
&= \bigcup_{1<j\le n+1} d^j d^0[\Delta^{n-1}] \\
&= \bigcup_{1<j\le n+1} d^0 d^{j-1}[\Delta^{n-1}] \\
&= \bigcup_{0<j\le n} d^0 d^j[\Delta^{n-1}] \\
&\cong_{d_0} \bigcup_{0<j\le n} d^j[\Delta^{n-1}]) \\
&= \Lambda^n_0.
\end{aligned}$$

Thus we have isomorphisms

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\cong", "d^0"'] \arrow[d, hook, "h^n_0"] & d^0[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{1<j\le n+1} d^j[\Delta^n]) \arrow[d, hook] \\
\Delta^n \arrow[r, "\cong", "d^0"'] & d^0[\Delta^n]
\end{tikzcd}$$

and induced inclusions

$$\begin{tikzcd}
C_{0,n} \arrow[r, hook, "k_n"] \arrow[dr, "\iota_{0,n}"'] & \Lambda^n_0 \arrow[r, hook] \arrow[d, hook, "h^n_0"] & C_{0,n+1} \cup \bigcup_{1<j\le n+1} d^j[\Delta^n] \arrow[d, hook] \\
& \Delta^n \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & C_{0,n+1} \cup \bigcup_{0\le j\le n+1, j\ne 1} d^j[\Delta^n] = \Lambda^{n+1}_1.
\end{tikzcd} \tag{4.5.2}$$

The maps $\iota_{0,m}$ are contained in A for $m \leq n$ by assumption and so is the map k_n by the proof of Lemma 4.4.1. Hence, by the right cancellation property of A, the inclusion h_0^n is contained in A, too, and so is the pushout along the bottom map in Diagram (4.5.2). Lastly, the inner horn inclusions $h_i^{n+1}\colon \Lambda_i^{n+1} \hookrightarrow \Delta^{n+1}$ for $0 < i < n+1$ are contained in A by [**32**, Lemma 3.5] and hence the composition

$$\iota_{0,n+1}\colon C_{0,n+1} \to C_{0,n+1} \cup \bigcup_{1<j\leq n+1} d^j[\Delta^n] \to \Lambda_1^{n+1} \xrightarrow{h_1^{n+1}} \Delta^{n+1}$$

is contained in A, too, since every component of it is contained in A. ☐

In other words, every map in a given Segal space X is invertible if and only if every map in X is left-invertible. Similar to the choice of fraction operations $\_/\_$ for B-spaces, giving a Segal space X and a section $\eta_2\colon X_1 \times_{X_0}^S X_1 \to X_2$ to the Segal map ξ_2 determines a composition operation $\_\circ\_$ via

$$d_1\eta_2\colon X_1(x,y) \times_{X_0}^S X_1(y,z) \to X_1(x,z)$$

as we have seen already in Corollary 4.4.8. This yields a commuting triangle

$$\begin{tikzcd} & X_2 \arrow[dr, "\beta_2"] & \\ X_1 \times_{X_0}^S X_1 \arrow[ur, "\eta_2"] \arrow[rr, "\lambda_2"'] & & X_1 \times_{X_0}^B X_1 \end{tikzcd}$$

for the map $\lambda_2(f,g) = (g\circ f, f)$. Let $X_{\mathrm{hoequiv}} \subseteq X_1$ denote the full sub-Kan complex of *homotopy equivalences* in X whose edges are those which become isomorphisms in $\mathrm{Ho}X$.

Corollary 4.5.2. *A Segal space X is a B-space if and only if either of the following equivalent conditions is satisfied.*

(1) The map λ_2 is a weak homotopy equivalence.

(2) Its associated homotopy category $\mathrm{Ho}X$ *is a groupoid.*

Proof. Part (1) follows immediately from Proposition 4.5.1 and the 2-for-3 property. For part (2), let X be a Segal space and assume $\mathrm{Ho}X$ is a groupoid. By Proposition 4.5.1 it suffices to show that the Bousfield map

$$\beta_2\colon X_2 \twoheadrightarrow X_1 \times_{X_0} X_1$$

is a weak equivalence. But the fact that $\mathrm{Ho}X$ is a groupoid implies that $X_{\mathrm{hoequiv}} = X_1$ and so the statement follows immediately from [**44**, Lemma 11.6]. ☐

In other words, a Segal space X is a Bousfield-Segal space if and only if every edge $f \in X_1$ is a homotopy equivalence in X.

Remark 4.5.3. Note that the homotopy category $\mathrm{Ho}X$ of a given Segal space X is a groupoid if and only if the quasi-category $X_{\bullet 0}$ is a Kan complex. This in turn holds if and only if all rows $X_{\bullet n}$ are Kan complexes. So we see that Bousfield-Segal spaces are exactly the Segal spaces horizontally fibrant in the projective model structure over $(\mathbf{S}, \mathrm{Kan})$.

Example 4.5.4. Let X be a bisimplicial set, let $\partial\colon X_n \to \partial\Delta^n \setminus X$ denote its n-th matching object and $\mathrm{Sub}_{s_0}X_1$ denote those subobjects $Y \subseteq X_1$ which factor the degeneracy $s_0\colon X_0 \hookrightarrow X_1$. The evaluation

$$(\_)_1\colon \mathrm{Sub}(X) \to \mathrm{Sub}_{s_0}X_1$$

of subobjects of X has a fully faithful right adjoint G_1 whose value at a subobject $Y \subseteq X_1$ can be thought of as the largest subobject K of X such that $K_1 \subset Y$. For $Y \in \mathrm{Sub}_{s_0}(X_1)$, its values are recursively given by

$$G_1(Y)_0 := X_0, \quad G_1(Y)_1 := Y \subseteq X_1$$

and

$$\begin{tikzcd}
G_1(Y)_n \arrow[r, hook] \arrow[d, "\partial"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "\partial"] \\
\partial\Delta^n \setminus G_1(Y) \arrow[r, hook] & \partial\Delta^n \setminus X
\end{tikzcd}$$

for $n \geq 2$. The boundaries are directly inherited from X while $s_0\colon G_1(Y)_0 \to G_1(Y)_1$ is given by requiring that $s_0\colon X_0 \hookrightarrow X_1$ factors through Y. Assuming that the degeneracies $s_k\colon G_1(Y)_{n-1} \hookrightarrow G_1(Y)_n$ for $0 \leq k < n$ are defined, for $0 \leq i < n+1$ let $\partial s_i := (s_{i-1}d_0, \dots, s_{i-1}d_{i-1}, 1, 1, s_i d_{i+2}, \dots, s_i d_n)$, so we obtain

$$\begin{tikzcd}
G_1(Y)_n \arrow[rr, hook] \arrow[dr, dashed, "s_i"] \arrow[ddr, bend right, "\partial s_i"'] & & X_n \arrow[d, "s_i"] \\
& G_1(Y)_{n+1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+1} \arrow[d] \\
& \partial\Delta^n \setminus G_1(Y) \arrow[r, hook] & \partial\Delta^n \setminus X
\end{tikzcd} \tag{4.5.3}$$

according to the corresponding simplicial identities. Then $G_1(Y)$ satisfies all simplicial identities as they do hold for X and the natural map $G_1(Y)_n \to X_n$ is monic. Hence, $G_1(Y)$ is a simplicial object in $\mathbf{S}$.

For any Segal space $X \in s\mathbf{S}$, the subobject $\mathrm{hoequiv}X \subseteq X_1$ contains the degeneracy $s_0 \colon X_0 \hookrightarrow X_1$, so we can define

$$\mathrm{Core}(X) := G_1(X_{\mathrm{hoequiv}}) \subseteq X,$$

the core of X. Then it is easy to show that for any Segal space X, the bisimplicial set $\mathrm{Core}(X)$ is a Bousfield-Segal space.

Proof. If we can show that $\mathrm{Core}(X)$ is Segal space, then it follows that it is a Bousfield-Segal space by Lemma 4.5.2.(2) since

$$\mathrm{Core}(X)_1 = X_{\mathrm{hoequiv}} = \mathrm{Core}(X)_{\mathrm{hoequiv}}.$$

But note that the boundary map $(d_1, d_0) \colon X_{\mathrm{hoequiv}} \to X_0 \times X_0$ is a Kan fibration, because the boundary $(d_1, d_0) \colon X_1 \to X_0 \times X_0$ is a Kan fibration by assumption and $X_{\mathrm{hoequiv}} \subseteq X_1$ is closed under connected components. So the restriction $(d_1, d_0) \colon X_{\mathrm{hoequiv}} \to X_0 \times X_0$ is a Kan fibration, too. All other matching maps $\mathrm{Core}(X)_n \to \partial\Delta^n \setminus (\mathrm{Core}(X))$ are pullbacks of Kan fibrations and hence Kan fibrations themselves. So $\mathrm{Core}(X)$ is Reedy fibrant. Also the square

$$\begin{tikzcd}
G_1(Y)_2 \arrow[r, hook] \arrow[d, two heads, "\xi_2"'] & X_2 \arrow[d, two heads, "\xi_2"'] \\
Y \times^S_{X_0} Y \arrow[r, hook] & X_1 \times^S_{X_0} X_1
\end{tikzcd}$$

is a pullback square and we can show inductively that for all $0 < i < n$ the higher Segal maps factor as follows,

$$\begin{tikzcd}
G_1(Y)_n \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_n \arrow[d, two heads, "\wr"] \\
\Lambda^n_i \setminus G_1(Y) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Lambda^n_i \setminus X \arrow[d, two heads, "\wr"] \\
(Y_{/X_0})^S_n \arrow[r, hook] & (X_{1/X_0})^S_n,
\end{tikzcd}$$

since the boundary map $\partial \colon G_1(Y)_n \to \partial\Delta^n \setminus G_1(Y)$ also factors through $\Lambda^n_i \setminus G_1(Y)$. This shows that $\mathbf{Core}(X)$ is a Segal space whenever X is such. $\square$

Remark 4.5.5. For a quasi-category $\mathcal{C}$, let $\mathcal{C}^{\simeq} \subseteq \mathcal{C}_1$ denote the set of equivalences in $\mathcal{C}$. Given a Segal space X, recall that X_{hoequiv} is the sub-Kan complex of X_1

generated by the set

$$\{(f\colon x \to y) \in X_1 \mid \exists g, h \in X_1 : (gf \sim 1_x) \in X_1(x,x)\text{and}(fh \sim 1_y) \in X_1(y,y)\}.$$

This equals $X^{\simeq}_{\bullet 0}$ by Reedy fibrancy of X. In fact for each $i \in \mathbb{N}$ the sets $(X_{\mathrm{hoequiv}})_i$ and $X^{\simeq}_{\bullet i}$ coincide and thus, denoting the nerve of the free groupoid over the category $[n]$ by $F[n]$, we see that

$$\mathrm{Core}(X)_{nm} = s\mathbf{S}(NF[n] \,\square\, \Delta^m, X).$$

Corollary 4.5.6. *If X is a B-space, then X/A is a Kan complex for every $A \in \mathbf{S}$. In particular, every row of X is a Kan complex.*

Proof. Let $A \in \mathbf{S}$ and X be a B-space. X is a Segal space by Theorem 4.4.7 and hence the simplicial set X/A is a quasi-category by [**32**, Corollary 3.6]. We know that $h^n_0 \setminus X$ is an acyclic fibration for all $n \geq 2$ by Corollary 4.4.2 and thus has the right lifting property against the cofibration $\emptyset \hookrightarrow A$. By Proposition 4.2.1 it follows that X/A has the right lifting property against all left horn inclusions and thus is an $\{h^n_0 \mid n \geq 2\}$-local quasi-category, i.e. left fibrant. By Remark 3.1.5, X/A is a Kan complex. □

Notation 4.5.7. Let $J := N(I)$ be the nerve of the interval object

$$I = \begin{tikzcd} 0 \arrow[r, bend left, "\cong"] & 1 \arrow[l, bend left] \end{tikzcd}$$

in the category of groupoids, $c\colon 1 \to J$ the inclusion $* \mapsto 0$, $!_J\colon J \to 1$ its terminal map and $e_1\colon \Delta^1 \to J$ the canonical inclusion.

J is the "freely walking isomorphism" and maps out of it determine the core of a quasi-category. Rezk showed in [**44**, Theorem 6.2] that every Segal space W induces a weak equivalence $e_1 \setminus W\colon J \setminus W \to W_{\mathrm{hoequiv}}$, where $W_{\mathrm{hoequiv}} \subseteq W_1$ is the subsimplicial set of *homotopy equivalences* in W.

Proposition 4.5.8. *The bisimplicial map $p_1^* c\colon 1 \to p_1^* J$ is an acyclic cofibration in $(s\mathbf{S}, B)$. If X is a B-space, then $X_{\mathrm{hoequiv}} = X_1$ and the canonical inclusion $e_1\colon \Delta^1 \hookrightarrow J$ induces an acyclic fibration $e_1 \setminus X\colon J \setminus X \to X_1$.*

Proof. We have seen in Corollary 4.4.7 that the homotopy category of a B-space X is a groupoid, thus $X_{\mathrm{hoequiv}} = X_1$. Therefore the statement follows directly from [**44**, Theorem 6.2, Section 11]. Indeed, in his proof of [**44**, Theorem 6.2], Rezk actually gives an explicit description of the inclusion e_1 as a transfinite composition of pushouts of the class $\{h^n_0 \mid n \geq 2\}$. □

CHAPTER 5

Complete Bousfield-Segal spaces

In this chapter we resume the discussion about the model structure of Bousfield-Segal spaces from Chapter 4 and – in analogy to the study of complete Segal spaces in [**44**] and [**32**] – localize it further at a set of maps C to obtain a model structure $(s\mathbf{S}, \mathrm{CB})$ for complete Bousfield-Segal spaces. The completeness condition on Segal spaces was introduced to ensure that the associated homotopy theory CS of complete Segal spaces is equivalent to the homotopy theory of quasi-categories. Similarly, we will see that the completeness condition on B-spaces ensures that the resulting homotopy theory CB is equivalent to the homotopy theory of ∞-groupoids. In fact we will see that the completeness condition trivializes B-spaces in the sense that a B-space is complete if and only if it is homotopically constant as will be specified in Section 5.2. It follows that CB is contained in a class of well understood model structures treated from different angles in the literature of [**46**], [**16**] and [**14**]. Using these results it follows that $(s\mathbf{S}, \mathrm{CB})$ is a type theoretic model category with as many univalent fibrant universes as there are inaccessible cardinals.

In Section 5.1 we give the basic definitions, properties and characterizations of complete Bousfield-Segal spaces and show that the diagonal functor $d^*\colon \mathbf{S} \to s\mathbf{S}$ is part of a Quillen equivalence between $(s\mathbf{S}, \mathrm{CB})$ and the standard model of ∞-groupoids $(\mathbf{S}, \mathrm{Kan})$. In Section 5.2 we note that $(s\mathbf{S}, \mathrm{CB})$ coincides with the *canonical* model structure on $s\mathbf{S}$ as introduced in [**46**], so that it follows from work of Cisinski's in [**14**] that $(s\mathbf{S}, \mathrm{CB})$ yields a model of Homotopy Type Theory in the sense of [**52**]. We discuss this observation in Section 5.3 and furthermore give a direct proof of right properness. In Section 5.4 we deduce that $(s\mathbf{S}, \mathrm{CB})$ is a cartesian closed model category from Rezk's work in [**44**].

5.1. The model structure CB

Recall from Notation 4.5.7 that we write J for nerve of the walking isomorphism

$$I = \begin{tikzcd} 0 \arrow[r, bend left, "\cong"] & 1 \arrow[l, bend left] \end{tikzcd}$$

and $c\colon 1 \to J$ for the inclusion $* \mapsto 0$. Localizing the model structure $(s\mathbf{S}, \mathrm{S})$ for Segal spaces at the set

$$\mathrm{C} := \{p_1^* c\colon p_1^* J \to p_1^* 1\}$$

defines the model structure $(s\mathbf{S}, \mathrm{CS}) := \mathcal{L}_{\mathrm{C}}(s\mathbf{S}, \mathrm{S})$ which originally was presented in [**44**] and is further studied in [**32**, Section 4]. Its fibrant objects are the C-local Segal spaces – the *complete* Segal spaces – i.e. the Segal spaces X such that the map

$$c \setminus X\colon J \setminus X \twoheadrightarrow X_0$$

is an acyclic fibration. Analogously, localizing $(s\mathbf{S}, \mathrm{B})$ at the set C yields the simplicial, left-proper and combinatorial model category

$$(s\mathbf{S}, \mathrm{CB}) := \mathcal{L}_C(s\mathbf{S}, \mathrm{B}) = \mathcal{L}_{\mathrm{B}}(s\mathbf{S}, \mathrm{CS}).$$

Definition 5.1.1. We say that $X \in s\mathbf{S}$ is a complete B-space if X is a B-local complete Segal space. That is if and only if X is fibrant in $(s\mathbf{S}, \mathrm{CB})$.

As will become clear throughout this chapter, completeness is a technical condition which relates the horizontal categorical dimension of a Segal space to its vertical homotopical dimension. More precisely, Rezk shows in [**44**, Theorem 6.2] that for every Segal space X the fibration $J \setminus X \twoheadrightarrow X_1$ induced by the inclusion $e_1\colon \Delta^1 \hookrightarrow J$ factors through $X_{\mathrm{hoequiv}} \subseteq X_1$ and yields an acyclic fibration

$$J \setminus X \twoheadrightarrow X_{\mathrm{hoequiv}}.$$

The degeneracy $s_0\colon X_0 \to X_1$ factors through X_{hoequiv}, too, and we see by 2-for-3, that X is complete if and only if the degeneracy $s_0\colon X_0 \hookrightarrow X_{\mathrm{hoequiv}}$ is a weak homotopy equivalence. The boundaries $d_i\colon X_1 \twoheadrightarrow X_0$ induce a fibration

$$(d_0, d_1)\colon X_{\mathrm{hoequiv}} \twoheadrightarrow X_0 \times X_0$$

by [**44**, Lemma 5.8] and so X is complete if and only if the simplicial set X_{hoequiv} is a path object for X_0. If X is a B-space, we have seen that $X_{\mathrm{hoequiv}} = X_1$ in Proposition 4.5.8, so in that case X is complete if and only if the object X_1 is a path space for X_0.

Example 5.1.2. Rezk introduces the *classifying diagram* $N_R(\mathbb{C})$ of a category $\mathbb{C}$ in [**44**, Section 3.5]. For $I[n]$ the free groupoid generated by the category $[n]$, its formula is given by

$$N_R(\mathbb{C})_{mn} = \mathrm{Hom}_{\mathbf{Cat}}([m] \times I[n], \mathbb{C}) = \mathrm{Hom}_{s\mathbf{S}}(\Delta^m \square N(I[n]), d_* N(\mathbb{C}))$$

where d_* denotes the right adjoint to the diagonal $d^*\colon s\mathbf{S} \to \mathbf{S}$. Rezk shows in [**44**, Proposition 6.1] that the classifying diagram $N_R(\mathbb{C})$ of a category $\mathbb{C}$ is a complete Segal space. It follows that the classifying diagram $N_R(\mathbb{G})$ of a groupoid $\mathbb{G}$ is a complete B-space.

Indeed, it is only left to show that $N_R(\mathbb{G})$ is B-local. But for a groupoid $\mathbb{G}$, we have $\mathrm{Hom}_{\mathbf{Cat}}([m] \times I[n], \mathbb{G}) \cong \mathrm{Hom}_{\mathbf{Cat}}([m] \times [n], \mathbb{G})$, hence $N_R(\mathbb{G}) \cong d_* N(\mathbb{G})$. Therefore, $\iota_{0,n} \setminus N_R(\mathbb{G}) = N(\mathbb{G})^{\iota_{0,n}}$ by Lemma 4.2.2. But $N(\mathbb{G})$ is a Kan complex and $\iota_{0,n}$ is anodyne, hence $N(\mathbb{G})^{\iota_{0,n}}$ is an acyclic fibration and $N_R(\mathbb{G})$ is a B-space by Definition 4.3.2.

Remark 5.1.3. We have seen in Theorem 4.4.7 that $(s\mathbf{S}, \mathrm{B})$ and $\mathcal{L}_B(s\mathbf{S}, \mathrm{S})$ coincide, so the equality $(s\mathbf{S}, \mathrm{CB}) = \mathcal{L}_B(s\mathbf{S}, \mathrm{CS})$ obviously holds, too. Thus, informally understanding the localization $\mathcal{L}$ at a set of maps as a partial function on the collection of model categories $\mathbb{M}$ together with a set of maps in $\mathbb{M}$, the genealogy of the considered model structures so far looks as follows.

$$
\begin{tikzcd}
R_v \arrow[rr] \arrow[d] & & S \arrow[d] \arrow[dr] & \\
B \arrow[rr, equal] \arrow[dr] & & \mathcal{L}_B(S) \arrow[dr] & CS \arrow[d] \\
& CB \arrow[rr, equal] & & \mathcal{L}_B(CS)
\end{tikzcd}
$$

There is a much more direct and concise proof of the bottom equality as some diagram chasing – which will be omitted here – shows that the inner horn inclusions can be obtained from the set l of left horn inclusions together with the map $c\colon 1 \to J$ by closure under finite pushouts, compositions and left cancellation of monomorphisms, using the following lemma.

Lemma 5.1.4. *Let $X \in s\mathbf{S}$ be v-fibrant. Then the following are equivalent.*

(1) X is a complete B-space,
(2) $\iota_{0,n} \setminus X$ is an acyclic fibration for all $n \geq 2$ and $c \setminus X$ is a trivial fibration,
(3) $h^n_0 \setminus X$ is an acyclic fibration for all $n \geq 1$,
(4) $\iota_{0,n} \setminus X$ is acyclic fibration for all $n \geq 2$ and X is a complete Segal space,
(5) $h^n_k \setminus X$ is an acyclic fibration for all $0 \leq k < n$,
(6) $u \setminus X$ is an acyclic fibration for all anodyne maps $u \in \mathbf{S}$,
(7) X/δ_n is a Kan fibration for all $n \geq 0$,

(8) X/v is a Kan fibration for all monomorphisms $v \in \mathbf{S}$.

Proof. (1) $\Leftrightarrow$ (2) holds by definition. Towards proving (2) $\Leftrightarrow$ (3), observe that, by Lemma 4.4.2, both conditions (2) and (3) imply that X is a B-space and hence that $e_1 \setminus X$ is a weak equivalence by Lemma 4.5.8. So, keeping in mind that

$$\begin{tikzcd} 1 \arrow[rr, "c"] \arrow[dr, "h^1_0"'] & & J \\ & \Delta^1 \arrow[ur, "e_1"'] & \end{tikzcd} \tag{5.1.1}$$

commutes, the map $h^1_0 \setminus X$ is a weak homotopy equivalence if and only if its section $c \setminus X$ is such. This gives (2) $\Leftrightarrow$ (3).

The equivalence of conditions (2), (4) and (5) follows from Theorem 4.4.7 and Lemma 4.4.2 similarly.

The equivalence of conditions (5) to (8) follows from Proposition 4.2.1. Note here that whenever X is a complete B-space and $u \colon A \hookrightarrow B$ is a monomorphism in $\mathbf{S}$, the map X/u is a left fibration between the Kan complexes X/A and X/B by Corollary 4.5.6 and part (5). But left fibrations between Kan complexes are Kan fibrations, see [**36**, Lemma 2.1.3.3]. This gives (5) $\Leftrightarrow$ (7). The equivalence of conditions (6), (7) and (8) is a direct application of Proposition 4.2.1. $\square$

Remark 5.1.5. Lemma 5.1.4 proves the note in Remark 3.1.5 on the equivalence of localizing at the left, right or all outer horn inclusions. Indeed, the right horn inclusions are anodyne, so the maps $h^n_n \setminus X$ are acyclic fibrations for every complete B-space X. Hence, the maps $p_1^* h^n_n$ are B-equivalences already.

The map $s_0 \colon \Delta^1 \to \Delta^0$ is anodyne, hence $s_0 \setminus X \colon X_0 \to X_1$ is a weak homotopy equivalence for every complete B-space X. Vice versa, we have seen that the map $e_1 \setminus X \colon J \setminus X \to X_1$ is an acyclic fibration for every B-space X, and clearly the maps $\Delta^1 \xrightarrow{e_1} J \xrightarrow{!_J} \Delta^0$ compose to s_0. Thus we can factor the degeneracy $s_0 \setminus X$ via

$$\begin{tikzcd} X_0 \arrow[rr, "s_0 \setminus X"] \arrow[dr, "!_J \setminus X"'] & & X_1 \\ & J \setminus X \arrow[ur, "e_1 \setminus X"'] & \end{tikzcd}$$

and see that a B-space X is complete if and only if $s_0 \setminus X \colon X_0 \to X_1$ is a weak equivalence.

Corollary 5.1.6. *Let $X \in s\mathbf{S}$. Then the following are equivalent.*

(1) X is a complete B-space,

(2) X is a complete Segal space and $e_1 \backslash X\colon J \backslash X \to X_1$ is a weak equivalence,

(3) X is a complete Segal space and $\lambda_2\colon X_1 \times^S_{X_0} X_1 \to X_1 \times^B_{X_0} X_1$, $(f,g) \mapsto (g \circ f, f)$ is a weak equivalence for any choice of composition "$\circ$" as in Corollary 4.5.2,

(4) X is a complete Segal space and its associated homotopy category $\mathrm{Ho}X$ *is a groupoid.*

Proof. The equivalence of (1) and (3) is Corollary 4.5.2. It is clear that (1) implies (2), while the converse also follows from Proposition 4.5.2. Namely, it suffices to show that $h^2_0 \backslash X$ is a weak equivalence. But the functor $\_ \backslash X$ sends every map in the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "d^0"] \arrow[d, "d^1"'] & \Delta^1 \arrow[d] & \\ \Delta^1 \arrow[r] \arrow[ur, phantom, "\ulcorner\cdot"] & \Lambda^2_1 \arrow[r, "h^2_1"] & \Delta^2 \end{tikzcd}$$

to an acyclic fibration, since $h^2_1 \backslash X$ is the Segal map ζ_2 and $d^i = h^1_{i-1}$ is part of Diagram (5.1.1). Hence, the composition $\iota_1 \backslash X\colon \Delta^2 \backslash X \to \Delta^0 \backslash X$ is an acyclic fibration, too, and by 2-for-3, every retraction $\iota_i \backslash X$ of the map $!_{\Delta^2} \backslash X$ for $i \in \{0,1,2\}$ is an acyclic fibration. Thus, considering the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "d^1"] \arrow[d, "d^1"'] & \Delta^1 \arrow[d] & \\ \Delta^1 \arrow[r] \arrow[ur, phantom, "\ulcorner\cdot"] & \Lambda^2_0 \arrow[r, "h^2_0"] & \Delta^2, \end{tikzcd}$$

we see that $h^2_0 \backslash X$ is a weak equivalence indeed, again by 2-for-3. Clearly, (3) implies (4). Vice versa, Rezk noted in [**44**, Corollary 6.6] that (4) holds if and only if X is a complete Segal space and $s_0 \backslash X\colon X_0 \to X_1$ is a weak equivalence. $\square$

Remark 5.1.7. Corollary 5.1.6 yields both a neat bisimplicial analogy to Joyal's criterion for a quasi-category to be a Kan complex – for instance as presented in [**36**, Proposition 1.2.4.3 and 1.2.5.1] – and an ∞-categorical analogy to the fact that the category **Gpd** is the reflective localization of **Cat** at the map $e_1\colon [1] \hookrightarrow I$.

Remark 5.1.8. Along the lines of the characterization of v-fibrations in Proposition 4.2.3, one can obtain a characterization of h-fibrations by simply swapping the components in the brackets $\langle \_ / \_ \rangle$ and $\langle \_ \backslash \_ \rangle$ respectively. Indeed, Lemma 5.1.4 shows that a bisimplicial set X is a complete B-space if and only if it is simultaneously v-fibrant and h-fibrant. This observation is all it will take to show that $(s\mathbf{S}, \mathrm{CB})$ is right proper in Section 5.3.

Remark 5.1.9. A map $f\colon X \to Y$ between complete B-spaces X and Y is a weak equivalence in $(s\mathbf{S}, \mathrm{CB})$ if and only if it is a level-wise weak homotopy equivalence. Rezk's result in [**44**, Proposition 7.6] shows that this in turn holds if and only if f is a *Dwyer-Kan equivalence*, i.e. an equivalence on the associated homotopy categories and fully faithful on mapping spaces.

In analogy to [**32**, Lemma 4.3, Proposition 4.4] we have the following propositions.

Proposition 5.1.10. *Let $f\colon X \to Y$ be a v-fibration between complete B-spaces. Then the map*

$$\langle f/v\rangle\colon X/T \to Y/T \times_{Y/S} X/S$$

is a Kan fibration for every monomorphism $v\colon S \to T$ in $\mathbf{S}$.

Proof. By [**32**, Lemma 4.3], $\langle f, v\rangle$ is a quasi-fibration. But quasi-fibrations between Kan complexes are Kan fibrations. □

Proposition 5.1.11. *A bisimplicial set X is a complete B-space if and only if the following two conditions are satisfied.*

(1) X/δ_n is a Kan fibration for all $n \geq 0$,
(2) X is groupoidally constant.

Proof. If X is a complete B-space, then (1) holds by part (7) of Lemma 5.1.4 and (2) holds by Proposition 4.2.6. If (1) holds, then X is h-fibrant by definition. So it is left to show that X is also v-fibrant if we furthermore assume (2). This can be shown along the lines of the proof of [**32**, Proposition 4.4], replacing the occurrences of "quasi" by "Kan" and "categorical" by "groupoidal". To give a basic outline of the arguments, by Proposition 4.2.3 one has to show that X/v is an acyclic fibration for every anodyne map $v \in \mathbf{S}$. Therefore, one considers the class of all monomorphisms $v \in \mathbf{S}$ such that X/v is an acyclic fibration. It is easy to check that this class is saturated, has the right cancellation property for monomorphisms and contains every face map d^i. Then [**32**, Lemma 3.7] gives that this class contains all anodyne maps and hence v-fibrancy of X follows by Proposition 4.2.3. □

Corollary 5.1.12. *The model category* $(s\mathbf{S}, \mathrm{CB})$ *is also a left Bousfield localization of* $(s\mathbf{S}, R_h)$*. An h-fibrant* $X \in s\mathbf{S}$ *is a complete B-space if and only if it is groupoidally constant.*

Proof. The cofibrations in both cases are precisely the monomorphisms. In order to show that $\mathcal{W}_h \subseteq \mathcal{W}_{CB}$ holds, it suffices to show that the identity $\mathrm{id}\colon (s\mathbf{S}, R_h) \to (s\mathbf{S}, \mathrm{CB})$ is a left Quillen functor, because all objects in $(s\mathbf{S}, R_h)$ are cofibrant. Equivalently, we may show that the identity $\mathrm{id}\colon (s\mathbf{S}, \mathrm{CB}) \to (s\mathbf{S}, R_h)$ preserves fibrations between fibrant objects. These are exactly the v-fibrations between complete B-spaces, and such are h-fibrations by Proposition 5.1.10. Indeed, $(s\mathbf{S}, \mathrm{CB})$ is the Bousfield localization of $(s\mathbf{S}, R_h)$ at $\{p_2^*\iota_{0,n} \mid n \geq 2\} \cup \{p_2^* c\}$, although with respect to the enrichment $\mathrm{Hom}_1(X, Y) := \iota_1^*(Y^X)$ by "orthogonal" argumentation. The second statement is a reformulation of Proposition 5.1.11. □

In analogy to [**32**, Proposition 4.6], Corollary 5.1.12 implies the following.

Corollary 5.1.13. *The box product* $\Box'\colon (\mathbf{S}, \mathrm{Kan}) \times (\mathbf{S}, \mathrm{Kan}) \to (s\mathbf{S}, \mathrm{CB})$ *is a left Quillen bifunctor.*

Proof. Let $u, v \in \mathbf{S}$ be cofibrations. By general argumentation about Reedy model structures, specifically [**32**, Proposition 7.36], $u\Box' v$ is a cofibration. If furthermore v is anodyne, $u\Box' v$ is acyclic in $(s\mathbf{S}, R_v)$, and so it is acyclic in $(s\mathbf{S}, \mathrm{CB})$. Now, suppose u is anodyne. We shall show that $u\Box' v$ has the right lifting property with respect to fibrations between complete B-spaces. But given a v-fibration $f\colon X \to Y$ between complete B-spaces, the map $\langle f/v\rangle$ is a Kan fibration by Proposition 5.1.10. Therefore $u \pitchfork \langle f/v\rangle$ holds and hence $u\Box' v \pitchfork f$. □

In [**32**, Theorem 4.11] it is shown that the pair

$$(p_1^*, \iota_1^*)\colon (\mathbf{S}, \mathrm{Qcat}) \to (s\mathbf{S}, \mathrm{CS})$$

from (4.2.3) is a Quillen equivalence. That means a complete Segal space X is determined by the quasi-category $X_{\bullet 0}$ and the homotopy theory of complete Segal spaces is equivalent to the homotopy theory of quasi-categories.

Theorem 5.1.14. *The pair*

$$(p_1^*, \iota_1^*)\colon (\mathbf{S}, \mathrm{Kan}) \to (s\mathbf{S}, \mathrm{CB})$$

is a Quillen equivalence.

Proof. By [**32**, Theorem 4.11], Example 3.1.5 and Lemma 3.1.6 the pair

$$(p_1^*, \iota_1^*) \colon (\mathbf{S}, \mathrm{Kan}) = \mathcal{L}_l(\mathbf{S}, \mathrm{Qcat}) \to \mathcal{L}_{\mathbb{L}p_1^* l}(s\mathbf{S}, \mathrm{CS})$$

is a Quillen equivalence, where l denotes the set of left n-horn inclusions for $n \geq 2$. So we are left to show that the model structures $\mathcal{L}_{\mathbb{L}p_1^* l}(s\mathbf{S}, \mathrm{CS})$ and $(s\mathbf{S}, \mathrm{CB})$ coincide. Every object in $(\mathbf{S}, \mathrm{Kan})$ is cofibrant, so

$$\mathbb{L}p_1^* l = p_1^* l = \{p_1^* h_0^n \mid n \geq 2\} \subset (s\mathbf{S}, \mathrm{CB}).$$

By Lemma 4.4.2, a v-fibrant object $X \in s\mathbf{S}$ is $p_1^* l$-local if and only if it is B-local, and hence the model categories $\mathcal{L}_{p_1^* l}(s\mathbf{S}, \mathrm{CS})$ and $\mathcal{L}_B(s\mathbf{S}, \mathrm{CS}) = (s\mathbf{S}, \mathrm{CB})$ coincide. $\square$

Theorem 5.1.15. *The pair*

$$(d^*, d_*) \colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan})$$

is a Quillen equivalence.

Proof. We first have to show that the pair (d^*, d_*) is still a Quillen pair. Certainly, the diagonal d^* preserves cofibrations. Also, recall from Theorem 4.2.8 that

$$(d^*, d_*) \colon (s\mathbf{S}, R_v) \to (\mathbf{S}, \mathrm{Kan})$$

is a Quillen pair and hence d_* sends Kan fibrations to v-fibrations. Thus, in order to show that the right adjoint $d_* \colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan})$ maps Kan fibrations between Kan complexes to B-fibrations ([**32**, Proposition 7.15]), it suffices to show that d_* maps Kan complexes to complete B-spaces. Given a Kan complex A, the maps $\iota_{0,n} \setminus d_* A = A^{\iota_{0,n}}$ and $c \setminus d_* A = A^c$ are acyclic fibrations, because $\iota_{0,n}$ and c are anodyne and $(\mathbf{S}, \mathrm{Kan})$ is cartesian. Hence, $d_* A$ is a complete B-space.

So all three pairs (d^*, d_*), $(p_1^*, \iota_1^*$ and $(\mathrm{id}, \mathrm{id})$ are Quillen pairs, and note that $d^* p_1^* = \mathrm{id} \colon \mathbf{S} \to \mathbf{S}$ and $\iota_1^* d_* = \mathrm{id} \colon \mathbf{S} \to \mathbf{S}$. Therefore, the statement follows from Corollary 5.1.14 by 2-for-3. $\square$

Remark 5.1.16. The fact that the diagonal induces an equivalence on homotopy categories as shown in Proposition 5.1.15 is exactly the content of [**9**, Theorem 3.1] for "very special bisimplicial sets" of type $n = 0$.

5.2. The canonical model structure and symmetry

In this section we show that the model structure $(s\mathbf{S}, \mathrm{CB})$ coincides with the *canonical*, *realization* or *hocolim* model structure on $s\mathbf{S}$ as introduced in [**46**] and

[**17**] respectively. We show that the "orthogonal" process of localizing the *horizontal* Reedy model structure $(s\mathbf{S}, R_h)$ at *horizontally* constant versions of the B-, S- and C-maps yields the very same model structure $(s\mathbf{S}, \mathrm{CB})$. Note that this does not follow from Corollary 5.1.12, because $(s\mathbf{S}, R_h)$ is not $\mathbf{S}$-enriched via $\mathrm{Hom}_2(X, Y) = \iota_2^*(Y^X)$, but via $\mathrm{Hom}_1(X, Y) := \iota_1^*(Y^X)$.

Definition 5.2.1. Recall that a bisimplicial set X is said to be *homotopically (or locally) constant* if the map $X(f)\colon X_m \to X_n$ is a weak equivalence for every function $(f\colon n \to m) \in \Delta$.

Lemma 5.2.2. *A v-fibrant bisimplicial set X is a complete Bousfield-Segal space if and only if X is homotopically constant.*

Proof. Clearly, X is homotopically constant if and only if all boundary and degeneracy maps of X are weak homotopy equivalences. This in turn holds if and only if all boundary maps of X are weak homotopy equivalences (since the degeneracies are sections of the boundaries). If X is homotopically constant, it is easy to see that all the X_n and all pullbacks $(X_{1/X_0})^n_B$ are contractible by right properness of $(\mathbf{S}, \mathrm{Kan})$, so the Bousfield maps are weak homotopy equivalences. Completeness follows trivially. Vice versa, if X is a complete B-space, we have seen that the degeneracy $s_0\colon X_0 \to X_1$ is a weak homotopy equivalence, and hence so are the boundaries $d_i\colon X_1 \to X_0$. This implies contractibility of the pullbacks $(X_{1/X_0})^n_B$ and, since the Bousfield maps are weak homotopy equivalences, therefore contractibility of the Kan complexes X_n. Thus, all boundaries of X are weak homotopy equivalences. $\square$

In [**46**], given a model category $\mathbb{M}$, the model structure on $\mathbb{M}^{\Delta^{op}}$ whose fibrant objects are exactly the homotopically constant Reedy fibrant simplicial objects is called the *canonical model structure* on $\mathbb{M}^{\Delta^{op}}$. So Lemma 5.2.2 shows that $(s\mathbf{S}, \mathrm{CB})$ is the canonical model structure on $s\mathbf{S}$. By [**46**, Theorem 3.6] this implies that the projection $\iota_2^*\colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan})$ onto the first column is part of a Quillen equivalence. Also, recall the isomorphism $\sigma^*\colon s\mathbf{S} \to s\mathbf{S}$ induced by the permutation $\sigma\colon \Delta \times \Delta \to \Delta \times \Delta$ swapping the components $([n], [m]) \mapsto ([m], [n])$. Using the notation from Section 4.2, note that $\sigma^*[\mathcal{W}_v] = \mathcal{W}_h$, $\sigma^*[\mathcal{C}_v] = \mathcal{C}_h = \mathcal{C}$ and even $\sigma^*[\mathcal{I}_v] = \mathcal{I}_h$ and $\sigma^*[\mathcal{J}_v] = \mathcal{J}_h$ as σ^* preserves colimits. Furthermore, for all objects $A, B \in s\mathbf{S}$ the isomorphism satisfies

$$\mathrm{Hom}_2(\sigma^* A, \sigma^* B) := \iota_2^*(\sigma^* B^{\sigma^* A}) = \iota_2^* \sigma^*(B^A) = \iota_1^*(B^A) =: \mathrm{Hom}_1(A, B)$$

and Hom_1 turns $(s\mathbf{S}, R_h)$ into a simplicial model category. Let

$$CB^{\perp} := \{p_2^* c\} \cup \{p_2^* \iota_{0,n} \mid n \geq 2\},$$

so we can build the Bousfield localization $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$.

Note that the model structures $(s\mathbf{S}, \mathrm{CB})$ and $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$ are isomorphic, so that all arguments presented so far are symmetric with respect to the vertical and horizontal direction. Hence, the fact that the first row projection $i_1^* \colon s\mathbf{S} \to \mathbf{S}$ is part of a Quillen equivalence as stated in Theorem 5.1.14 also follows from Lemma 5.2.2 and the general observations in [**46**, Theorem 3.6] (or [**17**] respectively).

It is easy to show that the model structures $(s\mathbf{S}, \mathrm{CB})$ and $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$ in fact coincide. This means that a bisimplicial set X is a (vertical) complete B-space if and only if $\sigma^* X$ is a (vertical) complete B-space, yielding a neat symmetry.

Indeed, the model categories $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$ and $(s\mathbf{S}, \mathrm{CB}) := \mathcal{L}_{CB}(s\mathbf{S}, R_v)$ have the same class of cofibrations. Furthermore, we have $CB^{\perp} = \sigma^*[CB]$, because $\sigma^* p_2^* = p_1^*$. So X is fibrant in $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$ if and only if any of the following equivalent conditions hold.

(1) X is h-fibrant and every map $f \colon A \to B$ in $CB^{\perp}$ induces an acyclic fibration

$$\begin{tikzcd} \mathrm{Hom}_1(B, X) \arrow[r, two heads, "f^*", "\sim"'] \arrow[d, equal] & \mathrm{Hom}_1(A, X) \arrow[d, equal] \\ \mathrm{Hom}_2(\sigma^* B, \sigma^* X) \arrow[r, "(\sigma^* f)^*"'] & \mathrm{Hom}_2(\sigma^* A, \sigma^* X). \end{tikzcd}$$

(2) $\sigma^* X$ is v-fibrant and $\sigma^*[CB^{\perp}] = CB$-local.
(3) $\sigma^* X$ is fibrant in $(s\mathbf{S}, \mathrm{CB})$.
(4) $\sigma^* X$ is both v- and h-fibrant.
(5) X is both h- and v-fibrant.
(6) X is fibrant in $(s\mathbf{S}, \mathrm{CB})$.

Hence we see that the model categories also have the same fibrant objects. It follows that the model structures $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$ and $(s\mathbf{S}, \mathrm{CB})$ coincide by a general result in [**30**, Proposition 1.38] due to Joyal.

5.3. $(s\mathbf{S}, \mathrm{CB})$ is a model of univalent type theory

The model structure CB is a cofibrantly generated model structure on the presheaf category $s\mathbf{S}$ whose cofibrations are exactly the monomorphisms. This means that the simplicial model category ($s\mathbf{S}$, CB) defines a *Cisinski model category.* Therefore, by [**52**, Theorem 5.1], in order for ($s\mathbf{S}$, CB) to support a homotopy type theoretical interpretation, we only have to show that ($s\mathbf{S}$, CB) is right proper and that it supports an infinite sequence of univalent universes. In this section we discuss two ways to show this.

The first of these ways can be covered rather swiftly. In [**14**, Section 1], Cisinski introduces the *locally constant model structure* $([\mathcal{A}^{op}, \mathbf{S}], \mathrm{lc})$ on simplicial presheaves over any elegant Reedy category $\mathcal{A}$. It is a left Bousfield localization of the injective model structure whose fibrant objects are exactly the homotopically constant Reedy fibrant objects $X \in [\mathcal{A}^{op}, \mathbf{S}]$, i.e. those Reedy fibrant objects such that the f-action $X(f)\colon X(b) \to X(a)$ is a homotopy weak equivalence for all maps $f\colon a \to b$ in $\mathcal{A}$. Hence, Lemma 5.2.2 shows that $(s\mathbf{S}, \mathrm{lc}) = (s\mathbf{S}, \mathrm{CB})$. In [**12**], he shows that $([\mathcal{A}^{op}, \mathbf{S}], \mathrm{lc})$ is always right proper and in [**14**, Proposition 1.1] he shows that the model category contains a fibrant univalent universe classifying κ-small maps for every inaccessible cardinal κ large enough. The latter statement is given in Theorem 3.3.3 as it applies to a much broader variety of model structures.

The following presents the second way. We obtain a sequence of univalent fibrant universes for ($s\mathbf{S}$, CB) from Theorem 2.5.8 and Corollary 2.5.9 if we can show that there is a set of generating acyclic cofibrations for ($s\mathbf{S}$, CB) with representable codomain. Having obtained the model structure ($s\mathbf{S}$, CB) by left Bousfield localization, it a priori is very hard to present a well behaved set of generating acyclic cofibrations. But recall that the authors of [**46**] show that the fibrations in the canonical model structure ($s\mathbf{S}$, CB) are exactly the *equi-fibred Reedy fibrations.* For such, a set of generating acyclic cofibrations is given in [**46**, Proposition 8.5] by $\mathcal{J}_{CB} = \mathcal{J}_h \cup \mathcal{J}''$ for

$$\mathcal{J}_h = \{h_i^n \Box' \delta_m \colon (\Delta^n \Box \partial\Delta^m) \cup_{\Lambda_i^n \Box \partial\Delta^m} (\Lambda_i^n \Box \Delta^m) \to (\Delta^n \Box \Delta^m) \mid 0 \leq i \leq m, n\},$$

$$\mathcal{J}'' := \{\delta_n \Box' d_i^m \colon (\Delta^n \Box \Delta^{m-1}) \cup_{\partial\Delta^n \Box \Delta^{m-1}} (\partial\Delta^n \Box \Delta^m) \to (\Delta^n \Box \Delta^m) \mid n \geq 0, m \geq i \geq 0\}.$$

The box products $\Delta^n \Box \Delta^m$ are exactly the representables in $s\mathbf{S}$, thus a set of generating acyclic cofibrations with representable codomain exists indeed.

Also, even though right properness of ($s\mathbf{S}$, CB) follows from the general considerations on fundamental localizers in [**12**] as mentioned above, there is a direct hands on proof for right properness in this special case. The rest of this section presents this proof.

Therefore, we simply use the fact that fibrant objects in $(s\mathbf{S}, \mathrm{CB})$ are exactly the objects fibrant both in the horizontal and the vertical Reedy structures as noted in Remark 5.1.8, and that both these Reedy structures are right proper. Recall that a model category $\mathbb{M}$ is right proper if and only if the pullback of any acyclic cofibration with fibrant codomain along fibrations is a weak equivalence. This is shown in [**10**, Lemma 9.4] for example.

Recall the sets $\mathcal{J}_v$ and $\mathcal{J}_h$ from Section 4.2 which generate the acyclic cofibrations in $(s\mathbf{S}, R_v)$ and in $(s\mathbf{S}, R_h)$ respectively.

Lemma 5.3.1. *The class of acyclic cofibrations with fibrant codomain in* $(s\mathbf{S}, \mathrm{CB})$ *is exactly the class of maps in the saturation of* $\mathcal{J}_v \cup \mathcal{J}_h$ *with fibrant codomain, i.e.*

$$(\mathcal{W}_{CB} \cap \mathcal{C})/\mathrm{CB}-\text{spaces} = {}^{\pitchfork}((\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork})/\mathrm{CB}-\text{spaces}.$$

Proof. As $(s\mathbf{S}, \mathrm{CB})$ is a left Bousfield localization of both $(s\mathbf{S}, R_v)$ and $(s\mathbf{S}, R_h)$, we have

$$\mathcal{J}_v \cup \mathcal{J}_h \subseteq \mathcal{W}_{CB} \cap \mathcal{C},$$

so one direction is clear. Vice versa, let $j\colon A \hookrightarrow B$ be a weak CB-equivalence with B a complete B-space. Note that $(\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork}$ is the intersection of the set $\mathcal{F}_v$ of v-fibrations and the set $\mathcal{F}_h$ of h-fibrations, and hence the pair $({}^{\pitchfork}((\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork}), \mathcal{F}_v \cap \mathcal{F}_h)$ is a weak factorization system on $s\mathbf{S}$ by general category theory. Pick a factorization $A \xrightarrow{k} C \xrightarrow{q} B$ of j with $k \in {}^{\pitchfork}((\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork})$ and $q \in \mathcal{F}_v \cap \mathcal{F}_h$,

$$(*) \qquad \begin{tikzcd} A \arrow[r, "k"] \arrow[d, hook, "j"] & C \arrow[d, "q"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

Since B is a complete B-space, C is now both v-fibrant and h-fibrant, hence a complete B-space, too. But a map between complete B-spaces is a CB-fibration if and only if it is a v-fibration. This in turn holds if and only if it is an h-fibration as can be seen by [**32**, Proposition 7.21]. Hence, we obtain a lift for the square $(*)$ which exhibits j as retract of k. Therefore, $j \in {}^{\pitchfork}((\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork})$. $\square$

Corollary 5.3.2. *Every acyclic cofibration in* $(s\mathbf{S}, \mathrm{CB})$ *into a complete B-space is the transfinite composition of acyclic v- and h-cofibrations.*

$\square$

Lemma 5.3.3. *The class of morphisms which are mapped into a weak CB-equivalence via pullback along some fixed map p is saturated.*

Proof. In the language of [**52**, 3], this holds in virtue of the "exactness" properties of Grothendieck toposes, i.e. pullbacks in $s\mathbf{S}$ commute with pushouts, transfinite compositions and retracts in such a way that the proof becomes a straightforward induction. $\square$

Now, we easily can derive right properness as anticipated.

Theorem 5.3.4. *The model category* $(s\mathbf{S}, \mathrm{CB})$ *is right proper.*

Proof. By Lemma 5.3.1 and Lemma 5.3.3 it remains to check that a pullback square of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p^*j"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "j"] \\ X \arrow[r, "p"'] & \Delta^n \square \Delta^m \end{tikzcd}$$

with a CB-fibration p and $j \in J_v \cup J_h$ exhibits the arrow p^*j to be a weak CB-equivalence. But $\mathcal{F}_{CB}$ is a subset of $\mathcal{F}_v \cap \mathcal{F}_h$, so p is both a v-fibration and an h-fibration. Both Reedy structures $(s\mathbf{S}, R_v)$ and $(s\mathbf{S}, R_h)$ are right proper due to the right properness of $(\mathbf{S}, \mathrm{Kan})$. Therefore, $p^*j \in \mathcal{W}_v \cup \mathcal{W}_h$. But both $\mathcal{W}_v$ and $\mathcal{W}_h$ are contained in $\mathcal{W}_{CB}$, since the model structure CB is a left Bousfield localization of both. This finishes the proof. $\square$

5.4. Cartesian closedness

In this short section we prove cartesian closure of the simplicial model category $(s\mathbf{S}, \mathrm{CB})$. The result follows easily from Rezk's combinatorial arguments for cartesian closure of the model category $(s\mathbf{S}, \mathrm{CS})$ for complete Segal spaces.

Lemma 5.4.1. *If* X *and* Y *are complete B-spaces, then so is the exponential* Y^X.

Proof. Knowing that X and Y are in particular complete Segal spaces, the exponential Y^X is a complete Segal space by [**44**, Theorem 7.2]. We are left to show that Y^X is B-local. Equivalently, we may show that the maps $\langle p_1^*\iota_{0,n}, \mathrm{id}_X\rangle \colon p_1^*C_{0,n} \times X \to p_1^*\Delta^n \times X$ are weak CB-equivalences for every complete B-space X. Now, the maps $p_1^*\iota_{0,n}$ are weak CB-equivalences by Definition 5.1.1 and we have shown in Theorem 5.3.4 that $(s\mathbf{S}, \mathrm{CB})$ is right proper. Therefore $\langle p_1^*\iota_{0,n}, \mathrm{id}_X\rangle$ is a weak CB-equivalence due to the fibrancy of X. $\square$

Lemma 5.4.2. *If X is a complete B-space, then so is the exponential $X^{p_1^*\Delta^1}$.*

Proof. Recall from [**44**, Theorem 6.2] that the map $p_1^*e_1 \colon p_1^*\Delta^1 \to p_1^*J$ is an acyclic cofibration in $(s\mathbf{S}, \mathrm{CS})$ (and hence so it is in $(s\mathbf{S}, \mathrm{CB})$). The Reedy structure $(s\mathbf{S}, R_v)$ is cartesian closed and $p_1^*e_1$ is a cofibration, so $X^{p_1^*e_1}$ is clearly a v-fibration. Rezk shows in [**44**, Theorem 7.1] that the model structure $(s\mathbf{S}, \mathrm{CS})$ is cartesian closed, hence the objects $X^{p_1^*J}$ and $X^{p_1^*\Delta^1}$ are complete Segal spaces. The constant bisimplicial set p_1^*J is strictly B-local itself (i.e. its Bousfield maps are isomorphisms), because J is a Kan complex. Let

$$r_J \colon p_1^*J \xrightarrow{r_v} \mathbb{R}_v p_1^*J \xrightarrow{r_{CS}} \mathbb{R}_{CS} p_1^*J$$

be the composition of fibrant replacements in $(s\mathbf{S}, R_v)$ and $(s\mathbf{S}, \mathrm{CS})$ respectively. Then $\mathbb{R}_v p_1^*J$ is a B-space and r_{CS} is a Dwyer-Kan equivalence in the sense of [**44**, 7.4] by [**44**, Theorem 7.7]. Hence, the homotopy category of the complete Segal space $\mathbb{R}_{\mathrm{CS}} p_1^*J$ is a groupoid. Therefore, $\mathbb{R}_{CS} p_1^*J$ is a complete B-space by Corollary 5.1.6. It follows that the exponential $X^{\mathbb{R}_{CS} p_1^*J}$ is a complete B-space as we have just shown in Lemma 5.4.1. The maps r_J and $p_1^*e_1$ are acyclic cofibrations in $(s\mathbf{S}, \mathrm{CS})$, hence the exponential

$$X^{r_J \circ p_1^*e_1} \colon X^{\mathbb{R}_{CS} p_1^*J} \to X^{p_1^*\Delta^1}$$

is an acyclic fibration from a complete B-space to a complete Segal space. Hence, $X^{p_1^*\Delta^1}$ is B-local by Lemma 3.1.7 and thus a complete B-space.

□

Proposition 5.4.3. *The model structure $(s\mathbf{S}, \mathrm{CB})$ is cartesian closed.*

Proof. This follows immediately from [**44**, Proposition 9.2] and Lemma 5.4.2. □

CHAPTER 6

Univalence and completeness of Segal objects

6.1. Introduction and preliminaries

In this chapter we introduce a notion of univalence and a notion of completeness for Segal objects X in type theoretic model categories $\mathbb{M}$ such that all fibrant objects are cofibrant. The former is a straightforward generalization of univalence in the type theoretic fibration category $\mathbb{C} := \mathbb{M}^f$ as treated for example in Section 1.2 and reduces to this notion of univalence whenever the given Segal object X is the nerve associated to a fibration p in $\mathbb{C}$. The latter is a generalization of Rezk's original definition of completeness for Segal spaces. Both conditions share the heuristic purpose to contract a respective object of internal equivalences associated to X over the object of points X_0, turning that object of internal equivalences into a path object for X_0. A priori, these objects of internal equivalences do not necessarily coincide. On one hand, univalence, being a formula in the internal type theory $\mathcal{T}_{\mathbb{C}}$ of $\mathbb{C}$, makes a statement about the type of equivalences associated to the type families

$$X_2 \xrightarrow{\zeta_2} X_1 \xrightarrow{(d_0,d_1)} X_0 \times X_0.$$

On the other hand, completeness, being motivated by simplicial homotopy theory, considers the adjoint pair $(J\square\,\_\,, J\setminus\,\_\,)\colon \mathbb{M} \to s\mathbb{M}$ (to be defined in this generality in Section 6.3) and the induced object of equivalences $J \setminus X$ for $J \in \mathbf{S}$ the walking isomorphism (or at least this is the canonical way to define internal equivalences following Rezk's original work). Over the category of simplicial sets, completeness is a technical device necessary and sufficient to ensure that Segal spaces which satisfy this condition indeed model $(\infty, 1)$-categories - instead modelling something like ∞-double categories with a groupoidal dimension. The aim of Sections 6.2, 6.3 and 6.4 is to prove the following theorem and define all the notions involved.

Theorem 6.4.4. Let X be a Segal object in $\mathbb{M}$. Then the following are equivalent.

(1) X is univalent.
(2) For any Reedy fibrant replacement $\mathbb{R}X$ of X, $\mathbb{R}X$ is complete.

In Section 6.5 we go back to the special case of simplicial sets and use Theorem 6.4.4 in Section 6.6 to show that univalent completion of a Kan fibration p as introduced in [**6**] is a special case of Rezk completion of its associated Segal space.

The reader may observe that the process associating univalent fibrations to complete Segal objects as presented in this chapter is quite close to the ideas presented in [**43**, 6] which was published around the same time I finished the first version of this chapter. On the one hand, Rasekh develops in the given paper a theory of complete Segal objects within quasi-categories and *defines* univalence of a map p in a locally cartesian closed quasi-category $\mathcal{C}$ via completeness of its associated Segal object $\mathcal{N}(p)$ ([**43**, Definition 6.24]). On the other hand, in this chapter we compare an existing definition of univalence with a generalization of another existing definition of Rezk completeness and show they are equivalent. The definition of univalence in [**43**] coincides with Gepner and Kock's definition from [**23**, 3.2] whenever the quasi-category $\mathcal{C}$ is presentable by [**43**, Theorem 6.28] and [**23**, Proposition 3.8]. While the authors of [**23**] show that there is a connection between univalence in the presentable quasi-category $\mathcal{C}$ and univalence in a type theoretic model category as given in Definition 1.5.1 presenting $\mathcal{C}$, there is no such result for the general definition of [**43**]. Of course we can assign to a given type theoretic model category $\mathbb{M}$ its associated quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$. Assuming that the construction of Segal objects and completeness of such as presented in this chapter are mapped by the functor Ho_∞ to the respective constructions and notions from [**43**], one can understand Theorem 6.4.4 as a proof that Rasekh's definition of univalence in locally cartesian closed quasi-categories indeed coincides with the type theoretic definition of univalence whenever $\mathcal{C}$ comes from a type theoretic model category.

The reason why we choose to work with a type theoretic model category but a type theoretic fibration category is Lemma 6.4.1 which assumes the existence of a Reedy fibrant replacement functor in $s\mathbb{M}$. Therefore note that in fact all statements in Sections 6.2, 6.3 and 6.4 also apply to type theoretic fibration categories $\mathbb{C}$ with finite colimits. In light of the correspondence (2) from the Introduction and its conjectured extension to dependent products the latter set-up is closer to the generality assumed in [**43**, Section 6].

6.2. Univalence of simplicial objects

In all of this chapter $\mathbb{M}$ is a fixed type theoretic model category and $\mathbb{C}$ denotes its associated type theoretic fibration category $\mathbb{M}^f$ of fibrant objects. We assume that fibrant objects in $\mathbb{M}$ are cofibrant, in order to assure that weak equivalences and homotopy equivalences in $\mathbb{M}^f$ coincide by Lemma 1.3.4. The category $s\mathbb{M} = [\Delta^{op}, \mathbb{M}]$ denotes the category of simplicial objects in $\mathbb{M}$. Given a simplicial object X in $\mathbb{M}$, recall from Section 4.3 the Segal maps

$$\xi_n \colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

associated to X given by the boundary maps which project an n-simplex in X to its values on the essential edges $I_n \subset \Delta^n$.

Definition 6.2.1. Let X be a simplicial object in $\mathbb{C}$.

(1) X is *sufficiently fibrant* if both the 2-Segal map

$$\xi_2 \colon X_2 \to X_1 \times_{X_0} X_1$$

and the boundary map

$$X_1 \xrightarrow{(d_1,d_0)} X_0 \times X_0$$

are fibrations in $\mathbb{C}$.

(2) Let X be sufficiently fibrant and recall notation from (4.3.1). We say that X is a *(strict) Segal object* if the associated Segal maps

$$\xi_n \colon X_n \to (X_{1/X_0})^n_S$$

are homotopy equivalences (isomorphisms) in $\mathbb{C}$. So a Segal object X gives rise to a type family of morphisms fibred over pairs of objects and a type of compositions fibred over the two respective components.

We purposely do not assume Reedy fibrancy for the definition of general Segal objects here. Therefore in our notation and in the reference case when $\mathbb{C}$ is the category of Kan complexes, Segal spaces are exactly Reedy fibrant Segal objects.

Remark. Note that the codomains of the Segal maps in Definition 6.2.1 are the ordinary pullbacks of X_1 over X_0. For non sufficiently fibrant simplicial objects, these are the "homotopically wrong" objects to consider and should be replaced with the corresponding homotopy pullbacks. But both notions coincide whenever X is sufficiently fibrant.

The definition of sufficient fibrancy is chosen in such a way that all notions considered in this chapter are stable under homotopy equivalence between sufficiently fibrant objects while also encompassing the following example.

Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{C}$. Recall from Section 1.5 that p induces the generic function type $\mathrm{Fun}p := [\pi_1^*E, \pi_2^*E]_{B\times B}$ together with source and target fibrations

$$\mathrm{Fun}p \overset{(s,t)}{\twoheadrightarrow} B \times B.$$

The object $\mathrm{Fun}p$ interprets the parametrized function type

$$\vdash \sum_{a,b:B} (E_a \to E_b) : \mathtt{type}$$

with corresponding source and target maps specified by the terms $\lambda(a,b,f).a$ and $\lambda(a,b,f).b$ in the internal type theory of $\mathbb{C}$.

Recall e.g. from [**26**, Section 3] that, for every object $B \in \mathbb{C}$, the category $\mathbb{C}/(B \times B)$ of *graphs* with vertices in B has a tensor product $\otimes_B$ given by pullback

$$\begin{tikzcd}
(E \xrightarrow{(s,t)} B^2) \otimes_B (E' \xrightarrow{(s',t')} B^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "s'"'] \arrow[ddr, bend left, "t'"] & \\
E \arrow[r, "t"'] \arrow[drr, bend right, "s"'] & B & \\
& & B.
\end{tikzcd}$$

The monoids in the monoidal category $(\mathbb{C}/(B \times B), \otimes_B)$ are the internal category objects in $\mathbb{C}$ over B, the category of these monoids is denoted by $\mathrm{ICat}(\mathbb{C})_B$. The assignment $B \mapsto \mathrm{ICat}(\mathbb{C})_B$ induces a pseudo-functor $\mathrm{ICat}\colon \mathbb{C}^{op} \to \mathbf{Cat}$ whose Grothendieck construction is the category $\mathrm{ICat}(\mathbb{C})$ of internal category objects in $\mathbb{C}$.

Proposition 6.2.2. *For a fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$, the graph*

$$\mathrm{Fun}p \overset{s}{\underset{t}{\rightrightarrows}} B$$

comes equipped with a unit and a multiplication which turns this graph into an internal category object in $\mathbb{C}$.

Proof. This is shown e.g. in [**28**, Example 7.1.4.(ii)]. Although the author assumes local cartesian closedness, the construction can be carried out for every fibration in a type theoretic fibration category. In an nutshell, a unit is given by the map

$$\begin{tikzcd} B \arrow[r, "\eta"] \arrow[dr, "\Delta"'] & {[E \times B, B \times E]_{B^2}} \arrow[d, two heads] \\ & B \times B \end{tikzcd} \quad \text{adjoint to} \quad \begin{tikzcd} E \arrow[r, "{(p,\mathrm{id})}"] \arrow[dr, "{(p,p)}"'] & B \times E \arrow[d, two heads, "{(\mathrm{id},p)}"] \\ & B \times B \end{tikzcd}$$

and a multiplication by the adjoint of successive evaluation on exponentials. □

Proposition 6.2.3. *There is a nerve functor $N\colon ICat(\mathbb{C}) \to s\mathbb{C}$ which, when restricted to its image, yields an equivalence to the subcategory of objects in $s\mathbb{C}$ whose Segal maps are isomorphisms.*

Proof. The functor N has a straightforward definition, given on $C \in \mathrm{ICat}(\mathbb{C})$ with underlying reflexive graph

$$C_1 \underset{t}{\overset{s}{\underset{\longleftarrow \eta -}{\rightrightarrows}}} C_0 \tag{6.2.1}$$

by

$$(NC)_n := (C_1/C_0)^n_S$$

in the notation of Section 4.2. The object $(NC)_n$ is canonically isomorphic to the n-th monoidal power $\bigotimes^n_{C_0}(\ C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0\ )$ of the underlying graph in $\mathbb{C}/(C_0 \times C_0)$ such that the 1-skeleton $(NC)_{\leq 1}$ with its degeneracy and two boundaries is exactly the reflexive graph (6.2.1). Higher degeneracies are given by inserting the unit η into corresponding components of the monoidal power and the boundaries are given by the obvious combination of multiplications and projections. One can verify the simplicial relations one by one. They hold exactly due to the associativity and unitality laws satisfied by the multiplication μ and the unit η, N being just the free category comonad resolution in $\mathbb{C}$.

Vice versa, every simplicial object $X \in s\mathbb{C}$ whose Segal maps are isomorphisms yields its 2-skeleton $X_{\leq 2} \in \mathrm{ICat}(\mathbb{C})_{X_0}$ such that $N(X_{\leq 2}) \cong X$. □

So for every fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$, we obtain the simplicial object

$$Np := N(\ \mathrm{Fun}p \underset{t}{\overset{s}{\underset{\longleftarrow \eta -}{\rightrightarrows}}} B)\ \in s\mathbb{C}.$$

The nerve Np is sufficiently fibrant, because the boundary

$$\Big((d_1, d_0)\colon (Np)_1 \to (Np)_0 \times (Np)_0\Big) = \Big((s,t)\colon \mathrm{Fun}p \twoheadrightarrow B \times B\Big)$$

is a fibration and the 2-Segal map ξ_2 is an isomorphism (and in particular a fibration). Therefore for every fibration $p \in \mathbb{C}$, by Proposition 6.2.3 the simplicial object Np is a strict Segal object in $\mathbb{C}$.

Taking a step back towards the general case, let X be a sufficiently fibrant simplicial object in $\mathbb{C}$. In the spirit of Joyal's definition of biinvertible functions in type theory – these are functions together with a left-inverse and a (potentially distinct) right-inverse, see for example [**41**, Section 4.3] – the structure of X suggests the definition of a type family of equivalences associated to X as follows. We have a judgement

$$x, y : X_0, f, g : X_1(x, y) \vdash X_2(f, g) \text{ type}$$

via the 2-Segal map. Denoting the substitution of terms $a, b : X_0$ for the variables $x, y : X_0$ in the type family $X_1(x, y)$ by $X_1[\frac{a}{x}, \frac{b}{y}]$, such a type family of equivalences is given in context $(x, y : X_0, f : X_1(x, y))$ by

$$\begin{aligned}
\mathrm{Linv}(x, y, f) &:\equiv \sum_{g : X_1[\frac{y}{x}, \frac{x}{y}]} \sum_{\sigma : X_2[f, g]} d_1\sigma =_{X_1[\frac{x}{x}, \frac{x}{y}]} s_0 x,\\
\mathrm{Rinv}(x, y, f) &:\equiv \sum_{h : X_1[\frac{y}{x}, \frac{x}{y}]} \sum_{\sigma : X_2[h, f]} d_1\sigma =_{X_1[\frac{y}{x}, \frac{y}{y}]} s_0 y,\\
\mathrm{Equiv}(x, y, f) &:\equiv \mathrm{Linv}(x, y, f) \times \mathrm{Rinv}(x, y, f).
\end{aligned}$$

Translating this into categorical notions, for $i \in \{0, 1\}$ we obtain the fibrations of left- and right-invertible maps $\mathrm{Inv}_0 X := \mathrm{Linv} X$ and $\mathrm{Inv}_1 X := \mathrm{Rinv} X$ over X_1 respectively by the following sequence of constructions.

First recall that type theoretic substitution is modelled by pullback and hence the substitution $X_1[\frac{a}{x}, \frac{b}{y}]$ for any two generalized elements $a, b \colon C \to X_0$, $C \in \mathbb{C}$, corresponds to the pullback $X_1(a, b)$ of X_1 along $(a, b) \colon C \to X_0 \times X_0$. Then let $\mathcal{B}_i$ be the object of triples (f, g, h) for elements $x_0, x_1 : X$ and maps $f : X_1(x_0, x_1)$, $g : X_1(x_1, x_0)$ and $h : X(x_i, x_i)$ obtained from the diagram below.

(6.2.2)
$$\begin{tikzcd}
\mathcal{B}_i := \pi_1^* d_{1-i}^* \Delta^* X_1 \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & d_{1-i}^* \Delta^* X_1 \arrow[d, two heads] & \\
X_1 \times_{X_0^2} X_1 \arrow[r, two heads, "\pi_1"] \arrow[d, two heads, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & X_1 \arrow[d, two heads, "{(d_1, d_0)}"'] & \Delta^* X_1 \arrow[l] \arrow[d, two heads] \arrow[dl, phantom, "\llcorner"] \\
X_1 \arrow[r, two heads, "{(d_0, d_1)}"'] & X_0^2 & X_0 \arrow[l, "\Delta"]
\end{tikzcd}$$

So the objects of $\mathcal{B}_i$ essentially consist of not necessarily commuting triangles (f, g, h) in X such that the 1-boundary h is an endomorphism on either the domain of f or g.

Denote $\bar{\pi}_i \colon X_1 \times_{X_0^2} X_1 \to X_1 \times_{X_0} X_1$ the canonical map induced by the projection $\pi_i \colon X_0 \times X_0 \to X_0$ for $i \in \{0, 1\}$. We obtain parallel maps ∂_i and ρ_i as in the diagram

(6.2.3)
$$\begin{tikzcd}
\mathcal{B}_i \arrow[rrd, two heads, bend right] & \bar{\pi}_i^* X_2 \arrow[l, shift right, "\rho_i"'] \arrow[l, shift left, "\partial_i"] \arrow[rr, "p_1"] \arrow[d, two heads, "p_2"'] \arrow[drr, phantom, "\lrcorner" very near start] & & X_2 \arrow[d, two heads, "\zeta_2"] \\
& X_1 \times_{X_0^2} X_1 \arrow[rr, "\bar{\pi}_i"'] & & X_1 \times_{X_0} X_1
\end{tikzcd}$$

for $\partial_i = (p_2, (d_{2(1-i)}, d_1)p_1)$ mapping a 2-simplex in $\bar{\pi}_i^* X_2$ of the form

(6.2.4)
$$\begin{tikzcd}
& y \arrow[dr, bend left, "g"] \arrow[d, phantom, "\sigma"] & \\
x \arrow[ur, bend left, "f"] \arrow[rr, bend right, "h"'] & {} & x
\end{tikzcd}$$

to its boundary, and $\rho_i := (p_2, (\mathrm{id}, s_0 d_{1-i}) d_{2(1-i)} p_1)$ mapping such a 2-simplex to the tuple $(f, g, s_0 x)$ (or $(f, g, s_0 y)$ respectively). Then we define $\mathrm{Inv}_i X$ to be the object of paths from the "composition" ∂_i to the "degeneracy" ρ_i as given by the following pullback on the right hand side.

(6.2.5)
$$\begin{tikzcd}
\mathrm{Eq}(\partial_i, \rho_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Inv}_i X \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \bar{\pi}_i^* X_2 \arrow[d, "(\partial_i, \rho_i)"'] \\
\mathcal{B}_i \arrow[r, hook, "\sim"] \arrow[rr, bend right, "\Delta"'] & P_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_i \arrow[r, two heads] & \mathcal{B}_i \times_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_i
\end{tikzcd}$$

We define $\mathrm{Equiv}(X) := \mathrm{Linv}X \times_{X_1} \mathrm{Rinv}X$ and obtain a sequence of fibrations

$$\mathrm{Equiv}(X) \twoheadrightarrow X_1 \overset{(d_1, d_0)}{\twoheadrightarrow} X_0^2.$$

Remark 6.2.4. Note that pullback along $\pi_1 d_{i-1}$ preserves path objects and hence we also obtain two pullback squares

$$
\begin{tikzcd}
\mathrm{Eq}(\partial_i,\rho_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Inv}_i X \arrow[rr, two heads] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \bar{\pi}_i^* X_2 \arrow[d, "d_1(\partial_i,\rho_i)"'] \\
\Delta^* X_1 \arrow[r, hook, "\sim"] \arrow[rr, bend right, "\Delta"'] & P_{X_0}\Delta^* X_1 \arrow[r, two heads] & \Delta^* X_1 \times_{X_0} \Delta^* X_1. &
\end{tikzcd}
\tag{6.2.6}
$$

for

$$d_1(\partial_i,\rho_i)\colon \bar{\pi}_i^* X_2 \to \mathcal{B}_i \times_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_i \to \Delta^* X_1 \times_{X_0} \Delta^* X_1$$

the canonical composition which takes a triangle of the form (6.2.4) to the pair $(h, s_0 x)$ (respectively $(h, s_0 y)$).

If $X \in s\mathbb{C}$ is a Segal object and so the acyclic fibration ξ_2 allows a section ζ with induced composition κ given by $d_1\zeta\colon X_1 \times_{X_0} X_1 \to X_1$, we obtain an equivalence of the type $\mathrm{Equiv}X(x,y)$ to the type

$$\sum_{f:X_1(x,y)} \Big(\sum_{g:X_1[\frac{y}{x},\frac{x}{y}]} \big(\kappa(f,g) =_{X_1[x,\frac{x}{y}]} s_0 x\big) \times \sum_{h:X_1[\frac{y}{x},\frac{x}{y}]} \big(\kappa(h,f) =_{X_1[\frac{y}{x},y]} s_0 y\big)\Big).$$

Definition 6.2.5. Let X be a sufficiently fibrant simplicial object. We say X is *univalent* if the lift

$$s_0\colon X_0 \to \mathrm{Equiv}(X)$$

of the degeneracy map is a homotopy equivalence in $\mathbb{C}$.

By definition, we have the following comparison.

Lemma 6.2.6. *Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{C}$. Then p is univalent in the sense of Definition 1.5.1 if and only if the Segal object Np is univalent in the sense of Definition 6.2.5.*

Proof. Both univalence of Np and univalence of p exactly ask for the map

$$
\begin{tikzcd}
B \arrow[r, hook, "r"] \arrow[d, equal] & \mathrm{Eq}p \cong \left(\sum_{a,b:B} E_a \simeq E_b\right) \arrow[d, equal] \\
(Np)_0 \arrow[r, hook, "s_0"] & \mathrm{Equiv}(Np)
\end{tikzcd}
$$

to be a homotopy equivalence. The existence of the right hand side isomorphism follows along the lines of Remark 6.2.4 since the 2-Segal map ξ_2 is an isomorphism.

□

6.3. Completeness of simplicial objects

In this section we want to generalize the notion of completeness as defined by Rezk in [**44**] for Segal spaces to Segal objects in the type theoretic fibration category $\mathbb{C} = \mathbb{M}^f$. The chosen generalization of Rezk's original definition is motivated by the understanding that completeness in its very essence is a tool to interpret the homotopy theory of quasi-categories in the homotopy theory of simplicial objects over $\mathbb{M}$ in a sense to be elaborated in this section.

Refreshing some notation from Chapter 5, recall the walking isomorphism

$$I := \left(\bullet \underset{\cong}{\rightleftarrows} \bullet \right) \in \mathbf{Gpd}.$$

Its nerve $J \in \mathbf{S}$ is an interval object in the model category $(\mathbf{S}, \mathrm{Qcat})$ for quasi-categories and has the property that a mid fibration $p\colon X \to Y$ (in the language of [**32**, 1]) between quasi-categories is a fibration in $(\mathbf{S}, \mathrm{Qcat})$ if and only if it has the right lifting property against the endpoint inclusion $c\colon \Delta^0 \to J$. In light of the guiding understanding of completeness referred to above, it turns out that these are the two relevant properties of J which induce a sensible definition of completeness for Segal spaces.

Although J presents the walking isomorphism in the quasi-category of quasi-categories, its simplicial structure is rather complicated; Rezk has observed in [**44**, Section 11] that J is an $< \omega$-sequential colimit $\bigcup_{n\geq 1} J^{(n)}$ as will be explained in Section 6.5, where the finite trunks $J^{(n)}$ at least intuitively present the walking higher half adjoint equivalences with some degenerate side conditions as will be explained in Remark 6.5.2 .

We therefore consider the simplicial set $K := J^{(2)} \sqcup_{\Delta^1} J^{(2)}$ instead, where $J^{(2)} \subset J$ is given by the pushout

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, hook] \arrow[d, "{(f,s_0 0)}"'] & \Delta^2 \arrow[d] \\ \Delta^1 \arrow[r, hook, "f"'] & J^{(2)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{6.3.1}$$

and denotes the "walking" left invertible map

$$\begin{tikzcd} & 1 \arrow[dr, bend left, "f"] & \\ 0 \arrow[ur, bend left, "g"] \arrow[rr, bend right, "s_0 0"'] & & 0 \end{tikzcd}$$

with exactly one non-degenerate 2-simplex as depicted above. And so $K \in \mathbf{S}$ is the "walking" biinvertible map

$$\begin{tikzcd} & 1 \arrow[rr, bend left, "s_0 1"] \arrow[dr, "f"] & & 1 \\ 0 \arrow[ur, "g"] \arrow[rr, bend right, "s_0 0"'] & & 0. \arrow[ur, "h"'] & \end{tikzcd}$$

Lemma 6.3.1.

(1) *The end point inclusion $\Delta^0 \hookrightarrow K$ is a weak categorical equivalence. In particular, the inclusion $K \hookrightarrow J$ is a weak categorical equivalence.*

(2) *A mid fibration $p\colon X \to Y$ between quasi-categories is a fibration in* $(\mathbf{S}, \mathrm{Qcat})$ *if and only if it has the right lifting property against the endpoint inclusion $\Delta^0 \hookrightarrow K$.*

Proof. For part (1), note that the fundamental category $\tau_1(K)$ is a groupoid. Taking a quasi-categorical replacement $\mathbb{R}K$, by 2-for-3 it suffices to show that the induced inclusion $1 \hookrightarrow \mathbb{R}K$ is a weak categorical equivalence. Since the functor τ_1 takes weak categorical equivalences to equivalences of categories, we see that $\mathbb{R}K$ is a Kan complex. So it suffices to show that $\mathbb{R}K$ is contractible (in the Quillen model structure). But it is easy to see that $J^{(2)}$ is contractible and hence the pushout K of $J^{(2)}$ with itself over the contractible object Δ^1 is contractible, too.

For part (2), we know that a map $p\colon X \to Y$ between quasi-categories is a fibration in $(\mathbf{S}, \mathrm{Qcat})$ if and only if it has the right lifting property against the set $\{h_i^n \mid 0 < i < n\}$ of inner horn inclusions and the endpoint inclusion $\Delta^0 \to J$. The fact that a mid fibration between quasi-categories has the right lifting property against $\Delta^0 \hookrightarrow K$ if and only if it has the right lifting property against the composite endpoint inclusion $\Delta^0 \to K \hookrightarrow J$ follows directly from part (1). $\square$

Lemma 6.3.1 justifies a definition of completeness for Segal objects induced by K in the following way. Given a simplicial object X in $\mathbb{M}$, again consider its right Kan extension

$$\begin{tikzcd} \Delta^{op} \arrow[r, "X"] \arrow[d, "y^{op}"'] & \mathbb{M} \\ (\mathbf{S})^{op}, \arrow[ur, dashed, "\_ \setminus X := \mathrm{Ran}_{y^{op}} X"'] & \end{tikzcd}$$

given point-wise by the formula $A \setminus X := \lim_{(\Delta^n/A) \in \mathbf{S}} X_n$. The functor $\_ \setminus X$ comes with a left adjoint $X/\_$, and being the unique limit preserving extension of

X along y^{op}, it is easy to see that this formula for $\_ \setminus X$ coincides with the end

$$A \setminus X := \int_{[n]\in\Delta} X_n^{A_n}$$

as defined in [**32**, Section 7]. Hence, the functors $\_ \setminus X \colon \mathbf{S}^{op} \to \mathbb{M}$ and $X/\_ \colon \mathbb{M}^{op} \to \mathbf{S}$ are induced by the parametrized right adjoints of the box product $\Box\colon \mathbf{S}\times\mathbb{M} \to s\mathbb{M}$ defined in [**32**, Section 7].

Definition 6.3.2. Let X be a Reedy fibrant simplicial object in $\mathbb{M}$. We say that X is *complete* if the map $K \setminus X \to X_0$ induced by the endpoint inclusion $\Delta^0 \hookrightarrow K$ is an acyclic fibration.

By construction, for every simplicial object X in $\mathbb{M}$ we have isomorphisms $\Delta^n \setminus X \cong X_n$, $\partial\Delta^n \setminus X \cong M_n X$ where $M_n X$ is the n-th matching object of X, and $\Lambda^n_i \setminus X \cong (X_{1/X_0})^n_{\mathrm{S}}$. Hence, one can show that the right Kan extension $\_ \setminus X \colon \mathbf{S}^{op} \to \mathbb{M}$ takes

- boundary inclusions (and hence all monomorphisms) in $\mathbf{S}$ to fibrations in $\mathbb{M}$ if and only if X is Reedy fibrant;
- furthermore inner horn inclusions (and hence all mid anodyne morphisms) in $\mathbf{S}$ to acyclic fibrations if and only if X is a Reedy fibrant Segal object;
- furthermore $1 \to K$ to an acyclic fibration if and only if X is a Reedy fibrant complete Segal object;
- furthermore left horn inclusions to acyclic fibrations if and only if X is a Reedy fibrant complete Bousfield-Segal object (we can take this as a definition).

Since the functor $\_ \setminus X$ naturally comes with a left adjoint $X/\_$, we obtain a one to one correspondence between

(1) Reedy fibrant complete Segal objects X in $\mathbb{M}$ and Quillen adjunctions $(F, G) \colon (\mathbf{S}, \mathrm{Qcat}) \to \mathbb{M}^{op}$,
(2) Reedy fibrant complete Bousfield-Segal objects X in $\mathbb{M}$ and Quillen adjunctions $(F, G) \colon (\mathbf{S}, \mathrm{Kan}) \to \mathbb{M}^{op}$.

To see this in the case (1) for a given Reedy fibrant complete Segal object X in $\mathbb{M}$, by the above observation the left adjoint $\_ \setminus X \colon (\mathbf{S}, \mathrm{QCat}) \to \mathbb{M}^{op}$ preserves cofibrations and takes both inner horn inclusions and the endpoint inclusion $1 \to K$ to acyclic cofibrations. But since this set of maps generates quasi-fibrations between quasi-categories, this implies that the right adjoint $X/\_$ preserves fibrations between fibrant objects. This implies that $X/\_$ is a right Quillen functor e.g. by [**32**, Proposition 7.15].

In particular, whenever Reedy fibrant complete (Bousfield)-Segal spaces arise as the fibrant objects of a model structure on $s\mathbb{M}$ we obtain a Quillen equivalent model structure on the functor category $\mathrm{Fun}(\mathbf{S}, \mathbb{M}^{op})$ whose fibrant objects are exactly the Quillen pairs from the Joyal model structure (or the Quillen model structure respectively) to $\mathbb{M}^{op}$.

Moreover, the box product $\square\colon \mathbf{S}\times\mathbb{M}\to s\mathbb{M}$ induces adjoint pairs

$$1\,\square\,\_ :\ \mathbb{M} \longleftrightarrow s\mathbb{M}\ : 1/\_$$

and

$$\_\,\square\, 1:\ \mathbf{S} \longleftrightarrow s\mathbb{M}\ :\ \_\setminus 1.$$

The former is a Quillen pair between the Reedy model structure on $s\mathbb{M}$ and $\mathbb{M}$ by [**32**, Proposition 7.37].

Example 6.3.3. While Rezk defines completeness for Segal spaces via acyclicity of the map $c\setminus X\colon J\setminus X\to X_0$, it is not hard to see that this is equivalent to Definition 6.3.2. In this case the functor $\_\,\square\,1\colon \mathbf{S}\to s\mathbf{S}$ with right adjoint $1\setminus\_$ is the pair (p_1^*, ι_1^*) from Section 4.2. Left Bousfield localizing the Reedy model structure on $s\mathbf{S}$ at the set

$$\{h_i^n\,\square\,1 \mid 0<i<n\}$$

yields the model category $(s\mathbf{S},\mathrm{S})$ whose fibrant objects are the Segal spaces by [**32**, Proposition 3.4]. Localizing $(s\mathbf{S},\mathrm{S})$ additionally at either $C:=\{(1\to J)\,\square\,1\}$ or $K:=\{(1\to K)\,\square\,1\}$ yields model categories $(s\mathbf{S},\mathrm{CS})$ and $(s\mathbf{S},\mathrm{KS})$ respectively, in both cases with a right Quillen functor $1\setminus\_$ to the Joyal model structure $(\mathbf{S},\mathrm{Qcat})$ by Lemma 6.3.1.(2). But this implies that the maps $1\to K$ and $1\to J$ are mapped to weak equivalences in both localizations respectively by Lemma 6.3.1.(1) and so the model structures in fact coincide. In particular, a Segal space is complete in the sense of Definition 6.3.2 if and only if it is C-local. This recovers Rezk's original definition of completeness over simplicial sets.

6.4. Comparison of univalence and completeness

Let $X\in s\mathbb{C}$ be a Segal object. In this section we want to show that X is univalent if and only if some – and hence any – Reedy fibrant replacement $\mathbb{R}X$ of X is complete. As a corollary we will obtain a comparison between the usual notion of univalence for a fibration in $\mathbb{C}$ and completeness of its associated Reedy fibrant Segal object. Therefore we have to analyse the object

$$K \setminus X = \lim((\Delta/K)^{op} \twoheadrightarrow (\Delta)^{op} \xrightarrow{X} \mathbb{C}).$$

For every Segal object X in $\mathbb{C}$, we have isomorphisms $\Delta^0 \setminus X \cong X_0$, $\Delta^1 \setminus X \cong X_1$ and $J^{(2)} \setminus X \cong \mathrm{Eq}(d_1, s_0 d_1 d_2)$ for the two maps

(6.4.1)
$$\begin{tikzcd} X_2 \arrow[rrr, "d_1"] \arrow[dr, "d_2"'] & & & X_1 \\ & X_1 \arrow[r, "d_1"'] & X_0 \arrow[ur, "s_0"'] & \end{tikzcd}$$

in $\mathbb{C}$. Note that this equalizer is isomorphic to the equalizer $\mathrm{Eq}(\partial_0, j_0)$ of Diagram (6.2.6). Thus $J^{(2)} \setminus X$ consists of the maps in X which are postcomposable to the identity, while the object $\mathrm{Linv}X$ defined in Diagram (6.2.6) contains maps in X which are postcomposable to a map homotopic to the identity in X_1. We will see that this discrepancy vanishes for Reedy fibrant Segal objects. Therefore, consider the following intermediate lemma.

Lemma 6.4.1. *For every Segal object $X \in s\mathbb{C}$ there is a point-wise homotopy equivalent Reedy fibrant Segal object $\tilde{X}$ in $\mathbb{C}$ with a homotopy equivalence* $\mathrm{Linv}X \simeq J^{(2)}/\tilde{X}$ *over* $X_1 \times_{X_0^2} X_1 = \tilde{X}_1 \times_{\tilde{X}_0^2} \tilde{X}_1$.

Proof. We define $\tilde{X}$ following the general recursive construction of Reedy fibrant replacements. We start with $(\tilde{X})_0 = X_0$ and $(\tilde{X})_1 = X_1$ with the same boundaries and the same degeneracy, since X is sufficiently fibrant. At the next degree, instead of taking an arbitrary factorization of $\partial := (d_2, d_1, d_0)\colon X_2 \to M_2X$ with $M_2X = M_2(\tilde{X})$ as usual, we factor the map

$$(\partial, s_0 d_1 d_2)\colon X_2 \to M_2X \times_{X_0} \Delta^* X_1$$

into an acyclic cofibration followed by a fibration,

(6.4.2)
$$\begin{tikzcd} & (\tilde{X})_2 \arrow[dr, two heads, "p_2"] & & \\ X_2 \arrow[ur, hook, "\sim"] \arrow[rr, "(\partial, s_0 d_1 d_2)"'] & & M_2X \times_{X_0} \Delta^* X_1 \arrow[r, two heads, "\pi_1"'] & M_2X. \end{tikzcd}$$

Obviously, $(\tilde{X})_2$ also factors the boundary map $\partial\colon X_2 \to M_2X$. We continue choosing the $(\tilde{X})_n$ for $n \geq 3$ inductively by standard procedure with no further restrictions. Clearly, $\tilde{X}$ is still a Segal object in $\mathbb{C}$. Now, recall the object

$\mathcal{B}_0 := (\Delta d_1\pi_1)^* X_1$ of 2-boundaries between vertices of the form (x, y, x) from Diagram (6.2.2) and similarly the forgetful map

$$\bar{\pi}_0 \colon X_1 \times_{X_0^2} X_1 \to X_1 \times_{X_0} X_1$$

induced by the projection $\pi_0 \colon X_0 \times X_0 \to X_0$ onto the first component. It is easy to see that $\mathcal{B}_0$ fits into the diagram

$$\begin{tikzcd}
\bar{\pi}_0^*(\tilde{X})_2 \arrow[r, two heads] \arrow[dr, dashed, two heads] & \mathcal{B}_0 \times_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_0 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & M_2X \times_{X_0} \Delta^* X_1 \arrow[d, two heads] & (\tilde{X})_2 \arrow[l, two heads, "p_2"'] \arrow[dl, dashed, two heads, "\tilde{\partial}"] \\
\bar{\pi}_0^* X_2 \arrow[u, hook, "\wr"] \arrow[dr, two heads, "\sim"] & \mathcal{B}_0 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & M_2X \arrow[d, two heads, "\pi_{\hat{2}}"] & X_2 \arrow[u, hook, "\wr"'] \arrow[dl, two heads, "\sim"', "\xi_2"] \\
& X_1 \times_{X_0^2} X_1 \arrow[r, "\bar{\pi}_0"] & X_1 \times_{X_0} X_1.
\end{tikzcd}$$

where the four maps on the left hand side denote the pull back of the respective right hand side maps along $\bar{\pi}_0$. We make the following observations. First, the composition

$$\bar{\pi}_0^* X_2 \to \bar{\pi}_0^*(\tilde{X})_2 \to \mathcal{B}_0 \times_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_0$$

yields a factorization of (∂_0, j_0) as given in Diagram (6.2.5) by choice of the factorization in (6.4.2).

Second, again by Diagram (6.2.5) when considered for the Segal object $\tilde{X}$, we obtain objects $\tilde{\mathcal{B}}_i = \mathcal{B}_i$ and see that $J^{(2)} \setminus \tilde{X}$ is the equalizer of the maps

$$\begin{tikzcd}
\bar{\pi}_0^*(\tilde{X})_2 \arrow[r, two heads, "\bar{\pi}_0^* p_2"] \arrow[rr, bend left, "\tilde{\partial}_0"] \arrow[rr, bend right, "\tilde{j}_0"'] & \mathcal{B}_0 \times_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_0 \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathcal{B}_0.
\end{tikzcd}$$

Therefore, we see that whether we construct $\mathrm{Linv} X$ or $J^{(2)} \setminus \tilde{X}$ is a matter of factoring either of the two legs of the square spanned by $\mathcal{B}_0$, $\mathcal{B}_0 \times_{X_1 \times_{X_0^2} X_1} \mathcal{B}_0$ and

$X_1 \times_{X_0^2} X_1$ in the diagram

$$\begin{tikzcd}
\mathrm{Eq}(\partial_0, j_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Linv}X \arrow[r, two heads] \arrow[d, hook, "\wr"'] \arrow[dr, phantom, "\lrcorner"] & \bar{\pi}_0{}^* X_2 \arrow[d, hook, "\wr"'] \\
J^{(2)}/\tilde{X} \arrow[r, hook, "\sim"'] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \bar{\pi}_0{}^*(\tilde{X})_2 \arrow[d, two heads] \\
\mathcal{B}_0 \arrow[r, hook, "\sim"] \arrow[d] \arrow[rr, bend right, "\Delta"] \arrow[dr, phantom, "\lrcorner"] & P_{(X_1 \times_{X_0^2} X_1)}(\mathcal{B}_0) \arrow[r, two heads] & \mathcal{B}_0 \times_{(X_1 \times_{X_0^2} X_1)} \mathcal{B}_0 \arrow[d] \\
\Delta^* X_1 \arrow[rr, "\Delta"] & & \Delta^* X_1 \times_{X_0} \Delta^* X_1 .
\end{tikzcd}$$

The maps $J^{(2)} \setminus \tilde{X} \to \bullet \leftarrow \mathrm{Linv}X$ are homotopy equivalences (over $X_1 \times_{X_0^2} X_1$) by the Frobenius property of $\mathbb{C}$. We thus obtain a homotopy equivalence

$$J^{(2)} \setminus \tilde{X} \simeq \mathrm{Linv}X$$

over $X_1 \times_{X_0^2} X_1$. $\square$

It is reasonable to expect that the constructions used to define univalence and completeness are homotopy invariant, and they indeed are in the following sense.

Lemma 6.4.2. *Suppose X, $Y \in s\mathbb{C}$ are sufficiently fibrant and point-wise homotopy equivalent. Then*

(1) $\mathrm{Inv}_i X \simeq \mathrm{Inv}_i Y$ *for $i \in \{0, 1\}$;*

(2) if X and Y further are Reedy fibrant, then $J^{(2)} \setminus X \simeq J^{(2)} \setminus Y$.

Proof. Part (1) is easily seen by chasing through Diagram (6.2.2) and exploiting the Frobenius property. For part (2), given an equivalence $X \xrightarrow{\simeq} Y$, we obtain an equivalence between the pullbacks

$$\begin{tikzcd}[row sep=scriptsize, column sep=scriptsize]
& J^{(2)} \setminus Y \arrow[rr] \arrow[dd, two heads] \arrow[ddrr, phantom, "\lrcorner"] & & Y_2 \arrow[dd, two heads] \\
J^{(2)} \setminus X \arrow[ur, "\simeq"] \arrow[rr, crossing over] \arrow[dd, two heads] \arrow[ddrr, phantom, "\lrcorner"] & & X_2 \arrow[ur, "\simeq"] & \\
& Y_1 \arrow[rr] & & \Lambda_0^2 \setminus Y \\
X_1 \arrow[ur, "\simeq"] \arrow[rr] & & \Lambda_0^2 \setminus X . \arrow[ur, "\simeq"] \arrow[from=uu, crossing over, two heads] &
\end{tikzcd}$$

Just note that here we use that both functors $\_ \setminus X$ and $\_ \setminus Y : \mathbf{S}^{op} \to \mathbb{C}$ send the 2-horn inclusion $\Lambda^2_0 \hookrightarrow \Delta^2$ to a fibration, which does not follow from only sufficient fibrancy. $\square$

Proposition 6.4.3. *Let $X \in s\mathbb{C}$ be a Segal object in $\mathbb{C}$. Then for any Reedy fibrant replacement $\mathbb{R}X$ of X, we have a homotopy equivalence*

$$\mathrm{Linv}X \simeq J^{(2)} \setminus \mathbb{R}X.$$

Dually, we obtain a homotopy equivalence between $\mathrm{Rinv}X$ *and* $J^{(2)} \setminus \mathbb{R}X$ *over* $X_1 \times_{X_0^2} X_1$ *with correspondingly permuted source and target maps.*

Proof. Let X be a Segal object and let $\mathbb{R}X$ be a Reedy fibrant replacement of X. By Lemma 6.4.2.1, we obtain a homotopy equivalence

$$\mathrm{Linv}X \simeq \mathrm{Linv}RX.$$

Then consider $(\tilde{\mathbb{R}X})$, the object from Lemma 6.4.1 constructed from $\mathbb{R}X$ (instead from X) such that $\mathrm{Linv}\mathbb{R}X \simeq J^{(2)} \setminus (\tilde{\mathbb{R}X})$. But both $\mathbb{R}X$ and $(\tilde{\mathbb{R}X})$ are Reedy fibrant, hence Lemma 6.4.2.2 finishes the proof. $\square$

We conclude this section with the main result.

Theorem 6.4.4. *Let $X \in s\mathbb{C}$ be a Segal object. Then the following are equivalent.*

(1) X is univalent.
(2) For any Reedy fibrant replacement $\mathbb{R}X$ of X, $\mathbb{R}X$ is complete.

Proof. Given a Segal object $X \in s\mathbb{C}$ together with a Reedy fibrant replacement $X \overset{\sim}{\hookrightarrow} \mathbb{R}X$ in $s\mathbb{C}$, by Lemma 6.4.2 and Proposition 6.4.3 we obtain a diagram

$$\begin{tikzcd}
 & \mathrm{Rinv}X \arrow[rr, "\simeq"] \arrow[dd, two heads] & & \mathrm{Rinv}\mathbb{R}X \arrow[rr, "\simeq"] \arrow[dd, two heads] & & J^{(2)} \setminus \mathbb{R}X \arrow[dll, bend left, two heads, "d_2\setminus\mathbb{R}X"] \\
\mathrm{Linv}X \arrow[rr, "\simeq"] \arrow[dr, two heads] & & \mathrm{Linv}\mathbb{R}X \arrow[rr, "\simeq"] \arrow[dr, two heads] & & J^{(2)} \setminus \mathbb{R}X \arrow[dl, two heads, "d_0/\mathbb{R}X"'] & \\
 & X_1 \arrow[rr, hook, "\sim"] & & \mathbb{R}X_1 & &
\end{tikzcd}$$

of equivalences between fibrations over X_1 and $\mathbb{R}X_1$ respectively. This induces a homotopy equivalence on pullbacks

$$\mathrm{Equiv}X \xrightarrow{\simeq} \mathrm{Equiv}\mathbb{R}X \xrightarrow{\simeq} (J^{(2)} \setminus \mathbb{R}X \times_{\mathbb{R}X_1} J^{(2)} \setminus \mathbb{R}X) \cong K \setminus \mathbb{R}X$$

over $X_1 \overset{\sim}{\hookrightarrow} \mathbb{R}X_1$. Therefore

$$\begin{tikzcd} \mathrm{Equiv}X \arrow[r, "\simeq"] \arrow[d, two heads] & \mathrm{Equiv}\mathbb{R}X \arrow[r, "\simeq"] \arrow[d, two heads] & K \setminus \mathbb{R}X \arrow[dl, bend left, two heads] \\ X_0 \arrow[r, hook, "\sim"] & \mathbb{R}X_0 & \end{tikzcd}$$

commutes, too, and the statement follows directly from 2-for-3. □

Corollary 6.4.5. *Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{C}$. Then the following are equivalent.*

(1) p is univalent in $\mathbb{C}$;
(2) Np is univalent;
(3) For any Reedy fibrant replacement $\mathbb{R}Np$ of Np, $\mathbb{R}Np$ is complete.

Proof. Follows immediately from Lemma 6.2.6 and Theorem 6.4.4. □

Remark 6.4.6. We just have seen that every univalent fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}$ induces a complete Segal object Np and hence a Quillen pair

$$\_ \setminus \mathbb{R}Np\colon (\mathbf{S}, \mathrm{Qcat}) \rightleftarrows \mathbb{M}^{op}\colon \mathbb{R}Np/\_$$

by the general considerations from Section 6.3. It would be interesting to find out whether we can find conditions on the univalent fibration p such that the induced left Quillen functor $\_ \setminus \mathbb{R}Np\colon (\mathbf{S}, \mathrm{Qcat}) \to \mathbb{M}^{op}$ is part of a Quillen equivalence. This naturally arising question appears to be connected to the considerations in [**54**] if one resolves the contravariance which arises since Toën considers a cosimplicial version of $\_ \setminus X$.

6.5. The special case of Segal spaces

In the special case that the type theoretic fibration category $\mathbb{C}$ is the category of Kan complexes, there is a direct way to prove homotopy equivalence of the objects $J \setminus X$ for J the nerve of the walking isomorphism and $\mathrm{Equiv}X$ for every Segal space $X \in s\mathbf{S}$. In fact much of the technical aspects of this discussion is already contained in [**44**, Section 11].

Recall that the simplicial set J is the 0-coskeleton of the discrete category $\{0, 1\}$ and hence possesses exactly two non-degenerate n-simplices in each degree n, corresponding to the two distinct alternating sequences in the letters 0 and 1 of length n. Rezk observed in [**44**, 11] that J possesses a filtration $J = \bigcup_{n\in\mathbb{N}} J^{(n)}$

with $J^{(1)} = \Delta^1$ and

$$
\begin{tikzcd}
\Lambda_0^{n+1} \arrow[r, hook] \arrow[d] & \Delta^{n+1} \arrow[d] \\
J^{(n)} \arrow[r, hook] & J^{(n+1)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\tag{6.5.1}
$$

for $n \geq 1$ where the vertical map $\Lambda_0^{n+1} \to J^{(n)}$ is the unique left horn in $J^{(n)}$ whose spine corresponds to the unique alternating sequence in the letters 0 and 1 of length n starting with 0. Note that each $J^{(n)}$ consists of exactly two non-degenerate m-simplices for $m < n$ and exactly one non-degenerate n-simplex freely filling the left horn in (6.5.1). Indeed, the freely added 0-boundary of this n-cell gives the priorly missing second $(n-1)$-cell.

Furthermore, recall from Example 4.5.4 the B-space $\mathrm{Core}(X)$ associated to a Segal space X. By pullback stability of finite limits and the fact that the natural map $\mathrm{Core}X \to X$ is a monomorphism which factors the fibration $K \setminus X \twoheadrightarrow X_1$, we obtain diagrams

$$
\begin{tikzcd}
J^{(3)} \setminus \mathrm{Core}(X) \arrow[rrr, "\cong"] \arrow[drr] \arrow[ddr, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & J^{(3)} \setminus X \arrow[dr] \arrow[dd, two heads] & \\
& & K \setminus \mathrm{Core}(X) \arrow[rr, "\cong"] \arrow[dl, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & K \setminus X \arrow[dl, two heads] \\
& \mathrm{Core}(X)_1 \arrow[rr, hook] & & X_1 &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
J^{(n+1)} \setminus \mathrm{Core}(X) \arrow[r, "\cong"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & J^{(n+1)} \setminus X \arrow[d, two heads] \\
J^{(n)} \setminus \mathrm{Core}(X) \arrow[r, "\cong"] & J^{(n)} \setminus X
\end{tikzcd}
$$

for every $n \geq 3$. So we can reduce the comparison of $K \setminus X$ and $J \setminus X$ to the case when $X = \mathrm{Core}(X)$ is a B-space. But the maps $J^{(n+1)} \setminus \mathrm{Core}(X) \to J^{(n)} \setminus \mathrm{Core}(X)$ are acyclic fibrations by Diagram (6.5.1), since $\mathrm{Core}(X)$ is a Bousfield-Segal space so that the functor $\_ \setminus \mathrm{Core}(X)\colon \mathbf{S}^{op} \to \mathbf{S}$ takes left horn inclusions to acyclic fibrations. This shows that the homotopy limit $J \setminus \mathrm{Core}(X)$ of the inverse sequence $(J^{(n)} \setminus X | n \geq 2)$ is homotopy equivalent to $J^{(n)} \setminus \mathrm{Core}(X)$ for all $n \geq 3$ and so we are left to show that the map

$$J^{(3)} \setminus \mathrm{Core}(X) \to K \setminus \mathrm{Core}(X)$$

is a homotopy equivalence. But note that both objects are contractible over $\mathrm{Core}(X)_1$, because $\_ \setminus \mathrm{Core}(X)$ sends the inclusions $\Delta^1 \to J^{(3)}$ and $\Delta^1 \to K$ to acyclic fibrations. This proves that $J \setminus X$ and $\mathrm{Equiv}X$ are homotopy equivalent over X_1 having shown that $K \setminus X$ and $\mathrm{Equiv}X$ are homotopy equivalent in the proof of Theorem 6.4.4.

Corollary 6.5.1. *For every Segal space X, the objects $\mathrm{Equiv}X$, $J^{(n)} \setminus X$ for $n \geq 3$ and $J \setminus X$ are pairwise homotopy equivalent over X_1. In particular, X is complete if and only if it is univalent in the sense of Definition 6.2.5.*

Remark 6.5.2. Given a Kan fibration $p\colon E \twoheadrightarrow B$ we know from [**41**, Section 4.2] that the generic type family $\mathrm{Eq}p = \mathrm{Equiv}Np$ of equivalences associated to p is homotopy equivalent over $\mathrm{Fun}p$ to the generic type family $\mathrm{hae}(p)$ of half-adjoint equivalences over p. We get a homotopy equivalence $J^{(3)} \setminus \mathbb{R}Np \to \mathrm{hae}(p)$, thinking of $J^{(3)} \setminus \mathbb{R}Np$ as the type of "vertically strict" half-adjoint equivalences in $\mathbb{R}Np$. This instance suggests that we obtain equivalences between $J^{(n)} \setminus \mathbb{R}Np$ and the type of respective higher versions of half-adjoint equivalences over p. As the literature does not suggest a parametrized version of such definitions of higher half-adjoint equivalences, the $J^{(n)} \setminus \mathbb{R}Np$ in fact yield a way to define such types. Then the considerations above imply that all these versions are equivalent to one another as expected syntactically in [**41**, Section 4.2].

Remark 6.5.3. All of the arguments presented in this section proving Corollary 6.5.1 generalize to fibrations p and their associated Segal objects in every type theoretic model category $\mathbb{M}$ which satisfies a "semi-strict comprehension" condition. Say $\mathbb{M}$ has *semi-strict comprehension* if for every (-1)-truncated fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{M}$ there is a subobject $\iota\colon B_E \hookrightarrow B$ such that the pullback of p along ι is an acyclic fibration and the pullback of ι along p is an isomorphism as depicted in the diagram below.

$$\begin{tikzcd}
P_E \arrow[r, "\cong"] \arrow[d, two heads, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
B_E \arrow[r, hook] \arrow[u, bend left, dashed] & B
\end{tikzcd}$$

This allows the construction of the Bousfield-Segal object $\mathrm{Core}(X)$ associated to a Segal object X as used here in the case of Kan complexes and Segal spaces. This semi-strict comprehension condition is for example satisfied in the injective model structure (and localizations thereof) on any simplicial presheaf category.

6.6. Univalent completion as Segal completion

The authors of [**6**] introduced a procedure of *univalent completion* of a fibration q to a univalent fibration $u(q)$ in the Quillen model structure on simplicial sets which we recall very briefly below. We can use the correspondence between univalence of Kan fibrations p and completeness of the Segal space $\mathbb{R}Np$ in order to show that the map $Nq \to Nu(q)$ of Segal spaces associated to the univalent completion in [**6**] is a fibrant replacement in the model structure $(s\mathbf{S}, \mathrm{CS})$ of complete Segal spaces.

In a nutshell, given a fibration $p\colon E \twoheadrightarrow B$ in $\mathbf{S}$, we obtain the internal category object $\mathrm{Fun}p$ by Proposition 6.2.2 and a fibration

$$e(p)\colon \mathrm{Eq}p \twoheadrightarrow \mathrm{Fun}p$$

whose image in $\mathrm{Fun}p$ we denote by $\mathrm{Weq}(p)$ (this image is denoted $\mathrm{Eq}p$ in [**6**]). We obtain a subfibration

$$\mathrm{Weq}(p) \twoheadrightarrow B \times B$$

of $\mathrm{Fun}p \twoheadrightarrow B \times B$ which also yields an internal category object $\mathrm{Weq}(p)$ (note that the object $\mathrm{Eq}p$ on the other hand does not give an internal category object). Since the fibration $e(p)$ is (-1)-truncated, the factorisation $e(p)\colon \mathrm{Eq}p \twoheadrightarrow \mathrm{Weq}(p)$ is an acylic fibration. We choose a minimal fibration $m\colon M \twoheadrightarrow B$ inside E such that the inclusion $M \hookrightarrow E$ is a fibrewise deformation retract. Then $\mathrm{Weq}(m)$ is the generic object $\mathrm{Iso}(m)$ of isomorphisms associated to m. So the internal category $\mathrm{Weq}(m)$ is in fact an internal groupoid and comes together with the canonical projection

$$\pi_m\colon \mathrm{Act}(m) \to \mathrm{Iso}(m)$$

from its associated action category (see e.g. [**39**, Section 2] for the construction). This induces a map of Segal objects

$$N\pi_m\colon N\mathrm{Act}(m) \to N\mathrm{Iso}(m).$$

Pushfoward along the diagonal $d^*\colon s\mathbf{S} \to \mathbf{S}$ yields the classifying space construction $B := d^*N$ and a Kan fibration of Kan complexes

$$B(\mathrm{Act}(m)) \twoheadrightarrow B(\mathrm{Iso}(m)).$$

This is an explicit description of the universal $\mathrm{Iso}(m)$-bundle $E(\mathrm{Iso}(m)) \twoheadrightarrow B(\mathrm{Iso}(m))$. We obtain a cartesian square

$$\begin{tikzcd} M \arrow[r] \arrow[d, two heads, "m"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B(\mathrm{Act}(m)) \arrow[d, two heads, "B\pi_m"] \\ B \arrow[r, hook, "\iota"'] & B(\mathrm{Iso}(m)) \end{tikzcd}$$

where the fibration $B\pi_m$ is univalent and the map $\iota\colon B \to B(\mathrm{Iso}(m))$ is a monomorphism. In the following we denote the univalent fibration on the right hand side by $u(p)\colon E(p) \twoheadrightarrow B(p)$, so we obtain a homotopy cartesian square of the form

(6.6.1)
$$\begin{tikzcd} E \arrow[r] \arrow[d, two heads, "p"] & E(p) \arrow[d, two heads, "u(p)"] \\ B \arrow[r, hook, "\iota"'] & B(p). \end{tikzcd}$$

This univalent completion induces a Segal completion $Np \to Nu(p)$ in the following way.

Proposition 6.6.1. *For every Kan fibration $p\colon E \twoheadrightarrow B$, the square in Diagram (6.6.1) induces a map*

$$Np \to Nu(p)$$

of Segal objects which is a weak equivalence in the model structure $(s\mathbf{S}, \mathrm{CS})$ of complete Segal spaces. In other words, given Reedy fibrant replacements $\mathbb{R}Np$ and $\mathbb{R}Nu(p)$ of Np and $Nu(p)$ respectively, we obtain a weak equivalence

$$\mathbb{R}Np \to \mathbb{R}Nu(p)$$

in $(s\mathbf{S}, \mathrm{CS})$ between the Segal space $\mathbb{R}Np$ and the complete Segal space $\mathbb{R}Nu(p)$.

Proof. The homotopy cartesian square in (6.6.1) induces the diagram

$$\begin{tikzcd} \mathrm{Fun}(p) \arrow[dr, "\sim", "\eta"'] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, two heads] & & \\ & \mathrm{Fun}(m) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(u(p)) \arrow[d, two heads] \\ & B \times B \arrow[r, "\iota\times\iota"'] & B(p) \times B(p), \end{tikzcd}$$

essentially by the proof of Proposition 1.6.1 and hence a map

$$N(\iota, \eta)\colon Np \to Nu(p).$$

We note that on the sets of objects the map $N(\iota,\eta)_{00}\colon (Np)_{00} \to (Nu(p))_{00}$ is simply the identity

$$B_0 \xrightarrow{\iota_0} B_0$$

and for every pair $b, b' \in B_0$ we have a weak equivalence

$$(b,b')^*\eta\colon [E_b, E_{b'}]_{B\times B} \to [M_b, M_{b'}]_{B\times B}$$

isomorphic to

$$N(\iota,\eta)(b,b')\colon Np(b,b') \to Nu(p)(b,b').$$

We have to show that if we are given Reedy fibrant replacements $Np \overset{\sim}{\hookrightarrow} \mathbb{R}Np$ and $Nu(p) \overset{\sim}{\hookrightarrow} \mathbb{R}Nu(p)$, the induced map

$$\mathbb{R}N(\iota,\eta)\colon \mathbb{R}Np \to \mathbb{R}Nu(p)$$

is a DK-equivalence in the sense of [**44**, Section 7.4]. Indeed, this proves the proposition, because DK-equivalences between Segal spaces are exactly the weak equivalences between Segal spaces in $(s\mathbf{S}, \mathrm{CS})$ by [**44**, Theorem 7.7], and $\mathbb{R}Nu(p)$ is a complete Segal space by Corollary 6.4.5. Without loss of generality, we can choose a Reedy fibrant replacement $\mathbb{R}N(\iota,\eta)\colon \mathbb{R}Np \to \mathbb{R}Nu(p)$ which equals to $N(\iota,\eta)\colon Np \to Nu(p)$ on degrees 0 and 1, since $p\colon E \twoheadrightarrow B$ and $u(p)\colon E(p) \twoheadrightarrow B(p)$ are fibrations between fibrant objects. In order to show that $\mathbb{R}N(\iota,\eta)$ is fully faithful, let $b, b' \in B_0$. Then we have

$$\begin{tikzcd}
\mathbb{R}Np(b,b') \arrow[r, "\mathbb{R}N(\iota,\eta)(b,b')"] & \mathbb{R}Nu(p)(b,b') \\
Np(b,b') \arrow[u, phantom, "\cong" sloped] \arrow[r, "N(\iota,\eta)(b,b')"] & Nu(p)(b,b') \arrow[u, phantom, "\cong" sloped]
\end{tikzcd}$$

which is a weak equivalence as noted above.

It is left to show that $\mathbb{R}N(\iota,\eta)$ induces a bijection on the quotients of B_0 by homotopy equivalence in $\mathbb{R}Np$ and $\mathbb{R}Nu(p)$ respectively. Since $\mathbb{R}N(\iota,\eta)_{00}\colon B_0 \to B_0$ is the identity, its induced map on the quotients clearly is surjective. Towards injectivity, let $b, b' \in B_0$, so we have to show that if b, b' are homotopy equivalent in $\mathbb{R}u(p)$, they also are homotopy equivalent in $\mathbb{R}Np$. But the homotopy equivalence

$$N(\iota,\eta)(b,b')\colon \mathrm{Fun}(p)(b,b') \xrightarrow{\sim} \mathrm{Fun}(u(p))(b,b')$$

restricts to a homotopy equivalence

$$\mathrm{Eq}(Np)(b,b') \xrightarrow{\sim} \mathrm{Eq}(Nu(p))(b,b')$$

by the proof of Lemma 6.2.6 and [**33**, Proposition 3.2.9]. By Proposition 6.4.3, this induces a homotopy equivalence

$$\left(J^{(2)} \setminus \mathbb{R}Np\right)(b, b') \to \left(J^{(2)} \setminus \mathbb{R}Nu(p)\right)(b, b')$$

between the spaces of homotopy equivalences in $\mathbb{R}Np$ and $\mathbb{R}Nu(p)$ respectively. This concludes the proof. $\square$

CHAPTER 7

Universal homotopy colimits

The general objective of this chapter is to study a specific formulation of local cartesian closedness for "presentable homotopy theories" in both the model categorical and the quasi-categorical context. Here, the noun "presentation" has two different but related meanings, and so it has two different but related adjectival derivations. On the one hand, we say that a quasi-category $\mathcal{C}$ *is presented by a model category* $\mathbb{M}$ if there is an equivalence between $\mathcal{C}$ and the *underlying quasi-category* $\mathrm{Ho}_{\infty}(\mathbb{M})$ of $\mathbb{M}$ as defined for example in [**35**, Definition 1.3.4.15]. On the other hand we say that $\mathcal{C}$ is *presentable* if it is presentable in the sense of [**36**, Definition 5.5.0.1], i.e. if it is accessible and admits small colimits. Equivalently, by [**36**, Theorem 5.5.1.1], that is if $\mathcal{C}$ is equivalent to an accessible localization of a presheaf quasi-category.

By a classical result of Dugger in [**16**] and subsequent work of Lurie it is well known that combinatorial model categories present presentable quasi-categories as quoted in Theorem 7.1.1. We consider this correspondence as the foundation of this chapter, so that all statements in the following three sections assume combinatoriality on the one hand and presentability on the other.

In [**45**], Rezk introduced the class of *model toposes*. By [**45**, Theorem 6.9] these can be characterized as those combinatorial model categories which satisfy a homotopical version of the two classical descent properties of Grothendieck toposes. In [**36**, Chapter 6], Lurie studied the class of Grothendieck ∞-toposes which can be characterized analogously as those presentable quasi-categories which satisfy two suitably formulated descent properties. By combination of their work it is also known that a combinatorial model category $\mathbb{M}$ presents a Grothendieck ∞-topos $\mathcal{C}$ if and only if it is a model topos, and vice versa a presentable quasi-category $\mathcal{C}$ is a Grothendieck ∞-topos if and only if it is presented by a model topos.

The authors of [**23**] developed a theory of locally cartesian closed presentable quasi-categories and one of the aims of this chapter is to give a characterization of the class of combinatorial model categories $\mathbb{M}$ which present locally cartesian closed presentable quasi-categories, just as model toposes present Grothendieck ∞-toposes in the sense above. This class will be shown to consist of the combinatorial model categories with *universal homotopy colimits*, that is Toën and Vezzosi's notion in [**55**, Definition 4.9.1.2] or equivalently Rezk's descent property (P1) to be recalled in Definition 7.2.6.

Theorem 7.2.4. A combinatorial model category $\mathbb{M}$ has universal homotopy colimits if and only if its associated quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ has universal colimits.

Moreover, it is shown in [**36**, Chapter 6] that every Grothendieck ∞-topos (defined via descent) is equivalent to a left exact localization of a presheaf quasi-category; Rezk has shown a parallel result for model toposes, stating that a combinatorial model category satisfies his descent properties if and only if it is Quillen equivalent to a left exact left Bousfield localization of a simplicial presheaf category with the projective (or equivalently injective) model structure.

Gepner and Kock have shown in [**23**] that every locally cartesian closed presentable quasi-category is equivalent to a semi-left exact localization of a presheaf quasi-category. Another aim of this chapter is to give the analogous characterization of combinatorial model categories with universal homotopy colimits in the world of combinatorial model categories and left Bousfield localizations. This characterization is given in Theorem 7.3.7. Therefore we will introduce a notion of semi-left exactness for left Bousfield localizations, and show that such Bousfield localizations preserve right properness of model categories in Lemma 7.3.9. One application of this lemma is Corollary 7.3.17 which shows that a combinatorial model category has universal homotopy colimits if and only if its associated presentation, as obtained from Theorem 7.3.7, is right proper. The "injective" version of this statement gives a proof of [**23**, Theorem 7.10] and Cisinski's observation in [**13**], stating that a presentable quasi-category is locally cartesian closed if and only if it is presented by a right proper Cisinski model category.

7.1. Background and definitions

In this section we exclusively consider combinatorial model categories. Hence, $\mathbb{M}$ and $\mathbb{N}$ will always denote combinatorial model categories.

Recall that by the work of Dugger and Lurie combinatorial model categories corresponds to presentable $(\infty, 1)$-categories in the following way.

Theorem 7.1.1 ([**36**, A.3.7.6])**.** *Let $\mathcal{C}$ be a quasi-category. Then the following are equivalent.*

(1) The quasi-category $\mathcal{C}$ is presentable.
(2) There is a combinatorial simplicial model category $\mathbb{M}$ and an equivalence $\mathcal{C} \simeq Ho_\infty(\mathbb{M})$.

For simplicial model categories $\mathbb{M}$, the underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ can be described as the simplicial nerve $\mathrm{N}(\mathbb{M}^\circ)$ of the simplicial category $\mathbb{M}^\circ$ of bifibrant objects in $\mathbb{M}$ as shown in [**35**, Theorem 1.3.4.20]. The proof of the theorem makes use of the fact that localizations of presheaf quasi-categories are presented by left Bousfield localizations of simplicial presheaf categories, and the following fundamental result by Dugger. Therefore, recall that a Quillen pair (L, R) is called a *homotopy localization* if it induces a (reflective) localization on homotopy categories, see [**32**, Definition 7.16].

Theorem 7.1.2 (Dugger's Presentation Theorem, [**16**, Theorem 1.1])**.** *Let $\mathbb{M}$ be a combinatorial model category. Then there is a category $\mathbb{C}$, a homotopy localization*

$$(L, R)\colon \mathrm{sPsh}(\mathbb{C})_{\mathrm{proj}} \rightleftarrows \mathbb{M}$$

and a set T of maps in $\mathrm{sPsh}(\mathbb{C})$ such that the homotopy localization induces a Quillen equivalence

$$(L, R)\colon \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}} \rightleftarrows \mathbb{M}.$$

Recall that a category $\mathbb{C}$ is said to have universal colimits if for every arrow $f\colon C \to D$ in $\mathbb{C}$, the pullback functor

$$f^*\colon \mathbb{C}/D \to \mathbb{C}/C$$

preserves small colimits. In case the category $\mathbb{C}$ is locally presentable, its colimits are universal if and only if it is locally cartesian closed by the Adjoint Functor Theorem.

In order to give a homotopy theoretical version of this definition we want to express what it means for a functor between combinatorial model categories to preserve homotopy colimits. Therefore let $F\colon \mathbb{M} \to \mathbb{N}$ be a functor which preserves weak equivalences. Given a small indexing category I and a functor $X\colon I \to \mathbb{M}$ which is cofibrant in the projective model structure, we have maps

$$\begin{tikzcd}
\mathrm{hocolim}(F\circ X) \arrow[d, phantom, "\wr\vert"] & & F(\mathrm{hocolim}X) \arrow[d, phantom, "\wr\vert"] \\
\mathrm{colim}\mathbb{L}(F\circ X) \arrow[r, "\mathrm{colim}(L_{F\circ X})"] & \mathrm{colim}(F\circ X) \arrow[r] & F(\mathrm{colim}X)
\end{tikzcd}$$

where $L\colon \mathbb{L} \Rightarrow \mathrm{id}$ denotes some cofibrant replacement in $[I, \mathbb{M}]_{\mathrm{proj}}$. We denote the composition by η_X. Thus, given an arbitrary (not necessarily cofibrant) functor $X\colon I \to \mathbb{M}$, the weak equivalence $\mathbb{L}X \overset{\sim}{\twoheadrightarrow} X$ in $[I, \mathbb{M}]_{\mathrm{proj}}$ induces vertical weak equivalences

$$\begin{tikzcd} \mathrm{hocolim}(F \circ X) & F(\mathrm{hocolim} X) \\ \mathrm{hocolim}(F \circ \mathbb{L}X) \arrow[u, "\wr"] \arrow[r, "\eta_{\mathbb{L}X}"'] & F(\mathrm{hocolim} \mathbb{L}X). \arrow[u, "\wr"] \end{tikzcd}$$

Dually, the weak equivalence $X \overset{\sim}{\hookrightarrow} \mathbb{R}X$ in $([I, \mathbb{M}])_{\mathrm{inj}}$ induces maps

$$\begin{tikzcd} \mathrm{holim}(F \circ X) \arrow[d, "\wr"] & F(\mathrm{holim} X) \arrow[d, "\wr"] \\ \mathrm{holim}(F \circ \mathbb{R}X) & F(\mathrm{holim} \mathbb{R}X). \arrow[l, "\epsilon_{\mathbb{R}X}"] \end{tikzcd}$$

Definition 7.1.3. Let $\mathbb{M}$, $\mathbb{N}$ be combinatorial and $F\colon \mathbb{M} \to \mathbb{N}$ be a functor which preserves weak equivalences.

(1) We say that F *preserves homotopy colimits* if for every diagram $X\colon I \to \mathbb{M}$, the map

$$\eta_{\mathbb{L}X}\colon \mathrm{hocolim}(F \circ \mathbb{L}X) \to F(\mathrm{hocolim} \mathbb{L}X)$$

is a weak equivalence.

(2) We say that F *preserves homotopy limits* if for every diagram $X\colon I \to \mathbb{M}$ the map

$$\epsilon_{\mathbb{R}X}\colon F(\mathrm{holim} \mathbb{R}X) \to \mathrm{holim}(F \circ \mathbb{R}X)$$

is a weak equivalence.

Note that Definition 7.1.3 does not depend on the choice of cofibrant and fibrant replacements.

Remark 7.1.4. If $\alpha\colon F \to G$ is a natural transformation between functors which preserve weak equivalences such that all components α_C for $C \in \mathbb{M}$ are weak equivalences in $\mathbb{N}$, then F preserves homotopy (co)limits if and only if G does so. It then follows that F preserves homotopy colimits if and only if its left derived functor $\mathbb{L}F = F \circ \mathbb{L}$ does so.

Notation 7.1.5. We write Λ^2_2 for the free category over the graph

$$\begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

Given a map $f\colon X \to Z$ in $\mathbb{M}$, we write $(f,\cdot)\colon \mathbb{M}/Z \to \mathbb{M}^{\Lambda^2_2}$ for the functor sending a map $g\colon Y \to Z$ to the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd}$$

Note that for every map $f\colon X \to Z$ in $\mathbb{M}$ the homotopy pullback functor $f^*_h\colon \mathbb{M}/Z \to \mathbb{M}$ defined via the composition

$$\begin{tikzcd} \mathbb{M}^{\Lambda^2_2} \arrow[r, "\mathrm{holim}"] & \mathbb{M}^{[1]\times[1]} \arrow[r, "\iota_1^*"] & \mathbb{M}^{[1]} \arrow[r, "\mathrm{dom}"] & \mathbb{M} \\ \mathbb{M}/Z \arrow[u, "(f,\cdot)"] \arrow[rr, dashed] & & \mathbb{M}/\mathbb{R}X \arrow[u, hook] & \end{tikzcd}$$

preserves weak equivalences. By abuse of notation we will sometimes refer to the top composition $\mathbb{M}^{\Lambda^2_2} \to \mathbb{M}$ as holim only.

Definition 7.1.6. Say $\mathbb{M}$ has *universal homotopy colimits* if homotopy pullbacks in $\mathbb{M}$ preserve homotopy colimits. More precisely, $\mathbb{M}$ has universal homotopy colimits if for every arrow $f\colon X \to Z$ with fibrant codomain Z, the associated homotopy pullback functor

$$f^*_h\colon \mathbb{M}/Z \to \mathbb{M}$$

preserves homotopy colimits.

An equivalent version of this definition was introduced in [**55**, Definition 4.9.1.2] as part of the Giraud-style axioms characterizing model toposes. It is easy to see that the consideration of the homotopy pullback f^*_h only for maps f with fibrant codomain is an unessential choice.

Lemma 7.1.7. *For a combinatorial model category $\mathbb{M}$, the following are equivalent.*

(1) $\mathbb{M}$ has universal homotopy colimits, i.e. for every arrow $f\colon X \to Z$ with fibrant codomain, the homotopy pullback functor f^*_h preserves homotopy colimits.

(2) *For every arrow $f\colon X \to Z$ with cofibrant domain (and fibrant codomain), the homotopy pullback functor f_h^* preserves homotopy colimits.*

(3) *For every arrow $f\colon X \to Z$, the homotopy pullback functor f_h^* preserves homotopy colimits.*

Proof. Given a map $f\colon X \to Z$ with non-fibrant codomain, fibrant replacement $R_Z\colon Z \xrightarrow{\sim} \mathbb{R}Z$ yields a point-wise weak equivalence $(f, \cdot) \Rightarrow (R_Z \circ f, \cdot)$ between functors $\mathbb{M}/Z \to \mathbb{M}^{\Lambda_2^2}$, and thus a point-wise weak equivalence $f_h^* \Rightarrow (R_Z \circ f)_h^*$. So, by Remark 7.1.4, f_h^* preserves homotopy colimits if and only if $(R_Z \circ f)_h^*$ does so. This shows that $\mathbb{M}$ has universal colimits if and only if for every arrow $f \in \mathbb{M}$, the homotopy pullback functor f_h^* preserves homotopy colimits. The restriction to cofibrant domains (and fibrant codomains) is shown analogously. □

We will also see that post-composition with the domain functor in order to obtain codomain $\mathbb{M}$ rather than $\mathbb{M}/\mathbb{R}X$ in the definition of homotopy pullback functors is an unessential choice, too. Recall the following facts from [**36**].

(1) Let $N\colon \mathbf{Cat} \to \mathbf{S}$ denote the ordinary nerve functor. Let $\mathbb{M}^c$ be the subcategory of cofibrant objects, $\mathcal{W}$ be the class of weak equivalences in $\mathbb{M}^c$ and $\mathcal{W}^{-1}N(\mathbb{M}^c)$ be the $(\infty,1)$-localization of the quasi-category $N(\mathbb{M}^c)$ at $\mathcal{W} \subset N(\mathbb{M}^c)_1$. Then every combinatorial model category $\mathbb{M}$ has an underlying quasi-category
$$\mathrm{Ho}_\infty(\mathbb{M}) = \mathcal{W}^{-1}N(\mathbb{M}^c).$$
The restriction to cofibrant objects again is an arbitrary choice as remarked in [**35**, Remark 1.3.4.16], and only employed here to be coherent with the definitions in [**35**, Section 1.3] whose results we will use throughout this section.

(2) For every quasi-category $\mathcal{C}$ and every subset $\mathcal{W} \subset \mathcal{C}_1$, the localization $\mathcal{W}^{-1}\mathcal{C}$ always exists and is presented by the underlying simplicial set of any fibrant replacement of the pair $(\mathcal{C}, \mathcal{W})$ in the category of marked simplicial sets, see [**36**, Section 5.2.7].

(3) A functor $p\colon \mathcal{C} \to \mathcal{D}$ of quasi-categories preserves colimits if, for all $I \in \mathbf{S}$ and diagrams $X\colon I \to \mathcal{C}$, the functor p takes initial X-cocones to initial pX-cocones. Let $I^{\triangleright}$ denote the *right cone* $I \star \Delta^0$ over I as in [**36**, Notation 1.2.8.4], and $[I^{\triangleright}, \mathcal{C}]^{ini} \subset [I^{\triangleright}, \mathcal{C}]$ and $[I^{\triangleright}, \mathcal{D}]^{ini} \subset [I^{\triangleright}, \mathcal{D}]$ denote the subobjects of colimiting cocones in $\mathcal{C}$ and $\mathcal{D}$, respectively. Then, whenever $\mathcal{C}$ and $\mathcal{D}$ are both cocomplete, by [**36**, Proposition 4.3.2.15, Section 5.3.3] they

admit colimit functors “colim” given by a section to the acyclic fibrations $[I^{\triangleright}, \mathcal{C}]^{ini} \twoheadrightarrow [I, \mathcal{C}]$ and $[I^{\triangleright}, \mathcal{D}]^{ini} \twoheadrightarrow [I, \mathcal{D}]$ respectively. Then p preserves colimits if and only if the natural map $\mathrm{colim} \circ p^I \to p \circ \mathrm{colim}$ is an equivalence in $\mathcal{D}$.

Lemma 7.1.8. *Let $F\colon \mathbb{M} \to \mathbb{N}$ preserve weak equivalences. Then F preserves homotopy colimits if and only if $\mathrm{Ho}_{\infty}(F)$ preserves colimits.*

Proof. It suffices to consider $F' := \mathbb{L}^{\mathbb{N}} \circ F \circ \mathbb{L}^{\mathbb{M}}$. Then the “if” direction follows from [**35**, Proposition 1.3.4.24] and the “only if” direction along the lines of the proof of [**35**, Corollary 1.3.4.26]. □

7.2. The relation to presentable locally cartesian closed quasi-categories

Notation. Throughout this section $\mathbb{M}$ will denote a combinatorial model category. Given a simplicial category $\mathbf{C}$, the projective model structure on the category of simplicial presheaves over $\mathbf{C}$ is denoted by $\mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}}$, the corresponding injective model structure by $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$.

We want to show that a combinatorial model category $\mathbb{M}$ has universal homotopy colimits if and only if its underlying quasi-category $\mathrm{Ho}_{\infty}(\mathbb{M})$ has universal colimits (and hence is locally cartesian closed) as defined in [**36**, Definition 6.1.1.2]. To recall the latter definition, we adopt Lurie’s notation of the quasi-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the exponential $\mathcal{D}^{\mathcal{C}}$ in simplicial sets between two quasi-categories $\mathcal{C}$ and $\mathcal{D}$. Given a quasi-category $\mathcal{C}$ which admits pullbacks and a map $f\colon X \to Z$ in $\mathcal{C}$, the bicartesian fibration $\mathrm{codom}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ pulls back to a bicartesian fibration

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \Delta^1 \arrow[r, "f"'] & \mathcal{C}. \end{tikzcd}$$

In terms of [**36**, Definitions 2.3.1.3, 5.2.1.1 and 5.2.2.1, Section 6.1.1] this is a correspondence whose associated right adjoint is defined to be the pullback functor $f^*\colon \mathcal{C}^{/Z} \to \mathcal{C}^{/X}$. This definition will be elaborated in more detail in the proof of Lemma 7.2.3.

In the following we compare the homotopy pullback functor f_h^* in $\mathbb{M}$ as considered in Section 7.2 with this pullback functor f^* in $\mathrm{Ho}_\infty(\mathbb{M})$. Since the definitions of both notions involve slices over objects in $\mathbb{M}$, we need to compare the corresponding quasi-categories. The relevant calculations are presented in [**35**] in the case when $\mathbb{M}$ is simplicial and generalize easily to non-simplicial model categories using Theorem 7.1.2.

Lemma 7.2.1. *Let $Z \in \mathbb{M}$ be fibrant. Then the natural map*

$$\mathrm{Ho}_\infty(\mathbb{M}/Z) \to \mathrm{Ho}_\infty(\mathbb{M})_{/Z}$$

is a categorical equivalence.

Proof. Recall that every Quillen equivalence $(L, R)\colon \mathbb{M}_1 \to \mathbb{M}_2$ between model categories $\mathbb{M}_i$ induces a Quillen equivalence

$$(\Sigma_\epsilon \circ L^\to, R^\to)\colon \mathbb{M}_1/RB \to \mathbb{M}_2/B$$

for every fibrant $B \in \mathbb{M}_2$ as shown for example in [**49**, Proposition 3.1]. Now, let

$$(F, G)\colon \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}} \simeq \mathbb{M}$$

be a presentation of $\mathbb{M}$ as provided by Theorem 7.1.2 and define $\mathbb{P} := \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$. Let $Z \in \mathbb{M}$ be fibrant. We obtain a square

$$\begin{tikzcd} \mathbb{P}/GZ \arrow[r, shift left, "\sum_\epsilon \circ F^\to"] \arrow[d, "\mathrm{dom}"'] & \mathbb{M}/Z \arrow[l, shift left, "G^\to"] \arrow[d, "\mathrm{dom}"] \\ \mathbb{P} \arrow[r, shift left, "F"] & \mathbb{M} \arrow[l, shift left, "G"] \end{tikzcd}$$

where the horizontal pairs are Quillen equivalences. This induces a square

$$\begin{tikzcd} \mathrm{Ho}_\infty(\mathbb{P}/GZ) \arrow[d, "\mathrm{dom}"'] & \mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[l, "\mathrm{Ho}_\infty(G^\to)"'] \arrow[d, "\mathrm{dom}"] \\ \mathrm{Ho}_\infty(\mathbb{P}) & \mathrm{Ho}_\infty(\mathbb{M}) \arrow[l, "\mathrm{Ho}_\infty(G)"'] \end{tikzcd}$$

on underlying quasi-categories, where the horizontal arrows are equivalences by [**35**, Lemma 1.3.4.21]. The vertical arrows factor via the over-categories

$$\begin{tikzcd}
\mathrm{Ho}_\infty(\mathbb{P}/GZ) \arrow[dd, "\mathrm{dom}"'] \arrow[dr] & & \mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[ll, "\mathrm{Ho}_\infty(G^{\to})"'] \arrow[dd, "\mathrm{dom}"'] \arrow[dr] & \\
& \mathrm{Ho}_\infty(\mathbb{P})_{/GZ} \arrow[dl] & & \mathrm{Ho}_\infty(\mathbb{M})_{/Z} \arrow[ll, "\mathrm{Ho}_\infty(G)^{\to}"'] \arrow[dl] \\
\mathrm{Ho}_\infty(\mathbb{P}) & & \mathrm{Ho}_\infty(\mathbb{M}) \arrow[ll, "\mathrm{Ho}_\infty(G)"'] &
\end{tikzcd}$$

Here, the upper square is given by the corresponding adjoint square

$$\begin{tikzcd}
\mathrm{Ho}_\infty(\mathbb{M}/Z) \star \Delta^0 \arrow[r, "\mathrm{Ho}_\infty G^{\to} \star \Delta^0"] \arrow[d, "(* \mapsto \mathrm{id}_Z)"'] & \mathrm{Ho}_\infty(\mathbb{P}/GZ) \star \Delta^0 \arrow[d, "(* \mapsto \mathrm{id}_Z)"] \\
\mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[r, "\mathrm{Ho}_\infty(G^{\to})"] \arrow[d] & \mathrm{Ho}_\infty(\mathbb{P}/GZ) \arrow[d] \\
\mathrm{Ho}_\infty(\mathbb{M}) \arrow[r, "G"] & \mathrm{Ho}_\infty(\mathbb{P}).
\end{tikzcd}$$

The intermediate horizontal arrow $\mathrm{Ho}_\infty(G^{\to})$ is an equivalence by [**36**, 1.2.9.3]. Now GZ is fibrant-cofibrant in $\mathbb{P}$ and hence the map $\mathrm{Ho}_\infty(\mathbb{P}/GZ) \to \mathrm{Ho}_\infty(\mathbb{P})_{/GZ}$ is an equivalence by [**36**, Lemma 6.1.3.13] and [**35**, Theorem 1.3.4.20]. Thus, by 2-for-3, the comparison map $\mathrm{Ho}_\infty(\mathbb{M}/Z) \to \mathrm{Ho}_\infty(\mathbb{M})_{/Z}$ is an equivalence, too. $\square$

Furthermore, universality of colimits is a statement about all maps in the model category $\mathbb{M}$ and hence we need a correspondence between maps in $\mathbb{M}$ and maps in $\mathrm{Ho}_\infty(\mathbb{M})$ in order to relate the two notions of universality of colimits. Indeed, every arrow $f\colon X \to Z$ in $\mathbb{M}$ induces the edge $\mathbb{L}f\colon \Delta^1 \to \mathrm{Ho}_\infty(\mathbb{M})$ after cofibrant replacement. Vice versa, the following lemma holds.

Lemma 7.2.2. *Every edge $f\colon \Delta^1 \to \mathrm{Ho}_\infty(\mathbb{M})$ is presented by a map $\bar{f}\colon X \to Z$ in $\mathbb{M}^c$ up to equivalence, i.e. there is an equivalence $e \in \mathrm{Ho}_\infty(\mathbb{M})$ such that $e \circ f = \bar{f}$.*

Proof. Let $\tau_1\colon \mathbf{S} \to \mathbf{Cat}$ be the left adjoint to the nerve functor $N\colon \mathbf{Cat} \to \mathbf{S}$ which maps a quasi-category $\mathcal{C}$ to its underlying category $\tau_1(\mathcal{C})$. Then the category $\tau_1(\mathrm{Ho}_\infty(\mathbb{M}))$ is the homotopy category $\mathrm{Ho}(\mathbb{M})$ of $\mathbb{M}$. This can be seen by checking that $\tau_1(\mathrm{Ho}_\infty(\mathbb{M}))$ satisfies the universal property of the ordinary localization $\mathbb{M}^c[\mathcal{W}^{-1}]$. Let $f\colon X \to Z$ be an edge in $\mathrm{Ho}_\infty(\mathbb{M})$. As the fibrant replacement

$R_Z\colon Z \to \mathbb{R}Z$ has fibrant codomain, the composition $[R_Z \circ f] \in \mathrm{Ho}(\mathbb{M})$ has a representative $\bar{f}\colon X \to \mathbb{R}Z$ in $\mathbb{M}$. As $[r_Z \circ f] = [\bar{f}]$ in $\mathrm{Ho}(\mathbb{M})$, we obtain a 2-simplex

$$\begin{tikzcd} & \mathbb{R}Z \arrow[dr, "\mathrm{id}_{\mathbb{R}Z}"] & \\ X \arrow[ur, "R_Z\circ f"] \arrow[rr, "\bar{f}"'] & & \mathbb{R}Z \end{tikzcd}$$

in $\mathrm{Ho}_\infty(\mathbb{M})$ which in turn induces the desired 2-cell

$$\begin{tikzcd} & Z \arrow[dr, "R_Z"] & \\ X \arrow[ur, "f"] \arrow[rr, "\bar{f}"'] & & \mathbb{R}Z \end{tikzcd}$$

by filling a corresponding inner 3-horn in $\mathrm{Ho}_\infty(\mathbb{M})$. □

Lastly, we have to compare the homotopy pullback functor associated to maps in $\mathbb{M}$ and the corresponding pullback functor in $\mathrm{Ho}_\infty(\mathbb{M})$. Therefore, note that just as in the ordinary case, every object Z in a quasi-category $\mathcal{C}$ induces a map

$$(\cdot,\cdot)\colon \mathcal{C}_{/Z} \times \mathcal{C}_{/Z} \to \mathcal{C}^{\Lambda^2_2}$$

which attaches two Z-cones to each other at their cone point. On n-simplices, it is induced by a contracting homotopy $H_n\colon \Delta^n \times \Delta^1 \to \Delta^n$ to $\{0\}$. Indeed, $H_n \times \mathrm{pr}_2\colon \Delta^n \times \Delta^1 \to \Delta^n \times \Delta^1$ factors through $\Delta^n \star \{0\} \subset \Delta^n \times \Delta^1$ and hence yields a map $k_n\colon \Delta^n \times \Delta^1 \to \Delta^{n+1}$. Then the associated pair

$$(k_n, k_n)\colon (\Delta^n \times \Delta^1) \sqcup_{\Delta^n \times \Delta^0} (\Delta^n \times \Delta^1) \to (\Delta^n \star \Delta^0) \sqcup_{\Delta^0} (\Delta^n \star \Delta^0)$$

gives the attaching map

$$(k_n, k_n)^*\colon [(\Delta^n \star \Delta^0) \sqcup_{\Delta^0} (\Delta^n \star \Delta^0), \mathcal{C}]_{\mathbf{S}} \to [(\Delta^n \times \Delta^1) \sqcup_{\Delta^n \times \Delta^0} (\Delta^n \times \Delta^1), \mathcal{C}]_{\mathbf{S}}$$

which we denote by $(\cdot,\cdot)_n$. Then in particular, for every edge $f\colon X \to Z$ in $\mathcal{C}$, the attaching map restricts to

$$(f,\cdot)\colon \Delta^0 \times \mathcal{C}_{/Z} \xrightarrow{(f,\mathrm{id})} \mathcal{C}_{/Z} \times \mathcal{C}_{/Z} \xrightarrow{(\cdot,\cdot)} \mathcal{C}^{\Lambda^2_2}.$$

We obtain a sequence of equivalences connecting the homotopy pullback functor in $\mathbb{M}$ as defined in Section 7.1 and the pullback functor in the quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ following the definition of Lurie as follows.

Here, $\mathcal{C}^{/Z}$ denotes Joyal's alternative slice construction as defined in [**36**, Section 4.2.1] and used in Lurie's definition in [**36**, Section 6.1.1.(ii)] of the pullback functor f^*. It can be constructed as the pullback

$$\begin{tikzcd}
\mathcal{C}^{/Z} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, two heads, "\mathrm{codom}"] \\
\Delta^0 \arrow[r, "Z"'] & \mathcal{C}.
\end{tikzcd}$$

Lemma 7.2.3.

(1) Let $\mathcal{C}$ be a finitely complete quasi-category and $f\colon X \to Z$ be a map in $\mathcal{C}$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/Z} \arrow[r, "{(f,\cdot)}"] \arrow[d, "\simeq"'] & \mathcal{C}^{\Lambda^2_2} \arrow[r, "\lim"] & \mathcal{C} \\
\mathcal{C}^{/Z} \arrow[r, "f^*"'] & \mathcal{C}^{/X} \arrow[ur, bend right, "\mathrm{dom}"'] &
\end{tikzcd}$$

of quasi-categories.

(2) For every map $f\colon X \to Z$ with fibrant codomain in $\mathbb{M}$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[r, "{\mathrm{Ho}_\infty(f,\cdot)}"] \arrow[d, "\simeq"'] & \mathrm{Ho}_\infty(\mathbb{M}^{\Lambda^2_2}) \arrow[d, "\simeq"'] \arrow[dr, bend left, "\mathrm{Ho}_\infty(\mathrm{holim})"] & \\
\mathrm{Ho}_\infty(\mathbb{M})_{/Z} \arrow[r, "{(f,\cdot)}"] \arrow[d, "\simeq"'] & \mathrm{Ho}_\infty(\mathbb{M})^{\Lambda^2_2} \arrow[r, "\lim"'] & \mathrm{Ho}_\infty(\mathbb{M}) \\
\mathrm{Ho}_\infty(\mathbb{M})^{/Z} \arrow[r, "f^*"'] & \mathrm{Ho}_\infty(\mathbb{M})^{/X} \arrow[ur, bend right, "\mathrm{dom}"'] &
\end{tikzcd}$$

of quasi-categories.

Proof. First, we prove part (1). The vertical equivalence between the two slice constructions $\mathcal{C}_{/Z}$ and $\mathcal{C}^{/Z}$ is given in [**36**, Proposition 4.2.1.5] and will be denoted by

$$\kappa_Z\colon \mathcal{C}_{/Z} \to \mathcal{C}^{/Z}.$$

We have to check that the resulting diagram commutes. Denoting the subobject of cartesian squares in $\mathcal{C}$ by $\mathrm{Car}(\mathcal{C}) \subseteq \mathcal{C}^{\Delta^1\times\Delta^1}$ and the canonical inclusion of simplicial sets $\Lambda^2_2 \hookrightarrow \Delta^1 \times \Delta^1$ by j, recall from [**36**, Proposition 4.3.2.15] that the fibration

$$j^* \colon \mathcal{C}^{\Delta^1 \times \Delta^1} \twoheadrightarrow \mathcal{C}^{\Lambda^2_2}$$

restricts to an acyclic fibration

$$j^* \colon \mathrm{Car}(\mathcal{C}) \overset{\sim}{\twoheadrightarrow} \mathcal{C}^{\Lambda^2_2},$$

because $\mathcal{C}$ has all pullbacks. Hence it has a section $s \colon \mathcal{C}^{\Lambda^2_2} \to \mathrm{Car}(\mathcal{C})$ and so every span in $\mathcal{C}$ can be completed to a cartesian square. The map s restricts to a section s_f of the pullback $(\kappa_Z^{-1}(f,\cdot))^*(j^*)$ and so we obtain a diagram of the form

$$\begin{tikzcd}
 & & \mathcal{C} & \\
\mathcal{C}^{/X} \arrow[rr, hook] \arrow[urr, bend left, "\mathrm{dom}"] & & \mathcal{C}^{\Delta^{\{0\}} \times \Delta^1} \arrow[u, "(d^1)^*"'] & \\
\mathrm{Car}(\mathcal{C})_f \arrow[rr, hook] \arrow[u, "L_f := (d^1 \times \mathrm{id})^*"] \arrow[d, two heads, "\wr"] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Car}(\mathcal{C}) \arrow[u, "(d^1 \times \mathrm{id})^*"'] \arrow[d, two heads, "\wr"] \arrow[r, phantom, "\subseteq"] & \mathcal{C}^{\Delta^1 \times \Delta^1} \\
\mathcal{C}^{/Z} \arrow[r, shift left, "\kappa_Z^{-1}"] \arrow[u, bend left, "s_f"] & \mathcal{C}_{/Z} \arrow[l, shift left, "\kappa_Z"] \arrow[r, hook, "(f,\cdot)"'] & \mathcal{C}^{\Lambda^2_2} \arrow[u, bend left, "s"] &
\end{tikzcd} \tag{7.2.1}$$

where, informally, the pullback $\mathrm{Car}(\mathcal{C})_f$ consists of the cartesian squares in $\mathcal{C}$ whose underlying bottom edge is f. The vertical upwards oriented sequence of arrows on the right hand side completes spans (g,h) to cartesian squares (g^*h, h^*g, g, h) and projects to the (domain of the) edge g^*h. By construction this sequence is equivalent to the limit functor

$$\lim \colon \mathcal{C}^{\Lambda^2_2} \to \mathcal{C}$$

and so it remains to show that the composition $L_f \circ s_f$ is equivalent to the pullback functor f^* as defined in [**36**, Definition 6.1.1.2]. In this definitions terminology we have to show that $L_f \circ s_f$ is associated to the correspondence defining the adjoint pair $(\sum_f, f^*)$. Therefore, note that the composition

$$\mathcal{C}_{/Z} \xrightarrow{(f,\cdot)} \mathcal{C}^{\Lambda^2_2} \xrightarrow{s} \mathrm{Car}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\Delta^1 \times \Delta^1},$$

which we denote by $\mathrm{pbs}_f \colon \mathcal{C}_{/Z} \to \mathcal{C}^{\Delta^1 \times \Delta^1}$, corresponds to a map

$$\overline{\mathrm{pbs}_f} \colon \mathcal{C}_{/Z} \times \Delta^1 \to \mathcal{C}^{\Delta^1}$$

such that the outer square

$$\begin{tikzcd}
\mathcal{C}_{/Z} \times \Delta^1 \arrow[drr, bend left, "\overline{\mathrm{pbs}_f}"] \arrow[dr, dashed, "\eta"] \arrow[ddr, bend right, "\pi_2"'] & & \\
& M_f \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{codom}"] \\
& \Delta^1 \arrow[r, "f"'] & \mathcal{C}.
\end{tikzcd}$$

commutes. Hence we obtain the dotted factorization η through the pullback M_f. First, recall that the target fibration $\mathrm{codom}\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$ is both cartesian and cocartesian since $\mathcal{C}$ is closed under pullbacks. Then, observe that for every vertex $g \in \mathcal{C}_{/Z}$, the edge $\eta|_{g\times\Delta^1}\colon \Delta^1 \to M_f$ is p-cartesian. This holds because pbs_f factors over the subobject of cartesian squares, which constitute exactly the codom-cartesian arrows in $\mathcal{C}^{\Delta^1}$ by [**36**, Lemma 6.1.1.1]. This means that every $\eta|_{g\times\Delta^1}$ is codom-cartesian and hence every such $\eta|_{g\times\Delta^1}$ is p-cartesian by [**36**, Proposition 2.4.1.3]. Furthermore, denote $\{0\},\{1\}\colon \Delta^0 \to \Delta^1$ the respective endpoint inclusions. Then, on the one hand, the maps

$$\{1\}^*\eta\colon \mathcal{C}_{/Z} \to f(1)^*(\mathcal{C}^{\Delta^1})$$
$$\kappa_Z\colon \mathcal{C}_{/Z} \to \mathcal{C}^{/Z}$$

coincide. On the other hand, $\{0\}^*\eta\colon \mathcal{C}_{/Z} \to \mathcal{C}^{/X}$ is $L_f \circ s_f \circ \kappa_Z$ by definition of $\overline{\mathrm{pbs}_f}$ and commutativity of Diagram (7.2.1). This means, by [**36**, Definition 5.2.1.1], that $L_f \circ s_f \circ \kappa_Z$ is associated to the correspondence $(M_f, \mathrm{id}, \kappa_Z)$ and so, by [**36**, Proposition 5.2.1.4], $L_f \circ s_f$ in turn is associated to the correspondence $(M_f, \mathrm{id}, \mathrm{id})$. The same proposition and [**36**, Section 6.1.1.(ii)] now imply that $L_f \circ s_f$ is equivalent to the pullback functor $f^*\colon \mathcal{C}^{/Z} \to \mathcal{C}^{/X}$.

For part (2), we note that the lower half of the diagram is given directly by part (1). For the top half, the existence of an equivalence $\mathrm{Ho}_\infty(\mathbb{M}/Z) \to \mathrm{Ho}_\infty(\mathbb{M})_{/Z}$ was shown in Lemma 7.2.1 and the equivalence $\mathrm{Ho}_\infty(\mathbb{M}^{\Lambda^2_2}) \to \mathrm{Ho}_\infty(\mathbb{M})^{\Lambda^2_2}$ is given by [**35**, 1.3.4.25]. Thus, we have to show that this top half commutes. We do this by showing commutativity of its two components separately. First, the triangle

$$\begin{tikzcd}
\mathrm{Ho}_\infty(\mathbb{M}^{\Lambda^2_2}) \arrow[r, "\sim"] \arrow[rr, bend right, "\mathrm{Ho}_\infty(\mathrm{holim})"'] & \mathrm{Ho}_\infty(\mathbb{M})^{\Lambda^2_2} \arrow[r, "\lim"] & \mathrm{Ho}_\infty(\mathbb{M})
\end{tikzcd}$$

commutes by [**35**, 1.3.4.23]. Second, to show that the square

$$\begin{tikzcd}
\mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[r, "\mathrm{Ho}_\infty(f,\cdot)"] \arrow[d, "\simeq"'] & \mathrm{Ho}_\infty(\mathbb{M}^{\Lambda^2_2}) \arrow[d, "\simeq"'] \\
\mathrm{Ho}_\infty(\mathbb{M})_{/Z} \arrow[r, "(f,\cdot)"] & \mathrm{Ho}_\infty(\mathbb{M})^{\Lambda^2_2}
\end{tikzcd} \tag{7.2.2}$$

commutes, by the universal property of localizations, it suffices to check it commutes after precomposition with the natural map

$$\iota_{\mathbb{M}/Z} \colon N(\mathbb{M}^c/Z) \to \mathrm{Ho}_\infty(\mathbb{M}/Z).$$

The composition

$$\begin{tikzcd}
N(\mathbb{M}^c/Z) \arrow[d, dashed, "\cong"'] \arrow[dr, "\iota_{\mathbb{M}/Z}"] & \\
N(\mathbb{M}^c)/Z \arrow[dr, dashed, "\iota_{\mathbb{M}/Z}"'] & \mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[d, "\simeq"] \\
 & \mathrm{Ho}_\infty(\mathbb{M})_{/Z}
\end{tikzcd} \tag{7.2.3}$$

depicted by the solid arrows factors via the dotted arrows up to equivalence by definition of the arrow $\mathrm{Ho}_\infty(\mathbb{M}/Z) \to \mathrm{Ho}_\infty(\mathbb{M})_{/Z}$ subject to Lemma 7.2.1. Also the squares

$$\begin{tikzcd}
N(\mathbb{M}/Z) \arrow[r, "N(f,\cdot)"] \arrow[d, "\cong"'] & N(\mathbb{M}^{\Lambda^2_2}) \arrow[d, "\cong"'] & \\
N(\mathbb{M})_{/Z} \arrow[r, "(f,\cdot)"] \arrow[dr, "\iota_{\mathbb{M}/Z}"'] & N(\mathbb{M})^{\Lambda^2_2} \arrow[dr, "\iota_{\mathbb{M}}^{\Lambda^2_2}"] & \\
 & \mathrm{Ho}_\infty(\mathbb{M})_{/Z} \arrow[r, "(f,\cdot)"] & \mathrm{Ho}_\infty(\mathbb{M})^{\Lambda^2_2}
\end{tikzcd} \tag{7.2.4}$$

commute by construction of the maps involved; indeed commutativity of the lower square follows from functoriality of the construction of $(\cdot,\cdot)$ and the upper square can be directly checked to commute on n-simplices. It is easy to see that the diagrams (7.2.3) and (7.2.4) can be pasted into a cube

$$\begin{tikzcd}
N(\mathbb{M}/Z) \arrow[rr, "N(f,\cdot)"] \arrow[dd, "\cong"'] \arrow[dr, "\iota_{\mathbb{M}/Z}"'] & & N(\mathbb{M}^{\Lambda^2_2}) \arrow[dd, "\cong"' near end] \arrow[dr, "\iota_{\mathbb{M}^{\Lambda^2_2}}"] & \\
& \mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[rr, "(f,\cdot)" near start] \arrow[dd, "\simeq" near start] & & \mathrm{Ho}_\infty(\mathbb{M}^{\Lambda^2_2}) \arrow[dd, "\simeq"] \\
N(\mathbb{M})/Z \arrow[rr, "N(f,\cdot)"' near end] \arrow[dr, "\iota_{\mathbb{M}/Z}"'] & & N(\mathbb{M})^{\Lambda^2_2} \arrow[dr, "\iota_{\mathbb{M}}^{\Lambda^2_2}"] & \\
& \mathrm{Ho}_\infty(\mathbb{M})_{/Z} \arrow[rr, "(f,\cdot)"'] & & \mathrm{Ho}_\infty(\mathbb{M})^{\Lambda^2_2}
\end{tikzcd}$$

where all faces but the front face are known to commute. But then the front face precomoposed with the upper left edge

$$N(\mathbb{M}/Z) \to \mathrm{Ho}_\infty(\mathbb{M}/Z)$$

commutes, which is exactly what we had to show for the square (7.2.2) to commute.

□

We can now prove the following statement.

Theorem 7.2.4. *A combinatorial model category* $\mathbb{M}$ *has universal homotopy colimits if and only if its associated quasi-category* $\mathrm{Ho}_\infty(\mathbb{M})$ *has universal colimits.*

Proof. For every edge $f \in \mathrm{Ho}_\infty(\mathbb{M})$ and every arrow $\bar{f}\colon X \to Z$ in $\mathbb{M}$ representing f, we have that the homotopy pullback functor $\bar{f}^*_h = \mathrm{holim} \circ (\bar{f}, \cdot)$ preserves homotopy colimits if and only if $\mathrm{Ho}_\infty(\bar{f}^*_h) = \mathrm{Ho}_\infty(\mathrm{holim}) \circ \mathrm{Ho}_\infty(\bar{f}, \cdot)$ preserves colimits by Lemma 7.1.8. This in turn holds if and only if $\mathrm{dom} \circ \bar{f}^*$ preserves colimits by Lemma 7.2.3. But the functor $\mathrm{dom}\colon \mathrm{Ho}_\infty(\mathbb{M})^{/X} \to \mathrm{Ho}_\infty(\mathbb{M})$ both preserves and reflects colimits by [**36**, Proposition 1.2.13.8], so this holds if and only if $\bar{f}^*$ – and hence f^* – preserves colimits in $\mathrm{Ho}_\infty(\mathbb{M})$. Then the "if"-direction follows from Lemma 7.2.2. Vice versa, the other direction follows as for every arrow $f\colon X \to Z$ in $\mathbb{M}$ with fibrant codomain and cofibrant replacement $\lambda_Z\colon \mathbb{L}Z \to Z$, the triangle

$$\begin{tikzcd}
& \mathrm{Ho}_\infty(\mathbb{M}/\mathbb{L}Z) \arrow[dr, "\mathrm{Ho}_\infty(\mathbb{L}f,\cdot)"] & \\
\mathrm{Ho}_\infty(\mathbb{M}/Z) \arrow[ur, "\lambda_Z", "\simeq"'] \arrow[rr, "\mathrm{Ho}_\infty(f,\cdot)"'] & & \mathrm{Ho}_\infty(\mathbb{M}^{\Lambda^2_2})
\end{tikzcd}$$

commutes up to equivalence, so that $\mathrm{Ho}_\infty(f_h^*)$ preserves homotopy colimits if and only if $\mathrm{Ho}_\infty((\mathbb{L}f)_h^*)$ does. □

Corollary 7.2.5. *Universality of homotopy colimits is invariant under Quillen equivalence, i.e. given a Quillen equivalence $\mathbb{M} \simeq \mathbb{N}$ between combinatorial model categories, then $\mathbb{M}$ has universal homotopy colimits if and only if $\mathbb{N}$ does.*

□

A major example of a class of model categories with universal homotopy colimits are model toposes in the sense of Rezk ([**45**]) and Toën and Vezzosi ([**55**]). Indeed, model toposes can be defined equivalently in terms of two descent properties as shown in [**45**, Theorem 6.9].

Recall from [**45**, 6.5] that a natural transformation $F\colon X \to Y$ between functors $X, Y\colon I \to \mathbb{M}$ is called *equifibred* if for every map $i \to j$ in I, the square

$$\begin{array}{ccc} X(i) & \longrightarrow & X(j) \\ {\scriptstyle F_i}\downarrow & & \downarrow{\scriptstyle F_j} \\ Y(i) & \longrightarrow & Y(j) \end{array}$$

is a homotopy pullback.

Definition 7.2.6 (Theorem 6.9, [**45**]). A combinatorial model category $\mathbb{M}$ is a *model topos* if it satisfies the following two "descent" properties.

(P1) Let I be a category, $X\colon I \to \mathbb{M}$ a functor and $f\colon \bar{Y} \to \mathrm{hocolim}X$ a map in $\mathbb{M}$. Let $Y\colon I \to \mathbb{M}$ be the functor defined point-wise via homotopy pullback

$$Y(i) := X(i) \times^h_{\mathrm{hocolimX}} \bar{Y}$$

Then the natural map $\mathrm{hocolim}Y \to \bar{Y}$ is a weak equivalence in $\mathbb{M}$. (To be more precise, we define $Y(i) := (\mathbb{L}X)(i) \times^h_{\mathrm{hocolim}X} \bar{Y}$ and require the natural map $\mathrm{hocolim}Y \to \mathrm{colim}Y \to R\bar{Y}$ for a factorization $\bar{Y} \overset{\sim}{\hookrightarrow} R\bar{Y} \twoheadrightarrow \mathrm{hocolim}X$ to be a weak equivalence.)

(P2) Let I be a category, $F\colon X \to Y$ be an equifibred natural transformation between functors $X, Y\colon I \to \mathbb{M}$ and $\mathrm{hocolim}F\colon \mathrm{hocolim}X \to \mathrm{hocolim}Y$ be the induced map between the corresponding homotopy colimits. Then, for every $i \in I$, the natural map

$$Y(i) \to X(i) \times^h_{\mathrm{hocolim}X} \mathrm{hocolim}Y$$

is a weak equivalence in $\mathbb{M}$. (Again, more precisely, we require that the natural map $(\mathbb{L}Y)(i) \to (\mathbb{L}X)(i) \times^h_{\mathrm{hocolim}X} \mathrm{hocolim}Y$ is a weak equivalence.)

As noted in [**45**] without proof, (P1) is equivalent to [**55**, Definition 4.9.1.2]; in other words, the following correspondence holds.

Lemma 7.2.7. *A combinatorial model category* $\mathbb{M}$ *satisfies (P1) if and only if it has universal homotopy colimits.*

Proof. Suppose (P1) holds, let $f\colon X \to Z$ be a map and $Y\colon I \to \mathbb{M}/Z$ be a diagram. Let

$$\bar{P} := Z \times^h_X \mathrm{hocolim}Y$$

and define $P(i) := (\mathbb{L}Y)(i) \times^h_{R\mathrm{hocolim}Y} \bar{P}$ for a factorization $\mathrm{hocolim}Y \overset{\sim}{\hookrightarrow} R\mathrm{hocolim}Y \twoheadrightarrow X$. Then the natural map

$$\mathrm{hocolim} f_h^* Y = \mathrm{hocolim}P \to \bar{P} = f_h^*(\mathrm{hocolim}Y)$$

is a weak equivalence by $(P1)$.
Vice versa, suppose $\mathbb{M}$ has universal homotopy colimits, let $X\colon I \to \mathbb{M}$ be a diagram and $f\colon \bar{Y} \to \mathrm{hocolim}X$ be a map. Define $Y(i) := (\mathbb{L}X)(i) \times^h_{\mathrm{hocolim}X} \bar{Y}$. Then the natural map

$$\mathrm{hocolim}Y = \mathrm{hocolim} f_h^* \mathbb{L}X \to f_h^*(\mathrm{hocolim}X) \simeq Y$$

is a weak equivalence by universality of homotopy colimits $\square$

In particular, a combinatorial model category with universal homotopy colimits is a model topos if and only if it satisfies (P2).

7.3. Semi-left exact localizations

Notation. In order to avoid stating the results of this section in an unnecessarily bulky fashion, in the following we consider a set of maps T in a model category $\mathbb{M}$ always to be given together with the left Bousfield localization $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ of T-local fibrant objects and T-local weak equivalences. That means whenever we let T to be a set of arrows in $\mathbb{M}$, we implicitly assume that the corresponding left Bousfield localization exists, and vice versa, whenever we claim existence of a set T of arrows in $\mathbb{M}$, we implicitly claim existence of the corresponding left Bousfield localization, too.

Recall the following definition and the subsequent presentation result for model toposes by Rezk.

Definition 7.3.1 ([**45**, Section 5.5])**.** Let $\mathbb{M}$ be a model category. A left Bousfield localization $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ is *left-exact* if the left derived functor $\mathbb{L}\mathrm{id}\colon \mathbb{M} \to \mathcal{L}_T\mathbb{M}$ preserves homotopy pullbacks.

Theorem 7.3.2 ([**45**, Corollary 6.10])**.** *Let* $\mathbb{M}$ *be a model topos. Then there is a simplicial category* $\mathbf{C}$*, a homotopy localization*

$$(L,R)\colon \mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}} \rightleftarrows \mathbb{M}$$

and a set $T \subset \mathrm{sPsh}(\mathbf{C})$ *of maps such that the left Bousfield localization* $\mathrm{sPsh}(\mathbf{C}) \to \mathcal{L}_T\mathrm{sPsh}(\mathbf{C})$ *is left exact and the homotopy localization induces a Quillen equivalence*

$$(L,R)\colon \mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{proj}} \rightleftarrows \mathbb{M}.$$

In fact, Rezk defines a model topos to be a combinatorial model category $\mathbb{M}$ together with a presentation as in Theorem 7.3.2 and proves the equivalence to Definition 7.2.6 in [**45**, Theorem 6.9]. By Theorem 7.1.2, the fact that left exact localizations of presheaf quasi-categories are presented by left exact Bousfield localizations of simplicial presheaf categories and Theorem 7.3.2 we have the following correspondence.

Theorem 7.3.3. *Let* $\mathcal{C}$ *be a quasi-category. Then the following are equivalent.*

(1) The quasi-category $\mathcal{C}$ *is a Grothendieck* ∞*-topos as defined in* [**36**, Definition 6.1.0.4].
(2) There is a model topos $\mathbb{M}$ *and an equivalence* $\mathcal{C} \simeq \mathrm{Ho}_\infty(\mathbb{M})$.

□

Generalizing the corresponding ordinary categorical notions and following the $(\infty,1)$-categorical constructions of [**23**], consider the following weaker condition on left Bousfield localizations.

Definition 7.3.4. A left Bousfield localization $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ is *semi-left exact* if the left derived functor $\mathbb{L}\mathrm{id}\colon \mathbb{M} \to \mathcal{L}_T\mathbb{M}$ preserves homotopy pullback along maps $f\colon X \to Z$ between T-local objects X and Z. That is if it preserves the homotopy limit of diagrams of the form

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

whenever X and Z are T-local.

Lemma 7.3.5. *Let $\mathbb{M}$ be a combinatorial model category and T be a set of maps in $\mathbb{M}$. The Bousfield localization $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ is semi-left exact if and only if the corresponding reflective localization $L_T\colon \mathrm{Ho}_\infty(\mathbb{M}) \to T^{-1}\mathrm{Ho}_\infty(\mathbb{M})$ is semi-left exact in the sense of* [**23**].

Proof. The map $\mathrm{Ho}_\infty(\mathrm{id})\colon \mathrm{Ho}_\infty(\mathbb{M}) \to \mathrm{Ho}_\infty(\mathcal{L}_T\mathbb{M})$ is part of a reflective localization and determined by the class of maps it sends to equivalences which consists exactly of the saturation of T. We hence have a commutative triangle

$$\begin{tikzcd} \mathrm{Ho}_\infty(\mathbb{M}) \arrow[r, "\mathrm{Ho}_\infty(\mathrm{id})"] \arrow[dr, "L_T"'] & \mathrm{Ho}_\infty(\mathcal{L}_T\mathbb{M}) \arrow[d, phantom, "\wr"] \\ & T^{-1}\mathrm{Ho}_\infty(\mathbb{M}) \end{tikzcd}$$

so that L_T is semi-left exact if and only if $\mathrm{Ho}_\infty(\mathrm{id})$ preserves pullbacks along maps $f\colon X \to Z$ with X and Z equivalent (in $\mathbb{M}$) to fibrant objects in $\mathcal{L}_T(\mathbb{M})$. In other words, with T-local objects X and Z in the sense of Definition 7.3.4. The transition between corresponding homotopy pullbacks in $\mathbb{M}$ and pullbacks in $\mathrm{Ho}_\infty(\mathbb{M})$ again follows from [**35**, Proposition 1.3.4.23]. □

Corollary 7.3.6. *Let $\mathbb{M}$ be a combinatorial model category with universal homotopy colimits and T be a set of maps in $\mathbb{M}$ such that the left Bousfield localization $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ is semi-left exact. Then the model category $\mathcal{L}_T\mathbb{M}$ has universal homotopy colimits.*

Proof. Under the given assumptions, the associated reflective localization $\mathrm{Ho}_\infty(\mathbb{M}) \to T^{-1}\mathrm{Ho}_\infty(\mathbb{M})$ on underlying quasi-categories is semi-left exact by Lemma 7.3.5 and the presentable quasi-category $T^{-1}\mathrm{Ho}_\infty(\mathbb{M})$ has universal colimits by [**23**, Proposition 1.4] and Theorem 7.2.4. But $T^{-1}\mathrm{Ho}_\infty(\mathbb{M})$ is equivalent to $\mathrm{Ho}_\infty(\mathcal{L}_T\mathbb{M})$, so that the model category $\mathcal{L}_T\mathbb{M}$ has universal homotopy colimits again by Theorem 7.2.4. □

We obtain the following representation theorem.

Theorem 7.3.7. *Let $\mathbb{M}$ be a combinatorial model category. Then there is a simplicial category $\mathbf{C}$, a homotopy localization*

$$(L,R)\colon \mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}} \rightleftarrows \mathbb{M}$$

and a set $T \subset \mathrm{sPsh}(\mathbf{C})$ of maps such that

(1) the left Bousfield localization $\mathrm{sPsh}(\mathbf{C}) \to \mathcal{L}_T\mathrm{sPsh}(\mathbf{C})$ is semi-left exact if and only if $\mathbb{M}$ has universal homotopy colimits,

(2) the homotopy localization induces a Quillen equivalence

$$(L,R)\colon \mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{proj}} \rightleftarrows \mathbb{M}.$$

Proof. Although Rezk's work in [**45**] is solely concerned with model toposes, the first part of his proof of [**45**, Corollary 6.10] shows that $\mathbf{C}$ and $T \subset \mathrm{sPsh}(\mathbf{C})$ exist and that the respective localization is semi-left exact if $\mathbb{M}$ satisfies $(P1)$. But $(P1)$ is equivalent to universality of homotopy colimits by Lemma 7.2.7. Vice versa, if the Bousfield localization is semi-left exact, universality of homotopy colimits follows from Corollary 7.3.6 and Corollary 7.2.5. □

Together with the constructions associated to Theorem 7.1.1 and Lemma 7.3.5, we also obtain the following analogue of Theorem 7.1.1 and Theorem 7.3.3.

Theorem 7.3.8. *Let $\mathcal{C}$ be a quasi-category. Then the following are equivalent.*

(1) The quasi-category $\mathcal{C}$ is presentable and locally cartesian closed as defined in [**36**, Definition 6.1.0.4].

(2) There is a combinatorial model category $\mathbb{M}$ with universal homotopy colimits and an equivalence $\mathcal{C} \simeq \mathrm{Ho}_\infty(\mathbb{M})$.

□

Cisinski has shown in [**13**] that presentable locally cartesian closed quasi-categories are presented by right proper localizations of simplicial presheaf categories with the injective model structure. This yields a type theoretic model category as noted by Gepner and Kock in [**23**, 7]. Their observation follows from Theorem 7.3.7 in light of the following generalization of [**23**, Proposition 7.8] (and the fact that the injective model structure on any simplicial presheaf category is always right proper).

Lemma 7.3.9. *Let $\mathbb{M}$ be a model category and $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ a left Bousfield localization.*

(1) *Suppose* $\mathcal{L}_T\mathbb{M}$ *is right proper. Then the Bousfield localization* $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ *is semi-left exact.*

(2) *Suppose* $\mathbb{M}$ *is right proper. Then the Bousfield localization* $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ *is semi-left exact if and only if the model category* $\mathcal{L}_T\mathbb{M}$ *is right proper.*

Proof. For arrows $f\colon A \to B$ and $g\colon C \to B$ in $\mathbb{M}$, successively replacing the objects and arrows fibrantly first in $\mathbb{M}$ and then in $\mathcal{L}_T\mathbb{M}$ gives a sequence of pullbacks

$$\begin{tikzcd}
P \arrow[rrr] \arrow[dr] \arrow[ddd] \arrow[dddr, phantom, "\lrcorner", very near start] & & & C \arrow[dr, hook, "\sim"] \arrow[ddd, "g"] & & \\
& Q \arrow[rrr, two heads] \arrow[dr] \arrow[ddd, two heads] \arrow[dddr, phantom, "\lrcorner", very near start] & & & \mathbb{R}C \arrow[dr, hook, "\sim_T"] \arrow[ddd, two heads, "\mathbb{R}g"] & \\
& & S \arrow[rrr, two heads] \arrow[ddd, two heads] \arrow[dddr, phantom, "\lrcorner", very near start] & & & \mathbb{R}_T C \arrow[ddd, two heads, "\mathbb{R}_T g"] \\
A \arrow[rrr, "f"'] \arrow[dr, hook, "\sim"'] & & & B \arrow[dr, hook, "\sim"] & & \\
& \mathbb{R}A \arrow[rrr, two heads, "\mathbb{R}f"'] \arrow[dr, hook, "\sim_T"'] & & & \mathbb{R}B \arrow[dr, hook, "\sim_T"] & \\
& & \mathbb{R}_T A \arrow[rrr, two heads, "\mathbb{R}_T f"'] & & & \mathbb{R}_T B.
\end{tikzcd} \tag{7.3.1}$$

where " $\sim$ "-arrows denote weak equivalences in $\mathbb{M}$ and " $\sim_T$ "-arrows denote weak equivalences in $\mathcal{L}_T\mathbb{M}$.

For part (1), assume $\mathcal{L}_T\mathbb{M}$ is right proper and let $f\colon A \to B$ be a map between T-local objects and $g\colon C \to B$. Then, in Diagram (7.3.1), the fibrant replacements $\mathbb{R}A$ and $\mathbb{R}B$ are T-local, too, and hence in fact already fibrant in the localization $\mathcal{L}_T\mathbb{M}$. So the fibration $\mathbb{R}f$ is a fibration in $\mathcal{L}_T\mathbb{M}$, too. But this implies that the map $Q \to S$ is a weak equivalence in $\mathcal{L}_T\mathbb{M}$, because $\mathcal{L}_T\mathbb{M}$ is right proper.

For part (2), suppose the localization is semi-left exact, let $f\colon A \twoheadrightarrow B$ be a fibration and $g\colon C \to B$ be a weak equivalence in $\mathcal{L}_T\mathbb{M}$. By [**10**, Lemma 9.4], without loss of generality assume that B is fibrant in $\mathcal{L}_T\mathbb{M}$. Then the map $P \to Q$ is a weak equivalence in $\mathbb{M}$ by right properness of $\mathbb{M}$ and fibrancy of f. Also, because B was assumed to be T-local, so are $\mathbb{R}B$ and hence $\mathbb{R}A$, thus the map $Q \to S$ is a weak equivalence by semi-left exactness of the localization. So all diagonal arrows in (7.3.1) are weak equivalences in $\mathcal{L}_T\mathbb{M}$. Since g was assumed to be a weak equivalence, by 2-for-3, the map $\mathbb{R}_T g$ is an acyclic fibration. Therefore, so is $S \to \mathbb{R}_T A$. But then, again by 2-for-3, the map $f^*g\colon P \to A$ is a weak equivalence in $\mathcal{L}_T\mathbb{M}$.

The other direction follows immediately from part (1). □

Remark 7.3.10. Lemma 7.3.9.(2) is intuitive in the sense that right properness assures that ordinary pullbacks along fibrations between fibrant objects are homotopy pullbacks, and both model categories $\mathbb{M}$ and $\mathcal{L}_T\mathbb{M}$ have the same underlying ordinary categorical structure. Lemma 7.3.9.(1) on the contrary states no compatibility conditions on any ∞-categorical structure in $\mathbb{M}$ and $\mathcal{L}_T\mathbb{M}$ and neither does it state any conditions which relate right properness and universal homotopy colimits. Nevertheless we obtain the following simple corollary relating right properness and universal homotopy colimits for a class of model categories obtained via left Bousfield localization.

Corollary 7.3.11. *Let $\mathbb{M}$ be a combinatorial model category with universal homotopy colimits and T be a set of maps in $\mathbb{M}$. If $\mathcal{L}_T\mathbb{M}$ is right proper, then $\mathcal{L}_T\mathbb{M}$ has universal homotopy colimits.*

Proof. Let $\mathbb{M}$ and $T \subseteq \mathbb{M}$ be as stated such that the localization $\mathcal{L}_T\mathbb{M}$ is right proper. By Lemma 7.3.9.(1) it follows that the localization $\mathbb{M} \to \mathcal{L}_T\mathbb{M}$ is semi-left exact. Hence, by Corollary 7.3.6 the model category $\mathcal{L}_T\mathbb{C}$ has universal homotopy colimits. □

The correspondence between semi-left exactness and right properness as phrased in Lemma 7.3.9.(2) was also observed by Balchin and Garner in [**3**, Lemma 40] for 1-dimensional model categories. Their lemma is a special case as 1-dimensional model categories are always right proper.

More generally, Shulman presented in [**13**] that right properness of $\mathbb{M}$ implies local cartesian closedness of $\mathrm{Ho}_\infty(\mathbb{M})$ whenever $\mathbb{M}$ has pullback stable cofibrations. In other words,

Lemma 7.3.12. *Let $\mathbb{M}$ be a model category with pullback stable cofibrations. If $\mathbb{M}$ is right proper, it has universal homotopy colimits.*

Proof. Let $f\colon X \to Z$ be a map with fibrant codomain in $\mathbb{M}$. Take a factorization $X \overset{\sim}{\hookrightarrow} \mathbb{R}X \overset{\mathbb{R}f}{\twoheadrightarrow} Z$ such that, if $\mathbb{M}$ is right proper, we have $f_h^* \simeq (\mathbb{R}f)^*$. But, assuming pullback stability of cofibrations, the functor

$$(\mathbb{R}f)^*\colon \mathbb{M}/Z \to \mathbb{M}/\mathbb{R}X$$

is a left Quillen functor which further preserves weak equivalences. Hence, it preserves homotopy colimits. □

Obviously, if the class of cofibrations in $\mathbb{M}$ is exactly the class of monomorphisms, it is pullback stable. This coincidence of cofibrations and monomorphisms is a standing assumption in all model categorical considerations in [**23**, Section 7], and hence, in their examples on simplicial presheaf categories, right properness and universality of homotopy colimits are synonymous notions by Lemma 7.3.12 and Lemma 7.3.9.

Example 7.3.13. Obviously, the category $\mathbf{S}$ of simplicial sets with the standard Quillen model structure is the prototype of a model topos, just as the category of sets is the prototype of an ordinary Grothendieck topos. In particular, every combinatorial model category Quillen equivalent to $\mathbf{S}$ is a model topos, including the category Grpd_Δ of simplicial groupoids equipped with the Dwyer-Kan model structure and the category of small categories equipped with the Thomason model structure.

Example 7.3.14. Recall the model structure $(s\mathbf{S}, \mathrm{CB})$ on bisimplicial sets for complete Bousfield-Segal spaces from Chapter 5. This is a model topos in virtue of the Quillen equivalence to $\mathbf{S}$ from Theorem 5.1.15, but note that the localization $(s\mathbf{S}, R_v) \to (s\mathbf{S}, \mathrm{CB})$ is not left exact.

Proposition 7.3.15. *The localization* $(s\mathbf{S}, R_v) \to (s\mathbf{S}, \mathrm{CB})$ *is not left exact.*

Proof. Since every map between non-empty (discrete simplicial) sets is a Kan fibration, every map $S \to T$ of simplicial sets induces a Reedy fibration $p_1^*S \to p_1^*T$ of bisimplicial sets. Let

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\wr"] \\ A \arrow[r] & B \end{tikzcd}$$

be a cartesian square in $\mathbf{S}$ such that $C \to B$ is a weak homotopy equivalence and its pullback $P \to A$ is not. Then

$$\begin{tikzcd} p_1^*P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & p_1^*C \arrow[d, "\wr"] \\ p_1^*A \arrow[r, two heads] & p_1^*B \end{tikzcd}$$

is cartesian in $s\mathbf{S}$, $p_1^*C \to p_1^*B$ is a weak equivalence in $(s\mathbf{S}, \mathrm{CB})$ and $p_1^*A \to p_1^*B$ is a Reedy fibration (although $A \to B$ is not a Kan fibration). Then $p_1^*P \to p_1^*A$ cannot be a weak equivalence in $(s\mathbf{S}, \mathrm{CB})$, because p_1^* is the left adjoint of a Quillen equivalence and hence reflects weak equivalences between cofibrant objects. In particular, the square cannot be homotopy cartesian in $(s\mathbf{S}, \mathrm{CB})$. But it certainly is homotopy cartesian in $(s\mathbf{S}, R_v)$, because $p_1^*A \to p_1^*B$ is a Reedy fibration and the Reedy model structure is right proper. $\square$

The fact that the localization associated to complete B-spaces is not left exact is not peculiar to the Reedy model structure, it is easy to induce that the localization $\mathcal{L}_{CB}(s\mathbf{S}_{\mathrm{proj}})$ of projective complete Bousfield-Segal spaces is not left exact either.

The localization nevertheless is semi-left exact, since we have shown that $(s\mathbf{S}, \mathrm{CB})$ is right proper in Section 5.3. But note that semi-left exactness also follows from commutativity of the square of left Quillen functors

$$\begin{tikzcd}
(\mathbf{S}\mathbb{C}, \mathrm{Qcat}) \arrow[r, "p_1^*"] \arrow[d, "\mathrm{id}"'] & (s\mathbf{S}, \mathrm{CS}) \arrow[d, "\mathrm{id}"'] \\
(\mathbf{S}, \mathrm{Kan}) \arrow[r, "p_1^*"] & (s\mathbf{S}, \mathrm{CB})
\end{tikzcd}$$

and the fact that, first, both horizontal maps are Quillen equivalences by [**32**, Theorem 4.11] and Theorem 5.1.14, and second, that the localization $\mathrm{id}\colon (\mathbf{S}, \mathrm{Qcat}) \to (\mathbf{S}, \mathrm{Kan})$ is semi-left exact by Lemma 7.3.9.(1) since $(\mathbf{S}, \mathrm{Kan})$ is right proper. Thus, conversely, this gives another proof of right properness of $(s\mathbf{S}, \mathrm{CB})$.

Remark 7.3.16. Classically, given toposes $\mathbb{C}$ and $\mathbb{D}$ and a reflective localization $L\colon \mathbb{C} \to \mathbb{D}$, clearly the localization is not necessarily left exact; mere existence of finite limits on both sides does not imply preservation of such under the left adjoint L. In other words, not every reflective localization between toposes is a geometric embedding. In the homotopical setting, this potential discrepancy shows in the example above; although both homotopy theories $(s\mathbf{S}, R_v)$ and $(s\mathbf{S}, \mathrm{CS})$ are ∞-toposes, their logical structure is not compatible.

An analogous situation arises for reflective localizations between locally cartesian closed categories. While it is shown in [**22**, Lemma 4.3] that the semi-left exact localization of a locally cartesian closed category is locally cartesian closed, it is noted in [**22**, Remark 4.4] that an arbitrary reflective localization $(L, R)\colon \mathbb{C} \to \mathbb{D}$ between locally cartesian closed categories is semi-left exact if and only if exponentials in the slices of $\mathbb{D}$ are preserved by the reflection R.

It is reasonable to expect a corresponding criterion for semi-left exactness and local cartesian closedness to arise in the ∞-categorical setting – indeed one direction is shown in [**23**, Proposition 1.4] – and it is interesting to note that the respective relationship between semi-left exactness and right properness does not require any such criteria, as we have seen in Lemma 7.3.9.(2).

We conclude this chapter restating Theorem 7.3.7 in light of Lemma 7.3.9.

Corollary 7.3.17. *Let* $\mathbb{M}$ *be a combinatorial model category. Then there is a simplicial category* $\mathbf{C}$*, a homotopy localization*

$$(L, R)\colon \mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}} \rightleftarrows \mathbb{M}$$

and a set $T \subset \mathrm{sPsh}(\mathbf{C})$ *of maps such that*

(1) $\mathcal{L}_T\mathrm{sPsh}(\mathbf{C})$ *is right proper if and only if* $\mathbb{M}$ *has universal homotopy colimits,*

(2) the homotopy localization induces a Quillen equivalence

$$(L, R)\colon \mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{proj}} \rightleftarrows \mathbb{M}.$$

□

Proving that semi-left exact localizations of presentable quasi-categories have a right proper model categorical presentation of the form $\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}})$ for some simplicial category $\mathbf{C}$ and some $T \subset \mathrm{sPsh}(\mathbf{C})$, it was shown in [**23**, 7] that every presentable locally cartesian closed quasi-category $\mathcal{C}$ can be presented by a type theoretic model category $\mathbb{M}$. The authors stated in the papers introduction that, whenever the quasi-category $\mathcal{C}$ is an ∞-topos and hence exhibits object classifiers, the model category $\mathbb{M}$ exhibits weakly universal fibrations. The main goal of the next chapter is to prove the correspondence of universal maps in $\mathrm{Ho}_\infty(\mathbb{M})$ and univalent weakly universal fibrations in $\mathbb{M}$, and further explore whether in fact we can present every ∞-topos by a type theoretic model category $\mathbb{M}$ which exhibits univalent strictly universal fibrations as needed to interpret Homotopy Type Theory as presented in [**41**].

CHAPTER 8

Comparing universes in quasi-categories and model categories

8.1. Statement of the goals

In the previous chapter we have seen how to present combinatorial model categories $\mathbb{M}$ as left Bousfield localizations of simplicial presheaf categories such that

- if $\mathbb{M}$ has universal homotopy colimits (Definition 7.1.6), the corresponding Bousfield localization is semi-left exact;
- if $\mathbb{M}$ has descent (Definition 7.2.6), the corresponding Bousfield localization is left exact.

The construction of such a presentation can be understood as replacing $\mathbb{M}$ with a Quillen equivalent model category $\mathrm{Pres}(\mathbb{M})$ such that some of the homotopy coherent structure in $\mathrm{Ho}_\infty(\mathbb{M})$ is presented by its ordinary categorical and hence strictly functorial counterpart in $\mathrm{Pres}(\mathbb{M})$. And it is such strict functoriality which is required in the categorical semantics of type theory. For example, 1-categorical exponentials in $\mathbb{M}$ (if they exist) do not necessarily model Π-types if the Frobenius property is not satisfied, as they will not necessarily be compatible with the construction rules of identity types. And neither do they necessarily present exponentials in the underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$. But both potential discrepancies are rectified in $\mathrm{Pres}(\mathbb{M})$ as constructed in Theorem 7.3.7 – when equipped with the injective model structure instead the projective one – because it is a Cisinski model category. In this sense, when $\mathbb{M}$ has universal homotopy colimits, we can understand the presentation result as a strictification procedure of homotopy coherent exponentials in $\mathrm{Ho}_\infty(\mathbb{M})$.

As noted in [**23**, Section 7], this presentation $\mathrm{Pres}(\mathbb{M})$ of the locally cartesian closed presentable quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ is in fact a type theoretic model category and supports all type constructors listed in [**41**, A.2] in the sense of [**51**] except univalent Tarski universes, whose existence in $\mathrm{Pres}(\mathbb{M})$ is yet to be verified. So, there is a natural interest in additionally lifting universes from $\mathrm{Ho}_\infty(\mathbb{M})$ – these are classifying objects, or more specifically object classifiers, of the form [**36**, Definition

6.1.6.1] – to Tarski universes in $\mathrm{Pres}(\mathbb{M})$. This comparison of universal maps is one of the guiding motivations of this chapter.

Therefore, in Section 8.2 we show that up to DK-equivalence every simplicial category can be replaced by the localization of a well founded poset as already observed by Shulman in [**53**]. It sets up prerequisite material for Sections 8.3 and 8.4 which otherwise are independent of each other. Section 8.3 is concerned with the comparison of κ-small fibrations in $\mathrm{Pres}(\mathbb{M})$ and relative κ-compact maps in $\mathcal{C}$. We will show the following two statements.

Theorem 8.3.11. Let $\mathbb{C}$ be a small category, $T \subset \mathrm{sPsh}(\mathbb{C})$ be a set of maps and $\mathbb{M} = \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$. Then for every sufficiently large inaccessible cardinal κ, a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact if and only if there is a κ-small fibration $p \in \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$ between fibrant objects such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.

Theorem 8.3.14. Let $\mathbf{C}$ be a small simplicial category and T a set of arrows in $\mathrm{sPsh}(\mathbf{C})$. Let $\mathbb{M}$ be the left Bousfield localization $\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}$.

(1) Every κ-small fibration $p \in \mathbb{M}$ between fibrant objects is relative κ-compact in the underlying quasi-category.
(2) If a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact, then there is a κ-small map $g \in \mathrm{sPsh}(\mathbf{C})$ such that $g \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.

The relevance of these results for the comparison of object classifiers in $\mathrm{Ho}_\infty(\mathbb{M})$ and (weak) Tarski universes in $\mathrm{Pres}(\mathbb{M})$ is explained in the end of Section 8.3.

Ultimately, this comparison is motivated by the informal question whether every Grothendieck ∞-topos $\mathcal{M}$ can presented by a model category $\mathbb{M}$ whose underlying fibration category $\mathbb{M}^f$ has an internal type theory $\mathcal{T}_{\mathbb{M}^f}$ in the sense of [**51**] which supports all type constructors listed in [**41**, A.2] and can be considered to be an *internal language of* $\mathcal{M}$. This requires some clarification. First, we briefly recall the definition of Grothendieck ∞-toposes.

The quasi-category of spaces is still denoted by $\mathcal{S}$. A quasi-category of presheaves is a quasi-category of the form $\mathcal{P}(K) := \mathrm{Fun}(K^{op}, \mathcal{S})$ for a simplicial set K as defined in [**36**, Definition 5.1.0.1]. Every such quasi-category is presented by a simplicial presheaf category $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ (or equivalently $\mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}}$) over some simplicial category $\mathbf{C}$ by [**36**, Proposition 5.1.1.1]. Following [**36**, Definition 6.1.0.4], a Grothendieck ∞-topos is a quasi-category equivalent to a left exact localization of a quasi-category of presheaves. Every such quasi-category is presented by a left exact left Bousfield localization $\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}$ over some simplicial category $\mathbf{C}$ in the sense of Definition 7.3.1. This follows directly from [**36**, Proposition 5.1.1.1], from [**36**, Theorem 4.2.4.1] which relates homotopy pullbacks in $\mathbb{M}$ with pullbacks in $\mathrm{Ho}_\infty(\mathbb{M})$, and from the bijective correspondence between left Bousfield localizations of a simplicial combinatorial model category $\mathbb{M}$ and reflective localizations of $\mathrm{Ho}_\infty(\mathbb{M})$.

Second, the notion of an internal language of an ∞-topos has not been made mathematically precise yet in the literature and therefore is somewhat problematic. The only rigorous way to date to assign a type theory to an ∞-topos $\mathcal{M}$ factors through homotopical algebra, that is for example by choosing a model categorical presentation $\mathbb{M}$ of $\mathcal{M}$ which is type theoretic in the sense of [**51**].

In that sense, recall the "compiler" (1) between the three languages from the Introduction. Then we say that $\mathbb{M}$ yields an internal language for its associated quasi-category $\mathcal{M}$ if the homotopical algebraic (and hence 1-categorical) constructions in $\mathbb{M}$ yield a strictly functorial model of two different languages at the same time. To pick one basic example, on the one hand, we want that ordinary strict pullbacks of fibrations in $\mathbb{M}$ interpret reindexing of type families in the type theory $\mathcal{T}_{\mathbb{M}^f}$. But, on the other hand, we also want such pullbacks to model pullbacks in the underlying quasi-category of $\mathbb{M}$. If $\mathcal{T}_{\mathbb{M}^f}$ is an "internal language" of $\mathcal{M}$, we would expect a corresponding simultaneous interpretation to be supported by basically all categorical structure of $\mathbb{M}$ that is expressible in the type theory $\mathcal{T}_{\mathbb{M}^f}$ and the quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$. That includes composition (of fibrations), dependent products (of fibrations) and, to come back to our initially expressed interest, Tarski universes.

In Section 8.4 we show that, even if we find a presentation $\mathbb{M}$ of $\mathcal{M}$ which comes equipped with a type theory $\mathcal{T}_{\mathbb{M}^f}$, a twofold interpretation of its categorical structure in the type theoretical and the higher categorical sense as referred to above may fail. More precisely, we show that every presheaf ∞-topos is presented by a model category $\mathbb{M}$ that comes equipped with an underlying Homotopy Type Theory, which in a large class of examples cannot be considered to be an internal language of $\mathcal{M}$ because of a discrepancy between homotopy equivalences and weak equivalences between fibrant objects in these model categories $\mathbb{M}$.

8.2. Replacing simplicial categories with direct posets

Mike Shulman noted in [**53**, Lemma 0.2] that every quasi-category can be presented by the localization of a direct – in other words, well founded – poset.[1] In this section we present a slight variation of his observation and discuss the resulting presentations of associated presheaf ∞-categories. Although the following sections only will require the fact that every quasi-category can be presented by the localization of an Eilenberg-Zilber Reedy category, proving the stronger condition of posetality only requires about as much work as the Eilenberg-Zilber Reedy condition itself.

Recall the following constructions and notation from [**5**]. A *relative category* is a pair $(\mathbb{C}, V)$ such that $\mathbb{C}$ is a category and V is a subcategory of $\mathbb{C}$. A *relative functor* $F\colon (\mathbb{C}, V) \to (\mathbb{D}, W)$ is a functor $F\colon \mathbb{C} \to \mathbb{D}$ of categories such that $F[V] \subseteq W$. The relative functor F is a *relative inclusion* if its underlying functor of categories is an inclusion and $V = W \cap \mathbb{C}$. The category of small relative categories and relative functors is denoted by **RelCat**.

There are two canonical inclusions of the category **Cat** of small categories into **RelCat**; for a category $\mathbb{C}$ and its discrete wide subcategory $\mathbb{C}_0$, we obtain the associated minimal relative category $\check{\mathbb{C}} := (\mathbb{C}, \mathbb{C}_0)$ and the associated maximal relative category $\hat{\mathbb{C}} := (\mathbb{C}, \mathbb{C})$.

In [**5**, Section 5.3], Barwick and Kan introduce a combinatorial sub-division operation $\xi\colon \mathbf{RelCat} \to \mathbf{RelCat}$ and an associated bisimplicial nerve construction $N_\xi\colon \mathbf{RelCat} \to s\mathbf{S}$ giving rise to the adjoint pair

$$s\mathbf{S} \underset{N_\xi}{\overset{K_\xi}{\rightleftarrows}} \mathbf{RelCat}.$$

The left adjoint K_ξ is given by $K_\xi(\Delta^m \square \Delta^n) = \xi(\check{[m]} \times \hat{[n]})$ and left Kan extension along the Yoneda embedding. The authors of [**5**] have shown that the category **RelCat** inherits a transferred model structure $(\mathbf{RelCat}, \mathrm{BK})$ from the Reedy model structure $(s\mathbf{S}, R_v)$ which turns the pair (K_ξ, N_ξ) into a Quillen equivalence. By construction, the set $K_\xi[\mathcal{I}_v]$ forms a set of generating cofibrations for the model structure in question, where

$$\mathcal{I}_v := \{\delta_n \square' \delta_m \mid n, m \in \mathbb{N}\}$$

[1] Shulman in fact argues for a presentation by inverse posets. But since localization commutes with taking opposite categories, this amounts to the same statement.

is the generating set of monomorphisms in $s\mathbf{S}$ defined in (4.2.1).

A central notion of [**5**] is that of "Dwyer maps" in **RelCat**. A relative functor $F\colon (\mathbb{C}, V) \to (\mathbb{D}, W)$ is a *Dwyer inclusion* if F is a relative inclusion such that $(\mathbb{C}, V)$ is a sieve in $(\mathbb{D}, W)$ and such that the cosieve $Z\mathbb{C}$ generated by $(\mathbb{C}, V)$ in $(\mathbb{D}, W)$ comes equipped with a strong deformation retraction $Z\mathbb{C} \to (\mathbb{C}, V)$. The relative functor F is a *Dwyer map* if it factors as an isomorphism followed by a Dwyer inclusion, see [**5**, Section 3.5] for more details.

A major insight of the authors was that the generating cofibrations

$$K_\xi(\delta_m\Box'\delta_n)\colon K_\xi((\Delta^m\Box\partial\Delta^n)\cup_{\partial\Delta^m\Box\partial\Delta^n}(\partial\Delta^m\Box\Delta^n)) \to K_\xi(\Delta^m\Box\Delta^n)$$

of the model category $(\mathbf{RelCat}, \mathrm{BK})$ are Dwyer maps of relative posets ([**5**, Proposition 9.5]). It follows that every cofibration in $(\mathbf{RelCat}, \mathrm{BK})$ is a Dwyer map as shown in [**5**, Theorem 6.1].

Proposition 8.2.1. *The underlying category of a cofibrant object in* $(\mathbf{RelCat}, \mathrm{BK})$ *is a direct (i.e. well founded) poset.*

Proof. Since the empty relative category $\emptyset$ is a relative direct poset, it suffices to show that for every cofibration $(\mathbb{P}, V) \hookrightarrow (\mathbb{Q}, W)$ where $(\mathbb{P}, V)$ is a relative direct poset also $(\mathbb{Q}, W)$ is a relative direct poset. We show this by "induction along the small object argument" as follows.

The generating cofibrations $K_\xi(\delta_m\Box'\delta_n)$ are maps between finite relative posets and such are clearly direct. Both Dwyer maps and relative posets are closed under coproducts and under pushouts along Dwyer maps between relative posets by [**5**, Proposition 9.2], and it is easy to see that both constructions preserve well foundedness, too. Suppose we are given a transfinite composition of Dwyer maps $A_\alpha \to A_\beta$ for $\alpha < \beta \leq \lambda$ ordinals and A_α relative inverse posets. Again by [**5**, Proposition 9.2], the colimit A_λ is a relative poset. Suppose $a = (a_i \mid i < \omega)$ is a descending sequence of arrows in A_λ and let $\alpha < \lambda$ such that $a_0 \in A_\alpha$. Then the whole sequence a is contained in A_α, because the inclusion $A_\alpha \hookrightarrow A_\lambda$ is a Dwyer map by [**5**, Proposition 9.3] and so $A_\alpha \subseteq A_\lambda$ is a sieve (see [**5**, 3.5]). Therefore, the sequence a is finite.

In particular, every free cofibration $\emptyset \hookrightarrow (\mathbb{P}, V)$ – that is every transfinite composition of pushouts of generating cofibrations with domain $\emptyset$ – yields a relative direct poset $(\mathbb{P}, V)$. But every cofibration $\emptyset \hookrightarrow (\mathbb{Q}, W)$ is a retract of such, and hence every cofibrant object in **RelCat** is a relative direct poset. $\Box$

Remark 8.2.2. The same proof shows that the cofibrant objects in the Thomason model structure on **Cat** are direct posets, using Thomason's original observation that the cofibrant objects in the Thomason model structure are posetal in the first place.

Let $F_\Delta \colon \mathbf{Cat} \to \mathbf{S}\text{-}\mathbf{Cat}$ be the Bar construction obtained in the standard way by comonad resolution of the free category functor F from reflexive Graphs to **Cat**. Recall that F_Δ is not the left adjoint to the underlying-category functor $(\cdot)_0$, but, as often remarked in the literature, a cofibrant replacement of this left adjoint. Furthermore, for example from [**19**], recall the (standard) simplicial localization functor

$$\mathcal{L}_\Delta \colon \mathbf{RelCat} \to \mathbf{S}\text{-}\mathbf{Cat}$$

which takes a relative category $(\mathbb{C}, V)$ to the simplicial category given in degree $n < \omega$ by

$$\mathcal{L}_\Delta(\mathbb{C}, V)_n = F_\Delta(\mathbb{C})_n[F_\Delta(V)_n^{-1}].$$

The simplicial category $\mathcal{L}_\Delta(\mathbb{C}, V)$ is in fact the enriched localization of $F_\Delta(\mathbb{C})$ at $F_\Delta(V)$ in the sense that, for every simplicial category $\mathbb{D}$ and $\mathbf{S}\text{-}\mathrm{Cat}(F_\Delta(\mathbb{C}), \mathbb{D})^{F_\Delta V \mapsto \mathrm{Iso}}$ the category of simplicial functors which map $F_\Delta V$ to the core of $\mathbb{D}$, we obtain a natural isomorphism

$$\mathbf{S}\text{-}\mathbf{Cat}(F_\Delta(\mathbb{C}), \mathbb{D})^{F_\Delta W \mapsto \mathrm{Iso}} \cong \mathbf{S}\text{-}\mathbf{Cat}(L_\Delta(\mathbb{C}, V), \mathbb{D}).$$

This universal property together with the corresponding observation that presheaves $X \colon \mathcal{L}_\Delta(\mathbb{C}, V)^{op} \to \mathbf{S}$ are exactly the presheaves $F_\Delta(\mathbb{C})^{op} \to \mathbf{S}$ which take maps in V to isomorphisms in $\mathbf{S}$, enables us to prove the following proposition in the same way as we would prove it for localizations in ordinary category theory.

Proposition 8.2.3. *For $(\mathbb{C}, V) \in \mathbf{RelCat}$ and $j \colon F_\Delta(\mathbb{C}) \to \mathcal{L}_\Delta(\mathbb{C}, V)$ the associated localization functor, the induced restriction*

$$j^* \colon \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V)) \to \mathrm{sPsh}(F_\Delta(\mathbb{C}))$$

is fully faithful.

Proof. Let $\mathbf{S}\text{-}\mathbf{Cat}(\mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V)), \mathbb{D})^{\mathrm{cocont.}}$ denote the full subcategory of colimit preserving simplicial functors in $\mathbf{S}\text{-}\mathbf{Cat}(\mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V)), \mathbb{D})$. We note that for every cocomplete simplicial category $\mathbb{D}$, by the universal property of simplicial localizations as stated above and (point-wise) left Kan extension we obtain a natural isomorphism

$$\begin{aligned}
\mathbf{S\text{-}Cat}(\mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C},V)),\mathbb{D})^{\text{cocont.}} &\cong \mathbf{S\text{-}Cat}(\mathcal{L}_\Delta(\mathbb{C},V)),\mathbb{D})\\
&\cong \mathbf{S\text{-}Cat}(F_\Delta(\mathbb{C}),\mathbb{D})^{W\mapsto\mathrm{Iso}}\\
&\cong \mathbf{S\text{-}Cat}(\mathrm{sPsh}(F_\Delta(\mathbb{C})),\mathbb{D})^{\text{cocont.},y[W]\mapsto\mathrm{Iso}}\\
&\cong \mathbf{S\text{-}Cat}(\mathrm{sPsh}(F_\Delta(\mathbb{C}))[y[V]^{-1}],\mathbb{D})^{\text{cocont.}}
\end{aligned}$$

This induces an equivalence $\mathrm{up}(j_!)$ in the triangle

$$\begin{tikzcd}
\mathrm{sPsh}(F_\Delta(\mathbb{C})) \arrow[r, "j_!"] \arrow[d, "L"'] & \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C},V)) \\
\mathrm{sPsh}(F_\Delta(\mathbb{C}))[y[V]^{-1}]. \arrow[ur, "\mathrm{up}(j_!)"'] &
\end{tikzcd}$$

which without loss of generality we assume to be an adjoint equivalence. But the localization $L\colon \mathrm{sPsh}(F_\Delta(\mathbb{C})) \to \mathrm{sPsh}(F_\Delta(\mathbb{C}))[y[V]^{-1}]$ is reflective and hence exhibits a fully faithful right adjoint ι. Commutativity of the triangle implies commutativity of the corresponding right adjoints, so that $\iota\circ(j_!)^{-1} = j^*$ holds. Thus, j^* is the composition of two fully faithful functors and hence fully faithful itself. □

Therefore, the map $j\colon F_\Delta(\mathbb{C}) \to \mathcal{L}_\Delta(\mathbb{C},V)$ induces both a localization

$$(j_!, j^*)\colon \mathrm{sPsh}(F_\Delta(\mathbb{C})) \to \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C},V))$$

and a colocalization

$$(j^*, j_*)\colon \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C},V)) \to \mathrm{sPsh}(F_\Delta(\mathbb{C}))$$

between simplicial presheaf categories. Equipping both sides with the injective model structure, the pair (j^*, j_*) becomes a Quillen pair. Its derived adjoint pair on underlying quasi-categories is the fully faithful left Kan extension $j^*\colon \mathcal{P}(N(\mathbb{C})[V^{-1}]) \to \mathcal{P}(N(\mathbb{C}))$ for $j\colon N(\mathbb{C}) \to N(\mathbb{C})[V^{-1}]$ together with its right adjoint j_*. It hence also gives rise to a colocalization of underlying quasi-categories. Hence, equipping both sides with the injective model structure, the pair (j^*, j_*) becomes a homotopy colocalization. Dually, equipping both sides with the projective model structure, the pair $(j_!, j^*)$ becomes a homotopy localization.

Remark 8.2.4. If one chooses to work with any other homotopical localization of $(\mathbb{C},V)$ such as the hammock localization $\mathcal{L}_H(\mathbb{C},V)$ an analogue of the functor

j^* still exists and also induces a localization $(j_!, j^*)$ and colocalization (j^*, j_*) on underlying quasi-categories.

Lemma 8.2.5. *For* $(\mathbb{C}, V) \in \mathbf{RelCat}$*, the functor* $j^* \colon \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V)) \to \mathrm{sPsh}(\mathbb{C})$ *induces a Quillen equivalence*

$$(j_!, j^*) \colon \mathcal{L}_{y[V]}\mathrm{sPsh}(\mathbb{C})_{\mathrm{proj}} \to \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))_{\mathrm{proj}}.$$

Proof. The $(\infty, 1)$-categorical content of this statement seems to be folklore and was also used in [**53**, Lemma 0.1]. The Quillen pair

$$(j_!, j^*) \colon \mathrm{sPsh}(\mathbb{C})_{\mathrm{proj}} \to \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))_{\mathrm{proj}}$$

is a homotopy localization as noted above, and it takes every map in $y[V]$ to a weak equivalence. By [**25**, Proposition 3.3.18.(1)], we hence obtain a homotopy localization

$$(j_!, j^*) \colon \mathcal{L}_{y[V]}\mathrm{sPsh}(\mathbb{C})_{\mathrm{proj}} \to \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))_{\mathrm{proj}}.$$

The fact that this Quillen pair is a Quillen equivalence can be seen on underlying quasi-categories, where it follows that the induced reflective localization is an equivalence by essentially the same computations we performed in the proof of Proposition 8.2.3. $\square$

We will show the dual of this lemma in Section 8.4 for the injective model structure. The localization functor $\mathcal{L}_\Delta \colon \mathbf{RelCat} \to \mathbf{S\text{-}Cat}$ has a homotopy inverse, the “delocalization” or “flattening”

$$\flat \colon \mathbf{S\text{-}Cat} \to \mathbf{RelCat},$$

given by the Grothendieck construction of its input $\mathbf{C}^{op} \colon \Delta \to \mathrm{Cat}$. This functor was introduced in [**20**, Theorem 2.5] and is analysed in detail in [**4**].

Now, given a simplicial category $\mathbf{C}$, consider its delocalization $\flat(\mathbf{C}) \in \mathbf{RelCat}$. Cofibrantly replacing $\flat(\mathbf{C})$ by some pair (I, V) in $\mathbf{RelCat}$ yields a direct relative poset (I, V) weakly equivalent – i.e. Rezk-equivalent in the language of [**4**] – to $\flat(\mathbf{C})$. Hence, by [**4**, Theorem 1.8], the simplicial localization $\mathcal{L}_\Delta(I, V) \in \mathrm{Cat}_\Delta$ is DK-equivalent to the original simplicial category $\mathbf{C}$, i.e. there is a zig-zag of DK-equivalences

$$(*) \qquad \mathbf{C} \xrightarrow{f_1} \dots \xleftarrow{f_n} \mathcal{L}_\Delta(I, V).$$

By [**36**, Proposition A.3.3.8] or [**20**, Theorem 2.1] and the sequence of maps in $(*)$, we obtain a zig-zag of Quillen equivalences

$$\mathrm{sPsh}(\mathcal{L}_\Delta(I,V))_{\mathrm{inj}} \underset{(f_n)_*}{\overset{f_n^*}{\rightleftarrows}} \dots \underset{(f_1)_*}{\overset{f_1^*}{\leftrightarrows}} (\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}.$$

Further recall from [**19**, Proposition 2.6] that for every category $\mathbb{C}$ the canonical projection $\varphi\colon F_\Delta\mathbb{C} \to \mathbb{C}$ is a DK-equivalence of simplicial categories. So, to summarize, we have seen the following.

Proposition 8.2.6. *Let $\mathbf{C}$ be a simplicial category. Then there is a direct relative poset (I,V) together with a zig-zag of DK-equivalences*

$$\mathbf{C} \to \dots \leftarrow \mathcal{L}_\Delta(I,V)$$

in Cat_Δ which induces a zig-zag of Quillen pairs

$$\mathrm{sPsh}(I)_{\mathrm{inj}} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} (\mathrm{sPsh}(F_\Delta I))_{\mathrm{inj}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{sPsh}(\mathcal{L}_\Delta(I,V))_{\mathrm{inj}} \underset{(f_n)_*}{\overset{f_n^*}{\rightleftarrows}} \dots \underset{(f_1)_*}{\overset{f_1^*}{\leftrightarrows}} \mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$$

such that (j^*, j_*) is a homotopy colocalization and all other pairs are Quillen equivalences.

□

Dually, on projective model structures we obtain the following chain of Quillen pairs.

Proposition 8.2.7. *Let $\mathbf{C}$ be a simplicial category. Then there is a direct relative poset (I,V) together with a zig-zag of DK-equivalences*

$$\mathbf{C} \to \dots \leftarrow \mathcal{L}_\Delta(I,V)$$

in Cat_Δ which induces a zig-zag of Quillen pairs

$$\mathrm{sPsh}(I)_{\mathrm{proj}} \underset{\varphi^*}{\overset{\varphi_!}{\leftrightarrows}} (\mathrm{sPsh}(F_\Delta I))_{\mathrm{proj}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathrm{sPsh}(\mathcal{L}_\Delta(I,V))_{\mathrm{proj}} \underset{f_n^*}{\overset{(f_n)_!}{\leftrightarrows}} \dots \underset{f_1^*}{\overset{(f_1)_!}{\rightleftarrows}} (\mathrm{sPsh}(\mathbf{C}))_{\mathrm{proj}}$$

such that (j^*, j_*) is a homotopy localization and all other pairs are Quillen equivalences.

□

8.3. Comparing compactness in quasi-categories and model categories

In order to show that small object classifiers in quasi-categories of the form [**36**, Definition 6.1.6.1, Definition 6.1.6.4] yield univalent fibrations which are weakly universal for the class of small fibrations in suitable model categorical presentations – and vice versa – we have to translate between two notions of smallness. This translation is the goal of this section.

We start by stating some ordinary categorical facts about compactness in presheaf categories. Given a small category $\mathbb{C}$ and a cardinal $\kappa > |\mathbb{C}|$, recall that a (set-valued) presheaf $X \in \widehat{\mathbb{C}}$ is κ-small if all its values $X(C)$ have cardinality smaller than κ.

Lemma 8.3.1.

(1) Let κ be an infinite regular cardinal. Then a set X is κ-compact if and only if $|X| < \kappa$.

(2) Let $\mathbb{C}$ be a small category and $\kappa > |\mathbb{C}|$ an infinite regular cardinal. Then

(a) An object $X \in \widehat{\mathbb{C}}$ is κ-compact if and only if it is κ-small.

(b) A map $f \in \widehat{\mathbb{C}}$ is relative κ-compact if and only if it is κ-small.

Proof. To prove part (1), suppose X is κ-compact. Since κ is regular, it is the κ-filtered colimit $\bigcup_{\mu<\kappa} \mu$. Hence, every map $|X| \to \kappa$ factors through a cardinal $\mu < \kappa$. In particular, there is no cofinal map from $|X|$ to κ and so $|X| < \kappa$ follows. The other direction is straightforward.

For part ($2a$), by the given assumptions and part (1), it suffices to show that an object $X \in \widehat{\mathbb{C}}$ is κ-compact if and only if every set $X(C)$ for $C \in \mathbb{C}$ is κ-compact. Therefore, assume that every set $X(C)$ is κ-compact. Then, using that $\kappa \geq |\mathbb{C}|$ is infinite, it is straightforward to see that, for every κ-filtered small category I and every functor $F\colon I \to \widehat{\mathbb{C}}$, the natural map

$$\mathrm{colim}_I \widehat{\mathbb{C}}(X, F(\cdot)) \to \widehat{\mathbb{C}}(X, \mathrm{colim}_I F)$$

is a bijection. Vice versa, suppose that $X \in \widehat{\mathbb{C}}$ is κ-compact and let $C \in \mathbb{C}$. Since both categories $\widehat{\mathbb{C}}$ and Set are locally κ-presentable ([**1**, Remark 1.20]), the evaluation $\mathrm{ev}_C\colon \widehat{\mathbb{C}} \to \mathrm{Set}$ has a κ-accessible right adjoint R_C given by $R_C(A) := \mathrm{Set}(\mathbb{C}(C, (\cdot)), A)$. Therefore, for every κ-filtered small category I and every functor $F\colon I \to \widehat{\mathbb{C}}$, we have

$$\widehat{\mathbb{C}}(X(C), \mathrm{colim}_I F) \cong \widehat{\mathbb{C}}(X, R_C(\mathrm{colim}_I F)$$

$$
\begin{aligned}
&\cong \widehat{\mathbb{C}}(X, \mathrm{colim}_I R_C F) \\
&\cong \mathrm{colim}_I \widehat{\mathbb{C}}(X, R_C F(\cdot)) \\
&\cong \mathrm{colim}_I \widehat{\mathbb{C}}(X(C), F(\cdot)).
\end{aligned}
$$

Part $(2b)$ is immediate by $(2a)$ since pullbacks are computed point-wise. □

Lemma 8.3.2. *Let $\mathbb{C}$ and $\mathbb{D}$ be small categories and let*

$$F\colon \widehat{\mathbb{C}} \rightleftarrows \widehat{\mathbb{D}}\colon G$$

be an adjoint pair. Let $\kappa > |\mathbb{C}| \cdot |\mathbb{D}|$ be regular (and inaccessible) and suppose F takes representables to representables (preserves κ-small objects). Then G preserves κ-small maps.

Proof. Let $\theta\colon [FX, Y]_{\widehat{\mathbb{C}}} \to [X, GY]_{\widehat{\mathbb{D}}}$ be the natural isomorphism associated to the adjunction $F \dashv G$. Let $f\colon X \to Y$ be a κ-small map in $\widehat{\mathbb{D}}$ and $g\colon yC \to GY$ be an element of Y. We have to show that for every $C' \in \mathbb{C}$ the hom-set $[yC', g^*GX]_{\widehat{\mathbb{C}}}$ is κ-small, but

$$
\begin{aligned}
[yC', g^*GX]_{\widehat{\mathbb{C}}} &\cong [yC', yC]_{\widehat{\mathbb{C}}} \times_{[yC', GY]_{\widehat{\mathbb{C}}}} [yC', GX]_{\widehat{\mathbb{C}}} \\
&\cong [C', C]_{\mathbb{C}} \times_{[FyC', Y]_{\widehat{\mathbb{D}}}} [FyC', X]_{\widehat{\mathbb{D}}} \\
&\cong \bigcup_{h \in [C', C]_{\mathbb{C}}} \left\{ \begin{tikzcd} yFC' \arrow[rr, dashed, "k"] \arrow[dr, "\theta(yh)\circ Gg"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd} \right\}.
\end{aligned}
$$

The hom-set $[C', C]_{\mathbb{C}}$ is κ-small by assumption. If F preserves representables, the object FyC' is a representable and hence $[FyC', X]_{[\mathbb{D}/Y]}$ is κ-small, too. If F preserves κ-compact objects, the object FyC' is κ-compact and hence $[FyC', X]_{\widehat{\mathbb{D}}/Y}$ is κ-small by inaccessibility of κ. Either way, it follows that $[yC', g^*GX]_{\widehat{\mathbb{C}}}$ is κ-small by regularity of κ. □

Given a small simplicial category $\mathbf{C}$, we say that a simplicial presheaf $X \in \mathrm{sPsh}(\mathbf{C})$ is κ-small if for every object $C \in \mathbb{C}$ the simplicial set $X(C)$ is a κ-small (set-valued) presheaf in the sense of Section 2.5. A map $f\colon X \to Y$ in $\mathrm{sPsh}(\mathbf{C})$ is κ-small if for every object $C \in \mathbf{C}$ and every element $g\colon yC \to Y$, the pullback g^*X is a κ-small simplicial set. We denote the cardinality of $\mathbf{C}$ by

$$|\mathbf{C}| := \bigcup_{C, D \in \mathbf{C}} |[C, D]_{\mathbf{C}}|$$

where $[C, D]_{\mathbf{C}} \in \mathbf{S}$ is the hom-space of $\mathbf{C}$. The cardinality of a simplicial set was defined in Section 2.5.

Remark 8.3.3. It is easy to show that Lemma 8.3.1 and Lemma 8.3.2 also hold with respect to this notion of smallness when we replace "ordinary" categories and set-valued presheaves with simplicial categories and simplicial presheaves.

The aim of the rest of this section is to compare this ordinary notion of compactness in a combinatorial model category $\mathbb{M}$ with the notion of compactness in its underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ as defined in [**36**, Definition 5.3.4.5] and [**36**, Definition 6.1.6.4]. The validity of this comparison was addressed in a question posted in [**37**] by Shulman, versions of it are given in Proposition 8.3.5, Theorem 8.3.11 and Theorem 8.3.14. A proof of the object-wise statement – that is Proposition 8.3.5 – was outlined by Lurie in the same post which in one direction coincides with our proof given in Proposition 8.3.5. Before we state the theorems, we make the following ad hoc construction and give one auxiliary lemma.

Given a λ-accessible quasi-category $\mathcal{C}$ with generating set A and a regular cardinal $\mu \geq \lambda$, define the full subcategory $J^\mu \subseteq \mathcal{C}$ recursively as follows. Let

$$J_0^{\mu,0} := A$$

and $J^{\mu,0}$ be the full subcategory of $\mathcal{C}$ generated by $J_0^{\mu,0}$. Whenever $\beta < \mu$ is a limit ordinal, let

$$J_0^{\mu,\beta} = \bigcup_{\alpha<\beta} J_0^{\mu,\alpha}$$

and $J^{\mu,\beta}$ the full subcategory generated by $J_0^{\mu,\beta}$. On successors, given $J^{\mu,\alpha}$, let

$$J_0^{\mu,\alpha+1} := \{\mathrm{colim} F \mid F\colon I \to J^{\mu,\alpha}, I \in \mathrm{QCat} \text{ is } \mu\text{-small and } \lambda\text{-filtered}\} \tag{8.3.1}$$

(so we choose a set of representatives $V_\mu^{\Delta^{op}}$ for μ-small simplicial sets) and $J^{\mu,\alpha+1}$ be the corresponding full subcategory. Eventually, we define the full subcategory J^μ of $\mathcal{C}$ to have the set of objects

$$J_0^\mu := \bigcup_{\alpha<\mu} J_0^{\mu,\alpha}.$$

Notation. For two regular cardinals μ, κ recall the relation $\mu \ll \kappa$ from [**36**, Definition 5.4.2.8] which holds if and only if for all cardinals $\kappa_0 < \kappa$ and $\mu_0 < \mu$ we have $\kappa_0^{\mu_0} < \kappa$.

The relation "$\ll$" is a special case of the *sharply smaller* relation from [**1**, Definition 2.12] as explained in [**1**, Examples 2.13.(4)]. The order "$\ll$" is chosen in such a way that whenever $\mu \ll \kappa$ holds, then $\mu \leq \kappa$ and μ-accessibility of a quasi-category $\mathcal{C}$ implies κ-accessibility of $\mathcal{C}$ ([**36**, Proposition 5.4.2.11]). As noted in [**36**, Section 5.4.2], the order is unbounded in the class of cardinals as for any cardinal κ we have $\kappa \ll \sup(\tau^\kappa \mid \tau < \kappa)^+$. Furthermore, whenever we have $\lambda < \kappa$ and $\kappa \ll \mu$, then also $\lambda \ll \mu$. Thus, it is easy to see that for any set X of cardinals there is a regular cardinal μ such that $\kappa \ll \mu$ for all $\kappa \in X$.

The following lemma is noted in [**36**, Section 5.4.2] and a generalization of the corresponding 1-categorical statement that can be found in [**1**, Remark 2.15] for accessible categories.

Lemma 8.3.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable quasi-categories.*

(1) Suppose $\mathcal{C}$ is λ-presentable. Then, for every regular $\mu \gg \lambda$, the μ-compact objects in $\mathcal{C}$ are, up to equivalence, exactly the retracts of objects in J^μ.

(2) Let $F\colon \mathcal{C} \to \mathcal{D}$ be an accessible functor. Then there is a cardinal μ such that F preserves κ-compact objects for all regular $\kappa \gg \mu$.

Proof. We show part (1). Let A be a generating set for $\mathcal{C}$, $\mu \gg \lambda$. For one direction, as μ-compact objects are closed under retracts ([**36**, Remark 5.3.4.16]), it suffices to show that objects in J^μ are μ-compact. We proceed by induction. Every object in $J^{\mu,0} = A$ is λ-compact by [**36**, Proposition 5.3.5.5], so is μ-compact in particular. The limit step is trivial. Suppose all objects in $J^{\mu,\alpha}$ are μ-compact. Then, being μ-small colimits of μ-compact objects, all objects in $J^{\mu,\alpha+1}$ are μ-compact, too, as μ-filtered colimits commute with μ-small limits in $\mathcal{S}$ by [**36**, Proposition 5.3.3.3]. Vice versa, suppose $X \in \mathcal{C}$ is μ-compact. We have $X \simeq \operatorname{colim}(A_{/X} \to \mathcal{C})$ and so

$$X \simeq \operatorname{colim}(J^\mu_{/X} \xrightarrow{\iota} \mathcal{C}),$$

too. This can be seen by noting that the quasi-category J^μ is the smallest quasi-category which contains A and is closed under μ-small λ-filtered colimits. It hence follows by [**36**, Lemma 5.3.5.8, Remark 5.3.5.9] that the continuous functor $J^\mu_{/X} \to \mathcal{C}$ is the left Kan extension of $A_{/X} \to \mathcal{C}$ along $\iota_\mu\colon A \hookrightarrow J^\mu$. But this extension is an evaluation of the "global" Kan extension $\operatorname{Lan}_{\iota_\mu}\colon \mathcal{C}^A \to \mathcal{C}^{J^\mu}$ which is left adjoint to the restriction ι_μ^* and exists due to the Adjoint Functor Theorem [**36**, Corollary

5.5.2.9]. Since the triangle

$$
\begin{tikzcd}
\mathcal{C}^{J^{\mu}_{/X}} \arrow[rr, "\iota_{\mu}^{*}"] & & \mathcal{C}^{A/X} \\
& \mathcal{C} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

of right adjoints commutes, the corresponding triangle of left adjoints commutes, too, which is to say that $\mathrm{Lan}_{\iota_{\mu}}$ commutes with the corresponding colimit functors. This proves $X \simeq \mathrm{colim}(J^{\mu}_{/X} \xrightarrow{\iota} \mathcal{C})$.

But note that J^{μ} is μ-filtered. To see that, in a nutshell, let K be a μ-small quasi-category and $G\colon K \to J^{\mu}$ be a functor. Let $\mathrm{Ind}_{\lambda}K$ be the Ind-completion of K and $\mathrm{Lan}G\colon \mathrm{Ind}_{\lambda}K \to J^{\mu}$ be the corresponding left Kan extension of G ([**36**, Lemma 5.3.5.8, Remark 5.3.5.9]). Then by the same reasoning as above, we have $\mathrm{colim}G \simeq \mathrm{colim}\mathrm{Lan}G$. But $\mathrm{Ind}_{\lambda}K$ is λ-filtered, and it is essentially μ-small because $\lambda \ll \mu$ (this can be proven by a similar inductive argument or along the lines of [**36**, Proposition 5.3.5.12]). So, by regularity of μ, there is $\alpha < \mu$ such that $\mathrm{Lan}G\colon \mathrm{Ind}_{\lambda}K \to J^{\mu}$ factors through $J^{\mu,\alpha}$ and hence $\mathrm{colim}\mathrm{Lan}G \in J^{\mu,\alpha+1}$ by definition of the sequence $(J^{\mu,\alpha})_{\alpha<\mu}$.

Therefore, since X is assumed to be μ-compact, the natural map

$$\mathrm{colim}\mathcal{C}(X, \iota\,\_) \to \mathcal{C}(X, X)$$

is an equivalence and hence gives a bijection on homotopy categories. Thus we obtain $Y \in J^{\mu}$ with $j_Y\colon Y \to X$ and $f\colon X \to Y$ such that $f^{*}j_Y = j_Y \circ f = \mathrm{id}_X$ in $\mathrm{ho}(\mathcal{C}(X, X))$. So X is a retract of $Y \in J^{\mu}$ which proves the claim.

We now show part (2). Let λ be a regular cardinal such that $\mathcal{C}$ and $\mathcal{D}$ are λ-presentable and F is λ-accessible. In particular, we obtain a set A of vertices in $\mathcal{C}$ such that $\mathrm{Ind}_{\lambda}A \simeq \mathcal{C}$. Now, every object $D \in \mathcal{D}$ is λ_D-compact for some cardinal $\lambda_D \geq \lambda$, so let

$$\mu := \sup(\lambda_{F(C)} | C \in A).$$

Let $\kappa \geq \mu$. As κ-compactness is stable under retracts, by part (1) it suffices to show that every object in J^{κ} is mapped to a κ-compact object in $\mathcal{D}$. We proceed by induction. By construction, for every object $C \in J^{\kappa,0} = A$ the vertex $F(C) \in \mathcal{D}$ is κ-compact. The limit stage is trivial again. Now suppose $F(C)$ is κ-compact for every object in $C \in J^{\kappa,\alpha}$, let $C' \in J^{\kappa,\alpha+1}$ and $G\colon I \to J^{\kappa,\alpha}$ for I a λ-filtered and κ-small simplicial set and such that $C' = \mathrm{colim}G$. Then

$$F(C') = F(\mathrm{colim}G) \simeq \mathrm{colim}FG$$

is a κ-small colimit of κ-compact objects and hence κ-compact (again because κ-filtered colimits commute with κ-small limits in $\mathcal{S}$ by [**36**, Proposition 5.3.3.3]). $\square$

The following group of statements will in each case claim that a certain comparison holds for all κ "sufficiently large" or "large enough". That means in each case there is a cardinal μ such that for all $\kappa \gg \mu$ the given statement holds true. Since we are not interested in a precise formula for the lower bound μ, we generally will not make the cardinal μ explicit. Instead, we note that we will have to impose the condition on κ to be "large enough" only finitely often and eventually have to take the corresponding supremum.

Proposition 8.3.5. *Let $\mathbb{M}$ be a combinatorial model category. Then for all sufficiently large regular cardinals κ, an object $C \in \mathrm{Ho}_\infty(\mathbb{M})$ is κ-compact if and only if there is a κ-compact $D \in \mathbb{M}$ such that $C \simeq D$ in $\mathrm{Ho}_\infty(\mathbb{M})$.*

Proof. As observed in [**37**], one direction can be shown directly for every combinatorial model category. Indeed, for κ large enough, κ-filtered colimits in $\mathbb{M}$ are homotopy colimits and the κ-compact objects in $\mathbb{M}$ are exactly the κ-compact objects in $N(\mathbb{M})$. So the localization $N(\mathbb{M}) \to \mathrm{Ho}_\infty(\mathbb{M})$ preserves κ-filtered colimits and hence is κ-accessible. The statement now follows from Lemma 8.3.4.

For the other direction, we note that by Dugger's Presentation Theorem 7.1.2 it suffices to consider left Bousfield localizations of simplicial presheaf categories $\mathrm{sPsh}(\mathbb{C})$. Indeed, given a combinatorial model category $\mathbb{M}$ together with a category $\mathbb{C}$, a set $T \subset \mathrm{sPsh}(\mathbb{C})$ of arrows and a Quillen equivalence

$$\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{M},$$

suppose we have shown the statement for $\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$ (or for $\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{inj}}$ equivalently as both model structures have the same underlying category and equivalent underlying quasi-categories). Then, as both categories $\mathbb{M}$ and $\mathrm{sPsh}(\mathbb{C})$ are presentable, we find $\kappa \gg |\mathbb{C}|$ large enough such that the right adjoint R preserves κ-compact objects. Certainly $\mathbb{L}L$ and $\mathbb{R}R$ preserve κ-compactness in $\mathrm{Ho}_\infty(\mathbb{M})$, so given $X \in \mathrm{Ho}_\infty(\mathbb{M})$ κ-compact, choose $Y \in \mathrm{sPsh}(\mathbb{C})$ κ-small weakly equivalent to $\mathbb{R}RX$. Without loss of generality Y is cofibrant by [**16**, Proposition 2.3.(iii)] and so $L(Y)$ is κ-compact in $\mathbb{M}$ and presents X in $\mathrm{Ho}_\infty(\mathbb{M})$. Therefore, assume $\mathbb{M} = \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{inj}}$. Furthermore, by Lemma 8.3.1 we can use the words "compact" and "small" interchangeably.

Now, if $X \in \mathrm{Ho}_\infty(\mathbb{M})$ is κ-compact, we have seen in Lemma 8.3.4.(1) that there is a $Y \in J^\kappa$ such that X is a homotopy retract of Y. But the representatives for the colimits in the construction of $(J^{\kappa,\alpha}|\alpha < \kappa)$ can be chosen to be homotopy colimits of strict diagrams $F\colon I \to \mathbf{S}$ for κ-small categories I in $\mathbb{M}$ by [**36**, Proposition 4.2.3.14] and [**35**, Proposition 1.3.4.25]. Hence, they can be computed according to the Bousfield-Kan formula

$$\mathrm{hocolim}F = \mathrm{coeq}\left(\coprod_{i\to j} F(i) \otimes N(j/I)^{\mathrm{op}} \rightrightarrows \coprod_i F(i) \otimes N(i/I)^{\mathrm{op}} \right)$$

because $\mathbb{M} = \mathcal{L}_T\mathrm{sPsh}(\mathbb{C})$ is a simplicial model category. But this choice of homotopy colimit is κ-small whenever F and I are κ-small, and hence, by induction, every object contained in J^κ is in fact a κ-small object in $\mathrm{sPsh}(\mathbb{C})$. Therefore, as X is homotopy equivalent to a subobject of $Y \in J^\kappa$, it is homotopy equivalent to a κ-small presheaf. This proves the proposition. □

In the following we generalize Proposition 8.3.5 to relative κ-compact maps. We begin with a special class of model categories.

Definition 8.3.6. Let $\mathbb{M}$ be a model category such that all cofibrations are monomorphisms. Say $\mathbb{M}$ has a *theory of minimal fibrations* if there is a pullback stable class $\mathcal{F}_{\mathbb{M}}^{\min}$ of fibrations in $\mathbb{M}$ – the class of *minimal fibrations* – such that the following hold.

(1) Let $p\colon X \twoheadrightarrow Y$ and $q\colon X' \twoheadrightarrow Y$ be minimal fibrations. Then every homotopy equivalence between X and X' over Y is an isomorphism.
(2) For every fibration $p\colon X \twoheadrightarrow Y$ in $\mathbb{M}$ there is an acyclic cofibration $M \overset{\sim}{\hookrightarrow} X$ such that the restriction $M \to Y$ is a minimal fibration.

Lemma 8.3.7. *Let $\mathbb{M}$ be a model category such that all cofibrations are monomorphisms. Suppose $\mathbb{M}$ has a theory of minimal fibrations. Let T be a class of maps in $\mathbb{M}$ such that the left Bousfield localization $\mathcal{L}_T(\mathbb{M})$ exists. Then the model category $\mathcal{L}_T(\mathbb{M})$ has a theory of minimal fibrations.*

Proof. Given a model category $\mathbb{M}$ and a class T of maps in $\mathbb{M}$ as stated, simply define the class $\mathcal{F}_{\mathbb{M}}^{\min}$ of minimal fibrations in $\mathcal{L}_T\mathbb{M}$ to be the class of fibrations in $\mathcal{L}_T\mathbb{M}$ which are minimal fibrations in $\mathbb{M}$. Pullback stability of $\mathcal{F}_{\mathbb{M}}^{\min}$ and Property (1) are immediate. For Property (2), let $p\colon X \twoheadrightarrow Y$ be a fibration in $\mathcal{L}_T\mathbb{M}$. By the assumption that $\mathbb{M}$ has a theory of minimal fibrations, there is an acyclic cofibration $M \overset{\sim}{\hookrightarrow} X$ in $\mathbb{M}$ such that the restriction $M \twoheadrightarrow Y$ is a minimal fibration in $\mathbb{M}$. But

$M \to X$ is a weak equivalence from the fibration $M \twoheadrightarrow Y$ to the fibration $p\colon X \twoheadrightarrow Y$ over Y. The latter is a fibration in $\mathcal{L}_T\mathbb{M}$ and it hence follows by Lemma 3.1.7 that $M \twoheadrightarrow Y$ is a fibration in $\mathcal{L}_T\mathbb{M}$, too. □

Proposition 8.3.8. *Let $\mathbb{C}$ be a small category and $\mathbb{M}$ be a cofibrantly generated model structure on the presheaf category $\widehat{\mathbb{C}}$ such that all cofibrations are monomorphisms. Suppose $\mathbb{M}$ has a theory of minimal fibrations. Then for all sufficiently large regular cardinals κ, a morphism $f\colon C \to D$ in $\mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact if and only if there is a κ-small fibration $p \in \mathbb{M}$ between fibrant objects such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$*

Proof. For one direction, let $p\colon X \twoheadrightarrow Y$ be a κ-small fibration between fibrant objects in $\mathbb{M}$. Given a map $g\colon A \to Y$ with κ-compact domain A in $\mathrm{Ho}_\infty(\mathbb{M})$, in order to show that the (strict) pullback of X along g is κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$, by part (1) we can present A by a κ-small object A'. Without loss of generality A' is bifibrant by [**16**, Proposition 2.3.(iii)], so we obtain a map $g'\colon A' \to Y$ in $\mathbb{M}$ presenting g. Also the pullback $(g')^*X$ is a homotopy pullback and it is κ-small by assumption. Hence, it is κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$ by part (1). This shows that p is relative κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$.

For the converse direction, assume that $f\colon C \to D$ is relative κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$ and $p\colon X \twoheadrightarrow Y$ is a fibration in $\mathbb{M}$ such that Y is fibrant in $\mathbb{M}$ and $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$. By Definition 8.3.6.(2) there is a subobject $M \subseteq X$ such that the restriction $m\colon M \twoheadrightarrow Y$ of p is a minimal fibration. As m and p are homotopy equivalent over Y, the fibration m is relative κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$, too. We want to show that m is a κ-small fibration. Therefore, for $C \in \mathbb{C}$ let $g\colon yC \to Y$ be an element of Y, so that we have to show that the pullback g^*M as depicted in the diagram

$$\begin{tikzcd} g^*M \arrow[r] \arrow[d, "g^*m"'] \arrow[dr, phantom, "\lrcorner" very near start] & M \arrow[d] \\ yC \arrow[r, "g"'] & Y. \end{tikzcd}$$

is a κ-small object in $\widehat{\mathbb{C}}$. By [**16**, Proposition 2.3.(iii)] there is a κ-small fibrant replacement RC of the representable yC. Since the object Y is fibrant, we obtain an extension $g'\colon RC \to Y$ of g along the acylic cofibration $yC \overset{\sim}{\hookrightarrow} RC$ and hence a factorization of the following form.

$$\begin{tikzcd}
g^*M \arrow[rr] \arrow[dr] \arrow[dd, two heads, "g^*m"'] & & M \arrow[dd, two heads, "m"] \\
& (g')^*M \arrow[ur] \arrow[dd, two heads] & \\
yC \arrow[rr, "g"] \arrow[dr, hook, "\sim"'] & & Y \\
& RC \arrow[ur, "g'"'] &
\end{tikzcd}$$

All three faces of the diagram are pullback squares, hence, in order to show that the object g^*M is κ-small, it suffices to show that the object $(g')^*M$ is κ-small.

We know that RC is also κ-compact in the underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ by part (1) and so is the map $(g')^*m$ by our assumption on the morphism f. The underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{M}/RC)$ is equivalent to the overcategory $\mathrm{Ho}_\infty(\mathbb{M})_{/RC}$ by Lemma 7.2.1, and is generated under (homotopy) colimits by the κ-small collection of maps with codomain RC and representable domain by [**18**, Proposition 2.9]. Recall that the domain functor

$$\mathrm{dom}\colon \mathrm{Ho}_\infty(\mathbb{M})_{/RC} \to \mathrm{Ho}_\infty(\mathbb{M})$$

both preserves and reflects colimits by [**36**, Proposition 1.2.13.8]. Therefore, by Lemma 8.3.4.(2) the functor dom preserves κ-small objects. Dually, using the recursive definition of J^κ from (8.3.1), it is easy to see that the functor dom also reflects κ-compact objects. We obtain that the map $r^*m \in \mathrm{Ho}_\infty(\mathbb{M})_{/RC}$ is a κ-compact object. By part (1) applied to the combinatorial model category $\mathbb{M}/RC$ (observing that the κ for $\mathbb{M}$ also works for $\mathbb{M}/RC$), we obtain a κ-small fibration $q\colon Z \twoheadrightarrow RC$ together with a homotopy equivalence $Z \simeq (g')^*M$ over RC. Again by [**14**, Theorem 2.14] there is a subobject $N \subseteq Z$ such that the restriction $n\colon N \twoheadrightarrow RC$ of $q\colon Z \to RC$ is a minimal fibration. Clearly n is still κ-small. But the induced homotopy equivalence $N \simeq (g')^*M$ over RC is a homotopy equivalence between minimal fibrations and hence turns out to be an isomorphism by [**14**, Proposition 2.16]. Therefore, $(g')^*M$ is κ-small. □

Corollary 8.3.9. *Let I be an Eilenberg-Zilber Reedy category in the sense of* [**14**, Section 2.1] *and* $\mathbb{M} = \mathrm{sPsh}(I)_{\mathrm{inj}}$ *the category of simplicial presheaves on I equipped with the injective model structure. Then for all sufficiently large regular cardinals*

κ, a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact if and only if there is a κ-small fibration $p \in \mathbb{M}$ such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.

Proof. Let $p\colon X \twoheadrightarrow Y$ be a κ-small fibration in $\mathbb{M}$ and $X \overset{\sim}{\hookrightarrow} RX$ be a fibrant replacement of X. By Theorem 2.5.4 and Theorem 2.5.8 we know that $\mathbb{M}$ satisfies the fibration extension property, hence we obtain a κ-small fibration $q\colon Z \twoheadrightarrow RX$ that is weakly equivalent to p. Hence it suffices to show that q is relative κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$. But q is a κ-small fibration between fibrant objects, so we can proceed now exactly as we did in the proof of Proposition 8.3.8.

The converse direction follows immediately from Proposition 8.3.8 and the fact that $\widehat{I \times \Delta}$ supports a theory of minimal fibrations as shown in [**14**, 2.13-2.16]. $\square$

Remark 8.3.10. Let I be an Eilenberg-Zilber Reedy category. Then Lemma 8.3.7 yields for every left Bousfield localization of the form $\mathcal{L}_T(\mathrm{sPsh}(I))_{\mathrm{inj}}$ a comparison between κ-small fibrations p between fibrant objects in $\mathcal{L}_T(\mathrm{sPsh}(I))_{\mathrm{inj}}$ and relative κ-compact maps f in the underlying quasi-category for large enough cardinals κ. But whenever the Bousfield localization is left exact, the condition on fibrancy of the codomain of p is also not necessary. Indeed, every κ-small fibration in $\mathcal{L}_T(\mathrm{sPsh}(I))_{\mathrm{inj}}$ is a κ-small fibration in $\mathrm{sPsh}(I)_{\mathrm{inj}}$ and hence relative κ-compact in the underlying quasi-category of $\mathrm{sPsh}(I)_{\mathrm{inj}}$ by Proposition 8.3.8. But, given that the localization preserves homotopy pullbacks, it is easy to see that it also preserves relative κ-compact maps on underlying quasi-categories.

We now make use of the observations in Section 8.2 to generalize Corollary 8.3.9 to simplicial presheaves over arbitrary small categories and small simplicial categories.

Theorem 8.3.11. *Let $\mathbb{C}$ be a small category, $T \subset \mathrm{sPsh}(\mathbb{C})$ be a set of maps and $\mathbb{M} = \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$. Then for all sufficiently large inaccessible cardinals κ, a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact if and only if there is a κ-small fibration $p \in \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$ between fibrant objects such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.*

Proof. Let $\mathbb{C}$ be a small category and $T \subset \mathrm{sPsh}(\mathbb{C})$ a set of maps. Combining Lemma 3.1.6, Lemma 8.2.5 and Proposition 8.2.7, we obtain a zig-zag of Quillen equivalences

$$\mathcal{L}_{\varphi_![y[V]]}\mathrm{sPsh}(I)_{\mathrm{inj}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \mathcal{L}_{\varphi_![y[V]]}\mathrm{sPsh}(I)_{\mathrm{proj}} \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \mathcal{L}_{y[V]}(\mathrm{sPsh}(F_\Delta I))_{\mathrm{proj}}$$

$$\mathcal{L}_{y[V]}(\mathrm{sPsh}(F_\Delta I))_{\mathrm{proj}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathrm{sPsh}(\mathcal{L}_\Delta(I,V))_{\mathrm{proj}} \underset{f_n^*}{\overset{(f_n)_!}{\leftrightarrows}} \dots \underset{f_1^*}{\overset{(f_1)_!}{\leftrightarrows}} (\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$$

such that I is an Eilenberg-Zilber Reedy category. This yields a zig-zag of Quillen equivalences

$$\mathcal{L}_{(\varphi_![y[V]]\cup\bar{T})}\mathrm{sPsh}(I)_{\mathrm{inj}} \rightleftarrows \dots \rightleftarrows \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$$

where $\bar{T} \subset \mathrm{sPsh}(I)$ is obtained from $T \subset \mathrm{sPsh}(\mathbb{C})$ by transferring T along the finitely many Quillen equivalences successively according to Lemma 3.1.6. We denote the union $\varphi_![y[V]] \cup \bar{T} \subset \mathrm{sPsh}(I)$ short-handedly by U.

By [**18**, Proposition 5.10, Corollary 6.5] this chain of Quillen equivalences induces a single Quillen equivalence

$$\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{L}_U(\mathrm{sPsh}(I))_{\mathrm{inj}}. \tag{8.3.2}$$

The Bousfield localization $\mathcal{L}_U(\mathrm{sPsh}(I))_{\mathrm{inj}}$ has a theory of minimal fibrations by Lemma 8.3.7. Now, let $\kappa \gg |\mathbb{C}|, |\mathbb{I}|$ be inaccessible large enough such that Corollary 8.3.9 applies to I and large enough such that Proposition 8.3.8 applies to $\mathcal{L}_U(\mathrm{sPsh}(I))_{\mathrm{inj}}$.

For one direction, let $f \in \mathrm{Ho}_\infty(\mathbb{M})$ be relative κ-compact. Since the pair (8.3.2) is a Quillen equivalence, the quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ is equivalent to the underlying quasi-category of $\mathcal{L}_U(\mathrm{sPsh}(I)_{\mathrm{inj}}$. Then, by Proposition 8.3.8, there is a κ-small fibration $p\colon X \twoheadrightarrow Y$ between fibrant objects in $\mathcal{L}_U(\mathrm{sPsh}(I))_{\mathrm{inj}}$ presenting f in $\mathrm{Ho}_\infty(\mathbb{M})$. By Lemma 8.3.4 (or its ordinary categorical analogon if favoured), we know that for $\kappa > |\mathbb{C}|, |\mathbb{I}|$ the left adjoint F preserves κ-compact objects. Hence, since $\kappa > |\mathbb{C}|, |\mathbb{I}|$ is inaccessible, by Lemma 8.3.2 the right Quillen functor G preserves κ-small maps. Thus, $Gp\colon GX \twoheadrightarrow GY$ is a κ-small fibration between fibrant objects presenting f in $\mathrm{Ho}_\infty(\mathbb{M})$.

The proof of other direction is exactly as in Proposition 8.3.8. □

Remark 8.3.12. The reason why in Theorem 8.3.11 we assume $\mathbb{M}$ to not only be Quillen equivalent but isomorphic to a left Bousfield localization of a simplicial presheaf category is that there is no obvious reason why the Quillen equivalence

$$\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{proj}} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{M}$$

given by Dugger's Presentation Theorem should preserve κ-small maps. While the right adjoint certainly does preserve such maps, the left adjoint does not seem to exhibit any properties with that respect.

Corollary 8.3.13. *Let $\mathbb{M}$ be a combinatorial model category. Let $\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$ be the presentation of $\mathbb{M}$ from Dugger's Representation Theorem 7.1.2. Then for all sufficiently large inaccessible cardinals κ, a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact if and only if there is a κ-small fibration $p \in \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$ between fibrant objects such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.*

□

More generally, with the changes mentioned in Remark 8.3.3, essentially the same proof of Theorem 8.3.11 also shows the following theorem.

Theorem 8.3.14. *Let $\mathbf{C}$ be a small simplicial category and T a set of arrows in $\mathrm{sPsh}(\mathbf{C})$. Let $\mathbb{M}$ be the left Bousfield localization $\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}$. Then for all sufficiently large inaccessible cardinals κ, the following two statements hold.*

(1) Every κ-small fibration $p \in \mathbb{M}$ between fibrant objects is relative κ-compact in the underlying quasi-category.

(2) If a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relative κ-compact, then there is a κ-small map $g \in \mathrm{sPsh}(\mathbf{C})$ such that $g \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.

Proof. Let $\mathbf{C}$ be a simplicial category. By Proposition 8.2.6 we obtain a zig-zag of Quillen pairs

(8.3.3)

$$\mathrm{sPsh}(I)_{\mathrm{inj}} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} (\mathrm{sPsh}(F_\Delta I))_{\mathrm{inj}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{sPsh}(\mathcal{L}_\Delta(I,V))_{\mathrm{inj}} \underset{(f_n)_*}{\overset{f_n^*}{\rightleftarrows}} \dots \underset{(f_1)_*}{\overset{f_1^*}{\leftrightarrows}} \mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$$

such that (j^*, j_*) is a homotopy colocalization and all other pairs are Quillen equivalences. Choose μ large enough such that

(1) all involved adjoints preserve (strictly) μ-compact objects in both directions;

(2) their derived functors preserve μ-compact objects on underlying quasi-categories;

(3) since the left Quillen functors $(f_i)^*$, j^* and φ^* are right adjoints themselves, the corresponding left adjoints $(f_i)_!$, $j_!$ and $\varphi_!$ preserve μ-compact objects, too;

(4) the localization $\mathrm{Ho}_\infty(\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}) \to \mathrm{Ho}_\infty(\mathrm{sPsh}(\mathcal{L}_T\mathbf{C})_{\mathrm{inj}})$ and its right adjoint preserve μ-compact objects.

Let $\kappa \gg \mu$ be an inaccessible cardinal. Then part (1) is shown in the same way as in Proposition 8.3.8 and Theorem 8.3.11. For part (2), let $f \in \mathrm{Ho}_\infty(\mathbb{M})$ be relative κ-compact. Then, by condition (4), it is easy to see that f is relative κ-compact in $\mathrm{Ho}_\infty(\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}})$, because the reflector preserves pullbacks. For the same reason we see that $\mathrm{Ho}_\infty(j^*)(f)$ is relative κ-compact in $\mathrm{Ho}_\infty(\mathrm{sPsh}(I)_{\mathrm{inj}})$. By Proposition 8.3.8 we find a fibration $p \in \mathrm{sPsh}(I)_{\mathrm{inj}}$ presenting f. Transferring p along the zig-zag of adjoint pairs in Diagram (8.3.3) yields a map $\bar{p}$ in $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ presenting f. The map $\bar{p}$ is κ-small by Lemma 8.3.2, Remark 8.3.3 and by choice of κ. $\square$

Remark 8.3.15. If [**18**, Section 9.6] generalizes to simplicial presheaf categories over small simplicial categories (Dugger only considered ordinary categories), the proof of Theorem 8.3.11 generalizes, too. In that case we can show that for every small simplicial category $\mathbf{C}$ and every set of arrows $T \in \mathrm{sPsh}(\mathbf{C})$, a morphism $f\colon C \to D$ in $\mathrm{Ho}_\infty(\mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}})$ is relative κ-compact if and only if there is a κ-small fibration $p\colon X \twoheadrightarrow Y$ between fibrant objects in $\mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}}$ such that $p \simeq f$ in its underlying quasi-category.

If one additionally can show that every κ-small projective fibration in $\mathrm{sPsh}(\mathbf{C})$ is point-wise weakly equivalent to a κ-small injective fibration, then the same comparison result holds true for the type theoretic model category $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$.

If furthermore the class S_κ of κ-small maps in $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ satisfies the fibration extension property (which is guaranteed if $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ can be shown to exhibit a universal fibration for S_κ satisfying the stratification property from Section 2.3), we can show along the lines of Remark 8.3.10 that for every left exact localization $\mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ and every inaccessible cardinal κ sufficiently large, a map $f \in \mathrm{Ho}_\infty(\mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}})$ is relative κ-compact if and only if there is a κ-small fibration $p \in \mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ such that $p \simeq f$ in the underlying ∞-topos.

We conclude this section by commenting briefly on the relevance of these results for Homotopy Type Theory.

Let $\mathcal{C}$ be a presentable quasi-category and let $\mathbf{C}$ be a small simplicial category and T a set of arrows in $\mathrm{sPsh}(\mathbf{C})$ such that the localization $\mathbb{M} := \mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ presents $\mathcal{C}$. Then $\mathbb{M}$ is a type theoretic model category as shown in [**23**, Section 7]. It is an open problem to date whether this presentation $\mathbb{M}$ exhibits an infinite sequence of univalent strict Tarski universes whenever $\mathcal{C}$ is an ∞-topos. But it is claimed in the Introduction of [**23**] that $\mathbb{M}$ does exhibit a sequence of univalent *weak* Tarski universes whenever $\mathcal{M}$ is a Grothendieck ∞-topos. Here, by "weak Tarski universe" we understand an inaccessible cardinal κ together with a fibration that is weakly universal for the class of κ-small fibrations. Weak universality of a fibration $p\colon E \twoheadrightarrow B$ for a class S of fibrations in turn means that p is univalent and that for all fibrations $q\colon X \twoheadrightarrow Y$ in S there is a map $w\colon X \to B$ such that q is the homotopy pullback of p along w.

Then it is easy to see that a univalent (weakly) universal fibration for a pullback stable class S of fibrations in $\mathbb{M}$ yields a classifying object for the class $\mathrm{Ho}_\infty[S]$ of morphisms in $\mathcal{C}$ and that, vice versa, every classifying object for a pullback stable class T of morphisms in $\mathcal{C}$ yields a univalent weakly universal fibration for the class

$$\bar{T} := \{f \in \mathcal{F}_{\mathbb{M}} \mid f \in \mathrm{Ho}_\infty(\mathbb{M}) \text{ is in } T\}$$

of maps in $\mathbb{M}$. There is one class of maps in each case which is relevant for the construction of strict Tarski universes in $\mathbb{M}$ on the one hand, and the construction of object classifiers in $\mathcal{C}$ on the other. That is, given a "sufficiently large" inaccessible cardinal κ, the class S_κ of κ-small fibrations in $\mathrm{sPsh}(\mathbf{C})$ and the class T_κ of relative κ-compact maps in $\mathcal{C}$. In the former case, the common constructions of univalent universal fibrations $\pi_\kappa\colon \tilde{U}_\kappa \twoheadrightarrow U_\kappa$ use various functorial closure properties of S_κ and the fact that an infinite sequence of inaccessible cardinals yields a cumulative hierarchy of universal fibrations in this way. In the latter case, [**36**, Theorem 6.1.6.8] characterizes ∞-toposes in terms of classifying objects V_κ for T_κ for all sufficiently large cardinals κ.

While it is clear that the associated classifying map $p_\kappa\colon \tilde{V}_\kappa \to V_\kappa$ lifts to a fibration in $\mathbb{M}$ which is weakly universal for $\bar{T}_\kappa$, and that U_κ descends to a classifying object for the class $\mathrm{Ho}_\infty[S_\kappa]$, it is a priori unclear whether $S_\kappa = \bar{T}_\kappa$ or $T_\kappa = \mathrm{Ho}_\infty[S_\kappa]$ hold. In other words, without a comparison of smallness notions as considered in this section and in Remark 8.3.15 in particular, it is not clear whether the categorical construction of universal κ-small fibrations in $\mathbb{M}$ – which models Tarski universes in the associated type theory – also models universes in the underlying quasi-category.

8.4. Presenting presheaf ∞-toposes via right Bousfield localizations

In this section we use the constructions from Section 8.2 to show that every presheaf ∞-topos $\mathcal{M}$ is presented by a model category $\mathbb{M}$ that comes equipped with an underlying Homotopy Type Theory. We show that in a large class of examples this specific construction of a Homotopy Type Theory associated to $\mathcal{M}$ cannot be considered to be an internal language of $\mathcal{M}$, because the type constructors do not describe the higher categorical structure in $\mathcal{M}$.

Let $\mathbf{C}$ be a simplicial category. Recall that by Proposition 8.2.6 we find a well founded poset I together with a subset $V \subseteq I$ of arrows such that $\mathrm{sPsh}(\mathcal{L}_\Delta(I, V))_{\mathrm{inj}}$ presents $(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}$. The localization $j\colon F_\Delta I \to \mathcal{L}_\Delta(I, V)$ induces an adjoint triple $(j_!, j^*, j_*)$ with fully faithful restriction j^* on associated simplicial presheaf categories by Proposition 8.2.3. This yields both a localization and colocalization of simplicial presheaf categories, and when equipping both sides with the injective model structure the colocalization (j^*, j_*) is a Quillen pair. In fact, it is a homotopy colocalization as explained in Section 8.2, and homotopy colocalizations correspond to right Bousfield localizations in the same way as homotopy localizations correspond to left Bousfield localizations. We therefore recall some basic facts about right Bousfield localizations.

Definition 8.4.1. Let $\mathbb{M}$ be a bicomplete category with two model structures $\mathbb{M}_i$ for $i \in \{0, 1\}$. Then $\mathbb{M}_2$ is said to be a right Bousfield localization of $\mathbb{M}_1$ if $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. If this holds, the identity $\mathrm{id}\colon \mathbb{M}_1 \to \mathbb{M}_2$ is a right Quillen functor.

Definition 8.4.2 (Definition 3.1.4, [**25**])**.** Given a set T of arrows in a model category $\mathbb{M}$, an object $X \in \mathbb{M}$ is said to be T-*colocal* if

$$f_*\colon [X, A]_h \to [X, B]_h$$

is a weak equivalence for all $f\colon A \to B$ in T. An arrow $g\colon A \to B$ is said to be a T-*colocal equivalence* if

$$g_*\colon [X, A]_h \to [X, B]_h$$

is a weak equivalence for all T-local objects X in $\mathbb{M}$. We say that a model structure $\mathbb{M}_T$ on the same underlying category is the *right Bousfield localization* of $\mathbb{M}$ at T if

(1) $\mathcal{F}_T = \mathcal{F}_{\mathbb{M}}$,
(2) $\mathcal{W}_T = \{T\text{-colocal equivalences}\}$.

Clearly, if the right Bousfield localization of $\mathbb{M}$ at P exists it is unique, because model structures are uniquely determined by their fibrations and weak equivalences.

Theorem 8.4.3 ([**25**, Theorem 5.1.1])**.** *Given a right proper combinatorial simplicial model category $\mathbb{M}$ and a set P of arrows in $\mathbb{M}$, the right Bousfield localization $\mathcal{R}_P\mathbb{M}$ of $\mathbb{M}$ at P exists and is again a right proper combinatorial simplicial model category (with the same function complexes). The cofibrant objects in $\mathbb{M}_P$ are exactly the P-colocal objects which are cofibrant in $\mathbb{M}$*

Proposition 8.4.4 ([**25**, Propositions 3.3.18.2, 3.3.20])**.** *Let $\mathbb{M}$ and $\mathbb{N}$ be model categories and let $(F,G)\colon \mathbb{M}\to\mathbb{N}$ be a Quillen pair. If P is a class of maps in $\mathbb{N}$, then,*

(1) if $\mathbb{R}G$ takes every element in $P\subseteq\mathbb{N}$ to a weak equivalence in $\mathbb{M}$, then $(F,G)\colon\mathbb{N}\to\mathcal{R}_P\mathbb{M}$ is a Quillen pair,
(2) (F,G) is also a Quillen pair when considered as functors $(F,G)\colon\mathcal{R}_{\mathbb{R}GP}\mathbb{M}\to\mathcal{R}_P\mathbb{N}$ between the right localizations of $\mathbb{M}$ and $\mathbb{N}$, and
(3) if $(F,G)\colon\mathbb{M}\to\mathbb{N}$ is a Quillen equivalence, then $(F,G)\colon\mathcal{R}_{\mathbb{R}GP}\mathbb{M}\to\mathcal{R}_P\mathbb{N}$ is also a Quillen equivalence.

Corollary 8.4.5. *Let $\mathbb{M}$ and $\mathbb{N}$ be model categories and let $(F,G)\colon\mathbb{M}\to\mathbb{N}$ be a homotopy colocalization, i.e. (F,G) is a Quillen pair such that the left derived functor $\mathbb{L}F$ is fully faithful on homotopy categories. Let*

$$M := \{(f\colon X\to Y)\in\mathbb{N}\mid \mathbb{R}Gf \text{ is a weak equivalence in } \mathbb{M}\}$$

be the set class of all "$\mathbb{M}$-colocal" maps in $\mathbb{N}$. Then,

(1) if $\mathcal{R}_M\mathbb{N}$ exists, the pair $(F,G)\colon\mathbb{M}\to\mathcal{R}_M\mathbb{N}$ is a Quillen equivalence.
(2) If $\mathbb{M}$ and $\mathbb{N}$ are combinatorial and $\mathbb{N}$ further is simplicial, the localization $\mathcal{R}_M\mathbb{N}$ exists.

Proof. We prove part (1). By Proposition 8.4.4.(1), the pair (F,G) induces a Quillen pair $(F,G)\colon\mathbb{M}\to\mathcal{R}_M\mathbb{N}$ which still is a homotopy colocalization. By choice of the class $M\subseteq\mathbb{N}$, the unit of the derived adjunction on homotopy categories is an isomorphism, so the pair $(F,G)\colon\mathbb{M}\to\mathcal{R}_M\mathbb{N}$ is a Quillen equivalence.

The proof of part (2) also is a sequence of standard arguments. If $\mathbb{M}$ is combinatorial, the subcategory $\mathcal{W}\subseteq\mathbb{M}^{[1]}$ of weak equivalences in $\mathbb{M}$ is accessible by Smith's original work on combinatorial model categories ([**36**, Corollary A.2.6.6]). But the functor $\mathbb{R}G = G\circ\mathbb{R}$ is accessible, too, because G is a right adjoint between presentable categories and $\mathbb{R}$ is accessible by [**16**, Proposition 2.3.(i)]. Hence, the class $M=\mathbb{R}G^{-1}[\mathcal{W}]$ is accessible by [**36**, Corollary A.2.6.5]. We therefore find a cardinal κ and a set A_M such that A_M generates $M \subseteq \mathbb{N}^{[1]}$ under κ-filtered colimits. Then $\mathcal{R}_{A_M}\mathbb{N}$ exists by Theorem 8.4.3. Without loss of generality κ is such that the class of weak equivalences in $\mathcal{R}_{A_M}\mathbb{N}$ is closed under κ-filtered colimits, as the localization $\mathcal{R}_{A_M}\mathbb{N}$ is combinatorial again. But then every map in M is a weak equivalence in $\mathcal{R}_{A_M}\mathbb{N}$ and clearly every map in A_M is a weak equivalence in $\mathcal{R}_M\mathbb{N}$. It follows that the model structures $\mathcal{R}_{A_M}\mathbb{N}$ and $\mathcal{R}_M\mathbb{N}$ coincide, which proves the corollary. □

By definition, the right Bousfield localization of a model category $\mathbb{M}$ has the same class of fibrations and hence the same associated weak factorization system of acyclic cofibrations and fibrations as $\mathbb{M}$. It is therefore easy to see that right Bousfield localizing a type theoretic model category $\mathbb{M}$ always yields a model category whose underlying fibration category is the same model of HoTT.

Proposition 8.4.6. *Let $\mathbb{M}$ be a type theoretic model category and $\mathcal{R}\mathbb{M}$ a right Bousfield localization of $\mathbb{M}$. Then the model category $\mathcal{R}\mathbb{M}$ has the Frobenius property. Furthermore, the underlying category $(\mathcal{R}\mathbb{M})^f$ of fibrant objects is a type theoretic fibration category supporting all type constructors among those listed in* [**41**, A.2] *that exist in $\mathbb{M}$. In particular it has strict universal fibrations classifying small fibrations whenever $\mathbb{M}$ does, and these universes are univalent whenever they are univalent in $\mathbb{M}$.*

Proof. The proof is nearly trivial since every right Bousfield localization preserves the weak factorization system of acylic cofibrations and fibrations. Hence, if $\mathbb{M}$ satisfies the Frobenius property, so does $\mathcal{R}\mathbb{M}$. Furthermore, the fibration categories $\mathbb{M}^f$ and $(\mathcal{R}\mathbb{M})^f$ coincide. In particular both fibration categories have the same path objects and preservation of all basic type constructors follows. This means that the model category $\mathcal{R}\mathbb{M}$ yields a model of the same intensional type theory. Universes for $\mathbb{M}^f$ still are universes for the right Bousfield localization, so there is only left to show that these universes still are univalent in $\mathcal{R}\mathbb{M}$. But as noted above both model categories do have the same path objects for fibrant objects and therefore the same notion of right homotopy between fibrant objects. It follows that they do have the same notion of homotopy sections and homotopy retractions and therefore the same type of (homotopy) equivalences. This implies that if a fibration in $\mathbb{M}^f = (\mathcal{R}\mathbb{M})^f$ is univalent in $\mathbb{M}$, it also is univalent in $\mathcal{R}\mathbb{M}$. This finishes the proof. □

Remark 8.4.7. Gepner and Kock have shown in [**23**, Proposition 3.14] that colocalizations of presentable locally cartesian closed $(\infty,1)$-categories preserve univalence of maps. Together with [**23**, Proposition 7.12] – which relates univalence of fibrations in a combinatorial type theoretic model category with univalence of maps in its underlying quasi-category – this implies that right Bousfield localizations of type theoretic model categories preserve univalence of fibrations whenever the right Bousfield localization is type theoretic, too. But right Bousfield localizations of type theoretic model categories will fail to yield a pullback stable class of cofibrations and hence the work of [**23**] does not apply here without alteration. In fact it is not hard to show that, whenever a right Bousfield localization $\mathcal{R}\mathbb{M}$ has pullback stable cofibrations, then the homotopy colocalization $\mathcal{R}\mathbb{M} \to \mathbb{M}$ is a Quillen equivalence and hence in fact the identity. The reason is that under given pullback stability the strict dependent product in $s\mathbb{M}$ presents both the dependent product in $\mathrm{Ho}_\infty(\mathbb{M})$ and its colocalization $\mathcal{R}\mathrm{Ho}_\infty(\mathbb{M})$. This means that the coreflector preserves dependent products and, in particular, is fully faithful and therefore part of an equivalence.

But we have seen that it is quite straightforward to prove inheritance of univalence directly, keeping in mind that univalence involves the type of homotopy equivalences between fibrations – which is invariant under (right) Bousfield localizations – and generally not the object of weak equivalences which is not invariant under Bousfield localizations.

Using Proposition 8.4.6, the following theorem constructs a class of model categories $\mathbb{M}$ presenting presheaf ∞-toposes which are well behaved enough for their category $\mathbb{M}^f$ of fibrant objects to model an intensional type theory $\mathcal{T}_{\mathbb{M}^f}$ in the sense of [**51**]. But we will see in Theorem 8.4.9 that these model categories $\mathbb{M}$ are not well behaved enough to ensure that the homotopical algebraic structure in $\mathbb{M}^f$ interpreting the type theory $\mathcal{T}_{\mathbb{M}^f}$ also presents the corresponding quasi-categorical structure in $\mathrm{Ho}_\infty(\mathbb{M})$.

Theorem 8.4.8. *For every presheaf ∞-topos $\mathcal{M} \simeq \mathcal{P}(S)$ for some simplicial set S, there is a combinatorial model category $\mathbb{M}$ presenting $\mathcal{M}$ such that $\mathbb{M}$ satisfies the Frobenius property and such that the underlying category $\mathbb{M}^f$ of fibrant objects is a type theoretic fibration category which interprets Homotopy Type Theory with all type constructors listed in* [**41**, A.2] *in the sense of* [**51**].

Proof. Let $\mathbf{C}$ be a small simplicial category such that $\mathrm{Ho}_\infty(\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}) \simeq \mathcal{M}$. By Proposition 8.2.6, there is a direct relative poset (I, V) such that $\mathrm{sPsh}(\mathcal{L}_\Delta(I,V)_{\mathrm{inj}}$ is Quillen equivalent to $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$. In the same proposition we have seen that the canonical maps $j\colon F_\Delta I \to \mathcal{L}_\Delta(I,V)$ and $\varphi\colon F_\Delta I \to I$ induce Quillen pairs

$$\mathrm{sPsh}(I)_{\mathrm{inj}} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} \mathrm{sPsh}(F_\Delta I)_{\mathrm{inj}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{sPsh}(\mathcal{L}_\Delta(I,V))_{\mathrm{inj}}$$

such that (j^*, j_*) is a homotopy colocalization and (φ^*, φ_*) is a Quillen equivalence. By Corollary 8.4.5 there is a set $T \subset \mathrm{sPsh}(F_\Delta I)$ of maps such that

$$\mathcal{R}_T\mathrm{sPsh}(F_\Delta I)_{\mathrm{inj}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{sPsh}(\mathcal{L}_\Delta(I,V))$$

is a Quillen equivalence. By Proposition 8.4.4.(3), the pair (φ^*, φ_*) descends to a Quillen equivalence

$$\mathcal{R}_{\mathbb{R}\varphi_* T}\mathrm{sPsh}(I)_{\mathrm{inj}} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} \mathcal{R}_T\mathrm{sPsh}(F_\Delta I)_{\mathrm{inj}}$$

and so we see that the model category $\mathcal{R}_{\mathbb{R}\varphi_* T}\mathrm{sPsh}(I)_{\mathrm{inj}}$ presents $\mathcal{M}$. The category I is an elegant Reedy category, so by [**52**, Theorem 6.4], the model category $\mathrm{sPsh}(I)_{\mathrm{inj}}$ is type theoretic and supports a model of intensional type theory with dependent sums and products, identity types, and as many univalent universes as there are inaccessible cardinals large enough. This is in fact to say that the type theoretic fibration category $\mathrm{sPsh}(I)^f_{\mathrm{inj}}$ is a model of such an intensional type theory. Thus, Proposition 8.4.6 concludes the proof. $\square$

It is natural to expect that the type theory $\mathcal{T}_{\mathbb{M}^f}$ constructed in Theorem 8.4.8 is a candidate for an "internal language" of the presheaf ∞-topos $\mathcal{M}$. However, the following theorem shows that this is not the case for the majority of presheaf ∞-toposes.

Theorem 8.4.9. *Let $\mathcal{M}$ be as in Theorem 8.4.8 and $\mathbb{M}$ be as constructed in the proof. Then, if any one of the following conditions is true,*

(1) $\mathbb{M}$ is a type theoretic model category (in either the sense of Definition 1.2.3 or [**23**]*);*

(2) the dependent product of two fibrations in $\mathbb{M}^f$ presents the dependent product of the two corresponding maps in $\mathcal{M}$;

(3) weak equivalences between fibrant objects in $\mathbb{M}$ are equivalences in $\mathcal{T}_{\mathbb{M}^f}$;

then $\mathcal{M}$ is equivalent to the quasi-category of presheaves over a well founded poset.

Proof. Let $\mathcal{M}$ and $\mathbb{M}$ be as in Theorem 8.4.8. First, if $\mathbb{M}$ is a type theoretic model category, its cofibrations are pullback stable. Hence, by Remark 8.4.7 this implies that $\mathbb{M} = \mathrm{sPsh}(I)_{\mathrm{inj}}$. But analogously, by Remark 8.4.7 the conditions (2) and (3) also imply that $\mathbb{M} = \mathrm{sPsh}(I)_{\mathrm{inj}}$. So in all three cases we obtain that $\mathrm{Ho}_\infty(\mathbb{M})$ is equivalent to $\mathcal{P}(NI)$ by [**35**, Proposition 1.3.4.25]. $\square$

Thus, let $\mathcal{M}$ be a presheaf ∞-topos which is not equivalent to the quasi-category of presheaves over a well founded poset. Then the model categorical presentation $\mathbb{M} := \mathcal{R}_P\mathrm{sPsh}(I)_{\mathrm{inj}}$ constructed in the proof of Theorem 8.4.8 is not type theoretic, although its associated fibration category certainly is type theoretic in the sense of Definition 1.2.1. In fact, the category $\mathbb{M}^f$ is only a type theoretic fibration category, because $\mathrm{sPsh}(I)_{\mathrm{inj}}$ is a type theoretic model category. Indeed, recall that every type theoretic fibration category equipped with the class of homotopy equivalences yields a category of fibrant objects in the sense of Brown (Lemma 1.2.2.(3)). But homotopy equivalences in $\mathbb{M}^f$ are exactly weak equivalences between fibrant objects in $\mathrm{sPsh}(I)_{\mathrm{inj}}$, while there are more weak equivalences between fibrant objects in $\mathbb{M}$ than there are homotopy equivalences in $\mathbb{M}^f$. Equivalently, by Lemma 1.3.4, this means that not all fibrant objects are cofibrant in $\mathbb{M}$. Consequently, the homotopy theory $\mathrm{Ho}_\infty(\mathbb{M}^f)$ associated to $\mathbb{M}^f$ as a category of fibrant objects is not equivalent to the homotopy theory $\mathcal{M}$ presented by $\mathbb{M}$. In the terminology of the introductory chapter, one might say that this class of examples yields a compilation error when plugged into Diagram (2).

In conclusion, this class of examples shows that the choice of a model categorical presentation of an ∞-topos $\mathcal{M}$ with the aim to construct an “internal language” for $\mathcal{M}$ has difficulties that go beyond the construction of enough well behaved homotopical algebraic structure to model Homotopy Type Theory.

Conclusion

In terms of the landscape depicted in Diagram (3), the course we pursued in this thesis was a more or less continuous transition from the bottom left corner to the bottom right corner of the diagram; we started from an almost purely syntactical framework and arrived at an almost entirely homotopical and higher categorical setting. As the reader will have noted, the mathematics in this thesis builds on a plethora of results from the literature and provides a range of results which often leave room for further analysis. We therefore conclude this thesis with a short summary of the material and give a brief outlook on related directions and problems open to further research.

In Chapter 1 we recalled the basic homotopy theoretical semantics of intensional type theory from [**51**] and discussed univalence and related notions like function extensionality and (-1)-truncatedness therein. The formalization of homotopy theoretical constructions as higher inductive types in such type theoretic fibration categories is the subject of synthetic homotopy theory and constitutes a thriving research program. Its goal is to formalize homotopy theoretical results and constructions in intensional type theory, many of which are implemented in functional programming languages such as Agda or Coq and are archived in various online libraries subject to constant development.

In Chapter 2 we have discussed diagrammatical properties of local classes in certain model categories $\mathbb{M}$, related to the construction of univalent universes in associated type theories. Given a fibration $p\colon E \twoheadrightarrow B$, we have seen that the weak equivalence extension property and the fibration extension property of the class F_p are characterized by univalence and fibrancy of the base B whenever p satisfies the stratification property. But both properties of F_p and both corresponding properties of p have homotopical interpretations as follows. For a class S of arrows in $\mathbb{M}$, consider the slice functor

$$\begin{aligned}(F_p/\_)^{\simeq}\colon \mathbb{M} &\to \mathcal{S}\\ X &\mapsto (F_p/X)^{\simeq}\end{aligned}$$

where $(F_p/X)^{\simeq}$ denotes the core of the overcategory F_p/X. For an object $B \in \mathbb{M}$, consider the representable

$$\begin{aligned}[\_, B]_{\mathbb{M}}\colon \mathbb{M} &\to \mathcal{S}\\ X &\mapsto [X, B]_h.\end{aligned}$$

Then, under suitable assumptions on the model category $\mathbb{M}$, for $F_p \subseteq S$ it is not hard to see that univalence of p, universality for S and fibrancy of the base B have interpretations in terms of injectivity and surjectivity of the natural transformation

$$(\_)^*p\colon [X, B]_{\mathbb{M}} \to (S/X)^{\simeq}.$$

Similarly, the weak equivalence extension property and the fibration extension property of S have interpretations in terms of the functor $(S/\_)^{\simeq}$. Indeed, S satisfies the weak equivalence extension property if and only if $(S/\_)^{\simeq}$ maps cofibrations to maps with the right lifting property with respect to the endpoint-inclusion $d^0\colon \Delta^0 \hookrightarrow \Delta^1$, and S satisfies the fibration extension property if and only if $(S/\_)^{\simeq}$ maps acyclic cofibrations to maps with the right lifting property to the map $\emptyset \hookrightarrow \Delta^0$. Much of the discussion from Chapter 2 can be carried out in terms of these interpretations and it would be interesting to gain an understanding of these type theoretically inspired notions in this homotopical setting.

After recalling the foundation of left Bousfield localizations and discussing some of the earlier notions in this context in Chapter 3, in Chapters 4 and 5 we studied the model structures for (complete) Bousfield-Segal spaces $(s\mathbf{S}, \mathrm{B})$ and $(s\mathbf{S}, \mathrm{CB})$ on the category of bisimplicial sets. Building on the work of Rezk, Schwede and Shipley [**46**] and Cisinski [**14**], in Section 5.3 we have seen that the model category $(s\mathbf{S}, \mathrm{CB})$ yields a model of Homotopy Type Theory in the sense of [**51**] and exhibits infinitely many univalent universal fibrations $\pi_\kappa\colon \tilde{U}_\kappa \twoheadrightarrow U_\kappa$. In light of the comparison of univalence and completeness in Chapter 6, we anticipate a similar comparison between univalence and completeness of fibrations in $(s\mathbf{S}, \mathrm{CB})$ in the following way. For a Reedy fibration $p\colon E \twoheadrightarrow B$ in $s\mathbf{S}$, let $\mathrm{Eq}p \in s\mathbf{S}$ be the generic type of equivalences associated to p as defined in Section 1.5. Then every Reedy fibration

$p\colon E \twoheadrightarrow B$ between Reedy fibrant objects induces a square of simplicial sets of the form

$$\begin{tikzcd} B_0 \arrow[r, hook, "r_0"] \arrow[d, hook, "s_0"'] & (\mathrm{Eq}p)_0 \arrow[d, two heads, "(s,t)_0"] \\ B_1 \arrow[r, two heads, "(d_1,d_0)"'] & B_0 \times B_0. \end{tikzcd}$$

Whenever p is a CB-fibration between B-spaces, the B-space B is complete in the sense of Definition 5.1.1 if and only if $s_0\colon B_0 \to B_1$ is an acyclic cofibration, and p is univalent in the sense of Definition 1.5.1 if and only if the map $r\colon B \to \mathrm{Eq}p$ is an acyclic cofibration. It is easy to see that we obtain a lift from B_1 to $(\mathrm{Eq}p)_0$ or vice versa either way, which corresponds to a bisimplicial version of Lurie's (un)straightening functor. Thus, we hope to be able to obtain a 2-for-3 property between univalence, completeness and a cartesian-like lifting property for such fibrations p.

Furthermore, it would be interesting to study generalizations of (C)B-objects in other model categories, especially with a view towards a covariant theory of such to establish the "theory of ∞-groupoids" in the sense of Toën's "théorie des catégories supérieures" studied in [**54**].

In Chapter 6 we expressed univalence of fibrations in suitable type theoretic model categories in terms of completeness of associated Segal objects. We chose the setting to be type theoretic model categories $\mathbb{M}$ with $\mathbb{M}^f = \mathbb{M}^{cf}$, because it seems to be the most general homotopical setting suitable for the interpretation of intensional type theory in which the notion of completeness as considered in this chapter still makes sense. Generalizing the setting further towards type theoretic fibration categories only leaves to define completeness via univalence, since there is no well behaved homotopy theory of Reedy fibrations in such generality. Such a definition would have been against the guiding idea of the chapter, but may be considered reasonable in light of the results in this chapter and Rasekh's work in [**43**, Section 6].

In Chapter 7 we defined and studied combinatorial model categories with universal homotopy colimits and semi-left exact Bousfield localizations. We compared these notions to locally cartesian closed presentable quasi-categories, semi-left exact localizations of quasi-categories and right properness. It is well known that presentable (quasi-)categories have universal homotopy colimits if and only if they are locally cartesian closed, i.e. if and only if the pullback functor along any morphism has a right adjoint "dependent product" functor. Therefore, it is left to introduce a theory of "locally homotopy cartesian closed" model categories which satisfy a comparison theorem with respect to locally cartesian closed quasi-categories along the lines of Theorem 7.2.4. In the context of combinatorial model categories it then follows directly that local homotopy cartesian closedness is equivalent to universality of homotopy colimits.

Finally, in Chapter 8 we compared κ-small fibrations in simplicial presheaf categories and relative κ-compact maps in the underlying quasi-categories for suitably large cardinals κ. The obvious open task for future studies in this context is to (rigorously formulate and) find an answer to the hypothesis stated in Section 8.1. As explained in Section 8.1, this entails a successful comparison of κ-small fibrations in left exact Bousfield localizations of model categories of the form $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ for simplicial categories $\mathbf{C}$ and suitable large inaccessible cardinals κ. In light of Theorem 8.3.14 and Remark 8.3.15, such a comparison essentially reduces to a replacement of κ-small projective fibrations by κ-small injective fibrations in $\mathrm{sPsh}(\mathbf{C})$. Indeed, as noted in Remark 8.3.15, we then obtain a successful comparison of κ-small fibrations in any left exact Bousfield localization $\mathcal{L}_T(\mathrm{sPsh}(\mathbf{C}))_{\mathrm{inj}}$ and relative κ-compact maps in the underlying ∞-topos for suitably large inaccessible cardinals κ. We would therefore obtain a successful transition result between (weak) universal small fibrations in such model categories and object classifiers in the sense of Lurie.

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