Type-Two Well-Ordering Principles, Admissible Sets, and Π^1_1-Comprehension

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Abstract

This thesis introduces a well-ordering principle of type two, which we call the Bachmann-Howard principle. The main result states that the Bachmann-Howard principle is equivalent to the existence of admissible sets and thus to Π^1_1-comprehension. This solves a conjecture of Rathjen and Montalbán. The equivalence is interesting because it relates "concrete" notions from ordinal analysis to "abstract" notions from reverse mathematics and set theory.

A type-one well-ordering principle is a map T which transforms each well-order X into another well-order $T[X]$. If T is particularly uniform then it is called a dilator (due to Girard). Our Bachmann-Howard principle transforms each dilator T into a well-order $\mathrm{BH}(T)$. The latter is a certain kind of fixed-point: It comes with an "almost" monotone collapse $\vartheta : T[\mathrm{BH}(T)] \to \mathrm{BH}(T)$ (we cannot expect full monotonicity, since the order-type of $T[X]$ may always exceed the order-type of X). The Bachmann-Howard principle asserts that such a collapsing structure exists. In fact we define three variants of this principle: They are equivalent but differ in the sense in which the order $\mathrm{BH}(T)$ is "computed".

On a technical level, our investigation involves the following achievements: a detailed discussion of primitive recursive set theory as a basis for set-theoretic reverse mathematics; a formalization of dilators in weak set theories and second-order arithmetic; a functorial version of the constructible hierarchy; an approach to deduction chains (Schütte) and β-completeness (Girard) in a set-theoretic context; and a β-consistency proof for Kripke-Platek set theory.

Independently of the Bachmann-Howard principle, the thesis contains a series of results connected to slow consistency (introduced by S.-D. Friedman, Rathjen and Weiermann): We present a slow reflection statement and investigate its consistency strength, as well as its computational properties. Exploiting the latter, we show that instances of the Paris-Harrington principle can only have extremely long proofs in certain fragments of arithmetic.

Contents

INTRODUCTION

Explaining the Main Result

In this thesis we explore how well-ordering statements from proof theory can be lifted to higher logical complexity. More specifically, we establish a conjecture of Rathjen [**67, 68**] and Montalbán [**55**, Section 4.5]: A certain type-two well-ordering principle is equivalent to the existence of admissible sets (i.e. transitive models of Kripke-Platek set theory) and thus to Π^1_1-comprehension. Our work uses techniques from proof theory, reverse mathematics and set theory. We hope that it will be interesting for researchers from many areas.

To understand well-ordering principles of higher type, it is good to start with a classical result of proof theory. As formulated, the following theorem can be found in [**9**, Theorem 4.6] and [**25**, Theorem 4.5]. Certainly, it can be traced back to the work of Gentzen. One should also mention the contributions of Kreisel and Lévy [**49**] (connections between reflection and transfinite induction), as well as Mints [**54**] and Schwichtenberg [**82**] (use of the repetition rule resp. improper occurrences of the ω-rule).

Theorem. *The following statements are equivalent over primitive recursive arithmetic:*

(i) Any Π^0_2-theorem of Peano arithmetic is true.

(ii) The usual notation system for ε_0 is primitive recursively well-founded.

Statement (i) is known as the uniform Π^0_2-reflection principle over Peano arithmetic. It is easy to see that Π^0_2-reflection implies consistency. Thus Gödel's theorem implies that (i) is unprovable in Peano arithmetic, and a fortiori in the weaker theory of primitive recursive arithmetic. Concerning statement (ii), the intuitive meaning of ε_0 is the smallest ordinal α with $\omega^\alpha = \alpha$. As no actual ordinals are available in arithmetic, we officially work with a notation system: Inductively build terms $\omega^{s_1} + \cdots + \omega^{s_n}$ representing ordinals in Cantor normal form, simultaneously defining an ordering $\prec$ on these terms. Note that the resulting set of terms and the ordering are primitive recursive. More precisely, then, statement (ii) asserts that any primitive recursive sequence $s_0, s_1, \dots$ of ordinal terms has entries $s_{n+1} \not\prec s_n$

for some number n. One cannot emphasize enough that the above theorem concerns statements of low logical complexity. Recall that Π^0_2-formulas have a very concrete interpretation: They assert that a certain algorithm terminates on all inputs. The philosophical position associated with Hilbert's programme regards Π^0_1-sentences as particularly meaningful (see e.g. [**96**]). It is a great achievement of proof theory that it can characterize the Π^0_2-theorems of rather strong theories (many other methods do not apply, because true Π^1_1-statements hold in all ω-models of second-order arithmetic). As a famous application, H. Friedman has shown that Kruskal's theorem on embeddings of finite trees (which is a Π^1_1-statement) and its Π^0_2-miniaturization are unprovable in the theory of arithmetical transfinite recursion (see the presentation by Simpson [**86**], as well as the precise bounds determined by Rathjen and Weiermann [**72**]).

Despite the foundational importance of Π^0_2-statements, it is interesting to lift proof-theoretic concepts and methods to higher logical complexity. At the level of Π^1_2-statements we can investigate questions of set existence, and certain methods from computability theory become available. Indeed, Marcone and Montalbán [**52**] have used such methods to establish the following result. Afshari and Rathjen [**4**] gave a second proof (despite the order of publication), relying on a proof-theoretic argument.

THEOREM. *The following are equivalent over* $\mathbf{ACA_0}$*:*

(i) Arithmetical recursion along $\mathbb{N}$ *(usually denoted* $\mathbf{ACA_0^+}$*).*
(ii) Any subset of $\mathbb{N}$ *is contained in a countable coded* ω*-model of* $\mathbf{ACA_0}$*.*
(iii) Whenever X *is a countable well-order, so is* ε_X*.*

With this result we have entered the domain of reverse mathematics: Broadly speaking, this research programme investigates connections between different foundational and mathematical statements in the language of second-order arithmetic. We refer to Simpson's book [**87**] for more information. Concerning the above theorem, recall that $\mathbf{ACA_0}$ stands for the theory of arithmetical comprehension. The base theory can be weakened to $\mathbf{RCA_0}$ (recursive comprehension) if one is only interested in the equivalence between (i) and (iii). In part (i) we could just as well speak of ω iterations of the Turing jump. To understand statement (ii), recall that an ω-model of second-order arithmetic interprets the first-order quantifiers as ranging over the standard structure $\mathbb{N}$ of natural numbers. The second-order quantifiers range over some subset of the power set of $\mathbb{N}$. Working in second-order arithmetic one can consider ω-models with countable second-order part, coded into a single subset of $\mathbb{N}$. Finally, the expression ε_X in (iii) refers to a relativized ordinal notation system: It contains terms ε_σ for all elements $\sigma \in X$, ordered as in X. The space between these (morally speaking) ε-numbers is filled up with terms that correspond to Cantor normal forms, such as $\omega^{\omega^{\varepsilon_\sigma}+\omega^0}$. Crucially, the term system ε_X and its ordering are primitive recursive relative to X, and $\mathbf{RCA_0}$ proves that these collections exist as sets. Thus the power of (iii) lies solely in the assertion that well-foundedness is preserved. Note that this is expressed by a Π^1_2-formula, as well-foundedness itself is a Π^1_1-property. A statement such as (iii) will be called a type-one well-ordering principle. It is worth observing that the contrapositive of (iii), rather than (iii) itself, is used in the proofs: Marcone and Montalbán [**52**, Theorem 5.21] show that, for some computable linear order X, there is a computable descending sequence in ε_X, while any descending sequence in X allows to compute the ω-jump. It would be interesting to investigate well-ordering principles in an intuitionistic setting, which is sensitive to the difference between a statement and its contrapositive, but this is not the topic of the present thesis.

The literature contains several results which have the same form as the theorem above. They characterize the following in terms of type-one well-ordering principles: arithmetical comprehension [**30, 35**], ω^α iterations of the Turing jump [**52**], arithmetical transfinite recursion [**24, 73, 52**], ω-models of arithmetical transfinite recursion [**68**], ω-models of bar induction [**71**], and ω-models of Π^1_1-comprehension (the latter with [**91**] and without [**92**] bar induction). At least in principle, there is no bound on the consistency strength of statements that can be reached in this way. In contrast, there is a limitation in terms of logical complexity: Principles of the forms (ii) and (iii) are expressed by Π^1_2-formulas. Thus they cannot be equivalent to a genuine Π^1_3-statement, such as Π^1_1-comprehension. To overcome this limitation, Rathjen [**67, 68**] and Montalbán [**55**, Section 4.5] suggest to consider well-ordering principles of type two: These take a type-one well-ordering principle as input and give a well-order as output. Alternatively, one could consider transformations of type-one well-ordering principles into type-one well-ordering principles. However, the type of the output is less important, because it can be lowered by Currying. To increase the logical complexity of part (ii) above, Rathjen suggests to replace ω-models by β-models. Recall that a β-model is an ω-model which satisfies all true Σ^1_1-statements. It is known that Π^1_1-comprehension is equivalent to the statement that any set is contained in a countably coded β-model (see [**87**, Theorem VII.2.10]). Rathjen [**67**] already describes a general strategy to construct β-models from higher well-ordering principles: To prove the theorem above, Afshari and Rathjen build ω-models by applying Schütte's [**77, 79**] method of deduction chains to ω-proofs (infinite proof trees where one can infer $\forall_x \varphi(x)$ once one has proved $\varphi(n)$ for each $n \in \mathbb{N}$). In order to construct β-models, Rathjen proposes to extend the relevant arguments to Girard's [**29**] β-proofs. These are compatible families of infinite proofs, indexed by the class of ordinals. Similar applications of deduction chains have been considered by Buchholz [**8**] and Jäger [**40**]. The present author found it difficult to directly build β-models of second-order arithmetic in this way. The construction of transitive models of set theories (which are related to β-models, but not always equivalent) turned out more feasible. Let us now state the main results of this thesis (labelled according to their appearance in the text). All notions will be explained below.

Theorem 4.4.6. The following are equivalent over the set theory $\mathbf{ATR_0^{set}}$:

(i) The principle of Π^1_1-comprehension.
(ii) The statement that each set is an element of some admissible set.
(iii) The abstract Bachmann-Howard principle: For any proto-dilator $\alpha \mapsto T^u_\alpha$, there is an ordinal α with a Bachmann-Howard collapse $\vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha$.
(iv) The predicative Bachmann-Howard principle: For an arbitrary dilator T^u, the Bachmann-Howard order $\mathrm{BH}(T^u)$ is well-founded.
(v) The computable Bachmann-Howard principle: For any coded dilator T, the relativized notation system $\vartheta(T)$ is well-founded.

While statements (ii), (iii) and (iv) rely on set-theoretic terminology, statement (v) can be formulated in second-order arithmetic. We will show that (v) implies arithmetical transfinite recursion, to get the following result:

Corollary 4.4.7. The following are equivalent over $\mathbf{RCA_0}$:

- The principle of Π^1_1-comprehension.
- The computable Bachmann-Howard principle.

Let us explain the notions that appear in these results: The base theory for most of this thesis will be primitive recursive set theory with infinity ($\mathbf{PRS}\omega$). For some arguments we need axiom beta and the axiom of countability as additional assumptions. This leads to $\mathbf{ATR_0^{set}}$, the set-theoretic version of arithmetical transfinite recursion due to Simpson [**85, 87**]. Chapter 1 presents these theories in detail. To make sense of statement (i) in a set-theoretic context, note that there is a natural way to interpret formulas of second-order arithmetic as formulas of set theory (see [**87**, Section VII.3]). Also recall that there is a single instance of Π^1_1-comprehension which implies all other instances of this axiom schema. As for statement (ii), an admissible set is defined as a transitive model of Kripke-Platek set theory. The equivalence between (i) and (ii) is known (see [**41**, Section 7]). In Corollary 1.4.13 we verify that it holds over $\mathbf{ATR_0^{set}}$.

Concerning statement (iii), a proto-dilator is a compatible choice of well-orders $T_\alpha = (T_\alpha, <_{T_\alpha})$ for all ordinals α (see Definition 2.1.1). In view of compatibility we get rank functions $|\cdot|_{T_\alpha} : T_\alpha \to \max\{\alpha, 1\}$. Crucially, the order-types of T_α do not need to form a normal function. Indeed, it is possible that the order-type of T_α is always bigger than α, so that no order-preserving collapse $T_\alpha \to \alpha$ can exist. However, we can take inspiration from ordinal analysis: The construction of the Bachmann-Howard ordinal involves a "sufficiently" order-preserving collapse of $\varepsilon_{\Omega+1}$ (the first ε-number above $\Omega = \aleph_1$) into the countable ordinals. Inspired by Rathjen's version of the collapsing construction (see [**72**, Section 1]) we introduce the notion of Bachmann-Howard collapse, denoted by $\vartheta : T_\alpha \xrightarrow{\text{BH}} \alpha$. The main requirement is that the order is preserved under a side condition, namely

$$\sigma <_{T_\alpha} \tau \Rightarrow \vartheta(\sigma) < \vartheta(\tau) \qquad \text{for } \sigma, \tau \in T_\alpha \text{ with } |\sigma|_{T_\alpha} < \vartheta(\tau).$$

The other requirement is $|\sigma|_{T_\alpha} < \vartheta(\sigma)$, which excludes some trivial choices of collapsing functions. The abstract Bachmann-Howard principle in (iii) states that such a collapse does always exist. It is relatively straightforward to see that this is true: Remark 2.1.6 provides a proof in **ZFC**. In Section 2.4 we show that a similar construction goes through on the basis of an admissible set, establishing (ii)⇒(iii) of the theorem above. We should mention that some restriction on the growth of the function $\alpha \mapsto T_\alpha$ is necessary: Our definition of proto-dilator will contain the condition that this is a primitive recursive set function (cf. Chapter 1). More precisely, we allow primitive recursive set functions with parameters; in particular the parameter ω is always allowed. As a consequence, any admissible set that contains the parameters will be closed under $\alpha \mapsto T_\alpha$. The restriction to primitive recursion is convenient in another respect: We formulate $\mathbf{ATR_0^{set}}$ in a language with a function symbol for each primitive recursive set function. Using a binary function symbol $(u, \alpha) \mapsto T^u_\alpha$ we can quantify over the family of functions $\alpha \mapsto T^u_\alpha$, by quantifying over the set-sized parameter u. In this way, the abstract Bachmann-Howard principle for a parametrized family of functions can be expressed by a single formula. More precisely, then, part (iii) is a schema which refers to the collection of these formulas.

To view the abstract Bachmann-Howard principle as a type-two well-ordering principle, note that it takes the type-one well-ordering principle $\alpha \mapsto T_\alpha$ as input and yields a well-order as output, namely an ordinal α which is sufficiently large to admit a collapse $\vartheta : T_\alpha \xrightarrow{\text{BH}} \alpha$. However, the transformation of input into output is not underpinned by computation. — In this sense the principle is abstract. While this is unsatisfactory from a foundational perspective, the abstract Bachmann-Howard principle has the advantage that it is easy to state, and that it can be appreciated independently of ordinal notation systems. In view of (iii)⇒(ii) it can be seen as a new characterization of the Church-Kleene ordinal ω_1^{CK} (as the latter is the minimal height of an admissible set, see [**6**, Corollary V.5.11]). This also sheds some light on the interpretation of Ω in notation systems for the Bachmann-Howard ordinal: Semantically, the obvious choices are $\Omega = \aleph_1$ and $\Omega = \omega_1^{\text{CK}}$. However, Pohlers [**60**, Section 9.7] shows that interpretations of Ω by ordinals below ω_1^{CK} can also be justified. In a sense, our result reinstates ω_1^{CK} as the smallest ordinal which supports the collapsing construction.

In order to remove the foundational deficits of the abstract Bachmann-Howard principle we would like to give an explicit construction of the ordinal α and the collapse $\vartheta : T_\alpha \xrightarrow{\text{BH}} \alpha$. To make this work we have to restrict to a class of particularly uniform well-ordering principles, which has been singled out by Girard [**28**]: Dilators are endo-functors of the category of well-orders that preserve direct limits and pull-backs. For details about our formulation of dilators we refer to Definition 2.2.1 and Remark 2.2.2. Let us remark that Aczel [**2, 3**] has already considered a concept similar to that of dilator. Jervell's [**45**] related notion of homogeneous tree is also relevant (his idea to view dilators as trees of finite sequences has inspired the presentation of search trees in Chapter 3). Proto-dilators were introduced by the present author, as a less sophisticated but much simpler notion, which is sufficient for some purposes.

Coming to statement (iv), the predicative Bachmann-Howard principle explicitly constructs a well-order $\text{BH}(T)$ which corresponds to the ordinal α in the abstract Bachmann-Howard principle. The idea is that an ordinal α with a collapse $\vartheta : T_\alpha \xrightarrow{\text{BH}} \alpha$ is a certain kind of fixed-point. To get such a fixed-point we define BH(T) as a direct limit

$$\begin{array}{ccccccc} \mathrm{BH}_0(T) & \longrightarrow & \mathrm{BH}_1(T) & \longrightarrow & \mathrm{BH}_2(T) & \longrightarrow & \cdots \\ & \searrow & \downarrow & \swarrow & & & \\ & & \mathrm{BH}(T). & & & & \end{array}$$

Roughly speaking, the construction guarantees a collapse of $T_{\mathrm{BH}_n(T)}$ into $\mathrm{BH}_{n+1}(T)$. In the limit we get the desired collapse of $T_{\mathrm{BH}(T)}$ into BH(T). Let us point out that well-foundedness is not, in general, preserved under direct limits: Indeed, any linear order is the direct limit of its finite (hence well-founded) sub-orders. The fact that the particular limit BH(T) is well-founded accounts for the strength of our type-two well-ordering principle. We will see that BH(T) can be constructed by a primitive recursive set function, with the parameters of T as inputs. Arguably, such functions are acceptable from a predicativist standpoint (cf. [**17**]). For this reason we describe the present variant of the Bachmann-Howard principle as "predicative". Of course, the implication (iv)$\Rightarrow$(i) of Theorem 4.4.6 means that the well-foundedness of BH(T) cannot be justified by predicative means.

We point out that our definition of BH(T) as the limit of a directed system of ω approximations is very similar to a construction due to Aczel [**2, 3**]. However, there is a significant difference: Aczel assumes that the functor T preserves initial segments (Girard calls such dilators "flowers"). Then the associated order-types form a normal function and we have an isomorphism $\alpha \cong T_\alpha$ for some ordinal α. In the present thesis we are most interested in dilators which do not preserve initial segments and do not admit fully order-preserving maps $T_\alpha \to \alpha$. The fact that we still have an "almost" order-preserving collapse of $T_{\mathrm{BH}(T)}$ into BH(T) seems to be responsible for the proof-theoretic strength of the Bachmann-Howard principle.

The computable Bachmann-Howard principle constructs a relativized ordinal notation system $\vartheta(T)$ by even more elementary means. Parallel to the type-one well-ordering principle $X \mapsto \varepsilon_X$ discussed above, the point is that the existence of $\vartheta(T)$ as a set (and indeed a linear order) can be proved in $\mathbf{RCA_0}$. Thus the entire strength of the computable Bachmann-Howard principle lies in the assertion that $\vartheta(T)$ is well-founded. Before we can construct $\vartheta(T)$ we need a formalization of dilators in second-order arithmetic. This relies on Girard's result that dilators are determined by their restrictions to the category of natural numbers. In Section 2.3 we give a primitive recursive set function which reconstructs any dilator from its restriction (up to natural isomorphism, cf. Remark 2.3.7). Let us point out two advantages with respect to the predicative Bachmann-Howard principle: Firstly, as observed by Montalbán [**55**, Section 4.5], we can now quantify over all dilators. Thus the predicative Bachmann-Howard principle can be expressed by a single formula, rather than a schema (see Proposition 2.3.10). Secondly, in the case of proto-dilators the restriction to primitive recursive set functions was somewhat ad hoc. For dilators this restriction is automatic.

Back to the computable Bachmann-Howard principle, assume that $T \subseteq \mathbb{N}$ codes the restriction of a dilator to the category of natural numbers. We will construct a term system $\vartheta(T)$ which, by its very structure, comes with a collapse

$$\vartheta(T) \to \vartheta(T) \cap \Omega.$$

Here the sub-order $\vartheta(T) \cap \Omega \subseteq \vartheta(T)$ corresponds to the order $\mathrm{BH}(T)$ from the predicative Bachmann-Howard principle. To turn $\vartheta(T) \cap \Omega$ into the desired fixed point, it is enough to ensure that $\vartheta(T)$ contains a copy of $T_{\vartheta(T)\cap\Omega}$. This can be achieved in the following way: Assume that we have terms $s_0 <_{\vartheta(T)} \cdots <_{\vartheta(T)} s_{n-1}$ in $\vartheta(T) \cap \Omega$, and consider the embedding $h : n \to \vartheta(T) \cap \Omega$ with $h(i) = s_i$. As T is functorial, any $\sigma \in T_n$ yields an element $T_h(\sigma) \in T_{\vartheta(T)\cap\Omega}$. Now the idea is to add a term $\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}} \in \vartheta(T)$, which acts as the representative of $T_h(\sigma)$ in $\vartheta(T)$. If we have an ordinal α with a Bachmann-Howard collapse then this interpretation of terms can be made official (see Theorem 2.4.9). We thus get an embedding of $\vartheta(T) \cap \Omega$ into α, establishing direction (iii)⇒(v) of Theorem 4.4.6.

So far we have mentioned the implications (i)⇔(ii)⇒(iii)⇒(v) of Theorem 4.4.6. To show direction (v)⇒(iv) we will construct an embedding of $\mathrm{BH}(T)$ into $\vartheta(T)\cap\Omega$, by glueing a compatible family of embeddings $\mathrm{BH}_n(T) \to \vartheta(T) \cap \Omega$ (see Corollary 2.4.18). In order to complete the proof, it suffices to establish (iv)⇒(ii). To get a glimpse of the argument, recall that Jäger [**39**] deduces the consistency of Kripke-Platek set theory from the well-foundedness of the Bachmann-Howard ordinal. By the completeness theorem one gets a model of Kripke-Platek set theory. There is, however, no reason why this model should be an admissible set: The symbol $\in$ does not have to be interpreted by a well-founded relation. Our task will be to build a well-founded (and indeed transitive) model under the assumption that $\mathrm{BH}(T)$ is well-founded for any dilator T. This is worked out in Chapters 3 and 4. A more detailed explanation can be found at the beginning of these chapters.

After completing the proof of Theorem 4.4.6, the present author has learned that a related result was announced in the unpublished second part of Girard's book on proof theory [**31**]: Girard claims that Π^1_1-comprehension is equivalent to the statement that his functor Λ maps dilators to dilators. However, a full proof of this equivalence is not given: Girard presents an argument (via functorial cut elimination for theories of inductive definitions) but notes that certain intermediate results are missing. We would also like to point out that Girard's functor Λ is quite different from our Bachmann-Howard principle (it would be interesting to determine the precise connection, possibly via Girard and Vauzeilles' [**32, 94**] functorial construction of the Bachmann hierarchy). In spite of these differences, the present thesis was strongly influenced by Girard's published work on Π^1_2-logic, in particular by the conceptual insights in [**28, 29**].

The author would also like to mention a draft version [**21**] of this thesis, which he has made available on the arXiv repository. It contains most material related to the abstract Bachmann-Howard principle, while the predicative and the computable versions are not yet included. The present thesis has been entirely rewritten, but some similar formulations may remain.

In the first paragraph of this introduction we have stated that our results solve a conjecture of Rathjen and Montalbán. Let us now assess this claim in more detail: Rathjen [**68**, Conjecture 6.1] writes that "[type-two well-ordering principles F], where F comes from some ordinal representation system used for an ordinal analysis of a theory T_F, [should be] equivalent to statements of the form 'every set belongs to a countable coded β-model of T_F'." He also points out that Π^1_1-comprehension is equivalent to such a statement about β-models. In Theorem 4.4.6 we do not directly construct β-models. However, we do construct admissible sets (which are rather similar to β-models), and we do reach Π^1_1-comprehension. Montalbán [**55**, Question 26] conjectures that Π^1_1-comprehension is equivalent to the statement that $\vartheta(f(\Omega+1))$ is well-ordered for every dilator f. He does not give the definition of $\vartheta(f(\Omega+1))$ in detail, but it seems to be somewhat different from our constructions. In any case, Montalbán writes that "the precise definitions of the notions involved might have to be adjusted to get interesting results." We think that Theorem 4.4.6 and Corollary 4.4.7 fall under this specification. It would be very interesting to know whether our results have computability-theoretic proofs as well, just as the aforementioned results [**35, 24, 52**] on well-ordering principles of type one.

Chapter 5 can be read independently of the rest of the thesis. It is concerned with the notion of slow consistency, introduced by S.-D. Friedman, Rathjen and Weiermann [**26**]. The results of this chapter were also obtained during the author's PhD studies and have been published in two papers [**19, 20**] in the *Annals of Pure* *and Applied Logic*. A precise definition of the slow consistency statement $\mathrm{Con}^\diamond(\mathbf{PA})$ can be found in the introduction of Chapter 5. The point is that it weakens the usual consistency statement $\mathrm{Con}(\mathbf{PA})$ for Peano arithmetic: Friedman, Rathjen and Weiermann [**26**] show that we have

$$\mathbf{PA} + \mathrm{Con}^\diamond(\mathbf{PA}) \nvdash \mathrm{Con}(\mathbf{PA})$$

but still $\mathbf{PA} \nvdash \mathrm{Con}^\diamond(\mathbf{PA})$. The present author has extended the idea by introducing slow reflection statements $\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_n)$. These have interesting properties, both on the level of computational content and of consistency strength: On the one hand $\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2)$ extends Peano arithmetic by a new provably total function. On the other hand we have

$$\mathbf{PA} + \mathrm{Con}(\mathbf{PA}) \vdash \mathrm{Con}(\mathbf{PA} + \{\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_n) \,|\, n \in \mathbb{N}\}).$$

Our results on slow reflection imply that the usual consistency statement for Peano arithmetic is equivalent to ε_0 iterations of slow consistency. This has been conjectured by S.-D. Friedman, Rathjen and Weiermann [**26**]. It was independently established by Henk and Pakhomov [**34**]. Surprisingly, slow reflection can also be used to prove a refinement of Paris and Harrington's [**58**] famous independence result: We will show that Σ^0_1-instances of the Paris-Harrington principle can only have extremely long proofs in certain fragments of Peano arithmetic.

CHAPTER 1

Preliminaries: The Base Theory

The present chapter is concerned with the base theory in which we are going to work for the rest of the thesis. As explained in the introduction this will be a reasonably weak set theory.

In second-order arithmetic the choice of base theory is — nowadays — relatively straightforward. The most common axiom systems are linearly ordered in a very strong sense: Given two of these systems, the stronger will prove all axioms of the weaker (rather than just their Π^0_1-consequences). As a base theory one will then choose the weakest possible system (from a theoretical or practical perspective). In contrast, natural set theories are not always comparable in this strong sense. For example, Simpson's set-theoretic version of arithmetical transfinite recursion and Kripke-Platek set theory each have well-founded models of height ω_1^{CK} (the Church-Kleene ordinal), but the two theories have no joint model of that height (see [**85**, Lemma 5.1]).

As an — admittedly rather construed — example of possible pitfalls, assume that we adopt Zermelo-Fraenkel set theory without infinity as our base theory. Since this theory is interpretable in Peano arithmetic (due to Ackermann [**1**]) it may, from a superficial standpoint, look like a legitimate choice of base theory. Now, the principle of Π^1_1-comprehension (which asserts that any Π^1_1-definable subclass of the natural numbers is a set) and the existence of admissible sets both imply the axiom of infinity. Thus when we add either statement the other one becomes provable, simply because it is a theorem of Zermelo-Fraenkel set theory with infinity. However, this does not establish any interesting connection between Π^1_1-comprehension and admissible sets.

For similar if more subtle reasons Kripke-Platek set theory, even with restrictions on induction, is no suitable basis for our investigation. This is because some of our arguments rely on axiom beta (which states that any well-founded relation can be collapsed to the $\in$-relation). However, Kripke-Platek set theory together with axiom beta already yields Δ^1_2-comprehension (see [**41**, Theorem 8.2]).

Guided by these negative examples we choose primitive recursive set theory with infinity (**PRS**ω) as our default base theory. This rests on the primitive recursive set functions investigated by Jensen and Karp [**44**] (who name Gandy, Platek, and Takeuti and Kino as further originators of this or similar notions). Following Rathjen [**62**], we formulate **PRS**ω in a language with a function symbol for each (definition of a) primitive recursive set function. Details can be found in Section 1.1. For the moment, let us just stress that "primitive recursive" will by default refer to primitive recursive set functions relative to the parameter ω (see Convention 1.2.9). For some arguments it will be necessary to extend **PRS**ω by axiom beta, or by the axiom of countability (which states that every set is countable). Where this is the case it will be indicated (see Convention 1.4.10).

Primitive recursive set theory has several advantages: Firstly, many fundamental constructions which are familiar from Kripke-Platek set theory are available in **PRS**ω as well. This is because the Σ-recursion theorem (see [**6**, Theorem I.5.4]) is mostly used to construct primitive recursive functions. Indeed, Rathjen [**62**] shows that **PRS**ω proves the same Π_2-theorems as Kripke-Platek set theory with Σ_1-foundation (unfortunately the same principle is sometimes called Π_1-foundation, due to the duality between well-foundedness and $\in$-induction). We will review some relevant constructions in Section 1.2 and Section 1.3.

Summarizing the previous paragraph, one might say that primitive recursive set theory keeps the concrete computational content of the Σ-recursion theorem. At the same time it avoids the abstract principle — Δ_0-collection — on which that theorem rests in Kripke-Platek set theory. This makes **PRS**ω more robust: As we have seen above, the strength of Kripke-Platek set theory increases dramatically when we add axiom beta. Extending **PRS**ω by axiom beta (and the axiom of countability) has a more moderate effect: Modulo the elimination of function symbols it leads us to Simpson's set-theoretic version of arithmetical transfinite recursion. Thus **PRS**ω + "beta" + "countability" has a natural interpretation in the second-order system $\mathbf{ATR}_0$. Moreover, this interpretation is robust under extensions, in the sense that we may add Π^1_1-comprehension and other second-order axioms on both sides (see [**87**, VII.3.34]). Note that the interpretation of Zermelo-Fraenkel set theory without infinity in Peano arithmetic (see above) does not enjoy this property. Details on the connection between **PRS**ω and Simpson's set theory can be found in Section 1.4. There we also review the known fact that Π^1_1-comprehension is equivalent to the statement that any set is an element of some admissible set. Note that admissible sets are defined as transitive models of Kripke-Platek set theory (see Definition 1.3.11).

Another advantage of primitive recursive set theory is that we can represent class-sized functions without coding. In particular this applies to the (proto-) dilators that appear in our main result. Also, primitive recursion singles out a natural class of functions with good properties: For example, admissible sets are closed under primitive recursive set functions. The ability to speak about proper classes becomes less important in view of Section 2.3: There we will see that dilators are naturally represented by set-sized objects.

Finally, most mathematical arguments seem to involve functional dependencies. With function symbols at our disposal we can formalize them in a straightforward yet rigorous way. In this respect we are reminded of primitive recursive arithmetic, with its great tradition in finitistic and proof-theoretic investigations.

1.1. A Set Theory Based on Primitive Recursion

In this section we introduce primitive recursive set theory with infinity (**PRS**ω), which will be the default base theory for the rest of this thesis (occasionally extended by axiom beta and the axiom of countability, see Section 1.4). Our exposition draws on Jensen and Karp's [**44**] investigation of primitive recursive set functions, and on Rathjen's [**62**] definition of a corresponding logical theory. As far as the present author is aware, the existing literature does not contain a detailed introduction to the formalization of set-theoretic arguments in **PRS**ω. Recovering known facts about primitive recursion inside this formal theory is mostly standard, but a few subtleties do arise: For example it will be important that Δ_0-separation is a theorem, rather than an axiom. Also, the interplay between transitive closures and $\in$-induction requires some attention (see the discussion before Proposition 1.1.11). Let us begin with the language of **PRS**ω:

Definition 1.1.1. We inductively define primitive recursive (p.r.) function symbols and their arities. Intended interpretations are given to guide the reader but play no official role.

(i) We have a unary p.r. function symbol Z with intended interpretation

$$Z(x) = 0.$$

(ii) We have a binary p.r. function symbol M with intended interpretation

$$M(x_1, x_2) = x_1 \cup \{x_2\}.$$

(iii) For each $n \geq 1$ and each $i = 1, \dots, n$ we have an n-ary p.r. function symbol P^n_i with intended interpretation

$$P^n_i(x_1, \dots, x_n) = x_i.$$

(iv) We have a quaternary p.r. function symbol C with intended interpretation

$$C(x_1, x_2, x_3, x_4) = \begin{cases} x_1 & \text{if } x_3 \in x_4, \\ x_2 & \text{otherwise.} \end{cases}$$

(v) Given p.r. function symbols G of arity k and $H_1, \dots, H_k$ of a common arity n we have an n-ary p.r. function symbol $K[G; H_1, \dots, H_k]$ with intended interpretation

$$K[G; H_1, \dots, H_k](\vec{x}) = G(H_1(\vec{x}), \dots, H_k(\vec{x})).$$

(vi) For each $(n{+}2)$-ary p.r. function symbol G we have an $(n{+}1)$-ary p.r. function symbol $R[G]$. The intended interpretation is the unique function which satisfies the recursion

$$R[G](x, \vec{y}) = G\left(\bigcup\{R[G](v, \vec{y}) \mid v \in x\}, x, \vec{y}\right).$$

By $\mathcal{L}_{\text{pr}}$ we denote the first order language with (binary) relation symbols $\in, =$ and all primitive recursive function symbols.

Note that all primitive recursive function symbols have non-zero arity. It would seem natural to conceive Z as a nullary function (constant symbol). The n-ary function with constant value 0 could then we recovered as $K[Z;]$, where the semicolon is followed by zero n-ary functions (one should write $K^n[Z;]$ to indicate the arity). The author finds it attractive to include nullary functions (primitive recursive sets) but does not want to deviate from the existing literature at this point. Note that nullary functions are dispensable for the purpose of formalization: Simply argue under the assumption that a fixed variable assumes a particular value (cf. Convention 1.2.9). Let us also point out that other forms of the recursion principle can be derived, in particular course-of-values recursion (see Lemma 1.2.3). The above variant of recursion is natural if one thinks of the stages of an inductive definition. We have said that the intended interpretations of our function symbols have no official status. Instead, their meaning is fixed as in [**62**, Section 6]:

Definition 1.1.2. To each primitive recursive function symbol F we associate an $\mathcal{L}_{\mathrm{pr}}$-formula $\mathcal{A}_F$, called the defining axiom of F:

$$\mathcal{A}_Z :\equiv \forall_x \forall_{y \in Z(x)} y \neq y,$$

$$\mathcal{A}_M :\equiv \forall_{x_1,x_2}(\forall_{y \in M(x_1,x_2)}(y \in x_1 \vee y = x_2) \wedge \forall_{y \in x_1} y \in M(x_1,x_2) \wedge x_2 \in M(x_1,x_2)),$$

$$\mathcal{A}_{P_i^n} :\equiv \forall_{x_1,\dots,x_n} P_i^n(x_1,\dots,x_n) = x_i,$$

$$\mathcal{A}_C :\equiv \forall_{x_1,x_2,x_3,x_4}((x_3 \in x_4 \wedge C(x_1,x_2,x_3,x_4) = x_1) \vee (x_3 \notin x_4 \wedge C(x_1,x_2,x_3,x_4) = x_2)),$$

$$\mathcal{A}_{K[G;H_1,\dots,H_k]} :\equiv \forall_{\vec{x}} K[G;H_1,\dots,H_k](\vec{x}) = G(H_1(\vec{x}),\dots,H_k(\vec{x})),$$

$$\mathcal{A}_{R[G]} :\equiv \forall_{x,\vec{y}} \exists_u (u = \bigcup\{R[G](v,\vec{y}) \mid v \in x\} \wedge R[G](x,\vec{y}) = G(u,x,\vec{y})).$$

In the last line, $u = \bigcup\{R[G](v,\vec{y}) \mid v \in x\}$ abbreviates

$$\forall_{w \in u} \exists_{v \in x} w \in R[G](v,\vec{y}) \wedge \forall_{v \in x} \forall_{w \in R[G](v,\vec{y})} w \in u.$$

Some readers may regret the unbounded existential quantifier in $\mathcal{A}_{R[G]}$. To avoid it one could compute the witness u by a primitive recursive function, as in Corollary 1.1.7 below. In any case, adding some fundamental facts about sets we obtain the desired formal theory:

Definition 1.1.3. Primitive recursive set theory with infinity (**PRS**ω) is the $\mathcal{L}_{\mathrm{pr}}$-theory with the following axioms:

(i) The usual equality axioms (in particular $=$ is compatible with each primitive recursive function symbol).

(ii) The axiom of extensionality, i.e. the formula

$$\forall_{x,y}(\forall_{z \in x} z \in y \wedge \forall_{z \in y} z \in x \to x = y).$$

(iii) The axiom of foundation (also called regularity), i.e. the formula

$$\forall_x(\exists_{y \in x} y = y \to \exists_{y \in x} \forall_{z \in y} z \notin x).$$

(iv) The axiom of infinity, in the form

$$\exists_x(Z(x) \in x \wedge \forall_{y \in x} M(y,y) \in x).$$

(v) The defining axiom $\mathcal{A}_F$ of each primitive recursive function symbol F.

To remove the unbounded existential quantifier in the axiom of infinity one could add a function symbol with constant value ω. We avoid such a function symbol to facilitate the comparison with Simpson's set theory $\mathbf{ATR_0^{set}}$ in Section 1.4. Axiom (iii) is, more specifically, called set foundation, to distinguish it from the scheme of foundation ($\in$-induction) for classes. The former implies the latter if we have transitive closures and a sufficient amount of separation (see Proposition 1.1.11). In addition to our axioms, Rathjen [**62**] lists pairing, union, and Δ_0-separation. We will show that these can in fact be derived.

PROPOSITION 1.1.4. *In* $\mathbf{PRS}\omega$ *we can prove the following:*

(i) *The pairing axiom. In fact,*

$$\{x, y\} := K\big[M; K[M; K[Z; P^2_1], P^2_1], P^2_2\big]\,(x, y)$$

defines a p.r. function symbol for which $\mathbf{PRS}\omega$ *shows*

$$z \in \{x, y\} \leftrightarrow z = x \vee z = y.$$

(ii) *The union axiom. In fact,*

$$\bigcup x := K\big[R\,\big[K[C; P^3_2, P^3_1, P^3_2, P^3_3]\big]\,; P^1_1, P^1_1\big]\,(x)$$

defines a p.r. function symbol for which $\mathbf{PRS}\omega$ *shows*

$$w \in \bigcup x \leftrightarrow \exists_{v\in x} w \in v.$$

Note that we can combine (i) and (ii) to construct the union $x \cup y = \bigcup\{x, y\}$ of two sets.

PROOF. (i) By the defining axioms for compositions and projections, together with the equality axioms, we get

$$K\big[M; K[M; K[Z; P^2_1], P^2_1], P^2_2\big]\,(x, y) = M(M(Z(x), x), y).$$

The axioms for M and Z make $z \in M(M(Z(x), x), y)$ equivalent to $z = x \vee z = y$.
(ii) We remark that the given definition of $\bigcup x$ is extracted from [**62**, 2.2(iii)]. Abbreviate $K[C; P^3_2, P^3_1, P^3_2, P^3_3]$ by G and observe

$$G(u, v, x) = C(v, u, v, x).$$

As a first step, let us show

$$v \in x \to R[G](v, x) = v.$$

The defining axiom for $R[G]$ provides an (irrelevant) witness u_0 such that we have

$$R[G](v, x) = G(u_0, v, x) = C(v, u_0, v, x).$$

For $v \in x$ the defining axiom of C yields $C(v, u_0, v, x) = v$, as desired. Next, by the definition of the function symbol $\bigcup \cdot$ we have

$$\bigcup x = K\left[R[G]; P^1_1, P^1_1\right](x) = R[G](x, x).$$

By the defining axiom of $R[G]$ we get a (now relevant) witness u with

$$\forall_w (w \in u \leftrightarrow \exists_{v \in x} w \in R[G](v, x)) \wedge R[G](x, x) = G(u, x, x).$$

In view of the above the first conjunct amounts to

$$w \in u \leftrightarrow \exists_{v \in x} w \in v.$$

To conclude it is thus enough to establish that u is equal to

$$\bigcup x = R[G](x, x) = G(u, x, x) = C(x, u, x, x).$$

By the defining axiom for C this reduces to $x \notin x$: Foundation for $\{x\} := \{x, x\} \neq \emptyset$ yields $\forall_{z \in x} z \notin \{x\}$. Thus $x \in x$ would imply $x \notin \{x\}$, which is false. □

To approach Δ_0-separation we first consider primitive recursive separation, which is prima facie weaker but turns out equivalent. It may be helpful to think in terms of primitive recursive classes, i.e. classes of the form

$$\{(v_0, \ldots, v_n) \,|\, Z(v_0) \in F(v_0, \ldots, v_n, \vec{y})\}$$

with a p.r. function symbol F. To be precise, one should say that this class is primitive recursive in $\vec{y}$, but we will often leave the parameters implicit. We shall also speak of primitive recursive relations or properties.

Lemma 1.1.5. *In* **PRS**ω *we can show primitive recursive separation. More precisely, for each $(n+1)$-ary p.r. function symbol F there is an $(n+1)$-ary p.r. function symbol F^S such that* **PRS**ω *proves*

$$v \in F^S(x, \vec{y}) \leftrightarrow v \in x \wedge Z(v) \in F(v, \vec{y}).$$

We will also write $\{v \in x \,|\, Z(v) \in F(v, \vec{y})\}$ at the place of $F^S(x, \vec{y})$.

Proof. The given requirement can be re-written as

$$F^S(x, \vec{y}) = \bigcup \{\{v\} \,|\, v \in x \wedge Z(v) \in F(v, \vec{y})\}.$$

This reveals the similarity with the function $\bigcup x = \bigcup\{v \mid v \in x\}$ that we have constructed in Proposition 1.1.4(ii). Indeed, it is not hard to adapt the argument given there to the construction of F^S. Details can be found in [**62**, 2.2(iii)]. □

Next, we observe that images of p.r. functions can themselves be computed by primitive recursion. This fact will replace some applications of Δ_0-collection, e.g. in the proof of Proposition 1.2.1. An analogous result holds for rudimentary functions, where it has a deeper proof via the Gandy-Jensen Lemma (see [**53**, Section 2]).

PROPOSITION 1.1.6. *For any $(n+1)$-ary p.r. function symbol F there is an $(n+1)$-ary p.r. function symbol rng_F such that* **PRS**ω *proves*

$$w \in \operatorname{rng}_F(x, \vec{y}) \leftrightarrow \exists_{v \in x} w = F(v, \vec{y}).$$

We will also write $\operatorname{rng}(F(\cdot, \vec{y}) \restriction x)$ *or* $\{F(v, \vec{y}) \mid v \in x\}$ *at the place of* $\operatorname{rng}_F(x, \vec{y})$.

PROOF. Similarly to Lemma 1.1.5, we can re-write the given requirement as

$$\operatorname{rng}_F(x, \vec{y}) = \bigcup\{\{F(v, \vec{y})\} \mid v \in x\}.$$

So again, this is a variant of the argument used to construct $\bigcup x = \bigcup\{v \mid v \in x\}$ in Proposition 1.1.4(ii). Details can be found in [**62**, 2.2(v)]. □

As promised, the defining axiom of $R[G]$ can now be reformulated without the unbounded quantifier:

COROLLARY 1.1.7. *The equation*

$$R[G](x, \vec{y}) = G\left(\bigcup \operatorname{rng}_{R[G]}(x, \vec{y}), x, \vec{y}\right)$$

is an atomic $\mathcal{L}_{\mathrm{pr}}$-formula, and indeed a theorem of **PRS**ω.

PROOF. The axiom $\mathcal{A}_{R[G]}$ yields $R[G](x, \vec{y}) = G(u, x, \vec{y})$ for a witness u with

$$w \in u \leftrightarrow \exists_{v \in x} w \in R[G](v, \vec{y}).$$

By Proposition 1.1.4 and Proposition 1.1.6 we get $u = \bigcup \operatorname{rng}_{R[G]}(x, \vec{y})$. □

The defining condition $Z(v_0) \in F(v_0, \ldots, v_n, \vec{y})$ of a primitive recursive class looks somewhat arbitrary, and it is cumbersome to apply. The following notion is an improvement in both respects:

DEFINITION 1.1.8. A Δ_0-formula (or bounded formula) in the language $\mathcal{L}_{\text{pr}}$ is a formula which only contains bounded quantifiers. This means that any occurrence of a universal resp. existential quantifier must be of the form

$$\forall_{x\in t}\theta \equiv \forall_x(x \in t \to \theta),$$
$$\exists_{x\in t}\theta \equiv \exists_x(x \in t \wedge \theta),$$

where t is a term in which the bound variable x does not occur.

Since we allow primitive recursive functions as bounds on quantifiers, the notion of Δ_0-formula is more liberal in $\mathcal{L}_{\text{pr}}$ than in the language of pure set theory. This need not concern us too much, as Δ_0-formulas are not mentioned in our axiomatization of $\mathbf{PRS}\omega$ (in contrast to [**62**], where Δ_0-separation is included). Also, the following result allows us to avoid reference to Δ_0-formulas of $\mathcal{L}_{\text{pr}}$: We will often speak of primitive recursive properties instead.

PROPOSITION 1.1.9. *Over primitive recursive set theory, a class is Δ_0 precisely if it is primitive recursive. More precisely, for each Δ_0-formula $\theta \equiv \theta(x_0, \ldots, x_n)$ of $\mathcal{L}_{\text{pr}}$ there is an $(n+1)$-ary p.r. function symbol F_θ such that $\mathbf{PRS}\omega$ proves*

$$\theta \leftrightarrow Z(x_0) \in F_\theta(x_0, \ldots, x_n).$$

PROOF. First, for each term $t \equiv t(x_0, \ldots, x_n)$ (in which not all displayed variables need to occur) there is an $(n+1)$-ary function symbol F_t (which also depends on the variable list $x_0, \ldots, x_n$) such that $\mathbf{PRS}\omega$ proves

$$t = F_t(x_0, \ldots, x_n).$$

It is straightforward to check this by induction on t: If t is a variable one uses the projections. Otherwise one writes

$$t \equiv G(t_0, \ldots, t_k)$$

and takes

$$F_t :\equiv K[G; F_{t_0}, \ldots, F_{t_k}].$$

Now, following [**62**, 2.2(iv)], we invoke extensionality and equality to see

$$s = t \leftrightarrow \forall_{z\in s} z \in t \wedge \forall_{z\in t} z \in s.$$

This means that we can eliminate all occurrences of the relation symbol $=$, without leaving the realm of Δ_0-formulas. Also, we can reduce the stock of logical symbols to negation, disjunction, and bounded existential quantifier. Under these assumptions we argue by induction on the build-up of formulas. Concerning the remaining prime formulas, observe that we have

$$s \in t \leftrightarrow Z(x_0) \in C(M(Z(x_0), Z(x_0)), Z(x_0), s, t).$$

Let F_s and F_t be the functions constructed above, relative to the given list of free variables $x_0, \ldots, x_n$. Writing Z^k for $K[Z; P_1^k]$ (the k-ary function with constant value zero) we can thus put

$$F_{s \in t} :\equiv K\big[C; K[M; Z^{n+1}, Z^{n+1}], Z^{n+1}, F_s, F_t\big] .$$

Next, in view of

$$Z(x_0) \notin F_\theta(x_0, \ldots, x_n) \leftrightarrow$$
$$Z(x_0) \in C(Z(x_0), M(Z(x_0), Z(x_0)), Z(x_0), F_\theta(x_0, \ldots, x_n)),$$

negation is covered by

$$F_{\neg\theta} :\equiv K[C; Z^{n+1}, K[M; Z^{n+1}, Z^{n+1}], Z^{n+1}, F_\theta].$$

Similarly, we stipulate

$$F_{\varphi\vee\psi}(x_0, \ldots, x_n) := F_\varphi(x_0, \ldots, x_n) \cup F_\psi(x_0, \ldots, x_n)$$

in the case of a disjunction. Finally, consider a bounded quantification

$$\theta \equiv \exists_{y \in t}\varphi.$$

By induction hypothesis we have an $(n+2)$-ary p.r. function symbol F_φ with

$$\varphi \leftrightarrow Z(y) \in F_\varphi(y, x_0, \ldots, x_n).$$

Using the $(n+2)$-ary p.r. function symbol rng_{F_φ} from Proposition 1.1.6 we set

$$F_\theta :\equiv K\left[\bigcup; K[\mathrm{rng}_{F_\varphi}; F_t, P_1^{n+1}, \ldots, P_{n+1}^{n+1}]\right].$$

Computing

$$F_\theta(x_0, \ldots, x_n) = \bigcup\{F_\varphi(y, x_0, \ldots, x_n) \,|\, y \in F_t(x_0, \ldots, x_n)\}$$

we obtain

$$Z(x_0) \in F_\theta(x_0, \ldots, x_n) \leftrightarrow \exists_{y \in t} Z(x_0) \in F_\varphi(y, x_0, \ldots, x_n).$$

By the above the right side is equivalent to θ, as required. □

In particular we can restate the following result, now for Δ_0-classes:

COROLLARY 1.1.10. *Primitive recursive set theory proves Δ_0-separation. More precisely, for each Δ_0-formula θ in the language $\mathcal{L}_{\mathrm{pr}}$ we have a* **PRS**ω*-proof of*

$$v \in F_\theta^S(x, y_1, \ldots, y_n) \leftrightarrow v \in x \wedge \theta(v, y_1, \ldots, y_n),$$

where $F_\theta^S = (F_\theta)^S$ is constructed according to Proposition 1.1.9 and Lemma 1.1.5. We will also write $\{v \in x \mid \theta(v, y_1, \ldots, y_n)\}$ at the place of $F_\theta^S(x, y_1, \ldots, y_n)$.

Let us turn to $\in$-induction. To derive this principle from the axiom of foundation we will use transitive closures, which is a somewhat subtle point: It would be tempting to define $\mathrm{TC}(x) = x \cup \bigcup\{\mathrm{TC}(y) \mid y \in x\}$, as in [**62**, 2.2(viii)]. However, to verify that the set $\mathrm{TC}(x)$ is transitive we then seem to need $\in$-induction, the very principle that we want to justify. As we shall see, the axiom of infinity offers a way out. This complication stresses the importance of $\in$-induction as a basic principle of **PRS**ω: We can define functions by $\in$-recursion, but to verify their properties we will often need $\in$-induction.

PROPOSITION 1.1.11. *Primitive recursive set theory with infinity establishes $\in$-induction for primitive recursive properties. Indeed, we have a* **PRS**ω*-proof of*

$$\forall_x(\forall_{y \in x}\theta(y, \vec{z}) \to \theta(x, \vec{z})) \to \forall_x \theta(x, \vec{z}),$$

for each Δ_0-formula $\theta(x, \vec{z})$ in the language $\mathcal{L}_{\mathrm{pr}}$.

PROOF. First, use primitive recursion to define

$$\mathrm{TC}_z(x) = x \cup \bigcup\left(\bigcup\{\mathrm{TC}_y(x) \mid y \in z\}\right).$$

Given a witness w to the axiom of infinity, we want to see that $\mathrm{TC}_w(x)$ is transitive. So consider arbitrary sets $u \in v \in \mathrm{TC}_w(x)$. It suffices to establish $v \in \mathrm{TC}_z(x)$ for some $z \in w$, for then we have $v \in \bigcup\{\mathrm{TC}_z(x) \mid z \in w\}$ and thus

$$u \in \bigcup\left(\bigcup\{\mathrm{TC}_z(x) \mid z \in w\}\right) \subseteq \mathrm{TC}_w(x).$$

If $v \in \mathrm{TC}_w(x)$ holds by virtue of $v \in x$ we take an arbitrary $z \in w$ (note that w is non-empty according to the axiom of infinity) and observe

$$v \in x \subseteq \mathrm{TC}_z(x).$$

Otherwise $v \in \mathrm{TC}_w(x)$ holds by virtue of $v \in \bigcup(\bigcup\{\mathrm{TC}_y(x) \mid y \in w\})$, so that we have $v \in \bigcup \mathrm{TC}_y(x)$ for some $y \in w$. Setting $z := M(y, y) \ni y$ we get

$$v \in \bigcup(\bigcup\{\mathrm{TC}_y(x) \mid y \in z\}) \subseteq \mathrm{TC}_z(x),$$

and the axiom of infinity ensures $z \in w$, as desired. Let us stress that we will not use $\mathrm{TC}_w(x) \subseteq u$ for transitive $u \supseteq x$. This would require $\in$-induction (on w), the very principle that we want to establish. Now consider the instance of $\in$-induction in the statement of the proposition. Aiming at its contrapositive, assume $\neg\theta(x, \vec{z})$ for some x. Corollary 1.1.10 allows us to form

$$\{y \in \mathrm{TC}_w(\{x\}) \mid \neg\theta(y, \vec{z})\}.$$

By assumption this set is non-empty, so that the axiom of foundation provides an $\in$-minimal $y \in \mathrm{TC}(\{x\})$ with $\neg\theta(y, \vec{z})$. In particular we have $\theta(y', \vec{z})$ for any $y' \in y$ with $y' \in \mathrm{TC}_w(\{x\})$. The last condition is redundant as $\mathrm{TC}_w(\{x\})$ is transitive. Thus we see

$$\forall_{y' \in y} \theta(y', \vec{z}) \land \neg\theta(y, \vec{z}),$$

and the contrapositive of $\in$-induction is established. □

To conclude this section, let us repeat that our axiomatization of **PRS**ω is equivalent to Rathjen's axiomatization in [**62**, Section 6], as pairing, union and Δ_0-separation are derivable by Proposition 1.1.4 and Corollary 1.1.10. Thus Rathjen's result [**62**, Theorem 1.4] applies to our setting: Any Π_2-theorem of Kripke-Platek set theory, with $\in$-induction restricted to Σ_1-formulas, is already provable in **PRS**ω. We will not use this fact, but it is certainly reassuring.

1.2. Basic Constructions in Primitive Recursive Set Theory

The present section recovers basic set-theoretic constructions in primitive recursive set theory with infinity (**PRS**ω), as introduced in the previous section. Largely we follow Barwise's development of Kripke-Platek set theory in [**6**, Chapter I]. As in the previous section, most relevant facts about primitive recursion are known and can be found in [**44, 62**].

First, the ordered pair of two sets can be computed by a primitive recursive function (we will no longer differentiate between functions and function symbols). Namely, using Proposition 1.1.4 we set

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

The claim that $\langle x, y \rangle$ is an ordered pair has an "external" and an "internal" justification: Externally, there is a universally accepted formula which characterizes ordered pairs in the language of pure set theory (i.e. without p.r. function symbols, cf. [**6**, Section I.3]). The set $\langle x, y \rangle$ defined above is the unique witness to that formula, provably in **PRS**ω. This connection with established set theories will be important in Section 1.4 (for example, concerning axiom beta we will want to know that "relation" means the same in **PRS**ω and in $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$). Internally, the role of $\langle x, y\rangle$ in **PRS**ω-proofs reveals its status as an ordered pair. Namely, Corollary 1.1.10 allows us to define the p.r. functions

$$\pi_1(p) = \bigcup\{x \in \bigcup p \mid \exists_{y \in \bigcup p}\, p = \langle x, y\rangle\},$$
$$\pi_2(p) = \bigcup\{y \in \bigcup p \mid \exists_{x \in \bigcup p}\, p = \langle x, y\rangle\},$$

and **PRS**ω proves $\pi_i(\langle x_1, x_2\rangle) = x_i$. A given set p is an ordered pair if and only if we have

$$p = \langle \pi_1(p), \pi_2(p)\rangle.$$

As this is a Δ_0-formula of $\mathcal{L}_{\mathrm{pr}}$, Proposition 1.1.9 tells us that being an ordered pair is a primitive recursive property. To indicate the intended domains and ranges of p.r. functions we will use expressions like

$$\langle\cdot,\cdot\rangle : \mathbb{V}^2 \to \text{"ordered pairs"},$$
$$\pi_i : \text{"ordered pairs"} \to \mathbb{V}.$$

As usual $\mathbb{V}$ stands for the universe of sets. Using Proposition 1.1.9 we may write it as the primitive recursive class

$$\mathbb{V} = \{x \mid x = x\} = \{x \mid Z(x) \in F_{x=x}(x)\}.$$

In particular, $x \in \mathbb{V}$ will be an abbreviation for the atomic formula $Z(x) \in F_{x=x}(x)$ (this will be more relevant for p.r. classes different from $\mathbb{V}$). Of course, $\forall_{x\in\mathbb{V}}\cdot$ is then an unbounded quantifier. Now the first line above expresses that $\langle\cdot,\cdot\rangle$ is a binary function and that all its values are ordered pairs. The second line conveys that the intended domain of the function π_i is the class of ordered pairs. Officially, π_i is still defined on the entire universe of sets, just as any primitive recursive function. All we want to express is that we are only interested in the values of π_i on ordered pairs. We only use this notation for domains (and ranges) which are primitive recursive classes. In that case we can, if we wish, assign a given default value outside of the intended domain. For example, let F be a p.r. function such that $Z(x) \in F(x)$ holds if and only if x is an ordered pair. Then

$$x \mapsto C(\pi_i(x), Z(x), Z(x), F(x))$$

is a primitive recursive function which coincides with π_i on ordered pairs and assigns the default value $\emptyset$ elsewhere (incidentally π_i already does the same). Case distinctions with several (primitive recursive) cases are available by composition. In particular, this covers domains which are dependent products, such as

$$\prod_{x \in \mathbb{V}} F(x) = \{(x_0, x_1) \,|\, x_0 \in \mathbb{V} \wedge x_1 \in F(x_0)\}.$$

Using our new terminology, we show that product sets exist (cf. [**62**, 2.2(vii)]):

PROPOSITION 1.2.1. *There is a primitive recursive function* $\times : \mathbb{V}^2 \to \mathbb{V}$ *such that* $\mathbf{PRS}\omega$ *shows*

$$p \in x \times y \leftrightarrow \text{“}p \text{ is an ordered pair”} \wedge \pi_1(p) \in x \wedge \pi_2(p) \in y.$$

PROOF. By Proposition 1.1.6 the function

$$F(x, w) = \{\langle v, w\rangle \,|\, v \in x\}$$

is primitive recursive. Using Proposition 1.1.6 again we get the p.r. function

$$x \times y = \bigcup \{F(x, w) \,|\, w \in y\}.$$

The required property is straightforward to verify. □

It is interesting to compare the previous proof with the argument in [**6**, Section I.3]: As advertised before, Proposition 1.1.6 replaces the applications of Δ_0-collection. Next, a set r is a relation if we have

$$\forall_{p \in r} \text{“}p \text{ is an ordered pair”},$$

which shows that this is a primitive recursive property of r. Using Proposition 1.1.6, the domain, range and field of a relation can be computed by the p.r. functions

$$\begin{aligned} \mathrm{dom}(r) &= \{\pi_1(p) \,|\, p \in r\}, \\ \mathrm{rng}(r) &= \{\pi_2(p) \,|\, p \in r\}, \\ \mathrm{field}(r) &= \mathrm{dom}(r) \cup \mathrm{rng}(r). \end{aligned}$$

A set f is a function if we have

$$\text{“}f \text{ is a relation”} \wedge \forall_{p,p' \in f}(\pi_1(p) = \pi_1(p') \to p = p'),$$

which is again a primitive recursive property. We speak of a function $f : x \to y$ if we have $\mathrm{dom}(f) = x$ and $\mathrm{rng}(f) \subseteq y$. In case $\mathrm{rng}(f) = y$ we have a surjection. Being injective is also a p.r. property, as it is defined by the Δ_0-formula

$$\forall_{p,p' \in f}(\pi_2(p) = \pi_2(p') \to p = p').$$

Using Corollary 1.1.10 and Proposition 1.1.6 we get p.r. functions

$$(x, f) \mapsto f(x) := \bigcup\{\pi_2(p) \,|\, p \in f \wedge \pi_1(p) = x),$$
$$(x, f) \mapsto f \restriction x := \{p \in f \,|\, \pi_1(p) \in x\}.$$

In view of Proposition 1.1.6 the equation

$$\operatorname{rng}(f) = \{f(x) \,|\, x \in \operatorname{dom}(f)\}$$

is an atomic formula of $\mathcal{L}_{\text{pr}}$, and indeed a theorem of $\mathbf{PRS}\omega$. It is of course important to distinguish between a function in this sense and a primitive recursive function (symbol). The latter is a class-sized object. The former may be called a set-sized function, if its status is not clear from the context. In particular, the expression $f(x)$ above is misleading: It might be better to write $\operatorname{Eval}(f, x)$ to make clear that this involves the p.r. evaluation function, applied to the set-sized function f; yet we keep writing $f(x)$ for its intuitive appeal. As important as this distinction between class-sized and set-sized functions is, there is also a connection:

Proposition 1.2.2. *The restriction of a p.r. function to a set is a set-sized function. More precisely, for each primitive recursive function $(v, \vec{y}) \mapsto F(v, \vec{y})$ there is a primitive recursive function $(x, \vec{y}) \mapsto F(\cdot, \vec{y}) \restriction x$ such that* $\mathbf{PRS}\omega$ *proves*

$$F(\cdot, \vec{y}) \restriction x = \{\langle v, F(v, \vec{y})\rangle \,|\, v \in x\}.$$

In particular we have $(F(\cdot, \vec{y}) \restriction x)(v) = F(v, \vec{y})$ for all $v \in x$.

Proof. In view of Proposition 1.1.6 the given equation can be read as a p.r. definition of $(x, \vec{y}) \mapsto F(\cdot, \vec{y}) \restriction x$. □

In particular, we can construct set-sized functions by $\in$-recursion: Simply define the corresponding class-sized function by primitive recursion and restrict it to the intended set-sized domain. Also, the terminology of the proposition allows a concise statement of a familiar variant of the recursion principle (which will be superseded by Proposition 1.2.10):

Lemma 1.2.3. *For each $(n+2)$-ary p.r. function G there is an $(n+1)$-ary p.r. function F such that* $\mathbf{PRS}\omega$ *proves*

$$F(x, \vec{y}) = G(F(\cdot, \vec{y}) \restriction x, x, \vec{y}).$$

Proof. Similarly to [**44**, 1.3(7)], but arguably somewhat simpler, we define

$$G'(u, x, \vec{y}) := \{\langle x, G(u, x, \vec{y})\rangle\},$$

$$F(x,\vec{y}) := \pi_2(\bigcup R[G'](x,\vec{y})).$$

For an appropriate set u we have $R[G'](x,\vec{y}) = G'(u,x,\vec{y})$. Thus $\bigcup R[G'](x,\vec{y})$ is indeed an ordered pair and we see

$$R[G'](x,\vec{y}) = \{\langle x, F(x,\vec{y})\rangle\}.$$

We can deduce

$$\bigcup\{R[G'](v,\vec{y}) \mid v \in x\} = F(\cdot,\vec{y}) \restriction x$$

and then

$$R[G'](x,\vec{y}) = G'\left(\bigcup\{R[G'](v,\vec{y}) \mid v \in x\}, x, \vec{y}\right) = \{\langle x, G(F(\cdot,\vec{y}) \restriction x, x, \vec{y})\rangle\}.$$

By the definition of F we get $F(x,\vec{y}) = G(F(\cdot,\vec{y}) \restriction x, x, \vec{y})$, as desired. □

An important application of these considerations is the Mostowski collapse. In $\mathbf{PRS}\omega$ we cannot collapse arbitrary well-founded relations (this requires axiom beta, cf. Section 1.4). However, we can collapse (the restriction of the $\in$-relation to) a non-transitive set onto a transitive set. First, x is transitive precisely if it satisfies the Δ_0-formula

$$\mathrm{Trans}(x) :\equiv \forall_{y\in x}\forall_{z\in y} z \in x.$$

Recall the subtle connection between transitive closures and $\in$-induction (cf. the proof of Proposition 1.1.11). Once the latter is established we may return to the recursive definition

$$\mathrm{TC}(x) = x \cup \bigcup\{\mathrm{TC}(y) \mid y \in x\},$$

and use $\in$-induction to show that $\mathrm{TC}(x)$ is the smallest transitive set that contains x. Now, following [**6**, Section I.7] we define C to fulfill the recursion

$$C(x,y) = \mathrm{rng}((C(\cdot,y) \restriction x) \restriction y) = \{C(v,y) \mid v \in x \cap y\}.$$

We are most interested in the set-sized functions

$$c_y := C(\cdot,y) \restriction y.$$

Let us recall their crucial properties (cf. [**6**, Section I.7]):

PROPOSITION 1.2.4. *The following is provable in* $\mathbf{PRS}\omega$*:*

(i) The function c_y *is a surjection from* y *onto the transitive set* $C(y,y)$*.*

(ii) If $x \subseteq y$ *is transitive then* $c_y \restriction x$ *is the identity on* x*.*

(iii) Assume that y is extensional, in the sense that we have

$$\forall_{x,x'\in y}(\forall_{z\in y}(z\in x\leftrightarrow z\in x')\to x=x').$$

Then $c_y : y\to C(y,y)$ is an $\in$-isomorphism, i.e. a bijection with

$$\forall_{x,x'\in y}(x\in x'\leftrightarrow c_y(x)\in c_y(x')).$$

PROOF. (i) By definition we have $\mathrm{rng}(c_y)=\{C(v,y)\,|\,v\in y\}=C(y,y)$. To see that $C(y,y)$ is transitive, consider $z\in x\in C(y,y)$. So we have $x=C(v,y)$ for some $v\in y$, and then $z=C(w,y)$ for some $w\in v\cap y$. In particular, $w\in y$ means $z=C(w,y)\in C(y,y)$, as desired.
(ii) We prove the Δ_0-formula $z\in x\to c_y(z)=z$ by $\in$-induction on z. As x is transitive $z'\in z\in x$ implies $z'\in x\subseteq y$. So for $z\in x$ the induction hypothesis does indeed yield

$$c_y(z)=\{c_y(z')\,|\,z'\in z\cap y\}=\{z'\,|\,z'\in z\}=z.$$

(iii) We already know that c_y has range $C(y,y)$. It is clear that $x,x'\in y$ and $x\in x'$ imply $c_y(x)\in c_y(x')$. For the converse and the fact that c_y is injective we follow the proof in [**6**, Theorem I.7.4]: On a superficial level it may look like this proof uses $\in$-induction for a Π_1-formula, which is unavailable in **PRS**ω. Fortunately, it is easy to restrict the unbounded quantifier. Namely, it suffices to establish the Δ_0-formula

$$x\in y\to\forall_{x'\in y}((c_y(x)=c_y(x')\to x=x')\wedge(c_y(x)\in c_y(x')\to x\in x'))$$

by (main) induction on x. In the step we fix $x\in y$ and show

$$x'\in y\to(c_y(x)=c_y(x')\to x=x')\wedge \\ (c_y(x)\in c_y(x')\to x\in x')\wedge(c_y(x')\in c_y(x)\to x'\in x)$$

by (side) induction on x'. Concerning the side induction step, assume first that we have $c_y(x)=c_y(x')$. For any $z\in x\cap y$ we have $c_y(z)\in c_y(x)=c_y(x')$, so that the main induction hypothesis gives $z\in x'$. Similarly, by the side induction hypothesis $z\in x'\cap y$ implies $z\in x$. As y is extensional this yields $x=x'$, as desired. Next, assume $c_y(x)\in c_y(x')$. By definition we have $c_y(x)=c_y(z)$ for some $z\in x'\cap y$. The side induction hypothesis gives $x=z\in x'$. Similarly, the main induction hypothesis shows that $c_y(x')\in c_y(x)$ implies $x'\in x$. This completes the side and main induction step. □

Next, being an ordinal is a p.r. property, defined by the Δ_0-formula

$$\mathrm{Ord}(x) :\equiv \mathrm{Trans}(x) \wedge \forall_{y \in x} \mathrm{Trans}(y).$$

In keeping with our above discussion of $x \in \mathbb{V}$, the expression $x \in \mathrm{Ord}$ is an abbreviation for the atomic $\mathcal{L}_{\mathrm{pr}}$-formula $Z(x) \in F_{\mathrm{Ord}}(x)$, equivalent to $\mathrm{Ord}(x)$ by Proposition 1.1.9. As usual, we use lower case greek letters for ordinals, often leaving subformulas $\alpha \in \mathrm{Ord}$ implicit. Also, we use $\alpha < \beta$ synonymous with $\alpha \in \beta$; by $\alpha \leq \beta$ we abbreviate $\alpha < \beta \vee \alpha = \beta$. As $x \in \mathrm{Ord}$ is atomic and $<$ coincides with $\in$, Proposition 1.1.11 allows us to establish primitive recursive properties of ordinals by induction. In Section 1.4 it will be important that "ordinal" means the same in $\mathbf{PRS}\omega$ and $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$. This is ensured by the following fundamental fact:

PROPOSITION 1.2.5. *The class of ordinals is well-ordered by $<$, provably in the theory $\mathbf{PRS}\omega$. Thus a set is an ordinal precisely if it is transitive and well-ordered by $\in$ (i.e. a von Neumann ordinal).*

PROOF. As in the proof of Proposition 1.1.4 we have $\alpha \notin \alpha$, i.e. the relation $<$ is irreflexive. That ordinals are transitive sets does indeed make $<$ transitive. To see that $<$ is total we try to compare two given ordinals α, β: In view of $\beta \in M(\beta, \beta)$ it suffices to show

$$\forall_{\gamma \in M(\beta,\beta)}(\alpha < \gamma \vee \alpha = \gamma \vee \gamma < \alpha).$$

As this is a Δ_0-formula we may argue by (main) $\in$-induction on α. For the induction step we establish

$$\gamma \in M(\beta, \beta) \rightarrow \alpha < \gamma \vee \alpha = \gamma \vee \gamma < \alpha$$

by (side) induction on γ. If some $\gamma' \in \gamma \cap M(\beta, \beta)$ satisfies $\alpha < \gamma'$ or $\alpha = \gamma'$ then we get $\alpha < \gamma$. Otherwise the side induction hypothesis yields $\gamma' < \alpha$ for all $\gamma' \in \gamma$ with $\gamma' \in \mathrm{Ord} \cap M(\beta, \beta)$. The last condition is redundant, as elements of ordinals are ordinals and $M(\beta, \beta)$ is transitive (indeed an ordinal). Thus we have $\gamma \subseteq \alpha$. In case $\gamma = \alpha$ we are done. Otherwise there is an (ordinal) $\alpha' \in \alpha$ with $\alpha' \notin \gamma$. By the (main) induction hypothesis we have $\alpha' = \gamma$ or $\gamma < \alpha'$. Both alternatives imply $\gamma < \alpha$, which completes the side and main induction step. As the relation $<$ is a restriction of $\in$ it is well-founded. $\square$

It is easy to check that the successor operation

$$\mathrm{Succ}(\alpha) := M(\alpha, \alpha)$$

maps ordinals to ordinals. We call α a successor ordinal if $\alpha = M(\beta, \beta)$ holds for some β. In view of $\beta \in M(\beta, \beta)$ this property is primitive recursive. We clearly have $\alpha < \mathrm{Succ}(\alpha)$, and also

$$\alpha < \beta \rightarrow \mathrm{Succ}(\alpha) \leq \beta,$$

as $\beta \in \mathrm{Succ}(\alpha)$ would imply $\beta < \alpha$ or $\beta = \alpha$. A limit ordinal is one that is neither a successor nor zero (the empty set). Limits are closed under the successor operation, by the implication that we have just established. If x is a set of ordinals we write

$$\sup x := \bigcup x.$$

It is straightforward to see that $x \subseteq \mathrm{Ord}$ (i.e. $\forall_{y \in x} y \in \mathrm{Ord}$) implies $\sup x \in \mathrm{Ord}$. We have $\alpha \leq \sup x$ for any $\alpha \in x$, and $\beta < \sup x$ implies $\beta < \alpha$ for some $\alpha \in x$. In view of $\sup \mathrm{Succ}(\alpha) = \bigcup \mathrm{Succ}(\alpha) = \alpha$ the successor relation is injective; and $\sup \lambda = \lambda$ holds for any limit ordinal λ. To define primitive recursive functions on the ordinals one often uses the following recursion schema (cf. [**62**, 2.2(x)]):

Proposition 1.2.6. *Given an $(n+1)$-ary p.r. function I and $(n+2)$-ary p.r. functions S, L there is an $(n+1)$-ary p.r. function F such that* $\mathbf{PRS}\omega$ *proves*

$$\begin{aligned} F(0, \vec{y}) &= I(0, \vec{y}), \\ F(\mathrm{Succ}(\alpha), \vec{y}) &= S(F(\alpha, \vec{y}), \alpha, \vec{y}), \\ F(\lambda, \vec{y}) &= L(F(\cdot, \vec{y}) \restriction \lambda, \lambda, \vec{y}) \quad \textit{if } \lambda \textit{ is a limit ordinal.} \end{aligned}$$

The argument 0 in $I(0, \vec{y})$ is simply a dummy to increase the arity (recall that we have no nullary functions).

Proof. As we have seen above, we can define a p.r. function by distinguishing several p.r. cases. Thus the recursion schema from Lemma 1.2.3 allows us to construct a primitive recursive function F with

$$F(x, \vec{y}) = \begin{cases} I(x, \vec{y}) & \text{if } x = 0, \\ S\left((F(\cdot, \vec{y}) \restriction x)(\bigcup x), \bigcup x, \vec{y}\right) & \text{if } x \text{ is a successor ordinal,} \\ L(F(\cdot, \vec{y}) \restriction x, x, \vec{y}) & \text{if } x \text{ is a limit ordinal,} \\ 0 & \text{if } x \text{ is not an ordinal.} \end{cases}$$

To see that this yields the desired result, recall that we have $\bigcup \mathrm{Succ}(\alpha) = \alpha$. In view of Proposition 1.2.2 we get $(F(\cdot, \vec{y}) \restriction \mathrm{Succ}(\alpha))(\bigcup \mathrm{Succ}(\alpha)) = F(\alpha, \vec{y})$. □

As an example, we can define a function $+ : \mathrm{Ord}^2 \to \mathrm{Ord}$ by setting

$$\begin{aligned} \alpha + 0 &= \alpha, \\ \alpha + \mathrm{Succ}(\beta) &= \mathrm{Succ}(\alpha + \beta), \\ \alpha + \lambda &= \sup\{\alpha + \gamma \mid \gamma < \lambda\} \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

Basic properties of ordinal addition (such as monotonicity; see the list in [**6**, Exercise I.6.9] for more examples) are provable by $\in$-induction over primitive recursive properties. Ordinal multiplication and exponentiation are defined analogously. Let us come to a very important object: the first infinite ordinal. A set x is called a natural number if it has the p.r. property

"x is an ordinal and every non-zero element of $\mathrm{Succ}(x)$ is a successor".

As usual we use the letters n, m, k for natural numbers. A priori the natural numbers form a class, just like the ordinals. However, in the presence of the axiom of infinity their status turns out quite different:

Proposition 1.2.7. *Provably in* $\mathbf{PRS}\omega$, *there is a unique set* ω *such that we have*

$$x \in \omega \leftrightarrow \text{“}x \text{ is a natural number”}.$$

This set ω *is the smallest limit ordinal.*

Proof. Uniqueness is clear by extensionality. As for existence, let u be a witness to the axiom of infinity, i.e. we have $0 \in u$ and $x \in u \to M(x, x) \in u$. Set

$$\omega := \{x \in u \mid \text{“}x \text{ is a natural number”}\}.$$

It remains to check that ω contains all natural numbers. This follows by induction over ordinals, observing that α is a natural number if $\mathrm{Succ}(\alpha) = M(\alpha, \alpha)$ is. As ω consists of ordinals it is itself an ordinal as soon as it is transitive: Given $\alpha \in n \in \omega$ we have $\mathrm{Succ}(\alpha) \leq n < \mathrm{Succ}(n)$. So every element of $\mathrm{Succ}(\alpha)$ lies in $\mathrm{Succ}(n)$, and α is a natural number in ω. To see that ω is a limit, observe that 0 is a natural number and that the class of natural numbers is closed under successors. $\square$

An important application of ω is in the context of finite sequences: A set x is called a finite sequence if and only if it satisfies the p.r. condition

"x is a function" $\wedge$ "$\mathrm{dom}(x)$ is a natural number".

We write $\mathbb{V}^{<\omega}$ for the class of finite sequences and use the letters σ, τ for elements of this class. Clearly, basic operations on finite sequences, such as

$$\begin{aligned} &\mathbb{V}^{<\omega} \to \omega, && \sigma \mapsto \operatorname{len}(\sigma) := \operatorname{dom}(\sigma), \\ &\mathbb{V}^{<\omega} \times \mathbb{V} \to \mathbb{V}^{<\omega}, && (\sigma, x) \mapsto \sigma^\frown x := \sigma \cup \{\langle \operatorname{len}(\sigma), x\rangle\}, \end{aligned}$$

are primitive recursive. On an informal level, $\langle x_0, \ldots, x_{n-1}\rangle$ denotes the sequence σ with $\operatorname{len}(\sigma) = n$ and $\sigma(i) = x_i$. In particular, $\langle\rangle = \emptyset$ is the unique sequence with empty domain. If we have $\operatorname{rng}(\sigma) \subseteq X$ for a set or p.r. class X then we say that σ is a sequence with entries in X, written as $\sigma \in X^{<\omega}$. The ordinal ω ensures that sets of finite sequences exist:

Proposition 1.2.8. *There is a primitive recursive function $G : \mathbb{V}^2 \to \mathbb{V}$ such that* **PRS**ω *proves the following: If ω is the set of natural numbers then we have*

$$y \in G(\omega, x) \leftrightarrow \text{“}y \textit{ is a finite sequence with entries in } x\text{”}.$$

We will write $x^{<\omega}$ at the place of $G(\omega, x)$ and speak of $x \mapsto x^{<\omega}$ as a unary primitive recursive function with parameter ω (cf. Convention 1.2.9 below).

Proof. We use the recursion schema from Proposition 1.2.6 to define

$$\begin{aligned} G(0, x) &:= \{\langle\rangle\}, \\ G(n+1, x) &:= \{\sigma^\frown y \mid \sigma \in G(n, x) \wedge y \in x\}, \\ G(\omega, x) &:= \bigcup\{G(n, x) \mid n \in \omega\}. \end{aligned}$$

The successor clause relies on two applications of Proposition 1.1.6, similar to the construction of the Cartesian product in Proposition 1.2.1. Given a finite sequence σ with entries in x one establishes the p.r. property

$$n \leq \operatorname{len}(\sigma) \to \sigma \restriction n \in G(n, x)$$

by induction on n. In particular $\sigma = \sigma \restriction \operatorname{len}(\sigma)$ lies in $G(\operatorname{len}(\sigma), x) \subseteq G(\omega, x)$. Conversely, a straightforward induction on n shows that any element of $G(n, x)$ is a finite sequence with entries in x. □

The formulation of the previous proposition was somewhat awkward because we do not have a constant symbol ω. Luckily, constants (as opposed to function symbols) behave very much like variables:

CONVENTION 1.2.9. Once and for all we reserve a variable which we denote by ω. All proofs will be under the assumption

$$x \in \omega \leftrightarrow \text{“}x \text{ is a natural number”}.$$

Proposition 1.2.7 implies that this global assumption is harmless: In particular it can be removed from any **PRS**ω-proof of a statement in which the free variable ω does not occur. Also, “primitive recursive” will from now on mean “primitive recursive in the parameter ω”. More precisely, when we say that F is an n-ary p.r. function (in ω) we officially refer to an $(n+1)$-ary p.r. function (in the strict sense) with one argument-position reserved for ω. Just like all results, properties of F are stated under the global assumption. The situation is exemplified by the previous proposition.

So far we have only seen recursion over the $\in$-relation. To define functions by recursion over finite sequences we will use the following flexible principle:

PROPOSITION 1.2.10. *Assume that we have p.r. functions $P : \mathbb{V}^{n+1} \to \mathbb{V}$ (“predecessors”) and $R : \mathbb{V}^{n+1} \to \mathrm{Ord}$ (“rank”) which satisfy*

$$v \in P(x, \vec{y}) \to R(v, \vec{y}) < R(x, \vec{y}),$$

provably in **PRS**ω. *For each p.r. function $G : \mathbb{V}^{n+2} \to \mathbb{V}$ we then have a p.r. function $F : \mathbb{V}^{n+1} \to \mathbb{V}$ (with parameter ω) such that* **PRS**ω *proves*

$$F(x, \vec{y}) = G(F(\cdot, \vec{y}) \restriction P(x, \vec{y}), x, \vec{y}).$$

PROOF. First, we need a notion of transitive closure with respect to P. By recursion over the (finite) ordinals we can set

$$\begin{aligned} \mathrm{TC}^P_0(x, \vec{y}) &= P(x, \vec{y}), \\ \mathrm{TC}^P_{n+1}(x, \vec{y}) &= \bigcup\{P(v, \vec{y}) \mid v \in \mathrm{TC}^P_n(x, \vec{y})\}, \\ \mathrm{TC}^P_\omega(x, \vec{y}) &= \bigcup\{\mathrm{TC}^P_n(x, \vec{y}) \mid n \in \omega\}. \end{aligned}$$

So we have $P(x, \vec{y}) \subseteq \mathrm{TC}^P_\omega(x, \vec{y})$, and $v \in \mathrm{TC}^P_\omega(x, \vec{y})$ yields $\mathrm{TC}^P_\omega(v, \vec{y}) \subseteq \mathrm{TC}^P_\omega(x, \vec{y})$. Now, by the recursion schema from Lemma 1.2.3 we can stipulate

$$H(\alpha, x, \vec{y}) = \{\langle v, G(H(R(v, \vec{y}), x, \vec{y}) \restriction P(v, \vec{y}), v, \vec{y})\rangle \mid v \in \mathrm{TC}^P_\omega(x, \vec{y}) \wedge R(v, \vec{y}) < \alpha\}.$$

More precisely, $H(R(v, \vec{y}), x, \vec{y})$ refers to the evaluation of the set-sized function $H(\cdot, x, \vec{y}) \restriction \alpha$ at the argument $R(v, \vec{y}) < \alpha$. Having defined H we put

$$F(x, \vec{y}) = G(H(R(x, \vec{y}), x, \vec{y}) \restriction P(x, \vec{y}), x, \vec{y}).$$

To see that F is as desired we must check

$$H(R(x,\vec{y}),x,\vec{y})\restriction P(x,\vec{y}) = F(\cdot,\vec{y})\restriction P(x,\vec{y}).$$

By the assumption that predecessors have lower rank we see that $H(R(x,\vec{y}),x,\vec{y})$ is a (set-sized) function with domain

$$\{v\in \mathrm{TC}^P_\omega(x,\vec{y})\,|\,R(v,\vec{y})<R(x,\vec{y})\}\supseteq P(x,\vec{y}).$$

Also, applying the function $H(R(x,\vec{y}),x,\vec{y})$ to the argument $v\in P(x,\vec{y})$ we get the value $G(H(R(v,\vec{y}),x,\vec{y})\restriction P(v,\vec{y}),v,\vec{y})$. Thus it remains to establish

$$G(H(R(v,\vec{y}),x,\vec{y})\restriction P(v,\vec{y}),v,\vec{y}) = F(v,\vec{y})$$

for arbitrary $v\in P(x,\vec{y})$. This is almost the definition of F, except that the expression $H(R(v,\vec{y}),v,\vec{y})\restriction P(v,\vec{y})$ is replaced by $H(R(v,\vec{y}),x,\vec{y})\restriction P(v,\vec{y})$. To conclude one verifies

$$H(\alpha,v,\vec{y}) = H(\alpha,x,\vec{y})\restriction \mathrm{TC}^P_\omega(v,\vec{y})$$

by a straightforward induction on α. □

It is easy to deduce the usual recursion principle for sequences. Note that the parameter ω is superfluous in this application (and in many others), as explained in the following proof:

Corollary 1.2.11. *Given p.r. functions G,H of appropriate arity there is a p.r. function $F:\mathbb{V}^{<\omega}\times\mathbb{V}^n\to\mathbb{V}$ such that* **PRS**ω *proves*

$$F(\langle\rangle,\vec{y}) = G(\vec{y}),$$
$$F(\sigma^\frown x,\vec{y}) = H(F(\sigma,\vec{y}),\sigma,x,\vec{y}).$$

Proof. Apply the previous proposition with $P(\sigma^\frown x,\vec{y})=\{\sigma\}$ and $P(z,\vec{y})=\emptyset$ for arguments z of a different form. The corresponding rank is $R(\sigma,\vec{y})=\mathrm{len}(\sigma)$. We point out that the parameter ω is not actually needed for this instance of the proposition: The relevant closure $\mathrm{TC}^P_\omega(\sigma,y)=\{\sigma\restriction n\,|\,n<\mathrm{len}(\sigma)\}$ can be constructed without it. □

Recursion over the length is also available, as long as we are only interested in a set-sized collection y of sequences: Simply apply Proposition 1.2.10 with

$$P(\sigma,y)=\{\tau\in y\,|\,\mathrm{len}(\tau)<\mathrm{len}(\sigma)\},$$

and again $R(\sigma, y) = \operatorname{len}(\sigma)$. In Proposition 1.3.3 we will use a similar principle to define the satisfaction of formulas in a model. As another application of Proposition 1.2.10 one can recover the familiar principle of recursion over transitive closures (see [**6**, Theorem I.6.4]): Take the predecessors $P(x) = \operatorname{TC}(x)$ and the primitive recursive rank function

$$\operatorname{rk} : \mathbb{V} \to \operatorname{Ord}, \qquad \operatorname{rk}(x) = \sup\{\operatorname{Succ}(\operatorname{rk}(y)) \,|\, y \in x\},$$

known as von Neumann rank. Induction over transitive closures (see [**6**, Theorem I.6.3]) is also available.

To conclude this section we discuss the notions of being finite resp. countable: As usual, a set x is called finite if and only if there is a bijection $f : n \to x$, for some natural number n. Any such function f is an element of $x^{<\omega}$, the set of finite sequences with entries in x. Conversely, a given sequence can (recursively) be transformed into an injective sequence, i.e. one that lists each entry exactly once. Setting

$$[x]^{<\omega} = \{\operatorname{rng}(\sigma) \,|\, \sigma \in x^{<\omega}\}$$

we thus have

$$y \in [x]^{<\omega} \leftrightarrow \text{“}y \text{ is a finite subset of } x\text{”},$$

provably in $\mathbf{PRS}\omega$. In particular, x is finite if and only if we have $x \in [x]^{<\omega}$, which makes this notion primitive recursive (in ω). Basic facts are easily verified: For example, the finite ordinals are precisely the natural numbers. Following [**87**, Definition VII.3.6] we call a set x countable if and only if there is a (set-sized) injection $f : x \to \omega$. Note that countability is not a primitive recursive property. The following observation will be needed later:

Proposition 1.2.12. *Provably in* $\mathbf{PRS}\omega$*, a non-empty set* x *is countable if and only if there exists a surjection* $g : \omega \to x$*.*

Proof. Assuming that x is non-empty and countable, fix an element $x_0 \in x$ and an injection $f : x \to \omega$. Define

$$g = \{\langle n, y\rangle \in \omega \times x \,|\, n = f(y) \vee (n \notin \operatorname{rng}(f) \wedge y = x_0)\}.$$

A straightforward verification shows that g is a surjective function with domain ω and range x. Conversely, if $g : \omega \to x$ is a surjection then

$$f = \{\langle y, n\rangle \in x \times \omega \,|\, y = g(n) \wedge \forall_{m<n} y \neq g(m)\}$$

is an injective function from x to ω. □

1.3. Around the Constructible Hierarchy

In this section we develop basic syntactic and semantic concepts in the base theory **PRS**ω. Our exposition culminates in the constructible hierarchy, which will play an important role in the following chapters. The material is of course well-known, even though the author is not aware of a presentation in **PRS**ω. Let us point out that we approach the constructible hierarchy via iterated definability rather than the Gödel operations. This is possible because the ordinal ω is at our disposal, as discussed in [**6**, Apologia I.5.1].

We begin with the notion of formula: As object language we consider the language $\mathcal{L}_\in$ of pure set theory, with relation symbols $\in, =$ and no function symbols. In contrast, recall that the language $\mathcal{L}_{\mathrm{pr}}$ of our base theory **PRS**ω contains a symbol for each primitive recursive function. That being said, we do consider object formulas with parameters (constant symbols); alternatively, one could work with parameter-free formulas and variable assignments. To implement formulas (and other syntactical objects) we use finite sequences, as discussed in the previous section. For example, the formula $x_i \in x_j$ could be represented by the sequence $\langle 0, i, j\rangle$. How exactly this is done will not matter, as long as we can construct and deconstruct formulas by primitive recursive functions. With this in mind, let us give our official definition of (object) formulas:

Definition 1.3.1. We define a p.r. function $\mathrm{For}: \omega \times \mathbb{V} \to \mathbb{V}$ by recursion over the natural numbers (see Proposition 1.2.6):

(i) The set $\mathrm{For}(0, M)$ consists of all expressions $s \in t, s \notin t, s = t, s \neq t$ (called prime formulas or atomic formulas) where s, t are variables or parameters (constant symbols) from the set M.

(ii) The set $\mathrm{For}(n+1, M)$ consists of all expressions of the form $\varphi, \varphi \land \psi$, $\varphi \lor \psi, \forall_{x \in s}\varphi, \exists_{x\in s}\varphi, \forall_x \varphi, \exists_x\varphi$, where φ, ψ lie in $\mathrm{For}(n, M)$ and s is a variable other than x or a parameter in M.

By a formula with parameters in M (short: M-formula) we mean an element of

$$\mathrm{For}(M) := \bigcup_{n\in\omega} \mathrm{For}(n, M).$$

In view of Convention 1.2.9 we speak of $M \mapsto \mathrm{For}(M)$ as a p.r. function (in ω).

Let us explain two points that may not be standard. First, when considering the object language, we view bounded quantifiers as logical symbols in their own right. This means that the formulas $\forall_{y\in x} y \neq y$ and $\forall_y (y \notin x \lor y \neq y)$ are different:

The former contains a bounded quantifier while the latter contains (a bounded occurrence of) an unbounded quantifier. This distinction will be important in some proof-theoretic arguments. At the same time, Lemma 1.3.4 will allow us to neglect the difference in semantic considerations. The second point is also connected with proof-theoretic requirements: We only consider formulas in negation normal form, i.e. negations may only occur in front of prime formulas. Negating an arbitrary formula then becomes a defined operation. As a blueprint for recursion over (the height of) formulas, let us present this operation in detail:

LEMMA 1.3.2. *There is a p.r. function*

$$\mathrm{Neg}:\prod_{M\in\mathbb{V}}\text{“}M\text{-formulas”}\to\mathbb{V}$$

such that **PRS**ω *proves the following: If* φ *is an* M*-formula then so is* $\mathrm{Neg}(M,\varphi)$*, and we have (writing* $\equiv$ *for equality of formulas)*

$$\mathrm{Neg}(M,s\in t)\equiv s\notin t,\qquad \mathrm{Neg}(M,s\notin t)\equiv s\in t,$$

$$\mathrm{Neg}(M,s=t)\equiv s\neq t,\qquad \mathrm{Neg}(M,s\neq t)\equiv s=t,$$

$$\mathrm{Neg}(M,\varphi\wedge\psi)\equiv\mathrm{Neg}(M,\varphi)\vee\mathrm{Neg}(M,\psi),$$

$$\mathrm{Neg}(M,\varphi\vee\psi)\equiv\mathrm{Neg}(M,\varphi)\wedge\mathrm{Neg}(M,\psi),$$

$$\mathrm{Neg}(M,\forall_{x\in s}\varphi)\equiv\exists_{x\in s}\,\mathrm{Neg}(M,\varphi),\quad \mathrm{Neg}(M,\exists_{x\in s}\varphi)\equiv\forall_{x\in s}\,\mathrm{Neg}(M,\varphi),$$

$$\mathrm{Neg}(M,\forall_x\varphi)\equiv\exists_x\,\mathrm{Neg}(M,\varphi),\qquad \mathrm{Neg}(M,\exists_x\varphi)\equiv\forall_x\,\mathrm{Neg}(M,\varphi).$$

We will usually write $\neg\varphi$ *for* $\mathrm{Neg}(M,\varphi)$*. This is justified because* $\varphi\in\mathrm{For}(M)$ *and* $M\subseteq M'$ *yield* $\varphi\in\mathrm{For}(M')$ *and* $\mathrm{Neg}(M,\varphi)\equiv\mathrm{Neg}(M',\varphi)$*.*

Before we give a proof, let us remark that the statement of the lemma already involves p.r. functions such as $(\varphi,\psi)\mapsto\varphi\wedge\psi$. These rely on basic operations on sequences. To deduce $\varphi\wedge\psi\in\mathrm{For}(M)$ from $\varphi\in\mathrm{For}(m,M)$ and $\psi\in\mathrm{For}(n,M)$ we must show that $n\geq m$ (similarly for $m\geq n$) implies $\varphi\in\mathrm{For}(n,M)$. This is established by an easy induction on n, justified by Proposition 1.1.11.

PROOF. We want to read the equations in the lemma as the clauses of a recursive definition in the sense of Proposition 1.2.10. To do so we define the height of a formula $\varphi\in\mathrm{For}(M)$ as the number

$$\min\{n\in\omega\,|\,\varphi\in\mathrm{For}(n,M)\}=\bigcup\{n\in\omega\,|\,\varphi\in\mathrm{For}(n,M)\wedge\forall_{m<n}\varphi\notin\mathrm{For}(m,M)\}.$$

In view of Corollary 1.1.10, the right hand side of the equation shows that the height of a formula can be computed by a p.r. function. Note that the height of φ resp. ψ is smaller than the height of $\varphi \wedge \psi$, and similarly for the other connectives and quantifiers. So we can indeed apply Proposition 1.2.10, with

$$P(\varphi, M) := \{\psi \in \mathrm{For}(M) \mid \text{“}\psi \text{ has smaller height than } \varphi\text{”}\},$$
$$R(\varphi, M) := \text{“the height of the } M\text{-formula } \varphi\text{”}.$$

The remaining claims follow by straightforward inductions. □

The reader should observe that $\neg\neg\varphi$ is the very same formula as φ. Building on negation, it is easy to conceive implication as a p.r. family (with parameter M) of operations on M-formulas, namely

$$\mathrm{For}(M) \ni \varphi, \psi \quad \mapsto \quad (\varphi \to \psi) :\equiv \mathrm{Neg}(M, \varphi) \vee \psi.$$

Other syntactic operations can also be constructed by recursion over (the height of) M-formulas. As in the proof of the lemma, this recursion principle is justified by Proposition 1.2.10. In particular, there is a p.r. family of functions which computes the free variables of an M-formula; we will call a formula closed if it has no free variables. Also, there is a p.r. family of functions $\mathrm{Subst}(M, \cdot)$ which take an M-formula φ, a finite list $\langle x_1, \dots, x_n\rangle$ of variables and a list $\langle a_1, \dots, a_n\rangle \in M^{<\omega}$ of parameters, and return the result of substituting a_i for x_i in φ. Usually we use the notation $\varphi \equiv \varphi(x_1, \dots, x_n)$ to convey both the formula and the variable list. We then write

$$\varphi(a_1, \dots, a_n) \equiv \mathrm{Subst}(M, \varphi, \langle x_1, \dots, x_n\rangle, \langle a_1, \dots, a_n\rangle)$$

for the result of substitution. It is important to keep in mind that $\varphi(a_1, \dots, a_n)$ is an $\mathcal{L}_{\mathrm{pr}}$-term which involves the primitive recursive function symbol Subst, applied to the arguments $M, \varphi, \langle x_1, \dots, x_n\rangle$ and $\langle a_1, \dots, a_n\rangle$. To distinguish the object formula $\varphi(x_1, \dots, x_n)$ from the $\mathcal{L}_{\mathrm{pr}}$-term $\varphi(a_1, \dots, a_n)$ we reserve, for the rest of this section, the letters x, y, z resp. a, b, c for use in the object resp. meta language (alternatively one could use dot notation similar to that of Feferman [**15**] and write $\varphi(\dot{x}_1, \dots, \dot{x}_n)$ in place of $\varphi(a_1, \dots, a_n)$). For example,

$$\forall_{\varphi(x) \in \mathrm{For}(M)} \forall_{a \in M} \varphi(a) \in \mathrm{For}(M)$$

is a (bounded) $\mathcal{L}_{\mathrm{pr}}$-formula which expresses that the substitution of an arbitrary M-parameter into an M-formula results in an M-formula. Working in **PRS**ω, it can be established by induction on (the height of) φ. Recursion over the height of formulas also allows us to define a crucial semantic concept: satisfaction in a model. We will only be interested in standard models, i.e. models where $\in$ is interpreted as actual set membership.

PROPOSITION 1.3.3. *There is a p.r. relation*

$$\vDash \ \subseteq \prod_{M \in \mathbb{V}} \text{“closed } M\text{-formulas”}$$

such that **PRS**ω *proves (writing* $M \vDash \varphi$ *for* $(M, \varphi) \in \vDash$*): For all* M*-formulas* φ, ψ *(closed resp. with the indicated free variable) and all parameters* $a, b \in M$ *we have*

$$(M \vDash a \in b) \leftrightarrow a \in b, \qquad (M \vDash a \notin b) \leftrightarrow a \notin b,$$

$$(M \vDash a = b) \leftrightarrow a = b, \qquad (M \vDash a \neq b) \leftrightarrow a \neq b,$$

$$(M \vDash \varphi \wedge \psi) \leftrightarrow (M \vDash \varphi) \wedge (M \vDash \psi), \quad (M \vDash \varphi \vee \psi) \leftrightarrow (M \vDash \varphi) \vee (M \vDash \psi),$$

$$(M \vDash \forall_{x \in a} \varphi(x)) \leftrightarrow \forall_{c \in a \cap M} (M \vDash \varphi(c)),$$

$$(M \vDash \exists_{x \in a} \varphi(x)) \leftrightarrow \exists_{c \in a \cap M} (M \vDash \varphi(c)),$$

$$(M \vDash \forall_x \varphi(x)) \leftrightarrow \forall_{c \in M} (M \vDash \varphi(c)), \qquad (M \vDash \exists_x \varphi(x)) \leftrightarrow \exists_{c \in M} (M \vDash \varphi(c)).$$

We will refer to these equivalences as Tarski's conditions.

PROOF. Induction on n shows that $\varphi \in \mathrm{For}(n, M)$ implies $\varphi(c) \in \mathrm{For}(n, M)$. Thus the height of $\varphi(c)$ is (at most) the height of $\varphi(x)$, which is smaller than the height of $\forall_{x \in a} \varphi(x)$ or $\forall_x \varphi(x)$. Thus the defining clauses of $\vDash$ (or rather: of its characteristic function) fall under the recursion principle from Proposition 1.2.10, as in the proof of Lemma 1.3.2. □

Writing $\nvDash$ for the complement of $\vDash$, a straightforward induction on the height of φ shows

$$M \nvDash \varphi \quad \Leftrightarrow \quad M \vDash \neg \varphi.$$

As promised, the distinction between bounded quantifiers (as logical symbols in their own right) and bounded occurrences of usual quantifiers is harmless from the semantic viewpoint:

LEMMA 1.3.4. *Working in* **PRS**ω*, consider an* M*-formula* φ *and a set* $a \in M$*. Then we have*

$$(M \vDash \forall_{x \in a} \varphi(x)) \leftrightarrow (M \vDash \forall_x (x \in a \rightarrow \varphi(x))).$$

The same holds for existential quantifiers.

PROOF. Recall that implication is a defined operation, so that $x \in a \to \varphi(x)$ stands for $x \notin a \vee \varphi(x)$. By Tarski's conditions (and basic properties of substitution) the claim reduces to

$$\forall_{c \in a \cap M} M \vDash \varphi(c) \leftrightarrow \forall_{c \in M}(c \notin a \vee M \vDash \varphi(c)),$$

which is obvious. □

Next, we consider the relativization of a formula to a set. In the present section we only need this concept for formulas of the meta language: Given a formula φ and a (fresh) variable c we write φ^c for the result of replacing each occurrence of an unbounded quantifier $\forall_a \cdot$ resp. $\exists_a \cdot$ in φ by the bounded quantifier $\forall_{a \in c} \cdot$ resp. $\exists_{a \in c} \cdot$. Later we will need the same construction for the object language: Working in **PRS**ω, there is a p.r. function which, given an M-formula φ and a parameter $c \in M$, computes the relativized M-formula φ^c. An important justification for the definition of $\vDash$ is its behaviour on standard formulas (i.e. $\mathcal{L}_\in$-formulas of the meta language). Each standard formula φ corresponds to a $\emptyset$-formula $\ulcorner \varphi \urcorner$ of the object language, its Gödel code. In this context, $\varphi(a_1, \dots, a_n)$ denotes the meta formula φ with free variables $a_1, \dots, a_n$. On the other hand, $\ulcorner \varphi \urcorner(a_1, \dots, a_n)$ is the result of replacing the free (object) variables of $\ulcorner \varphi \urcorner$ by constant symbols for the values of the (meta) variables $a_1, \dots, a_n$. As above, $\ulcorner \varphi \urcorner(a_1, \dots, a_n)$ is thus an $\mathcal{L}_{\mathrm{pr}}$-term with free variables $a_1, \dots, a_n$; for any values of these variables it denotes a closed object formula. Sometimes we will write φ at the place of $\ulcorner \varphi \urcorner$, relying on the context for clarification.

PROPOSITION 1.3.5. *For each $\mathcal{L}_\in$-formula $\varphi(a_1, \dots, a_n)$ of the meta language (with all free variables displayed) there is a* **PRS**ω*-proof of*

$$\mathrm{Trans}(M) \to \forall_{a_1, \dots, a_n \in M}(\varphi(a_1, \dots, a_n)^M \leftrightarrow M \vDash \ulcorner \varphi \urcorner(a_1, \dots, a_n)).$$

PROOF. One argues by (meta) induction on φ, using Tarski's conditions. The most interesting cases are the bounded quantifiers, say $\varphi(a) \equiv \forall_{b \in a} \psi(a, b)$. Working in **PRS**$\omega$, assume that $M \ni a$ is transitive. The induction hypothesis yields

$$\forall_{b \in M}(\psi(a, b)^M \leftrightarrow M \vDash \ulcorner \psi \urcorner(a, b)).$$

By Tarski's conditions $M \vDash \ulcorner \varphi \urcorner(a)$ is equivalent to $\forall_{b \in a \cap M} M \vDash \ulcorner \psi \urcorner(a, b)$, yielding

$$\forall_{b \in a \cap M} \psi(a, b)^M \leftrightarrow M \vDash \ulcorner \varphi \urcorner(a).$$

As M is transitive the condition $b \in a \cap M$ reduces to $b \in a$. The claim follows in view of $\forall_{b\in a}\psi(a,b)^M \equiv \varphi(a)^M$. Let us also consider the case of an unbounded quantifier, say $\varphi(a) \equiv \exists_b \psi(a,b)$. Similarly to the above we get

$$\exists_{b\in M}\psi(a,b)^M \leftrightarrow M \vDash \ulcorner\varphi\urcorner(a).$$

The claim follows by $\varphi(a)^M \equiv \exists_{b\in M}\psi(a,b)^M$, with the quantifier restricted. □

If φ contains no unbounded quantifiers then φ^M is the same formula as φ, and thus $M \vDash \ulcorner\varphi\urcorner(a_1,\dots,a_n)$ does not depend on M. Let us extend this insight to non-standard formulas: If we have $M \subseteq M'$ then any M-formula is an M'-formula. By a $\Delta_0(M)$-formula (resp. $\Sigma(M)$-formula resp. $\Pi(M)$-formula) we shall mean an M-formula which contains no unbounded quantifiers (resp. no unbounded universal resp. existential quantifiers). These notions can be defined by recursion over the formula height, and are thus primitive recursive. We obtain the following result:

Lemma 1.3.6. *Working in* **PRS**ω, *assume that M is transitive in $M' \supseteq M$, i.e. that $a \in M$ implies $a \cap M' \subseteq M$. For a $\Sigma(M)$-formula φ resp. $\Pi(M)$-formula ψ we then have*

$$M \vDash \varphi \;\Rightarrow\; M' \vDash \varphi \qquad \textit{resp.} \qquad M' \vDash \psi \;\Rightarrow\; M \vDash \psi.$$

In particular, a $\Delta_0(M)$-formula is satisfied in M if and only if it is satisfied in M'.

Note that a transitive set M is transitive in any $M' \supseteq M$.

Proof. Proposition 1.1.11 allows us to argue by induction on the height of the formula φ resp. ψ (note that the induction statement is primitive recursive). For the induction step one uses Tarski's conditions. The most interesting case is that of a bounded quantifier, say $\varphi \equiv \forall_{x\in a}\psi(x)$. From $M \vDash \varphi$ we get $\forall_{c\in a\cap M} M \vDash \psi(c)$. The height of $\psi(c)$ is smaller than the height of φ, so that the induction hypothesis yields $\forall_{c\in a\cap M} M' \vDash \psi(c)$. As M is transitive in M' we get $a \cap M = a \cap M'$, and then $M' \vDash \varphi$. Let us also consider the case of an unbounded existential quantifier, say $\varphi \equiv \exists_x \psi(x)$. From $M \vDash \varphi$ we obtain $\exists_{a\in M} M \vDash \psi(a)$, and the induction hypothesis gives $\exists_{a\in M} M' \vDash \psi(a)$. In view of $M \subseteq M'$ we obtain $\exists_{a\in M'} M' \vDash \psi(a)$ and thus $M' \vDash \varphi$. □

With a reasonable satisfaction relation at our disposal, we can now approach the constructible hierarchy. The following step is crucial:

LEMMA 1.3.7. *There is a p.r. function*

$$\mathrm{Def} : \prod_{M \in \mathbb{V}} \text{“}M\text{-formulas with a single free variable”} \to \mathbb{V}$$

such that $\mathbf{PRS}\omega$ *proves* $\mathrm{Def}(M, \varphi) = \{a \in M \mid M \vDash \varphi(a)\}$ *for any* M*-formula* $\varphi(x)$*. Also, there is a p.r. function* $\mathrm{Def}_0 : \mathbb{V} \to \mathbb{V}$ *such that*

$$\mathrm{Def}_0(M) = \{\mathrm{Def}(M, \varphi) \mid \text{“}\varphi \text{ a } \Delta_0(M)\text{-formula with a single free variable”}\}$$

is provable in $\mathbf{PRS}\omega$*.*

PROOF. The function Def is defined by p.r. separation, and is thus primitive recursive by Corollary 1.1.10. As for Def_0, the set of $\varphi \in \Delta_0(M) \subseteq \mathrm{For}(M)$ with a single free variable can be computed by a p.r. function, again using Corollary 1.1.10. As $\mathrm{Def}_0(M)$ is the range of $\mathrm{Def}(M, \cdot)$ on that set, the function Def_0 is primitive recursive by Proposition 1.1.6. □

The reader will have noticed that $\mathrm{Def}_0(M)$ only collects the subsets that are defined by a bounded formula. This is not essential, but it will be convenient later. For the definition of the constructible hierarchy it does not seem to make a big difference: As $\mathbb{L}_{\alpha+1} \ni \mathbb{L}_\alpha$ is transitive $\mathbb{L}_\alpha \vDash \varphi$ is equivalent to $\mathbb{L}_{\alpha+1} \vDash \varphi^{\mathbb{L}_\alpha}$, and $\varphi^{\mathbb{L}_\alpha}$ is a $\Delta_0(\mathbb{L}_{\alpha+1})$-formula, whether φ itself is bounded or not. In any case, we will work with the following notion:

DEFINITION 1.3.8. We define a primitive recursive function

$$\mathrm{Ord} \times \text{“transitive sets”} \ni (\alpha, u) \mapsto \mathbb{L}_\alpha^u \in \mathbb{V}$$

by recursion along the ordinals (Proposition 1.2.6), setting

$$\begin{aligned} \mathbb{L}_0^u &:= u, \\ \mathbb{L}_{\alpha+1}^u &:= \mathrm{Def}_0(\mathbb{L}_\alpha^u), \\ \mathbb{L}_\lambda^u &:= \bigcup \{\mathbb{L}_\gamma^u \mid \gamma < \lambda\} \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

The function $\alpha \mapsto \mathbb{L}_\alpha^u$ is called the constructible hierarchy relative to u.

To avoid confusion we point out that the term “relativized constructible hierarchy” is also used for a different construction (considered by Hajnal, Lévy and Shoenfield, see in particular [**50**]): Rather than starting with u as a set of “urelements” one insists that all sets are to be constructed from the empty set, but allows u as a predicate that can be used to define more subsets. This variant of the constructible hierarchy will not play any role in the present thesis. Let us show some basic properties:

LEMMA 1.3.9. *The following holds for any transitive u, provably in $\mathbf{PRS}\omega$:*

(i) The sets $\mathbb{L}^u_\alpha$ are transitive for all ordinals α.
(ii) If $\alpha < \beta$ then $\mathbb{L}^u_\alpha \in \mathbb{L}^u_\beta$.

PROOF. (i) Arguing by induction on α, it clearly suffices to show that $\mathrm{Def}_0(M)$ is transitive if M is. So assume $b \in c \in \mathrm{Def}_0(M)$. Then we have $c = \mathrm{Def}(M, \varphi)$ for some M-formula φ, and in particular $b \in c \subseteq M$. If M is transitive we get

$$b = \{a \in M \mid M \vDash a \in b\} = \mathrm{Def}(M, x \in b) \in \mathrm{Def}_0(M),$$

as required.
(ii) Observe

$$\mathbb{L}^u_\alpha = \{a \in \mathbb{L}^u_\alpha \mid \mathbb{L}^u_\alpha \vDash a = a\} = \mathrm{Def}(\mathbb{L}^u_\alpha, x = x) \in \mathrm{Def}_0(\mathbb{L}^u_\alpha) = \mathbb{L}^u_{\alpha+1}.$$

To conclude one shows $\mathbb{L}^u_{\alpha+1} \subseteq \mathbb{L}^u_\beta$ by induction on $\beta \geq \alpha + 1$. The successor step uses $M \subseteq \mathrm{Def}_0(M)$ for transitive M, as established in the proof of (i). ☐

Next, we want to extend the familiar equation $\mathbb{L}^\emptyset_\alpha \cap \mathrm{Ord} = \alpha$ to the relativized constructible hierarchy. Write

$$o(u) := u \cap \mathrm{Ord} \in \mathrm{Ord}$$

for the height of a transitive set u. The function $o(\cdot)$ is primitive recursive by Corollary 1.1.10. We obtain the following result:

LEMMA 1.3.10. *Provably in $\mathbf{PRS}\omega$, we have*

$$o(\mathbb{L}^u_\alpha) = o(u) + \alpha$$

for any transitive set u and any ordinal α.

PROOF. We argue by induction on α: For $\alpha = 0$ the claim is immediate. Concerning the successor case, recall that the class of ordinals is defined by a bounded formula $\mathrm{Ord}(a)$ in the (meta) language $\mathcal{L}_\in$. Using the induction hypothesis and Proposition 1.3.5 we get

$$o(u) + \alpha = \{a \in \mathbb{L}^u_\alpha \mid \mathrm{Ord}(a)\} = \{a \in \mathbb{L}^u_\alpha \mid \mathbb{L}^u_\alpha \vDash \ulcorner \mathrm{Ord} \urcorner (a)\} =$$

$$= \mathrm{Def}(\mathbb{L}^u_\alpha, \ulcorner \mathrm{Ord} \urcorner) \in \mathrm{Def}_0(\mathbb{L}^u_\alpha) = \mathbb{L}^u_{\alpha+1}.$$

Together with the fact that $\mathbb{L}^u_{\alpha+1}$ is transitive we get $o(u)+\alpha+1 \subseteq \mathbb{L}^u_{\alpha+1} \cap \mathrm{Ord}$. The converse inclusion holds as $\beta \in \mathbb{L}^u_{\alpha+1} = \mathrm{Def}_0(\mathbb{L}^u_\alpha)$ implies $\beta \subseteq \mathbb{L}^u_\alpha \cap \mathrm{Ord} = o(u)+\alpha$. For a limit ordinal α the claim is immediate from the induction hypothesis. $\square$

One of the main goals of this thesis is the construction of admissible sets, i.e. transitive models of Kripke-Platek set theory. Let us give an official definition:

Definition 1.3.11. Kripke-Platek set theory (**KP**) is the $\mathcal{L}_\in$-theory whose non-logical axioms are extensionality, pairing and union, as well as all instances of the axiom scheme of Δ_0-separation

$$\forall_{\vec{z}}\forall_x\exists_y\forall_v(v \in y \leftrightarrow v \in x \wedge \theta(v,\vec{z})) \qquad (\theta \text{ a } \Delta_0\text{-formula of } \mathcal{L}_\in),$$

all instances of Δ_0-collection

$$\forall_{\vec{z}}\forall_v(\forall_{x\in v}\exists_y\theta(x,y,\vec{z}) \to \exists_w\forall_{x\in v}\exists_{y\in w}\theta(x,y,\vec{z})) \qquad (\theta \text{ a } \Delta_0\text{-formula of } \mathcal{L}_\in),$$

and all instances of $\in$-induction

$$\forall_{\vec{z}}(\forall_x(\forall_{y\in x}\varphi(y,\vec{z}) \to \varphi(x,\vec{z})) \to \forall_x\varphi(x,\vec{z})) \qquad (\varphi \text{ any formula of } \mathcal{L}_\in).$$

We will refer to "the set of **KP**-axioms" both in the meta theory and when we work in **PRS**ω; in the latter case one has instances of the axiom schemes for all object formulas. Using the definition inside **PRS**ω we get an $\mathcal{L}_{\mathrm{pr}}$-formula

$$\mathrm{Ad}(\mathbb{A}) :\equiv \mathbb{A} \neq \emptyset \wedge \mathrm{Trans}(\mathbb{A}) \wedge \forall_\varphi(\text{“}\varphi \text{ a } \mathbf{KP}\text{-axiom”} \to \mathbb{A} \vDash \varphi).$$

If $\mathrm{Ad}(\mathbb{A})$ holds, then we call $\mathbb{A}$ an admissible set.

We remark that some authors demand $\omega \in \mathbb{A}$. This makes no difference for our result, as we will construct admissible sets that contain a given set — which one may choose to contain ω. For a general introduction to Kripke-Platek set theory we refer to Barwise's book [**6**]. In particular, the principle of Σ-recursion (see [**6**, Theorem I.6.4]) reveals that **KP**, extended by the axiom of infinity, is an extension of **PRS**ω. Many proof-theoretic investigations (e.g. [**41**]) implement admissible sets in a different way, avoiding the use of a satisfaction relation: Namely, they extend the language by a new relation symbol $\mathrm{Ad}(\cdot)$ and add an axiom

$$\forall_{\mathbb{A}}(\mathrm{Ad}(\mathbb{A}) \to \varphi^{\mathbb{A}})$$

for each **KP**-axiom φ (as above, $\varphi^{\mathbb{A}}$ is the relativization of φ to $\mathbb{A}$). Note that this refers to **KP**-axioms in the sense of the meta theory. Nevertheless it leads to an equivalent notion of admissible set: In one direction, if φ is an actual **KP**-axiom then $\ulcorner\varphi\urcorner$ is a **KP**-axiom in the sense of **PRS**ω. So if $\mathbb{A}$ is admissible in the "internal" sense then we have $\mathbb{A} \vDash \ulcorner\varphi\urcorner$. By Proposition 1.3.5 this implies $\varphi^{\mathbb{A}}$, as desired. For the other direction we must first get $\in$-induction out of the way:

LEMMA 1.3.12. *Working in* **PRS**ω*, we have*

$$M \vDash \forall_x(\forall_{y\in x}\varphi(y) \to \varphi(x)) \to \forall_x\varphi(x)$$

for any set M and any M-formula $\varphi(x)$.

PROOF. By Tarski's conditions the claim reduces to

$$\forall_{a\in M}(\forall_{b\in a\cap M}M \vDash \varphi(b) \to M \vDash \varphi(a)) \to \forall_{a\in M}M \vDash \varphi(a).$$

This amounts to $\in$-induction over the p.r. property $a \in M \to M \vDash \varphi(a)$, which is available by Proposition 1.1.11. □

Now we can show that finitely many standard formulas suffice to make a set admissible in the sense of Definition 1.3.11:

PROPOSITION 1.3.13. *There are Kripke-Platek axioms $\varphi_1, \dots, \varphi_n$ (in the sense of the meta theory) such that* **PRS**ω *proves*

$$\mathbb{A} \neq \emptyset \wedge \mathrm{Trans}(\mathbb{A}) \wedge {\varphi_1}^{\mathbb{A}} \wedge \cdots \wedge {\varphi_n}^{\mathbb{A}} \to \mathrm{Ad}(\mathbb{A}).$$

PROOF. First, arguing in the meta theory, let us show that there are finitely many **KP**-axioms $\varphi_1, \dots, \varphi_n$ which imply all other **KP**-axioms, with the exception of $\in$-induction: By [**6**, Section V.I] there is a Δ_1-formula $\mathrm{True}_{\Delta_0}(\cdot)$ which defines truth for Δ_0-formulas, provably in **KP**. Rather than including separation and collection for all Δ_0-formulas we thus use the "universal" versions of these axiom schemes, namely

$$\forall_\theta\forall_c(\forall_{a\in c}\exists_b \,\mathrm{True}_{\Delta_0}(\theta(a,b)) \to \exists_d\forall_{a\in c}\exists_{b\in d}\,\mathrm{True}_{\Delta_0}(\theta(a,b)))$$

in the case of collection. As $\mathrm{True}_{\Delta_0}(\cdot)$ is a Δ_1-formula this is not a **KP**-axiom. Nevertheless, it is a theorem of **KP** by [**6**, Theorem I.4.4] (for separation one uses [**6**, Theorem I.4.5]). To obtain collection for "actual" Δ_0-formulas we need the equivalences

$$\theta(a_1, \dots, a_k) \leftrightarrow \mathrm{True}_{\Delta_0}(\ulcorner\theta\urcorner(a_1, \dots, a_k)),$$

as established in [**6**, Proposition V.1.6]. The point is that these can be proved in a finite core theory, independent of the (actual) formula θ. Contemplating the proof of Proposition 1.3.5 above, it is indeed enough to know the Tarski conditions and properties of substitution (both independent of θ), and to verify some basic connections (e.g. that $\ulcorner\exists_{b\in a_i}\psi\urcorner$ is an existential formula with matrix ψ). Now, aiming at the claim of the proposition, let us work in $\mathbf{PRS}\omega$. Assume that $\mathbb{A}\neq\emptyset$ is transitive and that ${\varphi_i}^{\mathbb{A}}$ holds for $i=1,\dots,n$. By Proposition 1.3.5 this implies

$$\mathbb{A}\vDash\ulcorner\varphi_1\urcorner\land\dots\land\ulcorner\varphi_n\urcorner.$$

We need to verify $\mathbb{A}\vDash\varphi$ for an arbitrary object formula φ that is a $\mathbf{KP}$-axiom in the sense of $\mathbf{PRS}\omega$. If φ is an instance of $\in$-induction then it suffices to invoke the previous lemma. Otherwise, formalizing the argument above, φ can be proved from the assumptions $\ulcorner\varphi_i\urcorner$. The Tarski conditions guarantee that the satisfaction relation is correct, so that we get $\mathbb{A}\vDash\varphi$ as desired. □

In particular the proposition allows us to place a bound on the number of parameters in collection formulas. This will be convenient later.

Convention 1.3.14. Let $\varphi_1,\dots,\varphi_n$ be the Kripke-Platek axioms from the previous proposition. We reserve C for a number which is large enough to make the following true for $i=1,\dots,n$: If

$$\varphi_i\equiv\forall_{z_1,\dots,z_k}\forall_v(\forall_{x\in v}\exists_y\theta(x,y,z_1,\dots,z_k)\rightarrow\exists_w\forall_{x\in v}\exists_{y\in w}\theta(x,y,z_1,\dots,z_k))$$

is a Δ_0-collection axiom with k parameters then we have $k\leq C$.

We conclude the section with a well-known result, which reveals Δ_0-collection as the crucial axiom of Kripke-Platek set theory:

Proposition 1.3.15. *Working in* $\mathbf{PRS}\omega$*, consider a transitive set* u *and a limit ordinal* λ*. If* $\mathbb{L}_\lambda^u$ *satisfies* Δ_0*-collection then it is an admissible set. Indeed it suffices to consider instances of* Δ_0*-collection with at most* C *parameters.*

Proof. As λ is a limit we have $\mathbb{L}_\lambda^u\neq\emptyset$. Also recall that $\mathbb{L}_\lambda^u$ is transitive by Lemma 1.3.9. Thus it suffices to establish $\mathbb{L}_\lambda^u\vDash\ulcorner\varphi_i\urcorner$ for the axioms $\varphi_1,\dots,\varphi_n$ from Proposition 1.3.13. The instances of Δ_0-collection are satisfied by assumption and the choice of C, so that it remains to consider the other axioms of Kripke-Platek set theory: Extensionality is satisfied in any transitive set, and $\in$-induction is covered by Lemma 1.3.12. As for pairing, consider arbitrary sets $a,b\in\mathbb{L}_\lambda^u$ and choose an ordinal $\alpha<\lambda$ with $a,b\in\mathbb{L}_\alpha^u$. Then

$$c=\{d\in\mathbb{L}_\alpha^u\,|\,d=a\lor d=b\}=\{d\in\mathbb{L}_\alpha^u\,|\,\mathbb{L}_\alpha^u\vDash d=a\lor d=b\}$$

is an element of $\mathbb{L}_{\alpha+1}^u\subseteq\mathbb{L}_\lambda^u$. This establishes

$$\forall_{a\in\mathbb{L}_\lambda^u}\forall_{b\in\mathbb{L}_\lambda^u}\exists_{c\in\mathbb{L}_\lambda^u}(a\in c\land b\in c\land\forall_{d\in c}(d=a\lor d=b)).$$

By Proposition 1.3.5 it follows that $\mathbb{L}^u_\lambda$ satisfies the pairing axiom. The union axiom is established similarly (using the fact that any $\mathbb{L}^u_\alpha$ is transitive). It remains to verify separation for a Δ_0-formula $\theta(x,\vec{z})$: Consider arbitrary parameters $a,\vec{c}\in\mathbb{L}^u_\lambda$, say $a,\vec{c}\in\mathbb{L}^u_\alpha$ with $\alpha<\lambda$. We have

$$b:=\{d\in\mathbb{L}^u_\alpha\,|\,\mathbb{L}^u_\alpha\vDash d\in a\wedge\theta(d,\vec{c})\}\in\mathbb{L}^u_{\alpha+1}\subseteq\mathbb{L}^u_\lambda.$$

Using Lemma 1.3.6 we get $d\in b\leftrightarrow\mathbb{L}^u_\lambda\vDash d\in a\wedge\theta(d,\vec{c})$ for any $d\in\mathbb{L}^u_\lambda$, and then

$$\forall_{a,\vec{c}\in\mathbb{L}^u_\lambda}\exists_{b\in\mathbb{L}^u_\lambda}\forall_{d\in\mathbb{L}^u_\lambda}\mathbb{L}^u_\lambda\vDash d\in b\leftrightarrow d\in a\wedge\theta(d,\vec{c}).$$

Tarski's conditions allow us to internalize the quantifiers, which yields

$$\mathbb{L}^u_\lambda\vDash\forall_{x,\vec{z}}\exists_y\forall_v(v\in y\leftrightarrow v\in x\wedge\theta(v,\vec{z})).$$

This is the separation axiom for θ. □

1.4. Connecting with Second-Order Arithmetic

The last three sections were devoted to our default base theory **PRS**ω. Some arguments, however, will require additional assumptions: axiom beta and the axiom of countability (cf. Definition 1.4.2). This leads us to Simpson's [**85, 87**] set theory $\mathbf{ATR_0^{set}}$, which we review in the present section. As $\mathbf{ATR_0^{set}}$ is bi-interpretable with arithmetical transfinite recursion (the subsystem of second-order arithmetic) it is still a suitable base theory for our analysis of Π^1_1-comprehension. Note that Simpson formulates $\mathbf{ATR_0^{set}}$ in the language $\mathcal{L}_\in$ of pure set theory, with relation symbols $\in,=$ and no function symbols. A main (if standard) result of this section is that function symbols for all primitive recursive set functions can be added. Modulo the new function symbols, $\mathbf{ATR_0^{set}}$ is an extension of **PRS**ω. Towards the end of the section we review another fact that connects our results with second-order arithmetic: the equivalence between Π^1_1-comprehension and the existence of admissible sets.

Let us point out that Simpson in [**85**] resp. [**87**] gives two equivalent (modulo countability) but rather different axiomatizations of $\mathbf{ATR_0^{set}}$. We adopt the version in [**85**], which is closer to **PRS**ω. The connection with [**87**] will be explained in Remark 1.4.3. To describe $\mathbf{ATR_0^{set}}$ we need $\mathcal{L}_\in$-definitions of the primitive recursive functions. Simpson refers to [**44**], where one finds the following (modulo a harmless change regarding composition, cf. [**44**, 1.3(1)]):

DEFINITION 1.4.1. To each primitive recursive function symbol F of arity n (without additional parameter ω, contrary to Convention 1.2.9) we associate an $\mathcal{L}_\in$-formula $\mathcal{D}_F(x_1, \ldots, x_n, z)$, intended to express $F(x_1, \ldots, x_n) = z$:

$$\begin{aligned}
\mathcal{D}_Z(x, z) &:\equiv \forall_{y \in z} y \neq y, \\
\mathcal{D}_M(x_1, x_2, z) &:\equiv \forall_{y \in z}(y \in x_1 \vee y = x_2) \wedge \forall_{y \in x_1} y \in z \wedge x_2 \in z, \\
\mathcal{D}_{P_i^n}(x_1, \ldots, x_n, z) &:\equiv z = x_i, \\
\mathcal{D}_C(x_1, x_2, x_3, x_4, z) &:\equiv (x_3 \in x_4 \wedge z = x_1) \vee (x_3 \notin x_4 \wedge z = x_2), \\
\mathcal{D}_{K[G;H_1,\ldots,H_k]}(\vec{x}, z) &:\equiv \exists_{y_1,\ldots,y_k} \left(\bigwedge_{i=1}^{k} \mathcal{D}_{H_i}(\vec{x}, y_i) \wedge \mathcal{D}_G(y_1, \ldots, y_k, z) \right).
\end{aligned}$$

As for recursion, $\mathcal{D}_{R[G]}(x, \vec{y}, z)$ is defined as the formula

$$\begin{aligned}
&\exists_{f,w}(\text{“}f \text{ is a function with transitive domain } w \supseteq x\text{”} \wedge \\
&\quad \exists_u(u = \bigcup\{f(v) \mid v \in x\} \wedge \mathcal{D}_G(u, x, \vec{y}, z)) \wedge \\
&\quad \forall_{x_0 \in w} \exists_{u_0, z_0}(u_0 = \bigcup\{f(v) \mid v \in x_0\} \wedge z_0 = f(x_0) \wedge \mathcal{D}_G(u_0, x_0, \vec{y}, z_0))).
\end{aligned}$$

We can now present the axiomatization of $\mathbf{ATR_0^{set}}$ from [**85**]. Strictly speaking, this version of $\mathbf{ATR_0^{set}}$ does not include countability, which is described as an “optional extra axiom”. In any case, countability is covered by the crucial [**85**, Theorem 3.6], so that we can safely include it:

DEFINITION 1.4.2. The theory $\mathbf{ATR_0^{set}}$ is formulated in the language $\mathcal{L}_\in$ of pure set theory and has the following non-logical axioms:

(i) The axiom of extensionality.
(ii) The axiom of foundation (also called regularity).
(iii) The axiom of infinity, formulated as

$$\exists_x(\emptyset \in x \wedge \forall_{y \in x} y \cup \{y\} \in x).$$

(iv) The axiom

$$\forall_{x_1,\ldots,x_n} \exists_z \mathcal{D}_F(x_1, \ldots, x_n, z),$$

for each primitive recursive function symbol F.

(v) Axiom beta, stating that for any well-founded relation r (given as a set of ordered pairs) there exists a collapse, i.e. a set-sized function f which is defined on the field of r and satisfies

$$f(y) = \{f(x) \mid \langle x, y \rangle \in r\}.$$

Recall that r is well-founded (or regular) if we have

$$\forall_z (z \neq \emptyset \to \exists_{y \in z} \forall_{x \in z} \langle x, y \rangle \notin r).$$

(vi) The axiom of countability, stating that for any set x there exists an injection from x into the finite ordinals.

As promised, we now compare this axiomatization with the version in [**87**]:

REMARK 1.4.3. The axiomatization of $\mathbf{ATR_0^{set}}$ in [**87**, Section VII.3] does not demand the totality of all primitive recursive functions (axioms (iv) above) explicitly. It only demands the totality of (a finite number of) rudimentary functions, and strengthens the other axioms appropriately: Countability is replaced by hereditary countability, which includes the existence of transitive closures; infinity now requires a set of all finite sets, rather than a set of all finite ordinals. The primitive recursive functions can be recovered according to [**85**, Theorem 3.1], in conjunction with [**87**, Lemma VII.4.2]: The latter tells us that the (relativized) constructible hierarchy up to any ordinal exists. At the same time, axiom beta provides relatively large ordinals (e.g. for each provable well-ordering of $\mathbf{ATR}_0$). Once we have sufficiently long segments of the constructible hierarchy, the Stability Theorem [**44**, 2.5] ensures the totality of all primitive recursive functions.

Our next goal is to extend the language of $\mathbf{ATR_0^{set}}$ by primitive recursive function symbols. The following is a preparation:

LEMMA 1.4.4. *For each primitive recursive function symbol F we have*

$$\mathbf{ATR_0^{set}} \vdash \mathcal{D}_F(x_1, \dots, x_n, z) \wedge \mathcal{D}_F(x_1, \dots, x_n, z') \to z = z'.$$

The same holds with $\mathbf{PRS}\omega$ at the place of $\mathbf{ATR_0^{set}}$.

PROOF. We argue by (meta) induction on the build-up of F. The basic function symbols Z, M, P_i^n, C are covered by extensionality and the fact that $=$ is an equivalence relation. In case $F \equiv K[G; H_1, \dots, H_k]$ the assumptions $\mathcal{D}_F(\vec{x}, z)$ and $\mathcal{D}_F(\vec{x}, z')$ yield $\mathcal{D}_G(y_1, \dots, y_k, z)$ resp. $\mathcal{D}_G(y_1', \dots, y_k', z')$, for witnesses y_i, y_i' that satisfy $\mathcal{D}_{H_i}(\vec{x}, y_i)$ resp. $\mathcal{D}_{H_i}(\vec{x}, y_i')$. The induction hypothesis for H_i gives $y_i = y_i'$. By the equality axioms we get $\mathcal{D}_G(y_1, \dots, y_k, z')$, and the induction hypothesis for G yields $z = z'$, as desired. The final and most interesting case is $F \equiv R[G]$: Let f, w, u resp. f', w', u' be the witnesses provided by the assumptions $\mathcal{D}_F(x, \vec{y}, z)$ and $\mathcal{D}_F(x, \vec{y}, z')$. In particular we have $\mathcal{D}_G(u, x, \vec{y}, z)$ and $\mathcal{D}_G(u', x, \vec{y}, z')$. Invoking the induction hypothesis for G we can conclude $z = z'$ once we have checked $u = u'$.

As we have $u = \bigcup\{f(v) \mid v \in x\}$ and $u' = \bigcup\{f'(v) \mid v \in x\}$ it suffices to show that f and f' agree on x. We argue that they do agree on the transitive set $w \cap w' \supseteq x$, the intersection of their domains: Note first that the condition $f(x_0) \neq f'(x_0)$ can be expressed by a Δ_0-formula (which asserts that there are pairs $p \in f, p' \in f'$ such that p and p' both have first component x_0 but disagree on their second component). From [**87**, Lemma VII.3.5] and Corollary 1.1.10 we know that Δ_0-comprehension is available in $\mathbf{ATR_0^{set}}$ and in $\mathbf{PRS}\omega$, respectively. Thus we can form the set

$$\{x_0 \in w \cap w' \mid f(x_0) \neq f'(x_0)\}.$$

We must show that this set is empty. If it is not, foundation yields an $\in$-minimal element x_0. For this x_0 the assumption $\mathcal{D}_F(x, \vec{y}, z)$ provides witnesses u_0, z_0 with

$$u_0 = \bigcup\{f(v) \mid v \in x_0\} \wedge z_0 = f(x_0) \wedge \mathcal{D}_G(u_0, x_0, \vec{y}, z_0).$$

Write u_0', z_0' for the analogous witnesses provided by $\mathcal{D}_F(x, \vec{y}, z')$. By the minimality of x_0 and the fact that $w \cap w'$ is transitive we have $f(v) = f'(v)$ for any $v \in x_0$. This implies $u_0 = u_0'$ and then $z_0 = z_0'$, by the induction hypothesis for G. Thus we have $f(x_0) = z_0 = f'(x_0)$, contradicting the assumption $f(x_0) \neq f'(x_0)$. $\Box$

We can now add primitive recursive function symbols, and present the desired connection with second-order arithmetic:

Definition 1.4.5. The $\mathcal{L}_{\mathrm{pr}}$-theory $\mathbf{ATR_0^{set}}$ results from the $\mathcal{L}_\in$-theory $\mathbf{ATR_0^{set}}$ (the homonymy is justified in view of the following corollary) by adding all primitive recursive function symbols, the corresponding equality axioms, and the axioms

$$\forall_{x_1,\ldots,x_n} \mathcal{D}_F(x_1, \ldots, x_n, F(x_1, \ldots, x_n))$$

for all primitive recursive function symbols F.

Corollary 1.4.6. *The $\mathcal{L}_{\mathrm{pr}}$-theory $\mathbf{ATR_0^{set}}$ is a conservative extension of the $\mathcal{L}_\in$-theory $\mathbf{ATR_0^{set}}$. Furthermore, any $\mathcal{L}_{\mathrm{pr}}$-formula φ provable in the former theory corresponds to an $\mathcal{L}_\in$-formula φ^* provable in the latter.*

Proof. Recall that the $\mathcal{L}_\in$-theory $\mathbf{ATR_0^{set}}$ contains the axiom $\forall_{\vec{x}} \exists_z \mathcal{D}_F(\vec{x}, z)$, for each p.r. function symbol F. This suffices for the conservativity result (by an easy model-theoretic argument or, following the more constructive [**84**, Section 4.5], via Herbrand's theorem). The translation of φ into φ^* is described in [**84**, Section 4.6]. It relies on the uniqueness result of Lemma 1.4.4. $\Box$

Having established the $\mathcal{L}_{\mathrm{pr}}$-theory $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$ we want to connect it to primitive recursive set theory:

PROPOSITION 1.4.7. *The $\mathcal{L}_{\mathrm{pr}}$-theory $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$ is an extension of $\mathbf{PRS}\omega$.*

PROOF. We must show that $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$ proves the $\mathbf{PRS}\omega$-axiom $\mathcal{A}_F$ (see Definition 1.1.2), for each p.r. function symbol F. If F is one of the basic functions Z, M, P_i^n, C this holds because $\mathcal{A}_F$ and $\forall_{x_1,\dots,x_n}\mathcal{D}_F(x_1,\dots,x_n,F(x_1,\dots,x_n))$ are the very same formula. Concerning $F \equiv K[G;H_1,\dots,H_k]$, the $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$-axiom $\mathcal{D}_F(\vec{x},K[G;H_1,\dots,H_k](\vec{x}))$ states that there are witnesses $y_1,\dots,y_k$ which satisfy

$$\bigwedge_{i=1}^{k}\mathcal{D}_{H_i}(\vec{x},y_i)\wedge\mathcal{D}_G(y_1,\dots,y_k,K[G;H_1,\dots,H_k](\vec{x})).$$

The axioms $\mathcal{D}_{H_i}(\vec{x},H_i(\vec{x}))$ together with Lemma 1.4.4 force $y_i = H_i(\vec{x})$. Using the equality axioms we get

$$D_G(H_1(\vec{x}),\dots,H_1(\vec{x}),K[G;H_1,\dots,H_k](\vec{x})).$$

On the other hand, the axiom $\forall_{y_1,\dots,y_k}\mathcal{D}_G(y_1,\dots,y_k,G(y_1,\dots,y_k))$ yields

$$D_G(H_1(\vec{x}),\dots,H_k(\vec{x}),G(H_1(\vec{x}),\dots,H_k(\vec{x}))).$$

Again by Lemma 1.4.4 we can conclude

$$K[G;H_1,\dots,H_k](\vec{x}) = G(H_1(\vec{x}),\dots,H_k(\vec{x})),$$

which is the desired $\mathbf{PRS}\omega$-axiom $\mathcal{A}_{K[G;H_1,\dots,H_k]}$. Finally, assume that $F\equiv R[G]$ is defined by recursion. Consider the witnesses f,w,u provided by the $\mathbf{ATR}_{\mathbf{0}}^{\mathrm{set}}$-axiom $\mathcal{D}_{R[G]}(x,\vec{y},R[G](x,\vec{y}))$. In particular we have $\mathcal{D}_G(u,x,\vec{y},R[G](x,\vec{y}))$. Together with the axiom $\mathcal{D}_G(u,x,\vec{y},G(u,x,\vec{y}))$ and Lemma 1.4.4 this implies

$$R[G](x,\vec{y}) = G(u,x,\vec{y}).$$

To conclude that u is the witness required by $\mathcal{A}_{R[G]}$ it remains to establish

$$u = \bigcup\{R[G](v,\vec{y}) \mid v\in x\}.$$

As $\mathcal{D}_{R[G]}(x,\vec{y},R[G](x,\vec{y}))$ includes $u=\bigcup\{f(v)\mid v\in x\}$ this reduces to the equalities

$$f(x_0) = R[G](x_0,\vec{y}) \qquad \text{for all } x_0\in w\supseteq x.$$

By the third line of $\mathcal{D}_{R[G]}(x,\vec{y},R[G](x,\vec{y}))$ the set $u_0 = \bigcup\{f(v)\mid v\in x_0\}$ exists and satisfies $\mathcal{D}_G(u_0,x_0,\vec{y},f(x_0))$. Then it is immediate that $\mathcal{D}_{R[G]}(x_0,\vec{y},f(x_0))$ holds with witnesses f,w,u_0. Together with the axiom $\mathcal{D}_{R[G]}(x_0,\vec{y},R[G](x_0,\vec{y}))$ and Lemma 1.4.4 this implies $f(x_0)=R[G](x_0,\vec{y})$, as desired. □

Conversely, we will later need the following:

LEMMA 1.4.8. *For each primitive recursive function symbol F we have*

$$\mathbf{PRS}\omega \vdash \forall_{x_1,\ldots,x_n,z}(\mathcal{D}_F(x_1,\ldots,x_n,z) \leftrightarrow F(x_1,\ldots,x_n) = z).$$

PROOF. In view of Lemma 1.4.4 it is enough to establish

$$\mathbf{PRS}\omega \vdash \forall_{x_1,\ldots,x_n}\mathcal{D}_F(x_1,\ldots,x_n,F(x_1,\ldots,x_n)).$$

One argues by meta-induction on F. In the crucial case $F \equiv R[G]$, let us show that $\mathcal{D}_{R[G]}(x,\vec{y},R[G](x,\vec{y}))$ holds with witnesses $w = \mathrm{TC}(x)$ and $f = R[G](\cdot,\vec{y})\restriction w$. Note that f is a set, due to Proposition 1.2.2. First, we need $\mathcal{D}_G(u,x,\vec{y},R[G](x,\vec{y}))$ for $u = \bigcup\{f(v)\,|\,v \in x\}$. This follows from $\mathcal{D}_G(u,x,\vec{y},G(u,x,\vec{y}))$ (by the induction hypothesis for G) and $G(u,x,\vec{y}) = R[G](x,\vec{y})$ (by the axiom $\mathcal{A}_{R[G]}$ of $\mathbf{PRS}\omega$). By the same argument one has $\mathcal{D}_G(\bigcup\{f(v)\,|\,v \in x_0\},x_0,\vec{y},f(x_0))$ for any $x_0 \in w$, completing the verification of $\mathcal{D}_{R[G]}(x,\vec{y},R[G](x,\vec{y}))$. □

Now that we have seen how $\mathbf{PRS}\omega$ relates to Simpson's set theory $\mathbf{ATR_0^{set}}$, let us present known connections with second-order arithmetic. Following [**87**, Theorem VII.3.9], there is a "natural translation" which allows us to identify formulas of second-order arithmetic with formulas of set theory (not involving p.r. function symbols): The natural numbers are identified with the set ω. Concerning addition and multiplication on ω, rather weak set theories show that the graphs of these operations exist as sets, denoted by $\mathrm{Add},\mathrm{Mult} \subseteq \omega\times\omega\times\omega$. Using these sets as parameters, the desired translation is straightforward: For example, the formula

$$\exists_X\forall_n n+n \in X$$

of second-order arithmetic corresponds to the set-theoretic formula

$$\exists_x(x \subseteq \omega \wedge \forall_{y\in\omega}\exists_{z\in\omega}(\langle y,y,z\rangle \in \mathrm{Add} \wedge z \in x)).$$

Note that arithmetical formulas are translated into Δ_0-formulas of $\mathcal{L}_\in$. Thus the second-order theory $\mathbf{ACA_0}$ becomes a subtheory of $\mathbf{PRS}\omega$, and of much weaker set theories (all this is in the proof of [**87**, Theorem VII.3.9]; cf. also Corollary 1.1.10 above). In fact, Simpson's result yields the following connection:

COROLLARY 1.4.9. *The $\mathcal{L}_{\mathrm{pr}}$-theory $\mathbf{ATR_0^{set}}$ is conservative over $\mathbf{ATR}_0$, the subtheory of second-order arithmetic. Furthermore, any result of the $\mathcal{L}_{\mathrm{pr}}$-theory $\mathbf{ATR_0^{set}}$ can be translated into a corresponding result of $\mathbf{ATR}_0$.*

Proof. Corollary 1.4.6 reduces the first claim to the fact that the $\mathcal{L}_\in$-theory $\mathbf{ATR_0^{set}}$ is conservative over the second-order theory $\mathbf{ATR}_0$, which is famously due to Simpson [**85**, Theorem 3.6]. To exhibit the desired translation we consider an $\mathcal{L}_{\text{pr}}$-formula φ that is provable in the $\mathcal{L}_{\text{pr}}$-theory $\mathbf{ATR_0^{set}}$. Corollary 1.4.6 yields a $\mathcal{L}_\in$-formula φ^* that is provable in the $\mathcal{L}_\in$-theory $\mathbf{ATR_0^{set}}$. By [**87**, Lemma VII.3.16] a corresponding second-order formula $|\varphi^*|$ is also provable in $\mathbf{ATR_0^{set}}$ (modulo the identification of second-order formulas with $\mathcal{L}_\in$-formulas). Conservativity implies that $|\varphi^*|$ is provable in $\mathbf{ATR}_0$, as desired. $\square$

Recall that the aim of this thesis is an analysis of Π^1_1-comprehension in terms of type-two well-ordering principles. In view of the previous result the $\mathcal{L}_{\text{pr}}$-theory $\mathbf{ATR_0^{set}}$ and its subtheory $\mathbf{PRS}\omega$ are reasonable base theories for this undertaking. We adopt the following terminology:

Convention 1.4.10. The default base theory for the rest of this thesis is primitive recursive set theory with infinity ($\mathbf{PRS}\omega$). When an argument uses axiom beta or the axiom of countability this will be indicated (see e.g. Proposition 1.4.12 and Corollary 1.4.13). If not indicated otherwise, $\mathbf{ATR_0^{set}}$ will denote the $\mathcal{L}_{\text{pr}}$-theory of that name.

In the rest of this section we review the connection between Π^1_1-comprehension and the existence of admissible sets. The relevant arguments are known (see in particular [**41**, Section 7]), but we want to check that they apply in our setting. Let us begin with the easier direction (following Convention 1.4.10, this result is established in the base theory $\mathbf{PRS}\omega$):

Proposition 1.4.11. *If each countable set is an element of some admissible set then (each instance of) Π^1_1-comprehension holds.*

Note that there is an instance of Π^1_1-comprehension which implies all other instances over $\mathbf{ACA_0}$. To see this, one can for example use a Δ^1_1-definition of truth for arithmetical formulas (cf. [**87**, Lemma VII.2.2]).

Proof. We follow the argument in [**41**, Section 7]: Consider a Π^1_1-formula $\varphi(n,\vec{Y})$ and fix values for the parameters $\vec{Y}$. By Kleene's normal form theorem (see e.g. [**87**, Lemma V.1.4, Lemma V.1.8]) and arithmetical comprehension there is a countable set $X=(X_n)_{n\in\omega}$ such that X_n is a linear order with field ω, and such that we have

$$\forall_{n\in\omega}(\varphi(n,\vec{Y})\leftrightarrow \text{“}X_n \text{ is well-founded”}).$$

Now pick an admissible set $\mathbb{A}$ with $X \in \mathbb{A}$. If X_n is well-founded then, by [**41**, Theorem 4.6], there exists a collapsing function $f \in \mathbb{A}$ for X_n, i.e. a function $f : \omega \to \mathbb{V}$ which satisfies

$$\forall_{k,m\in\omega}(\langle k,m\rangle \in X_n \to f(k) \in f(m)).$$

Conversely, the existence of such a collapsing function ensures that X_n is well-founded: Given a non-empty set $Z \subseteq \omega$, form $\{f(m) \,|\, m \in Z\}$. By foundation this set has an $\in$-minimal element a. Then any $m \in Z$ with $f(m) = a$ is X_n-minimal in Z. Now Corollary 1.1.10 allows us to construct the set

$$\{n \in \omega \,|\, \exists_{f\in\mathbb{A}}\text{“}f : \omega \to \mathbb{V} \text{ is collapsing for } X_n\text{”}\} = \{n \in \omega \,|\, \varphi(n, \vec{Y})\},$$

as required for Π^1_1-comprehension. □

For the converse implication we rely on the known fact (see [**41**, Corollary 7.2]) that the statement “each set is an element of some admissible set” is interpretable in the subtheory $\mathbf{\Pi^1_1{-}CA_0}$ of second-order arithmetic. Transferring this result into a set-theoretic context, our task will be to show that we do not only get an interpretation but an actual admissible set. First, let us give an overview of the representation of admissible sets in $\mathbf{\Pi^1_1{-}CA_0}$: As in Simpson's interpretation of $\mathbf{ATR_0^{set}}$ in $\mathbf{ATR}_0$, the idea is to code the membership relation on a given set by a well-founded tree $T \subseteq \omega^{<\omega}$ (which can in turn be coded as a subset of ω). Note that Jäger [**41**, Section 1.7] only admits “representation trees” which satisfy the uniqueness condition

$$\sigma^\frown n \in T \wedge \sigma^\frown m \in T \wedge T_{\sigma^\frown n} \simeq T_{\sigma^\frown m} \to n = m.$$

Here $T_\tau = \{\tau' \,|\, \tau^\frown\tau' \in T\}$ is the subtree of T at $\tau \in T$, and $S \simeq T$ expresses that there is a tree-isomorphism $f : S \to T$. Jäger also reserves odd numbers to encode urelements, but this is not needed in our context. Now the idea is that the representation tree T stands for the set $|T| = c_T(\langle\rangle)$, where the collapsing function

$$c_T : T \to \mathbb{V}, \qquad c_T(\sigma) = \{c_T(\sigma^\frown n) \,|\, \sigma^\frown n \in T\}$$

is defined by recursion over T. When we switch back to a set-theoretic context we will make this interpretation official. As long as we work in second-order arithmetic there is no set $|T|$, and its definition can only serve as informal guidance. Instead, one officially argues in terms of the relations $\simeq$ and

$$S \,\widetilde{\in}\, T :\equiv \text{“}S, T \text{ are representation trees”} \wedge \exists_{n\in\omega}(\langle n\rangle \in T \wedge S \simeq T_{\langle n\rangle}).$$

To construct an admissible representation tree one uses properties of hyperarithmetical sets. Very roughly, the argument goes as follows: Assume the premise of Δ_0-collection in the form

$$\forall_{S\widetilde{\in}S'}\exists_T(\text{“}T\text{ a hyperarithmetical representation tree”}\land\varphi(S,T)).$$

Following [**41**, Lemma 5.2] we can use numerical Π^1_1-uniformization (due to H. Friedman and Simpson) to choose a unique numerical code for each of the hyperarithmetical trees T. This allows us to collect the relevant codes into a Δ^1_1-definable set. By the Suslin-Kleene theorem (see [**41**, Theorem 5.1]) any Δ^1_1-set is hyperarithmetical. Thus we can build a hyperarithmetical representation tree T' with

$$\forall_{S\widetilde{\in}S'}\exists_{T\widetilde{\in}T'}\varphi(S,T).$$

This means that the hyperarithmetical representation trees form a model of Kripke-Platek set theory. On the other hand, being a code of a hyperarithmetical representation tree is an arithmetical-in-Π^1_1-property. Thus Π^1_1-comprehension allows us to define a representation tree which contains all hyperarithmetical representation trees as immediate subtrees (see [**41**, Lemma 7.5]). This tree represents an admissible set. Details for all steps of this argument can be found in [**41**]. We shall be content to transfer the result back into set theory. Following Convention 1.4.10, the base theory for the following result is $\mathbf{PRS}\omega$ extended by axiom beta. Also recall that a set is hereditarily countable if and only if it is a subset of some countable transitive set.

Proposition 1.4.12 (Beta). *If Π^1_1-comprehension holds then each hereditarily countable set is an element of some admissible set.*

Proof. We begin by making the above semantics for representation trees official (cf. [**41**, Section 1.7] resp. [**87**, Section VII.3]): Each tree $T\subseteq\omega^{<\omega}$ is associated with the relation

$$r_T=\{\langle\sigma,\tau\rangle\in T\times T\,|\,\exists_{n\in\omega}\sigma=\tau^\frown n\},$$

which is well-founded if T has no infinite branch (see [**87**, Lemma V.1.3] or the proof of Lemma 3.3.4 below). In that case axiom beta provides a collapsing function $c_T:T\to\mathbb{V}$ for r_T, which satisfies

$$c_T(\sigma)=\{c_T(\sigma^\frown n)\,|\,\sigma^\frown n\in T\}.$$

The notation c_T is justified because the collapse is unique: If c'_T is a competitor, consider an r_T-minimal element $\sigma\in\{\tau\in T\,|\,c_T(\tau)\neq c'_T(\tau)\}$ and deduce the contradiction $c_T(\sigma) = c'_T(\sigma)$. On the other hand, there is no primitive recursive function which computes c_T from T. This will not affect the present proof, but it is an important distinction to make. In the same sense we write $|T| := c_T(\langle\rangle)$. Observe that $S \simeq T$ implies $|S| = |T|$: If $h : S \to T$ is an isomorphism then we have $c_S(\sigma) = c_T(h(\sigma))$ for all $\sigma \in S$, as an r_S-minimal counterexample would yield a contradiction. It follows that $S \mathrel{\widetilde{\in}} T$ implies $|S| \in |T|$. To establish the proposition, consider a hereditarily countable set a, witnessed by an injection $f : \mathrm{TC}(a) \to \omega$. As in the proof of Proposition 1.2.12 we get a bijection $g : \mathrm{rng}(f) \to \mathrm{TC}(a)$. Put

$$S := \{\langle n_1, \ldots, n_k\rangle \in \mathrm{rng}(f)^{<\omega} \,|\, g(n_{i+1}) \in g(n_i) \text{ for } i = 1, \ldots, k-1\}$$

and observe that $c_S(\langle\rangle) := \mathrm{TC}(a)$ and $c_S(\sigma^\frown n) := g(n)$ defines the collapse of S. In particular $S_{\sigma^\frown n} \simeq S_{\sigma^\frown m}$ implies $g(n) = c_S(\sigma^\frown n) = c_S(\sigma^\frown m) = g(m)$ and then $n = m$, so that S is a representation tree. Setting

$$S' := \{\langle n_1, \ldots, n_k\rangle \in S \,|\, g(n_1) \in a\}$$

yields a representation tree with $|S'| = a$. Now we can apply Jäger's result: By [**41**, Lemma 7.5] there is a representation tree T which satisfies $S' \mathrel{\widetilde{\in}} T$ as well as a certain formula $\widetilde{\mathrm{Ad}}(T)$ of second-order arithmetic. In view of $a = |S'| \in |T|$ it remains to check that $|T|$ is indeed an admissible set. The most difficult part is to prove the equivalence

$$S \simeq T \leftrightarrow |S| = |T|$$

for representation trees S and T. Note that Jäger [**41**, Lemma 7.7] proves this in the theory $\mathbf{KPl^r}$, which is stronger than $\mathbf{ATR_0^{set}}$ (namely as strong as $\mathbf{\Pi^1_1{-}CA_0}$); Simpson [**87**, Lemma VII.3.14] gives a proof in $\mathbf{ATR_0^{set}}$, but with a different definition of $\simeq$. We have already seen that $S \simeq T$ implies $|S| = |T|$. To prepare the converse direction, we first show a special case: For any representation tree T and any node $\tau \in T$ we have

$$\forall_{n,m\in\omega}(\tau^\frown n \in T \wedge \tau^\frown m \in T \wedge c_T(\tau^\frown n) = c_T(\tau^\frown m) \to n = m).$$

Assume that τ is a counter-example which is minimal with respect to the transitive closure of r_T. Consider witnesses n, m for the failure at τ. We want to deduce the contradiction $n = m$. By the definition of representation trees it suffices to construct an isomorphism $g : T_{\tau^\frown n} \to T_{\tau^\frown m}$. We do this by recursion on sequences in $T_{\tau^\frown n}$, justified by Corollary 1.2.11 and Proposition 1.2.2. Throughout the construction we will ensure

$$c_T(\tau^\frown n^\frown \sigma) = c_T(\tau^\frown m^\frown g(\sigma)).$$

Note that this holds at $g(\langle\rangle) = \langle\rangle$ by the assumption on τ. Now assume that $g(\sigma)$ is already defined. For each k with $\sigma^\frown k \in T_{\tau^\frown n}$ we have

$$c_T(\tau^\frown n^\frown \sigma^\frown k) \in c_T(\tau^\frown n^\frown \sigma) = \\ = c_T(\tau^\frown m^\frown g(\sigma)) = \{c_T(\tau^\frown m^\frown g(\sigma)^\frown l) \,|\, g(\sigma)^\frown l \in T_{\tau^\frown m}\}.$$

In other words, there is an $l \in \omega$ with $g(\sigma)^\frown l \in T_{\tau^\frown m}$ and

$$c_T(\tau^\frown n^\frown \sigma^\frown k) = c_T(\tau^\frown m^\frown g(\sigma)^\frown l).$$

Crucially, the minimality of τ ensures that this number l is unique. It can then be computed by a primitive recursive function: Simply form the set of all $l \in \omega$ with the required property and extract its only element. In the recursion step we can thus put $g(\sigma^\frown k) := g(\sigma)^\frown l$ for said number l. Injectivity is ensured by the fact that $k \neq k'$ implies $c_T(\tau^\frown n^\frown \sigma^\frown k) \neq c_T(\tau^\frown n^\frown \sigma^\frown k')$, again by minimality of τ. Surjectivity is straightforward. Now we can show that $|S| = |T|$ implies $S \simeq T$. Indeed, the construction of an isomorphism $f : S \to T$ is very similar to the construction of $g : T_{\tau^\frown n} \to T_{\tau^\frown m}$: In each step, make the unique choice that satisfies $c_S(\sigma) = c_T(f(\sigma))$. Using the equivalence between $\simeq$ and $=$ we also get

$$S \mathrel{\tilde{\in}} T \leftrightarrow |S| \in |T|$$

for representation trees S, T. Next, call a representation tree T transitive if $\sigma \in T$ implies $T_\sigma \mathrel{\tilde{\in}} T$. In that case we have $|T_\sigma| \in |T|$, and it is easy to see that $|T|$ is a transitive set. To each (meta) formula φ of pure set theory and any representation tree T Jäger [**41**, Section 7] associates a formula $\varphi^{(T)}$ of second-order arithmetic. As above, we identify $\varphi^{(T)}$ with its natural translation into the language of set theory. For $\varphi(x_1, \dots, x_n)$ as described we can show that

$$\varphi^{(T)}(T_{\sigma_1}, \dots, T_{\sigma_n}) \leftrightarrow |T| \vDash \varphi(|T_{\sigma_1}|, \dots, |T_{\sigma_n}|)$$

holds for any transitive representation tree T and all nodes $\sigma_1, \dots, \sigma_n \in T$. This follows by (meta) induction on φ, using the above equivalences between $\simeq$ resp. $\tilde{\in}$ and $=$ resp. $\in$ to cover the base cases. To complete the proof, let us now show that $|T|$ is an admissible set if the second-order formula $\widetilde{\mathrm{Ad}}(T)$ holds. In particular, this formula demands that T is a transitive tree, so that $|T|$ is a transitive set. Also, $\widetilde{\mathrm{Ad}}(T)$ states $\mathrm{Tr}_\Pi(T, \langle\rangle, \varphi)$ for all **KP**-axioms φ in the sense of the object language, using a certain truth definition $\mathrm{Tr}_\Pi(\cdot)$. By the truth condition (Tr4) in [**41**, Section 7] we get $\varphi^{(T)}$ for each **KP**-axiom in the sense of the meta language.

The above equivalence yields $|T| \vDash \varphi$. By Proposition 1.3.13 these standard axioms (i.e. axioms in the sense of the meta theory) suffice to make $|T|$ admissible. $\square$

Adding the axiom of countability we can summarize these (known) results as follows:

Corollary 1.4.13 ($\mathbf{ATR_0^{set}}$). *The following are equivalent:*

(i) The statement that each set is an element of some admissible set.
(ii) The principle of Π^1_1*-comprehension.*

Concerning (ii), recall that there is an instance of Π^1_1-comprehension which implies all other instances.

Proof. The direction from (i) to (ii) is a weakening of Proposition 1.4.11. Concerning the other direction, as $\mathbf{ATR_0^{set}}$ proves the existence of transitive closures and states that all sets are countable, it shows that all sets are hereditarily countable. Then the claim holds by Proposition 1.4.12. $\square$

CHAPTER 2

Type-Two Well-Ordering Principles

In this chapter we define the central new notions of the present thesis: the abstract, the predicative, and the computable Bachmann-Howard principle. The name derives from the fact that these principles relativize the construction of the Bachmann-Howard ordinal. Crucially, our Bachmann-Howard principles are well-ordering principles of type-two. This distinguishes them from Rathjen and Valencia Vizcaíno's [**71**] type-one well-ordering principle $X \mapsto \vartheta_X$, which is also based on the Bachmann-Howard construction.

Section 2.1 is devoted to the abstract Bachmann-Howard principle. The latter states that any proto-dilator $\alpha \mapsto T_\alpha$ admits an ordinal α with a Bachmann-Howard collapse $\vartheta : T_\alpha \xrightarrow{\text{BH}} \alpha$. For a detailed explanation of these notions we refer to the introduction of the thesis. To release the full power of the abstract Bachmann-Howard principle we need a particularly strong proto-dilator. In the second half of Section 2.1 we will thus transform a given proto-dilator $\alpha \mapsto T_\alpha$ into a proto-dilator $\alpha \mapsto \varepsilon(T)_\alpha$. Intuitively, $\varepsilon(T)_\alpha$ contains an uncountable ε-number for each element of T_α. This strengthening allows us to recover the usual construction of the Bachmann-Howard ordinal. It will be needed to prove the direction (iii)$\Rightarrow$(v) of Theorem 4.4.6.

We begin Section 2.2 by introducing dilators and prae-dilators. Note that the latter are not quite equivalent to Girard's pre-dilators (thus the different spelling). More details can be found in Remark 2.2.2. In the rest of the section we construct the relativized Bachmann-Howard order $\text{BH}(T)$ needed to state the predicative Bachmann-Howard principle. As explained in the introduction of the thesis, $\text{BH}(T)$ will be defined as a direct limit of approximations $\text{BH}_n(T)$. As a starting point we choose $\text{BH}_0(T) = \varepsilon_0$ (other reasonable choices would be ω or even 0). Following Rathjen and Valencia Vizcaíno's [**71**] construction of relativized Bachmann-Howard ordinals (but with one crucial difference, see Remark 2.2.4) we build a term system $\vartheta(T_{\text{BH}_0(T)})$. The structure of these terms immediately yields a sufficiently order-preserving collapse of $\vartheta(T_{\text{BH}_0(T)})$ into its "countable" part $\vartheta(T_{\text{BH}_0(T)}) \cap \Omega$. Thus we continue the construction with $\text{BH}_1(T) = \vartheta(T_{\text{BH}_0(T)}) \cap \Omega$, recursively building term systems $\mathrm{BH}_n(T)$ for all $n \in \mathbb{N}$. The choice of $\mathrm{BH}_0(T)$ yields an embedding $i_0 : \mathrm{BH}_0(T) \to \vartheta(T_{\mathrm{BH}_0(T)}) \cap \Omega = \mathrm{BH}_1(T)$. Using the fact that T is functorial we get $T_{i_0} : T_{\mathrm{BH}_0(T)} \to T_{\mathrm{BH}_1(T)}$, and then

$$i_1 : \mathrm{BH}_1(T) = \vartheta(T_{\mathrm{BH}_0(T)}) \cap \Omega \to \vartheta(T_{\mathrm{BH}_1(T)}) \cap \Omega = \mathrm{BH}_2(T).$$

Recursively one obtains embeddings $i_n : \mathrm{BH}_n(T) \to \mathrm{BH}_{n+1}(T)$ for all $n \in \mathbb{N}$. Now $\mathrm{BH}(T)$ can be defined as the direct limit of the resulting system, as promised. The construction also yields embeddings $\mathrm{BH}_n(T) \to \vartheta(T_{\mathrm{BH}_n(T)}) \cap \Omega \to \vartheta(T_{\mathrm{BH}(T)}) \cap \Omega$. These glue to an embedding of $\mathrm{BH}(T)$ into $\vartheta(T_{\mathrm{BH}(T)}) \cap \Omega$, which will be an isomorphism because T preserves direct limits. As before, the structure of the term system $\vartheta(T_{\mathrm{BH}(T)})$ yields a collapse

$$\vartheta(T_{\mathrm{BH}(T)}) \to \vartheta(T_{\mathrm{BH}(T)}) \cap \Omega \cong \mathrm{BH}(T),$$

so that $\mathrm{BH}(T)$ is the desired fixed-point. The predicative Bachmann-Howard principle asserts that this fixed-point is well-founded. We will see that the construction of $\mathrm{BH}(T)$ can be formalized in primitive recursive set theory. As that theory is predicatively reducible this justifies the specification "predicative". To assess the strength of the predicative Bachmann-Howard principle, recall the result of Rathjen and Valencia Vizcaíno [**71**]: If any set is contained in an ω-model of bar induction then $X \mapsto \vartheta(X)$ is a well-ordering principle (of type one). Assuming that T preserves well-foundedness as well we infer that $\vartheta(T_{\mathrm{BH}_n(T)}) \cap \Omega = \mathrm{BH}_{n+1}(T)$ is well-founded if $\mathrm{BH}_n(T)$ is. So inductively, all orders $\mathrm{BH}_n(T)$ are well-founded. The additional strength of our type-two well-ordering principle seems to lie in the fact that well-foundedness is preserved in the direct limit. Recall that this is not the case in general (indeed, any linear order is the direct limit of its finite suborders).

In the introduction to [**29**], Girard writes that "[t]he general aim of Π^1_2-logic [i.e. the study of dilators and related concepts] is to rebuild some parts of mathematics [...] by making more explicit the finitary contents of such and such construction of the actual infinite kind." There seem to be two ways to do this: Either one compares finitistic and infinitary concepts in a setting which can officially speak about both. Or one works in a finitist setting and treats infinite constructions as ideal objects, which are not officially part of the ontology. In Section 2.3 we develop both these approaches. First, we show that dilators are determined by their restrictions to the category of natural numbers. This result is due to Girard [**28**, Remark 2.1.6(ii)], but we find it very fruitful to re-work it in primitive recursive set theory. Note that Girard [**28**, Remarks 2.3.5 and 2.3.6] himself hints at the connection between dilators and primitive recursive set functions. Our presentation is greatly simplified by using an alternative characterization of dilators in terms of supports (cf. Remark 2.2.2). As a result, we learn that the predicative Bachmann-Howard principle can be given as a single statement, rather than an axiom scheme (see Proposition 2.3.10). In the second half of Section 2.3 we study dilators in the subtheory $\mathbf{RCA_0}$ of second-order arithmetic. This may be closer to Girard's original intention, who stresses the "'finitistic' character" of dilators (see [**28**, Section 0.2.1], as well as the introduction of [**29**]). To formulate the computable Bachmann-Howard principle we define a term system $\vartheta(T)$ which corresponds to the order $\vartheta(T_{\mathrm{BH}(T)})$ considered above. The point is that $\vartheta(T)$ is computable relative to the coded prae-dilator $T \subseteq \mathbb{N}$. Thus $\mathbf{RCA_0}$ proves that $\vartheta(T)$ exists as a linear order. The computable Bachmann-Howard principle states that this order is well-founded. More information on the definition of $\vartheta(T)$ can be found in the introduction of the thesis. To conclude Section 2.3 we use a result of Rathjen and Valencia Vizcaíno to show that the computable Bachmann-Howard principle implies arithmetical transfinite recursion.

In Section 2.4 we prove the directions (ii)⇒(iii)⇒(v)⇒(iv) of Theorem 4.4.6: First, we deduce the abstract Bachmann-Howard principle from the existence of admissible sets. Our argument is similar to the usual construction of the Bachmann-Howard ordinal (cf. Remark 2.1.6) and in particular to Rathjen's well-ordering proof in [**61**, Section 4]. Assuming the axiom of countability, we then show that the abstract Bachmann-Howard principle implies its computable counterpart, which in turn implies the predicative version.

2.1. An Abstract Bachmann-Howard Principle

In the first part of this section we introduce the new notions of proto-dilator and Bachmann-Howard collapse, and the abstract Bachmann-Howard principle. In the second part we build particularly strong proto-dilators, essentially reconstructing a notation system for the Bachmann-Howard ordinal.

To understand the formal side of the following definitions, recall that our base theory $\mathbf{PRS}\omega$ contains a symbol for each primitive recursive set function. According to Convention 1.2.9 the parameter ω is always allowed. While second-order quantification is not available, we can use Currying to quantify over parametrized families of functions: Given a primitive recursive function $(u, x) \mapsto F(u, x)$, quantification over the family $(x \mapsto F(u,x))_{u\in\mathbb{V}}$ can be implemented as first-order quantification over the parameter u.

DEFINITION 2.1.1. Consider a primitive recursive function

$$T : \mathbb{V} \times \mathrm{Ord} \to \mathbb{V}, \qquad (u,\alpha) \mapsto T^u_\alpha.$$

We say that T^u, for a fixed value of the parameter u, is a proto-dilator if the following holds:

(i) For any ordinal α, the value $T^u_\alpha = (T^u_\alpha, <_{T^u_\alpha})$ is a well-order.
(ii) If $\alpha < \beta$ then we have $T^u_\alpha \subseteq T^u_\beta$ and $<_{T^u_\alpha} = <_{T^u_\beta} \cap (T^u_\alpha \times T^u_\alpha)$ (i.e. T^u_α is a sub-order of T^u_β, but not necessarily an initial segment).
(iii) We have $T^u_\lambda = \bigcup_{\gamma<\lambda} T^u_\gamma$ for any limit ordinal λ.

The (new) notion of proto-dilator is a non-functorial version of Girard's [**28**] notion of dilator: Condition (ii) demands "functoriality" for inclusion maps $\alpha \hookrightarrow \beta$, but not for arbitrary order embeddings $\alpha \to \beta$. As we shall see, proto-dilators keep precisely those properties that are needed for the abstract Bachmann-Howard principle. We remark that Girard's notion of prae-dilator (cf. Definition 2.2.1 below) weakens the notion of dilator in a completely different way.

A proto-dilator T^u can be seen as approximating the class-sized well-order

$$\mathbb{T}^u = \bigcup_{\alpha\in\mathrm{Ord}} T^u_\alpha.$$

Note that $\mathbb{T}^u$ is a Σ_1-class. In primitive recursive set theory it is better to work with the primitive recursive class

$$\mathbf{T}^u = \{(\alpha,\sigma)\,|\,\sigma \in T^u_\alpha \wedge \forall_{\gamma<\alpha}\, \sigma \notin T^u_\gamma\},$$

as in the preprint [**21**]. In the present thesis, the interpretation of a proto-dilator as a class-sized well-order will not play an official role. Indeed, the process of approximation is as important as the approximated class. This is made explicit in the following definition. Note that if $\alpha > 0$, then $\sigma \in T^u_\alpha$ implies $\sigma \in T^u_{\gamma+1}$ for some $\gamma < \alpha$, using condition (iii) in the definition of proto-dilator.

DEFINITION 2.1.2. Given a proto-dilator T^u, we define rank functions

$$|\cdot|_{T^u_\alpha} : T^u_\alpha \to \max\{\alpha, 1\}$$

by setting

$$|\sigma|_{T^u_\alpha} := \begin{cases} \min\{\gamma < \alpha\,|\,\sigma \in T^u_{\gamma+1}\} & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Observe that we have

$$\min\{\gamma < \alpha \mid \sigma \in T^u_{\gamma+1}\} = \bigcup\{\gamma < \alpha \mid \sigma \in T^u_{\gamma+1} \wedge \forall_{\delta<\gamma}\, \sigma \notin T^u_{\delta+1}\}.$$

Thus, by Corollary 1.1.10 and Proposition 1.1.4, the function $(u, \alpha, \sigma) \mapsto |\sigma|_{T^u_\alpha}$ is primitive recursive. From Proposition 1.2.2 we learn that $|\cdot|_{T^u_\alpha} : T^u_\alpha \to \max\{\alpha, 1\}$ is a set-sized function, and that $(u, \alpha) \mapsto |\cdot|_{T^u_\alpha}$ is a primitive recursive operation. As we will see in Example 2.1.5, it is possible that the order type of T^u_α is always bigger than α. Thus we cannot expect to get an order embedding $T^u_\alpha \to \alpha$. Instead, the construction of the Bachmann-Howard ordinal suggests the following notion of "mostly" order-preserving collapse. We are particularly influenced by Rathjen's notation system, as presented in [**72**, Section 1].

Definition 2.1.3. Consider a proto-dilator T^u and an ordinal α. A function $\vartheta : T^u_\alpha \to \alpha$ is called a Bachmann-Howard collapse if the following conditions are satisfied for all $\sigma, \tau \in T^u_\alpha$:

(i) $\sigma <_{T^u_\alpha} \tau \wedge |\sigma|_{T^u_\alpha} < \vartheta(\tau) \Rightarrow \vartheta(\sigma) < \vartheta(\tau)$,

(ii) $|\sigma|_{T^u_\alpha} < \vartheta(\sigma)$.

We write $\vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha$ to express that these conditions hold.

Condition (i) expresses that the order is preserved under a side condition. Note that the side condition would always fail if ϑ was the constant zero function. This trivial choice is excluded by condition (ii).

Definition 2.1.4. The abstract Bachmann-Howard principle is the collection of statements

$$\forall_u(\text{"}T^u \text{ is a proto-dilator"} \to \exists_\alpha \exists_\vartheta\, \vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha)$$

for all primitive recursive set functions $(u, \alpha) \mapsto T^u_\alpha$.

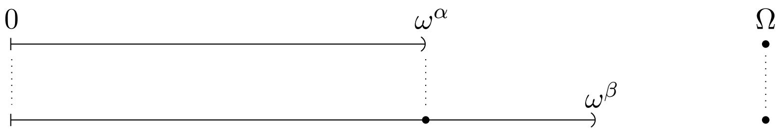
A proto-dilator is a special kind of type-one well-ordering principle: Each ordinal α (i.e. essentially each well-order) is transformed into a well-order T^u_α. Thus the abstract Bachmann-Howard principle can be seen as a type-two well-ordering principle: It takes the type-one well-ordering principle T^u as input and yields a certain well-order α as output. Virtues and shortcomings of the abstract Bachmann-Howard principle have been discussed in the introduction of the thesis. Concerning logical complexity, for each primitive recursive set function $(u, \alpha) \mapsto T^u_\alpha$, the statement that T^u is a proto-dilator can be expressed by a Π_1-formula in the language $\mathcal{L}_{\text{pr}}$ of primitive recursive set theory, with free variable u. Given a proto-dilator T^u

and an ordinal α, the statement that $\vartheta : T^u_\alpha \to \alpha$ is a Bachmann-Howard collapse is expressed by a bounded $\mathcal{L}_{\mathrm{pr}}$-formula. In other words, being a Bachmann-Howard collapse is a primitive recursive property (see Proposition 1.1.9). Thus each instance of the abstract Bachmann-Howard principle is a Π_2-statement. Recall that Π_2-statements of set theory correspond to Π^1_3-statements of second-order arithmetic (see [**87**, Theorem VII.3.24]). Theorem 4.4.6 will show that the abstract Bachmann-Howard principle is equivalent to Π^1_1-comprehension, which is a Π^1_3-statement as well. The following illustrates some important points:

Example 2.1.5. Define a proto-dilator $\alpha \mapsto T_\alpha$ by setting

$$T_\alpha := \omega^\alpha \cup \{\Omega\},$$

where ω^α is ordered as usual and Ω is a new biggest element. For $\alpha < \beta$ the inclusion of T_α into T_β can be pictured as follows:



Looking at one order T_α individually, it is natural to interpret Ω as the ordinal ω^α. However, when we consider $\alpha \mapsto T_\alpha$ as a compatible family, it is better to think of Ω as the order-type of the ordinals. In other words, we can think of T as approximating the class-sized well-order $\mathrm{Ord} \cup \{\Omega\}$. The same class could be approximated by $\alpha \mapsto \alpha \cup \{\Omega\}$, but $\alpha \mapsto \omega^\alpha \cup \{\Omega\}$ has the "stronger" rank functions

$$|\sigma|_{T_\alpha} = \begin{cases} \min\{\gamma < \alpha \,|\, \beta < \omega^{\gamma+1}\} & \text{if } \sigma = \beta < \omega^\alpha \text{ (and } \alpha > 0\text{)}, \\ 0 & \text{if } \sigma = \Omega \text{ (or } \alpha = 0\text{)}. \end{cases}$$

In particular, observe $|\omega^\beta|_{T_\alpha} = \beta$ for $\beta < \alpha$. As required of a proto-dilator, we do have

$$T_\lambda = \omega^\lambda \cup \{\Omega\} = \bigcup_{\gamma<\lambda} \omega^\gamma \cup \{\Omega\} = \bigcup_{\gamma<\lambda} T_\gamma$$

for any limit ordinal λ. However, on the level of order types we have

$$\mathrm{otyp}(T_\lambda) = \omega^\lambda + 1 > \omega^\lambda = \sup_{\gamma<\lambda}(\omega^\gamma + 1) = \sup_{\gamma<\lambda} \mathrm{otyp}(T_\gamma).$$

Thus the function $\alpha \mapsto T_\alpha$ is continuous in some sense, but not with respect to the order topology on the ordinals (cf. the discussion in [**28**, Section 0.2.7]). In other words, $\alpha \mapsto \mathrm{otyp}(T_\alpha)$ is not a normal function. As $\mathrm{otyp}(T_\alpha) = \omega^\alpha + 1 > \alpha$ holds for all α, there can be no fully order-preserving collapse of T_α into α. Nevertheless, we can define a Bachmann-Howard collapse $\vartheta : T_{\varepsilon_1} \xrightarrow{\mathrm{BH}} \varepsilon_1$ by setting

$$\begin{aligned}
&\vartheta(\beta) = \beta + 1 \quad \text{for } \beta < \omega^{\varepsilon_1} = \varepsilon_1,\\
&\vartheta(\Omega) = \varepsilon_0.
\end{aligned}$$

To see that condition (i) of Definition 2.1.3 is satisfied, note that the side condition $|\beta|_{T_{\varepsilon_1}} < \vartheta(\Omega) = \varepsilon_0$ implies $\beta < \omega^{|\beta|_{T_{\varepsilon_1}}+1} < \varepsilon_0$, and thus $\vartheta(\beta) = \beta + 1 < \varepsilon_0 = \vartheta(\Omega)$ as required. The inequalities $\varepsilon_0 <_{T_{\varepsilon_1}} \Omega$ and $\vartheta(\varepsilon_0) > \vartheta(\Omega)$ are compatible, because the side condition is violated by $|\varepsilon_0|_{T_{\varepsilon_1}} = \varepsilon_0 \not< \vartheta(\Omega)$. As for condition (ii), in view of $\beta < \omega^{\beta+1}$ we do have $|\beta|_{T_{\varepsilon_1}} \leq \beta < \vartheta(\beta)$. To hint at the strength of the abstract Bachmann-Howard principle, let us consider an arbitrary Bachmann-Howard collapse $\vartheta : T_\alpha \xrightarrow{\mathrm{BH}} \alpha$ (still for $T_\alpha = \omega^\alpha \cup \{\Omega\}$). We show that α and $\vartheta(\Omega)$ must be ε-numbers. First, observe that $\vartheta \restriction \omega^\alpha$ is order-preserving: Given ordinals $\beta < \gamma < \omega^\alpha$, condition (ii) of Definition 2.1.3 yields $|\beta|_{T_\alpha} \leq |\gamma|_{T_\alpha} < \vartheta(\gamma)$. Together with $\beta <_{T_\alpha} \gamma$, we infer $\vartheta(\beta) < \vartheta(\gamma)$ by condition (i). As for any order-preserving function, we inductively get $\gamma \leq \vartheta(\gamma)$ for all $\gamma < \omega^\alpha$. In view of $\operatorname{rng}(\vartheta) \subseteq \alpha$ this forces $\omega^\alpha = \alpha$. Next, we show that $\vartheta(\Omega)$ must be a limit ordinal: Condition (ii) ensures $0 = |\Omega|_{T_\alpha} < \vartheta(\Omega)$. Given $\beta < \vartheta(\Omega)$, observe that $\beta + 1 < \omega^{\beta+1}$ implies $|\beta+1|_{T_\alpha} \leq \beta < \vartheta(\Omega)$. Together with $\beta + 1 <_{T_\alpha} \Omega$ we get $\beta + 1 \leq \vartheta(\beta+1) < \vartheta(\Omega)$, by condition (i). To conclude that $\vartheta(\Omega)$ is an ε-number it remains to show that $\beta < \vartheta(\Omega)$ implies $\omega^\beta < \vartheta(\Omega)$. Indeed, the assumption ensures $|\omega^\beta|_{T_\alpha} = \beta < \vartheta(\Omega)$. Together with $\omega^\beta <_{T_\alpha} \Omega$ condition (i) yields $\omega^\beta \leq \vartheta(\omega^\beta) < \vartheta(\Omega)$, as desired. While $\vartheta(\Omega)$ has to be an ε-number, the values $\vartheta(\beta)$ for $\beta < \omega^\alpha$ are not very informative. The reason is that there are many order embeddings of $\omega^\alpha = \alpha$ into α, many of which qualify as a choice for $\vartheta \restriction \omega^\alpha$. On the other hand, there is no order embedding of $\omega^\alpha \cup \{\Omega\} \cong \omega^\alpha + 1$ into α.

The example shows that there may not be a fully order-preserving collapse of T_α into α. In contrast, we now prove that a Bachmann-Howard collapse does always exist. The following argument adapts the usual construction of the Bachmann-Howard ordinal (see e.g. [**72**, Lemma 1.1]). It is particularly easy to state in a strong base theory, relying on cardinality arguments. In Section 2.4 we will see that a similar proof goes through on the basis of an admissible set $\mathbb{A}$: The elements resp. sub-classes of $\mathbb{A}$ will play the role of the countable resp. uncountable sets.

REMARK 2.1.6. Working in **ZFC**, consider a proto-dilator T^u. In particular T^u is primitive recursive, say with hereditarily countable parameter u. It follows that T^u_α is countable for any $\alpha < \aleph_1$ (by the Gödel-Lévy Theorem this holds for any Σ_1-definable function, cf. the introduction of [**44**]). We want to define a Bachmann-Howard collapse $\vartheta : T^u_{\aleph_1} \to \aleph_1$ by recursion over the well-order $T^u_{\aleph_1}$. Assuming that the values $\vartheta(\sigma)$ are already defined for $\sigma <_{T^u_{\aleph_1}} \tau$ we put

$$C(\tau,\alpha) := \{|\tau|_{T^u_{\aleph_1}}\} \cup \{\vartheta(\sigma)\,|\,\sigma <_{T^u_{\aleph_1}} \tau \text{ and } |\sigma|_{T^u_{\aleph_1}} < \alpha\} \subseteq \aleph_1$$

for all $\alpha < \aleph_1$. By the definition of ranks, $|\sigma|_{T^u_{\aleph_1}} < \alpha$ implies $\sigma \in T^u_\alpha$. As T^u_α is countable this means that $C(\tau,\alpha)$ is countable as well. So we can construct a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \aleph_1$ with $C(\tau,\alpha_k) \subseteq \alpha_{k+1}$ for each $k \in \omega$. Setting $\alpha := \sup_{k\in\omega} \alpha_k < \aleph_1$ one easily sees

$$C(\tau,\alpha) = \bigcup_{k\in\omega} C(\tau,\alpha_k) \subseteq \sup_{k\in\omega} \alpha_{k+1} = \alpha.$$

We can thus set

$$\vartheta(\tau) := \min\{\alpha < \aleph_1 \,|\, C(\tau,\alpha) \subseteq \alpha\},$$

completing the recursive definition of $\vartheta : T^u_{\aleph_1} \to \aleph_1$. It remains to verify that this is a Bachmann-Howard collapse: For $\sigma,\tau \in T^u_{\aleph_1}$ with $\sigma <_{T^u_{\aleph_1}} \tau$ and $|\sigma|_{T^u_{\aleph_1}} < \vartheta(\tau)$ we get $\vartheta(\sigma) \in C(\tau,\vartheta(\tau)) \subseteq \vartheta(\tau)$, i.e. $\vartheta(\sigma) < \vartheta(\tau)$, as required by condition (i) of Definition 2.1.3. Also, the construction gives $|\tau|_{T^u_{\aleph_1}} \in C(\tau,\vartheta(\tau)) \subseteq \vartheta(\tau)$, which is condition (ii).

The following property of Rathjen's notation system is preserved:

LEMMA 2.1.7. *Any Bachmann-Howard collapse* $\vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha$ *is injective.*

PROOF. Consider two different elements $\sigma,\tau \in T^u_\alpha$. As T^u_α is linearly ordered we may assume $\sigma <_{T^u_\alpha} \tau$. If we have $|\sigma|_{T^u_\alpha} < \vartheta(\tau)$ then condition (i) of Definition 2.1.3 yields $\vartheta(\sigma) < \vartheta(\tau)$. If we have $|\sigma|_{T^u_\alpha} \geq \vartheta(\tau)$ then condition (ii) implies $\vartheta(\tau) < \vartheta(\sigma)$. In both cases the values $\vartheta(\sigma)$ and $\vartheta(\tau)$ are different, as required. □

The next remark shows that axiom beta (see Definition 1.4.2) is encoded in our formulation of the abstract Bachmann-Howard principle, at least for linear orders. The predicative Bachmann-Howard principle of the next section will not share this somewhat dubious feature.

REMARK 2.1.8. Put $T^u_\alpha := u$, independently of α. If $u = (u, <_u)$ is a well-order then T^u is a proto-dilator. As we have $T^u_\alpha = T^u_0$ for all α, we see that all ranks are zero. By the abstract Bachmann-Howard principle, assume that there is a collapse $\vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha$ for some ordinal α. For $\sigma, \tau \in T^u_\alpha$ we have

$$|\sigma|_{T^u_\alpha} = 0 = |\tau|_{T^u_\alpha} < \vartheta(\tau),$$

using condition (ii) of Definition 2.1.3. Thus $\sigma <_{T^u_\alpha} \tau$ implies $\vartheta(\sigma) < \vartheta(\tau)$, by condition (i). In other words, ϑ is a fully order-preserving embedding of u into α. Write $\operatorname{rng}(\vartheta)$ for the range of ϑ. In Section 1.2 we have seen that the transitive collapse $c : \operatorname{rng}(\vartheta) \to \mathbb{V}$ with

$$c(\gamma) = \{c(\beta) \mid \beta \in \operatorname{rng}(\vartheta) \text{ and } \beta < \gamma\}$$

for all $\gamma \in \operatorname{rng}(\vartheta)$ is available in primitive recursive set theory. Then the function $f := c \circ \vartheta : u \to \mathbb{V}$ satisfies

$$f(\tau) = \{f(\sigma) \mid \sigma <_u \tau\}$$

for all $\tau \in u$, making it the collapse demanded by axiom beta.

In the functorial setting of the next section, it will be useful to replace ranks by finite sets of "supports". The formulation of the abstract Bachmann-Howard principle is more elegant in terms of ranks, as these are fully determined by the function $\alpha \mapsto T^u_\alpha$. We now show that a formulation in terms of supports is equivalent:

Definition 2.1.9. Consider primitive recursive functions

$$T : \mathbb{V} \times \mathrm{Ord} \to \mathbb{V}, \qquad (u, \alpha) \mapsto T^u_\alpha,$$
$$\operatorname{supp} : \mathbb{V} \times \mathrm{Ord} \to \mathbb{V}, \qquad (u, \alpha) \mapsto \operatorname{supp}^u_\alpha.$$

Fix a value of the parameter u and assume that T^u is a proto-dilator. We say that supp^u is a support for T^u if the following holds:

(i) For any ordinal α, the value $\operatorname{supp}^u_\alpha$ is a function from T^u_α to $[\alpha]^{<\omega}$ (the set of finite subsets of α, see Section 1.2).

(ii) For any $\sigma \in T^u_\beta$ and any $\alpha < \beta$ we have

$$\operatorname{supp}^u_\beta(\sigma) \subseteq \alpha \quad \Rightarrow \quad \sigma \in T^u_\alpha.$$

Note that the definition of proto-dilator ensures $T^u_\alpha \subseteq T^u_\beta$.

(iii) If $\alpha < \beta$ then $\operatorname{supp}^u_\alpha$ is the restriction of $\operatorname{supp}^u_\beta$ to T^u_α.

Observe that the condition $T^u_\lambda = \bigcup_{\gamma<\lambda} T^u_\gamma$ in the definition of proto-dilator is automatic if one has a support: Given $\sigma \in T^u_\lambda$, the finite set $\operatorname{supp}^u_\lambda(\sigma) \subseteq \lambda$ will be contained in some $\gamma < \lambda$, which then implies $\sigma \in T^u_\gamma$.

LEMMA 2.1.10. *Consider a proto-dilator T^u with support* supp^u. *For any function $\vartheta : T^u_\alpha \to \alpha$ the following are equivalent:*

(i) The function ϑ is a Bachmann-Howard collapse.

(ii) For all $\sigma, \tau \in T^u_\alpha$ we have $\vartheta(\sigma) > 0$, as well as
- $\sigma <_{T^u_\alpha} \tau \land \operatorname{supp}^u_\alpha(\sigma) \subseteq \vartheta(\tau) \Rightarrow \vartheta(\sigma) < \vartheta(\tau)$,
- $\operatorname{supp}^u_\alpha(\sigma) \subseteq \vartheta(\sigma)$.

PROOF. The crucial observation is

$$|\sigma|_{T^u_\alpha} = \max(\operatorname{supp}^u_\alpha(\sigma) \cup \{0\}).$$

For $\operatorname{supp}^u_\alpha(\sigma) = \emptyset \subseteq 0$ (in particular for $\alpha = 0$) the definition of support yields $\sigma \in T^u_0$ and thus $|\sigma|_{T^u_\alpha} = 0$. For $\operatorname{supp}^u_\alpha(\sigma) \neq \emptyset$ (which implies $\alpha > 0$) we get

$$|\sigma|_{T^u_\alpha} \leq \gamma \Leftrightarrow \sigma \in T^u_{\gamma+1} \Leftrightarrow \operatorname{supp}^u_\alpha(\sigma) = \operatorname{supp}^u_{\gamma+1}(\sigma) \subseteq \gamma + 1 \Leftrightarrow$$
$$\Leftrightarrow \max(\operatorname{supp}^u_\alpha(\sigma) \cup \{0\}) \leq \gamma.$$

As a consequence, $|\sigma|_{T^u_\alpha} < \vartheta(\tau)$ is equivalent to $\operatorname{supp}^u_\alpha \subseteq \vartheta(\tau)$, provided that we have $\vartheta(\tau) > 0$. Note that $\vartheta(\tau) > 0$ is automatic if ϑ is a Bachmann-Howard collapse, because of the condition $\vartheta(\tau) > |\tau|_{T^u_\alpha}$. Now the equivalence between (i) and (ii) is immediate. □

We have seen a relatively simple proto-dilator in Example 2.1.5. It admitted a Bachmann-Howard collapse for the ordinal ε_1, which falls dramatically short of the actual Bachmann-Howard ordinal. In the rest of this section we construct a much stronger proto-dilator $\varepsilon(T)^u$, relative to a given proto-dilator T^u. This prepares the predicative Bachmann-Howard principle of the next section, and ultimately the construction of admissible sets. Intuitively, the order $\varepsilon(T)^u_\alpha$ consists of ε-numbers $\mathfrak{E}_\sigma > \Omega \geq \alpha$ for all $\sigma \in T^u_\alpha$, and of the ordinals in between. Formally, we begin with the following preliminary term systems:

DEFINITION 2.1.11. Consider a primitive recursive function $(u, \alpha) \mapsto T^u_\alpha$. Given values of u and α, assume that $T^u_\alpha = (T^u_\alpha, <_{T^u_\alpha})$ is a linear order. Then $\varepsilon^0(T)^u_\alpha$ is defined as the following set of terms:

(i) We have terms $0, \Omega \in \varepsilon^0(T)^u_\alpha$.

(ii) For each ε-number $\gamma < \alpha$ we have a term $\mathfrak{e}_\gamma \in \varepsilon^0(T)^u_\alpha$.

(iii) For each element $\sigma \in T^u_\alpha$ we have a term $\mathfrak{E}_\sigma \in \varepsilon^0(T)^u_\alpha$.

(iv) If $t_0, \ldots, t_n$ are terms in $\varepsilon^0(T)^u_\alpha$ then so is the expression $\omega^{t_0} + \cdots + \omega^{t_n}$.

Note that this is an inductive definition with closure ordinal ω, similar to the construction of M-formulas in Section 1.3. Thus the function $(u,\alpha) \mapsto \varepsilon^0(T)^u_\alpha$ is primitive recursive. Also, the stages of the inductive definition yield a notion of height for $\varepsilon^0(T)^u_\alpha$-terms, as in the proof of Lemma 1.3.2. Recursion over this height can be used to define further functions, such as the primitive recursive family of length functions

$$L^u_\alpha : \varepsilon^0(T)^u_\alpha \to \omega$$

with

$$L^u_\alpha(0) = L^u_\alpha(\Omega) = L^u_\alpha(\mathfrak{e}_\gamma) = L^u_\alpha(\mathfrak{E}_\sigma) = 0,$$
$$L^u_\alpha(\omega^{t_0} + \cdots + \omega^{t_n}) = L^u_\alpha(t_0) + \cdots + L^u_\alpha(t_n) + 1.$$

Using these lengths we single out ordered sets $\varepsilon(T)^u_\alpha \subseteq \varepsilon^0(T)^u_\alpha$ of terms in "Cantor normal form":

Lemma 2.1.12. *There is a primitive recursive function*

$$(u,\alpha) \mapsto (\varepsilon(T)^u_\alpha, <_{\varepsilon(T)^u_\alpha})$$

such that $<_{\varepsilon(T)^u_\alpha}$ is a binary relation on $\varepsilon(T)^u_\alpha \subseteq \varepsilon^0(T)^u_\alpha$ and the following holds:

(i) We have $0, \Omega \in \varepsilon(T)^u_\alpha$, as well as $\mathfrak{e}_\gamma, \mathfrak{E}_\sigma \in \varepsilon(T)^u_\alpha$ for any ε-number $\gamma < \alpha$ resp. any element $\sigma \in T^u_\alpha$.

(ii) A term $\omega^{t_0} + \cdots + \omega^{t_n} \in \varepsilon^0(T)^u_\alpha$ lies in $\varepsilon(T)^u_\alpha$ if and only if we have $\{t_0, \ldots, t_n\} \subseteq \varepsilon(T)^u_\alpha$ and one of the following conditions is satisfied:
- *Either we have $n = 0$ and t_0 is not of the form $\Omega, \mathfrak{e}_\gamma$ or $\mathfrak{E}_\sigma$;*
- *or we have $n > 0$ and $t_n \leq_{\varepsilon(T)^u_\alpha} \cdots \leq_{\varepsilon(T)^u_\alpha} t_0$ (where $s \leq_{\varepsilon(T)^u_\alpha} t$ means $s <_{\varepsilon(T)^u_\alpha} t \vee s = t$, the latter denoting syntactic equality).*

For $s, t \in \varepsilon(T)^u_\alpha$ we have $s <_{\varepsilon(T)^u_\alpha} t$ if and only if one of the following holds:

(i) $s = 0$ and $t \neq 0$;

(ii) $s = \mathfrak{e}_\gamma$ and
- *either t is of the form $\Omega, \mathfrak{E}_\tau$ or $\mathfrak{e}_\delta$ with $\gamma < \delta$,*
- *or we have $t = \omega^{t_0} + \cdots + \omega^{t_n}$ with $s \leq_{\varepsilon(T)^u_\alpha} t_0$;*

(iii) $s = \Omega$ and
- *either t is of the form $\mathfrak{E}_\tau$,*
- *or we have $t = \omega^{t_0} + \cdots + \omega^{t_n}$ with $s \leq_{\varepsilon(T)^u_\alpha} t_0$;*

(iv) $s = \mathfrak{E}_\sigma$ and
- *either t is of the form $\mathfrak{E}_\tau$ with $\sigma <_{T^u_\alpha} \tau$,*
- *or we have $t = \omega^{t_0} + \cdots + \omega^{t_n}$ with $s \leq_{\varepsilon(T)^u_\alpha} t_0$;*

(v) $s=\omega^{s_0}+\cdots+\omega^{s_m}$ *and*
- *either* t *is of the form* $\Omega,\mathfrak{e}_\delta$ *or* $\mathfrak{E}_\tau$ *and* $s_0<_{\varepsilon(T)^u_\alpha}t$,
- *or we have* $t=\omega^{t_0}+\cdots+\omega^{t_n}$ *and one of the following holds:*
  - $m<n$ *and* $s_i=t_i$ *for all* $i\leq m$,
  - *there is a number* $j\leq\min\{m,n\}$ *with* $s_j<_{\varepsilon(T)^u_\alpha}t_j$ *and* $s_i=t_i$ *for all* $i<j$.

Furthermore, these conditions determine $\varepsilon(T)^u_\alpha$ *and* $<_{\varepsilon(T)^u_\alpha}$ *uniquely.*

PROOF. The idea is to decide $r\in\varepsilon(T)^u_\alpha$ and $s<_{\varepsilon(T)^u_\alpha}t$ by simultaneous recursion on $L^u_\alpha(r)$ resp. $L^u_\alpha(s)+L^u_\alpha(t)$. In fact, everything can be formulated in terms of the relation $<_{\varepsilon(T)^u_\alpha}$, as $r\in\varepsilon(T)^u_\alpha$ should be equivalent to $r=0\vee 0<_{\varepsilon(T)^u_\alpha}r$. We will describe $<_{\varepsilon(T)^u_\alpha}$ via its characteristic function χ^u_α. Formally, we define

$$\chi:\prod_{(u,\alpha)\in\mathbb{V}\times\mathrm{Ord}}\varepsilon^0(T)^u_\alpha\times\varepsilon^0(T)^u_\alpha\to\{0,1\},\qquad(u,\alpha,\langle s,t\rangle)\mapsto\chi^u_\alpha(s,t)$$

by the recursion principle from Proposition 1.2.10. For this purpose we introduce the "rank function"

$$R:\prod_{(u,\alpha)\in\mathbb{V}\times\mathrm{Ord}}\varepsilon^0(T)^u_\alpha\times\varepsilon^0(T)^u_\alpha\to\omega,$$
$$R(u,\alpha,\langle s,t\rangle):=L^u_\alpha(s)+L^u_\alpha(t)$$

and the "predecessor function"

$$P:\prod_{(u,\alpha)\in\mathbb{V}\times\mathrm{Ord}}\varepsilon^0(T)^u_\alpha\times\varepsilon^0(T)^u_\alpha\to\mathbb{V},$$
$$P(u,\alpha,\langle s,t\rangle):=\{\langle s',t'\rangle\in\varepsilon^0(T)^u_\alpha\times\varepsilon^0(T)^u_\alpha\,|\,L^u_\alpha(s')+L^u_\alpha(t')<L^u_\alpha(s)+L^u_\alpha(t)\}.$$

The latter is primitive recursive by Corollary 1.1.10. For $\langle s',t'\rangle\in P(u,\alpha,\langle s,t\rangle)$ we have $R(u,\alpha,\langle s',t'\rangle)<R(u,\alpha,\langle s,t\rangle)$ by definition. In this situation, Proposition 1.2.10 allows us to construct χ by recursion, appealing to $\chi^u_\alpha\restriction P(u,\alpha,\langle s,t\rangle)$ in order to define the value $\chi^u_\alpha(s,t)$. It is straightforward to implement the clauses from the lemma in this form. For example, $\omega^{t_0}+\omega^{t_1}\in\varepsilon(T)^u_\alpha$ should be equivalent to $\bigwedge_{i=0,1}t_i\in\varepsilon(T)^u_\alpha\wedge t_1\leq_{\varepsilon(T)^u_\alpha}t_0$, which translates into

$$\chi^u_\alpha(0,\omega^{t_0}+\omega^{t_1})=1\;\Leftrightarrow\;\bigwedge_{i=0,1}(t_i=0\vee\chi^u_\alpha(0,t_i)=1)\wedge(t_0=t_1\vee\chi^u_\alpha(t_1,t_0)=1).$$

This equivalence qualifies as a recursive clause for χ, because we have

$$L^u_\alpha(0)+L^u_\alpha(t_i)\leq L^u_\alpha(t_1)+L^u_\alpha(t_0)<L^u_\alpha(0)+L^u_\alpha(\omega^{t_0}+\omega^{t_1}).$$

As a second example, $\Omega <_{\varepsilon(T)^u_\alpha} \omega^{t_0}+\omega^{t_1}$ should be equivalent to the statement $\omega^{t_0}+\omega^{t_1}\in\varepsilon(T)^u_\alpha\wedge\Omega\leq_{\varepsilon(T)^u_\alpha} t_0$, which becomes

$$\chi^u_\alpha(\Omega,\omega^{t_0}+\omega^{t_1})=1\;\Leftrightarrow\;\chi^u_\alpha(0,\omega^{t_0}+\omega^{t_1})\wedge(\Omega=t_0\vee\chi^u_\alpha(\Omega,t_0)=1)\;\Leftrightarrow$$
$$\Leftrightarrow\;\bigwedge_{i=0,1}(t_i=0\vee\chi^u_\alpha(0,t_i)=1)\wedge(t_0=t_1\vee\chi^u_\alpha(t_1,t_0)=1)\wedge(\Omega=t_0\vee\chi^u_\alpha(\Omega,t_0)=1).$$

The second equivalence qualifies as a recursive clause for χ. Note that we cannot take the first equivalence, because of $\langle 0,\omega^{t_0}+\omega^{t_1}\rangle\notin P(u,\alpha,\langle\Omega,\omega^{t_0}+\omega^{t_1}\rangle)$. Having constructed χ, we set

$$\varepsilon(T)^u_\alpha:=\{r\in\varepsilon^0(T)^u_\alpha\,|\,r=0\vee\chi^u_\alpha(0,r)=1\},$$
$$<_{\varepsilon(T)^u_\alpha}:=\{\langle s,t\rangle\in\varepsilon^0(T)^u_\alpha\times\varepsilon^0(T)^u_\alpha\,|\,\chi^u_\alpha(s,t)=1\}.$$

Again, Corollary 1.1.10 tells us that the maps $(u,\alpha)\mapsto\varepsilon(T)^u_\alpha$ and $(u,\alpha)\mapsto <_{\varepsilon(T)^u_\alpha}$ are primitive recursive. The conditions from the lemma hold by construction. To establish uniqueness, consider a competitor $(\varepsilon^*(T)^u_\alpha,<_{\varepsilon^*(T)^u_\alpha})$ which satisfies the same clauses. By induction on n one verifies, simultaneously, that we have

$$r\in\varepsilon^*(T)^u_\alpha\;\Leftrightarrow\;r\in\varepsilon(T)^u_\alpha,$$
$$s<_{\varepsilon^*(T)^u_\alpha}t\;\Leftrightarrow\;s<_{\varepsilon(T)^u_\alpha}t$$

for all $r,s,t\in\varepsilon^0(T)^u_\alpha$ with $L^u_\alpha(r)\leq n$ resp. $L^u_\alpha(s)+L^u_\alpha(t)\leq n$. □

It is standard to establish the following (cf. [**79**, Theorem 14.2]):

LEMMA 2.1.13. *If $(T^u_\alpha,<_{T^u_\alpha})$ is a linear order then so is $(\varepsilon(T)^u_\alpha,<_{\varepsilon(T)^u_\alpha})$.*

PROOF. First, refute $s<_{\varepsilon(T)^u_\alpha}s$ by induction on $L^u_\alpha(s)$. Next, use induction on $L^u_\alpha(r)+L^u_\alpha(s)+L^u_\alpha(t)$ to show that $r<_{\varepsilon(T)^u_\alpha}s$ and $s<_{\varepsilon(T)^u_\alpha}t$ imply $r<_{\varepsilon(T)^u_\alpha}t$. Finally, establish $s<_{\varepsilon(T)^u_\alpha}t\vee s=t\vee t<_{\varepsilon(T)^u_\alpha}s$ by induction on $L^u_\alpha(s)+L^u_\alpha(t)$. □

The next result is more subtle, but also familiar from proof theory:

PROPOSITION 2.1.14. *If $(T^u_\alpha,<_{T^u_\alpha})$ is a well-order then so is $(\varepsilon(T)^u_\alpha,<_{\varepsilon(T)^u_\alpha})$.*

We will use a syntactic construction of Gentzen to prove the result in primitive recursive set theory. Before, let us point out a simple semantic argument, which relies on axiom beta: The latter provides an embedding $c:T^u_\alpha\to\operatorname{Ord}$. Note that the enumerating function $\gamma\mapsto\varepsilon_\gamma$ of the class of ε-numbers is primitive recursive. It is straightforward to extend the assignment $\mathfrak{e}_\gamma\mapsto\gamma$, $\Omega\mapsto\varepsilon_\alpha$, $\mathfrak{E}_\sigma\mapsto\varepsilon_{\alpha+1+c(\sigma)}$ to an embedding of $\varepsilon(T)^u_\alpha$ into the ordinals.

PROOF. Consider the set

$$X = \{\mathfrak{e}_\gamma \mid \gamma \text{ an } \varepsilon\text{-number below } \alpha\} \cup \{\Omega\} \cup \{\mathfrak{E}_\sigma \mid \sigma \in T^u_\alpha\},$$

and let $<_X$ be the restriction of $<_{\varepsilon(T)^u_\alpha}$ to X. Clearly $(X, <_X)$ is a well-order. Now $(\varepsilon(T)^u_\alpha, <_{\varepsilon(T)^u_\alpha})$ is the order $(\varepsilon_X, <_{\varepsilon_X})$ considered by Afshari and Rathjen in [**4**] (except that they work in second-order arithmetic, where X is represented by a subset of ω; but this is not essential). According to [**4**, Section 5] the well-foundedness proof for $(\varepsilon_X, <_{\varepsilon_X})$, as presented in [**79**, Section 21], goes through in the theory $\mathbf{ACA_0^+}$ of second-order arithmetic, which is considerably weaker than primitive recursive set theory. For the reader's convenience we reproduce the main steps of the argument: First, observe that well-foundedness is the contrapositive of transfinite induction, i.e. it suffices to show

$$\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(z) \to \varepsilon(T)^u_\alpha \subseteq z,$$

where we abbreviate

$$\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(z) \equiv \forall_{t \in \varepsilon(T)^u_\alpha}(\forall_{s \in \varepsilon(T)^u_\alpha}(s <_{\varepsilon(T)^u_\alpha} t \to s \in z) \to t \in z).$$

In the following, quantifiers with bound variables r, s, t should always be read as restricted to $\varepsilon(T)^u_\alpha$; also, we sometimes write $<$ at the place of $<_{\varepsilon(T)^u_\alpha}$. To proceed, one defines counterparts of ordinal addition and exponentiation to the base ω on the term system $\varepsilon(T)^u_\alpha$, and proves basic properties (cf. [**79**, Section 14]). Iterated exponentiation is defined by $\omega^t_0 = t$ and $\omega^t_{n+1} = \omega^{\omega^t_n}$. It is straightforward to see that any term $s \in \varepsilon(T)^u_\alpha$ is $<_{\varepsilon(T)^u_\alpha}$-smaller than a term of the form ω^{x+1}_n, with $x \in X$, $n \in \omega$ and $1 := \omega^0$. Thus it is enough to show

$$\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(z) \to \forall_{s < \omega^{x+1}_n}\, s \in z$$

for all $x \in X$ and $n \in \omega$. We would like to argue by induction over $(x, n) \in X \times \omega$ with the alphabetical ordering. As X is well-ordered, this is possible in $\mathbf{PRS}\omega$, provided that the induction statement is primitive recursive (as the set of counter-examples can then be formed by Lemma 1.1.5). For fixed z, the above statement is indeed primitive recursive (cf. Proposition 1.1.9), but in the induction step we will have to vary z. To allow some variation while keeping the statement primitive recursive, we use Gentzen's jump operator

$$J(0, z) := z,$$
$$J(k+1, z) := \{r \in \varepsilon(T)^u_\alpha \mid \forall_t(\forall_{s<t}\, s \in J(k, z) \to \forall_{s < t + \omega^r}\, s \in J(k, z))\}.$$

Since all quantifiers are restricted to $\varepsilon(T)^u_\alpha$ the function J is primitive recursive. A crucial property of these jumps is the implication

$$\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k,z)) \to \mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k+1,z)).$$

To establish $\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k+1,z))$, consider $r \in \varepsilon(T)^u_\alpha$ with $\forall_{r_0<r}\, r_0 \in J(k+1,z)$. We have to deduce $r \in J(k+1,z)$, or equivalently

$$\forall_t(\forall_{s<t}\, s \in J(k,z) \to \forall_{s<t+\omega^r}\, s \in J(k,z)).$$

For $r = 0$ this is immediate by $\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k,z))$. Otherwise any $s < t + \omega^r$ is smaller than $t + \omega^{r_0} \cdot m$, for some $r_0 < r$ and $m \in \omega$. The primitive recursive statement $\forall_{s<t+\omega^{r_0}\cdot m}\, s \in J(k,z)$ can be shown by induction on m, using the assumption $r_0 \in J(k+1,z)$ for the induction step. After these preparations, we can use induction over $(x,n) \in (\{0\} \cup X) \times \omega$ to prove the primitive recursive statement

$$\forall_{k\in\omega}(\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k,z)) \to \forall_{s<\omega^{x+1}_n}\, s \in J(k,z)).$$

Note that the open claim above is the special case $k = 0$. First consider the induction step for $n = 0$, which means $\omega^{x+1}_n = x+1$. The case $x = 0$ is immediate. For $x > 0$, any $s < x$ is smaller than some term $\omega^{x_0+1}_m$ with $\{0\} \cup X \ni x_0 < x$. As we have $(x_0, m) < (x, 0)$ in the alphabetical ordering, we get $\forall_{s<x}\, s \in J(k,z)$ by induction hypothesis. Using $\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k,z))$ we obtain $\forall_{s<x+1}\, s \in J(k,z)$, as desired. Now consider the induction step for a pair of the form $(x, n+1)$. Assuming that we have $\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k,z))$ and thus $\mathrm{Prog}_{<_{\varepsilon(T)^u_\alpha}}(J(k+1,z))$, the induction hypotesis for (x,n) gives $\forall_{s<\omega^{x+1}_n}\, s \in J(k+1,z)$, and then $\omega^{x+1}_n \in J(k+1,z)$. The definition of $J(k+1,z)$, with $t = 0$, yields $\forall_{s<\omega^{x+1}_{n+1}}\, s \in J(k,z)$, as required. ☐

In the previous results, we have looked at the orders $\varepsilon(T)^u_\alpha = (\varepsilon(T)^u_\alpha, <_{\varepsilon(T)^u_\alpha})$ individually, i.e. for a fixed value of α. Let us now establish compatibility:

Proposition 2.1.15. *If $\alpha \mapsto T^u_\alpha$ is a proto-dilator then so is $\alpha \mapsto \varepsilon(T)^u_\alpha$.*

Proof. Condition (i) of Definition 2.1.1 holds by the previous proposition. To check condition (ii), consider arbitrary ordinals $\alpha < \beta$ and set

$$\varepsilon^*(T)^u_\alpha := \varepsilon(T)^u_\beta \cap \varepsilon^0(T)^u_\alpha,$$
$$<_{\varepsilon^*(T)^u_\alpha} := <_{\varepsilon(T)^u_\beta} \cap (\varepsilon^*(T)^u_\alpha \times \varepsilon^*(T)^u_\alpha).$$

It is straightforward to see that $\varepsilon^*(T)^u_\alpha$ and $<_{\varepsilon^*(T)^u_\alpha}$ satisfy the defining conditions for $\varepsilon(T)^u_\alpha$ resp. $<_{\varepsilon(T)^u_\alpha}$, as given in Lemma 2.1.12. Thus the uniqueness result in that lemma gives

$$\varepsilon(T)^u_\alpha = \varepsilon^*(T)^u_\alpha \subseteq \varepsilon(T)^u_\beta,$$
$$<_{\varepsilon(T)^u_\alpha} = <_{\varepsilon^*(T)^u_\alpha} = <_{\varepsilon(T)^u_\beta} \cap (\varepsilon(T)^u_\alpha \times \varepsilon(T)^u_\alpha),$$

as required by condition (ii) of Definition 2.1.1. Concerning condition (iii), assume that λ is a limit. By induction on terms $s \in \varepsilon(T)^u_\lambda$ we see that there is a $\gamma < \lambda$ with $s \in \varepsilon^0(T)^u_\gamma$. Together with the above we obtain $s \in \varepsilon(T)^u_\lambda \cap \varepsilon^0(T)^u_\gamma = \varepsilon(T)^u_\gamma$, as required. □

By Definition 2.1.2, the proto-dilator $\varepsilon(T)^u$ induces rank functions

$$|\cdot|_{\varepsilon(T)^u_\alpha} : \varepsilon(T)^u_\alpha \to \max\{\alpha, 1\}.$$

It will be helpful to have an explicit characterization:

LEMMA 2.1.16. *The rank functions $|\cdot|_{\varepsilon(T)^u_\alpha}$ of the proto-dilator $\varepsilon(T)^u$ satisfy*

$$|0|_{\varepsilon(T)^u_\alpha} = 0, \qquad |\Omega|_{\varepsilon(T)^u_\alpha} = 0, \qquad |\mathfrak{e}_\gamma|_{\varepsilon(T)^u_\alpha} = \gamma,$$
$$|\mathfrak{E}_\sigma|_{\varepsilon(T)^u_\alpha} = |\sigma|_{T^u_\alpha}, \qquad |\omega^{s_0} + \cdots + \omega^{s_n}|_{\varepsilon(T)^u_\alpha} = \max_{i \leq n} |s_i|_{\varepsilon(T)^u_\alpha}.$$

PROOF. Straightforward from Definition 2.1.2. □

The notation system $\varepsilon(T)^u_\alpha$ contains three different kinds of "ε-numbers", denoted by terms of the form $\mathfrak{e}_\gamma$, Ω and $\mathfrak{E}_\sigma$, respectively. So far, they have all played the same role. This changes with the following observation:

LEMMA 2.1.17. *Write α^+ for the smallest ε-number bigger than or equal to α. We can construct order isomorphisms*

$$i_\alpha : \alpha^+ \to \varepsilon(T)^u_\alpha \cap \Omega := \{s \in \varepsilon(T)^u_\alpha \,|\, s <_{\varepsilon(T)^u_\alpha} \Omega\}.$$

PROOF. By recursion over the ordinals below α^+ we can define i_α as a function with values in $\varepsilon^0(T)^u_\alpha \supseteq \varepsilon(T)^u_\alpha \cap \Omega$:

$$i_\alpha(0) = 0,$$
$$i_\alpha(\beta) = \mathfrak{e}_\beta \qquad \text{if } \beta < \alpha^+ \text{ is an } \varepsilon\text{-number (thus } \beta < \alpha),$$
$$i_\alpha(\beta) = \omega^{i_\alpha(\beta_0)} + \cdots + \omega^{i_\alpha(\beta_n)} \qquad \text{if } \beta = \omega^{\beta_0} + \cdots + \omega^{\beta_n} \text{ with } \beta > \beta_0 \geq \cdots \geq \beta_n.$$

In the last clause, $\omega^{\beta_0} + \cdots + \omega^{\beta_n}$ refers to exponentiation and addition on the actual ordinals; the expression $\omega^{i_\alpha(\beta_0)} + \cdots + \omega^{i_\alpha(\beta_n)}$ is a term in $\varepsilon^0(T)^u_\alpha$. We transfer the length function $L^u_\alpha:\varepsilon^0(T)^u_\alpha\to\omega$ to α^+, by setting $L:=L^u_\alpha\circ i_\alpha$. It is straightforward to verify

$$\beta<\alpha^+\ \Rightarrow\ i_\alpha(\beta)\in\varepsilon(T)^u_\alpha,$$
$$\gamma<\delta<\alpha^+\ \Rightarrow\ i_\alpha(\gamma)<_{\varepsilon(T)^u_\alpha} i_\alpha(\delta)$$

by simultaneous induction on $L(\beta)$ resp. $L(\gamma)+L(\delta)$. As $<$ and $<_{\varepsilon(T)^u_\alpha}$ are linear orders, this implies that $i_\alpha:\alpha^+\to\varepsilon(T)^u_\alpha$ is an order embedding. It is easy to check $i_\alpha(\beta)<_{\varepsilon(T)^u_\alpha}\Omega$ by induction on β. By induction on $s\in\varepsilon(T)^u_\alpha\cap\Omega$ we show that s lies in the range of i_α: For $s=0$ and $s=\mathfrak{e}_\gamma$ this is immediate. The cases $s=\Omega$ and $s=\mathfrak{E}_\sigma$ are excluded by $s<_{\varepsilon(T)^u_\alpha}\Omega$. Finally, consider a term of the form $s=\omega^{s_0}+\dots+\omega^{s_n}$. The induction hypothesis provides ordinals $\beta_i<\alpha^+$ with $s_i=i_\alpha(\beta_i)$. Set $\beta:=\omega^{\beta_0}+\dots+\omega^{\beta_n}<\alpha^+$. Since i_α is an order embedding we get $\beta_n\leq\dots\leq\beta_0$. In case $n=0$, the definition of $\varepsilon(T)^u_\alpha$ ensures that s_0 is not of the form $\mathfrak{e}_\gamma$. This means that β_0 cannot be an ε-number, so that we have $\beta>\beta_0$. Thus we see $i_\alpha(\beta)=s$, as desired. □

The following was foreshadowed in Example 2.1.5:

LEMMA 2.1.18. *Assume that $\alpha\mapsto T^u_\alpha$ and thus $\alpha\mapsto\varepsilon(T)^u_\alpha$ is a proto-dilator. For any Bachmann-Howard collapse $\vartheta:\varepsilon(T)^u_\alpha\xrightarrow{\text{BH}}\alpha$ we have the following:*

(i) If $\beta<\gamma<\alpha^+$ then $\vartheta(i_\alpha(\beta))<\vartheta(i_\alpha(\gamma))$.
(ii) If $\gamma<\alpha^+$ then $\gamma\leq\vartheta(i_\alpha(\gamma))$.
(iii) We have $\alpha^+=\alpha$, which means that α is an ε-number.

PROOF. (i) Using Lemma 2.1.16, it is straightforward to show

$$|i_\alpha(\delta)|_{\varepsilon(T)^u_\alpha}=\min\{\rho\,|\,\delta<(\rho+1)^+\}$$

by induction on $\delta<\alpha^+$. Thus $\beta<\gamma$ implies $|i_\alpha(\beta)|_{\varepsilon(T)^u_\alpha}\leq|i_\alpha(\gamma)|_{\varepsilon(T)^u_\alpha}$. Together with Definition 2.1.3(ii) we obtain $|i_\alpha(\beta)|_{\varepsilon(T)^u_\alpha}<\vartheta(i_\alpha(\gamma))$. As i_α is an order embedding we also have $i_\alpha(\beta)<_{\varepsilon(T)^u_\alpha}i_\alpha(\gamma)$. Then Definition 2.1.3(i) yields $\vartheta(i_\alpha(\beta))<\vartheta(i_\alpha(\gamma))$, as desired.
(ii) Using (i), we inductively get

$$\vartheta(i_\alpha(\gamma))\geq\sup\{\vartheta(i_\alpha(\beta))+1\,|\,\beta<\gamma\}\geq\sup\{\beta+1\,|\,\beta<\gamma\}=\gamma.$$

(iii) In view of $\operatorname{rng}(\vartheta)\subseteq\alpha$ we have $\beta\leq\vartheta(i_\alpha(\beta))<\alpha$ for any $\beta<\alpha^+$. □

The following property of the usual Bachmann-Howard construction can also be recovered, at least for terms above Ω:

LEMMA 2.1.19. *Assume that* $\vartheta : \varepsilon(T)^u_\alpha \xrightarrow{\text{BH}} \alpha$ *is a Bachmann-Howard collapse. Then* $\vartheta(t)$ *is an* ε*-number for any* $t \in \varepsilon(T)^u_\alpha$ *with* $\Omega \leq_{\varepsilon(T)^u_\alpha} t$.

PROOF. First, we show that $\vartheta(t)$ is a limit: By condition (ii) of Definition 2.1.3 we have $0 \leq |t|_{\varepsilon(T)^u_\alpha} < \vartheta(t)$. Given $\beta < \vartheta(t)$ we observe

$$|i_\alpha(\beta+1)|_{\varepsilon(T)^u_\alpha} = \min\{\rho \,|\, \beta+1 < (\rho+1)^+\} \leq \beta < \vartheta(t).$$

Together with $i_\alpha(\beta+1) <_{\varepsilon(T)^u_\alpha} \Omega \leq_{\varepsilon(T)^u_\alpha} t$ we get $\vartheta(i_\alpha(\beta+1)) < \vartheta(t)$, by Definition 2.1.3(ii). Part (ii) of the previous lemma yields $\beta+1 < \vartheta(t)$, making $\vartheta(t)$ a limit. To deduce that $\vartheta(t)$ is an ε-number we assume $\beta < \vartheta(t)$ and show $\omega^\beta < \vartheta(t)$. Crucially, we have

$$|i_\alpha(\omega^\beta)|_{\varepsilon(T)^u_\alpha} = \min\{\rho \,|\, \omega^\beta < (\rho+1)^+\} \leq \beta < \vartheta(t).$$

Together with $i_\alpha(\omega^\beta) <_{\varepsilon(T)^u_\alpha} t$ this does indeed yield $\omega^\beta \leq \vartheta(i_\alpha(\omega^\beta)) < \vartheta(t)$. □

As indicated in Example 2.1.5, we cannot expect much from the values $\vartheta(t)$ for $t <_{\varepsilon(T)^u_\alpha} \Omega$. To repair this, we will instead consider the value $\vartheta(\Omega + t)$.

DEFINITION 2.1.20. The map $\varepsilon(T)^u_\alpha \ni t \mapsto \Omega + t \in \varepsilon(T)^u_\alpha$ is given by the following clauses:

$$\Omega + 0 := \Omega, \quad \Omega + \mathfrak{e}_\gamma := \omega^\Omega + \omega^{\mathfrak{e}_\gamma}, \quad \Omega + \Omega := \omega^\Omega + \omega^\Omega, \quad \Omega + \mathfrak{E}_\sigma := \mathfrak{E}_\sigma,$$

$$\Omega + (\omega^{s_0} + \cdots + \omega^{s_n}) := \begin{cases} \omega^\Omega + \omega^{s_0} + \cdots + \omega^{s_n} & \text{if } s_0 \leq_{\varepsilon(T)^u_\alpha} \Omega, \\ \omega^{s_0} + \cdots + \omega^{s_n} & \text{if } \Omega <_{\varepsilon(T)^u_\alpha} s_0. \end{cases}$$

Let us list the required properties of this operation:

LEMMA 2.1.21. *The following holds for all* $s, t \in \varepsilon(T)^u_\alpha$*:*

(i) We have $\Omega \leq_{\varepsilon(T)^u_\alpha} \Omega + s$.
(ii) If $s <_{\varepsilon(T)^u_\alpha} t$ *then* $\Omega + s <_{\varepsilon(T)^u_\alpha} \Omega + t$.
(iii) We have $|\Omega + s|_{\varepsilon(T)^u_\alpha} = |s|_{\varepsilon(T)^u_\alpha}$.

PROOF. For all possible forms of s and t the claims are straightforward by definition. As an example for (ii), let us consider $s = \mathfrak{e}_\gamma <_{\varepsilon(T)^u_\alpha} \omega^{t_0} + \cdots + \omega^{t_n} = t$ with $\mathfrak{e}_\gamma \leq_{\varepsilon(T)^u_\alpha} t_0$. In case $t_0 = \mathfrak{e}_\gamma <_{\varepsilon(T)^u_\alpha} \Omega$ we must have $n > 0$, and thus

$$\Omega + s = \omega^\Omega + \omega^{\mathfrak{e}_\gamma} <_{\varepsilon(T)^u_\alpha} \omega^\Omega + \omega^{\mathfrak{e}_\gamma} + \omega^{t_1} + \cdots + \omega^{t_n} = \Omega + t.$$

For $\mathfrak{e}_\gamma <_{\varepsilon(T)^u_\alpha} t_0 \leq \Omega$ we have

$$\Omega + s = \omega^\Omega + \omega^{\mathfrak{e}_\gamma} <_{\varepsilon(T)^u_\alpha} \omega^\Omega + \omega^{t_0} + \cdots + \omega^{t_n} = \Omega + t.$$

For $\Omega <_{\varepsilon(T)^u_\alpha} t_0$ we obtain

$$\Omega + s = \omega^{\Omega} + \omega^{\mathfrak{e}_\gamma} <_{\varepsilon(T)^u_\alpha} \omega^{t_0} + \cdots + \omega^{t_n} = \Omega + t.$$

Part (iii) relies on the characterization of ranks in Lemma 2.1.16. In particular, recall $|\Omega|_{\varepsilon(T)^u_\alpha} = 0$. □

Together with Lemma 2.1.19, it follows that $\vartheta(\Omega + s)$ is an ε-number for all terms $s \in \varepsilon(T)^u_\alpha$. The following notation will help to recover the usual Bachmann-Howard construction:

Definition 2.1.22. Given a Bachmann-Howard collapse $\vartheta : \varepsilon(T)^u_\alpha \xrightarrow{\text{BH}} \alpha$, we define the function

$$\bar{\vartheta} : \varepsilon(T)^u_\alpha \to \varepsilon(T)^u_\alpha \cap \Omega, \qquad \bar{\vartheta}(s) := i_\alpha \circ \vartheta(\Omega + s) = \mathfrak{e}_{\vartheta(\Omega+s)}.$$

For $s \in \varepsilon(T)^u_\alpha$, we abbreviate

$$s^* := i_\alpha(|s|_{\varepsilon(T)^u_\alpha}) \in \varepsilon(T)^u_\alpha \cap \Omega.$$

The properties of a Bachmann-Howard collapse can now be stated as follows:

Proposition 2.1.23. *Assume that T^u and thus $\varepsilon(T)^u$ is a proto-dilator, and that $\vartheta : \varepsilon(T)^u_\alpha \xrightarrow{\text{BH}} \alpha$ is a Bachmann-Howard collapse. For $s, t \in \varepsilon(T)^u_\alpha$ we have*

$$\bar{\vartheta}(s) <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t) \Leftrightarrow \begin{cases} \text{either } s <_{\varepsilon(T)^u_\alpha} t \text{ and } s^* <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t), \\ \text{or } t <_{\varepsilon(T)^u_\alpha} s \text{ and } \bar{\vartheta}(s) \leq_{\varepsilon(T)^u_\alpha} t^*. \end{cases}$$

Proof. We begin with "$\Leftarrow$": First assume $s <_{\varepsilon(T)^u_\alpha} t$ and $s^* <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t)$. As i_α is an embedding, the latter implies $|s|_{\varepsilon(T)^u_\alpha} < \vartheta(\Omega + t)$. Using Lemma 2.1.21(iii) we get $|\Omega + s|_{\varepsilon(T)^u_\alpha} < \vartheta(\Omega + t)$. Also, Lemma 2.1.21(ii) yields $\Omega + s <_{\varepsilon(T)^u_\alpha} \Omega + t$. Condition (i) of Definition 2.1.3 gives $\vartheta(\Omega+s) < \vartheta(\Omega+t)$, and then $\bar{\vartheta}(s) <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t)$, as desired. Now assume $\bar{\vartheta}(s) \leq_{\varepsilon(T)^u_\alpha} t^*$ (the condition $t <_{\varepsilon(T)^u_\alpha} s$ is redundant, but we keep it for symmetry). Using condition (ii) of Definition 2.1.3 we see

$$|t|_{\varepsilon(T)^u_\alpha} = |\Omega + t|_{\varepsilon(T)^u_\alpha} < \vartheta(\Omega + t).$$

Applying i_α to both sides gives $t^* <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t)$, and then $\bar{\vartheta}(s) <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t)$. To show "$\Rightarrow$", we may assume $s \neq t$. Aiming at the contrapositive, assume that the right side of the equivalence fails. Then we have either $t <_{\varepsilon(T)^u_\alpha} s$ and $t^* <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(s)$; or we have $s <_{\varepsilon(T)^u_\alpha} t$ and $\bar{\vartheta}(t) \leq_{\varepsilon(T)^u_\alpha} s^*$. In both cases direction "$\Leftarrow$" (with s and t interchanged) yields $\bar{\vartheta}(t) <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(s)$. The latter implies $\bar{\vartheta}(s) \not<_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t)$, completing the proof of the contrapositive. □

We can now embed the usual notation system for the Bachmann-Howard ordinal. This will play no official role, but similar constructions will be crucial in Sections 2.3 and 2.4.

REMARK 2.1.24. Consider the constant proto-dilator $\gamma \mapsto T_\gamma^X = X$ for a well-order $X = (X, <_X)$ (cf. Remark 2.1.8). Recall that we have $|\sigma|_{T_\gamma^X} = 0$ for any $\sigma \in X = T_\gamma^X$. Form the associated proto-dilator $\gamma \mapsto \varepsilon(T)_\gamma^X$ and assume that $\vartheta : \varepsilon(T)_\alpha^X \xrightarrow{\text{BH}} \alpha$ is a Bachmann-Howard collapse. I claim that the relativized Bachmann-Howard order ϑ_X of Rathjen and Valencia Vizcaíno [**71**, Definition 2.6] can be embedded into $\varepsilon(T)_\alpha^X$: The term $\mathfrak{E}_\sigma \in \vartheta_X$ with $\sigma \in X = T_\alpha^X$ is identified with $\mathfrak{E}_\sigma \in \varepsilon(T)_\alpha^X$. Note that our rank

$$\mathfrak{E}_\sigma{}^* = i_\alpha(|\mathfrak{E}_\sigma|_{\varepsilon(T)_\alpha^X}) = i_\alpha(|\sigma|_{T_\alpha^X}) = i_\alpha(0) = 0$$

coincides with Rathjen and Valencia Vizcaíno's rank assignment $\mathfrak{E}_\sigma{}^* = 0$. Assuming that $s \in \vartheta_X$ is identified with $s \in \varepsilon(T)_\alpha^X$, we can identify $\vartheta s \in \vartheta_X$ with the element $\bar{\vartheta}(s) \in \varepsilon(T)_\alpha^X$. Again, our rank assignment

$$\bar{\vartheta}(s)^* = \mathfrak{e}_{\vartheta(\Omega+s)}{}^* = i_\alpha(|\mathfrak{e}_{\vartheta(\Omega+s)}|_{\varepsilon(T)_\alpha^X}) = i_\alpha(\vartheta(\Omega + s)) = \mathfrak{e}_{\vartheta(\Omega+s)} = \bar{\vartheta}(s)$$

is as required. Also, $\bar{\vartheta}(s) = \mathfrak{e}_{\vartheta(\Omega+s)} \in \varepsilon(T)_\alpha^X$ behaves like an ε-number below Ω, just as $\vartheta s \in \vartheta_X$. Using the previous proposition, it is straightforward to see that this yields an order embedding of ϑ_X into $\varepsilon(T)_\alpha^X$. Thus the well-foundedness of $\varepsilon(T)_\alpha^X$ (see Proposition 2.1.14) implies that ϑ_X is a well-order. In other words, the abstract Bachmann-Howard principle implies that $X \mapsto \vartheta_X$ is a type-one well-ordering principle, over primitive recursive set theory. Also, the given embedding restricts to an embedding of $\vartheta_X \cap \Omega$ into $\varepsilon(T)_\alpha^X \cap \Omega \cong \alpha$ (the isomorphism comes from Lemma 2.1.17, as Lemma 2.1.18 gives $\alpha = \alpha^+$). For $X = \emptyset$ the structure $\vartheta_X \cap \Omega$ is the usual notation system for the Bachmann-Howard ordinal. This shows that any ordinal α which admits a Bachmann-Howard collapse $\vartheta : \varepsilon(T)_\alpha^X \xrightarrow{\text{BH}} \alpha$ is at least as big as the Bachmann-Howard ordinal. From the assumption that $X \mapsto \vartheta_X$ is a type-one well-ordering principle, Rathjen and Valencia Vizcaíno deduce that there are ω-models of bar induction. Recall that we want to construct transitive set models of Kripke-Platek set theory. Thus we have reached the correct proof-theoretic strength: The theory of bar induction and Kripke-Platek set theory both correspond to the Bachmann-Howard ordinal. On the other hand, transitive set models are stronger than ω-models. For this reason we have constructed a type-two well-ordering principle: The point is that we do not only get a collapse of the "constant" order ϑ_X into some ordinal $\alpha \cong \vartheta_X \cap \Omega$, but rather a "fixed-point" $\alpha \cong \varepsilon(T)^u_\alpha \cap \Omega$ with collapsing structure $\vartheta : \varepsilon(T)^X_\alpha \xrightarrow{\text{BH}} \alpha$.

The following is also worth pointing out:

REMARK 2.1.25. There is considerable freedom in the choice of $\varepsilon(T)^u_\alpha$. For example, we could strengthen $\varepsilon(T)^u_\alpha$ by adding a term ε_s for each $s \in \varepsilon(T)^u_\alpha$; the terms ε_σ for $\sigma \in T^u_\alpha$ would then we replaced by terms $\varphi_2(\sigma)$, where φ_2 refers to the second branch of the Veblen function, which enumerates the fixed-points of $\alpha \mapsto \varepsilon_\alpha$. Note that Proposition 2.1.14 for this stronger system would still be provable in our base theory **PRS**ω. Conversely, it may be possible to weaken $\varepsilon(T)^u_\alpha$: As $\in$-induction is automatic in standard models of set theory (see Lemma 1.3.12), we only need to consider Kripke-Platek axioms of bounded complexity. Thus we may not need full ordinal exponentiation in our proof-theoretic arguments. Observe that there is a similar degree of freedom in the formulation of Theorem 4.4.6(ii): The statement "any set is contained in an admissible set" is, of course, equivalent to "any set is contained in an admissible set that is itself contained in an admissible set" — even though the corresponding theories **KP** and **KP** + "there is an admissible set" have different proof-theoretic strength. So for our purpose, the precise order-type of $\varepsilon(T)^u_\alpha$ seems less important than the collapsing structure. The given definition of $\varepsilon(T)^u_\alpha$ has the advantage that it relates to the familiar notation systems ϑ_X (from [**71**], cf. the previous remark) and ε_X (from [**52**] resp. [**4**]).

In the rest of this section we reformulate Proposition 2.1.23 in terms of supports, rather than ranks (cf. Definition 2.1.9 and Lemma 2.1.10). This formulation will be more suitable for the functorial approach in the next sections.

DEFINITION 2.1.26. Consider primitive recursive functions $(u, \alpha) \mapsto T^u_\alpha$ and $(u, \alpha) \mapsto \operatorname{supp}^u_\alpha$. For a fixed value of u, assume that T^u is a proto-dilator and that supp^u is a support for T^u. Define functions

$$E^u_\alpha : \varepsilon(T)^u_\alpha \to [\alpha]^{<\omega}$$

by the following recursion over terms:

$$E^u_\alpha(0) = \emptyset, \qquad E^u_\alpha(\Omega) = \emptyset, \qquad E^u_\alpha(\mathfrak{e}_\gamma) = \{\gamma\},$$
$$E^u_\alpha(\mathfrak{E}_\sigma) = \operatorname{supp}^u_\alpha(\sigma), \qquad E^u_\alpha(\omega^{s_0} + \cdots + \omega^{s_n}) = \bigcup_{i \leq n} E^u_\alpha(s_i).$$

For $s \in \varepsilon(T)^u_\alpha$ we write $\bar{E}^u_\alpha(s) := \{i_\alpha(\gamma) \mid \gamma \in E^u_\alpha(s)\} \subseteq \varepsilon(T)^u_\alpha \cap \Omega$.

Note that the function $(u, \alpha, s) \mapsto E^u_\alpha(s)$ is primitive recursive (just as the function $(u, \alpha, s) \mapsto L^u_\alpha(s)$, cf. the discussion after Definition 2.1.11). Proposition 1.2.2 tells us that $(u, \alpha) \mapsto E^u_\alpha$ is primitive recursive as well (and that E^u_α is set-sized).

Lemma 2.1.27. *If* supp^u *is a support for* T^u *then* E^u *is a support for* $\varepsilon(T)^u$*.*

Proof. Condition (i) of Definition 2.1.9 is immediate. To establish condition (ii), consider ordinals $\alpha < \beta$. The required implication

$$E^u_\beta(s) \subseteq \alpha \quad \Rightarrow \quad s \in \varepsilon(T)^u_\alpha$$

can be verified by induction on $s \in \varepsilon(T)^u_\beta$. The most interesting case is $s = \mathfrak{E}_\sigma$. By assumption we have $\operatorname{supp}^u_\beta(\sigma) = E^u_\beta(\mathfrak{E}_\sigma) \subseteq \alpha$. As supp^u is a support for T^u this implies $\sigma \in T^u_\alpha$, and then $\mathfrak{E}_\sigma \in \varepsilon(T)^u_\alpha$. For condition (iii) we also consider $\alpha < \beta$. We need $E^u_\alpha(s) = E^u_\beta(s)$ for $s \in \varepsilon(T)^u_\alpha$, which is also shown by induction on s. In the crucial case $s = \mathfrak{E}_\sigma$ with $\sigma \in T^u_\alpha$ we have

$$E^u_\alpha(\mathfrak{E}_\sigma) = \operatorname{supp}^u_\alpha(\sigma) = \operatorname{supp}^u_\beta(\sigma) = E^u_\beta(s),$$

as supp^u is a support for T^u. □

We can now reformulate Proposition 2.1.23 as promised:

Corollary 2.1.28. *Assume that* T^u *is a proto-dilator with support* supp^u*, so that* $\varepsilon(T)^u$ *is a proto-dilator with support* E^u*. Consider a Bachmann-Howard collapse* $\vartheta : \varepsilon(T)^u_\alpha \xrightarrow{\text{BH}} \alpha$ *and the associated function* $\bar{\vartheta} : \varepsilon(T)^u_\alpha \to \varepsilon(T)^u_\alpha \cap \Omega$*. Then, for all* $s, t \in \varepsilon(T)^u_\alpha$*, we have*

$$\bar{\vartheta}(s) <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t) \;\Leftrightarrow\; \begin{cases} \text{either } s <_{\varepsilon(T)^u_\alpha} t \text{ and } r <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t) \text{ for all } r \in \bar{E}^u_\alpha(s), \\ \text{or } t <_{\varepsilon(T)^u_\alpha} s \text{ and } \bar{\vartheta}(s) \leq_{\varepsilon(T)^u_\alpha} r \text{ for some } r \in \bar{E}^u_\alpha(t). \end{cases}$$

Proof. By the proof of Lemma 2.1.10 we have

$$|s|_{\varepsilon(T)^u_\alpha} = \max(E^u_\alpha(s) \cup \{0\}).$$

Applying the embedding i_α to both sides yields

$$s^* = \max\nolimits_{<_{\varepsilon(T)^u_\alpha}}(\bar{E}^u_\alpha(s) \cup \{0\}).$$

As the term $\bar{\vartheta}(t) = \mathfrak{e}_{\vartheta(\Omega+t)}$ is different from zero, this implies

$$s^* <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t) \quad \Leftrightarrow \quad r <_{\varepsilon(T)^u_\alpha} \bar{\vartheta}(t) \text{ for all } r \in \bar{E}^u_\alpha(s).$$

Similarly, we have

$$\bar{\vartheta}(s) \leq_{\varepsilon(T)^u_\alpha} t^* \quad \Leftrightarrow \quad \bar{\vartheta}(s) \leq_{\varepsilon(T)^u_\alpha} r \text{ for some } r \in \bar{E}^u_\alpha(t).$$

Then the claim follows from Proposition 2.1.23. □

Finally, it will be useful to have an explicit description of $\bar{E}^u_\alpha$:

LEMMA 2.1.29. *Consider a support* supp^u *for* T^u. *The associated function*

$$\bar{E}^u_\alpha : \varepsilon(T)^u_\alpha \to [\varepsilon(T)^u_\alpha \cap \Omega]^{<\omega}$$

satisfies

$$\bar{E}^u_\alpha(0) = \emptyset, \qquad \bar{E}^u_\alpha(\Omega) = \emptyset, \qquad \bar{E}^u_\alpha(\bar{\vartheta}(s)) = \{\bar{\vartheta}(s)\},$$
$$\bar{E}^u_\alpha(\mathfrak{E}_\sigma) = [i_\alpha]^{<\omega} \circ \mathrm{supp}^u_\alpha(\sigma), \qquad \bar{E}^u_\alpha(\omega^{s_0} + \cdots + \omega^{s_n}) = \bigcup_{i \leq n} \bar{E}^u_\alpha(s_i),$$

where $[i_\alpha]^{<\omega} \circ \mathrm{supp}^u_\alpha(\sigma)$ *abbreviates* $\{i_\alpha(\gamma) \mid \gamma \in \mathrm{supp}^u_\alpha(\sigma)\}$ *(this will coincide with a general definition of* $[f]^{<\omega}$ *in the next section).*

PROOF. We have

$$E^u_\alpha(\bar{\vartheta}(s)) = E^u_\alpha(\mathfrak{e}_{\vartheta(\Omega+s)}) = \{\vartheta(\Omega + s)\}$$

and thus

$$\bar{E}^u_\alpha(\bar{\vartheta}(s)) = \{i_\alpha(\gamma) \mid \gamma \in E^u_\alpha(\bar{\vartheta}(s))\} = \{i_\alpha(\vartheta(\Omega + s))\} = \{\mathfrak{e}_{\vartheta(\Omega+s)}\} = \{\bar{\vartheta}(s)\}.$$

The other cases are straightforward. □

2.2. A Predicative Bachmann-Howard Principle

As explained in the introduction of the thesis, the predicative Bachmann-Howard principle improves on its abstract counterpart in a certain respect: It underpins the transformation of a type-one well-ordering principle T into a well-order $\mathrm{BH}(T)$ by an explicit construction. For an informal explanation of $\mathrm{BH}(T)$ we again refer to the introduction. The present section presents the construction of $\mathrm{BH}(T)$ in detail.

Recall that the base theory of this thesis is primitive recursive set theory, as presented in Chapter 1. Let us discuss some notions of category theory in this context: Our base theory does not include the existence of power sets. Equivalently, it does not prove that the functions between two given sets form a set. Thus the category of sets, and many related categories, are not recognized as locally small. Nevertheless, we will see that many basic constructions are available. A functor will be given by two primitive recursive functions, acting on objects resp. morphisms. As an important example, recall the primitive recursive function

$$x \mapsto [x]^{<\omega} = \text{“the set of finite subsets of } x\text{”}$$

constructed in Section 1.2. To turn this into an endo-functor of the category of sets, consider a (set-sized) function $f : x \to y$ and an element $a \in [x]^{<\omega}$, and put

$$[f]^{<\omega}(a) = \{f(\sigma)\,|\,\sigma \in a\} \in [y]^{<\omega}.$$

The function $(f, \sigma) \mapsto f(\sigma)$ is primitive recursive by Section 1.2. From Proposition 1.1.6 we learn that $(f, a) \mapsto [f]^{<\omega}(a)$ is primitive recursive as well. Then Proposition 1.2.2 tells us that the restriction $[f]^{<\omega}(\cdot) \restriction z$ to a set z exists as a set, and that $(f, z) \mapsto [f]^{<\omega}(\cdot) \restriction z$ is primitive recursive. Taking $z = [x]^{<\omega}$ we finally see that the transformation of $f : x \to y$ into $[f]^{<\omega} : [x]^{<\omega} \to [y]^{<\omega}$ is a primitive recursive set function. Functoriality is easy to check. Thus the functions $[\cdot]^{<\omega}$ on objects resp. morphisms do indeed form a (primitive recursive) functor. Quantification over (class-sized) functors is not directly available. As in the previous section, we can use Currying to quantify over parametrized families: Consider primitive recursive functions $(u, x) \mapsto T^u_x$ and $(u, f) \mapsto T^u_f$. The statement “T^u is a functor” can then be expressed by a $\mathcal{L}_{\text{pr}}$-formula with free variable u. Quantification over the family $(T^u)_{u\in\mathbb{V}}$ can be implemented as quantification over the set-sized parameter u. Natural transformations between (primitive recursive) functors S and T can be handled in a similar way: Given a primitive recursive function η which transforms x and $\sigma \in S_x$ into $\eta_x(\sigma) \in T_x$, Proposition 1.2.2 yields a primitive recursive function $x \mapsto \eta_x$, where $\eta_x : S_x \to T_x$ is again a set-sized function. Apart from the category of sets, we are most interested in the category of linear orders, with order embeddings as morphisms. The forgetful functor from linear orders to sets will be left implicit. Conversely, a subset $y \subseteq x$ of an order $(x, <_x)$ can be seen as a suborder.

To construct $\text{BH}(T)$, we will need the type-one well-ordering principle T to be particularly uniform. As it turns out, the required properties lead precisely to Girard’s [**28**] notion of dilator; note that a similar notion was already considered by Aczel [**2, 3**].

Definition 2.2.1. A prae-dilator consists of

(i) an endo-functor T^u on the category of linear orders (where the actions $X \mapsto T^u_X = (T^u_X, <_{T^u_X})$ and $f \mapsto T^u_f$ on objects resp. morphisms are primitive recursive with parameter u), and
(ii) a natural transformation $\operatorname{supp}^u : T^u \Rightarrow [\cdot]^{<\omega}$ (given by a primitive recursive function $X \mapsto \operatorname{supp}^u_X : T^u_X \to [X]^{<\omega}$ with parameter u) which computes supports, in the sense that any $\sigma \in T^u_X$ lies in the range of $T^u_{\iota_\sigma} : T^u_{\operatorname{supp}^u_X(\sigma)} \to T^u_X$, where $\iota_\sigma : \operatorname{supp}^u_X(\sigma) \hookrightarrow X$ is the inclusion.

A dilator is a prae-dilator which preserves well-foundedness, i.e. T^u_X must be a well-order whenever X is.

In fact, the given definition is half-way between Girard's dilators and his systems of ordinal denotations. In the following we observe that the different notions of dilator are equivalent (but our prae-dilators are not quite equivalent to Girard's pre-dilators, hence the difference in spelling). As this equivalence does not play an official role in the present thesis, we will not worry about formalization in $\mathbf{PRS}\omega$.

Remark 2.2.2. Similar to [**28**, Section 0.1.2], we verify that (prae-)dilators preserve direct limits and pull-backs: Consider a direct system $(X_i, f_{ij})_{i,j\in I}$ of linear orders X_i and embeddings $f_{ij} : X_i \to X_j$. Assume that $(X, f_i)_{i\in I}$ with $f_i : X_i \to X$ is a direct limit. In the case of linear orders, the universal property amounts to

$$X = \bigcup_{i\in I} \operatorname{rng}(f_i),$$

as pointed out in [**28**, Example 1.3.6]. Assume that T and supp form a prae-dilator in the sense of Definition 2.2.1. To show that T preserves our direct limit, we must establish

$$T_X = \bigcup_{i\in I} \operatorname{rng}(T_{f_i}).$$

As T_{f_i} is an embedding of T_{X_i} into T_X, the inclusion $\supseteq$ is immediate. Concerning the other inclusion, consider an element $\sigma \in T_X$. Our definition of prae-dilator yields a $\sigma_0 \in T_{\operatorname{supp}_X(\sigma)}$ with $\sigma = T_{\iota_\sigma}(\sigma_0)$. As $\operatorname{supp}_X(\sigma) \subseteq X$ is finite, we get $\operatorname{supp}_X(\sigma) \subseteq \operatorname{rng}(f_i)$ for some $i \in I$. Then the inclusion $\iota_\sigma : \operatorname{supp}_X(\sigma) \hookrightarrow X$ factors as $\iota_\sigma = f_i \circ g$, with an embedding $g : \operatorname{supp}_X(\sigma) \to X_i$. So $\sigma = T_{\iota_\sigma}(\sigma_0) = T_{f_i}(T_g(\sigma_0))$ lies in the range of T_{f_i}, as desired. Next, consider order embeddings $f_i : X_i \to X$ for $i = 0, 1$. According to [**28**, Theorem 1.5.5], the conditions for $g : Y \to X$ to be a pull-back amount to

$$\operatorname{rng}(g) = \operatorname{rng}(f_0) \cap \operatorname{rng}(f_1).$$

To see that the prae-dilator (T, supp) preserves pull-backs we must thus show

$$\operatorname{rng}(T_g) = \operatorname{rng}(T_{f_0}) \cap \operatorname{rng}(T_{f_1}).$$

From $\operatorname{rng}(g) \subseteq \operatorname{rng}(f_i)$ we learn that g factors as $g = f_i \circ h_i$, with $h_i : Y \to X_i$. For an arbitrary element $T_g(\sigma)$ of $\operatorname{rng}(T_g)$ we then have $T_g(\sigma) = T_{f_i}(T_{h_i}(\sigma)) \in \operatorname{rng}(T_{f_i})$, as required for the inclusion $\subseteq$. Conversely, for any element $\tau = T_{f_i}(\sigma_i)$ of $\operatorname{rng}(T_{f_i})$ we have $\operatorname{supp}_X(\tau) = \operatorname{supp}_X(T_{f_i}(\sigma_i)) = [f_i]^{<\omega}(\operatorname{supp}_{X_i}(\sigma_i)) \subseteq \operatorname{rng}(f_i)$, using the naturality of supp. If this holds for $i = 0, 1$ we get $\operatorname{supp}_X(\tau) \subseteq \operatorname{rng}(g)$. Then the inclusion $\iota_\tau : \operatorname{supp}_X(\tau) \to X$ factors as $\iota_\tau = g \circ h$ with $h : \operatorname{supp}_X(\tau) \to Y$. By the definition of prae-dilator there is a $\tau_0 \in T_{\operatorname{supp}_X(\tau)}$ with $\tau = T_{\iota_\tau}(\tau_0)$. We thus get $\tau = T_g(T_h(\tau_0)) \in \operatorname{rng}(T_g)$, as required for the inclusion $\supseteq$. We have established that any dilator in the sense of Definition 2.2.1 preserves direct limits and pull-backs, and is thus a dilator in the sense of Girard [**28**, Definition 2.3.1]. Note that Girard [**28**, Definition 4.4.1] does also have a notion of pre-dilator: It involves a certain monotonicity condition which is automatic in the well-founded case, i.e. for dilators. We have not included this monotonicity condition in our definition of prae-dilator, as it will not be relevant for us. Thus there are prae-dilators in the sense of Definition 2.2.1 which are not pre-dilators in the sense of Girard (note the different spelling). The converse direction resembles Girard's "normal form theorem" [**28**, Theorem 2.3.12]: Consider an endo-functor T of linear orders which preserves direct limits and pull-backs. We want to show that this induces a natural family of support functions $\operatorname{supp}_X : T_X \to [X]^{<\omega}$. If such a family exists, then $\operatorname{supp}_X(\sigma)$ is uniquely determined as the smallest set $a \in [X]^{<\omega}$ such that we have $\sigma \in \operatorname{rng}(T_{\iota_a})$, where $\iota_a : a \hookrightarrow X$ denotes the inclusion. Indeed, $\sigma = T_{\iota_a}(\sigma_0)$ implies

$$\operatorname{supp}_X(\sigma) = \operatorname{supp}_X(T_{\iota_a}(\sigma_0)) = [\iota_a]^{<\omega}(\operatorname{supp}_a(\sigma_0)) \subseteq a,$$

using naturality. Guided by this observation, write $X = \bigcup_{a \in [X]^{<\omega}} \operatorname{rng}(\iota_a)$ as a direct limit. Since T preserves direct limits we obtain

$$T_X = \bigcup_{a \in [X]^{<\omega}} \operatorname{rng}(T_{\iota_a}).$$

Consider an element $\sigma \in \operatorname{rng}(T_{\iota_a}) \cap \operatorname{rng}(T_{\iota_{a'}})$, for some $a, a' \in [X]^{<\omega}$. Note that $\iota_{a \cap a'}$ is the pull-back of ι_a and $\iota_{a'}$. As T preserves pull-backs we get $\sigma \in \operatorname{rng}(T_{\iota_{a \cap a'}})$. Thus we can indeed define $\operatorname{supp}_X(\sigma)$ as the unique minimal $a \in [X]^{<\omega}$ with $\sigma \in \operatorname{rng}(T_{\iota_a})$. To see that this yields a natural transformation we must establish

$$\operatorname{supp}_Y(T_f(\sigma)) = [f]^{<\omega}(\operatorname{supp}_X(\sigma)) =: b,$$

for any embedding $f : X \to Y$ and any $\sigma \in T_X$. It suffices to verify that b is minimal with $T_f(\sigma) \in \operatorname{rng}(T_{\iota_b})$. For $a := \operatorname{supp}_X(\sigma)$ we have $\sigma \in \operatorname{rng}(T_{\iota_a})$, say $\sigma = T_{\iota_a}(\sigma_0)$. This yields

$$T_f(\sigma) = T_f(T_{\iota_a}(\sigma_0)) = T_{\iota_b}(T_{f\restriction a}(\sigma_0)) \in \operatorname{rng}(T_{\iota_b}).$$

Aiming at minimality, assume $T_f(\sigma) = T_{\iota_{b_0}}(\sigma_0)$ for some $b_0 \subseteq b$ and $\sigma_0 \in T_{b_0}$. Consider the set $a_0 := \{x \in a \,|\, f(x) \in b_0\}$. As $f \restriction a_0 : a_0 \to b_0$ is surjective (recall that we have $b = [f]^{<\omega}(a)$), we can form the inverse embedding $(f \restriction a_0)^{-1} : b_0 \to a_0$. Then we get

$$T_f(\sigma) = T_{\iota_{b_0}}(\sigma_0) = T_{\iota_{b_0} \circ f\restriction a_0 \circ (f\restriction a_0)^{-1}}(\sigma_0) = T_f(T_{\iota_{a_0} \circ (f\restriction a_0)^{-1}}(\sigma_0)),$$

and thus $\sigma = T_{\iota_{a_0}}(T_{(f\restriction a_0)^{-1}}(\sigma_0)) \in \operatorname{rng}(T_{\iota_{a_0}})$. By the minimality of $a = \operatorname{supp}_X(\sigma)$ this implies $a_0 = a$ and then $b_0 = b$, as required. We have seen that each (pre-)dilator T in the sense of Girard induces a unique (prae-)dilator in the sense of Definition 2.2.1, over the same functor T. Even though the supports $\operatorname{supp}_X(\sigma)$ are uniquely determined by T (provided that T preserves direct limits and pull-backs), it is very helpful to make them explicit, at least for our purpose.

Formalizing Definition 2.2.1 yields a Π_1-definition of prae-dilators. The notion of dilator is expressed by a Π_2-formula but becomes Π_1 in the presence of axiom beta (which turns well-foundedness into a Δ_1-property). Recall that Π_1-formulas of the set theory $\mathbf{ATR_0^{set}}$ correspond to Π^1_2-formulas of second-order arithmetic (see [**87**, Theorem VII.3.24]). In Section 2.3 we will recover a result of Girard: Prae-dilators are determined by certain set-sized approximations, at least up to isomorphism. So essentially, being a prae-dilator becomes a primitive recursive property. Preservation of well-foundedness can be tested on the countable ordinals (see [**28**, Theorem 2.1.15]). However, the latter do not form a set in $\mathbf{PRS}\omega$ or $\mathbf{ATR_0^{set}}$, so that the notion of dilator remains of complexity Π^1_2. Also, the construction in Section 2.3 will yield a single primitive recursive family which comprises all prae-dilators. For the rest of this section, we simply fix some primitive recursive family of functions $X \mapsto T^u_X$ and $X \mapsto \operatorname{supp}^u_X$ with parameter u. We want to transform this into a primitive recursive function $u \mapsto \operatorname{BH}(T^u)$. If $(T^u, \operatorname{supp}^u)$, for some value of u, is a prae-dilator then $\operatorname{BH}(T^u)$ will be a linear order. As described in the introduction of the thesis, we construct $\mathrm{BH}(T^u)$ as the "ω-th iterate" of a certain transformation $X \rightsquigarrow \vartheta(T^u_X) \cap \Omega$. The following is a preliminary step:

DEFINITION 2.2.3. For each linear order X we define a set $\vartheta^0(T^u_X)$ of terms:

(i) The symbol 0 is a term in $\vartheta^0(T^u_X)$.
(ii) The symbol Ω is a term in $\vartheta^0(T^u_X)$.
(iii) For each $\sigma \in T^u_X$ the expression $\mathfrak{E}_\sigma$ is a term in $\vartheta^0(T^u_X)$.
(iv) If s is a term in $\vartheta^0(T^u_X)$ then so is the expression ϑs.
(v) If $s_0, \ldots, s_n$ are terms in $\vartheta^0(T^u_X)$ then so is the expression $\omega^{s_0} + \cdots + \omega^{s_n}$.

Note that $(u, X) \mapsto \vartheta^0(T^u_X)$ is a primitive recursive function. Also, primitive recursion and induction over terms in $\vartheta^0(T^u_X)$ are available (cf. the discussion after Definition 2.1.11). We would like to single out a subset $\vartheta(T^u_X) \subseteq \vartheta^0(T^u_X)$ of terms in normal form. To recognize Cantor normal forms in (v), one must simultaneously define an order relation $<^\vartheta_X$ on the terms. Deciding the order between terms of the form ϑs does, in turn, rely on the definition of finite sets $E^\vartheta_X(s) \subseteq \vartheta(T_X)$. These sets come from the usual construction of the Bachmann-Howard ordinal (see [**72**, Section 1], as well as Corollary 2.1.28 above). Intuitively, $E^\vartheta_X(s)$ consists of "countable ordinals" on which s depends. In the present case, the crucial question is: What should $E^\vartheta_X(\mathfrak{E}_\sigma)$ be? The only choice that seems immediately available is $E^\vartheta_X(\mathfrak{E}_\sigma) = \emptyset$. However, this turns out to be a dead end:

REMARK 2.2.4. Our construction of $\vartheta(T^u_X)$ is inspired by Rathjen and Valencia Vizcaíno's relativized Bachmann-Howard construction in [**71**, Section 2]. As suggested above, they do indeed set $E^\vartheta_X(\mathfrak{E}_\sigma) = \emptyset$. In view of [**71**, Lemma 2.3] this makes $\vartheta\mathfrak{E}_\sigma <^\vartheta_X \vartheta\mathfrak{E}_\tau$ equivalent to $\sigma <_{T^u_X} \tau$. In other words, $\sigma \mapsto \vartheta\mathfrak{E}_\sigma$ becomes a fully order-preserving collapse of T^u_X into $\vartheta(T^u_X) \cap \Omega$. This is appropriate in Rathjen and Valencia Vizcaíno's case, but not in ours: The construction of admissible sets will rely on a certain fixed-point, namely a well-order $\mathrm{BH}(T^u)$ for which we have $\mathrm{BH}(T^u) \cong \vartheta(T^u_{\mathrm{BH}(T^u)}) \cap \Omega$. Together with the above, we would get an order-preserving collapse of $T^u_{\mathrm{BH}(T^u)}$ into $\mathrm{BH}(T^u)$. This is incompatible with the observation in Example 2.1.5: It can happen that, for any well-order X, the order-type of T^u_X is bigger than the order-type of X. In other words, if we set $E^\vartheta_X(\mathfrak{E}_\sigma) = \emptyset$ then we may still be able to construct a fixed-point $\mathrm{BH}(T^u)$, but we cannot expect it to be well-founded.

Ruminating possible definitions of $E^\vartheta_X(\mathfrak{E}_\sigma)$, we are lead to consider $\mathrm{supp}^u_X(\sigma)$, which is indeed a finite set on which $\mathfrak{E}_\sigma$ depends. Unfortunately, $\mathrm{supp}^u_X(\sigma)$ is a subset of X rather than $\vartheta^0(T^u_X)$. On the other hand, the fixed-point construction described in the introduction of the thesis did involve embeddings $X \hookrightarrow \vartheta(T^u_X)\cap\Omega$. This motivates the following notion:

Definition 2.2.5. Consider a linear order X and functions $E_X : X \to [X]^{<\omega}$, $L_X : X \to \omega$ and $i_X : X \to \vartheta^0(T^u_X)$. Define functions $E^\vartheta_X : \vartheta^0(T^u_X) \to [\vartheta^0(T^u_X)]^{<\omega}$ and $L^\vartheta_X : \vartheta^0(T^u_X) \to \omega$ by recursion over terms in $\vartheta^0(T^u_X)$, setting

(i) $E^\vartheta_X(0) = \emptyset$ and $L^\vartheta_X(0) = 0$,
(ii) $E^\vartheta_X(\Omega) = \emptyset$ and $L^\vartheta_X(\Omega) = 0$,
(iii) $E^\vartheta_X(\mathfrak{E}_\sigma) = [i_X]^{<\omega} \circ \operatorname{supp}^u_X(\sigma)$ and $L^\vartheta_X(\mathfrak{E}_\sigma) = \max\{L_X(x) \mid x \in \operatorname{supp}_X(\sigma)\}$,
(iv) $E^\vartheta_X(\vartheta s) = \{\vartheta s\}$ and $L^\vartheta_X(\vartheta s) = L^\vartheta_X(s) + 1$,
(v) $E^\vartheta_X(\omega^{s_0} + \cdots + \omega^{s_n}) = E^\vartheta_X(s_0) \cup \cdots \cup E^\vartheta_X(s_n)$ and
$L^\vartheta_X(\omega^{s_0} + \cdots + \omega^{s_n}) = L^\vartheta_X(s_0) + \cdots + L^\vartheta_X(s_n) + 1$.

If these functions are extensions of E_X and L_X, in the sense that we have

$$E^\vartheta_X \circ i_X = [i_X]^{<\omega} \circ E_X \qquad \text{and} \qquad L^\vartheta_X \circ i_X = L_X,$$

then $X = (X, E_X, L_X, i_X)$ is called a BH-system (for the prae-dilator $(T^u, \operatorname{supp}^u)$).

Note that the functions $(u, X) \mapsto E^\vartheta_X$ and $(u, X) \mapsto L^\vartheta_X$ are primitive recursive. In particular, the statement "X is a BH-system for $(T^u, \operatorname{supp}^u)$" is primitive recursive with parameter u. The function L^ϑ_X will control the recursive definition of a subset $\vartheta(T^u_X) \subseteq \vartheta^0(T^u_X)$. To play this role, it must have the following property:

Lemma 2.2.6. *Consider a* BH-*system* (X, E_X, L_X, i_X) *for* $(T^u, \operatorname{supp}^u)$. *For any* $s \in \vartheta^0(T^u_X)$ *and any* $r \in E^\vartheta_X(s)$ *we have* $L^\vartheta_X(r) \leq L^\vartheta_X(s)$.

Proof. One argues by induction on the build-up of s. The only interesting case is $s = \mathfrak{E}_\sigma$. By the definition of E^ϑ_X we have $r = i_X(x)$ for some $x \in \operatorname{supp}^u_X(\sigma)$. The definition of L^ϑ_X ensures $L_X(x) \leq L^\vartheta_X(s)$. To conclude we observe $L^\vartheta_X(r) = L_X(x)$, due to the condition $L^\vartheta_X \circ i_X = L_X$ in the definition of BH-system. □

As promised, we can now single out a set of terms in "normal form":

Lemma 2.2.7. *There is a primitive recursive function which transforms a value of u and a* BH-*system* (X, E_X, L_X, i_X) *for* $(T^u, \operatorname{supp}^u)$ *into a set* $\vartheta(T^u_X) \subseteq \vartheta^0(T^u_X)$ *and a binary relation* $<^\vartheta_X$ *on* $\vartheta(T^u_X)$, *satisfying the following properties:*

(i) We have $0, \Omega \in \vartheta(T^u_X)$, *as well as* $\mathfrak{E}_\sigma \in \vartheta(T^u_X)$ *for any* $\sigma \in T^u_X$.

(ii) *A term* $\vartheta s \in \vartheta^0(T_X^u)$ *lies in* $\vartheta(T_X^u)$ *if and only if* s *does.*

(iii) *We have* $\omega^{s_0} + \cdots + \omega^{s_n} \in \vartheta(T_X^u)$ *if and only if* $\{s_0, \ldots, s_n\} \subseteq \vartheta(T_X^u)$ *and*

- *either* $n = 0$ *and* s_0 *is not of the form* $\Omega, \mathfrak{E}_\sigma$ *or* ϑs,
- *or* $n > 0$ *and* $s_n \leq_X^\vartheta \cdots \leq_X^\vartheta s_0$ *(where* $s \leq_X^\vartheta t$ *means* $s <_X^\vartheta t \vee s = t$, *with the second disjunct referring to equality as terms).*

Given $s, t \in \vartheta(T_X^u)$, *we have* $s <_X^\vartheta t$ *if and only if one of the following holds:*

(i) $s = 0$ *and* $t \neq 0$,

(ii) $s = \Omega$ *and we have*

- *either* $t = \mathfrak{E}_\sigma$ *for some* $\sigma \in T_X^u$,
- *or* $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and* $s \leq_X^\vartheta t_0$,

(iii) $s = \mathfrak{E}_\sigma$ *for some* $\sigma \in T_X^u$ *and we have*

- *either* $t = \mathfrak{E}_\tau$ *for some* $\tau \in T_X^u$ *with* $\sigma <_{T_X^u} \tau$,
- *or* $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and* $s \leq_X^\vartheta t_0$,

(iv) $s = \vartheta s'$ *and one of the following holds:*

- $t = \vartheta t'$ *and we have*
  - *either* $s' <_X^\vartheta t'$ *and* $r <_X^\vartheta t$ *for all* $r \in E_X^\vartheta(s') \cap \vartheta(T_X^u)$,
  - *or* $t' <_X^\vartheta s'$ *and* $s \leq_X^\vartheta r$ *for some* $r \in E_X^\vartheta(t') \cap \vartheta(T_X^u)$,
- $t = \Omega$,
- $t = \mathfrak{E}_\tau$ *for some* $\tau \in T_X^u$,
- $t = \omega^{t_0} + \cdots + \omega^{t_n}$ *and* $s \leq_X^\vartheta t_0$,

(v) $s = \omega^{s_0} + \cdots + \omega^{s_n}$ *and one of the following holds:*

- $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and*
  - *either* $n < m$ *and* $s_i = t_i$ *for all* $i \leq n$,
  - *or there is a number* $j \leq \min\{m, n\}$ *with* $s_j <_X^\vartheta t_j$ *and* $s_i = t_i$ *for all* $i < j$,
- t *is of the form* Ω, $\mathfrak{E}_\tau$ *or* $\vartheta t'$ *and we have* $s_0 <_X^\vartheta t$.

Proof. The idea is to decide $s <_X^\vartheta t$ by recursion over $L_X^\vartheta(s) + L_X^\vartheta(t)$. In doing so, we read $r \in \vartheta(T_X^u)$ as an abbreviation for $r = 0 \vee 0 <_X^\vartheta r$. In other words, the defining clauses of $r \in \vartheta(T_X^u)$ are included in the guise of recursive conditions for $0 <_X^\vartheta r$. Formally, the described approach falls under the recursion principle from Proposition 1.2.10, as detailed in the proof of Lemma 2.1.12. To demonstrate how $s <_X^\vartheta t$ is decided recursively, let us look at the case of $s = \vartheta s'$ and $t = \vartheta t'$. In particular we must decide conditions $r \in \vartheta(T_X^u)$ and $r <_X^\vartheta t$ for $r \in E_X^\vartheta(s')$. Crucially, the previous lemma ensures $L_X^\vartheta(r) \leq L_X^\vartheta(s') < L_X^\vartheta(s)$. As this implies

$L^\vartheta_X(0) + L^\vartheta_X(r) \leq L^\vartheta_X(r) + L^\vartheta_X(t) < L^\vartheta_X(s) + L^\vartheta_X(t)$ our recursive procedure does indeed decide $r <^\vartheta_X t$ and $0 <^\vartheta_X r$, i.e. $r \in \vartheta(T^u_X)$. The other cases are similar. □

As suggested by the notation, we indeed have the following:

PROPOSITION 2.2.8. *Assume that* (X, E_X, L_X, i_X) *is a* BH-*system for the praedilator* $(T^u, \operatorname{supp}^u)$. *Then* $(\vartheta(T^u_X), <^\vartheta_X)$ *is a linear order.*

PROOF. First, by induction on $L^\vartheta_X(s)$ one refutes $s <^\vartheta_X s$. Next, by induction on $L^\vartheta_X(s) + L^\vartheta_X(t)$ one shows that one of the alternatives $s <^\vartheta_X t$, $s = t$ and $t <^\vartheta_X s$ must hold for any $s, t \in \vartheta(T^u_X)$. The only interesting case is $s = \vartheta s'$, $t = \vartheta t'$. If $s' = t'$ then $s = t$ and we are done. So, by the induction hypothesis and symmetry, we may assume $s' <^\vartheta_X t'$. For any $r \in E^\vartheta_X(s') \cap \vartheta(T^u_X)$ we have $L^\vartheta_X(r) \leq s'$. Thus the induction hypothesis yields $r <^\vartheta_X t$ or $t \leq_{\vartheta(T_X)} r$. If the former holds for all terms $r \in E^\vartheta_X(s') \cap \vartheta(T^u_X)$ then we have $s <^\vartheta_X t$. On the other hand, $t \leq_{\vartheta(T_X)} r$ for some $r \in E^\vartheta_X(s') \cap \vartheta(T^u_X)$ yields $t <^\vartheta_X s$. Finally, by induction on $L^\vartheta_X(r) + L^\vartheta_X(s) + L^\vartheta_X(t)$ one shows that $r <^\vartheta_X s$ and $s <^\vartheta_X t$ imply $r <^\vartheta_X t$. We only consider $r = \vartheta r'$, $s = \vartheta s'$, $t = \vartheta t'$: There are four possibilities to look at, the most interesting of which is

$$r' <^\vartheta_X s' \quad \text{and} \quad r'' <^\vartheta_X s \text{ for all } r'' \in E^\vartheta_X(r') \cap \vartheta(T^u_X),$$
$$t' <^\vartheta_X s' \quad \text{and} \quad s \leq^\vartheta_X t'' \text{ for some } t'' \in E^\vartheta_X(t') \cap \vartheta(T^u_X).$$

By trichotomy one of the following cases applies: *Case* $r' <^\vartheta_X t'$. For any term $r'' \in E^\vartheta_X(r') \cap \vartheta(T^u_X)$ we have $L^\vartheta_X(r'') \leq L^\vartheta_X(r') < L^\vartheta_X(r)$. From $r'' <^\vartheta_X s$ and $s <^\vartheta_X t$ we thus get $r'' <^\vartheta_X t$ by induction hypothesis. This yields $r <^\vartheta_X t$. *Case* $r' = t'$. We get a term $t'' \in E^\vartheta_X(t') \cap \vartheta(T^u_X) = E^\vartheta_X(r') \cap \vartheta(T^u_X)$ with $s \leq^\vartheta_X t''$ and $t'' <^\vartheta_X s$. As above we have $L^\vartheta_X(t'') < L^\vartheta_X(t)$ and $L^\vartheta_X(t'') < L^\vartheta_X(r)$, and the induction hypothesis yields $t'' <^\vartheta_X t''$. This contradicts our result above, which means that $r' = t'$ is not actually possible. *Case* $t' < r'$. Pick $t'' \in E^\vartheta_X(t') \cap \vartheta(T^u_X)$ with $s \leq^\vartheta_X t''$. By induction hypothesis we get $r <^\vartheta_X t''$, and then $r <^\vartheta_X t$. The other asymmetric case is easier: Assume that we have

$$s' <^\vartheta_X r' \quad \text{and} \quad r \leq^\vartheta_X s'' \text{ for some } s'' \in E^\vartheta_X(s') \cap \vartheta(T^u_X),$$
$$s' <^\vartheta_X t' \quad \text{and} \quad s'' <^\vartheta_X t \text{ for all } s'' \in E^\vartheta_X(s') \cap \vartheta(T^u_X).$$

Pick an $s'' \in E^\vartheta_X(s') \cap \vartheta(T^u_X)$ with $r \leq^\vartheta_X s''$. We also have $s'' <^\vartheta_X t$, so the induction hypothesis gives $r <^\vartheta_X t$. □

We are particularly interested in the sub-order

$$\vartheta(T^u_X)\cap\Omega:=\{s\in\vartheta(T^u_X)\,|\,s<^\vartheta_X\Omega\}.$$

The definition of BH-system simply asked for a function $i_X:X\to\vartheta^0(T^u_X)$. Now that we have structure on the co-domain, we do want i_X to respect it:

DEFINITION 2.2.9. A BH-system $X=(X,E_X,L_X,i_X)$ for $(T^u,\operatorname{supp}^u)$ is called good if the range of i_X is contained in $\vartheta(T^u_X)\cap\Omega$, and if

$$i_X:(X,<_X)\to(\vartheta(T^u_X)\cap\Omega,<^\vartheta_X)$$

is an order embedding.

As in the case of (plain) BH-systems, the statement "X is a good BH-system for $(T^u,\operatorname{supp}^u)$" is primitive recursive in X and u. To construct a good BH-system requires some amount of "divination": We must write down a function $i_X:X\to\vartheta(T^u_X)$ that becomes an embedding with respect to an order relation which i_X itself brings into being. Nevertheless, some initial good BH-systems are easy to write down. The following resembles Lemma 2.1.17:

LEMMA 2.2.10. *There is a good* BH-*system* $(\varepsilon_0,E_{\varepsilon_0},L_{\varepsilon_0},i_{\varepsilon_0})$ *over the ordinal* ε_0, *independently of the prae-dilator* $(T^u,\operatorname{supp}^u)$.

PROOF. We put $E_{\varepsilon_0}(\alpha)=\emptyset$ for all $\alpha<\varepsilon_0$. The functions $L_{\varepsilon_0}:\varepsilon_0\to\omega$ and $i_{\varepsilon_0}:\varepsilon_0\to\vartheta^0(T^u_{\varepsilon_0})$ are defined by recursion over Cantor normal forms: Set $L_{\varepsilon_0}(0)=0$ and $i_{\varepsilon_0}(0)=0$, as well as

$$L_{\varepsilon_0}(\omega^{\alpha_0}+\cdots+\omega^{\alpha_n})=L_{\varepsilon_0}(\alpha_0)+\cdots+L_{\varepsilon_0}(\alpha_n)+1,$$
$$i_{\varepsilon_0}(\omega^{\alpha_0}+\cdots+\omega^{\alpha_n})=\omega^{i_{\varepsilon_0}(\alpha_0)}+\cdots+\omega^{i_{\varepsilon_0}(\alpha_n)}$$

if $\varepsilon_0>\alpha_0\geq\cdots\geq\alpha_n$. By induction over (the Cantor normal form of) $\alpha<\varepsilon_0$ one verifies $E^\vartheta_{\varepsilon_0}\circ i_{\varepsilon_0}(\alpha)=\emptyset=[i_{\varepsilon_0}]^{<\omega}\circ E_{\varepsilon_0}(\alpha)$ and $L^\vartheta_{\varepsilon_0}\circ i_{\varepsilon_0}(\alpha)=L_{\varepsilon_0}(\alpha)$. This means that $(\varepsilon_0,E_{\varepsilon_0},L_{\varepsilon_0},i_{\varepsilon_0})$ is a BH-system. To prove that the latter is good, we must show $i_{\varepsilon_0}(\alpha)\in\vartheta(T^u_{\varepsilon_0})\cap\Omega$ and $\beta<\gamma\to i_{\varepsilon_0}(\beta)<^\vartheta_{\varepsilon_0}i_{\varepsilon_0}(\gamma)$. This is straightforward by simultaneous induction on $L_{\varepsilon_0}(\alpha)$ resp. $L_{\varepsilon_0}(\beta)+L_{\varepsilon_0}(\gamma)$. □

Next, we want to show that the structure of good BH-system carries over from X to $\vartheta(T^u_X)\cap\Omega$. The following result is a preparation. Note that it makes the intersection with $\vartheta(T^u_X)$ in clause (iv) of Lemma 2.2.7 redundant.

LEMMA 2.2.11. *Consider a good* BH-*system* (X,E_X,L_X,i_X). *For* $s\in\vartheta(T^u_X)$ *we have* $E^\vartheta_X(s)\subseteq\vartheta(T^u_X)\cap\Omega$.

PROOF. One argues by induction on the build-up of s. First consider $s = \mathfrak{E}_\sigma$. As X is good the range of i_X is contained in $\vartheta(T^u_X) \cap \Omega$, and we indeed have

$$E^\vartheta_X(s) = [i_X]^{<\omega} \circ \operatorname{supp}^u_X(\sigma) \subseteq \vartheta(T^u_X) \cap \Omega.$$

In case $s = \vartheta s'$ we have $E^\vartheta_X(s) = \{\vartheta s'\}$. Note that $\vartheta s' <^\vartheta_X \Omega$ holds by definition. The other cases are straightforward. □

Let us now equip $\vartheta(T^u_X) \cap \Omega$ with the structure of a BH-system:

DEFINITION 2.2.12. Assume that (X, E_X, L_X, i_X) is a good BH-system for a prae-dilator $(T^u, \operatorname{supp}^u)$. As T^u is a functor, the embedding $i_X : X \to \vartheta(T^u_X) \cap \Omega$ yields an embedding $T^u(i_X) : T^u_X \to T^u_{\vartheta(T^u_X)\cap\Omega}$ (we write $T^u(i_X)$ rather than $T^u_{i_X}$, to save subscripts). Construct a function $i^\vartheta_X : \vartheta^0(T^u_X) \to \vartheta^0(T^u_{\vartheta(T^u_X)\cap\Omega})$ by recursion over terms in $\vartheta^0(T^u_X)$, setting

(i) $i^\vartheta_X(0) = 0$,
(ii) $i^\vartheta_X(\Omega) = \Omega$,
(iii) $i^\vartheta_X(\mathfrak{E}_\sigma) = \mathfrak{E}_{T^u(i_X)(\sigma)}$ for $\sigma \in T^u_X$,
(iv) $i^\vartheta_X(\vartheta s) = \vartheta i^\vartheta_X(s)$,
(v) $i^\vartheta_X(\omega^{s_0} + \cdots + \omega^{s_n}) = \omega^{i^\vartheta_X(s_0)} + \cdots + \omega^{i^\vartheta_X(s_n)}$.

We write $E_{\vartheta(T^u_X)\cap\Omega}, L_{\vartheta(T^u_X)\cap\Omega}$ and $i_{\vartheta(T^u_X)\cap\Omega}$ for the restrictions of $E^\vartheta_X, L^\vartheta_X$ and i^ϑ_X to $\vartheta(T^u_X) \cap \Omega$. In view of the previous lemma this yields functions

$$E_{\vartheta(T^u_X)\cap\Omega} : \vartheta(T^u_X) \cap \Omega \to [\vartheta(T^u_X) \cap \Omega]^{<\omega},$$
$$L_{\vartheta(T^u_X)\cap\Omega} : \vartheta(T^u_X) \cap \Omega \to \omega,$$
$$i_{\vartheta(T^u_X)\cap\Omega} : \vartheta(T^u_X) \cap \Omega \to \vartheta^0(T^u_{\vartheta(T^u_X)\cap\Omega}).$$

As promised, we have the following result:

THEOREM 2.2.13. *If (X, E_X, L_X, i_X) is a good* BH*-system for a prae-dilator $(T^u, \operatorname{supp}^u)$ then so is*

$$(\vartheta(T^u_X) \cap \Omega, E_{\vartheta(T^u_X)\cap\Omega}, L_{\vartheta(T^u_X)\cap\Omega}, i_{\vartheta(T^u_X)\cap\Omega}).$$

PROOF. Recall that Definition 2.2.5 yields functions

$$E^\vartheta_{\vartheta(T^u_X)\cap\Omega} : \vartheta^0(T^u_{\vartheta(T^u_X)\cap\Omega}) \to [\vartheta^0(T^u_{\vartheta(T^u_X)\cap\Omega})]^{<\omega},$$
$$L^\vartheta_{\vartheta(T^u_X)\cap\Omega} : \vartheta^0(T^u_{\vartheta(T^u_X)\cap\Omega}) \to \omega.$$

To show that $(\vartheta(T^u_X) \cap \Omega, E_{\vartheta(T^u_X)\cap\Omega}, L_{\vartheta(T^u_X)\cap\Omega}, i_{\vartheta(T^u_X)\cap\Omega})$ is a BH-system we verify

$$E^\vartheta_{\vartheta(T^u_X)\cap\Omega} \circ i^\vartheta_X(s) = [i^\vartheta_X]^{<\omega} \circ E^\vartheta_X(s),$$

$$L^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}\circ i^{\vartheta}_X(s)=L^{\vartheta}_X(s)$$

by induction over the term $s\in\vartheta^0(T^u_X)$. The crucial case is $s=\mathfrak{E}_\sigma$ with $\sigma\in T^u_X$. As supp^u is a natural transformation we indeed get

$$E^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}\circ i^{\vartheta}_X(\mathfrak{E}_\sigma)=E^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}(\mathfrak{E}_{T^u(i_X)(\sigma)})=[i^{\vartheta}_X]^{<\omega}\circ\operatorname{supp}^u_{\vartheta(T^u_X)\cap\Omega}\circ T^u(i_X)(\sigma)=$$
$$=[i^{\vartheta}_X]^{<\omega}\circ[i_X]^{<\omega}\circ\operatorname{supp}^u_X(\sigma)=[i^{\vartheta}_X]^{<\omega}\circ E^{\vartheta}_X(\mathfrak{E}_\sigma).$$

Similarly, using the condition $L^{\vartheta}_X\circ i_X=L_X$, we have

$$L^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}\circ i^{\vartheta}_X(\mathfrak{E}_\sigma)=\max\{L^{\vartheta}_X(y)\,|\,y\in\operatorname{supp}^u_{\vartheta(T^u_X)\cap\Omega}\circ T^u(i_X)(\sigma)\}=$$
$$=\max\{L^{\vartheta}_X(y)\,|\,y\in[i_X]^{<\omega}\circ\operatorname{supp}^u_X(\sigma)\}=\max\{L^{\vartheta}_X\circ i_X(x)\,|\,x\in\operatorname{supp}^u_X(\sigma)\}=$$
$$=\max\{L_X(x)\,|\,x\in\operatorname{supp}^u_X(\sigma)\}=L^{\vartheta}_X(\mathfrak{E}_\sigma).$$

The other cases are straightforward. Once we know that $\vartheta(T^u_X)\cap\Omega$ is a BH-system, Lemma 2.2.7 yields a subset $\vartheta(T^u_{\vartheta(T^u_X)\cap\Omega})\subseteq\vartheta^0(T^u_{\vartheta(T^u_X)\cap\Omega})$ and an order relation $<^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}$ on that subset. For $r,s,t\in\vartheta(T^u_X)$ we prove

$$i^{\vartheta}_X(r)\in\vartheta(T^u_{\vartheta(T^u_X)\cap\Omega})$$
$$s<^{\vartheta}_X t\to i^{\vartheta}_X(s)<^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}i^{\vartheta}_X(t)$$

by simultaneous induction on $L^{\vartheta}_X(r)$ resp. $L^{\vartheta}_X(s)+L^{\vartheta}_X(t)$. Note that the two claims together imply $i^{\vartheta}_X(r)\in\vartheta(T^u_{\vartheta(T^u_X)\cap\Omega})\cap\Omega$ for $r\in\vartheta(T^u_X)\cap\Omega$, making $\vartheta(T^u_X)\cap\Omega$ a good BH-system. The first interesting case of the induction is $s=\mathfrak{E}_\sigma<^{\vartheta}_X\mathfrak{E}_\tau=t$ with $\sigma,\tau\in T^u_X$ and $\sigma<_{T^u_X}\tau$. As the function $T^u(i_X):T^u_X\to T^u_{\vartheta(T^u_X)\cap\Omega}$ is an order embedding we indeed get $T^u(i_X)(\sigma)<_{T^u(\vartheta(T^u_X)\cap\Omega)}T^u(i_X)(\tau)$ and thus

$$i^{\vartheta}_X(\mathfrak{E}_\sigma)=\mathfrak{E}_{T^u(i_X)(\sigma)}<^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}\mathfrak{E}_{T^u(i_X)(\tau)}=i^{\vartheta}_X(\mathfrak{E}_\tau).$$

The other interesting case is $s=\vartheta s'<^{\vartheta}_X\vartheta t'=t$. This inequality can hold for two reasons: First assume that we have $s'<^{\vartheta}_X t'$ and $r<^{\vartheta}_X t$ for all $r\in E^{\vartheta}_X(s')\cap\vartheta(T^u_X)$. The induction hypothesis provides

$$i^{\vartheta}_X(s')<^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}i^{\vartheta}_X(t').$$

To conclude $i^{\vartheta}_X(s)<^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}i^{\vartheta}_X(t)$ it remains to establish

$$r'<^{\vartheta}_{\vartheta(T_X)\cap\Omega}i^{\vartheta}_X(t)\quad\text{for all}\quad r'\in E^{\vartheta}_{\vartheta(T^u_X)\cap\Omega}(i^{\vartheta}_X(s'))\cap\vartheta(T^u_{\vartheta(T^u_X)\cap\Omega}).$$

By the above any such r' is an element of $[i^{\vartheta}_X]^{<\omega}\circ E^{\vartheta}_X(s')$, i.e. of the form $r'=i^{\vartheta}_X(r)$ for some $r\in E^{\vartheta}_X(s')$. Lemma 2.2.11 ensures $r\in\vartheta(T^u_X)$, so that the above assumption yields $r<^{\vartheta}_X t$. By Lemma 2.2.6 we have $L^{\vartheta}_X(r)\leq L^{\vartheta}_X(s')<L^{\vartheta}_X(s)$. Thus the

induction hypothesis gives $r' = i^\vartheta_X(r) <^\vartheta_{\vartheta(T^u_X)\cap\Omega} i^\vartheta_X(t)$, which is what we needed. Still concerning the case $s = \vartheta s' <^\vartheta_X \vartheta t' = t$, the other possibility is that we have $t' <^\vartheta_X s'$ and $s \leq^\vartheta_X r$ for some $r \in E^\vartheta_X(t') \cap \vartheta(T^u_X)$. First, the induction hypothesis yields $i^\vartheta_X(t') <^\vartheta_{\vartheta(T^u_X)\cap\Omega} i^\vartheta_X(s')$. Using the above we also get

$$i^\vartheta_X(r) \in [i^\vartheta_X]^{<\omega}(E^\vartheta_X(t')) = E^\vartheta_{\vartheta(T^u_X)\cap\Omega}(i^\vartheta_X(t')).$$

Thus, to conclude $i^\vartheta_X(s) <^\vartheta_{\vartheta(T_X)\cap\Omega} i^\vartheta_X(t)$ it suffices to establish $i^\vartheta_X(r) \in \vartheta(T^u_{\vartheta(T^u_X)\cap\Omega})$ and $i^\vartheta_X(s) \leq^\vartheta_{\vartheta(T^u_X)\cap\Omega} i^\vartheta_X(r)$. In view of

$$L^\vartheta_X(r) \leq L^\vartheta_X(s) + L^\vartheta_X(r) \leq L^\vartheta_X(s) + L^\vartheta_X(t') < L^\vartheta_X(s) + L^\vartheta_X(t)$$

both claims hold by induction hypothesis. The other cases are straightforward. □

By Lemma 2.2.7, the function $(u, X, E_X, L_X, i_X) \mapsto (\vartheta(T^u_X), <^\vartheta_X)$ is primitive recursive. It is easy to see that the transformation of (u, X, E_X, L_X, i_X) into $(u, \vartheta(T^u_X) \cap \Omega, E_{\vartheta(T^u_X)\cap\Omega}, L_{\vartheta(T^u_X)\cap\Omega}, i_{\vartheta(T^u_X)\cap\Omega})$ is primitive recursive as well. The following iterations are central for our fixed-point construction:

Definition 2.2.14. We define a primitive recursive function

$$(u, n) \mapsto \mathrm{BH}_n(T^u) = (\mathrm{BH}_n(T^u), E_{\mathrm{BH}_n(T^u)}, L_{\mathrm{BH}_n(T^u)}, i_{\mathrm{BH}_n(T^u)})$$

by recursion over $n \in \omega$: Invoking Lemma 2.2.10, we set

$$\mathrm{BH}_0(T^u) := (\varepsilon_0, E_{\varepsilon_0}, L_{\varepsilon_0}, i_{\varepsilon_0}).$$

In the recursion step, we define $\mathrm{BH}_{n+1}(T^u)$ as the tuple

$$(\vartheta(T^u_{\mathrm{BH}_n(T^u)}) \cap \Omega, E_{\vartheta(T^u(\mathrm{BH}_n(T^u)))\cap\Omega}, L_{\vartheta(T^u(\mathrm{BH}_n(T^u)))\cap\Omega}, i_{\vartheta(T^u(\mathrm{BH}_n(T^u)))\cap\Omega}),$$

relying on the notation from Definition 2.2.12.

Let us record the following:

Corollary 2.2.15. *If* (T^u, supp^u) *is a prae-dilator then* $\mathrm{BH}_n(T^u)$ *is a good* BH*-system for all* $n \in \mathbb{N}$.

Proof. One argues by induction on n, as justified by Proposition 1.1.11. Crucially, the induction statement is primitive recursive. Initial case and step are covered by Lemma 2.2.10 and Theorem 2.2.13, respectively. □

Note that $i_{\mathrm{BH}_n(T)}$ embeds $\mathrm{BH}_n(T^u)$ into $\vartheta(T^u_{\mathrm{BH}_n(T^u)}) \cap \Omega = \mathrm{BH}_{n+1}(T^u)$. We want to define $\mathrm{BH}(T^u)$ as the direct limit of the resulting directed system. To see that this is permissible in our base theory, we give an explicit construction:

DEFINITION 2.2.16. We put

$$\mathrm{BH}(T^u) := \{(n,\sigma) \mid \sigma \in \mathrm{BH}_n(T^u) \wedge (n = 0 \vee \sigma \notin \mathrm{rng}(i_{\mathrm{BH}_{n-1}(T^u)}))\}.$$

The function $u \mapsto \omega \times \bigcup_{k\in\omega} \mathrm{BH}_k(T^u)$ is primitive recursive by Proposition 1.1.6, Proposition 1.1.4 and Proposition 1.2.1. As we have $\mathrm{BH}(T^u) \subseteq \omega \times \bigcup_{k\in\omega} \mathrm{BH}_k(T^u)$, Corollary 1.1.10 tells us that $\mathrm{BH}(T^u)$ exists as a set, and that $u \mapsto \mathrm{BH}(T^u)$ is a primitive recursive function. In order to formulate the predicative Bachmann-Howard principle, we need to define the "limit order" on $\mathrm{BH}(T^u)$. To avoid explicit constructions in each case, we recover a version of the universal property:

LEMMA 2.2.17. *There is a (set-sized) surjection*

$$j : \prod_{n\in\mathbb{N}} \mathrm{BH}_n(T^u) \to \mathrm{BH}(T^u), \qquad (n,\sigma) \mapsto j_n(\sigma)$$

such that we have $j_{n+1} \circ i_{\mathrm{BH}_n(T^u)} = j_n$ *for each* $n \in \mathbb{N}$. *Furthermore, for each number* $k \in \mathbb{N}$ *and any (set-sized) function*

$$f : \prod_{n\in\mathbb{N}} \mathrm{BH}_n(T^u)^k \to a, \qquad (n,\sigma_1,\dots,\sigma_k) \mapsto f_n(\sigma_1,\dots,\sigma_k)$$

with

$$f_{n+1}(i_{\mathrm{BH}_n(T^u)}(\sigma_1),\dots,i_{\mathrm{BH}_n(T^u)}(\sigma_k)) = f_n(\sigma_1,\dots,\sigma_k) \quad \text{for } \sigma_1,\dots,\sigma_k \in \mathrm{BH}_n(T^u),$$

there is a function

$$f_\omega : \mathrm{BH}(T^u)^k \to a$$

such that we have

$$f_\omega(j_n(\sigma_1),\dots,j_n(\sigma_k)) = f_n(\sigma_1,\dots,\sigma_k) \quad \text{for } \sigma_1,\dots,\sigma_k \in \mathrm{BH}_n(T^u).$$

The transformation of f *into* f_ω *is primitive recursive.*

PROOF. By recursion over $n \geq m$ we define functions

$$i^n_m := i_{\mathrm{BH}_{n-1}(T^u)} \circ \dots \circ i_{\mathrm{BH}_m(T^u)} : \mathrm{BH}_m(T^u) \to \mathrm{BH}_n(T^u).$$

In particular, i^m_m is the identity on $\mathrm{BH}_m(T^u)$. As each $i_{\mathrm{BH}_k(T^u)}$ is an order embedding, the functions i^n_m are injective. Primitive recursive "ranks"

$$\mathrm{rk} : \prod_{n\in\mathbb{N}} \mathrm{BH}_n(T^u) \to \mathbb{N}$$

can be defined by

$$\operatorname{rk}(n,\sigma) := \min\{m \leq n \,|\, \sigma \in \operatorname{rng}(i^n_m)\} = \\ = \bigcup\{m \leq n \,|\, \sigma \in \operatorname{rng}(i^n_m) \wedge \forall_{l<m} \sigma \notin \operatorname{rng}(i^n_l)\}.$$

Note that there is a unique $\tau \in \mathrm{BH}_{\operatorname{rk}(n,\sigma)}(T^u)$ with $i^n_{\operatorname{rk}(n,\sigma)}(\tau) = \sigma$. Furthermore, the minimality of $\operatorname{rk}(n,\sigma)$ ensures $(\operatorname{rk}(n,\sigma),\tau) \in \mathrm{BH}(T^u)$. Thus we can put

$$j_n(\sigma) = \bigcup\{(\operatorname{rk}(n,\sigma),\tau) \,|\, \tau \in \mathrm{BH}_{\operatorname{rk}(n,\sigma)}(T^u) \wedge i^n_{\operatorname{rk}(n,\sigma)}(\tau) = \sigma\}.$$

From $i^{n+1}_m = i_{\mathrm{BH}_n(T^u)} \circ i^n_m$ one readily deduces $\operatorname{rk}(n+1, i_{\mathrm{BH}_n(T^u)}(\sigma)) = \operatorname{rk}(n,\sigma)$ and then $j_{n+1} \circ i_{\mathrm{BH}_n(T^u)}(\sigma) = j_n(\sigma)$. As for surjectivity, for $(n,\sigma) \in \mathrm{BH}(T^u)$ we clearly have $j_n(\sigma) = (n,\sigma)$. To establish the second claim, set

$$f_\omega((m_1,\tau_1),\dots,(m_k,\tau_k)) := f_m(i^m_{m_1}(\tau_1),\dots,i^m_{m_k}(\tau_k)) \\ \text{with } m = \max\{m_1,\dots,m_k\}.$$

The claim that $f \mapsto f_\omega$ is primitive recursive follows from Proposition 1.2.2. Given elements $\sigma_1,\dots,\sigma_k \in \mathrm{BH}_n(T^u)$, write $j_n(\sigma_i) = (m_i,\tau_i)$ for $i = 1,\dots,k$. Note that this entails $m := \max\{m_1,\dots,m_k\} \leq n$ and $i^n_{m_i}(\tau_i) = \sigma_i$. Thus we get

$$f_\omega(j_n(\sigma_1),\dots,j_n(\sigma_k)) = f_m(i^m_{m_1}(\tau_1),\dots,i^m_{m_k}(\tau_k)) = \\ = f_n(i^n_{m_1}(\tau_1),\dots,i^n_{m_k}(\tau_k)) = f_n(\sigma_1,\dots,\sigma_k),$$

as required. □

In particular, if $f_n : \mathrm{BH}_n(T^u)^2 \to \{0,1\}$ is the characteristic function of the order relation $<_{\mathrm{BH}_n(T^u)}$, then we get a primitive recursive function $u \mapsto <_{\mathrm{BH}(T^u)}$ such that $<_{\mathrm{BH}(T^u)}$ is a binary relation on $\mathrm{BH}(T)$ and we have

$$j_n(\sigma) <_{\mathrm{BH}(T^u)} j_n(\tau) \quad \Leftrightarrow \quad \sigma <_{\mathrm{BH}_n(T^u)} \tau \quad \text{for } \sigma,\tau \in \mathrm{BH}_n(T^u).$$

Let us verify the following:

Lemma 2.2.18. *If $(T^u, \operatorname{supp}^u)$ is a prae-dilator then $(\mathrm{BH}(T^u), <_{\mathrm{BH}(T^u)})$ is a linear order, and the functions*

$$j_n : (\mathrm{BH}_n(T^u), <_{\mathrm{BH}_n(T^u)}) \to (\mathrm{BH}(T^u), <_{\mathrm{BH}(T^u)})$$

are order embeddings.

PROOF. Consider arbitrary elements $j_n(\sigma), j_m(\tau) \in \mathrm{BH}(T^u)$, say with $m \leq n$. Using the notation from the previous proof, we have $\sigma, i^n_m(\tau) \in \mathrm{BH}_n(T^u)$. Invoking the trichotomy of $<_{\mathrm{BH}_n(T^u)}$ we have, say, $\sigma <_{\mathrm{BH}_n(T^u)} i^n_m(\tau)$. By the defining condition of $<_{\mathrm{BH}(T^u)}$ we get

$$j_n(\sigma) <_{\mathrm{BH}(T^u)} j_n(i^n_m(\tau)) = j_m(\tau),$$

establishing that $<_{\mathrm{BH}(T^u)}$ is trichotomous. The other conditions on a linear order carry over in a similar way. That j_n is an order embedding is immediate by the definition of $<_{\mathrm{BH}(T^u)}$. In particular, it follows that j_n is injective. □

Assume that (T^u, supp^u) is a dilator. Then the well-foundedness of $\mathrm{BH}_n(T^u)$ implies the well-foundedness of $T^u_{\mathrm{BH}_n(T^u)}$. Essentially by the type-one well-ordering principle $X \mapsto \vartheta_X$ of Rathjen and Valencia Vizcaíno (cf. Remark 2.1.24), it follows that $\vartheta(T^u_{\mathrm{BH}_n(T^u)}) \cap \Omega = \mathrm{BH}_{n+1}(T^u)$ is well-founded as well. So inductively, $\mathrm{BH}_n(T^u)$ is well-founded for all n. In general, well-foundedness is not preserved under direct limits (indeed, any linear order is a direct limit of finite orders). In Section 2.4 we will construct compatible embeddings of the orders $\mathrm{BH}_n(T^u)$ into a sufficiently large well-order. These can be glued to an embedding of $\mathrm{BH}(T^u)$ into that well-order, witnessing the well-foundedness of $\mathrm{BH}(T^u)$. The fact that this particular limit preserves well-foundedness constitutes the strength of our type-two well-ordering principle:

DEFINITION 2.2.19. The predicative Bachmann-Howard principle is the collection of statements

$$\forall_u(\text{“}(T^u, \mathrm{supp}^u) \text{ is a dilator”} \rightarrow \text{“}(\mathrm{BH}(T^u), <_{\mathrm{BH}(T^u)}) \text{ is well-founded”})$$

for all pairs of primitive recursive functions $(u, X) \mapsto T^u_X$ and $(u, X) \mapsto \mathrm{supp}^u_X$.

As observed before, axiom beta turns the definition of dilator into a Π_1-formula. Thus the predicative Bachmann-Howard principle becomes a collection of Π_2-formulas. As Π_2-formulas of $\mathbf{ATR}_\mathbf{0}^{\mathrm{set}}$ correspond to Π^1_3-formulas of second-order arithmetic, the predicative Bachmann-Howard principle has the same logical complexity as Π^1_1-comprehension. In Section 2.4 we will see that all prae-dilators form a single primitive recursive family. This will allow to state the predicative Bachmann-Howard principle as a single formula, rather than a schema. By direction (iv)⇒(i) of Theorem 4.4.6, the predicative Bachmann-Howard principle cannot be established in a predicative theory. Rather, the specification "predicative" refers to the fact that $u \mapsto (\mathrm{BH}(T^u), <_{\mathrm{BH}(T^u)})$ is a primitive recursive set function. One can argue that such functions are acceptable for a predicativist (cf. [**17**]).

Recall that we wanted a well-founded fixed-point $\mathrm{BH}(T^u) \cong \vartheta(T^u_{\mathrm{BH}(T^u)}) \cap \Omega$. Well-foundedness cannot be established in a weak base theory, and is instead postulated by the predicative Bachmann-Howard principle. In the rest of this section we show that $\mathrm{BH}(T^u)$ is a fixed point, working in our usual base theory $\mathbf{PRS}\omega$. To make sense of this claim, we have to endow $\mathrm{BH}(T^u)$ with the structure of a BH-system. The following is a preparation:

Definition 2.2.20. Construct functions

$$j^\vartheta_n : \vartheta^0(T^u_{\mathrm{BH}_n(T^u)}) \to \vartheta^0(T^u_{\mathrm{BH}(T^u)})$$

by recursion over terms in $\vartheta^0(T^u_{\mathrm{BH}_n(T^u)})$, setting

(i) $j^\vartheta_n(0) = 0$,
(ii) $j^\vartheta_n(\Omega) = \Omega$,
(iii) $j^\vartheta_n(\mathfrak{E}_\sigma) = \mathfrak{E}_{T^u(j_n)(\sigma)}$ (using the embedding $T^u(j_n) : T^u_{\mathrm{BH}_n(T^u)} \to T^u_{\mathrm{BH}(T^u)}$),
(iv) $j^\vartheta_n(\vartheta s) = \vartheta j^\vartheta_n(s)$,
(v) $j^\vartheta_n(\omega^{s_0} + \cdots + \omega^{s_n}) := \omega^{j^\vartheta_n(s_0)} + \cdots + \omega^{j^\vartheta_n(s_n)}$.

In order to glue these functions we must verify the following:

Lemma 2.2.21. *The families of functions*

$$[j_n]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)} : \mathrm{BH}_n(T^u) \to [\mathrm{BH}(T^u)]^{<\omega},$$
$$L_{\mathrm{BH}_n(T^u)} : \mathrm{BH}_n(T^u) \to \omega,$$
$$j^\vartheta_n \circ i_{\mathrm{BH}_n(T^u)} : \mathrm{BH}_n(T^u) \to \vartheta^0(T^u_{\mathrm{BH}(T^u)})$$

each satisfy the compatibility condition from Lemma 2.2.17.

Proof. In the first case we must show

$$[j_{n+1}]^{<\omega} \circ E_{\mathrm{BH}_{n+1}(T^u)} \circ i_{\mathrm{BH}_n(T^u)} = [j_n]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)}.$$

Recall that $E_{\mathrm{BH}_{n+1}(T^u)}$ is the restriction of $E^\vartheta_{\mathrm{BH}_n(T^u)}$ to $\mathrm{BH}_{n+1}(T^u) \subseteq \vartheta^0(T^u_{\mathrm{BH}_n(T^u)})$. As $\mathrm{BH}_n(T^u)$ is a BH-system we have $E^\vartheta_{\mathrm{BH}_n(T^u)} \circ i_{\mathrm{BH}_n(T^u)} = [i_{\mathrm{BH}_n(T^u)}]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)}$. Thus, for $s \in \mathrm{BH}_n(T^u)$, we obtain

$$[j_{n+1}]^{<\omega} \circ E_{\mathrm{BH}_{n+1}(T^u)} \circ i_{\mathrm{BH}_n(T^u)}(s) = [j_{n+1}]^{<\omega} \circ [i_{\mathrm{BH}_n(T^u)}]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)}(s) =$$
$$[j_{n+1} \circ i_{\mathrm{BH}_n(T^u)}]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)}(s) = [j_n]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)}(s),$$

as required. Similarly, we get $L_{\mathrm{BH}_{n+1}(T^u)} \circ i_{\mathrm{BH}_n(T^u)} = L_{\mathrm{BH}_n(T^u)}$. Finally, we must show

$$j^\vartheta_{n+1} \circ i_{\mathrm{BH}_{n+1}(T^u)} \circ i_{\mathrm{BH}_n(T^u)} = j^\vartheta_n \circ i_{\mathrm{BH}_n(T^u)}.$$

Recall that $i_{\mathrm{BH}_{n+1}(T^u)}$ is a restriction of $i^\vartheta_{\mathrm{BH}_n(T^u)} : \vartheta^0(T^u_{\mathrm{BH}_n(T^u)}) \to \vartheta^0(T^u_{\mathrm{BH}_{n+1}(T^u)})$. Thus it suffices to establish

$$j^\vartheta_{n+1} \circ i^\vartheta_{\mathrm{BH}_n(T^u)}(s) = j^\vartheta_n(s)$$

by induction on $s \in \vartheta^0(T^u_{\mathrm{BH}_n(T^u)})$. The interesting case is $s = \mathfrak{E}_\sigma$ for $\sigma \in T^u_{\mathrm{BH}_n(T^u)}$. We have $T^u(j_{n+1}) \circ T^u(i_{\mathrm{BH}_n(T^u)})(\sigma) = T^u(j_{n+1} \circ i_{\mathrm{BH}_n(T^u)})(\sigma) = T^u(j_n)(\sigma)$ by functoriality, and thus

$$\begin{aligned} j^\vartheta_{n+1} \circ i^\vartheta_{\mathrm{BH}_n(T^u)}(\mathfrak{E}_\sigma) = j^\vartheta_{n+1}(\mathfrak{E}_{T^u(i_{\mathrm{BH}_n(T^u)})(\sigma)}) = \\ = \mathfrak{E}_{T^u(j_{n+1}) \circ T^u(i_{\mathrm{BH}_n(T^u)})(\sigma)} = \mathfrak{E}_{T^u(j_n)(\sigma)} = j^\vartheta_n(\mathfrak{E}_\sigma), \end{aligned}$$

as desired. □

Now we can glue these functions as follows:

Definition 2.2.22. The previous lemma and Lemma 2.2.17 yield functions

$$\begin{aligned} E_{\mathrm{BH}(T^u)} &: \mathrm{BH}(T^u) \to [\mathrm{BH}(T^u)]^{<\omega}, \\ L_{\mathrm{BH}(T^u)} &: \mathrm{BH}(T^u) \to \omega, \\ i_{\mathrm{BH}(T^u)} &: \mathrm{BH}(T^u) \to \vartheta^0(T^u_{\mathrm{BH}(T^u)}) \end{aligned}$$

with

$$\begin{aligned} E_{\mathrm{BH}(T^u)} \circ j_n &= [j_n]^{<\omega} \circ E_{\mathrm{BH}_n(T^u)}, \\ L_{\mathrm{BH}(T^u)} \circ j_n &= L_{\mathrm{BH}_n(T^u)}, \\ i_{\mathrm{BH}(T^u)} \circ j_n &= j^\vartheta_n \circ i_{\mathrm{BH}_n(T^u)}. \end{aligned}$$

Let us check that the following structure is preserved in the limit:

Lemma 2.2.23. *The tuple* $(\mathrm{BH}(T^u), E_{\mathrm{BH}(T^u)}, L_{\mathrm{BH}(T^u)}, i_{\mathrm{BH}(T^u)})$ *is a* BH-*system, whenever* (T^u, supp^u) *is a prae-dilator.*

Proof. As a preparation, we establish

$$E^\vartheta_{\mathrm{BH}(T^u)} \circ j^\vartheta_n(s) = [j^\vartheta_n]^{<\omega} \circ E^\vartheta_{\mathrm{BH}_n(T^u)}(s)$$

by induction on $s \in \vartheta^0(T^u_{\mathrm{BH}_n(T^u)})$. The crucial case is $s = \mathfrak{E}_\sigma$ with $\sigma \in T^u_{\mathrm{BH}_n(T^u)}$. Using the fact that $\mathrm{supp}^u : T^u \Rightarrow [\cdot]^{<\omega}$ is a natural transformation one gets

$$E^\vartheta_{\mathrm{BH}(T^u)}\circ j^\vartheta_n(\mathfrak{E}_\sigma)=E^\vartheta_{\mathrm{BH}(T^u)}(\mathfrak{E}_{T^u(j_n)(\sigma)})=[i_{\mathrm{BH}(T^u)}]^{<\omega}\circ\mathrm{supp}^u_{\mathrm{BH}(T^u)}(T^u(j_n)(\sigma))=$$
$$=[i_{\mathrm{BH}(T^u)}]^{<\omega}\circ[j_n]^{<\omega}\circ\mathrm{supp}^u_{\mathrm{BH}_n(T^u)}(\sigma)=$$
$$=[j^\vartheta_n]^{<\omega}\circ[i_{\mathrm{BH}_n(T^u)}]^{<\omega}\circ\mathrm{supp}^u_{\mathrm{BH}_n(T^u)}(\sigma)=[j^\vartheta_n]^{<\omega}\circ E^\vartheta_{\mathrm{BH}_n(T^u)}(\mathfrak{E}_\sigma),$$

as required. Since $\mathrm{BH}_n(T^u)$ is a BH-system we can deduce

$$E^\vartheta_{\mathrm{BH}(T^u)}\circ i_{\mathrm{BH}(T^u)}(j_n(s))=E^\vartheta_{\mathrm{BH}(T^u)}\circ j^\vartheta_n\circ i_{\mathrm{BH}_n(T^u)}(s)=$$
$$=[j^\vartheta_n]^{<\omega}\circ E^\vartheta_{\mathrm{BH}_n(T^u)}\circ i_{\mathrm{BH}_n(T^u)}(s)=[j^\vartheta_n]^{<\omega}\circ[i_{\mathrm{BH}_n(T^u)}]^{<\omega}\circ E_{\mathrm{BH}_n(T^u)}(s)=$$
$$=[i_{\mathrm{BH}(T^u)}]^{<\omega}\circ[j_n]^{<\omega}\circ E_{\mathrm{BH}_n(T^u)}(s)=[i_{\mathrm{BH}(T^u)}]^{<\omega}\circ E_{\mathrm{BH}(T^u)}(j_n(s)).$$

By Lemma 2.2.17, any element of $\mathrm{BH}(T^u)$ is of the form $j_n(s)$ for some $n\in\mathbb{N}$ and some $s\in\mathrm{BH}_n(T^u)$. Thus we have $E^\vartheta_{\mathrm{BH}(T^u)}\circ i_{\mathrm{BH}(T^u)}=[i_{\mathrm{BH}(T^u)}]^{<\omega}\circ E_{\mathrm{BH}(T^u)}$. Similarly, one shows $L^\vartheta_{\mathrm{BH}(T^u)}\circ j^\vartheta_n=L^\vartheta_{\mathrm{BH}_n(T^u)}$ and then $L^\vartheta_{\mathrm{BH}(T^u)}\circ i_{\mathrm{BH}(T^u)}=L_{\mathrm{BH}(T^u)}$, making $\mathrm{BH}(T^u)$ a BH-system. □

Now Lemma 2.2.7 singles out the "normal forms" $\vartheta(T^u_{\mathrm{BH}(T^u)})\subseteq\vartheta^0(T^u_{\mathrm{BH}(T^u)})$, as well as a binary relation $<^\vartheta_{\mathrm{BH}(T^u)}$ on $\vartheta(T^u_{\mathrm{BH}(T^u)})$. Proposition 2.2.8 ensures that this is a linear order. As for the finite iterations we have the following:

Proposition 2.2.24. *The map* $i_{\mathrm{BH}(T^u)}$ *is an embedding of* $(\mathrm{BH}(T^u),<_{\mathrm{BH}(T^u)})$ *into* $(\vartheta(T^u_{\mathrm{BH}(T^u)})\cap\Omega,<^\vartheta_{\mathrm{BH}(T^u)})$. *In other words,* $\mathrm{BH}(T^u)$ *is a good* BH*-system for the prae-dilator* (T^u,supp^u).

Proof. The crucial step is to establish

$$r\in\vartheta(T^u_{\mathrm{BH}_n(T^u)})\quad\Rightarrow\quad j^\vartheta_n(r)\in\vartheta(T^u_{\mathrm{BH}(T^u)}),$$
$$s<^\vartheta_{\mathrm{BH}_n(T^u)}t\quad\Rightarrow\quad j^\vartheta_n(s)<^\vartheta_{\mathrm{BH}(T^u)}j^\vartheta_n(t),$$

by simultaneous induction on $L^\vartheta_{\mathrm{BH}_n(T^u)}(r)$ resp. $L^\vartheta_{\mathrm{BH}_n(T^u)}(s)+L^\vartheta_{\mathrm{BH}_n(T^u)}(t)$. The only difficult case is

$$s=\vartheta s'<^\vartheta_{\mathrm{BH}_n(T^u)}\vartheta t'=t.$$

First, assume that this holds because we have $s'<^\vartheta_{\mathrm{BH}_n(T^u)}t'$ and $r_0<^\vartheta_{\mathrm{BH}_n(T^u)}t$ for all $r_0\in E^\vartheta_{\mathrm{BH}_n(T^u)}(s')\cap\vartheta(T^u_{\mathrm{BH}_n(T^u)})$. Then $j^\vartheta_n(s')<^\vartheta_{\mathrm{BH}(T^u)}j^\vartheta_n(t')$ holds by induction hypothesis. To obtain $j^\vartheta_n(s)<^\vartheta_{\mathrm{BH}(T^u)}j^\vartheta_n(t)$ it suffices to establish

$$r<^\vartheta_{\mathrm{BH}(T^u)}j^\vartheta_n(t)\quad\text{for all}\quad r\in E^\vartheta_{\mathrm{BH}(T^u)}(j^\vartheta_n(s'))\cap\vartheta(T^u_{\mathrm{BH}(T^u)}).$$

As we have $E^\vartheta_{\mathrm{BH}(T^u)}\circ j^\vartheta_n(s')=[j^\vartheta_n]^{<\omega}\circ E^\vartheta_{\mathrm{BH}_n(T^u)}(s')$ by the previous proof, any such r can be written as $r=j^\vartheta_n(r_0)$ with $r_0\in E^\vartheta_{\mathrm{BH}_n(T^u)}(s')$. Lemma 2.2.11 ensures $r_0\in\vartheta(T^u_{\mathrm{BH}_n(T^u)})$, so that the above assumption yields $r_0<^\vartheta_{\mathrm{BH}_n(T^u)}t$.

Since $L^{\vartheta}_{\mathrm{BH}_n(T^u)}(r_0) \leq L^{\vartheta}_{\mathrm{BH}_n(T^u)}(s') < L^{\vartheta}_{\mathrm{BH}_n(T^u)}(s)$ holds by Lemma 2.2.6, the induction hypothesis yields $r = j^{\vartheta}_n(r_0) <^{\vartheta}_{\mathrm{BH}(T^u)} j^{\vartheta}_n(t)$, as required. Still concerning the case $s = \vartheta s' <^{\vartheta}_{\mathrm{BH}_n(T^u)} \vartheta t' = t$, let us now assume that we have $t' <^{\vartheta}_{\mathrm{BH}_n(T^u)} s'$ and $s \leq^{\vartheta}_{\mathrm{BH}_n(T^u)} r_0$ for some $r_0 \in E^{\vartheta}_{\mathrm{BH}_n(T^u)}(t') \cap \vartheta(T^u_{\mathrm{BH}_n(T^u)})$. The induction hypothesis provides $j^{\vartheta}_n(t') <^{\vartheta}_{\mathrm{BH}(T^u)} j^{\vartheta}_n(s')$, as well as $j^{\vartheta}_n(s) \leq^{\vartheta}_{\mathrm{BH}(T^u)} j^{\vartheta}_n(r_0)$ and $j^{\vartheta}_n(r_0) \in \vartheta(T^u_{\mathrm{BH}(T^u)})$. By the proof of the previous lemma we also have

$$j^{\vartheta}_n(r_0) \in [j^{\vartheta}_n]^{<\omega}(E^{\vartheta}_{\mathrm{BH}_n(T^u)}(t')) = E^{\vartheta}_{\mathrm{BH}(T^u)}(j^{\vartheta}_n(t')).$$

Together this implies

$$j^{\vartheta}_n(s) = \vartheta j^{\vartheta}_n(s') <^{\vartheta}_{\mathrm{BH}(T^u)} \vartheta j^{\vartheta}_n(t') = j^{\vartheta}_n(t),$$

as needed. To deduce that the BH-system $\mathrm{BH}(T^u)$ is good, consider an arbitrary element $j_n(s) \in \mathrm{BH}(T^u)$. As $\mathrm{BH}_n(T^u)$ is good we have $i_{\mathrm{BH}_n(T^u)}(s) \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$ and $i_{\mathrm{BH}_n(T^u)}(s) <^{\vartheta}_{\mathrm{BH}_n(T^u)} \Omega$. By the above this implies

$$i_{\mathrm{BH}(T^u)}(j_n(s)) = j^{\vartheta}_n(i_{\mathrm{BH}_n(T^u)}(s)) \in \vartheta(T^u_{\mathrm{BH}(T^u)}) \cap \Omega.$$

This means that the range of $i_{\mathrm{BH}(T^u)}$ is as desired. To see that $i_{\mathrm{BH}(T^u)}$ is an order embedding, consider two elements $j_n(s) <_{\mathrm{BH}(T^u)} j_m(t)$ of $\mathrm{BH}(T^u)$. In view of $j_k(r) = j_{k+1}(i_{\mathrm{BH}_k(T^u)}(r))$ we may assume $m = n$. By the defining equivalence of $<_{\mathrm{BH}(T^u)}$ we get $s <_{\mathrm{BH}_n(T^u)} t$, and then $i_{\mathrm{BH}_n(T^u)}(s) <^{\vartheta}_{\mathrm{BH}_n(T^u)} i_{\mathrm{BH}_n(T^u)}(t)$. Using the above we obtain

$$i_{\mathrm{BH}(T^u)}(j_n(s)) = j^{\vartheta}_n(i_{\mathrm{BH}_n(T^u)}(s)) <^{\vartheta}_{\mathrm{BH}(T^u)} j^{\vartheta}_n(i_{\mathrm{BH}_n(T^u)}(t)) = i_{\mathrm{BH}(T^u)}(j_n(t)),$$

so that $i_{\mathrm{BH}(T^u)}$ is indeed order preserving. □

Finally, the limit construction yields the following fixed-point property. This reveals the power of the predicative Bachmann-Howard principle:

THEOREM 2.2.25. *Consider a prae-dilator (T^u, supp^u). The embedding*

$$i_{\mathrm{BH}(T^u)} : (\mathrm{BH}(T^u), <_{\mathrm{BH}(T^u)}) \to (\vartheta(T^u_{\mathrm{BH}(T^u)}) \cap \Omega, <^{\vartheta}_{\mathrm{BH}(T^u)})$$

is surjective, and thus an order isomorphism.

PROOF. The main step is to show that any $s \in \vartheta(T^u_{\mathrm{BH}(T^u)})$ can be written as $s = j^{\vartheta}_n(t)$, for some $n \in \mathbb{N}$ and $t \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$. Arguing by induction on s, the crucial case is $s = \mathfrak{E}_\sigma$ with $\sigma \in T^u_{\mathrm{BH}(T^u)}$. By the definition of prae-dilator, σ lies in the range of $T^u_{\iota_\sigma} : T^u_{\mathrm{supp}^u_{\mathrm{BH}(T^u)}(\sigma)} \to T^u_{\mathrm{BH}(T^u)}$, where $\iota_\sigma : \mathrm{supp}^u_{\mathrm{BH}(T^u)}(\sigma) \hookrightarrow \mathrm{BH}(T^u)$ is the inclusion. Pick $\tau \in T^u_{\mathrm{supp}^u_{\mathrm{BH}(T^u)}(\sigma)}$ with $\sigma = T^u_{\iota_\sigma}(\tau)$. By Lemma 2.2.17 we have

$\mathrm{BH}(T^u) = \bigcup_{k\in\mathbb{N}} \mathrm{rng}(j_k)$. Also, $j_{k+1} \circ i_{\mathrm{BH}_k(T^u)} = j_k$ implies $\mathrm{rng}(j_k) \subseteq \mathrm{rng}(j_{k+1})$. Thus the finite set $\mathrm{supp}^u_{\mathrm{BH}(T^u)}(\sigma)$ is contained in the range of j_n, for some $n \in \mathbb{N}$. It follows that i_σ factors as $i_\sigma = j_n \circ h$, where $h : \mathrm{supp}^u_{\mathrm{BH}(T^u)}(\sigma) \to \mathrm{BH}_n(T^u)$ is an order embedding. Using the functoriality of T we get

$$\sigma = T^u(i_\sigma)(\tau) = T^u(j_n \circ h)(\tau) = T^u(j_n) \circ T^u(h)(\tau).$$

Setting $t := \mathfrak{E}_{T^u(h)(\tau)} \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$ we obtain

$$j^\vartheta_n(t) = \mathfrak{E}_{T^u(j_n)\circ T^u(h)(\tau)} = \mathfrak{E}_\sigma = s,$$

as desired. Before we tackle the next case, note that we can increase n: If we have $s = j^\vartheta_n(t)$ with $t \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$ then the proof of Lemma 2.2.21 yields

$$s = j^\vartheta_n(t) = j^\vartheta_{n+1}(i^\vartheta_{\mathrm{BH}_n(T^u)}(t)),$$

and the proof of Theorem 2.2.13 ensures

$$i^\vartheta_{\mathrm{BH}_n(T^u)}(t) \in \vartheta(T^u_{\vartheta(T^u_{\mathrm{BH}_n(T^u)})\cap\Omega}) = \vartheta(T^u_{\mathrm{BH}_{n+1}(T^u)}).$$

Now consider the case

$$s = \omega^{s_0} + \cdots + \omega^{s_k} \qquad \text{with } k > 0 \text{ and } s_k \leq^\vartheta_{\mathrm{BH}(T^u)} \cdots \leq^\vartheta_{\mathrm{BH}(T^u)} s_0.$$

By induction hypothesis we can write $s_i = j^\vartheta_n(t_i)$ with $t_i \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$ and, as we have just seen, the same n for all $i \leq k$. In the previous proof we have seen that the restriction of j^ϑ_n to $\vartheta(T^u_{\mathrm{BH}_n(T^u)})$ is order preserving. Thus we must have $t_k \leq^\vartheta_{\mathrm{BH}_n(T^u)} \cdots \leq^\vartheta_{\mathrm{BH}_n(T^u)} t_0$, and $t := \omega^{t_0} + \cdots + \omega^{t_k} \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$ is a term with $s = j^\vartheta_n(t)$. The remaining cases are straightforward. To deduce the claim of the theorem, consider an arbitrary element $s \in \vartheta(T^u_{\mathrm{BH}(T^u)}) \cap \Omega$. As shown, we can write $s = j^\vartheta_n(t)$ for some $n \in \mathbb{N}$ and $t \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$. Since j^ϑ_n is order preserving we get $t \in \vartheta(T^u_{\mathrm{BH}_n(T^u)}) \cap \Omega = \mathrm{BH}_{n+1}(T^u)$. Recall from Definition 2.2.12 that $i_{\mathrm{BH}_{n+1}(T^u)}$ is the restriction of $i^\vartheta_{\mathrm{BH}_n(T^u)}$ to this set. We can deduce that

$$s = j^\vartheta_n(t) = j^\vartheta_{n+1}(i^\vartheta_{\mathrm{BH}_n(T^u)}(t)) = j^\vartheta_{n+1}(i_{\mathrm{BH}_{n+1}(T^u)}(t)) = i_{\mathrm{BH}(T^u)}(j_{n+1}(t))$$

lies in the range of $i_{\mathrm{BH}(T^u)}$, as required. □

Let us point out that our definition of $\mathrm{BH}(T^u)$ is similar to Aczel's [**2, 3**] functorial construction of the derivative of a normal function. For a more detailed comparison we refer to the introduction of the present thesis.

2.3. A Computable Bachmann-Howard Principle

In the first part of this section we re-establish a result of Girard [**28**, Remark 2.1.6(ii)] in our context: Up to natural isomorphism, a prae-dilator is uniquely determined by its restriction to (the category of) natural numbers. Indeed, there is a single primitive recursive function which reconstructs (an isomorphic copy of) any given prae-dilator from this restriction. With respect to the predicative Bachmann-Howard principle of the previous section, this has two advantages: Firstly, we can quantify over (up to isomorphism) all prae-dilators, so that the predicative Bachmann-Howard principle becomes a single statement in the language of set theory, rather than a schema (as pointed out by Montalbán [**55**, Section 4.5]). Secondly, the seemingly ad-hoc condition that prae-dilators must be primitive recursive becomes automatic. Under the axiom of countability, prae-dilators can be coded by subsets of the natural numbers. In the second part of this section we will show that such a coded prae-dilator T can be transformed into a relativized notation system $\vartheta(T)$, also coded by a subset of the natural numbers. The transformation $T \mapsto \vartheta(T)$ will be primitive recursive in the usual sense, so that $\mathbf{RCA_0}$ proves the existence of $\vartheta(T)$ as a set, and indeed as a linear order. The computable Bachmann-Howard principle asserts that $\vartheta(T)$ is well-founded for any coded dilator T. In the next section we will see that the fixed-point $\mathrm{BH}(T)$ from the predicative Bachmann-Howard principle corresponds to a subset $\vartheta(T) \cap \Omega \subseteq \vartheta(T)$.

Any linear order is the direct limit of its finite sub-orders. As prae-dilators preserve direct limits (see Remark 2.2.2), they can (up to isomorphism) be reconstructed from their restrictions to the category of finite orders. This category is not small, but it is equivalent to a small category (cf. [**51**, Section IV.4]): the category of natural numbers, with strictly increasing maps

$$n = \{0, \dots, n-1\} \to \{0, \dots, m-1\} = m$$

as morphisms. Indeed, this category is small from the viewpoint of our base theory $\mathbf{PRS}\omega$, as all morphisms are finite subsets of $\omega \times \omega$, i.e. elements of the set $[\omega \times \omega]^{<\omega}$ (cf. Section 1.2). By Proposition 1.2.2, any primitive recursive function with domain ω resp. $[\omega \times \omega]^{<\omega}$ exists as a set. Thus functors from the category of natural numbers into some other category will be set-sized. The same applies to natural transformations between such functors. For example, the restriction of the "finite subset functor" $[\cdot]^{<\omega}$ to the category of natural numbers exists as a set. Of course $[n]^{<\omega}$ is the full power set of $n = \{0, \dots, n-1\}$. We need some notation concerning the equivalence of categories: Write $|a|$ resp. $\operatorname{en}_a:|a|\to a$ for the cardinality resp. increasing enumeration of a finite order a. As they are unique, these objects can be computed by primitive recursive functions (with parameter ω, as always). For example,

$$a\mapsto\operatorname{en}_a=\bigcup\{f\in[\omega\times a]^{<\omega}\mid \text{“}f:n\to a \text{ is an isomorphism for some } n\in\omega\text{”}\}$$

is primitive recursive by Corollary 1.1.10. Given an embedding $f:a\to b$, we write $|f|:|a|\to|b|$ for the unique increasing function with

$$\operatorname{en}_b\circ|f|=f\circ\operatorname{en}_a.$$

This makes $|\cdot|$ a functor from linear orders to natural numbers, and en. a natural transformation from $|\cdot|$ to the identity.

Definition 2.3.1. A set-sized prae-dilator consists of

(i) a set-sized functor $n\mapsto T_n=(T_n,<_{T_n})$, $f\mapsto T_f$ from the category of natural numbers to the category of linear orders and
(ii) a set-sized natural transformation $\operatorname{supp}:T\Rightarrow[\cdot]^{<\omega}$ that yields supports, in the sense that any $\sigma\in T_n$ lies in the range of $T_{\iota_\sigma\circ\operatorname{en}_\sigma}:T_{|\operatorname{supp}_n(\sigma)|}\to T_n$, where $\iota_\sigma:\operatorname{supp}_n(\sigma)\hookrightarrow n$ is the inclusion and en_σ abbreviates $\operatorname{en}_{\operatorname{supp}_n(\sigma)}$.

The notion of set-sized prae-dilator can be formalized by a bounded $\mathcal{L}_{\text{pr}}$-formula with parameter ω. Thus, by Proposition 1.1.9, being a set-sized prae-dilator is a primitive-recursive property. Following Girard [**28**, Theorem 2.1.5], a set-sized prae-dilator can be extended to an endo-functor of linear orders. In the following we give an explicit construction, to show that this extension is primitive recursive. The idea is that the pair $\langle a,\sigma\rangle$ with $a\in[X]^{<\omega}$ and $\sigma\in T_{|a|}$ represents the element $T_{\iota_a^X\circ\operatorname{en}_a}(\sigma)$ of T_X, where $\iota_a^X:a\hookrightarrow X$ is the inclusion. To make sure that every element is only represented once, we want to choose a minimal. This can be expressed by the condition $\operatorname{supp}_{|a|}(\sigma)=\{0,\dots,|a|-1\}=|a|$.

Definition 2.3.2. Given a set-sized prae-dilator $T=(T,\operatorname{supp})$ and a linear order X we define the set

$$D_X^T:=\{\langle a,\sigma\rangle\mid a\in[X]^{<\omega}\wedge\sigma\in T_{|a|}\wedge\operatorname{supp}_{|a|}(\sigma)=|a|\}.$$

The binary relation $<_X^T$ on D_X^T is given by

$$\langle a,\sigma\rangle<_X^T\langle b,\tau\rangle\quad\Leftrightarrow\quad T_{|\iota_a^{a\cup b}|}(\sigma)<_{T_{|a\cup b|}}T_{|\iota_b^{a\cup b}|}(\tau).$$

With each order embedding $f : X \to Y$ we associate the function

$$D_f^T : D_X^T \to D_Y^T, \qquad D_f^T(\langle a, \sigma\rangle) = \langle [f]^{<\omega}(a), \sigma\rangle.$$

Finally, we define the family of functions

$$\operatorname{supp}_X^T : D_X^T \to [X]^{<\omega}, \qquad \operatorname{supp}_X^T(\langle a, \sigma\rangle) = a.$$

To see that $D_f^T(\langle a, \sigma\rangle)$ is an element of D_Y^T it suffices to note $|[f]^{<\omega}(a)| = |a|$. The functions $(T, X) \mapsto D_X^T = (D_X^T, <_X^T)$, $(T, f) \mapsto D_f^T$ and $(T, X) \mapsto \operatorname{supp}_X^T$ are primitive recursive by the results of Sections 1.1 and 1.2. The following justifies the term "set-sized prae-dilator":

LEMMA 2.3.3. *If $T = (T, \operatorname{supp})$ is a set-sized prae-dilator then $(D^T, \operatorname{supp}^T)$ is a prae-dilator in the sense of Definition 2.2.1.*

PROOF. We begin by showing that $(D_X^T, <_X^T)$ is a linear order. Irreflexivity and transitivity are straightforward (for the latter, compose with $\iota_{a\cup b}^{a\cup b\cup c}$ and observe $|\iota_{a\cup b}^{a\cup b\cup c}| \circ |\iota_a^{a\cup b}| = |\iota_a^{a\cup b\cup c}|$ by functoriality). The claim that $<_X^T$ is trichotomous is easily reduced to the implication

$$T_{|\iota_a^{a\cup b}|}(\sigma) = T_{|\iota_b^{a\cup b}|}(\tau) \quad \Rightarrow \quad \langle a, \sigma\rangle = \langle b, \sigma\rangle.$$

Using the naturality of supp and the condition $\operatorname{supp}_{|a|}(\sigma) = |a|$ we get

$$\begin{aligned}[][\mathrm{en}_{a\cup b}]^{<\omega} \circ \operatorname{supp}_{|a\cup b|} \circ T_{|\iota_a^{a\cup b}|}(\sigma) = \\ = [\mathrm{en}_{a\cup b}]^{<\omega} \circ [|\iota_a^{a\cup b}|]^{<\omega} \circ \operatorname{supp}_{|a|}(\sigma) = [\iota_a^{a\cup b} \circ \mathrm{en}_a]^{<\omega}(|a|) = a.\end{aligned}$$

Thus a can be recovered from $T_{|\iota_a^{a\cup b}|}(\sigma)$, and $T_{|\iota_a^{a\cup b}|}(\sigma) = T_{|\iota_b^{a\cup b}|}(\tau)$ implies $a = b$. Then $|\iota_a^{a\cup b}| = |\iota_b^{a\cup b}|$ is the identity on $|a| = |a \cup b|$, and we get $\sigma = \tau$ as desired. Next, consider an embedding $f : X \to Y$. The claim that D_f^T is an embedding of $(D_X^T, <_X^T)$ into $(D_Y^T, <_Y^T)$ is easily reduced to $|[f]^{<\omega}(a) \cup [f]^{<\omega}(b)| = |a \cup b|$ and $|\iota_{[f]^{<\omega}(a)}^{[f]^{<\omega}(a)\cup[f]^{<\omega}(b)}| = |\iota_a^{a\cup b}|$. By uniqueness, the latter follows from

$$\begin{aligned}\mathrm{en}_{[f]^{<\omega}(a)\cup[f]^{<\omega}(b)} \circ |\iota_a^{a\cup b}| = (f \restriction a \cup b) \circ \mathrm{en}_{a\cup b} \circ |\iota_a^{a\cup b}| = (f \restriction a \cup b) \circ \iota_a^{a\cup b} \circ \mathrm{en}_a = \\ = \iota_{[f]^{<\omega}(a)}^{[f]^{<\omega}(a)\cup[f]^{<\omega}(b)} \circ (f \restriction a) \circ \mathrm{en}_a = \iota_{[f]^{<\omega}(a)}^{[f]^{<\omega}(a)\cup[f]^{<\omega}(b)} \circ \mathrm{en}_{[f]^{<\omega}(a)}\,.\end{aligned}$$

Functoriality is easy to observe. The naturality of supp^T amounts to

$$\operatorname{supp}_Y^T \circ D_f^T(\langle a, \sigma\rangle) = \operatorname{supp}_Y^T(\langle [f]^{<\omega}(a), \sigma\rangle) = [f]^{<\omega}(a) = [f]^{<\omega} \circ \operatorname{supp}_X^T(\langle a, \sigma\rangle).$$

Finally, we must see that supp^T_X computes supports: Consider $\langle a,\sigma\rangle\in D^T_X$ and the inclusion map $\iota:\operatorname{supp}^T_X(\langle a,\sigma\rangle)=a\hookrightarrow X$. We clearly have $\langle a,\sigma\rangle\in D^T_a$ and

$$D^T_\iota(\langle a,\sigma\rangle)=\langle[\iota]^{<\omega}(a),\sigma\rangle=\langle a,\sigma\rangle.$$

Thus $\langle a,\sigma\rangle$ lies in the range of D^T_ι, as required. □

Conversely, if $T^u=(T^u,\operatorname{supp}^u)$ is a prae-dilator in the sense of Definition 2.2.1 then we write $T^u\restriction\omega$ for its restriction to the category of natural numbers. Proposition 1.2.2 tells us that $T^u\restriction\omega$ exists as a set, and that $u\mapsto T^u\restriction\omega$ is a primitive recursive operation.

LEMMA 2.3.4. *If $T^u=(T^u,\operatorname{supp}^u)$ is a prae-dilator in the sense of Definition 2.2.1 then the restriction $T^u\restriction\omega$ is a set-sized prae-dilator.*

PROOF. The only slight change concerns the supports: By Definition 2.2.1, any $\sigma\in T^u_n$ lies in the range of $T^u_{\iota_\sigma}$, where $\iota_\sigma:\operatorname{supp}^u_n(\sigma)\hookrightarrow n$ is the inclusion. As $\operatorname{en}_\sigma:|\operatorname{supp}^u_n(\sigma)|\to\operatorname{supp}^u_n(\sigma)$ is an order isomorphism, σ does also lie in the range of $T^u_{\iota_\sigma\circ\operatorname{en}_\sigma}$, as required for a set-sized prae-dilator. Note that we could not refer to $T^u_{\iota_\sigma}$ in the definition of set-sized prae-dilator, because the domain of ι_σ is not a natural number. □

Of course, we cannot expect $D^{T^u\restriction\omega}_X$ and T^u_X to be equal as sets. Nevertheless we can relate them as follows:

DEFINITION 2.3.5. Given a prae-dilator $T^u=(T^u,\operatorname{supp}^u)$ and a linear order X we define the function

$$\eta^{T^u}_X:D^{T^u\restriction\omega}_X\to T^u_X,\qquad \eta^{T^u}_X(\langle a,\sigma\rangle)=T^u_{\iota^X_a\circ\operatorname{en}_a}(\sigma).$$

Given that $(u,f)\mapsto T^u_f$ is primitive recursive, it is clear that $\eta^{T^u}:X\mapsto\eta^{T^u}_X$ is primitive recursive with parameter u. Note that the definition of η^{T^u} does depend on the primitive recursive definition of T^u; in contrast, $(T,X)\mapsto D^T_X$ for set-sized T was given by a single primitive recursive definition.

PROPOSITION 2.3.6. *Consider a prae-dilator $T^u=(T^u,\operatorname{supp}^u)$ and the prae-dilator $(D^{T^u\restriction\omega},\operatorname{supp}^{T^u\restriction\omega})$ reconstructed from the set-sized prae-dilator $T^u\restriction\omega$. Then $\eta^{T^u}:D^{T^u\restriction\omega}\Rightarrow T^u$ is a natural isomorphism and we have*

$$\operatorname{supp}^u_X\circ\eta^{T^u}_X=\operatorname{supp}^{T^u\restriction\omega}_X.$$

PROOF. To see that $\eta^{T^u}_X : D^{T^u\restriction\omega}_X \to T^u_X$ is an order embedding, consider elements $\langle a,\sigma\rangle <^{T^u\restriction\omega}_X \langle b,\tau\rangle$ of $D^{T^u\restriction\omega}_X$. Then $T^u_{|\iota^{a\cup b}_a|}(\sigma) <_{T^u_{|a\cup b|}} T^u_{|\iota^{a\cup b}_b|}(\tau)$ holds by the definition of $<^{T^u\restriction\omega}_X$. Using $\iota^X_a\circ\operatorname{en}_a = \iota^X_{a\cup b}\circ\iota^{a\cup b}_a\circ\operatorname{en}_a = \iota^X_{a\cup b}\circ\operatorname{en}_{a\cup b}\circ|\iota^{a\cup b}_a|$ we get

$$\begin{aligned}\eta^{T^u}_X(\langle a,\sigma\rangle) = T^u_{\iota^X_a\circ\operatorname{en}_a}(\sigma) = T^u_{\iota^X_{a\cup b}\circ\operatorname{en}_{a\cup b}}\circ T^u_{|\iota^{a\cup b}_a|}(\sigma) <_{T^u_X} \\ <_{T^u_X} T^u_{\iota^X_{a\cup b}\circ\operatorname{en}_{a\cup b}}\circ T^u_{|\iota^{a\cup b}_b|}(\tau) = T^u_{\iota^X_b\circ\operatorname{en}_b}(\tau) = \eta^{T^u}_X(\langle b,\tau\rangle),\end{aligned}$$

as required. Next, consider an embedding $f : X \to Y$. For $a\in[X]^{<\omega}$ we have $(f\restriction a)\circ\operatorname{en}_a = \operatorname{en}_{[f]^{<\omega}(a)}$, as both functions enumerate $[f]^{<\omega}(a)$ in increasing order. This yields

$$f\circ\iota^X_a\circ\operatorname{en}_a = \iota^Y_{[f]^{<\omega}(a)}\circ(f\restriction a)\circ\operatorname{en}_a = \iota^Y_{[f]^{<\omega}(a)}\circ\operatorname{en}_{[f]^{<\omega}(a)}$$

and then

$$\begin{aligned}T^u_f\circ\eta^{T^u}_X(\langle a,\sigma\rangle) = T^u_{f\circ\iota^X_a\circ\operatorname{en}_a}(\sigma) = T^u_{\iota^Y_{[f]^{<\omega}(a)}\circ\operatorname{en}_{[f]^{<\omega}(a)}}(\sigma) = \\ = \eta^{T^u}_Y(\langle [f]^{<\omega}(a),\sigma\rangle) = \eta^{T^u}_Y\circ D^{T^u\restriction\omega}_f(\langle a,\sigma\rangle).\end{aligned}$$

To infer that η^{T^u} is a natural isomorphism it suffices to show that the components $\eta^{T^u}_X : D^{T^u\restriction\omega}_X \to T^u_X$ are surjective. Consider $\sigma\in T^u_X$ and set $a:=\operatorname{supp}^u_X(\sigma)$. By the definition of prae-dilator, σ lies in the range of $T^u_{\iota^X_a}$. As $\operatorname{en}_a : |a|\to a$ is an isomorphism we can write $\sigma = T^u_{\iota^X_a\circ\operatorname{en}_a}(\sigma_0)$ for some $\sigma_0\in T^u_{|a|}$. Using the naturality of supp^u we get

$$[\iota^X_a\circ\operatorname{en}_a]^{<\omega}\circ\operatorname{supp}^u_{|a|}(\sigma_0) = \operatorname{supp}^u_X\circ T^u_{\iota^X_a\circ\operatorname{en}_a}(\sigma_0) = \operatorname{supp}^u_X(\sigma) = a.$$

For cardinality reasons this implies $\operatorname{supp}^u_{|a|}(\sigma_0) = |a|$, which means $\langle a,\sigma_0\rangle\in D^{T^u\restriction\omega}_X$. In view of $\eta^{T^u}_X(\langle a,\sigma_0\rangle) = T^u_{\iota^X_a\circ\operatorname{en}_a}(\sigma_0) = \sigma$ we have established that $\eta^{T^u}_X$ is surjective. Using the naturality of supp^u and the condition $\operatorname{supp}^u_{|a|}(\sigma) = |a|$ we also get

$$\begin{aligned}\operatorname{supp}^u_X\circ\eta^{T^u}_X(\langle a,\sigma\rangle) = \operatorname{supp}^u_X\circ T^u_{\iota^X_a\circ\operatorname{en}_a}(\sigma) = [\iota^X_a\circ\operatorname{en}_a]^{<\omega}\circ\operatorname{supp}^u_{|a|}(\sigma) = \\ = [\iota^X_a\circ\operatorname{en}_a]^{<\omega}(|a|) = a = \operatorname{supp}^{T^u\restriction\omega}_X(\langle a,\sigma\rangle),\end{aligned}$$

as demanded by the proposition. □

By Lemma 2.3.3 and the previous proposition, the notions of prae-dilator and set-sized prae-dilator are essentially equivalent. An important application of this result can be found in Proposition 2.3.10 below. It also follows that prae-dilators are automatically primitive recursive. However, one has to be careful to interpret this claim correctly:

Remark 2.3.7. Working in a stronger base theory, we may come to consider prae-dilators which are not given by primitive recursive set functions. If T is such a prae-dilator, the previous result is still valid: It yields an equivalent prae-dilator $D^{T\restriction\omega}$ which is primitive recursive in the set-sized parameter $T\restriction\omega$ (the natural isomorphism $\eta^T : D^{T\restriction\omega} \Rightarrow T$ will no longer be primitive recursive). To point out one consequence of this result, assume that $T\restriction\omega$ has hereditary cardinality at most κ. For any cardinal μ we then have

$$\operatorname{card}(T_\mu) = \operatorname{card}(D_\mu^{T\restriction\omega}) \leq \max\{\mu, \kappa\},$$

as primitive recursive set functions cannot raise infinite cardinalities (see the introduction of [**44**]). In particular, the function $\mu \mapsto 2^\mu$ (cardinal arithmetic) cannot be a dilator. A similar argument applies to admissible ordinals at the place of cardinals (see [**28**, Remark 2.3.6]). It is important to note that the associated order-types of a dilator do not need to be primitive recursive: Assume that T_X is the relativized notation system $\varphi X0$ to be found in [**73**] and [**52**]. As a notation system, $\varphi X0$ has a primitive recursive construction. If X has order-type α then $\varphi X0$ has order-type $\varphi\alpha0$, which now denotes the corresponding value of the Veblen function. However, as a function on ordinals, $F(\alpha) := \varphi\alpha0$ is not primitive recursive: Aiming at a contradiction, assume that F is primitive recursive with parameters in $\mathbb{V}_\beta$ (von Neumann hierarchy). By induction over primitive recursive definitions one finds a number n such that $\alpha, \beta < \gamma$ implies $F(\alpha) \in \mathbb{V}_{\varphi n\gamma}$. Pick an ordinal $\alpha > n$ with $\alpha, \beta < \varphi\alpha0$. Then we get $\varphi\alpha0 = F(\alpha) < \varphi n(\varphi\alpha0) = \varphi\alpha0$, which is the desired contradiction. This is linked to the fact that the collapse of a well-founded relation cannot be computed primitive recursively (unless the relation is set membership, cf. Proposition 1.2.4 and Definition 1.4.2). We point out that Girard [**28**, Definition 2.3.1] defines dilators as endo-functors on the category of ordinals (rather than arbitrary well-orders). This has the advantage that equivalent dilators are equal (since isomorphic ordinals are equal, cf. [**28**, Remark 2.1.6(ii)]). However, if one only allows ordinals as values then it is no longer possible to find a primitive recursive copy of each dilator, as just seen.

Preservation of well-foundedness cannot be tested on the category of natural numbers. It can be tested on the countable ordinals (cf. [**28**, Theorem 2.1.15]), but these do not form a set in our base theory **PRS**ω. Indeed, the notion of dilator is Π^1_2-universal (as claimed in the foreword of [**28**]), so that its logical complexity cannot be reduced.

DEFINITION 2.3.8. A set-sized prae-dilator T is called a set-sized dilator if $(D^T_X, <^T_X)$ is well-founded for any well-order X.

Let us relate this to dilators in the sense of Definition 2.2.1:

COROLLARY 2.3.9. *If T is a set-sized dilator then $(D^T, \operatorname{supp}^T)$ is a dilator. Conversely, if $T^u = (T^u, \operatorname{supp}^u)$ is a dilator then $T^u \restriction \omega$ is a set-sized dilator.*

PROOF. From Lemma 2.3.3 we know that $(D^T, \operatorname{supp}^T)$ is a prae-dilator in the sense of Definition 2.2.1. By assumption the function $X \mapsto D^T_X$ preserves well-foundedness, which makes $(D^T, \operatorname{supp}^T)$ a dilator. Conversely, Lemma 2.3.4 tells us that $T^u \restriction \omega$ is a set-sized prae-dilator. By the definition of dilator, T^u_X is well-founded for any well-order X. By Proposition 2.3.6 the order $D^{T^u \restriction \omega}_X$ is isomorphic to T^u_X. Thus $D^{T^u \restriction \omega}_X$ is also well-founded, making $T^u \restriction \omega$ a set-sized dilator. $\square$

Recall the primitive recursive construction $u \mapsto \operatorname{BH}(T^u) = (\operatorname{BH}(T^u), <_{\operatorname{BH}(T^u)})$ from the previous section. With D^T at the place of T^u we get a primitive recursive function $T \mapsto \operatorname{BH}(D^T)$. If $T = (T, \operatorname{supp})$ is a set-sized prae-dilator then D^T is a prae-dilator in the sense of Definition 2.2.1, and Lemma 2.2.18 tells us that $\operatorname{BH}(D^T)$ is a linear order. If T is a set-sized dilator then D^T is a dilator, and the predicative Bachmann-Howard principle of Definition 2.2.19 asserts that $\operatorname{BH}(D^T)$ is well-founded. Indeed, this instance of the predicative Bachmann-Howard principle is universal:

PROPOSITION 2.3.10. *The statement*

$$\forall_T(\text{“}T = (T, \operatorname{supp}) \textit{ a set-sized dilator”} \to \text{“}(\operatorname{BH}(D^T), <_{\operatorname{BH}(D^T)}) \textit{ is well-founded”})$$

implies any other instance of the predicative Bachmann-Howard principle.

PROOF. Given primitive recursive functions $(u, X) \mapsto T^u_X$, $(u, f) \mapsto T^u_f$ and $(u, X) \mapsto \operatorname{supp}^u_X$, consider a value of u and assume that $T^u = (T^u, \operatorname{supp}^u)$ is a dilator. To deduce the corresponding instance of the predicative Bachmann-Howard principle, we must show that the order $\operatorname{BH}(T^u)$ is well-founded. The previous corollary ensures that $T^u \restriction \omega$ is a set-sized dilator. By the statement in the proposition, this implies that $\operatorname{BH}(D^{T^u \restriction \omega})$ is well-founded. Thus it suffices to give an embedding of $\operatorname{BH}(T^u)$ into $\operatorname{BH}(D^{T^u \restriction \omega})$. From the previous section, recall that $\operatorname{BH}(T^u)$ and $\operatorname{BH}(D^{T^u \restriction \omega})$ are the direct limits of approximations $\operatorname{BH}_n(T^u)$ resp. $\operatorname{BH}_n(D^{T^u \restriction \omega})$. The idea is to construct embeddings

$$h_n : \operatorname{BH}_n(T^u) \to \operatorname{BH}_n(D^{T^u \restriction \omega})$$

which are compatible in the sense that they satisfy

$$h_{n+1}\circ i_{\mathrm{BH}_n(T^u)}=i_{\mathrm{BH}_n(D^{T^u\restriction\omega})}\circ h_n:\mathrm{BH}_n(T^u)\to\mathrm{BH}_{n+1}(D^{T^u\restriction\omega}).$$

Assuming that we have these, recall the functions $j_n:\mathrm{BH}_n(D^{T^u\restriction\omega})\to\mathrm{BH}(D^{T^u\restriction\omega})$ with $j_{n+1}\circ i_{\mathrm{BH}_n(D^{T^u\restriction\omega})}=j_n$, and observe $j_{n+1}\circ h_{n+1}\circ i_{\mathrm{BH}_n(T^u)}=j_n\circ h_n$. Then Lemma 2.2.17 allows us to glue the embeddings $j_n\circ h_n:\mathrm{BH}_n(T^u)\to\mathrm{BH}(D^{T^u\restriction\omega})$ to an embedding of $\mathrm{BH}(T^u)$ into $\mathrm{BH}(D^{T^u\restriction\omega})$, as desired. Intuitively, it should be clear that the natural isomorphism from Proposition 2.3.6 allows to construct the maps h_n, but the details are somewhat tedious: For $n=0$ we take the identity on $\mathrm{BH}_0(T^u)=\varepsilon_0=\mathrm{BH}_0(D^{T^u\restriction\omega})$. To construct h_{n+1} from h_n, write $\mu:T^u\Rightarrow D^{T^u\restriction\omega}$ for the inverse of the natural isomorphism η^{T^u} from Proposition 2.3.6. For an element $\sigma\in T^u_{\mathrm{BH}_n(T^u)}$ we then have

$$\sigma':=\mu_{\mathrm{BH}_n(D^{T^u\restriction\omega})}\circ T^u_{h_n}(\sigma)\in D^{T^u\restriction\omega}_{\mathrm{BH}_n(D^{T^u\restriction\omega})}.$$

By the obvious recursion on terms, extend the stipulation $h_n^\vartheta(\mathfrak{E}_\sigma):=\mathfrak{E}_{\sigma'}$ to a map

$$h_n^\vartheta:\vartheta^0(T^u_{\mathrm{BH}_n(T^u)})\to\vartheta^0(D^{T^u\restriction\omega}_{\mathrm{BH}_n(D^{T^u\restriction\omega})}).$$

Let h_{n+1} be the restriction of h_n^ϑ to $\mathrm{BH}_{n+1}(T^u)=\vartheta(T^u_{\mathrm{BH}_n(T^u)})\cap\Omega\subseteq\vartheta^0(T^u_{\mathrm{BH}_n(T^u)})$. To see that this is an order embedding into $\mathrm{BH}_{n+1}(D^{T^u\restriction\omega})$, one first verifies the compatibility condition $h_{n+1}\circ i_{\mathrm{BH}_n(T^u)}=i_{\mathrm{BH}_n(D^{T^u\restriction\omega})}\circ h_n$ and, relying on it, the equation

$$[h_n^\vartheta]^{<\omega}\circ E^\vartheta_{\mathrm{BH}_n(T^u)}=E^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})}\circ h_n^\vartheta.$$

The interesting case is that of a term $\mathfrak{E}_\sigma$ for $n=m+1>0$. Here, the compatibility condition is readily reduced to the following computation, which uses the induction hypothesis $h_n\circ i_{\mathrm{BH}_m(T^u)}=i_{\mathrm{BH}_m(D^{T^u\restriction\omega})}\circ h_m$ and the naturality of μ:

$$\begin{aligned}\mu_{\mathrm{BH}_n(D^{T^u\restriction\omega})}\circ T^u_{h_n}\circ T^u_{i_{\mathrm{BH}_m(T^u)}}(\sigma)=\mu_{\mathrm{BH}_n(D^{T^u\restriction\omega})}\circ T^u_{i_{\mathrm{BH}_m(D^{T^u\restriction\omega})}}\circ T^u_{h_m}(\sigma)=\\ =D^{T^u\restriction\omega}_{i_{\mathrm{BH}_m(D^{T^u\restriction\omega})}}\circ\mu_{\mathrm{BH}_m(D^{T^u\restriction\omega})}\circ T^u_{h_m}(\sigma).\end{aligned}$$

Using compatibility, the equation $\operatorname{supp}^u_{\mathrm{BH}_n(T^u)}=\operatorname{supp}^{T^u\restriction\omega}_{\mathrm{BH}_n(T^u)}\circ\mu_{\mathrm{BH}_n(T^u)}$ from Proposition 2.3.6, the naturality of $\operatorname{supp}^{T^u\restriction\omega}$, and the naturality of μ we also get

$$\begin{aligned}&[h_n^\vartheta]^{<\omega}\circ E^\vartheta_{\mathrm{BH}_n(T^u)}(\mathfrak{E}_\sigma)=\\ &\quad=[h_{n+1}]^{<\omega}\circ[i_{\mathrm{BH}_n(T^u)}]^{<\omega}\circ\operatorname{supp}^u_{\mathrm{BH}_n(T^u)}(\sigma)=\\ &\quad=[i_{\mathrm{BH}_n(D^{T^u\restriction\omega})}]^{<\omega}\circ[h_n]^{<\omega}\circ\operatorname{supp}^{T^u\restriction\omega}_{\mathrm{BH}_n(T^u)}(\mu_{\mathrm{BH}_n(T^u)}(\sigma))=\\ &\quad=[i_{\mathrm{BH}_n(D^{T^u\restriction\omega})}]^{<\omega}\circ\operatorname{supp}^{T^u\restriction\omega}_{\mathrm{BH}_n(D^{T^u\restriction\omega})}(D^{T^u\restriction\omega}_{h_n}\circ\mu_{\mathrm{BH}_n(T^u)}(\sigma))=\end{aligned}$$

$$= [i_{\mathrm{BH}_n(D^{T^u\restriction\omega})}]^{<\omega} \circ \mathrm{supp}^{T^u\restriction\omega}_{\mathrm{BH}_n(D^{T^u\restriction\omega})}(\mu_{\mathrm{BH}_n(D^{T^u\restriction\omega})} \circ T^u_{h_n}(\sigma)) =$$
$$= E^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} \circ h^\vartheta_n(\mathfrak{E}_\sigma).$$

Using these results one can establish

$$r \in \vartheta(T^u_{\mathrm{BH}_n(T^u)}) \quad\Rightarrow\quad h^\vartheta_n(r) \in \vartheta(D^{T^u\restriction\omega}_{\mathrm{BH}_n(D^{T^u\restriction\omega})}),$$
$$s <^\vartheta_{\mathrm{BH}_n(T^u)} t \quad\Rightarrow\quad h^\vartheta_n(s) <^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} h^\vartheta_n(t)$$

by simultaneous induction on $L^\vartheta_{\mathrm{BH}_n(T^u)}(r)$ resp. $L^\vartheta_{\mathrm{BH}_n(T^u)}(s) + L^\vartheta_{\mathrm{BH}_n(T^u)}(t)$. The most interesting case is $s = \vartheta s' <^\vartheta_{\mathrm{BH}_n(T^u)} \vartheta t' = t$. Assume that this holds because we have $s' <^\vartheta_{\mathrm{BH}_n(T^u)} t'$ and $r' <^\vartheta_{\mathrm{BH}_n(T^u)} t$ for all $r' \in E^\vartheta_{\mathrm{BH}_n(T^u)}(s')$ (note that $r \in \vartheta(T^u_{\mathrm{BH}_n(T^u)})$ is automatic by Lemma 2.2.11). By induction hypothesis we have $h^\vartheta_n(s') <^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} h^\vartheta_n(t')$. To get $h^\vartheta_n(s) = \vartheta h^\vartheta_n(s') <^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} \vartheta h^\vartheta_n(t') = h^\vartheta_n(t)$ we have to show

$$r <^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} h^\vartheta_n(t) \qquad \text{for all } r \in E^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} \circ h^\vartheta_n(s').$$

By the above, any such r can be written as $r = h^\vartheta_n(r')$ with $r' \in E^\vartheta_{\mathrm{BH}_n(T^u)}(s')$. Lemma 2.2.6 ensures $L^\vartheta_{\mathrm{BH}_n(T^u)}(r') \leq L^\vartheta_{\mathrm{BH}_n(T^u)}(s') < L^\vartheta_{\mathrm{BH}_n(T^u)}(s)$, so that the induction hypothesis yields $r = h^\vartheta_n(r') <^\vartheta_{\mathrm{BH}_n(D^{T^u\restriction\omega})} h^\vartheta_n(t)$, as required. It follows that the restriction $h_{n+1} = h^\vartheta_n \restriction (\vartheta(T^u_{\mathrm{BH}_n(T^u)}) \cap \Omega)$ is indeed an embedding into $\vartheta(D^{T^u\restriction\omega}_{\mathrm{BH}_n(D^{T^u\restriction\omega})}) \cap \Omega = \mathrm{BH}_{n+1}(D^{T^u\restriction\omega})$. Of course, one can use the inverse η^{T^u} of μ to get an inverse embedding $\mathrm{BH}(D^{T^u\restriction\omega}) \to \mathrm{BH}(T^u)$, but this was not required. □

Our next goal is to formalize the concept of set-sized dilator in the subtheory $\mathbf{RCA_0}$ (recursive comprehension) of second-order arithmetic. For an introduction to this theory we refer to [**87**]; as $\mathbf{RCA_0}$ extends the first-order theory $\mathbf{I\Sigma_1}$, the information from [**37**] is also very useful. In the following, "primitive recursive" has the usual number-theoretic (rather than set-theoretic) sense. It is quite natural to formalize dilators in $\mathbf{RCA_0}$, given that Girard [**28**, Section 0.2.1] has stressed their finitistic nature.

To represent a functor from natural numbers to linear orders we first need a map $n \mapsto T_n = (T_n, <_{T_n})$ on objects. Using a suitable coding of tuples, this can be represented by the set

$$T^0 = \{\langle 0, n, \sigma\rangle \mid \sigma \in T_n\} \cup \{\langle 1, n, \sigma, \tau\rangle \mid \sigma <_{T_n} \tau\}.$$

Officially, we have the converse relation between T^0 and $(T_n, <_{T_n})$: The expressions $\sigma \in T_n$ and $\sigma <_{T_n} \tau$ are to be read as abbreviations for $\langle 0, n, \sigma\rangle \in T^0$ resp. $\langle 1, n, \sigma, \tau\rangle \in T^0$. Thus these relations are Δ^0_1 in $\mathbf{RCA_0}$, and they can freely feature in comprehension or induction formulas. In contrast, the statement "T_n is a linear order" has complexity Π^0_1. This is different from the set-theoretic context in which we have worked so far: There, the set T_n counted as a bound on quantifiers, and the statement "T_n is a linear order" was expressed by a bounded formula.

Next, we need a map $f \mapsto T_f$ on morphisms. Finite subsets of $\mathbb{N}$ and functions between such sets can be represented by (unique) natural numbers, using a primitive recursive coding of sequences (see [**87**, Section II.2] or [**37**, Section I.1(b)]). The function $n \mapsto$ "code of $n = \{0, \ldots, n-1\}$" is primitive recursive; here $n = \{0, \ldots, n-1\}$ is merely an abbreviation, in contrast to the set-theoretic setting. The statements "f codes a strictly increasing function (i.e. a morphism) from $n = \{0, \ldots, n-1\}$ to $m = \{0, \ldots, m-1\}$" and "h is the composition of g and f" are Δ^0_1. Furthermore, $\mathbf{RCA_0}$ shows that a unique composition $g \circ f$ exists. We can now represent the map $f \mapsto T_f$ by the set

$$T^1 = \{\langle f, \sigma, \tau\rangle \,|\, T_f(\sigma) = \tau\}.$$

Officially, $T_f(\sigma) = \tau$ becomes an abbreviation for the Δ^0_1-relation $\langle f, \sigma, \tau\rangle \in T^1$. Clearly, "$T_f$ is an embedding of T_n into T_m" is an arithmetical statement (of complexity Π^0_2, because of totality). The conditions $T_{\mathrm{Id}_n} = \mathrm{Id}_{T_n}$ and $T_g \circ T_f = T_{g\circ f}$ are also arithmetical. Overall, we have an arithmetical formalization of the statement "$T = (T^0, T^1)$ is a functor from natural numbers to (countable) linear orders".

To get a prae-dilator, we also need a natural transformation $\mathrm{supp} : T \Rightarrow [\cdot]^{<\omega}$. The relation $a \in [n]^{<\omega}$ is primitive recursive, and the code of any $a \in [n]^{<\omega}$ is primitive recursively bounded. Thus the function $n \mapsto [n]^{<\omega}$ is $\mathbf{RCA_0}$-provably total, by bounded comprehension (see [**87**, Theorems II.3.9 and II.2.5] or [**37**, Theorems I.1.36 and I.1.39]). For $f : n \to m$, the relation $[f]^{<\omega}(a) = b$ is primitive recursive, and the elements of $[f]^{<\omega} \subseteq [n]^{<\omega} \times [m]^{<\omega}$ are primitive recursively bounded. So again, the function $f \mapsto [f]^{<\omega}$ is $\mathbf{RCA_0}$-provably total. Now, our natural transformation is represented by the set

$$\mathrm{supp} = \{\langle n, \sigma, a\rangle \,|\, \mathrm{supp}_n(\sigma) = a\}.$$

Again, $\mathrm{supp}_n(\sigma) = a$ officially becomes an abbreviation for $\langle n, \sigma, a\rangle \in \mathrm{supp}$. By the above, the conditions "supp_n is a total function from T_n to $[n]^{<\omega}$" and $\mathrm{supp}_m \circ T_f = [f]^{<\omega} \circ \mathrm{supp}_n$, which make supp a natural transformation, are arithmetical. Given a (code of a) finite set a of numbers, we can primitive recursively compute its size $|a|$ and (a code of) the enumeration $\mathrm{en}_a : |a| \to a$. Given n and a subset $a \subseteq \{0, \ldots, n-1\} = n$ we can also compute the inclusion $\iota^n_a : a \hookrightarrow n$.

The condition $\sigma \in \operatorname{rng}(T_{\iota_\sigma^n \circ \mathrm{en}_\sigma})$ from Definition 2.3.1 can then be expressed by the Σ_1-formula

$$\exists_a(\operatorname{supp}_n(\sigma) = a \wedge \exists_{\sigma_0}(\sigma_0 \in T_{|a|} \wedge T_{\iota_a^n \circ \mathrm{en}_a}(\sigma_0) = \sigma)).$$

Note that the quantifier over $a \in [n]^{<\omega}$ is bounded, but not the quantifier over σ_0. We can now restate Definition 2.3.1 in second-order arithmetic:

DEFINITION 2.3.11 ($\mathbf{RCA_0}$). A coded prae-dilator consists of

(i) a functor $n \mapsto T_n = (T_n, <_{T_n})$, $f \mapsto T_f$ from natural numbers to linear orders (given by sets $T^0, T^1 \subseteq \mathbb{N}$, as explained above) and
(ii) a natural transformation $\operatorname{supp} : T \Rightarrow [\cdot]^{<\omega}$ (given by a set $\operatorname{supp} \subseteq \mathbb{N}$) which computes supports, in the sense that any $\sigma \in T_n$ lies in the range of $T_{\iota_\sigma^n \circ \mathrm{en}_\sigma}$ (as formalized above).

By the above, the statement that $T = (T^0, T^1)$ and supp form a coded prae-dilator is arithmetical. To define coded dilators we need the ordered sets $(D_X^T, <_X^T)$, as in Definition 2.3.8 (there will be no need to reconstruct the functions D_f^T and supp_X^T from Definition 2.3.2). Given $X \subseteq \mathbb{N}$, we can form the set

$$D_X^T = \{\langle a, \sigma\rangle \mid \text{“}a \text{ codes a finite subset of } X\text{”} \wedge \sigma \in T_{|a|} \wedge \operatorname{supp}_{|a|}(\sigma) = |a|\}$$

by Δ_1^0-comprehension. Above, we have considered enumerations $\mathrm{en}_a : |a| \to a$ with respect to the usual order on $a \subseteq \mathbb{N}$. Given a linear order $X = (X, <_X)$ and a finite set $a \subseteq X$ we can also consider the enumeration $\mathrm{en}_a^X : |a| \to a$ which is strictly increasing with respect to $<_X$. Note that $a \mapsto \mathrm{en}_a^X$ is primitive recursive relative to X. For finite sets $a \subseteq b \subseteq X$ we use bounded comprehension to construct

$$\mathrm{en}_{a,b}^X = \{\langle i, j\rangle \mid i < |a| \wedge j < |b| \wedge \mathrm{en}_a^X(i) = \mathrm{en}_b^X(j)\}.$$

In other words, the function $\mathrm{en}_{a,b}^X : |a| \to |b|$ satisfies $\mathrm{en}_b^X(\mathrm{en}_{a,b}^X(i)) = \mathrm{en}_a^X(i)$, with the inclusion $a \hookrightarrow b$ left implicit. It follows that $\mathrm{en}_{a,b}^X$ is strictly increasing. Note that $\mathrm{en}_{a,b}^X$ coincides with the function $|\iota_a^b|$ from Definition 2.3.2. Using the totality of the involved functions, the statement $T_{\mathrm{en}_{a,a\cup b}^X}(\sigma) <_{T_{|a\cup b|}} T_{\mathrm{en}_{b,a\cup b}^X}(\tau)$ is equivalent to both

$$\exists_{\sigma',\tau'}(T_{\mathrm{en}_{a,a\cup b}^X}(\sigma) = \sigma' \wedge T_{\mathrm{en}_{b,a\cup b}^X}(\tau) = \tau' \wedge \sigma' <_{T_{|a\cup b|}} \tau')$$

and

$$\forall_{\sigma',\tau'}(T_{\mathrm{en}_{a,a\cup b}^X}(\sigma) = \sigma' \wedge T_{\mathrm{en}_{b,a\cup b}^X}(\tau) = \tau' \to \sigma' <_{T_{|a\cup b|}} \tau').$$

Thus this condition is Δ^0_1, and $\mathbf{RCA_0}$ shows that the relation

$$\langle a,\sigma\rangle <^T_X \langle b,\tau\rangle \quad\Leftrightarrow\quad T_{\mathrm{en}^X_{a,a\cup b}}(\sigma) <_{T_{|a\cup b|}} T_{\mathrm{en}^X_{b,a\cup b}}(\tau)$$

on D^T_X exists as a set. Also, $\mathbf{RCA_0}$ proves that $(D^T_X,<^T_X)$ is a linear order whenever T is a coded prae-dilator and X is a linear order, as in Lemma 2.3.3. Before we define coded dilators, let us reflect on the definition of well-ordering (cf. [**87**, Definition I.6.1]):

LEMMA 2.3.12 ($\mathbf{RCA_0}$). *For any linear order $(X,<_X)$ the following characterizations of well-foundedness are equivalent:*

(i) There is no function $f:\mathbb{N}\to X$ with $f(n+1)<_X f(n)$ for all $n\in\mathbb{N}$.
(ii) Any non-empty $Z\subseteq X$ has a $<_X$-minimal element.

PROOF. Aiming at the contrapositive of (i)⇒(ii), assume that $\emptyset\neq Z\subseteq X$ has no $<_X$-minimal element. Say that a sequence $s=\langle s_0,\dots,s_{n-1}\rangle$ is minimally descending in Z (abbreviated Z-m.d.) if s_0 is the $<_{\mathbb{N}}$-minimal element of Z and s_{i+1} is the $<_{\mathbb{N}}$-minimal element of $\{z\in Z\,|\,z<_X s_i\}$, for all $i<n-1$. Clearly, this is a Δ^0_1-property of sequences. If both s and t are Z-m.d., one verifies $s_i=t_i$ by induction on i (up to the length of the shorter sequence). Given that Z has no $<_X$-minimal element, a straightforward induction produces Z-m.d. sequences of arbitrary length. By Δ^0_1-comprehension this yields the function

$$\begin{aligned} f:&=\{\langle n,m\rangle\,|\,\exists_s(\text{“}s\text{ is }Z\text{-m.d. of length }n+1\text{”}\land m=s_n)\}=\\ &=\{\langle n,m\rangle\,|\,\forall_s(\text{“}s\text{ is }Z\text{-m.d. of length }n+1\text{”}\to m=s_n)\},\end{aligned}$$

which satisfies $f(n+1)<_X f(n)$ for all $n\in\mathbb{N}$. Aiming at the contrapositive of (ii)⇒(i), assume that $f:\mathbb{N}\to X$ satisfies $f(n+1)<_X f(n)$ for all $n\in\mathbb{N}$. The difficulty is that we cannot form the set $\{f(n)\,|\,n\in\mathbb{N}\}$ in $\mathbf{RCA_0}$. Instead, we build an infinite recursive subset by the usual construction: A sequence $s=\langle s_0,\dots,s_{n-1}\rangle$ is called minimally increasing for f (abbreviated f-m.i.) if we have $s_0=0$ and if s_{i+1} is $<_{\mathbb{N}}$-minimal with $f(s_i)<_{\mathbb{N}} f(s_{i+1})$ and $s_i<_{\mathbb{N}} s_{i+1}$. Again, this is a Δ^0_1-property of s, and any two f-m.i. sequences agree (up to the lenght of the shorter sequence). As f must have infinite range, we can inductively construct f-m.i. sequences of arbitrary length. By Δ^0_1-comprehension we define

$$\begin{aligned} Z:&=\{n\,|\,\exists_s(\text{“}s\text{ is }f\text{-m.i. of length }n+1\text{”}\land\exists_{i\leq n}\,n=f(s_i))\}=\\ &=\{n\,|\,\forall_s(\text{“}s\text{ is }f\text{-m.i. of length }n+1\text{”}\to\exists_{i\leq n}\,n=f(s_i))\}.\end{aligned}$$

Note that Z contains $f(0)$, as witnessed by the f-m.i. sequence of length $f(0)+1$. Finally, no element $n = f(s_i) \in Z$ can be $<_X$-minimal: Extend s to length $n+2$, to ensure that the entry s_{i+1} exists. In view of $f(s_{i+1}) >_{\mathbb{N}} f(s_i) = n \geq i$, further extend s to length $f(s_{i+1})+1 \geq n+2$. The resulting sequence witnesses $f(s_{i+1}) \in Z$. In view of $s_{i+1} >_{\mathbb{N}} s_i$ we have $f(s_{i+1}) <_X f(s_i)$. □

Using either definition of well-foundedness, we can now restate Definition 2.3.8:

DEFINITION 2.3.13 ($\mathbf{RCA_0}$). A coded prae-dilator (T, supp) is called a coded dilator if the linear order $(D^T_X, <^T_X)$ (as constructed above) is well-founded for any well-order X.

As well-foundedness is a Π^1_1-property, being a coded dilator has complexity Π^1_2, just as expected. Our next goal is to define a notation system $\vartheta(T)$ relative to a coded prae-dilator T. This is supposed to be a fixed-point, in the sense that any $\sigma \in D^T_{\vartheta(T)\cap\Omega}$ gives rise to a term $\mathfrak{E}_\sigma \in \vartheta(T)$, similar to the previous two sections. In the present set-up, such a σ is of the form $\langle a, \sigma_0\rangle$, where a is a finite subset of $\vartheta(T)\cap\Omega$ and we have $\sigma_0 \in T_{|a|}$ with $\operatorname{supp}_{|a|}(\sigma_0) = |a|$. Writing $a = \{s_0, \dots, s_{n-1}\}$ with $s_0 <_{\vartheta(T)} \cdots <_{\vartheta(T)} s_{n-1}$, we will represent $\mathfrak{E}_\sigma$ by the term $\mathfrak{E}^{s_0,\dots,s_{n-1}}_{\sigma_0}$. Note that we do not need to have completed the construction of $\vartheta(T)$ in order to write down this term; in contrast, in the previous section we had to construct the order $\mathrm{BH}(T)$ before we could consider elements $\sigma \in T_{\mathrm{BH}(T)}$ and terms $\mathfrak{E}_\sigma \in \vartheta(T_{\mathrm{BH}(T)})$.

LEMMA 2.3.14 ($\mathbf{RCA_0}$). *For any coded prae-dilator $T = (T, \operatorname{supp})$ there is a set $\vartheta(T)$ of terms, a binary relation $<_{\vartheta(T)}$ on the set $\vartheta(T)$, and a function $E_{\vartheta(T)} : \vartheta(T) \to [\vartheta(T)]^{<\omega}$, which are simultaneously generated as follows:*

- *(i) We have a term $0 \in \vartheta(T)$.*
- *(ii) We have a term $\Omega \in \vartheta(T)$.*
- *(iii) Given terms $s_0, \dots, s_{n-1} \in \vartheta(T)$ with $s_0 <_{\vartheta(T)} \cdots <_{\vartheta(T)} s_{n-1} <_{\vartheta(T)} \Omega$ and an element $\sigma \in T_n$ with $\operatorname{supp}_n(\sigma) = n$ we have a term $\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma \in \vartheta(T)$.*
- *(iv) For each term $s \in \vartheta(T)$ we have a term $\vartheta s \in \vartheta(T)$.*
- *(v) Given terms $s_0, \dots, s_n \in \vartheta(T)$ we have a term $\omega^{s_0} + \cdots + \omega^{s_n} \in \vartheta(T)$, provided that*
    - *either $n = 0$ and s_0 is not of the form Ω, $\mathfrak{E}^{t_0,\dots,t_{m-1}}_\sigma$ or ϑs,*
    - *or $n > 0$ and we have $s_n \leq_{\vartheta(T)} \cdots \leq_{\vartheta(T)} s_0$ (where $s \leq_{\vartheta(T)} t$ abbreviates $s <_{\vartheta(T)} t \vee s = t$, the latter referring to equality as terms).*

Given $s, t \in \vartheta(T)$, we have $s <_{\vartheta(T)} t$ if and only if one of the following holds:

(i) $s = 0$ *and* $t \neq 0$,

(ii) $s = \Omega$ *and we have*

- *either* $t = \mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$,
- *or* $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and* $s \leq_{\vartheta(T)} t_0$,

(iii) $s = \mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}$ *and we have*

- *either* $t = \mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$ *and* $T_f(\sigma) <_{T_k} T_g(\tau)$ *for some strictly increasing functions* $f : n \to k := |\{s_0, \dots, s_{n-1}, t_0, \dots, t_{m-1}\}|$ *and* $g : m \to k$ *which satisfy* $f(i) < g(j) \Leftrightarrow s_i <_{\vartheta(T)} t_j$ *and* $f(i) = g(j) \Leftrightarrow s_i = t_j$ *for all* $i < n$ *and* $j < m$,
- *or* $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and* $s \leq_{\vartheta(T)} t_0$,

(iv) $s = \vartheta s'$ *and one of the following holds:*

- $t = \vartheta t'$ *and we have*
  - *either* $s' <_{\vartheta(T)} t'$ *and* $r <_{\vartheta(T)} t$ *for all* $r \in E_{\vartheta(T)}(s')$,
  - *or* $t' <_{\vartheta(T)} s'$ *and* $s \leq_{\vartheta(T)} r$ *for some* $r \in E_{\vartheta(T)}(t')$,
- $t = \Omega$,
- $t = \mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$,
- $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and* $s \leq_{\vartheta(T)} t_0$,

(v) $s = \omega^{s_0} + \cdots + \omega^{s_n}$ *and one of the following holds:*

- $t = \omega^{t_0} + \cdots + \omega^{t_m}$ *and*
  - *either* $n < m$ *and* $s_i = t_i$ *for all* $i \leq n$,
  - *or there is a number* $j \leq \min\{m, n\}$ *with* $s_j <_{\vartheta(T)} t_j$ *and* $s_i = t_i$ *for all* $i < j$,
- t *is of the form* Ω, $\mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$ *or* $\vartheta t'$ *and we have* $s_0 <_{\vartheta(T)} t$.

The function $E_{\vartheta(T)}$, *mapping each* $s \in \vartheta(T)$ *to a finite set* $E_{\vartheta(T)}(s) \subseteq \vartheta(T)$, *satisfies*

(i) $E_{\vartheta(T)}(0) = \emptyset$,

(ii) $E_{\vartheta(T)}(\Omega) = \emptyset$,

(iii) $E_{\vartheta(T)}(\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}) = \{s_0, \dots, s_{n-1}\}$,

(iv) $E_{\vartheta(T)}(\vartheta s) = \{\vartheta s\}$,

(v) $E_{\vartheta(T)}(\omega^{s_0} + \cdots + \omega^{s_n}) = E_{\vartheta(T)}(s_0) \cup \cdots \cup E_{\vartheta(T)}(s_n)$.

Furthermore, the set $\vartheta(T)$, *the relation* $<_{\vartheta(T)}$ *and the function* $E_{\vartheta(T)}$ *are uniquely determined by these clauses.*

A similar construction (relative to a well-order, rather than a prae-dilator) can be found in [**71**, Section 2.1].

PROOF. It is well-known that $\mathbf{RCA_0}$ can encode (finite) terms as natural numbers (see e.g. [**37**, Section I.1(d)]). In particular, we can construct a preliminary term system $\vartheta^0(T)\supseteq\vartheta(T)$ which ignores the conditions involving $<_{\vartheta(T)}$ (cf. Definition 2.2.3). Note that the conditions $\sigma\in T_n$ and $\operatorname{supp}_n(\sigma)=n$ are Δ^0_1. By recursion over terms, we can define $E_{\vartheta(T)}$ as a function from $\vartheta^0(T)$ to $[\vartheta^0(T)]^{<\omega}$. It will be clear that the restriction $E_{\vartheta(T)}\restriction\vartheta(T)$ has range in $[\vartheta(T)]^{<\omega}$. Writing $\ulcorner s\urcorner$ for the code of s as a natural number (having arithmetized terms, this is of course the number s itself), we define a "length function" $L_{\vartheta(T)}:\vartheta^0(T)\to\mathbb{N}$ by

$$L_{\vartheta(T)}(s)=\begin{cases}\ulcorner s\urcorner & \text{if } s\in\{0,\Omega\}\\ \max\{\ulcorner s\urcorner, 2\cdot L_{\vartheta(T)}(s_0)+\cdots+2\cdot L_{\vartheta(T)}(s_{n-1})+L_{\vartheta(T)}(\Omega)+1\} & \\ & \text{if } s=\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma\\ \max\{\ulcorner s\urcorner, L_{\vartheta(T)}(s')+1\} & \text{if } s=\vartheta s'\\ \max\{\ulcorner s\urcorner, L_{\vartheta(T)}(s_0)+\cdots+L_{\vartheta(T)}(s_n)+1\} & \text{if } s=\omega^{s_0}+\cdots+\omega^{s_n}.\end{cases}$$

The occurrences of $\ulcorner s\urcorner$ have the effect that $\forall_{s\in\vartheta^0(T)}(L_{\vartheta(T)}(s)\leq n\to\dots)$ is a bounded quantifier (of course, $s\in\vartheta^0(T)$ does not count as a bound). The factor 2 will be relevant in the next proof. We can decide $r\in\vartheta(T)$ and $s<_{\vartheta(T)}t$ by simultaneous recursion over $L_{\vartheta(T)}(r)$ resp. $L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)$. Note that we have $L_{\vartheta(T)}(r)\leq L_{\vartheta(T)}(s)$ for all $r\in E_{\vartheta(T)}(s)$, by induction over s. To formalize the decision procedure in $\mathbf{RCA_0}$, consider finite partial functions

$$d:\vartheta^0(T)\cup(\vartheta^0(T)\times\vartheta^0(T))\xrightarrow{p}\{0,1\},$$

coded by natural numbers. The idea is that $d(r)=1$ resp. $d(s,t)=1$ should be equivalent to $r\in\vartheta(T)$ resp. $s<_{\vartheta(T)}t$. Call d a decision function if it satisfies the corresponding clauses. For example, $\langle\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma,\mathfrak{E}^{t_0,\dots,t_{m-1}}_\tau\rangle\in\operatorname{dom}(d)$ must imply

$$\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma\in\operatorname{dom}(d)\land\mathfrak{E}^{t_0,\dots,t_{m-1}}_\tau\in\operatorname{dom}(d)\land\forall_{i<n,j<m}\langle s_i,t_j\rangle\in\operatorname{dom}(d),$$

and $d(\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma,\mathfrak{E}^{t_0,\dots,t_{m-1}}_\tau)=1$ must be equivalent to the conjunction of the statements $d(\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma)=d(\mathfrak{E}^{t_0,\dots,t_{m-1}}_\tau)=1$ and

$$\begin{aligned}\exists_{f,g}(\text{“}f:n\to k,\ g:m\to k\text{ strictly increasing”}\land T_f(\sigma)<_{T_k}T_g(\tau)\land\\ \land\forall_{i<n,j<m}((f(i)<f(j)\leftrightarrow d(s_i,t_j)=1)\land(f(i)=g(j)\leftrightarrow s_i=t_j))),\end{aligned}$$

with $k=|\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}|$. Assuming that T is a prae-dilator, the functions T_f and T_g are total, so that $T_f(\sigma)<_{T_k}T_g(\tau)$ has complexity Δ^0_1. Then the statement "d is a decision function" is Δ^0_1 as well (note the similarity with the "partial satisfactions" of [**37**, Definition I.1.71]). Call d a decision for r resp. $\langle s,t\rangle$ if d is a decision function with $r\in\operatorname{dom}(d)$ resp. $\langle s,t\rangle\in\operatorname{dom}(d)$. Given decision functions d and d', one can show the uniqueness result

$$\forall_n\forall_{r,s,t\in\vartheta^0(T)}((L_{\vartheta(T)}(r)\leq n\wedge r\in\operatorname{dom}(d)\cap\operatorname{dom}(d')\to d(r)=d'(r))\wedge\\ (L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)\leq n\wedge\langle s,t\rangle\in\operatorname{dom}(d)\cap\operatorname{dom}(d')\to d(s,t)=d'(s,t)))$$

by a straightforward induction on n. The existence of decision functions can be established in the form

$$\forall_n\forall_{r,s,t\in\vartheta^0(T)}((L_{\vartheta(T)}(r)\leq n\to\exists_d\text{ "}d\text{ is a decision for }r\text{"})\wedge\\ (L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)\leq n\to\exists_d\text{ "}d\text{ is a decision for }\langle s,t\rangle\text{"})),$$

again by induction on n (recall that Σ_1-formulas are provably closed under bounded quantification). To see how this works, let us take up the example of $s=\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}$ and $t=\mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$. In view of $L_{\vartheta(T)}(s_i)+L_{\vartheta(T)}(t_j)<L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)$, the induction hypothesis provides a decision function d_{ij} for each pair $\langle s_i,t_j\rangle$. Inductively, we get decision functions d'_k which cover all pairs $\langle s_i,t_j\rangle$ with $\langle i,j\rangle<k$: In the step, set $d'_{\langle i,j\rangle+1}:=d'_{\langle i,j\rangle}\cup d_{ij}$, which is a decision function by uniqueness. Once we have a decision function that covers $\langle s_i,t_j\rangle$ for all $i<n$ and $j<m$, we can simply add $\langle s,t\rangle$ to the domain (with the correct value). Having established uniqueness and existence, we can define $\vartheta(T)$ by Δ^0_1-comprehension, stipulating

$$s\in\vartheta(T)\Leftrightarrow\exists_d(\text{"}d\text{ is a decision for }s\text{"}\wedge d(s)=1)\\ \Leftrightarrow\forall_d(\text{"}d\text{ is a decision for }s\text{"}\to d(s)=1).$$

In the same way, we define $<_{\vartheta(T)}$. The uniqueness of $\vartheta(T)$ and $<_{\vartheta(T)}$ is established as the uniqueness of decision functions. □

As usual, linearity can be established in a weak base theory:

PROPOSITION 2.3.15 ($\mathbf{RCA_0}$). *If T is a prae-dilator then $<_{\vartheta(T)}$ is a linear order on the term system $\vartheta(T)$.*

PROOF. By simultaneous induction on n one proves that

$$L_{\vartheta(T)}(s)\leq n\to s\not<_{\vartheta(T)}s,\\ L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)\leq n\to s<_{\vartheta(T)}t\vee s=t\vee t<_{\vartheta(T)}s,\\ L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)+L_{\vartheta(T)}(r)\leq n\to(s<_{\vartheta(T)}t\wedge t<_{\vartheta(T)}r\to s<_{\vartheta(T)}r)$$

holds for all $s,t,r\in\vartheta(T)$. Concerning antisymmetry for $s=\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}$, any strictly increasing function $f:n\to k:=|\{s_0,\dots,s_{n-1}\}|\leq n$ must be the identity on $k=n$. Thus $\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}<_{\vartheta(T)}\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}$ would require $\sigma<_{T_n}\sigma$, which fails because $<_{T_n}$ is a linear order (by the definition of prae-dilator). Next, let us establish trichotomy for $s=\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}$ and $t=\mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$. The induction hypothesis makes $<_{\vartheta(T)}$ linear on $\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}$ (note that the factor 2 in the definition of $L_{\vartheta(T)}$ yields $L_{\vartheta(T)}(s_i)+L_{\vartheta(T)}(t_j)+L_{\vartheta(T)}(s_i)<L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)$, which covers the implication $s_i<_{\vartheta(T)}t_j\land t_j<_{\vartheta(T)}s_i\to s_i<_{\vartheta(T)}s_i$). The definition of $<_{\vartheta(T)}$ asks for strictly increasing functions $f:n\to k:=|\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}|$ and $g:m\to k$ which satisfy $f(i)<g(j)\Leftrightarrow s_i<_{\vartheta(T)}t_j$ and $f(i)=g(j)\Leftrightarrow s_i=t_j$ for all $i<n$ and $j<m$. Given that $<_{\vartheta(T)}$ is linear on $\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}$, this is equivalent to the condition that f and g make the following diagram commute, where the horizontal arrows are the increasing enumerations with respect to $<_{\vartheta(T)}$ (to deduce commutativity from the conditions on f and g, observe that $s_i\mapsto f(i)$, $t_j\mapsto g(j)$ defines the inverse of the isomorphism in the middle row):

$$\begin{array}{ccc}
n & \longrightarrow & \{s_0,\dots,s_{n-1}\}\\
{\scriptstyle f}\big\downarrow & & \big\downarrow\\
k & \longrightarrow & \{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}\\
{\scriptstyle g}\big\uparrow & & \big\uparrow\\
m & \longrightarrow & \{t_0,\dots,t_{m-1}\}.
\end{array}$$

Clearly, there are unique f and g which do make the diagram commute. If we have $T_f(\sigma)<_{T_k}T_g(\tau)$ or $T_g(\tau)<_{T_k}T_f(\sigma)$ then we get $s<_{\vartheta(T)}t$ resp. $t<_{\vartheta(T)}s$ and we are done. So now assume $T_f(\sigma)=T_g(\tau)$. Similar to the proof of Lemma 2.3.3, the naturality of supp and the condition $\operatorname{supp}_n(\sigma)=n$ yield

$$\begin{aligned}[f]^{<\omega}(n)=[f]^{<\omega}\circ\operatorname{supp}_n(\sigma)=\operatorname{supp}_k\circ T_f(\sigma)=&\\
=\operatorname{supp}_k\circ T_g(\tau)=[g]^{<\omega}\circ\operatorname{supp}_m(\tau)=[g]^{<\omega}(m).&\end{aligned}$$

Observing $[f]^{<\omega}(n)\cup[g]^{<\omega}(m)=k$ we get $[f]^{<\omega}(n)=[g]^{<\omega}(m)=k$, so that $f=g$ is the identity on $n=m=k$. This yields $\sigma=\tau$ and $\langle s_0,\dots,s_{n-1}\rangle=\langle t_0,\dots,t_{m-1}\rangle$, hence $s=t$. To get transitivity for $s=\mathfrak{E}_\sigma^{s_0,\dots,s_{n-1}}$, $t=\mathfrak{E}_\tau^{t_0,\dots,t_{m-1}}$ and $r=\mathfrak{E}_\rho^{r_0,\dots,r_{l-1}}$, compose with the embeddings into $\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1},r_0,\dots,r_{l-1}\}$. Again, the induction hypothesis ensures that $<_{\vartheta(T)}$ is linear on this set. Transitivity for $s=\vartheta s',t=\vartheta t'$ and $r=\vartheta r'$ is established as in the proof of Proposition 2.2.8. $\square$

We can now state our type-two well-ordering principle in second-order arithmetic:

Definition 2.3.16 ($\mathbf{RCA_0}$). The computable Bachmann-Howard principle is the statement

$$\forall_T(\text{“}T=(T^0,T^1,\mathrm{supp})\text{ a coded dilator”}\to\text{“}(\vartheta(T),<_{\vartheta(T)})\text{ is well-founded”}).$$

As observed above, being a coded dilator is a Π^1_2-property. Thus the computable Bachmann-Howard principle is a Π^1_3-statement. In view of Corollary 4.4.7 its logical complexity cannot be lowered. Our next goal is to show that the computable Bachmann-Howard principle implies arithmetical transfinite recursion, using a result of Rathjen and Valencia Vizcaíno [**71**]. This will allow us to switch back to the base theory $\mathbf{ATR}_0$, or indeed to its set-theoretic variant $\mathbf{ATR_0^{set}}$. As a preparation, we show that the condition $t<_{\vartheta(T)}s$ in the definition of $\vartheta s<_{\vartheta(T)}\vartheta t$ is redundant (we have included it for symmetry):

Lemma 2.3.17 ($\mathbf{RCA_0}$). *If $\vartheta s\leq_{\vartheta(T)}r$ holds for some $r\in E_{\vartheta(T)}(t)$ then we have $\vartheta s<_{\vartheta(T)}\vartheta t$.*

Proof. Invoking the transitivity of $<_{\vartheta(T)}$, it suffices to establish $r<_{\vartheta(T)}\vartheta t$ for all $r\in E_{\vartheta(T)}(t)$. Note that any such r is smaller than Ω. Also, r must be a subterm of t, and thus a proper subterm of ϑt. We prove the implication

$$\text{“if } r<_{\vartheta(T)}\Omega \text{ is a proper subterm of } t \text{ then we have } r<_{\vartheta(T)}t\text{”}$$

by induction on $L_{\vartheta(T)}(r)+L_{\vartheta(T)}(t)$. The case $t=\omega^{t_0}+\dots+\omega^{t_m}$ is easily reduced to $t_0<_{\vartheta(T)}\omega^{t_0}+\dots+\omega^{t_m}$, which holds by induction on t_0. The other interesting case concerns $r=\vartheta r_0$ and $t=\vartheta t_0$. As r_0 and t_0 must be different terms, there are two possibilities: First assume $r_0<_{\vartheta(T)}t_0$. To conclude $r<_{\vartheta(T)}t$ we need $r_1<_{\vartheta(T)}t$ for all $r_1\in E_{\vartheta(T)}(r_0)$. In view of $L_{\vartheta(T)}(r_1)\leq L_{\vartheta(T)}(r_0)<L_{\vartheta(T)}(r)$ this follows from the induction hypothesis. Now assume $t_0<_{\vartheta(T)}r_0$. To get $r<_{\vartheta(T)}t$ we need $r\leq_{\vartheta(T)}t_1$ for some $t_1\in E_{\vartheta(T)}(t_0)$. Observe that there is some $t_1\in E_{\vartheta(T)}(t_0)$ such that $r=\vartheta r_0$ is a subterm of t_1. In case $r=t_1$ we are done. Otherwise r is a proper subterm of t_1 and the induction hypothesis yields $r<_{\vartheta(T)}t_1$. □

We can now connect the computable Bachmann-Howard principle with a known well-ordering principle of type one:

PROPOSITION 2.3.18 ($\mathbf{RCA_0}$). *The computable Bachmann-Howard principle implies that ϑ_X is well-founded for any well-order X, where ϑ_X is the relativized notation system of Rathjen and Valencia Vizcaíno* [**71**, Section 2.1].

PROOF. The idea is to define T as the constant dilator with value $X = (X, <_X)$. As we shall see, $\vartheta(T)$ and ϑ_X are essentially the same term system. Working in the base theory $\mathbf{RCA_0}$, we invoke Δ^0_1-comprehension to define the sets

$$\begin{aligned} T^0 &= \{\langle 0, n, \sigma\rangle \mid n \in \mathbb{N} \land \sigma \in X\} \cup \{\langle 1, n, \sigma, \tau\rangle \mid n \in \mathbb{N} \land \sigma <_X \tau\}, \\ T^1 &= \{\langle f, \sigma, \tau\rangle \mid \text{“}f \text{ a morphism of natural numbers”} \land \sigma = \tau \in X\}, \\ \operatorname{supp} &= \{\langle n, \sigma, \emptyset\rangle \mid n \in \mathbb{N} \land \sigma \in X\}. \end{aligned}$$

In other words, T_f is the identity on X and all supports are empty. It is easy to see that $T = (T^0, T^1, \operatorname{supp})$ is a coded prae-dilator. The condition $\operatorname{supp}_{|a|}(\sigma) = |a|$ forces $a = \emptyset$, so that we have

$$D^T_Y = \{\langle \emptyset, \sigma\rangle \mid \sigma \in T_0 = X\} \quad \text{and} \quad \langle \emptyset, \sigma\rangle <^T_Y \langle \emptyset, \tau\rangle \Leftrightarrow \sigma <_X \tau.$$

Given that X is a well-order, this implies that $(D^T_Y, <^T_Y)$ is well-founded for any (well-)order Y. Thus T is a coded dilator, and the computable Bachmann-Howard principle tells us that $(\vartheta(T), <_{\vartheta(T)})$ is well-founded. Define a function

$$h : \vartheta_X \to \vartheta^0(T)$$

by the obvious recursion on terms. In particular, we have

$$h(\mathfrak{E}_\sigma) = \mathfrak{E}^{\langle\rangle}_\sigma \qquad \text{for } \sigma \in X = T_0.$$

To show that h is an embedding, define a length function $L_{\vartheta_X} : \vartheta_X \to \mathbb{N}$ by

$$L_{\vartheta_X}(s) = \begin{cases} \max\{\ulcorner s \urcorner, \ulcorner 0 \urcorner\} & \text{if } s \in \{0, \Omega\} \text{ or } s = \mathfrak{E}_\sigma, \\ \max\{\ulcorner s \urcorner, L_{\vartheta_X}(s') + 1\} & \text{if } s = \vartheta s', \\ \max\{\ulcorner s \urcorner, L_{\vartheta_X}(s_0) + \dots + L_{\vartheta_X}(s_n) + 1\} & \text{if } s = \omega^{s_0} + \dots + \omega^{s_n}. \end{cases}$$

Recall that the notation system ϑ_X works with a function $\cdot^* : \vartheta_X \to \vartheta_X$ at the place of our $E_{\vartheta(T)} : \vartheta(T) \to [\vartheta(T)]^{<\omega}$ (but note the sets $E_\Omega(\alpha)$ in [**71**, Section 2]). The occurrences of $\ulcorner 0 \urcorner$ ensure $L_{\vartheta_X}(s^*) \leq L_{\vartheta_X}(s)$, by an easy induction on s. As before, $\ulcorner s \urcorner \leq L_{\vartheta_X}(s)$ means that $\forall_{s \in \vartheta_X}(L_{\vartheta_X}(s) \leq n \to \dots)$ is a bounded quantifier. By simultaneous induction on $L_{\vartheta_X}(r)$ resp. $L_{\vartheta_X}(s) + L_{\vartheta_X}(t)$ one can show

$$\begin{gathered} r \in \vartheta_X \Rightarrow h(r) \in \vartheta(T), \\ h(r^*) = \max_{<_{\vartheta(T)}}(E_{\vartheta(T)}(h(r)) \cup \{0\}), \end{gathered}$$

$$s <_{\vartheta_X} t \Rightarrow h(s) <_{\vartheta(T)} h(t).$$

As an example, assume that $s = \vartheta s' <_{\vartheta_X} \vartheta t' = t$ holds because of $s \leq_{\vartheta_X} (t')^*$ (note that the definition of ϑ_X does not contain the condition $t' <_{\vartheta_X} s'$). In view of $L_{\vartheta_X}((t')^*) \leq L_{\vartheta_X}(t') < L_{\vartheta_X}(t)$ the induction hypothesis gives

$$\vartheta h(s') = h(s) \leq_{\vartheta(T)} h((t')^*) = \max_{<_{\vartheta(T)}}(E_{\vartheta(T)}(h(t')) \cup \{0\}).$$

As $\vartheta h(s') \leq_{\vartheta(T)} 0$ fails we must have $\vartheta h(s') \leq_{\vartheta(T)} r$ for some $r \in E_{\vartheta(T)}(h(t'))$. By the previous lemma we get $h(s) = \vartheta h(s') <_{\vartheta(T)} \vartheta h(t') = h(t)$, as required. We have thus shown that h is an order embedding of ϑ_X into $\vartheta(T)$. By recursion on terms we can also construct a section $h' : \vartheta(T) \to \vartheta_X$ such that $h' \circ h$ is the identity on ϑ_X (in the present case h' is indeed the inverse of h, but in general this is not necessary). To deduce that ϑ_X is well-founded, consider an inhabited set $Z \subseteq \vartheta_X$. Use Δ^0_1-comprehension to form the set

$$Z' = \{h(s) \,|\, s \in Z\} = \{t \in \vartheta(T) \,|\, h'(t) \in Z \wedge t = h \circ h'(t)\}.$$

As $\vartheta(T)$ is well-founded we get a $<_{\vartheta(T)}$-minimal element $h(s)$ of Z'. Then s must be $<_{\vartheta_X}$-minimal in Z. □

As promised, we can now use a result of Rathjen and Valencia Vizcaíno to boost the base theory:

COROLLARY 2.3.19. *Over* $\mathbf{RCA_0}$*, the computable Bachmann-Howard principle implies all axioms of* $\mathbf{ATR_0}$.

PROOF. Assuming the computable Bachmann-Howard principle, let us first boot up to $\mathbf{ACA_0^+}$: By [**52**, Theorem 5.23] or [**4**, Theorem 4.1] it suffices to show that the relativized notation system ε_X is well-founded for any well-order X. As in the previous proof, ε_X can be embedded into $\vartheta(T)$, where T is the constant dilator with value X (see also the last paragraph of [**71**, Section 4]). The point of this first step is that valuations for countably coded ω-models are now available (cf. [**87**, Lemma VII.2.2]). As usual, we write $\mathcal{M} \vDash \varphi$ to express that φ is satisfied under some (and thus any) valuation. Recall that $\mathbf{ATR_0}$ is axiomatized by a Π^1_2-sentence $\forall_X \exists_Y \varphi(X,Y)$ which is provable by bar induction (see [**87**, Corollary VII.2.19]). To deduce $\forall_X \exists_Y \varphi(X,Y)$ from the computable Bachmann-Howard principle, consider an arbitrary set X. In view of the previous proposition, [**71**, Theorem 4.1] yields a countably coded ω-model $\mathcal{M}$ of bar induction which contains X (in its second-order part). By meta-induction along the aforementioned proof we get $\mathcal{M} \vDash \forall_U \exists_V \varphi(U,V)$. As X is contained in $\mathcal{M}$ this implies $\mathcal{M} \vDash \exists_V \varphi(X,V)$. Finally, we obtain $\exists_Y \varphi(X,Y)$ by upward absoluteness. $\square$

Back in our default base theory $\mathbf{PRS}\omega$, let us relate coded and set-sized dilators. Given a coded prae-dilator $(T^0, T^1, \operatorname{supp})$, consider the primitive recursive set function $n \mapsto (T_n, <_{T_n})$ defined by

$$\begin{aligned} T_n &= \{\sigma \in \omega \mid \langle 0, n, \sigma\rangle \in T^0\},\\ <_{T_n} &= \{(\sigma,\tau) \in \omega \times \omega \mid \langle 1, n, \sigma, \tau\rangle \in T^0\}. \end{aligned}$$

Here (σ,τ) is the pair in the set-theoretic sense, while $\langle 1, n, \sigma, \tau\rangle$ refers to the encoding of tuples as natural numbers. Proposition 1.2.2 ensures that the function $n \mapsto (T_n, <_{T_n})$ exists as a set. In the context of $\mathbf{RCA_0}$, the expression $\sigma \in T_n$ was used as an abbreviation for $\langle 0, n, \sigma\rangle \in T^0$. Our definition of T_n turns this abbreviation into an equivalence. Similarly, the sets $T^1, \operatorname{supp} \subseteq \mathbb{N}$ give rise to set-sized functions $f \mapsto T_f$ and $n \mapsto \operatorname{supp}_n$. Clearly these functions form a set-sized prae-dilator. For dilators the situation is slightly less straightforward: Definition 2.3.8 tests the well-foundedness of D^T_X for all well-orders X, while Definition 2.3.13 does only consider well-orders $X \subseteq \mathbb{N}$ (or equivalently countable well-orders). In the presence of choice the definitions coincide: Consider an arbitrary well-order X. Due to the characterization of well-foundedness in terms of descending sequences, it suffices to check that any countable set $Z \subseteq D^T_X$ has a minimal element. It is easy to see $Z \subseteq D^T_Y$ for some countable $Y \subseteq X$ (cf. [**28**, Theorem 2.1.15]). Instead of choice, we will assume the axiom of countability, which also ensures that the two definitions of dilator coincide. Conversely, consider a set-sized (prae-)dilator (T, supp) such that we have $T_n \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$. Set

$$T^0 = \{\langle 0, n, \sigma\rangle \mid \sigma \in T_n\} \cup \{\langle 1, n, \sigma, \tau\rangle \mid \sigma <_{T_n} \tau\}$$

and define $T^1, \operatorname{supp} \subseteq \mathbb{N}$ in a similar way. It is easy to see that $(T^0, T^1, \operatorname{supp})$ is a coded (prae-)dilator. This justifies the following terminology:

Convention 2.3.20. Under the axiom of countability we identify coded (prae-) dilators and set-sized (prae-)dilators (T, supp) which satisfy $T_n \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$.

In fact, countability makes any set-sized dilator equivalent to a coded dilator. This leads to the following strengthening of Proposition 2.3.10:

PROPOSITION 2.3.21 (Countability). *The statement*

$$\forall_T(\text{“}T = (T, \operatorname{supp}) \text{ a coded dilator”} \to \text{“}(\mathrm{BH}(D^T), <_{\mathrm{BH}(D^T)}) \text{ is well-founded”})$$

implies any other instance of the predicative Bachmann-Howard principle.

PROOF. By Proposition 2.3.10 it suffices to show that $(\mathrm{BH}(D^S), <_{\mathrm{BH}(D^S)})$ is well-founded for any set-sized dilator $S = (S, \operatorname{supp}')$. Observe that $\bigcup\{S_n \mid n \in \omega\}$ exists as a set, as $n \mapsto S_n$ is a set-sized function. By the axiom of countability (see Definition 1.4.2) we get a bijection

$$d : c \xrightarrow{\cong} \bigcup\{S_n \mid n \in \omega\}$$

for some set $c \subseteq \omega$. Define

$$T_n = \{\sigma \in c \mid d(\sigma) \in S_n\}, \qquad T_f(\sigma) = d^{-1}(S_f(d(\sigma))),$$
$$<_{T_n} = \{(\sigma, \tau) \in c \times c \mid d(\sigma) <_{S_n} d(\tau)\}, \qquad \operatorname{supp}_n(\sigma) = \operatorname{supp}'_n(d(\sigma)).$$

By Corollary 1.1.10 and Proposition 1.2.2 these stipulations yield set-sized functions $n \mapsto (T_n, <_{T_n}), f \mapsto T_f$ and $n \mapsto \operatorname{supp}_n$. It is straightforward to check that (T, supp) is a coded prae-dilator. Recall the associated (prae-)dilators $(D^S, \operatorname{supp}^S)$ and $(D^T, \operatorname{supp}^T)$ from Definition 2.3.2. Consider the primitive recursive family of functions

$$\eta_X : D^T_X \to D^S_X, \qquad \eta_X(\langle a, \sigma\rangle) := \langle a, d(\sigma)\rangle.$$

It is straightforward to check that $\eta : D^T \Rightarrow D^S$ is a natural isomorphism. In particular D^T_X is well-founded whenever D^S_X is, which makes (T, supp) a coded dilator. By assumption this implies that $(\mathrm{BH}(D^T), <_{\mathrm{BH}(D^T)})$ is well-founded. Observe

$$\operatorname{supp}^S_X \circ \eta_X(\langle a, \sigma\rangle) = \operatorname{supp}^S_X(\langle a, d(\sigma)\rangle) = a = \operatorname{supp}^T_X(\langle a, \sigma\rangle),$$

and note the similarity with Proposition 2.3.6. As in the proof of Proposition 2.3.10 we get an embedding of $(\mathrm{BH}(D^S), <_{\mathrm{BH}(D^S)})$ into $(\mathrm{BH}(D^T), <_{\mathrm{BH}(D^T)})$. This implies that $(\mathrm{BH}(D^S), <_{\mathrm{BH}(D^S)})$ is well-founded, as desired. □

2.4. Well-Ordering Proofs

In this section we show that the abstract Bachmann-Howard principle holds if every set is contained in an admissible set. We also show that the abstract Bachmann-Howard principle implies its computable counterpart, which in turn implies the predicative version. This amounts to the directions (ii)⇒(iii)⇒(v)⇒(iv) of Theorem 4.4.6.

Recall that a proto-dilator consists of a primitive recursive function $\alpha \mapsto T^u_\alpha$ with parameter u (and parameter ω, cf. Convention 1.2.9). The abstract Bachmann-Howard principle demands a certain collapse $\vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha$ for some ordinal α. Assume that we have an admissible set $\mathbb{A}$ with $u, \omega \in \mathbb{A}$. Put

$$o(\mathbb{A}) := \mathbb{A} \cap \mathrm{Ord}$$

and observe that this is an ordinal because $\mathbb{A}$ is transitive. For such an ordinal, we will be able to construct the desired collapse $\vartheta_{\mathbb{A}} : T^u_{o(\mathbb{A})} \xrightarrow{\text{BH}} o(\mathbb{A})$. This approach combines Rathjen's well-ordering proof in [**61**, Section 4] (where $\mathbb{A}$ is the set-theoretic universe, i.e. class-sized) with the proof that admissible sets yield axiom beta (see e.g. [**41**, Theorem 4.6]). As a preparation, we need to show that $\mathbb{A}$ is closed under $\alpha \mapsto T^u_\alpha$, and that this function can be defined in $\mathbb{A}$. Recall that Definition 1.4.1 associates each primitive recursive function symbol $F : \mathbb{V}^n \to \mathbb{V}$ (again with implicit parameter ω) with a Σ-formula $\mathcal{D}_F(\omega, x_1, \dots, x_n, z)$ in the language $\mathcal{L}_\in$ of pure set theory (where the parameter ω is made explicit). We also write $\mathcal{D}_F$ for the Gödel code of this formula (cf. the discussion before Proposition 1.3.5).

Lemma 2.4.1. *For each primitive recursive function $F : \mathbb{V}^n \to \mathbb{V}$ the following is provable in* **PRS**ω*: If $\mathbb{A} \ni \omega$ is an admissible set then we have*

$$x_1, \dots, x_n \in \mathbb{A} \to F(x_1, \dots, x_n) \in \mathbb{A},$$

as well as

$$\forall_{x_1,\dots,x_n,z\in\mathbb{A}}(F(x_1, \dots, x_n) = z \leftrightarrow \mathbb{A} \vDash \mathcal{D}_F(\omega, x_1, \dots, x_n, z)).$$

Proof. By meta induction on the build-up of F it is straightforward to construct proofs of

$$\forall_{x_1,\dots,x_n}\exists_z \mathcal{D}_F(\omega, x_1, \dots, x_n, z)$$

in Kripke-Platek set theory, treating ω as a free variable. The crucial case $F \equiv R[G]$ is covered by the Σ-recursion theorem (see [**6**, Theorem I.6.4]). By meta induction over these proofs, using the Tarski conditions from Proposition 1.3.3, one obtains

$$\forall_{x_1,\dots,x_n\in\mathbb{A}}\exists_{z\in\mathbb{A}}\, \mathbb{A} \vDash \mathcal{D}_F(\omega, x_1, \dots, x_n, z),$$

where ω may now be given the intended interpretation. Both the closure of $\mathbb{A}$ under F and the direction "$\to$" of the equivalence are thus reduced to the direction "$\leftarrow$". To verify the latter, consider $\vec{x}, z \in \mathbb{A}$ with $\mathbb{A} \vDash \mathcal{D}_F(\omega, \vec{x}, z)$. By Proposition 1.3.5 we get $\mathcal{D}_F(\omega, \vec{x}, z)^{\mathbb{A}}$, which implies $\mathcal{D}_F(\omega, \vec{x}, z)$ because $\mathcal{D}_F$ is a Σ-formula. Finally, Lemma 1.4.8 yields $F(\vec{x}) = z$. $\square$

In particular, the lemma shows that $\mathcal{D}_F$ is a Δ-formula from the viewpoint of any admissible $\mathbb{A}$, namely

$$\mathbb{A} \vDash (\mathcal{D}_F(\omega, \vec{x}, z) \leftrightarrow \forall_{z' \neq z} \neg \mathcal{D}_F(\omega, \vec{x}, z')).$$

Under the scope of the satisfaction relation, we will write $F(\vec{x}) = z$ to denote either $\mathcal{D}_F(\omega, \vec{x}, z)$ or $\forall_{z' \neq z} \neg \mathcal{D}_F(\omega, \vec{x}, z')$, depending on the context. Note that the lemma ensures

$$\mathbb{A} \vDash F(\vec{x}) = z \quad \Leftrightarrow \quad F(\vec{x}) = z.$$

Similarly, we write $y \in F(\vec{x})$ to abbreviate the Σ-formula $\exists_z(\mathcal{D}_F(\omega, \vec{x}, z) \land y \in z)$ or the Π-formula $\forall_z(\mathcal{D}_F(\omega, \vec{x}, z) \to y \in z)$ of the language $\mathcal{L}_\in$. For both variants we have

$$\mathbb{A} \vDash y \in F(\vec{x}) \quad \Leftrightarrow \quad y \in F(\vec{x}).$$

Let us also recall the second recursion theorem in the context of admissible sets:

LEMMA 2.4.2. *Let $\varphi(x_1, \ldots, x_n, \vec{y}, R)$ be a Σ-formula in the language $\mathcal{L}_\in \cup \{R\}$, in which all occurrences of the n-ary relation symbol R are positive. Then there is a Σ-formula $\theta(x_1, \ldots, x_n, \vec{y})$ of $\mathcal{L}_\in$ such that* **PRS**ω *proves the following: We have*

$$\mathbb{A} \vDash (\theta(x_1, \ldots, x_n, \vec{y}) \leftrightarrow \varphi(x_1, \ldots, x_n, \vec{y}, \{x_1, \ldots, x_n \,|\, \theta(x_1, \ldots, x_n, \vec{y})\}))$$

for any admissible set $\mathbb{A}$ and any elements $x_1, \ldots, x_n, \vec{y} \in \mathbb{A}$.

PROOF. According to [**6**, Theorem 2.3] we can prove the equivalence in Kripke-Platek set theory, for a suitable formula θ. By meta induction along this proof we see that the equivalence holds in $\mathbb{A}$. □

One could prove the previous result for non-standard formulas, replacing meta induction by induction inside **PRS**ω, but this will not be needed. For the following, we fix a primitive recursive function $(u, \alpha) \mapsto T^u_\alpha = (T^u_\alpha, <_{T^u_\alpha})$. The idea is to define a Bachmann-Howard collapse $\vartheta_{\mathbb{A}} : T^u_{o(\mathbb{A})} \xrightarrow{\text{BH}} o(\mathbb{A})$ by stipulating

$$\vartheta_{\mathbb{A}}(\sigma) = \alpha \leftrightarrow \mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha),$$

where the formula θ_T is defined as follows. It may be helpful to recall the construction from Remark 2.1.6.

Definition 2.4.3. Invoking the second recursion theorem, let θ_T be a Σ-formula of $\mathcal{L}_\in$ such that $\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha)$ is equivalent to

$$\begin{aligned}\mathbb{A} \vDash \exists_\gamma(\sigma \in T^u_{\gamma+1} \wedge \forall_{\beta<\gamma}\, \sigma \notin T^u_{\beta+1} \wedge \\ \exists_f(\text{“}f : \omega \to \mathrm{Ord} \text{ is a function”} \wedge f(0) = \gamma + 1 \wedge \\ \forall_{n\in\omega}\exists_d(\text{“}d : \{\tau \in T^u_{f(n)} \,|\, \tau <_{T^u_{f(n)}} \sigma\} \to \mathrm{Ord} \text{ is a function”} \wedge \\ \forall_{\tau\in\mathrm{dom}(d)}\, \theta_T(\omega, u, \tau, d(\tau)) \wedge \\ f(n + 1) = \sup\{d(\tau) + 1 \,|\, \tau \in \mathrm{dom}(d)\}) \wedge \\ \alpha = \sup_{n\in\omega} f(n))),\end{aligned}$$

for any admissible set $\mathbb{A} \ni \omega$ and any $u, \sigma, \alpha \in \mathbb{A}$.

Assume that $\alpha \mapsto T^u_\alpha$ is a proto-dilator and that $\mathbb{A} \supseteq \{\omega, u\}$ is an admissible set. Then $o(\mathbb{A})$ is a limit ordinal and we have $T^u_{o(\mathbb{A})} = \bigcup_{\gamma<o(\mathbb{A})} T^u_\gamma$, by condition (iii) of Definition 2.1.1. From Lemma 2.4.1 we know that $T^u_\gamma \in \mathbb{A}$ holds for all $\gamma < o(\mathbb{A})$. Since $\mathbb{A}$ is transitive this yields

$$T^u_{o(\mathbb{A})} \subseteq \mathbb{A}.$$

The following prepares our definition of $\vartheta_{\mathbb{A}} : T^u_{o(\mathbb{A})} \xrightarrow{\mathrm{BH}} o(\mathbb{A})$.

Lemma 2.4.4. *If T^u is a proto-dilator then we have*

$$\forall_{\sigma\in T^u_{o(\mathbb{A})}}\forall_{\alpha_0,\alpha_1<o(\mathbb{A})}(\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha_0) \wedge \mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha_1) \to \alpha_0 = \alpha_1)$$

for any admissible set $\mathbb{A} \supseteq \{\omega, u\}$.

Proof. By assumption $(T^u_{o(\mathbb{A})}, <_{T^u_{o(\mathbb{A})}})$ is a well-order. As the satisfaction relation is primitive recursive (see Proposition 1.3.3) we can argue by (main) induction over $\sigma \in T^u_{o(\mathbb{A})}$. More precisely, Corollary 1.1.10 allows us to form the set

$$\{\sigma \in T^u_{o(\mathbb{A})} \,|\, \neg\forall_{\alpha_0,\alpha_1<o(\mathbb{A})}(\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha_0) \wedge \theta_T(\omega, u, \sigma, \alpha_1) \to \alpha_0 = \alpha_1)\}.$$

To show that this set is empty, one deduces a contradiction from the assumption that σ is a minimal element with respect to $<_{T^u_{o(\mathbb{A})}}$. So assume that the claim holds for all $\tau <_{T^u_{o(\mathbb{A})}} \sigma$, and that we have $\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha_i)$ for $i = 0, 1$. Let $\gamma_i, f_i \in \mathbb{A}$ be witnesses for the latter. To obtain $\alpha_0 = \alpha_1$ it suffices to prove $f_0(n) = f_1(n)$ by (side) induction over $n \in \omega$. Clearly, the defining equivalence of θ_T gives $\gamma_0 = \gamma_1$ and thus $f_0(0) = f_1(0)$. In the step $n \rightsquigarrow n + 1$ we consider functions

$$d_i : \{\tau \in T^u_{f_i(n)} \,|\, \tau <_{T^u_{f_i(n)}} \sigma\} \to \mathrm{Ord}$$

as demanded by $\theta_T(\omega, u, \sigma, \alpha_i)$. By induction hypothesis we have $f_0(n) = f_1(n)$, so that the domains of d_0 and d_1 are equal. For τ in the joint domain we have

$$\mathbb{A} \vDash \tau \in T^u_{f_i(n)} \wedge \tau <_{T^u_{f_i(n)}} \sigma \wedge \theta_T(\omega, u, \tau, d_i(\tau)).$$

As observed above this does indeed give $\tau \in T^u_{f_i(n)}$ and $\tau <_{T^u_{f_i(n)}} \sigma$. By condition (ii) of Definition 2.1.1 we obtain $\tau \in T^u_{o(\mathbb{A})}$ and $\tau <_{T^u_{o(\mathbb{A})}} \sigma$. Thus the main induction hypothesis yields $d_0(\tau) = d_1(\tau)$. From $d_0 = d_1$ we get $f_0(n+1) = f_1(n+1)$, as required for the side induction step. Once we know $f_0 = f_1$ we immediately get $\alpha_0 = \alpha_1$, contradicting the assumption that σ lies in the above set. □

In the previous result, $\mathbb{A}$ was used to obtain the set-sized well-order $T^u_{o(\mathbb{A})}$, and to make the induction statement primitive recursive. At the same time, we did not need the full power of admissible sets. The latter is used in the next result, in the form of Σ-replacement (see [**6**, Theorem I.4.6]).

Proposition 2.4.5. *If T^u is a proto-dilator then we have*

$$\forall_{\sigma \in T^u_{o(\mathbb{A})}} \exists_{\alpha < o(\mathbb{A})}\, \mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha)$$

for any admissible set $\mathbb{A} \supseteq \{\omega, u\}$.

Proof. As in the previous proof, we argue by (main) induction over $\sigma \in T^u_{o(\mathbb{A})}$. To establish the claim for σ, we need suitable witnesses $\gamma, f \in \mathbb{A}$. For the former we can take $\gamma = |\sigma|_{T^u_{o(\mathbb{A})}} < o(\mathbb{A})$ (cf. Definition 2.1.2). The values of f are constructed recursively. More formally, we argue by (side) induction over k to establish

$$\begin{aligned} \mathbb{A} \vDash \exists_{f_k}(\text{“}f_k : k+1 \to \text{Ord is a function”} \wedge f_k(0) = \gamma \wedge \\ \forall_{n<k}\exists_d(\text{“}d : \{\tau \in T^u_{f_k(n)} \mid \tau <_{T^u_{f_k(n)}} \sigma\} \to \text{Ord is a function”} \wedge \\ \forall_{\tau \in \operatorname{dom}(d)}\, \theta_T(\omega, u, \tau, d(\tau)) \wedge \\ f_k(n+1) = \sup\{d(\tau) + 1 \mid \tau \in \operatorname{dom}(d)\})). \end{aligned}$$

For $k = 0$ we have $f_0 = \{\langle 0, \gamma\rangle\}$. Inductively, assume that $f_k \in \mathbb{A}$ is given. By Δ-separation in $\mathbb{A}$ (see [**6**, Theorem I.4.5]) we can form the set

$$D = \{\tau \in T^u_{f_k(k)} \mid \tau <_{T^u_{f_k(k)}} \sigma\} \in \mathbb{A}.$$

As above $\mathbb{A} \vDash \tau <_{T^u_{f_k(k)}} \sigma$ is equivalent to $\tau <_{T^u_{f_k(k)}} \sigma$, which implies $\tau <_{T^u_{o(\mathbb{A})}} \sigma$. Thus the main induction hypothesis and the previous lemma yield

$$\mathbb{A} \vDash \forall_{\tau \in D} \exists!_\delta\, \theta_T(\omega, u, \tau, \delta).$$

By Σ-replacement in $\mathbb{A}$ we get a $d \in \mathbb{A}$ with

$$\mathbb{A} \vDash \text{“}d : D \to \mathrm{Ord} \text{ is a function”} \land \forall_{\tau \in D}\, \theta_T(\omega, u, \tau, d(\tau)).$$

Setting

$$f_{k+1} = f_k \cup \{\langle k+1, \sup\{d(\tau)+1 \,|\, d \in D\}\rangle\}$$

completes the side induction step. By the proof of the previous lemma, f_k is unique. Thus another application of Σ-replacement tells us that the function $k \mapsto f_k$ lies in $\mathbb{A}$. Finally, $\mathbb{A}$ contains the function $f : \omega \to \mathrm{Ord}$ with $f(n) = f_n(n)$. It is easy to see that f is a witness for

$$\theta_T(\omega, u, \sigma, \sup_{n\in\omega} f(n)),$$

completing the main induction step. □

Now we are ready to define the desired collapsing function:

Definition 2.4.6. Assume that T^u is a proto-dilator and that $\mathbb{A} \supseteq \{\omega, u\}$ is an admissible set. Proposition 2.4.5 and Lemma 2.4.4 allow us to define a function

$$\vartheta_{\mathbb{A}} : T^u_{o(\mathbb{A})} \to o(\mathbb{A})$$

by putting

$$\vartheta_{\mathbb{A}} = \{\langle \sigma, \alpha\rangle \in T^u_{o(\mathbb{A})} \times o(\mathbb{A}) \,|\, \mathbb{A} \vDash \theta_T(\omega, u, \sigma, \alpha)\}.$$

This is a set by Corollary 1.1.10 and Proposition 1.3.3.

It remains to verify the properties of a collapse:

Theorem 2.4.7. *If T^u is a proto-dilator and $\mathbb{A} \supseteq \{\omega, u\}$ is an admissible set then $\vartheta_{\mathbb{A}} : T^u_{o(\mathbb{A})} \to o(\mathbb{A})$ is a Bachmann-Howard collapse.*

Proof. We have to verify the two conditions from Definition 2.1.3. First, consider $\tau, \sigma \in T^u_{o(\mathbb{A})}$ with $\tau <_{T^u_{o(\mathbb{A})}} \sigma$ and $|\tau|_{T^u_{o(\mathbb{A})}} < \vartheta_{\mathbb{A}}(\sigma)$. By the latter, if γ, f witness $\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \vartheta_{\mathbb{A}}(\sigma))$ then we have $|\tau|_{T^u_{o(\mathbb{A})}} < f(n)$ for some $n \in \omega$. In view of Definition 2.1.2 and condition (ii) of Definition 2.1.1 this implies $\tau \in T^u_{f(n)}$. Switching to $n = 0$ if necessary we may also assume $\gamma + 1 = f(0) \leq f(n)$, to get $\sigma \in T^u_{\gamma+1} \subseteq T^u_{f(n)}$ and thus $\tau <_{T^u_{f(n)}} \sigma$. Then, if

$$d : \{\tau \in T^u_{f(n)} \,|\, \tau <_{T^u_{f(n)}} \sigma\} \to \mathrm{Ord}$$

witnesses $\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \vartheta_{\mathbb{A}}(\sigma))$, we have $\tau \in \mathrm{dom}(d)$ and $d(\tau) < f(n+1) \leq \vartheta_{\mathbb{A}}(\sigma)$. From $\mathbb{A} \vDash \theta_T(\omega, u, \tau, d(\tau))$ we get $d(\tau) = \vartheta_{\mathbb{A}}(\tau)$, so that we have $\vartheta_{\mathbb{A}}(\tau) < \vartheta_{\mathbb{A}}(\sigma)$ as required. Also, if γ, f witness $\mathbb{A} \vDash \theta_T(\omega, u, \sigma, \vartheta_{\mathbb{A}}(\sigma))$ then we get $\sigma \in T^u_{\gamma+1}$ and thus $|\sigma|_{T^u_{o(\mathbb{A})}} \leq \gamma < f(0) \leq \vartheta_{\mathbb{A}}(\sigma)$, as demanded by condition (ii) of Definition 2.1.3. □

The following is direction (ii)⇒(iii) of Theorem 4.4.6:

COROLLARY 2.4.8. *If every set is an element of an admissible set then (each instance of) the abstract Bachmann-Howard principle holds.*

PROOF. An instance of the abstract Bachmann-Howard principle is given by a primitive recursive function $(u, \alpha) \mapsto T^u_\alpha = (T^u_\alpha, <T^u_\alpha)$. Consider an arbitrary value of the parameter u and assume that T^u is a proto-dilator. By assumption there is an admissible set $\mathbb{A}$ with $\{\omega, u\} \in \mathbb{A}$ and thus $\{\omega, u\} \subseteq \mathbb{A}$. The previous theorem yields a Bachmann-Howard collapse $\vartheta_{\mathbb{A}} : T^u_{o(\mathbb{A})} \xrightarrow{\text{BH}} o(\mathbb{A})$, as required by the abstract Bachmann-Howard principle. □

We continue with direction (iii)⇒(v) of Theorem 4.4.6:

THEOREM 2.4.9 (Countability). *The computable Bachmann-Howard principle follows from (an appropriate instance of) the abstract Bachmann-Howard principle.*

PROOF. Let $T = (T, \operatorname{supp})$ be a coded dilator. To establish the computable Bachmann-Howard principle we must show that the term system $(\vartheta(T), <_{\vartheta(T)})$ from Lemma 2.3.14 is well-founded. In the presence of countability we may assume that T is a set-sized dilator (see Convention 2.3.20 and the discussion before). Then Corollary 2.3.9 implies that $(D^T, \operatorname{supp}^T)$ is a dilator in the sense of Definition 2.2.1. In particular $D^T_\alpha = (D^T_\alpha, <^T_\alpha)$ is a well-order for any ordinal α. Considering the definition of D^T, it is clear that $\alpha < \beta$ implies $D^T_\alpha \subseteq D^T_\beta$ and $<^T_\alpha = <^T_\beta \cap (D^T_\alpha \times D^T_\alpha)$, and that we have $D^T_\lambda = \bigcup_{\gamma<\lambda} D^T_\gamma$ for λ limit. This means that $\alpha \mapsto D^T_\alpha$ is a proto-dilator. By Lemma 2.1.12 and Proposition 2.1.15 we can form the strengthened proto-dilator $\alpha \mapsto \varepsilon(D)^T_\alpha$. The abstract Bachmann-Howard principle for the primitive recursive function $(T, \alpha) \mapsto \varepsilon(D)^T_\alpha$ yields an ordinal α with a collapse

$$\vartheta : \varepsilon(D)^T_\alpha \xrightarrow{\text{BH}} \alpha.$$

To conclude it suffices to embed $(\vartheta(T), <_{\vartheta(T)})$ into the well-order $(\varepsilon(D)^T_\alpha, <_{\varepsilon(D)^T_\alpha})$. Recall the preliminary term system $\varepsilon^0(D)^T_\alpha \supseteq \varepsilon(D)^T_\alpha$ from Definition 2.1.11. By recursion over terms we construct a function

$$h : \vartheta(T) \to \varepsilon^0(D)^T_\alpha.$$

The stipulations $h(0) = 0, h(\Omega) = \Omega$ and $h(\omega^{s_0} + \cdots + \omega^{s_n}) = \omega^{h(s_0)} + \cdots + \omega^{h(s_n)}$ are straightforward. Now consider a term $\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma \in \vartheta(T)$ where we have $\sigma \in T_n$

and $\operatorname{supp}_n(\sigma)=n$, as well as $s_0<_{\vartheta(T)}\cdots<_{\vartheta(T)}s_{n-1}<_{\vartheta(T)}\Omega$. Note that the first two conditions imply $\langle n,\sigma\rangle\in D^T_n$. Recall the isomorphism $i_\alpha:\alpha\xrightarrow{\cong}\varepsilon(D)^T_\alpha\cap\Omega$ from Lemma 2.1.17 (also observe Lemma 2.1.18(iii)). Assuming that we have $h(s_0)<_{\varepsilon(D)^T_\alpha}\cdots<_{\varepsilon(D)^T_\alpha}h(s_{n-1})<_{\varepsilon(D)^T_\alpha}\Omega$ (in particular $h(s_j)\in\varepsilon(D)^T_\alpha\subseteq\varepsilon^0(D)^T_\alpha$), consider the embedding $h^*:n\to\alpha$ defined by $h^*(j)=i_\alpha^{-1}(h(s_j))$. Then $D^T_{h^*}$ maps $\langle n,\sigma\rangle\in D^T_n$ to $D^T_{h^*}(\langle n,\sigma\rangle)\in D^T_\alpha$, and we can set

$$h(\mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma)=\mathfrak{E}_{D^T_{h^*}(\langle n,\sigma\rangle)}\in\varepsilon(D)^T_\alpha.$$

Explicitly, Definition 2.3.2 tells us

$$D^T_{h^*}(\langle n,\sigma\rangle)=\langle\{i_\alpha^{-1}(h(s_0)),\dots,i_\alpha^{-1}(h(s_{n-1}))\},\sigma\rangle.$$

The case where $h(s_0)<_{\varepsilon(D)^T_\alpha}\cdots<_{\varepsilon(D)^T_\alpha}h(s_{n-1})<_{\varepsilon(D)^T_\alpha}\Omega$ fails will not be relevant; one may assign 0 as a default value. Finally, consider a term $\vartheta s\in\vartheta(T)$. By Definition 2.1.22 the Bachmann-Howard collapse $\vartheta:\varepsilon(D)^T_\alpha\xrightarrow{\text{BH}}\alpha$ yields a function

$$\bar\vartheta:\varepsilon(D)^T_\alpha\to\varepsilon(D)^T_\alpha\cap\Omega.$$

In case $h(s)\in\varepsilon(D)^T_\alpha\subseteq\varepsilon^0(D)^T_\alpha$ we may thus set

$$h(\vartheta s)=\bar\vartheta(h(s)).$$

For $h(s)\notin\varepsilon(D)^T_\alpha$ we may again assign the default value $h(\vartheta s)=0$. Observe that supp^T (cf. Definition 2.3.2) is a support for D^T, in the sense of Definition 2.1.9. From Definition 2.1.26 and Lemma 2.1.27 we get a support E^T for $\varepsilon(D)^T$, and associated functions

$$\bar E^T_\alpha:\varepsilon(D)^T_\alpha\to[\varepsilon(D)^T_\alpha]^{<\omega}.$$

Now one verifies

$$\begin{gathered}r\in\vartheta(T)\to h(r)\in\varepsilon(D)^T_\alpha,\\ [h]^{<\omega}\circ E_{\vartheta(T)}(r)=\bar E^T_\alpha\circ h(r),\\ s<_{\vartheta(T)}t\to h(s)<_{\varepsilon(D)^T_\alpha}h(t)\end{gathered}$$

by simultaneous induction on $L_{\vartheta(T)}(r)$ resp. $L_{\vartheta(T)}(s)+L_{\vartheta(T)}(t)$ (cf. the proof of Lemma 2.3.14). This relies on Corollary 2.1.28, Lemma 2.1.29, and the observation that terms of the form $\bar\vartheta(s)=\mathfrak{e}_{\vartheta(\Omega+s)}$ behave like ε-numbers of $\varepsilon(D)^T_\alpha$ (cf. Definition 2.1.22). Let us look at three cases in detail: First, consider $r=\mathfrak{E}^{r_0,\dots,r_{n-1}}_\sigma$. In view of $L_{\vartheta(T)}(r_i)+L_{\vartheta(T)}(r_{i+1})<L_{\vartheta(T)}(r)$ and $L_{\vartheta(T)}(r_{n-1})+L_{\vartheta(T)}(\Omega)<L_{\vartheta(T)}(r)$

(see the proof of Lemma 2.3.14) we get $h(r_0) <_{\varepsilon(D)^T_\alpha} \cdots <_{\varepsilon(D)^T_\alpha} h(r_{n-1}) <_{\varepsilon(D)^T_\alpha} \Omega$ by induction hypothesis, so that we have

$$h(r) = \mathfrak{E}_{\langle\{i_\alpha^{-1}(h(r_0)),\dots,i_\alpha^{-1}(h(r_{n-1}))\},\sigma\rangle}.$$

Using Lemma 2.1.29 and Definition 2.3.2 we obtain

$$\bar{E}^T_\alpha \circ h(r) = [i_\alpha]^{<\omega} \circ \operatorname{supp}^T_\alpha(\langle\{i_\alpha^{-1}(h(r_0)),\dots,i_\alpha^{-1}(h(r_{n-1}))\},\sigma\rangle) =$$
$$= [i_\alpha]^{<\omega}(\{i_\alpha^{-1}(h(r_0)),\dots,i_\alpha^{-1}(h(r_{n-1}))\}) = \{h(r_0),\dots,h(r_{n-1})\} = [h]^{<\omega} \circ E_{\vartheta(T)}(r).$$

Next, consider $s = \mathfrak{E}^{s_0,\dots,s_{n-1}}_\sigma <_{\vartheta(T)} \mathfrak{E}^{t_0,\dots,t_{m-1}}_\tau = t$. Then we have $T_f(\sigma) <_{T_k} T_g(\tau)$ for some strictly increasing functions $f : n \to k = |\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}|$ and $g : m \to k$ which satisfy $f(i) < g(j) \Leftrightarrow s_i <_{\vartheta(T)} t_j$ and $f(i) = g(j) \Leftrightarrow s_i = t_j$ for all $i < n$ and $j < m$. The induction hypothesis tells us that h is order preserving on $\{s_0,\dots,s_{n-1},t_0,\dots,t_{m-1}\}$. Thus the conditions on f and g remain valid when we replace s_i and t_j by $i_\alpha^{-1}(h(s_i))$ resp. $i_\alpha^{-1}(h(t_j))$. As in the proof of Proposition 2.3.15 we infer that the following diagram commutes, where the horizontal arrows are the increasing enumerations:

$$\begin{array}{ccc}
n & \longrightarrow & \{i_\alpha^{-1}(h(s_0)),\dots,i_\alpha^{-1}(h(s_{n-1}))\} \\
{\scriptstyle f}\downarrow & & \downarrow \\
k & \longrightarrow & \{i_\alpha^{-1}(h(s_0)),\dots,i_\alpha^{-1}(h(s_{n-1})),i_\alpha^{-1}(h(t_0)),\dots,i_\alpha^{-1}(h(t_{m-1}))\} \\
{\scriptstyle g}\uparrow & & \uparrow \\
m & \longrightarrow & \{i_\alpha^{-1}(h(t_0)),\dots,i_\alpha^{-1}(h(t_{m-1}))\}.
\end{array}$$

This means that f and g correspond to the maps $|\iota^{a\cup b}_a|$ resp. $|\iota^{a\cup b}_b|$ from Definition 2.3.2, so that $T_f(\sigma) <_{T_k} T_g(\tau)$ implies

$$\langle\{i_\alpha^{-1}(h(s_0)),\dots,i_\alpha^{-1}(h(s_{n-1}))\},\sigma\rangle <^T_\alpha \langle\{i_\alpha^{-1}(h(t_0)),\dots,i_\alpha^{-1}(h(t_{m-1}))\},\tau\rangle.$$

By the definition of $<_{\varepsilon(D)^T_\alpha}$ we obtain

$$h(s) = \mathfrak{E}_{\langle\{i_\alpha^{-1}(h(s_0)),\dots,i_\alpha^{-1}(h(s_{n-1}))\},\sigma\rangle} <_{\varepsilon(D)^T_\alpha} \mathfrak{E}_{\langle\{i_\alpha^{-1}(h(t_0)),\dots,i_\alpha^{-1}(h(t_{m-1}))\},\tau\rangle} = h(t),$$

as desired. Finally, consider $s = \vartheta s' <_{\vartheta(T)} \vartheta t' = t$. Assume this holds because we have $s' <_{\vartheta(T)} t'$ and $r' <_{\vartheta(T)} t$ for all $r' \in E_{\vartheta(T)}(s')$. The induction hypothesis provides $h(s') <_{\varepsilon(D)^T_\alpha} h(t')$. To infer $h(s) = \bar{\vartheta}(h(s')) <_{\varepsilon(D)^T_\alpha} \bar{\vartheta}(h(t')) = h(t)$ by Corollary 2.1.28 we need $r <_{\varepsilon(D)^T_\alpha} \bar{\vartheta}(h(t'))$ for all $r \in \bar{E}^T_\alpha(h(s'))$. As the simultaneous induction makes $\bar{E}^T_\alpha(h(s')) = [h]^{<\omega} \circ E_{\vartheta(T)}(s')$ available, any such r can be written as $r = h(r')$ with $r' \in E_{\vartheta(T)}(s')$. In view of $L_{\vartheta(T)}(r') \leq L_{\vartheta(T)}(s') < L_{\vartheta(T)}(s)$ the induction hypothesis yields $r = h(r') <_{\varepsilon(D)^T_\alpha} h(t) = \bar{\vartheta}(h(t'))$, as required. In case $t' <_{\vartheta(T)} s'$ one argues similarly. We have thus shown that h is an order embedding of $(\vartheta(T), <_{\vartheta(T)})$ into $(\varepsilon(D)^T_\alpha, <_{\varepsilon(D)^T_\alpha})$. This implies that $(\vartheta(T), <_{\vartheta(T)})$ is well-founded, as demanded by the computable Bachmann-Howard principle. □

Let us discuss a potential simplification of the computable and the predicative Bachmann-Howard principle:

REMARK 2.4.10. In the context of the abstract Bachmann-Howard principle we have separated the general notion of collapse $\vartheta : T_\alpha \xrightarrow{\text{BH}} \alpha$ from the construction of a particular proto-dilator $\alpha \mapsto \varepsilon(T)_\alpha$. Combining the two, we obtained a particularly powerful collapse $\vartheta : \varepsilon(T)_\alpha \xrightarrow{\text{BH}} \alpha$, as exploited in the previous proof. This separation has two advantages: Firstly, the type-two aspect (existence of collapsing functions) and the type-one aspect (well-foundedness of $\varepsilon(T)_\alpha$, cf. Proposition 2.1.14) are transparent. Secondly, the general notion of Bachmann-Howard collapse becomes very simple and can be appreciated independently of ordinal notation systems. In the computable and the predicative Bachmann-Howard principle the two aspects have been merged: Recall that we have embedded $\vartheta(T)$ into $\varepsilon(D)^T_\alpha$, rather than D^T_α, in the previous proof. The main advantage is that $\vartheta(T)$ thus becomes similar to Rathjen and Valencia Vizcaíno's [**71**] notation system ϑ_X (which is relativized to a well-order X, rather than a dilator T). In particular, we have constructed an embedding of ϑ_X into $\vartheta(T)$ in order to boot up from $\mathbf{RCA_0}$ to $\mathbf{ATR_0}$ (see Lemma 2.3.18 and Corollary 2.3.19). Nevertheless it may be attractive to construct variants of $\vartheta(T)$ and $\mathrm{BH}(T)$ which separate the two aspects. Presumably, the modified term system $\vartheta(T)$ would consist of a single clause:

- Given terms $s_0 < \cdots < s_{n-1}$ and an element $\sigma \in T_n$ with $\operatorname{supp}_n(\sigma) = n$ we get a new term $\vartheta^{s_0,\dots,s_{n-1}}_\sigma$.

Assume that $\vartheta : T_\alpha \to \alpha$ is a Bachmann-Howard collapse. If s_i is interpreted by $h^*(i) \in \alpha$ (cf. the previous proof) then $\vartheta^{s_0,\dots,s_{n-1}}_\sigma$ will be interpreted by $\vartheta(T_{h^*}(\sigma))$. Conversely, if the modified term system $\vartheta(T)$ is well-founded with order type α, then one should be able to read off a Bachmann-Howard collapse $\vartheta : T_\alpha \to \alpha$. Over the base theory $\mathbf{ATR_0^{set}}$ the simplified version of $\vartheta(T)$ should thus be sufficient (but the details remain to be checked). Over $\mathbf{RCA_0}$ the simplification will lead to new complications: To show that a strengthened notation system like $\varepsilon(T)_\alpha$ is well-founded we seem to need $\mathbf{ACA_0^+}$ (cf. [**52, 4**]). Possibly, the well-foundedness of $\varepsilon(T)_\alpha$ could be deduced from an appropriate instance of the collapsing principle as well (recall that Schütte and Simpson [**80**] construct ε_0 from collapsing functions alone). We will not pursue this idea in the present thesis.

In the rest of this section we show that the computable Bachmann-Howard principle implies its predicative counterpart, again under the axiom of countability. This amounts to direction (v)⇒(iv) of Theorem 4.4.6. Let us fix a coded prae-dilator $T=(T,\operatorname{supp})$. By the discussion before Convention 2.3.20 we may view T as a set-sized prae-dilator. Consider the associated prae-dilator $D^T=(D^T,\operatorname{supp}^T)$ provided by Definition 2.3.2 and Lemma 2.3.3. The order $\operatorname{BH}(D^T)$ from the predicative Bachmann-Howard principle was constructed as the direct limit of approximations $\operatorname{BH}_n(D^T)$. Recall that this relied on the notion of good BH-system, introduced in Definitions 2.2.5 and 2.2.9. In the following we define compatible embeddings $h_n:\operatorname{BH}_n(D^T)\to\vartheta(T)\cap\Omega$, which glue to an embedding of $\operatorname{BH}(D^T)$ into

$$\vartheta(T)\cap\Omega=\{s\in\vartheta(T)\,|\,s<_{\vartheta(T)}\Omega\}.$$

Using the computable Bachmann-Howard principle, we learn that $\operatorname{BH}(D^T)$ is well-founded whenever T is a coded dilator. By Proposition 2.3.21 this suffices to establish the predicative Bachmann-Howard principle. To give a recursive construction of the embeddings h_n we introduce the following concept (note that $\vartheta^0(D^T_X)$ refers to the term system from Definition 2.2.3, while $\vartheta^0(T)$ is the term system constructed in the proof of Lemma 2.3.14):

Definition 2.4.11. Let $X=(X,E_X,L_X,i_X)$ be a good BH-system for D^T. Given an embedding $h_X:X\to\vartheta(T)\cap\Omega$, we construct a map $h_X^\vartheta:\vartheta^0(D^T_X)\to\vartheta^0(T)$ by the following recursion over terms:

(i) $h_X^\vartheta(0)=0$,
(ii) $h_X^\vartheta(\Omega)=\Omega$,
(iii) $h_X^\vartheta(\mathfrak{E}_\sigma)=\mathfrak{E}_{\sigma_0}^{h_X(s_0),\dots,h_X(s_{n-1})}$, where the element $\sigma\in D^T_X$ is written as $\sigma=\langle\{s_0,\dots,s_{n-1}\},\sigma_0\rangle$ with $s_0<_X\cdots<_X s_{n-1}$,
(iv) $h_X^\vartheta(\vartheta s)=\vartheta\, h_X^\vartheta(s)$,
(v) $h_X^\vartheta(\omega^{s_0}+\cdots+\omega^{s_n})=\omega^{h_X^\vartheta(s_0)}+\cdots+\omega^{h_X^\vartheta(s_n)}$.

If we have

$$h_X^\vartheta\circ i_X=h_X$$

then h_X is called a BH-embedding of X.

As a starting point, recall the good BH-system $\operatorname{BH}_0(D^T)=(\varepsilon_0,E_{\varepsilon_0},L_{\varepsilon_0},i_{\varepsilon_0})$ from Lemma 2.2.10 and Definition 2.2.14.

LEMMA 2.4.12. *There is a* BH-*embedding* $h_{\varepsilon_0} : \varepsilon_0 \to \vartheta(T) \cap \Omega$.

PROOF. The definition of h_{ε_0} coincides with the definition of the embedding $i_{\varepsilon_0} : \varepsilon_0 \to \vartheta(D^T_{\varepsilon_0}) \cap \Omega$ in the proof of Lemma 2.2.10. It is straightforward to verify the conditions of a BH-embedding. □

If X is a good BH-system then Lemma 2.2.7 and Proposition 2.2.8 yield a set $\vartheta(D^T_X) \subseteq \vartheta^0(D^T_X)$ of terms in normal form and a linear order $<^\vartheta_X$ on $\vartheta(D^T_X)$. Also recall the function $E^\vartheta_X : \vartheta^0(D^T_X) \to [\vartheta^0(D^T_X)]^{<\omega}$ from the definition of BH-system. The following is needed to construct h_{n+1} from h_n:

LEMMA 2.4.13. *Assume that* $h_X : X \to \vartheta(T) \cap \Omega$ *is a* BH-*embedding of the good* BH-*system* $X = (X, E_X, L_X, i_X)$. *Then the following holds for all* $r, s, t \in \vartheta^0(D^T_X)$*:*

(i) If $r \in \vartheta(D^T_X)$ *then* $h^\vartheta_X(r) \in \vartheta(T)$ *and* $E_{\vartheta(T)} \circ h^\vartheta_X(r) = [h^\vartheta_X]^{<\omega} \circ E^\vartheta_X(r)$.

(ii) If $s, t \in \vartheta(D^T_X)$ *and* $s <^\vartheta_X t$ *then* $h^\vartheta_X(s) <_{\vartheta(T)} h^\vartheta_X(t)$.

PROOF. We show parts (i) and (ii) by simultaneous induction over $L^\vartheta_X(r)$ and $L^\vartheta_X(s) + L^\vartheta_X(t)$, respectively. In (i), the only interesting case is $r = \mathfrak{E}_{\langle\{s_0,\dots,s_{n-1}\},\sigma_0\rangle}$. Using Definition 2.2.5, Definition 2.3.2, the fact that h_X is a BH-embedding, and the definition of $E_{\vartheta(T)}$ in Lemma 2.3.14, we indeed get

$$\begin{aligned}&[h^\vartheta_X]^{<\omega} \circ E^\vartheta_X(r) = [h^\vartheta_X]^{<\omega} \circ [i_X]^{<\omega} \circ \operatorname{supp}^T_X(\langle\{s_0,\dots,s_{n-1}\},\sigma_0\rangle) = \\ &\quad = [h^\vartheta_X \circ i_X]^{<\omega}(\{s_0,\dots,s_{n-1}\}) = \{h_X(s_0),\dots,h_X(s_{n-1})\} = E_{\vartheta(T)} \circ h^\vartheta_X(r).\end{aligned}$$

In (ii), the first interesting case is $s = \mathfrak{E}_{\langle\{s_0,\dots,s_{n-1}\},\sigma_0\rangle} <^\vartheta_X \mathfrak{E}_{\langle\{t_0,\dots,t_{m-1}\},\tau_0\rangle} = t$. This is equivalent to

$$\langle\{s_0,\dots,s_{n-1}\},\sigma_0\rangle <_{D^T_X} \langle\{t_0,\dots,t_{m-1}\},\tau_0\rangle.$$

By Lemma 2.3.3 the map $D^T_{h_X} : D^T_X \to D^T_{\vartheta(T)\cap\Omega}$ is an embedding, so that we get

$$\langle\{h_X(s_0),\dots,h_X(s_{n-1})\},\sigma_0\rangle <_{D^T_{\vartheta(T)\cap\Omega}} \langle\{h_X(t_0),\dots,h_X(t_{m-1})\},\tau_0\rangle.$$

According to Definition 2.3.2 this means

$$T_f(\sigma_0) <_{T_k} T_g(\tau_0),$$

where k is the cardinality of $\{h_X(s_0),\dots,h_X(s_{n-1}),h_X(t_0),\dots,h_X(t_{m-1})\}$ and f, g make the following diagram commute (the horizontal arrows are the increasing enumerations with respect to $<_{\vartheta(T)}$):

$$\begin{array}{ccc}
n & \longrightarrow & \{h_X(s_0),\dots,h_X(s_{n-1})\} \\
{\scriptstyle f}\downarrow & & \downarrow \\
k & \longrightarrow & \{h_X(s_0),\dots,h_X(s_{n-1}),h_X(t_0),\dots,h_X(t_{m-1})\} \\
{\scriptstyle g}\uparrow & & \uparrow \\
m & \longrightarrow & \{h_X(t_0),\dots,h_X(t_{m-1})\}.
\end{array}$$

It is easy to deduce the equivalences $f(i) < g(j) \Leftrightarrow h_X(s_i) <_{\vartheta(T)} h_X(t_j)$ and $f(i) = g(j) \Leftrightarrow h_X(s_i) = h_X(t_j)$. The definition of $<_{\vartheta(T)}$ in Lemma 2.3.14 yields

$$h_X^\vartheta(s) = \mathfrak{E}_{\sigma_0}^{h_X(s_0),\dots,h_X(s_{n-1})} <_{\vartheta(T)} \mathfrak{E}_{\tau_0}^{h_X(t_0),\dots,h_X(t_{m-1})} = h_X^\vartheta(t),$$

as desired. The other interesting case is $s = \vartheta s' <_X^\vartheta \vartheta t' = t$. Assume that this holds because we have $s' <_X^\vartheta t'$ and $r' <_X^\vartheta t$ for all $r' \in E_X^\vartheta(s')$ (recall that $r' \in \vartheta(D_X^T)$ is automatic by Lemma 2.2.11). The induction hypothesis provides $h_X^\vartheta(s') <_{\vartheta(T)} h_X^\vartheta(t')$. To conclude $h_X^\vartheta(s) = \vartheta\, h_X^\vartheta(s') <_{\vartheta(T)} \vartheta\, h_X^\vartheta(t') = h_X^\vartheta(t)$ we need $r <_{\vartheta(T)} h_X^\vartheta(t)$ for all terms $r \in E_{\vartheta(T)}(h_X^\vartheta(s'))$. By part (i) of the induction hypothesis any such r can be written as $r = h_X^\vartheta(r')$ with $r' \in E_X^\vartheta(s')$. Note that Lemma 2.2.6 ensures $L_X^\vartheta(r') \leq L_X^\vartheta(s') < L_X^\vartheta(s)$. Thus the induction hypothesis yields $r = h_X^\vartheta(r') <_{\vartheta(T)} h_X^\vartheta(t)$, as required. The case $\vartheta s' <_X^\vartheta \vartheta t'$ with $t' <_X^\vartheta s'$ is similar. □

We can now lift the BH-embedding $h_X : X \to \vartheta(T) \cap \Omega$ as follows:

Definition 2.4.14. With each BH-embedding $h_X : X \to \vartheta(T) \cap \Omega$ we associate the function

$$h_{\vartheta(D_X^T)\cap\Omega} := h_X^\vartheta \restriction (\vartheta(D_X^T) \cap \Omega) : \vartheta(D_X^T) \cap \Omega \to \vartheta(T) \cap \Omega.$$

In view of the previous lemma this is an order embedding.

By Theorem 2.2.13 a good BH-system over X induces a good BH-system

$$(\vartheta(D_X^T) \cap \Omega, E_{\vartheta(D_X^T)\cap\Omega}, L_{\vartheta(D_X^T)\cap\Omega}, i_{\vartheta(D_X^T)\cap\Omega})$$

over $\vartheta(D_X^T) \cap \Omega$. The following completes the lifting of BH-embeddings:

Proposition 2.4.15. *If $h_X : X \to \vartheta(T) \cap \Omega$ is a* BH-*embedding of the good* BH-*system X then $h_{\vartheta(D_X^T)\cap\Omega} : \vartheta(D_X^T) \cap \Omega \to \vartheta(T) \cap \Omega$ is a* BH-*embedding of the good* BH-*system $\vartheta(D_X^T) \cap \Omega$.*

PROOF. According to the definition of BH-embedding we must establish

$$h^{\vartheta}_{\vartheta(D^T_X)\cap\Omega}\circ i_{\vartheta(D^T_X)\cap\Omega}(s)=h_{\vartheta(D^T_X)\cap\Omega}(s)$$

for $s\in\vartheta(D^T_X)\cap\Omega$. Recall from Definition 2.2.12 that $i_{\vartheta(D^T_X)\cap\Omega}$ was defined as the restriction of a map $i^{\vartheta}_X:\vartheta^0(D^T_X)\to\vartheta^0(D^T_{\vartheta(D^T_X)\cap\Omega})$. Thus it suffices to show

$$h^{\vartheta}_{\vartheta(D^T_X)\cap\Omega}\circ i^{\vartheta}_X(s)=h^{\vartheta}_X(s)$$

for $s\in\vartheta^0(D^T_X)$. Arguing by induction over the build-up of s, the only interesting case is $s=\mathfrak{E}_\sigma$. Write $\sigma=\langle\{s_0,\dots,s_{n-1}\},\sigma_0\rangle$ and observe

$$i^{\vartheta}_X(s)=\mathfrak{E}_{D^T_{i_X}(\sigma)}\text{ with }D^T_{i_X}(\sigma)=\langle\{i_X(s_0),\dots,i_X(s_{n-1})\},\sigma_0\rangle.$$

As h_X is a BH-embedding we have $h_{\vartheta(D^T_X)\cap\Omega}\circ i_X(s_i)=h^{\vartheta}_X\circ i_X(s_i)=h_X(s_i)$. Now it is easy to see

$$h^{\vartheta}_{\vartheta(D^T_X)\cap\Omega}\circ i^{\vartheta}_X(s)=\mathfrak{E}^{h_X(s_0),\dots,h_X(s_{n-1})}_{\sigma_0}=h^{\vartheta}_X(s),$$

as required. □

Recall the recursively constructed BH-systems $\mathrm{BH}_n(D^T)$ from Definition 2.2.14. As promised, our construction yields corresponding BH-embeddings:

DEFINITION 2.4.16. Let

$$h_0:\mathrm{BH}_0(D^T)=\varepsilon_0\to\vartheta(T)\cap\Omega$$

be the BH-embedding from Lemma 2.4.12. Assuming that the BH-embedding $h_n:\mathrm{BH}_n(D^T)\to\vartheta(T)\cap\Omega$ is already defined, invoke Definition 2.4.14 to get

$$h_{n+1}:=h^{\vartheta}_n\restriction(\vartheta(D^T_{\mathrm{BH}_n(D^T)})\cap\Omega):\mathrm{BH}_{n+1}(D^T)=\vartheta(D^T_{\mathrm{BH}_n(D^T)})\cap\Omega\to\vartheta(T)\cap\Omega.$$

This is a BH-embedding by Proposition 2.4.15.

Observe that there is a primitive recursive function which computes $h^{\vartheta}_X(s)$ from h_X and s. By Proposition 1.2.2 this implies that the transformation of h_X into the set-sized function h^{ϑ}_X is primitive recursive. It follows that $n\mapsto h_n$ is primitive recursive (with parameter T). Another application of Proposition 1.2.2 tells us that $(n,s)\mapsto h_n(s)$ is a set-sized function. As promised, we can now glue the constructed BH-embeddings:

THEOREM 2.4.17. *For any coded prae-dilator T we have an order embedding*

$$h:(\mathrm{BH}(D^T),<_{\mathrm{BH}(D^T)})\to(\vartheta(T)\cap\Omega,<_{\vartheta(T)}).$$

Proof. By the definition of BH-embedding we have $h_n^\vartheta \circ i_{\mathrm{BH}_n(D^T)} = h_n$. As h_{n+1} is the restriction of h_n^ϑ this implies

$$h_{n+1} \circ i_{\mathrm{BH}_n(D^T)} = h_n.$$

Thus Lemma 2.2.17 yields a function

$$h : \mathrm{BH}(D^T) \to \vartheta(T) \cap \Omega$$

with $h \circ j_n = h_n$, where $j_n : (\mathrm{BH}_n(D^T), <_{\mathrm{BH}_n(D^T)}) \to (\mathrm{BH}(D^T), <_{\mathrm{BH}(D^T)})$ are order embeddings (see Lemma 2.2.18). As the BH-embeddings h_n are order preserving, it is easy to deduce that h is an order embedding as well. □

Finally, we can deduce direction (v)⇒(iv) of Theorem 4.4.6:

Corollary 2.4.18 (Countability). *The computable Bachmann-Howard principle implies (any instance of) the predicative Bachmann-Howard principle.*

Proof. By Proposition 2.3.21 it suffices to show that $\mathrm{BH}(D^T)$ is well-founded for any coded dilator T. The computable Bachmann-Howard principle tells us that $\vartheta(T)$ is well-founded. In particular, the sub-order $\vartheta(T) \cap \Omega$ is well-founded. We can conclude by the previous theorem. □

CHAPTER 3

Admissible Sets via Search Trees

In this chapter we present a construction of admissible sets, i.e. transitive models of Kripke-Platek set theory. This is needed to prove the crucial direction (iv)⇒(ii) of Theorem 4.4.6.

The main ingredient for our construction of admissible sets will be Schütte's method of deduction chains. In [**79**, Section II.5.3] this method is used to prove the completeness theorem for predicate logic: Given a formula φ one tries to construct a proof of φ in reverse order. If the process terminates then this results in a finite proof tree with φ at the root. Otherwise the attempted proof of φ has an infinite branch. This branch determines an interpretation of the non-logical symbols which makes any formula on the branch false. In particular it yields a countermodel to φ, as needed for the completeness theorem. Schütte calls a path between the root and any node of this attempted proof a deduction chain. We use the term "search tree" to refer to the attempted proof as a whole.

Historically, the first use of deduction chains seems to occur in [**77**]. In this paper Schütte introduces his positive and negative forms and uses them to formulate the completeness proof described in the previous paragraph. Remarkably, he goes on to apply the same method to a system with the ω-rule, proving what is now known as the Shoenfield completeness theorem for recursive ω-logic (cf. [**83**]). The recent literature contains more applications of deduction chains to ω-logic: Jäger and Strahm [**43**] prove that ω-model reflection for Π^1_{n+1}-formulas is equivalent to Π^1_n-bar induction, extending previous results of H. Friedman and Simpson. Afshari and Rathjen [**4**] combine deduction chains and the method of cut elimination to show that a certain well-ordering principle $X \mapsto \varepsilon_X$ implies the principle of arithmetical recursion along the natural numbers. Many other Π^1_2-statements have been characterized in a similar way, as described in the introduction to the present thesis. There are also some applications of deduction chains to β-logic: Buchholz [**8**] constructs a functorial family of search trees in order to show that certain arithmetical functions on Kleene's $\mathcal{O}$ are dominated by dilators (originally due to Girard). Jäger [**40**] extends this approach to general α-recursive functions.

In the present chapter we develop the method of search trees (deduction chains) in a set-theoretic context. The main difference to ω-logic is that the domain of the intended models is not fixed in advance: The first-order part of any ω-model is the standard structure of natural numbers, but there is no canonical model of set theory. To make the situation more manageable one can work inside a given stage $\mathbb{L}^u_\alpha$ of the relativized constructible hierarchy. Search trees are then constructed in $\mathbb{L}^u_\alpha$-logic, which is characterized by the infinitary rule

$$\frac{\cdots \qquad \Gamma,\varphi(a) \qquad \cdots \quad (a\in\mathbb{L}^u_\alpha)}{\Gamma,\forall_x\varphi(x)}$$

with a premise for each element $a\in\mathbb{L}^u_\alpha$. From an infinite branch of the resulting search tree we will be able to read off a model $M\subseteq\mathbb{L}^u_\alpha$ which, as before, makes all formulas on the given branch false. The main idea is quite simple: If a branch contains the conclusion $\forall_x\varphi(x)$ of an infinitary rule then it contains the premise $\varphi(a)$ for some $a\in\mathbb{L}^u_\alpha$. We stipulate that the model M determined by the branch contains this element a. By induction on the formula complexity we may assume $M\nvDash\varphi(a)$, which is enough to guarantee $M\nvDash\forall_x\varphi(x)$. Of course the existence of a suitable model $M\subseteq\mathbb{L}^u_\alpha$ relies on α being large enough. In our application we are not given a large ordinal α in advance. Instead we construct search trees S^u_α for all ordinals α simultaneously. Similar to the case of ω-logic one then argues as follows: A priori it is possible that all search trees S^u_α are well-founded. In this case the family of search trees can be extended to a β-proof of contradiction. In the next chapter we will use methods of ordinal analysis to show that such a proof cannot exist. Thus one of the search trees S^u_α must have an infinite branch after all. From this branch we obtain the desired model $M\subseteq\mathbb{L}^u_\alpha$, as described above.

To conclude this introduction, let us summarize the different sections of the present chapter: In order to get the β-proof mentioned in the previous paragraph we have to ensure that the construction of search trees is functorial. For this purpose we need a functorial version of the constructible hierarchy, which will be introduced in Section 3.1. The idea is to define a canonical term system $\mathbf{L}^u_X$ for any linear order X. If $X\cong\alpha$ is well-founded we can give an interpretation of $\mathbf{L}^u_X$ in the stage $\mathbb{L}^u_\alpha$ of the actual constructible hierarchy. We remark that syntactic versions of the constructible hierarchy are well-known in ordinal analysis, in particular from the work of Jäger [**39**]. Earlier, Schütte [**78**] had considered a syntactic version of ramified analysis. Section 3.2 contains a detailed definition of the search tree S^u_X, where X is an arbitrary linear order. For well-founded X we spell out how an infinite branch in S^u_X determines a transitive model of Kripke-Platek set theory. In Section 3.3 we verify that the construction $X \mapsto S^u_X$ yields a prae-dilator. If S^u_X is well-founded for any well-order X then we have a dilator. In Section 3.4 we show how this dilator can be transformed into the aforementioned β-proof. More precisely, β-proofs are usually defined as compatible families of infinite proofs indexed by the ordinals (see e.g. [**29**, Section 6]). We will instead construct a single infinite proof which represents the entire family: Invoking the predicative Bachmann-Howard principle from the previous chapter, the dilator $X \mapsto S^u_X$ can be transformed into a well-order $\mathrm{BH}(S^u)$. The order-type of $\mathrm{BH}(S^u)$ is so large that the single search tree $S^u_{\mathrm{BH}(S^u)}$ reflects many relevant properties of the general construction $X \mapsto S^u_X$. It will thus be enough to consider a single infinite proof based on $S^u_{\mathrm{BH}(S^u)}$, rather than an entire β-proof. This will simplify the ordinal analysis in the next chapter considerably: We can now use the usual methods for single infinite proofs, instead of generalizing them to compatible families.

3.1. A Functorial Version of the Constructible Hierarchy

In Section 1.3 we have considered the constructible hierarchy $\alpha \mapsto \mathbb{L}^u_\alpha$ over a transitive set u of "urelements". The present section introduces a functorial version $X \mapsto \mathbf{L}^u_X$ of this construction. Semantically we can only make sense of $\mathbf{L}^u_X$ if X is a well-order, and indeed isomorphic to an ordinal. From the functorial viewpoint, however, it is natural to construct $\mathbf{L}^u_X$ for an arbitrary linear order X. To achieve this we give a syntactic definition of $\mathbf{L}^u_X$ as a set of terms. For each order embedding $f : X \to Y$ we will be able to define a function $\mathbf{L}^u_f : \mathbf{L}^u_X \to \mathbf{L}^u_Y$ by recursion over the terms in $\mathbf{L}^u_X$. We shall also define linear order relations on the sets $\mathbf{L}^u_X$, in such a way that $\mathbf{L}^u_f$ becomes an order embedding. This will allow us to view $\mathbf{L}^u_\cdot$ as an endofunctor of linear orders. To begin, let us collect some properties that will be used at various points of the present chapter:

Assumption 3.1.1. Throughout the following we fix a set u and assume that

(i) u is transitive,
(ii) u is countable, with a fixed enumeration $u = \{u_i \mid i \in \omega\}$,
(iii) we have $0, 1 \in u$,
(iv) the height $o(u) = u \cap \mathrm{Ord}$ of u is a successor ordinal.

In view of (ii) one should officially think of u as the function $\omega \ni i \mapsto u_i$.

Condition (i) is standard in the definition of the constructible hierarchy. Condition (ii) will be crucial in the context of search trees, as each branch of such a tree can only accomodate a countable amount of information. Note that any non-empty countable set has an enumeration, by Proposition 1.2.12. Conditions (iii) and (iv) are pure convenience. To satisfy them we can simply replace u by $u' := u \cup \{0, 1\}$ and then further by $u'' := u' \cup \{o(u')\}$, which makes $o(u'') = o(u') + 1$ a successor.

To prepare the definition of $\mathbf{L}^u_\cdot$, recall that $[X]^{<\omega}$ denotes the set of finite subsets of a given set X. In Section 1.2 we have seen that the map $X \mapsto [X]^{<\omega}$ is primitive recursive. As in Section 2.2 we can turn this map into an endofunctor on the category of sets, setting

$$[f]^{<\omega}(x) = \{f(s) \mid s \in x\} \in [Y]^{<\omega}$$

for a finite set $x \in [X]^{<\omega}$ and a function $f : X \to Y$. We also apply $[\cdot]^{<\omega}$ to linear orders, omitting the forgetful functor to their underlying sets. Conversely, we may view a subset of a linear order as a sub-order. If $(X, <_X)$ is a linear order (or at least a partial order) then we define a partial order $<^{\text{fin}}_X$ on $[X]^{<\omega}$ by stipulating

$$x <^{\text{fin}}_X x' \quad :\Leftrightarrow \quad \text{"for any } s \in x \text{ there is an } s' \in x' \text{ with } s <_X s' \text{"}.$$

If s, s' are elements of X we write $x <^{\text{fin}}_X s'$ and $s <^{\text{fin}}_X x'$ rather than $x <^{\text{fin}}_X \{s'\}$ and $\{s\} <^{\text{fin}}_X x'$, respectively. We can now define the functor $\mathbf{L}^u_\cdot$ on objects:

Definition 3.1.2. For each linear order $(X, <_X)$ we define a set $\mathbf{L}^u_X$ of terms and a support function $\operatorname{supp}^{\mathbf{L}}_X : \mathbf{L}^u_X \to [X]^{<\omega}$, by simultaneous recursion:

(i) Any element $u_i \in u$ is an $\mathbf{L}^u_X$-term with support $\operatorname{supp}^{\mathbf{L}}_X(u_i) = \emptyset$,
(ii) for each $s \in X$ we have an $\mathbf{L}^u_X$-term $\mathbf{L}^u_s$ with support $\operatorname{supp}^{\mathbf{L}}_X(\mathbf{L}^u_s) = \{s\}$,
(iii) given $s \in X$, a Δ_0-formula $\varphi(x, y_1, \dots, y_n)$ and $\mathbf{L}^u_X$-terms $a_1, \dots, a_n$ with $\operatorname{supp}^{\mathbf{L}}_X(a_i) <^{\text{fin}}_X s$ we get an $\mathbf{L}^u_X$-term $\{x \in \mathbf{L}^u_s \mid \varphi(x, a_1, \dots, a_n)\}$ with

$$\operatorname{supp}^{\mathbf{L}}_X(\{x \in \mathbf{L}^u_s \mid \varphi(x, a_1, \dots, a_n)\}) = \{s\} \cup \operatorname{supp}^{\mathbf{L}}_X(a_1) \cup \dots \cup \operatorname{supp}^{\mathbf{L}}_X(a_n).$$

Observe that $\mathbf{L}^u_X$ is given by an inductive definition with closure ordinal ω. In this respect the situation is similar to the definition of M-formulas in Section 1.3. As in that case we see that the function $(u, X) \mapsto \mathbf{L}^u_X$ is primitive recursive (with parameter ω, cf. Convention 1.2.9). Parallel to Lemma 1.3.2 we can define primitive recursive functions by recursion over $\mathbf{L}^u_X$-terms. As a first application, let us show that $\mathbf{L}^u_X$ can be related to the usual constructible hierarchy if X is well-founded:

PROPOSITION 3.1.3. *Given an isomorphism* $X \xrightarrow{\cong} \alpha$, $s \mapsto \alpha_s$ *between* X *and an ordinal* α *we can construct a surjection*

$$[\![\cdot]\!] : \mathbf{L}^u_X \to \mathbb{L}^u_\alpha.$$

More precisely, we have $[\![a]\!] \in \mathbb{L}^u_{\alpha_s}$ *for each* $a \in \mathbf{L}^u_X$ *with* $\operatorname{supp}^{\mathbf{L}}_X(a) <^{\text{fin}}_X s$. *Conversely,* $x \in [\![a]\!]$ *implies* $x = [\![a_0]\!]$ *for some* $a_0 \in \mathbf{L}^u_X$ *with* $\operatorname{supp}^{\mathbf{L}}_X(a_0) <^{\text{fin}}_X \operatorname{supp}^{\mathbf{L}}_X(a)$.

Recall that our default base theory $\mathbf{PRS}\omega$ does not prove axiom beta, i.e. it does not prove that any well-order is isomorphic to an ordinal. The existence of an isomorphism $X \cong \alpha$ is an assumption of the above proposition. If such an isomorphism exists then it is, of course, unique.

PROOF. As explained above we can give a primitive recursive definition of $[\![\cdot]\!]$ by recursion over $\mathbf{L}^u_X$-terms, setting

$$\begin{aligned} [\![u_i]\!] &:= u_i, \\ [\![\mathbf{L}^u_s]\!] &:= \mathbb{L}^u_{\alpha_s}, \\ [\![\{x \in \mathbf{L}^u_s \,|\, \varphi(x, a_1, \dots, a_n)\}]\!] &:= \{x \in \mathbb{L}^u_{\alpha_s} \,|\, \mathbb{L}^u_{\alpha_s} \vDash \varphi(x, [\![a_1]\!], \dots, [\![a_n]\!])\}. \end{aligned}$$

The second clause uses the primitive recursive function $\gamma \mapsto \mathbb{L}^u_\gamma$ from Definition 1.3.8. The third clause relies on the fact that Δ_0-separation is primitive recursive (see Corollary 1.1.10), and on the primitive recursive satisfaction relation $\vDash$ from Proposition 1.3.3. By induction on $a \in \mathbf{L}^u_X$ one verifies that $\operatorname{supp}^{\mathbf{L}}_X(a) <^{\text{fin}}_X s$ implies $[\![a]\!] \in \mathbb{L}^u_{\alpha_s}$: The most interesting case is $a = \{x \in \mathbf{L}^u_s \,|\, \varphi(x, a_1, \dots, a_n)\}$. By the definition of $\mathbf{L}^u_X$ we have $\operatorname{supp}^{\mathbf{L}}_X(a_i) <^{\text{fin}}_X s$, so the induction hypothesis provides $[\![a_i]\!] \in \mathbb{L}^u_{\alpha_s}$. Thus $\varphi(x, [\![a_1]\!], \dots, [\![a_n]\!])$ is a $\Delta_0(\mathbb{L}^u_{\alpha_s})$-formula. Invoking Definition 1.3.7 and Definition 1.3.8 we get

$$[\![a]\!] = \operatorname{Def}(\mathbb{L}^u_{\alpha_s}, \varphi(x, [\![a_1]\!], \dots, [\![a_n]\!])) \in \operatorname{Def}_0(\mathbb{L}^u_{\alpha_s}) = \mathbb{L}^u_{\alpha_s+1}.$$

Now $\operatorname{supp}^{\mathbf{L}}_X(a) <^{\text{fin}}_X t$ implies $s <_X t$ and thus $\alpha_s + 1 \leq \alpha_t$. Using Lemma 1.3.9 we obtain $[\![a]\!] \in \mathbb{L}^u_{\alpha_t}$, as required. In particular it follows that the values of $[\![\cdot]\!]$ lie in $\mathbb{L}^u_\alpha$. Before we show that $[\![\cdot]\!]$ is surjective, let us establish the last claim of the proposition, i.e. the statement

$$\forall_{a \in \mathbf{L}^u_X} \forall_{x \in [\![a]\!]} \exists_{a_0 \in \mathbf{L}^u_X} ([\![a_0]\!] = x \wedge \operatorname{supp}^{\mathbf{L}}_X(a_0) <^{\text{fin}}_X \operatorname{supp}^{\mathbf{L}}_X(a)).$$

First consider $a = u_i$. As u is transitive $x \in [\![a]\!] = u_i \in u$ implies $x \in u$, say $x = u_j$. Taking $a_0 := u_j \in \mathbf{L}^u_X$ we have $[\![a_0]\!] = x$ and $\operatorname{supp}^{\mathbf{L}}_X(a_0) = \emptyset <^{\text{fin}}_X \operatorname{supp}^{\mathbf{L}}_X(a)$, as required. If a is not a term u_i then we have $\operatorname{supp}^{\mathbf{L}}_X(a) \neq \emptyset$. In the remaining cases we may thus argue by induction on $s = \max_{<_X}(\operatorname{supp}^{\mathbf{L}}_X(a)) \in X$. Clearly a is either the term $\mathbf{L}^u_s$ or a term of the form $\{y \in \mathbf{L}^u_s \mid \varphi(y, a_1, \ldots, a_n)\}$. In both cases the assumption $x \in [\![a]\!]$ implies $x \in \mathbb{L}^u_{\alpha_s}$. There are two possibilities: If x lies in u, say $x = u_j$, then $a_0 := u_j$ works as above. If x does not lie in u then we must have

$$x = \{y \in \mathbb{L}^u_{\alpha_0} \mid \mathbb{L}^u_{\alpha_0} \vDash \psi(y, z_1, \ldots, z_k)\}$$

for some $\alpha_0 < \alpha_s$ and $z_1, \ldots, z_k \in \mathbb{L}^u_{\alpha_0}$. As X is isomorphic to α we get an $s_0 \in X$ with $\alpha_{s_0} = \alpha_0$. In view of $s_0 <_X s$ we can apply the induction hypothesis to the term $\mathbf{L}^u_{s_0}$, which satisfies $\max_{<_X}(\operatorname{supp}^{\mathbf{L}}_X(\mathbf{L}^u_{s_0})) = s_0$ and $[\![\mathbf{L}^u_{s_0}]\!] = \mathbb{L}^u_{\alpha_0}$. This yields $c_1, \ldots, c_k \in \mathbf{L}^u_X$ with $[\![c_i]\!] = z_i$ and $\operatorname{supp}^{\mathbf{L}}_X(c_i) <^{\text{fin}}_X s_0$. We can thus form the term

$$a_0 := \{y \in \mathbf{L}^u_{s_0} \mid \psi(y, c_1, \ldots, c_k)\},$$

which satisfies

$$[\![a_0]\!] = \{y \in \mathbb{L}^u_{\alpha_0} \mid \mathbb{L}^u_{\alpha_0} \vDash \psi(y, [\![c_1]\!], \ldots, [\![c_k]\!])\} = x$$

as well as

$$\operatorname{supp}^{\mathbf{L}}_X(a_0) \leq^{\text{fin}}_X s_0 <_X s \leq^{\text{fin}}_X \operatorname{supp}^{\mathbf{L}}_X(a).$$

By a similar argument one deduces that $[\![\cdot]\!] : \mathbf{L}^u_X \to \mathbb{L}^u_\alpha$ is surjective. □

We will later establish a much deeper connection between $\mathbf{L}^u_X$ and $\mathbb{L}^u_\alpha$, via a verification calculus for $\mathbf{L}^u_X$-formulas. Before that, let us look at functorial aspects of the construction. The first step is to define $\mathbf{L}^u_\cdot$ on morphisms:

Definition 3.1.4. Given an order embedding $f : X \to Y$ we define a map $\mathbf{L}^u_f : \mathbf{L}^u_X \to \mathbf{L}^u_Y$ (cf. the following lemma) by recursion on terms, setting

$$\mathbf{L}^u_f(u_i) = u_i,$$
$$\mathbf{L}^u_f(\mathbf{L}^u_s) = \mathbf{L}^u_{f(s)},$$
$$\mathbf{L}^u_f(\{x \in \mathbf{L}^u_s \mid \varphi(x, a_1, \ldots, a_n)\}) = \{x \in \mathbf{L}^u_{f(s)} \mid \varphi(x, \mathbf{L}^u_f(a_1), \ldots, \mathbf{L}^u_f(a_n))\}.$$

As the domain $\mathbf{L}^u_X$ of $\mathbf{L}^u_f$ is a set, Proposition 1.2.2 tells us that $\mathbf{L}^u_f$ is a set-sized function. Furthermore the map $f \mapsto \mathbf{L}^u_f$ is primitive recursive. Recall that we represent functors and natural transformations by primitive recursive functions, as discussed at the beginning of Section 2.2. Indeed we can verify the following:

Lemma 3.1.5. *The functions $X \mapsto \mathbf{L}^u_X$, $f \mapsto \mathbf{L}^u_f$ form a functor from linear orders to sets (in particular we have $\operatorname{rng}(\mathbf{L}^u_f) \subseteq \mathbf{L}^u_Y$ for $f : X \to Y$). The family of functions $\operatorname{supp}^{\mathbf{L}}_X : \mathbf{L}^u_X \to [X]^{<\omega}$ is a natural transformation from $\mathbf{L}^u_\cdot$ to $[\cdot]^{<\omega}$.*

Proof. First, let us establish $\mathbf{L}^u_f(a) \in \mathbf{L}^u_Y$ for an order embedding $f : X \to Y$ and an arbitrary $a \in \mathbf{L}^u_X$. This can be shown by induction on the term a if we simultaneously check $\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) = [f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a))$. The only interesting case is $a = \{x \in \mathbf{L}^u_s \mid \varphi(x, a_1, \ldots, a_n)\}$: From $a \in \mathbf{L}^u_X$ we know $\operatorname{supp}^{\mathbf{L}}_X(a_i) <^{\text{fin}}_X s$. Using the simultaneous induction hypothesis and the fact that f is an order embedding we get

$$\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a_i)) = [f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a_i)) <^{\text{fin}}_Y f(s).$$

This yields $\mathbf{L}^u_f(a) \in \mathbf{L}^u_Y$, as desired. The equality $\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) = [f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a))$ is easily deduced from the induction hypothesis. Functoriality of $\mathbf{L}^u_\cdot$ is checked by a straightforward induction. That $\operatorname{supp}^{\mathbf{L}}_\cdot$ is a natural transformation was part of the simultaneous induction above. □

The following specifies in which sense $\operatorname{supp}^{\mathbf{L}}_\cdot$ is a family of support functions:

Lemma 3.1.6. *For each order embedding $f : X \to Y$ we have*

$$\operatorname{rng}(\mathbf{L}^u_f) = \{b \in \mathbf{L}^u_Y \mid \operatorname{supp}^{\mathbf{L}}_Y(b) \subseteq \operatorname{rng}(f)\}.$$

Proof. For $b = \mathbf{L}^u_f(a) \in \operatorname{rng}(\mathbf{L}^u_f)$ we have

$$\operatorname{supp}^{\mathbf{L}}_Y(b) = \operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) = [f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a)) \subseteq \operatorname{rng}(f),$$

as $\operatorname{supp}^{\mathbf{L}}_\cdot$ is a natural transformation. Conversely, if we have $\operatorname{supp}^{\mathbf{L}}_Y(b) \subseteq \operatorname{rng}(f)$ then the inclusion $i : \operatorname{supp}^{\mathbf{L}}_Y(b) \hookrightarrow Y$ factors as $i = f \circ g$ with $g : \operatorname{supp}^{\mathbf{L}}_Y(b) \to X$. It is easy to observe that $\mathbf{L}^u_i$ is the inclusion of $\mathbf{L}^u_{\operatorname{supp}^{\mathbf{L}}_Y(b)}$ into $\mathbf{L}^u_Y$, and that b is contained in $\mathbf{L}^u_{\operatorname{supp}^{\mathbf{L}}_Y(b)}$. Thus we have $b = \mathbf{L}^u_i(b) = \mathbf{L}^u_f(\mathbf{L}^u_g(b))$, which shows that b lies in the range of $\mathbf{L}^u_f$. □

It will be of central importance to have natural enumerations of the sets $\mathbf{L}^u_X$:

Proposition 3.1.7. *There are primitive recursive families of functions*

$$\operatorname{en}_X : [X]^{<\omega} \times \omega \to \mathbf{L}^u_X, \qquad \operatorname{code}_X : [X]^{<\omega} \times \mathbf{L}^u_X \to \omega$$

such that we have

$$\operatorname{en}_X(x, \operatorname{code}_X(x, a)) = a$$

whenever $\operatorname{supp}^{\mathbf{L}}_X(a) \subseteq x$. *These functions are natural in the sense that the equations*

$$\operatorname{en}_Y([f]^{<\omega}(x), n) = \mathbf{L}^u_f(\operatorname{en}_X(x, n)),$$
$$\operatorname{code}_Y([f]^{<\omega}(x), \mathbf{L}^u_f(a)) = \operatorname{code}_X(x, a)$$

hold for any order embedding $f : X \to Y$.

PROOF. The idea, inspired by Girard's work on dilators [**28**], is to represent terms in $\mathbf{L}^u_X$ by terms in $\mathbf{L}^u_n$, for a suitable $n \in \omega$. Here $n = \{0, \ldots, n-1\}$ is ordered as usual. According to Assumption 3.1.1 the set u is countable, with a fixed enumeration $u = \{u_i \,|\, i \in \omega\}$. By Gödel numbering we get a family of functions

$$\begin{aligned}\mathrm{en}^0_n &: \omega \to \mathbf{L}^u_n,\\ \mathrm{code}^0_n &: \mathbf{L}^u_n \to \omega\end{aligned}$$

such that we have

$$\mathrm{en}^0_n(\mathrm{code}^0_n(a)) = a$$

for all $n \in \omega$ and $a \in \mathbf{L}^u_n$. Given a finite linear order x we write

$$c_x : |x| \xrightarrow{\cong} x$$

for the unique increasing enumeration of x. By the previous lemma any $a \in \mathbf{L}^u_X$ with $\mathrm{supp}^{\mathbf{L}}_X(a) \subseteq x$ lies in the range of $\mathbf{L}^u_{i_x}$, where $i_x : x \hookrightarrow X$ is the inclusion. Thus a can be written as $a = \mathbf{L}^u_{i_x \circ c_x}(b)$ with $b \in \mathbf{L}^u_{|x|}$. This representation is unique, since $\mathbf{L}^u_{c_x}$ is a bijection and $\mathbf{L}^u_{i_x}$ is the inclusion of $\mathbf{L}^u_x$ into $\mathbf{L}^u_X$, hence injective. It follows that b can be computed by a primitive recursive function, namely as

$$b = \bigcup\{c \in \mathbf{L}^u_{|x|} \,|\, a = \mathbf{L}^u_{i_x \circ c_x}(c)\}.$$

Now we define

$$\begin{aligned}\mathrm{en}_X(x, n) &:= \mathbf{L}^u_{i_x \circ c_x}(\mathrm{en}^0_{|x|}(n)),\\ \mathrm{code}_X(x, a) &:= \begin{cases}\mathrm{code}^0_{|x|}(b) & \text{if } \mathrm{supp}^{\mathbf{L}}_X(a) \subseteq x \text{ and } a = \mathbf{L}^u_{i_x \circ c_x}(b),\\ 0 & \text{if } \mathrm{supp}^{\mathbf{L}}_X(a) \not\subseteq x.\end{cases}\end{aligned}$$

An easy computation yields $\mathrm{en}_X(x, \mathrm{code}_X(x, a)) = a$ in case $\mathrm{supp}^{\mathbf{L}}_X(a) \subseteq x$. For an embedding $f : X \to Y$ we have $|[f]^{<\omega}(x)| = |x|$ and $i_{[f]^{<\omega}(x)} \circ c_{[f]^{<\omega}(x)} = f \circ i_x \circ c_x$, where $i_{[f]^{<\omega}(x)}$ is the inclusion of $[f]^{<\omega}(x)$ into Y. Thus we can compute

$$\begin{aligned}\mathrm{en}_Y([f]^{<\omega}(x), n) = \mathbf{L}^u_{i_{[f]^{<\omega}(x)} \circ c_{[f]^{<\omega}(x)}}&(\mathrm{en}^0_{|[f]^{<\omega}(x)|}(n)) =\\ &= \mathbf{L}^u_f(\mathbf{L}^u_{i_x \circ c_x}(\mathrm{en}^0_{|x|}(n))) = \mathbf{L}^u_f(\mathrm{en}_X(x, n)).\end{aligned}$$

As $\mathrm{supp}^{\mathbf{L}}_{\cdot}$ is a natural transformation we see

$$\mathrm{supp}^{\mathbf{L}}_X(a) \subseteq x \;\Leftrightarrow\; [f]^{<\omega}(\mathrm{supp}^{\mathbf{L}}_X(a)) \subseteq [f]^{<\omega}(x) \;\Leftrightarrow\; \mathrm{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) \subseteq [f]^{<\omega}(x).$$

So if $\operatorname{supp}^{\mathbf{L}}_X(a) \subseteq x$ and $a = \mathbf{L}^u_{i_x \circ c_x}(b)$ then $\mathbf{L}^u_f(a) = \mathbf{L}^u_{i_{[f]^{<\omega}(x)} \circ c_{[f]^{<\omega}(x)}}(b)$ and

$$\operatorname{code}_Y([f]^{<\omega}(x), \mathbf{L}^u_f(a)) = \operatorname{code}^0_{|x|}(b) = \operatorname{code}_X(x, a).$$

If $\operatorname{supp}^{\mathbf{L}}_X(a) \nsubseteq x$ then we have $\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) \nsubseteq [f]^{<\omega}(x)$ and

$$\operatorname{code}_Y([f]^{<\omega}(x), \mathbf{L}^u_f(a)) = 0 = \operatorname{code}_X(x, a),$$

as required. □

As a first application of our enumerations we define natural order relations on the sets $\mathbf{L}^u_X$, similar to the usual order on the actual constructible hierarchy:

Definition 3.1.8. Given a linear order $(X, <_X)$ we write $<^*_X$ for the anti-lexicographic order on $[X]^{<\omega}$, i.e. for $x \neq x'$ we have

$$x <^*_X x' \quad \Leftrightarrow \quad \max_{<_X}(x \triangle x') \in x',$$

where $x \triangle x'$ denotes the symmetric difference. The relation $<^{\mathbf{L}}_X$ on $\mathbf{L}^u_X$ is given by

$$a <^{\mathbf{L}}_X b \quad \Leftrightarrow \quad \begin{cases} \text{either } \operatorname{supp}^{\mathbf{L}}_X(a) <^*_X \operatorname{supp}^{\mathbf{L}}_X(b), \\ \text{or } \operatorname{supp}^{\mathbf{L}}_X(a) = \operatorname{supp}^{\mathbf{L}}_X(b) \text{ and} \\ \qquad \operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(a), a) < \operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(b), b). \end{cases}$$

Let us verify an important non-functorial property of $<^{\mathbf{L}}_X$:

Lemma 3.1.9. *If $(X, <_X)$ is a linear order resp. well-order then so is $(\mathbf{L}^u_X, <^{\mathbf{L}}_X)$.*

Proof. It is immediate that the relation $<^*_X$ is trichotomous. To see that it is transitive, i.e. that $x_0 <^*_X x_1$ and $x_1 <^*_X x_2$ imply $x_0 <^*_X x_2$, one can argue by induction on $|x_2|$: If x_0 is empty the claim is immediate. Otherwise we have $\max x_0 \leq_X \max x_1 \leq_X \max x_2$ (all maxima with respect to $<_X$). In case $\max x_0 <_X \max x_2$ we have $\max(x_0 \triangle x_2) = \max x_2 \in x_2$ and thus $x_0 <^*_X x_2$. Now assume $\max x_0 = \max x_1 = \max x_2$. Set $x'_i := x_i \backslash \{\max x_i\}$ and observe that we have $x'_i \triangle x'_j = x_i \triangle x_j$. From this one gets $x'_0 <^*_X x'_1$ and $x'_1 <^*_X x'_2$. The induction hypothesis yields $x'_0 <^*_X x'_2$, which implies

$$\max(x_0 \triangle x_2) = \max(x'_0 \triangle x'_2) \in x'_2 \subseteq x_2,$$

as required for $x_0 <^*_X x_2$. By a similar but easier argument one deduces that the relation $<^{\mathbf{L}}_X$ is transitive. To see that $<^{\mathbf{L}}_X$ is trichotomous it suffices to observe that

$\operatorname{supp}^{\mathbf{L}}_X(a)=\operatorname{supp}^{\mathbf{L}}_X(b)$ and $\operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(a),a)=\operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(b),b)$ imply

$$\begin{aligned}a=\operatorname{en}_X(\operatorname{supp}^{\mathbf{L}}_X(a),\operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(a),a))=&\\ =\operatorname{en}_X(\operatorname{supp}^{\mathbf{L}}_X(b),\operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(b),b))=b.\end{aligned}$$

Now assume that $<_X$ is a well-order. Then the primitive recursive function

$$\min_{<_X}(Z):=\bigcup\{s\in Z\cap X\,|\,\forall_{t\in Z\cap X}\, s\leq_X t\}$$

computes the $<_X$-minimal element of any non-empty $Z\subseteq X$. Our first goal is to show that $<^*_X$ is a well-order. Aiming at a contradiction, assume that $\mathcal{Z}\subseteq[X]^{<\omega}$ is non-empty and has no $<^*_X$-minimal element. Using Proposition 1.1.6 we can define a sequence of sets $\mathcal{Z}_n\subseteq[X]^{<\omega}$ and elements $s_n\in X$ by the recursion

$$\begin{aligned}\mathcal{Z}_0&:=\mathcal{Z},\\ s_n&:=\min_{<_X}\{\max_{<_X}x\,|\,x\in\mathcal{Z}_n\backslash\{\emptyset\}\},\\ \mathcal{Z}_{n+1}&:=\{x\backslash\{\max_{<_X}x\}\,|\,x\in\mathcal{Z}_n\backslash\{\emptyset\}\text{ and }\max_{<_X}x=s_n\}.\end{aligned}$$

By induction on n we show that $\mathcal{Z}_n\subseteq[X]^{<\omega}$ is non-empty and has no $<^*_X$-minimal element. In the step, note first that $\mathcal{Z}_n$ cannot contain the empty set, which is the $<^*_X$-minimal element of $[X]^{<\omega}$. Thus we have $s_n=\max_{<_X}x\in X$ for some $x\in\mathcal{Z}_n$, and $\mathcal{Z}_{n+1}$ is non-empty. It is easy to observe that x with $\max_{<_X}x=s_n$ is minimal in $\mathcal{Z}_n$ if $x\backslash\{\max_{<_X}x\}$ is minimal in $\mathcal{Z}_{n+1}$. So in view of the induction hypothesis $\mathcal{Z}_{n+1}$ cannot have a minimal element. Now that we know $\mathcal{Z}_n\neq\emptyset$ and $\emptyset\notin\mathcal{Z}_n$ for all numbers n it is easy to see

$$s_0>_X s_1>_X\dots.$$

This contradicts the well-foundedness of $<_X$, so $<^*_X$ must be a well-order after all. By a similar but easier argument one deduces that $<^{\mathbf{L}}_X$ is a well-order. □

The following result generalizes a familiar fact about the actual constructible hierarchy: If we have $\alpha<\beta$ then the usual order on $\mathbb{L}_\beta$ extends the order on $\mathbb{L}_\alpha$.

LEMMA 3.1.10. *For each order embedding $f:X\to Y$ the function $\mathbf{L}^u_f$ is an order embedding of $(\mathbf{L}^u_X,<^{\mathbf{L}}_X)$ into $(\mathbf{L}^u_Y,<^{\mathbf{L}}_Y)$.*

PROOF. We must show that $a<^{\mathbf{L}}_X b$ implies $\mathbf{L}^u_f(a)<^{\mathbf{L}}_Y\mathbf{L}^u_f(b)$. First assume that $a<^{\mathbf{L}}_X b$ holds because of $\operatorname{supp}^{\mathbf{L}}_X(a)<^*_X\operatorname{supp}^{\mathbf{L}}_X(b)$. It is straightforward to verify

$$[f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a))<^*_Y[f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(b)).$$

As $\operatorname{supp}^{\mathbf{L}}_{\cdot}$ is a natural transformation we get

$$\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) <^*_Y \operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(b)),$$

which implies $\mathbf{L}^u_f(a) <^{\mathbf{L}}_Y \mathbf{L}^u_f(b)$ as desired. Now assume that $a <^{\mathbf{L}}_X b$ holds because of $\operatorname{supp}^{\mathbf{L}}_X(a) = \operatorname{supp}^{\mathbf{L}}_X(b)$ and $\operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(a), a) < \operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(b), b)$. Similar to the above we get

$$\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) = \operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(b)).$$

Using Proposition 3.1.7 we can also compute

$$\begin{aligned}\operatorname{code}_Y(\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)), \mathbf{L}^u_f(a)) &= \operatorname{code}_Y([f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a)), \mathbf{L}^u_f(a)) = \\ &= \operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(a), a) < \operatorname{code}_X(\operatorname{supp}^{\mathbf{L}}_X(b), b) = \\ &= \operatorname{code}_Y([f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(b)), \mathbf{L}^u_f(b)) = \operatorname{code}_Y(\operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(b)), \mathbf{L}^u_f(b)),\end{aligned}$$

which again yields $\mathbf{L}^u_f(a) <^{\mathbf{L}}_Y \mathbf{L}^u_f(b)$. □

Let us summarize our results in the terminology of the previous chapter:

Corollary 3.1.11. *We may consider* $\mathbf{L}^u_{\cdot}$ *as an endofunctor on the category of linear orders. Together with the natural transformation* $\operatorname{supp}^{\mathbf{L}}_{\cdot} : \mathbf{L}^u_{\cdot} \Rightarrow [\cdot]^{<\omega}$ *it forms a dilator.*

Proof. In Lemma 3.1.5 we have established $\mathbf{L}^u_{\cdot}$ as a functor from linear orders to sets. By Lemma 3.1.10 we may view it as a functor into linear orders. Also from Lemma 3.1.5 we know that $\operatorname{supp}^{\mathbf{L}}_{\cdot}$ is a natural transformation. To see that we have a prae-dilator in the sense of Definition 2.2.1, consider an arbitrary order X and an element $a \in \mathbf{L}^u_X$. Write ι_a for the inclusion of $\operatorname{supp}^{\mathbf{L}}_X(a)$ into X. In view of $\operatorname{supp}^{\mathbf{L}}_X(a) \subseteq \operatorname{rng}(\iota_a)$ Lemma 3.1.6 yields $a \in \operatorname{rng}(\mathbf{L}^u_{\iota_a})$, as required for the support function of a prae-dilator. Finally, Lemma 3.1.9 ensures that $\mathbf{L}^u_{\cdot}$ preserves well-foundedness, which makes it a dilator. □

The corollary completes the definition of a functor $\mathbf{L}^u_{\cdot}$, but many questions remain open from the functorial standpoint: What is the role of the formula φ in an $\mathbf{L}^u_X$-term of the form $\{x \in \mathbf{L}^u_s \,|\, \varphi(x, a_1, \dots, a_n)\}$? How would we decide whether $a \in b$ holds for given $\mathbf{L}^u_X$-terms a and b? Such questions can be answered with the help of a syntactic verification calculus, as considered by Jäger [**38, 39**]. In the rest of this section we verify that such a calculus can be set up in a functorial way. Let us begin with the class of formulas to which our verifications will apply:

Definition 3.1.12. An $\mathbf{L}^u_X$-formula is a formula that may contain elements of $\mathbf{L}^u_X$ as parameters (cf. the notion of M-formula in Definition 1.3.1). Given an embedding $f : X \to Y$ we define a map $\varphi \mapsto \varphi[f]$ from $\mathbf{L}^u_X$-formulas to $\mathbf{L}^u_Y$-formulas: Replace any parameter $a \in \mathbf{L}^u_X$ in φ by the parameter $\mathbf{L}^u_f(a) \in \mathbf{L}^u_Y$.

It is easy to observe that the maps $X \mapsto$ "$\mathbf{L}^u_X$-formulas" and $f \mapsto (\cdot)[f]$ form a functor. Recall from Section 1.3 that we only consider formulas in negation normal form: Negation is a defined operation and $\neg\neg\varphi$ is (syntactically) the same formula as φ. By induction over $\mathbf{L}^u_X$-formulas one sees

$$(\neg\varphi)[f] \equiv \neg(\varphi[f])$$

for any $\mathbf{L}^u_X$-formula φ and any embedding $f : X \to Y$. In other words, negation is a natural isomorphism from the functor $X \mapsto$ "$\mathbf{L}^u_X$-formulas" to itself. Similarly, the substitution of a parameter $a \in \mathbf{L}^u_X$ for the free variable of an $\mathbf{L}^u_X$-formula $\varphi(x)$ is natural: Writing $\varphi[f] \equiv \varphi[f](x)$ to designate the free variable of the formula $\varphi[f]$ we have

$$\varphi(a)[f] \equiv \varphi[f](\mathbf{L}^u_f(a)).$$

Now we are ready to describe the promised verification calculus. We adopt notation used by Buchholz [**10**], who attributes it to Tait. Note that the indices 0 and 1 are elements of $u \subseteq \mathbf{L}^u_X$, according to Assumption 3.1.1.

Definition 3.1.13. To each closed $\mathbf{L}^u_X$-formula φ we associate a (possibly infinite) disjunction $\varphi \simeq \bigvee_{a\in\iota_X(\varphi)} \varphi_a$ or conjunction $\varphi \simeq \bigwedge_{a\in\iota_X(\varphi)} \varphi_a$. More precisely, φ is assigned a type (disjunctive or conjunctive), an index set $\iota_X(\varphi) \subseteq \mathbf{L}^u_X$, and a sequence of closed $\mathbf{L}^u_X$-formulas φ_a for $a \in \iota_X(\varphi)$. The disjunctive types are

$$b \in u_i \simeq \begin{cases} \text{“the empty disjunction”} & \text{if } b \equiv u_j \text{ and } u_j \notin u_i, \\ \bigvee_{a\in\{u_j \,|\, u_j \in u_i\}} a = b & \text{if } b \text{ is not of the form } u_j, \end{cases}$$

$$b \notin u_i \simeq \text{“the empty disjunction”} \qquad \text{if } b \equiv u_j \text{ and } u_j \in u_i,$$

$$b \in \mathbf{L}^u_s \simeq \bigvee_{\mathrm{supp}^{\mathbf{L}}_X(a) <^{\mathrm{fin}}_X s} a = b,$$

$$b \in \{x \in \mathbf{L}^u_s \,|\, \theta(x,\vec{c})\} \simeq \bigvee_{\mathrm{supp}^{\mathbf{L}}_X(a) <^{\mathrm{fin}}_X s} a = b \wedge \theta(a,\vec{c}),$$

$$b_0 \neq b_1 \simeq \bigvee_{a\in\{0,1\}} \exists_{x\in b_a} x \notin b_{1-a},$$

$$\varphi_0 \vee \varphi_1 \simeq \bigvee_{a\in\{0,1\}} \varphi_a,$$

$$\exists_{x\in b}\psi(x) \simeq \begin{cases} \bigvee_{a\in\{u_j\,|\,u_j\in u_i\}}\psi(a) & \text{if } b\equiv u_i,\\ \bigvee_{\operatorname{supp}^{\mathbf{L}}_X(a)<^{\text{fin}}_X s}\psi(a) & \text{if } b\equiv \mathbf{L}^u_s,\\ \bigvee_{\operatorname{supp}^{\mathbf{L}}_X(a)<^{\text{fin}}_X s}\theta(a,\vec{c})\wedge\psi(a) & \\ & \text{if } b\equiv\{y\in\mathbf{L}^u_s\,|\,\theta(y,\vec{c})\},\end{cases}$$
$$\exists_x\psi(x)\simeq \bigvee_{a\in\mathbf{L}^u_X}\psi(a).$$

Observe that these disjunctive clauses cover precisely one of the formulas φ and $\neg\varphi$, for each $\mathbf{L}^u_X$-formula φ. In view of $\varphi\equiv\neg\neg\varphi$ we may assume that φ itself is the disjunctive formula. Then the formula $\neg\varphi$ is covered by the stipulation

$$\neg\varphi\simeq\bigwedge_{a\in\iota_X(\varphi)}\neg\varphi_a \qquad \text{if} \qquad \varphi\simeq\bigvee_{a\in\iota_X(\varphi)}\varphi_a.$$

In other words, if the formula φ is disjunctive then the formula $\neg\varphi$ is conjunctive and we have $\iota_X(\neg\varphi)=\iota_X(\varphi)$ as well as $(\neg\varphi)_a\equiv\neg(\varphi_a)$ for $a\in\iota_X(\varphi)$.

Let us show that the verification calculus is natural:

LEMMA 3.1.14. *Given any order embedding $f:X\to Y$, the $\mathbf{L}^u_X$-formula φ and the $\mathbf{L}^u_Y$-formula $\varphi[f]$ have the same type (disjunctive or conjunctive), we have*

$$a\in\iota_X(\varphi)\quad\Leftrightarrow\quad \mathbf{L}^u_f(a)\in\iota_Y(\varphi[f])$$

for all $a\in\mathbf{L}^u_X$, and

$$\varphi[f]_{\mathbf{L}^u_f(a)}\equiv\varphi_a[f]$$

holds for all $a\in\iota_X(\varphi)$.

PROOF. First assume that φ is disjunctive. We look at some representative cases: Consider a formula of the form

$$\varphi\equiv b\in u_i,$$

where b is not of the form u_j, and observe that we have

$$\varphi[f]\equiv\mathbf{L}^u_f(b)\in u_i.$$

The function $\mathbf{L}^u_f$ is injective, as it is an order embedding by Lemma 3.1.10. Also note that $\mathbf{L}^u_f$ is the identity on u. This implies that $\mathbf{L}^u_f(b)$ is still not of the form u_j, so that both φ and $\varphi[f]$ are disjunctive. We also get

$$a\in\iota_X(\varphi)\;\Leftrightarrow\; a\in\{u_j\,|\,u_j\in u_i\}\;\Leftrightarrow\;\mathbf{L}^u_f(a)\in\{u_j\,|\,u_j\in u_i\}\;\Leftrightarrow\;\mathbf{L}^u_f(a)\in\iota_Y(\varphi[f]),$$

as well as

$$\varphi[f]_{\mathbf{L}^u_f(a)}\equiv\mathbf{L}^u_f(a)=\mathbf{L}^u_f(b)\equiv(a=b)[f]\equiv\varphi_a[f].$$

Next, assume that we have

$$\varphi \equiv b \in \{x \in \mathbf{L}^u_s \,|\, \theta(x, c_1, \dots, c_k)\}$$

and thus

$$\varphi[f] \equiv \mathbf{L}^u_f(b) \in \{x \in \mathbf{L}^u_{f(s)} \,|\, \theta(x, \mathbf{L}^u_f(c_1), \dots, \mathbf{L}^u_f(c_k))\}.$$

Both formulas are disjunctive. As f is an order embedding and $\operatorname{supp}^{\mathbf{L}}_{\cdot}$ is a natural transformation we can deduce

$$a \in \iota_X(\varphi) \Leftrightarrow \operatorname{supp}^{\mathbf{L}}_X(a) <^{\text{fin}}_X s \Leftrightarrow [f]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_X(a)) <^{\text{fin}}_Y f(s) \Leftrightarrow$$
$$\Leftrightarrow \operatorname{supp}^{\mathbf{L}}_Y(\mathbf{L}^u_f(a)) <^{\text{fin}}_Y f(s) \Leftrightarrow \mathbf{L}^u_f(a) \in \iota_Y(\varphi[f]).$$

Since $\theta(a, c_1, \dots, c_k)$ does only contain the displayed parameters we also see

$$\varphi_a[f] \equiv \mathbf{L}^u_f(a) = \mathbf{L}^u_f(b) \wedge \theta(\mathbf{L}^u_f(a), \mathbf{L}^u_f(c_1), \dots, \mathbf{L}^u_f(c_k)) \equiv \varphi[f]_{\mathbf{L}^u_f(a)}.$$

Concerning the cases $\varphi \equiv b_0 \neq b_1$ and $\varphi \equiv \varphi_0 \vee \varphi_1$ we remark that $\mathbf{L}^u_f$ is the identity on $\{0, 1\} \subseteq u$. As a last example, assume that we have

$$\varphi \equiv \exists_x \psi(x)$$

and thus

$$\varphi[f] \equiv \exists_x \psi[f](x).$$

Both formulas are disjunctive. In view of $\iota_X(\varphi) = \mathbf{L}^u_X$ and $\iota_Y(\varphi[f]) = \mathbf{L}^u_Y$ we have $a \in \iota_X(\varphi)$ and $\mathbf{L}^u_f(a) \in \iota_Y(\varphi[f])$ for all terms $a \in \mathbf{L}^u_X$. Above we have observed that substitution is natural. Using this fact we get

$$\varphi_a[f] \equiv \psi(a)[f] \equiv \psi[f](\mathbf{L}^u_f(a)) \equiv \varphi[f]_{\mathbf{L}^u_f(a)},$$

as required. Having checked all disjunctive cases, assume now that φ is a conjunctive formula. Then φ is of the form $\varphi \equiv \neg\psi$ where ψ is disjunctive. By the naturality of negation we have $\varphi[f] \equiv (\neg\psi)[f] \equiv \neg(\psi[f])$, so we may savely omit the parentheses. From the disjunctive case we learn that $\psi[f]$ is disjunctive. According to the second part of Definition 3.1.13 this makes $\varphi[f] \equiv \neg\psi[f]$ conjunctive, as required. In the same way we get

$$a \in \iota_X(\varphi) \Leftrightarrow a \in \iota_X(\psi) \Leftrightarrow \mathbf{L}^u_f(a) \in \iota_Y(\psi[f]) \Leftrightarrow \mathbf{L}^u_f(a) \in \iota_Y(\varphi[f])$$

and

$$\varphi[f]_{\mathbf{L}^u_f} \equiv (\neg\psi[f])_{\mathbf{L}^u_f(a)} \equiv \neg(\psi[f]_{\mathbf{L}^u_f(a)}) \equiv \neg\psi_a[f] \equiv \varphi_a[f],$$

just as needed. □

Our next goal is to show that the verification calculus is sound if X is isomorphic to an ordinal α. Proposition 3.1.3 interprets each term $a \in \mathbf{L}^u_X$ as a set $[\![a]\!] \in \mathbb{L}^u_\alpha$ in the actual constructible hierarchy. We can extend this interpretation to formulas: To transform the $\mathbf{L}^u_X$-formula φ into an $\mathbb{L}^u_\alpha$-formula $[\![\varphi]\!]$, replace any parameter $a \in \mathbf{L}^u_X$ in φ by the parameter $[\![a]\!] \in \mathbb{L}^u_\alpha$. From Proposition 1.3.3 we know that the satisfaction relation $\mathbb{L}^u_\alpha \vDash [\![\varphi]\!]$ is primitive recursive.

LEMMA 3.1.15. *Assume that X is isomorphic to an ordinal α. Given a closed $\mathbf{L}^u_X$-formula φ, if we have $\varphi \simeq \bigvee_{a\in\iota_X(\varphi)} \varphi_a$ (resp. $\varphi \simeq \bigwedge_{a\in\iota_X(\varphi)} \varphi_a$) then $\mathbb{L}^u_\alpha \vDash [\![\varphi]\!]$ is equivalent to $\exists_{a\in\iota_X(\varphi)} \mathbb{L}^u_\alpha \vDash [\![\varphi_a]\!]$ (resp. $\forall_{a\in\iota_X(\varphi)} \mathbb{L}^u_\alpha \vDash [\![\varphi_a]\!]$).*

PROOF. One begins with the disjunctive clauses from Definition 3.1.13. As a first example, consider a formula

$$\varphi \equiv b \in \{x \in \mathbf{L}^u_s \mid \theta(x,c)\} \simeq \bigvee_{\operatorname{supp}^{\mathbf{L}}_X(a) <^{\operatorname{fin}}_X s} a = b \wedge \theta(a,c).$$

Let d be the $\mathbf{L}^u_X$-term $\{x \in \mathbf{L}^u_s \mid \theta(x,c)\}$, so that $\mathbb{L}^u_\alpha \vDash [\![\varphi]\!]$ is equivalent to $[\![b]\!] \in [\![d]\!]$. Writing $\alpha_s \in \alpha$ for the image of s under the isomorphism $X \cong \alpha$ we must verify

$$[\![b]\!] \in [\![d]\!] \quad \Leftrightarrow \quad \exists_{a\in\mathbf{L}^u_X}(\operatorname{supp}^{\mathbf{L}}_X(a) <^{\operatorname{fin}}_X s \wedge [\![a]\!] = [\![b]\!] \wedge \mathbb{L}^u_\alpha \vDash \theta([\![a]\!],[\![c]\!])).$$

Concerning direction $\Rightarrow$, Proposition 3.1.3 does ensure that there is a term $a \in \mathbf{L}^u_X$ with $[\![a]\!] = [\![b]\!]$ and $\operatorname{supp}^{\mathbf{L}}_X(a) <^{\operatorname{fin}}_X \operatorname{supp}^{\mathbf{L}}_X(d)$. As we have $\operatorname{supp}^{\mathbf{L}}_X(d) = \{s\} \cup \operatorname{supp}^{\mathbf{L}}_X(c)$ with $\operatorname{supp}^{\mathbf{L}}_X(c) <^{\operatorname{fin}}_X s$ the latter implies $\operatorname{supp}^{\mathbf{L}}_X(a) <^{\operatorname{fin}}_X s$. Also,

$$[\![a]\!] = [\![b]\!] \in [\![d]\!] = \{x \in \mathbb{L}^u_{\alpha_s} \mid \mathbb{L}^u_{\alpha_s} \vDash \theta(x,[\![c]\!])\}$$

yields $\mathbb{L}^u_{\alpha_s} \vDash \theta([\![a]\!],[\![c]\!])$. As θ is a Δ_0-formula (cf. Definition 3.1.2) the missing conjunct $\mathbb{L}^u_\alpha \vDash \theta([\![a]\!],[\![c]\!])$ follows from Lemma 1.3.6. As for direction $\Leftarrow$, by Proposition 3.1.3 the condition $\operatorname{supp}^{\mathbf{L}}_X(a) <^{\operatorname{fin}}_X s$ implies $[\![a]\!] \in \mathbb{L}^u_{\alpha_s}$. To deduce that $[\![b]\!] = [\![a]\!]$ is an element of $[\![d]\!]$ it remains to establish $\mathbb{L}^u_{\alpha_s} \vDash \theta([\![a]\!],[\![c]\!])$. The latter follows from the assumption $\mathbb{L}^u_\alpha \vDash \theta([\![a]\!],[\![c]\!])$, again by Lemma 1.3.6. Next, assume that φ is of the form $b_0 \neq b_1$. In this case the claim amounts to

$$\mathbb{L}^u_\alpha \vDash [\![b_0]\!] \neq [\![b_1]\!] \quad \Leftrightarrow \quad \mathbb{L}^u_\alpha \vDash \exists_{x\in[\![b_0]\!]}\, x \notin [\![b_1]\!] \vee \mathbb{L}^u_\alpha \vDash \exists_{x\in[\![b_1]\!]}\, x \notin [\![b_0]\!].$$

This holds because the transitive set $\mathbb{L}^u_\alpha$ (cf. Lemma 1.3.9) satisfies the axiom of extensionality. As a last disjunctive case, consider a formula

$$\varphi \equiv \exists_x \psi(x) \simeq \bigvee_{a\in\mathbf{L}^u_X} \psi(a).$$

Here we have to verify

$$\mathbb{L}^u_\alpha \vDash \exists_x [\![\psi]\!](x) \quad \Leftrightarrow \quad \exists_{a\in\mathbf{L}^u_X} \mathbb{L}^u_\alpha \vDash [\![\psi(a)]\!].$$

Concerning direction $\Rightarrow$, Tarski's conditions (see Proposition 1.3.3) yield a set $x \in \mathbb{L}^u_\alpha$ with $\mathbb{L}^u_\alpha \vDash [\![\psi]\!](x)$. By Proposition 3.1.3 the interpretation $[\![\cdot]\!] : \mathbf{L}^u_X \to \mathbb{L}^u_\alpha$ is surjective. Thus we get a term $a \in \mathbf{L}^u_X$ with $\mathbb{L}^u_\alpha \vDash [\![\psi]\!]([\![a]\!])$. To conclude it suffices to observe that $[\![\psi]\!]([\![a]\!])$ and $[\![\psi(a)]\!]$ are the same $\mathbb{L}^u_\alpha$-formula. In direction $\Leftarrow$ one first infers $\mathbb{L}^u_\alpha \vDash [\![\psi]\!]([\![a]\!])$. By Proposition 3.1.3 we have $[\![a]\!] \in \mathbb{L}^u_\alpha$. Thus we get $\exists_{x \in \mathbb{L}^u_\alpha} \mathbb{L}^u_\alpha \vDash [\![\psi]\!](x)$, and finally $\mathbb{L}^u_\alpha \vDash \exists_x [\![\psi]\!](x)$ by Tarski's conditions. Having checked all disjunctive cases, let us now consider a conjunctive formula φ. As before we write $\varphi \equiv \neg\psi$ where ψ is disjunctive, to get

$$\varphi \simeq \textstyle\bigwedge_{a \in \iota_X(\psi)} \neg\psi_a.$$

Also observe $[\![\neg\psi]\!] \equiv \neg[\![\psi]\!]$. Using the claim for the disjunctive case we obtain

$$\mathbb{L}^u_\alpha \vDash [\![\varphi]\!] \Leftrightarrow \mathbb{L}^u_\alpha \nvDash [\![\psi]\!] \Leftrightarrow \forall_{a \in \iota_X(\psi)} \mathbb{L}^u_\alpha \nvDash [\![\psi_a]\!] \Leftrightarrow \forall_{a \in \iota_X(\psi)} \mathbb{L}^u_\alpha \vDash [\![\neg\, \psi_a]\!],$$

just as required. □

To make use of soundness we will need to know that the relevant verifications are well-founded. As a first step, let us make the verification order explicit. Functoriality of this order will not be required.

Definition 3.1.16. The order $<_\iota$ on closed $\mathbf{L}^u_X$-formulas is defined as the transitive closure of the relation

$$\varphi_a <_\iota \varphi \qquad \text{for each } a \in \iota_X(\varphi).$$

Note that the transitive closure of a binary relation can be constructed in primitive recursive set theory, similar to the proof of Proposition 1.1.11. As promised we have the following:

Proposition 3.1.17. *If X is isomorphic to an ordinal then the verification relation $<_\iota$ on $\mathbf{L}^u_X$-formulas is well-founded.*

We could prove the same result for any well-order X, whether or not we have an isomorphism with an ordinal. However, such an isomorphism will be given in our application and it makes the presentation of the proof easier.

Proof. To accomodate the elements of u we shift the given isomorphism by one, so that we get an order embedding $X \ni s \mapsto \alpha_s \in \alpha$ with $\alpha_s > 0$ for all $s \in X$. Now we define a notion of ordinal height for $\mathbf{L}^u_X$-terms, setting

$$\mathrm{ht}(u_i) := 0, \qquad \mathrm{ht}(\mathbf{L}^u_s) := \mathrm{ht}(\{x \in \mathbf{L}^u_s \mid \varphi(x, \vec{a})\}) := \omega \cdot \alpha_s.$$

Some readers may have expected the height of $\{x \in \mathbf{L}_s^u \,|\, \varphi(x, \vec{a})\}$ to depend on the formula $\varphi(x, \vec{a})$. In our set-up it does not, because we have required $\varphi(x, \vec{a})$ to be a Δ_0-formula with parameters "below s" (cf. Definition 3.1.2). Following the proof-theoretic literature (see in particular [**38**]) we extend the notion of ordinal height to closed $\mathbf{L}_X^u$-formulas. Assumption 3.1.1 ensures that the parameter $0 \in u \subseteq \mathbf{L}_X^u$ is available for any order X. By recursion over the length of formulas we define

$$\begin{aligned}
\mathrm{ht}(a \in b) := \mathrm{ht}(a \notin b) &:= \max\{\mathrm{ht}(a) + 6, \mathrm{ht}(b) + 1\},\\
\mathrm{ht}(a = b) := \mathrm{ht}(a \neq b) &:= \max\{\mathrm{ht}(a), \mathrm{ht}(b), 5\} + 4,\\
\mathrm{ht}(\varphi_0 \vee \varphi_1) := \mathrm{ht}(\varphi_0 \wedge \varphi_1) &:= \max\{\mathrm{ht}(\varphi_0), \mathrm{ht}(\varphi_1)\} + 1,\\
\mathrm{ht}(\exists_{x \in b} \varphi(x)) := \mathrm{ht}(\forall_{x \in b} \varphi(x)) &:= \max\{\mathrm{ht}(b), \mathrm{ht}(\varphi(0)) + 2\},\\
\mathrm{ht}(\exists_x \varphi(x)) := \mathrm{ht}(\forall_x \varphi(x)) &:= \max\{\omega \cdot \alpha, \mathrm{ht}(\varphi(0)) + 1\}.
\end{aligned}$$

To establish the proposition it suffices to show

$$\mathrm{ht}(\varphi_a) < \mathrm{ht}(\varphi) \qquad \text{for any } \mathbf{L}_X^u\text{-formula } \varphi \text{ and any } a \in \iota_X(\varphi).$$

As a preparation, observe $\mathrm{ht}(a) < \omega \cdot \alpha_s$ for any $a \in \mathbf{L}_X^u$ with $\mathrm{supp}_X^{\mathbf{L}}(a) <_X^{\mathrm{fin}} s$. Note that this implies $\mathrm{ht}(a) + n < \omega \cdot \alpha_s$ for any $n \in \mathbb{N}$, because $\omega \cdot \alpha_s$ is a limit ordinal. If φ is a Δ_0-formula such that $\mathrm{supp}_X^{\mathbf{L}}(a) <_X^{\mathrm{fin}} s$ holds for any parameter a in φ, then we get $\mathrm{ht}(\varphi) < \omega \cdot \alpha_s$. For a general $\mathbf{L}_X^u$-formula $\varphi \equiv \varphi(x)$ one shows

$$\mathrm{ht}(\varphi(a)) < \max\{\omega \cdot \alpha_s, \mathrm{ht}(\varphi(0)) + 1\} \qquad \text{for } a \in \mathbf{L}_X^u \text{ with } \mathrm{supp}_X^{\mathbf{L}}(a) <_X^{\mathrm{fin}} s,$$

by induction over the length of φ (cf. [**38**, Lemma 3]). Similarly one sees

$$\mathrm{ht}(\varphi(a)) < \max\{\omega \cdot \alpha, \mathrm{ht}(\varphi(0)) + 1\} \qquad \text{for all } a \in \mathbf{L}_X^u.$$

Based on these facts it is straightforward to verify $\mathrm{ht}(\varphi_a) < \mathrm{ht}(\varphi)$ for any disjunctive formula φ. As a representative example, let us consider

$$\varphi \equiv \exists_{x \in \{y \in \mathbf{L}_s^u \,|\, \theta(y,c)\}} \psi(x) \simeq \bigvee\nolimits_{\mathrm{supp}_X^{\mathbf{L}}(a) <_X^{\mathrm{fin}} s} \theta(a, c) \wedge \psi(a).$$

Observe that we have

$$\mathrm{ht}(\varphi) = \max\{\omega \cdot \alpha_s, \mathrm{ht}(\psi(0)) + 2\}$$

and

$$\mathrm{ht}(\varphi_a) = \mathrm{ht}(\theta(a, c) \wedge \psi(a)) = \max\{\mathrm{ht}(\theta(a, c)) + 1, \mathrm{ht}(\psi(a)) + 1\}.$$

By the definition of $\mathbf{L}_X^u$-terms we have $\mathrm{supp}_X^{\mathbf{L}}(c) <_X^{\mathrm{fin}} s$, so that we get

$$\mathrm{ht}(\theta(a, c)) + 1 < \omega \cdot \alpha_s \leq \mathrm{ht}(\varphi).$$

We also have

$$\mathrm{ht}(\psi(a)) + 1 < \max\{\omega \cdot \alpha_s, \mathrm{ht}(\varphi(0)) + 2\} = \mathrm{ht}(\varphi),$$

as required. Having checked all disjunctive cases, consider a conjunctive formula $\varphi \equiv \neg\psi$ where ψ is disjunctive. According to Definition 3.1.13 we have

$$\varphi \simeq \textstyle\bigwedge_{a \in \iota_X(\psi)} \neg\, \psi_a.$$

By the disjunctive case we obtain

$$\mathrm{ht}(\varphi_a) = \mathrm{ht}(\neg\, \psi_a) = \mathrm{ht}(\psi_a) < \mathrm{ht}(\psi) = \mathrm{ht}(\neg\psi) = \mathrm{ht}(\varphi)$$

for any $a \in \iota_X(\psi) = \iota_X(\varphi)$, just as needed. □

3.2. From Search Tree to Admissible Set

In the introduction to this chapter we have sketched how Schütte's method of search trees (deduction chains) can be used to construct well-founded models of set theories. The details of this construction are worked out in the present section.

Our aim is to construct a search tree S^u_X for any linear order X and any set u which satisfies Assumption 3.1.1. Recall the syntactic version $\mathbf{L}^u_X$ of the constructible hierarchy that was defined in the previous section. According to Proposition 1.2.8 we can form the set $(\mathbf{L}^u_X)^{<\omega}$ of finite sequences with entries in $\mathbf{L}^u_X$. As usual we obtain a tree if we order these sequences by end extension. The search tree S^u_X will be defined as a labelled subtree of $(\mathbf{L}^u_X)^{<\omega}$.

Let us discuss some terminology that will be needed in the definition of S^u_X: First, recall that we write $\mathrm{len}(\sigma)$ for the length of a sequence $\sigma = \langle a_0, \dots, a_{\mathrm{len}(\sigma)-1}\rangle$. If $n \leq \mathrm{len}(\sigma)$ then $\sigma \restriction n$ denotes the restriction of σ to its first n entries, i.e. we have $\sigma \restriction n = \langle a_0, \dots, a_{n-1}\rangle$. In the previous section we have considered support functions $\mathrm{supp}^{\mathbf{L}}_X : \mathbf{L}^u_X \to [X]^{<\omega}$. To get functions $\mathrm{supp}^{\mathrm{S}}_X : (\mathbf{L}^u_X)^{<\omega} \to [X]^{<\omega}$ we set

$$\mathrm{supp}^{\mathrm{S}}_X(\langle a_0, \dots, a_{n-1}\rangle) := \mathrm{supp}^{\mathbf{L}}_X(a_0) \cup \dots \cup \mathrm{supp}^{\mathbf{L}}_X(a_{n-1}).$$

Note that the family of functions $\mathrm{supp}^{\mathrm{S}}_{\cdot}$ is primitive recursive by Corollary 1.2.11. In the next section we will see that the functions $\mathrm{supp}^{\mathrm{S}}_X$ compute the support of a certain prae-dilator.

Next, recall the notion of $\mathbf{L}^u_X$-formula from Definition 3.1.12. By an $\mathbf{L}^u_X$-sequent we mean a finite sequence $\Gamma = \langle \varphi_0, \dots, \varphi_{\mathrm{len}(\Gamma)-1}\rangle$ of closed $\mathbf{L}^u_X$-formulas φ_i. Intuitively Γ should be interpreted as the disjunction $\varphi_0 \vee \dots \vee \varphi_{\mathrm{len}(\Gamma)-1}$. In particular the empty sequent $\langle\rangle$ is a canonical way to express falsity. As usual we write Γ, φ

for the sequent $\Gamma^\frown\varphi = \langle\varphi_0, \ldots, \varphi_{\mathrm{len}(\Gamma)-1}, \varphi\rangle$. Each node of our search tree S^u_X will be labelled by an $\mathbf{L}^u_X$-sequent. It is helpful to think of S^u_X as an attempted (infinitary) proof in which these sequents are deduced. The order of the formulas in a sequent will be crucial for the definition of our search trees. Later we will be able to ignore it, i.e. we will view sequents as finite sets.

As we aim at a model of Kripke-Platek set theory we will need an enumeration of some relevant axioms: Let

$$\mathrm{Ax}_0 \equiv \forall_x\exists_y\, y = x \cup \{x\} \equiv \forall_x\exists_y(\forall_{z\in x}\, z \in y \land x \in y \land \forall_{z\in y}(z \in x \lor z = x))$$

be a formula which ensures that the height of any transitive model is a limit ordinal. Furthermore, let $\mathrm{Ax}_1, \mathrm{Ax}_2, \ldots$ be a primitive recursive enumeration of all instances

$$\mathrm{Ax}_{n+1} \equiv \forall_{z_1,\ldots,z_k}\forall_v(\forall_{x\in v}\exists_y\theta(x,y,z_1,\ldots,z_k) \to \exists_w\forall_{x\in v}\exists_{y\in w}\theta(x,y,z_1,\ldots,z_k))$$

of Δ_0-collection which have at most $k \leq C$ parameters (cf. Convention 1.3.14). The bound on the number of parameters is harmless by Proposition 1.3.15. It will be convenient for the ordinal analysis in the next chapter, even though it is not strictly necessary (cf. Remark 4.2.8). Observe that the axioms Ax_n are formulas of pure set theory. We may view them as $\mathbf{L}^u_X$-formulas without any $\mathbf{L}^u_X$-parameters.

Finally, we need a surjective function

$$\mathbb{N} \ni n \mapsto \langle\pi_0(n), \pi_1(n), \pi_2(n)\rangle \in \mathbb{N}^3,$$

which should additionally satisfy $\pi_i(n) \leq n$ for all $n \in \mathbb{N}$. It is straightforward to construct such a surjection, for example via the Cantor pairing function. Now we have all ingredients to define our search trees:

Definition 3.2.1. For any linear order X we define a tree $S^u_X \subseteq (\mathbf{L}^u_X)^{<\omega}$ and a labelling $l_X : S^u_X \to$ "$\mathbf{L}^u_X$-sequents" by recursion over $\sigma \in (\mathbf{L}^u_X)^{<\omega}$: In the base case $\sigma = \langle\rangle$ we set

$$\langle\rangle \in S^u_X \quad \text{and} \quad l_X(\langle\rangle) = \langle\rangle \quad \text{(the empty sequent).}$$

The recursion step is only interesting for $\sigma \in S^u_X$, as S^u_X is to become a tree. We distinguish odd and even stages: If $\mathrm{len}(\sigma) = 2n$ is even then we set

$$\sigma^\frown a \in S^u_X \Leftrightarrow a = 0 \quad \text{and} \quad l_X(\sigma^\frown 0) = l_X(\sigma), \neg\,\mathrm{Ax}_n\,.$$

To see $\sigma^\frown 0 \in (\mathbf{L}^u_X)^{<\omega}$, recall that we have $0 \in u \subseteq \mathbf{L}^u_X$ by Assumption 3.1.1. If $\mathrm{len}(\sigma) = 2n+1$ is odd then we write φ for the $\pi_0(n)$-th formula in $l_X(\sigma)$ (due to the axioms added at even stages this sequent contains at least $n+1 > \pi_0(n)$ formulas).

Now we distinguish cases according to the type of φ (cf. Definition 3.1.13): If the formula $\varphi \simeq \bigwedge_{a\in\iota_X(\varphi)} \varphi_a$ is conjunctive, we set

$$\sigma^\frown a \in S^u_X \Leftrightarrow a \in \iota_X(\varphi) \quad \text{and} \quad l_X(\sigma^\frown a) = l_X(\sigma), \varphi_a.$$

If $\varphi \simeq \bigvee_{a\in\iota_X(\varphi)} \varphi_a$ is disjunctive, we use the enumeration $\mathrm{en}_X : [X]^{<\omega} \times \omega \to \mathbf{L}^u_X$ from Proposition 3.1.7 to compute

$$b := \mathrm{en}_X(\mathrm{supp}^{\mathrm{S}}_X(\sigma \restriction \pi_1(n)), \pi_2(n)).$$

Then we set

$$\sigma^\frown a \in S^u_X \Leftrightarrow a = 0 \quad \text{and} \quad l_X(\sigma^\frown 0) = \begin{cases} l_X(\sigma), \varphi_b & \text{if } b \in \iota_X(\varphi), \\ l_X(\sigma) & \text{otherwise,} \end{cases}$$

completing the definition of our search tree.

It may be helpful to observe that a node $\sigma \in S^u_X$ with $\mathrm{len}(\sigma) = 2n + 1$ is a leaf if the $\pi_0(n)$-th formula of $l_X(\sigma)$ corresponds to the empty conjunction. Let us also discuss the formalization of the definition in our base theory $\mathbf{PRS}\omega$: By recursion over finite sequences (see Corollary 1.2.11) we get a primitive recursive family of functions $(\mathbf{L}^u_X)^{<\omega} \ni \sigma \mapsto \langle F_X(\sigma), l_X(\sigma)\rangle$ such that $\sigma \in S^u_X$ is equivalent to $F_X(\sigma) = 1$. According to Corollary 1.1.10 this makes the function

$$X \mapsto S^u_X = \{\sigma \in (\mathbf{L}^u_X)^{<\omega} \mid F_X(\sigma) = 1\}$$

primitive recursive. In particular S^u_X exists as a set. Using Proposition 1.2.2 we infer that $l_X : S^u_X \to$ "$\mathbf{L}^u_X$-sequents" is set-sized and that the map $X \mapsto l_X$ is primitive recursive.

Consider a (set-sized) function $f : \omega \to \mathbf{L}^u_X$ and recall that $f \restriction n$ denotes the sequence $\langle f(0), \ldots, f(n-1)\rangle \in (\mathbf{L}^u_X)^{<\omega}$. As usual, f is called a branch of S^u_X if we have $f \restriction n \in S^u_X$ for all numbers n. Let us put

$$X_f := \bigcup_{n\in\omega} \mathrm{supp}^{\mathbf{L}}_X(f(n)) = \bigcup_{n\in\omega} \mathrm{supp}^{\mathrm{S}}_X(f \restriction n) \subseteq X.$$

We say that a formula occurs on f if it is an entry of some sequent $l_X(f \restriction n)$. The following result is characteristic for the method of search trees (deduction chains):

Proposition 3.2.2. *For any branch f of the search tree S^u_X the following holds:*

(i) If the $\mathbf{L}^u_X$-formula $\varphi \simeq \bigwedge_{a\in\iota_X(\varphi)} \varphi_a$ occurs on f, then so does the formula φ_a, for some element $a \in \iota_X(\varphi) \cap \mathbf{L}^u_{X_f}$.

(ii) If the $\mathbf{L}^u_X$-formula $\varphi \simeq \bigvee_{a\in\iota_X(\varphi)}\varphi_a$ occurs on f, then so does the formula φ_a, for all elements $a \in \iota_X(\varphi)\cap\mathbf{L}^u_{X_f}$.

(iii) Any formula that occurs on f is an $\mathbf{L}^u_{X_f}$-formula.

To understand the proposition it may help to make the inclusion $\iota_f : X_f \hookrightarrow X$ explicit. In the previous section we have observed that $\mathbf{L}^u_{\iota_f}$ is the inclusion of $\mathbf{L}^u_{X_f}$ into $\mathbf{L}^u_X$. In claims (i) and (ii) one might thus write $\iota_X(\varphi)\cap\operatorname{rng}(\mathbf{L}^u_{\iota_f})$ at the place of $\iota_X(\varphi)\cap\mathbf{L}^u_{X_f}$. The map $\varphi\mapsto\varphi[\iota_f]$ of Definition 3.1.12 is the inclusion from $\mathbf{L}^u_{X_f}$-formulas into $\mathbf{L}^u_X$-formulas. Thus claim (iii) states that any $\mathbf{L}^u_X$-formula that occurs on f is of the form $\varphi[\iota_f]$, for some $\mathbf{L}^u_{X_f}$-formula φ.

Proof. (i) Assume that φ is the k-th formula of the sequent $l_X(f\restriction m)$. Pick an $n\in\mathbb{N}$ with $\pi_0(n)=k$ and $n\geq m$ (this can be ensured by demanding $\pi_1(n)=m$). As the sequents in the search tree are only ever extended at the end we see that φ is still the k-th formula of $l_X(f\restriction(2n+1))$. Since f is a branch we have

$$f\restriction(2n+2)=f\restriction(2n+1)^\frown f(2n+1)\in S^u_X.$$

By definition of the search tree we have $f(2n+1)\in\iota_X(\varphi)$, and the formula $\varphi_{f(2n+1)}$ occurs in $l_X(f\restriction(2n+2))$. It remains to show $f(2n+1)\in\mathbf{L}^u_{X_f}=\operatorname{rng}(\mathbf{L}^u_{\iota_f})$. In view of $\operatorname{supp}^{\mathbf{L}}_X(f(2n+1))\subseteq X_f=\operatorname{rng}(\iota_f)$ this follows from Lemma 3.1.6.
(ii) Let φ be the k_0-th formula of $l_X(f\restriction m)$. For $a\in\iota_X(\varphi)\cap X_f=\iota_X(\varphi)\cap\operatorname{rng}(\mathbf{L}^u_{\iota_f})$ we have $\operatorname{supp}^{\mathbf{L}}_X(a)\subseteq\operatorname{rng}(\iota_f)=X_f$ by Lemma 3.1.6. As $\operatorname{supp}^{\mathbf{L}}_X(a)$ is finite and the union $X_f=\bigcup_{n\in\omega}\operatorname{supp}^{\mathrm{S}}_X(f\restriction n)$ is increasing this yields $\operatorname{supp}^{\mathbf{L}}_X(a)\subseteq\operatorname{supp}^{\mathrm{S}}_X(f\restriction k_1)$ for some $k_1\geq m$. Setting $k_2:=\operatorname{code}_X(\operatorname{supp}^{\mathrm{S}}_X(f\restriction k_1),a)$ we get

$$a=\operatorname{en}_X(\operatorname{supp}^{\mathrm{S}}_X(f\restriction k_1),k_2)$$

by Proposition 3.1.7. Now pick a number n with $\pi_i(n)=k_i$. In view of $m\leq k_1\leq n$ we see that φ is still the k_0-th formula of $l_X(f\restriction(2n+1))$. By definition of the search tree the formula φ_a occurs in $l_X(f\restriction(2n+2))$, as required.
(iii) We prove a somewhat finer result, which will be needed in the next chapter:

> "For $\sigma\in S^u_X$, any formula in $l_X(\sigma)$ is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formula. More precisely, any such formula can be written as $\varphi[\iota_\sigma]$, where φ is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formula and $\iota_\sigma:\operatorname{supp}^{\mathrm{S}}_X(\sigma)\hookrightarrow X$ is the inclusion."

In particular, any formula in $l_X(f\restriction m)$ is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(f\restriction m)}$-formula. In view of $\operatorname{supp}^{\mathrm{S}}_X(f\restriction m)\subseteq X_f$ we see that it is an $\mathbf{L}^u_{X_f}$-formula, as required by claim (iii). More precisely, any formula in $l_X(f\restriction m)$ can be written as $\varphi[\iota_{f\restriction m}]$. The inclusion

$\iota_{f\restriction m}:\operatorname{supp}^{\mathrm{S}}(f\restriction m)\hookrightarrow X$ factors as $\iota_{f\restriction m}=\iota_f\circ\iota_0$ with $\iota_0:\operatorname{supp}^{\mathrm{S}}(f\restriction m)\hookrightarrow X_f$. Thus we obtain $\varphi[\iota_{f\restriction m}]=(\varphi[\iota_0])[\iota_f]$, where $\varphi[\iota_0]$ is an $\mathbf{L}^u_{X_f}$-formula. To establish the claim above, we show that any formula in $l_X(\tau)$ is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formula, by induction over the initial segment τ of σ. In view of $l_X(\langle\rangle)=\langle\rangle$ the claim is void for $\tau=\langle\rangle$. To deduce the claim for $\tau^\frown a$, assume first that $\operatorname{len}(\tau)=2n$ is even. Then we must have $a=0$ and the only new formula in $l_X(\tau^\frown a)$ is the negated axiom $\neg\,\mathrm{Ax}_n$. The axioms are formulas of pure set theory, so we may view them as $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formulas without parameters. Now assume that $\operatorname{len}(\tau)=2n+1$ is odd. Let φ be the $\pi_0(n)$-th formula of $l_X(\tau)$. The induction hypothesis implies that φ is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formula; we write $\varphi[\iota_\sigma]$ when we consider φ as an $\mathbf{L}^u_X$-formula. If $\varphi[\iota_\sigma]$ is conjunctive then the definition of S^u_X yields $a\in\iota_X(\varphi[\iota_\sigma])$, and $l_X(\tau^\frown a)$ contains the new formula $\varphi[\iota_\sigma]_a$. In view of $\operatorname{supp}^{\mathbf{L}}_X(a)\subseteq\operatorname{supp}^{\mathrm{S}}_X(\sigma)=\operatorname{rng}(\iota_\sigma)$ Lemma 3.1.6 guarantees $a\in\operatorname{rng}(\mathbf{L}^u_{\iota_\sigma})$, or in other words $a=\mathbf{L}^u_{\iota_\sigma}(a)$ with $a\in\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$. Using Lemma 3.1.14 we get $a\in\iota_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}(\varphi)$, and

$$\varphi[\iota_\sigma]_{\mathbf{L}^u_{\iota_\sigma}(a)}\equiv\varphi_a[\iota_\sigma]$$

is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formula. Now assume that $\varphi[\iota_\sigma]$ is disjunctive. If

$$b=\operatorname{en}_X(\operatorname{supp}^{\mathrm{S}}_X(\tau\restriction\pi_1(n)),\pi_2(n))\in\mathbf{L}^u_X$$

does not lie in $\iota_X(\varphi[\iota_\sigma])$ then we have $l_X(\tau^\frown a)=l_X(\tau)$ and the claim is immediate by induction hypothesis. If we do have $b\in\iota_X(\varphi[\iota_\sigma])$ then $l_X(\tau^\frown a)$ contains the new formula $\varphi[\iota_\sigma]_b$. Consider $\operatorname{supp}^{\mathrm{S}}_X(\tau\restriction\pi_1(n))$ as a subset of $\operatorname{supp}^{\mathrm{S}}_X(\sigma)$ and invoke Proposition 3.1.7 to see that

$$\begin{aligned}b=\operatorname{en}_X([\iota_\sigma]^{<\omega}(\operatorname{supp}^{\mathrm{S}}_X(\tau\restriction\pi_1(n))),\pi_2(n))=&\\ =\mathbf{L}^u_{\iota_\sigma}(\operatorname{en}_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}(\operatorname{supp}^{\mathrm{S}}_X(\tau\restriction\pi_1(n)),\pi_2(n)))&\end{aligned}$$

lies in the range of $\mathbf{L}^u_{\iota_\sigma}$. Since the latter is the inclusion of $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$ into $\mathbf{L}^u_X$ we may write $b=\mathbf{L}^u_{\iota_\sigma}(b)$ with $b=\operatorname{en}_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}(\operatorname{supp}^{\mathrm{S}}_X(\tau\restriction\pi_1(n)),\pi_2(n))\in\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$. As in the previous case Lemma 3.1.14 yields $b\in\iota_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}(\varphi)$, and

$$\varphi[\iota_\sigma]_{\mathbf{L}^u_{\iota_\sigma}(b)}\equiv\varphi_b[\iota_\sigma]$$

is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}_X(\sigma)}$-formula. □

As in other applications of search trees we can deduce that any formula on a branch is false in a corresponding model. In our case this relies on the connection between $\mathbf{L}^u_{\cdot}$ and the actual constructible hierarchy $\mathbb{L}^u_{\cdot}$ (cf. Lemma 3.1.15):

PROPOSITION 3.2.3. *Assume that f is a branch in the search tree S^u_X and that X_f, ordered as a subset of X, is isomorphic to an ordinal α. Then we have $\mathbb{L}^u_\alpha \nvDash [\![\varphi]\!]$ for any $\mathbf{L}^u_{X_f}$-formula φ that occurs on f.*

PROOF. As in the previous proposition it is helpful to write $\varphi \mapsto \varphi[\iota_f]$ for the inclusion map from $\mathbf{L}^u_{X_f}$-formulas to $\mathbf{L}^u_X$-formulas. Then the proposition claims that we have $\mathbb{L}^u_\alpha \nvDash [\![\varphi]\!]$ for any $\mathbf{L}^u_{X_f}$-formula φ such that $\varphi[\iota_f]$ occurs on the branch f. Observe that this is a primitive recursive statement about φ. In view of Proposition 3.1.17 it can be established by induction over the well-order $<_\iota$ on the set of $\mathbf{L}^u_{X_f}$-formulas. In the induction step we assume that $\varphi[\iota_f]$ occurs on f. If φ is conjunctive then so is the $\mathbf{L}^u_X$-formula $\varphi[\iota_f]$, by Lemma 3.1.14. The previous proposition yields an $a \in \iota_X(\varphi[\iota_f]) \cap \operatorname{rng}(\mathbf{L}^u_{\iota_f})$ such that $\varphi[\iota_f]_a$ occurs on f. Since $\mathbf{L}^u_{\iota_f} : \mathbf{L}^u_{X_f} \to \mathbf{L}^u_X$ is the inclusion we can write $a = \mathbf{L}^u_{\iota_f}(a)$ with $a \in \mathbf{L}^u_{X_f}$. From Lemma 3.1.14 we learn $a \in \iota_{X_f}(\varphi)$, and the formula $\varphi_a[\iota_f] \equiv \varphi[\iota_f]_{\mathbf{L}^u_{\iota_f}(a)}$ occurs on f. Also observe $\varphi_a <_\iota \varphi$, by definition of the verification order. Thus the induction hypothesis provides $\mathbb{L}^u_\alpha \nvDash [\![\varphi_a]\!]$, which implies $\mathbb{L}^u_\alpha \nvDash [\![\varphi]\!]$ by Lemma 3.1.15. Now assume that φ is disjunctive. To infer $\mathbb{L}^u_\alpha \nvDash [\![\varphi]\!]$ by Lemma 3.1.15 it suffices to establish $\mathbb{L}^u_\alpha \nvDash [\![\varphi_a]\!]$ for all $a \in \iota_{X_f}(\varphi)$. For any such a we have

$$\mathbf{L}^u_{\iota_f}(a) \in \iota_X(\varphi[\iota_f]) \cap \operatorname{rng}(\mathbf{L}^u_{\iota_f}) = \iota_X(\varphi[\iota_f]) \cap \mathbf{L}^u_{X_f},$$

by Lemma 3.1.14. Given that $\varphi[\iota_f]$ occurs on f the previous proposition tells us that $\varphi_a[\iota_f] \equiv \varphi[\iota_f]_{\mathbf{L}^u_{\iota_f}(a)}$ occurs on f as well. From the induction hypothesis we learn $\mathbb{L}^u_\alpha \nvDash [\![\varphi_a]\!]$, as required. □

Since the Kripke-Platek axioms have been included in our search trees we get the following conditional construction of admissible sets:

COROLLARY 3.2.4 (Beta). *If there is a well-order X such that the search tree S^u_X has a branch, then there is an admissible set $\mathbb{A}$ with $u \subseteq \mathbb{A}$.*

As indicated we extend our default base theory **PRS**ω by axiom beta in order to prove this result. The author does not see how to avoid axiom beta at this point (most other applications of axiom beta in the present thesis are mere convenience).

PROOF. Let f be the given branch. Since X is a well-order, so is $X_f \subseteq X$. Using axiom beta we get an isomorphism $X_f \cong \alpha$ with an ordinal α. We want to use Proposition 1.3.15 to show that $\mathbb{L}^u_\alpha \supseteq u$ is admissible. For this purpose we must first verify that α is a limit ordinal: By construction of the search tree the formula $\neg\,\mathrm{Ax}_0$ occurs on f, namely in the sequent $l_X(f \restriction 1)$. Invoking the previous proposition we obtain $\mathbb{L}^u_\alpha \nvDash \neg\,\mathrm{Ax}_0$ (observe $[\![\neg\,\mathrm{Ax}_0]\!] \equiv \neg\,\mathrm{Ax}_0$, as Ax_0 contains no parameters). Thus $\mathbb{L}^u_\alpha$ does satisfy the formula

$$\mathrm{Ax}_0 \equiv \forall_x \exists_y\, y = x \cup \{x\}.$$

It is straightforward to infer that the height

$$o(\mathbb{L}^u_\alpha) = \mathbb{L}^u_\alpha \cap \mathrm{Ord}$$

of $\mathbb{L}^u_\alpha$ is a limit ordinal or zero. By Lemma 1.3.10 we have $o(\mathbb{L}^u_\alpha) = o(u) + \alpha$, and $o(u) = u \cap \mathrm{Ord}$ is a successor ordinal according to Assumption 3.1.1. It follows that α must indeed be a limit. To conclude by Proposition 1.3.15 we have to show that $\mathbb{L}^u_\alpha$ satisfies all Δ_0-collection axioms with at most C parameters. Any such axiom appears in the list $\mathrm{Ax}_1, \mathrm{Ax}_2, \dots$ fixed above. Again the construction of S^u_X ensures that the negated axiom $\neg\,\mathrm{Ax}_n$ occurs on any branch f, namely in the sequent $l_X(f \restriction (2n{+}1))$. The previous proposition yields $\mathbb{L}^u_\alpha \vDash \mathrm{Ax}_n$, as required. $\square$

Applying the corollary to the transitive closure $u = \mathrm{TC}(\{x\})$ we can construct an admissible set $\mathbb{A}$ with $x \in u \subseteq \mathbb{A}$. This will be used to prove direction (iv)⇒(ii) of Theorem 4.4.6.

3.3. Search Trees as Dilators

In this section we show that the construction $X \mapsto S^u_X$ of search trees is functorial. We obtain a prae-dilator by considering S^u_X with the Kleene-Brouwer order (also called Lusin-Sierpiński order) and the support function $\mathrm{supp}^{\mathrm{S}}_X$. If the assumption of Corollary 3.2.4 fails then $X \mapsto S^u_X$ preserves well-foundedness and the search trees form a dilator. This yields a conditional construction of admissible sets.

The first step is to extend the construction of search trees to morphisms, i.e. to embeddings $f : X \to Y$. Lemma 3.1.10 provides an embedding $\mathbf{L}^u_f : \mathbf{L}^u_X \to \mathbf{L}^u_Y$. We obtain a function $S^u_f : (\mathbf{L}^u_X)^{<\omega} \to (\mathbf{L}^u_Y)^{<\omega}$ between the corresponding trees of finite sequences by setting

$$S^u_f(\langle a_0, \dots, a_{n-1}\rangle) := \langle \mathbf{L}^u_f(a_0), \dots, \mathbf{L}^u_f(a_{n-1})\rangle.$$

This definition is easily cast as a recursion over sequences, so that $(f, \sigma) \mapsto S^u_f(\sigma)$ is primitive recursive by Corollary 1.2.11. From Proposition 1.2.2 we learn that S^u_f is a set-sized function and that the map $f \mapsto S^u_f$ is primitive recursive. Also recall the natural transformation $\mathrm{supp}^{\mathbf{L}}_\cdot : \mathbf{L}^u_\cdot \Rightarrow [\cdot]^{<\omega}$ from Lemma 3.1.5. In the previous section we have extended $\mathrm{supp}^{\mathbf{L}}_\cdot$ to a family of functions $\mathrm{supp}^{\mathrm{S}}_X : (\mathbf{L}^u_X)^{<\omega} \to [X]^{<\omega}$

on finite sequences. It is straightforward to deduce that this family is natural as well, in the sense that we have

$$\operatorname{supp}^{\mathrm{S}}_Y(S^u_f(\sigma)) = [f]^{<\omega}(\operatorname{supp}^{\mathrm{S}}_X(\sigma))$$

for any $\sigma \in (\mathbf{L}^u_X)^{<\omega}$. We show that S^u_f respects our construction of search trees:

LEMMA 3.3.1. *Consider an order embedding $f : X \to Y$. For any $\sigma \in (\mathbf{L}^u_X)^{<\omega}$ we have $\sigma \in S^u_X$ if and only if $S^u_f(\sigma) \in S^u_Y$.*

In particular the restriction of S^u_f to S^u_X has range in S^u_Y. This restricted function will also be denoted by $S^u_f : S^u_X \to S^u_Y$.

PROOF. We argue by induction over the sequence σ. Simultaneously we check

$$l_Y(S^u_f(\sigma)) = l_X(\sigma)[f]$$

in case $\sigma \in S^u_X$. To understand this equation, recall that the operation $\varphi \mapsto \varphi[f]$ transforms $\mathbf{L}^u_X$-formulas into $\mathbf{L}^u_Y$-formulas. Its extension to sequents is defined by

$$\Gamma[f] := \langle \varphi_0[f], \ldots, \varphi_{n-1}[f] \rangle \quad \text{if} \quad \Gamma = \langle \varphi_0, \ldots, \varphi_{n-1} \rangle.$$

In the base case $\sigma = \langle\rangle$ of our simultaneous induction we observe $\langle\rangle \in S^u_X$ and $S^u_f(\langle\rangle) = \langle\rangle \in S^u_Y$, as well as

$$l_Y(S^u_f(\langle\rangle)) = l_Y(\langle\rangle) = \langle\rangle = \langle\rangle[f] = l_X(\langle\rangle)[f].$$

In the induction step we consider a sequence $\sigma^\frown a \in (\mathbf{L}^u_X)^{<\omega}$. By induction hypothesis $\sigma \in S^u_X$ is equivalent to $S^u_f(\sigma) \in S^u_Y$. As S^u_X and S^u_Y are trees the induction step is only interesting if $\sigma \in S^u_X$ and $S^u_f(\sigma) \in S^u_Y$ hold. Concerning this case, suppose first that σ has even length $\operatorname{len}(\sigma) = 2n$. Recall that we have $0 \in u$ by Assumption 3.1.1, which implies $\mathbf{L}^u_f(0) = 0$. Also note that $\mathbf{L}^u_f$ is injective because it is an order embedding. By construction of the search trees we get

$$\sigma^\frown a \in S^u_X \Leftrightarrow a = 0 \Leftrightarrow \mathbf{L}^u_f(a) = 0 \Leftrightarrow S^u_f(\sigma^\frown a) = S^u_f(\sigma)^\frown \mathbf{L}^u_f(a) \in S^u_Y.$$

As the axiom Ax_n is a formula of pure set theory (i.e. without parameters from $\mathbf{L}^u_X$) it is not affected by f. Using the induction hypothesis we thus obtain

$$l_Y(S^u_f(\sigma^\frown 0)) = l_Y(S^u_f(\sigma)), \neg\,\mathrm{Ax}_n = l_X(\sigma)[f], (\neg\,\mathrm{Ax}_n)[f] = l_X(\sigma^\frown 0)[f].$$

Now suppose that σ has odd length $\operatorname{len}(\sigma) = 2n+1$. If φ is the $\pi_0(n)$-th formula of $l_X(\sigma)$ then the simultaneous induction hypothesis ensures that $\varphi[f]$ is the $\pi_0(n)$-th formula of $l_Y(S^u_f(\sigma))$. Assume first that φ is conjunctive. Then $\varphi[f]$ is conjunctive as well, by Lemma 3.1.14. Using the same lemma we get

$$\sigma^\frown a \in S^u_X \;\Leftrightarrow\; a \in \iota_X(\varphi) \;\Leftrightarrow\; \mathbf{L}^u_f(a) \in \iota_Y(\varphi[f]) \;\Leftrightarrow\; S^u_f(\sigma^\frown a) = S^u_f(\sigma)^\frown \mathbf{L}^u_f(a) \in S^u_Y.$$

For $a \in \iota_X(\varphi)$ Lemma 3.1.14 also gives

$$l_Y(S^u_f(\sigma^\frown a)) = l_Y(S^u_f(\sigma)), \varphi[f]_{\mathbf{L}^u_f(a)} = l_X(\sigma)[f], \varphi_a[f] = l_X(\sigma^\frown a)[f].$$

Now assume that φ is disjunctive. As above we have

$$\sigma^\frown a \in S^u_X \;\Leftrightarrow\; a = 0 \;\Leftrightarrow\; \mathbf{L}^u_f(a) = 0 \;\Leftrightarrow\; S^u_f(\sigma^\frown a) = S^u_f(\sigma)^\frown \mathbf{L}^u_f(a) \in S^u_Y.$$

To describe the sequents $l_X(\sigma^\frown 0)$ and $l_Y(S^u_f(\sigma^\frown 0))$ we write

$$b = \mathrm{en}_X(\mathrm{supp}^{\mathrm{S}}_X(\sigma \restriction \pi_1(n)), \pi_2(n)).$$

The naturality of $\mathrm{supp}^{\mathrm{S}}_{\cdot}$ and Proposition 3.1.7 yield

$$\mathrm{en}_Y(\mathrm{supp}^{\mathrm{S}}_Y(S^u_f(\sigma) \restriction \pi_1(n)), \pi_2(n)) = \mathrm{en}_Y([f]^{<\omega}(\mathrm{supp}^{\mathrm{S}}_X(\sigma \restriction \pi_1(n))), \pi_2(n)) = \mathbf{L}^u_f(b).$$

By Lemma 3.1.14 we have $b \in \iota_X(\varphi)$ if and only if $\mathbf{L}^u_f(b) \in \iota_Y(\varphi[f])$. In case $b \in \iota_X(\varphi)$ we thus obtain

$$l_Y(S^u_f(\sigma^\frown 0)) = l_Y(S^u_f(\sigma)), \varphi[f]_{\mathbf{L}^u_f(b)} = l_X(\sigma)[f], \varphi_b[f] = l_X(\sigma^\frown 0)[f].$$

In case $b \notin \iota_X(\varphi)$ we have

$$l_Y(S^u_f(\sigma^\frown 0)) = l_Y(S^u_f(\sigma)) = l_X(\sigma)[f] = l_X(\sigma^\frown 0)[f],$$

completing the simultaneous induction. □

Next, we want to equip the search tree S^u_X with a linear order. Recall that Lemma 3.1.9 provides a linear order $<^{\mathbf{L}}_X$ on $\mathbf{L}^u_X$. Based on this order it is standard to define the Kleene-Brouwer order on the tree $(\mathbf{L}^u_X)^{<\omega}$ of sequences, namely by

$$\sigma_0 <^{\mathrm{S}}_X \sigma_1 \quad :\Leftrightarrow \quad \begin{cases} \text{either the sequence } \sigma_0 \text{ is a proper end extension of } \sigma_1, \\ \text{or we can write } \sigma_i = \sigma^\frown a_i{}^\frown \sigma_i' \text{ with } a_0 <^{\mathbf{L}}_X a_1. \end{cases}$$

Here the notation $\sigma_i = \sigma^\frown a_i{}^\frown \sigma_i'$ refers to the concatenation of sequences, i.e. the terms $a_i \in \mathbf{L}^u_X$ are the first entries on which σ_0 and σ_1 disagree. Using the linearity of $<^{\mathbf{L}}_X$ it is straightfoward to see that $<^{\mathrm{S}}_X$ is a linear order on $(\mathbf{L}^u_X)^{<\omega}$. Let us observe a functorial property:

Lemma 3.3.2. *Consider an order embedding $f : X \to Y$ and sequences σ_0, σ_1 in $(\mathbf{L}^u_X)^{<\omega}$. Then $\sigma_0 <^{\mathrm{S}}_X \sigma_1$ implies $S^u_f(\sigma_0) <^{\mathrm{S}}_Y S^u_f(\sigma_1)$.*

PROOF. If σ_0 is a proper end extension of σ_1 then $S^u_f(\sigma_0)$ is a proper end extension of $S^u_f(\sigma_1)$. Now assume $\sigma_i = \sigma^\frown a_i^\frown \sigma_i'$ with $a_0 <^{\mathbf{L}}_X a_1$. By definition of S^u_f we see $S^u_f(\sigma_i) = S^u_f(\sigma)^\frown \mathbf{L}^u_f(a_i)^\frown S^u_f(\sigma_i')$. Using Lemma 3.1.10 we obtain $\mathbf{L}^u_f(a_0) <^{\mathbf{L}}_Y \mathbf{L}^u_f(a_1)$, which implies the desired inequality $S^u_f(\sigma_0) <^{\mathrm{S}}_Y S^u_f(\sigma_1)$. □

The restriction of the order $<^{\mathrm{S}}_X$ and the function $\mathrm{supp}^{\mathrm{S}}_X : (\mathbf{L}^u_X)^{<\omega} \to [X]^{<\omega}$ to the search tree $S^u_X \subseteq (\mathbf{L}^u_X)^{<\omega}$ will also be denoted by $<^{\mathrm{S}}_X$ resp. $\mathrm{supp}^{\mathrm{S}}_X$ (recall that the same has already been declared for the function S^u_f). The following summarizes our functorial investigation of search trees:

PROPOSITION 3.3.3. *The maps $X \mapsto (S^u_X, <^{\mathrm{S}}_X)$, $f \mapsto S^u_f$ and $X \mapsto \mathrm{supp}^{\mathrm{S}}_X$ form a prae-dilator.*

PROOF. Lemma 3.3.1 and Lemma 3.3.2 ensure that each embedding $f : X \to Y$ is transformed into an embedding S^u_f of $(S^u_X, <^{\mathrm{S}}_X)$ into $(S^u_Y, <^{\mathrm{S}}_Y)$. Functoriality is easily deduced from the corresponding properties of $\mathbf{L}^u_\cdot$, established in Lemma 3.1.5. The naturality of $\mathrm{supp}^{\mathrm{S}}_\cdot$ has been observed at the beginning of the present section. It remains to show that $\mathrm{supp}^{\mathrm{S}}_X$ computes supports, in the sense that any $\sigma \in S^u_X$ lies in the range of $S^u_{\iota_\sigma} : S^u_{\mathrm{supp}^{\mathrm{S}}_X(\sigma)} \to S^u_X$, where $\iota_\sigma : \mathrm{supp}^{\mathrm{S}}_X(\sigma) \hookrightarrow X$ is the inclusion. Let us write $\sigma = \langle a_0, \dots, a_{n-1}\rangle$. In view of $\mathrm{supp}^{\mathbf{L}}_X(a_i) \subseteq \mathrm{supp}^{\mathrm{S}}_X(\sigma) = \mathrm{rng}(\iota_\sigma)$ Lemma 3.1.6 yields $a_i \in \mathrm{rng}(\mathbf{L}^u_{\iota_\sigma})$. As observed in Section 3.1 the function $\mathbf{L}^u_{\iota_\sigma}$ is the inclusion map from $\mathbf{L}^u_{\mathrm{supp}^{\mathrm{S}}_X(\sigma)}$ into $\mathbf{L}^u_X$. Thus we may consider $\sigma = \langle a_0, \dots, a_{n-1}\rangle$ as a sequence in $(\mathbf{L}^u_{\mathrm{supp}^{\mathrm{S}}_X(\sigma)})^{<\omega}$, with

$$S^u_{\iota_\sigma}(\sigma) = \langle \mathbf{L}^u_{\iota_\sigma}(a_0), \dots, \mathbf{L}^u_{\iota_\sigma}(a_{n-1})\rangle = \sigma \in S^u_X.$$

From Lemma 3.3.1 we get $\sigma \in S^u_{\mathrm{supp}^{\mathrm{S}}_X(\sigma)} \subseteq (\mathbf{L}^u_{\mathrm{supp}^{\mathrm{S}}_X(\sigma)})^{<\omega}$, so that $\sigma = S^u_{\iota_\sigma}(\sigma)$ lies in the range of the restricted function $S^u_{\iota_\sigma} : S^u_{\mathrm{supp}^{\mathrm{S}}_X(\sigma)} \to S^u_X$. □

Finally, we show that $X \mapsto (S^u_X, <^{\mathrm{S}}_X)$ preserves well-foundedness if the assumption of Corollary 3.2.4 fails. This relies on the following well-known property of the Kleene-Brouwer order, which we need to establish in our base theory $\mathbf{PRS}\omega$:

LEMMA 3.3.4. *Consider a well-order X and a tree $T \subseteq (\mathbf{L}^u_X)^{<\omega}$. If T has no branch then the restriction of $<^{\mathrm{S}}_X$ to T is a well-order.*

PROOF. From Lemma 3.1.9 we know that $<^{\mathbf{L}}_X$ is a well-order. Thus the primitive recursive function

$$\min_{<^{\mathbf{L}}_X}(Z) := \bigcup\{a \in Z \cap \mathbf{L}^u_X \mid \forall_{b \in Z \cap \mathbf{L}^u_X}\, a \leq^{\mathbf{L}}_X b\}$$

computes the minimal element of each non-empty $Z \subseteq \mathbf{L}^u_X$. In particular we can use $\min_{<^{\mathbf{L}}_X}$ as a choice function on $\mathbf{L}^u_X$. We will also need a choice function on the set $(\mathbf{L}^u_X)^{<\omega}$ of finite sequences. For that purpose we use the length-lexicographic order $\lhd^S_X$, defined by

$$\langle a_0, \dots, a_{n-1}\rangle \lhd^S_X \langle b_0, \dots, b_{m-1}\rangle \quad :\Leftrightarrow \quad \begin{cases} \text{either } n < m, \text{ or } n = m \text{ and } a_i <^{\mathbf{L}}_X b_i \\ \text{for the smallest } i < n \text{ with } a_i \neq b_i. \end{cases}$$

It is straightforward to see that $\lhd^S_X$ is a well-order: To find a $\lhd^S_X$-minimal element, first single out the sequences of minimal length. Amongst these, single out the sequences with $<^{\mathbf{L}}_X$-minimal i-th entry by recursion over i. As above this makes

$$\min_{\lhd^S_X}(\mathcal{Z}) := \bigcup\{\sigma \in \mathcal{Z} \cap (\mathbf{L}^u_X)^{<\omega} \mid \forall_{\tau \in \mathcal{Z} \cap (\mathbf{L}^u_X)^{<\omega}}\, \sigma \unlhd^S_X \tau\}$$

the desired choice function on $(\mathbf{L}^u_X)^{<\omega}$. Now the lemma is shown by contraposition: Assume that $\mathcal{Z} \subseteq T$ is a non-empty subset without $<^{\mathrm{S}}_X$-minimal element. Starting with some value $g(0) \in \mathcal{Z}$ we define a $<^S_X$-descending sequence $g : \omega \to \mathcal{Z}$ by

$$g(n+1) := \min_{\lhd^S_X}(\{\sigma \in \mathcal{Z} \mid \sigma <^{\mathrm{S}}_X g(n)\}).$$

To transform g into a branch f of T we recursively define

$$f(n) := \min_{<^{\mathbf{L}}_X}(\{a \in \mathbf{L}^u_X \mid (f{\restriction}n)^\frown a \text{ is an initial segment of } g(m) \text{ for infinitely many } m \in \omega\}).$$

Note that the property "is an infinite subset of ω" is primitive recursive. To conclude it suffices to show that the required element $a \in \mathbf{L}^u_X$ always exists: Inductively we assume that $f \restriction n$ is a proper initial segment of infinitely many nodes of the form $g(m)$. We can define a strictly increasing function $k \mapsto m_k$ such that all nodes in $\{g(m_k) \mid k \in \omega\}$ have this property. Let a_k be the unique set in $\mathbf{L}^u_X$ such that $(f{\restriction}n)^\frown a_k$ is an initial segment of $g(m_k)$. From $g(m_{k+1}) <^{\mathrm{S}}_X g(m_k)$ and the definition of the Kleene-Brouwer order we get $a_{k+1} \leq^{\mathbf{L}}_X a_k$. As $<^{\mathbf{L}}_X$ is well-founded there must be a bound K such that $a_k = a_K$ holds for all $k \geq K$. It follows that $(f{\restriction}n)^\frown a_K$ is an initial segment of $g(m_k)$ for all numbers $k \geq K$. Thus $a := a_K$ is as required for the definition of $f(n)$. □

The results of the previous sections culminate in the following conditional construction of admissible sets:

THEOREM 3.3.5 (Beta). *Consider a set u which satisfies Assumption 3.1.1. Then there is either an admissible set $\mathbb{A}$ with $u \subseteq \mathbb{A}$, or the construction of search trees results in a dilator $(S^u_\cdot, \operatorname{supp}^{\mathrm{S}}_\cdot)$.*

PROOF. First assume that there is a well-order X such that the search tree S^u_X has a branch. Then Corollary 3.2.4 (which requires axiom beta) provides the desired admissible set. If no such branches exist, then the previous lemma ensures that the map $X \mapsto (S^u_X, <^{\mathrm{S}}_X)$ preserves well-foundedness. Together with Proposition 3.3.3 this means that $(S^u_\cdot, \operatorname{supp}^{\mathrm{S}}_\cdot)$ is a dilator. □

To obtain an unconditional construction of admissible sets we must show that $(S^u_\cdot, \operatorname{supp}^{\mathrm{S}}_\cdot)$ cannot be a dilator: If it was, then the search trees would form a β-proof of contradiction in Kripke-Platek set theory, as we shall see in the next section. The ordinal analysis in Chapter 4 will show that such a β-proof cannot exist, provided that the predicative Bachmann-Howard principle from Section 2.2 holds. This will complete the proof of direction (iv)⇒(ii) of Theorem 4.4.6.

3.4. From Search Tree to Proof Tree

In the previous sections we have defined a family of search trees S^u_X, which form a prae-dilator according to Proposition 3.3.3. The construction from Section 2.2 transforms this prae-dilator into a linear order $\mathrm{BH}(S^u)$. This order can be seen as a fixed-point of a "relativized Bachmann-Howard construction". For that reason the single tree $S^u_{\mathrm{BH}(S^u)}$ will reflect crucial properties of the entire family $X \mapsto S^u_X$. In the present section we show that $S^u_{\mathrm{BH}(S^u)}$ can be extended into an infinite proof of a certain kind — we shall speak of S^u-proofs. The use of infinite proofs in the ordinal analysis of set theories is due to Jäger [**38, 39**]. More background will be provided in the introduction of the next chapter.

Before we define the notion of S^u-proof, let us recall our overall goal: We want to construct an admissible set which contains the given set u. In view of Theorem 3.3.5 it remains to cover the case where $S^u = (S^u, \operatorname{supp}^{\mathrm{S}})$ is a dilator. In that case the predicative Bachmann-Howard principle from Definition 2.2.19 tells us that $\mathrm{BH}(S^u)$ is a well-order. Invoking axiom beta we will be able to justify the following (for the time being we simply treat it as an open assumption):

ASSUMPTION 3.4.1. In the following we argue under the assumption that the order $\mathrm{BH}(S^u) = (\mathrm{BH}(S^u), <_{\mathrm{BH}(S^u)})$ is isomorphic to an ordinal α.

As other instances of ordinal analysis suggest, the well-foundedness of $\mathrm{BH}(S^u)$ should not be required until the very end of our consistency proof, and the isomorphism $\mathrm{BH}(S^u)\cong\alpha$ should not be needed at all (see also Remark 3.4.7 below). Nevertheless, Assumption 3.4.1 will allow us to simplify many arguments. The point is that it makes a semantic interpretation of $\mathbf{L}^u_{\mathrm{BH}(S^u)}$ available: According to Proposition 3.1.3 each term $a\in\mathbf{L}^u_{\mathrm{BH}(S^u)}$ corresponds to a set $[\![a]\!]\in\mathbb{L}^u_\alpha$ in the actual constructible hierarchy. Lemma 3.1.15 extends this interpretation to a map $\varphi\mapsto[\![\varphi]\!]$ from $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formulas to $\mathbb{L}^u_\alpha$-formulas. We can then rely on the satisfaction relation $\mathbb{L}^u_\alpha\vDash[\![\varphi]\!]$ in order to define and analyze infinite proofs. In particular, this accounts for the "truth rule" $(\mathrm{True},\cdot)$ in the following list:

Definition 3.4.2. By an S^u-rule we mean an expression of the form

- (True,φ), where φ is a bounded $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula such that $\mathbb{L}^u_\alpha\vDash[\![\varphi]\!]$,
- $(\bigwedge,\varphi)$, where φ is a conjunctive $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula (cf. Definition 3.1.13),
- $(\bigvee,\varphi,a)$, where φ is a disjunctive $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula and $a\in\iota_{\mathrm{BH}(S^u)}(\varphi)$,
- (Cut,φ), where φ is an $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula,
- $(\mathrm{Ref},\exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y))$, where we have $a\in\mathbf{L}^u_{\mathrm{BH}(S^u)}$ and $\theta(x,y)$ is a bounded $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula (with no other free variables),
- (Rep,a), where we have $a\in\mathbf{L}^u_{\mathrm{BH}(S^u)}$.

Next, the construction from Section 2.2 yields a linearly ordered set

$$(\vartheta(S^u_{\mathrm{BH}(S^u)}),<^\vartheta_{\mathrm{BH}(S^u)})$$

of ordinal terms (this relies on the fact that $\mathrm{BH}(S^u)$ can be equipped with the structure of a BH-system, see Lemma 2.2.23 and Proposition 2.2.8). To view $\mathrm{BH}(S^u)$ as a fixed-point of the relativized Bachmann-Howard construction, recall that $\vartheta(S^u_{\mathrm{BH}(S^u)})$ contains a term Ω, which one may view as the first uncountable ordinal. Theorem 2.2.25 provides an order isomorphism

$$i_{\mathrm{BH}(S^u)}:\mathrm{BH}(S^u)\xrightarrow{\cong}\vartheta(S^u_{\mathrm{BH}(S^u)})\cap\Omega=\{s\in\vartheta(S^u_{\mathrm{BH}(S^u)})\,|\,s<^\vartheta_{\mathrm{BH}(S^u)}\Omega\}.$$

In the following we identify $\mathrm{BH}(S^u)$ with $\vartheta(S^u_{\mathrm{BH}(S^u)})\cap\Omega\subseteq\vartheta(S^u_{\mathrm{BH}(S^u)})$, leaving the isomorphism $i_{\mathrm{BH}(S^u)}$ implicit. Instead of $<_{\mathrm{BH}(S^u)}$ and $<^\vartheta_{\mathrm{BH}(S^u)}$ we simply write $<$. For $x\in[\vartheta(S^u_{\mathrm{BH}(S^u)})]^{<\omega}$ and $t\in\vartheta(S^u_{\mathrm{BH}(S^u)})$ we abbreviate

$$x<^{\mathrm{fin}}t\qquad\Leftrightarrow\qquad\text{“we have } s<t \text{ for all } s\in x\text{”},$$

as in the previous sections. The subscript $\mathrm{BH}(S^u)$ will be omitted in expressions such as $\iota_{\mathrm{BH}(S^u)}(\varphi)$ and $\mathrm{supp}^{\mathbf{L}}_{\mathrm{BH}(S^u)}(a)$. Finally, let us agree to view sequents as finite sets, rather than sequences (the order of formulas is no longer relevant). Now we have all ingredients for the notion of S^u-proof:

DEFINITION 3.4.3. An S^u-proof consists of a tree $P \subseteq (\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega}$ and labelling functions

$$l : P \to \text{“}\mathbf{L}^u_{\mathrm{BH}(S^u)}\text{-sequents”}, \qquad r : P \to \text{“}S^u\text{-rules”}, \qquad o : P \to \vartheta(S^u_{\mathrm{BH}(S^u)}),$$

such that the following local correctness conditions hold at each node $\sigma \in P$:

If $r(\sigma)$ is ...	... then ...
(True, φ)	we have $\varphi \in l(\sigma)$;
$(\bigwedge, \varphi)$	we have $\varphi \in l(\sigma)$; for all $a \in \iota(\varphi)$ we have $\sigma^\frown a \in P$ and $l(\sigma^\frown a) \subseteq l(\sigma), \varphi_a$ as well as $o(\sigma^\frown a) < o(\sigma)$;
$(\bigvee, \varphi, a)$	we have $\varphi \in l(\sigma)$, $\sigma^\frown 0 \in P$ and $l(\sigma^\frown 0) \subseteq l(\sigma), \varphi_a$ as well as $o(\sigma^\frown 0) < o(\sigma)$ and $\mathrm{supp}^{\mathbf{L}}(a) <^{\mathrm{fin}} o(\sigma)$;
(Cut, φ)	we have $\sigma^\frown i \in P$ for $i \in \{0, 1\} \subseteq \mathbf{L}^u_{\mathrm{BH}(S^u)}$; also, we have $l(\sigma^\frown 0) \subseteq l(\sigma), \neg\varphi$ and $l(\sigma^\frown 1) \subseteq l(\sigma), \varphi$ as well as $o(\sigma^\frown i) < o(\sigma)$;
$(\mathrm{Ref}, \exists_w \forall_{x \in a} \exists_{y \in w} \theta(x, y))$	we have $\exists_w \forall_{x \in a} \exists_{y \in w} \theta(x, y) \in l(\sigma)$; also, we have $\sigma^\frown 0 \in P$ and $l(\sigma^\frown 0) \subseteq l(\sigma), \forall_{x \in a} \exists_y \theta(x, y)$ as well as $o(\sigma^\frown 0) < o(\sigma)$ and $\Omega \leq o(\sigma)$;
(Rep, a)	we have $\sigma^\frown a \in P$, $l(\sigma^\frown a) \subseteq l(\sigma)$ and $o(\sigma^\frown a) < o(\sigma)$.

Note that the definition does not contain any condition of the form $\sigma \notin P$. Indeed we could always assume that P is the full tree $(\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega}$, as we can add arbitrary nodes σ with labels

$$l(\sigma) = \langle 0 = 0 \rangle, \qquad r(\sigma) = (\mathrm{True}, 0 = 0), \qquad o(\sigma) = 0.$$

Nevertheless it is more intuitive to allow proper subtrees $P \subseteq (\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega}$. The side condition $\mathrm{supp}^{\mathbf{L}}(a) <^{\mathrm{fin}} o(\sigma)$ (or $[i_{\mathrm{BH}(S^u)}]^{<\omega}(\mathrm{supp}^{\mathbf{L}}(a)) <^{\mathrm{fin}} o(\sigma)$, making the isomorphism $i_{\mathrm{BH}(S^u)}$ explicit) of the disjunction rule $(\bigvee, \varphi, a)$ helps to control the computational content of a proof. This will be important for the ordinal analysis in the next chapter. The side condition $\Omega \leq o(\sigma)$ ensures that no reflection rule $(\mathrm{Ref},\cdot)$ can occur in an S^u-proof of height $o(\langle\rangle) < \Omega$. For such a proof it is straightforward to verify soundness — and thus consistency:

LEMMA 3.4.4. *Consider an S^u-proof $P = (P, l, r, o)$. If P has height $o(\langle\rangle) < \Omega$ then we have $\mathbb{L}^u_\alpha \vDash [\![\psi]\!]$ for some formula $\psi \in l(\langle\rangle)$ in the end-sequent of P. In particular $l(\langle\rangle)$ cannot be the empty sequent.*

PROOF. We use induction over the well-order $\vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega \cong \mathrm{BH}(S^u)$ in order to establish

$$\forall_{s \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega} \forall_{\sigma \in P} (o(\sigma) = s \to \exists_{\psi \in l(\sigma)} \mathbb{L}^u_\alpha \vDash [\![\psi]\!]).$$

Note that the induction statement is primitive recursive by Proposition 1.3.3. In the induction step we distinguish cases according to the rule used at σ. As an example, let us consider a disjunction rule $r(\sigma) = (\bigvee, \varphi, a)$: Local correctness yields $\sigma^\frown 0 \in P$ and $o(\sigma^\frown 0) < o(\sigma) < \Omega$. By induction hypothesis we get $\mathbb{L}^u_\alpha \vDash [\![\psi]\!]$ for some formula $\psi \in l(\sigma^\frown 0) \subseteq l(\sigma), \varphi_a$. For $\psi \in l(\sigma)$ the claim is immediate. Now assume $\psi \equiv \varphi_a$. The definition of our S^u-rules ensures $a \in \iota(\varphi)$, so that we get $\mathbb{L}^u_\alpha \vDash [\![\varphi]\!]$ by Lemma 3.1.15. Since local correctness guarantees $\varphi \in l(\sigma)$ this completes the induction step in the disjunctive case. As a second example, consider a cut rule $r(\sigma) = (\mathrm{Cut}, \varphi)$: From the induction hypothesis for $\sigma^\frown 0$ we learn that $\mathbb{L}^u_\alpha$ satisfies some formula $\psi \in l(\sigma^\frown 0) \subseteq l(\sigma), \neg\varphi$. For $\psi \in l(\sigma)$ the claim is immediate, so we may assume $\mathbb{L}^u_\alpha \vDash [\![\neg\varphi]\!]$. Since $[\![\neg\varphi]\!]$ and $\neg[\![\varphi]\!]$ are the same formula this amounts to $\mathbb{L}^u_\alpha \nvDash [\![\varphi]\!]$. The induction hypothesis for $\sigma^\frown 1$ yields $\mathbb{L}^u_\alpha \vDash [\![\psi]\!]$ for some formula $\psi \in l(\sigma^\frown 1) \subseteq l(\sigma), \varphi$. Since we have excluded the case $\psi \equiv \varphi$ this leaves us with $\psi \in l(\sigma)$, as needed for the induction step. The remaining cases are similar or easy. Note that $r(\sigma)$ cannot be a reflection rule $(\mathrm{Ref},\cdot)$, because local correctness would require $\Omega \leq o(\sigma)$, contrary to the assumption $o(\sigma) = s \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$. □

In the rest of this section we extend the search tree $S^u_{\mathrm{BH}(S^u)}$ to an S^u-proof of the empty sequent. This proof will have height $o(\langle\rangle) > \Omega$, so that the consistency result above does not apply. Consistency for S^u-proofs with height above Ω does not follow by a simple semantical argument, because we do not know whether $\mathbb{L}^u_\alpha$ satisfies reflection. Instead, the extended consistency result requires methods of impredicative ordinal analysis, which will be presented in the next chapter. The repetition rule $(\mathrm{Rep},\cdot)$, which is trivial from a semantical viewpoint, will play an important role there. In order to extend $S^u_{\mathrm{BH}(S^u)}$ to an S^u-proof we need to give infinite proofs of the Kripke-Platek axioms. As a preparation, let us observe that the isomorphism $\mathrm{BH}(S^u) \cong \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ allows us to use the term structure of $\vartheta(S^u_{\mathrm{BH}(S^u)})$ in order to define functions on $\mathrm{BH}(S^u)$. In particular we can define a map $\vartheta(S^u_{\mathrm{BH}(S^u)}) \ni s \mapsto s+1 \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ by the stipulations

$$0+1 := \omega^0, \qquad \Omega+1 := \omega^\Omega + \omega^0, \qquad \mathfrak{E}_\sigma + 1 := \omega^{\mathfrak{E}_\sigma} + \omega^0,$$
$$\vartheta s + 1 := \omega^{\vartheta s} + \omega^0, \qquad (\omega^{s_0} + \cdots + \omega^{s_n}) + 1 := \omega^{s_0} + \cdots + \omega^{s_n} + \omega^0.$$

It is straightforward to observe $s < s+1$. Also, $s < \Omega$ implies $s+1 < \Omega$, so that the given operation on terms induces a map $\mathrm{BH}(S^u) \ni s \mapsto s+1 \in \mathrm{BH}(S^u)$. It follows that $\mathrm{BH}(S^u)$ does not have a maximal element. This allows us to prove the "limit axiom" used in the construction of search trees:

LEMMA 3.4.5. *There is an S^u-proof $P_0 = (P_0, l_0, r_0, o_0)$ with height $o_0(\langle\rangle) = \Omega$ and end-sequent $l_0(\langle\rangle) = \langle \mathrm{Ax}_0 \rangle$, where*

$$\mathrm{Ax}_0 \equiv \forall_x \exists_y\, y = x \cup \{x\} \equiv \forall_x \exists_y (\forall_{z \in x}\, z \in y \land x \in y \land \forall_{z \in y}(z \in x \lor z = x)).$$

PROOF. For each $a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$ we compute

$$s_a = \max(\{s+1 \,|\, s \in \mathrm{supp}^{\mathbf{L}}(a)\} \cup \{0\}) \in \mathrm{BH}(S^u).$$

In view of $\mathrm{supp}^{\mathbf{L}}(a) <^{\mathrm{fin}} s_a$ we can form the $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-terms

$$b_a \equiv \{z \in \mathbf{L}^u_{s_a} \,|\, z \in a \lor z = a\}.$$

Now the required S^u-proof P_0 can be visualized as

$$\dfrac{\cdots \qquad \dfrac{\vdash^0 b_a = a \cup \{a\}}{\vdash^{s_a+1} \exists_y\, y = a \cup \{a\}}\,(\bigvee) \qquad \cdots}{\vdash^\Omega \forall_x \exists_y\, y = x \cup \{x\}}\,(\bigwedge).$$

This means that the leaves of P_0 have the form $\langle a, 0\rangle$, for arbitrary $a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$. They receive the labels

$$l_0(\langle a,0\rangle) = \langle b_a = a \cup \{a\}\rangle,$$
$$r_0(\langle a,0\rangle) = (\mathrm{True}, b_a = a \cup \{a\}),$$
$$o_0(\langle a,0\rangle) = 0.$$

To see that $(\mathrm{True}, b_a = a \cup \{a\})$ is an S^u-rule we must verify $\mathbb{L}^u_\alpha \vDash [\![b_a = a \cup \{a\}]\!]$, which amounts to $[\![b_a]\!] = [\![a]\!] \cup \{[\![a]\!]\}$. By the definition of $[\![\cdot]\!]$ (see the proof of Proposition 3.1.3) we have

$$[\![b_a]\!] = \{z \in \mathbb{L}^u_{\alpha_a} \mid z \in [\![a]\!] \vee z = [\![a]\!]\},$$

where α_a is the image of s_a under the isomorphism $\mathrm{BH}(S^u) \cong \alpha$. Also note that $\mathrm{supp}^{\mathbf{L}}(a) <^{\mathrm{fin}} s_a$ implies $[\![a]\!] \in \mathbb{L}^u_{\alpha_a}$, by Proposition 3.1.3. Since $\mathbb{L}^u_{\alpha_a}$ is transitive we can infer $[\![b_a]\!] = [\![a]\!] \cup \{[\![a]\!]\}$, as required. On the next level we have

$$\begin{aligned} l_0(\langle a \rangle) &= \langle \exists_y\, y = a \cup \{a\} \rangle, \\ r_0(\langle a \rangle) &= (\textstyle\bigvee, \exists_y\, y = a \cup \{a\}, b_a), \\ o_0(\langle a \rangle) &= s_a + 1. \end{aligned}$$

Local correctness is satisfied in view of

$$\exists_y\, y = a \cup \{a\} \simeq \textstyle\bigvee_{b \in \mathbf{L}^u_{\mathrm{BH}(S^u)}} b = a \cup \{a\}$$

and $\mathrm{supp}^{\mathbf{L}}(b_a) <^{\mathrm{fin}} s_a + 1$. Finally, the root of P_0 is labelled by

$$\begin{aligned} l_0(\langle\rangle) &= \langle \forall_x \exists_y\, y = x \cup \{x\} \rangle, \\ r_0(\langle\rangle) &= (\textstyle\bigwedge, \forall_x \exists_y\, y = x \cup \{x\}), \\ o_0(\langle\rangle) &= \Omega. \end{aligned}$$

To see that this assignment is locally correct, observe that we have

$$\forall_x \exists_y\, y = x \cup \{x\} \simeq \textstyle\bigwedge_{a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}} \exists_y\, y = a \cup \{a\},$$

and that $s_a + 1 \in \mathrm{BH}(S^u) \cong \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ means $s_a + 1 < \Omega$. Working in our base theory $\mathbf{PRS}\omega$, the set

$$P_0 = \{\sigma \in (\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega} \mid \sigma = \langle\rangle \vee \exists_{a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}} (\sigma = \langle a \rangle \vee \sigma = \langle a, 0 \rangle)\}$$

and the set-sized functions l_0, r_0, o_0 exist by Δ_0-comprehension. □

Using the reflection rule $(\mathrm{Ref}, \cdot)$ we can also construct S^u-proofs of Δ_0-collection (note that $\Omega + \omega$ abbreviates the term $\omega^\Omega + \omega^{\omega^0} \in \vartheta(S^u_{\mathrm{BH}(S^u)})$):

LEMMA 3.4.6. *There are S^u-proofs $P_{n+1} = (P_{n+1}, l_{n+1}, r_{n+1}, o_{n+1})$ with ordinal height $o_{n+1}(\langle\rangle) = \Omega + \omega$ and end-sequent $l_{n+1}(\langle\rangle) = \langle \mathrm{Ax}_{n+1} \rangle$, where*

$$\mathrm{Ax}_{n+1} \equiv \forall_{z_1,\dots,z_k} \forall_v (\forall_{x \in v} \exists_y \theta(x, y, z_1, \dots, z_k) \to \exists_w \forall_{x \in v} \exists_{y \in w} \theta(x, y, z_1, \dots, z_k))$$

is the n-th instance of Δ_0-collection, according to the enumeration from Section 3.2. The map $n \mapsto P_{n+1}$ exists as a set-sized function.

Proof. The S^u-proof P_{n+1} starts with $k+1$ applications of the rule $(\bigwedge, \cdot)$, which introduce the universal quantifiers at the beginning of Ax_{n+1}. This means that we have $\langle a_1, \ldots, a_k, b\rangle \in P_{n+1}$ for arbitrary $a_1, \ldots, a_k, b \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$. To describe the labels, let us abbreviate

$$\psi(z_1, \ldots, z_k, v) \equiv \forall_{x\in v}\exists_y \theta(x, y, z_1, \ldots, z_k) \to \exists_w \forall_{x\in v}\exists_{y\in w}\theta(x, y, z_1, \ldots, z_k).$$

For $i \leq k$ we then have

$$\begin{aligned} l_{n+1}(\langle a_1, \ldots, a_i\rangle) &= \langle \forall_{z_{i+1},\ldots,z_k}\forall_v \psi(a_1, \ldots, a_i, z_{i+1}, \ldots, z_k, v)\rangle, \\ r_{n+1}(\langle a_1, \ldots, a_i\rangle) &= (\textstyle\bigwedge, \forall_{z_{i+1},\ldots,z_k}\forall_v \psi(a_1, \ldots, a_i, z_{i+1}, \ldots, z_k, v)), \\ o_{n+1}(\langle a_1, \ldots, a_i\rangle) &= \begin{cases} \Omega + \omega & \text{if } i = 0, \\ \Omega + 9 + k - i & \text{otherwise.} \end{cases} \end{aligned}$$

We also have

$$\begin{aligned} l_{n+1}(\langle a_1, \ldots, a_k, b\rangle) &= \langle \forall_{x\in b}\exists_y\theta(x, y, a_1, \ldots, a_k) \to \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, a_1, \ldots, a_k)\rangle, \\ o_{n+1}(\langle a_1, \ldots, a_k, b\rangle) &= \Omega + 8. \end{aligned}$$

The rule $r_{n+1}(\langle a_1, \ldots, a_k, b\rangle)$ will be given below. It is straightforward to observe local correctness at the nodes $\langle a_1, \ldots, a_i\rangle$ for $i \leq k$. The next part of the proof can be visualized as

$$\dfrac{\dfrac{\dfrac{\vdash^{\Omega+5} \neg\forall_{x\in b}\exists_y\theta(x, y, \vec{a}), \forall_{x\in b}\exists_y\theta(x, y, \vec{a})}{\vdash^{\Omega+6} \neg\forall_{x\in b}\exists_y\theta(x, y, \vec{a}), \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, \vec{a})}\,(\mathrm{Ref})}{\vdash^{\Omega+7} \neg\forall_{x\in b}\exists_y\theta(x, y, \vec{a}), \neg\forall_{x\in b}\exists_y\theta(x, y, \vec{a}) \vee \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, \vec{a})}\,(\textstyle\bigvee)}{\vdash^{\Omega+8} \neg\forall_{x\in b}\exists_y\theta(x, y, \vec{a}) \vee \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, \vec{a})}\,(\textstyle\bigvee).$$

To give a more explicit description we abbreviate

$$\begin{aligned} \varphi_0 &\equiv \exists_{x\in b}\forall_y\neg\theta(x, y, a_1, \ldots, a_k), \\ \varphi_1 &\equiv \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, a_1, \ldots, a_k), \end{aligned}$$

so that we get

$$\begin{aligned} &\forall_{x\in b}\exists_y\theta(x, y, a_1, \ldots, a_k) \to \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, a_1, \ldots, a_k) \equiv \\ &\qquad\qquad \equiv \neg\varphi_0 \to \varphi_1 \equiv \varphi_0 \vee \varphi_1 \simeq \textstyle\bigvee_{i\in\{0,1\}} \varphi_i. \end{aligned}$$

Thus we stipulate $\langle\vec{a}, b, 0\rangle \in P_{n+1}$ and $\langle\vec{a}, b, 0, 0\rangle \in P$, with labels

$$\begin{aligned}
r_{n+1}(\langle\vec{a}, b\rangle) &= (\bigvee, \varphi_0 \vee \varphi_1, 0),\\
l_{n+1}(\langle\vec{a}, b, 0\rangle) &= \langle\varphi_0 \vee \varphi_1, \varphi_0\rangle,\\
o_{n+1}(\langle\vec{a}, b, 0\rangle) &= \Omega + 7,\\
r_{n+1}(\langle\vec{a}, b, 0\rangle) &= (\bigvee, \varphi_0 \vee \varphi_1, 1),\\
l_{n+1}(\langle\vec{a}, b, 0, 0\rangle) &= \langle\exists_{x\in b}\forall_y\neg\theta(x, y, \vec{a}), \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, \vec{a})\rangle,\\
o_{n+1}(\langle\vec{a}, b, 0, 0\rangle) &= \Omega + 6.
\end{aligned}$$

Local correctness at $\langle\vec{a}, b\rangle$ and $\langle\vec{a}, b, 0\rangle$ is easily checked. At last, we are in a position to apply the reflection rule: Local correctness at $\langle\vec{a}, b, 0, 0\rangle$ is ensured if we stipulate $\langle\vec{a}, b, 0, 0, 0\rangle \in P_{n+1}$ and

$$\begin{aligned}
r_{n+1}(\langle\vec{a}, b, 0, 0\rangle) &= (\mathrm{Ref}, \exists_w\forall_{x\in b}\exists_{y\in w}\theta(x, y, \vec{a})),\\
l_{n+1}(\langle\vec{a}, b, 0, 0, 0\rangle) &= \langle\exists_{x\in b}\forall_y\neg\theta(x, y, \vec{a}), \forall_{x\in b}\exists_y\theta(x, y, \vec{a})\rangle,\\
o_{n+1}(\langle\vec{a}, b, 0, 0, 0\rangle) &= \Omega + 5.
\end{aligned}$$

In particular $\Omega \leq \Omega + 6$ validates the side condition of the reflection rule. Observe that the sequent at $\langle\vec{a}, b, 0, 0, 0\rangle$ has the form $\varphi, \neg\varphi$. To deduce it we distinguish cases according to the form of b: For notational convenience we only look at $b \equiv \mathbf{L}^u_s$. In this case we have

$$\forall_{x\in b}\exists_y\theta(x, y, \vec{a}) \simeq \bigwedge\nolimits_{\mathrm{supp}^{\mathbf{L}}(c)<^{\mathrm{fin}}s} \exists_y\theta(c, y, \vec{a}),$$

and the remaining proof can be visualized as

$$\dfrac{\cdots \qquad \dfrac{\cdots \qquad \dfrac{\dfrac{\vdash^0 \neg\theta(c, d, \vec{a}), \theta(c, d, \vec{a})}{\vdash^\Omega \neg\theta(c, d, \vec{a}), \exists_y\theta(c, y, \vec{a})}\,(\bigvee) \qquad \cdots}{\vdash^{\Omega+1} \forall_y\neg\theta(c, y, \vec{a}), \exists_y\theta(c, y, \vec{a})}\,(\bigwedge)}{\vdash^{\Omega+2} \exists_{x\in b}\forall_y\neg\theta(x, y, \vec{a}), \exists_y\theta(c, y, \vec{a})}\,(\bigvee) \qquad \cdots}{\vdash^{\Omega+5} \exists_{x\in b}\forall_y\neg\theta(x, y, \vec{a}), \forall_{x\in b}\exists_y\theta(x, y, \vec{a})}\,(\bigwedge).$$

The top sequent is covered by a truth rule (True, $\cdot$), since one of the Δ_0-formulas $\neg\theta(c, d, \vec{a})$ and $\theta(c, d, \vec{a})$ holds in $\mathbb{L}^u_\alpha$. The lowest sequent could be deduced with height $\Omega + 3$, but the ordinal $\Omega + 5$ is needed to cover the slightly more complicated case $b \equiv \{z \in \mathbf{L}^u_s \mid \psi(z)\}$. More explicitly, we stipulate that P_{n+1} contains the nodes

$\langle \vec{a}, b, 0, 0, 0, c\rangle$ and $\langle \vec{a}, b, 0, 0, 0, c, 0\rangle$ for all $c \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$ which satisfy $\operatorname{supp}^{\mathbf{L}}(c) <^{\mathrm{fin}} s$. These nodes receive the labels

$$\begin{aligned}
r_{n+1}(\langle \vec{a}, b, 0, 0, 0\rangle) &= (\textstyle\bigwedge, \forall_{x\in b}\exists_y \theta(x, y, \vec{a})),\\
l_{n+1}(\langle \vec{a}, b, 0, 0, 0, c\rangle) &= \langle \exists_{x\in b}\forall_y \neg\theta(x, y, \vec{a}), \exists_y \theta(c, y, \vec{a})\rangle,\\
o_{n+1}(\langle \vec{a}, b, 0, 0, 0, c\rangle) &= \Omega + 2,\\
r_{n+1}(\langle \vec{a}, b, 0, 0, 0, c\rangle) &= (\textstyle\bigvee, \exists_{x\in b}\forall_y \neg\theta(x, y, \vec{a}), c),\\
l_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0\rangle) &= \langle \forall_y \neg\theta(c, y, \vec{a}), \exists_y \theta(c, y, \vec{a})\rangle,\\
o_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0\rangle) &= \Omega + 1.
\end{aligned}$$

Note that the side condition $\operatorname{supp}^{\mathbf{L}}(c) <^{\mathrm{fin}} o_{n+1}(\langle \vec{a}, b, 0, 0, 0, c\rangle)$ of the rule $(\bigvee, \cdot)$ is trivial in view of $\operatorname{supp}^{\mathbf{L}}(c) \subseteq \mathrm{BH}(S^u) \cong \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$. Finally, P_{n+1} contains the nodes $\langle \vec{a}, b, 0, 0, 0, c, 0, d\rangle$ and $\langle \vec{a}, b, 0, 0, 0, c, 0, d, 0\rangle$ for arbitrary $d \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$. They are labelled by

$$\begin{aligned}
r_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0\rangle) &= (\textstyle\bigwedge, \forall_y \neg\theta(c, y, \vec{a})),\\
l_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0, d\rangle) &= \langle \neg\theta(c, d, \vec{a}), \exists_y \theta(c, y, \vec{a})\rangle,\\
o_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0, d\rangle) &= \Omega,\\
r_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0, d\rangle) &= (\textstyle\bigvee, \exists_y \theta(c, y, \vec{a}), d),\\
l_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0, d, 0\rangle) &= \langle \neg\theta(c, d, \vec{a}), \theta(c, d, \vec{a})\rangle,\\
o_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0, d, 0\rangle) &= 0,\\
r_{n+1}(\langle \vec{a}, b, 0, 0, 0, c, 0, d, 0\rangle) &= \begin{cases} (\mathrm{True}, \theta(c, d, \vec{a})) & \text{if } \mathbb{L}^u_\alpha \vDash [\![\theta(c, d, \vec{a})]\!],\\ (\mathrm{True}, \neg\theta(c, d, \vec{a})) & \text{otherwise.}\end{cases}
\end{aligned}$$

Note that the case distinction is primitive recursive, by Proposition 1.3.3. The explicit description of P_{n+1} shows that the proof tree and its labelling functions can be defined by Δ_0-separation (in the language of $\mathbf{PRS}\omega$). All parameters of the separation formula can be computed from n, as we work with a primitive recursive enumeration $n \mapsto \mathrm{Ax}_n$ of axioms. By Corollary 1.1.10 this implies that the function $n \mapsto P_{n+1}$ is primitive recursive. As the domain of this function is the set ω, Proposition 1.2.2 tells us that $n \mapsto P_{n+1}$ exists as a set-sized function. □

Let us comment on the role of Assumption 3.4.1 in the context of S^u-proofs:

Remark 3.4.7. The S^u-proofs constructed in the previous lemmas make use of the truth rule $(\mathrm{True}, \cdot)$. This rule relies on an interpretation of $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-terms as sets in the actual constructible hierarchy, and thus on the isomorphism $\mathrm{BH}(S^u) \cong \alpha$ provided by Assumption 3.4.1. It is well known from ordinal analysis (see [**39**]) that formulas such as $\forall_x \exists_y\, y = x \cup \{x\}$ and $\varphi \vee \neg\varphi$ can be derived by the rules $(\bigwedge, \cdot)$ and $(\bigvee, \cdot)$ alone, without using the truth rule. This alternative approach would be more elegant, in the sense that it does not require axiom beta (which is needed to justify Assumption 3.4.1). On the other hand, several arguments would become more difficult without the truth rule: For example, S^u-proofs of $\varphi \vee \neg\varphi$ would have to be constructed by recursion over the height of φ. This is somewhat technical, because even Δ_0-formulas can have transfinite height, as in the proof of Proposition 3.1.17. The semantical interpretation of $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-terms is also used in Lemma 3.4.4, which establishes consistency via soundness. It is well known from ordinal analysis that consistency can be proved without semantical concepts: For proofs of height below Ω the relevant syntactical method is predicative cut elimination (cf. the introduction of the next chapter). Again, this alternative approach would avoid axiom beta but add technical difficulty. In any case, it seems that axiom beta is essential for the construction of admissible sets via search trees (see Corollary 3.2.4). Given that we cannot avoid axiom beta at that point, we may as well use it again, to justify Assumption 3.4.1 and simplify our S^u-proofs.

As promised, we can now extend the search tree $S^u_{\mathrm{BH}(S^u)}$ into an S^u-proof of the empty sequent. According to Lemma 2.2.7, each node $\sigma \in S^u_{\mathrm{BH}(S^u)}$ gives rise to a term $\mathfrak{E}_\sigma \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ with $\Omega < \mathfrak{E}_\sigma$. In particular we have an ordinal term $\mathfrak{E}_{\langle\rangle}$ that is associated with the root $\langle\rangle$ of the search tree.

Proposition 3.4.8. *If Assumption 3.4.1 is satisfied then there is an S^u-proof $P_S = (P_S, l_S, r_S, o_S)$ with empty end-sequent $l_S(\langle\rangle) = \langle\rangle$ and height $o_S(\langle\rangle) = \mathfrak{E}_{\langle\rangle}$.*

Proof. By construction the search tree is locally correct at nodes of odd length. For $\sigma \in S^u_{\mathrm{BH}(S^u)}$ of even length $\mathrm{len}(\sigma) = 2n$, the negated axiom $\neg\,\mathrm{Ax}_n$ has been added at $\sigma^\frown 0 \in S^u_{\mathrm{BH}(S^u)}$. To restore local correctness we label σ by a cut rule and add the premise Ax_n at the new node $\sigma^\frown 1 \in P_S$. Above $\sigma^\frown 1$ we insert the proof (P_n, l_n, o_n, r_n) from Lemma 3.4.5 resp. Lemma 3.4.6. More formally, we put

$$P_S = S^u_{\mathrm{BH}(S^u)} \cup \{\sigma^\frown 1^\frown \tau \mid \sigma \in S^u_{\mathrm{BH}(S^u)} \wedge \exists_{n\in\omega}(\mathrm{len}(\sigma) = 2n \wedge \tau \in P_n)\}.$$

Observe that the decomposition of $\sigma^\frown 1^\frown \tau \in P_S$ is unique, because $\sigma^\frown 1 \in S^u_{\mathrm{BH}(S^u)}$ fails if σ has even length. Concerning the labels, recall that the search tree is already equipped with a function

$$l_{\mathrm{BH}(S^u)} : S^u_{\mathrm{BH}(S^u)} \to \text{“}\mathbf{L}^u_{\mathrm{BH}(S^u)}\text{-sequents”}.$$

We obtain a primitive recursive function $l_S : P_S \to$ “$\mathbf{L}^u_{\mathrm{BH}(S^u)}$-sequents” by setting

$$\begin{aligned} l_S(\sigma) &= l_{\mathrm{BH}(S^u)}(\sigma) && \text{for } \sigma \in S^u_{\mathrm{BH}(S^u)}, \\ l_S(\sigma^\frown 1^\frown \tau) &= l_n(\tau) && \text{for } \sigma \in S^u_{\mathrm{BH}(S^u)} \text{ with } \mathrm{len}(\sigma) = 2n. \end{aligned}$$

Since l_S has set-sized domain, it exists as a set-sized function, by Proposition 1.2.2. To define the functions $r_S : P_S \to$ “$\mathbf{L}^u_{\mathrm{BH}(S^u)}$-rules” and $o_S : P_S \to \vartheta(S^u_{\mathrm{BH}(S^u)})$ in the same way, it suffices to describe their restrictions to the search tree: The ordinal labels are given by

$$o_S(\sigma) = \mathfrak{E}_\sigma \qquad \text{for } \sigma \in S^u_{\mathrm{BH}(S^u)},$$

similar to the construction of Rathjen and Valencia Vizcaíno [**71**, Theorem 5.26]. Concerning the rules, we first set

$$r_S(\sigma) = (\mathrm{Cut}, \mathrm{Ax}_n) \qquad \text{for } \sigma \in S^u_{\mathrm{BH}(S^u)} \text{ with } \mathrm{len}(\sigma) = 2n.$$

To see that the labelling is locally correct for nodes of even length, observe that the construction of the search tree yields $\sigma^\frown 0 \in S^u_{\mathrm{BH}(S^u)} \subseteq P_S$ and

$$l_S(\sigma^\frown 0) = l_{\mathrm{BH}(S^u)}(\sigma^\frown 0) = l_{\mathrm{BH}(S^u)}(\sigma), \neg\,\mathrm{Ax}_n = l_S(\sigma), \neg\,\mathrm{Ax}_n\,.$$

The inequality $\sigma^\frown 0 <^S_{\mathrm{BH}(S^u)} \sigma$ in the Kleene-Brouwer order implies

$$o_S(\sigma^\frown 0) = \mathfrak{E}_{\sigma^\frown 0} < \mathfrak{E}_\sigma = o_S(\sigma),$$

by the definition of $\vartheta(S^u_{\mathrm{BH}(S^u)})$. We also have $\sigma^\frown 1 = \sigma^\frown 1^\frown \langle\rangle \in P_S$ and

$$l_S(\sigma^\frown 1) = l_n(\langle\rangle) = \langle \mathrm{Ax}_n \rangle \subseteq l_S(\sigma), \mathrm{Ax}_n,$$

as well as

$$o_S(\sigma^\frown 1) = o_n(\langle\rangle) \leq \Omega + \omega = \omega^\Omega + \omega^{\omega^0} < \mathfrak{E}_\sigma = o_S(\sigma).$$

Now consider a node $\sigma \in S^u_{\mathrm{BH}(S^u)}$ of odd length $2n+1$. Following the construction of the search tree, let φ be the $\pi_0(n)$-th formula in the sequent $l_{\mathrm{BH}(S^u)}(\sigma) = l_S(\sigma)$. If φ is conjunctive, we set $r_S(\sigma) = (\bigwedge, \varphi)$. Observe that we have $\sigma^\frown a \in S^u_{\mathrm{BH}(S^u)}$ and $l_{\mathrm{BH}(S^u)}(\sigma^\frown a) = l_{\mathrm{BH}(S^u)}(\sigma), \varphi_a$ for all $a \in \iota(\varphi)$, as needed for local correctness. If φ is disjunctive, we compute $b \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$ as in Definition 3.2.1. Then we set

$$r_S(\sigma) = \begin{cases} (\bigvee, \varphi, b) & \text{if } b \in \iota(\varphi), \\ (\mathrm{Rep}, 0) & \text{otherwise.} \end{cases}$$

Note that $\mathrm{supp}^{\mathbf{L}}(b) \subseteq \mathrm{BH}(S^u) \cong \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ guarantees the side condition

$$\mathrm{supp}^{\mathbf{L}}(b) <^{\mathrm{fin}} \Omega < \mathfrak{E}_\sigma = o(\sigma)$$

of the disjunction rule. The other local correctness conditions are straightforward. To complete the proof we point out that local correctness at $\sigma^\frown 1^\frown \tau \in P_S$ follows from local correctness at $\tau \in P_n$. $\square$

CHAPTER 4

A Consistency Proof

In this chapter we conclude the proof of Theorem 4.4.6, which is the main result of the present thesis. To establish the open implication (iv)⇒(ii) of the theorem, we must show that the predicative Bachmann-Howard principle implies the existence of admissible sets. Aiming at a contradiction, assume that the desired admissible sets do not exist. Then Theorem 3.3.5 tells us that the construction of search trees results in a dilator $X \mapsto S^u_X$. By the predicative Bachmann-Howard principle we infer that the linear order $\mathrm{BH}(S^u)$ constructed in Section 2.2 is well-founded. Invoking axiom beta we obtain an ordinal α with $\mathrm{BH}(S^u) \cong \alpha$. In other words, we are able to satisfy Assumption 3.4.1, on which we have based our investigation of infinite proof trees. In particular, Proposition 3.4.8 yields an S^u-proof of a contradiction (represented by the empty sequent). In the present chapter we adapt Jäger's [**39**] ordinal analysis of Kripke-Platek set theory in order to show that such an S^u-proof cannot exist. The resulting contradiction concludes our construction of admissible sets. In the following we give a brief historical survey of ordinal analysis, emphasizing concepts that will be important for our application. For a comprehensive expert introduction we refer to Rathjen's paper [**66**].

Ordinal analysis was invented by Gentzen [**27**] in his consistency proof for Peano arithmetic. Of course, an unconditional consistency proof is impossible in view of Gödel's incompleteness theorems. Nevertheless, Gentzen managed to reduce the consistency of Peano arithmetic to a principle which is finitistically meaningful and has some intuitive justification: the well-foundedness of ε_0 (cf. the introduction to this thesis). Subsequent investigations have shown that much more information can be extracted from this reduction: An important example are the provably total functions considered by Kreisel [**48**], Wainer [**95**] and Schwichtenberg [**81**]. The information from an ordinal analysis can be used to establish natural independence results, such as the unprovability of Goodstein's theorem in Peano arithmetic. This result is due to Kirby and Paris [**47**], who use model-theoretic methods. Rathjen [**69**] argues that a proof-theoretic argument via Gentzen's ordinal analysis was already anticipated by Goodstein [**33**]. Cichon [**14**] has shown that the result can be reduced to Schwichtenberg and Wainer's characterization of the provably total functions. As a second example we mention the independence of Kruskal's theorem from $\mathbf{ATR}_0$. This result has been proved by H. Friedman (see the presentation by Simpson [**86**]), relying on methods of ordinal analysis. Let us also refer to the work of Schmidt [**75**], which already contains the crucial observation, and to the precise bounds established by Rathjen and Weiermann [**72**]. To summarize one might say that Gentzen presented his investigation as a consistency proof in the spirit of Hilbert's programme, while modern ordinal analysis tends to emphasize questions of conservativity, computational content and independence.

On a technical level, Schütte [**76**] recognized that the use of infinite proofs with the ω-rule would greatly simplify the ordinal analysis of Peano arithmetic: It becomes a straightforward application of cut elimination, a method which removes detours through "complex" formulas in order to restrict the space of possible proofs. At the same time, infinite proofs are harder to formalize in a weak base theory, and they make it more difficult to extract computational information. A particularly elegant solution to this problem was provided by Buchholz [**9, 11**]: He denotes each infinite proof by a finite term that describes its role in the ordinal analysis. In Sections 4.1 and 4.2 we will present "Buchholz codes" and cut elimination for our S^u-proofs.

An important classification within ordinal analysis distinguishes between predicative and impredicative methods. Roughly speaking, models of predicative theories are built from below, while models of impredicative theories are characterized by a closure property that requires a large set (e.g. a non-recursive or uncountable ordinal). Cut elimination (with an extension to infinite formula ranks) is the main method of predicative ordinal analysis. Indeed, Feferman [**16**] and Schütte [**78**] have used it to analyze the notion of predicativity itself. The first impredicative result is Takeuti's [**90**] ordinal analysis of Π^1_1-comprehension. Many subsequent investigations have helped to make the ordinal analysis of impredicative theories more transparent: Let us mention the approach via inductive definitions, due to Buchholz, Feferman, Pohlers and Sieg [**12**], as well as Pohler's [**59**] notion of local predicativity and Buchholz' [**10**] operator controlled derivations. In Section 4.3 we will introduce operator control for our S^u-proofs.

A particularly important innovation for our application is due to Jäger: He was the first to give direct ordinal analyses for set-theoretic axiom systems (see [**38, 39, 41**], as well as the joint paper [**42**] with Pohlers). Specifically, the consistency proof in the present chapter is based on Jäger's ordinal analysis of Kripke-Platek set theory. To indicate how impredicativity is handled in this context, let us consider the reflection rule

$$\frac{\Gamma, \forall_{x\in a}\exists_y\theta(x,y)}{\Gamma, \exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y)}.$$

This rule is sound in the first uncountable stage $\mathbb{L}_{\aleph_1}$ of the constructible hierarchy: For each $x\in a\in\mathbb{L}_{\aleph_1}$, the premise $\mathbb{L}_{\aleph_1}\vDash\forall_{x\in a}\exists_y\theta(x,y)$ provides an ordinal $\alpha_x<\aleph_1$ such that we have $\mathbb{L}_{\aleph_1}\vDash\theta(x,y)$ for some $y\in\mathbb{L}_{\alpha_x}$. Since $a\in\mathbb{L}_{\aleph_1}$ is countable we have $\alpha:=\sup_{x\in a}\alpha_x<\aleph_1$. Then $w:=\mathbb{L}_\alpha$ is a witness for $\mathbb{L}_{\aleph_1}\vDash\exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y)$. We could replace $\aleph_1$ by the Church-Kleene ordinal ω_1^{CK}, but not by a recursive ordinal $>\omega$. This indicates that the reflection rule is indeed impredicative. Also, the rule is not admissible, since a proof of the premise $\Gamma,\forall_{x\in a}\exists_y\theta(x,y)$ cannot be transformed into a proof of the conclusion $\Gamma,\exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y)$. The situation changes when we restrict to countable proof trees (in general one has an uncountable rule, which deduces $\forall_x\varphi(x)$ from a premise $\varphi(a)$ for each $a\in\mathbb{L}_{\aleph_1}$): If $\Gamma,\forall_{x\in a}\exists_y\theta(x,y)$ has an infinite proof of countable height α then we can (under suitable assumptions) construct a proof of the sequent $\Gamma,\forall_{x\in a}\exists_{y\in\mathbb{L}_\alpha}\theta(x,y)$. One might thus describe reflection as a "countably admissible rule". In order to obtain an ordinal analysis of Kripke-Platek set theory one has to perform three tasks simultaneously: eliminate cuts, collapse proofs to countable height, and remove occurrences of the reflection rule. In Section 4.4 we will work out the details for our S^u-proofs.

To conclude our survey of ordinal analysis, let us mention the strongest results that are known today: Rathjen [**63, 64, 65**] has analyzed Π^1_2-comprehension via the extension of Kripke-Platek set theory by Σ_1-separation. Theories of similar strength have been studied by Arai [**5**]. These investigations require collapsing structures which go far beyond the methods used in the present thesis. For example, Rathjen's [**63**] ordinal analysis of Kripke-Platek set theory with Π_3-reflection collapses a single infinite proof into stationary many proofs below a weakly compact cardinal. More information can be found in [**66**, Section 3].

4.1. Buchholz Codes

To carry out the ordinal analysis sketched in the introduction of this chapter, we need to transform S^u-proofs in various ways. The most intuitive definition of these transformations uses recursion over the ordinal height of the infinite proofs. However, this approach is not available in our base theories $\mathbf{PRS}\omega$ and $\mathbf{ATR_0^{set}}$, because the S^u-proofs of a given height do not form a set. A very elegant alternative was developed by Buchholz [**9, 11**]: The collection of infinite proofs is replaced by a set of finite terms, which represent proofs according to their role in the ordinal analysis. For example, if the term P denotes an infinite proof $[P]$, then $\mathcal{E}P$ denotes the proof that results from $[P]$ by an application of cut elimination. If one works in a finitistic base theory, then the actual proofs $[P]$ do not play any official role and all arguments are carried out on the level of finite terms. On the other hand, if the base theory allows for the existence of infinite objects, then the interpretation of codes as actual proofs can be made official (cf. [**9**, Definition 5.4]). In our setting, $P \mapsto [P]$ will be a primitive recursive set function.

From a formal viewpoint it would be best to give the entire system of codes for infinite proofs at once. However, this would condense the whole ordinal analysis into a single technical definition, which would be difficult to present in an intuitive and readable way. For this reason we start with a system of basic S^u-codes. As we present different parts of the ordinal analysis, we will introduce new codes step by step, so that the full system of S^u-codes is defined by the end of the chapter. It is straightforward to transform this exposition into a more formal proof, as explained in Remark 4.1.6 below. The following term system recovers the S^u-proof $P_S = (P_S, l_S, r_S, o_S)$ from Proposition 3.4.8 (similar to [**9**, Definition 6.1]):

Definition 4.1.1. A basic S^u-code is an expression of the form $P_S\sigma$, where σ is a finite sequence in $(\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega}$. We define functions

$$\begin{aligned} l_{\langle\rangle} &: \text{“basic } S^u\text{-codes”} \to \text{“}\mathbf{L}^u_{\mathrm{BH}(S^u)}\text{-sequents”}, \\ r_{\langle\rangle} &: \text{“basic } S^u\text{-codes”} \to \text{“}S^u\text{-rules”}, \\ o_{\langle\rangle} &: \text{“basic } S^u\text{-codes”} \to \vartheta(S^u_{\mathrm{BH}(S^u)}), \\ n &: \text{“basic } S^u\text{-codes”} \times \mathbf{L}^u_{\mathrm{BH}(S^u)} \to \text{“basic } S^u\text{-codes”} \end{aligned}$$

by stipulating

$$l_{\langle\rangle}(P_S\sigma) = \begin{cases} l_S(\sigma) & \text{if } \sigma \in P_S, \\ \langle 0 = 0 \rangle & \text{otherwise,} \end{cases}$$

$$r_{\langle\rangle}(P_S\sigma) = \begin{cases} r_S(\sigma) & \text{if } \sigma \in P_S, \\ (\mathrm{True}, 0 = 0) & \text{otherwise,} \end{cases}$$

$$o_{\langle\rangle}(P_S\sigma) = \begin{cases} o_S(\sigma) & \text{if } \sigma \in P_S, \\ 0 & \text{otherwise,} \end{cases}$$
$$n(P_S\sigma, a) = P_S\, \sigma^\frown a.$$

The idea is that $P_S\sigma$ denotes the subtree of P_S above the node σ. The functions $l_{\langle\rangle}, r_{\langle\rangle}$ and $o_{\langle\rangle}$ read off the labels at the root of this subtree, i.e. at the node σ itself. The function n navigates to the immediate subproofs. Iterating this function, we can recover the entire proof tree. Note that the following definition is formulated for general S^u-codes, rather than just for basic codes. Indeed, it will extend to all S^u-codes that we introduce over the course of this chapter.

Definition 4.1.2. We extend n to a function

$$\bar{n} : \text{“}S^u\text{-codes”} \times (\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega} \to \text{“}S^u\text{-codes”}$$

by setting

$$\bar{n}(P, \langle\rangle) = P, \qquad \bar{n}(P, \sigma^\frown a) = n(\bar{n}(P,\sigma), a).$$

For each S^u-code P we define a tree $[P] \subseteq (\mathbf{L}^u_{\mathrm{BH}(S^u)})^{<\omega}$: Assuming that $\sigma \in [P]$ holds by recursion we stipulate

$$\sigma^\frown a \in [P] \quad \Leftrightarrow \quad a \in \iota(r_{\langle\rangle}(\bar{n}(P,\sigma))),$$

where the arity of the different S^u-rules is given by

$$\iota\left((\mathrm{True}, \varphi)\right) = \emptyset, \qquad \iota\left(\left(\bigwedge, \varphi\right)\right) = \iota(\varphi), \qquad \iota\left(\left(\bigvee, \varphi, a\right)\right) = \{0\},$$
$$\iota\left((\mathrm{Cut}, \varphi)\right) = \{0,1\}, \qquad \iota\left((\mathrm{Ref}, \exists_w \forall_{x\in a} \exists_{y \in w} \theta)\right) = \{0\}, \qquad \iota\left((\mathrm{Rep}, a)\right) = \{a\}.$$

We also consider the functions

$$\begin{aligned} l_P &: [P] \to \text{“}\mathbf{L}^u_{\mathrm{BH}(S^u)}\text{-sequents”}, & l_P(\sigma) &= l_{\langle\rangle}(\bar{n}(P,\sigma)), \\ r_P &: [P] \to \text{“}S^u\text{-rules”}, & r_P(\sigma) &= r_{\langle\rangle}(\bar{n}(P,\sigma)), \\ o_P &: [P] \to \vartheta(S^u_{\mathrm{BH}(S^u)}), & o_P(\sigma) &= o_{\langle\rangle}(\bar{n}(P,\sigma)). \end{aligned}$$

The tuple $[P] = ([P], l_P, r_P, o_P)$ is called the interpretation of P.

The function $\bar{n}$ and the characteristic function of the tree $[P]$ are primitive recursive by Corollary 1.2.11. From Corollary 1.1.10 we learn that $[P]$ is a set, which is primitive recursive in P. Proposition 1.2.2 ensures that l_P, r_P and o_P exist as set-sized functions and that the map $P \mapsto ([P], l_P, r_P, o_P)$ is primitive recursive. It is straightforward to see that $[P_S\langle\rangle]$ coincides with the S^u-proof P_S.

We will have no official use for this observation, because it is superseded by the following more general approach, which shows that the interpretation $[P]$ of any S^u-code P is a locally correct S^u-proof (in the sense of Definition 3.4.3).

Definition 4.1.3. We say that an S^u-code P is locally correct if it satisfies the relevant condition from the following list (we will later speak of condition (L) in order to refer to this requirement):

If $r_{\langle\rangle}(P)$ is ...	... then ...
(True,φ)	we have $\varphi\in l_{\langle\rangle}(P)$;
$(\bigwedge,\varphi)$	we have $\varphi\in l_{\langle\rangle}(P)$; for all elements $a\in\iota(\varphi)$ we have $l_{\langle\rangle}(n(P,a))\subseteq l_{\langle\rangle}(P),\varphi_a$ and $o_{\langle\rangle}(n(P,a))<o_{\langle\rangle}(P)$;
$(\bigvee,\varphi,a)$	we have $\varphi\in l_{\langle\rangle}(P)$ and $l_{\langle\rangle}(n(P,0))\subseteq l_{\langle\rangle}(P),\varphi_a$, as well as $o_{\langle\rangle}(n(P,0))<o_{\langle\rangle}(P)$ and $\mathrm{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}o_{\langle\rangle}(P)$;
(Cut,φ)	$l_{\langle\rangle}(n(P,0))\subseteq l_{\langle\rangle}(P),\neg\varphi$ and $l_{\langle\rangle}(n(P,1))\subseteq l_{\langle\rangle}(P),\varphi$, as well as $o_{\langle\rangle}(n(P,i))<o_{\langle\rangle}(P)$ for $i=0,1$;
$(\mathrm{Ref},\exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y))$	we have $\exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y)\in l_{\langle\rangle}(P)$; furthermore we have $l_{\langle\rangle}(n(P,0))\subseteq l_{\langle\rangle}(P),\forall_{x\in a}\exists_y\theta(x,y)$, as well as $o_{\langle\rangle}(n(P,0))<o_{\langle\rangle}(P)$ and $\Omega\leq o_{\langle\rangle}(P)$;
(Rep,a)	we have $l_{\langle\rangle}(n(P,a))\subseteq l_{\langle\rangle}(P)$ and $o_{\langle\rangle}(n(P,a))<o_{\langle\rangle}(P)$.

If the S^u-code P is locally correct then its interpretation $[P]$ is locally correct at the root. To infer that $[P]$ is locally correct at every node, we must consider the system of S^u-codes as a whole:

Lemma 4.1.4. *Any basic S^u-code is locally correct in the sense of condition (L).*

Proof. Let us first consider an S^u-code $P_S\sigma$ with $\sigma\notin P_S$. Then we have $r_{\langle\rangle}(P_S\sigma)=(\mathrm{True},0=0)$ and $l_{\langle\rangle}(P_S\sigma)=\langle 0=0\rangle$, from which local correctness is obvious. Now consider an S^u-code $P_S\sigma$ with $\sigma\in P_S$. In this case the local correctness of $P_S\sigma$ follows from the local correctness of the S^u-proof P_S at the node σ. As an example, let us look at the case $r_{\langle\rangle}(P_S\sigma)=r_S(\sigma)=(\mathrm{Rep},a)$:

Definition 3.4.3 ensures $\sigma^\frown a \in P_S$ and $l_S(\sigma^\frown a) \subseteq l_S(\sigma)$, from which we deduce

$$l_{\langle\rangle}(n(P_S\sigma, a)) = l_{\langle\rangle}(P_S\,\sigma^\frown a) = l_S(\sigma^\frown a) \subseteq l_S(\sigma) = l_{\langle\rangle}(P_S\sigma).$$

Similarly we obtain $o_{\langle\rangle}(n(P_S\sigma, a)) < o_{\langle\rangle}(P_S\sigma)$, as required. $\square$

As promised, this is enough to secure the following result, which will extend to all S^u-codes introduced in this chapter:

COROLLARY 4.1.5. *The interpretation $[P]$ of any S^u-code P is locally correct at every node. Thus $[P]$ is an S^u-proof in the sense of Definition 3.4.3.*

PROOF. The local correctness of $[P]$ at the node $\sigma \in [P]$ follows from the local correctness of the S^u-code $\bar{n}(P, \sigma)$. As an example, let us consider the case of a rule $r_P(\sigma) = r_{\langle\rangle}(\bar{n}(P, \sigma)) = (\bigwedge, \varphi)$. By definition we have $\sigma^\frown a \in [P]$ for any element $a \in \iota(\varphi) = \iota(r_{\langle\rangle}(\bar{n}(P, \sigma)))$. The local correctness of the code $\bar{n}(P, \sigma)$ yields $\varphi \in l_{\langle\rangle}(\bar{n}(P, \sigma)) = l_P(\sigma)$. For $a \in \iota(\varphi)$ we also get

$$l_P(\sigma^\frown a) = l_{\langle\rangle}(\bar{n}(P, \sigma^\frown a)) = l_{\langle\rangle}(n(\bar{n}(P, \sigma), a)) \subseteq l_{\langle\rangle}(\bar{n}(P, \sigma)), \varphi_a = l_P(\sigma), \varphi_a.$$

Similarly one shows $o_P(\sigma^\frown a) < o_P(\sigma)$, as required by Definition 3.4.3. $\square$

In particular we learn that $[P_S\langle\rangle]$ is an S^u-proof with end-sequent

$$l_{P_S\langle\rangle}(\langle\rangle) = l_{\langle\rangle}(\bar{n}(P_S\langle\rangle, \langle\rangle)) = l_{\langle\rangle}(P_S\langle\rangle) = l_S(\langle\rangle) = \langle\rangle.$$

In this sense we have recovered the S^u-proof (P_S, l_S, r_S, o_S) from Proposition 3.4.8 in terms of basic S^u-codes. As indicated above, the system of S^u-codes will be extended dynamically. Let us explain how this works from a formal standpoint:

REMARK 4.1.6. The interpretation $[P]$ of an S^u-code P depends on the functions $l_{\langle\rangle}, r_{\langle\rangle}, o_{\langle\rangle}$ and n, but not on any specific properties of the basic S^u-codes. Similarly, Corollary 4.1.5 does only depend on the fact that all S^u-codes satisfy condition (L). This means that any system of codes admits an interpretation $P \mapsto [P]$ by S^u-proofs, provided that the functions $l_{\langle\rangle}, r_{\langle\rangle}, o_{\langle\rangle}$ and n are defined and that condition (L) holds. As we present the different steps of our ordinal analysis we will introduce function symbols

$$\mathcal{I}_{\varphi,a}, \qquad \mathcal{R}_\varphi, \qquad \mathcal{E}, \qquad \mathcal{B}^\Sigma_{\varphi,s}, \qquad \mathcal{B}^\Pi_{\varphi,s}, \qquad \mathcal{C}_t,$$

where φ is a formula and we have $a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$, $s \in \mathrm{BH}(S^u)$ and $t \in \vartheta(S^u_{\mathrm{BH}(S^u)})$. At the end of the chapter we will have completed the definition of a set of S^u-codes, given by the following clauses:

- Any basic S^u-code $P_S\sigma$ is an S^u-code.
- If $\mathcal{F}$ is a k-ary function symbol from the list above and $P_1,\dots,P_k$ are S^u-codes, then the term $\mathcal{F}P_1\dots P_k$ is an S^u-code as well.

Note that this is an inductive definition with closure ordinal ω. Thus the set of S^u-codes exists in our base theory $\mathbf{PRS}\omega$, similarly to the set of M-formulas constructed in Definition 1.3.1. Primitive recursive functions can be defined by recursion over the build-up of S^u-codes as terms, parallel to the proof of Lemma 1.3.2. Induction over S^u-codes can be used to establish primitive recursive properties. The basic S^u-codes considered above are the constant symbols of the full term system. Thus they constitute the base case of any recursion resp. induction over S^u-codes. Once the definition of S^u-codes is completed, the functions $l_{\langle\rangle}, r_{\langle\rangle}, o_{\langle\rangle}$ and n from Definition 4.1.1 must be extended to all codes. In order to do so, we will need auxiliary functions

$$\begin{aligned}
d &: \text{“}S^u\text{-codes”} \to \omega,\\
h_0 &: \text{“}S^u\text{-codes”} \to \vartheta(S^u_{\mathrm{BH}(S^u)}),\\
h_1 &: \text{“}S^u\text{-codes”} \to [\vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega]^{<\omega},\\
o_{\langle a\rangle} &: \text{“}S^u\text{-codes”} \to \vartheta(S^u_{\mathrm{BH}(S^u)}) \qquad (\text{for } a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}).
\end{aligned}$$

Officially, the values $l_{\langle\rangle}(P), r_{\langle\rangle}(P), o_{\langle\rangle}(P), d(P), h_0(P), h_1(P)$ and the set-sized functions $a \mapsto n(P,a)$ and $a \mapsto o_{\langle a\rangle}(P)$ are defined by simultaneous recursion over the S^u-code P. Once the definition of these functions is complete, condition (L) must be established for all S^u-codes. In order to do so, we will need auxiliary correctness conditions (C1), (C2), (H1), (H2), (H3), (N1) and (N2), which govern the behaviour of the functions d, h_0, h_1 and $o_{\langle a\rangle}$. Officially, all these conditions are established by simultaneous induction over S^u-codes. However, this will not be the order of presentation: To give an intuitive and readable account of the ordinal analysis, we will state the various recursive clauses and induction steps as they become relevant. Nevertheless, it is straightforward to transform our exposition into a formal proof: First, define the set of S^u-codes as the set of terms that is generated by the basic S^u-codes as constants and all the function symbols listed above. Next, collect all recursive clauses, which are distributed over the following sections. Together they ensure that the functions $l_{\langle\rangle}, r_{\langle\rangle}, o_{\langle\rangle}$ and n and the auxiliary functions are defined on all S^u-codes. Finally, collect the proofs of all induction steps. They show that all S^u-codes satisfy condition (L) and the other correctness conditions listed above. Once this is accomplished we see that Definition 4.1.2 and Corollary 4.1.5 extend to the full system of S^u-codes. Thus any S^u-code P yields an S^u-proof $[P]$ with corresponding properties.

As a first extension of the system of basic S^u-codes we introduce the function symbols $\mathcal{I}_{\varphi,a}$. This implements a proof transformation known as inversion: The idea is that the proof of a universal formula $\forall_x\psi(x) \simeq \bigwedge_{a\in\mathbf{L}^u_{\mathrm{BH}(S^u)}}\psi(a)$ can be transformed into a proof of any instance $\psi(a)$.

LEMMA 4.1.7. *For any conjunctive* $\mathbf{L}^u_{\mathrm{BH}(S^u)}$*-formula* φ *and any* $a\in\iota(\varphi)$ *we can extend the system of* S^u*-codes by a unary function symbol* $\mathcal{I}_{\varphi,a}$*, such that we have*

$$l_{\langle\rangle}(\mathcal{I}_{\varphi,a}P) = (l_{\langle\rangle}(P)\backslash\{\varphi\}) \cup \{\varphi_a\},$$
$$o_{\langle\rangle}(\mathcal{I}_{\varphi,a}P) = o_{\langle\rangle}(P)$$

for any S^u*-code* P*.*

The meaning of the lemma may not be completely obvious, but it should become clear in view of Remark 4.1.6: The given equations are the recursive clauses for the functions $l_{\langle\rangle}$ and $o_{\langle\rangle}$ in the case of a code of the form $\mathcal{I}_{\varphi,a}P$. To prove that "we can extend the system of S^u-codes" in the specified way, we must state recursive clauses for the functions $r_{\langle\rangle}$ and n, and we must verify the corresponding induction step for condition (L). The auxiliary functions d, h_0, h_1 and $o_{\langle a\rangle}$ and the corresponding correctness conditions will be added over the course of the next sections.

PROOF. In order to explain the idea behind the required proof transformation, we begin with an informal description in terms of transfinite recursion: The crucial case is that of a proof P with last rule $(\bigwedge,\varphi)$. This means that the end-sequent of P has the form Γ,φ, and that the a-th immediate subproof $n(P,a)$ of P deduces the sequent Γ,φ,φ_a. Since this subproof has smaller ordinal height than P, we may recursively transform it into a proof $\mathcal{I}_{\varphi,a}n(P,a)$ with end-sequent Γ,φ_a. Using the repetition rule we obtain a proof $\mathcal{I}_{\varphi,a}P$ with the same end-sequent and the desired ordinal height. One should also consider the case of a proof P with last rule (True,φ) and end-sequent Γ,φ. Definition 3.4.2 ensures that φ is a bounded formula with $\mathbb{L}^u_\alpha\vDash[\![\varphi]\!]$. It is straightforward to observe that φ_a is bounded as well, and Lemma 3.1.15 yields $\mathbb{L}^u_\alpha\vDash[\![\varphi_a]\!]$. In this situation, the end-sequent Γ,φ_a of the desired proof $\mathcal{I}_{\varphi,a}P$ can be deduced by the rule $(\mathrm{True},\varphi_a)$, without a recursive call. Now we reproduce the same idea in terms of S^u-codes: The recursive clauses for

$l_{\langle\rangle}$ and $o_{\langle\rangle}$ can be found in the statement of the lemma. To extend $r_{\langle\rangle}$ and n we set

$$r_{\langle\rangle}(\mathcal{I}_{\varphi,a}P)=\begin{cases}(\mathrm{Rep},a) & \text{if } r_{\langle\rangle}(P)=(\bigwedge,\varphi),\\ (\mathrm{True},\varphi_a) & \text{if } r_{\langle\rangle}(P)=(\mathrm{True},\varphi),\\ r_{\langle\rangle}(P) & \text{otherwise.}\end{cases}$$

$$n(\mathcal{I}_{\varphi,a}P,b)=\mathcal{I}_{\varphi,a}n(P,b).$$

It remains to verify condition (L) for $\mathcal{I}_{\varphi,a}P$, assuming the same condition for P. This is done by case distinction over the last rule of P. Let us write out the details for the crucial case $r_{\langle\rangle}(P)=(\bigwedge,\varphi)$: In view of $a\in\iota(\varphi)$ condition (L) for P yields $l_{\langle\rangle}(n(P,a))\subseteq l_{\langle\rangle}(P)\cup\{\varphi_a\}$, which implies

$$l_{\langle\rangle}(n(\mathcal{I}_{\varphi,a}P,a))=l_{\langle\rangle}(\mathcal{I}_{\varphi,a}n(P,a))=l_{\langle\rangle}(n(P,a))\backslash\{\varphi\}\cup\{\varphi_a\}\subseteq$$
$$\subseteq(l_{\langle\rangle}(P)\cup\{\varphi_a\})\backslash\{\varphi\}\cup\{\varphi_a\}\subseteq l_{\langle\rangle}(P)\backslash\{\varphi\}\cup\{\varphi_a\}=l_{\langle\rangle}(\mathcal{I}_{\varphi,a}P).$$

Similarly, we get

$$o_{\langle\rangle}(n(\mathcal{I}_{\varphi,a}P,a))=o_{\langle\rangle}(\mathcal{I}_{\varphi,a}n(P,a))=o_{\langle\rangle}(n(P,a))<o_{\langle\rangle}(P)=o_{\langle\rangle}(\mathcal{I}_{\varphi,a}P).$$

In view of $r_{\langle\rangle}(\mathcal{I}_{\varphi,a}P)=(\mathrm{Rep},a)$ this means that $\mathcal{I}_{\varphi,a}P$ satisfies condition (L). The other cases are straightforward. Note that φ cannot be the main formula of a rule $(\mathrm{Ref},\cdot)$ or $(\bigvee,\cdot)$, because these formulas are disjunctive. □

Note that the recursive clauses which define $l_{\langle\rangle}(\mathcal{I}_{\varphi,a}P), o_{\langle\rangle}(\mathcal{I}_{\varphi,a}P), r_{\langle\rangle}(\mathcal{I}_{\varphi,a}P)$ and $n(\mathcal{I}_{\varphi,a}P,b)$ do not rely on any assumption about the specific form of the term P. For this reason, the same recursive clauses will apply to a term $P=\mathcal{E}P'$, even though the function symbol $\mathcal{E}$ has not yet been introduced. Similarly, the proof above will show that condition (L) for $\mathcal{I}_{\varphi,a}\mathcal{E}P'$ follows from condition (L) for $\mathcal{E}P'$, as soon as this term has been added. A more detailed explanation of this point can be found in Remark 4.1.6. To see an application of inversion, observe that $[P_S\langle 1\rangle]$ is an S^u-proof with end-sequent

$$l_{P_S\langle 1\rangle}(\langle\rangle)=l_{\langle\rangle}(\bar{n}(P_S\langle 1\rangle,\langle\rangle))=l_{\langle\rangle}(P_S\langle 1\rangle)=l_S(\langle 1\rangle)=\{\mathrm{Ax}_0\}=\{\forall_x\exists_y\, y=x\cup\{x\}\}.$$

In view of $\forall_x\exists_y\, y=x\cup\{x\}\simeq\bigwedge_{a\in\mathbf{L}^u_{\mathrm{BH}(S^u)}}\exists_y\, y=a\cup\{a\}$ the inversion operator allows us to construct an S^u-proof $[\mathcal{I}_{\forall_x\exists_y\, y=x\cup\{x\},a}P_S\langle 1\rangle]$ with end-sequent

$$l_{\mathcal{I}_{\forall_x\exists_y\, y=x\cup\{x\},a}P_S\langle 1\rangle}(\langle\rangle)=l_{\langle\rangle}(\mathcal{I}_{\forall_x\exists_y\, y=x\cup\{x\},a}P_S\langle 1\rangle)=$$
$$=l_{\langle\rangle}(P_S\langle 1\rangle)\backslash\{\forall_x\exists_y\, y=x\cup\{x\}\}\cup\{\exists_y\, y=a\cup\{a\}\}=\{\exists_y\, y=a\cup\{a\}\}.$$

Inversion is a simple proof transformation, which could have been implemented without the help of Buchholz codes. On the other hand, codes seem to be the most elegant way to formalize the more complex proof transformations that we will discuss in the following sections.

4.2. Cut Elimination

In the present section we show how certain occurrences of the cut rule $(\mathrm{Cut},\cdot)$ in our S^u-proofs can be eliminated. The required transformations of infinite proofs will be implemented with the help of Buchholz codes, as described in the previous section. The central ingredient for the ordinal analysis of impredicative theories is collapsing, rather than cut elimination, but the latter is still needed as an auxiliary construction. Specifically, our goal is to remove all cuts (Cut,φ) where φ is not bounded or of the form $\forall_x\theta$, for some bounded formula θ. To control the cuts that occur in a proof we introduce the following notion of formula rank:

Definition 4.2.1. The rank $\mathrm{rk}(\varphi)$ of a closed $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula φ is defined by the following recursion over the length of φ:

- If φ is a bounded formula then we set $\mathrm{rk}(\varphi)=0$.
- If at least one of the formulas φ_0 and φ_1 is unbounded then we set

$$\mathrm{rk}(\varphi_0\vee\varphi_1)=\mathrm{rk}(\varphi_0\wedge\varphi_1)=\max\{\mathrm{rk}(\varphi_0),\mathrm{rk}(\varphi_1)\}+1.$$

- If $\varphi\equiv\varphi(x)$ is unbounded then we set

$$\mathrm{rk}(\exists_{x\in a}\varphi(x))=\mathrm{rk}(\forall_{x\in a}\varphi(x))=\mathrm{rk}(\varphi(0))+2.$$

- We set $\mathrm{rk}(\exists_x\varphi(x))=\mathrm{rk}(\forall_x\varphi(x))=\mathrm{rk}(\varphi(0))+1$.

Observe that a formula φ is bounded if and only if it has rank $\mathrm{rk}(\varphi)=0$. If we have $\mathrm{rk}(\varphi)=1$ then φ must be of the form $\forall_x\theta$ or $\exists_x\theta$, where θ is bounded. Crucially, the "instances" of a formula have smaller rank than the formula itself:

Lemma 4.2.2. *Consider a closed $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula φ with $\mathrm{rk}(\varphi)>0$. Then we have $\mathrm{rk}(\varphi_a)<\mathrm{rk}(\varphi)$ for all $a\in\iota(\varphi)$.*

Proof. As a preparation one verifies $\mathrm{rk}(\psi(a))=\mathrm{rk}(\psi(0))$, by a straightforward induction over the length of $\psi\equiv\psi(x)$. Based on this fact one can check the claim for all disjunctive formulas from Definition 3.1.13. The most interesting case is

$$\varphi\equiv\exists_{x\in\{y\in\mathbf{L}^u_s\,|\,\theta(y,\vec{c})\}}\,\psi(x)\simeq\bigvee\nolimits_{\mathrm{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}s}\theta(a,\vec{c})\wedge\psi(a).$$

Since θ is a bounded formula (cf. Definition 3.1.2) we indeed get

$$\mathrm{rk}(\theta(a,\vec{c})\wedge\psi(a))=\mathrm{rk}(\psi(a))+1=\mathrm{rk}(\psi(0))+1<\mathrm{rk}(\psi(0))+2=\mathrm{rk}(\varphi).$$

To infer the claim for conjunctive formulas it suffices to observe $\mathrm{rk}(\neg\psi)=\mathrm{rk}(\psi)$. □

In Remark 3.4.7 we have explained that certain technical difficulties can be avoided because of Assumption 3.4.1. This also applies to the notion of rank: To obtain a finitistic ordinal analysis of Kripke-Platek set theory, one would have to assign transfinite formula ranks, similarly to the proof of Proposition 3.1.17. In particular, bounded formulas would receive non-zero ranks, and the previous lemma would apply to these formulas as well. In the presence of Assumption 3.4.1 and the truth rules $(\mathrm{True},\cdot)$ we can ignore the internal structure of bounded formulas, which allows us to work with finite ranks. Based on the notion of formula rank, we assign cut ranks to the S^u-codes that we have constructed in the previous section:

Definition 4.2.3. The function

$$d:\text{“}S^u\text{-codes”}\to\omega$$

is defined by the recursive clauses

$$\begin{aligned} d(P_S\sigma)&=C+6 \qquad \text{for any basic } S^u\text{-code } P_S\sigma,\\ d(\mathcal{I}_{\varphi,a}P)&=d(P), \end{aligned}$$

where C is the constant from Convention 1.3.14.

Clearly this yields $d(P)=C+6$ for any of the S^u-codes considered so far. Nevertheless it is important to state the recursive clause for $d(\mathcal{I}_{\varphi,a}P)$ explicitly: As explained in the previous section, the system of S^u-codes will be extended dynamically. Once we have introduced the function symbol $\mathcal{E}$, the recursive clause above will allow us to compute $d(\mathcal{I}_{\varphi,a}\mathcal{E}P)=d(\mathcal{E}P)$, which may well be different from $C+6$. Whenever we add a new function symbol, we will give a recursive clause which ensures that d is defined on the extended system of S^u-codes. We will also make sure that the next result is preserved, by adding a proof of the relevant induction step (see Remark 4.1.6 for more details):

Lemma 4.2.4. *The assignment of cut ranks is locally correct, in the sense that the following conditions hold for every S^u-code P:*

(C1) If $r_{\langle\rangle}(P)=(\mathrm{Cut},\varphi)$ is a cut rule then we have $\mathrm{rk}(\varphi)<d(P)$.

(C2) We have $d(n(P,a))\leq d(P)$ for any $a\in\iota(r_{\langle\rangle}(P))$.

Proof. The claim is established by induction over S^u-codes. First, we consider a basic S^u-code $P_S\sigma$. Condition (C2) is satisfied because $n(P_S\sigma, a) = P_S\,\sigma^\frown a$ is a basic S^u-code as well, so that we have $d(n(P_S\sigma, a)) = C + 6 = d(P_S\sigma)$. To verify condition (C1) we distinguish the following three cases: For $\sigma \notin P_S$ we observe that $r_{\langle\rangle}(P_S\sigma) = (\text{True}, 0 = 0)$ is not a cut rule, which means that condition (C1) is void. Next, assume that σ lies in the S^u-proof P_S but not in the search tree $S^u_{\text{BH}(S^u)}$. By the proof of Proposition 3.4.8 we have $\sigma = \sigma_0{}^\frown 1^\frown\tau$, where $\sigma_0 \in S^u_{\text{BH}(S^u)}$ has even length $\text{len}(\sigma_0) = 2n$ and τ lies in the S^u-proof $P_n = (P_n, l_n, r_n, o_n)$ constructed in Lemma 3.4.5 resp. Lemma 3.4.6. Together with Definition 4.1.1 we can compute

$$r_{\langle\rangle}(P_S\sigma) = r_S(\sigma) = r_n(\sigma).$$

It is straightforward to observe that the S^u-proofs P_n do not contain any cuts. In other words, $r_n(\sigma)$ is not a cut rule and condition (C1) is void once again. Finally, assume that σ does lie in the search tree $S^u_{\text{BH}(S^u)} \subseteq P_S$. If σ has odd length, then $r_{\langle\rangle}(P_S\sigma) = r_S(\sigma)$ is of the form $(\bigvee, \cdot), (\bigwedge, \cdot)$ or $(\text{Rep}, \cdot)$, by the proof of Proposition 3.4.8. If $\text{len}(\sigma) = 2n$ is even, then we see

$$r_{\langle\rangle}(P_S\sigma) = r_S(\sigma) = (\text{Cut}, \text{Ax}_n).$$

For $n = 0$ we have $\text{Ax}_0 \equiv \forall_x \exists_y\, y = x \cup \{x\}$. Since $y = x\cup\{x\}$ abbreviates a bounded formula this yields $\text{rk}(\text{Ax}_0) = 2 < C + 6 = d(P_S\sigma)$, as required by condition (C1). For $n > 0$ the formula Ax_n is one of the Δ_0-collection axioms

$$\forall_{z_1,\dots,z_k}\forall_v(\forall_{x\in v}\exists_y\theta(x, y, z_1, \dots, z_k) \to \exists_w\forall_{x\in v}\exists_{y\in w}\theta(x, y, z_1, \dots, z_k)).$$

As agreed in Section 3.2, we only list collection axioms with $k \leq C$ parameters (this is harmless by Proposition 1.3.15). Thus we have $\text{rk}(\text{Ax}_n) = k+5 < C+6 = d(P_S\sigma)$, which means that condition (C1) is satisfied. Now that we have covered all basic S^u-codes, let us consider a code of the form $\mathcal{I}_{\psi,b}P$. Concerning condition (C1), note that $r_{\langle\rangle}(\mathcal{I}_{\psi,b}P) = (\text{Cut}, \varphi)$ implies $r_{\langle\rangle}(P) = (\text{Cut}, \varphi)$, by the proof of Lemma 4.1.7. Using the induction hypothesis for P we obtain

$$\text{rk}(\varphi) < d(p) = d(\mathcal{I}_{\psi,b}P).$$

Also observe $\iota(r_{\langle\rangle}(\mathcal{I}_{\psi,b}P)) \subseteq \iota(r_{\langle\rangle}(P))$: The function symbol $\mathcal{I}_{\psi,b}$ was only introduced for $b \in \iota(\psi)$. In the crucial case of a rule $r_{\langle\rangle}(P) = (\bigwedge, \psi)$ this yields

$$\iota(r_{\langle\rangle}(\mathcal{I}_{\psi,b}P)) = \iota((\text{Rep}, b)) = \{b\} \subseteq \iota(\psi) = \iota(r_{\langle\rangle}(P)).$$

Thus condition (C2) for P applies to any $a \in \iota(r_{\langle\rangle}(\mathcal{I}_{\psi,b}P))$. We can deduce

$$d(n(\mathcal{I}_{\psi,b}P,a)) = d(\mathcal{I}_{\psi,b}n(P,a)) = d(n(P,a)) \leq d(P) = d(\mathcal{I}_{\psi,b}P),$$

as required by condition (C2) for $\mathcal{I}_{\psi,b}P$. □

In the previous section we have seen that any S^u-code P can be interpreted as an S^u-proof $[P] = ([P], l_P, r_P, o_P)$. It is instructive to observe that $d(P)$ bounds the complexity of all cut formulas in this proof (even though this fact will not play an official role): Iterating condition (C2) we get $d(\bar{n}(P,\sigma)) \leq d(P)$ for any $\sigma \in [P]$. So if $r_P(\sigma) = r_{\langle\rangle}(\bar{n}(P,\sigma)) = (\mathrm{Cut}, \varphi)$ is a cut, then condition (C1) for $\bar{n}(P,\sigma)$ yields

$$\mathrm{rk}(\varphi) < d(\bar{n}(P,\sigma)) \leq d(P).$$

The aim of this section is to transform each S^u-code P into an S^u-code P' such that any cut formula in the S^u-proof $[P']$ is bounded or of the form $\exists_x\theta$ resp. $\forall_x\theta$, for some bounded formula θ. By the above observations it suffices to ensure $d(P') \leq 2$. It is well-known that cut elimination increases the height of proofs. In order to describe the increased ordinal labels, we define the following operations on our notation system $\vartheta(S^u_{\mathrm{BH}(S^u)})$:

Lemma 4.2.5. *We can construct maps*

$$\hat{+} : \vartheta(S^u_{\mathrm{BH}(S^u)}) \times \vartheta(S^u_{\mathrm{BH}(S^u)}) \to \vartheta(S^u_{\mathrm{BH}(S^u)}),$$
$$\hat{\omega}^{(\cdot)} : \vartheta(S^u_{\mathrm{BH}(S^u)}) \to \vartheta(S^u_{\mathrm{BH}(S^u)})$$

with the following properties:

(a) *If* $s < t$ *then* $r \hat{+} s < r \hat{+} t$.
(b) *We have* $s \leq s \hat{+} t$ *and* $t \leq s \hat{+} t$.
(c) *We have* $(r \hat{+} s) \hat{+} t = r \hat{+} (s \hat{+} t)$.
(d) *If* $s < t$ *then* $\hat{\omega}^s < \hat{\omega}^t$.
(e) *We have* $s \leq \hat{\omega}^s$.
(f) *If* $s < \hat{\omega}^r$ *and* $t < \hat{\omega}^r$ *then* $s \hat{+} t < \hat{\omega}^r$.

Note that we write $\hat{+}$ and $\hat{\omega}$, because the terms in $\vartheta(S^u_{\mathrm{BH}(S^u)})$ already contain the symbols $+$ and ω. For example, we want to distinguish the expression $\hat{\omega}^0 \hat{+} \hat{\omega}^0$ from the term $\omega^0 + \omega^0$ (even though the former will denote the latter).

Proof. Given terms $s_n \leq \cdots \leq s_1$ in $\vartheta(S^u_{\mathrm{BH}(S^u)})$ we define

$$\omega\langle s_1,\dots,s_n\rangle = \begin{cases} 0 & \text{if } n=0,\\ s_1 & \text{if } n=1 \text{ and } s_1 \text{ is of the form } \Omega, \mathfrak{E}_\sigma \text{ or } \vartheta s,\\ \omega^{s_1}+\dots+\omega^{s_n} & \text{otherwise.}\end{cases}$$

Lemma 2.2.7 yields $\omega\langle s_1,\dots,s_n\rangle \in \vartheta(S^u_{\mathrm{BH}(S^u)})$. Conversely, any $s \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ can uniquely be written as $s = \omega\langle s_1,\dots,s_n\rangle$. We can thus define addition by

$$\omega\langle s_1,\dots,s_n\rangle \mathbin{\hat{+}} \omega\langle t_1,\dots,t_m\rangle = \omega\langle s_1,\dots,s_i,t_1,\dots,t_m\rangle,$$

where $i \leq n$ is maximal with $t_1 \leq s_i$ (set $i=0$ if $s_1 < t_1$, and $i=n$ if $m=0$). Exponentiation can be given as

$$\hat{\omega}^s = \omega\langle s\rangle.$$

In view of Lemma 2.2.7 it is straightforward to observe

$$\omega\langle s_1,\dots,s_n\rangle < \omega\langle t_1,\dots,t_m\rangle \Leftrightarrow \begin{cases} \text{either } n<m \text{ and } s_i = t_i \text{ for } i \leq n,\\ \text{or there is a } j \leq \min\{n,m\} \text{ with } s_j < t_j\\ \text{and } s_i = t_i \text{ for } i<j.\end{cases}$$

Building on this characterization we verify the claims of the lemma:

(a) Write $r = \omega\langle r_1,\dots,r_k\rangle$, $s = \omega\langle s_1,\dots,s_n\rangle$ and $t = \omega\langle t_1,\dots,t_m\rangle$, as well as

$$r \mathbin{\hat{+}} s = \omega\langle r_1,\dots,r_i,s_1,\dots,s_n\rangle,$$
$$r \mathbin{\hat{+}} t = \omega\langle r_1,\dots,r_j,t_1,\dots,t_m\rangle.$$

For $r=0$ we have $r \mathbin{\hat{+}} s = s < t = r \mathbin{\hat{+}} t$, and $t=0$ is impossible in view of $s<t$. To cover the case $s=0$ we infer $r < r \mathbin{\hat{+}} t$: For $r_1 < t_1$ we get $r < t = r \mathbin{\hat{+}} t$. Otherwise $j \leq k$ is maximal with $t_1 \leq r_j$. For $j=k$ we observe that $\langle r_1,\dots,r_j,t_1,\dots,t_m\rangle$ is a proper end extension of $\langle r_1,\dots,r_k\rangle$. For $j<k$ we note $r_{j+1} < t_1$, due to the maximality of j. Now assume that r, s and t are all different from zero: From $s<t$ we infer $s_1 \leq t_1$, which implies $i \geq j$. If we have $i=j$, then $r \mathbin{\hat{+}} s < r \mathbin{\hat{+}} t$ is straightforward. If we have $j < i \leq k$, then the choice of j implies $r_{j+1} < t_1$, which again yields $r \mathbin{\hat{+}} s < r \mathbin{\hat{+}} t$.

(b) In view of $0 \leq t$ we can infer $s = s \mathbin{\hat{+}} 0 \leq s \mathbin{\hat{+}} t$ by part (a). The inequality

$$t = \omega\langle t_1,\dots,t_m\rangle \leq \omega\langle s_1,\dots,s_i,t_1,\dots,t_m\rangle = s \mathbin{\hat{+}} t$$

follows from $t_m \leq \cdots \leq t_1 \leq s_i \leq \cdots \leq s_1$, which is due to the choice of i.

(c) The claim is immediate if we have $s=0$, $t=0$ or $r=0$. Otherwise we write

$r = \omega\langle r_1, \ldots, r_k\rangle$, $s = \omega\langle s_1, \ldots, s_n\rangle$ and $t = \omega\langle t_1, \ldots, t_m\rangle$ and distinguish two cases: If we have $s_1 < t_1$, then the reader can easily verify

$$(r \hat{+} s) \hat{+} t = \omega\langle r_1, \ldots, r_i, t_1, \ldots, t_m\rangle = r \hat{+} (s \hat{+} t),$$

where $i \leq k$ is maximal with $t_1 \leq r_i$ (or $i = 0$ if $r_1 < t_1$). For $t_1 \leq s_1$ we get

$$(r \hat{+} s) \hat{+} t = \omega\langle r_1, \ldots, r_i, s_1, \ldots, s_j, t_1, \ldots, t_m\rangle = r \hat{+} (s \hat{+} t),$$

where $i \leq k$ is maximal with $s_1 \leq r_i$ and $j \leq n$ is maximal with $t_1 \leq s_j$.
(d) By the above we have $\hat{\omega}^s = \omega\langle s\rangle < \omega\langle t\rangle = \hat{\omega}^t$.
(e) If s is of the form $\Omega, \mathfrak{E}_\sigma$ or $\vartheta s'$ then we have $\hat{\omega}^s = s$. Also note $0 < \omega^0 = \hat{\omega}^0$. To cover the remaining case we prove the statement

"if s is of the form $\omega^{s_0} + \cdots + \omega^{s_n}$ then we have $s_0 < s$",

by induction over s. If s_0 is of the form $0, \Omega, \mathfrak{E}$ or $\vartheta s'$, then the claim is immediate by Lemma 2.2.7. If we have $s_0 = \omega^{s_{0,0}} + \cdots + \omega^{s_{0,m}}$, then $s_0 < s$ follows from $s_{0,0} < s_0$, as provided by the induction hypothesis. Once we know $s_0 < s$ we can infer

$$s = \omega^{s_0} + \cdots + \omega^{s_n} < \omega^s = \hat{\omega}^s,$$

as required for claim (e).
(f) For $s = 0$ or $t = 0$ the claim is immediate. Otherwise we write $s = \omega\langle s_1, \ldots, s_n\rangle$ and $t = \omega\langle t_1, \ldots, t_m\rangle$. The assumptions $s < \hat{\omega}^r = \omega\langle r\rangle$ and $t < \hat{\omega}^r = \omega\langle r\rangle$ imply $s_1 < r$ resp. $t_1 < r$. We can infer $s \hat{+} t = \omega\langle s_1, \ldots, s_i, t_1, \ldots, t_m\rangle < \omega\langle r\rangle = \hat{\omega}^r$, as desired. □

Before the statement of Lemma 3.4.5 above, we have give an ad hoc definition of a map $s \mapsto s + 1$. It is straightforward to check that this is consistent with the present definition of addition, in the sense that we have $s + 1 = s \hat{+} \omega^0$. As a preparation for cut elimination, we present a proof transformation known as reduction: It allows to combine proofs of $\Gamma, \neg\varphi$ and Γ, φ into a proof of Γ, without increasing the cut rank.

LEMMA 4.2.6. *For any conjunctive $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula φ with $\mathrm{rk}(\varphi) \geq 2$ we can extend the system of S^u-codes by a binary function symbol $\mathcal{R}_\varphi$, such that we have*

$$\begin{aligned} l_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1) &= (l_{\langle\rangle}(P_0)\backslash\{\neg\varphi\}) \cup (l_{\langle\rangle}(P_1)\backslash\{\varphi\}), \\ o_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1) &= o_{\langle\rangle}(P_1) \hat{+} o_{\langle\rangle}(P_0), \\ d(\mathcal{R}_\varphi P_0 P_1) &= \max\{d(P_0), d(P_1), \mathrm{rk}(\varphi)\} \end{aligned}$$

for all S^u-codes P_0 and P_1.

To understand what exactly the lemma claims, the reader may wish to consult Remark 4.1.6, as well as the explanation after the statement of Lemma 4.1.7.

Proof. We begin with an informal description of the proof transformation in terms of transfinite recursion: Assume that P_0 and P_1 have end-sequent $\Gamma, \neg\varphi$ and Γ, φ, respectively. Our goal is to construct a proof of Γ. In the crucial case, the formula $\neg\varphi$ has been introduced by the last rule of P_0. The assumption $\operatorname{rk}(\varphi) \geq 2$ ensures that this cannot be a reflection rule. So P_0 ends with a rule $(\bigvee, \neg\varphi, b)$, and the immediate subproof $n(P_0, 0)$ has end-sequent $\Gamma, \neg\varphi, \neg\varphi_b$. Since $n(P_0, 0)$ has smaller height than P_0 we can recursively construct a proof $\mathcal{R}_\varphi n(P_0, 0)P_1$ of the sequent $\Gamma, \neg\varphi_b$. On the other hand we can apply inversion to the proof P_1, in order to get a proof $\mathcal{I}_{\varphi,b}P_1$ of the sequent Γ, φ_b. Combining these proofs by the rule $(\mathrm{Cut}, \varphi_b)$ we obtain the desired proof of Γ. Officially, this idea is implemented on the level of S^u-codes: The recursive clauses for $l_{\langle\rangle}, o_{\langle\rangle}$ and d can be found in the statement of the lemma. To extend the functions $r_{\langle\rangle}$ and n we set

$$r_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1) = \begin{cases} (\mathrm{Cut}, \varphi_b) & \text{if } r_{\langle\rangle}(P_0) = (\bigvee, \neg\varphi, b), \\ r_{\langle\rangle}(P_0) & \text{otherwise,} \end{cases}$$

$$n(\mathcal{R}_\varphi P_0 P_1, a) = \begin{cases} \mathcal{I}_{\varphi,b}P_1 & \text{if } r_{\langle\rangle}(P_0) = (\bigvee, \neg\varphi, b) \text{ and } a = 1, \\ \mathcal{R}_\varphi n(P_0, a)P_1 & \text{otherwise.} \end{cases}$$

To justify this extension we must verify condition (L) from Definition 4.1.3, as well as conditions (C1) and (C2). This is done by case distinction on the last rule of P_0. Let us write out the details for $r_{\langle\rangle}(P_0) = (\bigvee, \neg\varphi, b)$: Inductively we may assume condition (L) for P_0, which provides $l_{\langle\rangle}(n(P_0, 0)) \subseteq l_{\langle\rangle}(P_0) \cup \{\neg\varphi_b\}$ (note that $(\neg\varphi)_b$ and $\neg(\varphi_b)$ are the same formula, by Definition 3.1.13). This allows us to deduce

$$\begin{aligned} l_{\langle\rangle}(n(\mathcal{R}_\varphi P_0 P_1, 0)) = l_{\langle\rangle}(\mathcal{R}_\varphi n(P_0, 0)P_1) = l_{\langle\rangle}(n(P_0, 0)) \backslash \{\neg\varphi\} \cup l_{\langle\rangle}(P_1) \backslash \{\varphi\} \subseteq \\ \subseteq (l_{\langle\rangle}(P_0) \cup \{\neg\varphi_b\}) \backslash \{\neg\varphi\} \cup l_{\langle\rangle}(P_1) \backslash \{\varphi\} \subseteq l_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1) \cup \{\neg\varphi_b\} \end{aligned}$$

and

$$l_{\langle\rangle}(n(\mathcal{R}_\varphi P_0 P_1, 1)) = l_{\langle\rangle}(\mathcal{I}_{\varphi,b}P_1) = l_{\langle\rangle}(P_1) \backslash \{\varphi\} \cup \{\varphi_b\} \subseteq l_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1) \cup \{\varphi_b\},$$

as condition (L) demands in the case of the cut rule $r_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1) = (\mathrm{Cut}, \varphi_b)$. Since condition (L) for P_0 provides $o_{\langle\rangle}(n(P_0, 0)) < o_{\langle\rangle}(P_0)$ we can also infer

$$o_{\langle\rangle}(n(\mathcal{R}_\varphi P_0 P_1, 0)) = o_{\langle\rangle}(P_1) \hat{+} o_{\langle\rangle}(n(P_0, 0)) < o_{\langle\rangle}(P_1) \hat{+} o_{\langle\rangle}(P_0) = o_{\langle\rangle}(\mathcal{R}_\varphi P_0 P_1),$$

as well as

$$o_{\langle\rangle}(n(\mathcal{R}_\varphi P_0P_1,1)) = o_{\langle\rangle}(\mathcal{I}_{\varphi,b}P_1) = o_{\langle\rangle}(P_1) \leq o_{\langle\rangle}(P_1) \hat{+} o_{\langle\rangle}(n(P_0,0)) < o_{\langle\rangle}(\mathcal{R}_\varphi P_0P_1).$$

Condition (C1) for $\mathcal{R}_\varphi P_0P_1$ is satisfied, because Lemma 4.2.2 implies

$$\mathrm{rk}(\varphi_b) < \mathrm{rk}(\varphi) \leq d(\mathcal{R}_\varphi P_0P_1).$$

Using condition (C2) for P_0 we get

$$d(n(\mathcal{R}_\varphi P_0P_1,0)) = d(\mathcal{R}_\varphi n(P_0,0)P_1) = \max\{d(n(P_0,0)), d(P_1), \mathrm{rk}(\varphi)\} \leq$$
$$\leq \max\{d(P_0), d(P_1), \mathrm{rk}(\varphi)\} = d(\mathcal{R}_\varphi P_0P_1).$$

We can also observe

$$d(n(\mathcal{R}_\varphi P_0P_1,1)) = d(\mathcal{I}_{\varphi,b}P_1) = d(P_1) \leq d(\mathcal{R}_\varphi P_0P_1),$$

as required by condition (C2) for $\mathcal{R}_\varphi P_0P_1$. The other cases are established by similar computations. Concerning the side condition of the reflection rule, observe that $\Omega \leq o_{\langle\rangle}(P_0)$ implies $\Omega \leq o_{\langle\rangle}(P_1) \hat{+} o_{\langle\rangle}(P_0) = o_{\langle\rangle}(\mathcal{R}_\varphi P_0P_1)$, by part (b) of the previous lemma. Similarly, $\mathrm{supp}^{\mathbf{L}}(b) <^{\mathrm{fin}} o_{\langle\rangle}(P_0)$ implies $\mathrm{supp}^{\mathbf{L}}(b) <^{\mathrm{fin}} o_{\langle\rangle}(\mathcal{R}_\varphi P_0P_1)$, as needed to preserve the side condition of a rule $(\bigvee, \psi, b)$ with $\psi \not\equiv \neg\varphi$. □

As usual, we can now define a cut elimination process, which lowers the cut rank as long as it is bigger than two:

Proposition 4.2.7. *We can extend the system of S^u-codes by a unary function symbol $\mathcal{E}$, such that we have*

$$l_{\langle\rangle}(\mathcal{E}P) = l_{\langle\rangle}(P),$$
$$o_{\langle\rangle}(\mathcal{E}P) = \hat{\omega}^{o_{\langle\rangle}(P)},$$
$$d(\mathcal{E}P) = \max\{2, d(P)-1\}$$

for any S^u-code P.

Proof. As before, we begin with an informal presentation: In the crucial case, the last rule of P is a cut (Cut, φ) with $\mathrm{rk}(\varphi) = d(P) - 1 \geq 2$. We assume that φ is conjunctive. The immediate subproofs $n(P,0)$ and $n(P,1)$ deduce sequents $\Gamma, \neg\varphi$ resp. Γ, φ, where Γ is the end-sequent of P. Recursively we obtain proofs $\mathcal{E}n(P,0)$ and $\mathcal{E}n(P,1)$ with the same end-sequents but lower cut rank. Using the transformation from the previous lemma we can build a proof $\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))$ of the desired sequent Γ. The cut rank remains low, because we have avoided to reapply the rule (Cut, φ). Officially, the proof transformation has to be implemented in terms of S^u-codes: This is accomplished by the recursive clauses

$$r_{\langle\rangle}(\mathcal{E}P) = \begin{cases} (\mathrm{Rep}, 0) & \text{if } r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi) \text{ with } \mathrm{rk}(\varphi) \geq 2, \\ r_{\langle\rangle}(P) & \text{otherwise,} \end{cases}$$

$$n(\mathcal{E}P, a) = \begin{cases} \mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1)) & \text{if } r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi) \text{ where } \varphi \text{ is conjunctive and } \mathrm{rk}(\varphi) \geq 2, \\ \mathcal{R}_{\neg\varphi}(\mathcal{E}n(P,1))(\mathcal{E}n(P,0)) & \text{if } r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi) \text{ where } \varphi \text{ is disjunctive and } \mathrm{rk}(\varphi) \geq 2, \\ \mathcal{E}n(P,a) & \text{otherwise.} \end{cases}$$

Observe how the repetition rule $(\mathrm{Rep}, 0)$ is used to "call" the result of cut reduction: The informal presentation above suggests that $\mathcal{E}P$ and $\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))$ are the same proof. Taking this literally, we would have to define $r_{\langle\rangle}(\mathcal{E}P)$ as the rule $r_{\langle\rangle}(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1)))$. However, we are not allowed to refer to this rule, since we define $r_{\langle\rangle}$ by recursion over terms. The repetition rule allows us to state a valid recursive clause nevertheless, as discovered by Mints [**54**] (the improper ω-rule considered by Schwichtenberg [**82**] serves a similar purpose). To complete the proof we must establish conditions (L), (C1) and (C2) for $\mathcal{E}P$, assuming that the same conditions hold for P. Let us write out the details for the crucial case of a rule $r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi)$, where φ is a conjunctive formula with $\mathrm{rk}(\varphi) \geq 2$: Using condition (L) for P we can compute

$$l_{\langle\rangle}(\mathcal{E}n(P,0))\backslash\{\neg\varphi\} = l_{\langle\rangle}(n(P,0))\backslash\{\neg\varphi\} \subseteq (l_{\langle\rangle}(P) \cup \{\neg\varphi\})\backslash\{\neg\varphi\} \subseteq l_{\langle\rangle}(P).$$

Similarly we have $l_{\langle\rangle}(\mathcal{E}n(P,1))\backslash\{\varphi\} \subseteq l_{\langle\rangle}(P)$, so that we can deduce

$$\begin{aligned} l_{\langle\rangle}(n(\mathcal{E}P,0)) = l_{\langle\rangle}(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))) = \\ = l_{\langle\rangle}(\mathcal{E}n(P,0))\backslash\{\neg\varphi\} \cup l_{\langle\rangle}(\mathcal{E}n(P,1))\backslash\{\varphi\} \subseteq l_{\langle\rangle}(P) = l_{\langle\rangle}(\mathcal{E}P), \end{aligned}$$

as condition (L) demands in the case $r_{\langle\rangle}(\mathcal{E}P) = (\mathrm{Rep}, 0)$. For $i = 0, 1$, condition (L) for P yields $o_{\langle\rangle}(n(P,i)) < o_{\langle\rangle}(P)$, which implies $\hat{\omega}^{o_{\langle\rangle}(n(P,i))} < \hat{\omega}^{o_{\langle\rangle}(P)}$. By part (f) of Lemma 4.2.5 we obtain

$$\begin{aligned} o_{\langle\rangle}(n(\mathcal{E}P,0)) = o_{\langle\rangle}(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))) = o_{\langle\rangle}(\mathcal{E}n(P,1)) \hat{+} o_{\langle\rangle}(\mathcal{E}n(P,0)) = \\ = \hat{\omega}^{o_{\langle\rangle}(n(P,1))} \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,0))} < \hat{\omega}^{o_{\langle\rangle}(P)} = o_{\langle\rangle}(\mathcal{E}P). \end{aligned}$$

Condition (C1) is void because $r_{\langle\rangle}(\mathcal{E}P)$ is not a cut rule. To prepare condition (C2), observe that the same condition for P ensures

$$d(\mathcal{E}n(P,i)) = \max\{2, d(n(P,i)) - 1\} \leq \max\{2, d(P) - 1\} = d(\mathcal{E}P).$$

In view of $r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi)$, condition (C1) for P provides $\mathrm{rk}(\varphi) < d(P)$ and thus

$$\mathrm{rk}(\varphi) \leq d(P) - 1 \leq d(\mathcal{E}P).$$

Together we obtain

$$\begin{aligned} d(n(\mathcal{E}P,0)) = d(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))) = \\ = \max\{d(\mathcal{E}n(P,0)), d(\mathcal{E}n(P,1)), \mathrm{rk}(\varphi)\} \leq d(\mathcal{E}P), \end{aligned}$$

as required by condition (C2) for $\mathcal{E}P$. The other cases are verified by similar computations. Note that part (e) of Lemma 4.2.5 yields $o_{\langle\rangle}(P) \leq \hat{\omega}^{o_{\langle\rangle}(P)} = o_{\langle\rangle}(\mathcal{E}P)$, so that the side conditions of the rules $(\bigvee, \cdot)$ and $(\mathrm{Ref}, \cdot)$ are preserved. □

In the previous section, the S^u-proof P_S from Proposition 3.4.8 has been implemented in terms of S^u-codes: It can be recovered as the interpretation $P_S = [P_S\langle\rangle]$ of the S^u-code $P_S\langle\rangle$ with end-sequent $l_{\langle\rangle}(P_S\langle\rangle) = \langle\rangle$ and cut rank $d(P_S\langle\rangle) = C + 6$. The previous proposition allows us to form the S^u-code $\mathcal{E}^{C+4}P_S\langle\rangle = \mathcal{E}\cdots\mathcal{E}P_S\langle\rangle$ with $C+4$ occurrences of the function symbol $\mathcal{E}$. It is straightforward to compute

$$\begin{aligned} l_{\langle\rangle}(\mathcal{E}^{C+4}P_S\langle\rangle) = \langle\rangle, \\ d(\mathcal{E}^{C+4}P_S\langle\rangle) = 2, \end{aligned}$$

which means that $[\mathcal{E}^{C+4}P_S\langle\rangle]$ is an S^u-proof of the empty sequent. For any cut rule (Cut, φ) that occurs in this proof we must have $\mathrm{rk}(\varphi) < 2$, so that φ is bounded or of the form $\varphi \equiv \forall_x\theta$ resp. $\varphi \equiv \exists_x\theta$ with a bounded formula θ. We cannot eliminate the remaining cuts, because they may occur in the form

$$\dfrac{\dfrac{\Gamma, \forall_{x\in a}\exists_y\theta(x,y)}{\Gamma, \exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y)}\,(\mathrm{Ref}) \qquad \Gamma, \forall_w\exists_{x\in a}\forall_{y\in w}\neg\theta(x,y)}{\Gamma}\,(\mathrm{Cut}),$$

where the cut formula has rank $\mathrm{rk}(\exists_w\forall_{x\in a}\exists_{y\in w}\theta(x,y)) = 1$. Cut reduction cannot be extended to this situation, because we are unable to transform the proof of $\Gamma, \forall_w\exists_{x\in a}\forall_{y\in w}\neg\theta(x,y)$ into a proof of $\Gamma, \exists_{x\in a}\forall_y\neg\theta(x,y)$. Even if we had such a transformation this would not lower the cut rank, because the new cut formula would have rank $\mathrm{rk}(\forall_{x\in a}\exists_y\theta(x,y)) = 3$, which is even higher than before. In the following sections we will present a transformation known as collapsing, which removes occurrences of the reflection rule. To conclude the present section, let us comment on the role of the constant C:

REMARK 4.2.8. The usual axiomatization of Kripke-Platek set theory includes collection for all Δ_0-formulas. At the same time it is not hard to show that finitely many instances of collection are sufficient (see Proposition 1.3.15). This allowed us to work with an axiomatization $\mathrm{Ax}_0, \mathrm{Ax}_1, \dots$ in which the number of parameters that may occur in a collection axiom has a finite bound C, leading to the bound $\mathrm{rk}(\mathrm{Ax}_n) \leq C + 5$ on the rank of the axioms. As a consequence, we were able to extend our search tree $S^u_{\mathrm{BH}(S^u)}$ into an S^u-proof P_S with cut rank $C + 6$. If one admits axioms of unbounded rank, then the embedding of the search tree into the infinite proof system becomes considerably more difficult. Indeed, this is the situation that Rathjen and Valencia Vizcaíno [**71**] had to resolve in their analysis of bar induction. Their idea was to intertwine embedding and cut elimination: Recursively, we may assume that the search tree above the nodes $\sigma^\frown a$ has been embedded as a sequence of infinite proofs $P_{\sigma^\frown a}$ of finite cut rank. We would like to combine these proofs into a proof P_σ that embeds the search tree above σ. However, it is possible that there is no common bound for the cut ranks of the proofs $P_{\sigma^\frown a}$, so that P_σ would have infinite cut rank. The solution is to apply cut elimination to the proofs $P_{\sigma^\frown a}$, in order to reduce their cut ranks before the proofs are combined. We did not follow this approach, because the bound C is harmless in our situation and simplifies matters considerably. Let us point out that the role of the epsilon numbers $o_S(\sigma) = \mathfrak{E}_\sigma$ is much clearer in Rathjen and Valencia Vizcaíno's case: The point is that one can perform cut elimination above the node σ, without changing the ordinal label $\mathfrak{E}_\sigma = \hat{\omega}^{\mathfrak{E}_\sigma}$. This feature is not required in our proof, so potentially the notation system $\vartheta(S^u_{\mathrm{BH}(S^u)})$ could have been defined more economically. In any case, the type-two aspect of our well-ordering principles seems more important than the precise order-type of $\vartheta(S^u_{\mathrm{BH}(S^u)})$, as discussed in Remarks 2.1.25 and 2.4.10.

4.3. Operator Control

In order to obtain a consistency result for S^u-proofs we need to collapse proofs to countable height, as indicated in the introduction of the present chapter. The collapsing construction for proofs relies on a corresponding collapse on the ordinal notations: By the definition of the order $(\vartheta(S^u_{\mathrm{BH}(S^u)}), <)$ in Lemma 2.2.7, each term $s \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ gives rise to a term $\vartheta s \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ with $\vartheta s < \Omega$. Also recall that we identify the ordered sets $\vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ and $\mathrm{BH}(S^u)$, which are isomorphic by Theorem 2.2.25. To summarize, we have a collapsing function

$$\vartheta(S^u_{\mathrm{BH}(S^u)}) \ni s \mapsto \vartheta s \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega \cong \mathrm{BH}(S^u).$$

A complication arises from the fact that this function is not fully order preserving: According to Lemma 2.2.7 we have

$$s < t \quad \Rightarrow \quad \vartheta s < \vartheta t \qquad \text{under the side condition } E^\vartheta(s) <^{\mathrm{fin}} \vartheta t,$$

where $E^\vartheta(s) = E^\vartheta_{\mathrm{BH}(S^u)}(s)$ is computed as in Definition 2.2.5. Note that we have $E^\vartheta(s) \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ by Lemma 2.2.11. Collapsing functions which are partially order preserving are a characteristic feature of impredicative proof theory. Buchholz [**10**] has introduced controlling operators as an elegant way to handle them. In the present section we adapt his approach to the setting of S^u-proofs. The first step is to define the required operators themselves:

Definition 4.3.1. For a term $t \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ and a finite set $x \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ we construct a set $\mathcal{H}_t(x) \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)})$ by the stipulations

$$\begin{aligned}
\mathcal{H}^0_t(x) &= \bigcup_{s \in x} E^\vartheta(s),\\
\mathcal{H}^{n+1}_t(x) &= \{\vartheta s \mid s \in \mathcal{H}^n_t(x) \wedge s \leq t\} \cup \{s \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \mid E^\vartheta(s) \subseteq \mathcal{H}^n_t(x)\} \cup\\
&\qquad \cup \{s \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \mid \exists_{s'}(s' \in \mathcal{H}^n_t(x) \wedge s < s' < \Omega)\},\\
\mathcal{H}_t(x) &= \bigcup_{n \in \omega} \mathcal{H}^n_t(x).
\end{aligned}$$

It is straightforward to see that the function $(t, x, n) \mapsto \mathcal{H}^n_t(x)$ is primitive recursive. Proposition 1.1.6 ensures that $(t, x) \mapsto \mathcal{H}_t(x)$ is a primitive recursive set function (with parameter ω, cf. Convention 1.2.9). In fact, Proposition 1.2.2 shows that $(t, x) \mapsto \mathcal{H}_t(x)$ exists as a set-sized function. Let us establish basic properties:

Lemma 4.3.2. *The following holds for any number n:*

(a) If $s \in \mathcal{H}^n_t(x)$ then $E^\vartheta(s) \subseteq \mathcal{H}^n_t(x)$.
(b) We have $\mathcal{H}^n_t(x) \subseteq \mathcal{H}^{n+1}_t(x)$.

Since $E^\vartheta(s)$ is finite, it follows that $s \in \mathcal{H}_t(x)$ is equivalent to $E^\vartheta(s) \subseteq \mathcal{H}_t(x)$.

Proof. The claims are established by simultaneous induction over n:
(a) First assume $n = 0$: A straightforward induction over the term $s' < \Omega$ shows that $s \in E^\vartheta(s')$ implies $E^\vartheta(s) \subseteq E^\vartheta(s')$. Now $s \in \mathcal{H}^0_t(x)$ means $s \in E^\vartheta(s')$ for some $s' \in x \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$. Thus we get $E^\vartheta(s) \subseteq E^\vartheta(s') \subseteq \mathcal{H}^0_t(x)$, as desired.

Now assume that we have $n = m+1$: A straightforward induction over $s < \Omega$ shows $E^\vartheta(s) \leq^{\text{fin}} s$ (recall $s_0 < \omega^{s_0} + \cdots + \omega^{s_n}$ from the proof of Lemma 4.2.5(e)). So if $s \in \mathcal{H}_t^n(x)$ holds because of $s < s' < \Omega$ with $s' \in \mathcal{H}_t^m(x)$ then $E^\vartheta(s) <^{\text{fin}} s'$ yields $E^\vartheta(s) \subseteq \mathcal{H}_t^n(x)$. If $s \in \mathcal{H}_t^n(x)$ holds because of $E^\vartheta(s) \subseteq \mathcal{H}_t^m(x)$ then we get $E^\vartheta(s) \subseteq \mathcal{H}_t^n(x)$ by the induction hypothesis for (b). For $s = \vartheta s'$ the claim is trivial because of $E^\vartheta(\vartheta s') = \{\vartheta s'\}$.
(b) By part (a) we see that $s \in \mathcal{H}_t^n(x)$ implies $E^\vartheta(s) \subseteq \mathcal{H}_t^n(x)$, which yields $s \in \mathcal{H}_t^{n+1}(x)$ as desired. □

As observed by Buchholz [**10**], it is crucial that $\mathcal{H}_t(\cdot)$ has the properties of a closure operator:

Lemma 4.3.3. *The following holds for all finite sets $x, y \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$:*

(a) We have $x \subseteq \mathcal{H}_t(x)$.
(b) If $x \subseteq \mathcal{H}_t(y)$ then $\mathcal{H}_t(x) \subseteq \mathcal{H}_t(y)$.

Together, part (a) and (b) show that $x \subseteq y$ implies $\mathcal{H}_t(x) \subseteq \mathcal{H}_t(y)$.

Proof. (a) For $s \in x$ we have $E^\vartheta(s) \subseteq \mathcal{H}_t^0(x)$ and thus $s \in \mathcal{H}_t^1(x)$.
(b) By the previous lemma we have $\bigcup_{s \in x} E^\vartheta(s) \subseteq \mathcal{H}_t^n(y)$ for some n. A straightforward induction on m shows $\mathcal{H}_t^m(x) \subseteq \mathcal{H}_t^{n+m}(y)$. □

Next, we show that the values of $\mathcal{H}_t(\cdot)$ are closed under certain operations on the ordinal notations (i.e. the operator $\mathcal{H}_t(\cdot)$ is "nice" in the sense of Buchholz [**10**]):

Lemma 4.3.4. *The following holds for any $t \in \vartheta(S^u_{\mathrm{BH}(S^u)})$:*

(a) We have $0 \in \mathcal{H}_t(\emptyset)$ and $\Omega \in \mathcal{H}_t(\emptyset)$.
(b) For $\sigma \in S^u_{\mathrm{BH}(S^u)}$ we have $\mathfrak{E}_\sigma \in \mathcal{H}_t(\mathrm{supp}^{\mathrm{S}}(\sigma))$.
(c) If $s, s' \in \mathcal{H}_t(x)$ then $s \hat{+} s' \in \mathcal{H}_t(x)$ and $\hat{\omega}^s \in \mathcal{H}_t(x)$.

To understand part (b), recall that $i_{\mathrm{BH}(S^u)} : \mathrm{BH}(S^u) \xrightarrow{\cong} \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ is left implicit. Officially, $\mathrm{supp}^{\mathrm{S}}(\sigma)$ refers to $[i_{\mathrm{BH}(S^u)}]^{<\omega}(\mathrm{supp}^{\mathrm{S}}_{\mathrm{BH}(S^u)}(\sigma)) \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$.

Proof. (a) From $E^\vartheta(0) = E^\vartheta(\Omega) = \emptyset \subseteq \mathcal{H}_t^0(\emptyset)$ we get $0, \Omega \in \mathcal{H}_t^1(\emptyset)$.
(b) By Definition 2.2.5 we have $E^\vartheta(\mathfrak{E}_\sigma) = [i_{\mathrm{BH}(S^u)}]^{<\omega}(\mathrm{supp}^{\mathrm{S}}_{\mathrm{BH}(S^u)}(\sigma))$. Thus the previous lemma yields $E^\vartheta(\mathfrak{E}_\sigma) \subseteq \mathcal{H}_t(\mathrm{supp}^{\mathrm{S}}(\sigma))$. In view of Lemma 4.3.2(b) we can assume $E^\vartheta(\mathfrak{E}_\sigma) \subseteq \mathcal{H}_t^n(\mathrm{supp}^{\mathrm{S}}(\sigma))$ for some number n. The definition of our operators allows us to infer $\mathfrak{E}_\sigma \in \mathcal{H}_t^{n+1}(\mathrm{supp}^{\mathrm{S}}(\sigma)) \subseteq \mathcal{H}_t(\mathrm{supp}^{\mathrm{S}}(\sigma))$.
(c) Recall the notation $\omega\langle s_1, \dots, s_n\rangle$ from the proof of Lemma 4.2.5, and observe that we have $E^\vartheta(\omega\langle s_1, \dots, s_n\rangle) = E^\vartheta(s_1) \cup \cdots \cup E^\vartheta(s_n)$. Writing $s = \omega\langle s_1, \dots, s_n\rangle$

and $s' = \omega\langle s'_1, \dots, s'_m\rangle$ we have $s \hat{+} s' = \omega\langle s_1, \dots, s_i, s'_1, \dots, s'_m\rangle$ for some $i \leq n$, which yields

$$E^\vartheta(s \hat{+} s') = E^\vartheta(s_1) \cup \cdots \cup E^\vartheta(s_i) \cup E^\vartheta(s'_1) \cup \cdots \cup E^\vartheta(s'_m) \subseteq E^\vartheta(s) \cup E^\vartheta(s').$$

If we have $s, s' \in \mathcal{H}_t(x)$ then Lemma 4.3.2 yields $E^\vartheta(s) \cup E^\vartheta(s') \subseteq \mathcal{H}_t(x)$. Thus we get $E^\vartheta(s \hat{+} s') \subseteq \mathcal{H}_t(x)$, which implies $s \hat{+} s' \in \mathcal{H}_t(x)$. Also recall $\hat{\omega}^s = \omega\langle s\rangle$. Given $s \in \mathcal{H}_t(x)$ we can thus infer $E^\vartheta(\hat{\omega}^s) = E^\vartheta(s) \subseteq \mathcal{H}_t(x)$ and then $\hat{\omega}^s \in \mathcal{H}_t(x)$. □

Finally, we relate our operators to the collapsing function $s \mapsto \vartheta s$:

LEMMA 4.3.5. *The following holds:*

(a) If $t < t'$ then $\mathcal{H}_t(x) \subseteq \mathcal{H}_{t'}(x)$.
(b) If $s \in \mathcal{H}_t(x)$ and $s \leq t$ then $\vartheta s \in \mathcal{H}_t(x)$.
(c) If $s \in \mathcal{H}_t(\emptyset) \cap \Omega$ and $t < t'$ then $s < \vartheta t'$.
(d) If $s, t \in \mathcal{H}_t(\emptyset)$ and $s < s'$ then $\vartheta(t \hat{+} \hat{\omega}^s) < \vartheta(t \hat{+} \hat{\omega}^{s'})$.

Concerning (c), note that $\mathcal{H}_t(\emptyset) \cap \Omega$ is an abbreviation for $\{s \in \mathcal{H}_t(\emptyset) \mid s < \Omega\}$.

PROOF. (a) A straightforward induction on n shows $\mathcal{H}^n_t(x) \subseteq \mathcal{H}^n_{t'}(x)$.
(b) The claim is immediate by the definition of $\mathcal{H}_t(x)$.
(c) The claim for $s \in \mathcal{H}^n_t(\emptyset) \cap \Omega$ is established by induction on n: For $n = 0$ it suffices to observe $\mathcal{H}^0_t(\emptyset) = \emptyset$. Now assume that we have $\vartheta s \in \mathcal{H}^{n+1}_t(\emptyset)$ because of $s \in \mathcal{H}^n_t(\emptyset)$ and $s \leq t$. By Lemma 4.3.2 we obtain $E^\vartheta(s) \subseteq \mathcal{H}^n_t(\emptyset)$, so that the induction hypothesis yields $E^\vartheta(s) <^{\text{fin}} \vartheta t'$. Together with $s \leq t < t'$ we can infer $\vartheta s < \vartheta t'$ by Lemma 2.2.7. Next, assume that $s \in \mathcal{H}^{n+1}_t(\emptyset)$ holds because of $E^\vartheta(s) \subseteq \mathcal{H}^n_t(\emptyset)$. Then the induction hypothesis yields $E^\vartheta(s) <^{\text{fin}} \vartheta t'$. Given that we have $s < \Omega$ this implies $s < \vartheta t'$, by a straightforward induction on the term s. Finally, if $s \in \mathcal{H}^{n+1}_t(\emptyset)$ holds because we have $s < s' < \Omega$ for some $s' \in \mathcal{H}^n_t(\emptyset)$, then we get $s < s' < \vartheta t'$ by the induction hypothesis.
(d) By the previous lemma we get $t \hat{+} \hat{\omega}^s \in \mathcal{H}_t(\emptyset)$. Lemma 4.2.5 provides $t \leq t \hat{+} \hat{\omega}^s$, so that part (a) resp. (b) yield $t \hat{+} \hat{\omega}^s \in \mathcal{H}_{t \hat{+} \hat{\omega}^s}(\emptyset)$ and $\vartheta(t \hat{+} \hat{\omega}^s) \in \mathcal{H}_{t \hat{+} \hat{\omega}^s}(\emptyset)$. In view of $\vartheta(t \hat{+} \hat{\omega}^s) < \Omega$ we can infer $\vartheta(t \hat{+} \hat{\omega}^s) < \vartheta(t \hat{+} \hat{\omega}^{s'})$ by part (c). □

Recall that Lemma 3.4.4 shows consistency for S^u-proofs of height below Ω. In order to extend this result to a proof P of height $o_{\langle\rangle}(P) \geq \Omega$ we will collapse P into a proof $\mathcal{C}_t P$ with $o_{\langle\rangle}(\mathcal{C}_t P) = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}) < \Omega$. This is possible under certain conditions of operator control: For example, we will ensure $o_{\langle\rangle}(n(P, a)) \in \mathcal{H}_t(\emptyset)$ to get

$$\vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,a))}) < \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}),$$

which means that monotonicity of the ordinal labels is preserved. Similarly, we will ensure $\operatorname{supp}^{\mathbf{L}}(a) \subseteq \mathcal{H}_t(\emptyset)$ in order to obtain

$$\operatorname{supp}^{\mathbf{L}}(a) <^{\text{fin}} \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}) = o_{\langle\rangle}(\mathcal{C}_t P),$$

as the side condition of a rule $r_{\langle\rangle}(\mathcal{C}_t P) = (\bigvee, \varphi, a)$ demands. To track the required conditions we associate each proof P with an operator $x \mapsto \mathcal{H}_{h_0(P)}(h_1(P) \cup x)$:

Definition 4.3.6. We construct functions

$$h_0 : \text{“}S^u\text{-codes”} \to \vartheta(S^u_{\mathrm{BH}(S^u)}),$$
$$h_1 : \text{“}S^u\text{-codes”} \to [\vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega]^{<\omega}$$

by recursion over the S^u-codes considered so far:

$$h_0(P_S\sigma) = 0, \qquad h_1(P_S\sigma) = \operatorname{supp}^{\mathrm{S}}(\sigma),$$
$$h_0(\mathcal{I}_{\varphi,a}P) = h_0(P), \qquad h_1(\mathcal{I}_{\varphi,a}P) = h_1(P) \cup \operatorname{supp}^{\mathbf{L}}(a),$$
$$h_0(\mathcal{R}_\varphi P_0 P_1) = \max\{h_0(P_0), h_0(P_1)\}, \qquad h_1(\mathcal{R}_\varphi P_0 P_1) = h_1(P_0) \cup h_1(P_1),$$
$$h_0(\mathcal{E}P) = h_0(P), \qquad h_1(\mathcal{E}P) = h_1(P).$$

We will abbreviate $\mathcal{H}_P(x) = \mathcal{H}_{h_0(P)}(h_1(P) \cup x)$.

As before, the isomorphism $i_{\mathrm{BH}(S^u)} : \mathrm{BH}(S^u) \to \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ is left implicit. This means that $\operatorname{supp}^{\mathrm{S}}(\sigma)$ and $\operatorname{supp}^{\mathbf{L}}(a)$ refer to the sets $[i_{\mathrm{BH}(S^u)}]^{<\omega}(\operatorname{supp}^{\mathrm{S}}_{\mathrm{BH}(S^u)}(\sigma))$ and $[i_{\mathrm{BH}(S^u)}]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_{\mathrm{BH}(S^u)}(a))$, respectively. It is straightforward to see that the functions $P \mapsto h_i(P)$ for $i = 0, 1$ are primitive recursive. Whenever we extend the system of S^u-codes by a new function symbol, we will add a recursive clause to the definition of the functions h_i. The assignment of operators to proofs must satisfy certain local correctness conditions. To express these conditions, we associate each $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula φ with a (generally infinite) set $\operatorname{supp}(\varphi) \subseteq \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$, putting

$$\operatorname{supp}(\varphi) = \begin{cases} \emptyset & \text{if } \iota(\varphi) = \mathbf{L}^u_{\mathrm{BH}(S^u)}, \\ \bigcup_{a \in \iota(\varphi)} \operatorname{supp}^{\mathbf{L}}(a) & \text{otherwise.} \end{cases}$$

In view of Lemma 3.1.13 we have $\iota(\varphi) = \mathbf{L}^u_{\mathrm{BH}(S^u)}$ if and only if φ begins with an unbounded quantifier. We will use $\operatorname{supp}(\varphi)$ to keep track of the bounded universal formulas $\varphi \equiv \forall_{x \in a} \psi(x)$ that occur in a proof. Since $\operatorname{supp}(\varphi) = \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ would break operator control, we have no choice but to set $\operatorname{supp}(\varphi) = \emptyset$ in the case of an unbounded formula. Indeed, it is well known that collapsing does not apply to proofs of unbounded universal statements. We can now formulate local correctness conditions for our controlling operators (cf. [**10**, Section 3]):

LEMMA 4.3.7. *The following holds for any S^u-code P:*

(H1) We have $o_{\langle\rangle}(P) \in \mathcal{H}_P(\emptyset)$.

(H2) If $r_{\langle\rangle}(P)$ *is of the form* $(\bigwedge, \varphi)$ *then we have* $\operatorname{supp}(\varphi) \subseteq \mathcal{H}_P(\emptyset)$. *If* $r_{\langle\rangle}(P)$ *is of the form* $(\bigvee, \varphi, a)$ *or* (Rep, a) *then we have* $\operatorname{supp}^{\mathbf{L}}(a) \subseteq \mathcal{H}_P(\emptyset)$.

(H3) We have $h_0(n(P, a)) \leq h_0(P)$ *and* $h_1(n(P, a)) \subseteq \mathcal{H}_P(\operatorname{supp}^{\mathbf{L}}(a))$ *for any element* $a \in \iota(r_{\langle\rangle}(P))$.

PROOF. As explained in Remark 4.1.6, all local correctness conditions are established by induction over S^u-codes. For a basic S^u-code $P = P_S\sigma$ we observe

$$h_0(n(P_S\sigma, a)) = h_0(P_S\sigma^\frown a) = 0 = h_0(P_S\sigma)$$

and

$$\begin{aligned} h_1(n(P_S\sigma, a)) &= h_1(P_S\sigma^\frown a) = \operatorname{supp}^{\mathrm{S}}(\sigma^\frown a) = \operatorname{supp}^{\mathrm{S}}(\sigma) \cup \operatorname{supp}^{\mathbf{L}}(a) = \\ &= h_1(P_S\sigma) \cup \operatorname{supp}^{\mathbf{L}}(a) \subseteq \mathcal{H}_{h_0(P_S\sigma)}(h_1(P_S\sigma) \cup \operatorname{supp}^{\mathbf{L}}(a)) = \mathcal{H}_{P_S\sigma}(\operatorname{supp}^{\mathbf{L}}(a)), \end{aligned}$$

as required by condition (H3). To verify conditions (H1) and (H2) we distinguish three cases: First, assume that σ does not lie in the S^u-proof $P_S = (P_S, l_S, o_S, r_S)$ from Proposition 3.4.8. Then we have $o_{\langle\rangle}(P_S\sigma) = 0$ and $r_{\langle\rangle}(P_S\sigma) = (\mathrm{True}, 0 = 0)$, so conditions (H1) and (H2) are immediate. Next, assume that σ lies in the proof P_S but not in the search tree $S^u_{\mathrm{BH}(S^u)} \subseteq P_S$. This means that we have $\sigma = \sigma_0{}^\frown 1^\frown\tau$, where $\sigma_0 \in S^u_{\mathrm{BH}(S^u)}$ has even length $\operatorname{len}(\sigma_0) = 2n$ and τ is a node in the S^u-proof $P_n = (P_n, l_n, o_n, r_n)$ from Lemma 3.4.5 resp. Lemma 3.4.6. The ordinal label is computed as

$$o_{\langle\rangle}(P_S\sigma) = o_S(\sigma) = o_n(\tau),$$

and similarly we have $r_{\langle\rangle}(P_S\sigma) = r_n(\tau)$. For $n = 0$ the interesting nodes are those of the form $\tau = \langle a\rangle$, which are labelled by

$$\begin{aligned} &o_0(\langle a\rangle) = s_a + 1 = s_a \hat{+} \hat{\omega}^0, && \text{with } s_a = \max(\{s+1 \mid s \in \operatorname{supp}^{\mathbf{L}}(a)\} \cup \{0\}), \\ &r_0(\langle a\rangle) = (\bigvee, \exists_y\, y = a \cup \{a\}, b_a), && \text{with } b_a \equiv \{z \in \mathbf{L}^u_{s_a} \mid z \in a \vee z = a\}. \end{aligned}$$

It is straightforward to observe

$$\operatorname{supp}^{\mathbf{L}}(a) = \operatorname{supp}^{\mathrm{S}}(\tau) \subseteq \operatorname{supp}^{\mathrm{S}}(\sigma) \subseteq \mathcal{H}_{h_0(P_S\sigma)}(\operatorname{supp}^{\mathrm{S}}(\sigma)) = \mathcal{H}_{P_S\sigma}(\emptyset).$$

By Lemma 4.3.4 we get $s+1 = s \hat{+} \hat{\omega}^0 \in \mathcal{H}_{P_S\sigma}(\emptyset)$ for any $s \in \operatorname{supp}^{\mathbf{L}}(a)$. This implies $s_a \in \mathcal{H}_{P_S\sigma}(\emptyset)$ and then

$$o_{\langle\rangle}(P_S\sigma) = o_0(\langle a\rangle) = s_a \hat{+} \hat{\omega}^0 \in \mathcal{H}_{P_S\sigma}(\emptyset),$$

as required by condition (H1). We also obtain

$$\operatorname{supp}^{\mathbf{L}}(b_a) = \operatorname{supp}^{\mathbf{L}}(a) \cup \{s_a\} \subseteq \mathcal{H}_{P_S\sigma}(\emptyset),$$

as needed for condition (H2). For $n > 0$ all ordinal labels in P_n are of the form $\Omega + k = \omega^\Omega + \omega^0 + \cdots + \omega^0$ (with k summands ω^0), $\Omega + \omega = \omega^\Omega + \omega^{\omega^0}$ or 0. Thus we have $E^\vartheta(o_n(\tau)) = \emptyset$ for all $\tau \in P_n$, which implies $o_n(\tau) \in \mathcal{H}_t(x)$ for arbitrary values of t and x. In particular we get

$$o_{\langle\rangle}(P_S\sigma) = o_n(\tau) \in \mathcal{H}_{P_S\sigma}(\emptyset),$$

as required by condition (H1). Concerning condition (H2), the most interesting node is $\tau = \langle \vec{a}, b, 0, 0, 0\rangle \in P_n$ with label

$$r_{\langle\rangle}(P_S\sigma) = r_n(\langle \vec{a}, b, 0, 0, 0\rangle) = (\textstyle\bigwedge, \forall_{x\in b}\exists_y \theta(x, y, \vec{a})).$$

We need to establish $\operatorname{supp}^{\mathbf{L}}(c) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset)$ for all $c \in \iota(\forall_{x\in b}\exists_y\theta(x,y,\vec{a}))$. If b is an $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-term of the form u_i, then c must be a term u_j with $u_j \in u_i$. According to Definition 3.1.2 we have $\operatorname{supp}^{\mathbf{L}}(u_j) = \emptyset$, which makes the claim trivial. If b is of the form $\mathbf{L}^u_s$ or of the form $\{z \in \mathbf{L}^u_s \mid \psi(z)\}$, then $c \in \iota(\forall_{x\in b}\exists_y\theta(x,y,\vec{a}))$ implies

$$\operatorname{supp}^{\mathbf{L}}(c) <^{\mathrm{fin}} s \in \operatorname{supp}^{\mathbf{L}}(b) \subseteq \operatorname{supp}^{\mathrm{S}}(\tau) \subseteq \operatorname{supp}^{\mathrm{S}}(\sigma) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset) \cap \Omega.$$

In view of Definition 4.3.1 we obtain

$$\operatorname{supp}^{\mathbf{L}}(c) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset),$$

as required. The other relevant rules are of the form

$$r_{\langle\rangle}(P_S\sigma) = r_n(\tau) = (\textstyle\bigvee, \varphi, a),$$

where a occurs in the finite sequence τ. In this case we have

$$\operatorname{supp}^{\mathbf{L}}(a) \subseteq \operatorname{supp}^{\mathrm{S}}(\tau) \subseteq \operatorname{supp}^{\mathrm{S}}(\sigma) = h_1(P_S\sigma) \subseteq \mathcal{H}_{h_0(P_S\sigma)}(h_1(P_S\sigma)) = \mathcal{H}_{P_S\sigma}(\emptyset),$$

so that condition (H2) is satisfied. It remains to consider a basic S^u-code $P_S\sigma$ where σ does lie in the search tree $S^u_{\mathrm{BH}(S^u)} \subseteq P_S$. Using Lemma 4.3.4(b) we get

$$o_{\langle\rangle}(P_S\sigma) = o_S(\sigma) = \mathfrak{E}_\sigma \in \mathcal{H}_{h_0(P_S\sigma)}(\operatorname{supp}^{\mathrm{S}}(\sigma)) = \mathcal{H}_{P_S\sigma}(\emptyset),$$

as required by condition (H1). If σ has even length then $r_{\langle\rangle}(P_S\sigma) = r_S(\sigma)$ is a cut rule and condition (H2) is void. Now assume that σ has odd length $\operatorname{len}(\sigma) = 2n+1$.

Following Definition 3.2.1 and the proof of Proposition 3.4.8, let φ be the $\pi_0(n)$-th formula of the sequent $l_S(\sigma)$. If φ is conjunctive then we have

$$r_{\langle\rangle}(P_S\sigma) = r_S(\sigma) = (\bigwedge, \varphi).$$

The proof of Proposition 3.2.2(iii) shows that φ is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}(\sigma)}$-formula. More precisely, we can write $\varphi \equiv \varphi_0[\iota_\sigma]$, where $\iota_\sigma : \operatorname{supp}^{\mathrm{S}}(\sigma) \hookrightarrow \mathrm{BH}(S^u)$ is the inclusion and φ_0 is an $\mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}(\sigma)}$-formula. In view of Definition 3.1.12 any parameter b in φ is of the form $b = \mathbf{L}^u_{\iota_\sigma}(b_0)$ with $b_0 \in \mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}(\sigma)}$. The naturality of $\operatorname{supp}^{\mathbf{L}}_{\cdot}$ yields

$$\operatorname{supp}^{\mathbf{L}}(b) = \operatorname{supp}^{\mathbf{L}}_{\mathrm{BH}(S^u)}(\mathbf{L}^u_{\iota_\sigma}(b_0)) = [\iota_\sigma]^{<\omega}(\operatorname{supp}^{\mathbf{L}}_{\operatorname{supp}^{\mathrm{S}}(\sigma)}(b_0)) \subseteq \operatorname{supp}^{\mathrm{S}}(\sigma) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset).$$

It is straightforward to deduce $\operatorname{supp}(\varphi) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset)$, as needed for condition (H2): Above, this has already been verified for $\varphi \equiv \forall_{x\in b}\exists_y\theta$. As a second example, let us consider the formula $\varphi \equiv b \notin \mathbf{L}^u_s$. For any $a \in \iota(\varphi)$ we have

$$\operatorname{supp}^{\mathbf{L}}(a) <^{\mathrm{fin}} s \in \operatorname{supp}^{\mathbf{L}}(\mathbf{L}^u_s) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset) \cap \Omega,$$

because $\mathbf{L}^u_s$ is a parameter of φ. By definition, our operators are downwards closed below Ω. Thus we see $\operatorname{supp}^{\mathbf{L}}(a) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset)$ and then

$$\operatorname{supp}(\varphi) = \textstyle\bigcup_{a\in\iota(\varphi)} \operatorname{supp}^{\mathbf{L}}(a) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset),$$

as promised. Now assume that φ, the $\pi_0(n)$-th formula of the sequent $l_S(\sigma)$, is disjunctive. Following the proof of Proposition 3.4.8 we compute

$$b = \mathrm{en}_{\mathrm{BH}(S^u)}(\operatorname{supp}^{\mathrm{S}}(\sigma \restriction \pi_1(n)), \pi_2(n))$$

and observe

$$r_{\langle\rangle}(P_S\sigma) = r_S(\sigma) = \begin{cases} (\bigvee, \varphi, b) & \text{if } b \in \iota(\varphi), \\ (\mathrm{Rep}, 0) & \text{otherwise.} \end{cases}$$

As in the proof of Proposition 3.2.2(iii) we get $b = \mathbf{L}^u_{\iota_\sigma}(b_0)$ for some $b_0 \in \mathbf{L}^u_{\operatorname{supp}^{\mathrm{S}}(\sigma)}$. Above we have seen that this implies

$$\operatorname{supp}^{\mathbf{L}}(b) \subseteq \mathcal{H}_{P_S\sigma}(\emptyset),$$

which means that condition (H2) is satisfied. Having established the claim for all basic S^u-codes, let us look at a code of the form $\mathcal{I}_{\varphi,a}P$. To prove the corresponding induction step we must deduce conditions (H1) to (H3) for $\mathcal{I}_{\varphi,a}P$ from the same conditions for P. Indeed, condition (H1) for P and Lemma 4.3.3 yield

$$o_{\langle\rangle}(\mathcal{I}_{\varphi,a}P) = o_{\langle\rangle}(P) \in \mathcal{H}_{h_0(P)}(h_1(P)) \subseteq \mathcal{H}_{h_0(P)}(h_1(P) \cup \operatorname{supp}^{\mathbf{L}}(a)) = \mathcal{H}_{\mathcal{I}_{\varphi,a}P}(\emptyset),$$

as required by condition (H1) for $\mathcal{I}_{\varphi,a}P$. Condition (H2) is verified by case distinction on the last rule of P. Let us consider the most interesting case $r_{\langle\rangle}(P)=(\bigwedge,\varphi)$: Then we have $r_{\langle\rangle}(\mathcal{I}_{\varphi,a}P)=(\mathrm{Rep},a)$, and the definition of $h_1(\mathcal{I}_{\varphi,a}P)$ ensures

$$\operatorname{supp}^{\mathbf{L}}(a)\subseteq h_1(\mathcal{I}_{\varphi,a}P)\subseteq\mathcal{H}_{\mathcal{I}_{\varphi,a}P}(\emptyset).$$

In all other cases, condition (H2) is preserved in view of $\mathcal{H}_P(\emptyset)\subseteq\mathcal{H}_{\mathcal{I}_{\varphi,a}P}(\emptyset)$. Concerning condition (H3), we note that the function symbol $\mathcal{I}_{\varphi,a}$ was only introduced for $a\in\iota(\varphi)=\iota((\bigwedge,\varphi))$. It is straightforward to see that this implies $\iota(r_{\langle\rangle}(\mathcal{I}_{\varphi,a}P))\subseteq\iota(r_{\langle\rangle}(P))$ in all possible cases. Using condition (H3) for P we can thus infer

$$h_0(n(\mathcal{I}_{\varphi,a}P,b))=h_0(\mathcal{I}_{\varphi,a}n(P,b))=h_0(n(P,b))\leq h_0(P)=h_0(\mathcal{I}_{\varphi,a}P)$$

for all $b\in\iota(r_{\langle\rangle}(\mathcal{I}_{\varphi,a}P))$. With the help of Lemma 4.3.3 we also get

$$\begin{aligned}h_1(n(\mathcal{I}_{\varphi,a}P,b))=h_1(\mathcal{I}_{\varphi,a}n(P,b))=h_1(n(P,b))\cup\operatorname{supp}^{\mathbf{L}}(a)\subseteq\\ \subseteq\mathcal{H}_{h_0(P)}(h_1(P)\cup\operatorname{supp}^{\mathbf{L}}(b)\cup\operatorname{supp}^{\mathbf{L}}(a))\subseteq\mathcal{H}_{\mathcal{I}_{\varphi,a}P}(\operatorname{supp}^{\mathbf{L}}(b)),\end{aligned}$$

as required by condition (H3) for $\mathcal{I}_{\varphi,a}P$. Next, let us consider an S^u-code of the form $\mathcal{R}_\varphi P_0P_1$. In view of $h_0(P_i)\leq h_0(\mathcal{R}_\varphi P_0P_1)$ and $h_1(P_i)\subseteq h_1(\mathcal{R}_\varphi P_0P_1)$ we get $\mathcal{H}_{P_i}(x)\subseteq\mathcal{H}_{\mathcal{R}_\varphi P_0P_1}(x)$, by Lemma 4.3.5(a) and Lemma 4.3.3. Together with condition (H1) for P_i we see $o_{\langle\rangle}(P_i)\in\mathcal{H}_{P_i}(\emptyset)\subseteq\mathcal{H}_{\mathcal{R}_\varphi P_0P_1}(\emptyset)$. Using Lemma 4.3.4(c) this implies

$$o_{\langle\rangle}(\mathcal{R}_\varphi P_0P_1)=o_{\langle\rangle}(P_1)\mathbin{\hat{+}}o_{\langle\rangle}(P_0)\in\mathcal{H}_{\mathcal{R}_\varphi P_0P_1}(\emptyset),$$

as condition (H1) demands. In view of $\mathcal{H}_{P_0}(\emptyset)\subseteq\mathcal{H}_{\mathcal{R}_\varphi P_0P_1}(\emptyset)$ condition (H2) is preserved. Condition (H3) is most interesting if we have $r_{\langle\rangle}(P_0)=(\bigvee,\neg\varphi,b)$ and thus $r_{\langle\rangle}(\mathcal{R}_\varphi P_0P_1)=(\mathrm{Cut},\varphi_b)$: Using condition (H3) for P_0 we get

$$\begin{aligned}h_0(n(\mathcal{R}_\varphi P_0P_1,0))=h_0(\mathcal{R}_\varphi n(P_0,0)P_1)=\max\{h_0(n(P_0,0)),h_0(P_1)\}\leq\\ \leq\max\{h_0(P_0),h_0(P_1)\}=h_0(\mathcal{R}_\varphi P_0P_1),\end{aligned}$$

as well as

$$h_0(n(\mathcal{R}_\varphi P_0P_1,1))=h_0(\mathcal{I}_{\varphi,b}P_1)=h_0(P_1)\leq h_0(\mathcal{R}_\varphi P_0P_1).$$

We can also compute

$$\begin{aligned}h_1(n(\mathcal{R}_\varphi P_0P_1,0))=h_1(\mathcal{R}_\varphi n(P_0,0)P_1)=h_1(n(P_0,0))\cup h_1(P_1)\subseteq\\ \subseteq\mathcal{H}_{P_0}(\operatorname{supp}^{\mathbf{L}}(0))\cup h_1(\mathcal{R}_\varphi P_0P_1)\subseteq\mathcal{H}_{\mathcal{R}_\varphi P_0P_1}(\operatorname{supp}^{\mathbf{L}}(0)).\end{aligned}$$

Crucially, condition (H2) for P_0 provides $\operatorname{supp}^{\mathbf{L}}(b) \subseteq \mathcal{H}_{P_0}(\emptyset)$. This implies

$$h_1(n(\mathcal{R}_\varphi P_0 P_1, 1)) = h_1(\mathcal{I}_{\varphi,b} P_1) = h_1(P_1) \cup \operatorname{supp}^{\mathbf{L}}(b) \subseteq \mathcal{H}_{\mathcal{R}_\varphi P_0 P_1}(\emptyset),$$

as required by condition (H3) for $\mathcal{R}_\varphi P_0 P_1$. Finally, we look at an S^u-code of the form $\mathcal{E}P$. Condition (H1) for P yields $o_{\langle\rangle}(P) \in \mathcal{H}_P(\emptyset)$. By Lemma 4.3.4(c) we get

$$o_{\langle\rangle}(\mathcal{E}P) = \hat{\omega}^{o_{\langle\rangle}(P)} \in \mathcal{H}_P(\emptyset) = \mathcal{H}_{\mathcal{E}P}(\emptyset),$$

so that condition (H1) for $\mathcal{E}P$ is satisfied. In view of $\mathcal{H}_{\mathcal{E}P}(\emptyset) = \mathcal{H}_P(\emptyset)$ we see that condition (H2) is preserved. Condition (H3) is most interesting if $r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi)$ is a cut with $\mathrm{rk}(\varphi) \geq 2$. Assuming that φ is conjunctive, condition (H3) for P yields

$$h_0(n(\mathcal{E}P, 0)) = h_0(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))) = \max\{h_0(\mathcal{E}n(P,0)), h_0(\mathcal{E}n(P,1))\} =$$
$$= \max\{h_0(n(P,0)), h_0(n(P,1))\} \leq h_0(P) = h_0(\mathcal{E}P).$$

In view of $\operatorname{supp}^{\mathbf{L}}(0) = \operatorname{supp}^{\mathbf{L}}(1) = \emptyset$ we can also compute

$$h_1(n(\mathcal{E}P, 0)) = h_1(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))) = h_1(\mathcal{E}n(P,0)) \cup h_1(\mathcal{E}n(P,1)) =$$
$$h_1(n(P,0)) \cup h_1(n(P,1)) \subseteq \mathcal{H}_P(\emptyset) = \mathcal{H}_{\mathcal{E}P}(\operatorname{supp}^{\mathbf{L}}(0)),$$

as required by condition (H3) for $\mathcal{E}P$. □

Whenever we add a function symbol to the system of S^u-codes, we will extend the previous proof by the corresponding induction step. Recall that any S^u-code P yields an S^u-proof $[P] = ([P], l_P, r_P, o_P)$. Conditions (H1) and (H3) ensure that all ordinal labels in this proof are controlled by the operator $\mathcal{H}_P(\cdot)$, in the sense that we have

$$o_P(\sigma) \in \mathcal{H}_P(\operatorname{supp}^{\mathrm{S}}(\sigma))$$

for any sequence $\sigma \in [P]$. To see that this is the case, note that condition (H3) implies $\mathcal{H}_{n(P,a)}(x) \subseteq \mathcal{H}_P(\operatorname{supp}^{\mathbf{L}}(a) \cup x)$, using Lemma 4.3.5(a) and Lemma 4.3.3. By induction over σ we get $\mathcal{H}_{\bar{n}(P,\sigma)}(x) \subseteq \mathcal{H}_P(\operatorname{supp}^{\mathrm{S}}(\sigma) \cup x)$. Now condition (H1) for $\bar{n}(P,a)$ yields $o_P(\sigma) = o_{\langle\rangle}(\bar{n}(P,\sigma)) \in \mathcal{H}_{\bar{n}(P,\sigma)}(\emptyset) \subseteq \mathcal{H}_P(\operatorname{supp}^{\mathrm{S}}(\sigma))$, as promised. The parameters of the rule $r_P(\sigma)$ are controlled as well, using condition (H2).

To conclude this section we discuss a further complication: The last rule $r_{\langle\rangle}(\mathcal{C}_t P)$ of the collapsed proof $\mathcal{C}_t P$ will depend on the ordinal height $o_{\langle\rangle}(n(P,a))$ of the immediate subproof $n(P,a)$ of P. This is problematic because we define $r_{\langle\rangle}$ and $o_{\langle\rangle}$ by simultaneous recursion over terms and because $n(P,a)$ is not a subterm of P (cf. Remark 4.1.6). In order to solve this problem we add an auxiliary function $(P,a) \mapsto o_{\langle a\rangle}(P)$ to our simultaneous recursion. Since P is a subterm of $\mathcal{C}_t P$, the rule $r_{\langle\rangle}(\mathcal{C}_t P)$ is allowed to depend on the value $o_{\langle a\rangle}(P)$. We will ensure $o_{\langle a\rangle}(P) = o_{\langle\rangle}(n(P,a))$ in all relevant cases. First, let us define the (set-sized) function $a \mapsto o_{\langle a\rangle}(P)$ by recursion over the S^u-codes introduced so far:

DEFINITION 4.3.8. We construct a function

$$o_{\langle\cdot\rangle}(\cdot) : \text{“}S^u\text{-codes”} \times \mathbf{L}^u_{\mathrm{BH}(S^u)} \to \vartheta(S^u_{\mathrm{BH}(S^u)}), \qquad (P,a) \mapsto o_{\langle a\rangle}(P)$$

by recursion over S^u-codes, setting

$$\begin{aligned}
o_{\langle a\rangle}(P_S\sigma) &= o_{\langle\rangle}(P_S\,\sigma^\frown a),\\
o_{\langle a\rangle}(\mathcal{I}_{\varphi,b}P) &= o_{\langle a\rangle}(P),\\
o_{\langle a\rangle}(\mathcal{R}_\varphi P_0 P_1) &= \begin{cases} o_{\langle\rangle}(P_1) & \text{if } r_{\langle\rangle}(P_0) = (\bigvee, \neg\varphi, b) \text{ and } a = 1,\\ o_{\langle\rangle}(P_1) \hat{+}\, o_{\langle a\rangle}(P_0) & \text{otherwise,}\end{cases}\\
o_{\langle a\rangle}(\mathcal{E}P) &= \begin{cases} \hat{\omega}^{o_{\langle 1\rangle}(P)} \hat{+}\, \hat{\omega}^{o_{\langle 0\rangle}(P)} & \text{if } r_{\langle\rangle}(P) = (\mathrm{Cut},\varphi) \text{ where } \varphi \text{ is conjunctive and } \mathrm{rk}(\varphi) \geq 2,\\ \hat{\omega}^{o_{\langle 0\rangle}(P)} \hat{+}\, \hat{\omega}^{o_{\langle 1\rangle}(P)} & \text{if } r_{\langle\rangle}(P) = (\mathrm{Cut},\varphi) \text{ where } \varphi \text{ is disjunctive and } \mathrm{rk}(\varphi) \geq 2,\\ \hat{\omega}^{o_{\langle a\rangle}(P)} & \text{otherwise.}\end{cases}
\end{aligned}$$

The required properties of $o_{\langle a\rangle}(\cdot)$ are ensured by local correctness conditions:

LEMMA 4.3.9. *The following conditions hold for any S^u-code P:*

(N1) We have $o_{\langle a\rangle}(P) = o_{\langle\rangle}(n(P,a))$ for all $a \in \iota(r_{\langle\rangle}(P))$.

(N2) We have $o_{\langle a\rangle}(P) \in \mathcal{H}_{n(P,a)}(\emptyset)$ for all $a \in \iota(r_{\langle\rangle}(P))$.

PROOF. The two conditions are established by induction over S^u-codes. For a basic S^u-code $P_S\sigma$ we have

$$o_{\langle a\rangle}(P_S\sigma) = o_{\langle\rangle}(P_S\,\sigma^\frown a) = o_{\langle\rangle}(n(P_S\,\sigma, a))$$

for any $a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$. By condition (H1) for the basic S^u-code $P_S\sigma^\frown a$ we also get

$$o_{\langle a\rangle}(P_S\sigma) = o_{\langle\rangle}(P_S\,\sigma^\frown a) \in \mathcal{H}_{P_S\sigma^\frown a}(\emptyset) = \mathcal{H}_{n(P_S\sigma,a)}(\emptyset).$$

For an S^u-code of the form $\mathcal{I}_{\varphi,b}P$ we observe $\iota(r_{\langle\rangle}(\mathcal{I}_{\varphi,b}P)) \subseteq \iota(r_{\langle\rangle}(P))$. Thus condition (N1) for P implies

$$o_{\langle a\rangle}(\mathcal{I}_{\varphi,b}P) = o_{\langle a\rangle}(P) = o_{\langle\rangle}(n(P,a)) = o_{\langle\rangle}(\mathcal{I}_{\varphi,b}n(P,a)) = o_{\langle\rangle}(n(\mathcal{I}_{\varphi,b}P,a))$$

for any $a \in \iota(r_{\langle\rangle}(\mathcal{I}_{\varphi,b}P))$, as condition (N1) for $\mathcal{I}_{\varphi,b}P$ demands. By condition (N2) for P we also obtain

$$o_{\langle a\rangle}(\mathcal{I}_{\varphi,b}P)=o_{\langle a\rangle}(P)\in\mathcal{H}_{n(P,a)}(\emptyset)\subseteq$$
$$\subseteq\mathcal{H}_{n(P,a)}(\operatorname{supp}^{\mathbf{L}}(b))=\mathcal{H}_{\mathcal{I}_{\varphi,b}n(P,a)}(\emptyset)=\mathcal{H}_{n(\mathcal{I}_{\varphi,b}P,a)}(\emptyset).$$

For an S^u-code $\mathcal{R}_\varphi P_0P_1$ we consider the crucial case of a rule $r_{\langle\rangle}(P_0)=(\bigvee,\neg\varphi,b)$, which yields $r_{\langle\rangle}(\mathcal{R}_\varphi P_0P_1)=(\mathrm{Cut},\varphi_b)$. Using condition (N1) for P_0 we obtain

$$o_{\langle 0\rangle}(\mathcal{R}_\varphi P_0P_1)=o_{\langle\rangle}(P_1)\hat{+}o_{\langle 0\rangle}(P_0)=o_{\langle\rangle}(P_1)\hat{+}o_{\langle\rangle}(n(P_0,0))=$$
$$=o_{\langle\rangle}(\mathcal{R}_\varphi n(P_0,0)P_1)=o_{\langle\rangle}(n(\mathcal{R}_\varphi P_0P_1,0)),$$

as well as

$$o_{\langle 1\rangle}(\mathcal{R}_\varphi P_0P_1)=o_{\langle\rangle}(P_1)=o_{\langle\rangle}(\mathcal{I}_{\varphi,b}P_1)=o_{\langle\rangle}(n(\mathcal{R}_\varphi P_0P_1,1)).$$

By condition (H1) for P_1 we obtain $o_{\langle\rangle}(P_1)\in\mathcal{H}_{P_1}(\emptyset)\subseteq\mathcal{H}_{\mathcal{R}_\varphi n(P_0,0)P_1}(\emptyset)$, and condition (N2) for P_0 provides $o_{\langle 0\rangle}(P_0)\in\mathcal{H}_{n(P_0,0)}(\emptyset)\subseteq\mathcal{H}_{\mathcal{R}_\varphi n(P_0,0)P_1}(\emptyset)$. Together with Lemma 4.3.4(c) we can conclude

$$o_{\langle 0\rangle}(\mathcal{R}_\varphi P_0P_1)=o_{\langle\rangle}(P_1)\hat{+}o_{\langle 0\rangle}(P_0)\in\mathcal{H}_{\mathcal{R}_\varphi n(P_0,0)P_1}(\emptyset)=\mathcal{H}_{n(\mathcal{R}_\varphi P_0P_1,0)}(\emptyset).$$

Also by condition (H1) for P_1 we obtain

$$o_{\langle 1\rangle}(\mathcal{R}_\varphi P_0P_1)=o_{\langle\rangle}(P_1)\in\mathcal{H}_{P_1}(\emptyset)\subseteq\mathcal{H}_{\mathcal{I}_{\varphi,b}P_1}(\emptyset)=\mathcal{H}_{n(\mathcal{R}_\varphi P_0P_1,1)}(\emptyset),$$

as required by condition (N2) for $\mathcal{R}_\varphi P_0P_1$. Finally, we consider an S^u-code of the form $\mathcal{E}P$. Let us look at the crucial case of a rule $r_{\langle\rangle}(P)=(\mathrm{Cut},\varphi)$ with $\mathrm{rk}(\varphi)\geq 2$, which yields $r_{\langle\rangle}(\mathcal{E}P)=(\mathrm{Rep},0)$. Assuming that φ is conjunctive, we can use condition (N1) for P to deduce

$$o_{\langle 0\rangle}(\mathcal{E}P)=\hat{\omega}^{o_{\langle 1\rangle}(P)}\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)}=\hat{\omega}^{o_{\langle\rangle}(n(P,1))}\hat{+}\hat{\omega}^{o_{\langle\rangle}(n(P,0))}=$$
$$=o_{\langle\rangle}(\mathcal{E}n(P,1))\hat{+}o_{\langle\rangle}(\mathcal{E}n(P,0))=o_{\langle\rangle}(\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1)))=o_{\langle\rangle}(n(\mathcal{E}P,0)).$$

For $i=0,1$, condition (N2) for P implies

$$o_{\langle i\rangle}(P)\in\mathcal{H}_{n(P,i)}(\emptyset)=\mathcal{H}_{\mathcal{E}n(P,i)}(\emptyset)\subseteq\mathcal{H}_{\mathcal{R}_\varphi(\mathcal{E}n(P,0))(\mathcal{E}n(P,1))}(\emptyset)=\mathcal{H}_{n(\mathcal{E}P,0)}(\emptyset).$$

Together with Lemma 4.3.4(c) we obtain

$$o_{\langle 0\rangle}(\mathcal{E}P)=\hat{\omega}^{o_{\langle 1\rangle}(P)}\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)}\in\mathcal{H}_{n(\mathcal{E}P,0)}(\emptyset),$$

as required by condition (N2) for $\mathcal{E}P$. □

Whenever we add a function symbol to the system of S^u-codes, we will extend Definition 4.3.8 and Lemma 4.3.9 by a recursive clause and a proof of the corresponding induction step. Condition (N2) for P coincides with condition (H1) for $n(P,a)$, but the two conditions become available at different stages of our simultaneous induction. More specifically, we will need condition (N2) for P in order to ensure condition (L) for $\mathcal{C}_tP$. Condition (H1) for $n(P,a)$ has not been established at that point of the induction, since $n(P,a)$ is not a subterm of P. Let us point out that condition (N1) may — and will — fail for $a \notin \iota(r_{\langle\rangle}(P))$. We also remark that Buchholz [**11**, Section 5] follows a different approach, which avoids the redundancy that is inherent to the function $o_{\langle a\rangle}(\cdot)$: He begins by defining the function $o_{\langle\rangle}$ on all S^u-codes, observing that it does not depend on any of the other functions that are part of our simultaneous recursion. Only afterwards, these other functions are defined by a separate recursion. Of course, the rule $r_{\langle\rangle}(\mathcal{C}_tP)$ may now depend on the ordinal $o_{\langle\rangle}(n(P,a))$. Similarly, condition (H1) can be verified in a separate induction. Afterwards, condition (H1) for $n(P,a)$ may be used in order to establish condition (L) for $\mathcal{C}_tP$. This is a viable alternative, which avoids some redundancy. On the other hand it makes the structure of the proof yet more complicated (in view of Remark 4.1.6 it is already somewhat challenging to keep track of the official proof). We have chosen the approach via the function $o_{\langle a\rangle}(\cdot)$ in order to avoid this.

4.4. Collapsing

In this section we present the collapsing construction for S^u-proofs. Under suitable conditions, it transforms a proof of ordinal height above Ω into a proof with the same end-sequent and height below Ω. Lemma 3.4.4 tells us that the latter cannot prove a contradiction, so neither does the former. In this way, the consistency result is extended to proofs with height above Ω. This is the last piece that we need in order to establish Theorem 4.4.6 and Corollary 4.4.7, the main results of the present thesis. They will be proved at the end of the section.

Recall that we write φ^a for the relativization of the $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula φ to the term $a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$: To obtain φ^a from φ one replaces each occurrence of an unbounded quantifier $\exists_x\cdot$ or $\forall_x\cdot$ by the bounded quantifier $\exists_{x\in a}\cdot$ or $\exists_{x\in a}\cdot$, respectively. For $s \in \mathrm{BH}(S^u)$ we write φ^s rather than $\varphi^{\mathbf{L}^u_s}$. As before, a subscript φ_a refers to the verification calculus of Definition 3.1.13. Also recall that a $\Sigma(\mathbf{L}^u_{\mathrm{BH}(S^u)})$-formula (short: Σ-formula) is an $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula which does not contain any unbounded universal quantifier. Using this terminology we can describe a proof transformation known as boundedness (cf. the "Begrenzungslemma" in [**39**, Section 6]):

LEMMA 4.4.1. *For any* Σ*-formula* φ *and any* $s \in \mathrm{BH}(S^u)$ *we can extend the system of* S^u*-codes by a unary function symbol* $\mathcal{B}^{\Sigma}_{\varphi,s}$*, such that we have*

$$\begin{aligned} l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P) &= \begin{cases} (l_{\langle\rangle}(P)\backslash\{\varphi\}) \cup \{\varphi^s\} & \text{if } o_{\langle\rangle}(P) \leq s, \\ l_{\langle\rangle}(P) & \text{otherwise}, \end{cases} \\ o_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P) &= o_{\langle\rangle}(P), \\ d(\mathcal{B}^{\Sigma}_{\varphi,s}P) &= d(P), \\ h_0(\mathcal{B}^{\Sigma}_{\varphi,s}P) &= h_0(P), \\ h_1(\mathcal{B}^{\Sigma}_{\varphi,s}P) &= h_1(P) \end{aligned}$$

for any S^u*-code* P.

As before, the isomorphism $i_{\mathrm{BH}(S^u)} : \mathrm{BH}(S^u) \to \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ is left implicit. The condition in the case distinction should officially read $o_{\langle\rangle}(P) \leq i_{\mathrm{BH}(S^u)}(s)$.

PROOF. We begin with an informal description in terms of transfinite recursion: In the most interesting case, $\varphi \equiv \exists_x \psi(x)$ is an existential formula introduced by the last rule $(\bigvee, \exists_x\psi(x), b)$ of P. Thus the end-sequent of P has the form $\Gamma, \exists_x\psi(x)$ and the immediate subproof $n(P,0)$ establishes the sequent $\Gamma, \exists_x\psi(x), \psi(b)$. Recursively we may transform $n(P,0)$ into a proof $\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0)$ of $\Gamma, \exists_{x\in\mathbf{L}^u_s}\psi^s(x), \psi(b)$, and then into a proof $\mathcal{B}^{\Sigma}_{\psi(b),s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0)$ of $\Gamma, \exists_{x\in\mathbf{L}^u_s}\psi^s(x), \psi^s(b)$. Crucially, the assumption $o_{\langle\rangle}(P) \leq s$ and condition (L) for P imply $\mathrm{supp}^{\mathbf{L}}(b) <^{\mathrm{fin}} s$. In view of $\exists_{x\in\mathbf{L}^u_s}\psi^s(x) \simeq \bigvee_{\mathrm{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}s} \psi^s(a)$ we can use the rule $(\bigvee, \exists_{x\in\mathbf{L}^u_s}\psi^s(x), b)$ to deduce the desired sequent $\Gamma, \exists_{x\in\mathbf{L}^u_s}\psi^s(x)$. Officially, we have to implement this idea in terms of S^u-codes: Let us first consider the "unintended" case $o_{\langle\rangle}(P) > s$. Here we set $r_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P) = r_{\langle\rangle}(P)$ and $n(\mathcal{B}^{\Sigma}_{\varphi,s}P, a) = n(P,a)$, as well as $o_{\langle a\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P) = o_{\langle a\rangle}(P)$. Then the code $\mathcal{B}^{\Sigma}_{\varphi,s}P$ behaves as P itself, and it is straightforward to see that the local correctness conditions are preserved. In the "intended" case $o_{\langle\rangle}(P) \leq s$ we set

$$r_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P) = \begin{cases} (\bigvee, \varphi^s, b) & \text{if } r_{\langle\rangle}(P) = (\bigvee, \varphi, b), \\ (\bigwedge, \varphi^s) & \text{if } r_{\langle\rangle}(P) = (\bigwedge, \varphi), \\ r_{\langle\rangle}(P) & \text{otherwise}, \end{cases}$$

$$n(\mathcal{B}^{\Sigma}_{\varphi,s}P,a)=\begin{cases}\mathcal{B}^{\Sigma}_{\varphi_b,s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0) & \text{if } r_{\langle\rangle}(P)=(\bigvee,\varphi,b),\\ \mathcal{B}^{\Sigma}_{\varphi_a,s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,a) & \text{if } r_{\langle\rangle}(P)=(\bigwedge,\varphi) \text{ and } a\in\iota(\varphi),\\ \mathcal{B}^{\Sigma}_{\varphi,s}n(P,a) & \text{otherwise,}\end{cases}$$

$$o_{\langle a\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P)=o_{\langle a\rangle}(P).$$

Local correctness is verified by case distinction over the last rule of P. Let us first assume $r_{\langle\rangle}(P)=(\bigvee,\varphi,b)$: Then φ is a disjunctive formula with $b\in\iota(\varphi)$, and condition (L) for P provides $\operatorname{supp}^{\mathbf{L}}(b)<^{\text{fin}}o_{\langle\rangle}(P)\leq s$. Based on these facts one can verify that φ^s is disjunctive, that we have $b\in\iota(\varphi^s)$ and $(\varphi^s)_b\equiv(\varphi_b)^s$, and that φ_b is a Σ-formula. This means that $r_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P)=(\bigvee,\varphi^s,b)$ is an S^u-rule and that $n(\mathcal{B}^{\Sigma}_{\varphi,s}P,a)=\mathcal{B}^{\Sigma}_{\varphi_b,s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0)$ is an S^u-code. To establish condition (L) for $\mathcal{B}^{\Sigma}_{\varphi,s}P$ we first observe that $\varphi^s\in l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P)$ holds by definition. By condition (L) for P we obtain $o_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0))=o_{\langle\rangle}(n(P,0))<o_{\langle\rangle}(P)\leq s$, so that $\mathcal{B}^{\Sigma}_{\varphi_b,s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0)$ and $\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0)$ fall under the intended case. We can thus compute

$$l_{\langle\rangle}(n(\mathcal{B}^{\Sigma}_{\varphi,s}P,0))=l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi_b,s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,0))\subseteq l_{\langle\rangle}(n(P,0))\backslash\{\varphi,\varphi_b\}\cup\{\varphi^s,\varphi_b^s\}\subseteq$$
$$\subseteq(l_{\langle\rangle}(P)\cup\{\varphi_b\})\backslash\{\varphi,\varphi_b\}\cup\{\varphi^s,\varphi_b^s\}\subseteq l_{\langle\rangle}(P)\backslash\{\varphi\}\cup\{\varphi^s,\varphi_b^s\}=l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P)\cup\{\varphi_b^s\},$$

as required by condition (L) for $\mathcal{B}^{\Sigma}_{\varphi,s}P$. The other local correctness conditions follow from the same conditions for P, since $\mathcal{B}^{\Sigma}_{\cdot}$ does not change the ordinal height, the cut rank or the controlling operator. We also look at the case $r_{\langle\rangle}(P)=(\bigwedge,\varphi)$: Note that the conjunctive formula φ cannot begin with an unbounded universal quantifier, as it is a Σ-formula. Based on this fact one can check that φ^s is conjunctive, that we have $\iota(\varphi^s)=\iota(\varphi)$ and $(\varphi^s)_a\equiv(\varphi_a)^s$ for all $a\in\iota(\varphi)$, and that φ_a is a Σ-formula. In particular this implies that $r_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P)=(\bigwedge,\varphi^s)$ is an S^u-rule and that $\mathcal{B}^{\Sigma}_{\varphi_a,s}$ is one of the new function symbols. For $a\in\iota(\varphi^s)=\iota(\varphi)$ we get $o_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}n(P,a))=o_{\langle\rangle}(n(P,a))<o_{\langle\rangle}(P)\leq s$ by condition (L) for P. Similarly to the previous case we can infer

$$l_{\langle\rangle}(n(\mathcal{B}^{\Sigma}_{\varphi,s}P,a))=l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi_a,s}\mathcal{B}^{\Sigma}_{\varphi,s}n(P,a))\subseteq l_{\langle\rangle}(n(P,a))\backslash\{\varphi,\varphi_a\}\cup\{\varphi^s,\varphi_a^s\}\subseteq$$
$$\subseteq l_{\langle\rangle}(P)\backslash\{\varphi\}\cup\{\varphi^s,\varphi_a^s\}=l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\varphi,s}P)\cup\{\varphi_a^s\},$$

as required by condition (L) for $\mathcal{B}^{\Sigma}_{\varphi,s}P$. To see that condition (H2) is preserved, note that $\iota(\varphi)=\iota(\varphi^s)$ implies

$$\operatorname{supp}(\varphi^s)=\operatorname{supp}(\varphi)\subseteq\mathcal{H}_P(\emptyset)=\mathcal{H}_{\mathcal{B}^{\Sigma}_{\varphi,s}P}(\emptyset).$$

The remaining conditions and cases are straightforward. Note that $r_{\langle\rangle}(P) = (\mathrm{Ref}, \cdot)$ is impossible, since $o_{\langle\rangle}(P) \leq s \in \mathrm{BH}(S^u) \cong \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ means $o_{\langle\rangle}(P) < \Omega$. $\square$

To complement boundedness, we define a variant of inversion (cf. Lemma 4.1.7), which relativizes unbounded universal quantifiers. In Section 4.2 we have seen that any formula ψ of rank $\mathrm{rk}(\psi) = 0$ is bounded. Furthermore, $\mathrm{rk}(\psi) = 1$ implies that ψ is of the form $\exists_x \theta$ or $\forall_x \theta$, where $\theta \equiv \theta(x)$ is a bounded formula. If we also know that ψ is conjunctive, then we must have $\psi \equiv \forall_x \theta$. This explains the formulation of the following result:

LEMMA 4.4.2. *For any conjunctive $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula ψ with $\mathrm{rk}(\psi) \leq 1$ and any $s \in \mathrm{BH}(S^u)$ we can extend the system of S^u-codes by a unary function symbol $\mathcal{B}^\Pi_{\psi,s}$, such that we have*

$$\begin{aligned} l_{\langle\rangle}(\mathcal{B}^\Pi_{\psi,s} P) &= (l_{\langle\rangle}(P) \backslash \{\psi\}) \cup \{\psi^s\}, \\ o_{\langle\rangle}(\mathcal{B}^\Pi_{\psi,s} P) &= o_{\langle\rangle}(P), \\ d(\mathcal{B}^\Pi_{\psi,s} P) &= d(P), \\ h_0(\mathcal{B}^\Pi_{\psi,s} P) &= h_0(P), \\ h_1(\mathcal{B}^\Pi_{\psi,s} P) &= h_1(P) \cup \{s\} \end{aligned}$$

for any S^u-code P.

In fact, we could restrict the universal quantifiers in any $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula ψ. The case $\mathrm{rk}(\psi) \leq 1$ is somewhat simpler and will be sufficient for our purpose.

PROOF. As before, we begin with an informal description in terms of transfinite recursion: In the crucial case, the formula $\psi \equiv \forall_x \theta(x)$ is introduced by the last rule $r_{\langle\rangle}(P) = (\bigwedge, \psi)$ of P. Thus the end-sequent of P has the form $\Gamma, \forall_x \theta(x)$ and the immediate subproofs $n(P, a)$ deduce $\Gamma, \forall_x \theta(x), \theta(a)$ for all $a \in \mathbf{L}^u_{\mathrm{BH}(S^u)}$. Recursively we obtain proofs $\mathcal{B}^\Pi_{\psi,s} n(P, a)$ of $\Gamma, \forall_{x \in \mathbf{L}^u_s} \theta(x), \theta(a)$. In view of

$$\psi^s \equiv \forall_{x \in \mathbf{L}^u_s} \theta(x) \simeq \textstyle\bigwedge_{\mathrm{supp}^{\mathbf{L}}(a) <^{\mathrm{fin}} s} \theta(a)$$

we can use a rule $(\bigwedge, \psi^s)$ to deduce the desired sequent $\Gamma, \forall_{x \in \mathbf{L}^u_s} \theta(x)$. Observe that the premises $\Gamma, \forall_{x \in \mathbf{L}^u_s} \theta(x), \theta(a)$ with $\mathrm{supp}^{\mathbf{L}}(a) \not<^{\mathrm{fin}} s$ are simply discarded. In terms of S^u-codes, this idea is implemented by the clauses

$$r_{\langle\rangle}(\mathcal{B}^\Pi_{\psi,s} P) = \begin{cases} (\bigwedge, \psi^s) & \text{if } r_{\langle\rangle}(P) = (\bigwedge, \psi), \\ r_{\langle\rangle}(P) & \text{otherwise,} \end{cases}$$

$$n(\mathcal{B}^{\Pi}_{\psi,s}P,a)=\mathcal{B}^{\Pi}_{\psi,s}n(P,a),$$
$$o_{\langle a\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P)=o_{\langle a\rangle}(P).$$

To complete the proof we must deduce the local correctness conditions for $\mathcal{B}^{\Pi}_{\psi,s}P$ from the same conditions for P. This is straightforward in case $\mathrm{rk}(\psi)=0$: Then the formula ψ is bounded, we have $\psi^s\equiv\psi$, and $\mathcal{B}^{\Pi}_{\psi,s}P$ behaves similarly to P. Now assume that we have $\mathrm{rk}(\psi)=1$. As observed above this means that ψ is of the form $\forall_x\theta$, where $\theta\equiv\theta(x)$ is a bounded formula. Thus we have $\psi^s\equiv\forall_{x\in\mathbf{L}^u_s}\theta$ and

$$\iota(\psi^s)=\{a\in\mathbf{L}^u_{\mathrm{BH}(S^u)}\mid\mathrm{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}s\}\subseteq\mathbf{L}^u_{\mathrm{BH}(S^u)}=\iota(\psi).$$

In particular we see that $\iota(r_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P))\subseteq\iota(r_{\langle\rangle}(P))$ holds in all cases. In view of this fact it is straightforward to verify that conditions (C2), (H3), (N1) and (N2) are preserved. The same is true for conditions (C1) and (H1). The remaining conditions (L) and (H2) are established by case distinction over the last rule of P. Let us look at the crucial case $r_{\langle\rangle}(P)=(\bigwedge,\psi)$: Concerning condition (L) we note that $\psi^s\in l_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P)$ holds by definition. For $a\in\iota(\psi^s)\subseteq\iota(\psi)$ we observe $\psi_a\equiv\theta(a)\equiv(\psi^s)_a$ in order to deduce

$$\begin{aligned}l_{\langle\rangle}(n(\mathcal{B}^{\Pi}_{\psi,s}P,a))&=l_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}n(P,a))=l_{\langle\rangle}(n(P,a))\backslash\{\psi\}\cup\{\psi^s\}\subseteq\\&\subseteq(l_{\langle\rangle}(P)\cup\{\psi_a\})\backslash\{\psi\}\cup\{\psi^s\}\subseteq l_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P)\cup\{\psi_a\}=l_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P)\cup\{(\psi^s)_a\},\end{aligned}$$

as condition (L) demands for the rule $r_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P)=(\bigwedge,\psi^s)$. We can also compute

$$o_{\langle\rangle}(n(\mathcal{B}^{\Pi}_{\psi,s}P,a))=o_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}n(P,a))=o_{\langle\rangle}(n(P,a))<o_{\langle\rangle}(P)=o_{\langle\rangle}(\mathcal{B}^{\Pi}_{\psi,s}P).$$

Condition (H2) for P provides $\mathrm{supp}(\psi)\subseteq\mathcal{H}_P(\emptyset)$. However, this inclusion turns out to be trivial, since $\iota(\psi)=\mathbf{L}^u_{\mathrm{BH}(S^u)}$ implies $\mathrm{supp}(\psi)=\emptyset$. Thus we have to show

$$\mathrm{supp}(\psi^s)=\bigcup_{a\in\iota(\psi^s)}\mathrm{supp}^{\mathbf{L}}(a)\subseteq\mathcal{H}_{\mathcal{B}^{\Pi}_{\psi,s}P}(\emptyset)$$

without any input from the induction hypothesis. For $a\in\iota(\psi^s)$ we have

$$\mathrm{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}s\in h_1(\mathcal{B}^{\Pi}_{\psi,s}P)\subseteq\mathcal{H}_{\mathcal{B}^{\Pi}_{\psi,s}P}(\emptyset)\cap\Omega.$$

In view of Definition 4.3.1 this yields $\mathrm{supp}^{\mathbf{L}}(a)\subseteq\mathcal{H}_{\mathcal{B}^{\Pi}_{\psi,s}P}(\emptyset)$, as needed to establish condition (H2) for $\mathcal{B}^{\Pi}_{\psi,s}P$. In all other cases it is straightforward to see that conditions (L) and (H2) are preserved. □

Finally, we have all ingredients to present the collapsing construction. Let us begin by setting out the conditions under which an S^u-proof can be collapsed to countable height:

Definition 4.4.3. Consider an ordinal term $t \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ with $t \in \mathcal{H}_t(\emptyset)$. An S^u-code P is called t-controlled if the following conditions are satisfied:

- Any formula in the end-sequent $l_{\langle\rangle}(P)$ of P is a Σ-formula.
- We have $h_0(P) \leq t$ and $h_1(P) \subseteq \mathcal{H}_t(\emptyset)$. Thus the controlling operator of P satisfies $\mathcal{H}_P(\emptyset) \subseteq \mathcal{H}_t(\emptyset)$, by Lemma 4.3.5(a) and Lemma 4.3.3(b).

If P is t-controlled and has cut rank $d(P) \leq 2$, then P is called t-collapsing.

In view of Lemma 2.2.7 we have $\vartheta s < \Omega$ for any term $s \in \vartheta(S^u_{\mathrm{BH}(S^u)})$. Thus the following result shows that a t-collapsing proof can be transformed into a proof of height below Ω (cf. the "Kollabierungslemma" in Jäger's ordinal analysis [**39**]):

Theorem 4.4.4. *For any $t \in \vartheta(S^u_{\mathrm{BH}(S^u)})$ with $t \in \mathcal{H}_t(\emptyset)$ we can extend the system of S^u-codes by a unary function symbol $\mathcal{C}_t$, such that we have*

$$l_{\langle\rangle}(\mathcal{C}_t P) = l_{\langle\rangle}(P),$$

$$o_{\langle\rangle}(\mathcal{C}_t P) = \begin{cases} \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}) & \text{if } P \text{ is } t\text{-collapsing,} \\ o_{\langle\rangle}(P) & \text{otherwise} \end{cases}$$

for any S^u-code P.

Proof. Let us begin with an informal description of the proof transformation in terms of transfinite recursion: In the crucial case, the proof P ends with a reflection rule $r_{\langle\rangle}(P) = (\mathrm{Ref}, \exists_w \forall_{x\in a} \exists_{y\in w} \theta(x,y))$. Thus the end-sequent of P has the form $\Gamma, \exists_w \forall_{x\in a} \exists_{y\in w} \theta(x,y)$ and the immediate subproof $n(P,0)$ deduces

$$\Gamma, \exists_w \forall_{x\in a} \exists_{y\in w} \theta(x,y), \forall_{x\in a} \exists_y \theta(x,y).$$

Recursively we can transform $n(P,0)$ into a proof $\mathcal{C}_t n(P,0)$ of height

$$s := \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,0))}) \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega \cong \mathrm{BH}(S^u).$$

As agreed above, the isomorphism $i_{\mathrm{BH}(S^u)} : \mathrm{BH}(S^u) \to \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega$ from Theorem 2.2.25 is left implicit. By the boundedness result from Lemma 4.4.1 we obtain a proof $\mathcal{B}^{\Sigma}_{\forall_{x\in a}\exists_y \theta(x,y),s} \mathcal{C}_t n(P,0)$ of the sequent

$$\Gamma, \exists_w \forall_{x\in a} \exists_{y\in w} \theta(x,y), \forall_{x\in a} \exists_{y\in \mathbf{L}^u_s} \theta(x,y).$$

Using $\mathbf{L}^u_s$ as an existential witness for w we can recover the end-sequent

$$\Gamma, \exists_w \forall_{x\in a} \exists_{y\in w} \theta(x,y)$$

of P. The new proof does not use the reflection rule and has height below Ω.

Still on an informal level, we also discuss the case of a cut $r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi)$: Then the immediate subproofs $n(P,0)$ and $n(P,1)$ deduce $\Gamma, \neg\varphi$ resp. Γ, φ, where Γ is the end-sequent of P. By condition (C1) for P we have $\mathrm{rk}(\varphi) < d(P) \leq 2$. Assuming that φ is conjunctive this means that φ is bounded or of the form $\forall_x \theta$, for a bounded formula θ. The problem is that φ may not be a Σ-formula, so that $n(P,1)$ is not t-collapsing. To resolve this we collapse $n(P,0)$ into a proof $\mathcal{C}_t n(P,0)$ of height $\vartheta t'$, with

$$t' := t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,0))}.$$

By boundedness we obtain a proof $\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'} \mathcal{C}_t n(P,0)$ of the sequent

$$\Gamma, \neg\varphi^{\vartheta t'}.$$

On the other hand, Lemma 4.4.2 transforms $n(P,1)$ into a proof $\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)$ of

$$\Gamma, \varphi^{\vartheta t'}.$$

Now $\varphi^{\vartheta t'}$ is a bounded formula, and we can show that $\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)$ is t'-collapsing. Thus we get a proof $\mathcal{C}_{t'} \mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)$ with height below Ω and end-sequent $\Gamma, \varphi^{\vartheta t'}$. Finally, a cut over the formula $\varphi^{\vartheta t'}$ recovers the end-sequent Γ of P.

The requirements for a formal proof are explained in Remark 4.1.6: We must state recursive clauses which define the values $l_{\langle\rangle}(\mathcal{C}_t P), o_{\langle\rangle}(\mathcal{C}_t P), r_{\langle\rangle}(\mathcal{C}_t P), d(\mathcal{C}_t P), h_0(\mathcal{C}_t P), h_1(\mathcal{C}_t P)$ and the set-sized functions $a \mapsto n(\mathcal{C}_t P, a)$ and $a \mapsto o_{\langle a\rangle}(\mathcal{C}_t P)$ in terms of the corresponding values for P. Also, we must establish the local correctness conditions (L), (C1), (C2), (H1), (H2), (H3), (N1) and (N2) for $\mathcal{C}_t P$, assuming that the same conditions hold for P. Observe that the property "P is t-collapsing" is primitive recursive in the values $l_{\langle\rangle}(P), d(P), h_0(P)$ and $h_1(P)$. Thus the aforementioned recursive clauses may depend on this property. In particular, the statement of the theorem contains valid recursive clauses for $l_{\langle\rangle}(\mathcal{C}_t P)$ and $o_{\langle\rangle}(\mathcal{C}_t P)$. Concerning the other functions, we first assume that P is not t-collapsing. In this "unintended" case we set

$$r_{\langle\rangle}(\mathcal{C}_t P) = r_{\langle\rangle}(P), \qquad n(\mathcal{C}_t P, a) = n(P,a), \qquad d(\mathcal{C}_t P) = d(P),$$
$$h_0(\mathcal{C}_t P) = h_0(P), \qquad h_1(\mathcal{C}_t P) = h_1(P), \qquad o_{\langle a\rangle}(\mathcal{C}_t P) = o_{\langle a\rangle}(P).$$

This means that $\mathcal{C}_t P$ behaves just like P itself. As a consequence, it is completely straightforward to check that local correctness is preserved. In the "intended" case of a t-collapsing S^u-code P we put

$$d(\mathcal{C}_t P) = 1, \qquad h_0(\mathcal{C}_t P) = t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}, \qquad h_1(\mathcal{C}_t P) = \emptyset.$$

Still assuming that P is t-collapsing, the values of $r_{\langle\rangle}(\mathcal{C}_tP)$, $n(\mathcal{C}_tP,a)$ and $o_{\langle a\rangle}(\mathcal{C}_tP)$ are defined by case distinction over the last rule of P. We verify local correctness as we go along:

Case $r_{\langle\rangle}(P)=(\text{True},\varphi)$*:* We set

$$r_{\langle\rangle}(\mathcal{C}_tP)=(\text{True},\varphi),\qquad n(\mathcal{C}_tP,a)=P,\qquad o_{\langle a\rangle}(\mathcal{C}_tP)=o_{\langle\rangle}(P).$$

Using condition (L) for P we get $\varphi\in l_{\langle\rangle}(P)=l_{\langle\rangle}(\mathcal{C}_tP)$, as required by condition (L) for $\mathcal{C}_tP$. Conditions (C1) and (H2) are trivial in the case of a truth rule. Conditions (C2), (H3), (N1) and (N2) are void because of $\iota(r_{\langle\rangle}(\mathcal{C}_tP))=\iota((\text{True},\varphi))=\emptyset$ (so the values of $n(\mathcal{C}_tP,a)$ and $o_{\langle a\rangle}(\mathcal{C}_tP)$ are in fact irrelevant). It remains to check condition (H1): The same condition for P and the fact that P is t-collapsing imply

$$o_{\langle\rangle}(P)\in\mathcal{H}_P(\emptyset)\subseteq\mathcal{H}_t(\emptyset)\subseteq\mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)}}(\emptyset).$$

Also recall the assumption $t\in\mathcal{H}_t(\emptyset)$. By Lemma 4.3.4(c) and Lemma 4.3.5(b) we obtain $t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)}\in\mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)}}(\emptyset)$ and then

$$o_{\langle\rangle}(\mathcal{C}_tP)=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)})\in\mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)}}(\emptyset)=\mathcal{H}_{\mathcal{C}_tP}(\emptyset),$$

as required by condition (H1) for $\mathcal{C}_tP$.

Case $r_{\langle\rangle}(P)=(\bigwedge,\varphi)$*:* We set

$$r_{\langle\rangle}(\mathcal{C}_tP)=(\bigwedge,\varphi),\qquad n(\mathcal{C}_tP,a)=\mathcal{C}_tn(P,a),\qquad o_{\langle a\rangle}(\mathcal{C}_tP)=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle a\rangle}(P)}).$$

To prepare local correctness, let us show that $n(P,a)$ is t-collapsing for any $a\in\iota(\varphi)$: Condition (L) for P ensures $\varphi\in l_{\langle\rangle}(P)$. As P is t-collapsing this implies that φ is a Σ-formula. In view of Definition 3.1.13 we infer that φ_a is a Σ-formula as well. Using condition (L) for P it follows that $l_{\langle\rangle}(n(P,a))\subseteq l_{\langle\rangle}(P),\varphi_a$ consists of Σ-formulas. Condition (C2) for P ensures $d(n(P,a))\leq d(P)\leq 2$, and condition (H3) implies $h_0(n(P,a))\leq h_0(P)\leq t$. To conclude that $n(P,a)$ is t-collapsing it remains to establish $h_1(n(P,a))\subseteq\mathcal{H}_t(\emptyset)$: Since φ is a conjunctive Σ-formula we have $\iota(\varphi)\neq\mathbf{L}^u_{\mathrm{BH}(S^u)}$. The definition of $\operatorname{supp}(\varphi)$ and condition (H2) for P yield

$$\operatorname{supp}^{\mathbf{L}}(a)\subseteq\operatorname{supp}(\varphi)\subseteq\mathcal{H}_P(\emptyset).$$

Using condition (H3) for P and Lemma 4.3.3 we obtain the desired inclusion

$$h_1(n(P,a))\subseteq\mathcal{H}_P(\operatorname{supp}^{\mathbf{L}}(a))\subseteq\mathcal{H}_P(\emptyset)\subseteq\mathcal{H}_t(\emptyset).$$

Throughout the following we assume $a\in\iota(\varphi)=\iota(r_{\langle\rangle}(P))=\iota(r_{\langle\rangle}(\mathcal{C}_tP))$. We have just checked that $n(P,a)$ is t-collapsing. Crucially, this means that $\mathcal{C}_tn(P,a)$ is evaluated according to the intended case. As seen in Definition 4.4.3 we also get

$$\mathcal{H}_{n(P,a)}(\emptyset) \subseteq \mathcal{H}_t(\emptyset).$$

Based on these observations we can verify local correctness for $\mathcal{C}_tP$: Condition (L) for P implies $\varphi \in l_{\langle\rangle}(P) = l_{\langle\rangle}(\mathcal{C}_tP)$, as well as

$$l_{\langle\rangle}(n(\mathcal{C}_tP,a)) = l_{\langle\rangle}(\mathcal{C}_tn(P,a)) = l_{\langle\rangle}(n(P,a)) \subseteq l_{\langle\rangle}(P) \cup \{\varphi_a\} = l_{\langle\rangle}(\mathcal{C}_tP) \cup \{\varphi_a\}.$$

To see that the ordinal labels remain monotone we need $o_{\langle\rangle}(n(P,a)) \in \mathcal{H}_{n(P,a)}(\emptyset)$. It would be tempting to infer this from condition (H1) for $n(P,a)$. However, that condition is not available at the present stage of the induction, since $n(P,a)$ may not be a subterm of P. Instead, we invoke conditions (N1) and (N2) for P to obtain

$$o_{\langle\rangle}(n(P,a)) = o_{\langle a\rangle}(P) \in \mathcal{H}_{n(P,a)}(\emptyset) \subseteq \mathcal{H}_t(\emptyset).$$

Also, condition (L) for P yields $o_{\langle\rangle}(n(P,a)) < o_{\langle\rangle}(P)$. By Lemma 4.3.5(d) we get

$$o_{\langle\rangle}(n(\mathcal{C}_tP,a)) = o_{\langle\rangle}(\mathcal{C}_tn(P,a)) = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,a))}) < \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}) = o_{\langle\rangle}(\mathcal{C}_tP),$$

as required by condition (L) for $\mathcal{C}_tP$. Condition (C1) is trivial because $\mathcal{C}_tP$ does not end with a cut rule, and condition (C2) is satisfied in view of

$$d(n(\mathcal{C}_tP,a)) = d(\mathcal{C}_tn(P,a)) = 1 = d(\mathcal{C}_tP).$$

The proof of condition (H1) is the same as in the previous case. By condition (H2) for P we have $\mathrm{supp}(\varphi) \subseteq \mathcal{H}_P(\emptyset)$. Since P is t-collapsing this implies

$$\mathrm{supp}(\varphi) \subseteq \mathcal{H}_t(\emptyset) \subseteq \mathcal{H}_{t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}}(\emptyset) = \mathcal{H}_{\mathcal{C}_tP}(\emptyset),$$

as required by condition (H2) for $\mathcal{C}_tP$. To establish condition (H3) for $\mathcal{C}_tP$ we recall that condition (L) for P provides $o_{\langle\rangle}(n(P,a)) < o_{\langle\rangle}(P)$. By Lemma 4.2.5 we infer

$$h_0(n(\mathcal{C}_tP,a)) = h_0(\mathcal{C}_tn(P,a)) = t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,a))} < t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)} = h_0(\mathcal{C}_tP).$$

Since $n(P,a)$ is t-collapsing we also get

$$h_1(n(\mathcal{C}_tP,a)) = h_1(\mathcal{C}_tn(P,a)) = \emptyset \subseteq \mathcal{H}_{\mathcal{C}_tP}(\mathrm{supp}^{\mathbf{L}}(a)),$$

as required by condition (H3) for $\mathcal{C}_tP$. To show that condition (N1) is preserved we note that $o_{\langle\rangle}(n(P,a)) = o_{\langle a\rangle}(P)$ implies

$$o_{\langle\rangle}(n(\mathcal{C}_tP,a)) = o_{\langle\rangle}(\mathcal{C}_tn(P,a)) = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,a))}) = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle a\rangle}(P)}) = o_{\langle a\rangle}(\mathcal{C}_tP).$$

Finally, condition (N2) for P yields $o_{\langle a\rangle}(P) \in \mathcal{H}_{n(P,a)}(\emptyset) \subseteq \mathcal{H}_t(\emptyset)$. Lemma 4.3.4(c) allows us to infer $t \hat{+} \hat{\omega}^{o_{\langle a\rangle}(P)} \in \mathcal{H}_t(\emptyset) \subseteq \mathcal{H}_{t \hat{+} \hat{\omega}^{o_{\langle a\rangle}(P)}}(\emptyset)$. By Lemma 4.3.5(b) we get

$$o_{\langle a\rangle}(\mathcal{C}_tP)=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle a\rangle}(P)})\in\mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle a\rangle}(P)}}(\emptyset)=$$
$$=\mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle\rangle}(n(P,a))}}(\emptyset)=\mathcal{H}_{\mathcal{C}_tn(P,a)}(\emptyset)=\mathcal{H}_{n(\mathcal{C}_tP,a)}(\emptyset),$$

as required by condition (N2) for $\mathcal{C}_tP$.

Case $r_{\langle\rangle}(P)=(\bigvee,\varphi,a)$*:* We set

$$r_{\langle\rangle}(\mathcal{C}_tP)=(\bigvee,\varphi,a),\qquad n(\mathcal{C}_tP,a)=\mathcal{C}_tn(P,a),\qquad o_{\langle a\rangle}(\mathcal{C}_tP)=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle a\rangle}(P)}).$$

In view of Assumption 3.1.1 and Lemma 3.1.2 the element $0\in u\subseteq\mathbf{L}^u_{\mathrm{BH}(S^u)}$ has support $\operatorname{supp}^{\mathbf{L}}(0)=\emptyset\subseteq\mathcal{H}_t(\emptyset)$. One can deduce that $n(P,0)$ is t-collapsing, similarly to the previous case. As part of condition (L) we must show $\operatorname{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}o_{\langle\rangle}(\mathcal{C}_tP)$: Condition (H2) for P guarantees the inclusion $\operatorname{supp}^{\mathbf{L}}(a)\subseteq\mathcal{H}_P(\emptyset)\subseteq\mathcal{H}_t(\emptyset)$. In view of $\operatorname{supp}^{\mathbf{L}}(a)\subseteq\mathrm{BH}(S^u)\cong\vartheta(S^u_{\mathrm{BH}(S^u)})\cap\Omega$ we do in fact have $\operatorname{supp}^{\mathbf{L}}(a)\subseteq\mathcal{H}_t(\emptyset)\cap\Omega$. Together with $t<t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)}$ we can invoke Lemma 4.3.5(c) to get

$$\operatorname{supp}^{\mathbf{L}}(a)<^{\mathrm{fin}}\vartheta(t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)})=o_{\langle\rangle}(\mathcal{C}_tP),$$

as promised. The other conditions are handled as in the previous case.

Case $r_{\langle\rangle}(P)=(\mathrm{Cut},\varphi)$ *for a conjunctive formula* φ*:* We abbreviate

$$t':=t\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)}$$

and set

$$r_{\langle\rangle}(\mathcal{C}_tP)=(\mathrm{Cut},\varphi^{\vartheta t'}),$$
$$n(\mathcal{C}_tP,a)=\begin{cases}\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'}\,\mathcal{C}_t\,n(P,0) & \text{if } a=0,\\ \mathcal{C}_{t'}\,\mathcal{B}^{\Pi}_{\varphi,\vartheta t'}\,n(P,a) & \text{otherwise,}\end{cases}$$
$$o_{\langle a\rangle}(\mathcal{C}_tP)=\begin{cases}\vartheta t' & \text{if } a=0,\\ \vartheta(t'\hat{+}\hat{\omega}^{o_{\langle 1\rangle}(P)}) & \text{otherwise.}\end{cases}$$

Let us verify that these equations are valid recursive clauses: First, the clauses for $\mathcal{C}_tP$ may depend on the value $o_{\langle 0\rangle}(P)$. According to condition (N1) for P we have $o_{\langle 0\rangle}(P)=o_{\langle\rangle}(n(P,0))$. However, we had to avoid the expression $o_{\langle\rangle}(n(P,0))$ in the recursive clauses for $\mathcal{C}_tP$, because $n(P,0)$ is not a subterm of P. Next, in view of $\vartheta t'<\Omega$ we may consider $\vartheta t'$ as an element of $\mathrm{BH}(S^u)\cong\vartheta(S^u_{\mathrm{BH}(S^u)})\cap\Omega$. This allows us to form the relativized formula $\varphi^{\vartheta t'}$. Condition (C1) for the t-collapsing proof P yields $\mathrm{rk}(\varphi)<d(P)\leq 2$. As observed before the statement of Lemma 4.4.2 this means that φ is bounded or of the form $\forall_x\theta$, for some bounded formula θ. It follows that $\neg\varphi$ is a Σ-formula. Thus $\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'}$ and $\mathcal{B}^{\Pi}_{\varphi,\vartheta t'}$ have the form of the function

symbols introduced in Lemma 4.4.1 resp. Lemma 4.4.2. Finally, condition (N2) and condition (H3) for the t-collapsing proof P imply

$$o_{\langle 0\rangle}(P)\in\mathcal{H}_{n(P,0)}(\emptyset)\subseteq\mathcal{H}_P(\operatorname{supp}^{\mathbf{L}}(0))\subseteq\mathcal{H}_t(\emptyset).$$

Together with the assumption $t\in\mathcal{H}_t(\emptyset)$ we obtain

$$t'\in\mathcal{H}_t(\emptyset)\subseteq\mathcal{H}_{t'}(\emptyset),$$

which ensures that $\mathcal{C}_{t'}$ is one of the function symbols introduced by the present theorem. Having established that the definition of $r_{\langle\rangle}(\mathcal{C}_tP)$, $n(\mathcal{C}_tP,a)$ and $o_{\langle a\rangle}(\mathcal{C}_tP)$ is valid, let us verify the local correctness conditions for $\mathcal{C}_tP$: Condition (L) for P provides $l_{\langle\rangle}(n(P,0))\subseteq l_{\langle\rangle}(P)\cup\{\neg\varphi\}$, so that this sequent consists of Σ-formulas. As before we infer that $n(P,0)$ is t-collapsing. Together with condition (N1) for P we obtain

$$o_{\langle\rangle}(\mathcal{C}_tn(P,0))=\vartheta(t\mathbin{\hat{+}}\hat{\omega}^{o_{\langle\rangle}(n(P,0))})=\vartheta(t\mathbin{\hat{+}}\hat{\omega}^{o_{\langle 0\rangle}(P)})=\vartheta t'.$$

This ensures that the code $\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'}\mathcal{C}_tn(P,0)$ is evaluated according to the intended case $o_{\langle\rangle}(\mathcal{C}_tn(P,0))\leq\vartheta t'$: The transformation $\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'}$ does replace the Σ-formula $\neg\varphi$ by the bounded formula $(\neg\varphi)^{\vartheta t'}\equiv\neg(\varphi^{\vartheta t'})$. Using condition (L) for P we get

$$\begin{aligned}l_{\langle\rangle}(n(\mathcal{C}_tP,0))&=l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'}\mathcal{C}_tn(P,0))=l_{\langle\rangle}(\mathcal{C}_tn(P,0))\backslash\{\neg\varphi\}\cup\{\neg\varphi^{\vartheta t'}\}=\\&=l_{\langle\rangle}(n(P,0))\backslash\{\neg\varphi\}\cup\{\neg\varphi^{\vartheta t'}\}\subseteq l_{\langle\rangle}(P)\cup\{\neg\varphi^{\vartheta t'}\}=l_{\langle\rangle}(\mathcal{C}_tP)\cup\{\neg\varphi^{\vartheta t'}\},\end{aligned}$$

as required by condition (L) for $\mathcal{C}_tP$. Similarly to the previous cases we also have

$$\begin{aligned}o_{\langle\rangle}(n(\mathcal{C}_tP,0))&=o_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\neg\varphi,\vartheta t'}\mathcal{C}_tn(P,0))=o_{\langle\rangle}(\mathcal{C}_tn(P,0))=\\&=\vartheta(t\mathbin{\hat{+}}\hat{\omega}^{o_{\langle\rangle}(n(P,0))})<\vartheta(t\mathbin{\hat{+}}\hat{\omega}^{o_{\langle\rangle}(P)})=o_{\langle\rangle}(\mathcal{C}_tP).\end{aligned}$$

In order to establish condition (L) with respect to $n(\mathcal{C}_tP,1)$ we show that the S^u-code $\mathcal{B}^{\Pi}_{\varphi,\vartheta t'}n(P,1)$ is t'-collapsing: Since $\varphi^{\vartheta t'}$ is bounded the sequent

$$l_{\langle\rangle}(\mathcal{B}^{\Pi}_{\varphi,\vartheta t'}n(P,1))=l_{\langle\rangle}(n(P,1))\backslash\{\varphi\}\cup\{\varphi^{\vartheta t'}\}\subseteq l_{\langle\rangle}(P)\cup\{\varphi^{\vartheta t'}\}$$

consists of Σ-formulas. We can also compute

$$d(\mathcal{B}^{\Pi}_{\varphi,\vartheta t'}n(P,1))=d(n(P,1))\leq d(P)\leq 2,$$

as well as

$$h_0(\mathcal{B}^{\Pi}_{\varphi,\vartheta t'}n(P,1))=h_0(n(P,1))\leq h_0(P)\leq t\leq t'.$$

As before we get $h_1(n(P,1)) \subseteq \mathcal{H}_P(\emptyset) \subseteq \mathcal{H}_t(\emptyset) \subseteq \mathcal{H}_{t'}(\emptyset)$, by condition (H3) for P and the fact that P is t-collapsing. Together with $t' \in \mathcal{H}_{t'}(\emptyset)$ we obtain

$$h_1(\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)) = h_1(n(P,1)) \cup \{\vartheta t'\} \subseteq \mathcal{H}_{t'}(\emptyset).$$

Having established that $\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)$ is t'-collapsing, let us now verify condition (L) with respect to $n(\mathcal{C}_t P,1)$: Invoking the above inclusion we get

$$\begin{aligned} l_{\langle\rangle}(n(\mathcal{C}_t P,1)) = l_{\langle\rangle}(\mathcal{C}_{t'}\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)) = l_{\langle\rangle}(\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)) \subseteq \\ \subseteq l_{\langle\rangle}(P) \cup \{\varphi^{\vartheta t'}\} = l_{\langle\rangle}(\mathcal{C}_t P) \cup \{\varphi^{\vartheta t'}\}. \end{aligned}$$

To see that the ordinal labels remain monotone we observe $o_{\langle\rangle}(n(P,1)) \in \mathcal{H}_t(\emptyset)$, by conditions (N1), (N2) and (H3) for P and since P is t-collapsing. This implies

$$\vartheta(t' \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,1))}) \in \mathcal{H}_{t' \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,1))}}(\emptyset) \cap \Omega.$$

Conditions (N1) and (L) for P provide $o_{\langle i\rangle}(P) = o_{\langle\rangle}(n(P,i)) < o_{\langle\rangle}(P)$ for $i = 0,1$. Using Lemma 4.2.5 we can deduce

$$t' \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,1))} = t \hat{+} (\hat{\omega}^{o_{\langle 0\rangle}(P)} \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,1))}) < t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}.$$

Finally, Lemma 4.3.5(c) allows us to infer

$$\begin{aligned} o_{\langle\rangle}(n(\mathcal{C}_t P,1)) = o_{\langle\rangle}(\mathcal{C}_{t'}\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1)) = \vartheta(t' \hat{+} \hat{\omega}^{o_{\langle\rangle}(\mathcal{B}^{\Pi}_{\varphi,\vartheta t'} n(P,1))}) = \\ = \vartheta(t' \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,1))}) < \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}) = o_{\langle\rangle}(\mathcal{C}_t P), \end{aligned}$$

completing the proof of condition (L) for $\mathcal{C}_t P$. To establish condition (C1) for $\mathcal{C}_t P$ we observe that $\varphi^{\vartheta t'}$ is a bounded formula, which implies

$$\mathrm{rk}(\varphi^{\vartheta t'}) = 0 < 1 = d(\mathcal{C}_t P).$$

The other local correctness conditions are straightforward to verify, based on the facts that we have already established.

Case $r_{\langle\rangle}(P) = (\mathrm{Cut}, \varphi)$ for a disjunctive formula φ: The situation is symmetric to the previous case. Thus $\neg\varphi$ is conjunctive, we have $\mathrm{rk}(\neg\varphi) = \mathrm{rk}(\varphi) < d(P) \leq 2$, and φ is a Σ-formula. We abbreviate $t' := t \hat{+} \hat{\omega}^{o_{\langle 1\rangle}(P)}$ and observe $t' \in \mathcal{H}_{t'}(\emptyset)$ as well as $\vartheta t' \in \vartheta(S^u_{\mathrm{BH}(S^u)}) \cap \Omega \cong \mathrm{BH}(S^u)$. As in the previous case we obtain valid recursive clauses by setting

$$r_{\langle\rangle}(\mathcal{C}_t P) = (\mathrm{Cut}, \varphi^{\vartheta t'}),$$

$$n(\mathcal{C}_t P, a) = \begin{cases} \mathcal{B}^{\Sigma}_{\varphi,\vartheta t'}\, \mathcal{C}_t\, n(P,1) & \text{if } a = 1, \\ \mathcal{C}_{t'}\, \mathcal{B}^{\Pi}_{\neg\varphi,\vartheta t'}\, n(P,a) & \text{otherwise,} \end{cases}$$

$$o_{\langle a\rangle}(\mathcal{C}_t P)=\begin{cases}\vartheta t' & \text{if } a=1,\\ \vartheta(t'\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)}) & \text{otherwise.}\end{cases}$$

The local correctness conditions are verified as in the previous case.

Case $r_{\langle\rangle}(P)=(\mathrm{Ref},\exists_w\forall_{x\in a}\exists_{y\in w}\theta)$*:* We abbreviate

$$s:=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)})$$

and set

$$\begin{aligned}r_{\langle\rangle}(\mathcal{C}_t P)&=(\textstyle\bigvee,\exists_w\forall_{x\in a}\exists_{y\in w}\theta,\mathbf{L}^u_s),\\ n(\mathcal{C}_t P,a)&=\mathcal{B}^{\Sigma}_{\forall_{x\in a}\exists_y\theta,s}\,\mathcal{C}_t\,n(P,0),\\ o_{\langle a\rangle}(\mathcal{C}_t P)&=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)}).\end{aligned}$$

As in the case of a cut we observe that s is an element of $\vartheta(S^u_{\mathrm{BH}(S^u)})\cap\Omega\cong\mathrm{BH}(S^u)$. This allows us to form the $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-term $\mathbf{L}^u_s$. According to Definition 3.4.2 the formula θ must be bounded. Thus $\forall_{x\in a}\exists_y\theta$ is a Σ-formula and $\mathcal{B}^{\Sigma}_{\forall_{x\in a}\exists_y\theta,s}$ is one of the function symbols introduced in Lemma 4.4.1. Condition (L) for P yields

$$l_{\langle\rangle}(n(P,0))\subseteq l_{\langle\rangle}(P)\cup\{\forall_{x\in a}\exists_y\theta\},$$

so that $l_{\langle\rangle}(n(P,0))$ consists of Σ-formulas. As in the previous cases we can deduce that $n(P,0)$ is t-collapsing. Together with condition (N1) for P we infer

$$o_{\langle\rangle}(\mathcal{C}_t n(P,0))=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle\rangle}(n(P,0))})=\vartheta(t\hat{+}\hat{\omega}^{o_{\langle 0\rangle}(P)})=s.$$

Crucially, this ensures that $\mathcal{B}^{\Sigma}_{\forall_{x\in a}\exists_y\theta,s}\mathcal{C}_t n(P,0)$ is evaluated according to the intended case, which means that $\mathcal{B}^{\Sigma}_{\forall_{x\in a}\exists_y\theta,s}$ replaces $\forall_{x\in a}\exists_y\theta$ by the bounded formula

$$(\forall_{x\in a}\exists_y\theta)^s\equiv\forall_{x\in a}\exists_{y\in\mathbf{L}^u_s}\theta.$$

Since θ does not contain the variable w (see Definition 3.4.2) we also have

$$\exists_w\forall_{x\in a}\exists_{y\in w}\theta\simeq\textstyle\bigvee_{b\in\mathbf{L}^u_{\mathrm{BH}(S^u)}}\forall_{x\in a}\exists_{y\in b}\theta.$$

Putting these observations together we obtain

$$\begin{aligned}l_{\langle\rangle}(n(\mathcal{C}_t P,0))=l_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\forall_{x\in a}\exists_y\theta,s}\mathcal{C}_t n(P,0))=l_{\langle\rangle}(n(P,0))\backslash\{\forall_{x\in a}\exists_y\theta\}\cup\{(\forall_{x\in a}\exists_y\theta)^s\}\subseteq\\ \subseteq l_{\langle\rangle}(P)\cup\{\forall_{x\in a}\exists_{y\in\mathbf{L}^u_s}\theta\}=l_{\langle\rangle}(\mathcal{C}_t P)\cup\{(\exists_w\forall_{x\in a}\exists_{y\in w}\theta)_{\mathbf{L}^u_s}\},\end{aligned}$$

as condition (L) demands for the rule $r_{\langle\rangle}(\mathcal{C}_t P)=(\bigvee,\exists_w\forall_{x\in a}\exists_{y\in w}\theta,\mathbf{L}^u_s)$. Also note that $\exists_w\forall_{x\in a}\exists_{y\in w}\theta\in l_{\langle\rangle}(P)=l_{\langle\rangle}(\mathcal{C}_t P)$ follows from condition (L) for P. As in the previous cases we get

$$o_{\langle\rangle}(n(P,0))=o_{\langle 0\rangle}(P)\in\mathcal{H}_{n(P,0)}(\emptyset)\subseteq\mathcal{H}_t(\emptyset),$$

using conditions (N1) and (N2) for P and the fact that $n(P,0)$ is t-collapsing. Since condition (L) for P provides $o_{\langle\rangle}(n(P,0)) < o_{\langle\rangle}(P)$ we can conclude

$$o_{\langle\rangle}(n(\mathcal{C}_tP,0)) = o_{\langle\rangle}(\mathcal{B}^{\Sigma}_{\forall x\in a\exists_y\theta,s}\mathcal{C}_tn(P,0)) = o_{\langle\rangle}(\mathcal{C}_tn(P,0)) = \\ = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,0))}) < \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(P)}) = o_{\langle\rangle}(\mathcal{C}_tP)$$

by Lemma 4.3.5(d). In view of $s = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,0))})$ the same inequality shows

$$\operatorname{supp}^{\mathbf{L}}(\mathbf{L}^u_s) = \{s\} <^{\text{fin}} o_{\langle\rangle}(\mathcal{C}_tP),$$

completing the proof of condition (L) for $\mathcal{C}_tP$. Let us also observe

$$s = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(n(P,0))}) \in \mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle\rangle}(n(P,0))}}(\emptyset) \subseteq \mathcal{H}_{t\hat{+}\hat{\omega}^{o_{\langle\rangle}(P)}}(\emptyset) = \mathcal{H}_{\mathcal{C}_tP}(\emptyset).$$

This yields $\operatorname{supp}^{\mathbf{L}}(\mathbf{L}^u_s) = \{s\} \subseteq \mathcal{H}_{\mathcal{C}_tP}(\emptyset)$, as required by condition (H2) for $\mathcal{C}_tP$. The remaining local correctness conditions are straightforward to verify.

Case $r_{\langle\rangle}(P) = (\text{Rep}, b)$*:* We set

$$r_{\langle\rangle}(\mathcal{C}_tP) = (\text{Rep}, b), \qquad n(\mathcal{C}_tP, a) = \mathcal{C}_tn(P,a), \qquad o_{\langle a\rangle}(\mathcal{C}_tP) = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle a\rangle}(P)}).$$

Crucially, condition (H2) for P provides $\operatorname{supp}^{\mathbf{L}}(b) \subseteq \mathcal{H}_P(\emptyset)$, so that Lemma 4.3.3 yields $\mathcal{H}_P(\operatorname{supp}^{\mathbf{L}}(b)) \subseteq \mathcal{H}_P(\emptyset)$. Using condition (H3) for P and the fact that P is t-collapsing we get

$$h_1(n(P,b)) \subseteq \mathcal{H}_P(\operatorname{supp}^{\mathbf{L}}(b)) \subseteq \mathcal{H}_P(\emptyset) \subseteq \mathcal{H}_t(\emptyset).$$

This ensures that $n(P,b)$ is t-collapsing, as one can easily show. Based on this fact it is straightforward to verify the local correctness conditions, similarly to the case of a rule $r_{\langle\rangle}(P) = (\bigwedge, \varphi)$. □

In Lemma 3.4.4 we have established soundness for S^u-proofs of height below Ω. As promised, collapsing allows us to extend this result to suitable S^u-proofs of arbitrary ordinal height:

COROLLARY 4.4.5. *Consider an* S^u*-code* P *which is* t*-controlled, for some term* $t \in \vartheta(S^u_{\text{BH}(S^u)})$ *with* $t \in \mathcal{H}_t(\emptyset)$*. Then we have* $\mathbb{L}^u_\alpha \vDash [\![\varphi]\!]$ *for some formula* $\varphi \in l_{\langle\rangle}(P)$*. In particular the end-sequent* $l_{\langle\rangle}(P)$ *of* P *cannot be empty.*

To understand the soundness claim, recall that Assumption 3.4.1 provides an isomorphism between $\text{BH}(S^u)$ and an ordinal α. From Proposition 3.1.3 we get an interpretation $[\![\cdot]\!] : \mathbf{L}^u_{\text{BH}(S^u)} \to \mathbb{L}^u_\alpha$ of $\mathbf{L}^u_{\text{BH}(S^u)}$-terms by elements of the actual constructible hierarchy. Before the statement of Lemma 3.1.15 this has been extended to a map $\varphi \mapsto [\![\varphi]\!]$, which interprets each $\mathbf{L}^u_{\mathrm{BH}(S^u)}$-formula φ by a formula $[\![\varphi]\!]$ with parameters in $\mathbb{L}^u_\alpha$.

Proof. The idea is to collapse P to height below Ω, by the previous theorem. Then we can apply Lemma 3.4.4 to the collapsed proof. We have assumed that P is t-controlled, but it may still fail to be t-collapsing, due to its cut rank. This obstacle is removed by Proposition 4.2.7: The S^u-code $\mathcal{E}^iP = \mathcal{E}\cdots\mathcal{E}P$ with i-occurrences of the function symbol $\mathcal{E}$ has the same end-sequent

$$l_{\langle\rangle}(\mathcal{E}^iP) = l_{\langle\rangle}(P)$$

and the lower cut rank

$$d(\mathcal{E}^iP) \leq \max\{2, d(P) - i\},$$

as one can verify by induction over i. Definition 4.3.6 yields $h_0(\mathcal{E}^iP) = h_0(P)$ and $h_1(\mathcal{E}^iP) = h_1(P)$, which means that $\mathcal{E}^iP$ and P are controlled by the same operator. Invoking the previous theorem, let us now form the S^u-code $\mathcal{C}_t\,\mathcal{E}^{d(P)}P$. In view of $d(\mathcal{E}^{d(P)}P) \leq 2$ it is straightforward to check that $\mathcal{E}^{d(P)}P$ is t-collapsing, which ensures

$$o_{\langle\rangle}(\mathcal{C}_t\,\mathcal{E}^{d(P)}P) = \vartheta(t \hat{+} \hat{\omega}^{o_{\langle\rangle}(\mathcal{E}^{d(P)}P)}) < \Omega.$$

The inequality holds by definition of the order on ordinal terms (see Lemma 2.2.7). Before we can apply Lemma 3.4.4 we must transform the S^u-code $\mathcal{C}_t\,\mathcal{E}^{d(P)}P$ into an "actual" S^u-proof (cf. Definition 3.4.3). This is achieved by Corollary 4.1.5, which extends to all codes that have been introduced in the present chapter, as exlained in Remark 4.1.6: It yields an S^u-proof $[\mathcal{C}_t\,\mathcal{E}^{d(P)}P]$ with end-sequent

$$l_{\mathcal{C}_t\,\mathcal{E}^{d(P)}P}(\langle\rangle) = l_{\langle\rangle}(\mathcal{C}_t\,\mathcal{E}^{d(P)}P) = l_{\langle\rangle}(\mathcal{E}^{d(P)}P) = l_{\langle\rangle}(P)$$

and ordinal height

$$o_{\mathcal{C}_t\,\mathcal{E}^{d(P)}P}(\langle\rangle) = o_{\langle\rangle}(\mathcal{C}_t\,\mathcal{E}^{d(P)}P) < \Omega.$$

Now Lemma 3.4.4 implies $\mathbb{L}^u_\alpha \vDash [\![\varphi]\!]$ for some formula $\varphi \in l_{\mathcal{C}_t\,\mathcal{E}^{d(P)}P}(\langle\rangle) = l_{\langle\rangle}(P)$, just as the corollary claims. □

Finally we have all ingredients for the characterization of Π^1_1-comprehension in terms of type-two well-ordering principles. An explanation of this result can be found in the introduction to the present thesis. The different versions of the Bachmann-Howard principle have been introduced in Chapter 2. In particular we refer to Definitions 2.1.3 and 2.1.4 (abstract version), Definition 2.2.19 (predicative version), as well as Lemma 2.3.14 and Definition 2.3.16 (computable version). To avoid misunderstanding, let us stress that the following theorem does not rely on Assumptions 3.1.1 and 3.4.1, upon which some of the previous considerations were based. As part of the proof we will show that these standing assumptions are satisfied where they are required.

THEOREM 4.4.6. *The following are equivalent over* $\mathbf{ATR_0^{set}}$*:*

(i) The principle of Π^1_1*-comprehension.*
(ii) The statement that each set is an element of some admissible set.
(iii) The abstract Bachmann-Howard principle: For any proto-dilator $\alpha \mapsto T^u_\alpha$*, there is an ordinal* α *with a Bachmann-Howard collapse* $\vartheta : T^u_\alpha \xrightarrow{\text{BH}} \alpha$*.*
(iv) The predicative Bachmann-Howard principle: For an arbitrary dilator T^u*, the Bachmann-Howard order* $\mathrm{BH}(T^u)$ *is well-founded.*
(v) The computable Bachmann-Howard principle: For any coded dilator T*, the relativized notation system* $\vartheta(T)$ *is well-founded.*

Before we prove the theorem, let us point out that the abstract Bachmann-Howard principle in (iii) is a schema: We can quantify over parametrized families of proto-dilators (by quantifying over the parameter u), but not over all proto-dilators, since the latter are class-sized functions. The proof will show that (ii) implies any instance of (iii). Conversely, a single instance of (iii) implies (v) and thus (ii). As a consequence, there is a single instance of the abstract Bachmann-Howard principle which implies any other instance. Note that this fact is established in a rather indirect way. The argument is much more direct in the case of (iv) and (v): Up to natural isomorphism, a dilator is determined by its set-sized restriction to the category of natural numbers, as shown by Girard [**28**] (see also Proposition 2.3.6 above). This allows us to quantify over essentially all dilators. In particular, the predicative Bachmann-Howard principle is naturally expressed by a single instance, as elaborated in Proposition 2.3.10. Coded dilators are set-sized objects from the outset (see Definition 2.3.11). Thus we can quantify over all coded dilators, and the computable Bachmann-Howard principle is expressed by a single formula.

PROOF. The equivalence between (i) and (ii) is known (see [**41**, Section 7]). In Corollary 1.4.13 we have verified that it holds over the base theory $\mathbf{ATR_0^{set}}$. The implications (ii)⇒(iii)⇒(v)⇒(iv) have been established in Corollary 2.4.8, Theorem 2.4.9 and Corollary 2.4.18. It remains to prove the implication (iv)⇒(ii): Aiming at (ii) we consider an arbitrary set x. The first step is to construct a set

$u \ni x$ which satisfies Assumption 3.1.1. We begin by forming the transitive closure

$$u' := \mathrm{TC}(\{x, 0, 1\}).$$

Writing $o(u') = u' \cap \mathrm{Ord}$ for the ordinal height of u' we put

$$u := u' \cup \{o(u')\}.$$

Note that u is transitive and that $o(u) = o(u')+1$ is a successor ordinal. Also recall that $\mathbf{ATR_0^{set}}$ includes the axiom of countability, which ensures that u is a countable set. Proposition 1.2.12 yields a function $i \mapsto u_i$ with range $u = \{u_i \,|\, i \in \omega\}$. Having verified Assumption 3.1.1, we can consider the functorial constructible hierarchy $X \mapsto \mathbf{L}_X^u$ and the construction of search trees $X \mapsto S_X^u$, as presented in Sections 3.1 and 3.2. We have to consider two possibilities: First assume that there is a well-order X for which the search tree S_X^u has an (infinite) branch. In this case Corollary 3.2.4 provides an admissible set $\mathbb{A}$ with $x \in u \subseteq \mathbb{A}$, just as required by statement (ii) of the present theorem. Now assume that S_X^u does not have a branch for any well-order X. As we have seen in Section 3.3 this implies that $X \mapsto S_X^u$ is a dilator (with the Kleene-Brouwer order on the tree S_X^u and certain support functions $\mathrm{supp}_X^{\mathrm{S}} : S_X^u \to [X]^{<\omega}$). On an informal level we remark that the collection of search trees becomes a β-proof (cf. [**29**]) of contradiction, with the Kripke-Platek axioms as open assumptions. We want to show that this is impossible, so that one of the search trees must have a branch after all, completing the proof of (ii). In order to achieve this we invoke statement (iv) of the theorem, which tells us that the Bachmann-Howard order $\mathrm{BH}(S^u)$ for the dilator $X \mapsto S_X^u$ is well-founded. Since the base theory $\mathbf{ATR_0^{set}}$ includes axiom beta we learn that $\mathrm{BH}(S^u)$ is isomorphic to some ordinal α. This allows us to satisfy Assumption 3.4.1, upon which we have based our investigation of infinite proofs. According to Proposition 3.4.8, the search tree $S^u_{\mathrm{BH}(S^u)}$ can be extended to an S^u-proof P_S with empty end-sequent $l_S(\langle\rangle) = \langle\rangle$. Over the course of the present chapter we have reconstructed a certain class of S^u-proofs as a system of S^u-codes, which is more suitable for the formalization of ordinal analysis in a restricted base theory. In particular, Definition 4.1.1 provides an S^u-code $P_S\langle\rangle$ with end-sequent

$$l_{\langle\rangle}(P_S\langle\rangle) = l_S(\langle\rangle) = \langle\rangle.$$

In view of Definition 4.3.6 the controlling operator of $P_S\langle\rangle$ is given by

$$h_0(P_S\langle\rangle) = 0, \qquad h_1(P_S\langle\rangle) = \mathrm{supp}^{\mathrm{S}}_{\mathrm{BH}(S^u)}(\langle\rangle) = \emptyset.$$

We can now observe that $P_S\langle\rangle$ is 0-controlled in the sense of Definition 4.4.3. According to Corollary 4.4.5 this implies $l_{\langle\rangle}(P_S\langle\rangle) \neq \langle\rangle$. The resulting contradiction shows that $X \mapsto S^u_X$ cannot be a dilator. Thus there must be a well-order X for which the search tree S^u_X has a branch. As seen above this yields an admissible set $\mathbb{A}$ with $x \in u \subseteq \mathbb{A}$. This completes the proof of the missing implication (iv)$\Rightarrow$(ii). □

Statements (ii), (iii) and (iv) from the previous theorem involve set-theoretic notions. In contrast, the computable Bachmann-Howard principle in (v) has been formulated in the language of second-order arithmetic (see Section 2.3). We can conclude our investigation with a result in the usual setting of reverse mathematics:

Corollary 4.4.7. *The following are equivalent over* $\mathbf{RCA_0}$*:*

(i) The principle of Π^1_1*-comprehension.*

(ii) The computable Bachmann-Howard principle.

Proof. By the previous theorem the equivalence holds over $\mathbf{ATR_0^{\mathrm{set}}}$. Since (i) and (ii) are statements of second-order arithmetic they are equivalent over $\mathbf{ATR}_0$, by the conservativity result from Corollary 1.4.9 (originally due to Simpson [**85**]). Crucially, Corollary 2.3.19 shows that the computable Bachmann-Howard principle implies any axiom of $\mathbf{ATR}_0$, over the base theory $\mathbf{RCA_0}$ (this builds on a result of Rathjen and Valencia Vizcaíno [**71**]). It is well known that Π^1_1-comprehension implies any axiom of $\mathbf{ATR}_0$ as well (e.g. via Σ^1_1-separation, see [**87**, Theorem V.5.1]). Thus the equivalence between (i) and (ii) holds over $\mathbf{RCA_0}$, as desired. □

Finally, let us recall Remark 2.4.10: There we have sketched simplified versions of the predicative and the computable Bachmann-Howard principle, which make the type-two and the type-one aspects of these well-ordering principles more transparent. It seems very worthwhile to elaborate these ideas in greater detail, but this lies beyond the scope of the present thesis.

CHAPTER 5

Some Results about Slow Consistency

The notion of slow consistency for Peano arithmetic (**PA**) was introduced by S.-D. Friedman, Rathjen and Weiermann [**26**]. During his PhD the present author has written two papers about this notion, which have been published in the *Annals of Pure and Applied Logic* [**19, 20**]. This chapter summarizes the main results and ideas of this work, without repeating the proofs in full detail. It can be read independently of the rest of the thesis.

Before we can define slow consistency we need to recall the fast-growing hierarchy, which is due to Wainer [**95**] and Schwichtenberg [**81**]: To each limit ordinal $\lambda \leq \varepsilon_0$ one associates a "fundamental sequence", i.e. a strictly increasing sequence of ordinals $\{\lambda\}(n)$ with supremum λ. The idea is to compute $\{\lambda\}(n)$ from the Cantor normal form of λ, in a natural way. As the precise definition varies throughout the literature we fix the version used in [**26**]. In particular this yields $\{\varepsilon_0\}(n) := \omega_{n+1} := \omega^1_{n+1}$ with $\omega^\alpha_0 := \alpha$ and $\omega^\alpha_{k+1} := \omega^{\omega^\alpha_k}$. Now the fast-growing hierarchy of functions F_α is defined by recursion over the ordinals $\alpha \leq \varepsilon_0$, setting

$$\begin{aligned} F_0(n) &:= n+1, \\ F_{\alpha+1}(n) &:= F^{n+1}_\alpha(n), \\ F_\lambda(n) &:= F_{\{\lambda\}(n)}(n) \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

A superscript to a function denotes the number of iterations, i.e. we have $F^0(n) = n$ and $F^{k+1}(n) = F(F^k(n))$. Exploiting the Cantor normal form, the ordinals below ε_0 can be coded by finite terms and thus by natural numbers. The value $F_\alpha(n)$ can be computed by manipulating expressions of the form $F^{k_1}_{\alpha_1}(\cdots(F^{k_i}_{\alpha_i}(m))\cdots)$. Sommer [**89**, Section 5.2] has encoded the resulting computation sequences in a particularly efficient way, to obtain a Δ_0-formula $F^i_\alpha(x) = y$ with free variables α, x, y, i which defines (iterates of) the functions in the fast-growing hierarchy. To be precise, Sommer's formula only applies to $\alpha < \varepsilon_0$. To see that the graph of F_{ε_0} has a Δ_0-definition one observes that the code of ω_{n+1} is bounded by a polynomial in $F_{\varepsilon_0}(n)$, as verified in [**19**, Section 2]. We will write $F_\alpha(x){\downarrow}$ and $F_\alpha{\downarrow}$ to abbreviate the formulas $\exists_y F_\alpha(x) = y$ and $\forall_x \exists_y F_\alpha(x) = y$, respectively. Also recall that $\mathbf{I\Sigma}_n$ is the fragment of Peano arithmetic in which induction is restricted to Σ_n-formulas (we write Σ_n rather than Σ^0_n because the present chapter is only concerned with first-order arithmetic). By the usual arithmetization of syntax (see e.g. [**37**]) we obtain a Π_1-formula $\mathrm{Con}(\mathbf{I\Sigma}_x)$ with free variable x which expresses that the theory $\mathbf{I\Sigma}_x$ is consistent. Now slow consistency can be defined as the statement

$$\mathrm{Con}^\diamond(\mathbf{PA}) :\equiv \forall_x (F_{\varepsilon_0}(x){\downarrow} \to \mathrm{Con}(\mathbf{I\Sigma}_{x+1})).$$

Let us point out that the original slow consistency statement of Friedman, Rathjen and Weiermann [**26**] has $\mathrm{Con}(\mathbf{I\Sigma}_x)$ at the place of $\mathrm{Con}(\mathbf{I\Sigma}_{x+1})$ and is denoted by $\mathrm{Con}^*(\mathbf{PA})$. The stronger fragment $\mathbf{I\Sigma}_{x+1}$ is optimal in a certain sense, as we shall see in Section 5.2.

The usual consistency statement $\mathrm{Con}(\mathbf{PA})$ for Peano arithmetic is equivalent to the assertion $\forall_x \mathrm{Con}(\mathbf{I\Sigma}_x)$. This means that $\mathrm{Con}^\diamond(\mathbf{PA})$ implies $\mathrm{Con}(\mathbf{PA})$ if the statement $F_{\varepsilon_0}{\downarrow}$ is available. However, the results of Kreisel [**48**], Wainer [**95**] and Schwichtenberg [**81**] show that the totality of F_{ε_0} is not provable in Peano arithmetic. It is thus conceivable that we have

$$\mathbf{PA} + \mathrm{Con}^\diamond(\mathbf{PA}) \nvdash \mathrm{Con}(\mathbf{PA}).$$

Friedman, Rathjen and Weiermann [**26**, Theorem 3.10] show that this is indeed the case. Their proof uses a model-theoretic result of Sommer [**89**]. A proof-theoretic argument can be found in a recent preprint by Pakhomov and the present author [**23**]. As observed in [**26**, Proposition 3.3], an application of Gödel's theorem also yields

$$\mathbf{PA} \nvdash \mathrm{Con}^\diamond(\mathbf{PA}).$$

Thus $\mathbf{PA} + \mathrm{Con}^\diamond(\mathbf{PA})$ is a natural theory between $\mathbf{PA}$ and $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$ (also with respect to interpretability, as pointed out in [**26**]). In addition to this result, Friedman, Rathjen and Weiermann construct natural examples of incomparable theories and of an Orey sentence (see [**26**, Corollaries 3.15 and 3.19]). To explain the name "slow consistency", let us consider the function $F^{-1}_{\varepsilon_0}$ given by

$$F^{-1}_{\varepsilon_0}(n) := \max(\{m \mid F_{\varepsilon_0}(m) \leq n\} \cup \{0\}).$$

Note that the maximum is well-defined because $F_{\varepsilon_0}(m) \leq n$ implies $m \leq n$. Since F_{ε_0} grows extremely fast, its inverse grows extremely slow: The function $F^{-1}_{\varepsilon_0}$ is

PA-provably total, but **PA** does not show that it has infinite range. Friedman, Rathjen and Weiermann point out that slow consistency can be described as

$$\mathrm{Con}^\diamond(\mathbf{PA}) \equiv \mathrm{Con}(\bigcup\{\mathbf{I\Sigma}_{F_{\varepsilon_0}^{-1}(x)+1} \,|\, x \in \mathbb{N}\}).$$

In other words, the slow consistency statement corresponds to an extremely slow enumeration of the fragments of Peano arithmetic.

For each fixed number n there is an arithmetical formula $\mathrm{True}_{\Pi_n}(\cdot)$ which defines truth for Π_n-formulas (see e.g. [**37**, Section I.1(d)]). The statement that φ is provable in $\mathbf{I\Sigma}_x$ can be expressed by a formula $\mathrm{Prov}_{\mathbf{I\Sigma}_x}(\varphi)$ with free variables x and φ. Now the uniform Π_n-reflection principle over $\mathbf{I\Sigma}_x$ is given by the arithmetical formula

$$\mathrm{RFN}_{\mathbf{I\Sigma}_x}(\Pi_n) :\equiv \forall_\varphi(\text{“}\varphi \text{ a closed } \Pi_n\text{-formula”} \land \mathrm{Prov}_{\mathbf{I\Sigma}_x}(\varphi) \to \mathrm{True}_{\Pi_n}(\varphi))$$

with free variable x. Since Σ_n-reflection is equivalent to Π_{n+1}-reflection this covers all of the usual formula classes. Extending the notion of slow consistency, the present author [**19, 20**] has introduced the slow reflection principles

$$\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_n) :\equiv \forall_x(F_{\varepsilon_0}(x){\downarrow} \to \mathrm{RFN}_{\mathbf{I\Sigma}_{x+1}}(\Pi_n)).$$

Various properties of slow reflection will be summarized in Section 5.1, from the viewpoints of computational content and of consistency strength. To anticipate one of the most interesting results, extending Peano arithmetic by the slow reflection principle $\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2)$ adds a new provably total function $F^\diamond_{\varepsilon_0}$ (a somewhat slower version of F_{ε_0}), but at the same time we have

$$\mathbf{PA} + \mathrm{Con}(\mathbf{PA}) \vdash \mathrm{Con}(\mathbf{PA} + \{\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_n) \,|\, n \in \mathbb{N}\}).$$

In the final section of [**26**], Friedman, Rathjen and Weiermann show that finite iterations of slow consistency remain weaker than the usual consistency statement. They conjecture that the same is true for transfinite iterations below ε_0. This conjecture has been proved by the present author [**20**] and independently by Henk and Pakhomov [**34**]. In Section 5.2 we will summarize our proof, which relies on the analysis of slow reflection. Another application will be presented in Section 5.3: Our computational bounds on slow reflection imply that Σ_1-instances of the Paris-Harrington principle cannot have short proofs in certain fragments of Peano arithmetic. More results about slow consistency can be found in the papers of Henk and Pakhomov [**34**] and of Rathjen [**70**].

5.1. An Analysis of Slow Reflection

The slow reflection statements $\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_n)$ have been defined in the introduction of this chapter. In the present section we summarize results [**19, 20**] about their computational content and their consistency strength.

First recall that uniform Π_2-reflection over the fragment $\mathbf{I\Sigma}_n$ is equivalent to the totality of the function F_{ω_n} from the fast-growing hierarchy. This has been established by Paris [**57**], using model-theoretic methods. Beklemishev [**7**] gives a proof-theoretic argument based on transfinite iterations of reflection. It is not completely clear whether these papers prove the equivalence for each standard number n or whether n can be seen as a free variable. The present author [**18**] has provided an explicit proof of

$$\mathbf{I\Sigma}_1 \vdash \forall_x(\mathrm{RFN}_{\mathbf{I\Sigma}_x}(\Pi_2) \leftrightarrow F_{\omega_x}\!\downarrow),$$

by formalizing an ordinal analysis of Buchholz and Wainer [**13**]. Indeed, one can argue that the equivalence is already implicit in the work of Wainer [**95**] and Schwichtenberg [**81**]. By the definition of slow reflection we now see

$$\mathbf{I\Sigma}_1 \vdash \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2) \leftrightarrow \forall_x(F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow).$$

The function F_{ε_0} is defined by diagonalization over the previous functions in the fast-growing hierarchy: We have $F_{\varepsilon_0}(n) = F_{\omega_{n+1}}(n)$. To characterize slow reflection we diagonalize over the same functions, but in an extremely slow manner:

Definition 5.1.1 ([**19**]). The function $F^{\diamond}_{\varepsilon_0} : \mathbb{N} \to \mathbb{N}$ is defined by

$$F^{\diamond}_{\varepsilon_0}(n) := F_{\omega_{m+1}}(n) \quad \text{with} \quad m = F^{-1}_{\varepsilon_0}(n).$$

A Δ_0-definition of the graph of $F^{\diamond}_{\varepsilon_0}$ can be found in [**19**].

Uniform Π_2-reflection over Peano arithmetic is equivalent to the totality of F_{ε_0}. We can now establish a similar result for slow reflection:

Proposition 5.1.2 ([**19**]). *We have*

$$\mathbf{I\Sigma}_1 \vdash \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2) \leftrightarrow F^{\diamond}_{\varepsilon_0}\!\downarrow.$$

Proof. In view of the above it suffices to show

$$\mathbf{I\Sigma}_1 \vdash F^{\diamond}_{\varepsilon_0}\!\downarrow \leftrightarrow \forall_x(F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow).$$

To establish the direction from left to right we consider an arbitrary x with $F_{\varepsilon_0}(x)\!\downarrow$. It suffices to show $F_{\omega_{x+1}}(y)\!\downarrow$ for $y \geq F_{\varepsilon_0}(x)$. The latter implies $x \leq F^{-1}_{\varepsilon_0}(y)$. Using the assumption $F^{\diamond}_{\varepsilon_0}\!\downarrow$ we infer that $F_{\omega_{x+1}}(y) \leq F^{\diamond}_{\varepsilon_0}(y)$ is defined (see [**89**, Section 5] and [**26**, Section 2] for the required properties of the fast-growing hierarchy). For the other direction we have to show $F^{\diamond}_{\varepsilon_0}(y)\!\downarrow$ for arbitrary y. Setting $x := F^{-1}_{\varepsilon_0}(y)$ we see $F_{\varepsilon_0}(x)\!\downarrow$, by the definition of $F^{-1}_{\varepsilon_0}$. Using the right side of the equivalence we learn that $F^{\diamond}_{\varepsilon_0}(y) = F_{\omega_{x+1}}(y)$ is defined, as desired. □

Recall that $g : \mathbb{N} \to \mathbb{N}$ is called a provably total function of a theory $\mathbf{T}$ if we have $\mathbf{T} \vdash \forall_x \exists_y \theta(x, y)$ for some Σ_1-formula $\theta(x, y)$ which defines the relation $g(x) = y$ (note that the existence of a Σ_1-definition implies that g is computable). The previous result implies that $F^{\diamond}_{\varepsilon_0}$ is a provably total function of $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_n)$. Let us make the following observation:

Proposition 5.1.3 ([**19**]). *Any provably total function of Peano arithmetic is eventually dominated by* $F^{\diamond}_{\varepsilon_0}$.

Proof. By the result of Wainer [**95**] and Schwichtenberg [**81**] it suffices to show that $F^{\diamond}_{\varepsilon_0}$ dominates any of the functions F_{ω_m} for $m \in \mathbb{N}$. Set $N := F_{\varepsilon_0}(m)$ and observe $m \leq F^{-1}_{\varepsilon_0}(n)$ for $n \geq N$. The definition of $F^{\diamond}_{\varepsilon_0}$ yields $F_{\omega_m}(n) \leq F^{\diamond}_{\varepsilon_0}(n)$. □

We point out that the previous proof cannot be formalized in Peano arithmetic, which does not know that the required value $F_{\varepsilon_0}(m)$ is defined. Indeed, we will get

$$\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2) \nvdash \forall_x\, F_{\omega_x}\!\downarrow$$

by Theorem 5.1.5 below. Using the previous proposition we can show that slow reflection does not follow from usual reflection statements of lower logic complexity:

Proposition 5.1.4 ([**20**]). *For each number* $n > 0$ *we have*

$$\mathbf{PA} + \mathrm{RFN}_{\mathbf{PA}}(\Pi_n) \nvdash \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_{n+1}).$$

Proof. For $n = 1$ we recall that Π_1-reflection is equivalent to consistency (this also explains why the claim fails for $n = 0$). Thus it suffices to show

$$\mathbf{PA} + \mathrm{Con}(\mathbf{PA}) \nvdash \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2).$$

This holds because the Π_1-formula $\mathrm{Con}(\mathbf{PA})$ cannot add a provably total function, while $\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2)$ adds the function $F^{\diamond}_{\varepsilon_0}$. For $n \geq 2$ we observe that $\mathrm{RFN}_{\mathbf{PA}}(\Pi_n)$ implies the totality of F_{ε_0}. In the presence of $F_{\varepsilon_0}\!\downarrow$ slow reflection collapses into the usual reflection statement. So if $\mathrm{RFN}_{\mathbf{PA}}(\Pi_n)$ did imply $\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_{n+1})$, then it would also imply $\mathrm{RFN}_{\mathbf{PA}}(\Pi_{n+1})$. That implication is unprovable by a classical result of Kreisel and Lévy [**49**]. □

The previous results provide lower bounds on the strength of slow reflection. Next, we present an upper bound on the computational content of $\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2)$:

THEOREM 5.1.5 ([**19**]). *Any provably total function of* $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2)$ *is eventually dominated by* F_{ε_0}.

PROOF (SKETCH). In view of Proposition 5.1.2 we can replace $\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2)$ by the statement that $F^{\diamond}_{\varepsilon_0}$ is total. As mentioned above, any provably total function of Peano arithmetic is dominated by one of the functions F_α with $\alpha < \varepsilon_0$. An analogous result can be established with $F^{\diamond}_{\varepsilon_0}$ as a new base function: First define a hierarchy of functions $F^{\diamond}_{\varepsilon_0+\alpha}$ by setting

$$\begin{aligned} F^{\diamond}_{\varepsilon_0+0}(n) &:= F^{\diamond}_{\varepsilon_0}(n), \\ F^{\diamond}_{\varepsilon_0+\alpha+1}(n) &:= (F^{\diamond}_{\varepsilon_0+\alpha})^{n+1}(n), \\ F^{\diamond}_{\varepsilon_0+\lambda}(n) &:= F^{\diamond}_{\varepsilon_0+\{\lambda\}(n)}(n) \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

In [**19**, Proposition 3.9] we have shown that any provably total function of the theory $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2)$ is dominated by a function $F^{\diamond}_{\varepsilon_0+\alpha}$ with $\alpha < \varepsilon_0$, adapting Buchholz and Wainer's [**13**] proof of the result for Peano arithmetic. To deduce the claim of the theorem we must bound the functions $F^{\diamond}_{\varepsilon_0+\alpha}$ in terms of F_{ε_0}: According to [**19**, Lemma 3.7] we have

$$(F^{\diamond}_{\varepsilon_0+\alpha})^l(n) \leq (F_{\omega_m+\alpha})^l(n)$$

for any $m > 0$ and any $\alpha \leq \omega_m$ which satisfy $(F_{\omega_m+\alpha})^l(n) \leq F_{\varepsilon_0}(m)$. This is shown by transfinite induction over α, with a side induction over l. For $\alpha = 0$ and $l = 1$ the assumption $n < F_{\omega_m}(n) \leq F_{\varepsilon_0}(m)$ ensures $F^{-1}_{\varepsilon_0}(n) < m$. By the definition of $F^{\diamond}_{\varepsilon_0}$ we can infer $F^{\diamond}_{\varepsilon_0}(n) \leq F_{\omega_m}(n)$, as promised. To conclude that $F^{\diamond}_{\varepsilon_0+\alpha}$ is dominated by F_{ε_0} one shows that

$$F_{\omega_n+\alpha}(n) \leq F_{\omega_n+\omega_n}(n) \leq F_{\omega_{n+1}}(n) = F_{\varepsilon_0}(n)$$

holds for sufficiently large n. By the above we get $F^{\diamond}_{\varepsilon_0+\alpha}(n) \leq F_{\omega_n+\alpha}(n) \leq F_{\varepsilon_0}(n)$, as desired. □

Since the usual Π_2-reflection principle over Peano arithmetic is equivalent to the totality of F_{ε_0} the theorem implies

$$\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_2) \nvdash \mathrm{RFN}_{\mathbf{PA}}(\Pi_2).$$

As a consequence of Theorem 5.1.7 below we will in fact have

$$\mathbf{PA} + \{\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_n) \,|\, n \in \mathbb{N}\} \nvdash \mathrm{Con}(\mathbf{PA}).$$

We proceed with a model-theoretic construction due to Sommer [**89**], which plays an important role in Friedman, Rathjen and Weiermann's [**26**] investigation of slow consistency. In the case of slow reflection we need a non-standard version of Sommer's result (see the explanation below):

PROPOSITION 5.1.6. *Consider a model* $\mathcal{M} \vDash \mathbf{PA}$ *and assume that we have*

$$\mathcal{M} \vDash F_{\omega_{n-1}^{\omega^\alpha \cdot c}}(a) = b,$$

where $n > 0$ *is standard,* a, b, c *are non-standard, and* $\alpha \in \mathcal{M}$ *codes an ordinal* $< \varepsilon_0$ *(in the sense of* $\mathcal{M}$*). Then there is an initial segment* $\mathcal{I} \subseteq \mathcal{M}$ *with* $a \in \mathcal{I}, b \notin \mathcal{I}$ *and* $\mathcal{I} \vDash \mathbf{I\Sigma}_n$*. If we have* $\alpha \in \mathcal{I}$ *then* $\mathcal{I}$ *satisfies transfinite* Π_n*-induction up to* ω_2^α*.*

Sommer [**89**, Theorem 5.25] proves the same result for the case of a standard ordinal α (in other words, the code of α must lie in $\mathbb{N} \subseteq \mathcal{M}$). For our application it will be crucial that α can be a non-standard element of $\mathcal{M}$. Most of Sommer's proof carries over to the non-standard case, but some arguments need to be adapted (for example, the proof of [**89**, Lemma 2.24] constructs a number m which depends on α; Sommer relies on the fact that α is standard in order to guarantee $a \geq m + n + 1$ for any non-standard $a \in \mathcal{M}$). In [**20**, Section 2] we go through Sommer's entire proof, in order to verify that all necessary changes can be accommodated. We also check that the result can be formalized in $\mathbf{ACA_0}$ (if one is satisfied with a partial satisfaction relation for the model $\mathcal{I}$). The theory $\mathbf{ACA_0}$ is convenient because it allows to express model-theoretic concepts in a natural way. Sommer [**89**, Section 6.4] shows that his arguments can be formalized in a much weaker theory. To conclude this section we deduce an upper bound on the consistency strength of slow reflection:

THEOREM 5.1.7 ([**20**]). *We have*

$$\mathbf{PA} + \mathrm{Con}(\mathbf{PA}) \vdash \mathrm{Con}(\mathbf{PA} + \{\mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_n) \mid n \in \mathbb{N}\}).$$

PROOF (SKETCH). We can argue in $\mathbf{ACA_0}$, which is conservative over Peano arithmetic. The assumption $\mathrm{Con}(\mathbf{PA})$ provides a model $\mathcal{M} \vDash \mathbf{PA}$, which we may assume to be non-standard. Using Gentzen's proofs of transfinite induction we get $\mathbf{PA} \vdash F_{\varepsilon_0}(m){\downarrow}$ and thus $\mathcal{M} \vDash F_{\varepsilon_0}(m){\downarrow}$ for each standard number m (this does not follow by Σ_1-completeness, because our meta theory does not prove the totality of F_{ε_0}). Overspill provides non-standard elements $a, b \in \mathcal{M}$ with $\mathcal{M} \vDash F_{\varepsilon_0}(a) = b$. Let $n > 0$ be an arbitrary standard number. Exploiting provable relations between the functions in the fast-growing hierarchy (see [**89**, Section 5] and [**26**, Section 2]) we obtain a non-standard $b' \leq b$ with

$$\mathcal{M} \vDash F_{\omega_{n-1}^{\omega_{a-n+1}\cdot(a+1)}}(a) = b'.$$

By the previous proposition this yields an initial segment $\mathcal{I} \subseteq \mathcal{M}$ with $a \in \mathcal{I}, b' \notin \mathcal{I}$ and $\mathcal{I} \vDash \mathbf{I\Sigma}_n$. The code of ω_x is primitive recursive in x, which implies that ω_{a-n} lies in $\mathcal{I}$. Thus $\mathcal{I}$ satisfies Π_n-induction up to the ordinal ω_{a-n+2}. Kreisel and Lévy [**49**] have observed the equivalence between transfinite induction and uniform reflection principles. The precise relation for fragments of arithmetic has been determined by Ono [**56**]. In [**20**, Proposition 1.3] we provide a uniform proof, which applies to non-standard fragments. Using this result we can infer $\mathcal{I} \vDash \mathrm{RFN}_{\mathbf{I\Sigma}_a}(\Pi_n)$. In view of $\mathcal{I} \not\ni b' \leq b = F_{\varepsilon_0}(a)$ we have $\mathcal{I} \nvDash F_{\varepsilon_0}(a){\downarrow}$. Thus we see that $\mathcal{I}$ satisfies

$$\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_n) \equiv \forall_x (F_{\varepsilon_0}(x){\downarrow} \to \mathrm{RFN}_{\mathbf{I\Sigma}_{x+1}}(\Pi_n)),$$

which establishes the consistency of $\mathbf{I\Sigma}_n + \mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_n)$. Since $n > 0$ was arbitrary we learn that $\mathbf{PA} + \{\mathrm{RFN}^{\diamond}_{\mathbf{PA}}(\Pi_n) \mid n \in \mathbb{N}\}$ is consistent, as desired. □

5.2. Transfinite Iterations of Slow Consistency

In this section we apply our analysis of slow reflection in order to prove a conjecture of Friedman, Rathjen and Weiermann [**26**]: The usual consistency statement for Peano arithmetic is equivalent to ε_0 iterations of slow consistency. We will also see that this goes down to ω iterations if we change the definition of slow consistency by an "index shift".

Classical results on iterated consistency and reflection can be found in the work of Turing [**93**] and Feferman [**15**]. In particular, Turing has shown that any true Π_1-formula follows from iterations of consistency along a well-order of type $\omega + 1$. Crucially, this result relies on the fact that arbitrary paths in Kleene's $\mathcal{O}$ are admitted: The truth of the given Π_1-formula is simply encoded into a particular definition of the order ω. The situation changes considerably when we restrict to iterations along a "natural" description of a well-order, such as the usual notation system for ε_0. Schmerl [**74**] has shown that we can characterize the Π_1-consequences of Peano arithmetic by iterating consistency along ε_0, starting with primitive recursive arithmetic. This result has been refined and extended by Beklemishev [**7**]. In a recent preprint [**22**], the present author has observed that bounds on iterated consistency can be read off from an ordinal analysis in terms of infinite proofs (Schmerl and Beklemishev use different methods). Roughly speaking, a Π_1-formula with a cut-free ω-proof of height α follows from α iterations of slow consistency. It may seem that this fact requires transfinite induction up to α, but this is not the case: Statements about iterated consistency can be established by the reflexive induction rule introduced by Schmerl [**74**] (who acknowledges a simplification due to Girard). For the base theory $\mathbf{I\Sigma}_1$ this rule tells us that

$$\mathbf{I\Sigma}_1 \vdash \forall_\alpha(\mathrm{Prov}_{\mathbf{I\Sigma}_1}(\forall_{\beta<\dot\alpha}\varphi(\beta)) \to \varphi(\alpha)) \qquad \text{implies} \qquad \mathbf{I\Sigma}_1 \vdash \forall_\alpha \varphi(\alpha).$$

The statement of the rule involves Feferman's [**15**] dot notation: By $\psi(\dot\alpha)$ we denote the Gödel number of the formula which results from $\psi(\gamma)$ by substituting the numeral $\overline{\alpha}$ for the free variable γ. The reflexive induction rule resembles an instance of transfinite induction, but it is actually a consequence of Löb's theorem.

For a finite extension $\mathbf{PA}+\varphi$ of Peano arithmetic, slow consistency is defined by the formula

$$\mathrm{Con}^\diamond(\mathbf{PA}+\varphi) :\equiv \forall_x(F_{\varepsilon_0}(x){\downarrow} \to \mathrm{Con}(\mathbf{I\Sigma}_{x+1}+\varphi)).$$

To describe finite iterations of slow consistency we set $\mathrm{Con}^\diamond_1(\mathbf{PA}) \equiv \mathrm{Con}^\diamond(\mathbf{PA})$ and

$$\mathrm{Con}^\diamond_{n+1}(\mathbf{PA}) \equiv \mathrm{Con}^\diamond(\mathbf{PA}+\mathrm{Con}^\diamond_n(\mathbf{PA})).$$

As shown by Friedman, Rathjen and Weiermann [**26**], we have

$$\mathbf{PA}+\mathrm{Con}(\mathbf{PA}) \vdash \mathrm{Con}(\mathbf{PA}+\mathrm{Con}^\diamond_n(\mathbf{PA}))$$

for each number $n \geq 1$. By Gödel's theorem we conclude that the usual consistency statement does not follow from finitely many iterations of slow consistency. Friedman, Rathjen and Weiermann have conjectured that the same holds for transfinite iterations below ε_0. The first task is to express these iterations. As pointed out by Beklemishev [**7**], the diagonal lemma offers an elegant way to do this: We can use it to construct a Π_1-formula $\mathrm{Con}^\diamond_\alpha(\mathbf{PA})$ with

$$\mathbf{I\Sigma}_1 \vdash \mathrm{Con}^\diamond_\alpha(\mathbf{PA}) \leftrightarrow \forall_{\beta<\alpha}\mathrm{Con}^\diamond(\mathbf{PA}+\mathrm{Con}^\diamond_{\dot\beta}(\mathbf{PA})).$$

It is easy to see that this results in a strict hierarchy:

LEMMA 5.2.1 ([**20**]). *For $\alpha,\beta \leq \varepsilon_0$ we have*

$$\mathbf{PA}+\mathrm{Con}^\diamond_\alpha(\mathbf{PA}) \vdash \mathrm{Con}^\diamond_\beta(\mathbf{PA}) \quad \textit{if and only if} \quad \alpha \geq \beta.$$

PROOF. The direction from right to left follows from our definition of iteration. For the other direction we argue by contradiction: Assume that we have $\alpha < \beta$

and $\mathbf{PA} + \mathrm{Con}^\diamond_\alpha(\mathbf{PA}) \vdash \mathrm{Con}^\diamond_\beta(\mathbf{PA})$. The definition of iterated consistency yields

$$\mathbf{PA} + \mathrm{Con}^\diamond_\alpha(\mathbf{PA}) \vdash \mathrm{Con}^\diamond(\mathbf{PA} + \mathrm{Con}^\diamond_\alpha(\mathbf{PA})).$$

This leads to a contradiction with Gödel's theorem, as shown in [**26**]. □

According to [**20**, Proposition 3.3] the theories $\mathbf{T}_\alpha := \mathbf{PA} + \mathrm{Con}^\diamond_\alpha(\mathbf{PA})$ satisfy the recursion

$$\begin{aligned} \mathbf{T}_0 &\equiv \mathbf{PA}, \\ \mathbf{T}_{\alpha+1} &\equiv \mathbf{PA} + \mathrm{Con}^\diamond(\mathbf{T}_\alpha), \\ \mathbf{T}_\lambda &\equiv \mathbf{PA} + \forall_{\alpha<\lambda} \mathrm{Con}^\diamond(\mathbf{T}_\alpha) \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

Note that the finite iterations coincide with those above. Previous work on iterated consistency would suggest the weaker limit $\mathbf{T}'_\lambda = \mathbf{PA} \cup \{\mathrm{Con}^\diamond(\mathbf{T}'_\alpha) \mid \alpha < \lambda\}$. However, the definition of slow consistency for infinite extensions of Peano arithmetic is not obvious: Should $F_{\varepsilon_0}(x)\!\downarrow$ imply $\mathrm{Con}(\mathbf{I\Sigma}_{x+1} \cup \{\mathrm{Con}^\diamond(\mathbf{T}'_\alpha) \mid \alpha < \lambda\})$ or rather $\mathrm{Con}(\mathbf{I\Sigma}_{x+1} + \mathrm{Con}^\diamond(\mathbf{T}'_{\{\lambda\}(x)}))$, where $\{\lambda\}(x)$ refers to the fundamental sequence of λ? We avoid this question by working with finite extensions of $\mathbf{PA}$. Let us now present the main result of this section:

THEOREM 5.2.2 ([**20**]; independently [**34**]). *We have*

$$\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}) \leftrightarrow \mathrm{Con}^\diamond_{\varepsilon_0}(\mathbf{PA}).$$

PROOF (SKETCH). For the direction from left to right we observe that reflection allows iterations of consistency: If $\mathbf{T} \vdash \neg\varphi$ implies $\neg\varphi$, then φ implies $\mathbf{T} \nvdash \neg\varphi$, which means $\mathrm{Con}(\mathbf{T} + \varphi)$. Applying this idea to slow consistency we get

$$\mathbf{PA} + \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2) \vdash \mathrm{Con}^\diamond_\beta(\mathbf{PA}) \to \mathrm{Con}^\diamond(\mathbf{PA} + \mathrm{Con}^\diamond_\beta(\mathbf{PA})).$$

By the defining equivalence of $\mathrm{Con}^\diamond_\alpha(\mathbf{PA})$ this yields

$$\mathbf{PA} + \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2) \vdash \forall_{\beta<\alpha} \mathrm{Con}^\diamond_\beta(\mathbf{PA}) \to \mathrm{Con}^\diamond_\alpha(\mathbf{PA}).$$

As shown by Gentzen, Peano arithmetic proves transfinite induction up to each fixed ordinal $\alpha < \varepsilon_0$. For such an ordinal we can thus infer

$$\mathbf{PA} + \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2) \vdash \mathrm{Con}^\diamond_\alpha(\mathbf{PA}).$$

Now we invoke the assumption $\mathrm{Con}(\mathbf{PA})$: By Theorem 5.1.7 it implies the consistency of $\mathbf{PA} + \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2)$. Since consistency entails Π_1-reflection we learn that $\mathrm{Con}^\diamond_\alpha(\mathbf{PA})$ is true for any $\alpha < \varepsilon_0$. This implies $\mathrm{Con}^\diamond_{\varepsilon_0}(\mathbf{PA})$, as desired. The other direction relies on Schmerl's [**74**] result on iterated consistency, which we have already mentioned above. It tells us that we have

$$\mathbf{PA} \vdash \mathrm{Con}_{\varepsilon_0}(\mathbf{I\Sigma}_1) \to \mathrm{Con}(\mathbf{PA}),$$

where $\mathrm{Con}_\alpha(\mathbf{I\Sigma}_1)$ refers to iterations of usual consistency over $\mathbf{I\Sigma}_1$. We observe that $\mathrm{Con}^\diamond(\mathbf{PA})$ implies $\mathrm{Con}(\mathbf{I\Sigma}_1)$, since the premise $F_{\varepsilon_0}(0)\!\downarrow$ in the slow consistency statement is available by Σ_1-completeness. For the transfinite iterations we obtain

$$\mathbf{I\Sigma}_1 \vdash \forall_{\alpha\leq\varepsilon_0}(\mathrm{Con}^\diamond_\alpha(\mathbf{PA}) \to \mathrm{Con}_\alpha(\mathbf{I\Sigma}_1)).$$

Note that transfinite induction up to ε_0 is not available. Instead, the claim is established by the reflexive induction rule discussed above. Combining the previous implications we see that $\mathrm{Con}^\diamond_{\varepsilon_0}(\mathbf{PA})$ implies $\mathrm{Con}(\mathbf{PA})$, as desired. □

In particular we can conclude

$$\mathbf{PA} + \mathrm{Con}(\mathbf{PA}) \vdash \mathrm{Con}^\diamond_\alpha(\mathbf{PA})$$

for each $\alpha < \varepsilon_0$, as conjectured in [**26**, Remark 4.4]. In the introduction to the present chapter we have mentioned that the original slow consistency statement $\mathrm{Con}^*(\mathbf{PA})$ of Rathjen, Friedman and Weiermann and our statement $\mathrm{Con}^\diamond(\mathbf{PA})$ differ by an index shift. Since $\mathrm{Con}^*(\mathbf{PA})$ is slightly weaker than $\mathrm{Con}^\diamond(\mathbf{PA})$, it is straightforward to see that the previous results remain valid for the original definition of slow consistency. On the other hand, the picture changes considerably if we look at the stronger variants

$$\mathrm{Con}^\dagger(\mathbf{PA}+\varphi) \equiv \forall_x(F_{\varepsilon_0}(x)\!\downarrow \to \mathrm{Con}(\mathbf{I\Sigma}_{x+2}+\varphi))$$

and

$$\mathrm{RFN}^\dagger_{\mathbf{PA}}(\Pi_n) \equiv \forall_x(F_{\varepsilon_0}(x)\!\downarrow \to \mathrm{RFN}_{\mathbf{I\Sigma}_{x+2}}(\Pi_n)).$$

The change is particularly dramatic in the case of slow reflection:

Proposition 5.2.3 ([**20**]). *For each $n > 1$ we have*

$$\mathbf{PA} + \mathrm{RFN}^\dagger_{\mathbf{PA}}(\Pi_n) \vdash \mathrm{RFN}_{\mathbf{PA}}(\Pi_n).$$

Proof. To see that $\mathrm{RFN}^\dagger_{\mathbf{PA}}(\Pi_n)$ collapses into the usual reflection principle we show that the †-variant of slow reflection implies the totality of F_{ε_0}. Arguing by induction over x, let us assume that $F_{\varepsilon_0}(x)\!\downarrow$ holds. By the definition of $\mathrm{RFN}^\dagger_{\mathbf{PA}}(\Pi_n)$ this means that the reflection statement $\mathrm{RFN}_{\mathbf{I\Sigma}_{x+2}}(\Pi_n)$ is available. In $\mathbf{I\Sigma}_{x+2}$ one can establish Π_2-induction up to $\omega^{x+2}_{x+1} < \omega_{x+2}$, by Gentzen's classical construction (see [**89**] for the precise bounds in fragments of arithmetic). In particular $\mathbf{I\Sigma}_{x+2}$

proves that $F_{\varepsilon_0}(x+1)=F_{\omega_{x+2}}(x+1)=F_{\omega^{x+2}_{x+1}}(x+1)$ is defined. Using reflection we infer that $F_{\varepsilon_0}(x+1)\!\downarrow$ is true, as needed for the induction step. □

This explains why we focus on the $\diamond$-variant: It yields the strongest non-trivial reflection statements. In the next section we will use our analysis of slow reflection to establish optimal bounds on the length of certain proofs. These optimal bounds cannot be deduced from the $*$-variant of slow reflection, which is weaker. In the case of slow consistency the $\dagger$-variant does not collapse completely, but the hierarchy of iterated consistency statements (defined as above) becomes much shorter:

PROPOSITION 5.2.4 ([**20**]; independently [**34**]). *We have*

$$\mathbf{PA}\vdash\mathrm{Con}(\mathbf{PA})\leftrightarrow\mathrm{Con}^\dagger_\omega(\mathbf{PA}).$$

PROOF (SKETCH). Recall that Friedman, Rathjen and Weiermann [**26**] show

$$\mathbf{PA}+\mathrm{Con}(\mathbf{PA})\vdash\mathrm{Con}(\mathbf{PA}+\mathrm{Con}^\diamond_n(\mathbf{PA})),$$

by induction on n in the meta theory. Their argument is not affected by the index shift, and the induction is easily internalized. Thus we get

$$\mathbf{PA}+\mathrm{Con}(\mathbf{PA})\vdash\forall_n\,\mathrm{Con}(\mathbf{PA}+\mathrm{Con}^\dagger_{\dot n}(\mathbf{PA})).$$

The formula on the right implies $\mathrm{Con}^\dagger_\omega(\mathbf{PA})$, as needed for direction "$\rightarrow$" of the claim. To establish the other direction we prove

$$\mathbf{PA}\vdash\mathrm{Con}^\dagger_{n+1}(\mathbf{PA})\rightarrow\forall_{k\leq n}\,\mathrm{Con}(\mathbf{I\Sigma}_{k+1}+\mathrm{Con}^\dagger_{\dot n-\dot k}(\mathbf{PA})).$$

We argue by induction over k: Recall that $\mathbf{I\Sigma}_{k+1}$ shows $F_{\varepsilon_0}(k)\!\downarrow$, as in the previous proof. Also note that $\mathrm{Con}^\dagger_{n-k}(\mathbf{PA})$ is equivalent to $\mathrm{Con}^\dagger(\mathbf{PA}+\mathrm{Con}^\dagger_{n-(k+1)}(\mathbf{PA}))$. By definition of the $\dagger$-variant of slow consistency this yields

$$\mathbf{I\Sigma}_{k+1}+\mathrm{Con}^\dagger_{n-k}(\mathbf{PA})\vdash\mathrm{Con}(\mathbf{I\Sigma}_{k+2}+\mathrm{Con}^\dagger_{n-(k+1)}(\mathbf{PA})).$$

Since consistency entails Π_1-reflection, we learn that the statement on the right follows from $\mathrm{Con}(\mathbf{I\Sigma}_{k+1}+\mathrm{Con}^\dagger_{n-k}(\mathbf{PA}))$, as required for the induction step. In particular we have shown that $\mathrm{Con}^\dagger_{n+1}(\mathbf{PA})$ implies $\mathrm{Con}(\mathbf{I\Sigma}_{n+1})$, which means that $\mathrm{Con}^\dagger_\omega(\mathbf{PA})\equiv\forall_n\,\mathrm{Con}^\dagger_n(\mathbf{PA})$ implies $\forall_n\,\mathrm{Con}(\mathbf{I\Sigma}_n)\equiv\mathrm{Con}(\mathbf{PA})$. □

To conclude this section we mention that Henk and Pakhomov [**34**] introduce a "square root" variant of slow consistency, which reaches the usual consistency statement after two iterations. They do not consider the notion of slow reflection.

5.3. Proof Lengths for the Paris-Harrington Principle

In this section we present natural mathematical theorems which only have extremely long proofs in certain fragments of Peano arithmetic. Surprisingly, the notion of slow reflection plays a crucial role in our arguments, even though it does not occur in the statement of the results.

Recall that true Σ_1-statements can be established in very weak fragments of arithmetic: Choose a witness to the existential quantifier and prove the resulting Δ_0-formula by an explicit verification of finitely many cases. Let us observe that these trivial proofs are long if the existential witness is large. In this case we would like to find shorter proofs, which explain the truth of the Σ_1-formula in a more meaningful way. Refining the traditional independence question, we ask which theory is required to formalize a *feasible* proof of a given statement. In order to answer this question we will apply the following observation: If we have a computational bound on reflection, then we know that the smallest witness to an existential statement cannot be too much bigger than the size of its proof. Conversely, the proof must be long if the minimal witness is large. This has been observed before (see e.g. [**36**]), but the following application appears to be new:

PROPOSITION 5.3.1 ([**19**]). *For any fixed number k, there is no primitive recursive way to construct proofs of the statements $F_{\varepsilon_0}(n)\downarrow$ in the theory $\mathbf{I\Sigma}_k$.*

Note that we write n rather than $\overline{n}$ for the n-th numeral.

PROOF. Consider a function $g:\mathbb{N}\to\mathbb{N}$ such that $g(n)$ is the Gödel number of an $\mathbf{I\Sigma}_k$-proof of $F_{\varepsilon_0}(n)\downarrow$. We have to show that g cannot be primitive recursive. In fact we will prove that g dominates any provably total function of Peano arithmetic, which is a much stronger claim. The point is that Peano arithmetic proves uniform Σ_1-reflection over $\mathbf{I\Sigma}_k$. Prenexing quantifiers we obtain

$$\mathbf{PA}\vdash\forall_{x,p}\exists_y(\mathrm{Proof}_{\mathbf{I\Sigma}_k}(p,F_{\varepsilon_0}(\dot{x})\downarrow)\to F_{\varepsilon_0}(x)=y),$$

where $\mathrm{Proof}_{\mathbf{I\Sigma}_k}(p,\varphi)$ expresses that p codes an $\mathbf{I\Sigma}_k$-proof of φ. This shows that there is a $\mathbf{PA}$-provably total function $f:\mathbb{N}^2\to\mathbb{N}$ such that we have

$$F_{\varepsilon_0}(n)=f(p,n)\qquad\text{whenever } p \text{ is an } \mathbf{I\Sigma}_k\text{-proof of } F_{\varepsilon_0}(n)\downarrow.$$

It is straightforward to transform f into a $\mathbf{PA}$-provably total function $f':\mathbb{N}^2\to\mathbb{N}$ which is monotone in the first argument and dominates f. Now consider an arbitrary $\mathbf{PA}$-provably total function $h:\mathbb{N}\to\mathbb{N}$. Aiming at a contradiction we assume that $g(n) \leq h(n)$ holds for arbitrarily large values of n. As $g(n)$ codes an $\mathbf{I\Sigma}_k$-proof of $F_{\varepsilon_0}(n)\downarrow$ we obtain

$$F_{\varepsilon_0}(n) = f(g(n), n) \leq f'(g(n), n) \leq f'(h(n), n) < f'(h(n), n) + 1.$$

This is impossible, because the **PA**-provably total function $n \mapsto f'(h(n), n) + 1$ must eventually be dominated by F_{ε_0}. □

Let us point out that the previous result yields an asymptotic bound: It shows that the function $n \mapsto$ "the smallest $\mathbf{I\Sigma}_k$-proof of $F_{\varepsilon_0}(n)\downarrow$" grows very fast, but it does not say anything about the initial values of that function. This is both a defect and a virtue: Certainly it would be more satisfying to exhibit a single statement with a long proof. On the other hand, asymptotic results are more manageable, because we do not have to estimate any constants (which arise e.g. from the chosen arithmetization of syntax). A theorem of H. Friedman (presented by Smith [**88**]) states that Π^1_2-bar induction is not enough to formalize a feasible proof of one particular Σ_1-statement. This is much more impressive than the results in the present section, but it is hard to work out all details of the proof. To see how the previous proposition can be improved we make the following observation:

PROPOSITION 5.3.2. *There is a primitive recursive construction which maps each number n to a proofs of $F_{\varepsilon_0}(n)\downarrow$ in $\mathbf{I\Sigma}_{n+1}$.*

PROOF. As observed in the proof of Proposition 5.2.3 we get $\mathbf{I\Sigma}_{n+1} \vdash F_{\varepsilon_0}(n)\downarrow$ via Gentzen's proofs of ordinal induction. It is straightforward to see that these proofs are constructed in a primitive recursive way. □

We want to show that no primitive recursive construction yields proofs of $F_{\varepsilon_0}(n)\downarrow$ in $\mathbf{I\Sigma}_n$. Compared to Proposition 5.3.1 this means that we have to vary k alongside with n. At first sight it may seem that this cannot be achieved by the same method: To get an asymptotic bound for a sequence of proofs in a theory $\mathbf{T}$ we have worked with the Σ_1-reflection principle over $\mathbf{T}$. Which theory should we consider in order to cover all the fragments $\mathbf{I\Sigma}_n$ at once? As it turns out, slow reflection creates just the right interplay between proof length and the complexity of the induction axioms that are used in a proof. To elaborate this idea, let us begin with the slow provability predicate

$$\mathrm{Prov}^{\diamond}_{\mathbf{PA}}(\varphi) :\equiv \exists_x (F_{\varepsilon_0}(x)\downarrow \wedge \mathrm{Prov}_{\mathbf{I\Sigma}_{x+1}}(\varphi)).$$

It was first considered by Rathjen [**70**], who showed that it satisfies the axioms of Gödel-Löb provability logic. The joint provability logic of slow and usual provability was determined by Henk and Pakhomov [**34**]. One can easily verify

$$\mathbf{I\Sigma}_1 \vdash \mathrm{Con}^\diamond(\mathbf{PA}) \leftrightarrow \neg\,\mathrm{Prov}^\diamond_{\mathbf{PA}}(0=1),$$

as well as

$$\mathbf{I\Sigma}_1 \vdash \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_n) \leftrightarrow \forall_\varphi(\text{“}\varphi \text{ a closed } \Pi_n\text{-formula”} \wedge \mathrm{Prov}^\diamond_{\mathbf{PA}}(\varphi) \to \mathrm{True}_{\Pi_n}(\varphi)).$$

The present author has observed that slow provability can in turn be derived from a notion of slow proof (we write $\langle\cdot,\cdot\rangle$ for the Cantor pairing function):

Definition 5.3.3 ([**19**]). A slow proof of φ is a pair $p = \langle q, N\rangle$ where q codes an $\mathbf{I\Sigma}_{n+1}$-proof of φ and we have $F_{\varepsilon_0}(n) = N$, for some number n. This can be expressed by a formula $\mathrm{Proof}^\diamond_{\mathbf{PA}}(p,\varphi)$ which is Δ_1 in $\mathbf{I\Sigma}_1$ (because of $n \leq N$).

The point is that a slow proof becomes extremely large if it uses complex induction axioms, due to its second component. As expected we have

$$\mathbf{I\Sigma}_1 \vdash \mathrm{Prov}^\diamond_{\mathbf{PA}}(\varphi) \leftrightarrow \exists_p\ \mathrm{Proof}^\diamond_{\mathbf{PA}}(p,\varphi).$$

Using the notion of slow proof, Proposition 5.3.1 can be improved as follows:

Theorem 5.3.4 ([**19**]). *For sufficiently large values of n, any $\mathbf{I\Sigma}_n$-proof of the statement $F_{\varepsilon_0}(n)\downarrow$ must have code above $F_{\varepsilon_0}(n-1)$. In particular there is no primitive recursive way to construct proofs of $F_{\varepsilon_0}(n)\downarrow$ in $\mathbf{I\Sigma}_n$.*

Proof (Sketch). Uniform Σ_1-reflection for the notion of slow proof is (trivially) provable in $\mathbf{PA} + \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2)$. From Theorem 5.1.5 we know that any provably total function of that theory is dominated F_{ε_0}. In fact one can show a bit more (see [**19**, Theorem 3.10]): If g is a provably total function of $\mathbf{PA} + \mathrm{RFN}^\diamond_{\mathbf{PA}}(\Pi_2)$, then we have

$$g(F_{\varepsilon_0}(n-1)) \leq F_{\varepsilon_0}(n)$$

for sufficiently large values of n. Similarly to the proof of Proposition 5.3.1, the following can be deduced for any primitive recursive function h: If n is sufficiently large, then we have

$$p > h(F_{\varepsilon_0}(n-1)) \qquad \text{for any slow proof } p \text{ of } F_{\varepsilon_0}(n)\downarrow.$$

In the present proof we take $h(m) = \langle m, m\rangle$ to be the diagonal of the Cantor pairing function. Aiming at a contradiction, let us now assume that $q \leq F_{\varepsilon_0}(n-1)$

codes an $\mathbf{I\Sigma}_n$-proof of $F_{\varepsilon_0}(n)\downarrow$. Then $p = \langle q, F_{\varepsilon_0}(n-1)\rangle$ is a slow proof of the same statement. By the monotonicity of the Cantor pairing function we obtain

$$p \leq \langle F_{\varepsilon_0}(n-1), F_{\varepsilon_0}(n-1)\rangle = h(F_{\varepsilon_0}(n-1)).$$

For sufficiently large n this contradicts the inequality above. $\square$

In the rest of this section we present similar bounds for instances of the Paris-Harrington principle. Let us begin with some notation: By $|x|$ we denote the cardinality of a finite set $x \subseteq \mathbb{N}$. We say that x is large if it is non-empty and satisfies $|x| \geq \min(x)$. Let us also write

$$[x]^n = \{y \subseteq x\,;\, |y| = n\}.$$

Given a function $f : [z]^n \to k$, a subset $x \subseteq z$ is called f-homogeneous if f is constant on $[x]^n$. Where appropriate we write $N = \{0, \dots, N-1\}$. Following [**58**] we now consider the relation

$$\mathrm{PH}(k,m,n,N) \quad :\equiv \quad \begin{array}{c}\text{“for any function } f : [N]^n \to k \text{ there is a large} \\ f\text{-homogeneous subset } x \subseteq N \text{ with } |x| \geq m\text{”.}\end{array}$$

We will refer to k as the number of colours. The Paris-Harrington principle (or Strengthened Finite Ramsey Theorem) asserts $\forall_{k,m,n}\exists_N \mathrm{PH}(k,m,n,N)$. By the famous independence result of Paris and Harrington [**58**], this principle is true but unprovable in Peano arithmetic. For fixed values of k, m and n the Σ_1-formula $\exists_N \mathrm{PH}(k,m,n,N)$ can be established in a very weak theory, as explained above. We obtain the following result on proof length:

THEOREM 5.3.5 ([**19**]). *For sufficiently large values of n, any proof of the statement $\exists_N \mathrm{PH}(8, n+4, n+3, N)$ in the theory $\mathbf{I\Sigma}_n$ has code above $F_{\varepsilon_0}(n-1)$.*

For $i = 3, 4$ we assume that $\overline{n+i}$ and $\overline{n}+i$ are the same term. This assumption is convenient but not necessary (cf. the proof of Theorem 5.3.7 below).

PROOF (SKETCH). Ketonen and Solovay [**46**] have shown that the minimal witness for $\exists_N \mathrm{PH}(8, n+4, n+3, N)$ is at least as big as $F_{\varepsilon_0}(n)$. Now one can argue as in the proof of Theorem 5.3.4, with the formula $\mathrm{PH}(8, x+4, x+3, y)$ at the place of $F_{\varepsilon_0}(x) = y$. $\square$

Let us compare our bound with the following known result:

PROPOSITION 5.3.6. *There is a primitive recursive construction which maps k and n to a proof of $\forall_m\exists_N\ \mathrm{PH}(k,m,n+3,N)$ in the theory $\mathbf{I\Sigma}_{n+2}$.*

Proof. According to [**37**, Section II.2(c)] the theory $\mathbf{I\Sigma}_1$ shows that the desired sequence of proofs exists. Since the provably total functions of $\mathbf{I\Sigma}_1$ are primitive recursive this implies the claim. □

In view of this result it is not clear whether the fragment $\mathbf{I\Sigma}_n$ in Theorem 5.3.5 is optimal, or whether it can be strengthened to $\mathbf{I\Sigma}_{n+1}$. This question is still open. However, we can reach the optimal fragment if we vary the number of colours:

Theorem 5.3.7 ([**19**]). *There is an elementary function $c : \mathbb{N} \to \mathbb{N}$ with the following property: For sufficiently large n, any proof of $\exists_N \mathrm{PH}(c(n), n+4, n+3, N)$ in the theory $\mathbf{I\Sigma}_{n+1}$ has code above $F_{\varepsilon_0}(n)$.*

We clarify that $c(n)$ refers to the numeral $\overline{c(n)}$ rather than the expression $c(\overline{n})$.

Proof (Sketch). We set $c(n) = 10^{35(n+1)^2}$. Then the minimal witness for $\exists_N \mathrm{PH}(c(n), n+4, n+3, N)$ is at least as big as $F_{\varepsilon_0}(n+1)$, as shown by Ketonen and Solovay [**46**] (copying their result would suggest $c(n) = 10^{23(n+1)^2}$, but the bound needs to be adapted because of a slightly different definition of fundamental sequences, cf. [**19**, Lemma 2.4]). The result is deduced similarly to Theorem 5.3.4, but there is one new subtlety: The argument from Proposition 5.3.1 yields bounds for a sequence of formulas $\exists_y \theta(\overline{n}, y)$ which are parametrized by the n-th numeral. As in the case of the previous theorem we may assume that we have $\overline{n+4} \equiv \overline{n}+4$ and $\overline{n+3} \equiv \overline{n}+3$. However, there is no term $t(x)$ with $\overline{c(n)} \equiv t(\overline{n})$. Thus the statements $\exists_N \mathrm{PH}(c(n), n+4, n+3, N)$ are not of the required form. In order to prove the desired bound we need to "preprocess" proofs. For this purpose one constructs a Σ_1-formula $\theta(n, N)$ and a primitive recursive function $h_0 : \mathbb{N} \to \mathbb{N}$ with the following properties: Firstly, the statements $\theta(n, N)$ and $\mathrm{PH}(c(n), n+4, n+3, N)$ are equivalent (in the standard structure of natural numbers). Secondly, if q is an $\mathbf{I\Sigma}_{n+1}$-proof of $\exists_N \mathrm{PH}(c(n), n+4, n+3, N)$ then $h_0(q)$ is an $\mathbf{I\Sigma}_{n+1}$-proof of $\exists_N \theta(n, N)$. As in the proof of Theorem 5.3.4 one can show that the following holds for any primitive recursive function h: If n is sufficiently large then we have

$$p > h(F_{\varepsilon_0}(n)) \qquad \text{for any slow proof } p \text{ of } \exists_N \theta(n, N).$$

In the case of Theorem 5.3.4 we have considered $h(m) = \langle m, m \rangle$. For the present application we need to set

$$h(m) := \max\{\langle h_0(q), m \rangle) \mid q \leq m\}.$$

Now the claim can be established by contradiction: Assume that $q \le F_{\varepsilon_0}(n)$ is an $\mathbf{I\Sigma}_{n+1}$-proof of $\exists_N \mathrm{PH}(c(n), n+4, n+3, N)$. It follows that $h_0(q)$ is an $\mathbf{I\Sigma}_{n+1}$-proof of $\exists_N \theta(n, N)$, so that $p := \langle h_0(q), F_{\varepsilon_0}(n)\rangle$ is a slow proof of the same statement. By the definition of h we have

$$p = \langle h_0(q), F_{\varepsilon_0}(n)\rangle \le h(F_{\varepsilon_0}(n)).$$

For sufficiently large n this contradicts the inequality above. $\square$

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