Representations and Cohomology of Algebraic Groups

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Abstract

Let G be a semisimple simply connected linear algebraic group over an algebraically closed field k of characteristic p. In [11], Donkin gave a recursive description for the characters of cohomology of line bundles on the flag variety G/B with $G = \mathrm{SL}_3$. In chapter 2 of this thesis we try to give a non recursive description for these characters. In chapter 3, we give the first step of a version of formulae in [11] for $G = G_2$.

In his famous paper [7], Demazure introduced certain indecomposable modules and used them to give a short proof of the Borel-Weil-Bott theorem (characteristic zero). In chapter 5 we give the cohomology of these modules.

In a recent paper [17], Doty introduces the notion of r−minuscule weight and exhibits a tensor product factorization of a corresponding tilting module under the assumption $p \geq 2h - 2$, where h is the Coxeter number. In chapter 4, we remove the restriction on p and consider some variations involving the more general notion of (p, r)−minuscule weights.

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Declaration

I declare that all the work in this thesis is my original work unless stated otherwise. The contents of chapter 4 and chapter 5 have been published see [1] and [2].

Chapter 1

Preliminaries

In this chapter we will set up the notation and outline some of the basic concepts in the representation theory of algebraic groups. Most of the material is given in [18], [19] and [20]. The author has also consulted [26] and [25] for some details. Throughout this thesis k will denote an algebraically closed field. Moreover $\mathbb{Z}$ will denote the ring of integers, the field of real numbers will be denoted by $\mathbb{R}$ and $\mathbb{Q}$ will denote the field of rational numbers. Also G will always denote a group and our groups will be algebraic unless stated otherwise. The set of all $n \times n$ matrices with entries in k is denoted by $M_n(k)$ and we will write $\mathrm{GL}_n(k)$ for the group of $n \times n$ invertible matrices with entries in k and call it the general linear group.

1.1 Representation Theory

Definition 1.1.1. *Let k be a field. A $k-$algebra is a ring A that is also a $k-$vector space and*

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

for all $\lambda \in k$*,* $a, b \in A$*.*

We will now construct the group algebra that we will denote by kG. Let G be a group and k be a field. Consider a set $\{a_g | g \in G\}$ and define kG to be a vector space with basis $a_g, g \in G$. We will now define multiplication on kG. If $\sum_{x \in G} \lambda_x a_x$ and $\sum_{y \in G} \mu_y a_y$ are elements in kG then

$$(\sum_{x \in G} \lambda_x a_x)(\sum_{y \in G} \mu_y a_y) = \sum_{x,y \in G} \lambda_x \mu_y a_x a_y = \sum_{x,y \in G} \lambda_x \mu_y a_{xy}.$$

Here xy denotes the multiplication of x and y in G. Since G is a group we have $xy = g$ for some $g \in G$ and we can write

$$\sum_{x,y \in G} \lambda_x \mu_y = \sum_{g \in G} \lambda_{gx^{-1}} \mu_x = \sum_{g \in G} \tau_g$$

where $\tau_g = \sum_{x \in G} \lambda_{gx^{-1}} \mu_x$. It is easy to show that this operation makes kG into a $k-$algebra.

Definition 1.1.2. *Let* G *be a group. A matrix representation of* G *is a group homomorphism* $\rho : G \to \mathrm{GL}_n(k)$*. We call* n *the degree of representation* ρ*. Similarly a matrix representation of a group algebra* kG *is a* $k-$*algebra homomorphism*

$$\phi : kG \to M_n(k)$$

for some n*.*

Let $\phi : kG \to M_n(k)$ be a matrix representation of the algebra kG. Define V to be the space of column vectors of length n with entries in k then we can make V into a $kG-$module by defining $gv = \phi(g)v$, for $g \in kG$ and $v \in V$. The module V is called the module afforded by ϕ. Conversely let V be a $kG-$module with basis $v_1, v_2, ..., v_n$ then we have $gv_i \in V$ for all $g \in kG$. So gv_i

is a linear combination of $v_1, v_2, ..., v_n$ with the coefficients of v_j depending on i, j and g. We have

$$gv_i = \sum_{j=1}^{n} \phi_{ji}(g)v_j.$$

Define a map $\phi : G \to \mathrm{GL}_n(k)$ by $\phi(g) = (\phi_{ij}(g))$. It is easy to check that this is a group homomorphism and hence a matrix representation of G. We call this the matrix representation of G afforded by the module V with respect to the basis $v_1, ..., v_n$.

Definition 1.1.3. *A non-trivial $kG-$module V is called simple (irreducible) if its only submodules are $\{0\}$ and V itself. V is called completely reducible (semisimple) if it is the direct sum of simple submodules.*

Theorem 1.1.1. *(Maschke's Theorem) Let G be a finite group and k be a field. If the characteristic of k does not divide the order of G then every finite dimensional $kG-$module is completely reducible.*

Recall that the character of a matrix representation $\rho : G \to \mathrm{GL}_n(k)$ is the function $\chi : G \to k$ defined by

$$\chi(g) = \mathrm{trace}(\rho(g))$$

for $g \in G$.

Proposition 1.1.2. *(Schur's Lemma) Let V, W be finite dimensional simple $kG-$modules (k algebraically closed) and let $\theta : V \to W$ be a $kG-$module homomorphism.*

1. *If V is not isomorphic to W then $\theta = 0$.*

2. If $V = W$ *then there is a constant* $\lambda \in k$ *such that*

$$\theta(v) = \lambda v$$

for all $v \in V$.

1.2 Algebraic Groups

In this section we will give a very short overview of algebraic groups. Our emphasis here will be on linear algebraic groups. For further details of this rather dense subject the reader is encouraged to see [19] and [8]. Let V be a set. Define

$$\mathrm{Map}(V,k) = \{f \mid f : V \to k\}.$$

We can regard $\mathrm{Map}(V,k)$ as a $k-$algebra by point-wise operations, in particular the multiplication is defined by $fg(x) = f(x)g(x)$. Let A be a $k-$subalgebra of $\mathrm{Map}(V,k)$. We have a map $\epsilon_x : A \to k$, called evaluation at x, defined by $\epsilon_x(f) = f(x)$ for $f \in A$. For $k-$algebras A and B we denote by $\mathrm{Hom}_{k-\mathrm{alg}}(A,B)$ the set of $k-$algebra homomorphisms from A to B.

Definition 1.2.1. *An affine variety (over k) is a pair* (V,A) *consisting of a set* V *and a finitely generated* $k-$*subalgebra* A *of* $\mathrm{Map}(V,k)$ *such that the map*

$$V \to \mathrm{Hom}_{k-\mathrm{alg}}(A,k)$$

$$x \mapsto \epsilon_x$$

is a bijection.

From now on we will just say that V is an affine variety to mean that we have a pair (V, A) together with the above bijection. If V is an affine variety then we will denote A by $k[V]$ and call it the coordinate algebra of V.

Definition 1.2.2. *Let V, W be affine varieties. A morphism of affine varieties is a map $\phi : V \to W$ such that $g \circ \phi \in k[V]$, for every $g \in k[W]$.*

Definition 1.2.3. *A group G is called a linear algebraic group if G is also an affine variety such that the multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are morphism of affine varieties. Moreover a map $\phi : G \to H$, of algebraic groups is called a morphism of algebraic groups if it is a group homomorphism and a morphism of affine varieties.*

We will only be considering affine varieties in this thesis. All our algebraic groups will be linear so we will often drop the word linear.

We can assign to each affine variety a topology as follows. Let (V, A) be an affine variety and $S \subset A$. We define

$$\curlyvee(S) = \{x \in V | \, f(x) = 0 \text{ for all } f \in S\}.$$

then it is not very difficult to show that

1. $\curlyvee(1) = \varnothing$, $\curlyvee(0) = V$;
2. If we have a collection S_j of subsets of A then $\curlyvee(\bigcup_j S_j) = \bigcap_j \curlyvee(S_j)$.
3. For subsets S, T of A we have $\curlyvee(ST) = \curlyvee(S) \cup \curlyvee(T)$.

Here ST denotes the set $\{fg | \, f \in S, g \in T\}$. It is clear from the above conditions that the sets $\curlyvee(S)$ form the closed set of a topology on V. This topology is called the Zariski topology.

An algebraic group G is said to be connected if it is connected as a variety. Let G be an algebraic group and A, B be closed subgroups of G. We define the commutator group (A, B) to be the group generated by the commutators $xyx^{-1}y^{-1}$ where $x \in A$ and $y \in B$. We define the derived series of G inductively by $\mathfrak{D}^0 G = G$ and $\mathfrak{D}^{i+1} G = (\mathfrak{D}^i G, \mathfrak{D}^i G)$ for $i \geq 0$. If G is a connected algebraic group then $\mathfrak{D}^i G$ is a closed normal connected subgroup of G. We say that G is solvable if $\mathfrak{D}^n G = 1_G$ for some n. Similarly we define the descending central series of G by induction as $\mathfrak{C}^0 G = G$ and $\mathfrak{C}^{i+1} G = (G, \mathfrak{C}^i G)$ for $i \geq 0$. We call G nilpotent if $\mathfrak{C}^n G = 1_G$ for some n. We define a matrix $A \in GL_n$ to be unipotent if $A - I_n$ is nilpotent i.e. $(A - I_n)^r = 0$ for some r. An element $g \in G$ is called unipotent if for some (hence every) faithful (one to one) representation $\rho : G \to GL_n(k)$ the image $\rho(g)$ is unipotent. A subgroup of G is called unipotent if all its elements are unipotent.

Every algebraic group G has a unique maximal normal solvable subgroup say H. This subgroup of G is always closed. The connected component of H is denoted by $R(G)$ and is called the radical of G. Let $U(G)$ denote the subgroup of $R(G)$ consisting of the unipotent elements in $R(G)$. It is not difficult to show that $U(G)$ is normal in G. We call $U(G)$ the unipotent radical of G and it is the largest connected normal unipotent subgroup of G.

Definition 1.2.4. *(Semi-simple and Reductive Groups) An algebraic group G is called semi-simple if the radical of G is trivial. Similarly G is called reductive if $U(G)$ is trivial.*

An algebraic group G is called a torus if it is isomorphic to $D_n(k)$, the subgroup of GL_n consisting of the diagonal matrices, for some n. The maximal tori are conjugate in G. The dimension n of a maximal torus T is called the rank of G. A maximal, closed, connected, solvable subgroup B of G is called a Borel subgroup of G. All Borel subgroups of G are conjugate. We fix a Borel subgroup B of G and a maximal torus $T \subseteq B$ of G. Let $N_G(T)$ denote the normalizer of T in G and $Z_G(T)$ denotes the centralizer of T in G then the Weyl group W of G is given by $N_G(T)/Z_G(T)$. Note that W is a finite group.

1.2.1 Representation Theory of Algebraic Groups

Now we will say a few words about the representation theory of the algebraic groups. Let G be a group and V a finite dimensional $kG-$module with basis $v_1, ..., v_n$. We define the coefficient functions $f_{ij} : G \to k$ of V with respect to the given basis by the equations

$$gv_i = \sum_{j=1}^{n} f_{ji}(g)v_j.$$

By $\mathrm{cf}(V) \subseteq \mathrm{Map}(G, k)$ we denote the coefficient space of V and define it to be the $k-$span of $\{f_{ij} | 1 \leq i, j \leq n\}$.

Definition 1.2.5. *Let G be an algebraic group. A finite dimensional $kG-$module is said to be rational if $\mathrm{cf}(V) \subseteq k[G]$. If V is a finite dimensional $kG-$module and $\rho : G \to \mathrm{GL}_n$ is a matrix representation afforded by V then V is rational if and only if ρ is a morphism of algebraic groups. We will call ρ a rational representation of G in V.*

We will denote by $\mathrm{mod}(G)$ the category of finite dimensional rational $G-$modules. All the modules considered in this thesis are rational unless stated otherwise. We finish this section with a rather beautiful proposition.

Proposition 1.2.1. *Every (linear) algebraic group G is isomorphic to a closed subgroup of* GL_n*, for some n. This result justifies the name linear for linear algebraic groups.*

1.2.2 The Lie Algebra of an Algebraic Group

Suppose (V, A) is an affine variety. For $x \in V$ we define

$$T_x(V) = \{\alpha : A \to k \mid \alpha(fg) = f(x)\alpha(g) + \alpha(f)g(x) \text{ for all } f, g \in A\}$$

and call it the tangent space of V at a point x. Now we define $\mathrm{Lie}(G) = T_1(G)$ and we will give $\mathrm{Lie}(G)$ a Lie algebra structure. Let V, W be affine varieties and let $x \in V$, $y \in W$. Define a bijection $\Phi : T_x(V) \times T_y(W) \to T_{(x,y)}(V \times W)$ with $\Phi(\alpha, \beta)(a \otimes b) = \alpha(a)\beta(b)$, for $a \in k[V]$ and $b \in k[W]$. Using Φ we will identify $T_x(V) \times T_y(W)$ with $T_{(x,y)}(V \times W)$.

Finally define a map $\phi : G \times G \to G$ by $\phi(x, y) = xyx^{-1}y^{-1}$ for all $x, y \in G$. This map ϕ gives us a differential $\mathrm{d}\phi_{(1,1)} : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ and we define the bracket operation on $\mathrm{Lie}(G)$ by $[X, Y] = \mathrm{d}\phi_{(1,1)}(X, Y)$. We leave this to the reader to prove that this indeed defines a Lie bracket and makes $\mathrm{Lie}(G)$ a Lie algebras. Details are also given in [8]. We will usually denote the Lie algebra of G by $\mathfrak{g}$.

1.2.3 Weights and Roots

Let G_k be the multiplicative algebraic subgroup of k (considered as the affine line), then G_k is isomorphic to $k^\times$ as a group. Suppose also that G is reductive. If T is a maximal torus of G then we define $X(T)$ to be the set of morphisms of algebraic groups $\phi : T \to G_k$. For a T−module V and $\lambda \in X(T)$, we define $V^\lambda = \{v \in V | tv = \lambda(t)v\}$ to be the λ–weight space of V. Those λ's for which V^λ is non-zero are called weights of V. Any G–module V is completely reducible as a T–module. So V decomposes as a direct sum of its weight spaces and we have $V = \bigoplus_{\lambda \in X(T)} V^\lambda$ as a T–module.

We call $X(T)$ the weight lattice of G. It has a structure of a torsion free abelian group and is isomorphic to $\mathbb{Z}^n$ for some n. The Weyl group W acts on T by $wt = n_w t n_w^{-1}$ where $t \in T$ and $w = n_w T$. This action can be extended to an action on $X(T)$ by $w\lambda(t) = \lambda(w^{-1}t)$ for $\lambda \in X(T), w \in W, t \in T$. Let $e(\lambda), \lambda \in X(T)$ be the canonical basis for the integral group ring $\mathbb{Z}X(T)$. The character formula for V is defined to be the sum

$$\text{ch}\, V = \sum_{\lambda} (\dim V^\lambda) e(\lambda),\ \lambda \in X(T).$$

Suppose (,) is a real, positive definite, symmetric, W–invariant bilinear form on $\mathbb{R} \otimes X(T)$. Let Φ denote the set of nonzero weights for the action of T on Lie(G). The elements of Φ are then called the roots of G. We identify $X(T)$ with a subgroup of $\mathbb{R} \otimes X(T)$. Let $\mathbb{E}$ denote the $\mathbb{R}$–span of Φ in $\mathbb{R} \otimes X(T)$ then the induced bilinear form considered above make $(\mathbb{E}, \Phi)$ into a root system with Weyl group W, see e.g. [20] and [10]. A subset S of Φ is called a set of simple roots if every $\alpha \in \Phi$ can be written as a $\mathbb{Z}$ linear combination of elements of S i.e. $\alpha = \sum_{\beta} x_\beta \beta$ where $\beta \in S$ and $x_\beta \in \mathbb{Z}$ are either all nonnegative or all nonpositive.

A root α is called a positive root if $x_\beta \geq 0$ for all β and negative if $x_\beta \leq 0$ for all β. We will denote the set of positive roots by Φ^+ and Φ^- will denote the set of negative roots. For $\alpha \in \Phi$ we will denote by $\alpha^\vee$ the coroot of α and is given by $\frac{2\alpha}{(\alpha,\alpha)}$. The reflection of α denoted s_α is given by $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$.

It is easy to see that s_α sends α to $-\alpha$. We will denote the number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ by $\langle\beta,\alpha\rangle$. For $\alpha \in S$, we denote by P_α the parabolic subgroup containing B which has α as its only positive root.

As an example of root systems we take $G = GL_n$. Let $e_i = (0,0,...,0,1,0,...,0)$ with the ith entry 1 and 0 every where else. The maximal torus T of G is a diagonal matrix in G. Suppose $T = \text{diag}(t_1,t_2,...,t_n)$ then $e_i(\text{diag}(t_1,t_2,...,t_n)) = t_i$. The set of roots Φ of G is given by $\Phi = \{e_i - e_j \mid 1 \le i \le n, 1 \le j \le n\}$, the set of positive roots Φ^+ is given by $\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le n\}$, and the set of simple roots S is given by $S = \{e_i - e_{i+1} \mid 1 \le i \le n-1\}$.

The Weyl group W is generated by the set $\{s_\alpha \mid \alpha \in S\}$. The action of W on $X(T)$ called the dot action is given by

$$w.\lambda = w(\lambda + \rho) - \rho$$

where $w \in W$, $\lambda \in X(T)$ and ρ is the half sum of the positive roots. We can give $X(T)$ a partial order by defining

$$\mu \le \lambda \text{ if and only if } \lambda - \mu \text{ can be expressed as a sum of positive roots.}$$

We call a weight λ dominant if $\langle\lambda,\alpha^\vee\rangle \ge 0$ for all $\alpha \in \Phi^+$. We will denote the set of dominant weights by $X^+(T)$. We define the r−restricted weights to be the set

$$X_r(T) = \{\lambda \mid 0 \le \langle\lambda,\alpha^\vee\rangle < p^r \text{ for all } \alpha \in S\}$$

where p is the characteristic of the field k. The set $X_1(T)$ is simply called the set of restricted weights.

1.3 Hopf Algebras and Group Schemes

The main purpose of this section is to define the infinitesimal groups G_r. We will only be giving the necessary definitions here. For further details please see [9]. Let k be a field then we will denote the k tensor product by $\otimes_k$ or simply by $\otimes$ see e.g. [23]. Let A be an associative algebra over k with identity 1_A. We define a coalgebra over k as follows

Definition 1.3.1. *(Coalgebra) A coalgebra over k is defined to be a triple (C, δ, ϵ), where C is a vector space over k and $\delta : C \to C \otimes C$, $\epsilon : C \to k$ called comultiplication and counit respectively are linear maps satisfying*

$$(\delta \otimes \mathrm{id}_C)\delta = (\mathrm{id}_C \otimes \delta)\delta$$

$$(\epsilon \otimes \mathrm{id}_C)\delta = (\mathrm{id}_C \otimes \epsilon)\delta = \mathrm{id}_C$$

From now on we will just write C for the coalgebra and we will call δ and ϵ the structure maps of C. Let (C, δ, ϵ) be a coalgebra. A coideal of C is a subspace I such that $\delta(I) \leq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$.

Definition 1.3.2. *Let (C, δ, ϵ) be a coalgebra over k. A right $C-$comodule over k is a pair (V, τ), where V is a vector space over k and $\tau : V \to V \otimes C$ is a linear map satisfying the following conditions.*

$$(\tau \otimes \mathrm{id}_C)\tau = (\mathrm{id}_V \otimes \delta)\tau$$

$$(\mathrm{id}_V \otimes \epsilon)\tau = \mathrm{id}_V.$$

Definition 1.3.3. *(bialgebra) A bialgebra over k is a coalgebra (C, δ, ϵ) such that C is also an algebra and δ, ϵ are morphisms of algebra.*

Suppose (C, δ, ϵ), (C', δ', ϵ') are bialgebras over k. A morphism of bialgebras is a map $\phi : C \to C'$ such that ϕ is both an algebra and coalgebra morphism. We define an antipode of a bialgebra as follows.

Definition 1.3.4. *Let (C, δ, ϵ) be a bialgebra. An antipode of C is a linear map $\sigma : C \to C$ such that*

$$m(\sigma \otimes \mathrm{id}_C)\delta = m(\mathrm{id}_C \otimes \sigma)\delta = 1_C \epsilon$$

where m denotes multiplication.

We are now in a position to define Hopf algebras.

Definition 1.3.5. *(Hopf algebra) A Hopf algebra is a quadruple $(H, \delta, \epsilon, \sigma)$ where (H, δ, ϵ) is a bialgebra and σ is an antipode. Suppose $(H, \delta, \epsilon, \sigma)$, $(H', \delta', \epsilon', \sigma')$ are Hopf algebras. A morphism of Hopf algebras is a map $\phi : H \to H'$ such that ϕ is a bialgebra morphism and $\sigma' \circ \phi = \phi \circ \sigma$. Moreover an ideal $\mathfrak{m}$ of a Hopf algebra is an algebra ideal, a coideal and $\sigma(\mathfrak{m}) = \mathfrak{m}$.*

We have a relation between affine group schemes and Hopf algebras. Loosely speaking affine group schemes over k correspond to commutative Hopf algebras over k see e.g. [29].

In order to avoid giving all the details about affine group schemes we say that G is a group scheme over k to mean that we have in mind a commutative Hopf algebra over k denoted by $k[G]$. Let G, H be (affine) group schemes then we will call $\phi : G \to H$ a morphism of group schemes if the map $\phi^{\#} : k[G] \to k[H]$ is a morphism of Hopf algebras. We will call V a left G–module if we have a structure map $\tau : V \to V \otimes k[G]$ such that (V, τ) is a

$k[G]$–comodule. Suppose G is a group scheme then we will call H a subgroup scheme of G if there is a Hopf ideal I_H of $k[G]$ such that $k[H] = k[G]/I_H$.

Suppose (H, δ, ϵ) is a commutative Hopf algebra and $\mathfrak{m} = \mathrm{Ker}(\epsilon)$ then $\mathfrak{m}$ is a Hopf ideal. Moreover let $\mathfrak{m}^{[p]}$ be an ideal generated by f^p, $f \in \mathfrak{m}$ then $\mathfrak{m}^{[p]}$ is also a Hopf ideal. Now we are able to define the infinitesimal groups G_r.

Definition 1.3.6. *Suppose $r \geq 1$ then G_r is the affine group scheme such that*

$$k[G_r] = k[G]/\mathfrak{m}^{[p^r]},$$

where $\mathfrak{m}^{[p^r]}$ is the Hopf ideal generated by f^{p^r}, $f \in \mathfrak{m}$

The main result that we will be using from this section is the following

$$G_1 - \text{modules} \equiv \text{restricted } \mathfrak{g} - \text{modules}$$

see e.g. [21].

1.4 Induced Modules and Weyl Modules

Let G be a reductive algebraic group and V a B–module (B a Borel subgroup). We define the induced module $\mathrm{Ind}_B^G V$ to be the space of maps $f : G \to V$ with the following properties

1. $\mathrm{Im}(f)$ is contained in a finite dimensional subspace of V, say V_0, and the restriction $f : G \to V_0$ is a morphism of affine varieties. If f satisfies this condition then f is called a regular map.

2. $f(bx) = bf(x)$ for all $b \in B$ and $x \in G$. If f satisfies this condition then f is said to be B–equivariant.

The group G acts on $\mathrm{Ind}_B^G V$ by $(gf)(x) = f(xg)$, $g \in G$. The module $\mathrm{Ind}_B^G V$ is a rational G–module. Let k_λ be the one dimensional B–module with weight λ (B acts trivially) then $\mathrm{Ind}_B^G k_\lambda$ is nonzero if and only if $\lambda \in X^+(T)$ see [10, theorem 4.3.]. For $\lambda \in X^+(T)$ we will write $\nabla(\lambda) = \mathrm{Ind}_B^G k_\lambda$. We will denote by k_λ the one dimensional B–module with weight λ. We will often write λ for k_λ. The character of $\nabla(\lambda)$ is given by the Weyl character formula (See [21, II, proposition 5.10]) and we will write $\chi(\lambda)$ for ch $\nabla(\lambda)$.

For $\lambda \in X^+(T)$ there exists an irreducible (simple) G–module with highest weight λ that we will denote by $L(\lambda)$. Moreover the modules $L(\lambda)$, $\lambda \in X^+(T)$ form a complete set of pairwise non-isomorphic simple G–modules. Now the module $\nabla(\lambda)$ has simple socle $L(\lambda)$ with the highest weight λ and is the largest G–module with this property. We define the Weyl module $\Delta(\lambda) = \nabla(-w_0\lambda)^*$ where w_0 is the longest element of W and $*$ denotes the dual. It is not very difficult to see that both $\nabla(\lambda)$ and $\Delta(\lambda)$ have the same character. Note that due to duality the Weyl module has a simple head $L(\lambda)$. If k is a field of characteristic $p > 0$ then we will call the module $L((p-1)\rho)$ the Steinberg module and we will denote it by St.

1.5 Tilting Modules

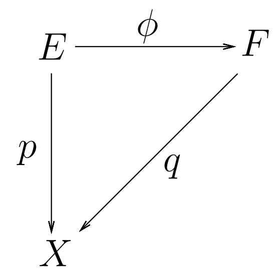
Let $V \in \mathrm{mod}(G)$ be a G–module. We define a good filtration of V to be a filtration $0 = V_0 \leq V_1 \leq V_2 \leq ... \leq V_n = V$ such that for each $0 < i \leq n$, V_i/V_{i-1} is either zero or isomorphic to $\nabla(\lambda_i)$ for some $\lambda_i \in X^+(T)$.

A tilting module of G is a finite dimensional G–module V such that V and its dual module V^* both admit good filtrations. For each $\lambda \in X^+(T)$ there is an indecomposable tilting module $T(\lambda)$ which has highest weight λ. Every tilting module is a direct sum of copies of $T(\lambda)$, $\lambda \in X^+(T)$ as in [21, E.6, proposition]. For $\lambda \in X^+(T)$ the tilting module $T((p-1)\rho + \lambda)$ is projective as a G_1–module, where G_1 is the first infinitesimal group and ρ is the half sum of positive roots.

1.6 Vector Bundles

We will now give a definition of vector bundles. Our definition is based on an account in [3], see also [28] and [5]. We will start by giving a definition of a family of vector spaces.

Definition 1.6.1. *Let X be a variety. A family of vector spaces over X is a variety E together with a morphism of varieties $p : E \to X$ such that for all $x \in X$ the set $E_x = p^{-1}(x)$ has the structure of a vector space. Let E, F be families of vector spaces over X. A morphism of families is a map $\phi : E \to F$ such that the map $\phi_x : E_x \to F_x$ is a vector space homomorphism and the following diagram commutes*



Here p, q are called projection maps, X is called the base space of the family and E its total space.

For every vector space V we have a product family i.e. if X is a variety then take $E = X \times V$ and define $p : E \to X$ by $p(x, v) = x$ so that $E_x = \{x\} \times V$. If F is any family then F is called a trivial family if it is isomorphic to some product family. Let E be a family of vector spaces over X and let Y be a subvariety of X. Then E is clearly a family over Y with projection p, $p : p^{-1}(Y) \to Y$. We call this family the restriction of E to Y and denote it by $E|Y$.

A faimly E of vector spaces over X is called locally trivial if for every $x \in X$ there exists an open cover U_x such that the restriction $E|U_x$ is trivial. We can now define a vector bundle.

Definition 1.6.2. *Let E be a family of vector spaces over X. Then E is called a vector bundle if E is locally trivial.*

It is worth mentioning that there is also a notion of an algebraic vector bundle. It is known that the two notions are equivalent. In this thesis we will be using the vector bundles defined as follows. Let G be an algebraic group and B a Borel subgroup of G. Let V be a $B-$module. We define a vector bundle denoted $\mathfrak{L}_V$ by $E = G \times^B V$ and $X = G/B$. Here $G \times^B V = G \times V / \sim$,where $\sim$ is the equivalence relation given by $(gb, v) \sim (g, bv)$ for all $g \in G, v \in V$ and $b \in B$. The projection map $p : E \to X$ is given by $p(gb, v) = gb$ for some $g \in G$ and for all $b \in B$. If V is one dimensional then the vector bundle is called a line bundle. We will denote a line bundle by $\mathfrak{L}_\lambda$, where λ is the weight of the one dimensional vector space.

1.7 Cohomology and a Spectral Sequence

We will now briefly discuss cohomology. A version of all the definitions given below is available for homology. The material is mainly from [4].

Let $\mathfrak{A}$ be an abelian category. A cochain complex of objects in $\mathfrak{A}$ consists of collection of objects $\mathbf{C} = \{C^n | \, n \in \mathbb{Z}\}$ and a collection of maps $\delta^n : C^n \to C^{n+1}$ satisfying $\delta^n \, o \, \delta^{n-1} = 0$. We say that x has degree n if $x \in C^n$. The cohomology of a cochain complex $\mathbf{C}$ denoted $H^n(\mathbf{C})$ is then given by

$$H^n(\mathbf{C}) = \frac{\mathrm{Ker}(\delta^n : C^n \to C^{n+1})}{\mathrm{Im}(\delta^{n-1} : C^{n-1} \to C^n)}.$$

A short exact sequence of chain complexes

$$0 \to \mathbf{C}' \to \mathbf{C} \to \mathbf{C}'' \to 0$$

gives rise to a long exact sequence of cohomology given by

$$... \to H^n(\mathbf{C}') \to H^n(\mathbf{C}) \to H^n(\mathbf{C}'') \to H^{n+1}(\mathbf{C}') \to H^{n+1}(\mathbf{C}) \to ...$$

Let G be an algebraic group and B a Borel subgroup of G. For the rest of the thesis we will be using the cohomology defined by the derived functor of induction i.e. if M is a B−module then $H^i(M) = R^i\mathrm{Ind}_B^G(M)$. As a k−vector space $H^i(M)$ is isomorphic to the vector bundle cohomology $H^i(G/B, G \times^B M)$ as in [27].

Now we will describe the Grothendieck spectral sequence as given in [21].

Proposition 1.7.1. *(Grothendieck Spectral Sequence) Let $\mathfrak{A}$,$\mathfrak{A}'$ and $\mathfrak{A}''$ be abelian categories with $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}'$ and $\mathfrak{F}' : \mathfrak{A}' \to \mathfrak{A}''$ the additive functors. If $\mathfrak{F}'$ is left exact and if $\mathfrak{F}$ maps injective objects in $\mathfrak{A}$ to objects acyclic for*

$\mathfrak{F}'$, then there is a spectral sequence for each object $M \in \mathfrak{A}$ with differential d_r of bidegree $(r, 1-r)$, and we have

$$E_2^{n,m} = (R^n\mathfrak{F}')(R^m\mathfrak{F})M \implies R^{n+m}(\mathfrak{F}' o \mathfrak{F})M.$$

We will be extensively using this spectral sequence in the form

$$R^n\mathrm{Ind}_{P_\alpha}^G R^m\mathrm{Ind}_B^{P_\alpha} M \implies R^{n+m}\mathrm{Ind}_B^G M.$$

1.8 On the Representation Theory of SL_3 and G_2

We will now consider the representation theory of SL_3 and G_2. We will discuss SL_3 in more depth and state the results for G_2 and leave them to the reader to verify. Let $G = \mathrm{SL}_3$ then we can take the maximal torus T of G to be the diagonal matrices in SL_3 so

$$T = \{g|\ g \in D_3 \text{ and } \det(g) = 1\}.$$

It is clear that $\dim(T) = 2$. Let $t_1, t_2 \in k^\times$ be the generators of T then a typical element $t \in T$ will look like

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & \frac{1}{t_1 t_2} \end{pmatrix}$$

We can take a Borel subgroup B of G to be the group of lower triangular matrices in SL_3. Recall that $\lambda \in X(T)$ is defined as a map $\lambda : T \to k^\times$. For SL_3 we define maps ω_1 and ω_2 in $X(T)$ by $\omega_1(t) = t_1$ and $\omega_2(t) = t_1 t_2$

then it is clear that $X(T) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ therefore we can identify $X(T)$ by $\mathbb{Z}^2$. We will write $\lambda = (r, s)$ where $r, s \in \mathbb{Z}$. Recall that the λ weight space for a T−module V is defined by

$$V^\lambda = \{v \in V |\, tv = \lambda(t)v \text{ for all } t \in T\}.$$

We will simply call λ's the weights of SL_3. So in particular we have

$$V^{(r,s)} = \{v \in V |\, tv = t_1^{r+s} t_2^s v\}.$$

Let $\alpha = (2, -1)$ and $\beta = (-1, 2)$ then the set of roots Φ for SL_3 is given by $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, the set of positive roots Φ^+ is given by $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$, and the set of negative roots Φ^- is given by $\Phi^- = \{-\alpha, -\beta, -(\alpha+\beta)\}$. Also SL_3 has two simple roots namely α and β. Moreover the Weyl group W of SL_3 is generated by s_α and s_β.

Now we consider $G = G_2$. The weights λ of G_2 are of the form $\lambda = (r, s)$ where $r, s \in \mathbb{Z}$. There are two simple roots of G_2 given by $\alpha = (2, -1)$ and $\beta = (-3, 2)$. The set of roots is then defined to be the $\mathbb{Z}$-span of α and β.

1.9 Formulation of the Problem

Finally we will formulate the problem that we will be considering throughout the chapters 2, 3 and 5. Let G be a reductive algebraic group and B a Borel subgroup of G. Then G/B is a flag variety. Moreover if $\mathfrak{B}$ is the collection of all Borel subgroups of G then we can identify $\mathfrak{B}$ with G/B via the correspondence $G/B \to \mathfrak{B}$ such that $gB \to gBg^{-1}$ see e.g. [19]. Let $\mathfrak{L}_\lambda$ denote a line bundle on the variety G/B then we consider the i−th cohomology groups $H^i(G/B, \mathfrak{L}_\lambda)$. In Chapter 2 and 3 we take $G = SL_3$ and $G = G_2$ respectively and try to find the character of $H^i(G/B, \mathfrak{L}_\lambda)$ when the characteristic of the field k is a prime p.

Chapter 5 gives cohomology of certain modules appearing in Demazure's proof of the Borel-Weil-Bott theorem (characteristic zero).

Chapter 2

Characters of Cohomology of Line Bundles on Flag Varieties

2.1 Introduction

In this chapter we will try to calculate the cohomology of line bundles on flag varieties. The main part of the chapter will consist of results about the three dimensional flag variety. We will start by giving some of the known results in this area and then we will give some new results. Most of the new results are based on Donkin's results in [11].

For the rest of this chapter G will be a semisimple, simply connected linear algebraic group over an algebraically closed field k.

2.2 Characteristic Zero Case

If the characteristic of the field k is zero then $H^i(G/B, \mathcal{L}(k_\lambda))$ is given by the following theorem.

Theorem 2.2.1. *(Borel-Weil-Bott)[21, II, chapter 5]*
If $\lambda \in X^+(T)$ *and* $w \in W$ *then* $H^i(w.\lambda) = 0$ *for* $i \neq l(w)$ *and* $H^{l(w)}(w.\lambda) \simeq H^0(\lambda)$.

Also the character of $H^0(\lambda)$ is given by Weyl character formula below.

Theorem 2.2.2. *(Weyl character formula)[21, II, proposition 5.10]*
If $\lambda \in X^+(T)$ *and* $w \in W$, *define*

$$A(\lambda) = \sum_{w \in W} \operatorname{sgn}(w) e(w\lambda)$$

then

$$\chi(\lambda) = \frac{A(\lambda + \rho)}{A(\rho)}$$

where ρ *is half sum of the positive roots.*

The above two theorems resolve the characteristic zero case completely.

2.3 Characteristic p Case

A natural question to ask is what will happen when characteristic of the field k is a prime p. One would expect the above results to generalize nicely in this case. But the truth is far from expectation. If $i = l(w)$ then it is not true in general that $H^i(w.\lambda) \simeq H^0(\lambda)$ (characteristic p). Moreover a weaker result that $\operatorname{ch}(H^i(w.\lambda)) = \operatorname{ch}(H^0(\lambda))$ is also false.

In fact we know very little in the case of characteristic p. Given below are some of the known results.

Theorem 2.3.1. *(Kempf's vanishing theorem)[21, II, proposition 4.5]*
If $\lambda \in X^+(T)$ *then* $H^i(\lambda) = 0$ *for all* $i > 0$.

Theorem 2.3.2. *(Grothendieck vanishing)* $H^i(G/B, \mathcal{L}_M) = 0$ *for all* $i > \dim(G/B)$.

Theorem 2.3.3. *(Serre duality)[21, II, 4.2(10)]*
Let $d = \dim(G/B)$*, then*

$$H^i(G/B, \mathcal{L}_M)^* \simeq H^{d-i}(G/B, \mathcal{L}_{(M^* \otimes k_{-2\rho})}),$$

where M^* *is dual of* M*.*

The following result by H. Andersen gives a necessary and sufficient condition for first cohomology modules to be non-zero. Moreover this result also gives the highest weight for the cohomology module if it is nonzero (see e.g [21, II, proposition 5.15].

Proposition 2.3.4. *(Andersen)*
Suppose char(k)= $p \neq 0$*. Let* $\alpha \in \Phi^+$ *be a simple root and* $\lambda \in X(T)$ *with* $\langle \lambda, \alpha^\vee \rangle \geq 0$*.*

1. *Suppose* $\langle \lambda, \alpha^\vee \rangle = ap^n - 1$ *for some positive integers* a, n *with* $0 < a < p$*. Then*
$$H^1(s_\alpha.\lambda) \neq 0 \Longleftrightarrow \lambda \in X^+(T).$$
2. *Let* $\langle \lambda, \alpha^\vee \rangle = \sum_{j=0}^{n} a_j p^j$ *with* $0 \leq a_j < p$ *and* $a_j \neq 0$*. Suppose there is some* $j < n$ *with* $a_j < p - 1$*. Then*
$$H^1(s_\alpha.\lambda) \neq 0 \Longleftrightarrow s_\alpha.\lambda + a_n p^n \alpha \in X^+(T).$$
Moreover If $\lambda \in X^+(T)$ *then* λ *is the highest weight of* $H^1(s_\alpha.\lambda)$*. If not let* m *be minimal with* $a_m < p - 1$*. Suppose* $m' \geq m$ *is minimal for* $\mu = s_\alpha.\lambda + \sum_{j=m'}^{n} a_j p^j \alpha \in X^+(T)$*. Then* μ *is the highest weight of* $H^1(s_\alpha.\lambda)$ *and occurs with multiplicity 1.*

The following result tells us when $R^i\mathrm{Ind}_B^{P_\alpha}\lambda$ is nonzero. We will be using this result extensively throughout the thesis.

Proposition 2.3.5. *[21, II, proposition 5.2]*
Let $\alpha \in S$ and $\lambda \in X(T)$.

1. *If $\langle \lambda, \alpha^\vee \rangle = -1$ then $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ for all i.*

2. *If $\langle \lambda, \alpha^\vee \rangle \geq 0$ then $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ for all $i \neq 0$.*

3. *If $\langle \lambda, \alpha^\vee \rangle \leq -2$ then $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ for all $i \neq 1$.*

We will often write $\nabla_\alpha(\lambda)$ for $\mathrm{Ind}_B^{P_\alpha}\lambda$.

Proposition 2.3.6. *(The linkage principle)[21, II, 6.17]*
Let $\lambda, \mu \in X^+(T)$. If $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$, then $\lambda \in W \cdot \mu + p\mathbb{Z}\Phi$.

Let $G = \mathrm{SL}_2$ and let B be a Borel subgroup of G then dim $(G/B) = 1$. By Grothendieck vanishing we have $H^i(M) = 0$, for all $i > 1$. Also by Serre duality

$$H^1(\lambda) \simeq H^0(\lambda - 2\rho)^*.$$

This gives us the complete result in this case.

2.4 $G = \mathrm{SL}_3$

For the rest of this chapter let $G = \mathrm{SL}_3$ unless stated otherwise. Let k be an algebraically closed field of characteristic p. We will denote the character of $H^i(\lambda)$ by $\chi^i(\lambda)$. Since $\dim G/B = 3$, by Grothendieck vanishing $H^i(M) = 0$, $i > 3$. Also by Serre duality $H^3(\lambda) \simeq H^0(\lambda - 2\rho)^*$ and $H^2(\lambda) \simeq H^1(\lambda - 2\rho)^*$. So it is sufficient to find ch $H^1(\lambda)$. Also by Kempf's vanishing theorem

$H^i(\lambda) = 0$ for all $\lambda \in X^+(T)$ and $i > 0$. For $\lambda = (r,s) \in X_1(T)$ we will denote the character of $L(r,s)$ by $\chi_p(r,s)$ and it is given by

$$\chi_p(r,s) = \begin{cases} \chi(r,s) - \chi(r-a, s-a), & \text{if } r+s+2 = p+a, a > 0 \\ \chi(r,s), & \text{otherwise.} \end{cases}$$

Let $F : G \to G$ be the Frobenius morphism of G defined by $F(g) = (g_{ij}^p)$ for all $g = (g_{ij}) \in G$. Let N_α denote the unique two-dimensional indecomposable B−module with character $e(0) + e(-\alpha)$. Thus there is a non-split short exact sequence $0 \to k_{-\alpha} \to N_\alpha \to k \to 0$. We denote by $N_\alpha(\lambda)$ the module $\lambda \otimes N_\alpha$ and we write $\chi_\alpha^i(\lambda)$ for $\chi^i(N_\alpha(\lambda))$. In [11], Donkin gave a recursive description for the characters of cohomology of line bundles on the flag variety G/B, where $G = \mathrm{SL}_3$. He gave the result in three parts, $p = 2$, $p = 3$ and $p \geq 5$. The result in characteristic $p = 2$ is given below.

Lemma 2.4.1. *(Donkin) For $i \geq 0$, integers r,s and α, β simple roots we have:*

1. $$\chi^i(2r,2s) = \chi^i(r,s)^F + \chi^i(r-1,s-1)^F + \chi(1,0)\chi^i(r,s-1)^F + \chi(0,1)\chi^i(r-1,s)^F;$$

2. $$\chi^i(1+2r,2s) = \chi_\alpha^i(r+1,s-1)^F + \chi(1,0)\chi^i(r,s)^F + \chi(0,1)\chi^i(r,s-1)^F;$$

3. $$\chi^i(2r,1+2s) = \chi_\beta^i(r-1,s+1)^F + \chi(1,0)\chi^i(r-1,s)^F + \chi(0,1)\chi^i(r,s)^F;$$

4. $$\chi^i(1+2r,1+2s) = \chi(1,1)\chi^i(r,s)^F.$$

Also the result for $\chi_\alpha^i(r,s)$ is given by

Lemma 2.4.2. *(Donkin) For $i \geq 0$, and integers r, s and a simple root α we have:*

1. $$\chi^i_\alpha(2r, 2s) = \chi^i(r-1, s-1)^F + \chi(1,0)^F\chi^i(r-1, s)^F + 2\chi(0,1)\chi^i(r-1, s)^F + \chi(1,0)\chi^i_\alpha(r, s-1)^F;$$

2. $$\chi^i_\alpha(1+2r, 2s) = \chi^i_\alpha(r+1, s-1)^F + \chi(1,0)\chi^i(r, s)^F + \chi(0,1)\chi^i(r, s-1)^F + \chi(1,1)\chi^i(r-1, s)^F;$$

3. $$\chi^i_\alpha(2r, 1+2s) = \chi(0,1)^F\chi^i(r-1, s)^F + \chi^i(r-1, s+1)^F + 2\chi(1,0)\chi^i(r-1, s)^F + \chi(0,1)\chi^i_\alpha(r, s)^F;$$

4. $$\chi^i_\alpha(1+2r, 1+2s) = \chi^i_\alpha(r, s)^F + \chi(1,0)\chi^i(r-1, s+1)^F + \chi(0,1)\chi^i(r-1, s)^F + \chi(1,1)\chi^i(r, s)^F.$$

The above result is the best result available so far which provides us with a complete description of the characters. But due to the recursive nature of the result it is very hard to use. In what follows we will try to give a non recursive description for these characters. The results given rely heavily on the recursion given in [11].

We will first give a few results on $H^i(N_\alpha(\lambda))$. The following theorem gives the cohomology of $H^i(N_\alpha(\lambda))$, for a dominant weight λ.

Theorem 2.4.3. *Let $\lambda \in X^+(T)$ and G be a semisimple group. Suppose α is a simple root.*

1. *If $\langle \lambda, \alpha^v \rangle > 0$ then $H^i(N_\alpha(\lambda)) = 0$ for $i > 0$.*

2. If $\langle \lambda, \alpha^v \rangle = 0$ then $H^i(N_\alpha(\lambda)) = 0$ for all i.

Proof. We have a short exact sequence

$$0 \to \lambda - \alpha \to N_\alpha(\lambda) \to \lambda \to 0.$$

This gives rise to a long exact sequence of induction given by

$$0 \to H^0(\lambda - \alpha) \to H^0(N_\alpha(\lambda)) \to H^0(\lambda) \to H^1(\lambda - \alpha)$$

$$\to H^1(N_\alpha(\lambda)) \to H^1(\lambda) \to ...$$

Now $\lambda \in X^+(T)$ so $H^i(\lambda) = 0$ for all $i > 0$ by Kempf's vanishing theorem. For $\langle \lambda, \alpha^v \rangle > 0$ we have $\langle \lambda - \alpha, \alpha^v \rangle \geq -1$ and $H^i(\lambda - \alpha) = 0$ because either $\langle \lambda - \alpha, \alpha^v \rangle = -1$ or $\lambda - \alpha \in X^+(T)$ and the result is true by Kempf's vanishing theorem. So $H^i(N_\alpha(\lambda)) = 0$. This proves 1.

For $\langle \lambda, \alpha^v \rangle = 0$ we use the spectral sequence given by
$R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(\rho) \otimes (\lambda - \rho)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho))$. Since $\langle \lambda, \alpha^v \rangle = 0$ so $\langle \lambda - \rho, \alpha^v \rangle = -1$ and hence $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho) = 0$.
This gives the result in 2. □

Suppose $\lambda \in X(T)$. Now we will ask the question: when is $H^i(N_\alpha(\lambda)) = H^i(\lambda) \oplus H^i(\lambda - \alpha)$? Recall that $N_\alpha(\lambda)$ is given by the short exact sequence $0 \to \lambda - \alpha \to N_\alpha(\lambda) \to \lambda \to 0$. So the above statement is definitely true when λ and $\lambda - \alpha$ are in different blocks. For a definition of blocks see [15]. The next two propositions give the precise condition when λ and $\lambda - \alpha$ are in the same block.

Proposition 2.4.4. *Let $G = GL_3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in X(T)$. Let $\alpha = (1, -1, 0)$ be a simple root then λ and $\lambda - \alpha$ are in the same block if and only if λ has one of following forms:*

1. $(\lambda_2 + pa_1, \lambda_2, \lambda_3)$

2. $(\lambda_1, \lambda_1 + pa_2, \lambda_1 + 1 - pa_1)$

Proof. Using the linkage principle if λ and $\lambda - \alpha$ are in the same block then $\lambda - \alpha \in W \cdot \lambda + p\mathbb{Z}\Phi$ where W is the Weyl group and Φ is the set of roots. So there exists some $w \in W$ and $\theta \in \mathbb{Z}\Phi$ such that $\lambda - \alpha = w \cdot \lambda + p\theta$. The Weyl group for GL_3 is the symmetric group S_3. We have the following possibilities

1. Let $w = I_3$ then λ and $\lambda - \alpha$ can not be in the same block because if they are we would have $\alpha \in p\mathbb{Z}\Phi$ which is clearly not possible.

2. Let $w = (12)$ then $w \cdot (\lambda_1, \lambda_2, \lambda_3) = w(\lambda_1 + 1, \lambda_2, \lambda_3 - 1) - (1, 0, -1)$. For $w = (12)$ if λ and $\lambda - \alpha$ are in the same block then $(\lambda_1 - 1, \lambda_2 + 1, \lambda_3) = (\lambda_2 - 1, \lambda_1 + 1, \lambda_3) + p(a_1, a_2, a_3)$. Therefore $\lambda_1 = \lambda_2 + pa_1$, $\lambda_1 = \lambda_2 - pa_2$ and $\lambda_3 = \lambda_3 + pa_3$. This is only possible when $a_2 = -a_1$ and $a_3 = 0$. Hence $\lambda = (\lambda_2 + pa_1, \lambda_2, \lambda_3)$.

3. Let $w = (123)$ then $(\lambda_1 - 1, \lambda_2 + 1, \lambda_3) = (\lambda_2 - 1, \lambda_3 - 1, \lambda_1 + 2) + p(a_1, a_2, a_3)$. Therefore $\lambda_1 = \lambda_2 + pa_1$, $\lambda_3 = \lambda_2 - pa_2 + 2$ and $\lambda_3 = \lambda_1 + pa_3 + 2$. So we get $a_1 = -a_2 - a_3$ which is always true. Hence $\lambda = (\lambda_2 + pa_1, \lambda_2, \lambda_2 + 2 - pa_2)$.

4. If $w = (23)$ then λ and $\lambda - \alpha$ can not be in the same block because if they are we will get $pa_1 = -1$ which is clearly not possible.

5. Suppose $w = (132)$ we get $(\lambda_1 - 1, \lambda_2 + 1, \lambda_3) = (\lambda_3 - 2, \lambda_1 + 1, \lambda_2 + 1) + p(a_1, a_2, a_3)$. Therefore $\lambda_3 = \lambda_1 - pa_1 + 1$, $\lambda_2 = \lambda_1 + pa_2$ and $\lambda_3 = \lambda_1 + 1 + p(a_2 + a_3)$. So we get $-a_1 = a_2 + a_3$. Hence $\lambda = (\lambda_1, \lambda_1 + pa_2, \lambda_1 + 1 - pa_1)$.

6. If $w = (13)$ then λ and $\lambda - \alpha$ can not be in the same block because if they are we will get $pa_2 = 1$ which is clearly not possible.

   We have the result.

□

From proposition 2.4.4, we get the following

Proposition 2.4.5. *Let $G = SL_3$ and $\lambda \in X(T)$. If p does not divide $\langle \lambda, \alpha^\vee \rangle$ then*

$$H^i(N_\alpha(\lambda)) \simeq H^i(\lambda) \oplus H^i(\lambda - \alpha).$$

Proof. Since $G = SL_3$ so $\lambda = (r, s)$, where r, s are integers. From proposition 2.4.4, λ and $\lambda - \alpha$ are in the same block if and only if λ is in one of the following forms

1. $(pa_1, \lambda_2 - \lambda_3)$
2. $(-pa_2, 1 + pa_2 - pa_1)$

From these conditions it is easy to see the result. □

The proposition 2.4.5 help us a great deal to simplify the recursion. For an application the results of Donkin given by Lemma 2.4.2 now become

Lemma 2.4.6. *For $i \geq 0$ integers r, s and a simple root α we have:*

1. $$\chi^i_\alpha(2r, 2s) = \chi^i(r-1, s-1)^F + \chi(1,0)^F \chi^i(r-1, s)^F + 2\chi(0,1)\chi^i(r-1, s)^F$$
$$+\chi(1,0)\chi^i_\alpha(r, s-1)^F;$$

2. $$\chi^i_\alpha(1 + 2r, 2s) = \chi^i(1 + 2r, 2s) + \chi^i(2r - 1, 1 + 2s);$$

3. $$\chi^i_\alpha(2r, 1+2s) = \chi(0,1)^F \chi^i(r-1, s)^F + \chi^i(r-1, s+1)^F + 2\chi(1,0)\chi^i(r-1, s)^F$$
$$+\chi(0,1)\chi^i_\alpha(r, s)^F;$$

4. $\chi^i_\alpha(1+2r, 1+2s) = \chi^i(1+2r, 1+2s) + \chi^i(2r-1, 2s+2).$

We will now give some results about $H^i(\lambda)$.

Proposition 2.4.7. *Suppose $n > 0$ and $0 \le m \le p-1$ then we have*

$$H^i(r, -p^n(m+1)-1) = \begin{cases} H^0(r - p^n(m+1), p^n(m+1)-1), & r \ge p^n(m+1) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Using the spectral sequence we get

$$\begin{aligned} H^i(r, -p^n(m+1)-1) &= R^i\mathrm{Ind}_B^G(r, -p^n(m+1)-1) \\ &= \mathrm{Ind}_{P_\beta}^G R^i\mathrm{Ind}_B^{P_\beta}(r, -p^n(m+1)-1) \qquad (2.1) \end{aligned}$$

Now $R^i\mathrm{Ind}_B^{P_\beta}(r, -p^n(m+1)-1) = 0$ for all $i \ne 1$. So from equation 2.1 we get

$$H^i(r, -p^n(m+1)-1) = \mathrm{Ind}_{P_\beta}^G R\mathrm{Ind}_B^{P_\beta}(r, -p^n(m+1)-1) \qquad (2.2)$$

Using the Serre duality we get

$$R\mathrm{Ind}_B^{P_\beta}(r, -p^n(m+1)-1) = \nabla_\beta(-r+1, p^n(m+1)-1)^*$$

and $\nabla_\beta(-r+1, p^n(m+1)-1)^* = \nabla_\beta(r - p^n(m+1), p^n(m+1)-1)$ (Using the SL_2 case). Replace the values back in equation 2.2 to get

$$\begin{aligned} H^i(r, -p^n(m+1)-1) &= \mathrm{Ind}_{P_\beta}^G \nabla_\beta(r - p^n(m+1), p^n(m+1)-1) \\ &= \mathrm{Ind}_B^G(r - p^n(m+1), p^n(m+1)-1) \end{aligned}$$

Finally $\mathrm{Ind}_B^G(r - p^n(m+1), p^n(m+1)-1) \ne 0$ if and only if $r \ge p^n(m+1)$. We have the result.

□

Proposition 2.4.8. *Suppose $n \geq 0$ then for all $r \geq 2$ we have*

$$H^1(p^n - 1, -p^n - r) = 0.$$

Proof. Let $\lambda = s_\beta.(p^n - 1, -p^n - r) = (-r, p^n + r - 2)$ then $\langle \lambda, \beta^\vee \rangle = p^n + r - 2$. For $r = 1$ the result is true by proposition 2.4.7. Now for $r > 1$ we write $p^n + r - 2 = \sum_{j=0}^{m} a_j p^j$. So by proposition 2.3.4 case 2, we have

$$H^1(p^n - 1, -p^n - r) \neq 0 \iff (p^n - 1, -p^n - r) + a_m p^m(-1, 2) \in X^+(T).$$

The above statement is true if and only if $p^n \geq a_m p^m + 1$ and $2a_m p^m \geq p^n + r$. From the first inequality we get $n > m$ but if $n > m$ then $2a_m p^m \geq p^n + r$ is never true. Hence the result. □

Proposition 2.4.9. *Suppose $n \geq 0$ and $r \geq 2$ then we have*

$$H^1(N_\beta(p^n - 1, -p^n - r)) = H^1(p^n, -p^n - r - 2).$$

Proof. We have the short exact sequence given by

$$0 \to (p^n, -p^n - r - 2) \to N_\beta(p^n - 1, -p^n - r) \to (p^n - 1, -p^n - r) \to 0.$$

Moreover $H^0(p^n - 1, -p^n - r) = 0$ and $H^1(p^n - 1, -p^n - r) = 0$ by proposition 2.4.8. So by the long exact sequence of induction we have

$$0 \to H^1(p^n, -p^n - r - 2) \to H^1(N_\beta(p^n - 1, -p^n - r)) \to 0.$$

and hence the result. □

Proposition 2.4.10. *Suppose $n \geq 0$ and $r > 0$ then we have*

$$H^i(N_\beta(p^n - r, -p^n - 1)) = H^i(p^n - r + 1, -p^n - 3).$$

Proof. We have the short exact sequence given by

$$0 \to (p^n - r + 1, -p^n - 3) \to N_\beta(p^n - r, -p^n - 1) \to (p^n - r, -p^n - 1) \to 0.$$

Moreover $H^i(p^n - r, -p^n - 1) = 0$ for all i by proposition 2.4.7. So by the long exact sequence of induction we have

$$0 \to H^i(p^n - r + 1, -p^n - 3) \to H^i(N_\beta(p^n - r, -p^n - 1)) \to 0.$$

and hence the result. □

We would like to point out here that Donkin's recursive formulas given [11] are also valid for $i = 0$. In this chapter we will often be using the phrase "the p–expansion of Weyl character of $\chi(u, v)$" to mean that we are considering the expansion of $\chi(u, v)$ using Donkin's formulas.

2.4.1 The case $p = 2$

Now we will consider the case $p = 2$. Kempf's vanishing theorem implies that $\chi^i(r, s) = 0$ when $r \geq 0$ and $s \geq 0$. Also $\chi^i(r, s) = 0$, if $r = 0, -1$ or$s = 0, -1$ see e.g. [11, lemma 1,2]. So if $\chi^i(r, s) \neq 0$, then either $r < -1$ or $s < -1$. Without loss of generality we will assume that $s < -1$. The following result gives us the condition when the result is the same as in characteristic zero.

Proposition 2.4.11. *Let $p = 2$ then $\chi^1(r, -s - 2) = \chi(r - s - 1, s)$ for all $r \geq 2s$, $r, s > 0$. Moreover*

$$\chi^1_\alpha(r, -s - 2) = \chi(r - s - 1, s) + \chi(r - s - 2, s - 1)$$

and

$$\chi^1_\beta(r, -s - 2) = \chi(r - s - 1, s) + \chi(r - s - 2, s + 2)$$

for all $r \geq 2s$, $r, s > 0$.

Proof. The argument is by induction on r and s. We will divide the proof into the following cases

1. Suppose $r = 1 + 2u$ and $s = 1 + 2v$ for some positive integers u and v then

$$\begin{aligned}\chi^1(1+2u, 1+2(-v-2)) &= \chi(1,1)\chi^1(u,-v-2)^F \\ &= \chi(1,1)\chi(u-v-1,v)^F \\ &= \chi(2u-2v-1, 2v+1) \\ &= \chi(r-s-1, s).\end{aligned}$$

2. Let $r = 2u$ and $s = 2v$ for some positive integers u and v then

$$\begin{aligned}\chi^1(2u, 2(-v-1)) = \chi^1(u,-v-1)^F + \chi^1(u-1,-v-2)^F + \chi(1,0)\chi^1(u,-v-2)^F \\ + \chi(0,1)\chi^1(u-1,-v-1)^F. \qquad (2.3)\end{aligned}$$

To apply induction we need $u \geq 2v-2$ and $u \geq 2v-1$. But by our assumption $u \geq 2v$ so both inequalities are clearly true. From equation 2.3 we get

$$\begin{aligned}\chi^1(2u, 2(-v-1)) = \chi(u-v, v-1)^F + \chi(u-v-2, v)^F + \chi(1,0)\chi(u-v-1,v)^F \\ + \chi(0,1)\chi(u-v-1, v-1)^F. \qquad (2.4)\end{aligned}$$

Now the p–expansion of Weyl character of $\chi(2u-2v-1, 2v)$ is given by

$$\begin{aligned}\chi(2u-2v-1, 2v) = \chi(u-v, v-1)^F + \chi(u-v-2, v)^F \\ + \chi(1,0)\chi(u-v-1, v)^F + \chi(0,1)\chi(u-v-1, v-1)^F. \\ (2.5)\end{aligned}$$

Compare equations 2.4 and 2.5 to get

$$\chi^1(2u, 2(-v-1)) = \chi(2u-2v-1, 2v).$$

Hence the result.

3. Now we take $r = 1 + 2u$ and $s = 2v$ for some positive integers u and v then

$$\begin{aligned}\chi^1(1+2u, 2(-v-1)) &= \chi^1_\alpha(u+1, -v-2)^F + \chi(1,0)\chi^1(u,-v-1)^F \\ &+ \chi(0,1)\chi^1(u,-v-2)^F \\ &= \chi^1_\alpha(u+1,-v-2)^F + \chi(1,0)\chi(u-v,v-1)^F \\ &+ \chi(0,1)\chi(u-v-1,v)^F. \end{aligned} \tag{2.6}$$

Now the p–expansion of Weyl character of $\chi(2u-2v, 2v)$ is given by

$$\begin{aligned}\chi(2u-2v,2v) &= \chi(u-v,v)^F + \chi(u-v-1,v-1)^F \\ &+ \chi(1,0)\chi(u-v,v-1)^F + \chi(0,1)\chi(u-v-1,v)^F. \end{aligned} \tag{2.7}$$

If equation 2.6 and 2.7 are the same we must have

$$\chi^1_\alpha(u+1,-v-2) = \chi(u-v,v) + \chi(u-v-1,v-1).$$

We will now prove that $\chi^1_\alpha(r,-s-2) = \chi(r-s-1,s) + \chi(r-s-2,s-1)$ for all $r \geq 2s$. If r is odd then the result is true by proposition 2.4.5 and the inductive hypothesis. We have the following two cases left

(a) Suppose $r = 2u$ and $s = 2v$ for some positive integers u and v we have

$$\begin{aligned}\chi^1_\alpha(2u, 2(-v-1)) &= \chi^1(u-1,-v-2)^F + \chi(1,0)^F\chi^1(u-1,-v-1)^F \\ &+ 2\chi(0,1)\chi^1(u-1,-v-1)^F + \chi(1,0)\chi^1_\alpha(u,-v-2)^F \\ &= \chi(u-v-2,v)^F + \chi(1,0)^F\chi(u-v-1,v-1)^F + 2\chi(0,1) \\ &\chi(u-v-1,v-1)^F + \chi(1,0)[\chi(u-v-1,v) + \chi(u-v-2,v-1)]^F. \end{aligned} \tag{2.8}$$

Now the p–expansion of Weyl character of $\chi(2u-2v-1,2v)$ is given by

$$\chi(2u-2v-1,2v) = \chi(u-v,v-1)^F + \chi(u-v-2,v)^F + \chi(1,0)\chi(u-v-1,v)^F + \chi(0,1)\chi(u-v-1,v-1)^F. \quad (2.9)$$

and the p–expansion of Weyl character of $\chi(2u-2v-2,2v-1)$ is given by

$$\chi(2u-2v-2,2v-1) = \chi(u-v-2,v)^F + \chi(u-v-1,v-2)^F + \chi(1,0)\chi(u-v-2,v-1)^F + \chi(0,1)\chi(u-v-1,v-1)^F. \quad (2.10)$$

Add equations 2.9, 2.10 and note that

$$\chi(1,0)\chi(u-v-1,v-1) = \chi(u-v,v-1)+\chi(u-v-2,v)+\chi(u-v-1,v-2).$$

to get the result.

(b) Now let $r = 2u$ and $s = 1+2v$ for some positive integers u and v we have

$$\chi^1_\alpha(2u,-3-2v) = \chi(0,1)^F\chi^1(u-1,-v-2)^F + \chi^1(u-1,-v-1)^F + 2\chi(1,0)\chi^1(u-1,-v-2)^F + \chi(0,1)\chi^1_\alpha(u,-v-2)^F. \quad (2.11)$$

Using the inductive hypothesis we get

$$\chi^1_\alpha(2u,-3-2v) = \chi(0,1)^F\chi(u-v-2,v)^F + \chi(u-v-1,v-1)^F + 2\chi(1,0) \chi(u-v-2,v)^F + \chi(0,1)[\chi(u-v-1,v) + \chi(u-v-2,v-1)]^F. \quad (2.12)$$

The p–expansion of Weyl character of $\chi(2u-2v-2,1+2v)$ is given by

$$\chi(2(u-v-1),1+2(v)) = \chi(u-v-2,v+1)^F + \chi(u-v-1,v-1)^F + \chi(1,0)\chi(u-v-2,v)^F + \chi(0,1)\chi(u-v-1,v)^F. \quad (2.13)$$

and the p–expansion of Weyl character of $\chi(2u-2v-3,2v)$ is

$$\chi(1+2(u-v-2),2(v)) = \chi(u-v-1,v-1)^F + \chi(u-v-3,v)^F + \chi(1,0)\chi(u-v-2,v)^F + \chi(0,1)\chi(u-v-2,v-1)^F. \tag{2.14}$$

Add equations 2.13,2.14 and note that

$$\chi(0,1)\chi(u-v-2,v) = \chi(u-v-2,v+1)+\chi(u-v-1,v-1)+\chi(u-v-3,v).$$

to get the result.

4. Finally suppose $r = 2u$ and $s = 1+2v$ for some positive integers u and v then

$$\begin{aligned}\chi^1(2u,1+2(-v-2)) &= \chi^1_\beta(u-1,-v-1)^F + \chi(1,0)\chi^1(u-1,-v-2)^F \\ &+ \chi(0,1)\chi^1(u,-v-2)^F \\ &= \chi^1_\beta(u-1,-v-1)^F + \chi(1,0)\chi(u-v-2,v)^F \\ &+ \chi(0,1)\chi(u-v-1,v)^F. \end{aligned} \tag{2.15}$$

Now the p–expansion of Weyl character of $\chi(2u-2v-2,1+2v)$ is given by

$$\begin{aligned}\chi(2u-2v-2,1+2v) &= \chi(u-v-2,v+1)^F + \chi(u-v-1,v-1)^F \\ &+ \chi(1,0)\chi(u-v-2,v)^F + \chi(0,1)\chi(u-v-1,v)^F.\end{aligned} \tag{2.16}$$

If equation 2.15 and 2.16 are the same we must have

$$\chi^1_\beta(u-1,-v-1) = \chi(u-v-2,v+1) + \chi(u-v-1,v-1).$$

We will now prove that $\chi^1_\beta(r,-s-2) = \chi(r-s-1,s) + \chi(r-s-2,s+2)$ for all $r \geq 2s$. If s is odd then the result is true by proposition 2.4.5 and the inductive hypothesis. We have the following two cases left

(a) First we take $r = 2u$ and $s = 2v$ for some positive integers u and v then

$$\chi^1_\beta(2u, 2(-v-1)) = \chi^1(u-1, -v-2)^F + \chi(0,1)^F \chi^1(u, -v-2)^F + 2\chi(1,0)\chi^1(u, -v-2)^F + \chi(0,1)\chi^1_\beta(u-1, -v-1)^F. \quad (2.17)$$

Using the inductive hypothesis we get

$$\chi^1_\beta(2u, 2(-v-1)) = \chi(u-v-2, v)^F + \chi(0,1)^F \chi(u-v-1, v)^F + 2\chi(1,0) \chi(u-v-1, v)^F + \chi(0,1)[\chi(u-v-1, v-1) + \chi(u-v-2, v+1)]^F. \quad (2.18)$$

Also the p–expansion of Weyl character of $\chi(2u-2v-1, 2v)$ is given by

$$\chi(1+2(u-v-1), 2v) = \chi(u-v, v-1)^F + \chi(u-v-2, v)^F + \chi(1,0)\chi(u-v-1, v)^F + \chi(0,1)\chi(u-v-1, v-1)^F. \quad (2.19)$$

and the p–expansion of Weyl character of $\chi(2u-2v-2, 2v+2)$ is

$$\chi(2(u-v-1), 2(v+1)) = \chi(u-v-1, v+1)^F + \chi(u-v-2, v)^F + \chi(1,0)\chi(u-v-1, v)^F + \chi(0,1)\chi(u-v-2, v+1)^F. \quad (2.20)$$

Add 2.19,2.20 and note that

$$\chi(0,1)\chi(u-v-1, v) = \chi(u-v-1, v+1) + \chi(u-v, v-1) + \chi(u-v-2, v). \quad (2.21)$$

to get the result.

(b) In our very last case we consider $r = 1 + 2u$ and $s = 2v$ for some positive integers u and v then

$$\chi^1_\beta(1+2u, -2(-v-1)) = \chi(1,0)^F \chi^1(u, -v-2)^F + \chi^1(u+1, -v-2)^F + 2\chi(0,1)\chi^1(u, -v-2)^F + \chi(1,0)\chi^1_\beta(u, -v-1)^F. \quad (2.22)$$

Using the inductive hypothesis we have

$$\chi^1_\beta(1+2u,2(-v-1)) = \chi(1,0)^F\chi(u-v-1,v)^F + \chi(u-v,v)^F + 2\chi(0,1)$$
$$\chi(u-v-1,v)^F + \chi(1,0)[\chi(u-v,v-1) + \chi(u-v-1,v+1)]^F. \tag{2.23}$$

The p–expansion of Weyl character for $\chi(2u-2v,2v)$ is given by

$$\chi(2(u-v),2v) = \chi(u-v,v)^F + \chi(u-v-1,v-1)^F$$
$$+\chi(1,0)\chi(u-v,v-1)^F + \chi(0,1)\chi(u-v-1,v)^F. \tag{2.24}$$

and the p–expansion of Weyl character of $\chi(2u-2v-1,2v+2)$ is

$$\chi(1+2(u-v-1),2(v+1)) = \chi(u-v,v)^F + \chi(u-v-2,v+1)^F$$
$$+\chi(1,0)\chi(u-v-1,v+1)^F + \chi(0,1)\chi(u-v-1,v)^F. \tag{2.25}$$

Add equations 2.24 and 2.25 and note that

$$\chi(1,0)\chi(u-v-1,v) = \chi(u-v,v)+\chi(u-v-2,v+1)+\chi(u-v-1,v-1). \tag{2.26}$$

to get the result.

This completes the proof.

□

Corollary 2.4.12. *Let $p=2$. Then the sequence*

$$0 \to H^1(r-2,-s-1) \longrightarrow H^1(N_\alpha(r,-s-2)) \longrightarrow H^1(r,-s-2) \longrightarrow 0$$

is exact for all $r \geq 2s$.

Proof. We know that the sequence

$$H^1(r-2,-s-1) \xrightarrow{\phi} H^1(N_\alpha(r,-s-2)) \xrightarrow{\psi} H^1(r,-s-2)$$

is exact meaning $\mathrm{Im}\phi = \mathrm{Ker}\psi$. Moreover by proposition 2.4.11, we have

$$\chi^1_\alpha(r,-s-2) = \chi^1(r,-s-2) + \chi^1(r-2,-s-1).$$

Let $X = H^1(r-2,-s-1)$, $Y = H^1(N_\alpha(r,-s-2))$ and $Z = H^1(r,-s-2)$. Now $\mathrm{Im}\psi \simeq Y/\mathrm{Ker}\psi \simeq Y/\mathrm{Im}\phi$, therefore $\dim Y = \dim \mathrm{Im}\phi + \dim \mathrm{Im}\psi$. It is clear that $\dim \mathrm{Im}\psi \leq \dim Z$. Also by the character result we have $\dim Y = \dim X + \dim Z$. Now $\dim \mathrm{Im}\phi = \dim Y - \dim \mathrm{Im}\psi \leq \dim Y - \dim Z = \dim X$. This implies $\dim \mathrm{Im}\phi \leq \dim X$. We have proved that $\dim \mathrm{Im}\phi \leq \dim X$ and $\dim \mathrm{Im}\psi \leq \dim Z$ but $\dim Y = \dim X + \dim Z$ so we must have the equality. This proves that ϕ is injective and ψ is surjective and hence the result. □

The only region left to consider is when $r < 2s$. We will give here a few special cases to give us an idea how the general case may look like. The proof is by repeated application of recursive formulas and is very easy but long. We will leave it to the reader to verify. The cases $s = -2, -3, -5$ are already given in [11].

$$\chi^1(r,-4) = \begin{cases} \chi(0), & r = 2 \\ \chi(r-3,2), & r \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-6) = \begin{cases} \chi(0,2), & r = 4 \\ \chi(0,4) + \chi(0,1), & r = 5 \\ \chi(1,4) + \chi(0), & r = 6 \\ \chi(2,4) + \chi(0,1)^F, & r = 7 \\ \chi(r-5,4), & r \geq 8 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-7) = \begin{cases} \chi(0,1), & r = 4 \\ \chi(1,1), & r = 5 \\ \chi(0,5) + \chi(1,0), & r = 6 \\ \chi(r-6,5), & r \geq 7 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-8) = \begin{cases} \chi(0), & r = 4 \\ \chi(1,0), & r = 5 \\ \chi(0,2)^F + \chi(0,1), & r = 6 \\ \chi(r-7,6), & r \geq 7 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-9) = \begin{cases} \chi(r-8,7), & r \geq 8 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-10) = \begin{cases} \chi(0,6), & r = 8 \\ \chi(r-9,8), & r \geq 9 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-11)=\begin{cases}\chi(0,5), & r=8\\ \chi(1,5), & r=9\\ \chi(0,9)+\chi(1,4), & r=10\\ \chi(1,9)+\chi(1,3), & r=11\\ \chi(2,9)+\chi(1,2), & r=12\\ \chi(3,9)+\chi(1,1), & r=13\\ \chi(4,9)+\chi(1,0)+\chi(0,1)\chi(0,1)^{F^2}, & r=14\\ \chi(5,9)+\chi(1,1)\chi(0,1)^{F^2}, & r=15\\ \chi(6,9)+\chi(1,0)\chi(0,1)^{F^2}, & r=16\\ \chi(r-10,9), & r\geq 17\\ 0, & \text{otherwise.}\end{cases}$$

Proposition 2.4.13. $\chi^1(2^n,-2^n-2)=\chi(0,2^n-2)$, *for all* $n\geq 0$.

Proof. The argument is by induction on n. The result is true for $n=1$ by above special cases. Suppose the result is true for $n-1$. Using the recursive formula we have

$$\begin{aligned}\chi^1(2^n,-2^n-2)&=\chi^1(2^{n-1},-2^{n-1}-1)^F+\chi^1(2^{n-1}-1,-2^{n-1}-2)^F\\&\quad+\chi(1,0)\chi^1(2^{n-1},-2^{n-1}-2)^F+\chi(0,1)\chi^1(2^{n-1}-1,-2^{n-1}-1)^F.\end{aligned}\tag{2.27}$$

Now by proposition 2.4.8 we have $\chi^1(2^{n-1}-1,-2^{n-1}-2)=0$. Also $\chi^1(2^{n-1}-1,-2^{n-1}-1)=0$ and $\chi^1(2^{n-1},-2^{n-1}-1)=\chi(0,2^{n-1}-1)$ by proposition 2.4.7. the equation 2.27 becomes

$$\chi^1(2^n,-2^n-2)=\chi(0,2^{n-1}-1)^F+\chi(1,0)\chi(0,2^{n-1}-2)^F.\tag{2.28}$$

Finally compare equation 2.28 with the p–expansion of Weyl character of $\chi(0, 2^n - 2)$ to get the result. □

Proposition 2.4.14. *Suppose* $n \geq 0$ *then we have* $\chi^1(2^n - 1, -2^n) = \chi(0, 2^n - 2)$. *Also*

$$\chi^1_\alpha(2^n, -2^n - 1) = \chi(0, 2^n - 1).$$

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}\chi^1(2^n - 1, -2^n) &= \chi^1_\alpha(2^{n-1}, -2^{n-1} - 1)^F + \chi(1,0)\chi^1(2^{n-1} - 1, -2^{n-1})^F \\ &\quad + \chi(0,1)\chi^1(2^{n-1} - 1, -2^{n-1} - 1)^F. \qquad (2.29)\end{aligned}$$

Now $\chi^1(2^{n-1} - 1, -2^{n-1} - 1) = 0$ by proposition 2.4.7. Also the p–expansion of Weyl character of $\chi(0, 2^n - 2)$ gives

$$\chi(0, 2^n - 2) = \chi(0, 2^{n-1} - 1)^F + \chi(1,0)\chi(0, 2^{n-1} - 2)^F.$$

So if the result is true we must have $\chi^1_\alpha(2^{n-1}, -2^{n-1} - 1) = \chi(0, 2^{n-1} - 1)$. We will show this by induction on n. The result is clearly true for $n = 1$. Now by the recursive formula we get

$$\begin{aligned}\chi^1_\alpha(2^n, -2^n - 1) &= \chi(0,1)^F\chi^1(2^{n-1} - 1, -2^{n-1} - 1)^F + \chi^1(2^{n-1} - 1, -2^{n-1})^F \\ &\quad + 2\chi(1,0)\chi^1(2^{n-1} - 1, -2^{n-1} - 1)^F + \chi(0,1)\chi^1_\alpha(2^{n-1}, -2^{n-1} - 1)^F. \\ &\qquad (2.30)\end{aligned}$$

Now $\chi^1(2^{n-1} - 1, -2^{n-1} - 1) = 0$ by proposition 2.4.7. Using the inductive hypothesis equation 2.30 becomes

$$\begin{aligned}\chi^1_\alpha(2^n, -2^n - 1) &= \chi(0, 2^{n-1} - 2)^F + \chi(0,1)\chi(0, 2^{n-1} - 1)^F \\ &= \chi(0, 2^n - 1). \qquad (2.31)\end{aligned}$$

Substituting from equation 2.31 to equation 2.29 and using the inductive hypothesis we get the result. □

Proposition 2.4.15. $\chi^1(2^n, -2^n) = \chi(1, 2^n - 2)$ *for all* $n \geq 0$.

Proof. We argue by induction on n. The result is clearly true for $n = 0$. Suppose the result is true for all $t < n$ then we have

$$\chi^1(2^t, -2^t) = \chi(1, 2^t - 2).$$

Now for $n = t + 1$ we get

$$\begin{aligned}\chi^1(2^{t+1}, -2^{t+1}) &= \chi^1(2^t, -2^t)^F + \chi^1(2^t - 1, -2^t - 1)^F \\ &\quad + \chi(1,0)\chi^1(2^t, -2^t - 1)^F + \chi(0,1)\chi^1(2^t - 1, -2^t)^F. \qquad (2.32)\end{aligned}$$

From propositions 2.4.14 and 2.4.7 we get that $\chi^1(2^t - 1, -2^t) = \chi(0, 2^t - 2)$ and $\chi^1(2^t - 1, -2^t - 1) = 0$. Also $\chi^1(2^t, -2^t - 1) = \chi(0, 2^t - 1)$ by proposition 2.4.7. The equation 2.32 becomes

$$\chi^1(2^{t+1}, -2^{t+1}) = \chi(1, 2^t - 2)^F + \chi(1,0)\chi(0, 2^t - 1)^F + \chi(0,1)\chi(0, 2^t - 2)^F. \quad (2.33)$$

Compare the p–expansion of Weyl character of $\chi(1, 2^{t+1} - 2)$ with equation 2.33 to get the required result. □

Proposition 2.4.16. *Suppose* $n \geq 0$ *then we have* $\chi^1(2^n + 1, -2^n) = \chi(1, 2^n - 2)$. *Also*

$$\chi^1_\alpha(2^n, -2^n - 1) = \chi(0, 2^n - 1).$$

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}\chi^1(2^n + 1, -2^n) &= \chi^1_\alpha(2^{n-1} + 1, -2^{n-1} - 1)^F + \chi(1,0)\chi^1(2^{n-1}, -2^{n-1})^F \\ &\quad + \chi(0,1)\chi^1(2^{n-1}, -2^{n-1} - 1)^F. \qquad (2.34)\end{aligned}$$

Now $\chi^1(2^{n-1}, -2^{n-1}-1) = \chi(0, 2^{n-1}-1)$ by proposition 2.4.7. Also by proposition 2.4.15 we get $\chi^1(2^{n-1}, -2^{n-1}) = \chi(1, 2^{n-1}-2)$. The proposition 2.4.5 gives

$$\chi^1_\alpha(2^{n-1}+1, -2^{n-1}-1) = \chi(1, 2^{n-1}-1) + \chi(0, 2^{n-1}-2).$$

The equation 2.34 becomes

$$\begin{aligned}\chi^1(2^n+1, -2^n) = [\chi(1, 2^{n-1}-1) + \chi(0, 2^{n-1}-2)]^F + \chi(1,0)\\ \chi(1, 2^{n-1}-2)^F + \chi(0,1)\chi(0, 2^{n-1}-1)^F.\end{aligned} \tag{2.35}$$

Also the $p-$expansion of Weyl character of $\chi(1, 2^n-2)$ gives

$$\begin{aligned}\chi(1, 2^n-2) = [\chi(1, 2^{n-1}-1) + \chi(0, 2^{n-1}-2)]^F + \chi(1,0)\\ \chi(1, 2^{n-1}-2)^F + \chi(0,1)\chi(0, 2^{n-1}-1)^F.\end{aligned} \tag{2.36}$$

Compare equations 2.35 and 2.36 to get the result. □

Proposition 2.4.17. *Suppose $n \geq 0$ then we have $\chi^1(2^n-1, -2^n+1) = \chi(1, 2^n-3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}\chi^1(2^n-1, -2^n+1) &= \chi(1,1)\chi^1(2^{n-1}-1, -2^{n-1})^F\\ &= \chi(1,1)\chi(0, 2^{n-1}-2)^F\\ &= \chi(1, 2^n-3).\end{aligned}$$

□

Proposition 2.4.18. *Suppose $n \geq 0$ then we have $\chi^1(2^n, -2^n+1) = \chi(2, 2^n-3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\chi^1(2^n, -2^n + 1) = \chi^1_\beta(2^{n-1} - 1, -2^{n-1} + 1)^F + \chi(1,0)\chi^1(2^{n-1} - 1, -2^{n-1})^F + \chi(0,1)\chi^1(2^{n-1}, -2^{n-1})^F. \tag{2.37}$$

Now $\chi^1(2^{n-1} - 1, -2^{n-1}) = \chi(0, 2^{n-1} - 2)$ by proposition 2.4.14. Also $\chi^1(2^{n-1}, -2^{n-1}) = \chi(1, 2^{n-1} - 2)$ by proposition 2.4.15. Moreover proposition 2.4.5 gives

$$\chi^1_\beta(2^{n-1} - 1, -2^{n-1} + 1) = \chi^1(2^{n-1} - 1, -2^{n-1} + 1) + \chi^1(2^{n-1}, -2^{n-1} - 1)$$

Using propositions 2.4.7 and 2.4.17, equation 2.37 becomes

$$\chi^1(2^n, -2^n + 1) = [\chi(1, 2^{n-1} - 3) + \chi(0, 2^{n-1} - 1)]^F + \chi(1,0)\chi^1(0, 2^{n-1} - 2)^F + \chi(0,1)\chi^1(1, 2^{n-1} - 2)^F. \tag{2.38}$$

Compare equation 2.38 with the $p-$expansion of Weyl character of $\chi(2, 2^n - 3)$ to get the result. □

Proposition 2.4.19. *Suppose $n \geq 0$ then we have $\chi^1(2^n + 1, -2^n + 1) = \chi(3, 2^n - 3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned} \chi^1(2^n + 1, -2^n + 1) &= \chi(1,1)\chi^1(2^{n-1}, -2^{n-1})^F \\ &= \chi(1,1)\chi(1, 2^{n-1} - 2)^F \\ &= \chi(3, 2^n - 3). \end{aligned}$$

□

Proposition 2.4.20. *Suppose $r, n > 0$ then we have*

$$\chi^1(2^n, -2^n - r) = \begin{cases} \chi(0, 2^n - r), & n \geq r - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We argue by induction on r. The result is true for $r = 1$ by proposition 2.4.7. Suppose the result is true for all $t-1 < r$. For $r = t$ we have the following two possibilities.

1. Suppose t is even then $t = 2u$ for some positive integer u we have

$$\begin{aligned} &\chi^1(2^n, -2^n - 2u) \\ &\quad = \chi^1(2^{n-1}, -2^{n-1} - u)^F + \chi^1(2^{n-1} - 1, -2^{n-1} - u - 1)^F \\ &\quad + \chi(1,0)\chi^1(2^{n-1}, -2^{n-1} - u - 1)^F + \chi(0,1)\chi^1(2^{n-1} - 1, -2^{n-1} - u)^F. \end{aligned} \tag{2.39}$$

By proposition 2.4.8 we have $\chi^1(2^{n-1}-1, -2^{n-1}-u-1) = 0$ and $\chi^1(2^{n-1} - 1, -2^{n-1} - u) = 0$. Using the inductive hypothesis we get

$$\chi^1(2^n, -2^n - 2u) = \chi(0, 2^{n-1} - u)^F + \chi(1,0)\chi^1(0, 2^{n-1} - u - 1)^F. \tag{2.40}$$

Compare equation 2.39 with the p–expansion of Weyl character of $\chi(0, 2^n - 2u)$ to get the result.

2. Now suppose $r = 2u + 1$ for some positive integer u then we have

$$\begin{aligned} &\chi^1(2^n, -2^n - 2u - 1) \\ &\quad = \chi^1_\beta(2^{n-1} - 1, -2^{n-1} - u)^F + \chi(1,0)\chi^1(2^{n-1} - 1, -2^{n-1} - u - 1)^F \\ &\quad + \chi(0,1)\chi^1(2^{n-1}, -2^{n-1} - u - 1)^F. \end{aligned} \tag{2.41}$$

By proposition 2.4.8 we have $\chi^1(2^{n-1}-1,-2^{n-1}-u-1)=0$. Also by proposition(16) we have $\chi^1_\beta(2^{n-1}-1,-2^{n-1}-u)=\chi^1(2^{n-1},-2^{n-1}-u-2)$. Using the inductive hypothesis we get

$$\chi^1(2^n,-2^n-2u-1)=\chi(0,2^{n-1}-u-2)^F+\chi(0,1)\chi^1(0,2^{n-1}-u-1)^F. \quad (2.42)$$

Compare equation 2.42 with the p–expansion of Weyl character of $\chi(0,2^n-2u-1)$ to get the result.

□

2.4.2 The case $p=3$

We will now give a version of the results given in the above section in $p=3$. The following proposition tells us when the result is the same as in characteristic zero.

Proposition 2.4.21. *Let $p=3$ then $\chi^1(r,-s-2)=\chi(r-s-1,s)$ for all $r\geq 3s$, $r,s>0$. Moreover*

$$\chi^1_\alpha(r,-s-2)=\chi(r-s-1,s)+\chi(r-s-2,s-1)$$

and

$$\chi^1_\beta(r,-s-2)=\chi(r-s-1,s)+\chi(r-s-2,s+2)$$

for all $r\geq 3s$, $r,s>0$.

Proof. The proof is again by induction and the recursive formulas given in [11]. As in characteristic $p=2$ there are many cases to consider. We will outline some of the cases below.

1. First let $r = 2 + 3u$ and $s = 2 + 3v$ for some positive integers u and v then we have

$$\begin{aligned}\chi^1(2+3u,-3v-4) &= \chi(2,2)\chi^1(u,-v-2)^F \\ &= \chi(2,2)\chi(u-v-1,v)^F \\ &= \chi(3u-3v-1,2+3v).\end{aligned}$$

This proves the result in this case.

2. Next we take $r = 1 + 3u$ and $s = 1 + 3v$ for some positive integers u and v then we have

$$\begin{aligned}\chi^1(1+3u,-3v-3) = {}& \chi(1,0)[\chi^1(u,-v-1) + \chi^1(u-1,-v-2)]^F \\ &+ \chi(0,2)\chi^1(u-1,-v-1)^F + \chi(2,1)\chi^1(u,-v-2)^F \\ = {}& \chi(1,0)[\chi(u-v,v-1) + \chi(u-v-2,v)]^F \\ &+ \chi(0,2)\chi(u-v-1,v-1)^F + \chi(2,1)\chi(u-v-1,v)^F. \quad (2.43)\end{aligned}$$

Now the p–expansion of Weyl character of $\chi(3u-3v-1,1+3v)$ is given by

$$\begin{aligned}\chi(3u-3v-1,1+3v) = {}& \chi(1,0)[\chi(u-v,v-1) + \chi(u-v-2,v)]^F \\ &+ \chi(0,2)\chi(u-v-1,v-1)^F + \chi(2,1)\chi(u-v-1,v)^F. \quad (2.44)\end{aligned}$$

Compare equations 2.44 and 2.43 to get the result.

3. Now consider $r = 3u$ and $s = 3v$ for some positive integers u and v then

we have

$$\begin{aligned}\chi^1(3u,-3v-2) &= \chi(0,1)[\chi^1(u,-v-1)+\chi^1(u-1,-v-2)]^F \\ &+\chi(2,0)\chi^1(u,-v-2)^F+\chi(1,2)\chi^1(u-1,-v-1)^F \\ &= \chi(0,1)[\chi(u-v,v-1)+\chi(u-v-2,v)]^F \\ &+\chi(2,0)\chi(u-v-1,v-1)^F+\chi(1,2)\chi(u-v-1,v-1)^F. \quad (2.45)\end{aligned}$$

Now the p–expansion of Weyl character of $\chi(3u-3v-1,3v)$ is given by

$$\begin{aligned}\chi(3u-3v-1,3v) &= \chi(0,1)[\chi(u-v,v-1)+\chi(u-v-2,v)]^F \\ &+\chi(2,0)\chi(u-v-1,v-1)^F+\chi(1,2)\chi(u-v-1,v-1)^F. \quad (2.46)\end{aligned}$$

Compare equations 2.45 and 2.46 to get the result.

4. Let $r = 2+3u$ and $s = 1+3v$ for some positive integers u and v then we have

$$\begin{aligned}\chi^1(2+3u,-3v-3) &= \chi(0,1)\chi^1_\alpha(u+1,-v-2)^F+\chi(2,0)\chi^1(u,-v-1)^F \\ &+\chi(1,2)\chi^1(u,-v-2)^F \\ &= \chi(0,1)\chi^1_\alpha(u+1,-v-2)^F+\chi(2,0)\chi(u-v-1,v-1)^F \\ &+\chi(1,2)\chi(u-v-1,v)^F. \quad (2.47)\end{aligned}$$

Now the p–expansion of Weyl character of $\chi(3u-3v,1+3v)$ is given by

$$\begin{aligned}\chi(3u-3v,1+3v) &= \chi(0,1)[\chi(u-v,v)+\chi(u-v-1,v-1)]^F \\ &+\chi(2,0)\chi(u-v,v-1)^F+\chi(1,2)\chi(u-v-1,v)^F. \quad (2.48)\end{aligned}$$

If equations 2.47 and 2.48 are equal we must have

$$\chi^1_\alpha(u+1,-v-2) = \chi(u-v,v)+\chi(u-v-1,v-1).$$

We will now show that $\chi^1_\alpha(r,-s-2) = \chi(r-s-1,s)+\chi(r-s-2,s-1)$. By proposition 2.4.5 the result is true whenever $r \neq 3u$ for some positive integer u. We will now cover the case when $r = 3u$ and we have the following cases for s.

(a) Let $s = 3v$ then we get

$$\begin{aligned}\chi^1_\alpha(3u,-3v-2) &= \chi(0,1)[\chi^1(u-1,-v-2)+\chi(1,0)\chi^1(u-1,-v-1)]^F \\ &+ \chi(2,0)\chi^1_\alpha(u,-v-2)^F + 2\chi(1,2)\chi^1(u-1,-v-1)^F \\ &= \chi(0,1)[\chi(u-v-2,v)+\chi(1,0)\chi(u-v-1,v-1)]^F + \chi(2,0) \\ &[\chi(u-v-1,v)+\chi(u-v-2,v-1)]^F + 2\chi(1,2)\chi(u-v-1,v)^F.\end{aligned} \tag{2.49}$$

Now the p–expansion of Weyl character of $\chi(3u-3v-1,3v)$ is given by

$$\begin{aligned}\chi(3u-3v-1,3v) &= \chi(0,1)[\chi(u-v,v-1)+\chi(u-v-2,v)]^F \\ &+ \chi(2,0)\chi(u-v-1,v)^F + \chi(1,2)\chi(u-v-1,v-1)^F.\end{aligned} \tag{2.50}$$

And the p–expansion of Weyl character for $\chi(3u-3v-2,3v-1)$ is

$$\begin{aligned}\chi(3u-3v-2,3v-1) &= \chi(0,1)[\chi(u-v-2,v)+\chi(u-v-1,v-2)]^F \\ &+ \chi(2,0)\chi(u-v-2,v-1)^F + \chi(1,2)\chi(u-v-1,v-1)^F.\end{aligned} \tag{2.51}$$

Add equations 2.50 and 2.51 and compare it with equation 2.49 to get the result.

(b) Now suppose $s = 3v + 2$ then we have

$$\chi_{\alpha}^{1}(3u, -3v-4) = \chi(1,0)[\chi^{1}(u-1, -v-1) + \chi(0,1)\chi^{1}(u-1, -v-2)]^{F}$$
$$+ \chi(0,2)\chi_{\alpha}^{1}(u, -v-2)^{F} + 2\chi(2,1)\chi^{1}(u-1, -v-2)^{F}$$
$$= \chi(1,0)[\chi(u-v-1, v-1) + \chi(0,1)\chi(u-v-2, v)]^{F} + \chi(0,2)$$
$$[\chi(u-v-1, v) + \chi(u-v-2, v-1)]^{F} + 2\chi(2,1)\chi(u-v-2, v)^{F}. \tag{2.52}$$

Now the p–expansion of Weyl character of $\chi(3u-3v-3, 2+3v)$ is given by

$$\chi(3u-3v-3, 2+3v) = \chi(1,0)[\chi(u-v-2, v+1) + \chi(u-v-1, v-1)]^{F}$$
$$+ \chi(0,2)\chi(u-v-1, v)^{F} + \chi(2,1)\chi(u-v-2, v)^{F}. \tag{2.53}$$

And the p–expansion of Weyl character for $\chi(3u-3v-4, 3v+1)$ is

$$\chi(3u-3v-4, 3v+1) = \chi(1,0)[\chi(u-v-1, v-1) + \chi(u-v-3, v)]^{F}$$
$$+ \chi(0,2)\chi(u-v-2, v-1)^{F} + \chi(2,1)\chi(u-v-2, v)^{F}. \tag{2.54}$$

Add equations 2.53 and 2.54 and compare it with equation 2.52 to get the result.

(c) Finally we have the case when $s = 1 + 3v$ which follows exactly the same as the above two cases. We will omit the details.

This completes the result in this case.

□

Corollary 2.4.22. *Let* $p = 3$*. Then the sequence*

$$0 \to H^1(r-2,-s-1) \longrightarrow H^1(N_\alpha(r,-s-2)) \longrightarrow H^1(r,-s-2) \longrightarrow 0$$

is exact for all $r \geq 3s$*.*

Proof. We know that the sequence

$$H^1(r-2,-s-1) \xrightarrow{\phi} H^1(N_\alpha(r,-s-2)) \xrightarrow{\psi} H^1(r,-s-2)$$

is exact meaning $\mathrm{Im}\phi = \mathrm{Ker}\psi$. Moreover by proposition 2.4.21, we have

$$\chi^1_\alpha(r,-s-2) = \chi^1(r,-s-2) + \chi^1(r-2,-s-1).$$

Let $X = H^1(r-2,-s-1)$, $Y = H^1(N_\alpha(r,-s-2))$ and $Z = H^1(r,-s-2)$. Now $\mathrm{Im}\psi \simeq Y/\mathrm{Ker}\psi \simeq Y/\mathrm{Im}\phi$, therefore $\dim Y = \dim \mathrm{Im}\phi + \dim \mathrm{Im}\psi$. It is clear that $\dim \mathrm{Im}\psi \leq \dim Z$. Also by the character result we have $\dim Y = \dim X + \dim Z$. Now $\dim \mathrm{Im}\phi = \dim Y - \dim \mathrm{Im}\psi \leq \dim Y - \dim Z = \dim X$. This implies $\dim \mathrm{Im}\phi \leq \dim X$. We have proved that $\dim \mathrm{Im}\phi \leq \dim X$ and $\dim \mathrm{Im}\psi \leq \dim Z$ but $\dim Y = \dim X + \dim Z$ so we must have the equality. This proves that ϕ is injective and ψ is surjective and hence the result. □

As explained in characteristic $p = 2$ the only region left to consider is when $r < 3s$. We will give here a few special cases to give us a sense of the general case.

$$\chi^1(r,-2) = \begin{cases} \chi(r-1,0), & r > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-3) = \begin{cases} \chi(0,1), & r = 2 \\ \chi(1,1), & r = 3 \\ \chi(2,1), & r = 4 \\ \chi(3,1) + \chi(0,1)\chi(0,2)^F, & r = 5 \\ \chi(r-2,1), & r \geq 6 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-4) = \begin{cases} \chi(r-3,2), & r \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi^1(r,-5) = \begin{cases} \chi(0,1), & r = 3 \\ \chi(0,3) + \chi(0), & r = 4 \\ \chi(r-4,3), & r \geq 5 \\ 0, & \text{otherwise.} \end{cases}$$

Now we will try to give a version of all the results given for $p = 2$ in the case of $p = 3$.

Proposition 2.4.23. $\chi^1(3^n, -3^n - 2) = \chi(0, 3^n - 2)$, *for all* $n \geq 0$.

Proof. The argument is by induction on n. The result is true for $n = 1$ by the special cases above. Suppose the result is true for $n - 1$. Using the recursive formula we have

$$\begin{aligned} \chi^1(3^n, -3^n - 2) &= \chi(0,1)[\chi^1(3^{n-1}, -3^{n-1} - 1) + \chi^1(3^{n-1} - 1, -3^{n-1} - 2)]^F \\ &+ \chi(2,0)\chi^1(3^{n-1}, -3^{n-1} - 2)^F + \chi(1,2)\chi^1(3^{n-1} - 1, -3^{n-1} - 1)^F. \quad (2.55) \end{aligned}$$

Now by proposition 2.4.8 we have $\chi^1(3^{n-1}-1,-3^{n-1}-2)=0$. Also $\chi^1(3^{n-1}-1,-3^{n-1}-1)=0$ and $\chi^1(3^{n-1},-3^{n-1}-1)=\chi(0,3^{n-1}-1)$ by proposition 2.4.7. The equation 2.55 becomes

$$\chi^1(3^n,-3^n-2)=\chi(0,1)\chi(0,3^{n-1}-1)^F+\chi(2,0)\chi(0,3^{n-1}-2)^F. \tag{2.56}$$

Finally compare equation 2.56 with the p–expansion of Weyl character of $\chi(0,3^n-2)$ to get the result.

□

Proposition 2.4.24. *Suppose $n\geq 0$ then we have $\chi^1(3^n-1,-3^n)=\chi(0,3^n-2)$. Also*

$$\chi^1_\alpha(3^n,-3^n-1)=\chi(0,3^n-1).$$

Proof. The argument is by induction on n. The result is clearly true for $n=0$. Using the recursive formulas we get

$$\begin{aligned}\chi^1(3^n-1,-3^n)&=\chi(0,1)\chi^1_\alpha(3^{n-1},-3^{n-1}-1)^F+\chi(2,0)\chi^1(3^{n-1}-1,-3^{n-1})^F\\&\quad+\chi(1,2)\chi^1(3^{n-1}-1,-3^{n-1}-1)^F.\end{aligned} \tag{2.57}$$

Now $\chi^1(3^{n-1}-1,-3^{n-1}-1)=0$ by proposition 2.4.7. Also the p–expansion of Weyl character of $\chi(0,3^n-2)$ gives

$$\chi(0,3^n-2)=\chi(0,1)\chi(0,3^{n-1}-1)^F+\chi(2,0)\chi(0,3^{n-1}-2)^F.$$

So if the result is true we must have $\chi^1_\alpha(3^{n-1},-3^{n-1}-1)=\chi(0,3^{n-1}-1)$. We will show this by induction on n. The result is clearly true for $n=1$. Now by the recursive formula we get

$$\begin{aligned}\chi^1_\alpha(3^n,-3^n-1)&=\chi(1,0)[\chi^1(3^{n-1}-1,-3^{n-1})+\chi(0,1)\\&\quad\chi^1(3^{n-1}-1,-3^{n-1}-1)]^F+\chi(0,2)\chi^1_\alpha(3^{n-1},-3^{n-1}-1)^F\\&\quad+2\chi(2,1)\chi^1(3^{n-1}-1,-3^{n-1}-1)^F.\end{aligned} \tag{2.58}$$

Now $\chi^1(3^{n-1}-1,-3^{n-1}-1) = 0$ by proposition 2.4.7. Using the inductive hypothesis equation 2.58 becomes

$$\begin{aligned}\chi^1_\alpha(3^n,-3^n-1) &= \chi(1,0)\chi(0,3^{n-1}-2)^F + \chi(0,2)\chi(0,3^{n-1}-1)^F \\ &= \chi(0,3^n-1). \end{aligned} \tag{2.59}$$

Replacing value from equation 2.59 to equation 2.57 and using the inductive hypothesis we get the result. □

Proposition 2.4.25. $\chi^1(3^n,-3^n) = \chi(1,3^n-2)$ *for all* $n \geq 0$*. Also*

$$\chi^1_\beta(3^n-1,-3^n) = \chi(0,3^n-2).$$

Proof. We argue by induction on n. The result is clearly true for $n = 0$. Suppose the result is true for all $t < n$ then we have

$$\chi^1(3^t,-3^t) = \chi(1,3^t-2).$$

Now for $n = t+1$ we get

$$\begin{aligned}\chi^1(3^{t+1},-3^{t+1}) &= [\chi^1(3^t,-3^t) + \chi^1(3^t-1,-3^t-1) \\ &\quad + \chi^1_\beta(3^t-1,-3^t) + \chi^1_\alpha(3^t,-3^t-1)]^F \\ &\quad + \mathrm{ch}L(\rho)[\chi^1(3^t,-3^t-1) + \chi^1(3^t-1,-3^t) + \chi^1(3^t-1,-3^t-1)]^F. \end{aligned} \tag{2.60}$$

From propositions 2.4.14 and 2.4.7 we get that $\chi^1(3^t-1,-3^t) = \chi(0,3^t-2)$ and $\chi^1(3^t-1,-3^t-1) = 0$. Also $\chi^1(3^t,-3^t-1) = \chi(0,3^t-1)$ by proposition 2.4.7. Moreover $\chi^1_\alpha(3^t,-3^t-1) = \chi(0,3^t-1)$ by proposition 2.4.24. The equation 2.60 becomes

$$\begin{aligned}\chi^1(3^{t+1},-3^{t+1}) &= [\chi(1,3^t-2) + \chi^1_\beta(3^t-1,-3^t) + \chi(0,3^t-1)]^F \\ &\quad + \mathrm{ch}L(\rho)[\chi(0,3^t-1) + \chi(0,3^t-2)]^F. \end{aligned} \tag{2.61}$$

Now the p–expansion of Weyl character of $\chi(1,3^{t+1}-2)$ is given by

$$\begin{aligned}\chi(1,3^{t+1}-2) &= [\chi(0,3^t-2)+\chi(1,3^t-2)+\chi(0,3^t-1)]^F\\ &+ \mathrm{ch}L(\rho)[\chi(0,3^t-1)+\chi(0,3^t-2)]^F. \qquad (2.62)\end{aligned}$$

Now if equations 2.61 and 2.62 are equal we must have $\chi^1_\beta(3^t-1,-3^t) = \chi(0,3^t-2)$. We will prove this by recursion. We have

$$\begin{aligned}&\chi^1_\beta(3^t-1,-3^t)\\ &= \chi(0,1)[\chi^1(3^{t-1}+1,-3^{t-1}-1)+\chi(1,0)\chi^1(3^{t-1}-1,-3^{t-1}-1)]^F\\ &+ \chi(2,0)\chi^1_\beta(3^{t-1}-1,-3^{t-1})^F + 2\chi(1,2)\chi^1(3^{t-1}-1,-3^{t-1}-1)^F. \qquad (2.63)\end{aligned}$$

From proposition 2.4.7 we get that $\chi^1(3^{t-1}+1,-3^{t-1}-1) = \chi(1,3^{t-1}-1)$ and $\chi^1(3^t-1,-3^t-1) = 0$. The equation 2.63 becomes

$$\chi^1_\beta(3^t-1,-3^t) = \chi(0,1)\chi(1,3^{t-1}-1)^F + \chi(2,0)\chi(0,3^{t-1}-2)^F. \qquad (2.64)$$

Compare this with the p–expansion of Weyl character $\chi(0,3^t-2)$ to get the required result.

□

Proposition 2.4.26. *Suppose $n \geq 0$ then we have $\chi^1(3^n+1,-3^n) = \chi(1,3^n-2)$.*

Proof. The argument is by induction on n. The result is clearly true for $n=0$. Using the recursive formulas we get

$$\begin{aligned}\chi^1(3^n+1,-3^n) &= \chi(1,0)[\chi^1(3^{n-1},-3^{n-1})+\chi^1(3^{n-1}-1,-3^{n-1}-1)]^F + \chi(0,2)\\ &\chi^1(3^{n-1}-1,-3^{n-1})^F + \chi(2,1)\chi^1(3^{n-1},-3^{n-1}-1)^F. \qquad (2.65)\end{aligned}$$

Now $\chi^1(3^{n-1}, -3^{n-1} - 1) = \chi(0, 3^{n-1} - 1)$ by proposition 2.4.7. Also by proposition 2.4.25 we get $\chi^1(3^{n-1}, -3^{n-1}) = \chi(1, 3^{n-1} - 2)$ and $\chi^1(3^{n-1} - 1, -3^{n-1}) = \chi(0, 3^{n-1} - 2)$ by proposition 2.4.24. The equation 2.65 becomes

$$\chi^1(3^n + 1, -3^n) = \chi(1,0)\chi(1, 3^{n-1} - 2)^F + \chi(0,2)\chi(0, 3^{n-1} - 2)^F + \chi(2,1)\chi(0, 3^{n-1} - 1)^F. \tag{2.66}$$

Also the p–expansion of Weyl character of $\chi(1, 3^n - 2)$ gives

$$\chi(1, 3^n - 2) = \chi(1,0)\chi(1, 3^{n-1} - 2)^F + \chi(0,2)\chi(0, 3^{n-1} - 2)^F + \chi(2,1)\chi(0, 3^{n-1} - 1)^F. \tag{2.67}$$

Compare equations 2.66 and 2.67 to get the result. □

Proposition 2.4.27. *Suppose $n \geq 0$ then we have $\chi^1(3^n-1, -3^n+1) = \chi(1, 3^n-3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\chi^1(3^n - 1, -3^n + 1) = \chi(1,0)\chi^1_\alpha(3^{n-1}, -3^{n-1} - 1)^F + \chi(0,2) \chi^1(3^{n-1} - 1, -3^{n-1} - 1)^F + \chi(2,1)\chi^1(3^{n-1} - 1, -3^{n-1})^F. \tag{2.68}$$

Now $\chi^1(3^{n-1}-1, -3^{n-1}-1) = 0$ by proposition 2.4.7. Also by proposition 2.9 we have $\chi^1_\alpha(3^{n-1}, -3^{n-1}-1) = \chi(0, 3^{n-1}-1)$ and $\chi^1(3^{n-1}-1, -3^{n-1}) = \chi(0, 3^{n-1}-2)$. The equation 2.68 now becomes

$$\chi^1(3^n - 1, -3^n + 1) = \chi(1,0)\chi(0, 3^{n-1} - 1)^F + \chi(2,1)\chi(0, 3^{n-1} - 2)^F. \tag{2.69}$$

Compare equation 2.69 with the p–expansion of Weyl character of $\chi(1, 3^n-3)$ to get the result.

□

Proposition 2.4.28. *Suppose* $n \geq 0$ *then we have* $\chi^1(3^n, -3^n+1) = \chi(2, 3^n - 3)$.

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}\chi^1(3^n, -3^n+1) &= \chi(0,1)[\chi^1(3^{n-1}, -3^{n-1}) + \chi^1(3^{n-1}-1, -3^{n-1}-1)]^F \\ &\quad + \chi(2,0)\chi^1(3^{n-1}, -3^{n-1}-1)^F + \chi(1,2)\chi^1(3^{n-1}-1, -3^{n-1})^F. \qquad (2.70)\end{aligned}$$

Now $\chi^1(3^{n-1}-1, -3^{n-1}) = \chi(0, 3^{n-1}-2)$ by proposition 2.4.24. Also $\chi^1(3^{n-1}, -3^{n-1}) = \chi(1, 3^{n-1}-2)$ by proposition 2.4.25. Moreover proposition 2.4.7 gives $\chi^1(3^{n-1}-1, -3^{n-1}-1) = 0$. Equation 2.70 becomes

$$\begin{aligned}\chi^1(3^n, -3^n+1) &= \chi(0,1)\chi(1, 3^{n-1}-2)^F + \chi(2,0)\chi(0, 3^{n-1}-1)^F \\ &\quad + \chi(1,2)\chi(0, 3^{n-1}-2)^F. \qquad (2.71)\end{aligned}$$

Compare equation 2.71 with the p−expansion of Weyl character of $\chi(2, 3^n-3)$ to get the result. □

Proposition 2.4.29. *Suppose* $n \geq 0$ *then we have* $\chi^1(3^n+1, -3^n+1) = \chi(3, 3^n - 3)$.

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}&\chi^1(3^n+1, -3^n+1) \\ &\quad = [\chi^1_\beta(3^{n-1}-1, -3^{n-1}+1) + \chi^1_\alpha(3^{n-1}+1, -3^{n-1}-1) + \chi^1(3^{n-1}, -3^{n-1}) \\ &\quad + \chi^1(3^{n-1}-1, -3^{n-1}-1)]^F + \mathrm{ch}L(\rho)[\chi^1(3^{n-1}, -3^{n-1}) + \chi^1(3^{n-1}, -3^{n-1}-1) \\ &\quad + \chi^1(3^{n-1}-1, -3^{n-1})]^F. \qquad (2.72)\end{aligned}$$

By proposition 2.4.5 we have

$$\chi^1_\beta(3^{n-1}-1,-3^{n-1}+1) = \chi^1(3^{n-1}-1,-3^{n-1}+1) + \chi^1(3^{n-1},-3^{n-1}-1)$$

and

$$\chi^1_\alpha(3^{n-1}+1,-3^{n-1}-1) = \chi^1(3^{n-1}+1,-3^{n-1}-1) + \chi^1(3^{n-1}-1,-3^{n-1}).$$

Also using proposition 2.4.7, 2.4.24 and 2.4.25 the equation 2.72 becomes

$$\begin{aligned}&\chi^1(3^n+1,-3^n+1)\\ &\quad= [\chi(1,3^{n-1}-3)+\chi(0,3^{n-1}-1)+\chi(1,3^{n-1}-1)\\ &\quad+\chi(0,3^{n-1}-2)+\chi(1,3^{n-1}-2)]^F+\mathrm{ch}L(\rho)\\ &\quad[\chi(1,3^{n-1}-2)+\chi(0,3^{n-1}-1)+\chi(0,3^{n-1}-2)]^F.\end{aligned} \tag{2.73}$$

The p–expansion of Weyl character of $\chi(3,3^n-3)$ is given by

$$\begin{aligned}&\chi(3,3^n-3)\\ &\quad= [\chi(1,3^{n-1}-3)+\chi(0,3^{n-1}-1)+\chi(1,3^{n-1}-1)\\ &\quad+\chi(0,3^{n-1}-2)+\chi(1,3^{n-1}-2)]^F+\mathrm{ch}L(\rho)\\ &\quad[\chi(1,3^{n-1}-2)+\chi(0,3^{n-1}-1)+\chi(0,3^{n-1}-2)]^F.\end{aligned} \tag{2.74}$$

Hence the result. □

Proposition 2.4.30. *Suppose $r,n>0$ then we have*

$$\chi^1(3^n,-3^n-r) = \begin{cases}\chi(0,3^n-r), & n\geq r-1\\ 0, & \text{otherwise.}\end{cases}$$

Proof. We argue by induction on r. The result is true for $r=1$ by proposition 2.4.7. Suppose the result is true for all $t-1<r$. For $r=t$ we have the following three possibilities

1. Suppose $t = 2 + 3u$ for some positive integer u we have

$$\begin{aligned}
&\chi^1((3^n, -3^n - 3u - 2)) \\
&\qquad = \chi(0,1)[\chi^1(3^{n-1}, -3^{n-1} - u - 1) + \chi^1(3^{n-1} - 1, -3^{n-1} - u - 2)]^F \\
&\qquad + \chi(2,0)\chi^1(3^{n-1}, -3^{n-1} - u - 2)^F + \chi(1,2)\chi^1(3^{n-1} - 1, -3^{n-1} - u - 1)^F.
\end{aligned} \tag{2.75}$$

By proposition 2.4.8 we have $\chi^1(3^{n-1} - 1, -3^{n-1} - u - 1) = 0$ and $\chi^1(3^{n-1} - 1, -3^{n-1} - u) = 0$. Using the inductive hypothesis we get

$$\chi^1(3^n, -3^n - 3u - 2) = \chi(0,1)\chi(0, 3^{n-1} - u - 1)^F + \chi(2,0)\chi(0, 3^{n-1} - u - 2)^F. \tag{2.76}$$

Compare equation 2.76 with the p–expansion of Weyl character of $\chi(0, 3^n - 3u - 2)$ to get the result.

2. Now suppose $r = 3u + 1$ for some positive integer u then we have

$$\begin{aligned}
&\chi^1(3^n, -3^n - 3u - 1) \\
&\qquad = \chi(1,0)\chi^1_\beta(3^{n-1} - 1, -3^{n-1} - u)^F + \chi(0,2)\chi^1(3^{n-1}, -3^{n-1} - u - 1)^F \\
&\qquad + \chi(2,1)\chi^1(3^{n-1} - 1, -3^{n-1} - u - 1)^F.
\end{aligned} \tag{2.77}$$

By proposition 2.4.8 we have $\chi^1(3^{n-1} - 1, -3^{n-1} - u) = 0$. Also by proposition 2.4.5 we have $\chi^1_\beta(3^{n-1} - 1, -3^{n-1} - u) = \chi^1(3^{n-1}, -3^{n-1} - u - 2)$. Using the inductive hypothesis we get

$$\chi^1(3^n, -3^n - 3u - 1) = \chi(1,0)\chi(0, 3^{n-1} - u - 2)^F + \chi(0,2)\chi(0, 3^{n-1} - u - 1)^F. \tag{2.78}$$

Compare equation 2.78 with the p–expansion of Weyl character of $\chi(0, 3^n - 3u - 1)$ to get the result.

3. Finally let $r = 3u$ for some positive integer u then we have

$$\begin{aligned}
&\chi^1(3^n, -3^n - 3u) \\
&\quad = [\chi^1(3^{n-1}, -3^{n-1} - u) + \chi^1(3^{n-1} - 1, -3^{n-1} - u - 1) \\
&\quad + \chi^1_\beta(3^{n-1} - 1, -3^{n-1} - u) + \chi^1_\alpha(3^{n-1}, -3^{n-1} - u - 1)]^F \\
&\quad + \mathrm{ch}L(\rho)[\chi^1(3^{n-1}, -3^{n-1} - u - 1) + \chi^1(3^{n-1} - 1, -3^{n-1} - u) \\
&\quad + \chi^1(3^{n-1} - 1, -3^{n-1} - u - 1)]^F. \qquad (2.79)
\end{aligned}$$

By proposition 2.4.8 we have $\chi^1(3^{n-1}-1, -3^{n-1}-u) = 0$. Also by proposition 2.4.5 we have $\chi^1_\beta(3^{n-1} - 1, -3^{n-1} - u) = \chi^1(3^{n-1}, -3^{n-1} - u - 2)$. Using the inductive hypothesis we get

$$\begin{aligned}
\chi^1(3^n, -3^n - 3u) &= [\chi(0, 3^{n-1} - u) + \chi(0, 3^{n-1} - u - 2) \\
&+ \chi^1_\alpha(3^{n-1}, -3^{n-1} - u - 1)]^F + \mathrm{ch}L(\rho)\chi(0, 3^{n-1} - u - 1)^F. \qquad (2.80)
\end{aligned}$$

Now the p–expansion of Weyl character of $\chi(0, 3^n - 3u)$ is given by

$$\begin{aligned}
\chi(0,3^n - 3u) &= [\chi(0, 3^{n-1} - u) + \chi(0, 3^{n-1} - u - 2) + \chi(0, 3^{n-1} - u - 1)]^F \\
&+ \mathrm{ch}L(\rho)\chi(0, 3^{n-1} - u - 1)^F. \qquad (2.81)
\end{aligned}$$

If the equations 2.80 and 2.81 are the same then we must have

$$\chi^1_\alpha(3^{n-1}, -3^{n-1} - u - 1) = \chi(0, 3^{n-1} - u - 1).$$

This completes the proof.

□

2.4.3 The case $p \geq 5$

In this section we take $p \geq 5$ try to give simplify the recursion in this case. The following proposition tells us when the result is same as in characteristic zero.

Proposition 2.4.31. *Let $p \geq 5$ then $\chi^1(r, -s-2) = \chi(r-s-1, s)$ for all $r \geq ps$, $r, s > 0$. Moreover*

$$\chi^1_\alpha(r, -s-2) = \chi(r-s-1, s) + \chi(r-s-2, s-1)$$

and

$$\chi^1_\beta(r, -s-2) = \chi(r-s-1, s) + \chi(r-s-2, s+2)$$

for all $r \geq ps$, $r, s > 0$.

Proof. The proof is again by induction and the recursive formulas given in [11]. As in characteristic $p = 2, 3$ there are many cases to consider. We will outline some of the cases below.

1. First let $r = a + pu$ and $s + 2 = -p + 2 + a + pv$ for some positive integers u and v and $0 \leq a \leq p-2$ then we have

$$\begin{aligned}&\chi^1(a+pu, p-2-a-pv)\\&\quad= \chi(p-1, a)\chi^1(u, -v-1)^F + \chi(a, p-2-a)[\chi^1(u, -v)\\&\quad+ \chi^1(u-1, -v-1)]^F + \chi(p-2-a, p-1)\chi^1(u-1, -v)^F.\end{aligned}$$

Apply the inductive hypothesis to get

$$\begin{aligned}&\chi^1(a+pu, p-2-a-pv)\\&\quad= \chi(p-1, a)\chi(u-v, v-1)^F + \chi(a, p-2-a)[\chi(u-v+1, v-2)\\&\quad+ \chi(u-v-1, v-1)]^F + \chi(p-2-a, p-1)\chi(u-v, v-2)^F. \quad (2.82)\end{aligned}$$

The p–expansion of Weyl character of $\chi(p-1+p(u-v), a+p(v-1))$ is given by

$$\begin{aligned}
&\chi(p-1+p(u-v), a+p(v-1)) \\
&\quad = \chi(p-1,a)\chi(u-v,v-1)^F + \chi(a,p-2-a)[\chi(u-v+1,v-2) \\
&\quad + \chi(u-v-1,v-1)]^F + \chi(p-2-a,p-1)\chi(u-v,v-2)^F. \quad (2.83)
\end{aligned}$$

Compare equations 2.82 and 2.83 to get the result.

2. Now suppose $r = p-1+pu$ and $s+2 = -a+pv$ for some positive integers u and v and $0 \le a \le p-2$ then we have

$$\begin{aligned}
&\chi^1(p-1+pu, a-pv) \\
&\quad = \chi(p-1,a)\chi^1(u,-v)^F + \chi(a,p-2-a)\chi^1_\alpha(u+1,-v-1)^F \\
&\quad + \chi(p-2-a,p-1)\chi^1(u,-v-1)^F.
\end{aligned}$$

Apply the inductive hypothesis to get

$$\begin{aligned}
&\chi^1(p-1+pu, a-pv) \\
&\quad = \chi(p-1,a)\chi(u-v+1,v-2)^F + \chi(a,p-2-a)\chi^1_\alpha(u+1,-v-1)^F \\
&\quad + \chi(p-2-a,p-1)\chi(u-v,v-1)^F. \quad (2.84)
\end{aligned}$$

The p–expansion of Weyl character of $\chi(a+p(u-v+1), p-2-a+p(v-1))$ is given by

$$\begin{aligned}
&\chi(a+p(u-v+1), p-2-a+p(v-1)) \\
&\quad = \chi(p-1,a)\chi(u-v+1,v-2)^F + \chi(a,p-2-a)[\chi(u-v+1,v-1) \\
&\quad + \chi(u-v,v-2)]^F + \chi(p-2-a,p-1)\chi(u-v,v-1)^F. \quad (2.85)
\end{aligned}$$

If the equations 2.84 and 2.85 are the same we must have

$$\chi^1_\alpha(u+1,-v-1) = \chi(u-v+1,v-1) + \chi(u-v,v-2).$$

We will now prove that

$$\chi_\alpha^1(r, -s-2) = \chi(r-s-1, s) + \chi(r-s-2, s-1).$$

By proposition 2.4.5 the result is true if $r \neq pu$ for some positive integer u. We will now take $r = pu$ and we have the following cases for $-s-2$:

(a) Let $-s-2 = p-2-pv$ for some positive integer v then

$$\begin{aligned}&\chi_\alpha^1(pu, p-2-pv)\\ &\quad = \chi(0, p-2)[\chi(1,0)\chi^1(u-1, -v) + \chi^1(u-1, -v-1)]^F + \chi(p-1, 0)\\ &\quad \chi_\alpha^1(u, -v-1)^F + 2\chi_p(p-2, p-1)\chi^1(u-1, -v)^F.\end{aligned}$$

Using the inductive hypothesis we have

$$\begin{aligned}&\chi_\alpha^1(pu, p-2-pv)\\ &\quad = \chi(0, p-2)[\chi(1,0)\chi(u-v, v-2) + \chi(u-v-1, v-1)]^F\\ &\quad + \chi(p-1, 0)[\chi(u-v, v-1) + \chi(u-v-1, v-2)]^F\\ &\quad + 2\chi_p(p-2, p-1)\chi(u-v-1, v-2)^F. \qquad (2.86)\end{aligned}$$

The p–expansion of Weyl character of $\chi(p-1+p(u-v), p(v-1))$ is given by

$$\begin{aligned}&\chi(p-1+p(u-v), p(v-1))\\ &\quad = \chi(p-1, 0)\chi(u-v, v-1)^F + \chi(0, p-2)[\chi(u-v+1, v-2)\\ &\quad + \chi(u-v-1, v-1)]^F + \chi(p-2, p-1)\chi(u-v, v-2)^F. \quad (2.87)\end{aligned}$$

and the p–expansion of Weyl character for $\chi(p-2+p(u-v), p-1+p(v-2))$ is

$$\begin{aligned}&\chi(p-2+p(u-v), p-1+p(v-2))\\ &\quad = \chi(p-1, 0)\chi(u-v-1, v-2)^F + \chi(0, p-2)[\chi(u-v-1, v-1)\\ &\quad + \chi(u-v, v-3)]^F + \chi(p-2, p-1)\chi(u-v, v-2)^F. \qquad (2.88)\end{aligned}$$

Add equations 2.87, 2.88 and note that

$$\chi(1,0)\chi(u-v,v-2) = \chi(u-v+1,v-2)+\chi(u-v-1,v-1)+\chi(u-v,v-3). \tag{2.89}$$

to get the result.

(b) Now let $-s-2 = p-2-b-pv$ for some positive integer v then

$$\begin{aligned}
&\chi^1_\alpha(pu, p-2-b-pv) \\
&\quad = 2\chi_p(p-2,p-1-b)\chi^1(u-1,-v)^F + \chi_p(0,p-2-b)[\chi^1(u-1,-v-1) \\
&\quad + \chi(1,0)\chi^1(u-1,-v)]^F + 2\chi_p(p-2-b,b-1)\chi^1_\alpha(u,-v-1)^F \\
&\quad + \chi_p(p-1-b,b)\chi^1_\alpha(u,-v-1)^F + 2\chi_p(b,p-2)\chi^1(u-1,-v-1)^F \\
&\quad + \chi_p(b-1,0)[\chi^1(u-1,-v) + \chi(0,1)\chi^1(u-1,-v-1)]^F.
\end{aligned}$$

Using the inductive hypothesis we have

$$\begin{aligned}
&\chi^1_\alpha(pu, p-2-b-pv) \\
&\quad = 2\chi_p(p-2,p-1-b)\chi(u-v,v-2)^F + \chi_p(0,p-2-b)[\chi(u-v-1,v-1) \\
&\quad + \chi(1,0)\chi(u-v,v-2)]^F + 2\chi_p(p-2-b,b-1)[\chi(u-v,v-1) \\
&\quad + \chi(u-v-1,v-2)]^F + \chi_p(p-1-b,b)[\chi(u-v,v-1) \\
&\quad + \chi(u-v-1,v-2)]^F + 2\chi_p(b,p-2)\chi(u-v-1,v-1)^F \\
&\quad + \chi_p(b-1,0)[\chi(u-v-1,v-2) + \chi(0,1)\chi(u-v-1,v-1)]^F.
\end{aligned} \tag{2.90}$$

The p–expansion of Weyl character of $\chi(p-1-b+p(u-v), b+p(v-1))$ is given by

$$\begin{aligned}
&\chi(p-1-b+p(u-v), b+p(v-1)) \\
&\quad = \chi_p(p-2,p-1-b)\chi(u-v,v-2)^F + \chi_p(0,p-2-b)[\chi(u-v+1,v-2)
\end{aligned}$$

$$
\begin{aligned}
&+ \chi(u-v-1, v-1)]^F + \chi_p(p-2-b, b-1)[\chi(u-v, v-1) \\
&+ \chi(u-v-1, v-2)]^F + \chi_p(p-1-b, b)\chi(u-v, v-1)^F \\
&+ \chi_p(b, p-2)\chi(u-v-1, v-1)^F + \chi_p(b-1, 0)[\chi(u-v-1, v) \\
&+ \chi(u-v, v-2)]^F. \qquad (2.91)
\end{aligned}
$$

and the p–expansion of Weyl character of $\chi(p-2-b+p(u-v), b-1+p(v-1))$ is

$$
\begin{aligned}
&\chi(p-2-b+p(u-v), b-1+p(v-1)) \\
&\quad = \chi_p(p-2, p-1-b)\chi(u-v, v-2)^F + \chi_p(0, p-2-b)[\chi(u-v-1, v-1) \\
&\quad + \chi(u-v, v-3)]^F + \chi_p(p-2-b, b-1)[\chi(u-v, v-1) \\
&\quad + \chi(u-v-1, v-2)]^F + \chi_p(p-1-b, b)\chi(u-v-1, v)^F \\
&\quad + \chi_p(b, p-2)\chi(u-v-1, v-1)^F + \chi_p(b-1, 0)[\chi(u-v, v-1) \\
&\quad + \chi(u-v-2, v-1)]^F. \qquad (2.92)
\end{aligned}
$$

Add equations 2.91, 2.92 and note that

$$\chi(0,1)\chi(u-v-1, v-1) = \chi(u-v-1, v)+\chi(u-v, v-2)+\chi(u-v-2, v-1).$$

and

$$\chi(1,0)\chi(u-v, v-2) = \chi(u-v+1, v-2)+\chi(u-v-1, v-1)+\chi(u-v, v-3).$$

to get the result.

The remaining two cases are very similar to the above case so we leave them to the reader.

3. Now suppose $r = p-2-a+pu$ and $s+2 = -p+1+pv$ for some positive

integers u and v and $0 \le a \le p-2$ then we have

$$\begin{aligned}&\chi^1(p-2-a+pu, p-1-pv)\\ &\quad = \chi(p-1,a)\chi^1(u-1,-v)^F + \chi(a,p-2-a)\chi^1_\beta(u-1,-v+1)^F\\ &\quad + \chi(p-2-a,p-1)\chi^1(u,-v)^F.\end{aligned}$$

Apply the inductive hypothesis to get

$$\begin{aligned}&\chi^1(p-2-a+pu, p-1-pv)\\ &\quad = \chi(p-1,a)\chi(u-v,v-2)^F + \chi(a,p-2-a)\chi^1_\beta(u-1,-v+1)^F\\ &\quad + \chi(p-2-a,p-1)\chi(u-v+1,v-2)^F.\end{aligned} \tag{2.93}$$

The p–expansion of Weyl character of $\chi(p-2-a+p(u-v+1), p-1+p(v-2))$ is given by

$$\begin{aligned}&\chi(p-2-a+p(u-v+1), p-1+p(v-2))\\ &\quad = \chi(p-1,a)\chi(u-v,v-2)^F + \chi(a,p-2-a)[\chi(u-v,v-1)\\ &\quad + \chi(u-v+1,v-3)]^F + \chi(p-2-a,p-1)\chi(u-v+1,v-2)^F.\end{aligned} \tag{2.94}$$

If the equations 2.93 and 2.94 are the same we must have

$$\chi^1_\beta(u-1,-v+1) = \chi(u-v,v-1) + \chi(u-v+1,v-3).$$

We will now prove that

$$\chi^1_\beta(r,-s-2) = \chi(r-s-1,s) + \chi(r-s-2,s+2).$$

By proposition 2.4.5 the result is true if $s+2 \ne pv$ for some positive integer v. We will now take $s+2 = pv$ and we have the following cases for r:

(a) Let $r = p - 2 + pu$ for some positive integer u then

$$\chi_\beta^1(p-2+pu,-pv)$$
$$= \chi(p-2,0)[\chi(0,1)\chi^1(u,-v-1) + \chi^1(u-1,-v-1)]^F + \chi(0,p-1)$$
$$\chi_\beta^1(u-1,-v)^F + 2\chi_p(p-1,p-2)\chi^1(u,-v-1)^F.$$

Using the inductive hypothesis we have

$$\begin{aligned}\chi_\beta^1(p-2+pu,-pv) &= \chi(p-2,0)[\chi(0,1)\chi(u-v,v-1) + \chi(u-v-1,v-1)]^F \\ &+ \chi(0,p-1)[\chi(u-v,v-2) + \chi(u-v-1,v)]^F \\ &+ 2\chi_p(p-1,p-2)\chi(u-v,v-1)^F. \qquad (2.95)\end{aligned}$$

The p–expansion of Weyl character of $\chi(p-1+p(u-v), p-2+p(v-1))$ is given by

$$\begin{aligned}\chi(p-1+p(u-v),p-2+p(v-1)) &= \chi(p-1,p-2)\chi(u-v,v-1)^F + \chi(p-2,0)[\chi(u-v+1,v-2) \\ &+ \chi(u-v-1,v-1)]^F + \chi(0,p-1)\chi(u-v,v-2)^F. \qquad (2.96)\end{aligned}$$

and the p–expansion of Weyl character for $\chi(p-2+p(u-v),pv)$ is

$$\begin{aligned}\chi(p-2+p(u-v),pv) &= \chi(p-1,p-2)\chi(u-v,v-1)^F + \chi(p-2,0)[\chi(u-v,v) \\ &+ \chi(u-v-1,v-1)]^F + \chi(0,p-1)\chi(u-v-1,v)^F. \qquad (2.97)\end{aligned}$$

Add equations 2.96, 2.97 and note that

$$\chi(0,1)\chi(u-v,v-1) = \chi(u-v,v) + \chi(u-v+1,v-2) + \chi(u-v-1,v-1). \qquad (2.98)$$

to get the result.

(b) Now let $r = p - 2 - b + pu$ for some positive integer u then

$$\chi^1_\beta(p-2-b+pu,-pv)$$
$$= 2\chi_p(p-1-b,p-2)\chi^1(u,-v-1)^F + \chi_p(p-2-b,0)$$
$$[\chi^1(u-1,-v-1)+\chi(0,1)\chi^1(u,-v-1)]^F$$
$$+ 2\chi_p(b-1,p-2-b)\chi^1_\beta(u-1,-v)^F + \chi_p(b,p-1-b)$$
$$\chi^1_\beta(u-1,-v)^F + 2\chi_p(p-2,b)\chi^1(u-1,-v-1)^F$$
$$+ \chi_p(0,b-1)[\chi^1(u,-v-1)+\chi(1,0)\chi^1(u-1,-v-1)]^F.$$

Using the inductive hypothesis we have

$$\chi^1_\beta(p-2-b+pu,-pv)$$
$$= 2\chi_p(p-1-b,p-2)\chi(u-v,v-1)^F + \chi_p(p-2-b,0)$$
$$[\chi(u-v-1,v-1)+\chi(0,1)\chi(u-v,v-1)]^F$$
$$+ 2\chi_p(b-1,p-2-b)[\chi(u-v,v-2)+\chi(u-v-1,v)]^F$$
$$+ \chi_p(b,p-1-b)[\chi(u-v,v-2)+\chi(u-v-1,v)]^F$$
$$+ 2\chi_p(p-2,b)\chi(u-v-1,v-1)^F + \chi_p(0,b-1)$$
$$[\chi(u-v,v-1)+\chi(1,0)\chi(u-v-1,v-1)]^F. \tag{2.99}$$

The $p-$expansion of Weyl character of $\chi(p-1-b+p(u-v),p-2+p(v-1))$ is given by

$$\chi(p-1-b+p(u-v),p-2+p(v-1))$$
$$= \chi_p(p-1-b,p-2)\chi(u-v,v-1)^F + \chi_p(b-1,p-2-b)$$
$$[\chi(u-v-1,v+1)+\chi(u-v,v-1)]^F$$
$$+ \chi_p(p-2-b,b-1)[\chi(u-v+1,v-2)+\chi(u-v-1,v-1)]^F$$
$$+ \chi_p(p-2,b)\chi(u-v-1,v-1)^F + \chi_p(b,p-1-b)$$

$$\chi(u-v,v-2)^F + \chi_p(0,b-1)[\chi(u-v,v-1) + \chi(u-v-1,v-2)]^F. \quad (2.100)$$

and the p–expansion of Weyl character for $\chi(p-2-b+p(u-v),pv)$ is

$$\begin{aligned}\chi(p-2-b+p(u-v),pv) &= \chi_p(p-1-b,p-2)\chi(u-v,v-1)^F + \chi_p(b-1,p-2-b) \\ &[\chi(u-v-1,v) + \chi(u-v,v-2)]^F \\ &+ \chi_p(p-2-b,0)[\chi(u-v,v) + \chi(u-v-1,v-1)]^F \\ &+ \chi_p(p-2,b)\chi(u-v-1,v+1)^F + \chi_p(b,p-1-b) \\ &\chi(u-v-1,v)^F + \chi_p(0,b-1)[\chi(u-v,v-1) \\ &+ \chi(u-v-2,v)]^F. \end{aligned} \quad (2.101)$$

Add equations 2.100, 2.101 and note that

$$\chi(0,1)\chi(u-v,v-1) = \chi(u-v,v)+\chi(u-v+1,v-2)+\chi(u-v-1,v-1).$$

and

$$\chi(1,0)\chi(u-v-1,v-1) = \chi(u-v,v-1)+\chi(u-v-2,v)+\chi(u-v-1,v-2).$$

to get the result.

The remaining two cases are very similar to the above case so we leave them to the reader.

4. Suppose $r = p-1-a+pu$ and $s+2 = -p+1+b+pv$ for some positive integers u and v with

$1 \le a, b < p$ and $a + 2b, 2a + b \le p$ then we have

$$\chi^1(p-1-a+pu, p-1-b-pv)$$
$$= \chi_p(p-1-a, p-1-b)\chi^1(u,-v)^F + \chi_p(a-1, p-1-a-b)$$
$$\chi^1_\beta(u-1,-v+1)^F + \chi_p(p-1-a-b, b-1)\chi^1_\alpha(u+1,-v-1)^F$$
$$+ \chi_p(p-1-b, a+b-1)\chi^1(u-1,-v)^F + \chi_p(a+b-1, p-1-a)$$
$$\chi^1(u,-v-1)^F + \chi_p(b-1, a-1)[\chi^1(u,-v) + \chi^1(u-1,-v-1)]^F.$$

Apply the inductive hypothesis and the results proved above for $\chi^1_\alpha(r,-s)$ and $\chi^1_\beta(r,-s)$ we get

$$\chi^1(p-1-a+pu, p-1-b-pv)$$
$$= \chi_p(p-1-a, p-1-b)\chi(u-v+1, v-2)^F + \chi_p(a-1, p-1-a-b)$$
$$[\chi(u-v, v-1) + \chi(u-v+1, v-3)]^F + \chi_p(p-1-a-b, b-1)$$
$$[\chi(u-v+1, v-1) + \chi(u-v, v-2)]^F + \chi_p(p-1-b, a+b-1)$$
$$\chi(u-v, v-2)^F + \chi_p(a+b-1, p-1-a)\chi(u-v, v-1)^F$$
$$+ \chi_p(b-1, a-1)[\chi(u-v+1, v-2) + \chi(u-v-1, v-1)]^F. \quad (2.102)$$

The p–expansion of Weyl character of $\chi(p-1-a-b+p(u-v+1), b-1+p(v-1))$ is given by

$$\chi(p-1-a-b+p(u-v+1), b-1+p(v-1))$$
$$= \chi_p(p-1-a, p-1-b)\chi(u-v+1, v-2)^F + \chi_p(a-1, p-1-a-b)$$
$$[\chi(u-v, v-1) + \chi(u-v+1, v-3)]^F + \chi_p(p-1-a-b, b-1)$$
$$[\chi(u-v+1, v-1) + \chi(u-v, v-2)]^F + \chi_p(p-1-b, a+b-1)$$
$$\chi(u-v, v-2)^F + \chi_p(a+b-1, p-1-a)\chi(u-v, v-1)^F$$

$$+\chi_p(b-1,a-1)[\chi(u-v+1,v-2)+\chi(u-v-1,v-1)]^F. \quad (2.103)$$

Compare equations 2.102 and 2.103 to get the result.

All the remaining cases are very much similar to the above case. We leave the details to the reader. □

Corollary 2.4.32. *Let $p \geq 5$. Then the sequence*

$$0 \to H^1(r-2,-s-1) \longrightarrow H^1(N_\alpha(r,-s-2)) \longrightarrow H^1(r,-s-2) \longrightarrow 0$$

is exact for all $r \geq ps$.

Proof. We know that the sequence

$$H^1(r-2,-s-1) \xrightarrow{\phi} H^1(N_\alpha(r,-s-2)) \xrightarrow{\psi} H^1(r,-s-2)$$

is exact meaning $\text{Im}\phi = \text{Ker}\psi$. Moreover by proposition 2.4.31, we have

$$\chi^1_\alpha(r,-s-2) = \chi^1(r,-s-2) + \chi^1(r-2,-s-1).$$

Let $X = H^1(r-2,-s-1)$, $Y = H^1(N_\alpha(r,-s-2))$ and $Z = H^1(r,-s-2)$. Now $\text{Im}\psi \simeq Y/\text{Ker}\psi \simeq Y/\text{Im}\phi$, therefore $\dim Y = \dim \text{Im}\phi + \dim \text{Im}\psi$. It is clear that $\dim \text{Im}\psi \leq \dim Z$. Also by the character result we have $\dim Y = \dim X + \dim Z$. Now $\dim \text{Im}\phi = \dim Y - \dim \text{Im}\psi \leq \dim Y - \dim Z = \dim X$. This implies $\dim \text{Im}\phi \leq \dim X$. We have proved that $\dim \text{Im}\phi \leq \dim X$ and $\dim \text{Im}\psi \leq \dim Z$ but $\dim Y = \dim X + \dim Z$ so we must have the equality. This proves that ϕ is injective and ψ is surjective and hence the result. □

The only region left to consider is when $r < ps$. We will list here a version of the propositions proved earlier for $p = 2$ and $p = 3$ in the case of $p \geq 5$.

Proposition 2.4.33. $\chi^1(p^n, -p^n - 2) = \chi(0, p^n - 2)$, *for all* $n \geq 0$.

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Suppose now $n > 1$ and the result is true for $n - 1$. Using the recursive formula we have

$$\begin{aligned}
&\chi^1(p^n, -p^n - 2) \\
&\quad = \chi(p-1, 0)\chi^1(p^{n-1}, -p^{n-1} - 2)^F + \chi(0, p-2)[\chi^1(p^{n-1}, -p^{n-1} - 1) \\
&\quad + \chi^1(p^{n-1} - 1, -p^{n-1} - 2)]^F + \chi(p-2, p-1)\chi^1(p^{n-1} - 1, -p^{n-1} - 1)^F.
\end{aligned} \tag{2.104}$$

Now by proposition 2.4.8 we have $\chi^1(p^{n-1} - 1, -p^{n-1} - 2) = 0$. Also $\chi^1(p^{n-1} - 1, -p^{n-1} - 1) = 0$ and $\chi^1(p^{n-1}, -p^{n-1} - 1) = \chi(0, p^{n-1} - 1)$ by proposition 2.4.7. The equation 2.104 becomes

$$\chi^1(p^n, -p^n - 2) = \chi(p-1, 0)\chi(0, p^{n-1} - 2)^F + \chi(0, p-2)\chi(0, p^{n-1} - 1)^F. \tag{2.105}$$

Finally compare equation 2.105 with the p–expansion of Weyl character of $\chi(0, p^n - 2)$ to get the result.

□

Proposition 2.4.34. *Suppose* $n \geq 0$ *then we have* $\chi^1(p^n - 1, -p^n) = \chi(0, p^n - 2)$. *Also*

$$\chi^1_\alpha(p^n, -p^n - 1) = \chi(0, p^n - 1).$$

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}
&\chi^1(p^n - 1, -p^n) \\
&\quad = \chi(p-1, 0)\chi^1(p^{n-1} - 1, -p^{n-1})^F + \chi(0, p-2)\chi^1_\alpha(p^{n-1}, -p^{n-1} - 1)^F \\
&\quad + \chi(p-2, p-1)\chi^1(p^{n-1} - 1, -p^{n-1} - 1)^F.
\end{aligned} \tag{2.106}$$

Now $\chi^1(p^{n-1}-1, -p^{n-1}-1) = 0$ by proposition 2.4.7. Also the p–expansion of Weyl character of $\chi(0, p^n - 2)$ gives

$$\chi(0, p^n - 2) = \chi(p-1, 0)\chi(0, p^{n-1} - 2)^F + \chi(0, p-2)\chi(0, p^{n-1} - 1)^F.$$

So if the result is true we must have $\chi^1_\alpha(p^{n-1}, -p^{n-1} - 1) = \chi(0, p^{n-1} - 1)$. Now by [11, 6.3, lemma1(iii)], we have

$$\chi^1_\alpha(p^{n-1}, -p^{n-1} - 1) = \chi^1(p^{n-1}, -p^{n-1} - 1) + \chi^1(p^{n-1} - 2, -p^{n-1} + 1).$$

Now $\chi^1(p^{n-1} - 2, -p^{n-1} + 1) = 0$ and $\chi^1(p^{n-1}, -p^{n-1} - 1) = \chi(0, p^{n-1} - 1)$ by proposition 2.4.7. This completes the proof. □

Proposition 2.4.35. $\chi^1(p^n, -p^n) = \chi(1, p^n - 2)$ *for all* $n \geq 0$.

Proof. We argue by induction on n. The result is clearly true for $n = 0$. Suppose the result is true for all $t < n$ then we have

$$\chi^1(p^t, -p^t) = \chi(1, p^t - 2).$$

Now for $n = t + 1$ we get

$$\begin{aligned}
&\chi^1(p^{t+1}, -p^{t+1}) \\
&\quad = \chi_p(p-2, p-2)\chi^1(p^t - 1, -p^t)^F + \chi_p(0, p-3)[\chi^1(p^t, -p^t) \\
&\quad + \chi^1(p^t - 1, -p^t - 1)]^F + \chi_p(p-3, 0)\chi^1_\beta(p^t - 1, -p^t + 1)^F \\
&\quad + \chi_p(p-2, 1)\chi^1(p^t, -p^t - 1)^F + \chi_p(1, p-2)\chi^1(p^t, -p^t)^F \\
&\quad + \chi_p(0, 0)\chi^1_\alpha(p^t + 1, -p^t - 1)^F. \qquad (2.107)
\end{aligned}$$

From propositions 2.4.34 and 2.4.7 we get that $\chi^1(p^t - 1, -p^t) = \chi(0, p^t - 2)$ and $\chi^1(p^t - 1, -p^t - 1) = 0$. Also $\chi^1(p^t, -p^t - 1) = \chi(0, p^t - 1)$ by proposition 2.4.7. Moreover

$$\chi^1_\alpha(p^t + 1, -p^t - 1) = \chi^1(p^t + 1, -p^t - 1) + \chi^1(p^t - 1, -p^t)$$

and

$$\chi^1_\beta(p^t-1,-p^t+1)=\chi^1(p^t-1,-p^t+1)+\chi^1(p^t,-p^t-1)$$

by proposition 2.4.5. The equation 2.107 becomes

$$\begin{aligned}
&\chi^1(p^{t+1},-p^{t+1})\\
&\quad=\chi_p(0,p-3)\chi(1,p^t-2)^F+\chi_p(p-3,0)[\chi(1,p^t-3)+\chi(0,p^t-1)]^F\\
&\quad+\chi_p(p-2,1)\chi(0,p^t-1)^F+\chi_p(1,p-2)\chi(1,p^t-2)^F\\
&\quad+\chi_p(0,0)\chi^1_\alpha(p^t+1,-p^t-1)^F. \qquad (2.108)
\end{aligned}$$

Compare the p–expansion of Weyl character of $\chi(1,p^{t+1}-2)$ with equation 2.33 to get the required result. □

Proposition 2.4.36. *Suppose $n\geq 0$ then we have $\chi^1(p^n+1,-p^n)=\chi(2,p^n-2)$. Also*

$$\chi^1_\beta(p^{n-1}-1,-p^{n-1})=\chi(0,p^{n-1}-2).$$

Proof. The argument is by induction on n. The result is clearly true for $n=0$. Using the recursive formulas we get

$$\begin{aligned}
&\chi^1(p^n+1,-p^n)\\
&\quad=\chi_p(p-1,p-3)\chi^1(p^{n-1}-1,-p^{n-1}-1)^F+\chi_p(0,p-4)\\
&\quad\chi^1_\alpha(p^{n-1},-p^{n-1}-1)^F+\chi_p(p-4,1)\chi^1_\beta(p^{n-1}-1,-p^{n-1})^F\\
&\quad+\chi_p(p-4,1)\chi^1(p^{n-1}-1,-p^{n-1})^F+\chi_p(2,p-2)\chi^1(p^{n-1},-p^{n-1}-1)^F\\
&\quad+\chi_p(1,0)[\chi^1(p^{n-1},-p^{n-1})+\chi^1(p^{n-1}-1,-p^{n-1}-1)]^F. \qquad (2.109)
\end{aligned}$$

Now $\chi^1(p^{n-1},-p^{n-1}-1)=\chi(0,p^{n-1}-1)$ by proposition 2.4.7. Also by proposition 2.4.35 we get $\chi^1(p^{n-1},-p^{n-1})=\chi(1,p^{n-1}-2)$ and $\chi^1(p^{n-1}-1,-p^{n-1})=\chi(0,p^{n-1}-2)$ by proposition 2.4.34. Moreover

$$\chi^1_\alpha(p^{n-1},-p^{n-1}-1)=\chi(0,p^{n-1}-1)$$

The equation 2.109 becomes

$$\begin{aligned}
&\chi^1(p^n+1,-p^n)\\
&\quad = \chi_p(0,p-4)\chi(0,p^{n-1}-1)^F + \chi_p(p-4,1)\chi^1_\beta(p^{n-1}-1,-p^{n-1})^F\\
&\quad + \chi_p(p-4,1)\chi(0,p^{n-1}-2)^F + \chi_p(2,p-2)\chi(0,p^{n-1}-1)^F\\
&\quad + \chi_p(1,0)\chi(1,p^{n-1}-2)^F. \qquad (2.110)
\end{aligned}$$

Also the p–expansion of Weyl character of $\chi(2,p^n-2)$ gives

$$\begin{aligned}
&\chi(2,p^n-2)\\
&\quad = \chi_p(0,p-4)\chi(0,p^{n-1}-1)^F + \chi_p(p-4,1)\chi(1,p^{n-1}-2)^F\\
&\quad + \chi_p(p-3,2)\chi(0,p^{n-1}-2)^F + \chi_p(2,p-2)\chi(0,p^{n-1}-1)^F\\
&\quad + \chi_p(1,0)\chi(1,p^{n-1}-2)^F. \qquad (2.111)
\end{aligned}$$

Compare equations 2.110 and 2.111 to get the result. □

Proposition 2.4.37. *Suppose $n \geq 0$ then we have $\chi^1(p^n-1,-p^n+1) = \chi(1,p^n-3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}
\chi^1(p^n-1,-p^n+1) &= \chi(p-1,1)\chi^1(p^{n-1}-1,-p^{n-1})^F + \chi(1,p-3)\\
&\chi^1_\alpha(p^{n-1},-p^{n-1}-1)^F + \chi(p-3,p-1)\chi^1(p^{n-1}-1,-p^{n-1}-1)^F. \qquad (2.112)
\end{aligned}$$

Now $\chi^1(p^{n-1}-1,-p^{n-1}-1) = 0$ by proposition 2.4.7. Also by proposition 2.4.34 we have $\chi^1_\alpha(p^{n-1},-p^{n-1}-1) = \chi(0,p^{n-1}-1)$ and $\chi^1(p^{n-1}-1,-p^{n-1}) = \chi(1,p^{n-1}-2)$. The equation 2.112 now becomes

$$\chi^1(p^n-1,-p^n+1) = \chi(p-1,1)\chi(1,p^{n-1}-2)^F+\chi(1,p-3)\chi(0,p^{n-1}-1)^F. \qquad (2.113)$$

Compare equation 2.113 with the p–expansion of Weyl character of $\chi(1, p^n - 3)$ to get the result.

□

Proposition 2.4.38. *Suppose $n \geq 0$ then we have $\chi^1(p^n, -p^n + 1) = \chi(2, p^n - 3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}
&\chi^1(p^n, -p^n + 1) \\
&\quad = \chi_p(p-3, p-2)\chi^1(p^{n-1} - 1, -p^{n-1} - 1)^F + \chi_p(1, p-4) \\
&\quad \chi^1_\alpha(p^{n-1}, -p^{n-1} - 1)]^F + \chi_p(p-4, 0)\chi^1_\beta(p^{n-1} - 1, -p^{n-1})^F \\
&\quad + \chi_p(p-4, 0)\chi^1(p^{n-1} - 1, -p^{n-1})^F + \chi_p(2, p-3)\chi^1(p^{n-1}, -p^{n-1} - 1)^F \\
&\quad + \chi_p(0, 1)[\chi^1(p^{n-1}, -p^{n-1}) + \chi^1(p^{n-1} - 1, -p^{n-1} - 1)]^F.
\end{aligned} \tag{2.114}$$

Now $\chi^1(p^{n-1} - 1, -p^{n-1}) = \chi(0, p^{n-1} - 2)$ and $\chi^1_\alpha(p^{n-1}, -p^{n-1} - 1) = \chi(0, p^n - 1)$ by proposition 2.4.34. Also
$\chi^1(p^{n-1}, -p^{n-1}) = \chi(1, p^{n-1} - 2)$ by proposition 2.4.25. Moreover proposition 2.4.7 gives $\chi^1(p^{n-1} - 1, -p^{n-1} - 1) = 0$. Equation 2.114 becomes

$$\begin{aligned}
&\chi^1(p^n, -p^n + 1) \\
&\quad = \chi_p(1, p-4)\chi(0, p^{n-1} - 1)^F + \chi_p(p-4, 0)[\chi(0, p^{n-1} - 2) \\
&\quad + \chi(0, p^{n-1} - 2)]^F + \chi_p(p-4, 0)\chi(0, p^{n-1} - 2)^F \\
&\quad + \chi_p(2, p-3)\chi(0, p^{n-1} - 1)^F + \chi_p(0, 1)\chi(1, p^{n-1} - 2)^F.
\end{aligned} \tag{2.115}$$

Compare equation 2.115 with the p–expansion of Weyl character of $\chi(2, p^n - 3)$ to get the result. □

Proposition 2.4.39. *Suppose $n \geq 0$ then we have $\chi^1(p^n+1, -p^n+1) = \chi(3, p^n-3)$.*

Proof. The argument is by induction on n. The result is clearly true for $n = 0$. Using the recursive formulas we get

$$\begin{aligned}
&\chi^1(p^n+1, -p^n+1) \\
&\quad = \chi_p(p-3, p-3)\chi^1(p^{n-1}-1, -p^{n-1}-1)^F + \chi_p(1, p-5) \\
&\quad \chi^1_\alpha(p^{n-1}, -p^{n-1}-1)^F + \chi_p(p-5, 1)\chi^1_\beta(p^{n-1}-1, -p^{n-1})^F \\
&\quad + \chi_p(p-5, 1)\chi^1(p^{n-1}-1, -p^{n-1})^F + \chi_p(3, p-3)\chi^1(p^{n-1}, -p^{n-1}-1)^F \\
&\quad + \chi_p(1, 1)[\chi^1(p^{n-1}, -p^{n-1}) + \chi^1(p^{n-1}-1, -p^{n-1}-1)]^F. \qquad (2.116)
\end{aligned}$$

By proposition 2.4.34 we have $\chi^1_\alpha(p^{n-1}, -p^{n-1}-1) = \chi(0, p^{n-1}-1)$. Also using proposition 2.4.7, 2.4.34 and 2.4.35 the equation 2.116 becomes

$$\begin{aligned}
&\chi^1(p^n+1, -p^n+1) \\
&\quad = \chi_p(1, p-5)\chi(0, p^{n-1}-1)^F + \chi_p(p-5, 1)[\chi(0, p^{n-1}-2) \\
&\quad + \chi(0, p^{n-1}-2)]^F + \chi_p(p-5, 1)\chi(0, p^{n-1}-2)^F \\
&\quad + \chi_p(3, p-3)\chi(0, p^{n-1}-1)^F + \chi_p(1, 1)\chi(1, p^{n-1}-2)^F. \qquad (2.117)
\end{aligned}$$

The p–expansion of Weyl character of $\chi(3, p^n-3)$ is given by

$$\begin{aligned}
&\chi(3, p^n-3) \\
&\quad = \chi_p(1, p-5)\chi(0, p^{n-1}-1)^F + \chi_p(p-5, 1)[\chi(0, p^{n-1}-2) \\
&\quad + \chi(0, p^{n-1}-2)]^F + \chi_p(p-5, 1)\chi(0, p^{n-1}-2)^F + \chi_p(3, p-3) \\
&\quad \chi(0, p^{n-1}-1)^F + \chi_p(1, 1)\chi(1, p^{n-1}-2)^F. \qquad (2.118)
\end{aligned}$$

Hence the result. □

Proposition 2.4.40. *Suppose $r, n > 0$ then we have*

$$\chi^1(p^n, -p^n - r) = \begin{cases} \chi(0, p^n - r), & n \geq r - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We argue by induction on r. The result is true for $r = 1$ by proposition 2.4.7. Suppose the result is true for all $t-1 < r$. For $r = t$ we have the following possibilities

1. Suppose $t = p - 2$ then

$$\begin{aligned} &\chi^1(p^n, -p^n - p + 2) \\ &\quad = \chi(p-1, 0)\chi^1(p^{n-1}, -p^{n-1} - 1)^F + \chi(0, p-2)[\chi^1(p^{n-1}, -p^{n-1}) \\ &\quad + \chi^1(p^{n-1} - 1, -p^{n-1} - 1)]^F + \chi(p-2, p-1)\chi^1(p^{n-1} - 1, -p^{n-1})^F. \end{aligned} \tag{2.119}$$

By proposition 2.4.7 we have $\chi^1(p^{n-1} - 1, -p^{n-1} - 1) = 0$ and $\chi^1(p^{n-1}, -p^{n-1} - 1) = \chi(0, p^{n-1} - 1)$. Also by proposition 2.4.34 we get $\chi^1(p^{n-1} - 1, -p^{n-1}) = \chi(0, p^{n-1} - 2)$ and $\chi^1(p^{n-1}, -p^{n-1}) = \chi(1, p^{n-1} - 2)$. The equation 2.119 becomes

$$\begin{aligned} &\chi^1(p^n, -p^n - p + 2) \\ &\quad = \chi(p-1, 0)\chi(0, p^{n-1} - 1)^F + \chi(0, p-2)\chi(1, p^{n-1} - 2)^F \\ &\quad + \chi(p-2, p-1)\chi(0, p^{n-1} - 2)^F. \end{aligned} \tag{2.120}$$

Compare equation 2.120 with the p–expansion of Weyl character of $\chi(0, p^n - p + 2)$ to get the result.

2. Now suppose $t = p - 1$

$$\begin{aligned}
&\chi^1(p^n, -p^n - p + 1) \\
&\quad = \chi(p-1, p-2)\chi^1(p^{n-1} - 1, -p^{n-1})^F + \chi(p-2, 0) \\
&\quad \chi^1_\beta(p^{n-1} - 1, -p^{n-1} + 1)^F + \chi(0, p-1)\chi^1(p^{n-1}, -p^{n-1})^F. \qquad (2.121)
\end{aligned}$$

By proposition 2.4.34 we get $\chi^1(p^{n-1} - 1, -p^{n-1}) = \chi(0, p^{n-1} - 2)$ and $\chi^1(p^{n-1}, -p^{n-1}) = \chi(1, p^{n-1} - 2)$. Moreover $\chi^1_\beta(p^{n-1} - 1, -p^{n-1} + 1) = \chi^1(p^{n-1} - 1, -p^{n-1} + 1) + \chi^1(p^{n-1}, -p^{n-1} - 1)$ by proposition 2.4.5. The equation 2.121 becomes

$$\begin{aligned}
&\chi^1(p^n, -p^n - p + 1) \\
&\quad = \chi(p-1, p-2)\chi(0, p^{n-1} - 2)^F + \chi(p-2, 0)[\chi(1, p^n - 3) \\
&\quad + \chi(0, p^n - 1)]^F + \chi(0, p-1)\chi(1, p^{n-1} - 2)^F. \qquad (2.122)
\end{aligned}$$

Compare equation 2.122 with the p–expansion of Weyl character of $\chi(0, p^n - p + 1)$ to get the result.

3. Now suppose $t = p - b - 2$, where $1 \le b < p$ and $1 + 2b, 2 + b \le p$ then

$$\begin{aligned}
&\chi^1(p^n, -p^n - p + b + 2) \\
&\quad = \chi_p(p-2, p-1-b)\chi^1(p^{n-1} - 1, -p^{n-1})^F + \chi_p(0, p-b-2) \\
&\quad [\chi^1(p^{n-1}, -p^{n-1}) + \chi^1(p^{n-1} - 1, -p^{n-1} - 1)]^F \\
&\quad + \chi_p(p-b-2, b-1)\chi^1_\alpha(p^{n-1}, -p^{n-1} - 1)^F + \chi_p(p-b-2, b-1) \\
&\quad \chi^1(p^{n-1}, -p^{n-1} - 1)^F + \chi_p(b, p-2)\chi^1(p^{n-1} - 1, -p^{n-1} - 1)^F \\
&\quad + \chi_p(b-1, 0)\chi^1_\beta(p^{n-1} - 1, -p^{n-1})^F. \qquad (2.123)
\end{aligned}$$

By proposition 2.4.34 and 2.4.35 we get $\chi^1(p^{n-1} - 1, -p^{n-1}) = \chi(0, p^{n-1} - 2)$ and $\chi^1(p^{n-1}, -p^{n-1}) = \chi(1, p^{n-1} - 2)$. Also $\chi^1(p^{n-1} - 1, -p^{n-1} - 1) = 0$

by proposition 2.4.7. Moreover $\chi^1_\alpha(p^{n-1}, -p^{n-1}-1) = \chi(0, p^{n-1}-1)$ by proposition 2.4.34. The equation 2.123 becomes

$$\begin{aligned}
&\chi^1(p^n, -p^n - p + b + 2)\\
&\quad = \chi_p(p-2, p-1-b)\chi(0, p^{n-1}-2)^F + \chi_p(0, p-b-2)\chi(1, p^{n-1}-2)^F\\
&\quad + \chi_p(p-b-2, b-1)\chi(0, p^{n-1}-1)^F + \chi_p(p-b-2, b-1)\chi(0, p^{n-1}-1)^F\\
&\quad + \chi_p(b-1, 0)\chi(0, p^{n-1}-2)^F. \qquad (2.124)
\end{aligned}$$

Compare equation 2.124 with the p–expansion of Weyl character of $\chi(0, p^n - p + b + 2)$ to get the result.

4. Finally let $t = a - 1$, where $1 \le a < p$ and $a + 2, 2a + 1 \le p$. The case is similar to the above case so we omit the details.

□

2.5 Summary of Results and Conclusion

In the last part we will give a list of all the results proved in this chapter. The following result helps us simplify the recursive formulas of $\chi^i_\alpha(\lambda)$.

Proposition 2.5.1. *Let $G = SL_3$ and $\lambda \in X(T)$. Suppose α is a simple root of G. If p does not divide $\langle\lambda, \alpha^\vee\rangle$ then*

$$H^1(N_\alpha(\lambda)) \simeq H^1(\lambda) \oplus H^1(\lambda - \alpha).$$

The next two results partially answer the question: When the result if the same as in characteristic zero? We would like to point out that there are some other instances when this condition holds.

Proposition 2.5.2. *We have* $\chi^1(r, -s-2) = \chi(r-s-1, s)$ *for all* $r \geq ps$, $r, s > 0$. *Moreover*

$$\chi^1_\alpha(r, -s-2) = \chi(r-s-1, s) + \chi(r-s-2, s-1)$$

and

$$\chi^1_\beta(r, -s-2) = \chi(r-s-1, s) + \chi(r-s-2, s+2)$$

for all $r \geq ps$, $r, s > 0$.

Corollary 2.5.3. *The sequence*

$$0 \to H^1(r-2, -s-1) \longrightarrow H^1(N_\alpha(r, -s-2)) \longrightarrow H^1(r, -s-2) \longrightarrow 0$$

is exact for all $r \geq ps$.

The remaining results outline some of the cases in which we were able to find a non-recursive result. All the results are valid for all $p > 0$. The reader might wonder why we gave the proves in separate cases earlier in the chapter. There are two reasons for this. One because Donkin's formulas are different in these cases. It is not yet clear if the formulas for $p = 2$ and $p = 3$ can be deduced from formulas for $p \geq 5$. Although we do believe that this might be possible (work in progress). But even if we know that the formulas are same in all positive characteristics, we found it almost impossible to get a sense of the results without working in a particular characteristic.

Proposition 2.5.4. *Suppose* $n > 0$ *and* $0 \leq m \leq p-1$ *then we have*

$$H^i(r, -p^n(m+1)-1) = \begin{cases} H^0(r - p^n(m+1), p^n(m+1)-1), & r \geq p^n(m+1) \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.5.5. *Suppose $n \geq 0$ then for all $r \geq 2$ we have*

$$H^1(p^n - 1, -p^n - r) = 0.$$

Proposition 2.5.6. $\chi^1(p^n, -p^n - 2) = \chi(0, p^n - 2)$*, for all $n \geq 0$.*

Proposition 2.5.7. *Suppose $n \geq 0$ then we have*

$$\chi^1(p^n + r, -p^n) = \chi(r + 1, p^n - 2), \quad \text{for } \; r = -1, 0, 1.$$

Also

$$\chi^1_\alpha(p^n, -p^n - 1) = \chi(0, p^n - 1).$$

Proposition 2.5.8. *Suppose $n \geq 0$ then we have*

$$\chi^1(p^n + r, -p^n + 1) = \chi(r + 2, p^n - 3), \quad \text{for } \; r = -1, 0, 1.$$

Also

$$\chi^1_\beta(p^n - 1, -p^n) = \chi(0, p^n - 1).$$

Proposition 2.5.9. *Suppose $r, n > 0$ then we have*

$$\chi^1(p^n, -p^n - r) = \begin{cases} \chi(0, p^n - r), & n \geq r - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Conclusion

The results of this chapter help us a great deal to simplify the recursive formulas of Donkin. A significant progress has been made towards finding a result which describes $\chi^i(r, s)$ completely. There are still many cases to handle but we were unable to spot a pattern in these cases.

Chapter 3

Towards a Recursive Description for Characters of Cohomology of Line Bundles On G_2 in Characteristic 2

3.1 Introduction

Let k be an algebraically closed field of characteristic p. Let G be an algebraic group of type G_2 and B a Borel subgroup of G. Suppose $M \in \text{mod}(B)$ and $P_\lambda \in \text{mod}(G)$, where P_λ is projective on restriction to the first infinitesimal subgroup G_1 (of G of type G_2). The representation theory of G has been discussed briefly in chapter 1. Let U_1 denote the radical of G_1. We will denote by $\hat{Q}_r(\lambda)$ the projective cover of $L(\lambda)$ as a G_rT- module. If $N = M^F$ then we will write $M = N^{-1}$. We will assume the following conjecture due to Donkin in this chapter see e.g. [13]. The conjecture is known to be true for

$p \geq 2h - 2$, where h is the Coxeter number of G.

Conjecture. *(Donkin)*
Let $\lambda \in X_1(T)$ then the restriction of the indecomposable tilting module $T(2(p-1)\rho + w_0\lambda)$ to G_1 is indecomposable. This implies that $T(2(p-1)\rho + w_0\lambda)$ is isomorphic to $\hat{Q}_1(\lambda)$ as G_1T-module.

We will also be extensively using the following result without reference see e.g. [21, II, propostion 5.20].

Proposition 3.1.1. *Suppose* $\mathrm{char}(k) = p \neq 0$.

1. *We have $H^1(B, -p^n\alpha) = k$ for all $\alpha \in S$ and $n \in \mathbb{N}$.*
2. *For all $\mu \in X(T)$ with $\mu \neq -p^n\alpha$ for all $\alpha \in S$ and $n \in \mathbb{N}$ we have $H^1(B, \mu) = 0$.*

In this chapter we will try to give a recursive description for the characters of cohomology of line bundles on G. The method we will use here was developed by Donkin in [12]. Application of the method for $G = \mathrm{SL}_3$ is given in [11] and has been discussed extensively in the previous chapter. First we will say a few words about the method and how it works and then we will try to apply it to G_2. We would like to remark here that although the method given in [12] is very impressive it relies mostly on finding the U_1-invariants of $M \otimes P_\lambda$ which is not always easy to find.

The precise result is given below see e.g. [12, proposition 1.1b].

Proposition 3.1.2. *(Donkin) Suppose that, for $\lambda \in X_1$, we have modules $P_\lambda, Q_\lambda \in \mathrm{mod}(G)$ which are projective on restriction to G_1 and that $P_\lambda \simeq \hat{Q}_1(\lambda) \oplus Q_\lambda$, as G_1T−modules. Then for all $M \in \mathrm{mod}(B)$ and $i \geq 0$ we have*

$$\chi^i(M) = \sum_{\lambda \in X_1} \mathrm{ch}L(\lambda)^* \chi^i(H^0(B_1, P_\lambda \otimes M)^{-1})^F - \sum_{\lambda \in X_1} \mathrm{ch}L(\lambda)^* \chi^i(H^0(B_1, Q_\lambda \otimes M)^{-1})^F \tag{3.1}$$

The result given in proposition 3.1.2 is not the most general case but it is sufficient for application to G_2. The most general formula is given in [12]. As remarked in [12, remark 1] if $Q_\lambda|_{G_1T} \simeq \hat{Q}_1(\lambda)$ then the proposition 3.1.2 becomes

$$\chi^i(M) = \sum_{\lambda \in X_1} \mathrm{ch}L(\lambda)^* \chi^i(H^0(B_1, Q_\lambda \otimes M)^{-1})^F.$$

It will become clear later on that the condition $Q_\lambda|_{G_1T} \simeq \hat{Q}_1(\lambda)$ does not always hold for G_2 and hence we will be using the original formula. For the rest of this chapter G will denote the algebraic group G_2 unless stated otherwise.

Recall that there are two simple roots of G given by $\alpha = (2, -1)$ and $\beta = (-3, 2)$. The Weyl group W is generated by s_α and s_β and is a finite group of order 12. We will denote by $s(\lambda)$ the orbit sum of λ under the action of the Weyl group W i.e. $s(\lambda) = \sum_{\mu \in W\lambda} e(\mu)$. We will also some times denote by $s(\lambda)$ the orbit of λ when no confusion arises. We will first take $p = 2$.

3.2 The case $p = 2$

Suppose that the characteristic of the field k is 2. We will now specialize proposition 3.1.2 for this case. Recall that for $p = 2$ on restriction to G_1, G

has four simple modules $L(0,0), L(1,0), L(0,1)$ and $L(1,1)$. Also for $\lambda \neq 0$ we have $P_\lambda = T(2\rho - \lambda)$ and $Q_\lambda = 0$. For $\lambda = 0$ we take $P_0 = T(2,2)'$, where $T(2,2)'$ denotes the non-Steinberg component of $St \otimes St$ and $Q_0 = 2T(2,1)$. Here $T(\mu)$ denotes the tilting module with highest weight μ see section 1.5. The details of this will becomes clear as we go on to find the tensor product of the simple modules. So proposition 3.1.2 now becomes

$$\begin{aligned}
\chi^i(a,b) = \text{ch } L(0,0)\chi^i(H^0(B_1, T(2,2)' \otimes (a,b))^{-1})^F \\
+ \text{ch } L(1,0)\chi^i(H^0(B_1, T(1,2) \otimes (a,b))^{-1})^F \\
+ \text{ch } L(0,1)\chi^i(H^0(B_1, T(2,1) \otimes (a,b))^{-1})^F \\
+ \text{ch } L(1,1)\chi^i(H^0(B_1, T(1,1) \otimes (a,b))^{-1})^F \\
- 2\text{ch } L(0,0)\chi^i(H^0(B_1, T(2,1) \otimes (a,b))^{-1})^F
\end{aligned} \tag{3.2}$$

In order to calculate all the terms in this formula we need to first calculate the tensor product of all the simple modules and then their U_1–invariants. We will finally incorporate all the information back into equation 3.2 to get the formulas.

3.2.1 Tensor product of simple modules

In this section we will calculate the tensor product of the simple modules of G. We will denote $L(1,1)$ by St.

1. $L(0,0) \otimes \text{St} = \text{St}$
2. $L(1,0) \otimes \text{St} = T(2,1) = Q_{(0,1)}$. This is true because

$$\text{Hom}_G(L(0,1), L(1,0) \otimes \text{St}) = \text{Hom}_G(L(0,1) \otimes L(1,0), \text{St}) = k.$$

3. Now consider $L(0,1) \otimes \mathrm{St}$. We need to calculate the following homomorphism spaces to decompose this module.

   (a) $\mathrm{Hom}_G(L(0,0), L(0,1) \otimes \mathrm{St}) = 0$.

   (b) $\mathrm{Hom}_G(L(1,0), L(0,1) \otimes \mathrm{St}) = k$.

   (c) $\mathrm{Hom}_G(L(0,1), L(0,1) \otimes \mathrm{St}) = \mathrm{Hom}_G(L(0,1) \otimes L(0,1), \mathrm{St})$. We need to calculate $\chi(0,1)^2$ to find the dimension of this Hom space.

$$\begin{aligned} s(0,1) &= s(0,1)\chi(0,0) \\ &= \chi(0,1) + \chi(-3,2) + \chi(3,-2) + \chi(-3,1) \\ &= \chi(0,1) - \chi(1,0) - \chi(0,0) \end{aligned}$$

   Therefore

$$\chi(0,1) = s(0,1) + s(1,0) + 2s(0,0)$$

   We have

$$\begin{aligned} \chi(0,1)^2 &= \chi(0,1)\chi(0,1) \\ &= \chi(0,2) + \chi(3,0) + \chi(-3,3) + \chi(-3,2) + \chi(0,0) \\ &\quad + \chi(1,1) + \chi(2,0) + \chi(-2,2) + \chi(1,0) + \chi(0,1) \\ &\quad + \chi(0,1) \\ &= \chi(0,2) + \chi(3,0) - \chi(1,1) - \chi(1,0) + \chi(0,0) \\ &\quad + \chi(1,1) + \chi(2,0) - \chi(0,1) + \chi(1,0) + \chi(0,1) \\ &\quad + \chi(0,1) \\ &= \chi(0,2) + \chi(3,0) + \chi(2,0) + \chi(0,1) + \chi(0,0) \end{aligned}$$

   Since ch St does not appear in the character of $L(0,1) \otimes L(0,1)$ we have $\dim \mathrm{Hom}_G(L(0,1), L(0,1) \otimes \mathrm{St}) = 0$.

(d) Finally we need to find $\mathrm{Hom}_G(\mathrm{St}, L(0,1) \otimes \mathrm{St})$. We will calculate $\chi(0,1)\chi(1,1)$ to get its dimension.

$$\begin{aligned}\chi(0,1)\chi(1,1) &= \chi(1,2) + \chi(4,0) + \chi(-2,3) + \chi(-2,2) + \chi(1,0) \\ &+ \chi(2,1) + \chi(0,2) + \chi(3,0) + \chi(2,0) + \chi(0,1) + 2\chi(1,1) \\ &= \chi(1,2) + \chi(4,0) - \chi(0,2) - \chi(0,1) + \chi(1,0) + \chi(2,1) \\ &+ \chi(0,2) + \chi(3,0) + \chi(2,0) + \chi(0,1) + 2\chi(1,1) \\ &= \chi(1,2) + \chi(4,0) + \chi(1,0) + \chi(2,1) + \chi(3,0) \\ &+ \chi(2,0) + 2\chi(1,1)\end{aligned}$$

Hence $\dim \mathrm{Hom}_G(\mathrm{St}, L(0,1) \otimes \mathrm{St}) = 2$. Combining the above results we get

$$L(0,1) \otimes \mathrm{St} = 2\mathrm{St} \oplus T(1,2) = 2\mathrm{St} \oplus Q_{(1,0)}.$$

4. Finally we need a decomposition of $\mathrm{St} \otimes \mathrm{St}$. We will first calculate $\chi(1,1)^2$. For this we need the orbits listed in table 3.1.

$$\begin{aligned}s(1,1) &= s(1,1)\chi(0,0) \\ &= \chi(1,1) + \chi(5,-2) + \chi(-5,3) + \chi(4,-3) + \chi(-4,1) \\ &+ \chi(1,-2) + \chi(-5,2) + \chi(5,-3) + \chi(-4,3) \\ &= \chi(1,1) - \chi(2,0) + \chi(0,0) + \chi(0,0) - \chi(2,0) \\ &= \chi(1,1) - 2\chi(2,0) + 2\chi(0,0)\end{aligned}$$

Therefore $\chi(1,1) = s(1,1) + 2\chi(2,0) - 2\chi(0)$. Also

$$\begin{aligned}s(2,0) &= \chi(2,0) + \chi(-2,2) + \chi(4,-2) + \chi(-4,2) + \chi(2,-2) + \chi(-2,0) \\ &= \chi(2,0) - \chi(0,1) - \chi(1,0)\end{aligned}$$

Hence $\chi(2,0) = s(2,0) + \chi(0,1) + \chi(1,0)$. Moreover $\chi(1,0) = s(1,0) + s(0,0)$ and $\chi(0,1) = s(0,1) + s(1,0) + 2s(0,0)$ so

$$\chi(2,0) = s(2,0) + s(0,1) + 2s(1,0) + 3s(0,0).$$

We have

$$\chi(1,1) = s(1,1) + 2s(2,0) + 2s(0,1) + 4s(1,0) + 4s(0,0)$$

$$\begin{aligned}
\chi(1,1)^2 &= \chi(2,2) + \chi(0,3) + \chi(-4,4) + \chi(5,-2) + \chi(-3,2) \\
&+ \chi(0,0) + \chi(-4,3) + \chi(6,-2) + \chi(-3,4) + \chi(5,0) \\
&+ 2\chi(3,1) + 2\chi(-3,3) + 2\chi(1,2) + 2\chi(4,0) + 2\chi(-2,3) \\
&+ 2\chi(-2,2) + 2\chi(1,0) + 4\chi(2,1) + 4\chi(0,2) + 4\chi(3,0) \\
&+ 4\chi(2,0) + 4\chi(0,1) + 4\chi(1,1) \\
&= \chi(2,2) + \chi(0,3) - \chi(2,1) - \chi(2,0) - \chi(1,0) \\
&+ \chi(0,0) - \chi(2,0) - \chi(3,0) - \chi(1,2) + \chi(5,0) \\
&+ 2\chi(3,1) - 2\chi(1,1) + 2\chi(1,2) + 2\chi(4,0) - 2\chi(0,2) \\
&- 2\chi(0,1) + 2\chi(1,0) + 4\chi(2,1) + 4\chi(0,2) + 4\chi(3,0) \\
&+ 4\chi(2,0) + 4\chi(0,1) + 4\chi(1,1) \\
&= \chi(2,2) + 2\chi(3,1) + \chi(5,0) + \chi(1,2) + \chi(0,3) \\
&+ 3\chi(2,1) + 2\chi(4,0) + 2\chi(0,2) + 3\chi(3,0) + 2\chi(1,1) \\
&+ 2\chi(2,0) + 2\chi(0,1) + \chi(1,0) + \chi(0,0).
\end{aligned}$$

Therefore $\dim \operatorname{Hom}_G(\mathrm{St}, \mathrm{St} \otimes \mathrm{St}) = 2$ and we get

$$\mathrm{St} \otimes \mathrm{St}|_{G_1} = Q_0 \oplus 2Q_{(0,1)} \oplus 16\mathrm{St}$$

s(1,1)		s(2,0)	s(0,1)	s(1,0)
(1,1)	(-1,-1)	(2,0)	(0,1)	(1,0)
(-1,2)	(1,-2)	(-2,2)	(3,-1)	(-1,1)
(5,-2)	(-5,2)	(4,-2)	(-3,2)	(2,-1)
(-5,3)	(5,-3)	(-4,2)	(3,-2)	(-2,1)
(4,-3)	(-4,3)	(2,-2)	(-3,1)	(1,-1)
(-4,1)	(4,-1)	(-2,0)	(0,-1)	(-1,0)

Table 3.1: Orbits of $(1,1),(2,0),(0,1)$ and $(1,0)$

3.2.2 The U_1–invariants

Now we will calculate $\mathrm{H}^0(U_1, X)$, where X is one of the modules given in 1,2,3 and 4. We will be using the notation $N_{\alpha\beta\alpha}(\lambda)$ to mean we have module with weights $\lambda, \lambda-\alpha, \lambda-\alpha-\beta, \lambda-\alpha-\beta-\alpha$. Similarly we will be using the notations $N_{\beta\alpha\beta}(\lambda)$, $N_{\alpha\beta}(\lambda)$, $N_{\beta\alpha}(\lambda)$ and their variations.

1. $\mathrm{H}^0(U_1, L(0,0)\otimes \mathrm{St}) = (-1,-1)$.

2. Now we consider $\mathrm{H}^0(U_1, L(1,0)\otimes \mathrm{St})$. We know that

$$\mathrm{ch}\ \mathrm{H}^0(U_1, L(1,0)\otimes \mathrm{St}) = (\mathrm{ch}\ L(1,0))e(-1,-1).$$

From table 3.2 we get

$s(1,0)$	$+(-1,-1)$	$\bigoplus_{\lambda\in X_1(T)} k_\lambda \otimes Z_\lambda^F$
$(1,0)$	$(0,-1)$	$(0,1)+2(0,-1)$
$(-1,1)$	$(-2,0)$	$(0,0)+2(-1,0)$
$(2,-1)$	$(1,-2)$	$(1,0)+2(0,-1)$
$(-2,1)$	$(-3,0)$	$(1,0)+2(-2,0)$
$(1,-1)$	$(0,-2)$	$(0,0)+2(0,-1)$
$(-1,0)$	$(-2,-1)$	$(0,1)+2(-1,-1)$

Table 3.2: Base two expansion of (ch $L(1,0))e(-1,-1)$

$$\begin{pmatrix}(-1,0)\\(0,-1)\end{pmatrix}^F \oplus (0,1)\otimes \begin{pmatrix}(0,-1)\\(-1,-1)\end{pmatrix}^F \oplus (1,0)\otimes \begin{pmatrix}(0,-1)\\(-2,0)\end{pmatrix}^F$$

Also the B−socle is given by

$$(0,-1)\oplus(-2,0)\oplus(-3,0)\oplus(0,-2)\oplus(-2,-1).$$

Moreover $\mathrm{Ext}^1_B((-1,0),(0,-1)) = 0$ and $\mathrm{Ext}^1_B((0,-1),(-1,-1)) = 0$. There is an extension from $(0,-1)$ to $(-2,0)$ given by $N_\alpha(0,-1)$. Therefore we have

$$\begin{aligned}\mathrm{H}^0(U_1, L(1,0)\otimes \mathrm{St}) =& [(-1,0)\oplus(0,-1)]^F \oplus (0,1)\otimes[(0,-1)\oplus(-1,-1)]^F\\ &\oplus (1,0)\otimes N_\alpha(0,-1)^F\end{aligned}$$

3. Now we take $\mathrm{H}^0(U_1, L(0,1)\otimes\mathrm{St})$. We know that

$$\mathrm{ch}\,\mathrm{H}^0(U_1, L(0,1)\otimes\mathrm{St}) = (\mathrm{ch}\,L(0,1))e(-1,-1).$$

From table 3.3 we get

$$\begin{pmatrix}(1,-1)\\(-1,0)\\(0,-1)\\(-2,0)\end{pmatrix}^F \oplus (0,1)\otimes\begin{pmatrix}(0,-1)\\(-2,0)\\(1,-2)\\(-1,-1)\end{pmatrix}^F \oplus (1,0)\otimes\begin{pmatrix}(-1,0)\\(0,-1)\\(-2,0)\\(-1,-1)\end{pmatrix}^F$$
$$\oplus 2(1,1)\otimes(-1,-1)^{[F]}$$

We will denote by $Z_{(i,j)}$ the (i,j)–part of $\mathrm{H}^0(U_1,M)$ when we write it in the above form, where $i,j=0,1$. We get

$$Z^F_{(0,0)}\oplus(0,1)\otimes Z^F_{(0,1)}\oplus(1,0)\otimes Z^F_{(1,0)}\oplus(1,1)\otimes Z^F_{(1,1)}.$$

The B–socle is given by

$$[(-2,0)\oplus(-1,0)]^F\oplus(0,1)\otimes[(-1,-1)]^F\oplus(1,0)\otimes$$
$$[(-1,0)\oplus(-2,0)\oplus(-1,-1)]^F\oplus 2(1,1)\otimes(-1,-1)^F$$

We now consider each module $Z_{(i,j)}$ separately.

(a) Consider the module $Z_{(0,0)}$. We can see that

$$\mathrm{Ext}^1_B((0,-1),(-2,0))=k$$

and this extension is given by $N_\alpha(0,-1)$. Now

$$\mathrm{Ext}^1_B((-1,0),N_\alpha(0,-1))=\mathrm{H}^1(B,N_\alpha(1,-1))=0$$

Moreover $\mathrm{Ext}^1_B((1,-1),(-1,0))=k$ and this extension is given by $N_\alpha(1,-1)$. We get

$$Z_{(0,0)}=N_\alpha(0,-1)\oplus N_\alpha(1,-1)$$

weights	$+(-1,-1)$	$\bigoplus_{\lambda\in X_1(T)} k_\lambda \otimes Z_\lambda^F$
$(0,1)$	$(-1,0)$	$(1,0)+2(-1,0)$
$(3,-1)$	$(2,-2)$	$(0,0)+2(1,-1)$
$(-3,2)$	$(-4,1)$	$(0,1)+2(-2,0)$
$(3,-2)$	$(2,-3)$	$(0,1)+2(1,-2)$
$(-3,1)$	$(-4,0)$	$(0,0)+2(-2,0)$
$(0,-1)$	$(-1,-2)$	$(1,0)+2(-1,-1)$
$(1,0)$	$(0,-1)$	$(0,1)+2(0,-1)$
$(-1,1)$	$(-2,0)$	$(0,0)+2(-1,0)$
$(2,-1)$	$(1,-2)$	$(1,0)+2(0,-1)$
$(-2,1)$	$(-3,0)$	$(1,0)+2(-2,0)$
$(1,-1)$	$(0,-2)$	$(0,0)+2(0,-1)$
$(-1,0)$	$(-2,-1)$	$(0,1)+2(-1,-1)$
$(0,0)$	$(-1,-1)$	$(1,1)+2(-1,-1)$
$(0,0)$	$(-1,-1)$	$(1,1)+2(-1,-1)$

Table 3.3: Base two expansion of $(\text{ch}\, L(0,1))e(-1,-1)$

(b) We will now consider the module $Z_{(0,1)}$. It is easy to see that

$$\mathrm{Ext}^1_B((1,-2),(-1,-1)) = k$$

and this extension is given by $N_\alpha(1,-2)$. Also

$$\mathrm{Ext}^1_B((-2,0),N_\alpha(1,-2)) = \mathrm{H}^1(B,N_\alpha(3,-2)) = k$$

So there is a unique extension from $N_\alpha(1,-2)$ to $(-2,0)$. This extension is given by $N_{\beta\alpha}(-2,0)$, where $N_{\beta\alpha}(-2,0)$ is a submodule of $\nabla(1,0)$. Lastly we want to check if there is an extension from $N_{\beta\alpha}(-2,0)$ to $(0,-1)$. Now

$$\mathrm{Ext}^1_B((0,-1),N_{\beta\alpha}(-2,0)) = \mathrm{H}^1(B,N_{\beta\alpha}(-2,1))$$

We have a short exact sequence of modules given by

$$0 \to (-3,0) \to N_{\beta\alpha}(-2,1) \to N_\beta(-2,1) \to 0.$$

Now $H^1(B,(-3,0)) = 0$ by [21, 5.20 proposition]. Also $H^1(B,N_\beta(-2,1)) = k$ therefore by the long exact sequence of induction we have $H^1(B,N_{\beta\alpha}(-2,1)) = k$. Hence there is a unique extension from $N_{\beta\alpha}(-2,0)$ to $(0,-1)$ and this extension is given by $N_{\alpha\beta\alpha}(0,-1)$, where $N_{\alpha\beta\alpha}(0,-1)$ is a submodule of $\nabla(1,0)$ and is self dual. We get

$$Z_{(0,1)} = N_{\alpha\beta\alpha}(0,-1).$$

(c) Now we will consider the module $Z_{(1,0)}$. We can see that

$$\mathrm{Ext}^1_B((-2,0),(-1,-1)) = \mathrm{H}^1(B,(1,-1)) = 0$$

and there is no extension from $(-1,-1)$ to $(-2,0)$. Also

$$\mathrm{Ext}^1_B((0,-1),(-2,0)) = \mathrm{H}^1(B,(-2,1)) = k$$

and

$$\mathrm{Ext}^1_B((0,-1),(-1,-1)) = \mathrm{H}^1(B,(-1,0)) = 0.$$

So there is a unique extension from $(-2,0)$ to $(0,-1)$ and this extension is given by $N_\alpha(0,-1)$. Lastly

$$\mathrm{Ext}^1_B((-1,0),N_\alpha(0,-1)) = \mathrm{H}^1(B,N_\alpha(1,-1)) = 0$$

So there is no extension from $N_\alpha(0,-1)$ to $(-1,0)$. We get

$$Z_{(1,0)} = (-1,-1) \oplus N_\alpha(0,-1) \oplus (-1,0)$$

Combine the above cases to get

$$\begin{aligned}\mathrm{H}^0(U_1,L(0,1)\otimes \mathrm{St}) = {} & [N_\alpha(0,-1)\oplus N_\alpha(1,-1)]^F \oplus (0,1)\otimes N_{\alpha\beta\alpha}(0,-1)^F \\ & \oplus (1,0)\otimes [(-1,-1)\oplus N_\alpha(0,-1)\oplus(-1,0)]^F \\ & \oplus 2(1,1)\otimes(-1,-1)^F\end{aligned}$$

4. Finally we consider $\mathrm{H}^0(U_1,\mathrm{St}\otimes\mathrm{St})$. We know that

$$\mathrm{ch}\,\mathrm{H}^0(U_1,\mathrm{St}\otimes\mathrm{St}) = (\mathrm{ch}\,L(1,1))e(-1,-1).$$

From table 3.4 we get

$$\begin{pmatrix} (0,0) \\ 2*(1,-1) \\ 4*(-1,0) \\ (-3,1),(2,-2) \\ 4*(0,-1) \\ 2*(-2,0) \\ (-1,-1) \end{pmatrix}^F \oplus (0,1)\otimes \begin{pmatrix} (-1,0) \\ (2,-2) \\ 4*(0,-1) \\ 2*(-2,0) \\ 2*(1,-2) \\ 4*(-1,-1) \\ (-3,0) \\ (0,-2) \end{pmatrix}^F \oplus (1,0)\otimes \begin{pmatrix} (1,-1) \\ 2*(-1,0) \\ (-3,1) \\ 4*(0,-1) \\ 4*(-2,0) \\ 2*(1,-2) \\ 2*(-1,-1) \\ (-3,0) \end{pmatrix}^F$$

$$\oplus(1,1)\otimes\begin{pmatrix} 2*(0,-1) \\ 2*(-2,0) \\ 2*(1,-2) \\ 2*(-3,0) \\ 2*(0,-2) \\ 2*(-2,-1) \\ 4*(-1,-1) \end{pmatrix}^F$$

we can write this as

$$Z^F_{(0,0)} \oplus (0,1) \otimes Z^F_{(0,1)} \oplus (1,0) \otimes Z^F_{(1,0)} \oplus (1,1) \otimes Z^F_{(1,1)}.$$

The B–socle is given by

$$\begin{aligned} &[(-1,-1) \oplus 2*(-2,0) \oplus 2*(0,-1) \oplus 2*(-1,0) \oplus (0,0)]^F \oplus \\ &(0,1) \otimes [(0,-2) \oplus 3*(-1,-1) \oplus 2*(0,-1)]^F \oplus \\ &(1,0) \otimes [(-3,0) \oplus (-1,-1) \oplus 3*(-2,0) \oplus (-1,0)]^F \oplus \\ &(1,1) \otimes [2*(-2,-1) \oplus 2*(-1,-1)]^F \end{aligned}$$

We will now consider each module $Z_{(i,j)}$ separately.

(a) Consider the module $Z_{(0,0)}$. Arranging the weights in descending order we get figure 3.1.

weights	$+(-1,-1)$	$\bigoplus_{\lambda\in X_1} k_\lambda \otimes Z^F_\lambda$	weights	$+(-1,-1)$	$\bigoplus_{\lambda\in X_1} k_\lambda \otimes Z^F_\lambda$
$(1,1)$	$(0,1)$	$(0,0)+2(0,0)$	$2*(2,-2)$	$(1,-3)$	$(1,1)+2(0,-2)$
$(-1,2)$	$(-2,1)$	$(0,1)+2(-1,0)$	$2*(-2,0)$	$(-3,-1)$	$(1,1)+2(-2,-1)$
$(5,-2)$	$(4,-3)$	$(0,1)+2(2,-2)$	$2*(0,1)$	$(-1,0)$	$(1,0)+2(-1,0)$
$(-5,3)$	$(-6,2)$	$(0,0)+2(-3,1)$	$2*(3,-1)$	$(2,-2)$	$(0,0)+2(1,-1)$
$(4,-3)$	$(3,-4)$	$(1,0)+2(1,-2)$	$2*(-3,2)$	$(-4,1)$	$(0,1)+2(-2,0)$
$(-4,1)$	$(-5,0)$	$(1,0)+2(-3,0)$	$2*(3,-2)$	$(2,-3)$	$(0,1)+2(1,-2)$
$(-1,-1)$	$(-2,-2)$	$(0,0)+2(-1,-1)$	$2*(-3,1)$	$(-4,0)$	$(0,0)+2(-2,0)$
$(1,-2)$	$(0,-3)$	$(0,1)+2(0,-2)$	$2*(0,-1)$	$(-1,-2)$	$(1,0)+2(-1,-1)$
$(-5,2)$	$(-6,1)$	$(0,1)+2(-3,0)$	$4*(1,0)$	$(0,-1)$	$(0,1)+2(0,-1)$
$(5,-3)$	$(4,-4)$	$(0,0)+2(2,-2)$	$4*(-1,1)$	$(-2,0)$	$(0,0)+2(-1,0)$
$(-4,3)$	$(-5,2)$	$(1,0)+2(-3,1)$	$4*(2,-1)$	$(1,-2)$	$(1,0)+2(0,-1)$
$(4,-1)$	$(3,-2)$	$(1,0)+2(1,-1)$	$4*(-2,1)$	$(-3,0)$	$(1,0)+2(-2,0)$
$2*(2,0)$	$(1,-1)$	$(1,1)+2(0,-1)$	$4*(1,-1)$	$(0,-2)$	$(0,0)+2(0,-1)$
$2*(-2,2)$	$(-3,1)$	$(1,1)+2(-2,0)$	$4*(-1,0)$	$(-2,-1)$	$(0,1)+2(-1,-1)$
$2*(4,-2)$	$(3,-3)$	$(1,1)+2(1,-2)$	$4*(0,0)$	$(-1,-1)$	$(1,1)+2(-1,-1)$
$2*(-4,2)$	$(-5,1)$	$(1,1)+2(-3,0)$			

Table 3.4: Base two expansion of (ch $L(1,1))e(-1,-1)$

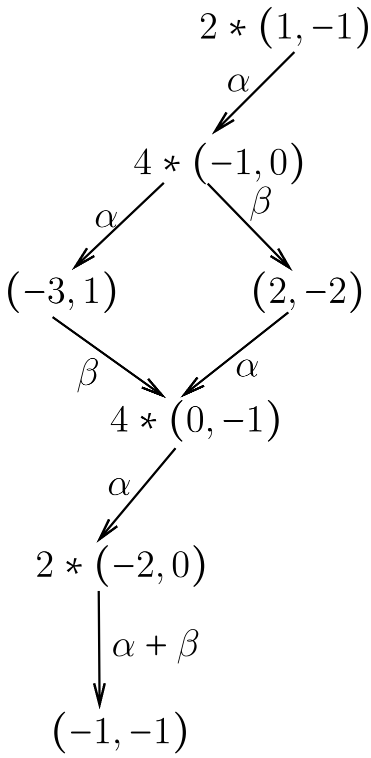


Figure 3.1:

It is easy to see that

$$\mathrm{Ext}^1_B((-2,0),(-1,-1)) = \mathrm{H}^1(B,(1,-1)) = 0,$$

so there is no extension from $(-1,-1)$ to $(-2,0)$. Also

$$\mathrm{Ext}^1_B((0,-1),(-2,0)) = \mathrm{H}^1(B,(-2,1)) = k,$$

hence there is a unique extension from $(-2,0)$ to $(0,-1)$ and this extension is given by $N_\alpha(0,-1)$. We get figure 3.2.

Now

$$\begin{aligned}\mathrm{Ext}^1_B((-3,1),N_\alpha(0,-1)) &= \mathrm{H}^1(B,N_\alpha(3,-2))\\ &= k.\end{aligned}$$

$$2 * (1,-1)$$

$$4 * (-1,0)$$

$$(-3,1) \qquad\qquad\qquad (2,-2)$$

$$2 * N_\alpha(0,-1) \oplus 2 * (0,-1)$$

Figure 3.2:

The above statement is true because we have a short exact sequence of modules given by

$$0 \to (1,-1) \to N_\alpha(3,-2) \to (3,-2) \to 0.$$

This gives rise to the long exact sequence of induction and we get $\mathrm{H}^1(B, N_\alpha(3,-2)) = \mathrm{H}^1(3,-2)$. By [11, 7.2(2)] we have $\mathrm{H}^1(-p^r\alpha) = k$ so $\mathrm{H}^1(B, N_\alpha(3,-2)) = k$. Similarly $\mathrm{Ext}^1_B((2,-2), N_\alpha(0,-1)) = \mathrm{H}^1(B, N_\alpha(-2,1)) = k$. Hence there are unique extensions from $N_\alpha(0,-1)$ to $(-3,1)$ and from $N_\alpha(0,-1)$ to $(2,-2)$ and these extensions are given by $N_{\beta\alpha}(-3,1)$ and $N_{\alpha\alpha}(2,-2)$ respectively. Moreover

$$\mathrm{Ext}^1_B((-1,0), N_{\alpha\alpha}(2,-2)) = k$$

and

$$\mathrm{Ext}^1_B((-1,0), N_{\beta\alpha}(-3,1)) = k.$$

Again we have a short exact sequence

$$0 \to N_\alpha(1,-1) \to N_{\beta\alpha}(-2,1) \to (-2,1) \to 0.$$

$$\frac{2 * (1,-1)}{(0,0) \oplus (-1,-1) \oplus 2 * (0,-1) \oplus N_{\alpha\beta\alpha}(-1,0) \oplus N_{\beta\alpha\alpha}(-1,0)}$$

Figure 3.3:

This gives rise to the long exact sequence of induction and we get

$$0 \to H^1(N_\alpha(1,-1)) \to H^1(N_{\beta\alpha}(-2,1)) \to H^1(-2,1) \to$$

$$H^2(N_\alpha(1,-1)) \to \ ...$$

Moreover we have a short exact sequence

$$0 \to (-1,0) \to N_\alpha(1,-1) \to (1,-1) \to 0.$$

Using the long exact sequence we get $\mathrm{H}^i(N_\alpha(1,-1)) = 0$ for all i. Using this we get $\mathrm{H}^1(N_{\beta\alpha}(-2,1)) = k$. Similarly we can show that $\mathrm{H}^1(N_{\alpha\alpha}(3,-2)) = k$ So there are unique extensions from $N_{\alpha\alpha}(2,-2)$ to $(-1,0)$ and from $N_{\beta\alpha}(-3,1)$ to $(-1,0)$ and these extensions are given by $N_{\beta\alpha\alpha}(-1,0)$ and $N_{\alpha\beta\alpha}(-1,0)$ respectively. We arrive at figure 3.3.

Now consider

$$\mathrm{Ext}^1_B((1,-1), N_{\beta\alpha\alpha}(-1,0)) = \mathrm{H}^1(B, N_{\beta\alpha\alpha}(-2,1)) = k$$

Also

$$\mathrm{Ext}^1_B((1,-1), N_{\alpha\beta\alpha}(-1,0)) = \mathrm{H}^1(B, N_{\alpha\beta\alpha}(-2,1)) = k$$

Hence there are unique extensions from $N_{\beta\alpha\alpha}(-1,0)$ to $(1,-1)$ and from $N_{\alpha\beta\alpha}(-1,0)$ to $(1,-1)$ and these extension are given by $N_{\alpha\beta\alpha\alpha}(1,-1)$ and $N_{\alpha\alpha\beta\alpha}(1,-1)$ respectively. Combine all the above results to get

$$Z_{(0,0)} = (0,0) \oplus (-1,-1) \oplus 2 * (0,-1) \oplus 2 * (-1,0)$$
$$\oplus N_{\alpha\alpha\beta\alpha}(1,-1) \oplus N_{\alpha\beta\alpha\alpha}(1,-1)$$

Remark. Note that there is another possible decomposition of $Z_{(0,1)}$ but that decomposition contradicts the famous conjecture of Donkin given in [13]. We assume that the conjecture is true hence we take the above decomposition.

(b) Now we will consider the module $Z_{(1,0)}$. Arranging the weights in descending order we get figure 3.4. There is a unique extension from $(-3,0)$ to $(-1,-1)$, because $\mathrm{Ext}^1_B((-1,-1),(-3,0)) = k$ and this extension is given by $N_\alpha(-1,-1)$. Also

$$\mathrm{Ext}^1_B((1,-2), N_\alpha(-1,-1)) = \mathrm{H}^1(B, N_\alpha(-2,1)) = k$$

So there is a unique extension from $N_\alpha(-1,-1)$ to $(1,-2)$ and this extension is given by $N_{\alpha\alpha}(1,-2)$, where $N_{\alpha\alpha}(1,-2)$ has weights $(1,-2),(-1,-1),(-3,0)$. Now we will check if there is an extension from $N_{\alpha\alpha}(1,-2)$ to $(-2,0)$.

$$\mathrm{Ext}^1_B((-2,0), N_{\alpha\alpha}(1,-2)) = \mathrm{H}^1(B, N_{\alpha\alpha}(3,-2)) = k$$

Hence there is a unique extension from $N_{\alpha\alpha}(1,-2)$ to $(-2,0)$ and this extension is given by $N_{\beta\alpha\alpha}(-2,0)$.

Now we will check to see if there is an extension from $N_{\beta\alpha\alpha}(-2,0)$ to $(0,-1)$. We have

$$\mathrm{Ext}^1_B((0,-1), N_{\beta\alpha\alpha}(-2,0)) = \mathrm{H}^1(B, N_{\beta\alpha\alpha}(-2,1)) = k.$$

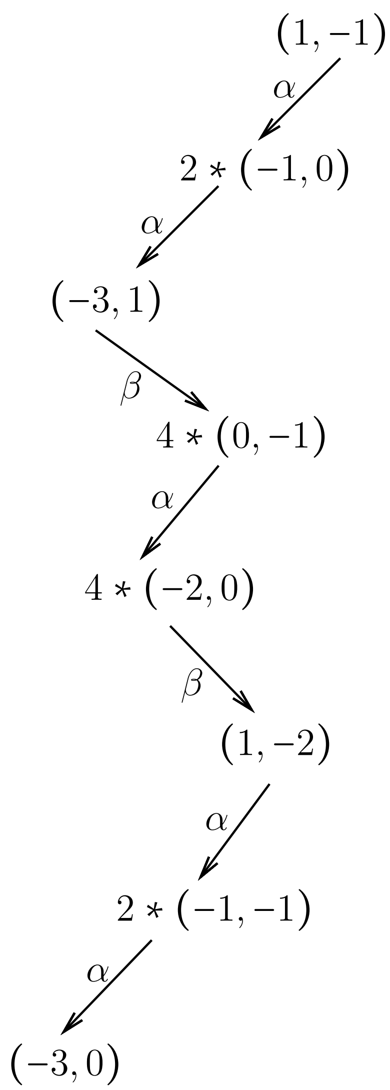


Figure 3.4:

$$(1,-1)$$

$$2*(-1,0)$$

$$(-3,1)$$

$$(-1,-1)\oplus N_{\alpha\beta\alpha\alpha}(0,-1)\oplus 3*N_{\alpha}(0,-1)$$

Figure 3.5:

This is true because we have a short exact sequence given by

$$0\to N_{\alpha}(-1,0)\to N_{\beta\alpha\alpha}(-2,1)\to N_{\beta}(-2,1)\to 0.$$

Also $\mathrm{H}^i(B,N_{\alpha}(-1,0))=0$ and $\mathrm{H}^1(B,N_{\beta}(-2,1))=k$, so using the long exact sequence of induction we get

$$\mathrm{H}^1(B,N_{\beta\alpha\alpha}(-2,1))=k,$$

so there is a unique extension from $N_{\beta\alpha\alpha}(-2,0)$ to $(0,-1)$ and this extension is given by $N_{\alpha\beta\alpha\alpha}(0,-1)$. Moreover

$$\mathrm{Ext}^1_B((0,-1),(-2,0))=\mathrm{H}^1(B,(-2,1))=k.$$

So there is a unique extension from $(-2,0)$ to $(0,-1)$ and this extension is given by $N_{\alpha}(0,-1)$. Combining these we get figure 3.5.

Now we will check if there is an extension from $N_{\alpha}(0,-1)$ to $(-3,1)$. We have

$$\mathrm{Ext}^1_B((-3,1),N_{\alpha}(0,-1))=\mathrm{H}^1(B,N_{\alpha}(3,-2))=k$$

The above statement is true because we have a short exact sequence of modules given by

$$0 \to (1,-1) \to N_\alpha(3,-2) \to (3,-2) \to 0.$$

This gives rise to the long exact sequence of induction and we get $\mathrm{H}^1(B, N_\alpha(3,-2)) = \mathrm{H}^1(3,-2)$. By [11, 7.2(2)] we have $\mathrm{H}^1(-p^r\alpha) = k$ so $\mathrm{H}^1(B, N_\alpha(3,-2)) = k$. Hence there is a unique extension from $N_\alpha(0,-1)$ to $(-3,1)$ and this extension is given by $N_{\beta\alpha}(-3,1)$. We also have

$$\mathrm{Ext}^1_B((-1,0), N_{\beta\alpha}(-3,1)) = \mathrm{H}^1(B, N_{\beta\alpha}(-2,1)).$$

Again we have a short exact sequence

$$0 \to N_\alpha(1,-1) \to N_{\beta\alpha}(-2,1) \to (-2,1) \to 0.$$

This gives rise to the long exact sequence of induction and we get

$$0 \to H^1(N_\alpha(1,-1)) \to H^1(N_{\beta\alpha}(-2,1)) \to H^1(-2,1) \to$$
$$H^2(N_\alpha(1,-1)) \to \;...$$

Moreover we have a short exact sequence

$$0 \to (-1,0) \to N_\alpha(1,-1) \to (1,-1) \to 0.$$

Using the long exact sequence we get $\mathrm{H}^i(N_\alpha(1,-1)) = 0$ for all i. Using this we get $\mathrm{H}^i(N_{\beta\alpha}(-2,1)) = k$. So there is a unique extension from $N_{\beta\alpha}(-3,1)$ to $(-1,0)$ and this extension is given by $N_{\alpha\beta\alpha}(-1,0)$, where $N_{\alpha\beta\alpha}(-1,0)$ is a submodules of $\nabla(1,0)$. Using this we get figure 3.6.

$$\frac{(1,-1)}{(-1,-1) \oplus (-1,0) \oplus N_{\alpha\beta\alpha\alpha}(0,-1) \oplus N_{\alpha\beta\alpha}(-1,0) \oplus 2 * N_{\alpha}(0,-1)}$$

Figure 3.6:

Finally

$$\mathrm{Ext}^1_B((1,-1), N_{\alpha\beta\alpha}(-1,0)) = \mathrm{H}^1(B, N_{\alpha\beta\alpha}(-2,1)) = k.$$

So there is a unique extension from $N_{\alpha\beta\alpha}(-1,0)$ to $(1,-1)$ and this extension is given by $N_{\alpha\alpha\beta\alpha}(1,-1)$, where $N_{\alpha\alpha\beta\alpha}(1,-1)$ is a submodules of $\nabla(1,0)$. Combining all the above results we get

$$Z_{(1,0)} = (-1,-1) \oplus (-1,0) \oplus N_{\alpha\beta\alpha\alpha}(0,-1)$$
$$\oplus N_{\alpha\alpha\beta\alpha}(1,-1) \oplus 2 * N_{\alpha}(0,-1)$$

(c) Now we will consider the module $Z_{(0,1)}$. Arranging the weights in descending order we get figure 3.7. There is a unique extension from $(0,-2)$ to $(-3,0)$, because $\mathrm{Ext}^1_B((-3,0),(0,-2)) = k$ and this extension is given by $N_{\beta}(-3,0)$. Also

$$\mathrm{Ext}^1_B((-1,-1), N_{\beta}(-3,0)) = \mathrm{H}^1(B, N_{\beta}(-2,1)) = k$$

So there is a unique extension from $N_{\beta}(-3,0)$ to $(-1,-1)$ and this extension is given by $N_{\alpha\beta}(-1,-1)$. Now we will check if there is an extension from $N_{\alpha\beta}(-1,-1)$ to $(1,-2)$.

$$\mathrm{Ext}^1_B((1,-2), N_{\alpha\beta}(-1,-1)) = \mathrm{H}^1(B, N_{\alpha\beta}(-2,1)) = k$$

Hence there is a unique extension from $N_{\alpha\beta}(-1,-1)$ to $(1,-2)$ and this extension is given by $N_{\alpha\alpha\beta}(1,-2)$. Also there is an extension

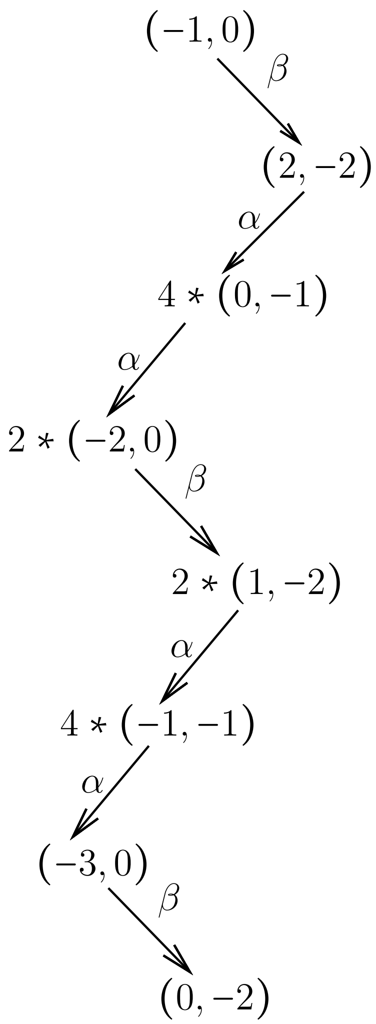


Figure 3.7:

$$(-1, 0)$$

$$(2, -2)$$

$$4 * (0, -1)$$

$$2 * (-2, 0)$$

$$2 * (-1, -1) \oplus N_{\alpha\alpha\beta}(1, -2) \oplus N_{\alpha}(1, -2)$$

Figure 3.8:

from $(-1, -1)$ to $(1, -2)$ and this extension is by given $N_{\alpha}(1, -2)$. Combining above results we arrive at figure 3.8.

Now we will check to see if there is an extension from $N_{\alpha\alpha\beta}(1, -2)$ to $(-2, 0)$. We have

$$\mathrm{Ext}^1_B((-2, 0), N_{\alpha\alpha\beta}(1, -2)) = \mathrm{H}^1(B, N_{\alpha\alpha\beta}(3, -2)) = k.$$

So there is a unique extension from $N_{\alpha\alpha\beta}(1, -2)$ to $(-2, 0)$ and this extension is given by $N_{\beta\alpha\alpha\beta}(-2, 0)$. Also

$$\mathrm{Ext}^1_B((-2, 0), N_{\alpha}(1, -2)) = \mathrm{H}^1(B, N_{\alpha}(3, -2)) = k$$

Hence there is a unique extension from $N_{\alpha}(1, -2)$ to $(-2, 0)$ and this extension is given by $N_{\beta\alpha}(-2, 0)$. We get figure 3.9.

We also have

$$\mathrm{Ext}^1_B((0, -1), N_{\beta\alpha}(-2, 0)) = \mathrm{H}^1(B, N_{\beta\alpha}(-2, 1)) = k.$$

$$(-1,0)$$

$$(2,-2)$$

$$4*(0,-1)$$

$$2*(-1,-1)\oplus N_{\alpha\alpha\beta}(1,-2)\oplus N_{\beta\alpha}(-2,0)$$

Figure 3.9:

$$(-1,0)$$

$$(2,-2)$$

$$2*(-1,-1)\oplus 2*(0,-1)\oplus((-1,-1)\otimes M)\oplus N_{\alpha\beta\alpha}(-2,0)$$

Figure 3.10:

So there is a unique extension from $N_{\beta\alpha}(-2,0)$ to $(0,-1)$ and this extension is given by $N_{\alpha\beta\alpha}(0,-1)$. Moreover

$$\mathrm{Ext}^1_B((0,-1),N_{\beta\alpha\alpha\beta}(-2,0))=\mathrm{H}^1(B,N_{\beta\alpha\alpha\beta}(-2,1))=k.$$

So there is a unique extension from $N_{\beta\alpha\alpha\beta}(-2,0)$ to $(0,-1)$ and this extension is given by $(-1,-1)\otimes M$, where $M=\nabla(1,0)/(-1,0)$. Using this we get figure 3.10.

Now we will check if there is an extension from $N_{\alpha\beta\alpha}(0,-1)$ to

$$\frac{(-1,0)}{2*(-1,-1)\oplus 2*(0,-1)\oplus((-1,-1)\otimes M)\oplus N_{\alpha\alpha\beta\alpha}(2,-2)}$$

Figure 3.11:

$(2,-2)$. We have

$$\mathrm{Ext}^1_B((2,-2),N_{\alpha\beta\alpha}(0,-1)) = \mathrm{H}^1(B,N_{\alpha\beta\alpha}(-2,1)) = k.$$

So there is a unique extension from $N_{\alpha\beta\alpha}(0,-1)$ to $(2,-2)$ and this extension is given by $N_{\alpha\alpha\beta\alpha}(2,-2)$. Combining all the above results we get figure 3.11.

Finally we will check if there is an extension from $N_{\alpha\alpha\beta\alpha}(2,-2)$ to $(-1,0)$. We have

$$\mathrm{Ext}^1_B((-1,0),N_{\alpha\alpha\beta\alpha}(2,-2)) = \mathrm{H}^1(B,N_{\alpha\alpha\beta\alpha}(3,-2)) = k.$$

So there is a unique extension from $N_{\alpha\alpha\beta\alpha}(2,-2)$ to $(-1,0)$ and this extension is given by $(0,-1)\otimes N$, where $\nabla(1,0)/N = (1,0)$. Combining all the above results we get

$$\begin{aligned} Z_{(0,1)} &= 2*(-1,-1)\oplus 2*(0,-1)\oplus((-1,-1)\otimes M) \\ &\oplus((0,-1)\otimes N) \end{aligned}$$

Remark. Note that there is another possible decomposition of $Z_{(0,1)}$ but that decomposition contradicts the famous conjecture of Donkin given in [13]. We assume that the conjecture is true hence we take the above decomposition.

(d) Finally for the module $Z_{(1,1)}$ we have

$$Z_{(1,1)} = (-1,-1) \oplus (-1,-1) \otimes T(1,0)$$

Combining the results in (a),(b),(c) and (d), we get

$$\begin{aligned} \mathrm{H}^0(U_1, \mathrm{St} \otimes \mathrm{St}) = {} & [(0,0) \oplus (-1,-1) \oplus 2*(0,-1) \oplus 2*(-1,0) \\ & \oplus N_{\alpha\alpha\beta\alpha}(1,-1) \oplus N_{\alpha\beta\alpha\alpha}(1,-1)]^F \\ & \oplus (0,1) \otimes [2*(-1,-1) \oplus 2*(0,-1) \oplus ((-1,-1) \otimes M) \\ & \oplus ((0,-1) \otimes N)]^F \oplus (1,0) \otimes [(-1,-1) \oplus (-1,0) \oplus N_{\alpha\beta\alpha\alpha}(0,-1) \\ & \oplus N_{\alpha\alpha\beta\alpha}(1,-1) \oplus 2*N_\alpha(0,-1)]^F \\ & \oplus (1,1) \otimes [(-1,-1) \oplus (-1,-1) \otimes T(1,0)]^F \end{aligned}$$

Lemma 3.2.1. *We have:*

1. $\mathrm{H}^0(U_1, L(0,0) \otimes \mathrm{St}) = (-1,-1)$.

2. $\mathrm{H}^0(U_1, L(1,0) \otimes \mathrm{St}) = [(-1,0) \oplus (0,-1)]^F \oplus (0,1) \otimes [(0,-1)$
$$\oplus(-1,-1)]^F \oplus (1,0) \otimes N_\alpha(0,-1)^F.$$

3. $\mathrm{H}^0(U_1, L(0,1) \otimes \mathrm{St}) = [N_\alpha(0,-1) \oplus N_\alpha(1,-1)]^F \oplus (0,1) \otimes N_{\alpha\beta\alpha}(0,-1)^F$
$$\oplus(1,0) \otimes [(-1,-1) \oplus N_\alpha(0,-1) \oplus (-1,0)]^F \oplus 2(1,1) \otimes (-1,-1)^F.$$

4. $\mathrm{H}^0(U_1, \mathrm{St}\otimes\mathrm{St}) = [(0,0)\oplus(-1,-1)\oplus 2*(0,-1)\oplus 2*(-1,0)\oplus N_{\alpha\alpha\beta\alpha}(1,-1)$
$$\oplus N_{\alpha\beta\alpha\alpha}(1,-1)]^F \oplus (0,1) \otimes [2*(-1,-1) \oplus 2*(0,-1) \oplus ((-1,-1) \otimes M)$$
$$\oplus((0,-1) \otimes N)]^F \oplus (1,0) \otimes [(-1,-1) \oplus (-1,0) \oplus N_{\alpha\beta\alpha\alpha}(0,-1)$$
$$\oplus N_{\alpha\alpha\beta\alpha}(1,-1)\oplus 2*N_\alpha(0,-1)]^F \oplus (1,1)\otimes[(-1,-1)\oplus(-1,-1)\otimes T(1,0)]^F.$$

Now we will write the recursive formulas obtained from the above lemma. We have the following cases:

1. Suppose $(a,b) = (2r,2s)$, where r,s are integers then the equation 3.2 becomes

$$\begin{aligned}\chi^i(2r,2s) &= \chi^i(H^0(B_1,T(2,2)'\otimes(2r,2s))^{-1})^F\\ &+\chi(1,0)\chi^i(H^0(B_1,T(1,2)\otimes(2r,2s))^{-1})^F\\ &+\chi(0,1)\chi^i(H^0(B_1,P_{(2,1)}\otimes(2r,2s))^{-1})^F\\ &+\chi(1,1)\chi^i(H^0(B_1,P_{(1,1)}\otimes(2r,2s))^{-1})^F\\ &-2\chi^i(H^0(B_1,T(2,1)\otimes(2r,2s))^{-1})^F\end{aligned}$$

We will calculate each term separately.

(a) First we will take $\chi^i(H^0(B_1,T(2,2)'\otimes(2r,2s))^{-1})$

$$\begin{aligned}\chi^i(H^0(B_1,T(2,2)'\otimes(2r,2s))^{-1}) &= \chi^i(H^0(U_1,P'_{(2,2)}\otimes(2r,2s))^{-1})^{T_1}\\ &=\chi^i((2r,2s)\otimes H^0(U_1,T(2,2)')^{T_1})^{-1}\\ &=\chi^i(r,s)+\chi^i(r-1,s-1)+2\chi^i(r,s-1)+2\chi^i(r-1,s)\\ &+\chi^i(N_{\alpha\alpha\beta\alpha}(r+1,s-1))+\chi^i(N_{\alpha\beta\alpha\alpha}(r+1,s-1)).\end{aligned}$$

(b) Now take $\chi^i(H^0(B_1,T(1,2)\otimes(2r,2s))^{-1})$

$$\begin{aligned}\chi^i(H^0(B_1,T(1,2)\otimes(2r,2s))^{-1}) &= \chi^i(H^0(U_1,T(1,2)\otimes(2r,2s))^{-1})^{T_1}\\ &=\chi^i((2r,2s)\otimes H^0(U_1,L(0,1)\otimes\mathrm{St})^{T_1})^{-1}\\ &=\chi^i_\alpha(r,s-1)+\chi^i_\alpha(r+1,s-1).\end{aligned}$$

(c) We will now calculate $\chi^i(H^0(B_1, T(2,1) \otimes (2r,2s))^{-1})$

$$\chi^i(H^0(B_1, T(2,1) \otimes (2r,2s))^{-1}) = \chi^i(H^0(U_1, T(2,1) \otimes (2r,2s))^{-1})^{T_1}$$
$$= \chi^i((2r,2s) \otimes H^0(U_1, L(1,0) \otimes \mathrm{St})^{T_1})^{-1}$$
$$= \chi^i(r-1,s) + \chi^i(r,s-1).$$

(d) Finally take $\chi^i(H^0(B_1, T(1,1) \otimes (2r,2s))^{-1})$

$$\chi^i(H^0(B_1, T(1,1) \otimes (2r,2s))^{-1}) = \chi^i(H^0(U_1, T(1,1) \otimes (2r,2s))^{-1})^{T_1}$$
$$= \chi^i((2r,2s) \otimes H^0(U_1, \mathrm{St})^{T_1})^{-1} = 0.$$

We will denote the character of $\chi^i(N_{\alpha\alpha\beta}(r,s))$ by $\chi^i_{\alpha\alpha\beta}$ and $\chi^i(N_{\alpha\beta\alpha}(r,s))$ by $\chi^i_{\alpha\beta\alpha}$. Combine the above results to get

$$\begin{aligned}\chi^i(2r,2s) &= \chi^i(r,s) + \chi^i(r-1,s-1) + 2\chi^i(r,s-1) + 2\chi^i(r-1,s)\\ &+ \chi^i(N_{\alpha\alpha\beta\alpha}(r+1,s-1)) + \chi^i(N_{\alpha\beta\alpha\alpha}(r+1,s-1)) + \chi^i_\alpha(r,s-1)^F\\ &+ \chi^i_\alpha(r+1,s-1)^F + \chi^i(r-1,s)^F + \chi^i(r,s-1)^F\\ &- 2\chi^i(r-1,s)^F - 2\chi^i(r,s-1)^F\\ &= \chi^i(r,s)^F + \chi^i_{\alpha\alpha\beta}(r+1,s-1)^F + \chi^i_{\alpha\beta\alpha}(r+1,s-1)^F\\ &+ \chi^i_\alpha(r,s-1)^F + \chi^i_\alpha(r+1,s-1)^F + \chi^i(r-1,s)^F + \chi^i(r,s-1)^F.\end{aligned}$$

Similarly we can get $\chi^i(2r,1+2s), \chi^i(1+2r,2s)$ and $\chi^i(1+2r,1+2s)$. The complete result is given in the following lemma.

Lemma 3.2.2. *For $i \geq 0$, integers r,s and α,β simple roots we have:*

1. $$\chi^i(2r,2s) = \chi^i(r,s)^F + \chi^i(r-1,s-1) + \chi^i_{\alpha\alpha\beta\alpha}(r+1,s-1)^F$$
$$+\chi^i_{\alpha\beta\alpha\alpha}(r+1,s-1)^F + \chi^i_\alpha(r,s-1)^F + \chi^i_\alpha(r+1,s-1)^F + \chi^i(r-1,s)^F$$
$$+\chi^i(r,s-1)^F;$$

2. $\chi^i(2r, 1+2s) = \chi^i((r-1, s-1) \otimes M)^F + \chi^i((r, s-1) \otimes N)^F$

$$+\chi^i(r, s-1)^F + \chi^i(r-1, s-1)^F + \chi^i_{\alpha\beta\alpha}(r, s-1)^F;$$

3. $\chi^i(1+2r, 2s) = 2\chi^i(r-1, s-1)^F + 2\chi^i(r-1, s)^F + \chi^i_{\alpha\beta\alpha\alpha}(r, s-1)^F$

$$+\chi^i_{\alpha\alpha\beta\alpha}(r+1, s-1)^F + 2\chi^i_{\alpha}(r, s-1)^F;$$

4. $\chi^i(1+2r, 1+2s) = 4\chi^i(r-1, s-1)^F + \chi^i((-1, -1) \otimes T(1, 0))^F.$

3.3 Conclusion

In this chapter we have given the first step of the recursive description for $\chi^i(r, s)$. As it is clear from the above lemma that there are some new modules appearing in this description. The next step is to find a description, recursive or otherwise, for these modules. Unfortunately it is very hard to find this description using the methods available to us. There is also scope to repeat this process for higher characteristics but we have not tried it yet.

Chapter 4

A Tensor Product Factorization for Certain Tilting Modules

4.1 Introduction

The contents of this chapter have been published see [1]. Let G be a semisimple, simply connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$. In this chapter we will give a generalization of the results given by Doty in [17]. In his paper [17], Doty observed that the tensor product of the Steinberg module with a minuscule module is always indecomposable tilting. We will show that the tensor product of the Steinberg module with a module whose dominant weights are modular minuscule is a tilting module, not always indecomposable. We also give the decomposition of such a module into indecomposable tilting modules. Doty also proved that if $p \geq 2h - 2$, then for $r-$minuscule weights the tilting module is isomorphic to a tensor product of two simple modules, usually in many ways. We remove the characteristic restriction on this result. A generalization of

[11, proposition 5.5(i)] for (p,r)−minuscule weights is also given. We start by setting up notation and stating some important definitions and results which will be useful later on. For further explanation please see chapter 1.

Let $F : G \to G$ be the Frobenius morphism of G. Let M be a $G-$ module. In this chapter we will denote M^{F^r} by $M^{[r]}$, $r \geq 1$. Let B be a Borel subgroup of G and $T \subset B$ be a maximal torus of G. Recall that $X_r(T)$ denotes the set of $r-$restricted weights. Moreover $\Delta(\lambda)$ and $L(\lambda)$ denote the Weyl module and the simple module of highest weight λ respectively. Let $T(\lambda)$ denote the tilting module with highest weight λ.

Recall also that for $\lambda \in X_r(T)$, the modules $L(\lambda)$ form a complete set of pairwise non-isomorphic irreducible G_r-modules. For $\mu \in X(T)$ let $\hat{Q}_r(\mu)$ denote the projective cover of $L(\mu)$ as a G_rT-module see e.g [21] and [22]. The modules $\hat{Q}_r(\lambda)$, $\lambda \in X_r(T)$, form a complete set of pairwise non-isomorphic projective G_r-modules.

A dominant weight λ is called minuscule if the weights of $\Delta(\lambda)$ form a single orbit under the (usual) action of W. Equivalently, by [6, chapter VIII, Section 7, proposition 6(iii)], λ is minuscule if $-1 \leq (\lambda, \alpha^\vee) \leq 1$ for all roots α. If $s(\lambda) = \sum_{\mu \in W\lambda} e(\mu)$ then λ minuscule implies $s(\lambda) = \operatorname{ch}\Delta(\lambda) = \operatorname{ch}\nabla(\lambda) = \operatorname{ch} L(\lambda)$. For $\lambda \in X^+(T)$ define λ to be modular minuscule if $\langle \lambda, \beta_0^\vee \rangle \leq p$, where β_0 is the highest short root. Moreover we define a weight $\lambda \in X_r(T)$ to be (p,r)-minuscule if $\lambda = \sum_{j=0}^{r-1} p^j \lambda^j$, where each λ^j is modular minuscule (and $\lambda^j \in X_1(T)$). In [17] Doty defines a weight λ to be $r-$minuscule if $\lambda = \sum_{j=0}^{r-1} p^j \lambda^j$, with each λ^j minuscule. Note that λ minuscule implies λ is modular minuscule. Similarly if λ is $r-$minuscule then λ is $(p,r)-$minuscule.

4.2 Main Results

Definition 4.2.1. *For* $\lambda = \sum_{j=0}^{r-1} p^j \lambda^j \in X_r(T)$, $\lambda^j \in X_1(T)$ *define*

$$s_r(\lambda) = s(\lambda^0)s(p\lambda^1)\dots s(p^r\lambda^{r-1}).$$

Proposition 4.2.1. *If* λ *is* $(p,r)-$*minuscule then*

$$\operatorname{ch} T((p^r-1)\rho+\lambda) = \chi((p^r-1)\rho)s_r(\lambda),$$

where $\chi(\lambda)$ *denotes the Weyl character corresponding to* $\lambda \in X^+(T)$ *as given in [21, II, proposition 5.10].*

Proof. By [10, theorem 5.3] we have if $\lambda \in X_1(T)$ and $T((p-1)\rho+\lambda)|_{G_1}$ is indecomposable then $T((p-1)\rho+\lambda)\otimes T(\mu)^{[1]} \simeq T((p-1)\rho+\lambda+p\mu)$ for all $\mu \in X^+(T)$. Also by the argument of [10, proposition 5.5] for modular minuscule (and restricted) λ we get that $T((p-1)\rho+\lambda)|_{G_1}$ is indecomposable. So we have $T((p^r-1)\rho+\lambda) = \bigotimes_{j=0}^{r-1} T((p-1)\rho+\lambda^j)^{[j]}$. So $\operatorname{ch} T((p^r-1)\rho+\lambda) = \Pi_{j=0}^{r-1} \operatorname{ch} T((p-1)\rho+\lambda^j)^{[j]}$. Since each λ^j is modular minuscule by [10, proposition 5.5] we get $\operatorname{ch} T((p-1)\rho+\lambda^j) = \chi((p-1)\rho)s(\lambda^j)$. Hence $\operatorname{ch} T((p^r-1)\rho+\lambda) = \Pi_{j=0}^{r-1}\chi((p-1)\rho)^{[j]}s(\lambda^j)^{[j]}$. Combine this with the above definition to get the result.

□

Remark. If λ is minuscule then $s(\lambda) = \operatorname{ch} L(\lambda)$ and hence $T((p-1)\rho+\lambda) \simeq \mathrm{St}\otimes L(\lambda)$ because these are tilting modules with the same character. This gives us [17, lemma].

Lemma 4.2.2. *We have the following:*

1. *If* $\mu \in X^+(T)$ *then* $T((p^r-1)\rho) \otimes T(\mu)^{[r]} \simeq T((p^r-1)\rho + p^r\mu)$.

2. *Suppose* λ *is minuscule then*

$$\mathrm{St} \otimes L(\lambda) \simeq \hat{Q}_1((p-1)\rho + w_0\lambda),$$

as G_1T*−modules, where* w_0 *is the longest element of* W*. In particular* $\mathrm{St} \otimes L(\lambda)|_{G_1}$ *is indecomposable.*

3. *If* λ *is minuscule and* $\mu \in X^+(T)$ *then*

$$\mathrm{St} \otimes L(\lambda) \otimes T(\mu)^{[r]} \simeq T((p-1)\rho + \lambda + p^r\mu).$$

Proof. By [21, II, 3.19] with $i = 0$ we have $\mathrm{St}_r \otimes \nabla(\mu)^{[r]} \simeq \nabla((p^r-1)\rho + p^n\mu)$ for every $\mu \in X^+(T)$. It follows that $\mathrm{St}_r \otimes V^{[r]}$ is tilting for every tilting module V. In particular $\mathrm{St}_r \otimes T(\mu)^{[r]}$ is tilting. By [13, 2.1], $\mathrm{St}_r \otimes T(\mu)^{[r]}$ is isomorphic to $T((p^r-1)\rho + p^r\mu)$. This proves part (1).

Since

$$\begin{aligned}\mathrm{Hom}_{G_1T}&(L((p-1)\rho + w_0\lambda), \mathrm{St} \otimes L(\lambda)) \\ &= \mathrm{Hom}_{G_1T}(L((p-1)\rho + w_0\lambda) \otimes L(\lambda)^*, \mathrm{St}) \\ &= \mathrm{Hom}_{G_1T}(L((p-1)\rho + w_0\lambda) \otimes L(-w_0\lambda), \mathrm{St}) \neq 0.\end{aligned}$$

we have

$$\mathrm{St} \otimes L(\lambda)|_{G_1} = \hat{Q}_1((p-1)\rho + w_0\lambda) \oplus Z.$$

Also by [11, 1.2(2)], $\mathrm{ch}\,\hat{Q}_1((p-1)\rho + w_0\lambda) = \chi((p-1)\rho)\psi$, where $\psi = \sum a_\xi e(\xi)$ and $a_\xi \geq 0$ for all ξ.
Also by [21, II, 11.7, lemma(c)], $\mathrm{ch}\,\hat{Q}_1((p-1)\rho + w_0\lambda)$ is W invariant. This implies ψ is W invariant. Moreover $\hat{Q}_1((p-1)\rho + w_0\lambda)$ has unique highest weight $(p-1)\rho + \lambda$, so $\psi = s(\lambda) + \theta$ where $\theta = \sum_{\mu \in X^+(T)} a_\mu s(\mu)$ with $\mu < \lambda$. But ψ

is W invariant and $\operatorname{ch}\hat{Q}_1((p-1)\rho+w_0\lambda)$ is divisible by $\chi((p-1)\rho)$ so we must have $\psi = s(\lambda)$. So we get $Z = 0$ and $\operatorname{ch}\hat{Q}_1((p-1)\rho + w_0\lambda) = \operatorname{ch}(\mathrm{St}\otimes L(\lambda))$. This proves that

$$\mathrm{St}\otimes L(\lambda) \simeq \hat{Q}_1((p-1)\rho + w_0\lambda).$$

Now by [22, 4.2, Satz], $\hat{Q}_1((p-1)\rho + w_0\lambda)$ is indecomposable as G_1–module, so $\mathrm{St}\otimes L(\lambda)$ is indecomposable as G_1–module. Hence $\mathrm{St}\otimes L(\lambda)|_{G_1}$ is indecomposable. This proves part (2).

Since $\mathrm{St}\otimes L(\lambda)|_{G_1}$ is indecomposable by [13, 2.1] we get

$$\mathrm{St}\otimes L(\lambda)\otimes T(\mu)^{[r]} \simeq T((p-1)\rho + \lambda + p^r\mu).$$

This gives us the result in part (3).

□

Proposition 4.2.3. *Suppose λ is r–minuscule and $\mu \in X^+(T)$ then*

$$\mathrm{St}_r\otimes L(\lambda)\otimes T(\mu)^{[r]} \simeq T((p^r-1)\rho + \lambda + p^r\mu).$$

Proof. Using Steinberg's tensor product theorem we get

$$\mathrm{St}_r\otimes L(\lambda) \simeq \bigotimes_{j=0}^{r-1}(\mathrm{St}\otimes L(\lambda^j))^{[j]}$$

where λ is r–minuscule. By remark 4.2, we have

$$\mathrm{St}_r\otimes L(\lambda) \simeq \bigotimes_{j=0}^{r-1}(T((p-1)\rho + \lambda^j))^{[j]}.$$

Apply lemma 4.2.2(3) inductively to get

$$\mathrm{St}_r\otimes L(\lambda) \simeq T((p^r-1)\rho + \lambda).$$

Now tensor both sides by $T(\mu)^{[r]}$ and apply lemma 4.2.2(3) again to get the result.

□

Corollary 4.2.4. *Let λ be $r-$minuscule and $\mu \in X^+(T)$ then:*

1. $T((p^r-1)\rho + p^r\mu) \otimes L(\lambda) \simeq T((p^r-1)\rho + \lambda + p^r\mu)$.

2. *If* $T(\mu)$ *is simple then* $\mathrm{St}_r \otimes L(p^r\mu + \lambda) \simeq T((p^r-1)\rho + p^r\mu + \lambda)$.

Proof. By lemma 4.2.2(1) we get $\mathrm{St}_r \otimes T(\mu)^{[r]} \simeq T((p^r-1)\rho + p^r\mu)$. Tensor this with $L(\lambda)$ to get the result in part (1).

If $T(\mu)$ is simple then $L(\mu) \simeq T(\mu)$. So $L(\lambda) \otimes T(\mu)^{[r]} \simeq L(\lambda) \otimes L(\mu)^{[r]}$. Using Steinberg's tensor product theorem we get $L(\lambda) \otimes L(\mu)^{[r]} \simeq L(\lambda + p^r\mu)$. Tensor this with the $r-$th Steinberg module to get the result in part (2). □

Remark. Note that proposition 4.2.4 and its corollary remove the restriction on p in the corresponding results in [17].

In case λ is modular minuscule it is of interest to determine the decomposition $\mathrm{St}\otimes\mathrm{L}(\lambda)$, $\mathrm{St}\otimes\Delta(\lambda)$ and $\mathrm{St}\otimes\nabla(\lambda)$ as a direct sum of indecomposable modules. In what follows we will show that these are all tilting modules and the direct sum decomposition is determined by the characters of $\nabla(\lambda)$ and $\mathrm{L}(\lambda)$. We will also show that if λ is $(p,r)-$minuscule then $\mathrm{St_r} \otimes \mathrm{L}(\lambda)$ is tilting. We will also give decomposition of $St_r \otimes \mathrm{L}(\lambda)$ into indecomposable tilting modules.

Lemma 4.2.5. *Suppose λ is modular minuscule. Then every weight μ of $V(\lambda)$ satisfies $p\rho + \mu \in X^+(T)$, where $V(\lambda) = \Delta(\lambda)$ or $\nabla(\lambda)$.*

Proof. If τ is a dominant weight of $V(\lambda)$ then τ is also modular minuscule because λ is the highest weight so $\tau \leq \lambda$ and we can write $\lambda = \tau + \theta$ where θ is a sum of positive roots. Also $p \geq \langle \lambda, \beta_0^\vee \rangle = \langle \tau, \beta_0^\vee \rangle + \langle \theta, \beta_0^\vee \rangle \geq \langle \tau, \beta_0^\vee \rangle$.

Let μ be a weight of $V(\lambda)$ then $w\mu = \tau$ for some modular minuscule $\tau \in X^+(T)$ and $w \in W$. Let α be a simple root then $\langle p\rho + \mu, \alpha^\vee \rangle = p + \langle w^{-1}\tau, \alpha^\vee \rangle = p + \langle \tau, (w\alpha)^\vee \rangle$. So we need to show that $p + \langle \tau, \gamma^\vee \rangle \geq 0$ for all roots γ.

Now $p + \langle \tau, \gamma^\vee \rangle \geq 0$ for all roots $\gamma \iff p + \langle \tau, (w_0\gamma)^\vee \rangle \geq 0$ for all roots γ. And this is true $\iff p + \langle w_0\tau, \gamma^\vee \rangle \geq 0 \iff p - \langle -w_0\tau, \gamma^\vee \rangle \geq 0 \iff p - \langle \tau, \gamma^\vee \rangle \geq 0$. From the last inequality we get $\langle \tau, \gamma^\vee \rangle \leq p$ and since $\langle \tau, \gamma^\vee \rangle \leq \langle \tau, \beta_0^\vee \rangle \leq p$ we have the required result.

□

Recall that if $0 = M_0 \leq M_1 \leq ... \leq M_t = M$ is a chain of B−modules and $R\mathrm{Ind}_B^G M_i/M_{i-1} = 0, 1 \leq i \leq t$ then for $\mathrm{Ind}_B^G M$ we have a sequence $0 = \mathrm{Ind}_B^G M_0 \leq \mathrm{Ind}_B^G M_1 \leq ... \leq \mathrm{Ind}_B^G M_t = \mathrm{Ind}_B^G M$ with $\mathrm{Ind}_B^G M_i / \mathrm{Ind}_B^G M_{i-1} \simeq \mathrm{Ind}_B^G M_i/M_{i-1}$. This follows by induction on t. Recall also that $R\mathrm{Ind}_B^G \mu = 0$ if $\langle \mu, \alpha^\vee \rangle \geq -1$ for all simple roots α. This follows by Kempf's vanishing theorem and [21, II, proposition 5.4(a)].

Proposition 4.2.6. *Assume λ is modular minuscule and let V be a finite dimensional G−module such that $\mu \leq \lambda$ for all weights μ of V. Then* $\mathrm{St} \otimes V$ *is a tilting module.*

Proof. We will show that $\mathrm{St} \otimes V$ has a ∇−filtration. Let μ be a weight of V, then μ is a weight of some composition factor $L(\nu)$ of V. Now $\nu \leq \lambda$, so $\langle \nu, \beta_0^\vee \rangle \leq \langle \lambda, \beta_0^\vee \rangle \leq p$, therefore ν is modular minuscule. Moreover μ is a weight of $L(\nu)$ and hence of $\nabla(\nu)$ and so by lemma 4.2.5 we have $p\rho + \mu \in X^+(T)$.

Now choose a B−module filtration of V given by $0 = V_0 \leq V_1 \leq ... \leq V_t = V$ with $V_i/V_{i-1} \simeq k_{\mu_i}$ where μ_i is a weight of V. Then $\mathrm{St} \otimes V = \mathrm{Ind}_B^G((p-1)\rho \otimes V)$ and $(p-1)\rho \otimes V$ has a filtration $0 = (p-1)\rho \otimes V_0 \leq (p-1)\rho \otimes V_1 \leq ... \leq (p-1)\rho \otimes V_t = (p-1)\rho \otimes V$.

Also for each section $(p-1)\rho \otimes V_i/V_{i-1}$ we have $R\mathrm{Ind}_B^G((p-1)\rho \otimes V_i/V_{i-1}) = R\mathrm{Ind}_B^G((p-1)\rho \otimes k_{\mu_i}) = R\mathrm{Ind}_B^G((p-1)\rho+\mu_i) = 0$ because $\langle (p-1)\rho+\mu_i, \alpha^\vee\rangle \geq -1$. So $\mathrm{St} \otimes V$ has filtration with section

$$\mathrm{Ind}_B^G((p-1)\rho \otimes V_i/V_{i-1}) = \begin{cases} \nabla(\mu_i), & \mu_i \in X^+(T) \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\mathrm{St} \otimes V$ has a ∇–filtration. Also $\mu^* \leq \lambda^*$ for all weights μ^* of V^* and λ^* is modular minuscule. So $\mathrm{St} \otimes V^*$ has a ∇–filtration. Therefore $(\mathrm{St} \otimes V^*)^* = \mathrm{St} \otimes V$ has a Δ–filtration. Hence $\mathrm{St} \otimes V$ is tilting.

□

Corollary 4.2.7. *Suppose λ is modular minuscule then* $\mathrm{St}\otimes\Delta(\lambda) \simeq \mathrm{St}\otimes\nabla(\lambda)$.

Proof. By proposition 4.2.6, $\mathrm{St} \otimes \Delta(\lambda)$ and $\mathrm{St} \otimes \nabla(\lambda)$ are tilting modules. Moreover $\mathrm{St} \otimes \Delta(\lambda)$ and $\mathrm{St} \otimes \nabla(\lambda)$ have the same character and hence are isomorphic. □

Theorem 4.2.8. *Let λ be modular minuscule and V be a finite dimensional G–module such that $\mu \leq \lambda$ for all weights μ of V. Then*

$$\mathrm{St} \otimes V \simeq \bigoplus_{\nu \in X^+(T)} a_\nu T((p-1)\rho + \nu)$$

where $\mathrm{ch}(V) = \sum_{\nu \in X^+(T)} a_\nu s(\nu)$.

Proof. By proposition 4.2.6 we have $\mathrm{St} \otimes V$ is a tilting module. Also by [11, proposition 5.5] we get $\mathrm{ch}\, T((p-1)\rho + \nu) = \chi((p-1)\rho) s(\nu)$. Write $\mathrm{ch}\,(V) = \sum_{\nu \in X^+(T)} a_\nu s(\nu)$ then the tilting modules $\mathrm{St} \otimes V$ and $\oplus_{\nu \in X^+(T)} a_\nu T((p-1)\rho+\nu)$ have the same character and hence are isomorphic.

□

Proposition 4.2.9. *Assume λ is (p,r)−minuscule then* $\mathrm{St_r} \otimes L(\lambda)$ *is a tilting module.*

Proof. Since λ is (p,r)−minuscule this implies $\lambda \in X_r(T)$ and $\lambda = \sum_{j=0}^{r-1} p^j \lambda^j$, where each λ^j is modular minuscule. Using Steinberg tensor product theorem we have $\mathrm{St_r} \otimes L(\lambda) = \bigotimes_{j=0}^{r-1} (\mathrm{St} \otimes L(\lambda^j))^{[j]}$. By proposition 4.2.6, $\mathrm{St} \otimes L(\lambda^j)$ is tilting for each λ^j. We will use mathematical induction to complete the proof.

Write $\mathrm{St_r} \otimes L(\lambda) = \mathrm{St} \otimes L(\lambda^0) \otimes (\mathrm{St} \otimes L(\lambda^1) \otimes \mathrm{St}^{[1]} \otimes L(\lambda^2)^{[1]} \otimes ... \otimes \mathrm{St}^{[r-2]} \otimes L(\lambda^{r-1})^{[r-2]})^{[1]}$. Then using inductive hypothesis and theorem 4.2.8 we have $\mathrm{St_r} \otimes L(\lambda) = \bigoplus_\mu a_\mu \mathrm{St} \otimes L(\lambda^0) \otimes T(\mu)^{[1]}$. Also by theorem 4.2.8, $\mathrm{St} \otimes L(\lambda^0) = \bigoplus_{\nu \in X^+(T)} b_\nu T((p-1)\rho+\nu)$. So $\mathrm{St_r} \otimes L(\lambda) = \bigoplus_{\mu,\nu} a_\mu b_\nu T((p-1)\rho+\nu) \otimes T(\mu)^{[1]}$. Hence $\mathrm{St_r} \otimes L(\lambda)$ is tilting. □

Theorem 4.2.10. *Let λ be (p,r)−minuscule then*

$$\mathrm{St_r} \otimes L(\lambda) \simeq \bigoplus_{\nu \in X^+(T)} b_\nu T((p^r-1)\rho+\nu)$$

where $\mathrm{ch}\, L(\lambda) = \sum_{\nu \in X^+(T)} b_\nu s_r(\nu)$.

Proof. $\mathrm{St_r} \otimes L(\lambda)$ is tilting by proposition 4.2.9. Also by proposition 4.2.1 we have $\mathrm{ch}\, T((p^r-1)\rho+\nu) = \chi((p^r-1)\rho)s_r(\nu)$. Write $\mathrm{ch}\, L(\lambda) = \sum_{\nu \in X^+(T)} b_\nu s_r(\nu)$ then the tilting modules $\mathrm{St}_r \otimes L(\lambda)$ and $\bigoplus_{\nu \in X^+(T)} b_\nu T((p^r-1)\rho+\nu)$ have the same character and hence are isomorphic. □

4.3 Conclusion

In this chapter we have removed the characteristic restriction on Doty's results given in [17]. We have also proved that if λ is modular minuscule and

V is finite dimensional G–module such that $\mu \leq \lambda$ for all weights μ of V then $\mathrm{St} \otimes V$ is a tilting module. We have also given the decomposition of $\mathrm{St} \otimes V$ into indecomposable tilting modules. This in particular given the decomposition of $\mathrm{St} \otimes \nabla(\lambda)$ and $\mathrm{St} \otimes \Delta(\lambda)$ (λ modular minuscule). We have also shown that if λ is (p, r)–minuscule then $\mathrm{St_r} \otimes L(\lambda)$ is a tilting module. We also give its decomposition into indecomposable tilting modules.

Chapter 5

On the Cohomology of Certain Homogeneous Vector Bundles of G/B in Characteristic Zero

5.1 Introduction

The contents of this chapter have been accepted for publication see [2]. Let k be an algebraically closed field of characteristic zero and let G be a reductive connected algebraic group over k. Let B be a Borel subgroup of G and $T \subset B$ be a maximal torus of G. Recall that for an algebraic group J, we write $\mathrm{mod}(J)$ for the category of finite dimensional rational J–modules. As explained in chapter 1, given an algebraic group J over a field k and a subgroup H we have the induction functor $\mathrm{Ind}_H^J : \mathrm{mod}(H) \to \mathrm{mod}(J)$ see e.g [21]. For $H \leq J \leq K$ and V an H–module we have a Grothendieck spectral sequence converging to $R^*\mathrm{Ind}_H^K V$, with E_2 page $R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha} V$. For $\lambda \in X(T)$ we denote by k_λ the one dimensional (rational) B–module on which T acts via λ. In what follows we will denote k_λ simply by λ. We will also write $H^i(M)$ for $R^i\mathrm{Ind}_B^G M$. We will denote by $P_\alpha/R_u(P_\alpha)$ the P_α–module on which the unipotent radical $R_u(P_\alpha)$ acts trivially.

In his famous paper [7] Demazure introduced the indecomposable modules $V_{\lambda,\alpha}$ with weights $\lambda, \lambda-\alpha, ..., s_\alpha(\lambda)$, where α is a simple root and s_α is the corresponding reflection. He used these modules to give a short proof of the Borel-Weil-Bott's theorem. In this chapter, we consider a generalization of the module $V_{\lambda,\alpha}$ by $M_{\alpha,r}(\lambda)$, where $M_{\alpha,r}(\lambda) = \nabla_\alpha(r\rho) \otimes (\lambda - r\rho)$ and $r \geq 0$. We will first show that $M_{\alpha,r}(\lambda)$ is the unique (up to isomorphism) indecomposable B–module with weights $\lambda, \lambda-\alpha, ..., \lambda-r\alpha$. We determine the i–th cohomology of $M_{\alpha,r}(\lambda)$ for all i. This in particular gives all cohomology of the modules $V_{\lambda,\alpha}$ appearing in Demazure's paper.

The following proposition along with its corollary shows that $M_{\alpha,r}(\lambda)$ is the unique (up to isomorphism) indecomposable B–module with the given weights.

Proposition 5.1.1. *If V has weights $\lambda, \lambda-\alpha, ..., \lambda-r\alpha$ then V is a direct sum of copies of $M_{\alpha,s}(\mu)$, where μ is of the form $\lambda - j\alpha$, $j \geq 0$.*

Proof. We will use induction on r to prove the result. The result is clearly true for $r = 0$. Now let $r \geq 1$ and suppose that the result is true for $r-1$. Let V' denote the sum of weight spaces $V^{\lambda-i\alpha}$, $i > 0$. Then V' is a B–submodule of V. By the inductive hypothesis V' is a direct sum of copies of $M_{\alpha,s}(\mu)$. We write V' as a direct sum of B–modules, $V' = X \oplus Y$, with $X \simeq M_{\alpha,s}(\lambda-\alpha)$ and $Y = \bigoplus M_{\alpha,t_i}(\lambda - q_i\alpha)$, $t_i \geq 0$, $q_i \geq 2$. We will now check to see whether there is an extension from Y to λ. Firstly $\mathrm{Ext}_B^1(k_\lambda, M_{\alpha,t_i}(\lambda-q_i\alpha)) = H^i(B, -\lambda \otimes M_{\alpha,t_i}(\lambda-q_i\alpha)) = 0$, for all $q_i \geq 2$. This is true because the weights of $-\lambda \otimes M_{\alpha,t_i}(\lambda-q_i\alpha)$ are $-q_i\alpha, -(q_i+1)\alpha, ... -(q_i+t_i)\alpha$ and $H^1(B, -j\alpha) = 0$

for all $j \geq 2$. This shows that there is no extension from Y to λ. Hence we can write $V = Z \oplus Y$ as B–modules, where Z is an extension of X by λ i.e. $X \leq Z$ and $Z/X \simeq \lambda$.

Now we will check whether there is an extension from X to λ. We have $\mathrm{Ext}^1_B(k_\lambda, X) = \mathrm{Ext}^1_B(k_\lambda, \nabla_\alpha((r-1)\rho) \otimes (\lambda - (r-1)\rho - \alpha) = H^1(B, \nabla_\alpha((r-1)\rho) \otimes (-(r-1)\rho - \alpha))$. We get $H^1(B, \nabla_\alpha((r-1)\rho) \otimes (-(r-1)\rho - \alpha)) = H^0(P_\alpha, \nabla_\alpha((r-1)\rho) \otimes \nabla_\alpha((r-1)\rho)^*) = k$. Hence there is a unique non-split extension from X to λ.

Using the above extensions we get that V is either $\lambda \oplus X \oplus Y$ or $M_{\alpha,s+1}(\lambda) \oplus Y$.

□

Corollary 5.1.2. *If V is indecomposable then $V = M_{\alpha,r}(\lambda)$.*

Proof. The result is clear from the above proposition. □

We will now give the cohomology of $M_{\alpha,r}(\lambda)$.

5.2 Main results

Consider the module $M_{\alpha,1}(\lambda)$, with weights $\lambda, \lambda - \alpha$. The following theorem gives the i–th cohomology of $M_{\alpha,1}(\lambda)$ for all i.

Theorem 5.2.1. *Let α be a simple root and $\lambda \in X(T)$ then*

$$H^i(M_{\alpha,1}(\lambda)) = \begin{cases} H^i(\lambda) \oplus H^i(\lambda - \alpha), & \langle \lambda, \alpha^\vee \rangle \neq 0 \\ 0, & \langle \lambda, \alpha^\vee \rangle = 0. \end{cases}$$

Proof. We will give the proof in separate cases.

1. Let $\langle \lambda, \alpha^\vee \rangle \leq -1$. On the second page of the Grothendieck spectral sequence we have

$$H^i(M_{\alpha,1}(\lambda)) = R^{i-1}\mathrm{Ind}_{P_\alpha}^G R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,1}(\lambda)).$$

Also since $\langle \lambda, \alpha^\vee \rangle \leq -1$, we have $\mathrm{Ind}_B^{P_\alpha}\lambda = 0$. Moreover P_α/B is one dimensional so $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ for all $i \geq 2$. Hence from the short exact sequence

$$0 \to \lambda - \alpha \to M_{\alpha,1}(\lambda) \to \lambda \to 0$$

we get

$$0 \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha) \to R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,1}(\lambda)) \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda) \to 0.$$

Since all modules for $P_\alpha/R_u(P_\alpha)$ are completely reducible (Weyl's complete reducibility theorem) we have $R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,1}(\lambda)) \simeq R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda)$. Therefore

$$H^i(M_{\alpha,1}(\lambda)) = R^{i-1}\mathrm{Ind}_{P_\alpha}^G(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda))$$

and we get the result.

2. For $\langle \lambda, \alpha^\vee \rangle = 1$ we get that $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho)$ is zero for all $j \neq 0$. Therefore $\nabla_\alpha(\rho) \otimes \nabla_\alpha(\lambda - \rho) = \nabla_\alpha(\lambda)$. Hence $H^i(M_{\alpha,1}(\lambda)) = H^i(\lambda)$.

3. Suppose $\langle \lambda, \alpha^\vee \rangle \geq 2$. On the second page of the spectral sequence we have

$$R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(\rho) \otimes (\lambda - \rho)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho)).$$

For $\langle \lambda, \alpha^\vee \rangle \geq 2$ we have that $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho)$ is zero for all $j \neq 0$. Therefore $H^i(M_{\alpha,1}(\lambda)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes \nabla_\alpha(\lambda - \rho))$. Since the weights of

$\nabla_\alpha(\rho)$ are ρ and $\rho - \alpha$, by a special case of the Clebsch-Gordan formula we get $\nabla_\alpha(\rho) \otimes \nabla_\alpha(\lambda - \rho) \simeq \nabla_\alpha(\lambda) \oplus \nabla_\alpha(\lambda - \alpha)$ (we are working in characteristic zero) and hence

$$H^i(M_{\alpha,1}(\lambda)) = H^i(\lambda) \oplus H^i(\lambda - \alpha).$$

4. Finally we consider the case $\langle \lambda, \alpha^\vee \rangle = 0$. We have $\langle \lambda - \rho, \alpha^\vee \rangle = -1$ and hence $R^j \mathrm{Ind}_B^{P_\alpha}(\lambda - \rho) = 0$ for all j. So we get $H^i(M_{\alpha,1}(\lambda)) = 0$ for all i.

This completes the proof.

□

Now take $M_{\alpha,r}(\lambda)$ for all $r \geq 0$. The i–th cohomology of this module is given by the following theorem.

Theorem 5.2.2. *Let $r \geq 0$, $\lambda \in X(T)$, $m = \langle \lambda, \alpha^\vee \rangle$ and $s = \langle \lambda - r\rho, \alpha^\vee \rangle$; then*

$$H^i(M_{\alpha,r}(\lambda)) = \begin{cases} \bigoplus_{t=0}^{r} H^i(\lambda - t\alpha), & \langle \lambda, \alpha^\vee \rangle \leq -1 \\ \bigoplus_{t=0}^{r} H^i(\lambda - t\alpha), & \langle \lambda, \alpha^\vee \rangle > r \\ & and r \leq s \\ \bigoplus_{t=0}^{s} H^i(\lambda - t\alpha), & \langle \lambda, \alpha^\vee \rangle > r \\ & and r > s \\ \bigoplus_{t=0}^{r} H^i(t\rho - (t - 2 - m)\alpha), & 0 \leq \langle \lambda, \alpha^\vee \rangle < r - 1 \\ H^i(\lambda), & \langle \lambda, \alpha^\vee \rangle = r \\ 0, & \langle \lambda, \alpha^\vee \rangle = r - 1. \end{cases}$$

Proof. We will use induction on r to prove the result. The result is true for

$r = 1$ by theorem 1. Suppose the result is true for $r-1$; then

$$H^i(M_{\alpha,r-1}(\lambda)) = \begin{cases} \bigoplus_{t=0}^{r-1} H^i(\lambda - t\alpha), & \langle\lambda,\alpha^\vee\rangle \leq -1 \\ \bigoplus_{t=0}^{r-1} H^i(\lambda - t\alpha), & \langle\lambda,\alpha^\vee\rangle > r-1 \\ & \text{and}\, r-1 \leq s \\ \bigoplus_{t=0}^{s} H^i(\lambda - t\alpha), & \langle\lambda,\alpha^\vee\rangle > r-1 \\ & \text{and}\, r-1 > s \\ \bigoplus_{t=0}^{r-1} H^i(t\rho - (t-2-m)\alpha), & 0 \leq \langle\lambda,\alpha^\vee\rangle < r-2 \\ H^i(\lambda), & \langle\lambda,\alpha^\vee\rangle = r-1 \\ 0, & \langle\lambda,\alpha^\vee\rangle = r-2. \end{cases}$$

Now for r we give the result in cases as in theorem 5.2.1.

1. Let $\langle\lambda,\alpha^\vee\rangle \leq -1$. On the second page of the spectral sequence we have

   $$H^i(M_{\alpha,r}(\lambda)) = R^{i-1}\mathrm{Ind}_{P_\alpha}^G R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r}(\lambda)).$$

   Moreover we have the short exact sequence

   $$0 \to M_{\alpha,r-1}(\lambda-\alpha) \to M_{\alpha,r}(\lambda) \to \lambda \to 0.$$

   Also since $\langle\lambda,\alpha^\vee\rangle \leq -1$, we have $\mathrm{Ind}_B^{P_\alpha}(\lambda) = 0$. Moreover P_α/B is one dimensional so $R^i\mathrm{Ind}_B^{P_\alpha} = 0$ for all $i \geq 2$. Using the above short exact sequence we get

   $$0 \to R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r-1}(\lambda-\alpha)) \to R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r}(\lambda)) \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda) \to 0.$$

   Since all modules for $P_\alpha/R_u(P_\alpha)$ are completely reducible we get

   $$R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r}(\lambda)) \simeq R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r-1}(\lambda-\alpha)) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda).$$

Therefore

$$H^i(M_{\alpha,r}(\lambda)) = R^{i-1}\mathrm{Ind}_{P_\alpha}^G(R^1\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r-1}(\lambda-\alpha)) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda)).$$

Since we are working in characteristic zero we can get

$$H^i(M_{\alpha,r}(\lambda)) = H^i(M_{\alpha,r-1}(\lambda)) \oplus H^i(\lambda).$$

Now use the inductive hypothesis to get the result.

2. Suppose $\langle \lambda, \alpha^\vee \rangle > r$. On the second page of the spectral sequence we have

$$R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(r\rho) \otimes (\lambda - r\rho))$$
$$= R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(r\rho) \otimes R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - r\rho)).$$

For $\langle \lambda, \alpha^\vee \rangle > r$ we have that $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - r\rho)$ is zero for all $j \neq 0$. Therefore

$$R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(M_{\alpha,r}(\lambda)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(r\rho) \otimes \nabla_\alpha(\lambda - r\rho)).$$

Now we have two cases here. Firstly let $r \leq s$ then we will get $H^i(M_{\alpha,r}(\lambda)) = \bigoplus_{t=0}^r H^i(\lambda - t\alpha)$. Now if $r > s$ then we have

$$H^i(M_{\alpha,r}(\lambda)) = \bigoplus_{t=0}^{s} H^i(\lambda - t\alpha).$$

3. For $0 \leq \langle \lambda, \alpha^\vee \rangle < r - 1$ we get that $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - r\rho)$ is zero for all $j \neq 1$. We get

$$R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(r\rho) \otimes (\lambda - r\rho))$$
$$= R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(r\rho) \otimes R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - r\rho)).$$

Using Serre duality we get $(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - r\rho))^* = \mathrm{Ind}_B^{P_\alpha}(-\lambda + r\rho - \alpha)$. Therefore $H^i(M_{\alpha,r}(\lambda)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(r\rho) \otimes \nabla_\alpha(-\lambda + r\rho - \alpha)^*)$.

Now let $\langle \lambda, \alpha^\vee \rangle = m$ then we get

$$H^i(M_{\alpha,r}(\lambda)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(r\rho) \otimes \nabla_\alpha(\lambda - r\rho + (r - 1 - m)\alpha)).$$

Apply the Clebsch-Gordan formula again to get the result.

4. For $\langle \lambda, \alpha^\vee \rangle = r$ we get that $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - r\rho)$ is zero for all $j \neq 0$. Therefore $\nabla_\alpha(r\rho) \otimes \nabla_\alpha(\lambda - r\rho) = \nabla_\alpha(\lambda)$. Hence $H^i(M_{\alpha,r}(\lambda)) = H^i(\lambda)$.

5. Finally we consider the case where $\langle \lambda, \alpha^\vee \rangle = r - 1$, so $\langle \lambda - r\rho, \alpha^\vee \rangle = -1$ and hence $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho) = 0$ for all j. So we get $H^i(M_{\alpha,r}(\lambda)) = 0$ for all i.

This completes the proof. □

5.3 Conclusion

In this chapter we have proved that in characteristic zero the modules $M_{\alpha,r}(\lambda)$ are unique (up to isomorphism). We have also given the i–th cohomology of these modules. The modules with weights $\lambda, \lambda - \alpha, ..., \lambda - r\alpha$ are of particular interest to us in characteristic p as they appear in recursion given in chapter 2 and 3. The module $M_{\alpha,1}(\lambda)$ is unique in positive characteristic p and is denoted by $N_\alpha(\lambda)$ in [11]. For $r > 1$ the modules $M_{\alpha,r}(\lambda)$ are not always unique in characteristic p hence we are using the different notation to that of $N_\alpha(\lambda)$.

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