A Comparison of Models for the Fulton-Macpherson Operads

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Summary

In this thesis we explore the structure of the Fulton-Macpherson operads $\bar{F}_N$ by providing two new models for them. It is shown in [22] that these operads are cofibrant by claiming the existence of an isomorphism of operads $W\bar{F}_N \to \bar{F}_N$. Here, W is a functor which, for a large class of topological operads, produces cofibrant replacements. It would be satisfying to be able to write down explicitly what these isomorphisms are. Our new models are an attempt to move towards this.

The building blocks of the first model appeared in [23] but they were not assembled into an operad here. This model has a more algebraic feel than others in the literature which gives it technical advantages. We use this to demonstrate many of the well-known properties of the Fulton-Macpherson operads. In particular, we are able to write down explicit isomorphisms between $\bar{F}_1$ and the Stasheff operad which we have not seen previously in the literature. This model is isomorphic to other models of the Fulton-Macpherson operads.

The second model is a realisation of an operad in posets. This poset operad is built from combinatorial objects called chains of preorders. These objects encode maps from a finite set A to some Euclidean space $\mathbb{R}^N$. In particular, we can impose restrictions to encode injective maps of this type. This model is equivalent up to homotopy to the Fulton-Macpherson operads in a way which we define. It is also homotopy equivalent to the Smith operads, another example of topological operads defined combinatorially. The main advantage of this model is that it has an obvious spine which may pave the way to writing down the desired isomorphisms $W\bar{F}_N \to \bar{F}_N$.

Contents

Introduction

An Introduction to Operads

An operad is an object in a symmetric monoidal category, $\mathcal{C}$, that encodes algebraic structure. It comprises of a functor $\mathcal{P}$ from the category of finite sets and bijections to $\mathcal{C}$, as well as maps γ_p in $\mathcal{C}$ indexed by maps of finite sets $p : A \to B$. More precisely, if A is a finite set, then the object $\mathcal{P}(A)$ should encode all of the algebraic operations with inputs labelled by A. The maps γ_p tell us how to compose such operations, with this composition satisfying associativity conditions that one would expect. One can draw analogy with the way groups encode symmetries, as in much the same way, an operad in $\mathcal{C}$ can act upon objects of $\mathcal{C}$. This action equips the object with the particular algebraic structure associated with the operad. Arguably, the easiest and most tractable examples of operads arise in the category of vector spaces. Here, we have operads that encode the structure of associative, commutative and Lie algebras which are all classical examples and easy to understand. We can even encode more complicated algebraic structures such as the Poisson algebra, which has two binary operations and a distributive law. It should be noted though that the operad is not an all powerful object. For example, it is impossible to create an operad encoding the Jordan algebra structure due to the inhomogeneous nature of its defining identities.

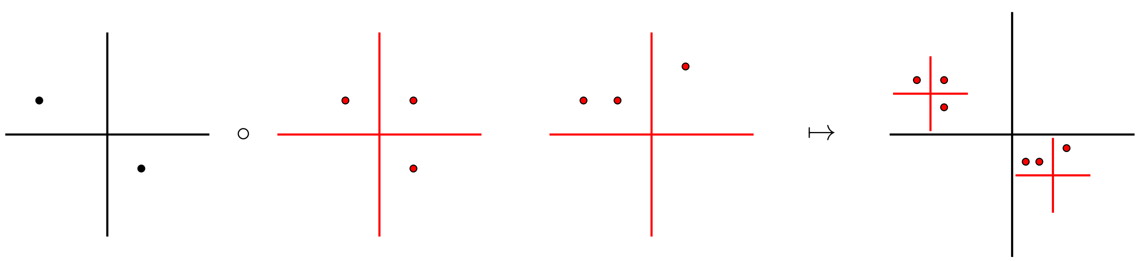
Operads were in fact first defined in the category of topological spaces by May in [19]. (Whenever we say topological spaces in this thesis, we will always mean the category of compactly generated topological spaces that one would normally do algebraic topology in.) Here, May describes the little N-cubes operads which encode operations in a much more abstract way than the operads of the algebras in the previous paragraph. Indeed, for a natural number N, a little N-cube is defined to be a parallel linear embedding of the N-cube $(0, 1)^N$ to itself. Then, the spaces of operations $\mathrm{C}_N(A)$ are the sets of little cubes labelled by A with pairwise disjoint images. Composition of these operations is simply given by composing the respective embeddings. For a much more detailed description, see section 1.2. It turns out that this encodes a commutative algebra structure that is satisfied up to N levels of coherent homotopies.

The most famous example of such a structure is the N-fold loop space $\Omega^N X$ of some based space X. Specifically, we may concatenate loops $f, g \in \Omega^N X$ to form a new loop $f * g$. This operation is not quite commutative, but we do have a homotopy between $f * g$ and $g * f$, as long as $N > 1$. ($N = 1$ is a special case as here we only really have the associative part of a commutative algebra structure.) There are then homotopies between these homotopies and so on, all the way up the N^{th} level. So $\Omega^N X$ is an algebra over the little N-cubes operad and then May's famous recognition principle tells us that in fact the converse is also true; any path-connected algebra over the little N-cubes operad actually has the homotopy type of an N-fold loop space for some topological space X. Another interesting fact is that one can make a link between the little N-cubes operads and the algebraic operads in the previous paragraph. Indeed, the homology functor preserves operads, and if one applies it to the little N-cubes operad, we in fact retrieve the operad of degree-N Poisson algebras [7], [8].

The importance of the little N-cubes operads leads to them being used to define a whole family of operads, namely the E_N operads for $N \in \mathbb{N} \cup \{\infty\}$. An E_N operad is one that is weakly equivalent to the little N-cubes operad. We can define the infinite little cubes operad as the direct limit of inclusions $C_N(A) \hookrightarrow C_{N+1}(A)$. Then an E_∞ operad can be defined as one that is equivalent to this, although there is a simpler definition, see 1.2.14. There have been many realisations of E_N operads, each with their own features. Instead of considering embeddings of cubes, one may consider embeddings of disks to form the little N-disks operads D_N. These have the property that there is a compatible action of the orthogonal group $\mathcal{O}(N)$ on the spaces $\mathrm{D}_N(A)$. There are also the Steiner operads described in [27]. These have the technical advantages of both the little cubes and little disks operads but the spaces of operations are infinite dimensional. Another example is the Fulton-Macpherson operads, the main object of study in this thesis.

The Fulton-Macpherson Operads

It is not difficult to show that the spaces $\mathrm{C}_N(A)$ in the little cubes operads are homotopy equivalent to Euclidean configuration spaces $\mathrm{Inj}(A, \mathbb{R}^N)$. It is therefore natural to ask whether we can form an E_N operad from these spaces. If we attempt to do this, then we find that the operad composition will be given by embedding configurations into some base configuration, as shown in the diagram below.



One issue is that the points in the base configuration can be arbitrarily close to one another. Therefore we must find a way to scale our embedded configurations in order to avoid collisions between the points. One finds though that there is no way of doing this that will be compatible with the operad composition. However, we can provide an alternative that is very close to our originally desired result. In [10], the authors create a compactification of configuration spaces using geometric blowups. Getzler and Jones slightly modify this in [12] to put operad structures on these compactifications which are now known as the Fulton-Macpherson operads. In [18], Markl provides another definition of the Fulton-Macpherson operads as an operadic completion of some quotient of the configuration spaces. This means that in some sense, the Fulton-Macpherson operads are the smallest operads that contain the configuration spaces, (modulo the aforementioned quotient).

These operads have proven to be a useful example of E_N operads. In [16] they are used in a zig-zag of weak equivalences to show that for $N \neq 2$, the little N-disks operads, (and hence all E_N-operads), are formal. This result then has applications in areas such as deformation quantization and knot theory. The interested reader may refer to [15] and [17] respectively for further details. The Fulton-Macpherson operads are also interesting objects of study in their own right. The spaces involved are compact, smooth manifolds with corners. The interiors of these manifolds are homotopy equivalent to $\mathrm{Inj}(A, \mathbb{R}^N)$ which as we shall see in the thesis, quite easily implies that the spaces of the operad are also homotopy equivalent to the configuration spaces. However, arguably the most useful property for these operads is found in [22]. Here, the author proves that the Fulton-Macpherson operads are cofibrant objects for the usual Quillen model structure on the category of operads, (corollary 4.8).

To show that the Fulton-Macpherson operads are cofibrant, Salvatore makes use of the W-construction. This is a functor that produces cofibrant replacements for topological operads satisfying some mild conditions. If one applies the W-construction to the Fulton-Macpherson operads, then intuitively, it adds a block collar to the manifolds with corners. More precisely, it glues a d-dimensional cube to each point in a d-codimensional face of the boundary of the manifold. The addition of this collar will not change the diffeomorphism type of the manifolds and so it is claimed that there exists an isomorphism between the Fulton-Macpherson operads and their cofibrant replacements, (proposition 4.7). However, no detail is given as to how these diffeomorphisms are compatible with the operad structure and so the proof is unsatisfying. The research that led to this thesis was an attempt to find explicit maps that definitively prove this result.

Outline of the Thesis

In chapter 1, we begin by introducing the basic definitions and constructions that will be used in the thesis, mainly to set notation but also to refresh the reader's memory. We give two definitions for an operad but show they are equivalent. We also define morphisms of operads and algebras over operads. We then go on to define several examples of operads, most of which will play a role later on in the thesis. Notable inclusions are the little cubes operads, the operads of trees and the Stasheff operad. Next, we define the reduced free operad in the category of sets as well as the related notion of a well-labelled operad. The latter gives us a way of saying when an operad which has an underlying operad in sets is in fact set-theoretically free. Finally we define the W-construction, although only for reduced topological operads. A much more general version can be found in [4]. This construction produces cofibrant replacements for topological operads $\mathcal{P}$ that are well-pointed and such that $\mathcal{P}(A)$ is cofibrant for all A. For all of the operads that we will consider, these conditions will be satisfied. At the end, we show that the Stasheff operad K is cofibrant by producing an isomorphism $WK \to K$. We do it this way as it will be useful for our study of the Fulton-Macpherson operads later in the thesis.

The next chapter focuses on what we call the Singh model for the Fulton-Macpherson operads. The spaces for the operad use an analogous construction from [23] although the author never assembled these into operads. Before we begin describing them, we first analyse the homotopy type of the configuration spaces $\mathrm{Inj}(A, \mathbb{R}^N)$ for some finite set A and natural number N. This is classical work but it will be useful to have explicit generators for the integral cohomology $H^*(\mathrm{Inj}(A, \mathbb{R}^N))$. We then proceed to define the spaces $\bar{F}_N(A)$ that will make up our operad as well as give a stratification by trees on A. This definition has a much more algebraic feel than some others and so it makes calculation reasonably tractable. With this in mind, we go on to give elementary proofs of some of the well-known properties of the Fulton-Macpherson compactifications, paying particular attention to the manifold with corners structure. We define the operad structure on the collection $\bar{F}_N = \{\bar{F}_N(A)\}$ and show that this is well-labelled. Once we have an operad in spaces, we can apply the homology functor to acquire an operad in graded abelian groups. We show that if one does this, then we acquire the same operad as when we apply homology to the little cubes operads, i.e. the operad of graded Poisson algebras. Obviously this is not surprising if we believe that the Fulton-Macpherson operads are E_N operads, but it is still nice to see. Finally, we restrict our attention to the case $N = 1$ and show that $\bar{F}_1$ is isomorphic to the Stasheff operad. From our work in chapter 1, this means that we have our first explicit isomorphism $W\bar{F} \to \bar{F}$.

In the relatively short chapter 3, we compare the Singh model for the Fulton-Macpherson operads with the one proposed by Sinha. The spaces for Sinha's operad are described and studied in [24] and the operad structure appears in [16]. It turns out that the Singh and the Sinha model are isomorphic and have compatible stratification by trees. This means that we can justify calling the Singh model the Fulton-Macpherson operads.

In the final chapter we introduce another model for the Fulton-Macpherson operads which arises as the realisation of operads in posets. To begin with we recall some basic facts about finite posets. As well as this, we outline how one can actually put a topology on a finite poset. This then allows us to apply a theorem of McCord from [20] to prove a statement concerning the homotopy type of the realisation of a poset. In particular, if we have a map $X \to P$ from a space to a finite poset with contractible open fibers, then this induces a weak equivalence $X \to |P|$. We also prove a corollary to this theorem which makes similar statements for operads. Once we have all of this machinery in place, we define the main combinatorial objects that we will use, chains of preorders. These objects encode maps from a finite set A to a Euclidean space $\mathbb{R}^N$ and can be modified to specifically encode injective maps. This means that when we take realisations, we are able to prove that we have combinatorial models for many of the spaces of maps we have seen previously in the thesis. We study the model for injective maps in particular and show that it has the same homological dimension as $\mathrm{Inj}(A, \mathbb{R}^N)$. This means that it can be thought of as a spine for $\mathrm{Inj}(A, \mathbb{R}^N)$ and all of the other homotopy equivalent spaces we have seen, in particular, $\bar{F}_N(A)$. Finally, we introduce what we call the Fulton-Macpherson posets $\bar{\mathcal{F}}_N(A)$. These are defined analogously to the spaces $\bar{F}_N(A)$. We show that they have much of the same structure and can be assembled into an operad. Our final results use the theorems we proved earlier in the chapter to conclude that $|\bar{\mathcal{F}}_N|$ is equivalent, up to homotopy, as an operad to $\bar{F}_N$. We also compare $\bar{\mathcal{F}}_N$ with other well-known combinatorial operads.

To define the homeomorphisms $W\bar{F}_N(A) \to \bar{F}_N(A)$ needed for our desired isomorphism of operads $W\bar{F}_N \to \bar{F}_N$, it would be useful to be able to embed the spine of $\bar{F}_N(A)$ in a way compatible with the operad structure. However, it is not immediately clear how the combinatorial spine we define above interacts with the operad composition. The motivation for defining the Fulton-Macpherson posets is that their realisations contain the previously mentioned spine in an easy to see way. It was expected that the dimension of the Fulton-Macpherson posets would be the same as the manifolds $\bar{F}_N(A)$. Therefore, it would have been likely that their realisations would have in fact been isomorphic and so collaring one would collar the other. However, in general the two spaces do not have the same dimension. We include a brief discussion of why this is the case and explain why we believe one can refine the posets to in fact have these desired properties.

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Chapter 1

Operads - Definitions and Examples

In this first chapter, we introduce the main objects and constructions that we will use throughout the thesis. This is as much to set notation as to refresh the reader's memory of these concepts. We begin by defining an operad in a symmetric monoidal category and the related notion of an algebra over an operad. We then give many examples of operads and elaborate on those that will play a large role in the thesis. Towards the end of the chapter we will introduce some more advanced concepts, namely well-labelled operads and the W-construction. The latter gives us a way of recognising cofibrant operads as we demonstrate using the Stasheff operad.

1.1 Definitions

In this first section, we begin by defining an operad. We in fact give two definitions of an operad and then show that they are in fact equivalent. The idea of an operad is that it encodes a collection of operations, satisfying certain composition axioms, that turn an object into an algebra. This is best understood by defining the endomorphism operad before then seeing the definition of an algebra over an operad.

Definition 1.1.1. Let $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category and $\mathrm{FSET}_{\cong}$ be the category of finite sets with arrows the bijections between them. An operad is a functor $\mathcal{P} : \mathrm{FSET}_{\cong} \to \mathcal{C}$, equipped with maps in $\mathcal{C}$ called composition and unit maps. Let $p : A \to B$ be a map of finite sets and define $A_b = p^{-1}(b)$ for all $b \in B$. Then for each such p, there is a composition map γ_p of the form

$$\gamma_p : \mathcal{P}(B) \otimes \bigotimes_{b \in B} \mathcal{P}(A_b) \to \mathcal{P}(A).$$

These maps should be natural for commutative squares of the form

$$\begin{tikzcd} A \arrow[r, "f", "\cong"'] \arrow[d, "p"'] & A' \arrow[d, "p'"] \\ B \arrow[r, "g", "\cong"'] & B' \end{tikzcd}$$

and satisfy an associativity condition: if $q : B \to C$ is another map of finite sets, $B_c = q^{-1}(c)$

and $A_c = (qp)^{-1}(c)$, then the following diagram must commute:

$$\begin{tikzcd}
\mathcal{P}(C) \otimes \left(\bigotimes_{c\in C} \mathcal{P}(B_c)\right) \otimes \left(\bigotimes_{b\in B} \mathcal{P}(A_b)\right) \arrow[r, "1\otimes\bigotimes_{c\in C}\gamma_{p_c}"] \arrow[d, "\gamma_q\otimes 1"'] & \mathcal{P}(C) \otimes \bigotimes_{c\in C} \mathcal{P}(A_c) \arrow[d, "\gamma_{qp}"] \\
\mathcal{P}(B) \otimes \bigotimes_{b\in B} \mathcal{P}(A_b) \arrow[r, "\gamma_p"] & \mathcal{P}(A)
\end{tikzcd}$$

where $p_c : A_c \to B_c$ is the restriction of p. The unit map is $\eta : 1 \to \mathcal{P}(\{*\})$ and the following diagram must also commute where c_A is the unique map $c_A : A \to \{*\}$ and $p : A \to B$ is any bijection:

$$\begin{tikzcd}
1 \otimes \mathcal{P}(A) \arrow[r, "\eta\otimes 1"] \arrow[dr, "1"'] & \mathcal{P}(\{*\}) \otimes \mathcal{P}(A) \arrow[d, "\gamma_{c_A}"] \\
& \mathcal{P}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(B) \otimes \bigotimes_{b\in B} \mathcal{P}(\{*\}) \arrow[d, "\gamma_p"] & \mathcal{P}(B) \otimes \bigotimes_{b\in B} 1 \arrow[l, "1\otimes\bigotimes_{b\in B}\eta"'] \arrow[dl, "1"] \\
\mathcal{P}(A) &
\end{tikzcd}$$

Most of the time, we will not be overly concerned with defining a unit in our operads as it will be obvious as to which element it should be. We refer to this definition as the traditional definition of an operad.

Definition 1.1.2. Let $\mathcal{C} = (\mathcal{C}, \times, 1)$ have a symmetric monoidal structure given by cartesian product. Assume that $\mathcal{C}$ has an initial object 0 such that $0 \times X = 0$ for all $X \in \mathcal{C}$, and terminal object 1. We call an operad $\mathcal{P}$ reduced if $\mathcal{P}(\emptyset) = 0$ and $\mathcal{P}(\{*\}) = 1$.

Remark 1.1.3. Many of the operads we will consider in this thesis will be reduced. In the case of some category $\mathcal{C}$ whose objects have an underlying set, to define composition maps, we only need to consider surjective maps of finite sets $p : A \to B$. This is because if we have $A_b = \emptyset$ for some $b \in B$, then the domain of γ_p will also be empty.

We now give an alternative definition for an operad but prove that they are in fact equivalent. However, some situations make one definition more convenient than the other. First we set some notation.

Definition 1.1.4. Let A be a finite set and $B \subseteq A$. We then define $A/B = (A \setminus B) \amalg \{*\}$, i.e. we collapse the subset B to a single point. Notice that if $B = \emptyset$ then $A/\emptyset = A \amalg \{*\}$. If $B_1, B_2 \subseteq A$ are disjoint subsets of A then we define

$$A/(B_1, B_2) = ((A \setminus B_1) \setminus B_2) \amalg \{*_1, *_2\}$$

i.e. we collapse B_1 and B_2 to distinct points. This definition is independent of the implied ordering of the subsets B_1 and B_2.

Definition 1.1.5. An operad is a functor $\mathcal{P} : \mathrm{FSET}_{\cong} \to \mathcal{C}$, equipped with maps in $\mathcal{C}$ called composition and unit maps. The unit map is defined exactly as before. For the composition maps, consider finite sets $B \subseteq A$. For each such pair we have maps

$$\gamma^A_B : \mathcal{P}(A/B) \otimes \mathcal{P}(B) \to \mathcal{P}(A)$$

such that the following hold for $B_1, B_2 \subseteq A$:

i) If $B_1 \cap B_2 = \emptyset$ then the following diagram should commute:

$$\begin{tikzcd}
\mathcal{P}(A/(B_1,B_2)) \otimes \mathcal{P}(B_1) \otimes \mathcal{P}(B_2) \arrow[r, "\gamma_{B_2}^{A/B_1} \otimes 1"] \arrow[d, "\gamma_{B_1}^{A/B_2} \otimes 1"'] & \mathcal{P}(A/B_1) \otimes \mathcal{P}(B_1) \arrow[d, "\gamma_{B_1}^{A}"] \\
\mathcal{P}(A/B_2) \otimes \mathcal{P}(B_2) \arrow[r, "\gamma_{B_2}^{A}"] & \mathcal{P}(A)
\end{tikzcd}$$

Note that we have omitted the necessary twist map to make the top map well-defined.

ii) If $B_1 \subseteq B_2$ then the following diagram should commute:

$$\begin{tikzcd}
\mathcal{P}(A/B_2) \otimes \mathcal{P}(B_2/B_1) \otimes \mathcal{P}(B_1) \arrow[r, "1 \otimes \gamma_{B_1}^{B_2}"] \arrow[d, "\gamma_{B_2/B_1}^{A/B_1} \otimes 1"'] & \mathcal{P}(A/B_2) \otimes \mathcal{P}(B_2) \arrow[d, "\gamma_{B_2}^{A}"] \\
\mathcal{P}(A/B_1) \otimes \mathcal{P}(B_1) \arrow[r, "\gamma_{B_1}^{A}"] & \mathcal{P}(A)
\end{tikzcd}$$

We refer to this definition as the combinatorial definition of an operad.

Proposition 1.1.6. *The traditional and combinatorial definitions of an operad create equivalent objects.*

Proof. Since the only difference in the definitions is how we define the composition maps, the only thing we need to check is that we have a traditional composition if and only if we have a combinatorial composition. First assume we have an operad by the traditional definition. If $B \subseteq A$ then we can define a map $p : A \to A/B$ by the obvious projection. This means we have an operad composition map $\gamma_p : \mathcal{P}(A/B) \otimes \bigotimes_{b \in A/B} \mathcal{P}(A_b) \to \mathcal{P}(A)$ and therefore we define

$$\gamma_B^A : \mathcal{P}(A/B) \otimes \mathcal{P}(B) \to \mathcal{P}(A)$$

by setting $\gamma_B^A(x,y) = \gamma_p(x, (y, \eta(1), \ldots, \eta(1))_{b \in A/B})$. We can do this since if $a \notin B$ then $|A_a| = 1$ and so $\eta(1) \in \mathcal{P}(A_a)$. To check that γ_B^A satisfies the necessary conditions, first let $B_1, B_2 \in A$ with $B_1 \cap B_2 = \emptyset$ and notice that we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "p_1"] \arrow[d, "p_2"'] & A/B_1 \arrow[d, "q_1"] \\
A/B_2 \arrow[r, "q_2"'] & A/(B_1, B_2)
\end{tikzcd}$$

where each map is the obvious projection. Our strategy is to apply the operad associativity axiom to this diagram:

$$\begin{aligned}
\gamma_{B_1}^A(\gamma_{B_2}^{A/B_1}(x,y),z) &= \gamma_{B_1}^A(\gamma_{q_1}(x,(y,\eta(1),\ldots,\eta(1))_{c \in A/B_1,B_2}), z) \\
&= \gamma_{p_1}(\gamma_{q_1}(x,(y,\eta(1),\ldots,\eta(1))_{c \in A/B_1,B_2}), (z,\eta(1),\ldots,\eta(1))_{b \in A/B_1}) \\
&= \gamma_{q_1 p_1}(x, (\gamma_{p_{B_1}}(y,(\eta(1),\ldots,\eta(1))_{b \in B_1}), \gamma_{p_{B_2}}(z,(\eta(1),\ldots,\eta(1))_{b \in B_2}), \\
&\qquad\qquad \eta(1),\ldots,\eta(1))_{c \in A/B_1,B_2}) \\
&= \gamma_{q_1 p_1}(x,(y,z,\eta(1),\ldots,\eta(1))_{c \in A/B_1,B_2})
\end{aligned}$$

Symmetrically, $\gamma^A_{B_2}(\gamma^{A/B_2}_{B_1}(x,z),y) = \gamma_{q_2p_2}(x,(y,z,\eta(1),\ldots,\eta(1))_{c\in A/B_1,B_2})$ but $q_1p_1 = q_2p_2$ and so this is equal to $\gamma^A_{B_1}(\gamma^{A/B_1}_{B_2}(x,y),z)$ as required.

Now let $B_1 \subseteq B_2 \subseteq A$ and again notice that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "qp"'] & A/B_1 \arrow[d, "q"] \\ & A/B_2 \end{tikzcd}$$

where each map is the projection map. If we let $r : B_2 \to B_2/B_1$ denote the projection then on the one hand we have

$$\begin{aligned} \gamma^A_{B_2}(x,\gamma^{B_2}_{B_1}(y,z)) &= \gamma^A_{B_2}(x,\gamma_r(y,(z,\eta(1),\ldots,\eta(1))_{c\in B_2/B_1})) \\ &= \gamma_{qp}(x,(\gamma_r(y,(z,\eta(1),\ldots,\eta(1))_{c\in B_2/B_1}),\eta(1),\ldots,\eta(1))_{b\in A/B_2}). \end{aligned}$$

On the other hand, if we apply the operad associativity to the first diagram we see that

$$\begin{aligned} \gamma^A_{B_1}(\gamma^{A/B_1}_{B_2/B_1}(x,y),z) &= \gamma^A_{B_1}(\gamma_q(x,(y,\eta(1),\ldots,\eta(1))_{c\in A/B_2}),z) \\ &= \gamma_q(\gamma_p(x,(y,\eta(1),\ldots,\eta(1))_{c\in A/B_2}),(z,\eta(1),\ldots,\eta(1))_{b\in A/B_1}) \\ &= \gamma_{qp}(x,(\gamma_r(y,(z,\eta(1),\ldots,\eta(1))_{c\in B_2/B_1}),\eta(1),\ldots,\eta(1))_{b\in A/B_2}) \\ &= \gamma^A_{B_2}(x,\gamma^{B_2}_{B_1}(y,z)) \end{aligned}$$

since $p|_{B_2} = r$.

Now assume that we have an operad by the combinatorial definition. Let $p : A \to B$ be a map of finite sets and $X = \{x_0,\ldots,x_m\} \subseteq B$ a subset. We then introduce the following notation

$$A(p,X) = X \amalg p^{-1}(B \setminus X) = A/(A_{x_0},\ldots,A_{x_m})$$

$$\mathcal{P}(p,X) = \mathcal{P}(A(p,X)) \otimes \bigotimes_{x\in X} \mathcal{P}(A_x).$$

Notice that we have $A(p,\emptyset) = A$ and $A(p,B) = B$. If $b \notin X$ then A_b identifies with a subset of $A(p,X)$ and we have

$$A(p,X \amalg \{b\}) = A(p,X)/A_b.$$

Our combinatorial operad therefore gives us a map

$$\gamma^{A(p,X)}_{A_b} : \mathcal{P}(A(p,X \amalg \{b\})) \otimes \mathcal{P}(A_b) \to \mathcal{P}(A(p,X))$$

and so by tensoring with appropriate identity maps we get a map

$$\mathcal{P}(p,X \amalg \{b\}) \to \mathcal{P}(p,X).$$

By composing maps of this type we then get a map

$$\mathcal{P}(p,B) = \mathcal{P}(B) \otimes \bigotimes_{b\in B} \mathcal{P}(A_b) \to \mathcal{P}(A) = \mathcal{P}(p,\emptyset)$$

and we set this to be our operad composition in the traditional sense. It is implicit in this definition that we have chosen an ordering on B. However, if we take $b_1, b_2 \notin X$ then the diagram

$$\begin{tikzcd}
\mathcal{P}(p, X \amalg \{b_1, b_2\}) \arrow[r] \arrow[d] & \mathcal{P}(p, X \amalg \{b_1\}) \arrow[d] \\
\mathcal{P}(p, X \amalg \{b_2\}) \arrow[r] & \mathcal{P}(p, X)
\end{tikzcd}$$

commutes by axiom i) in definition 1.1.5. Therefore our definition of the composition is independent of this implicit ordering.

Now we check that our defined composition is associative. Let $A \xrightarrow{p} B \xrightarrow{q} C$ be maps of finite sets. One side of the associativity diagram for composition using our definitions will looks as follows

$$\mathcal{P}(q, C) \otimes \bigotimes_{b \in B} \mathcal{P}(A_b) \to \cdots \to \mathcal{P}(q, \emptyset) \otimes \bigotimes_{b \in B} \mathcal{P}(A_b) = \mathcal{P}(p, B) \to \cdots \to \mathcal{P}(p, \emptyset).$$

The other side will have the following form:

$$\mathcal{P}(C) \otimes \bigotimes_{c \in C} \mathcal{P}(p_c, B_c) \to \cdots \to \mathcal{P}(C) \otimes \bigotimes_{c \in C} \mathcal{P}(p_c, \emptyset) = \mathcal{P}(qp, C) \to \cdots \to \mathcal{P}(qp, \emptyset).$$

We analyse a specific piece of this sequence. Choose some $c_0 \in C$ and then a $b_0 \in B_{c_0}$. We then have inclusions $A_{b_0} \subseteq A_{c_0} \subseteq A(qp, C \setminus \{c_0\})$. The piece of the sequence we are interested in looks as follows:

$$\mathcal{P}(C) \otimes \bigotimes_{c \in C \setminus \{c_0\}} \mathcal{P}(A_c) \otimes \mathcal{P}(p_c, b_0) \to \mathcal{P}(C) \otimes \bigotimes_{c \in C \setminus \{c_0\}} \mathcal{P}(A_c) \otimes \mathcal{P}(A_{c_0}) \to \mathcal{P}(qp, C \setminus c_0).$$

However, by axiom ii) of definition 1.1.5, this composition is equal to the composition

$$\mathcal{P}(r, \{c_0\}) \otimes \mathcal{P}(A_{b_0}) \otimes \bigotimes_{c \in C \setminus \{c_0\}} \mathcal{P}(A_c) \to \mathcal{P}(id \amalg p|_{A_{b_0}}, \{b_0\}) \otimes \bigotimes_{c \in C \setminus \{c_0\}} \mathcal{P}(A_c) \to \mathcal{P}(qp, C \setminus \{c_0\})$$

where $r : C \setminus \{c_0\} \amalg A_{c_0}/A_{b_0} \to C$ is defined to be the identity on $C \setminus \{c_0\}$ and sends A_{c_0}/A_{b_0} to c_0. If we repeatedly use this and the commutative diagram above, we can show that the two sides of the associativity diagram are equal as required. We will not spell out the details however. $\square$

The collection of operads in a symmetric monoidal category is itself a category. The morphisms in this category are defined as follows.

Definition 1.1.7. A morphism of operads $f : (\mathcal{P}, \eta, \gamma) \to (\mathcal{Q}, H, \Gamma)$ is a natural transformation that is compatible with the composition and unit maps. That is to say that the following diagrams should commute:

$$\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[dr, "H"'] & \mathcal{P}(\{*\}) \arrow[d, "f_{\{*\}}"] \\
& \mathcal{Q}(\{*\})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(B) \otimes \bigotimes_{b \in B} \mathcal{P}(A_b) \arrow[r, "\gamma_p"] \arrow[d, "f_B \otimes \bigotimes_{b \in B} f_{A_b}"'] & \mathcal{P}(A) \arrow[d, "f_A"] \\
\mathcal{Q}(B) \otimes \bigotimes_{b \in B} \mathcal{Q}(A_b) \arrow[r, "\Gamma_p"] & \mathcal{Q}(A)
\end{tikzcd}$$

for every map of finite sets $p : A \to B$.

We will now give our first example of an operad.

Example 1.1.8. Let $\mathcal{C}$ be a symmetric monoidal category and $C \in \mathcal{C}$ an object. We define $\mathrm{End}_C : \mathrm{FSET}_{\cong} \to \mathcal{C}$ by setting $\mathrm{End}_C(A) = Hom_{\mathcal{C}}(C^{\otimes A}, C)$. We can turn this into an operad as follows. Define $\eta : 1 \to \mathrm{End}_C(\{*\})$ to be the map that selects the identity map in $Hom_{\mathcal{C}}(C, C)$. For $p : A \to B$, define $\gamma_p : \mathrm{End}_C(B) \otimes \bigotimes_{b \in B} \mathrm{End}_C(A_b)$ by setting

$$\gamma_p(f, (g_b)_{b\in B}) = f \circ \left(\bigotimes_{b \in B} g_b \right).$$

We call this operad the endomorphism operad for C.

Remark 1.1.9. This is the canonical example of an operad and the example that one should keep in mind when thinking about an operad conceptually. The idea of an operad is that each object $\mathcal{P}(A)$ should encode the A-ary operations $C^{\otimes A} \to C$ on some object C. In the case above, we have simply set this to be all of the morphisms from $C^{\otimes A}$ to C. The unit map in the operad is supposed to select an identity operation and the composition maps are supposed to mimic the behaviour we have when we select an element in

$$Hom_{\mathcal{C}}(C^{\otimes B}, C) \otimes \bigotimes_{b \in B} Hom_{\mathcal{C}}(C^{\otimes A_b}, C),$$

and compose the elements according to some map $p : A \to B$.

With this idea of operads encoding operations in mind, we now define an algebra over an operad $\mathcal{P}$. We will see some examples of algebras over operads in the next section.

Definition 1.1.10. Let $C \in \mathcal{C}$ be an object in $\mathcal{C}$. For an operad $\mathcal{P}$ in $\mathcal{C}$, C is a $\mathcal{P}$-algebra if there exist maps $\theta_A : \mathcal{P}(A) \otimes C^{\otimes A} \to C$ that commute with the unit and composition maps in $\mathcal{P}$. More precisely, we should have that

$$C \longrightarrow 1 \otimes C \xrightarrow{\eta \otimes 1} \mathcal{P}(\{*\}) \otimes C \xrightarrow{\theta_{\{*\}}} C$$

is the identity. Then, for any map of finite sets $p : A \to B$, the diagram

$$\begin{tikzcd}
\mathcal{P}(B) \otimes \bigotimes_{b\in B} \mathcal{P}(A_b) \otimes C^{\otimes A} \arrow[r, "\gamma_p \otimes 1"] \arrow[d, "1 \otimes \bigotimes_{b \in B} \theta_{A_b}"'] & \mathcal{P}(A) \otimes C^{\otimes A} \arrow[d, "\theta_A"] \\
\mathcal{P}(B) \otimes C^{\otimes B} \arrow[r, "\theta_B"] & C^{\otimes A}
\end{tikzcd}$$

should commute. If $\mathcal{C}$ is a category with a tensor-hom adjunction, then we can simplify this to say an algebra $C \in \mathcal{C}$ over an operad $\mathcal{P}$ is a morphism of operads $\theta : \mathcal{P} \to \mathrm{End}_C$.

1.2 Some Useful Examples

In this section we will give some examples of operads. We begin by giving some very simple examples that will reinforce the intuition behind an operad. We then define as well as elaborate on three important examples that will be used throughout the thesis: the little cubes operads, the operad of trees and the Stasheff operad.

Example 1.2.1. The simplest example of an operad is one such that $\mathcal{P}(A) = \{*\}$ for $|A| = 1$ and $\mathcal{P}(A) = \emptyset$ otherwise. Algebras over this operad are trivial in the sense that the only operation we have is an identity operation.

Example 1.2.2. A slightly less trivial example of an operad is the commutative operad denoted by Com. Here $\mathrm{Com}(A) = \{*\}$ for every finite set A and therefore the composition is completely determined. Algebras over this operad are commutative, (and associative), monoids.

Example 1.2.3. Denote by $\mathrm{Ass}(A)$ the set of all linear orders on a finite set A. For clarity, one should note that there is a unique linear order on the empty set, $\emptyset \subseteq \emptyset^2$, and so $\mathrm{Ass}(\emptyset) \cong \{*\}$. We define composition for maps $p : A \to B$ as follows. If $R \in \mathrm{Ass}(B)$ and $S_b \in \mathrm{Ass}(A_b)$ for $b \in B$ then set

$$\gamma_p(R, (S_b)) = T$$

where for $a, a' \in A$, $a <_T a'$ if and only if $p(a) <_R p(a')$ or $p(a) = p(a')$ and $a <_{S_{p(a)}} a'$. One can check that this defines an operad that we call the associative operad. Its algebras are associative monoids.

The Little Cubes Operads

We introduce the little N-cubes operads which encode the operations on N-fold loop spaces. This leads to the little N-cubes operads being used to define an entire class of operads, namely the E_N operads.

Definition 1.2.4. Let N be a natural number. Set $J = (0, 1)$ and $\bar{J} = [0, 1]$. Say $u \prec v$ in $\bar{J}^N$ if $u_i < v_i$ for all i and then define $f_{uv} : J^N \to J^N$ for such u and v by setting

$$f_{uv}(t)_i = (1 - t_i)u_i + t_i v_i.$$

Notice that $image(f_{uv}) = (u, v)$ which is the N-dimensional cube with opposite corners u and v. Denote by $\mathrm{C}_N(\mathbf{1})$ the set of all maps of the form f_{uv} for $u, v \in \bar{J}^N$ and $u \prec v$. We call the image of such a map a little N-cube.

Definition 1.2.5. Let A be a finite set. Given any map $f : A \times J^N \to J^N$, for $a \in A$ define $f(a) : J^N \to J^N$ by setting $f(a)(t) = f(a, t)$. Then let $\mathrm{C}_N(A)$ be the set of all injective maps $f : A \times J^N \to J^N$ such that $f(a) \in \mathrm{C}_N(\mathbf{1})$ for all $a \in A$. This is a topological space by considering it to be a subspace of $\bar{J}^{|A| \cdot 2N}$.

Remark 1.2.6. If $A = \emptyset$ then $\emptyset \times J^N \cong \emptyset$ and so $\mathrm{C}_N(\emptyset)$ will be a single point, namely the empty embedding into J^N.

Lemma 1.2.7. *The space $\mathrm{C}_N(A)$ is homotopy equivalent to $\mathrm{Inj}(A, \mathbb{R}^N)$, the space of injective maps from A to $\mathbb{R}^N$.*

Proof. Obviously, we have a homeomorphism $\mathrm{Inj}(A, \mathbb{R}^N) \cong \mathrm{Inj}(A, J^N)$. Now let $m = (\frac{1}{2}, \ldots, \frac{1}{2})$ be the centre of J^N. We then define an embedding $\iota : A \to A \times J^N$ by setting $\iota(a) = (a, m)$, and this in turn defines a map

$$\iota^* : \mathrm{C}_N(A) \to \mathrm{Inj}(A, J^N)$$

given by $\iota^*(f)(a) = (f \circ \iota)(a)$. One sees that this is a homotopy equivalence by considering a homotopy inverse that we will not spell out the details for. One takes an element $g \in \mathrm{Inj}(A, J^N)$ and sends it to the element $f \in \mathrm{C}_N(A)$ such that $image(f(a))$ is centred at $g(a)$ and each little N-cube $f(a)$ is the same size and maximal such that f is injective. $\square$

Definition 1.2.8. Let $p : A \to B$ be a map of finite sets. We then define a map $\gamma_p : \mathrm{C}_N(B) \times \prod_{b\in B} \mathrm{C}_N(A_b) \to \mathrm{C}_N(A)$ by setting

$$\gamma_p(f, (g_b)_{b\in B}) = \coprod_{b\in B} (f(b) \circ g_b)\,.$$

It is a simple check to see that $\coprod_{b\in B} (f(b) \circ g_b) \in \mathrm{C}_N(A)$.

Proposition 1.2.9. *Definition 1.2.8 makes $\mathrm{C}_N = \{\mathrm{C}_N(A)\}$ into an operad which we call the little N-cubes operad.*

Proof. We set the unit to be the identity map in $\mathrm{C}_N(A)$ for $|A| = 1$. It is easy to see that this behaves as required. We then just need to check that the composition map is associative. However this is almost immediate as the definition can be realised as a composition of maps

$$A \times J^N \xrightarrow{(p, g_{p(a)})} B \times J^N \xrightarrow{\quad f \quad} J^N\,.$$

$\square$

Remark 1.2.10. The composition for the little N-cubes operad is best understood diagrammatically with an example. Set $N = 2$, $A = \{a_0, \ldots, a_4\}$, $B = \{b_0, b_1\}$ and $p : A \to B$ defined by $p(a_0) = p(a_1) = p(a_2) = b_0$ and $p(a_3) = p(a_4) = b_1$. We choose elements $f \in \mathrm{C}_N(B)$ and $g_{b_k} \in \mathrm{C}_N(A_{b_k})$ represented by the following diagrams:

Image of f

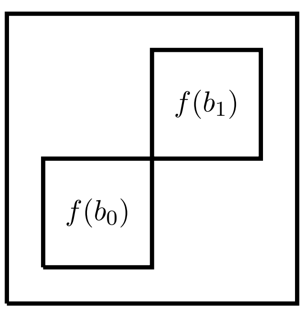


Image of g_{b_0}

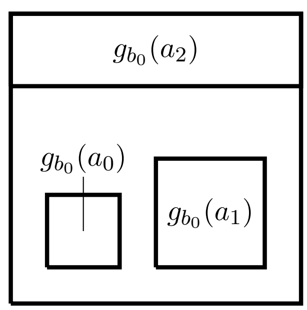
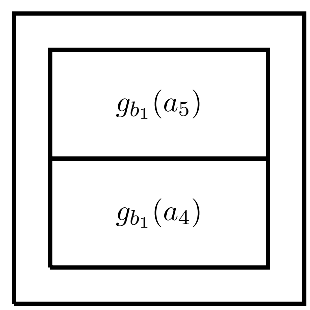
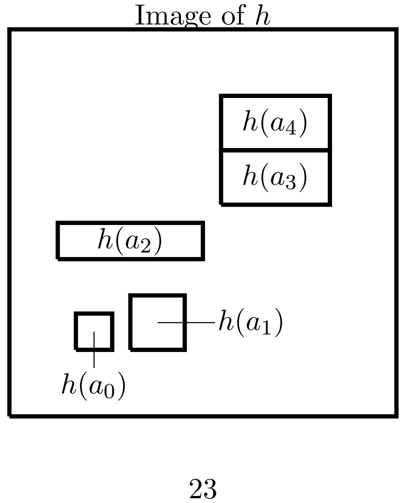


Image of g_{b_1}



The image of each component of the maps is marked with the appropriate label. The tuple (f, g_{b_0}, g_{b_1}) is then sent to the following configuration of little cubes in $\mathrm{C}_N(A)$ by the operad composition $h = \gamma_p(f, g_{b_0}, g_{b_1})$:

Image of h



Definition 1.2.11. Let $(X, *)$ be a based topological space and denote by

$$\Omega^N X = \{l : S^N \to X\},$$

the set of based N-fold loops in X. Let $q : \bar{J}^N \to S^N$ be the well-known quotient map that identifies the boundary of $\bar{J}^N$ to the basepoint of S^N. Notice that this map is a bijection on $J^N \subset \bar{J}^N$. We can therefore define maps

$$\theta_A : \mathrm{C}_N(A) \times (\Omega^N X)^A \to \Omega^N X$$

$$\theta_A(f, (l_a)_{a \in A}) = l : S^N \to X$$

where

$$l(v) = \begin{cases} l_a(q(u)) & \text{if } q^{-1}(v) = f(a)(u) \\ * & \text{otherwise} \end{cases}.$$

One can easily check that this makes $\Omega^N X$ an algebra over the little N-cubes operad.

The following classical theorem is due to May, ([19], theorem 1.3), and demonstrates the usefulness of the little cubes operads.

Theorem 1.2.12. *A path connected, based topological space X is an algebra over the little N-cubes operad C_N if and only if it has the homotopy type of an N-fold loop space $\Omega^N Y$, for some based space Y.*

This importance of the little cubes operads leads to definition 1.2.14 below. First though we define a particular type of equivalence between topological operads.

Definition 1.2.13. A map of topological operads $f : \mathcal{P} \to \mathcal{Q}$ is a weak equivalence if each map $f_A : \mathcal{P}(A) \to \mathcal{Q}(A)$ is a weak equivalence. Two topological operads are said to be weakly equivalent if there exists a zig-zag of weak equivalences between them.

Definition 1.2.14. An E_N operad is one that is weakly equivalent to C_N. An E_∞ operad is one such that each space $E_\infty(A)$ is contractible and has a free action of the permutation group Σ_A. It is possible to define this in terms of a related "infinite" little cubes operad C_∞ but we will not explore this here.

We also have the following interesting result if we consider the operad induced by taking the homology of the little N-cubes operad. We will elaborate on this construction in section 2.3.

Definition 1.2.15. An degree-N Poisson algebra is a graded abelian group P_* together with an associative product $P_i \otimes P_j \to P_{i+j}$ that is commutative in the graded sense, i.e.

$$x \cdot y = (-1)^{|x| \cdot |y|} y \cdot x,$$

and a bracket operation $[\ ,\ ] : P_i \otimes P_j \to P_{i+j+N-1}$ satisfying

- $[x, y] + (-1)^{\bar{x}\bar{y}}[y, x] = 0$,
- $(-1)^{\bar{x}\bar{y}}[x, [y, z]] + (-1)^{\bar{y}\bar{z}}[y, [z, x]] + (-1)^{\bar{z}\bar{x}}[z, [x, y]] = 0$,
- $[x, y \cdot z] = [x, y] \cdot z + (-1)^{\bar{x}|y|} y \cdot [x, z]$

where $\bar{x} = |x| + N - 1$. These are sometimes called Gerstenhaber algebras although more often this means the specific case when $N = 2$.

Theorem 1.2.16. *The operad $H_*(\mathrm{C}_N)$ is that whose algebras are unital degree-N Poisson algebras.*

It appears to be hard to properly attribute a reference to this theorem. Certainly the original description of the groups $H_*(\mathrm{C}_N(A))$ are in the classical work [7]. However there is no mention of operads here. The same author describes Poisson structures arising from these groups in [8]. The case $N = 2$ was done by Getzler in [11]. Although it is not the original source, an elementary treatment of theorem 1.2.16 can be found in [25].

Trees

The set of trees on a finite set A, which we define below, has an operad structure. We also highlight some of the features of a tree which we will use extensively throughout the thesis.

Definition 1.2.17. Let A be a finite set and $\mathbb{P}^*(A)$ denote the set of nonempty subsets of A. A tree on A is a subset $\mathcal{T} \subseteq \mathbb{P}^*(A)$ such that

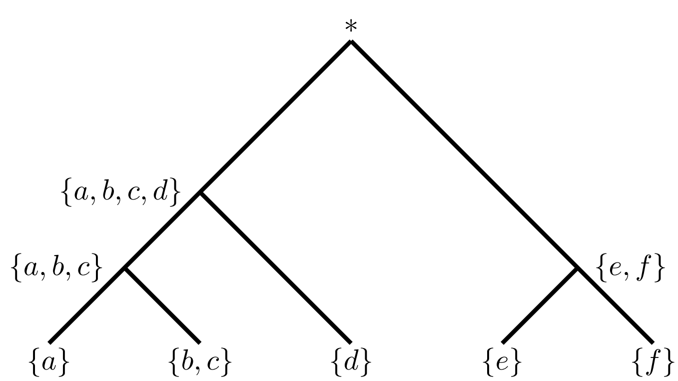
- If $T, U \in \mathcal{T}$ then either $T \subseteq U$, $U \subseteq T$ or $T \cap U = \emptyset$.
- The minimal sets in $\mathcal{T}$ form a partition of A.

We will write $\mathrm{Trees}(A)$ for the set of all trees on A. This can be regarded as a poset by setting $\mathcal{T} \preceq \mathcal{T}'$ if and only if $\mathcal{T} \supseteq \mathcal{T}'$.

Remark 1.2.18. Given a tree $\mathcal{T} \in \mathrm{Trees}(A)$, we can identify this with a connected graph that has no bivalent vertices or cycles, in other words a tree in the more traditional sense. To do this, first notice that $\mathcal{T}$ is a poset as it is a subset of $\mathbb{P}^*(A)$ which is ordered by inclusion of sets. Now adjoin a maximal element $*$ to $\mathcal{T}$, i.e. $T \prec *$ for all $T \in \mathcal{T}$. Then we define the vertex set of our graph to be $\mathcal{T} \cup \{*\}$ and we connect vertices $T, U \in \mathcal{T} \cup \{*\}$ with an edge if $T \prec U$ and there does not exist $V \in \mathcal{T} \cup \{*\}$ such that $T \prec V \prec U$. The vertex $*$ will be the root and the minimal sets of $\mathcal{T}$ the leaves. As an example, let $A = \{a, b, c, d, e, f\}$ and

$$\mathcal{T} = \{\{a, b, c, d\}, \{a, b, c\}, \{e, f\}, \{a\}, \{b, c\}, \{d\}, \{e\}, \{f\}\}.$$

Then the associated graph will be



One should note that this associated graph is the same as the Hasse diagram for the poset $\mathcal{T} \cup \{*\}$. Conversely, if one has a graph Γ that is a rooted tree, (root denoted $*$), with set of leaves isomorphic to A, one can associate to it a tree as in definition 1.2.17. For a vertex $v \in vert(\Gamma)$, we define T_v to be the set of $a \in A$ such that the shortest path from the leaf associated to a to $*$ goes through v. Then $\mathcal{T} = \{T_v \mid v \in vert(\Gamma)\}$ will be a tree.

Definition 1.2.19. We say that $\mathcal{T} \in \mathrm{Trees}(A)$ is separated if every singleton set lies in $\mathcal{T}$ and full if in addition $A \in \mathcal{T}$. Denote by $\mathrm{FTrees}(A)$ the set of full trees on A. This set has a maximal element corresponding to $\mathcal{C}_A = \{A\} \cup \{\text{all singleton sets}\}$ which we call the corolla on A.

Remark 1.2.20. Any tree can be thought of as being separated in the correct context. If $T \in \mathrm{Trees}(A)$ then the minimal sets in $\mathcal{T}$, denote them by B, partition A and every $T \in \mathcal{T}$ is formed by taking a union of some of the elements of B. Therefore $\mathcal{T}$ can be thought of as coming from a separated tree $\widetilde{\mathcal{T}} \in \mathrm{Trees}(B)$.

Definition 1.2.21. Let $\mathcal{T} \in \mathrm{Trees}(A)$ be a separated tree. We highlight the following features of $\mathcal{T}$:

- We define $\mathcal{T}' = \{T \in \mathcal{T} \mid |T| > 1\}$. We call this the set of internal vertices of $\mathcal{T}$.
- For $T \in \mathcal{T}$, we define a child of T to be a maximal element in the set $\{U \in \mathcal{T} \mid U \subset T\}$. (Note that for us, the symbol $\subset$ denotes a proper subset). Denote by $\delta_{\mathcal{T}} T$ the set of children of T and notice that this actually forms a partition of T. Most of the time we will only write δT as the tree $\mathcal{T}$ will be understood from context. We define a grown child of T to be an element in the set $\delta' T = \delta T \cap \mathcal{T}'$.
- We say that a tree $\mathcal{T}$ is binary if $|\delta T| = 2$ for all $T \in \mathcal{T}$ such that T is not minimal.
- For $T \in \mathcal{T}$, we define the parent of T, denoted by $\zeta(T)$, to be the minimal element in the set $\{U \in \mathcal{T} \mid U \supset T\}$.
- Finally, if $\mathcal{T}$ is a full tree on A and $B \subseteq A$ then denote by $\mathcal{T}(B)$ the smallest set in $\mathcal{T}$ that contains B.

Definition 1.2.22. Let $p : A \to B$ be a map of finite sets. Then for $\mathcal{T}_B \in \mathrm{Trees}(B)$ and $\mathcal{T}_b \in \mathrm{Trees}(A_b)$, $b \in B$, we define a map $\gamma_p : \mathrm{Trees}(B) \times \prod_{b \in B} \mathrm{Trees}(A_b) \to \mathrm{Trees}(A)$ by setting

$$\gamma_p(\mathcal{T}_B, (\mathcal{T}_b)_{b \in B}) = \mathcal{T} = p^{-1}\mathcal{T}_B \cup \bigcup_{b \in B} \mathcal{T}_b,$$

where $p^{-1}\mathcal{T}_B = \{p^{-1}(T) \mid T \in \mathcal{T}_B\} \setminus \{\emptyset\}$.

Proposition 1.2.23. *Definition 1.2.22 makes* $\mathrm{Trees} = \{\mathrm{Trees}(A)\}$ *into an operad.*

Proof. Since for $|A| = 1$, $\mathrm{Trees}(A)$ contains only one element, there is only one choice for the unit map. It is a simple check to see that this behaves as a unit with respect to composition. Therefore we only need to check that the map in definition 1.2.22 lands in the specified codomain and is associative. The former is a simple check of cases. Let $U, T \in \mathcal{T}$.

- If $U, T \in p^{-1}\mathcal{T}_B$ such that $U = p^{-1}(U')$ and $T = p^{-1}(T')$ then we will have either $T' \subseteq U'$, $U' \subseteq T'$ or $U' \cap T' = \emptyset$ since $U', T' \in \mathcal{T}_B$. It is then clear that this is preserved by taking preimages.

- If $U \in p^{-1}\mathcal{T}_B$ such that $U = p(U')$ and $T \in \mathcal{T}_b$ then either $b \in U'$ in which case $T \subseteq U$ or $b \notin U'$ which implies that $U \cap T = \emptyset$.

- If $U \in \mathcal{T}_b$ and $T \in \mathcal{T}_{b'}$ such that $b \neq b'$ then $U \cap T = \emptyset$.

- If $U, T \in \mathcal{T}_b$ then the check is trivial since $\mathcal{T}_b$ is a tree on A_b.

Now to check associativity, assume we have surjective maps of finite sets $A \xrightarrow{p} B \xrightarrow{q} C$ and elements $\mathcal{T}_C \in \mathrm{Trees}(C)$, $\mathcal{T}_c \in \mathrm{Trees}(B_c)$ for $c \in C$ and $\mathcal{T}_b \in \mathrm{Trees}(A_b)$ for $b \in B$. Then

$$\gamma_p(\gamma_q(\mathcal{T}_C, (\mathcal{T}_c)_{c\in C}), (\mathcal{T}_b)_{b\in B}) = p^{-1}\left(q^{-1}\mathcal{T}_C \cup \bigcup_{c\in C} \mathcal{T}_c\right) \cup \bigcup_{b\in B} \mathcal{T}_b.$$

Similarly

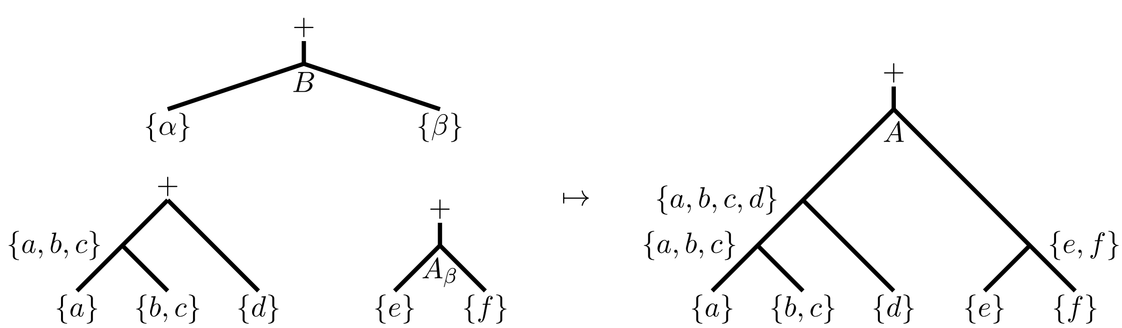
$$\gamma_{qp}(\mathcal{T}_C, (\gamma_p(\mathcal{T}_c, (\mathcal{T}_b)_{b\in B_c}))_{c\in C}) = (qp)^{-1}\mathcal{T}_C \cup \bigcup_{c\in C}\left(p^{-1}\mathcal{T}_c \cup \bigcup_{b\in B_c} \mathcal{T}_b\right),$$

and so $\gamma_p(\gamma_q(\mathcal{T}_C, (\mathcal{T}_c)_{c\in C}), (\mathcal{T}_b)_{b\in B}) = \gamma_{qp}(\mathcal{T}_C, (\gamma_{p_c}(\mathcal{T}_c, (\mathcal{T}_b)_{b\in B_c}))_{c\in C})$ as required. $\square$

Remark 1.2.24. The operad composition we have described above corresponds to the well-known grafting of trees as graphs. This is best illustrated with an example. Let $A = \{a, b, c, d, e, f\}$, $B = \{\alpha, \beta\}$ and define $p : A \to B$ by setting $p(a) = p(b) = p(c) = p(d) = \alpha$ and $p(e) = p(f) = \beta$. Then let

$$\mathcal{T}_B = \{B, \{\alpha\}, \{\beta\}\}$$
$$\mathcal{T}_\alpha = \{\{a, b, c\}, \{b, c\}, \{a\}, \{d\}\}$$
$$\mathcal{T}_\beta = \{A_\beta, \{e\}, \{f\}\}.$$

Then the operad composition produces the following:



Remark 1.2.25. It is not hard to check that if $\mathcal{T}_b$ is a full tree on A_b for all $b \in B$, and $\mathcal{T}_B$ is a full tree on B, then $\gamma_p(\mathcal{T}_B, (\mathcal{T}_b)_{b\in B}) \in \mathrm{FTrees}(A)$. Therefore $\mathrm{FTrees} = \{\mathrm{FTrees}(A)\}$ is a suboperad of Trees.

Remark 1.2.26. It is worth noting that $\mathrm{Trees}(\emptyset) = \mathrm{Trees}(\{*\}) = \mathrm{FTrees}(\{*\}) = \{*\}$ but $\mathrm{FTrees}(\emptyset) = \emptyset$. Therefore Trees is not a reduced operad but FTrees is.

Lemma 1.2.27. *The composition map* $\gamma_p : \mathrm{FTrees}(B) \times \prod_{b\in B} \mathrm{FTrees}(A_b) \to \mathrm{FTrees}(A)$ *is injective for any surjective* $p : A \to B$.

Proof. Let
$$\gamma_p(\mathcal{T}_B, (\mathcal{T}_b)_{b\in B}) = \mathcal{V} = \gamma_p(\mathcal{U}_B, (\mathcal{U}_b)_{b\in B}).$$
For $b \in B$ define $\mathcal{V}_b = \{V \in \mathcal{V} \mid V \subseteq A_b\}$. By the definition of γ_p it is easy to see that
$$\mathcal{T}_b = \mathcal{V}_b = \mathcal{U}_b.$$
Now let $p\mathcal{V} = \{p(V) \mid V \in \mathcal{V}\}$. If $V \in \mathcal{T}_b = \mathcal{U}_b$ for some $b \in B$ then $p(V) = b$. Therefore we see that
$$\mathcal{T}_B = p\mathcal{V} = \mathcal{U}_B$$
so γ_p is injective as required. □

The Stasheff Operad

One could argue that the Stasheff operad is the very first example of an operad, although it was not formulated in this way. However, the definition of the spaces involved and their application to A_∞ algebras predates May's original definition of an operad. We present one of many ways to define this operad.

Definition 1.2.28. Let A be a finite totally ordered set. A Stasheff tree on A is a full tree $\mathcal{T}$ such that each $T \in \mathcal{T}$ is an interval in A, i.e. if $a, c \in T$ then $b \in T$ for all $a \leq b \leq c$.

Definition 1.2.29. Let A be a finite set equipped with a total order R. Denote by $\mathcal{J}(A, R)$ the set of intervals $J \subseteq A$ with respect to the ordering R. Then we define $K(A, R)$ to be the set of maps $t : \mathcal{J}(A, R) \to [0, 1]$ such that

- $\mathrm{supp}(t) = \{J \in \mathcal{J}(A, R) \mid t(J) > 0\}$ is a Stasheff tree on A.
- $t(A) = 1$ and $t(\{a\}) = 1$ for all $a \in A$.

We then define $K(A) = \coprod_{R\in \mathrm{Ord}(A)} K(A, R)$.

Lemma 1.2.30. *The space $K(A, R)$ is contractible for any finite set A and total order R.*

Proof. Consider $t^* \in K(A, R)$ defined by
$$t^*(J) = \begin{cases} 1 & \text{if } J = A \text{ or } J = \{a\} \\ 0 & \text{otherwise} \end{cases}.$$
We can then produce a deformation retraction on to this point by defining
$$h : [0, 1] \times K(A, R) \to K(A, R),$$
$$h(\lambda, t)(J) = \lambda t(J) + (1 - \lambda)t^*(J)$$
which is equivalent to saying
$$h(\lambda, t)(J) = \begin{cases} 1 & \text{if } J = A \text{ or } J = \{a\} \\ \lambda \cdot t(J) & \text{otherwise} \end{cases}.$$
□

Definition 1.2.31. Let $p : A \to B$ be a surjective map of finite sets. We then define a map $\gamma_p : K(B) \times \prod_{b\in B} K(A_b) \to K(A)$ by setting $\gamma_p((r, R), (s_b, S_b)_{b\in B}) = (t, T)$ where

- $a <_T a'$ if and only if $(p(a) <_R p(a'))$ or $(p(a) = p(a') = b$ and $a <_{S_b} a')$.
- $t(J) = \begin{cases} s_b(J) & \text{if } p(J) = \{b\} \\ r(p(J)) & \text{if } J = p^{-1}p(J) \\ 0 & \text{otherwise} \end{cases}$.

The map t is well-defined since if $p(J) = \{b\}$ and $J = p^{-1}p(J)$ simultaneously, then $r(p(J)) = r(\{b\}) = 1$ and $J = p^{-1}\{b\} = A_b$ so $s_b(J) = s_b(A_b) = 1$ also.

Proposition 1.2.32. *Definition 1.2.31 makes the collection $K = \{K(A)\}$ into an operad which we call the Stasheff operad.*

Proof. We need to check that the composition map γ_p satisfies the associativity rule. On the second component, the total order on A, this is clear since the map acts in exactly the same way as the composition for the associative operad Ass. Therefore we only need to check on the first component. Let $A \xrightarrow{p} B \xrightarrow{q} C$ be maps between finite sets and $r \in K(C)$, $s_c \in K(B_c)$ and $t_b \in K(A_b)$. We need to check that

$$u = \gamma_p(\gamma_q(r, (s_c)_{c\in C}), (t_b)_{b\in B}) = \gamma_{qp}(r, (\gamma_{p_c}(s_c, (t_b)_{b\in B_c}))_{c\in C}) = u'.$$

One can check by considering cases that

$$u(J) = u'(J) = \begin{cases} t_b(J) & \text{if } p(J) = \{b\} \\ s_c(p(J)) & \text{if } J = p^{-1}p(J) \text{ and } qp(J) = \{c\} \\ r(qp(J)) & \text{if } J = (qp)^{-1}qp(J) \\ 0 & \text{otherwise} \end{cases}$$

as required. □

Definition 1.2.33. An A_∞ operad is an operad that is weakly equivalent to the associative operad Ass and is cofibrant. Equivalently, an A_∞ operad is a cofibrant E_1 operad, as defined in 1.2.14.

Proposition 1.2.34. *The Stasheff operad is an A_∞ operad.*

Proof. Since $K(A, R)$ is contractible, the projection maps $\theta_A : K(A) \to \mathrm{Ass}(A)$ defined by $\theta_A(r, R) = R$ are homotopy equivalences. It is then immediate from the definitions that the collection $\theta = \{\theta_A\}$ is a map of operads $\theta : K \to \mathrm{Ass}$. We will see in theorem 1.4.10 that the Stasheff operad is cofibrant. □

Remark 1.2.35. We have presented just one formulation of the Stasheff operad of which there are many. However, the one we demonstrate here will be more appropriate for our uses later on in the thesis.

1.3 Free Operads and Well-Labelled Operads

For now, let $\mathcal{C}$ be the category of sets. In this section we briefly outline the free operad construction for reduced operads in this category. We then introduce the notion of a well-labelled operad which allows us to give explicit conditions for when an operad is set-theoretically free.

Definition 1.3.1. A Σ-module M in $\mathcal{C}$ is a functor from $\text{FSET}_{\cong}$ to $\mathcal{C}$. Similar to operads, a Σ-module is called reduced if $M(\emptyset) = \emptyset$ and $M(\{*\}) = \{*\}$.

Definition 1.3.2. Let M be a reduced Σ-module. For any finite set A define

$$\Phi M(A) = \{(\mathcal{T}, (m_T)_{T\in\mathcal{T}'}) \mid \mathcal{T} \in \text{FTrees}(A),\ m_T \in M(\delta T)\}.$$

Since $\text{FTrees}(\emptyset) = \emptyset$ we have $\Phi M(\emptyset) = \emptyset$. Also, if A has only one point, then $\text{FTrees}(A)$ contains only one tree that has no internal vertices, and so $\Phi M(A) = \{*\}$. If $f : A \to B$ is a map in $\text{FSET}_{\cong}$ then define $\Phi(f)$ to be the map:

$$(\mathcal{T}, (m_T)_{T\in\mathcal{T}'}) \mapsto (f(\mathcal{T}), (Mf_T(m_T)_{T\in\mathcal{T}'})$$

where $f(\mathcal{T}) = \{f(T) \mid T \in \mathcal{T}\}$ and $f_T : \delta T \to \delta f(T)$ is the obvious map that identifies children. Therefore, ΦM is a reduced Σ-module which we call the module of decorated trees in M.

Definition 1.3.3. Let $p : A \to B$ be a surjective map of finite sets. We then define a map

$$\gamma_p : \Phi M(B) \times \prod_{b\in B} \Phi M(A_b) \to \Phi M(A)$$

by setting

$$\gamma_p((\mathcal{T}_B, (m_T)_{T\in\mathcal{T}_B'}), (\mathcal{T}_b, (n_{b,T})_{T\in\mathcal{T}_b'})_{b\in B}) = (\mathcal{T}, (o_T)_{T\in\mathcal{T}'})$$

where

- $\mathcal{T} = p^{-1}\mathcal{T}_B \cup \bigcup_{b\in B} \mathcal{T}_b$ and so is simply the grafting of trees.
- $o_T = n_{b,T}$ if $T \subseteq A_b$. If $T = p^{-1}U$ for some $U \in \mathcal{T}_B'$ then we have a bijection $f : \delta U \to \delta p^{-1}U$ and so we set $o_T = Mf(m_U) \in M(\delta p^{-1}U)$.

Proposition 1.3.4. *Definition 1.3.3 makes ΦM into an operad.*

Proof. Since $\Phi M(A)$ is a single point when $|A| = 1$, there is no choice for the unit map and it is easy to see that it behaves as expected with regards to γ_p. We have proved in proposition 1.2.23 that the grafting of trees is an associative composition and so it is easily seen that γ_p is associative as the new decorations are trivially defined. □

Definition 1.3.5. We define a morphism of Σ-modules $\iota : M \to \Phi M$ by setting, for any finite set A,

$$\iota_A(m) = (\mathcal{C}_A, m),$$

i.e. the corolla on A decorated by $m \in M(A)$.

We have the following universal property for the construction Φ and as such it is a left-adjoint to the forgetful functor from reduced operads to reduced Σ-modules. First though, we define an iterated composition for a general operad that we shall use in the proof.

Definition 1.3.6. Let $\mathcal{P}$ be an operad in $\mathcal{C}$ and $\mathcal{T}$ a full tree on A. We can then define an iterated composition map

$$\gamma_{\mathcal{T}} : \bigotimes_{T\in\mathcal{T}'} \mathcal{P}(\delta T) \to \mathcal{P}(A)$$

as follows. Firstly, if $|\mathcal{T}'| = 1$ then we simply let $\gamma_\mathcal{T}$ be the identity on $\mathcal{P}(\delta A) \cong \mathcal{P}(A)$. Now for each $U \in \delta A$, we define $\mathcal{T}_U = \{T \in \mathcal{T} \mid T \subseteq U\}$ which is a full tree on U. Since $|\mathcal{T}_U'| < |\mathcal{T}'|$, by an inductive hypothesis we have a definition for $\gamma_{\mathcal{T}_U}$. Let $p : A \to \delta A$ be the obvious projection map. We then set

$$\gamma_\mathcal{T}\left((x_T)_{T\in\mathcal{T}'}\right) = \gamma_p\left(x_A, \left(\gamma_{\mathcal{T}_U}\left((x_T)_{T\in\mathcal{T}_U'}\right)\right)_{U\in\delta A}\right).$$

Proposition 1.3.7. *Let $\mathcal{P}$ be a reduced operad in $\mathcal{C}$ and $f : M \to \mathcal{P}$ a morphism of Σ-modules. Then there is a unique morphism of operads $\tilde{f} : \Phi M \to \mathcal{P}$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "f"'] & \Phi M \arrow[d, "\tilde{f}"] \\ & \mathcal{P} \end{tikzcd}$$

Proof. Define $\Phi_0 M$ to be the image of ι, i.e. $\Phi_0 M(A) = \{(\mathcal{C}_A, m) \mid m \in M(A)\}$. It is then not difficult to show that every element in ΦM can be described uniquely, up to composition with the identity, as the image of an iterated composition map on elements in $\Phi_0 M(A)$. The rest of the proof is then a standard free object argument. $\square$

Now we present the notion of a well-labelled operad. We can define this for operads in any category for which the objects have an underlying set. It turns out that this gives a precise way of deciding when an operad is set-theoretically free.

Definition 1.3.8. Let I be a finite set. An I-labelled set is a set X along with subsets $L_i X \subseteq X$ for every $i \in I$. If $J \subseteq I$ then we define $L_J X = \bigcap_{j\in J} L_j X$.

Definition 1.3.9. Let $\mathcal{P}$ be a reduced operad and A a non-empty finite set. Then for all non-empty $B \subseteq A$, we have a composition map $\gamma_B^A : \mathcal{P}(A/B) \otimes \mathcal{P}(B) \to \mathcal{P}(A)$. We define

$$L_B\mathcal{P}(A) = image(\gamma_B^A)$$

which makes $\mathcal{P}(A)$ into a $\mathbb{P}^*(A)$-labelled set where $\mathbb{P}^*(A)$ is the set of non-empty subsets of A.

Remark 1.3.10. One should notice that $L_A\mathcal{P}(A) = L_{\{a\}}\mathcal{P}(A) = \mathcal{P}(A)$ for any reduced operad $\mathcal{P}$ and $a \in A$. Therefore, if $J \subseteq \mathbb{P}^*(A)$ and $J' = J \cup \mathcal{C}_A$ then $L_J\mathcal{P}(A) = L_{J'}\mathcal{P}(A)$.

Remark 1.3.11. It is easy to see, by the associativity of the composition, that the iterated composition map $\gamma_\mathcal{T}$ from definition 1.3.6 factors through γ_T^A for all $T \in \mathcal{T}$ and therefore

$$image(\gamma_\mathcal{T}) \subseteq \bigcap_{T\in\mathcal{T}} L_T\mathcal{P}(A) = L_\mathcal{T}\mathcal{P}(A).$$

This allows us to make definition 1.3.12 below.

Definition 1.3.12. An operad $\mathcal{P}$ is well-labelled if for every non-empty finite set A and $J \subseteq \mathbb{P}^*(A)$ we have the following:

- If $J \in \text{Trees}(A)$ then γ_J is injective and $image(\gamma_J) = L_J\mathcal{P}(A)$.

- If J is not a tree on A then $L_J\mathcal{P}(A)$ is empty.

Proposition 1.3.13. *Let $\mathcal{P}$ be a well-labelled operad. Then it is (set-theoretically) isomorphic to ΦM where the Σ-module M is defined by*

$$M(A) = \mathcal{P}(A) \setminus \bigcup_{1<|B|<|A|} L_B\mathcal{P}(A).$$

Proof. Clearly we have a map of Σ-modules $f : M \to \mathcal{P}$ simply defined by inclusion. Then by proposition 1.3.7, there is a unique map of operads $\tilde{f} : \Phi(M) \to \mathcal{P}$. We claim that $\tilde{f}_A$ is a bijection for every finite set A.

Firstly, if $|A| \le 2$ then $M(A) = \mathcal{P}(A)$ and $\Phi M(A) \cong M(A)$ since $\mathrm{FTrees}(A)$ only contains the corolla on A. So then f_A must be a bijection. We now proceed by induction on $|A|$. Let $x \in L_B\mathcal{P}(A)$ for some $B \subset A$ with $|B| > 1$. Then because $\mathcal{P}$ is well-labelled, $x = \gamma_B^A(y_A, y_B)$ for unique $y_A \in \mathcal{P}(A/B)$ and $y_B \in \mathcal{P}(B)$. Since $1 < |A/B| < |A|$, by induction $y_A = \tilde{f}_{A/B}(\mathcal{T}_A, (m_T)_{T\in\mathcal{T}'_A})$ and $y_B = \tilde{f}_B(\mathcal{T}_B, (n_T)_{T\in\mathcal{T}'_B})$ for unique $(\mathcal{T}_A, (m_T)_{T\in\mathcal{T}'_A}) \in \Phi M(A/B)$ and $(\mathcal{T}_B, (n_T)_{T\in\mathcal{T}'_B}) \in \Phi M(B)$. Therefore, because $\tilde{f}$ is a map of operads, we must have

$$x = \tilde{f}_A\left(\gamma_B^A\left(\left(\mathcal{T}_A, (m_T)_{T\in\mathcal{T}'_A}\right), \left(\mathcal{T}_B, (n_T)_{T\in\mathcal{T}'_B}\right)\right)\right)$$

so $\tilde{f}$ is surjective. It is also injective by the injectivity of $\gamma_{\mathcal{T}}$ for every $\mathcal{T} \in \mathrm{FTrees}(A)$ since intersections of $L_B\mathcal{P}(A)$'s are only non-empty if the collection of B's forms a tree. ☐

Example 1.3.14. The following operads are well-labelled:

- The operad of full trees, FTrees.
- The Stasheff operad K, (this will be immediate from the work in section 1.4).
- ΦM for any reduced Σ-module M, by proposition 1.3.13.

We will see other examples of well-labelled operads in later parts of the thesis. We end this section by giving an alternative viewpoint to a well-labelled operad.

Proposition 1.3.15. *$\mathcal{P}$ is a well-labelled operad if and only if there is a map of operads $\tau : \mathcal{P} \to \mathrm{FTrees}$ such that every square of type*

$$\begin{tikzcd}
\mathcal{P}(B) \times \prod_{b\in B} \mathcal{P}(A_b) \arrow[r, "\gamma_p"] \arrow[d, "{(\tau_B, (\tau_{A_b})_{b\in B})}"'] & \mathcal{P}(A) \arrow[d, "\tau_A"] \\
\mathrm{FTrees}(B) \times \prod_{b\in B} \mathrm{FTrees}(A_b) \arrow[r, "\gamma_p"'] & \mathrm{FTrees}(A)
\end{tikzcd}$$

is a pullback.

Proof. First assume that $\mathcal{P}$ is well-labelled. Then by proposition 1.3.13 it is isomorphic to ΦM for some Σ-module M. Define $\tau : \Phi M \to \mathrm{FTrees}$ by setting $\tau_A(\mathcal{T}, (m_T)_{T\in\mathcal{T}'}) = \mathcal{T}$. This is clearly a map of operads. To see that the square in the hypothesis is a pullback, we compare $\Phi M(B) \times \prod_{b\in B} \Phi M(A_b)$ with the pullback of the diagram

$$\begin{tikzcd}
 & \Phi M(A) \arrow[d, "\tau_A"] \\
\mathrm{FTrees}(B) \times \prod_{b\in B} \mathrm{FTrees}(A_b) \arrow[r, "\gamma_p"'] & \mathrm{FTrees}(A)
\end{tikzcd} .$$

This is equal to

$$\left\{\left((\mathcal{T}_B, (\mathcal{T}_b)_{b\in B}), \left(\mathcal{T}_A, (m_T)_{T\in\mathcal{T}'_A}\right)\right) \;\middle|\; \gamma_p(\mathcal{T}_B, (\mathcal{T}_B)_{b\in B}) = \mathcal{T}_A\right\}.$$

On the other hand $\Phi M(B) \times \prod_{b\in B} \Phi M(A_b)$ contains all elements of the form

$$((\mathcal{T}_B, (n_T)_{T\in\mathcal{T}'_B}), (\mathcal{T}_b, (o_{b,T})_{T\in\mathcal{T}'_b})_{b\in B})$$

with $n_T, o_{b,T} \in M(\delta T)$. When we compose these elements, these decorations simply become the decorations $m_T \in M(\delta T)$ of the tree $\mathcal{T}_A = \gamma_p(\mathcal{T}_B, (\mathcal{T}_B)_{b\in B})$ and so it is easy to see that these two sets are isomorphic.

Conversely, assume we have a map of operads $\tau : \mathcal{P} \to \mathrm{FTrees}$ such that every commutative diagram of compositions is a pullback. In particular, one can deduce that the combinatorial compositions:

$$\begin{tikzcd}
\mathcal{P}(A/B) \times \mathcal{P}(B) \arrow[r, "\gamma_B^A"] \arrow[d, "\tau_{A/B}\times\tau_B"'] & \mathcal{P}(A) \arrow[d, "\tau_A"] \\
\mathrm{FTrees}(A/B) \times \mathrm{FTrees}(B) \arrow[r, "\gamma_B^A"'] & \mathrm{FTrees}(A)
\end{tikzcd}$$

are also pullbacks. Recall that $\gamma_{\mathcal{T}}$ factors through γ_T^A for all $T \in \mathcal{T}$. We can therefore use the pasting lemma for pullbacks to show that

$$\begin{tikzcd}
\prod_{T\in\mathcal{T}'} \mathcal{P}(\delta T) \arrow[r, "\gamma_{\mathcal{T}}"] \arrow[d, "(\tau_T)_{T\in\mathcal{T}'}"'] & \mathcal{P}(A) \arrow[d, "\tau_A"] \\
\prod_{T\in\mathcal{T}'} \mathrm{FTrees}(\delta T) \arrow[r, "\gamma_{\mathcal{T}}"'] & \mathrm{FTrees}(A)
\end{tikzcd}$$

is also a pullback square. By a repeated use of lemma 1.2.27, we see that $\gamma_{\mathcal{T}}$, (bottom arrow), is injective and therefore $\gamma_{\mathcal{T}}$, (top arrow), is also injective.

Now define

$$\tau_A^{-1}(\mathcal{T} \subseteq) = \{p \in \mathcal{P}(A) \mid \tau_A(p) \supseteq \mathcal{T}\}.$$

First we claim that $\tau_A^{-1}(\mathcal{T} \subseteq) = image(\gamma_{\mathcal{T}})$ if $\mathcal{T}$ is not the corolla, (since this case is trivial anyway). Let $p \in \tau_A^{-1}(\mathcal{T} \subseteq)$. Then there exists $(\mathcal{T}_T)_{T\in\mathcal{T}'} \in \prod_{T\in\mathcal{T}'} \mathrm{FTrees}(\delta T)$ such that $\gamma_{\mathcal{T}}(\mathcal{T}_T) = \tau_A(p)$. Therefore $((\mathcal{T}_T)_{T\in\mathcal{T}'}, p)$ is in the pullback and so corresponds to some $(p_T)_{T\in\mathcal{T}'} \in \prod_{T\in\mathcal{T}'} \mathcal{P}(\delta T)$. Then because the composition square is a pullback, this means $\gamma_{\mathcal{T}}((p_T)_{T\in\mathcal{T}'}) = p$ and so $p \in image(\gamma_{\mathcal{T}})$. On the other hand, let $p \in image(\gamma_{\mathcal{T}})$ so that $p = \gamma_{\mathcal{T}}((p_T)_{T\in\mathcal{T}'})$ for some $(p_T)_{T\in\mathcal{T}'} \in \prod_{T\in\mathcal{T}'} \mathcal{P}(\delta T)$. Then $\tau_A(p) = \gamma_{\mathcal{T}}((\tau_T(p_T)))$. $(p_T)_{T\in\mathcal{T}'}$ corresponds to some $((\mathcal{T}_T)_{T\in\mathcal{T}'}, p)$ in the pullback and so $\tau_A(p) = \gamma_{\mathcal{T}}((\mathcal{T}_T))$. But each $\mathcal{T}_T$ contains T and so $\mathcal{T} \subseteq \gamma_{\mathcal{T}}((\mathcal{T}_T)) = \tau_A(p)$ as required.

Next we claim that $\tau_A^{-1}(\mathcal{T} \subseteq) = L_{\mathcal{T}}\mathcal{P}(A) = \bigcap_{T\in\mathcal{T}'} L_T\mathcal{P}(A)$ where $\mathcal{T}$ is not the corolla. We will then have an equality $image(\gamma_{\mathcal{T}}) = L_{\mathcal{T}}\mathcal{P}(A)$. Firstly we see that $\tau_A^{-1}(\mathcal{T} \subseteq) = image(\gamma_{\mathcal{T}}) \subseteq L_{\mathcal{T}}\mathcal{P}(A)$. Therefore take $p \in L_{\mathcal{T}}\mathcal{P}(A)$ which implies that for all $T \in \mathcal{T}' \setminus A$ there exists $(p_T) \in \mathcal{P}(A/T) \times \mathcal{P}(T)$ such that $\gamma(p_T) = p$. Therefore $p \in image(\gamma_T^A)$ and so $T \in \tau_A(p)$.

Finally, assume that we have some $J \subseteq \mathbb{P}^*(A)$ that is not a tree on A and such that $L_J\mathcal{P}(A)$ is non-empty. By the above argument, this would mean that $J \subseteq \tau_A(p)$ for all $p \in L_J\mathcal{P}(A)$ which cannot happen. Therefore $L_J\mathcal{P}(A)$ must be empty and so $\mathcal{P}$ is well-labelled. $\square$

1.4 The W-construction

In this final section we describe the W-construction originally due to Boardman and Vogt in [5]. We only do this for reduced topological operads although the original definition caters for all topological operads. For all cases that we will consider, this construction creates a cofibrant replacement for a topological operad and so it encodes the up to homotopy algebras of that operad.

Definition 1.4.1. Let $\mathcal{P}$ be a reduced operad in the category of topological spaces. For every finite set A we define a new set $\widetilde{W}\mathcal{P}(A)$. This is the set of triples $(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}})$ such that

- $\mathcal{T}$ is a full tree on A.
- For $T \in \mathcal{T}'$, $x_T \in \mathcal{P}(\delta T)$ which we refer to as the decoration of T.
- For $T \in \mathcal{T}$, $\lambda_T \in [0, 1]$ such that $\lambda_A = 1$ and $\lambda_{\{a\}} = 1$ for all $a \in A$. We refer to this as the edge length of T.

Now we define $W\mathcal{P}(A) = \widetilde{W}\mathcal{P}(A)/\sim$ where the equivalence relation removes edges with length 0. More precisely, let $V \in \mathcal{T} \setminus \{A\}$ such that $\lambda_V = 0$. Then set $\widetilde{\mathcal{T}} = \mathcal{T} \setminus \{V\}$ and $\widetilde{\lambda}_T = \lambda_T$ for $T \in \widetilde{\mathcal{T}}$. Let $U = \zeta(T) \in \mathcal{T}$ be the parent of V and denote by $\delta_{\widetilde{\mathcal{T}}}T$ the set of children of $T \in \widetilde{\mathcal{T}}$. Now, for $T \in \widetilde{\mathcal{T}} \setminus \{U\}$ we have $\delta_{\widetilde{\mathcal{T}}}T = \delta_{\mathcal{T}}T$ and so in this case we set $\widetilde{x}_T = x_T$. Next we see that

$$\delta_{\widetilde{\mathcal{T}}}U = (\delta_{\mathcal{T}}U \setminus \{V\}) \cup \delta_{\mathcal{T}}V.$$

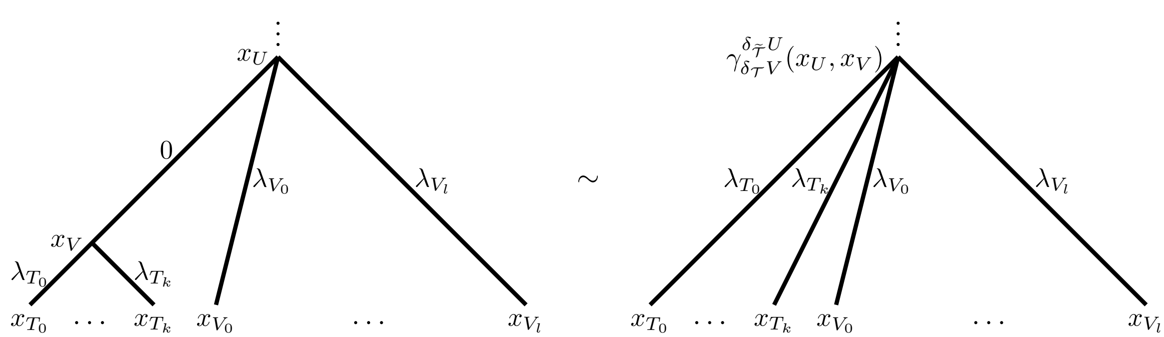
Therefore $\delta_{\mathcal{T}}U \cong \delta_{\widetilde{\mathcal{T}}}U/\delta_{\mathcal{T}}V$ and so we have an operad composition map

$$\gamma^{\delta_{\widetilde{\mathcal{T}}}U}_{\delta_{\mathcal{T}}V} : \mathcal{P}(\delta_{\mathcal{T}}U) \times \mathcal{P}(\delta_{\mathcal{T}}V) \to \mathcal{P}(\delta_{\widetilde{\mathcal{T}}}U).$$

Set $\widetilde{x}_U = \gamma^{\delta_{\widetilde{\mathcal{T}}}U}_{\delta_{\mathcal{T}}V}(x_U, x_V)$ and define the equivalence relation to be generated by

$$(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \quad \sim \quad \left(\widetilde{\mathcal{T}}, (\widetilde{x}_T)_{T\in\widetilde{\mathcal{T}}'}, (\widetilde{\lambda}_T)_{T\in\widetilde{\mathcal{T}}}\right).$$

This can be summarised pictorially as



Remark 1.4.2. Using the equivalence relation above, one sees that $W\mathcal{P}(A)$ maps bijectively to the set of $(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \in W\mathcal{P}(A)$ such that $\lambda_T > 0$ for all $T \in \mathcal{T}$. However, it is much harder to describe the topology with this definition, hence our original definition 1.4.1.

Definition 1.4.3. Let $p : A \to B$ be a surjective map of finite sets. We then define a composition map
$$\gamma_p : W\mathcal{P}(B) \times \prod_{b \in B} W\mathcal{P}(A_b) \to W\mathcal{P}(A)$$
by setting
$$\gamma_p((\mathcal{T}_B, (x_T)_{T \in \mathcal{T}'_B}, (\lambda_T)_{T \in \mathcal{T}_B}), ((\mathcal{T}_b, (y_{b,T})_{T \in \mathcal{T}'_b}, (\mu_{b,T})_{T \in \mathcal{T}_b}))_{b \in B}) = (\mathcal{T}, (z_T)_{T \in \mathcal{T}'}, (\omega_T)_{T \in \mathcal{T}})$$
where

- $\mathcal{T} = p^{-1}\mathcal{T}_B \cup \bigcup_{b \in B} \mathcal{T}_b$.
- $z_T = y_{b,T}$ if $T \subseteq A_b$. If $T = p^{-1}U$ for some $U \in \mathcal{T}_B$ then we have a bijection $f : \delta U \to \delta p^{-1}U$ and so we set $z_T = \mathcal{P}f(x_U) \in \mathcal{P}(\delta p^{-1}U)$.
- $$\omega_T = \begin{cases} \mu_{b,T} & \text{if } T \subseteq A_b \\ \lambda_U & \text{if } T = p(U) \text{ for some } U \in \mathcal{T}_B \end{cases}.$$

This construction is well defined since the edge lengths of A and the singleton sets are equal to 1. In particular this means that where we have overlap in the definition of T, i.e. when $p^{-1}\{b\} = A_b$, then there is no contradiction in the definition of ω_{A_b}. Also, edges of length 0 are contained in either $p^{-1}\mathcal{T}_B$ or $\mathcal{T}_b$ for some $b \in B$ so it is automatic that γ_p respects the equivalence relations.

Proposition 1.4.4. *Definition 1.4.3 makes $W\mathcal{P}$ into an operad. Moreover W is an endofunctor for operads in topological spaces. We henceforth refer to this functor as the W-construction.*

Proof. If $|A| = 1$ then $W\mathcal{P}(A)$ will be a single point, i.e. the unique full tree on A, with no decorations since it has no vertices of size greater than 1. Therefore there is no choice for the unit map and it is easy to see that it behaves as expected with regards to γ_p. One easily sees that the composition is associative as we have already proved in proposition 1.2.23 that the grafting of full trees is associative, and then everything else is trivially defined.

If $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of operads then we can define a morphism
$$f : W\mathcal{P} \to W\mathcal{Q}$$
by setting $f(\mathcal{T}, (x_T)_{T \in \mathcal{T}'}, (\lambda_T)_{T \in \mathcal{T}}) = (\mathcal{T}, (f(x_T))_{T \in \mathcal{T}'}, (\lambda_T)_{T \in \mathcal{T}})$. This is well-defined since f commutes with the compositions in $\mathcal{P}$ and $\mathcal{Q}$. It is then obvious by inspecting the definition that this will be functorial. □

It should be clear from the definitions that there is a connection between the W-construction and the free operad in sets. Indeed, we have the following easy result.

Proposition 1.4.5. *$W\mathcal{P}$ is a well-labelled operad for any operad $\mathcal{P}$ in topological spaces.*

Proof. Clearly we have a map of operads $\tau : W\mathcal{P} \to \text{FTrees}$ which is simply defined by projection,
$$\tau_A(\mathcal{T}, (x_T)_{T \in \mathcal{T}'}, (\lambda_T)_{T \in \mathcal{T}}) = \mathcal{T}.$$
Then, we can apply proposition 1.3.15 by showing that $W\mathcal{P}(B) \times \prod_{b \in B} W\mathcal{P}(A_b)$ is isomorphic to the appropriate pullback. But this is easy to see since the operad of full trees is well-labelled and the decorations in the W-construction are trivially defined. □

Remark 1.4.6. On the other hand, if we consider $\mathcal{P}$ as an operad in the category of sets, then there is an evident map

$$\widetilde{W}\mathcal{P}(A) \twoheadrightarrow \Phi\mathcal{P}(A)$$

given by forgetting the edge lengths in $\widetilde{W}\mathcal{P}(A)$. By the universal property of the free operad construction Φ, we also have a map $\Phi\mathcal{P}(A) \to \mathcal{P}(A)$ which comes from completing the diagram in proposition 1.3.7 when considering the identity map $\mathcal{P}(A) \to \mathcal{P}(A)$. These maps fit together into a commutative diagram

$$\begin{tikzcd} \widetilde{W}\mathcal{P}(A) \arrow[r, two heads] \arrow[d, two heads] & W\mathcal{P}(A) \arrow[d, dotted] \\ \Phi\mathcal{P}(A) \arrow[r] & \mathcal{P}(A). \end{tikzcd}$$

The right hand arrow is forced and is described in lemma 1.4.7 below. This is basically the same as the bottom arrow in the diagram above. The key point to remember is that points that are equivalent in $\widetilde{W}\mathcal{P}(A)$ have the same image in $\mathcal{P}(A)$.

Lemma 1.4.7. *The operad $W\mathcal{P}$ is homotopy equivalent to $\mathcal{P}$. More specifically, there is a map of operads $W\mathcal{P} \to \mathcal{P}$ such that each map $W\mathcal{P}(A) \to \mathcal{P}(A)$ is a homotopy equivalence.*

Proof. For every $(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \in W\mathcal{P}(A)$, we wish to assign an element $z \in \mathcal{P}(A)$. To begin with let $(\mathcal{C}_A, x_A, (1)) \in W\mathcal{P}(A)$. We then simply send $(\mathcal{C}_A, x_A, (1))$ to $x_A \in \mathcal{P}(\delta A) \cong \mathcal{P}(A)$. Now we perform induction on $|\mathcal{T}|$. Let $(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \in W\mathcal{P}(A)$ and for each $U \in \delta A$ define an element $(\mathcal{T}_U, (x_T)_{T\in\mathcal{T}'_U}, ((\lambda_T)_{T\in\mathcal{T}_U\setminus\{U\}}, 1_U))$. Here

$$\mathcal{T}_U = \{T \in \mathcal{T} \mid T \subseteq U\}$$

and is a full tree on U. Notice that $|\mathcal{T}_U| < |\mathcal{T}|$ for all $U \in \delta A$ and so by our inductive hypothesis we have assigned an element $z_U \in \mathcal{P}(U)$. We have a surjective map $p : A \to \delta A$ which sends $a \in A$ to the unique child that contains it. Therefore we may assign

$$(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \mapsto z = \gamma_p(x_A, (z_U)_{U\in\delta A}).$$

This gives us a continuous map $\epsilon_A : W\mathcal{P}(A) \to \mathcal{P}(A)$ which is in fact homotopic to the identity on $W\mathcal{P}(A)$ via the homotopy that linearly collapses edge lengths to 0, and therefore a homotopy equivalence. It is easy to see that the collection $\epsilon = \{\epsilon_A\}$ comes together to form a morphism of operads. $\square$

For symmetric monoidal categories $\mathcal{C}$ satisfying certain conditions, one can transfer a model structure from $\mathcal{C}$ to the category of operads in $\mathcal{C}$. This requires a certain amount of categorical machinery which we will not go through here. A full account can be found in [3]. However, we wish to highlight the following point. The cofibrant objects in the category of operads are those operads that encode algebras that satisfy conditions up to all coherent homotopies. The classic demonstrative example of such an algebra is the space of loops ΩX on some based space X. This is an algebra where the associativity conditions are satisfied up to all coherent homotopies and as such it is an A_∞ algebra. With this in mind, the following theorem, ([28], theorem 4.1), highlights the usefulness of the W-construction.

Theorem 1.4.8. *Let $\mathcal{P}$ be a well-pointed operad such that $\mathcal{P}(A)$ is cofibrant for all A. Then $W\mathcal{P}$ is a cofibrant replacement for $\mathcal{P}$ with respect to the model structure introduced in* [28].

Note that an operad is well-pointed if the inclusion of the identity operation is a cofibration in topological spaces, (with the usual Quillen model structure). So in other words, the theorem says that the W-construction produces an operad $W\mathcal{P}$ that encodes up to homotopy P-algebras. We can apply this to some of our previous examples.

Example 1.4.9. Consider the reduced commutative operad Com and recall that $\mathrm{Com}(A)$ is a single point for any non-empty A. If we apply the W-construction, then for some $(\mathcal{T}, (x_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \in W\,\mathrm{Com}(A)$, the decorations $(x_T)_{T\in\mathcal{T}'}$ are trivial and so we might as well omit them. We call the space $W\,\mathrm{Com}(A)$ the space of metric trees on A, as it is essentially the set of full trees on A where the edges are assigned a length.

It is easy to see that $W\,\mathrm{Com}(A)$ is contractible for any A since we can take $(\mathcal{T}, (\lambda_T)_{T\in\mathcal{T}})$ and shrink all of the internal edge lengths to 0, (except the one for A). The equivalence relation on $W\,\mathrm{Com}(A)$ then specifies that the element we are left with is $(\mathcal{C}_A, (1)_{T\in\mathcal{C}_A})$. The action of Σ_A is not free however as the corolla in $W\,\mathrm{Com}(A)$ is fixed by all elements of Σ_A. The point is though that E_∞ operads are supposed to encode algebras that are commutative up to all coherent homotopies and the operations have a free action of the symmetric group. Therefore, showing that our application of the W-construction to Com is "almost" an E_∞ operad is a nod in the right direction.

Theorem 1.4.10. *There exists an isomorphism of operads $\Phi : WK \to K$. In particular, this means that the Stasheff operad is cofibrant.*

Proof. Fix a finite non-empty set A. We then define a map $\widetilde{\Phi}_A : \widetilde{W}K(A) \to K(A)$ by setting

$$\widetilde{\Phi}_A(\mathcal{T}, (t_T, R_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) = (t, R) \in K(A)$$

where we define (t, R) as follows. For $T \in \mathcal{T}'$ let $\pi_T : T \to \delta T$ be the obvious projection map. Firstly for $a \neq a'$ in A, if $T = \mathcal{T}(\{a, a'\})$ then $\pi_T(a) \neq \pi_T(a')$ in δT. We have a total order R_T on δT and so we set

$$a <_R a' \qquad \Leftrightarrow \qquad \pi_T(a) <_{R_T} \pi_T(a').$$

It should be clear from this definition that each $T \in \mathcal{T}$ is then an interval in A with respect to R. Furthermore, for any $T \in \mathcal{T}'$, $\pi_T^{-1}(J)$ will be an interval for any $J \in \mathcal{J}(\delta T, R_T)$. We can then define $t : \mathcal{J}(A, R) \to [0, 1]$ by setting

$$t(J) = \begin{cases} \frac{1}{2}(\lambda_J + 1) & \text{if } J \in \mathcal{T} \\ \frac{1}{2} t_{\mathcal{T}(J)}(\pi_{\mathcal{T}(J)}(J)) & \text{if } J \notin \mathcal{T} \text{ and } \pi_{\mathcal{T}(J)}^{-1}\pi_{\mathcal{T}(J)}(J) = J \\ 0 & \text{otherwise} \end{cases}.$$

The map t is well-defined since $A \in \mathcal{T}$ with $\lambda_A = 1$ and $\{a\} \in \mathcal{T}$ for all $a \in A$ also with $\lambda_{\{a\}} = 1$. Therefore $t(A) = \frac{1}{2}(1 + 1) = 1$ and $t(\{a\}) = \frac{1}{2}(1 + 1) = 1$. Also

$$\mathrm{supp}(t) = \bigcup_{T\in\mathcal{T}'} \pi_T^{-1}\,\mathrm{supp}(t_T)$$

and it is an easy check to see that this is a full tree on A, and necessarily a Stasheff tree.

It is easy to see that $\widetilde{\Phi}$ is a continuous map. We then claim that this in fact factors through the quotient map $\widetilde{W}K(A) \to WK(A)$ by showing that equivalent elements have the same image. Indeed let $(\mathcal{T}, (t_T, R_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \in \widetilde{W}K(A)$ such that $\lambda_V = 0$ for some $V \in \mathcal{T}'$. If

$\widetilde{\Phi}_A(\mathcal{T}, (t_T, R_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) = (t, R)$ then $t(V) = \frac{1}{2}$. Now let $(\widetilde{\mathcal{T}}, (\widetilde{t}_T, \widetilde{R}_T)_{T\in\widetilde{\mathcal{T}}'}, (\widetilde{\lambda}_T)_{T\in\widetilde{\mathcal{T}}}) \in \widetilde{W}K(A)$ be the equivalent element with V removed and set $\widetilde{\Phi}_A(\widetilde{\mathcal{T}}, (\widetilde{t}_T, \widetilde{R}_T)_{T\in\widetilde{\mathcal{T}}'}, (\widetilde{\lambda}_T)_{T\in\widetilde{\mathcal{T}}}) = (t', R')$. Firstly, a simple inspection of the definition of composition in the Stasheff operad will reveal that $R = R'$. If $U = \zeta(V)$ is the parent of V then one can calculate that $\widetilde{t}_U(\widetilde{\pi}_U(V)) = 1$ and so $t'(V) = \frac{1}{2}$. Everything else remains the same and so $t = t'$ as required. This defines a map $\Phi_A : WK(A) \to K(A)$ which is continuous because $WK(A)$ has the quotient topology.

Next, we show that Φ_A is bijective by defining an inverse. This is sufficient to show Φ_A is a homeomorphism since it will be a continuous bijective map from a compact space to a Hausdorff space. We define $\Theta_A : K(A) \to WK(A)$ by setting

$$\Theta_A(t, R) = (\mathcal{T}, (t_T, R_T)_{T\in\mathcal{T}'}, (\lambda_T)_{T\in\mathcal{T}}) \in WK(A).$$

Here

- $\mathcal{T} = \{J \in \operatorname{supp}(t) \mid t(J) > \frac{1}{2}\}$.
- For $T \in \mathcal{T}'$ set

  $$\mathcal{J}(A;T) = \{J \in \mathcal{J}(A, R) \mid J \subseteq T \text{ and } J = \bigcup U_i \text{ for some } U_i \in \delta T\}.$$

  It is easy to see that $\mathcal{J}(\delta T, R|_{\delta T}) \cong \mathcal{J}(A;T)$ and so we can define $t_T = 2t|_{\mathcal{J}(A;T)} \wedge 1$. Also, because each $U \in \delta T$ is an interval with respect to R, it induces a total order on δT which we set to be R_T.
- For $T \in \mathcal{T}$ set $\lambda_T = 2t(T) - 1$.

It is an elementary check to see that this is indeed an inverse for Φ_A, so long as it is well-defined. This is not immediate only for checking that $t_T \in K(\delta T, R_T)$. First we see that $t_T(T) = 1$ since $2t(T) > 1$ and similarly $t_T(U) = 1$ for all $U \in \delta T$. To see that $\operatorname{supp}(t_T)$ is a tree first notice that $\operatorname{supp}(t) \subseteq \bigcup_{T\in\mathcal{T}'} \mathcal{J}(A;T)$ because $\operatorname{supp}(t)$ is itself a tree. Then $\operatorname{supp}(t_T) \cong \mathcal{J}(A;T) \cap \operatorname{supp}(t)$ and so is itself a tree.

Finally, we show that the collection $\Phi = \{\Phi_A\}$ is a map of operads. The preservation of the unit is trivial as $WK(A) \cong K(A)$ when $|A| = 1$ and is a single point. Therefore we only need to show that Φ_A preserves composition. For a surjective map of finite sets $p : A \to B$, let γ_p denote the composition in WK and Γ_p the composition in K. Let

$$((\mathcal{T}_B, (q_T, Q_T)_{T\in\mathcal{T}_B'}, (\lambda_T)_{T\in\mathcal{T}_B}), (\mathcal{T}_b, (r_{b,T}, R_{b,T})_{T\in\mathcal{T}_b'}, (\mu_{b,T})_{T\in\mathcal{T}_b})_{b\in B}) \in WK(B) \times \prod_{b\in B} WK(A_b),$$

$$\begin{aligned}&(\mathcal{T}, (s_T, S_T)_{T\in\mathcal{T}'}, (\omega_T)_{T\in\mathcal{T}}) = \\ &\qquad \gamma_p((\mathcal{T}_B, (q_T, Q_T)_{T\in\mathcal{T}_B'}, (\lambda_T)_{T\in\mathcal{T}_B}), (\mathcal{T}_b, (r_{b,T}, R_{b,T})_{T\in\mathcal{T}_b'}, (\mu_{b,T})_{T\in\mathcal{T}_b})_{b\in B}),\end{aligned}$$

$$\begin{aligned}&((q, Q), (r_b, R_b)_{b\in B}) = \\ &\qquad (\Phi_B, (\Phi_{A_b})_{b\in B})((\mathcal{T}_B, (q_T, Q_T)_{T\in\mathcal{T}_B'}, (\lambda_T)_{T\in\mathcal{T}_B}), (\mathcal{T}_b, (r_{b,T}, R_{b,T})_{T\in\mathcal{T}_b'}, (\mu_{b,T})_{T\in\mathcal{T}_b})_{b\in B}),\end{aligned}$$

$$(s, S) = \Phi_A(\mathcal{T}, (s_T, S_T)_{T\in\mathcal{T}'}, (\omega_T)_{T\in\mathcal{T}}),$$

$$(s', S') = \Gamma_p((q, Q), (r_b, R_b)_{b\in B}).$$

Therefore, we need to show that $(s, S) = (s', S')$. Firstly, one should notice that S is the result of a nested composition of $(S_T)_{T\in\mathcal{T}'}$ when considered as elements in the associative operad Ass. This clearly commutes with composing the orderings in the Stasheff operad and so $S = S'$. Now

$$s(J) = \begin{cases} \frac{1}{2}(\omega_J + 1) & \text{if } J \in \mathcal{T} \\ \frac{1}{2}s_{\mathcal{T}(J)}(\pi_{\mathcal{T}(J)}(J)) & \text{if } J \notin \mathcal{T} \text{ and } \pi^{-1}_{\mathcal{T}(J)}\pi_{\mathcal{T}(J)}(J) = J \\ 0 & \text{otherwise} \end{cases},$$

and even more explicitly

$$s(J) = \begin{cases} \frac{1}{2}(\mu_{b,J} + 1) & \text{if } J \in \mathcal{T}_b \\ \frac{1}{2}(\lambda_U + 1) & \text{if } J = p^{-1}(U) \text{ for some } U \in \mathcal{T}_B \\ \frac{1}{2}r_{b,\mathcal{T}_b(J)}(\pi_{\mathcal{T}_b(J)}(J)) & \text{if } \ p(J) = b,\ J \notin \mathcal{T}_b \text{ and } \pi^{-1}_{\mathcal{T}_b(J)}\pi_{\mathcal{T}(J)}(J) = J \\ \frac{1}{2}q_{\mathcal{T}_B(U)}(\pi_{\mathcal{T}_B(U)}(U)) & \text{if } J = p^{-1}(U),\ U \notin \mathcal{T}_B \text{ and } \pi^{-1}_{\mathcal{T}_B(U)}\pi_{\mathcal{T}_B(U)}(U) = U \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand

$$s'(J) = \begin{cases} r_b(J) & \text{if } p(J) = b \\ q(p(J)) & \text{if } J = p^{-1}p(J) \\ 0 & \text{otherwise} \end{cases},$$

which we can again write out more explicitly as

$$s'(J) = \begin{cases} \frac{1}{2}(\mu_{b,J} + 1) & \text{if } J \in \mathcal{T}_b \\ \frac{1}{2}r_{b,\mathcal{T}_b(J)}(\pi_{\mathcal{T}_b(J)}(J)) & \text{if } \ p(J) = b,\ J \notin \mathcal{T}_b \text{ and } \pi^{-1}_{\mathcal{T}_b(J)}\pi_{\mathcal{T}(J)}(J) = J \\ \frac{1}{2}(\lambda_{p(J)} + 1) & \text{if } J = p^{-1}p(J) \text{ and } p(J) \in \mathcal{T}_B \\ \frac{1}{2}q_{\mathcal{T}_B(p(J))}(\pi_{\mathcal{T}_B(p(J))}(p(J))) & \text{if } J = p^{-1}p(J),\ p(J) \notin \mathcal{T}(B) \\ & \qquad \text{and } \pi^{-1}_{\mathcal{T}_B(p(J))}\pi_{\mathcal{T}_B(p(J))}(p(J)) = p(J) \\ 0 & \text{otherwise} \end{cases}$$

and therefore $s = s'$ as required. □

Remark 1.4.11. We could in fact prove theorem 1.4.10 by instead showing that there is an isomorphism of operads $\Phi : W\,\mathrm{Ass}_r \to K$. However, the statement we choose to prove is more appropriate for work later in the thesis. Here, Ass_r is the reduced associative operad where $\mathrm{Ass}_r(\emptyset) = \emptyset$ and $\mathrm{Ass}_r(A) = \mathrm{Ass}(A)$ otherwise. The composition is defined in exactly the same way as for Ass.

Chapter 2

The Singh Model

In this chapter, we present an alternative definition for the Fulton-Macpherson operads. The Ph.D. thesis of Daniel Singh, [23], introduces some complex projective varieties which give new models for moduli spaces of stable n-pointed curves of genus zero. These spaces can be assembled into an operad but this was not considered by Singh. Here we introduce our model for the Fulton-Macpherson operads which is constructed in an analogous way. This approach has a more algebraic feel than others and as such makes calculation more tractable. Along the way we will give elementary proofs for useful properties of the Fulton-Macpherson operads as well as detail an explicit isomorphism of operads between K and $\bar{F}_1$. Although this is a well known equivalence, I have not seen such an explicit map anywhere in the literature previously.

2.1 Configuration Spaces

The spaces in the Fulton-Macpherson operads are canonical compactifications of Euclidean configuration spaces. We will see later in this chapter that these compactifications have the same homotopy type as their respective configuration spaces. The aim of this section is to give an explicit description of the homology of the configuration spaces, which will be useful when we come to consider the homology of the Fulton-Macpherson operads.

Definition 2.1.1. Let A be a finite set and N a natural number. Denote by $\mathrm{Inj}(A, \mathbb{R}^N)$ the space of injective maps from A to $\mathbb{R}^N$. We can define an equivalence relation on this space by setting for $f, g \in \mathrm{Inj}(A, \mathbb{R}^N)$

$$f \sim g \qquad \Leftrightarrow \qquad f = \lambda g + v,$$

where $\lambda > 0$ and $v \in \mathbb{R}^N$ represents the constant map $v : A \to \mathbb{R}^N$ taking the value v. Denote by $F_N(A)$ the space $\mathrm{Inj}(A, \mathbb{R}^N)/\sim$.

Remark 2.1.2. It is not hard to see that $\mathrm{Inj}(A, \mathbb{R}^N)$ is homeomorphic to $\mathbb{R}^N \times (0, \infty) \times F_N(A)$ and so in particular $F_N(A)$ is homotopy equivalent to $\mathrm{Inj}(A, \mathbb{R}^{\mathbb{N}})$.

Remark 2.1.3. If $|A| > 1$ and we require explicit representatives for elements in $F_N(A)$ then it will be natural to consider the space

$$F_N(A) = \left\{ f \in \mathrm{Inj}(A, \mathbb{R}^N) \;\middle|\; \sum_{a \in A} f(a) = 0, \sum_{a \in A} \|f(a)\|^2 = 1 \right\}.$$

Of course, this is not the only set of representatives that one can consider and sometimes we will specify alternatives.

We will now explore the homotopy type further by giving a description of the (integral) cohomology $H^*(F_N(A))$. Although this is classical work, it will be useful later on to have an explicit description of this ring.

Remark 2.1.4. If $N = 1$ then it is not difficult to show that $\mathrm{Inj}(A, \mathbb{R}^N)$ is simply the disjoint union of $|A|!$ contractible components. Each of these components is labelled by an element $R \in \mathrm{Ass}(A)$ and is equal to

$$\{f \in \mathrm{Inj}(A, \mathbb{R}^N) \mid f(a) \leq f(b) \Leftrightarrow a \leq_R b\}.$$

Therefore we can say easily what the cohomology of $F_N(A)$ is in this case.

In light of remark 2.1.4, we assume for now that $N > 1$.

Definition 2.1.5. Let $a, b \in A$ with $a \neq b$ and denote by S^{N-1} the $(N-1)$-dimensional sphere. We then define a map $\pi_{ab} : F_N(A) \to S^{N-1}$ by setting

$$\pi_{ab}(x) = \frac{x(a) - x(b)}{\|x(a) - x(b)\|}.$$

This definition requires a choice of representative but it is easy to see that it is independent of this choice. Now let u be the canonical generator of $H^{N-1}(S^{N-1})$ and set $u_{ab} = \pi_{ab}^*(u) \in H^{N-1}(F_N(A))$.

Lemma 2.1.6. $u_{ba} = (-1)^{N-1} u_{ab}$ *and* $u_{ab}^2 = 0$.

Proof. π_{ba} is π_{ab} composed with $N-1$ reflections, each having degree -1 which demonstrates the first relation. The second is clear because $H^{2N-2}(S^{N-1}) = 0$. □

If $|A| = 1$ then $F_N(A)$ is a single point and so $H^*(F_N(A)) = \mathbb{Z}$. Also, if $|A| = 2$ then $\pi_{ab} : F_N(A) \to S^{N-1}$ is in fact a homeomorphism and so

$$H^*(F_N(A)) = \frac{\mathbb{Z}[u_{ab}, u_{ba}]}{(u_{ba} + (-1)^{N-1} u_{ab}, u_{ab}^2)}.$$

Now for $|A| > 2$, we consider one final relation.

Definition 2.1.7. Let $A = \{a, b, c\}$ and then define

$$r_{abc} = u_{ab} u_{bc} + u_{bc} u_{ca} + u_{ca} u_{ab}.$$

Notice that $r_{abc} = r_{bca}$ and $r_{bac} = (-1)^{N-1} r_{abc}$ so that r_{abc} only depends on the set $\{a, b, c\}$ up to sign.

For the next part we work with $\mathrm{Inj}(A, \mathbb{R}^N)$ as opposed to $F_N(A)$. However what we discover in cohomology will still be valid for both since they are homotopy equivalent spaces.

Definition 2.1.8. Let $A = \{a, b, c\}$. We define several maps:

$$\pi = (\pi_{ab}, \pi_{bc}, \pi_{ca}) : \mathrm{Inj}(A, \mathbb{R}^N) \to (S^{N-1})^3$$

as well as $\Delta : S^{N-1} \to (S^{N-1})^2$, $f_+, f_- : (S^{N-1})^2 \to \mathrm{Inj}(A, \mathbb{R}^N)$ and $g_+, g_- : (S^{N-1})^2 \to S(N-1)^3$ given by

$$\Delta(v) = (v, v),$$
$$f_\pm(v, w) = (a \mapsto -v, b \mapsto v, c \mapsto \pm(v - w)),$$
$$g_+(v, w) = (v, -w, -v),$$
$$g_-(v, w) = (v, -v, -w).$$

Proposition 2.1.9. *Let $A = \{a, b, c\}$ and consider the following diagram:*

$$\begin{tikzcd}
S^{N-1} \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] & & (S^{N-1})^2 \arrow[dl, "f_+"'] \arrow[dd, "g_+"] \\
& \mathrm{Inj}(A, \mathbb{R}^N) \arrow[dr, "\pi"] & \\
(S^{N-1})^2 \arrow[ur, "f_-"] \arrow[rr, "g_-"'] & & (S^{N-1})^3
\end{tikzcd}$$

Then the top left triangle commutes on the nose and the rest of the diagram commutes up to homotopy. Moreover, the top left triangle is a homotopy pushout.

Proof. It is easy to check the stated commutativities of the diagram and so we simply show the final statement. Let W be the pushout of the diagram

$$(S^{N-1})^2 \xleftarrow{\Delta} S^{N-1} \xrightarrow{\Delta} (S^{N-1})^2 ,$$

i.e. $W = (S^{N-1})^2 \amalg (S^{N-1})^2 / ((v, v) \sim (v, v))$. Let $X = \{f \in \mathrm{Inj}(A, \mathbb{R}^N) \mid f(a) + f(b) = 0, \|f(a)\| = \|f(b)\| = 1\}$. One can see from slightly adapting the implied map in remark 2.1.2 that this is a deformation retract of $\mathrm{Inj}(A, \mathbb{R}^N)$. If we then consider $Y \subseteq X$ defined by

$$Y = \{f \in X \mid \|f(c) - f(a)\| = 1 \text{ or } \|f(c) - f(b)\| = 1\},$$

then one can check that $f_+ \amalg f_- : W \to Y$ is a well-defined homeomorphism. Therefore, we just need to show that Y is a deformation retract of X. To do this we define a map $r : X \to Y$ by setting $r(-v, v, w) = (-v, v, q(-v, v, w))$ where

$$q(-v, v, w) = \begin{cases} v + \frac{w-v}{\|w-v\|} & \text{if } 0 < \|w - v\| \le 1 \\ \frac{w+v}{\|w+v\|} - v & \text{if } 0 < \|w + v\| \le 1 \\ 0 & \text{if } \|w - v\|, \|w + v\| \ge 1 \text{ and } w = 0 \\ \frac{2|w \cdot v| w}{\|w\|^2} & \text{if } \|w - v\|, \|w + v\| \ge 1 \text{ and } w \ne 0 \end{cases}.$$

Notice that $\|w - v\|, \|w + v\| \ge 1$ implies that $2|w \cdot v| \le \|w\|^2$ and so in the fourth case, $\|q(-v, v, w)\| \le \|w\|$ which implies continuity as w tends to 0. Also notice that if any two of the conditions are satisfied then $q(-v, v, w) = w$. Therefore q is a well-defined continuous map. To see that it indeed lands in Y, one can check that

- If $0 < \|w - v\| \le 1$ then $\|q(-v, v, w) - v\| = 1$.
- If $0 < \|w + v\| \le 1$ then $\|q(-v, v, w) + v\| = 1$.
- If $\|w - v\|, \|w + v\| \ge 1$ and $w \cdot v \ge 0$ then $\|q(-v, v, w) - v\| = 1$.
- If $\|w - v\|, \|w + v\| \ge 1$ and $w \cdot v \le 0$ then $\|q(-v, v, w) + v\| = 1$.

One easily checks that if $(-v, v, w) \in Y$ then $q(-v, v, w) = w$ and so r is the identity on Y and therefore a retraction. It is clear that the line segment joining $(-v, v, w) \in X$ to $r(-v, v, w) \in Y \subseteq X$ lies wholly in X and so we can define a deformation retraction as required. ☐

Proposition 2.1.10. *Let $A = \{a, b, c\}$. Then $r_{abc} = 0$ in $H^{2N-2}(F_N(A))$.*

Proof. By proposition 2.1.9, we have a Mayer-Vietoris sequence

$$\xrightarrow{(\Delta^*,\Delta^*)} H^{2N-3}(S^{N-1}) \longrightarrow H^{2N-2}(F_N(A)) \xrightarrow{(f_+^*,f_-^*)} H^{2N-2}((S^{N-1})^2) \oplus H^{2N-2}((S^{N-1})^2) \longrightarrow .$$

We claim that (Δ^*, Δ^*) is surjective, which by the exactness of the sequence, will imply that (f_+^*, f_-^*) is injective. This is simple since we can easily define a retraction for Δ and so Δ^* is surjective. Then this surjectivity implies that one can use the restriction $(\Delta^*, \Delta^*)|_{H^d((S^{N-1})^2) \oplus \{e\}}$, where e is the neutral element, to demonstrate that (Δ^*, Δ^*) is also surjective.

Now it is sufficient to prove that $f_\pm^*(r_{abc}) = 0$ in $H^{2N-2}((S^{N-1})^2)$. Let μ_0 and μ_1 be the two canonical generators of $H^{N-1}((S^{N-1})^2)$. Then using the fact that $\pi \circ f_+ = g_+$ we see that $f_+^*(u_{ab}) = \mu_0$, $f_+^*(u_{bc}) = (-1)^N \mu_1$ and $f_+^*(u_{ca}) = (-1)^N \mu_0$. Putting this all together then gives

$$f_+^*(r_{abc}) = (-1)^N \mu_0\mu_1 + (-1)^{2N} \mu_1\mu_0 + (-1)^N \mu_0^2 = (-1)^N \mu_0\mu_1 + (-1)^{N-1} \mu_0\mu_1 = 0.$$

We can also perform a similar calculation for $f_-^*(r_{abc})$. $\square$

Corollary 2.1.11. *Let A be any finite set and $a, b, c \in A$ all distinct. Then $r_{abc} = 0$ in $H^{2N-2}(F_N(A))$.*

Proof. Let $\rho : F_N(A) \to F_N(\{a, b, c\})$ be the obvious restriction map which induces a ring homomorphism $\rho^* : H^*(F_N(\{a, b, c\})) \to H^*(F_N(A))$. Then r_{abc} in $H^{2N-2}(F_N(A))$ will be equal to ρ^* evaluated on r_{abc} in $H^{2N-2}(F_N(\{a, b, c\}))$ which is 0 by 2.1.10. $\square$

It turns out that the above generators and relations can be used to completely describe $H^*(F_N(A))$. We summarise this in the theorem below which we state without proof. This theorem applies for $N > 0$.

Theorem 2.1.12. *$H^*(F_N(A))$ is isomorphic to the free graded commutative ring over $\mathbb{Z}$ on elements u_{ab} for distinct $a, b \in A$ with order $N-1$. These elements are subject to the relations*

$$\begin{aligned} u_{ab}^2 &= 0, \\ u_{ba} &= (-1)^{N-1} u_{ab}, \\ r_{abc} = u_{ab}u_{bc} + u_{bc}u_{ca} + u_{ca}u_{ab} &= 0 \qquad \textit{for distinct } a, b, c \in A. \end{aligned}$$

Moreover, this means that the top dimension of $H^*(F_N(A))$ is equal to $(N-1)(|A|-1)$,

Proof. See [7] lemma 6.2. $\square$

Remark 2.1.13. It is in fact an intermediate step of the proof of theorem 2.1.12 that gives us the following alternative description. Let A be a finite set and $B = A \setminus \{a\}$ for any $a \in A$. Then we have an isomorphism of rings

$$H^*(F_N(A)) \cong H^*(F_N(B)) \otimes H^* \left(\bigvee_{i=0}^{|B|-1} S^{N-1} \right).$$

Alternatively we can interpret this as saying $H^*(F_N(A))$ is freely generated by $\{u_{ab} \mid b \in B\}$ as a module over $H^*(F_N(B))$. This isomorphism comes from the Fadell-Neuwirth fibrations in [9]. We discuss these fibrations further in section 4.4

2.2 The Compactifications $\bar{F}_N(A)$

In this section we will define the spaces $\bar{F}_N(A)$ which are compactifications of the configuration space $\mathrm{Inj}(A,\mathbb{R}^N)$ modulo translation and scaling. We will give elementary proofs of many of the well known results for the spaces $\bar{F}_N(A)$ before eventually assembling them into an operad $\bar{F}_N$.

Definition 2.2.1. Let A be a finite set and N a natural number. Denote by $\mathrm{Map}(A,\mathbb{R}^N)$ the space of maps from A to $\mathbb{R}^N$. We can define an equivalence relation on this space by setting for $f,g\in\mathrm{Map}(A,\mathbb{R}^N)$

$$f\sim g \qquad \Leftrightarrow \qquad f=g+v,$$

where $v\in\mathbb{R}^N$ represents the constant map $v:A\to\mathbb{R}^N$ taking the value v. Denote by $W_N(A)$ the space $\mathrm{Map}(A,\mathbb{R}^N)/\sim$.

Remark 2.2.2. One can show that $W_N(A)$ is still a vector space by setting $[f]+[g]=[f+g]$ for $f,g\in\mathrm{Map}(A,\mathbb{R}^N)$ and $\lambda\cdot[f]=[\lambda\cdot f]$ for $\lambda\in\mathbb{R}$. In certain scenarios we will want $W_N(A)$ to also be an inner product space. To achieve this we will have to identify $W_N(A)$ with the subspace

$$\left\{ f\in\mathrm{Map}(A,\mathbb{R}^N) \;\middle|\; \sum_{a\in A} f(a)=0 \right\}$$

and then use the obvious inner product induced by $\mathrm{Map}(A,\mathbb{R}^N)$, which is in turn the obvious one induced by the standard inner product on $\mathbb{R}^N$.

Definition 2.2.3. Let V be a finite dimensional vector space. A ray in V is a set of the form $\{\lambda v \mid \lambda\geq 0\}$ where $v\in V$ and is non-zero. Denote by $S(V)$ the set of all rays in V. Of course, if V is an inner product space then we can identify $S(V)$ with the unit sphere. Now suppose we have a surjective map $f:V\to W$. We then define

$$S(V,W)=\{(x,y)\in S(V)\times S(W)\mid x\subseteq f^{-1}(y)\}.$$

Lemma 2.2.4. *Let V and W be inner product spaces and $f:V\to W$ a surjective map. Then $S(V,W)\cong B(U)\times S(W)$ where $U\subseteq V$ is the kernel of f and $B(U)=\{u\in U\mid \|u\|\leq 1\}$.*

Proof. We may identify f with an orthogonal projection $V=U\oplus W\to W$ so then the definition of $S(V,W)$ becomes

$$\begin{aligned} S(V,W) &= \{(u_v,w_v,w)\in U\times W\times W \mid \|u_v\|^2+\|w_v\|^2=\|w\|^2=1 \\ &\qquad\qquad\qquad\qquad\qquad \text{and } w_v=\lambda w \text{ for some } \lambda\geq 0\} \\ &= \{(u,\sqrt{1-\|u\|^2}w,w)\mid (u,w)\in B(U)\times S(W)). \end{aligned}$$

$\square$

Remark 2.2.5. Despite the explicit nature of the description in lemma 2.2.4, this picture is not particularly useful for our purposes. This is because in what follows, we will be considering surjective maps $V\to W$ for a variety of vector spaces W and so needing an orthogonal decomposition of V for each W would be inconvenient.

Definition 2.2.6. Let A be a finite set with $|A| > 1$ and $x \in \prod_{B\subseteq A,|B|>1} S(W_N(B))$. If we have $C \subseteq B$ then we can define the obvious restriction map

$$\rho^B_C : W_N(B) \to W_N(C).$$

This is surjective and so $S(W_N(B), W_N(C))$ is defined. We say that x is coherent if for all $C \subseteq B$ we have $(x_B, x_C) \in S(W_N(B), W_N(C))$. Denote by $\bar{F}_N(A)$ the set of all coherent elements in $\prod_{B\subseteq A,|B|>1} S(W_N(B))$.

Remark 2.2.7. Instead of considering equivalence classes, we can alternatively set

$$S(W_N(A)) = \left\{ x : A \to \mathbb{R}^N \;\middle|\; \sum_{a\in A} x(a) = 0, \sum_{a\in A} \|x(a)\|^2 = 1 \right\}.$$

We then have a normalisation map $\eta : \mathrm{Map}(A, \mathbb{R}^N) \to S(W_N(A)) \cup \{0\}$ given by

$$x \mapsto \tilde{x},$$

where

$$\tilde{x}(a) = \frac{x(a) - \sum_{a'\in A} x(a')}{\sum_{a'\in A} \|x(a) - \sum_{a'\in A} x(a')\|^2}$$

if x is non-constant and $\tilde{x}$ is the zero map if x is constant. If we form an analogous definition of $\bar{F}_N(A)$ using these representatives then the coherence condition translates to requiring that for all $C \subseteq B$ we have

$$\widetilde{x_B|_C} \in \{x_C, 0\}.$$

I.e. we require that $x_B|_C$ is either x_C modulo translation and scaling or it is constant.

Remark 2.2.8. If we have a surjective map of finite sets $\pi : A \to B$ then this induces a map

$$\pi^* : S(W_N(B)) \to S(W_N(A))$$

which is given by precomposition with π.

We now give a stratification of $\bar{F}_N(A)$ by full trees on A. By this we mean that we will assign a full tree $\mathcal{T}(x) \in \mathrm{FTrees}(A)$ to every element $x \in \bar{F}_N(A)$. The combinatorics that this imposes will give us a lot of insight into the structure of $\bar{F}_N(A)$. The reader should recall the terminology and notation from definitions 1.2.19 and 1.2.21 as these will be used extensively throughout this section.

Definition 2.2.9. Let $x \in \bar{F}_N(A)$. We say that $T \subseteq A$ is x-critical if for all $T \subset U \subseteq A$ we have $\rho^U_T(x_U) = 0$. This is equivalent to requiring that for any representative $f : U \to \mathbb{R}^N$ of x_U, $f|_T$ is constant. Denote by $\mathcal{T}(x)$ the set of all x-critical sets.

Lemma 2.2.10. *$\mathcal{T}(x)$ is a full tree on A.*

Proof. All singleton sets are trivially in $\mathcal{T}(x)$ and $A \in \mathcal{T}(x)$ as it vacuously satisfies the conditions. Now let $T, T' \in \mathcal{T}(x)$ such that $T \cap T' \neq \emptyset$. We need to show that either $T \subseteq T'$ or $T' \subseteq T$. If neither holds then $U = T \cup T'$ is a strict superset of both T and T'. Therefore if $f : U \to \mathbb{R}^N$ is a representative of x_U then $f|_T$ and $f|_{T'}$ are both constant. This implies that f is in fact constant on all of U which contradicts $x_U \in S(W_N(U))$. $\square$

Lemma 2.2.11. *For any $x \in \bar{F}_N(A)$ and $B \subseteq A$ with $|B| > 1$, there exists a unique x-critical set $T \supseteq B$ with $x_T|_B = x_B$ or more precisely, $f_T|_B = \lambda f_B + v$ for some $\lambda \in (0, \infty)$ and $v \in \mathbb{R}^N$ where f_T and f_B are any representatives of x_T and x_B respectively. Moreover, $T = \mathcal{T}(x)(B)$.*

Proof. Let T be a set of largest possible size such that $x_T|_B \neq 0$ which means that by coherence $x_T|_B \sim x_B$. Now let $U \supset T$. Then since T is maximal we have $x_U|_B = x_U|_T|_B = 0$. This then implies that $x_U|_T = 0$ since $x_U|_T \not\sim x_T$. Therefore T is x-critical. Now let T' be another x-critical set containing B. Since $T, T' \in \mathcal{T}(x)$ we have to have either $T \subseteq T'$ or $T' \subset T$. In the latter case, $x_T|_{T'} = 0$ since T' is x-critical. But then $x_T|_B = x_T|_{T'}|_B = 0$ which contradicts our assumption about T. Therefore $T = \mathcal{T}(x)(B)$ as required. □

Corollary 2.2.12. *$F_N(A)$ is homeomorphic to the set $\{x \in \bar{F}_N(A) \mid \mathcal{T}(x) = \mathcal{C}_A\}$.*

Proof. We can define an embedding $\iota : F_N(A) \to \bar{F}_N(A)$ by setting

$$\iota(x) = (x|_B)_{B \subseteq A}.$$

It is clear that $\mathcal{T}(\iota(x)) = \mathcal{C}_A$ since any representative of x is injective. On the other hand, if $z \in \bar{F}_N(A)$ and $\mathcal{T}(z) = \mathcal{C}_A$ then for any $B \subseteq A$, $\mathcal{T}(z)(B) = A$ and so $z_A|_B \sim z_B$ by lemma 2.2.11. In particular $z_A|_B$ is non-constant for all two-element subsets $B \subseteq A$ which is equivalent to saying that z_A is injective. It is then easy to see that $z_A \in image(\iota)$. □

Lemmas 2.2.10 and 2.2.11 give us the basic stratification and structure for $\bar{F}_N(A)$. An element $x \in \bar{F}_N(A)$ is completely determined by its critical tree $\mathcal{T}(x)$ and the values x_T for $T \in \mathcal{T}(x)'$. We will now develop a generalisation of these elements that will help to explore the structure of $\bar{F}_N(A)$ even further, and in particular show that it is a manifold with corners.

Definition 2.2.13. Let $\mathcal{T}$ be a full tree on A with $1 < |A| < \infty$. We define a space $\bar{F}_N(A; \mathcal{T}) \subseteq \prod_{T \in \mathcal{T}'} S(W_N(T))$ by saying that $x \in \bar{F}_N(A; \mathcal{T})$ if it satisfies the coherence condition for all $T \subseteq U$ with $T, U \in \mathcal{T}'$. There is an obvious projection map $\tau : \bar{F}_N(A) \to \bar{F}_N(A; \mathcal{T})$ that simply forgets x_B for $B \notin \mathcal{T}'$.

Remark 2.2.14. Notice that for $B \subseteq A$ we have $S(W_N(A), W_N(B)) = \bar{F}_N(A; \mathcal{T})$ where

$$\mathcal{T} = \{A, B\} \cup \{\text{singleton sets}\}.$$

Definition 2.2.15. Let A be a finite set and V a finite-dimensional real inner product space such that we do not simultaneously have $A = \emptyset$ and $dim(V) = 0$. We define

$$D(A, V) = \left\{ (t, v) \in \text{Map}(A, \mathbb{R}_+) \times V \;\middle|\; \sum_{a \in A} t(a)^2 + \|v\|^2 = 1 \right\}.$$

This space is a closed subspace of a sphere and so it is a compact space.

Definition 2.2.16. An m-dimensional smooth manifold with corners is the same as a regular smooth manifold, except that the charts in the atlas should be diffeomorphic to some open subset of $\mathbb{R}_+^k \times \mathbb{R}^{m-k}$. The boundary of a manifold with corners M, denoted ∂M is the set of points

$$\partial M = \{m \in M \mid \phi(m)_i = 0 \text{ for some } i = 0, \ldots, k-1\},$$

where $\phi : U \to \mathbb{R}_+^k \times \mathbb{R}^{m-k}$ is some chart on an open set $U \subseteq M$ containing m. The interior of M is the complement of ∂M. The boundary can be stratified by separating it into pieces for which some fixed number of coordinates are equal to 0. The codimension of this boundary stratum is this fixed number.

Remark 2.2.17. We have the following implications

smooth manifold $\Rightarrow$ smooth manifold with boundary $\Rightarrow$ smooth manifold with corners.

Also notice that if M and M' are smooth manifolds with corners then their product $M \times M'$ is also a smooth manifold with corners. This is not the case if M and M' are only manifolds with boundary; then $M \times M'$ is not a manifold with boundary.

Lemma 2.2.18. *$D(A,V)$ is a $(|A|+d-1)$-dimensional manifold with corners where $d = dim(V)$.*

Proof. Here we will simply provide definitions of charts without going through all of the details as to why they are indeed charts. If $d = 0$ then $D(A,V)$ is homeomorphic to the simplex

$$\Delta(A) = \left\{ t : A \to [0,1] \;\middle|\; \sum_{a \in A} t(a) = 1 \right\}.$$

This is the prototypical example of a manifold with corners. We can define a chart for each $a \in A$ by setting $U_a = \{t \in \Delta(A) \mid t(a) \neq 0\}$ and then defining

$$\phi_a : U_a \to [0,1)^A$$

$$\phi_a(t)(a') = \begin{cases} t(a') & \text{if } a' \neq a \\ 1 - t(a') & \text{if } a' = a \end{cases}.$$

Therefore assume that $d > 0$. Then, we can choose a unit vector $e \in V$. For each such vector we can define a chart by setting $U_e = D(A,V) \setminus \{(0,e)\}$, (0 represents the 0-map in $\mathrm{Map}(A, \mathbb{R}_+)$), and then defining

$$\phi_e : U_e \to \mathrm{Map}(A, \mathbb{R}_+) \times e^{\perp} \cong \mathbb{R}_+^A \times \mathbb{R}^{d-1}$$

$$\phi_e(t,v) = \frac{(t, v - \langle (t,v), (0,e) \rangle \cdot e)}{1 - \langle (t,v), (0,e) \rangle}.$$

Notice that this is simply an adaptation of stereographic projection and so it is not hard to check the necessary conditions. $\square$

Lemma 2.2.19. *The boundary of $D(A,V)$ is the set of points*

$$\partial D(A,V) = \{(t,v) \in D(A,V) \mid t(a) = 0 \textit{ for some } a \in A\}.$$

Proof. If $dim(V) = 0$ then this is clear except in the case when $\phi_a(t)(a) = 0$ which implies that $t(a) = 1$. But then $t(a') = 0$ for all $a' \neq a$ so the condition is satisfied. If $dim(V) > 1$ then this is immediate from the definition of ϕ_e. $\square$

Proposition 2.2.20. *For $N \geq 1$, we have a natural homeomorphism*

$$\prod_{T \in \mathcal{T}'} D(\delta' T, W_N(\delta T)) \to \bar{F}_N(A; \mathcal{T}).$$

Proof. Let $(t_T, y_T)_{T\in\mathcal{T}'} \in \prod_{T\in\mathcal{T}'} D(\delta' T, W_N(\delta T))$ such that each y_T is the unique representative with centre of mass equal to 0. Recall the normalisation and precomposition constructions from remarks 2.2.7 and 2.2.8 respectively. Define a map

$$\theta : \prod_{T\in\mathcal{T}'} D(\delta' T, W_N(\delta T)) \to \bar{F}_N(A; \mathcal{T}),$$

by setting $\theta((t_T, y_T)_{T\in\mathcal{T}'}) = z = (z_T)_{T\in\mathcal{T}'}$ where

$$z_T(a) = \begin{cases} \|y_T\| \cdot \widetilde{\pi^* y_T}(a) + t_T(U)\cdot z_U(a) & \text{if } a \in U \in \delta' T \\ \|y_T\| \cdot \widetilde{\pi^* y_T}(a) & \text{otherwise} \end{cases},$$

and $\pi : T \to \delta T$ is the obvious projection map. If $T \in \mathcal{T}'$ is minimal then $\delta' T = \emptyset$ and so the recursive nature of z_T makes sense. It also easily implies that z is coherent. It is also clear that θ is a continuous map. Notice that if $\delta' T = \emptyset$ then $z_T = y_T$ and $\|y_T\| = 1$ since $\sum_{a\in\delta' T} t(a) = 0$, so z_T is a normalised representative. Then by induction, z_T will be a normalised representative for all $T \in \mathcal{T}'$. It is an easy check to see that the centre of mass of z_T will be equal to 0. Then

$$\sum_{a\in T} \|z_T(a)\|^2 = \sum_{a\in T} \|y_T\|^2 \cdot \|\widetilde{\pi^* y_T}(a)\|^2 + \sum_{U\in\delta' T}\sum_{a\in U} t_T(U)^2 \|z_U(a)\|^2 + \\ 2\sum_{U\in\delta' T}\sum_{a\in U} \langle \|y_T\| \cdot \widetilde{\pi^* y_T}(a), z_U(a)\rangle.$$

However, the final term is equal to 0 since $\pi^* y_T$ is constant on U and $\sum_{a\in U} z_U(a) = 0$. Therefore

$$\sum_{a\in T} \|z_T(a)\|^2 = \|y_T\|^2 + \sum_{U\in\delta' T} t_T(U)^2 = 1.$$

In the other direction, let $x = (x_T)_{T\in\mathcal{T}'} \in \bar{F}_N(A; \mathcal{T})$ such that each x_T is a normalised representative. We define a map

$$\varphi : \bar{F}_N(A; \mathcal{T}) \to \prod_{T\in\mathcal{T}'} D(\delta' T, W_N(\delta T))$$

by setting $\varphi(x) = (s_T, w_T)_{T\in\mathcal{T}'}$ where

$$s_T(U) = \sqrt{\sum_{a\in U} \left\| x_T(a) - |U|^{-1} \sum_{u\in U} x_T(u) \right\|^2}$$

$$\tilde{\omega}_T(U) = |U|^{-1} \sum_{a\in U} x_T(a)$$

and $w_T = (1 - \sum_{U\in\delta' T} s_T(U)^2)^{\frac{1}{2}} \cdot \tilde{\omega}_T$. It is a simple check of the definitions to see that θ and φ are mutually inverse maps. Then since $\prod_{T\in\mathcal{T}'} D(\delta' T, W_N(\delta T))$ is compact and $\bar{F}_N(A; \mathcal{T})$ is Hausdorff, these are in fact mutually inverse homeomorphisms. □

Corollary 2.2.21. *$\bar{F}_N(A; \mathcal{T})$ is a manifold with corners of dimension $N(|A| - 1) - 1$.*

Proof. Since $1 < |A| < \infty$ and $N \geq 1$, we eliminate the case where there exists $T \in \mathcal{T}'$ such that $\delta' T = \emptyset$ and $dim(W_N(\delta T)) = 0$. Therefore $D(\delta' T, W_N(\delta T))$ is a manifold with corners for all $T \in \mathcal{T}'$ and so the product

$$\prod_{T \in \mathcal{T}'} D(\delta' T, W_N(\delta T))$$

is also a manifold with corners.

The dimension of $\bar{F}_N(A; \mathcal{T})$ is equal to

$$\sum_{T \in \mathcal{T}'} dim(D(\delta' T, W_N(\delta T))) = \sum_{T \in \mathcal{T}'} (|\delta' T| + N(|\delta T| - 1) - 1).$$

First assume that $\mathcal{T}$ is binary. Then $|\delta T| = 2$ for all $T \in \mathcal{T}'$, $|\mathcal{T}'| = |A| - 1$ and therefore

$$\sum_{T \in \mathcal{T}'} |\delta' T| = 2(|A| - 1) - |A| = |A| - 2.$$

Putting this all together gives

$$\sum_{T \in \mathcal{T}'} dim(D(\delta' T, W_N(\delta T))) = |A| - 2 + N(|A| - 1) - |A| + 1 = N(|A| - 1) - 1$$

as required. We can then proceed by backwards induction on $|\mathcal{T}'|$. Assume that $dim(\bar{F}_N(A; \mathcal{T})) = N(|A| - 1) - 1$ and then consider $\mathcal{U} = \mathcal{T} \setminus \{T\}$ for some $T \in \mathcal{T}$. Let $\zeta(T) \in \mathcal{T}$ be the parent of T. It is then the case that if $U \in \mathcal{U} \setminus \{\zeta(T)\}$ then $|\delta U|$ and $|\delta' U|$ will be independent of being taken with respect to $\mathcal{T}$ or $\mathcal{U}$. Then

$$|\delta_{\mathcal{U}} \zeta(T)| = |\delta_{\mathcal{T}} T| + |\delta_{\mathcal{T}} \zeta(T)| - 1$$

$$|\delta'_{\mathcal{U}} \zeta(T)| = |\delta'_{\mathcal{T}} T| + |\delta'_{\mathcal{T}} \zeta(T)| - 1.$$

Therefore notice that

$$\begin{aligned} |\delta'_{\mathcal{U}} \zeta(T)| + N(|\delta_{\mathcal{U}} \zeta(T)| - 1) - 1 &= |\delta'_{\mathcal{T}} T| + |\delta'_{\mathcal{T}} \zeta(T)| - 1 + N(|\delta_{\mathcal{T}} T| + |\delta_{\mathcal{T}} \zeta(T)| - 2) - 1 \\ &= |\delta'_{\mathcal{T}} T| + N(|\delta_{\mathcal{T}} T| - 1) - 1 + |\delta'_{\mathcal{T}} \zeta(T)| + N(|\delta_{\mathcal{T}} \zeta(T)| - 1) - 1. \end{aligned}$$

Therefore one can conclude that

$$\sum_{U \in \mathcal{U}} |\delta' U| + N(|\delta U| - 1) - 1 = \sum_{T \in \mathcal{T}'} |\delta' T| + N(|\delta T| - 1) - 1 = N(|A| - 1) - 1$$

as required. □

Corollary 2.2.22. *The boundary of $\bar{F}_N(A; \mathcal{T})$ is the set of points*

$$\partial \bar{F}_N(A; \mathcal{T}) = \{x \in \bar{F}_N(A; \mathcal{T}) \mid x_A|_T \sim 0 \text{ for some } T \in \mathcal{T}'\}.$$

Proof. By lemma 2.2.19, the boundary of $D(\delta' T, W_N(\delta T))$ is the set of points

$$\{(t, v) \in D(\delta' T, W_N(\delta T)) \mid t(U) = 0 \text{ for some } U \in \delta' T\}.$$

An element is in the boundary of a product of manifolds with corners if its projection is in the boundary of one of the manifolds in the product. By proposition 2.2.20, an element in this set maps to some $z \in \bar{F}_N(A;\mathcal{T})$ such that for $a \in U$

$$z_T(a) = \sqrt{\|y_T\|} \cdot \widetilde{\pi^* y_T}(a) + t_T(U) \cdot z_U(a).$$

However $t_T(U) = 0$ implies that $z_T|_U \sim 0$ and so by coherence $z_A|_U \sim 0$ also. Therefore

$$\partial \bar{F}_N(A;\mathcal{T}) \subseteq \{x \in \bar{F}_N(A;\mathcal{T}) \mid x_A|_T \sim 0 \text{ for some } T \in \mathcal{T}'\}.$$

Conversely, take some $x \in \bar{F}_N(A;\mathcal{T})$ such that $x_A|_T \sim 0$ for some $T \in \mathcal{T}'$. Assume that T is maximal, i.e. there does not exist $T' \in \mathcal{T}'$ such that $T \subset T'$ and $x_A|_{T'} \sim 0$. Then in particular, $x_A|_{\zeta(T)}$ is not constant and so by coherence $x_{\zeta(T)}|_T \sim x_A|_T \sim 0$. By proposition 2.2.20, this will map to $(t_W, v_W)_{W \in \mathcal{T}'}$ such that $t_T(U) = 0$ and so

$$\{x \in \bar{F}_N(A;\mathcal{T}) \mid x_A|_T \sim 0 \text{ for some } T \in \mathcal{T}'\} \subseteq \partial \bar{F}_N(A;\mathcal{T})$$

as required. $\square$

Corollary 2.2.23. *$\bar{F}_N(A;\mathcal{T})$ is homotopy equivalent to $\prod_{T \in \mathcal{T}', \delta' T = \emptyset} S(W_N(T))$.*

Proof. We claim that if $A \neq \emptyset$ then $D(A,V)$ is contractible. Indeed, we define an element $b \in \mathrm{Map}(A, \mathbb{R}_+)$ by setting $b(a) = |A|^{-\frac{1}{2}}$ for all $a \in A$. We then define a map that contracts to $(b, 0)$ by setting

$$h_0(\lambda, (t,v)) = (\lambda b + (1-\lambda)t, (1-\lambda)v)$$

which we can then normalise

$$h(\lambda) = \|h_0(\lambda, (t,v))\|^{-\frac{1}{2}} \cdot h_0(\lambda, (t,v))$$

so that it lands in $D(A,V)$. $\square$

Now we compare $\bar{F}_N(A)$ with $\bar{F}_N(A;\mathcal{T})$. One should think of $\bar{F}_N(A;\mathcal{T})$ as $\bar{F}_N(A)$ with some strata collapsed, (not necessarily to a single point). This allows us to break up $\bar{F}_N(A)$ into more manageable pieces.

Definition 2.2.24. Let $\mathcal{T}$ be a full tree on A. We then define the following spaces:

$$\bar{F}_N(A; = \mathcal{T}) = \{x \in \bar{F}_N(A) \mid \mathcal{T}(x) = \mathcal{T}\}.$$

$$\bar{F}_N(A; \subseteq \mathcal{T}) = \{x \in \bar{F}_N(A) \mid \mathcal{T}(x) \subseteq \mathcal{T}\}.$$

$$\bar{F}_N(A; \supseteq \mathcal{T}) = \{x \in \bar{F}_N(A) \mid \mathcal{T}(x) \supseteq \mathcal{T}\}.$$

Remark 2.2.25. Let $T \subseteq A$. It is then not hard to see that the set $\{x \in \bar{F}_N(A) \mid T \text{ is } x\text{-critical}\}$ is closed in $\bar{F}_N(A)$. Therefore $\bar{F}_N(A; \supseteq \mathcal{T})$ is closed, $\bar{F}_N(A; \subseteq \mathcal{T})$ is open and $\bar{F}_N(A; = \mathcal{T})$ is locally closed.

Proposition 2.2.26. *The projection $\tau : \bar{F}_N(A) \to \bar{F}_N(A;\mathcal{T})$ restricts to give an open inclusion $\bar{F}_N(A; \subseteq \mathcal{T}) \to \bar{F}_N(A;\mathcal{T})$. The image is the set of elements*

$$\mathcal{C}\bar{F}_N(A;\mathcal{T}) = \{x \in \bar{F}_N(A;\mathcal{T}) \mid x_{\mathcal{T}(B)}|_B \not\sim 0 \textit{ for all } B \subseteq A\}.$$

Proof. First we show that $\tau(\bar{F}(A;\subseteq \mathcal{T})) \subseteq \mathcal{C}\bar{F}_N(A;\mathcal{T})$. Take $x \in \bar{F}_N(A;\subseteq \mathcal{T})$ and $B \notin \mathcal{T}$. Because $\mathcal{T}(x) \subseteq \mathcal{T}$ we have $B \subset \mathcal{T}(B) \subseteq \mathcal{T}(x)(B)$. Lemma 2.2.11 tells us that $x_{\mathcal{T}(x)(B)}|_B \sim x_B$ and so in particular $x_{\mathcal{T}(x)(B)}|_{\mathcal{T}(B)} \not\sim 0$ and so it has to be equivalent to $x_{\mathcal{T}(B)}$. Therefore $x_{\mathcal{T}(B)}|_B \sim x_{\mathcal{T}(x)(B)}|_B \sim x_B$ and so $\tau(x) \in \mathcal{C}BF_N(A;\mathcal{T})$.

We can now define a map $\sigma : \mathcal{C}\bar{F}_N(A;\mathcal{T}) \to \prod_{B\subseteq A,|B|>1} S(W_N(B))$ by setting $\sigma(z)_B = z_{\mathcal{T}(B)}|_B$. By lemma 2.2.11, it is immediate that $\sigma \circ \tau = id_{\bar{F}_N(A;\subseteq \mathcal{T})}$. We then claim that

$$image(\sigma) \subseteq \bar{F}_N(A;\subseteq \mathcal{T}).$$

If this is true then it is clear that $\tau \circ \sigma = id_{\mathcal{C}\bar{F}_N(A;\mathcal{T})}$ and we will be done.

Let $C \subseteq B \subseteq A$ and consider $\sigma(z)_B|_C = z_{\mathcal{T}(B)}|_B|_C$. We want this to be equivalent to either 0 or $\sigma(z)_C = z_{\mathcal{T}(C)}|_C$. Since $C \subseteq B$, we also have $\mathcal{T}(C) \subseteq \mathcal{T}(B)$. Therefore, since z is coherent $z_{\mathcal{T}(B)}|_{\mathcal{T}(C)} \in \{0, z_{\mathcal{T}(C)}\}$. If we restrict this to C then we acquire the desired result. So $\sigma(z)$ is coherent and therefore lies in $\bar{F}_N(A)$.

Now consider some $B \notin \mathcal{T}$. Then $\mathcal{T}(B)$ is a strict superset of B such that $\sigma(z)_{\mathcal{T}(B)}|_B = z_{\mathcal{T}(B)}|_B \not\sim 0$. Therefore B is not $\sigma(z)$-critical and so $\mathcal{T}(\sigma(z)) \subseteq \mathcal{T}$. So $\sigma(z) \in \bar{F}_N(A;\subseteq \mathcal{T})$ as required. □

Corollary 2.2.27. *$\bar{F}_N(A)$ is a manifold with corners of dimension $N(|A|-1)-1$.*

Proof. By proposition 2.2.26, every element $x \in \bar{F}_N(A)$ has an open neighbourhood $\bar{F}_N(A;\subseteq \mathcal{T}(x))$ that is homeomorphic to an open subset of $\bar{F}_N(A;\mathcal{T})$ which is a manifold with corners. Therefore $\bar{F}_N(A)$ is itself a manifold with corners. Since $dim(\bar{F}_N(A;\mathcal{T})) = N(|A|-1)-1$ for any $\mathcal{T}$, $dim(\bar{F}_N(A)) = N(|A|-1)-1$ also. □

Corollary 2.2.28. *The boundary of $\bar{F}_N(A)$ is the set*

$$\partial\bar{F}_N(A) = \{x \in \bar{F}_N(A) \mid \mathcal{T}(x) \neq \mathcal{C}_A\}.$$

The boundary is therefore stratified by full trees $\mathcal{T}$ on A that are not the corolla. In particular, the codimension of the boundary stratum indexed by the tree $\mathcal{T}$ is equal to $|\mathcal{T}'|-1$.

Proof. An element $x \in \bar{F}_N(A)$ is in $\partial\bar{F}_N(A)$ if and only if $\tau(x) \in \partial\bar{F}_N(A;\mathcal{T}(x))$ when τ is restricted to $\bar{F}_N(A;\subseteq \mathcal{T})$. If $x \in \bar{F}_N(A)$ such that $\mathcal{T}(x) \neq \mathcal{C}_A$ then $x_A|_T \sim 0$ for all $T \in \mathcal{T}'$. Then the same is true for $\tau(x)_A$ and so

$$\{x \in \bar{F}_N(A) \mid \mathcal{T}(x) \neq \mathcal{C}_A\} \subseteq \partial\bar{F}_N(A).$$

Working through the various homeomorphisms, the codimension of this element corresponds to the number of $T \in \mathcal{T}' \setminus \{A\}$ such that $x_{\zeta(T)}|_T \sim 0$. But since this is true for all $T \in \mathcal{T}' \setminus \{A\}$, the codimension is equal to $|\mathcal{T}'|-1$.

In the opposite direction, if $z \in \mathcal{C}\bar{F}_N(A;\mathcal{T}) \cap \partial\bar{F}_N(A;\mathcal{T})$ then in particular, z_A is not injective and so $\mathcal{T}(x) \neq \mathcal{C}_A$ for its preimage $x \in \bar{F}_N(A;\subseteq \mathcal{T})$. Therefore

$$\partial\bar{F}_N(A) \subseteq \{x \in \bar{F}_N(A) \mid \mathcal{T}(x) \neq \mathcal{C}_A\}$$

as required. □

Corollary 2.2.29. *$\bar{F}_N(A)$ is homotopy equivalent to $F_N(A)$.*

Proof. Topologically, a manifold with corners is the same as a manifold with boundary and via the collar-neighbourhood theorem for topological manifolds, ([6], theorem 2), a manifold with boundary is homotopy equivalent to its interior. Therefore

$$\bar{F}_N(A) \simeq (\partial \bar{F}_N(A))^c = \{x \in \bar{F}_N(A) \mid \mathcal{T}(x) = \mathcal{C}_A\} \cong F_N(A).$$

$\square$

We also can produce a useful description of another space from definition 2.2.24.

Proposition 2.2.30. *$\bar{F}_N(A; = \mathcal{T})$ is homeomorphic to $\prod_{T \in \mathcal{T}'} F_N(\delta T)$.*

Proof. Let $\pi_T : T \to \delta T$ be the obvious projection. We then define a map

$$\pi : \prod_{T \in \mathcal{T}'} F_N(\delta T) \to \prod_{T \in \mathcal{T}'} S(W_N(T)),$$

$$\pi(x) = (\pi_T^* x_T)_{T \in \mathcal{T}'}.$$

It is easy to see that this is a continuous injective map. We claim that $image(\pi) \subseteq \mathcal{C}\bar{F}_N(A; \mathcal{T})$. Firstly let $U, T \in \mathcal{T}$ with $U \subset T$. Then $\pi_T|_U$ is constant and so $\pi(x)_T|_U \sim 0$ so $\pi(x)$ is coherent. Now let $B \subseteq A$ with $|B| > 1$. Then $\pi_{\mathcal{T}(B)}(B)$ is not a singleton set as $\mathcal{T}(B)$ is minimal so $\pi_{\mathcal{T}(B)}^* x_{\mathcal{T}(B)}|_B \not\sim 0$ since $x_{\mathcal{T}(B)}$ is injective. This proves our first claim.

Now we can define an injective map $\sigma\pi : \prod_{T \in \mathcal{T}'} F_N(\delta T) \to \bar{F}_N(A; \subseteq \mathcal{T})$. Our next claim is that $\mathcal{T}(\sigma\pi(x)) = \mathcal{T}$ for all $x \in \prod_{T \in \mathcal{T}'} F_N(\delta T)$. Let $T \in \mathcal{T}$ and $T \subset B \subseteq A$. Then $\sigma\pi(x)_B|_T = \pi_{\mathcal{T}(B)}^* x_{\mathcal{T}(B)}|_B|_T$. But $\pi_{\mathcal{T}(B)}|_T$ is constant and so $\sigma\pi(x)_B|_T \sim 0$ and so T is critical.

Therefore $image(\sigma\pi) \subseteq \bar{F}_N(A; = \mathcal{T})$. Our final claim is that this is in fact an equality. We have

$$\tau(\bar{F}_N(A; = \mathcal{T})) = \mathcal{C}\bar{F}_N(A; \mathcal{T}) \cap \{x \in \bar{F}_N(A; \mathcal{T}) \mid x_T|_U \sim 0 \text{ for all } U, T \in \mathcal{T} \text{ such that } U \subset T\}.$$

If x is in this set then we claim that the well-defined map, (after choosing representatives)

$$z_T : \delta T \to \mathbb{R}^N$$

$$z_T(U) = x_T(u) \qquad \text{for any } u \in U$$

is injective. Indeed, take $U, U' \in \delta T$ with $U \neq U'$ as well as $u \in U$ and $u' \in U'$. Then $\mathcal{T}(\{u, u'\}) = T$ and so $x_T|_{\{u,u'\}} \not\sim 0$. This implies that $x_T(u) \neq x_T(u')$ and so $z_T(U) \neq z_T(U')$. It is now easy to see that $image(\pi) = \tau(\bar{F}_N(A; = \mathcal{T}))$ as required. $\square$

Remark 2.2.31. In light of proposition 2.2.30, we can think of $\bar{F}_N(A)$ as being the set of full trees on A with vertices decorated by $F_N(\delta T)$ for $T \in \mathcal{T}'$. This description has some advantages when we consider an operad structure below. However, it is not clear how the topology of $\bar{F}_N(A)$ should work in this case which is why our main definition is 2.2.6.

Corollary 2.2.32. *The action of the permutation group Σ_A is free on $\bar{F}_N(A)$.*

Proof. Let $x \in \bar{F}_N(A)$, $\sigma \in \Sigma_A$ and assume $\sigma \cdot x = x$. We claim that for $T \in \mathcal{T}(x)$ we have $\sigma(T) = T$. This is obviously true for A and so we can assume for induction that $T \in \delta U$ where $U \in \mathcal{T}$ such that $\sigma(U) = U$. By assumption, σ preserves x_U and by proposition 2.2.30, $x_U = \pi_U^* z_U$ for some $z_U \in F_N(\delta U)$. Since z_U is an injective map, σ must act as the identity on δU and so $\sigma(T) = T$. Because $\mathcal{T}(x)$ is a full tree on A, in particular this means that $\sigma(\{a\}) = \{a\}$ for all $a \in A$ and so $\sigma = id$ and therefore the action is free. $\square$

For any N define $\bar{F}_N(\emptyset) = \{*\}$ and $\bar{F}_N(A) = \{*\}$ for all A such that $|A| = 1$. This is consistent with previous definitions since if $|A| \leq 1$ then $\prod_{B \subseteq A, |B|>1} S(W_N(B))$ will be the product of no terms, i.e. a single point. Now, if we fix N, we can give the collection $\bar{F}_N = \{\bar{F}_N(A)\}$ an operad structure as we detail below.

Definition 2.2.33. Let $p : A \to B$ be a map of finite sets. Notice that this induces a map $p^* : S(W_N(B)) \to S(W_N(A))$ defined by $[x] \mapsto [p^*(x)]$ where $p^*x = x \circ p$. We then define a map $\gamma_p : \bar{F}_N(B) \times \prod_{b \in B} \bar{F}_N(A_b) \to \bar{F}_N(A)$ such that $\gamma_p(x, (y_b)_{b\in B}) = w$ and

$$w_T = \begin{cases} y_{b,T} & \text{if } p(T) = \{b\} \\ p^* x_{p(T)} & \text{if } |p(T)| > 1 \end{cases}.$$

Theorem 2.2.34. *This definition makes $\bar{F}_N$ into an operad. We refer to these operads as the Fulton-Macpherson operads.*

Proof. There is no choice for the unit map and it is an easy check of definition 2.2.33 that it will indeed act as a unit for the composition γ_p. We therefore only need to check that γ_p is suitably defined as a composition map. γ_p is clearly continuous as the cases in the definition only depend on p itself. To ensure that $\gamma_p(x, (y_b)_{b\in B}) = w$ is in $\bar{F}_N(A)$, we need to check that w is coherent. Consider $U \subseteq T \subseteq A$. If $p(U) = p(T) = \{b\}$ then $w_T = y_{b,T}$ and $w_U = y_{b,U}$ and so the coherence of y ensures that $w_T|_U$ also satisfies coherence. If $|p(U)| = 1$ but $|p(T)| > 1$ then $w_T = p^* x_{p(T)}$ will be constant on U and so again coherence is satisfied. Finally, if $|p(U)|, |p(T)| > 1$ then $w_T = p^* x_{p(T)}$ and $w_U = p^* x_{p(U)}$ and so again, the coherence of x ensures that $w_T|_U$ satisfies coherence.

Finally, we need to check the associativity condition. Let $A \xrightarrow{p} B \xrightarrow{q} C$ be maps between finite sets and consider elements $x \in \bar{F}_N(C)$, $y_c \in \bar{F}_N(B_c)$ and $z_b \in \bar{F}_N(A_b)$. We need to check that

$$w = \gamma_p(\gamma_q(x, (y_c)_{c\in C}), (z_b)_{b\in B}) = \gamma_{qp}(x, (\gamma_{p_c}(y_c, (z_b)_{b\in B_c}))_{c\in C}) = w'.$$

Considering the left hand side first,

$$w_T = \begin{cases} z_{b,T} & \text{if } p(T) = \{b\} \\ p^* \gamma_q(x, (y_c)_{c\in C})_{p(T)} & \text{if } |p(T)| > 1 \end{cases}$$

which we can write even more explicitly as

$$w_T = \begin{cases} z_{b,T} & \text{if } p(T) = \{b\} \\ p^* y_{c,p(T)} & \text{if } |p(T)| > 1 \text{ and } qp(T) = \{c\} \\ p^* q^* x_{qp(T)} & \text{if } |qp(T)| > 1 \end{cases}.$$

Now considering the right hand side

$$w'_T = \begin{cases} \gamma_{p_c}(y_c, (z_b)_{b\in B_c})_{c,T} & \text{if } qp(T) = \{c\} \\ (qp)^* x_{qp(T)} & \text{if } |qp(T)| > 1 \end{cases}$$

which we can write more explicitly as

$$w'_T = \begin{cases} z_{b,T} & \text{if } p(T) = \{b\} \\ p^* y_{c,p(T)} & \text{if } |p(T)| > 1 \text{ and } qp(T) = \{c\} \\ (qp)^* x_{qp(T)} & \text{if } |qp(T)| > 1 \end{cases}.$$

Then, one simply has to notice that $p^* q^* = (qp)^*$ to see that $w = w'$. $\square$

We can consider a reduced version of the Fulton-Macpherson operads by setting $\bar{F}_N(\emptyset) = \emptyset$. It will turn out that this operad is well-labelled and set-theoretically, $\bar{F}_N$ will be the free operad on the collection $F_N = \{F_N(A)\}$. In order to be able to differentiate later in the thesis, denote by $\bar{F}_N^+$ the non-reduced Fulton-Macpherson operads.

Lemma 2.2.35. *The collection $\mathcal{T} : \bar{F}_N \to \mathrm{FTrees}$ given by taking the critical tree is a map of operads in the category of sets.*

Proof. Let $(x, (y_b)_{b\in B}) \in \bar{F}_N(B) \times \prod_{b\in B} \bar{F}_N(A_b)$ and $\gamma_p(x, (y_b)_{b\in B}) = z$. Then we need to show that

$$\mathcal{T}(z) = p^{-1}\mathcal{T}(x) \cup \bigcup_{b\in B} \mathcal{T}(y_b)$$

where $p^{-1}\mathcal{T}(x) = \{p^{-1}T \mid T \in \mathcal{T}(x)\}$. First let $T \in \mathcal{T}(y_b)$ for some $b \in B$ and consider $T \subset U \subseteq A$. Notice that this implies $p(T) = \{b\}$. If $p(U) = \{b\}$ also, then $z_U = y_{b,U}$. But $y_{b,U}|_T \sim 0$ since $T \in \mathcal{T}(y_b)$. If $|p(U)| > 1$ then $z_U = p^* x_{p(U)}$ but since p is constant on T, $p^* x_{p(U)}|_T \sim 0$ and so $T \in \mathcal{T}(z)$. Now let $T \in p^{-1}\mathcal{T}(x)$ which means that $|p(T)| > 1$, and again consider $T \subset U \subseteq A$. We must have that $|p(U)| > 1$ also so that $z_U = p^* x_{p(U)}$. However, $x_{p(U)}|_{p(T)} \sim 0$ since $p(T) \in \mathcal{T}(x)$ and so $p^* x_{p(U)}|_T \sim 0$ meaning $T \in \mathcal{T}(z)$. So

$$p^{-1}\mathcal{T}(x) \cup \bigcup_{b\in B} \mathcal{T}(y_b) \subseteq \mathcal{T}(z).$$

Conversely, let $T \in \mathcal{T}(z)$. If $p(T) = \{b\}$ then the criticality of T implies that $z_U|_T = y_{b,U}|_T \sim 0$ for all $T \subset U \subseteq A_b$. Therefore $T \in \mathcal{T}(y_b)$. If instead $|p(T)| > 1$ then the criticality of T implies that $z_U|_T = p^* x_{p(U)}|_T \sim 0$ for all $T \subset U \subseteq A$. But $p^* x_{p(U)}|_T \sim 0$ if and only if $x_{p(U)}|_{p(T)} \sim 0$ and so $T \in p^{-1}\mathcal{T}(x)$. Therefore

$$\mathcal{T}(z) \subseteq p^{-1}\mathcal{T}(x) \cup \bigcup_{b\in B} \mathcal{T}(y_b)$$

and so the two sets are equal as required. $\square$

Proposition 2.2.36. *The operad composition map $\gamma_p : \bar{F}_N(B) \times \prod_{b\in B} \bar{F}_N(A_b) \to \bar{F}_N(A)$ can be described as the grafting of trees with vertices decorated by F_N.*

Proof. As noted in remark 2.2.31, every element in $\bar{F}_N(A)$ can be described as its critical tree $\mathcal{T}$ decorated by elements in $F_N(\delta T)$ for $T \in \mathcal{T}$. Then lemma 2.2.35 tell us that the composition map for $\bar{F}_N$ grafts the critical trees of its arguments. Therefore we just need to show that the decorations are preserved. Let $\gamma_p(x, (y_b)_{b\in B}) = z$ and first consider z_T. First assume that $T \in \mathcal{T}(y_b)$ for some $b \in B$. Then $z_T = y_b$ and so the claim is obvious. Now consider $T = p^{-1}U$ for some $U \in \mathcal{T}(x)$ so that $z_T = p^* x_U$. We know that z_T is constant on $p^{-1}(u)$ for $u \in U$ and so we can essentially undo p^* from x. Therefore the implied map

$$\bar{F}_N(A; \mathcal{T}(z)) \longrightarrow S(W_N(T)) \longrightarrow F_N(\delta T)$$

from the proof of proposition 2.2.30, factors as

$$\bar{F}_N(A; \mathcal{T}(z)) \longrightarrow S(W_N(T)) \longrightarrow S(W_N(U)) \longrightarrow F_N(\delta U)\ ,$$

since δT in $\mathcal{T}(z)$ is isomorphic to δU in $\mathcal{T}(x)$ by lemma 2.2.35. This shows that the decoration of x_U is preserved as required. $\square$

Corollary 2.2.37. *$\bar{F}_N$ is a well-labelled operad.*

Proof. One can deduce from proposition 2.2.36 that $x \in \bar{F}_N(A)$ is not in the image of a non-trivial composition map if and only if $x \in F_N(A)$. Now by proposition 1.3.7 there is a map of operads (in sets) $\tilde{f} : \Phi F_N \to \bar{F}_N$. This is in fact an isomorphism by the above statement and as a consequence of proposition 2.2.30. Then proposition 1.3.13 tells us that this is equivalent to $\bar{F}_N$ being well-labelled. $\square$

Remark 2.2.38. We could provide an alternative proof of corollary 2.2.37 by considering proposition 1.3.15. We already have a map of operads $\mathcal{T} : \bar{F}_N \to \mathrm{FTrees}$. Then if we have an element

$$((\mathcal{T}_B, (\mathcal{T}_b)_{b\in B}), x) \in \left(\mathrm{FTrees}(B) \times \prod_{b\in B} \mathrm{FTrees}(A_b)\right) \times \bar{F}_N(A)$$

such that $\gamma_p(\mathcal{T}_B, (\mathcal{T}_b)_{b\in B}) = \mathcal{T}(x)$, we can use proposition 2.2.36 to decompose x into an element in $\bar{F}_N(B) \times \prod_{b\in B} \bar{F}_N(A_b)$. One can then show that this produces the necessary isomorphism to conclude that the composition diagram is a pullback.

Remark 2.2.39. Corollary 2.2.37 means that set-theoretically, $\bar{F}_N$ is the free operad on the collection $F_N = \{F_N(A)\}$. This agrees with Markl's construction of the Fulton-Macpherson operads as an operadic completion as detailed in [18].

Corollary 2.2.40. *If $|A| \geq 1$ then $\bar{F}_N(A; \supseteq \mathcal{T})$ is homeomorphic to $\prod_{T\in\mathcal{T}'} \bar{F}_N(\delta T)$.*

Proof. Recall the labelling of an operad by $\mathbb{P}^*(A)$ from definition 1.3.9 which defines subsets $L_B\bar{F}_N(A)$ for some $B \in \mathbb{P}^*(A)$. By corollary 2.2.37, we have a homeomorphism

$$\gamma_{\mathcal{T}} : \prod_{T\in\mathcal{T}'} \bar{F}_N(\delta T) \to L_{\mathcal{T}}\bar{F}_N(A) = \bigcap_{T\in\mathcal{T}} L_T\bar{F}_N(A).$$

But for $B \subset A$ with $|B| > 1$, it is easy to see that $L_B\bar{F}_N(A) = \{x \in \bar{F}_N(A) \mid B \in \mathcal{T}(x)\}$ and therefore

$$L_{\mathcal{T}}\bar{F}_N(A) = \bigcap_{T\in\mathcal{T}} \{x \in \bar{F}_N(A) \mid T \in \mathcal{T}(x)\} = \bar{F}_N(A; \supseteq \mathcal{T}).$$

$\square$

2.3 Homology of the Operad $\bar{F}$

Recall the little N-cubes operads C_N from section 1.2. In this section, we will compare the operads that are produced when we apply the homology functor to both $\bar{F}_N^+$ and C_N. This will allow us to say explicitly what the operad $H_*(\bar{F}_N^+)$ is.

Definition 2.3.1. Let $\mathcal{P} = (\mathcal{P}, \gamma, \eta)$ be an operad in the category of topological spaces. We can then apply the (integral) homology functor H_* to acquire an operad $H_*(\mathcal{P})$ in the category of graded abelian groups. In particular $H_*(\mathcal{P})(A) = H_*(\mathcal{P}(A))$ and we create unit and composition maps by applying H_* to η and γ_p for any $p : A \to B$, and then precomposing with the Künneth map

$$H_*(\mathcal{P}(A)) \otimes \bigotimes_{b\in B} H_*(\mathcal{P}(A_b)) \to H_*\left(\mathcal{P}(B) \times \prod_{b\in B} \mathcal{P}(A_b)\right).$$

We intend to explore what happens when we apply H_* to $\bar{F}_N^+$. It is in fact easier, due to the extra structure, to apply the cohomology functor H^* and then draw conclusions about $H_*(\bar{F}_N^+)$. Applying H^* produces the dual notion of a cooperad but for what we do here, it is not really important to understand this object thoroughly.

Theorem 2.3.2. *For $|A| > 1$, $H^*(\bar{F}_N^+(A))$ is isomorphic to the free graded commutative ring over $\mathbb{Z}$ on elements u_{ab} for distinct $a, b \in A$ with order $N-1$. These elements are subject to the relations*

$$u_{ab}^2 = 0,$$
$$u_{ba} = (-1)^{N-1} u_{ab},$$
$$r_{abc} = u_{ab}u_{bc} + u_{bc}u_{ca} + u_{ca}u_{ab} = 0 \qquad \textit{for distinct } a, b, c \in A.$$

Proof. By corollary 2.2.29, $\bar{F}_N^+(A)$ is homotopy equivalent to $F_N(A)$ for $|A| > 1$. Therefore they have isomorphic cohomology which we described in theorem 2.1.12. ☐

Remark 2.3.3. By theorem 2.3.2, $H^*(\bar{F}_N^+(A))$ is generated by elements $u_{ab} \in H^{N-1}(\bar{F}_N^+(A))$ for $a, b \in A$ with $a \neq b$. Therefore the induced maps γ_p^* are completely determined by how they act on these generators. Since $H^{N-1}(\bar{F}_N^+(A))$ is free and finitely generated, the Künneth map will in fact be an isomorphism and so we can consider γ_p^* to produce a cocomposition map on the nose.

Lemma 2.3.4. *The graded rings $H^*(\bar{F}_N^+(A))$ and $H^*(\mathrm{C}_N(A))$ are isomorphic for any finite set A and natural number N.*

Proof. By lemma 1.2.7, remark 2.1.2 and 2.2.29 we have a chain of homotopy equivalences

$$\mathrm{C}_N(A) \xrightarrow{\simeq} \mathrm{Inj}(A, \mathbb{R}^N) \xrightarrow{\simeq} F_N(A) \xrightarrow{\simeq} \bar{F}_N^+(A)$$

which induces an isomorphism in cohomology. ☐

Definition 2.3.5. Let $\mathcal{P}$ be an operad in topological spaces such that $\mathcal{P}(\emptyset) = \mathcal{P}(\{*\}) = \{*\}$ and denote by $\mathbf{2} = \{0, 1\}$, a set with two points. Now take any finite set A such that $|A| > 1$ and $a_0, a_1 \in A$ with $a_0 \neq a_1$. We then have an inclusion map $\iota_{a_0a_1} : \mathbf{2} \to A$ with image $\{a_0, a_1\}$ and therefore a map

$$\gamma_{\iota_{a_0a_1}} : \mathcal{P}(A) \times \prod_{a \in A} \mathcal{P}(\mathbf{2}_a) \to \mathcal{P}(\mathbf{2}).$$

The domain of this map is in fact isomorphic to just $\mathcal{P}(A)$ and we label the map

$$\pi_{a_0a_1} : \mathcal{P}(A) \to \mathcal{P}(\mathbf{2}).$$

Lemma 2.3.6. *Let $p : A \to B$ be a map of finite sets and $a_0, a_1 \in A$ with $a_0 \neq a_1$. Then $\pi_{a_0a_1} \circ \gamma_p$ is equal to*

$$\mathcal{P}(B) \times \textstyle\prod_{b \in B} E(A_b) \xrightarrow{proj} \mathcal{P}(B) \xrightarrow{\pi_{p(a_0)p(a_1)}} \mathcal{P}(\mathit{2})$$

if $p(a_0) \neq p(a_1)$ or

$$\mathcal{P}(B) \times \textstyle\prod_{b \in B} E(A_b) \xrightarrow{proj} \mathcal{P}(A_b) \xrightarrow{\pi_{a_0a_1}} \mathcal{P}(\mathit{2})$$

if $p(a_0) = p(a_1) = b$.

Proof. Consider the following associativity diagram

$$\begin{tikzcd}
\mathcal{P}(B) \times \prod_{b\in B} E(A_b) \times \prod_{a\in A} \mathcal{P}(\mathbf{2}_a) \arrow[r, "\gamma_p \times id"] \arrow[d, "id\times\pi_{a_0a_1}|_b"'] & \mathcal{P}(A) \times \prod_{a\in A} \mathcal{P}(\mathbf{2}_a) \arrow[d, "\pi_{a_0a_1}"] \\
\mathcal{P}(B) \times \prod_{b\in B} \mathcal{P}(\mathbf{2}_b) \arrow[r, "\gamma_{p\iota_{a_0a_1}}"'] & \mathcal{P}(\mathbf{2})
\end{tikzcd}$$

If $p(a_0) \neq p(a_1)$ then $\mathcal{P}(\mathbf{2}_b) = \{*\}$ for all $b \in B$ and so $id \times \pi_{a_0a_1}|_b$ simply projects on to $\mathcal{P}(B)$. If $p(a_0) = p(a_1) = b$ then $\pi_{a_0a_1}|_b$ is the same as $\pi_{a_0a_1} : \mathcal{P}(A_b) \to \mathcal{P}(\mathbf{2})$ and $p \circ \iota_{a_0a_1} : \mathbf{2} \to B$ is the constant map. One can show, by decomposing this map as

$$\mathbf{2} \twoheadrightarrow \{*\} \hookrightarrow B$$

and then using operad associativity, that $\gamma_{p\iota_{a_0a_1}} : \mathcal{P}(B) \times \mathcal{P}(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$ is just the projection on to $\mathcal{P}(\mathbf{2})$ as required. $\square$

Proposition 2.3.7. *Let $p : A \to B$ be map of finite sets. Then the composition maps*

$$\gamma_p : \bar{F}_N^+(B) \times \prod_{b\in B} \bar{F}_N^+(A_b) \to \bar{F}_N^+(A)$$

and

$$\Gamma_p : \mathrm{C}_N(B) \times \prod_{b\in B} \mathrm{C}_N(A_b) \to C_N(A)$$

induce the same maps in cohomology.

Proof. Consider the restriction map $\rho^A_B : \bar{F}_N^+(A) \to \bar{F}_N^+(B)$. We first claim that this is simply γ_ι where $\iota : B \to A$ is the inclusion map. Indeed, for all $T \subseteq B$ with $|T| > 1$, we have $|\iota(T)| > 1$ also and so $\gamma_\iota(x, (*)_{a\in A})_T = \iota^* x_{\iota(T)} = x_{\iota(T)} \circ \iota$ which is simply x_T. It is similarly simple to show that the same is true for the restriction $\rho^A_B : \mathrm{C}_N(A) \to \mathrm{C}_N(B)$.

We have a chain of homotopy equivalences

$$\mathrm{C}_N(A) \to \mathrm{Inj}(A, \mathbb{R}^N) \to F_N(A) \to \bar{F}_N^+(A)$$

and it is easy to see that these are compatible with the restriction maps, i.e. the diagram

$$\begin{tikzcd}
\mathrm{C}_N(A) \arrow[r] \arrow[d, "\rho^A_B"'] & \mathrm{Inj}(A, \mathbb{R}^N) \arrow[r] \arrow[d, "\rho^A_B"'] & F_N(A) \arrow[r] \arrow[d, "\rho^A_B"] & \bar{F}_N^+(A) \arrow[d, "\rho^A_B"] \\
\mathrm{C}_N(B) \arrow[r] & \mathrm{Inj}(B, \mathbb{R}^N) \arrow[r] & F_N(B) \arrow[r] & \bar{F}_N^+(B)
\end{tikzcd}$$

commutes. Therefore we can choose a generator $u \in H^{N-1}(\bar{F}_N^+(\mathbf{2})) \cong H^{N-1}(S^{N-1})$ and this gives us compatible families of generators u_{ab} in each of

$$H^{N-1}(\bar{F}_N^+(A)) \cong H^{N-1}(F_N(A)) \cong H^{N-1}(\mathrm{Inj}(A, \mathbb{R}^N)) \cong H^{N-1}(\mathrm{C}_N(A)).$$

It is sufficient to show that the induced maps γ_p^* and Γ_p^* act in the same way on generators but this is in fact forced by our above analysis of the restriction maps and then lemma 2.3.6. $\square$

Theorem 2.3.8. *$H_*(\bar{F}_N^+)$ and $H_*(\mathrm{C}_N)$ are isomorphic as operads.*

Proof. Proposition 2.3.7 combines with the earlier results in this section to show that $H^*(\bar{F}_N^+)$ and $H^*(\mathrm{C}_N)$ are isomorphic as cooperads. Then since $H^{N-1}(\bar{F}_N^+(A)) \cong H^{N-1}(\mathrm{C}_N(A))$ is free and finitely generated, when we take the linear dual operads, i.e. $H_*(\bar{F}_N^+)$ and $H_*(\mathrm{C}_N)$, the composition maps will again be equal and so these are isomorphic as operads. □

Remark 2.3.9. We will see in chapter 3 that the model of the Fulton-Macpherson operads that we present here is equivalent to others in the literature. Therefore, the above theorem can be seen as a direct consequence of proposition 4.9 in [22] which states that the Fulton-Macpherson operads are weakly equivalent to the little disks operads, (which are themselves weakly equivalent to the little cubes operads).

Corollary 2.3.10. *The operad $H_*(\bar{F}_N^+)$ is that whose algebras are unital degree-N Poisson algebras.*

Proof. By theorem 2.3.8, $H_*(\bar{F}_N^+)$ and $H_*(\mathrm{C}_N)$ are isomorphic operads. The result is then a direct consequence of theorem 1.2.16. □

2.4 Comparison with the Stasheff Operad

In this section, N will always be equal to 1. As such we will drop this from our notation so that $\bar{F}(A)$ is understood to be $\bar{F}_1(A)$. We will also be considering the reduced Fulton-Macpherson operad so that $\bar{F}_N(\emptyset) = \emptyset$. It is a well-known fact that the 1-dimensional Fulton-Macpherson operad $\bar{F}$ is isomorphic to the Stasheff operad K described in section 1.2. We will use our description of $\bar{F}$ to give an explicit map demonstrating this isomorphism.

Definition 2.4.1. Let $x \in \bar{F}(A)$. We then define a relation $<_x$ on A generated by, (for $a \neq b$),

$$a <_x b \qquad \Leftrightarrow \qquad x_{\{a,b\}}(a) < x_{\{a,b\}}(b).$$

This definition is independent of the choice of representative for $x_{\{a,b\}}$. Denote by $\bar{F}(A, R)$ the set of all points $x \in \bar{F}(A)$ such that the induced relation is equal to $R \subseteq A^2$.

Lemma 2.4.2. *$<_x$ defines a total order on A.*

Proof. The totality and antisymmetry of $<_x$ is obvious from the definition. To see that it is transitive, let $a <_x b$ and $b <_x c$ and consider x_A where $A = \{a, b, c\}$. Let $A_a = \{b, c\}$, $A_b = \{a, c\}$ and $A_c = \{a, b\}$. Notice that because $x_A \in S(W_1(A))$, it can be constant on at most one of A_a, A_b or A_c, and then has to be equivalent to the appropriate x_{A_i} when restricted to one of the other two by coherence. If x_A is injective then $x_A(a) < x_A(b) < x_A(c)$ and so $x_{A_b}(a) < x_{A_b}(c)$. If $x_A|_{A_a} \sim 0$ then

$$x_A(a) < x_A(b) = x_A(c) \qquad \Rightarrow x_{A_b}(a) < x_{A_b}(c).$$

If $x_A|_{A_c} \sim 0$ then

$$x_A(a) = x_A(b) < x_A(c) \qquad \Rightarrow x_{A_b}(a) < x_{A_b}(c).$$

If $x_A|_{A_b} \sim 0$ then for coherence we will simultaneously require $x_A(a) = x_A(c) < x_A(b)$ and $x_A(b) < x_A(c) = x_A(a)$ which cannot happen. Therefore $<_x$ is transitive. □

Corollary 2.4.3. *For any finite set A we have $\bar{F}(A) \cong \coprod_{R \in \mathrm{Ord}(A)} \bar{F}(A, R)$.*

Proof. $\bar{F}(A,R)$ is easily seen to be an open set in $\bar{F}(A)$ and then it is closed because it is part of a disjoint cover of $\bar{F}(A)$. Therefore the obvious map

$$\bar{F}(A) \to \coprod_{R \in \mathrm{Ord}(A)} \bar{F}(A,R)$$

is a continuous bijection between compact Hausdorff spaces. □

Lemma 2.4.4. *Let $x \in \bar{F}(A,R)$. Then $\mathcal{T}(x)$ is a Stasheff tree on A with respect to the ordering R.*

Proof. We need to check that each $T \in \mathcal{T}(x)$ is an interval with respect to R. Notice that if $a <_x b$ then by coherence $x_B(a) \leq x_B(b)$ for any $B \supseteq \{a, b\}$. Let J be the smallest interval in A containing T, i.e. $J = [min\{T\}, max\{T\}]$. Assume $J \supset T$. Then because T is x-critical we have $x_J|_T \sim 0$. However, since this means that $x_J(min\{J\}) = x_J(max\{J\})$, it implies that $x_J \sim 0$ which is a contradiction. Therefore $J = T$ as required. □

With lemma 2.4.4 in mind, it seems plausible that $\bar{F}(A,R)$ is homeomorphic to $K(A,R)$, the connected component associated to the total order R on A of the A^{th} space of the Stasheff operad. This is indeed the case as we will show below. These homeomorphisms can then be brought together to form an isomorphism of operads $\bar{F} \to K$.

For now, fix an ordering R on A so that we may enumerate $A = \{a_0, \ldots, a_{k-1}\}$. We can similarly enumerate $B \subseteq A$ by setting for $a_i, a_j \in B$, $a_i <_B a_j$ if and only if $i < j$ and then enumerating $B = \{b_0, \ldots, b_{l-1}\}$. We will also fix representatives for $\bar{F}(A,R)$ by setting

$$S(W_1(B)) = \{x : B \to [0,1] \mid x(min\{B\}) = 0 \text{ and } x(max\{B\}) = 1\}.$$

Definition 2.4.5. Let $x \in \bar{F}(A,R)$ and $C \subseteq B \subseteq A$. Then define

$$g_C^B(x) = max\{x_B(c_{i+1}) - x_B(c_i) \mid c_i \in C \setminus \{max(C)\}\}.$$

If $|C| < 2$ then define $g_C^B(x) = 0$.

Remark 2.4.6. One should note the following properties:

- $g_B^B(x) > 0$ if $|B| > 1$.
- If C is an interval then $g_C^B(x) \leq g_B^B(x)$.
- For $C \subseteq B$ the ratio $\frac{g_C^B(x)}{g_B^B(x)}$ is invariant under translation and scaling of x.

Definition 2.4.7. Given $x \in \bar{F}(A,R)$ we can associate to it a Stasheff tree, denoted $\mathcal{S}(x)$ as follows. We define $\mathcal{S}(x)$ to be the set of all intervals $J \in \mathcal{J}(A,R)$ such that for all strictly larger intervals $J' \supset J$ we have $g_J^{J'}(x) < g_{J'}^{J'}(x)$.

Lemma 2.4.8. *$\mathcal{S}(x)$ is a Stasheff tree on A.*

Proof. Since $g_B^B > 0$ for all $B \subseteq A$ with $|B| > 1$, it is clear that all singleton sets lie in $\mathcal{S}(x)$. Also $A \in \mathcal{S}(x)$ as it vacuously satisfies the criteria. We therefore only need to check that $\mathcal{S}(x)$ is a tree. Let $J_0, J_1 \in \mathcal{S}(x)$ such that $J_0 \cap J_1 \neq \emptyset$. We need to show that either $J_0 \subseteq J_1$ or $J_1 \subseteq J_0$. If neither holds then $K = J_0 \cup J_1$ is a strict superset of both J_0 and J_1 and is itself an interval. Therefore by definition we have $g_{J_0}^K(x) < g_K^K(x)$ and $g_{J_1}^K(x) < g_K^K(x)$. However it is easily seen that $g_K^K(x) = max\{g_{J_0}^K(x), g_{J_1}^K(x)\}$ which gives us a contradiction. □

Remark 2.4.9. Let $x \in \bar{F}(A,R)$ and $B \subseteq A$. One can define a partition on B by taking classes of the equivalence relation generated by

$$b_i \equiv b_{i+1} \qquad \Leftrightarrow \qquad x_B(b_{i+1}) - x_B(b_i) < g_B^B(x).$$

Notice that this partition cannot be equal to $\{B\}$ and it only produces the singleton partition if $x_B(b_{i+1}) - x_B(b_i) = x_B(b_{j+1}) - x_B(b_j)$ for all $i, j \in \{0, \ldots, l-2\}$. If $B \in \mathcal{S}(x)$ then one can show that this partition is equal to δB.

Lemma 2.4.10. *For $x \in \bar{F}(A,R)$ we have $\mathcal{T}(x) \subseteq \mathcal{S}(x)$. In particular,*

$$\mathcal{T}(x) \setminus \{A\} = \{S \in \mathcal{S}(x) \mid g_S^{\zeta(S)}(x) = 0\},$$

where $\zeta(S)$ is the parent of S in $\mathcal{S}(x)$.

Proof. We know that A is in both $\mathcal{T}(x)$ and $\mathcal{S}(x)$ and so we can ignore this case. Therefore let $T \in \mathcal{T}(x)' \setminus \{A\}$ and $S \in \mathcal{S}(x)$ be the smallest interval that is a strict superset of T. Because T is x-critical, we have $x_S|_T \sim 0$. Therefore $x_S(a_{i+1}) - x_S(a_i) < g_S^S(x)$ for all $a_i \in T \setminus \{max(T)\}$. This means that $T \subseteq B$ for some $B \in \delta S$ but this has to be an equality by the minimality of S. So $T \in \{S \in \mathcal{S} \mid g_S^{\zeta(S)}(x) = 0\}$. Conversely, if we take $S \in \{S \in \mathcal{S} \mid g_S^{\zeta(S)}(x) = 0\}$ then we must have $x_{\zeta(S)}|_S \sim 0$. In particular S will be maximal in the sense that $\nexists S' \supset S$ such that $x_{\zeta(S)}|_{S'} \sim 0$. Since $\mathcal{T}(x) \subseteq \mathcal{S}(x)$, one can therefore deduce that $\mathcal{T}(x)(S) = S$ and so $S \in \mathcal{T}(x)$ as required. $\square$

Definition 2.4.11. We define a map $\theta_{A,R} : \bar{F}(A,R) \to K(A,R)$ by setting $\theta_{A,R}(x) = t$ where

$$t(J) = \begin{cases} 0 & \text{if } J \notin \mathcal{S}(x) \\ 1 & \text{if } J = A \text{ or } J = \{a\} \\ 1 - \frac{g_J^{\zeta(J)}(x)}{g_{\zeta(J)}^{\zeta(J)}(x)} & \text{if } J \in \mathcal{S}(x)' \setminus \{A\} \end{cases}.$$

Since $\mathcal{S}(x)$ is a Stasheff tree on A, and $t(J) > 0$ for all $J \in \mathcal{S}(x)$, it is clear that $t \in K(A,R)$.

Proposition 2.4.12. $\theta_{A,R} : \bar{F}(A,R) \to K(A,R)$ *is a homeomorphism.*

Proof. To prove that $\theta_{A,R}$ is a bijection, we define a map $\phi_{A,R} : K(A,R) \to \bar{F}(A,R)$ which we claim is an inverse. Firstly, let $a \in A$ and denote by a_- the element immediately less than a with respect to R and a_+ the element immediately greater than a with respect to R, (with the obvious restrictions on a being either a least or a greatest element). Now suppose that $J \subseteq A$ is an interval such that $\{a, a_+\} \subseteq J$ and $t \in K(A,R)$. We define $m_J(a)$ to be the product of all terms $(1 - t(J'))$ such that $J \nsubseteq J'$ and $\{a, a_+\} \subseteq J'$. We then define $y_J : J \to \mathbb{R}$ by setting

$$y_J(a) = \sum_{b \in J, b < a} m_J(b).$$

Now let $\mathcal{S} = \text{supp}(t)$. We then define $\phi_{A,R}(t) = z$ such that

$$z_B = \begin{cases} y_B & \text{if } B \in \mathcal{S} \\ z_{\mathcal{S}(B)}|_B & \text{otherwise} \end{cases}.$$

We first prove that $z \in \bar{F}(A, R)$. Let $\mathcal{T} = \{S \in \mathcal{S} \mid t(S) = 1\}$, $T \in \mathcal{T}$, $U \in \delta T$ and $a \in U$ such that a is not minimal. In particular this means that $a_- \in U$. Therefore $m_T(a_-) = 0$ since $T \not\subseteq U$, $\{a_-, a\} \subseteq U$ and $1 - t(U) = 1 - 1 = 0$. We then deduce

$$y_T(a) = \sum_{b \in T, b < a} m_T(b) = y_T(a_-) + m_T(a_-) = y_T(a_-)$$

and so by an inductive argument $z_T|_U$ is constant. Moreover, this is actually an if and only if statement. Notice that the product $m_J(a)$ does not contain $(1 - t(A))$ or $(1 - t(\{b\}))$ for all $b \in A$ and so $m_J(a) = 0$ if and only if there exists $T \in \mathcal{T}$ such that $J \not\subseteq T$ and $\{a, a_+\} \subseteq T$. In particular, for $T \in \mathcal{T}$, if $\{a, a_+\}$ is not contained in some child of T then $z_T|_{\{a,a_+\}}$ will be non-constant and so the projection $S(W_N(T)) \to S(W_N(\delta T))$ will produce an injective map. Therefore by proposition 2.2.30, $z \in \bar{F}(A; = \mathcal{T}) \subseteq \bar{F}(A)$. It is clear that the ordering R is preserved since any ordered pair $a < b$ with respect to R produces a map $z_{\{a,b\}}$ with $z_{\{a,b\}}(a) = 0$ and $z_{\{a,b\}}(b) = 1$.

We now prove that $\phi_{A,R}$ is inverse to $\theta_{A,R}$. First let $x \in \bar{F}(A, R)$, $t = \theta_{A,R}(x)$ and $z = \phi_{A,R}(t)$. By lemma 2.4.10 we have $z \in \bar{F}(A; = \mathcal{T}(x))$ and so if we prove $z_J \sim x_J$ for all $J \in \mathcal{S}(x) \supseteq \mathcal{T}(x)$ then $z = x$. First let $J \in \mathcal{S}(x)$ be minimal. From remark 2.4.9 this means that the image of x_J must be an equally spaced configuration of the points in J. Similarly, because the product $m_J(a)$ will simply be a product of 1's in this case, $z_J \sim x_J$ as required. Now assume an inductive hypothesis that $x_{J'} \sim z_{J'}$ for all $J' \subset J$. Note that one can show $g^J_J(z) = 1$ for all $J \in \mathcal{S}(x)$ and that this gap appears between $z_J(a)$ and $z_J(a_+)$ such that there does not exist $J' \in \delta J$ such that $\{a, a_+\} \subseteq J'$. Now for $J' \in \delta J$, one should also notice that

$$z_J|_{J'} = \frac{g^J_{J'}(x)}{g^J_J(x)} \cdot z_{J'} + c$$

for some constant map c. This is because the only non-trivial element that one must multiply $m_{J'}(b)$ by to get $m_J(b)$ is $1 - t(J') = \frac{g^J_{J'}(x)}{g^J_J(x)}$. Since both x_J and z_J are 0 when evaluated at the least element, one concludes that

$$z_J = g^J_J(x)^{-1} x_J$$

as required. For the reverse argument, let $t \in K(A, R)$, $z = \phi_{A,R}(t)$ and $s = \theta_{A,R}$. Then it is clear from the arguments above that

$$s(J) = 1 - \frac{g^{\zeta(J)}_J(z)}{g^{\zeta(J)}_{\zeta(J)}(z)} = 1 - \frac{1 - t(J)}{1} = t(J)$$

as required.

It is easy to see that $\phi_{A,R}$ is continuous. Therefore we have a continuous inverse to $\theta_{A,R}$ and we know that $K(A, R)$ is compact and $\bar{F}(A, R)$ is Hausdorff. Hence we can conclude that $\theta_{A,R}$ is a homeomorphism. $\square$

Theorem 2.4.13. *Let $\theta_A : \bar{F}(A) \to K(A)$ be the disjoint union of maps $\theta_A = \coprod_{R \in \mathrm{Ord}(A)} \theta_{A,R}$. Then the collection $\theta = \{\theta_A\}$ is an isomorphism of operads*

$$\theta : \bar{F} \to K.$$

Proof. Proposition 2.4.12 shows that each θ_A is a homeomorphism. Therefore, all that is left to show is that θ commutes with the operad composition maps. If $p : A \to B$ is a surjective map of finite sets, then to make notation easier we will denote the associated composition map in K by Γ_p and the composition map in $\bar{F}$ will remain γ_p. Let $(x, (y_b)_{b\in B}) \in \bar{F}(B) \times \prod_{b\in B} \bar{F}(A_b)$. We need to show that

$$(t, R) = \theta_A(\gamma_p(x, (y_b)_{b\in B})) = \Gamma_p(\theta_B(x), (\theta_{A_b}(y_b))_{b\in B}) = (t', R').$$

First we check the total orders on A agree. Let $a, a' \in A$ such that $a \neq a'$ and first assume that $p(\{a, a'\}) = \{b\}$. Then

$$a <_R a' \;\Leftrightarrow\; \gamma_p(x, (y_b)_{b\in B})_{\{a,a'\}}(a) < \gamma_p(x, (y_b)_{b\in B})_{\{a,a'\}}(a') \;\Leftrightarrow\; y_{b,\{a,a'\}}(a) < y_{b,\{a,a'\}}(a'),$$

and

$$a <_{R'} a' \;\Leftrightarrow\; a <_{\theta_{A_b}(y_b)} a' \;\Leftrightarrow\; y_{b,\{a,a'\}}(a) < y_{b,\{a,a'\}}(a')$$

as required. Similarly, if $p(a) \neq p(a')$ then

$$a <_R a' \;\Leftrightarrow\; \gamma_p(x, (y_b)_{b\in B})_{\{a,a'\}}(a) < \gamma_p(x, (y_b)_{b\in B})_{\{a,a'\}}(a') \;\Leftrightarrow\; x_{\{p(a),p(a')\}}(a) < x_{\{p(a),p(a')\}}(a'),$$

and

$$a <_{R'} a' \;\Leftrightarrow\; a <_{\theta_B(x)} a' \;\Leftrightarrow\; x_{\{p(a),p(a')\}}(a) < x_{\{p(a),p(a')\}}(a'),$$

and so $R = R'$.

Now we show that t and t' are equal. We begin by showing that they have the same support. If we let $z = \gamma_p(x, (y_b)_{b\in B})$ then $\mathrm{supp}(t) = \mathcal{S}(z)$. On the other hand, if we let $\theta_B(x) = (t_B, R_B)$ and $\theta_{A_b}(y, b) = (t_b, R_b)$ for $b \in B$ then

$$\begin{aligned} \mathrm{supp}(t') &= p^{-1}\,\mathrm{supp}(t_B) \cup \bigcup_{b\in B} \mathrm{supp}(t_b) \\ &= p^{-1}\mathcal{S}(x) \cup \bigcup_{b\in B} \mathcal{S}(y_b). \end{aligned}$$

Denote this simply by $\mathcal{S}$. Therefore, we need to show that the composition in $\bar{F}$ grafts the associated Stasheff trees. Since $B \in \mathcal{S}(x)$, we have $p^{-1}(B) = A \in \mathcal{S}$. We then claim that if T is in both $\mathcal{S}(z)$ and $\mathcal{S}$, then the two sets of children of T agree which would be sufficient by induction. First assume that $p(T) = \{b\}$ so that $z_T = y_{b,T}$. Then it is obvious that U is a child of T with respect to $\mathcal{S}(z)$ if and only if it is a child of T with respect to $\mathcal{S}(y_b)$ also. Now assume $|p(T)| > 1$ so that $z_T = p^* x_{p(T)}$. One should notice that if $f : T \to \mathbb{R}$ is a non-constant map such that $f|_U \sim 0$ then U will be a subset of one of the blocks of the partition associated to f, (from remark 2.4.9). This is because all of the gaps in U will be 0 and so are less than the maximum. Therefore the blocks in the partition of z_T will only depend on the partition of $x_{p(T)}$ and so U is a child of T with respect to $\mathcal{S}(z)$ if and only if it is a child of T with respect to $p^{-1}\mathcal{S}(x)$ also. To show that t and t' agree on their support one can calculate that

$$t'(J) = \begin{cases} 1 - \dfrac{g_J^{\zeta(J)}(y_b)}{g_{\zeta(J)}^{\zeta(J)}(y_b)} & \text{if } p(J) = \{b\} \text{ and } J \in \mathcal{S}(y_b) \\ 1 & \text{if } p(J) = \{b\} \text{ and } J = A_b \text{ or } J = \{a\} \\ 1 - \dfrac{g_{p(J)}^{\zeta(p(J))}(x)}{g_{\zeta(J)}^{\zeta(J)}(x)} & \text{if } J = p^{-1}p(J) \text{ and } p(J) \in \mathcal{S}(x) \\ 1 & \text{if } J = p^{-1}p(J) \text{ and } p(J) = B \text{ or } p(J) = \{b\} \\ 0 & \text{otherwise} \end{cases}.$$

It is obvious that this is the same as $t(J)$ except in the case $J = A_b$ so that we simultaneously have $p(J) = \{b\}$ and $J = p^{-1}p(J)$. Here, $\zeta(J)$ wrt $\mathcal{S}(z)$ will be equal to $T = p^{-1}(\zeta(b))$. Then $z_T = p^* x_{\zeta(b)}$. But z_T will therefore be constant on J and so $g_J^T(z) = 0$ and so $1 - \frac{g_J^T(z)}{g_T^T(z)} = 1$ as required. ☐

Corollary 2.4.14. *We have an explicit isomorphism of operads*

$$W\bar{F}_1 \to \bar{F}_1.$$

In particular, $\bar{F}_1$ is a cofibrant operad.

Proof. We simply compose using the isomorphism from theorem 1.4.10

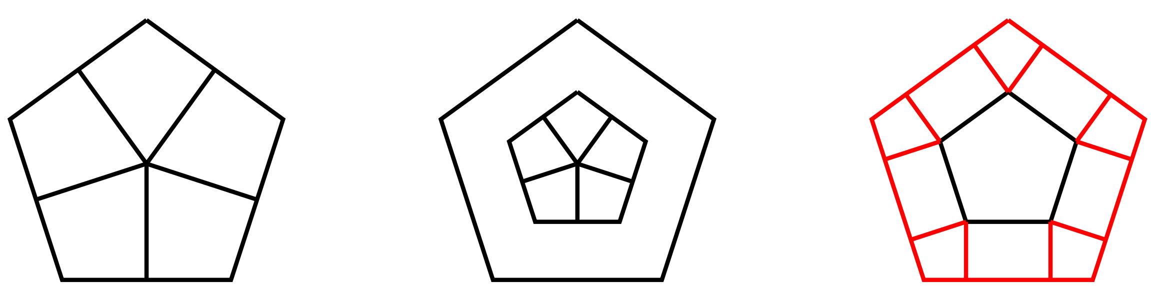
$$W\bar{F}_1 \xrightarrow{W\theta} WK \xrightarrow{\Phi} K \xrightarrow{\theta^{-1}} \bar{F}_1 .$$

☐

Remark 2.4.15. It is a result of Salvatore, [22] proposition 4.7, that in fact we have an isomorphism of operads $\Phi_N : W\bar{F}_N \to \bar{F}_N$ for any natural number N, and therefore the reduced Fulton-Macpherson operads are cofibrant. It would be desirable to be able to construct such isomorphisms explicitly.

If one applies the W-construction to $\bar{F}_N$ then this has the effect of gluing a block collar to the boundary of each space $\bar{F}_N(A)$. More precisely, it glues an m-dimensional cube to each point in each codimension m face of $\bar{F}_N(A)$. To see this, recall that by proposition 2.2.30, we can describe each element $x \in \bar{F}_N(A)$ by a full tree $\mathcal{T}$ on A decorated by points $x_T \in F_N(\delta T)$ for $T \in \mathcal{T}'$. Since the composition in $\bar{F}_N$ is equivalent to the grafting of these trees, this in fact means that the element $(x, \mathcal{C}_A, (-)) \in W\bar{F}_N(A)$ is equivalent to $((x_T), \mathcal{T}, (0)) \in W\bar{F}_N$. Therefore $W\bar{F}_N(A)$ is equivalent to the space $\widetilde{W}F_N(A)$ from definition 1.4.1. (Although F_N is not an operad, this makes sense since $\widetilde{W}$ only really requires a collection of spaces to act upon). So to each point in the boundary of $\bar{F}_N(A)$ with critical tree $\mathcal{T}$, we have glued a cube of dimension $|\mathcal{T}' \setminus \{A\}|$. But of course, this is exactly the codimension of the face that the point belongs to.

To define an isomorphism $W\bar{F}_N \to \bar{F}_N$ we need a family of homeomorphisms $W\bar{F}_N(A) \to \bar{F}_N(A)$ that commutes with the operad composition. To define these, we need a block collar for $\bar{F}_N(A)$ itself that we can then shrink to accommodate the collar added by W. This is essentially what we did for theorem 1.4.10 and why we chose to prove the theorem in this way. We can demonstrate an example of this using the following diagram:



Here we are concentrating on the case $|A| = 4$, where each connected component of $K(A)$ is classically drawn as a pentagon as above. The decomposition of quadrilaterals represents maps $t : \mathcal{J}(A, R) \to [0, 1]$ with different trees for support. As we can see, this decomposition provides the necessary collar. The second diagram represents how our map sends the image of $K(A) \hookrightarrow WK(A)$ to the shrunken down version in the interior of the pentagon. The third diagram then shows how this makes room for the collar that the W-construction adds.

Defining these collars is not too tricky when $|A| \leq 3$ but after this it becomes more complicated. This is because if we define the individual cubes that make up the collar, then we need them to agree where they intersect, and this is difficult. It would be useful if we had a spine of the manifold $\bar{F}_N(A)$ that we could define to be the subset of the collar with coordinates all 0. We could then attempt to linearly interpolate between this and the boundary, which would correspond to the subset of the collar with coordinates all 1. The work in chapter 4 is an attempt to make progress towards defining these spines and the associated collarings.

Chapter 3

A Comparison of Models

In this short chapter, we compare the Singh model of the Fulton-Macpherson operads with the one described by Sinha. It will turn out that these two models are isomorphic and have the same stratification via full trees. Therefore it is indeed appropriate to refer to the $\bar{F}_N$ as the Fulton-Macpherson operads.

3.1 Sinha's Model

We begin by giving an overview of the model for the Fulton-Macpherson operads explored by Sinha. The spaces are described in [24], but the only place we have been able to find an explicit description of the operad structure is in [16], (section 5.2), although we assume this is not the original source. Throughout this chapter, let $\mathbf{m} = \{0, \dots, m-1\}$.

Definition 3.1.1. Let N be a natural number, A a finite set with $|A| > 1$ and $(a_0, a_1) \in A^2$ such that $a_0 \neq a_1$. We then define a map $\pi_{a_0 a_1} : F_N(A) \to S^{N-1}$ by setting

$$\pi_{a_0 a_1}(x) = \frac{x(a_0) - x(a_1)}{\|x(a_0) - x(a_1)\|}.$$

We will refer to the output of $\pi_{a_0 a_1}$ as a relative direction. Also, let $(a_0, a_1, a_2) \in A^3$ such that a_0, a_1, a_2 are pairwise distinct. Then define a map $s_{a_0 a_1 a_2} : F_N(A) \to [0, \infty]$ by setting

$$s_{a_0 a_1 a_2}(x) = \frac{\|x(a_0) - x(a_1)\|}{\|x(a_0) - x(a_2)\|}.$$

We will refer to the output of $s_{a_0 a_1 a_2}$ as a relative distance. Notice that in both of these definitions we have had to make a choice of representatives. However, it is an easy check to see that both $\pi_{a_0 a_1}$ and $s_{a_0 a_1 a_2}$ are independent of this choice.

Definition 3.1.2. Define $\mathcal{A}_N(A)$ to be the product

$$\mathcal{A}_N(A) = \prod_{\omega \in \mathrm{Inj}(\mathbf{1}, A)} S^{N-1} \times \prod_{\eta \in \mathrm{Inj}(\mathbf{2}, A)} [0, \infty],$$

and then define the map $\alpha_N(A) : F_N(A) \to \mathcal{A}_N(A)$ to be the product

$$\alpha_N(A) = \prod_{\omega \in \mathrm{Inj}(\mathbf{1}, A)} \pi_{\omega(0), \omega(1)} \times \prod_{\eta \in \mathrm{Inj}(\mathbf{2}, A)} s_{\eta(0)\eta(1)\eta(2)}.$$

Finally, we define $\bar{D}_N(A) = cl(image(\alpha_N(A)))$ where $cl(X)$ is the closure of a subspace X.

Remark 3.1.3. It is immediate to see that $\bar{D}_N(A)$ is compact since it is a closed subspace of the compact space $\mathcal{A}_N(A)$.

Definition 3.1.4. We can stratify the space $\bar{D}_N(A)$ by full trees on A. Let

$$z = ((u_{a_0a_1}), (d_{a_0a_1a_2})) \in \bar{D}_N(A).$$

Then we associate a full tree $\mathcal{T}(z)$ which is the collection of sets

$$T_{a_0a_2} = \{a_0\} \cup \bigcup_{d_{a_0a_1a_2}=0} \{a_1\},$$

or empty if there is no such a_1, as well as the set A itself and all of the singleton sets.

Lemma 3.1.5. *$\mathcal{T}(z)$ is indeed a full tree on A.*

Proof. Firstly, take any $a_1 \in T_{a_0a_2}$ which means that $d_{a_0a_1a_2} = 0$. Notice that in general $d_{a_0a_1a_2}d_{a_0a_2a_1} = 1$ and so we also have $d_{a_0a_2a_1} = \infty$. (In the first equality, we are defining $0 \cdot \infty = \infty \cdot 0 = 1$). Now, representatives $x \in F_N(A)$ will satisfy the triangle inequality $\|x(a) - x(c)\| \leq \|x(a) - x(b)\| + \|x(b) - x(c)\|$ and so this will transfer to $\bar{D}_N(A)$ to become $d_{acb} \leq 1 + d_{bca}$. Therefore we have

$$\infty = d_{a_0a_2a_1} \leq 1 + d_{a_1a_2a_0}$$

which implies that $d_{a_1a_2a_0} = \infty$ also. This is in fact an if and only if statement: $d_{a_1a_2a_0} = d_{a_0a_2a_1}$ if and only if they are both equal to infinity. Therefore $d_{a_1a_0a_2} = 0$ and so $a_0 \in T_{a_1a_2}$. Now take any other $a_3 \in T_{a_0a_2}$. This means that $d_{a_0a_3a_2} = d_{a_3a_0a_2} = 0$ and $d_{a_3a_2a_0} = \infty$. Because $d_{a_0a_1a_2} = d_{a_1a_0a_2}$ it implies that $d_{a_2a_0a_1} = 1$. Notice that if this is the case in $F_N(A)$ we have $d_{a_1a_3a_2} = d_{a_0a_3a_2}d_{a_3a_2a_0}d_{a_3a_1a_2}$ but the first two terms multiply to 1 and so $d_{a_1a_3a_2} = d_{a_3a_1a_2}$. This passes to the limit and implies that $d_{a_1a_2a_3} = d_{a_3a_2a_1}$ but our statement above implies that $d_{a_1a_2a_3} = \infty$ and so $d_{a_1a_3a_2} = 0$. Therefore we can conclude that $T_{a_0a_2} \subseteq T_{a_1a_2}$. Because we know that $a_0 \in T_{a_1a_2}$ we can then rewrite all of these arguments to show that $T_{a_1a_2} \subseteq T_{a_0a_2}$ and therefore they are equal.

Now assume that we have $T_{a_0a_1}, T_{a_2a_3} \in \mathcal{T}(z)$ such that $T_{a_0a_1} \cap T_{a_2a_3} \neq \emptyset$. Therefore take $a' \in T_{a_0a_1} \cap T_{a_2a_3}$ then by the above we can relabel as $T_{a_0a_1} = T_{a'a_1}$ and $T_{a_2a_3} = T_{a'a_3}$. Then assume further that $T_{a'a_3} \not\subseteq T_{a'a_1}$. We then need to show that $T_{a'a_1} \subseteq T_{a'a_3}$. Now, our assumption means there exists $a'' \in A$ such that $d_{a'a''a_3} = 0$ but $d_{a'a''a_1} \neq 0$. Notice that for elements in $F_N(A)$ we have the identity $d_{ijk} = d_{ijl}d_{ilk}$ and then this passes to the closure by taking limits. Using this we see that

$$0 = d_{a'a''a_3} = d_{a'a_1a_3}d_{a'a''a_1}$$

so we have to have $d_{a'a_1a_3} = 0$ which implies that $a_1 \in T_{a'a_3}$. Now take any $a \in T_{a'a_1}$. Then $d_{a'aa_1} = 0$ and

$$d_{a'aa_1}d_{a'a_1a_3} = d_{a'aa_3}$$

so $d_{a'aa_3} = 0$ also. Therefore $a \in T_{a'a_3}$ and so $T_{a'a_1} \subseteq T_{a'a_3}$ as required. $\square$

Definition 3.1.6. Set $\bar{D}_N(A) = \{*\}$ if $|A| \leq 1$. The functor $\bar{D}_N : \mathrm{FSET}_{\cong} \to TOP$ given by $A \mapsto \bar{D}_N(A)$ can be given an operad structure as follows. The unit map is forced since

$\bar{D}_N(\{*\})$ is a point. Then let $p : A \to B$ be a map of finite sets, $((u_{b_0b_1}), (d_{b_0b_1b_2})) \in \bar{D}_N(B)$ and $\left((v^b_{a_0a_1}), (e^b_{a_0a_1a_2})\right) \in \bar{D}_N(A_b)$. We then define composition maps

$$\gamma_p : \bar{D}_N(B) \times \prod_{b\in B} \bar{D}_N(A_b) \to \bar{D}_N(A)$$

by setting

$$\gamma_p \left(((u_{b_0b_1}), (d_{b_0b_1b_2})) \, , \left((v^b_{a_0a_1}), (e^b_{a_0a_1a_2})\right)_{b\in B} \right) = ((w_{a_0a_1}), (f_{a_0a_1a_2}))$$

where

$$w_{a_0a_1} = \begin{cases} v^b_{a_0a_1} & \text{if } p(a_0) = p(a_1) = b \\ u_{p(a_0)p(a_1)} & \text{otherwise} \end{cases}$$

and

$$f_{a_0a_1a_2} = \begin{cases} e^b_{a_0a_1a_2} & \text{if } p(a_0) = p(a_1) = p(a_2) = b \\ d_{p(a_0)p(a_1)p(a_2)} & \text{if } p(a_0), p(a_1), p(a_2) \text{ are all distinct} \\ 0 & \text{if } p(a_0) = p(a_1) \neq p(a_2) \\ 1 & \text{if } p(a_0) \neq p(a_1) = p(a_2) \\ \infty & \text{if } p(a_0) = p(a_2) \neq p(a_1) \end{cases} .$$

The definition of this composition can alternatively be found in [16], section 5.2.

3.2 Equivalence to the Singh Model

Recall $W_N(-)$, $S(W_N(-))$ and $\bar{F}_N(-)$ from definitions 2.2.1, 2.2.3 and 2.2.6 respectively. In this section we will implicitly use a diagram that looks as follows:

$$\begin{tikzcd}
F_N(A) \arrow[r, "\iota"] \arrow[ddr, "\alpha_N(A)"'] & \bar{F}_N(A) \arrow[r, "\iota"] \arrow[d, "\alpha_N(A)"] & \prod S(W_N(B)) \arrow[ddl, "\alpha_N(A)"] \\
 & \bar{D}_N(A) \arrow[d, "\iota"] & \\
 & \mathcal{A}_N(A) &
\end{tikzcd}$$

where all of the arrows labelled ι are simply inclusions. We will define a map

$$\alpha_N(A) : \prod_{B\subseteq A, |B|>1} S(W_N(B)) \to \mathcal{A}_N(A)$$

which when restricted to $\bar{F}_N(A)$ gives a homeomorphism to $\bar{D}_N(A)$. It then turns out that the collection of these maps will be an isomorphism of operads $\alpha_N : \bar{F}_N \to \bar{D}_N$.

Definition 3.2.1. For $|A| > 1$ we define a map

$$\alpha_N(A) : \prod_{B\subseteq A, |B|>1} S(W_N(B)) \to \mathcal{A}_N(A),$$

by setting

$$\alpha_N(A)(x) = \big((\pi_{\omega(0)\omega(1)}(x_{\omega(\mathbf{1})}))_{\omega\in\mathrm{Inj}(\mathbf{1},A)}, (s_{\eta(0)\eta(1)\eta(2)}(x_{\eta(\mathbf{2})}))_{\eta\in\mathrm{Inj}(\mathbf{2},A)}\big)$$

We have had to make a choice of representative here for x but it is clear that the definition of $\alpha_N(A)$ is independent of that choice. Note also that here in the definition of s, the numerator and the denominator cannot both equal 0 since this would imply that $x_{\eta(\mathbf{2})}$ is constant. Therefore there is no problem in interpreting this as an element of $[0, \infty]$.

We state without proof the following lemma which we will use in the proof of proposition 3.2.3.

Lemma 3.2.2. *Let Z be a subspace of X and $f : X \to Y$ be a continuous function. Then $f(cl_X(Z)) \subseteq cl_Y(f(Z))$. If $cl_X(Z)$ is compact then this inclusion is an equality.*

Proposition 3.2.3. *The image of $\alpha_N(A)$ restricted to $\bar{F}_N(A) \subseteq \prod_{B\subseteq A,|B|>1} S(W_N(B))$ is $\bar{D}_N(A)$.*

Proof. We know that $cl(F_N(A)) = \bar{F}_N(A)$ and this is a compact space, and $cl(\alpha_N(A)(F_N(A)))$ $= \bar{D}_N(A)$ by definition. Since $\alpha_N(A)$ is clearly continuous, we conclude by lemma 3.2.2 that

$$\alpha_N(A)(\bar{F}_N(A)) = \bar{D}_N(A)$$

as required. □

Lemma 3.2.4. *Let $x \in \bar{F}_N(A)$. Then $\mathcal{T}(\alpha_N(A)(x)) = \mathcal{T}(x)$.*

Proof. Let $\alpha_N(A)(x) = ((u_{a_0a_1}), (d_{a_0a_1a_2}))$. The lemma is obviously true if $|A| \leq 2$ and so we assume that $|A| > 2$. First take $T \in \mathcal{T}(x)' \setminus \{A\}$ which means that for all $T \subset U \subseteq A$ we have $x_U|_T \sim 0$. In particular, this means that $x_{\zeta(T)}|_T \sim 0$ where $\zeta(T)$ is the parent of T. Take $a \in T$ and $c \in \zeta(T) \setminus T$. Then let b be any other element in T that is not equal to a. Such an element exists since $|T| > 1$. It is then the case that $\mathcal{T}(x)(B) = \zeta(T)$, where $B = \{a, b, c\}$ and so it is easy to see that $d_{abc} = 0$. Therefore $b \in T_{ac}$ and so $T \subseteq T_{ac}$. Conversely, if $d_{abc} = 0$ for some $b \in A$ then $x_{\mathcal{T}(x)(B)}(a) = x_{\mathcal{T}(x)(B)}(b)$. Note that we must have $\mathcal{T}(x)(B) \supseteq \zeta(T)$ but if this is a strict superset then $x_{\mathcal{T}(x)(B)}(a) = x_{\mathcal{T}(x)(B)}(c)$ by the criticality of $\zeta(T)$. This contradicts $x_{\mathcal{T}(x)(B)}|_B$ being non-constant and so we must have $\mathcal{T}(x)(B) = \zeta(T)$. Hence $b \in T$ as a consequence of 2.2.30 so $T_{ac} \subseteq T$ and therefore they are equal. This means that $\mathcal{T}(x) \subseteq \mathcal{T}(\alpha_N(A)(x))$.

On the other hand, let $T_{ac} \in \mathcal{T}(\alpha_N(A)(x))$ and $T = \mathcal{T}(x)(T_{ac})$. First notice that we cannot have $c \in T$. If it were then $T = \mathcal{T}(x)(\{a, b, c\})$ for any $b \in T_{ac}$ with $b \neq a$ but then this would imply that $x_T|_{T_{ac}} \sim 0$ which is a contradiction. Consequently, we must then have $c \in \zeta(T)$. If it is not then $\mathcal{T}(x)(\{c\} \cup T_{ac}) \supset \zeta(T)$ and in particular for all $b \in \zeta(T)$ we would have $d_{abc} = 0$. This in turn would mean that $\zeta(T) \subseteq T_{ac}$ which contradicts $T = \mathcal{T}(x)(T_{ac})$. Now, if $c \in \zeta(T)$ then we can argue the same way again to show that $T \subseteq T_{ac}$ and therefore they are equal. This means that $\mathcal{T}(\alpha_N(A)(x)) \subseteq \mathcal{T}(x)$ as required. □

Proposition 3.2.5. *The restriction $\alpha_N(A) : \bar{F}_N(A) \to \bar{D}_N(A)$ is an injective map.*

Proof. Let

$$z = ((u_{a_0a_1}), (d_{a_0a_1a_2})) = \alpha_N(A)(x) \in \bar{D}_N(A).$$

We show that we can retrieve x by knowing z. By lemma 3.2.4 we know that $\mathcal{T}(z) = \mathcal{T}(x)$, denote this simply by $\mathcal{T}$. Now take $B \in \mathcal{T}'$ and for every $V \in \delta B$, select an element $a_V \in V$.

Note that $|\delta B| > 1$ so there will exist at least one pair of distinct elements a_{V_0} and a_{V_1}. We then define a map $y_B : B \to \mathbb{R}^N$ by setting

$$y_B(a_{V_0}) = 0,$$

$$y_B(a_{V_1}) = u_{a_{V_1} a_{V_0}},$$

$$y_B(a_V) = d_{a_{V_0} a_V a_{V_1}} \cdot u_{a_V a_{V_0}} \qquad \text{for all } a_V \neq a_{V_0}, a_{V_1}.$$

The last line makes sense since $d_{a_{V_0} a_V a_{V_1}} = \infty$ if and only if $d_{a_{V_0} a_{V_1} a_V} = 0$ which in turn happens if and only if a_{V_0} and a_{V_1} are in the same child of B which does not happen here by construction. Finally we set

$$y_B(a) = y_B(a_V) \qquad \Leftrightarrow \qquad a \in V \in \delta B.$$

If $B \subseteq A$ with $|B| > 1$ but $B \notin \mathcal{T}'$ then we define $y_B : B \to \mathbb{R}^N$ by setting $y_B = y_{\mathcal{T}(B)}|_B$. We claim that y_B is in the equivalence class x_B for all $B \subseteq A$, i.e. that y_B is equivalent to some representative x_B modulo translation and scaling. First let $B \in \mathcal{T}'$ and notice that for any $a_V, a_{V'} \in B$ with V and V' distinct, $\mathcal{T}(\{a_V, a_{V'}\}) = B$. Then

$$y_B(a_{V_0}) = \lambda(x_B(a_{V_0}) - x_B(a_{V_0})) \qquad \text{for any } \lambda > 0,$$

$$y_B(a_{V_1}) = u_{a_{V_1} a_{V_0}} = \frac{x_B(a_{V_1}) - x_B(a_{V_0})}{\|x_B(a_{V_1}) - x_B(a_{V_0})\|}$$

$$\begin{aligned} y_B(a_V) &= d_{a_{V_0} a_V a_{V_1}} \cdot u_{a_V a_{V_0}} \\ &= \frac{\|x_B(a_{V_0}) - x_B(a_V)\|}{\|x_B(a_{V_0}) - x_B(a_{V_1})\|} \cdot \frac{x_B(a_V) - x_B(a_{V_0})}{\|x_B(a_V) - x_B(a_{V_0})\|} \\ &= \frac{x_B(a_V) - x_B(a_{V_0})}{\|x_B(a_{V_0}) - x_B(a_{V_1})\|}. \end{aligned}$$

Finally, notice that if $a, a' \in V \in \delta B$ then $x_B(a) = x_B(a')$. Therefore we conclude that

$$y_B = \frac{x_B - x_B(a_{V_0})}{\|x_B(a_{V_0}) - x_B(a_{V_1})\|} \sim x_B.$$

This is also true for $B \notin \mathcal{T}'$ since $x_B \sim x_{\mathcal{T}(B)}|_B$ which is also the case for y_B. □

Remark 3.2.6. The algorithm for computing the preimage in proposition 3.2.5 in fact gives us a well-defined map $\alpha_N(A)^{-1} : \bar{D}_N(A) \to \bar{F}_N(A)$. The proof demonstrates that the choices made for a_{V_0} and a_{V_1} only affect the translation and scale factor when comparing y_B to x_B and so in fact have no effect on the class of y_B.

Corollary 3.2.7. *The restriction $\alpha_N(A) : \bar{F}_N(A) \to \bar{D}_N(A)$ is a homeomorphism.*

Proof. By propositions 3.2.3 and 3.2.5 we see that $\alpha_N(A)$ is a continuous bijective map. Then, because $\bar{F}_N(A)$ is compact and $\bar{D}_N(A)$ is Hausdorff, we may conclude that $\alpha_N(A)$ is a homeomorphism. □

Theorem 3.2.8. *If $|A| \leq 1$ then set $\alpha_N(A) : \bar{F}_N(A) \to \bar{D}_N(A)$ to be the unique map from a point to itself. Then the collection $\alpha_N = \{\alpha_N(A)\}$ is an isomorphism of operads $\alpha_N : \bar{F}_N \to \bar{D}_N$.*

Proof. Corollary 3.2.7 tells us that each $\alpha_N(A) : \bar{F}_N(A) \to \bar{D}_N(A)$ is a homeomorphism. It is clear that if $|A| = 1$ then $\alpha_N(A)$ will preserve the unit as there is only one choice for it in both $\bar{F}_N$ and $\bar{D}_N$. Therefore, we only need to prove that α_N commutes with the composition maps. To make notation easier, if $p : A \to B$ is a map of finite sets, then we will denote the associated composition map in $\bar{D}_N$ by Γ_p, and the composition map in $\bar{F}_N$ will remain as γ_p. Let $(x, (y_b)_{b\in B}) \in \bar{F}_N(B) \times \prod_{b\in B} \bar{F}_N(A_b)$. We then want to show that

$$\alpha_N(A)(\gamma_p(x, (y_b)_{b\in B})) = \Gamma_p(\alpha_N(B)(x), (\alpha_N(A_b)(y_b))_{b\in B}).$$

First we analyse the left hand side. Let $z = \gamma_p(x, (y_b)_{b\in B})$ so that

$$z_T = \begin{cases} y_{b,T} & \text{if } p(T) = \{b\} \\ p^* x_{p(T)} & \text{if } |p(T)| > 1 \end{cases}.$$

Then set $\alpha_N(A)(z) = ((\tilde{u}_{a_0a_1}), (\tilde{d}_{a_0a_1a_2}))$. Now we turn to the right hand side. Let

$$\alpha_N(B)(x) = ((v_{b_0b_1}), (e_{b_0b_1b_2})),$$

$$\alpha_N(A_b)(y_b) = ((w^b_{a_0a_1}), (f^b_{a_0a_1a_2}))$$

and $\Gamma_p(\alpha_N(B)(x), (\alpha_N(A_b)(y_b))_{b\in B}) = ((u_{a_0a_1}), (d_{a_0a_1a_2}))$. Recall that $\mathcal{T}(z) = p^{-1}\mathcal{T}(x) \cup \bigcup_{b\in B} \mathcal{T}(y_b)$ and so for $T \subseteq A$

$$\mathcal{T}(z)(T) = \begin{cases} \mathcal{T}(y_b)(T) & \text{if } p(T) = \{b\} \\ p^{-1}\mathcal{T}(x)(p(T)) & \text{if } |p(T)| > 1 \end{cases}.$$

We first check the relative directions. If $p(\{a_0, a_1\}) = \{b\}$ then

$$\begin{aligned} \tilde{u}_{a_0a_1} &= \pi_{a_0a_1}(z_{\{a_0,a_1\}}) \\ &= \pi_{a_0a_1}(y_{b,\{a_0,a_1\}}) \\ &= w^b_{a_0a_1} \\ &= u_{a_0a_1}. \end{aligned}$$

Similarly, if $p(a_0) \neq p(a_1)$ then

$$\begin{aligned} \tilde{u}_{a_0a_1} &= \pi_{a_0a_1}(z_{\{a_0,a_1\}}) \\ &= \pi_{a_0a_1}(p^* x_{p(T)}) \\ &= v_{p(a_0)p(a_1)} \\ &= u_{a_0a_1}. \end{aligned}$$

We can also go through and check all of the cases for the relative distances. If $p(\{a_0, a_1, a_2\}) = \{b\}$ then

$$\tilde{d}_{a_0a_1a_2} = s_{a_0a_1a_2}(y_{b,\{a_0,a_1,a_2\}}) = f^b_{a_0a_1a_2} = d_{a_0a_1a_2}.$$

Otherwise, if $|p(\{a_0, a_1, a_2\})| > 1$ then

$$\tilde{d}_{a_0a_1a_2} = s_{a_0a_1a_2}(p^* x_{p\{a_0,a_1,a_2\}}).$$

We then have to check subcases:

- If $p(a_0)$, $p(a_1)$ and $p(a_2)$ are all distinct then $\tilde{d}_{a_0a_1a_2} = e_{p(a_0)p(a_1)p(a_2)} = d_{a_0a_1a_2}$.
- If $p(a_0) = p(a_1) \neq p(a_2)$ then $\tilde{d}_{a_0a_1a_2} = 0 = d_{a_0a_1a_2}$ since
$$p^*x_{p\{a_0,a_1,a_2\}}(a_0) = p^*x_{p\{a_0,a_1,a_2\}}(a_1).$$
- If $p(a_0) \neq p(a_1) = p(a_2)$ then $\tilde{d}_{a_0a_1a_2} = 1 = d_{a_0a_1a_2}$ since
$$p^*x_{p\{a_0,a_1,a_2\}}(a_1) = p^*x_{p\{a_0,a_1,a_2\}}(a_2).$$
- If $p(a_0) = p(a_2) \neq p(a_1)$ then $\tilde{d}_{a_0a_1a_2} = \infty = d_{a_0a_1a_2}$ since
$$p^*x_{p\{a_0,a_1,a_2\}}(a_0) = p^*x_{p\{a_0,a_1,a_2\}}(a_2).$$

So $((\tilde{u}_{a_0a_1}), (\tilde{d}_{a_0a_1a_2})) = ((u_{a_0a_1}), (d_{a_0a_1a_2}))$ as required. $\square$

Chapter 4

The Fulton-Macpherson Posets

In this chapter, we introduce partially ordered sets that are combinatorial models for spaces we have seen earlier in the thesis. By this we mean that the geometric realisations of these posets are homotopy equivalent to those spaces. At the beginning of the chapter we remind the reader of some of the facts concerning posets. We then prove a statement that allows one to induce a homotopy equivalence of spaces, given a map from a space to a poset satisfying certain conditions. Next we introduce the main family of posets that we will use, as well as interesting subsets that will form the basis of our models. We will pay particular attention to the posets of injective chained linear preorders. These posets are indexed by natural numbers N and finite sets A. It will turn out that the dimension of their geometric realisations are $(N-1)(|A|-1)$, which one should recall was the homological dimension of $F_N(A)$. Therefore, the realisations of these posets are spines for $F_N(A)$'s. Finally, we will explore a family of posets with an operad structure that is a combinatorial model for the Fulton-Macpherson operads.

4.1 Revision of Finite Posets

Here we recall some constructions and facts concerning finite posets, mostly to set notation.

Definition 4.1.1. A partially ordered set, or poset for short, is a set P equipped with a relation $R \subseteq P^2$ such that

- $(a, a) \in R$ for all $a \in P$.
- If $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.
- If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

If $(a, b) \in R$ then we will usually write $a \preceq_P b$ or $a \prec_P b$ if we know that $a \neq b$. A chain in P is a subset $C \subseteq P$ that is totally ordered, i.e.

$$a, b \in C \qquad \Rightarrow \qquad a \preceq_P b \text{ or } b \preceq_P a.$$

Remark 4.1.2. The category of posets, where the morphisms are the order preserving maps, is symmetric monoidal with product the cartesian product of sets. If P and Q are posets, then the relation on $P \times Q$ is defined by setting

$$(a, c) \preceq_{P \times Q} (b, d) \qquad \Leftrightarrow \qquad a \preceq_P b \text{ and } c \preceq_Q d.$$

Definition 4.1.3. We can associate to every finite poset P, a simplicial complex whose vertices are the points in P, and simplices are the non-empty chains in P. If $t : P \to [0,1]$ is a map, then define the support of t to be $\mathrm{supp}(t) = \{a \in P \mid t(a) \neq 0\}$. Then the geometric realisation of P is the realisation of the simplicial complex:

$$|P| = \left\{ t : P \to [0,1] \;\middle|\; \mathrm{supp}(t) \text{ is a chain and } \sum_{a \in P} t(a) = 1 \right\}.$$

Notice that this construction is a functor from finite posets to compact Hausdorff topological spaces.

Lemma 4.1.4. *Geometric realisation preserves cartesian products, that is $|P \times Q|$ is homeomorphic to $|P| \times |Q|$.*

Proof. We will provide the necessary maps without spelling out the details of why they are inverse homeomorphisms. We define a map $\theta : |P \times Q| \to |P| \times |Q|$ by setting $\theta(r) = (s,t)$ where

$$s(p) = \sum_{q \in Q} r(p,q) \qquad\qquad t(q) = \sum_{p \in P} r(p,q).$$

Conversely we define $\theta^{-1} : |P| \times |Q| \to |P \times Q|$ as follows. Let $s \in |P|$ and $t \in |Q|$ and define

$$\tilde{s}(p) = \sum_{p' \preceq p} s(p') \qquad\qquad \tilde{t}(q) = \sum_{q' \preceq q} t(q').$$

We then define

$$\tilde{r}(p,q) = min\{\tilde{s}(p), \tilde{t}(q)\}$$

$$r(p,q) = \tilde{r}(p,q) - max\{\tilde{r}(p',q') \mid (p',q') \prec (p,q)\}$$

and set $\theta^{-1}(s,t) = r$. □

Remark 4.1.5. The previous lemma demonstrates that if we have an operad $\mathcal{P}$ in the category of posets, then we have an associated operad $|\mathcal{P}|$ in the category of spaces.

Geometric realisation does not commute in general with taking unions. A subset $U \subseteq P$ is said to be closed upwards if $p \in U$ and $p \preceq q$ in P implies that $q \in U$ also. We can then use this to rectify the problem above.

Lemma 4.1.6. *Let P be a partially ordered set and $U, V \subseteq P$ be subsets that are closed upwards. Then $U \cup V$ is closed upwards and $|U \cup V| = |U| \cup |V|$.*

Proof. The first claim and that $|U| \cup |V| \subseteq |U \cup V|$ are clear and so we only need to show the opposite inclusion. Let $t \in |U \cup V|$ and p be the smallest element in $\mathrm{supp}(t)$. If $p \in U$ then the whole chain is in U since U is closed upwards. Therefore $t \in |U|$. Similarly, if $p \in V$ then $t \in |V|$ and so $|U \cup V| \subseteq |U| \cup |V|$ as required. □

Definition 4.1.7. A ranked poset P is one equipped with a rank function $\rho : P \to \mathbb{N}_0$ such that

- If $a \prec_P b$ then $\rho(a) < \rho(b)$.
- If $a \prec_P b$ and there does not exist $c \in P$ such that $a \prec_P c \prec_P b$ then $\rho(a) = \rho(b) + 1$.

Lemma 4.1.8. *Let P be a finite ranked poset such that $max_{a\in P}\{\rho(a)\} - min_{b\in P}\{\rho(b)\} = d$ and there exists a chain $C \subseteq P$ such that $|C| = d+1$. Then the dimension of $|P|$ is equal to d.*

Proof. The existence of C shows that the dimension of $|P|$ is at least d. If $C' \subseteq P$ is a chain such that $|C'| > d+1$ then

$$\rho(max\{a \in C\}) - \rho(min\{b \in C\}) > d$$

which contradicts the first condition of the lemma. Therefore the dimension of $|P|$ is at most d. □

It is possible to make inferences about the homotopy type of realised maps and spaces based on properties of the posets themselves and the maps between them. The following two results are examples of this. The first is very well known.

Proposition 4.1.9. *Let $f, g : P \to Q$ be two maps of finite posets. Suppose that $f(p) \preceq g(p)$ for all $p \in P$. Then $|f|$ and $|g|$ are homotopic.*

The second is often known as the "McCord-Quillen" theorem for finite posets, although the reference we provide contains a more elementary proof.

Theorem 4.1.10. *If $f : P \to Q$ is a map of finite posets and for all $q \in Q$ the poset*

$$W(q) = \{p \in P \mid f(p) \succeq q\}$$

has contractible realisation, then $|f| : |P| \to |Q|$ is a homotopy equivalence ([1] theorem 1.1).

Using these two results, we can make conclusions about the homotopy types of fibers of maps under realisation. To facilitate this, we first give a description of these fibers as a poset.

Definition 4.1.11. Let $f : P \to Q$ be a map of finite posets, $q_0 \prec q_1 \in Q$ and $p_0 \in P$ such that $f(p_0) = q_0$. We then define

$$\mathcal{U}(p_0; q_0, q_1) = \{p_1 \in P \mid p_0 \prec p_1 \text{ and } f(p_1) = q_1\}.$$

Definition 4.1.12. Let $f : P \to Q$ be a map of finite posets and $\underline{q} = q_0 \prec \cdots \prec q_k$ a chain in Q. Also, let $t \in |Q|$ such that $\mathrm{supp}(t) = \underline{q}$. We then define $F(t)$ to be the set of chains of the form $(p_0 \prec \cdots \prec p_k)$ such that $f(p_i) = q_i$ for all $0 \leq i \leq k$. We give this set a partial order by viewing $F(t)$ as a subset of $P^{\mathrm{op}} \times P \times P^{\mathrm{op}} \times P \times \cdots$.

Proposition 4.1.13. *$|F(t)|$ is homeomorphic to $|f|^{-1}(t)$.*

Proof. First we define a map $\theta_t : F(t) \to |f|^{-1}(t)$ by setting $\theta_t(p_0, \ldots, p_k) = s$ where $s(p) = t(q_i)$ for $p = p_i$ and $s(p) = 0$ otherwise. We then take all convex combinations to give a map $|\theta_t| : |F(t)| \to \mathrm{Map}(P, [0,1])$ which we claim is in fact a homeomorphism to $|f|^{-1}(t)$. Let

$$\underline{p} = (p_{00}, \cdots, p_{0k}) \prec \cdots \prec (p_{l0}, \cdots, p_{lk})$$

be a chain in $F(t)$. Then we in fact have a chain in P:

$$p_{l0} \preceq \cdots \preceq p_{00} \prec p_{01} \preceq \cdots \preceq p_{l1} \prec p_{l2} \preceq \cdots.$$

We cannot specify the end of the chain as this depends on whether k is odd or even, but we do know that it will be finite. If $s \in |P|$ is supported on this chain then $|f|(s)$ will be supported on $\underline{q}$. If $r \in |F(t)|$ is supported on $\underline{p}$ then $\sum_{l'=0}^{l} |\theta(t)|(r)(p_{l'i}) = t(q_i)$ for all $0 \leq i \leq k$ and so $image(|\theta_t|)$ is in fact contained in $|f|^{-1}(t)$.

Now notice that we can consider $|f|^{-1}(t)$ as a subset of $\prod_{i=0}^{k} |P|$. By definition, $|F(t)|$ is a subset of $|P^{\mathrm{op}} \times P \times P^{\mathrm{op}} \times P \times \cdots|$. By lemma 4.1.4 and by seeing that realisation is invariant when we take the opposite poset, we have homeomorphisms

$$|P^{\mathrm{op}} \times P \times P^{\mathrm{op}} \times P \times \cdots| \cong |P^{\mathrm{op}}| \times |P| \times |P^{\mathrm{op}}| \times |P| \cdots \cong |P| \times |P| \times |P| \times |P| \cdots .$$

Then by checking the definitions, we see that $|\theta_t|$ is in fact the restriction of these homeomorphisms to $|F(t)|$ and so $|\theta_t|$ is injective. Finally, we check that $|f|^{-1}(t) \subseteq image(|\theta_t|)$ as this will imply that they are in fact equal and so $|\theta_t| : |F(t)| \to |f|^{-1}(t)$ will be surjective also. If $\underline{q} = \{q_0\}$ is only a single point then this is obvious since $F(t) = f^{-1}(q_0)$ in this case. Otherwise, we may assume by induction that we can find a preimage $r' \in |F(t)|$ for some $s' \in |f|^{-1}(t')$ where t' is supported on $\underline{q}' = \{q_0, \ldots, q_{k-1}\}$. Then if we have $s \in |f|^{-1}(t)$, we should first notice that if $p \in \mathrm{supp}(s)$ such that $f(p) = q_k$ then $p \succ p'$ for all $p' \in \mathrm{supp}(s)$ such that $f(p') \neq q_k$. Therefore we can take these p's and append them to the end of the chains in the preimage r', either in order or opposite order depending on whether k is odd or even. If we have too few elements then we can repeat elements as this will still be a chain. Equally, if we have too many, we can repeat elements in r' as necessary. This construction is completely analogous to the inverse of the homeomorphism between products above and creates a preimage $r \in |F(t)|$ as required. $\square$

We are now ready to prove the following:

Proposition 4.1.14. *Let $f : P \to Q$ be a map of finite posets. Given $p_0 \in P$ and $q_0 \prec q_1$ in Q with $f(p_0) = q_0$, put*

$$\mathcal{U}(p_0; q_0, q_1) = \{p_1 \in P \mid p_0 \prec p_1,\ f(p_1) = q_1\}.$$

as in definition 4.1.11. If $|\mathcal{U}(p_0; q_0, q_1)|$ is contractible for all $(p_0; q_0, q_1)$ then all fibers of $|f|$ have the same homotopy type as the fiber of some vertex.

Proof. Using the fact $|f|^{-1}(t) \cong |F(t)|$ for $t \in |Q|$ and t supported on $\underline{q} = q_0 \prec \cdots \prec q_k$, we define a map of posets

$$\rho : F(t)^{\mathrm{op}} \to F(t')$$
$$\rho(s, p) = s,$$

where as before t' is supported on $\underline{q}' = q_0 \prec \cdots \prec q_{k-1}$ and appropriately scaled. For $s' \in F(t')$ we consider the set

$$\begin{aligned} W(s') &= \{(s,p) \in F(t)^{\mathrm{op}} \mid \rho(s,p) \succeq s'\} \\ &= \{(s,p) \in F(t)^{\mathrm{op}} \mid s \succeq s'\}. \end{aligned}$$

Define $\phi : \mathcal{U}(s'_{max}; q_{k-1}, q_k)^{\mathrm{op}} \to W(s')$ by

$$\phi(p) = (s', p)$$

and $\sigma : W(s') \to \mathcal{U}(s'_{max}; q_{k-1}, q_k)^{\mathrm{op}}$ by

$$\sigma(s, p) = p.$$

Then one sees that $\sigma\phi = 1$ and $\phi\sigma \preceq 1$ so $|\phi|$ and $|\sigma|$ are homotopy equivalences by proposition 4.1.9. Therefore

$$|W(s')| \simeq |\mathcal{U}(s'_{max}; q_{k-1}, q_k)^{\mathrm{op}}| \cong |\mathcal{U}(s'_{max}; q_{k-1}, q_k)|$$

is contractible by hypothesis and so $|\rho|$ is also a homotopy equivalence by theorem 4.1.10. If q_0 is the least element in $\mathrm{supp}(t)$ then by induction we have

$$|f|^{-1}(t) \cong |F(t)| \cong |F(t)^{\mathrm{op}}| \simeq |F(t')| \simeq \cdots \simeq |f|^{-1}(q_0)$$

as required. ☐

4.2 Topology of Finite Posets

Here we describe how one can give a finite poset a topology in a non-trivial way. It turns out that this makes a finite poset weakly equivalent to its realisation. Then, if one has a map from a space to a finite poset satisfying certain conditions, we can actually show that the space is homotopy equivalent to the realisation of the poset.

Definition 4.2.1. Let P be a finite poset. We can give P a topology where

- $U \subseteq P$ is open if $x \in U$ and $x \preceq y \Rightarrow y \in U$.
- $F \subseteq P$ is closed if $x \in F$ and $y \preceq x \Rightarrow y \in F$.

Now define $\pi : |P| \to P$ by setting $\pi(t) = max\{\mathrm{supp}(t)\}$.

Theorem 4.2.2. *π is a weak equivalence and therefore for any Q, the map*

$$[|Q|, |P|] \to [|Q|, P]$$

$$f \mapsto \pi \circ f$$

is bijective.

Proof. See [20], theorem 2 to prove that π is a weak equivalence. The final conclusion is then an application of the Whitehead theorem. ☐

Let

$$U(p) = \{q \in P \mid q \succeq p\}$$

$$D(p) = \{q \in P \mid q \preceq p\}$$

Definition 4.2.3. Let $\mathcal{U}$ be an open cover of a space X. Then $\mathcal{U}$ is called basis-like if whenever we have $x \in U \cap V$ for $U, V \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $x \in W \subseteq U \cap V$.

Lemma 4.2.4. *The open cover $\{U(p) \mid p \in P\}$ of a poset P is basis-like.*

Proof. Let $x \in U(p) \cap U(q)$ for $p, q \in P$. Then it is obvious that $U(x) \subseteq U(p) \cap U(q)$ as required. ☐

We will now prove the following:

Theorem 4.2.5. *Suppose we have a map $f : X \to P$ where X is a space with the homotopy type of a CW-complex and $\forall p \in P$, $f^{-1}(U(p))$ is open and contractible. Then f is a weak equivalence.*

Proof. We aim to apply the following theorem:

Theorem 4.2.6. *Suppose $p : E \to B$ is a map for which there exists a basis-like open cover $\mathcal{U}$ of B satisfying: For each $U \in \mathcal{U}$, the restriction $p : p^{-1}(U) \to U$ is a weak equivalence. Then p is also a weak equivalence, ([20], theorem 6).*

Therefore it is enough to show that the restriction $f : f^{-1}(U(p)) \to U(p)$ is a weak equivalence for all $p \in P$. Firstly, to show that f is indeed continuous, we note that any open set in P is a union of $U(p)$'s. Therefore its preimage is a union of open sets and so f is continuous.

Now let $U = U(p)$. We claim that U is contractible via $h : [0,1] \times U \to U$,

$$h(t,x) = \begin{cases} p & \text{if } t = 0 \\ x & \text{if } t > 0 \end{cases}.$$

We need to check that h is continuous. Take $V \subseteq U$ open. If $p \notin V$ then $h^{-1}V = (0,1] \times V$ which is open. If $p \in V$ then $h^{-1}V = (0,1] \times V \cup \{0\} \times U$. But $p \in V$ implies that $V = U$ and so in fact $h^{-1}V = [0,1] \times U$ which is open. Since $f^{-1}(U)$ is contractible by assumption, we conclude that $f : f^{-1}(U(p)) \to U(p)$ is a weak equivalence as required. □

Corollary 4.2.7. *Suppose $f : X \to P$ is a map where X has the homotopy type of a CW-complex and $\forall p \in P$, $f^{-1}(U(p))$ is open and contractible. Then there exists $\tilde{f} : X \to |P|$ such that $\pi \circ \tilde{f} \simeq f$ and $\tilde{f}$ is a homotopy equivalence.*

Proof. Because $\pi_* : [X, |P|] \to [X, P]$ is bijective, this shows that $\tilde{f}$ exists with $\pi \circ \tilde{f} \simeq f$. Then $\tilde{f}$ will be a weak equivalence of CW-complexes by the two-out-of-three property and in fact it is a homotopy equivalence. □

If we have a collection of spaces with an operad structure, then we can make further conclusions.

Corollary 4.2.8. *Let P be an operad in finite posets and X an operad in spaces such that for every finite set A, $X(A)$ has the homotopy type of a CW-complex. Suppose we have a map of operads in sets, $f : X \to P$, such that for every finite set A, $f_A^{-1}(U(p))$ is open and contractible for all $p \in P(A)$. Then there exists a collection of homotopy equivalences $\tilde{f}_A : X(A) \to |P(A)|$ which form a map of operads up to homotopy.*

Proof. Firstly, we will denote by γ_{Top}, γ_{Pos} and γ_{Real} the operad composition maps of X, P and $|P|$ respectively. We first claim that $\pi : |P| \to P$ given by

$$\pi_A(t) = max\{\text{supp}(t)\}$$

is a map of operads. Let $t \in |P(A/B) \times P(B)|$. Then

$$(\pi \circ \gamma_{Real})(t) = \pi\left(\sum_{(p,q)} t(p,q)\gamma_{Pos}(p,q)\right) = \gamma_{Pos}(p_{max}, q_{max})$$

where $(p_{max}, q_{max}) = max\{\text{supp}(t)\}$. This is true since γ_{Pos} is a map of posets. We also have

$$(\gamma_{Pos} \circ (\pi \times \pi))(t) = \gamma_{Pos}(p_{max}, q_{max})$$

as required. Now consider the following diagram:

$$\begin{tikzcd}
X(A/B)\times X(B) \arrow[rrr, "\gamma_{Top}"] \arrow[dr, "f\times f"'] \arrow[dd, "\tilde{f}\times\tilde{f}"'] & & & X(A) \arrow[dl, "f"] \arrow[dd, "\tilde{f}"] \\
& P(A/B)\times P(B) \arrow[r, "\gamma_{Pos}"] & P(A) & \\
|P(A/B)|\times|P(B)| \arrow[rrr, "\gamma_{Real}"] \arrow[ur, "\pi\times\pi"] & & & |P(A)| \arrow[ul, "\pi"']
\end{tikzcd}$$

where the $\tilde{f}$'s are the homotopy equivalences resulting from corollary 4.2.7. The outer triangles commute up to homotopy and the upper and lower quadrilaterals commute on the nose. Therefore we have the following chain of equivalences

$$\pi \circ \gamma_{Real} \circ (\tilde{f} \times \tilde{f}) = \gamma_{Pos} \circ (\pi \times \pi) \circ (\tilde{f} \times \tilde{f}) \simeq \gamma_{Pos} \circ (f \times f) = \mu \circ \gamma_{Top} \simeq \pi \circ \tilde{f} \circ \gamma_{Top}.$$

Since π is a weak equivalence this implies that

$$\gamma_{Real} \circ (\tilde{f} \times \tilde{f}) \simeq \tilde{f} \circ \gamma_{Top}$$

and so $\tilde{f}$ induces an isomorphism of operads between X and $|P|$ in the homotopy category. $\square$

4.3 Combinatorial Models for Spaces of Maps

Now we will create combinatorial models for some of the spaces of maps that we have seen previously in the thesis. By this we mean that we will describe posets whose realisations have the homotopy type of those spaces of maps. To summarise, we will create the following:

Poset	Homotopy Type
$\mathrm{ACP}_N(A)$	$W_N(A)$
$\mathrm{SCP}_N(A)$	$S(W_N(A))$
$\mathrm{ICP}_N(A)$	$F_N(A)$

To begin with, we need to introduce the concept of a preorder, but first we fix some notation for relations.

Definition 4.3.1. Denote by $1_A \subseteq A^2$ the relation $1_A := \{(a, a) \mid a \in A\}$ on A. If R is a relation on A, then denote by R^{op} the relation $R^{\mathrm{op}} := \{(b, a) \mid (a, b) \in R\}$.

Definition 4.3.2. A preorder on a set A is a relation $P \subseteq A^2$ such that

i) $(a, a) \in P$ for all $a \in A$.

ii) If $(a, b) \in P$ and $(b, c) \in P$ then $(a, c) \in P$.

A preorder is called total if in addition it satisfies

iii) For all $a, b \in A$, either $(a, b) \in P$ or $(b, a) \in P$,

and it is called separated if it puts a poset structure on A. We will use the following notation:

- $a \leq_P b$ if $(a, b) \in P$.
- $a \equiv_P b$ if $(a, b) \in P$ and $(b, a) \in P$.
- $a <_P b$ if $(a, b) \in P$ and $(b, a) \notin P$.

Remark 4.3.3. Given a preorder P on A, one can easily show that $\equiv_P$ defines an equivalence relation on A. We will often call an element in $A/\equiv_P$ a block. P then induces a poset structure on the set of equivalence classes $A/\equiv_P$ which is a total order if P is total.

Definition 4.3.4. Let A be a finite set with $|A| > 1$ and N a natural number. We write $\mathrm{ACP}_N(A)$ for the set of lists $Q = (Q_0, \ldots, Q_{N-1})$ such that

(a) Each Q_i is a preorder on A.

(b) Q_0 is total.

(c) $Q_i \cup Q_i^{\mathrm{op}} = Q_{i-1} \cap Q_{i-1}^{\mathrm{op}}$ for $0 < i < N$.

We call such a list a chain of preorders. $\mathrm{ACP}_N(A)$ has a poset structure such that $Q \preceq Q'$ if and only if $Q_i \supseteq Q'_i$ for all i. We let the subset $\mathrm{SCP}_N(A) \subseteq \mathrm{ACP}_N(A)$ be the set of elements also satisfying

(d) $Q_{N-1} \neq A^2$.

We give this the subset poset structure and call it the set of non-constant chains of preorders. Additionally, we let the subset $\mathrm{ICP}_N(A) \subseteq \mathrm{SCP}_N(A) \subseteq \mathrm{ACP}_N(A)$ be the set of elements also satisfying

e) Q_{N-1} is separated.

Again, we give this the subset poset structure and we call it the set of injective chains of preorders.

Remark 4.3.5. Given a preorder P on A, we will say that $a, b \in A$ are comparable if $(a, b) \in P$ or $(b, a) \in P$. If $Q = (Q_0, \ldots, Q_{N-1}) \in \mathrm{ACP}_N(A)$ then comparability with respect to Q_i is an equivalence relation on A for all $0 \leq i < N$. This follows from axiom (c) in definition 4.3.4. The poset induced by $\equiv_P$ is a chain on each equivalence class of comparable elements.

Remark 4.3.6. If we have a map $f : A \to B$, there is an induced map $f^* : \mathrm{ACP}_N(B) \to \mathrm{ACP}_N(A)$ given by $f^*(R) = Q$ where

$$(a_1, a_2) \in Q_i \Leftrightarrow (f(a_1), f(a_2)) \in R_i.$$

If f is surjective then f^* preserves SCP_N and if it is injective then it preserves ICP_N.

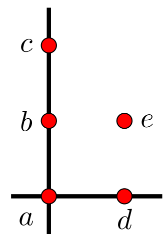
Definition 4.3.7. Let P be a preorder on A and $B \subseteq A$. We can then define a preorder $P|_B$ by setting $P|_B = P \cap B^2$. It is an easy check to see that this is indeed a preorder. Similarly, if $Q \in \mathrm{ACP}_N(A)$ then define $Q|_B = (Q_0|_B, \ldots, Q_{N-1}|_B) \in \mathrm{ACP}_N(B)$.

Now we can begin to compare our posets to the spaces of maps we highlighted at the start of this section.

Definition 4.3.8. We define a map $\mu : W_N(A) \to \mathrm{ACP}_N(A)$ by setting $\mu(x) = Q$ where

$$Q_i = \{(a, b) \in A^2 \mid x(a)_i \leq x(b)_i, \text{ and } x(a)_j = x(b)_j \text{ for all } j < i\}.$$

Example 4.3.9. We give an example of how the map μ works. Set $A = \{a, b, c, d, e\}$ and $N = 2$. Then consider the following representation of an element of $W_N(A)$:



This would then be sent to $Q = (a \equiv b \equiv c < d \equiv e; a < b < c, d < e)$ by μ. Notice that a, b and c are not comparable to d and e in Q_1 because they became separate in Q_0.

Lemma 4.3.10. *Let $x \in W_N(A)$. Then*

1. *$\mu(x) = (A^2)^N$ if and only if any representative of x is constant.*

2. *$\mu(x) \in \mathrm{ICP}_N(A)$ if and only if any representative of x is injective.*

Proof. Firstly we claim that $a \equiv_{\mu(x)_{N-1}} b$ if and only if $x(a) = x(b)$. The first direction is obvious. For the converse, if $a \equiv_{\mu(x)_{N-1}} b$ then by definition $x(a)_j = x(b)_j$ for all $j < N - 1$. Then we also have to have $x(a)_{N-1} \leq x(b)_{N-1}$ and $x(b)_{N-1} \leq x(a)_{N-1}$ which implies that $x(a)_{N-1} = x(b)_{N-1}$ and so $x(a) = x(b)$ as claimed. Now the two claims in the lemma are simple.

1. $\mu(x) = (A^2)^N$ if and only if $\mu(x)_{N-1} = A^2$. Therefore, by the above this happens if and only if for all $a, b \in A$ we have $x(a) = x(b)$, i.e. if x is constant.

2. $\mu(x) \in \mathrm{ICP}_N(A)$ if and only if for all $a, b \in A$ distinct we have $a \not\equiv_{\mu(x)_{N-1}} b$. By the above, this happens if and only for all distinct $a, b \in A$ we have $x(a) \neq x(b)$, i.e. if x is injective.

$\square$

Definition 4.3.11. In the other direction, we define $\sigma : \mathrm{ACP}_N(A) \to \mathrm{Map}(A, \mathbb{R}^N)$ by setting $\sigma(Q) = x$ where

$$x(a)_i = |\{B \in A/\equiv_{Q_i} \mid B \prec [a]\}|.$$

Remark 4.3.12. The image of σ lies in the subset of elements in $\mathrm{Map}(A, \mathbb{R}^{\mathbb{N}})$ whose image lies on the integer lattice of $\mathbb{R}^N$. If we apply σ to Q from example 4.3.9 then we actually produce the configuration in the diagram in that example. However, it is not true in general that $\sigma \circ \mu$ is the identity.

Lemma 4.3.13. *$\mu \circ \sigma : \mathrm{ACP}_N(A) \to \mathrm{ACP}_N(A)$ is the identity.*

Proof. Let $Q \in \mathrm{ACP}_N(A)$, $x = \sigma(Q)$ and $P = \mu(x)$. We proceed by induction on i. Firstly, if $i = 0$ then $(a, b) \in P_i$ if and only if $x(a)_i \leq x(b)_i$ which in turn happens if and only if $a \leq_{Q_0} b$ or in other words, $(a, b) \in Q_0$. Now for $i > 0$ we may assume that $P_j = Q_j$ for all $j < i$. In particular, this means that $P_{i-1} = Q_{i-1}$ and so P_i and Q_i have the same comparability equivalence classes by condition c) in definition 4.3.4. Both P_i and Q_i are total preorders when restricted to some comparability equivalence class. Therefore we can repeat the argument from the base case for each of these classes to prove that $P_i = Q_i$ as required. □

We denote by $\bar{\sigma} : |\mathrm{ACP}_N(A)| \to \mathrm{Map}(A, \mathbb{R}^N)$ the map induced by taking all convex combinations. That is to say

$$\bar{\sigma}(t) = \sum_{Q \in \mathrm{ACP}_N(A)} t(Q) \cdot \sigma(Q).$$

Lemma 4.3.14. *Let $t \in |\mathrm{ACP}_N(A)|$. Then $\mu(\bar{\sigma}(t)) = max\{\mathrm{supp}(t)\}$.*

Proof. Let $Q = max\{\mathrm{supp}(t)\}$, $x = \bar{\sigma}(t)$ and $P = \mu(x)$. Firstly, let $(a, b) \in Q_i$. This means that for all $Q' \in \mathrm{supp}(t)$, $(a, b) \in Q'_i$ and $a \equiv_{Q'_j} b$ for all $j < i$. Therefore $\sigma(Q')(a)_j = \sigma(Q')(b)_j$ for all $Q' \in \mathrm{supp}(t)$ and $j < i$ and so $x(a)_j = x(b)_j$ for all $j < i$. Also $\sigma(Q')(a)_i \leq \sigma(Q')(b)_i$ for all $Q' \in \mathrm{supp}(t)$ and so $x(a)_i \leq x(b)_i$ meaning that $(a, b) \in P_i$ and so $Q_i \subseteq P_i$ for all i.

Now take any $(a, b) \in P_i \setminus Q_i$. If $(b, a) \in Q_i$ then the above argument shows that $x(b)_i < x(a)_i$. This contradicts $a \equiv_{P_i} b$ and so a and b must be Q_i-incomparable. Therefore there must exist $i' < i$ such that a and b are $Q_{i'}$-comparable but $a \not\equiv_{Q_{i'}} b$. Then $x(a)_{i'} \neq x(b)_{i'}$ which contradicts $(a, b) \in P_i$. Therefore $P_i \setminus Q_i$ must be empty and so $P_i = Q_i$ for all i. □

Definition 4.3.15. We define the following set of representatives for $W_N(A)$:

$$W_N(A) := \left\{x : A \to \mathbb{R}^N \mid min_{a \in A}\{x(a)_i\} = 0 \text{ for all } i\right\}.$$

Definition 4.3.16. Let $x \in W_N(A)$ and define $C(x) = \{(a, b, i) \mid a <_{\mu(x)_i} b\}$. Notice that $C(x)$ is empty if and only if x is constant. Therefore, if x is non-constant set $x_0 = x$ and define $\tau_0(x) = min\{x_0(b)_i - x_0(a)_i \mid (a, b, i) \in C(x_0)\}$. If x is constant then we set $\tau_0(x) = 0$. Then inductively define

$$x_{k'} = x_{k'-1} - \tau_{k'-1}(x)\sigma(\mu(x_{k'-1}))$$

$$\tau_{k'}(x) = min\{x_{k'}(b)_i - x_{k'}(a)_i \mid (a, b, i) \in C(x_{k'})\}.$$

Let k be such that $\mu(x_k) = (A^2)^N$. Note that k is always finite. We then define

$$\tau(x) = \sum_{k'=0}^{k} \tau_{k'}(x).$$

Remark 4.3.17. If $x, z \in W_N(A)$ and $z = \lambda x$ for some $\lambda > 0$ then $\tau(z) = \lambda\tau(x)$. Therefore we may define a set of representatives for $S(W_N(A))$:

$$S(W_N(A)) := \{x \in W_N(A) \mid \tau(x) = 1\}.$$

Proposition 4.3.18. *$\bar{\sigma}$ is an injective map.*

Proof. It is clear that $image(\bar{\sigma}) \subseteq W_N(A)$ by considering the representatives we define in 4.3.15. Let $x \in image(\bar{\sigma})$ with $x = \bar{\sigma}(t)$. We will assume x is non-constant since lemmas 4.3.10 and 4.3.14 combine to show that if x is constant then it must have preimage the vertex $(A^2)^N$. We then claim that $t(\mu(x)) = \tau_0(x)$. Note that $(a, a, i) \notin C(x)$ for all $a \in A$ since we always have $a \equiv_{\mu(x)_i} a$. Therefore $(a, b, i) \in C(x)$ means that $a \neq b$ and $x_i(a) < x_i(b)$. Take $(a, b, i) \in C(x)$ such that $\nexists c \in A$ with $a <_{\mu(x)_i} c <_{\mu(x)_i} b$ and $a \equiv_{Q'_i} b$ for all $Q' \in \mathrm{supp}(t)$ with $Q' \neq \mu(x)$. We can do this since $\mu(x)$ is the largest element in $\mathrm{supp}(t)$ which is a chain. Therefore, if Q' is say the second largest element, then there exists some i such that $\mu(x)_i \subset Q'_i$. This means that in particular $\mu(x)_i \neq A^2$ and so we can choose $a, b \in A$ such that $a \equiv_{Q'_i} b$ but $(b, a) \notin \mu(x)_i$. To ensure there is no $c \in A$ in between a and b, we simply choose a and b from equivalence classes that are adjacent in the poset $A/\equiv_{Q_i}$. Having made this choice, we then have

$$\sigma(\mu(x))(b)_i - \sigma(\mu(x))(a)_i = 1$$
$$\sigma(Q')(b)_i - \sigma(Q')(a)_i = 0 \qquad \text{for all } Q' \in \mathrm{supp}(t),\ Q' \neq \mu(x)$$

which implies that $x(b)_i - x(a)_i = t(\mu(x))$. To see that this is minimal, note that for any $(a, b, i) \in C(x)$ not satisfying the conditions, we have at least one of the following occur:

$$\sigma(\mu(x))(b)_i - \sigma(\mu(x))(a)_i > 1 \text{ so } x(b)_i - x(a)_i > t(\mu(x)).$$

$$\sigma(Q')(b)_i - \sigma(Q')(a)_i \geq 1 \text{ so } x(b)_i - x(a)_i \geq t(\mu(x)) + t(Q')$$

for some $Q' \in \mathrm{supp}(t)$ not equal to $\mu(x)$.

This means that from the above we may calculate $max\{\mathrm{supp}(t)\} = \mu(x)$ and $t(\mu(x)) = \tau_0(x)$. We can then consider $x_1 = x - t(\mu(x))\sigma(\mu(x))$ and repeat this process since x_1 will be a scalar multiple of some element in $image(\bar{\sigma})$. We terminate the process at some finite point k when $\mu(x_k) = (A^2)^N$. It will then be the case that

$$t((A^2)^N) = 1 - \left(\sum_{k'=1}^{k-1} t(\mu(x_{k'})) + t(\mu(x)) \right),$$

meaning we can completely determine t from its image and so $\bar{\sigma}$ is injective. □

Proposition 4.3.19. *$\bar{\sigma}$ restricts to give a homeomorphism*

$$\bar{\sigma} : |\operatorname{SCP}_N(A)| \to S(W_N(A)).$$

Proof. Firstly one should notice that the restriction $\bar{\sigma} : |\operatorname{SCP}_N(A)| \to S(W_N(A))$ is well-defined by considering the representatives in remark 4.3.17, and injective by proposition 4.3.18. To see that it is surjective, we define a $\bar{\mu} : S(W_N(A)) \to |\operatorname{SCP}_N(A)|$ by setting $\bar{\mu}(x) = s$ where

$$s(Q) = \begin{cases} \tau_{k'}(x) & \text{if } Q = \mu(x_{k'}) \text{ for some } 0 \leq k' < k \\ 0 & \text{otherwise} \end{cases}.$$

We need to check that $\bar{\mu}$ is well-defined which comes down to checking that $\mathrm{supp}(s)$ is a chain. Let $(a, b) \in \mu(x_{k'})_i$ for $k' < k - 1$ which means that $x_{k'}(a)_i \leq x_{k'}(b)_i$ and $x_{k'}(a)_j = x_{k'}(b)_j$ for all $j < i$. Now

$$x_{k'+1}(a)_j = x_{k'}(a)_j - \tau_{k'}(x)\sigma(\mu(x_{k'}))(a)_j = x_{k'}(b)_j - \tau_{k'}(x)\sigma(\mu(x_{k'}))(b)_j = x_{k'+1}(b)_j$$

$$x_{k'+1}(a)_i = x_{k'}(a)_i - \tau_{k'}(x)\sigma(\mu(x_{k'}))(a)_i \leq x_{k'}(b)_i - \tau_{k'}(x)\sigma(\mu(x_{k'}))(b)_i = x_{k'+1}(b)_i$$

with the latter being true because $\tau_{k'}(x)$ is picked to be minimal. Therefore $(a,b) \in \mu(x_{k'+1})_i$ and so $\mu(x_{k'+1}) \preceq \mu(x_{k'})$.

To see that $\bar{\sigma} \circ \bar{\mu} = id$, notice that we must have that x_k is the 0 map and so

$$x - \sum_{k'=0}^{k-2} \tau_{k'}(x)\sigma(\mu(x_{k'})) = \tau_{k-1}(x)\sigma(\mu(x_{k-1})).$$

Therefore

$$\begin{aligned} \bar{\sigma}(\bar{\mu}(x)) &= \sum_{k'=0}^{k-1} \tau_{k'}(x)\sigma(\mu(x_{k'})) \\ &= \sum_{k'=0}^{k-2} \tau_{k'}(x)\sigma(\mu(x_{k'})) + x - \sum_{k'=0}^{k-2} \tau_{k'}(x)\sigma(\mu(x_{k'})) \\ &= x \end{aligned}$$

as required. □

Corollary 4.3.20. $|\operatorname{ACP}_N(A)|$ *is homotopy equivalent to* $W_N(A)$.

Proof. Since $\operatorname{ACP}_N(A) \setminus \operatorname{SCP}_N(A) = \{(A^2)^N\}$ and this is a maximal element in $\operatorname{ACP}_N(A)$, we have

$$|\operatorname{ACP}_N(A)| \cong C|\operatorname{SCP}_N(A)|$$

where CX is the cone of some general space X. Therefore $|\operatorname{ACP}_N(A)|$ is contractible, as is $W_N(A)$. □

To show the final homotopy equivalence from our table at the start of this section, we will employ the methods we developed in section 4.2. It is more than likely that we could have used these methods in one fell swoop to show all of the homotopy equivalences, but then we would have missed out on the explicit nature of the map $\bar{\mu} : S(W_N(A)) \to |\operatorname{SCP}_N(A)|$ and of course the stronger fact that this is a homeomorphism.

Proposition 4.3.21. $|\operatorname{ICP}_N(A)|$ *is homotopy equivalent to* $F_N(A)$.

Proof. We will apply corollary 4.2.7 by showing that the restriction $\mu : \operatorname{Inj}(A, \mathbb{R}^N) \to \operatorname{ICP}_N(A)$ satisfies the hypotheses. This is sufficient since $\operatorname{Inj}(A, \mathbb{R}^N)$ deformation retracts onto $F_N(A)$. Firstly, $F_N(A)$ has the homotopy type of a CW-complex since it is homotopy equivalent to $\bar{F}_N(A)$ which is a compact real semi-algebraic set and so is triangulable. The fact that any compact real semi-algebraic set is triangulable is shown in section 2 of [13]. Now let $Q \in \operatorname{ICP}_N(A)$ and let $X_Q = \mu^{-1}(Q)$. Then

$$\mu^{-1}(U(Q)) = \coprod_{Q \preceq Q'} X_{Q'},$$

which is the complement of

$$\bigcup_{Q' \notin U(Q)} \mu^{-1}(D(Q')),$$

which we claim is a closed set and so will show that $\mu^{-1}(U(Q))$ is open. In particular, we claim that the closure of $X_{Q'}$ is equal to

$$\bigcup_{Q'' \preceq Q'} X_{Q''} = \mu^{-1}(D(Q'')).$$

To see this, first take x in the closure $\overline{X_{Q'}}$, and then consider distinct $a, b \in A$ such that $a <_{Q'_i} b$ for some i. For all $y \in X_{Q'}$ we have $y(a)_j = y(b)_j$ for $j < i$ and $y(a)_i < y(b)_i$. Therefore, by taking limits we see that $x(a)_j = x(b)_j$ for $j < i$ and $y(a)_i \leq y(b)_i$. This implies that $(a, b) \in \mu(x)_i$ and so $\mu(x) \preceq Q'$. Conversely, let $Q'' \preceq Q'$ and $x \in X_{Q''}$. Then set $y = x + \lambda\sigma(Q')$ for $\lambda > 0$. Now $(a, b) \in \mu(y)_0$ if and only if $y(a)_0 \leq y(b)_0$. However, because $Q''_0 \supseteq Q'_0$ and both preorders are total, one can show that this happens if and only if $(a, b) \in Q'_0$ and so $\mu(y)_0 = Q'_0$. Then inductively, we will know that the equivalence classes of comparability will be the same for $\mu(y)_i$ and Q'_i and so the same arguments show that $\mu(y)_i = Q'_i$ for all i. Therefore $y \in X_{Q'}$ for $\lambda > 0$ and then by taking the limit as $\lambda \to 0$ we see that x is in the closure of $X_{Q'}$ as required.

To show that $\mu^{-1}(U(Q))$ is contractible, we in fact show that it is star-shaped. Let $x \in \mu^{-1}(U(Q))$ and then consider $y = (1 - \lambda)x + \lambda\sigma(Q)$ for $\lambda \in [0, 1]$. Then for $\lambda \neq 1$, the same argument as above shows that $\mu(y) = \mu(x)$ and so in particular $y \in \mu^{-1}(U(Q))$ for all $\lambda \in [0, 1]$. □

Lemma 4.3.22. *The restriction $\bar{\sigma} : |\operatorname{ICP}_N(A)| \to S(W_N(A))$ actually lands in $F_N(A)$.*

Proof. Lemma 4.3.14 tells us that $\mu(\bar{\sigma}(t)) = max\{\operatorname{supp}(t)\}$ which in this case is in $\operatorname{ICP}_N(A)$. Then lemma 4.3.10 tells us that this implies $\bar{\sigma}(t)$ is injective and so it is in $F_N(A)$. □

Proposition 4.3.23. *The restriction of $\bar{\sigma}$ is a homotopy inverse for $\tilde{\mu} : F_N(A) \to |\operatorname{ICP}_N(A)|$ from proposition 4.3.21.*

Proof. Let $\pi : |\operatorname{ICP}_N(A)| \to \operatorname{ICP}_N(A)$ be the weak equivalence given by $\pi(t) = max\{\operatorname{supp}(t)\}$. By lemma 4.3.14, we see that $\mu \circ \bar{\sigma} = \pi$. Therefore

$$\pi \circ \tilde{\mu} \circ \bar{\sigma} \simeq \mu \circ \bar{\sigma} = \pi$$

which implies that $\tilde{\mu} \circ \bar{\sigma} \simeq id$ since π is a weak equivalence. In the other direction

$$\mu \circ \bar{\sigma} \circ \tilde{\mu} = \pi \circ \tilde{\mu} \simeq \mu$$

which implies that $\bar{\sigma} \circ \tilde{\mu} \simeq id$ since μ is a weak equivalence. □

Finally, we can use the theory of ranked posets to say what the dimension of $|\operatorname{ICP}_N(A)|$ is.

Definition 4.3.24. Let P be a preorder on A. Then define $\xi(P) = |A/\equiv_P| - 1$. Now take $Q \in \operatorname{ICP}_N(A)$ and define the signature of Q to be $\xi^*(Q)$ where

$$\xi^* : \operatorname{ICP}_N(A) \to \mathbb{N}_0^N$$

$$\xi^*(Q)_i = \xi(Q_i).$$

Remark 4.3.25. The following are true for ξ^*:

- $\xi^*(Q)_{N-1} = |A| - 1$ for any Q.
- $\xi^*(Q)_i \leq \xi^*(Q)_{i+1}$ for $i < N - 1$.
- $Q \preceq Q' \Rightarrow \xi^*(Q)_i \leq \xi^*(Q')_i$ for all i with equality if and only if $\xi^*(Q)_i = \xi^*(Q')_i$ for all i.

Definition 4.3.26. We define a rank function $\rho : \mathrm{ICP}_N(A) \to \mathbb{N}_0$ by setting

$$\rho(Q) = \sum_{i=0}^{N-2} \xi^*(Q)_i.$$

Proposition 4.3.27. *ρ is indeed a rank function. Moreover, the minimal rank an element can take is 0, the maximal rank an element can take is $(N-1)(|A|-1)$ and there exists a chain $C \subseteq \mathrm{ICP}_N(A)$ such that $|C| = (N-1)(|A|-1)+1$.*

Proof. Remark 4.3.25 shows us that $Q \prec Q'$ implies that $\rho(Q) < \rho(Q')$. It is clear that the minimal value $\xi(P)$ can be is 0 and so the minimal value that $\rho(Q)$ can be is also 0. This is achieved by setting $Q = (A^2, \dots, A^2, Q_{N-1})$ where Q_{N-1} is any linear order on A. Similarly, the maximal value that $\xi(P)$ can take is $|A|-1$ and so the maximal value that $\rho(Q)$ can be is $(N-1)(|A|-1)$. This is achieved by setting $Q = (Q_0, 1_A, \dots, 1_A)$ where Q_0 is any linear order on A.

We now claim that given $Q \in \mathrm{ICP}_N(A)$ such that $\xi^*(Q)_i = l_i$ and $\rho(Q) = l < (N-1)(|A|-1)$, we can produce an element $Q' \in \mathrm{ICP}_N(A)$ such that $Q \prec Q'$ and $\rho(Q') = l+1$. This will be enough to prove the rest of the proposition. Let i' be the largest index such that $Q_{i'}$ is not separated. This always exists since $\xi(Q_0) \neq |A|-1$. This means there exists $a \in A$ such that $[a] \neq \{a\}$ in $A/\equiv_{Q_{i'}}$. We then define our element Q' by setting

$$Q'_i = \begin{cases} Q_i & \text{if } i < i' \\ Q'_{i'} & \text{if } i = i' \\ Q_i \cap Q'_{i-1} \cap Q'^{\mathrm{op}}_{i-1} & \text{if } i = i'+1 \\ 1_A & \text{otherwise} \end{cases}$$

where

$$Q'_{i'} = \{(b,c) \in Q_{i'} \mid b, c \neq a\} \cup \{(b,a) \mid b <_{Q_{i'}} a\} \cup \{(a,b) \mid (a,b) \in Q_{i'}\}.$$

In words, Q'_i has the same blocks as Q_i except that we take $[a]_{Q_i}$ and split it into two: one block that only contains the element a and then another containing all of the other elements of $[a]_{Q_i}$ with this block being greater than $\{a\}$ in the poset $A/\equiv_{Q'_i}$. It is then a simple check to see that

$$\xi^*(Q')_i = \begin{cases} l_i + 1 & \text{if } i = i' \\ l_i & \text{otherwise} \end{cases}$$

so that $\rho(Q') = l+1$ as required. □

As a direct consequence of lemma 4.1.8 we can conclude:

Corollary 4.3.28. *The dimension of* $\mathrm{ICP}_N(A)$ *is* $(N-1)(|A|-1)$.

This is a satisfying result since we saw earlier in the thesis that the homological dimension of $F_N(A)$ was also $(N-1)(|A|-1)$. Therefore, we can consider $|\mathrm{ICP}_N(A)|$ to be a combinatorial model for $F_N(A)$ of minimal dimension, or in other words, a spine.

4.4 Fadell-Neuwirth Fibrations

In this section we will always have $|A| > 1$. It is well known that there exists a family of fiber bundles [9], often referred to as the Fadell-Neuwirth fibrations,

$$\mathbb{R}^N \setminus \{(|A|-1)\text{-points}\} \to \mathrm{Inj}(A, \mathbb{R}^N) \to \mathrm{Inj}(A \setminus \{a\}, \mathbb{R}^N)$$

where the map on the right is given by simply restricting $x \in \mathrm{Inj}(A, \mathbb{R}^N)$ to $A \setminus \{a\}$. Each fiber bundle in this family has a continuous section $g : \mathrm{Inj}(A \setminus \{a\}, \mathbb{R}^N) \to \mathrm{Inj}(A, \mathbb{R}^N)$ by setting

$$g(y)(b) = \begin{cases} y(b) & \text{if } b \in A \setminus \{a\} \\ (m_0(y)+1, 0, 0, \ldots, 0) & \text{if } b = a \end{cases}$$

where $m_0(y) = max\{y(b)_0 \mid b \in A \setminus \{a\}\}$. We would like to produce similar maps for the posets $\mathrm{ICP}_N(A)$ and their realisations. First though, we use the Fadell-Neuwirth fibrations to conclude the following:

Lemma 4.4.1. *For $N > 2$, $\mathrm{Inj}(A, \mathbb{R}^N)$ is a simply connected space.*

Proof. The Fadell-Neuwirth fibrations induce long exact sequences of homotopy groups. In particular we can look at the following segment

$$\cdots \pi_1(\textstyle\bigvee_{b\in B} S^{N-1}) \longrightarrow \pi_1(\mathrm{Inj}(A, \mathbb{R}^N)) \longrightarrow \pi_1(\mathrm{Inj}(B, \mathbb{R}^N)) \longrightarrow \pi_0(\textstyle\bigvee_{b\in B} S^{N-1}) \longrightarrow \cdots$$

where $B = A \setminus \{a\}$ for some $a \in A$. If $N > 2$ then $\pi_1(\bigvee_{b\in B} S^{N-1}) \cong \pi_0(\bigvee_{b\in B} S^{N-1}) \cong \{*\}$ and so the induced map

$$\pi_1(\mathrm{Inj}(A, \mathbb{R}^N)) \to \pi_1(\mathrm{Inj}(B, \mathbb{R}^N))$$

is an isomorphism. Therefore by induction, $\pi_1(\mathrm{Inj}(A, \mathbb{R}^N)) \cong \pi_1(\mathrm{Inj}(\{*\}, \mathbb{R}^N)) \cong \{*\}$ as required. ☐

Definition 4.4.2. Let $Q \in \mathrm{ICP}_N(A)$. For a subset $B \subseteq A$ define $Q|_B$ to be the injective chain of preorders in $\mathrm{ICP}_N(B)$ given by

$$Q_i|_B = Q_i \cap B^2.$$

It is simple to check that we do indeed have $Q|_B \in \mathrm{ICP}_N(B)$.

Definition 4.4.3. Choose $a \in A$. We can then define a map $f : \mathrm{ICP}_N(A) \to \mathrm{ICP}_N(A \setminus \{a\})$ by

$$f(Q) = Q|_{A\setminus\{a\}}.$$

It is clear that this is a map of posets so the extension $\bar{f} : |\,\mathrm{ICP}_N(A)| \to |\,\mathrm{ICP}_N(A \setminus \{a\})|$ by taking all convex combinations is also well-defined. The map f has a section $g : \mathrm{ICP}_N(A \setminus \{a\}) \to \mathrm{ICP}_N(A)$ defined by

$$g(Q)_i = \begin{cases} Q_i \cup \{(b, a) \mid b \in A\} & \text{if } i = 0 \\ Q_i \cup \{(a, a)\} & \text{otherwise} \end{cases}.$$

By the functoriality of the geometric realisation, this extends to a continuous section $\bar{g} : |\,\mathrm{ICP}_N(A \setminus \{a\})| \to |\,\mathrm{ICP}_N(A)|$.

The maps in definition 4.4.3 are analogous to the Fadell-Neuwirth fibrations and so we would like to show that they have similar properties. In particular, we will show that the fibers of $\bar{f}$ are all homotopy equivalent to $\mathbb{R}^N \setminus \{(|A|-1)\text{-points}\}$, the fibers of the Fadell-Neuwirth fibrations. We also conjecture that $\bar{f}$ is a quasifibration.

Remark 4.4.4. If $|A| = 2$ then $|\operatorname{ICP}_N(A \setminus \{a\})|$ is a single point. Therefore the preimage of this point under the map $\bar{f}$ is all of $|\operatorname{ICP}_N(A)|$. In this case, for some $Q \in \operatorname{ACP}_N(A)$, axioms (d) and (e) in definition 4.3.4 are equivalent and therefore $\operatorname{ICP}_N(A) \cong \operatorname{SCP}_N(A)$. Then by proposition 4.3.19, $|\operatorname{SCP}_N(A)|$ is homeomorphic to the sphere S^{N-1}. This is in direct analogy with the Fadell-Neuwirth fibrations as required above since $\mathbb{R}^N \setminus \{(|A|-1)\text{-points}\} \simeq \bigvee_{A\setminus\{a\}} S^{N-1}$.

Definition 4.4.5. Let $b \in A \setminus \{a\}$. Then for $Q \in \operatorname{ICP}_N(A \setminus \{a\})$ we define a subset $S_b(Q) \subseteq f^{-1}(Q)$ such that for any $P \in S_b(Q)$ we have

- If a and b are P_i-comparable then $a \not\equiv_{P_i} c$ for any $c \not\equiv_{Q_i} b$.
- $a \leq_{P_i} b \Rightarrow \nexists c$ such that $a <_{P_i} c <_{P_i} b$.
- The symmetric condition for $b \leq_{P_i} a$.

In other words, $P \in S_b(Q)$ if for the least i such that $a \not\equiv_{P_i} b$, we have $[a]_{P_i} = \{a\}$ and $[a]_{P_i}$ is adjacent to $[b]_{P_i}$ with respect to the ordering on $A/\equiv_{P_i}$.

Lemma 4.4.6. $f^{-1}(Q) = \bigcup_{b \in A\setminus\{a\}} S_b(Q)$.

Proof. Obviously we have that $\bigcup_{b\in A\setminus\{a\}} S_b(Q) \subseteq f^{-1}(Q)$. Now take some $P \in f^{-1}(Q)$. Choose the lowest index i such that $[a]_{P_i} = \{a\}$ in $A/\equiv_{P_i}$. Assume we have

$$B_0 <_{P_i} [a]_{P_i} <_{P_i} B_1$$

as our ordering of blocks in $A/\equiv_{P_i}$ and there are no blocks in between these ones. If no such B_0 or B_1 exists then we allow them to be the empty set. However at least one of these must be non-empty. If $i = 0$ then P_i is total so a shares a relation with every other element of A. If $i > 0$ then $[a]_{P_{i-1}} \neq \{a\}$ in $A/\equiv_{P_{i-1}}$ and P_i is a total preorder on $[a]_{P_{i-1}}$ with at least two blocks. Therefore we may choose an element $b \in B_0 \cup B_1$ and we claim that $P \in S_b(Q)$. If $i = 0$ then the statement is obviously true since a and b are only comparable in P_0. Otherwise, we see that we must have $B_0 \equiv_{P_j} [a] \equiv_{P_j} B_1$ for all $j < i$ and so all of the conditions for $S_b(Q)$ are satisfied. $\square$

Lemma 4.4.7. $S_b(Q) \cong \operatorname{SCP}_N(\{a,b\})$. *In particular we have* $|S_b(Q)| \cong S^{N-1}$.

Proof. Define a map

$$g : S_b(Q) \to \operatorname{SCP}_N(\{a,b\})$$

by $g(P)_i = P_i|_{\{a,b\}}$. It is clear that this is a map of posets. Let $R \in \operatorname{SCP}_N(A)$ and i the least index such that $a \not\equiv_{R_i} b$. We define a map

$$h : \operatorname{SCP}_N(\{a,b\}) \to S_b(Q)$$

in the opposite direction by setting $h(R)_j = \{(a,a)\} \cup Q_j \cup T_j$ where T_j is defined as follows:

$$T_j = \begin{cases} \{(b',a) \mid b' \leq_{Q_j} b\} \cup \{(a,b') \mid b \leq_{Q_j} b'\} & \text{if } j < i \\ \{(b',a) \mid b' <_{Q_j} b\} \cup \{(a,b') \mid b \leq_{Q_j} b'\} & \text{if } j = i \text{ and } a <_{R_i} b \\ \{(b',a) \mid b' \leq_{Q_j} b\} \cup \{(a,b') \mid b <_{Q_j} b'\} & \text{if } j = i \text{ and } a >_{R_i} b \\ \emptyset & \text{if } j > i \end{cases}.$$

It is an elementary check to see that $h(R) \in S_b(Q)$ and that h is inverse to g. $\square$

Definition 4.4.8. Let $Q \in \mathrm{ICP}_N(A)$. There is then an associated total order on A generated by, for $a \neq b$,

$$a < b \quad \Leftrightarrow \quad a <_{Q_i} b \text{ for some } 0 \leq i \leq N-1.$$

It is a simple check to see that this is indeed a total order.

In what follows, let $b_0 < \cdots < b_m$ be the total order on $B = A \setminus \{a\}$ associated with Q. We also ease notation by defining $S_b = S_b(Q)$ from now on.

Lemma 4.4.9. *$S_{b_\alpha} \cap S_{b_{\alpha+1}} \neq \emptyset$ for any $0 \leq \alpha < m$. Moreover, $S_{b_\alpha} \cap S_{b_{\alpha+1}}$ has a minimum element so in particular $|S_{b_\alpha} \cap S_{b_{\alpha+1}}|$ is contractible.*

Proof. Let i be the maximal index such that $b_\alpha \equiv_{Q_i} b_{\alpha+1}$, or if no such i exists, then set $i = -1$. Let $P \in \mathrm{ICP}_N(A)$ be the element generated by

- $P|_B = Q$.
- $a \equiv_{P_j} b_\alpha \equiv_{P_j} b_{\alpha+1}$ for $j \leq i$.
- $b_\alpha <_{P_{i+1}} a <_{P_{i+1}} b_{\alpha+1}$.

It is clear that $P \in S_{b_\alpha} \cap S_{b_{\alpha+1}}$. We claim that P is minimal in this poset. Take any $P' \in S_{b_\alpha} \cap S_{b_{\alpha+1}}$. Since comparability in P'_j is a transitive relation, a can only be P'_j-comparable to both b_α and $b_{\alpha+1}$ if b_α and $b_{\alpha+1}$ are themselves Q_j-comparable which means $j \leq i+1$. If $j = i+1$ then clearly the only option for a to be adjacent to both b_α and $b_{\alpha+1}$ is for it to lie in between them as for P_{i+1}. If $j < i+1$ then $P_j|_{\{b_\alpha, b_{\alpha+1}, a\}}$ is the total relation, so it is clear that $P'_j \subseteq P_j \ \forall j$ so $P' \succeq P$. $\square$

Lemma 4.4.10. *If $S_{b_\alpha} \cap S_{b_\beta} \neq \emptyset$ with $\alpha < \beta$ then $S_{b_\alpha} \cap S_{b_\beta} \subseteq S_{b_{\beta-1}} \cap S_{b_\beta}$.*

Proof. Let $P \in S_{b_\alpha} \cap S_{b_\beta}$. There are two ways that such an element can exist:

1. $a <_{P_j} b_\alpha \equiv_{P_j} b_\beta$ (or symmetrically $b_\alpha \equiv_{P_j} b_\beta <_{P_j} a$) for some j, but then we must have $b_\alpha \equiv_{P_j} b_{\beta-1} \equiv_{P_j} b_\beta$ so $P \in S_{b_{\beta-1}}$ as well.
2. $[b_\alpha] <_{P_j} [a] <_{P_j} [b_\beta]$ in $A/\equiv_{P_j}$ which implies that either $b_\alpha \equiv_{P_j} b_{\beta-1}$ or $b_{\beta-1} \equiv_{P_j} b_\beta$ but both of these imply that $P \in S_{b_{\beta-1}}$.

$\square$

Lemma 4.4.11. *S_b is closed upwards for all $b \in B$. In particular this means $|f^{-1}(Q)| = \bigcup_{b \in A\setminus\{a\}} |S_b|$.*

Proof. Let $P \in S_b$ and $P' \in f^{-1}(Q)$ with $P \preceq P'$. Let i be the least index such that $a \not\equiv_{P'_i} b$ and assume that $a <_{P'_i} b$, (the case $b <_{P'_i} a$ is symmetric). If $P' \notin S_b$ then there exists some $c \in B$ such that $a \leq_{P'_i} c <_{P'_i} b$. But since $P'_i \subseteq P_i$ and $P_i|_B = P'_i|_B = Q$ this means that $a \leq_{P_i} c <_{P_i} b$ which contradicts $P \in S_b$. $\square$

Proposition 4.4.12. *$|f^{-1}(Q)|$ is homotopy equivalent to $\bigvee_{b \in A\setminus\{a\}} S^{N-1}$.*

Proof. Consider $\left(\bigcup_{\alpha'<\alpha} S_{b_{\alpha'}}\right) \cap S_{b_\alpha}$. Now lemmas 4.4.9 and 4.4.10 combine to show that this is equal to $S_{b_{\alpha-1}} \cap S_{b_\alpha}$ which has contractible realisation. Therefore

$$|f^{-1}(Q)| = \bigcup_{b \in A\setminus\{a\}} |S_b(Q)| \cong \bigvee_{b \in A\setminus\{a\}} S^{N-1}.$$

$\square$

So we have shown that for vertices $Q \in |\operatorname{ICP}_N(B)|$, the fiber $\bar{f}^{-1}(Q)$ is homotopy equivalent to a wedge of spheres, in direct analogy with the Fadell-Neuwirth fibrations. We can show with more work that this in fact extends to all of $|\operatorname{ICP}_N(B)|$.

Definition 4.4.13. Let $Q(0) \prec Q(1)$ in $\operatorname{ICP}_N(B)$. For $P(0) \in f^{-1}(Q(0))$ define

$$\mathcal{U}(P(0); Q(0), Q(1)) = \{P(1) \in \operatorname{ICP}_N(A) \mid P(0) \prec P(1) \text{ and } f(P(1)) = Q(1)\}.$$

Since $\mathcal{U}(P(0); Q(0), Q(1)) \subseteq f^{-1}(Q(1)) = \bigcup_{b \in B} S_b(Q_1)$, we also define

$$\widetilde{S}_b = \mathcal{U}(P(0); Q(0), Q(1)) \cap S_b(Q_1).$$

Remark 4.4.14. One should notice that the poset $\operatorname{SCP}_N(\{a,b\})$ is isomorphic to the poset

$$\{(k,\alpha) \mid 0 \leq k \leq N-1,\ \alpha \in \{a,b\}\}.$$

The correspondence comes from noticing that for a typical element $Q \in \operatorname{SCP}_N(\{a,b\})$ we have that $Q_i = \{a,b\}^2$ for i less than some k. Then Q_k will be a total order on $\{a,b\}$, with α denoting the least element in this order. Finally $Q_i = 1_{\{a,b\}}$ for $i > k$. The partial order is given by setting $(k,\alpha) \prec (k',\alpha')$ if and only if $k' < k$. We will make use of this description in what follows.

Lemma 4.4.15. *$|\widetilde{S}_b|$ is either a sphere or a ball.*

Proof. If $\widetilde{S}_b = \emptyset$ then we consider this to be a sphere of dimension -1. First we claim that if $P \in \widetilde{S}_b$ then $P' \in \widetilde{S}_b$ for any $P' \in S_b$ such that $P \preceq P'$. However this is clear since transitivity then implies that $P(0) \prec P'$ which shows $P' \in \mathcal{U}(P(0); Q(0), Q(1))$. Now let k be the greatest integer such that $(k,\alpha) \in \widetilde{S}_b$ for some $\alpha \in \{a,b\}$. If $(k,\alpha') \in \widetilde{S}_b$ also then the realisation will be a sphere S^{k-1}. If not then the realisation is a ball as it will be the cone of the sphere S^{k-2}. $\square$

Definition 4.4.16. Define $k(b)$ to be the least integer such that $b \not\equiv_{P(0)_{k(b)}} a$. Then define

$$L_b = \{h(k,\alpha) \in S_b \mid k < k(b)\}$$

$$U_b = \{h(k,\alpha) \in S_b \mid k(b) < k\},$$

where $h : \operatorname{SCP}_N(\{a,b\}) \to S_b(Q(1))$ is the isomorphism from lemma 4.4.7.

Lemma 4.4.17. *We claim the following two statements:*

- $U_b \cap \widetilde{S}_b = \emptyset$.
- $L_b \subseteq \widetilde{S}_b$.

Proof. For the first statement, by the definition of $k(b)$ and $\mathrm{ICP}_N(A)$ we have $P(0)_i \cap \{(a,b),(b,a)\} = \emptyset$ for $i > k(b)$. However, for any $P \in U_b$ we will have $P_{k(b)+1} \cap \{(a,b),(b,a)\} \neq \emptyset$ and so we cannot possibly have $P_{k(b)+1} \subseteq P(0)_{k(b)+1}$.

For the second statement, first notice that if $k(b) = 0$ then $L_b = \emptyset$ and so the claim is trivial. If $k(b) > 0$ then $a \equiv_{P(0)_i} b$ for all $i < k(b)$. Any $P \in L_b$ takes the form $P_j = \{(a,a)\} \cup Q(1)_j \cup T_j$. The only real check to do is that $T_j \subseteq P(0)_j$. But this is true since $a \equiv_{P(0)_{k(b)}} b$ means

$$b' <_{Q(1)_j} b \;\Rightarrow b' \leq_{Q(0)_j} b \;\Rightarrow b' \leq_{P(0)_j} b \;\Rightarrow b' \leq_{P(0)_j} a$$

and similarly for $b <_{Q(1)_j} b'$. ☐

Lemma 4.4.18. *Assume that $a <_{P(0)_{k(b)}} b$, (the case when $b <_{P(0)_{k(b)}} a$ is completely symmetric). The realisation $|\widetilde{S_b}|$ is a ball if and only if*

- *b is a minimal point in the preorder $Q(1)_{k(b)}|_C$ where $C = [b]_{P(0)_{k(b)}}$, and*
- *b is not comparable in $Q(1)_{k(b)}$ to any element $b' \in B$ such that $a <_{P(0)_{k(b)}} b' <_{P(0)_{k(b)}} b$.*

Proof. It is easy to see that the element $h(k(b), b)$ cannot be in $\widetilde{S_b}$ since $(b,a) \notin P(0)_{k(b)}$. Therefore, deciding whether $|\widetilde{S_b}|$ is a ball or not is equivalent to deciding whether the element $h(k(b), a) \in \widetilde{S_b}$. If it is, then it will be a minimal element and $|\widetilde{S_b}|$ will be the cone of $|L_b|$ which is itself a sphere. If it is not then $|\widetilde{S_b}| = |L_b|$.

First assume that we do have $P = h(k(b), a) \in \widetilde{S_b}$ so in particular $P_i \subseteq P(0)_i$ for all i. Assume that there exists $b' \in C$ such that $b' <_{Q(1)_{k(b)}} b$. Then we must have $b' <_{P_{k(b)}} a$ which in turn implies that $b' \leq_{P(0)_{k(b)}} a$ which is a contradiction. Similarly, assume that b is comparable to some $b' \in B$ such that $a <_{P(0)_{k(b)}} b' <_{P(0)_{k(b)}} b$. In particular this means that $b' <_{Q(0)_{k(b)}} b$ and so $b' <_{Q(1)_{k(b)}} b$ also. But again, this will imply that $b' <_{P_{k(b)}} a$ which in turn implies that $b' \leq_{P(0)_{k(b)}} a$ which is a contradiction. Therefore the two conditions are satisfied.

Conversely, assume that we have a b satisfying the conditions of the lemma. We then need to show that $P = h(k(b), a) \succ P(0)$. Each P_i is of the form $P_i = \{(a,a)\} \cup Q(1)_i \cup T_i$ such that

$$T_i = \begin{cases} \{(b',a) \mid b' \leq_{Q(1)_i} b\} \cup \{(a,b') \mid b \leq_{Q(1)_i} b'\} & \text{if } i < k(b) \\ \{(b',a) \mid b' <_{Q(1)_i} b\} \cup \{(a,b') \mid b \leq_{Q(1)_i} b'\} & \text{if } i = k(b) \\ \emptyset & \text{if } i > k(b) \end{cases}.$$

We only really need to check $T_{k(b)} \subseteq P(0)_{k(b)}$ since $i < k(b)$ is taken care of by knowing that $L_b \subseteq \widetilde{S_b}$ from lemma 4.4.17. First take a pair from the set $\{(a,b') \mid b \leq_{Q(1)_{k(b)}} b'\}$. Now $b \leq_{Q(1)_{k(b)}} b'$ implies that $b \leq_{Q(0)_{k(b)}} b'$ and so $b \leq_{P(0)_{k(b)}} b'$ as well. By transitivity this gives $a <_{P(0)_{k(b)}} b'$ and so $(a,b') \in P(0)_{k(b)}$ as required. Now take a pair from the set $\{(b',a) \mid b' <_{Q(1)_{k(b)}} b\}$. Again we have that $b' \leq_{Q(1)_{k(b)}} b$ implies that $b' \leq_{Q(0)_{k(b)}} b$ and so $b' \leq_{P(0)_{k(b)}} b$ as well. This means that a and b' will be $P(0)_{k(b)}$-comparable since comparability is an equivalence relation. However, if $a <_{P(0)_{k(b)}} b'$ then this will contradict one of our conditions and so we must have $(b',a) \in P(0)_{k(b)}$ as required. ☐

Lemma 4.4.19. *Assume that $a <_{P(0)_{k(b)}} b$, (the case when $b <_{P(0)_{k(b)}} a$ is completely symmetric), and that $|\widetilde{S_b}|$ is a sphere. Then there exists $b' \in B$ such that $a <_{P(0)_{k(b)}} b'$ and $|\widetilde{S_{b'}}|$ is a ball. In particular, if we let b' be a greatest element such that $b' < b$ with respect to the preordering induced by $Q(1)_{k(b)}$ then $\widetilde{S_b} \subseteq \widetilde{S_{b'}}$.*

Proof. Let $C = \{c \in B \mid a <_{P(0)_{k(b)}} c \leq_{P(0)_{k(b)}} b\}$. We know C is non-empty because if it were then $|\widetilde{S_b}|$ would be a ball. Let b' be a least element with respect to $Q(1)_{k(b)}$ in the set $C \cap [b]_{Q(1)_{k(b)-1}}$. Then one can check that $|\widetilde{S_{b'}}|$ will be a ball by applying lemma 4.4.18. It should also be clear that b' will be a greatest element with this property such that $b' < b$. Now, since $|\widetilde{S_b}|$ is a sphere, we have seen already that $\widetilde{S_b} = L_b$. Then, because we have $b' \equiv_{Q(1)_{k(b)-1}} b$ we can deduce that $L_b = L_{b'} \subseteq \widetilde{S_{b'}}$ as required. ☐

Corollary 4.4.20. *Let $\tilde{B} \subseteq B$ be the set of all $b \in B$ such that $|\widetilde{S_b}|$ is homeomorphic to a ball. Then $|\mathcal{U}(P(0); Q(0), Q(1))| \cong \bigcup_{b \in \tilde{B}} |\widetilde{S_b}|$.*

Proof. This is clear from lemma 4.4.19 and that each $\widetilde{S_b}$ is closed upwards. ☐

Lemma 4.4.21. *Let $b, c \in \tilde{B}$ such that $b <_{Q(1)_m} c$. Then $k(b), k(c) \geq m$.*

Proof. Let $i < m$. It is enough to show that none of the inequalities $a <_{P(0)_i} b$, $a <_{P(0)_i} c$, $b <_{P(0)_i} a$ or $c <_{P(0)_i} a$ can hold. If $a <_{P(0)_i} b$ then we have

$$a <_{P(0)_i} b \leq_{P(0)_i} c.$$

If $b \equiv_{P(0)_i} c$ then c will contradict the first condition of lemma 4.4.18 and so $c \notin \tilde{B}$. If $b <_{P(0)_i} c$ then it will contradict the second and so again $c \notin \tilde{B}$. We can argue similarly to show that $c <_{P(0)_i} a$ implies that $b \notin \tilde{B}$.

Now, because $b <_{Q(1)_m} c$ we must have $b \equiv_{Q(1)_i} c$. This then implies that $b \equiv_{P(0)_i} c$ since $Q(1)_i \subseteq Q(0)_i \subseteq P(0)_i$. Therefore $c <_{P(0)_i} a$ is equivalent to $b <_{P(0)_i} a$ which we have just proved to be impossible. Similarly, $a <_{P(0)_i} b$ is equivalent to $a <_{P(0)_i} c$ which we have also proved to be impossible. ☐

Proposition 4.4.22. *Let b and c be adjacent in $\tilde{B}$ with respect to the total ordering induced by $Q(1)$. Then $\widetilde{S_b} \cap \widetilde{S_c}$ has contractible realisation.*

Proof. Let m be such that $b <_{Q(1)_m} c$. By lemma 4.4.21, it is enough to just consider when $k(b), k(c) \geq m$. If $k(b) = m = k(c)$ then we must have $b <_{P(0)_m} a <_{P(0)_m} c$ and that b and c are adjacent in B with respect to the total order induced by $Q(1)$. Therefore we see that $\widetilde{S_b} = L_b \cup \{h(m, b)\}$ and $\widetilde{S_c} = L_c \cup \{h(m, a)\}$. These two sets are equal and so the realisation of their intersection is homeomorphic to a ball since $b, c \in \tilde{B}$.

Now assume that $k(b) > m$. Notice that this means $\{h(i, \alpha) \mid i < m\} \cup \{h(m, b)\} \subseteq \widetilde{S_b}$ and we claim that this is equal to the intersection $\widetilde{S_b} \cap \widetilde{S_c}$. First notice that if $m < i \leq k(b)$ then $P = h(i, \alpha) \notin \widetilde{S_c}$. This is because if it were then in particular $P \in S_c$ and so a would be P_i-comparable to both b and c. Since comparability is an equivalence relation, this implies that b and c are P_i-comparable but this is a contradiction since b and c are not $Q(1)_i$-comparable. Next we claim that $P = h(m, a) \in S_b$ is not in $\widetilde{S_c}$ since one sees easily that in fact $P \notin S_c$. However, we do have $P = h(m, b) \in \widetilde{S_c}$. Indeed, we will have $b <_{P_m} a <_{P_m} c$. We need to show that there does not exist $d \in B$ such that $a <_{P_m} d <_{P_m} c$. However, if there is then this will violate $c \in \tilde{B}$. Finally, if $i < m$ then $b \equiv_{Q(1)_i} c$ and so $h(i, \alpha)$ is clearly in $\widetilde{S_c}$ as required. We argue similarly that if $k(c) > m$ then $\widetilde{S_b} \cap \widetilde{S_c} = \{h(i, \alpha) \mid i < m\} \cup \{h(m, a)\}$. These intersections have contractible realisation as required. ☐

Finally, we can prove the result we have been building to.

Proposition 4.4.23. $|\mathcal{U}(P(0); Q(0), Q(1))|$ *is contractible.*

Proof. By corollary 4.4.20, $|\mathcal{U}(P(0);Q(0),Q(1))|$ is a union of contractible spaces. For $b \in \tilde{B}$, let V_b be the union of the sets $\widetilde{S_c}$ for $c \in \tilde{B}$ with $c \leq b$ with respect to the total order on B induced by $Q(1)$. It will then be sufficient to prove that each $|V_b|$ is contractible which we prove by induction. Let b' be the successor of b in $\tilde{B}$ so that $V_{b'} = \widetilde{S_{b'}} \cup V_b$. It is then sufficient to prove that $|V_b| \cap |\widetilde{S_{b'}}|$ is contractible. If $c \leq b$ with respect to $Q(1)$ then by lemmas 4.4.9 and 4.4.10 we have $S_c \cap S_{b'} \subseteq S_b$. Therefore $V_b \cap \widetilde{S_{b'}} = \widetilde{S_b} \cap \widetilde{S_{b'}}$ and this has contractible realisation by proposition 4.4.22. $\square$

The above allows us to apply a result from section 4.1 to conclude the following.

Corollary 4.4.24. *$\bar{f}^{-1}(t) \simeq \bigvee_{b \in B} S^{N-1}$ for all $t \in |\operatorname{ICP}_N(B)|$.*

Proof. By proposition 4.4.23, $f : \operatorname{ICP}_N(A) \to \operatorname{ICP}_N(B)$ satisfies the hypotheses of proposition 4.1.14. Therefore $\bar{f}^{-1}(t)$ is homotopy equivalent to $\bar{f}^{-1}(Q)$ for some vertex $Q \in \operatorname{ICP}_N(B)$. But proposition 4.4.12 tells us that $\bar{f}^{-1}(Q) \simeq \bigvee_{b \in B} S^{N-1}$. $\square$

So now that we have a map $\bar{f} : \operatorname{ICP}_N(A) \to \operatorname{ICP}_N(B)$ with all homotopy equivalent fibers, it would be satisfying if we could find even more similarity with the Fadell-Neuwirth fibrations. We conjecture the following.

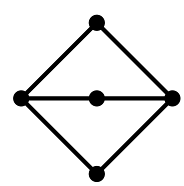
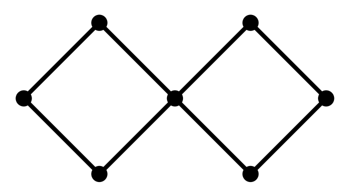
Conjecture 4.4.25. *The map $\bar{f} : |\operatorname{ICP}_N(A)| \to |\operatorname{ICP}_N(A \setminus \{a\})|$ is a quasifibration for all N, any finite set A and $a \in A$.*

It should first be noted that $\bar{f}$ will not be a trivial fiber bundle in general since the fibers are not homeomorphic. Indeed let $N = 2$, $A = \{a, b, c\}$ and $B = \{b, c\}$. Now let $Q(0), Q(1) \in \operatorname{ICP}_N(B)$ with these elements generated by

$$Q(0) = (b \equiv c, b < c),$$

$$Q(1) = (b < c, 1),$$

then we can draw pictures for $\bar{f}^{-1}(Q(i))$.

$\bar{f}^{-1}(Q(0))$  $\bar{f}^{-1}(Q(1))$ 

There are somewhat standard techniques to try and prove that a map is a quasifibration by showing that certain maps are weak equivalences. For example, the following is proved in [21], (page 98).

Theorem 4.4.26. *Let $f : P \to Q$ be a map of posets and for $q \in Q$ define $E(q) = \{p \in P \mid f(p) \preceq q\}$. Then for $q \preceq q'$ we have an inclusion map $g : E(q) \to E(q')$. If for all $q \preceq q'$ the map $|g| : |E(q)| \to |E(q')|$ is a weak equivalence, then $|f|$ is a quasifibration.*

However, it is still not clear how one might try to apply this in the example above. If $q \prec q'$ then $|g|$ will be a strict inclusion. All of our previous techniques require us to show that certain preimages are contractible, but in this case some of them will be empty. Instead we could try to define a homotopy inverse to $|g|$ and use one of these techniques as part of the proof. To do this we would either need a good map $E(q') \to E(q)$ or $|E(q')| \to E(q)$. However, there do not seem to be any obvious candidates.

4.5 A Combinatorial Model for $\bar{F}_N$

Using the combinatorics developed in section 4.3, we now go on to define an operad in posets that is a combinatorial model for the Fulton-Macpherson operads $\bar{F}_N$.

Definition 4.5.1. Let $B \subseteq A$. Then define

$$\mathrm{SCP}_N(A, B) = \{(Q, P) \in \mathrm{SCP}_N(A) \times \mathrm{SCP}_N(B) \mid Q|_B \in \{P, (B^2)^N\}\}.$$

Definition 4.5.2. Consider the set $\prod \mathrm{SCP}_N(B)$ where the product runs over all $B \subseteq A$ such that $|B| > 1$. We say that $Q = (Q(B))$ is coherent if for all $C \subseteq B \subseteq A$ we have

$$(Q(B), Q(C)) \in \mathrm{SCP}_N(B, C).$$

Denote by $\bar{\mathcal{F}}_N(A)$ the set of all coherent elements. We give this the product poset structure and call it the Fulton-Macpherson poset.

We can prove that $\bar{\mathcal{F}}_N(A)$ has much of the same structure as $\bar{F}_N(A)$.

Definition 4.5.3. Let $Q = (Q(B)) \in \bar{\mathcal{F}}_N(A)$. We say that $T \subseteq A$ is Q-critical if for all $T \subset U \subseteq A$ we have $Q(U)|_T = (T^2)^N$. Write $\mathcal{T}(Q)$ for the set of Q-critical sets.

Lemma 4.5.4. *$\mathcal{T}(Q)$ is a full tree on A.*

Proof. It is clear that all of the singleton sets are in $\mathcal{T}(Q)$ as well as A, (as it vacuously satisfies the conditions). Let $T, T' \in \mathcal{T}(Q)$ such that $T \cap T' \neq \emptyset$. We need to show that either $T \subseteq T'$ or $T' \subseteq T$. If neither holds then $U = T \cup T'$ is a strict superset of both T and T' and therefore

$$Q(U)|_T = (T^2)^N \qquad Q(U)|_{T'} = (T'^2)^N.$$

However, because $T \cap T' \neq \emptyset$, this means that

$$(U^2)^N = Q(U)|_{T \cup T'} = Q(U)|_U = Q(U)$$

which contradicts $Q(U) \in \mathrm{SCP}_N(U)$. □

Lemma 4.5.5. *Let $P, Q \in \bar{\mathcal{F}}_N(A)$ such that $P \preceq Q$. Then $\mathcal{T}(Q) \subseteq \mathcal{T}(P)$. In particular, this means that the map $\mathcal{T} : \bar{\mathcal{F}}_N(A) \to \mathrm{FTrees}(A)$ is a map of posets.*

Proof. Let $T \in \mathcal{T}(Q)$ which happens if and only if $Q(U)|_T = (T^2)^N$ for all $T \subset U \subseteq A$. Since $P(U) \preceq Q(U)$, this means $Q(U)_i \subseteq P(U)_i$ for all $0 \leq i \leq N-1$ and therefore $P(U)|_T = (T^2)^N$ and so $T \in \mathcal{T}(P)$. □

Lemma 4.5.6. *For any $Q \in \bar{\mathcal{F}}_N(A)$ and any $B \subseteq A$ with $|B| > 1$, there exists a unique Q-critical set $T \supseteq B$ with $Q(B) = Q(T)|_B$. Moreover, this is $\mathcal{T}(Q)(B)$.*

Proof. Let T be a set of largest possible size such that $Q(T)|_B \neq (B^2)^N$. We claim that T is Q-critical. Indeed, if $U \supset T$ then $Q(U)|_T|_B = Q(U)|_B = (B^2)^N$ since T was maximal. However this means $Q(U)|_T \neq Q(T)$ so we must have $Q(U)|_T = (T^2)^N$, so T is Q-critical. If T' is also Q-critical and $B \subseteq T'$ then either $T' \subset T$ or $T \subseteq T'$. In the former case $Q(T)|_{T'} = (T'^2)^N$ so $Q(T)|_B = Q(T)|_{T'}|_B = (B^2)^N$ contrary to hypothesis. Therefore $T \subseteq T'$ so it is indeed the smallest Q-critical set containing B. □

There is a canonical subset of $\bar{\mathcal{F}}_N(A)$ analogous to an interior. In fact, it is a poset that we have seen before.

Lemma 4.5.7. *Let $\mathcal{F}_N(A) = \{Q \in \bar{\mathcal{F}}_N(A) \mid Q(A)|_B = Q(B) \quad \forall B \subseteq A\}$. Then $\mathcal{F}_N(A) \cong \mathrm{ICP}_N(A)$.*

Proof. Let $\varphi : \mathcal{F}_N(A) \to \mathrm{ICP}_N(A)$ be the map defined by

$$\varphi(Q) = Q(A).$$

We claim that this is an isomorphism of posets. Firstly we need to show that $Q(A) \in \mathrm{ICP}_N(A)$, which reduces to showing that $Q(A)_{N-1}$ is separated. Take $a, b \in A$ distinct. Since $Q(\{a,b\}) \in \mathrm{SCP}_N(A)$, we must have that $Q(\{a,b\})_{N-1} \neq \{a,b\}^2$ and so it has to be separated, i.e. $[a] \neq [b]$ in the set $A/\equiv_{Q(\{a,b\})_{N-1}}$. Then because $Q(A)|_{\{a,b\}} = Q(\{a,b\})$, the same is also true in $A/\equiv_{Q(A)_{N-1}}$. Since our choice of a, b was arbitrary, this is true for all pairs and so $A/\equiv_{Q(A)_{N-1}} \cong A$ meaning $Q(A)_{N-1}$ is separated.

It is obvious that φ is a map of posets. One can then define a map $\phi : \mathrm{ICP}_N(A) \to \mathcal{F}_N(A)$ in the other direction by setting

$$\phi(P) = (P|_B)_{B \subseteq A}$$

and it is immediate to check that this is inverse to φ. $\square$

Corollary 4.5.8. *$|\mathcal{F}_N(A)| \simeq F_N(A)$.*

Proof. This is an immediate consequence of proposition 4.3.21. $\square$

It is easy to see that $\mathcal{F}_N(A)$ is the set of elements in $\bar{\mathcal{F}}_N(A)$ that have critical tree equal to the corolla on A. Similarly to $\bar{F}_N(A)$, we define subsets of $\bar{\mathcal{F}}_N(A)$ related to the stratification by trees.

Definition 4.5.9. We define the following subsets of $\bar{\mathcal{F}}_N(A)$:

$$\bar{\mathcal{F}}_N(A; = \mathcal{T}) = \{Q \in \bar{\mathcal{F}}_N(A) \mid \mathcal{T}(Q) = \mathcal{T}\}.$$

$$\bar{\mathcal{F}}_N(A; \subseteq \mathcal{T}) = \{Q \in \bar{\mathcal{F}}_N(A) \mid \mathcal{T}(Q) \subseteq \mathcal{T}\}.$$

$$\bar{\mathcal{F}}_N(A; \supseteq \mathcal{T}) = \{Q \in \bar{\mathcal{F}}_N(A) \mid \mathcal{T}(Q) \supseteq \mathcal{T}\}.$$

Lemma 4.5.10. *$\bar{\mathcal{F}}_N(A; = \mathcal{T}) \cong \prod_{T \in \mathcal{T}'} \mathcal{F}_N(\delta T)$.*

Proof. Take $Q \in \bar{\mathcal{F}}_N(A; = \mathcal{T})$ and let $\pi_T : T \to \delta T$ be the obvious projection. Define a map

$$\theta : \bar{\mathcal{F}}_N(A; = \mathcal{T}) \to \prod_{T \in \mathcal{T}'} \mathcal{F}_N(\delta T) = \prod_{T \in \mathcal{T}'} \mathrm{ICP}_N(\delta T),$$

$$\theta(Q) = (P(T))_{T \in \mathcal{T}'},$$

where $(U, V) \in P(T)_i \Leftrightarrow (u, v) \in Q(T)_i$ for some $u \in \pi_T^{-1}(U)$ and $v \in \pi_T^{-1}(V)$. Notice that this is independent of the choice of (u, v) since $Q(T)_i|_U = U^2$ for any $U \in \delta T$. To see that $P(T) \in \mathrm{ICP}_N(\delta T)$, take U, V distinct in δT and let $W = U \cup V$. Then T is the smallest Q-critical set containing W so $Q(T)|_W = Q(W) \in \mathrm{SCP}_N(W)$ meaning $Q(W) \neq (W^2)^N$. Therefore we must have $u \not\equiv_{Q(T)_{N-1}} v$ for all $u \in U$ and $v \in V$ and so one can deduce that $P(T) \in \mathrm{ICP}_N(\delta T)$.

Now define $\omega : \prod_{T\in\mathcal{T}'} \mathrm{ICP}_N(A) \to \bar{\mathcal{F}}_N(A; = \mathcal{T})$ by setting $\omega(P(T))_{T\in\mathcal{T}'} = Q$ where

$$Q(B) = \begin{cases} \pi_B^* P(B) & \text{if } B \in \mathcal{T}' \\ Q(\mathcal{T}(B))|_B & \text{otherwise} \end{cases}.$$

One easily checks that θ and ω are mutually inverse. □

We now put an operad structure on $\bar{\mathcal{F}}_N$ using definition 1.1.5 for an operad. Because geometric realisation preserves products, this automatically puts an operad structure on $|\bar{\mathcal{F}}_N|$ as well.

Definition 4.5.11. Let $B \subseteq A$, $Q \in \bar{\mathcal{F}}_N(A/B)$, $P \in \bar{\mathcal{F}}_N(B)$ and $\pi : A \to A/B$ the obvious projection. We then define

$$\gamma_B^A : \bar{\mathcal{F}}_N(A/B) \times \bar{\mathcal{F}}_N(B) \to \bar{\mathcal{F}}_N(A)$$

by setting $\gamma_B^A(Q, P) = S$ where

$$S(T) = \begin{cases} P(T) & T \subseteq B \\ \pi^* Q(\pi(T)) & \text{otherwise} \end{cases}.$$

Here, $\pi^* : \mathrm{SCP}_N(A/B) \to \mathrm{SCP}_N(A)$ is the map induced by π as in remark 4.3.6.

Proposition 4.5.12. *The maps γ_B^A above make $\bar{\mathcal{F}}_N$ an operad in posets.*

Proof. There are three things that we need to check:

1. $\gamma_B^A(Q, P) = S \in \bar{\mathcal{F}}_N(A)$.
2. γ_B^A is a map of posets.
3. The collection of maps γ_B^A satisfy the operad conditions.

For the first statement, we need to check that S is coherent. If $U \subseteq T \subseteq B$ then $S(U) = P(U)$ and $S(T) = P(T)$ so by the coherence of P, we have $(P(T), P(U)) \in \mathrm{SCP}_N(T, U)$. If $U \subseteq B$ but $T \not\subseteq B$ then now $S(T) = \pi^* Q(\pi(T))$. However this means that $S(T)|_{T\cap B} = ((T \cap B)^2)^N$ so $S(T)|_U = S(T)|_{T\cap B}|_U = (U^2)^N$. Finally if $B \not\subseteq U$ also then $S(U) = \pi^* Q(\pi(U))$. Then by the coherence of Q, we have $Q(\pi(T))|_{\pi(U)} \in \{(\pi(U)^2, Q(\pi(U))\}$ and since $\pi^* Q(\pi(U))|_{U\cap B} = ((U \cap B)^2)^N$, this implies that $(\pi^* Q(\pi(T)), \pi^* Q(\pi(U))) \in \mathrm{SCP}_N(T, U)$, so this proves (1).

To prove that these are maps of posets, take $(Q, P) \preceq (Q', P')$ in $\bar{\mathcal{F}}_N(A/B) \times \bar{\mathcal{F}}_N(B)$ and let $\gamma_B^A(Q, P) = S$ and $\gamma_B^A(Q', P') = S'$. If $T \subseteq B$ then $S(T) = P(T)$ and $S'(T) = P'(T)$ and so $P(T) \preceq P'(T)$ implies that $S(T) \preceq S'(T)$. If $T \not\subseteq B$ then $S(T) = \pi^* Q(\pi(T))$ and $S'(T) = \pi^* Q'(\pi(T))$. Therefore

$$\begin{aligned} (t_1, t_2) \in S'(T)_i \quad &\Rightarrow \quad (\pi(t_1), \pi(t_2)) \in Q'(\pi(T))_i \\ &\Rightarrow \quad (\pi(t_1), \pi(t_2)) \in Q(\pi(T))_i \\ &\Rightarrow \quad (t_1, t_2) \in S(T)_i \end{aligned}$$

so $S'(T)_i \subseteq S(T)_i$ as required.

Finally, we prove that these maps give an operad structure. Firstly let $B_1, B_2 \subseteq A$ and $B_1 \cap B_2 = \emptyset$ as well as

$$S = \gamma_{B_1}^A(\gamma_{B_2}^{A/B_1}(Q, P), R) \qquad S' = \gamma_{B_2}^A(\gamma_{B_1}^{A/B_2}(Q, R), P).$$

If $T \subseteq B_1$ then $T \not\subseteq B_2$. Therefore we know that $S(T) = R(T)$ and then also

$$\begin{aligned} S'(T) &= \pi^* \gamma_{B_1}^{A/B_2}(Q, R)(\pi(T)) \\ &= \pi^* R(T) \\ &= R(T) \end{aligned}$$

since $T \cap B_2 = \emptyset$. The case $T \subseteq B_2$ is completely symmetric. Finally, if $T \not\subseteq B_1$ and $T \not\subseteq B_2$ then

$$S(T) = \pi^*(\pi^*(Q((\pi \circ \pi)(T)))) \qquad S'(T) = \pi^*(\pi^*(Q((\pi \circ \pi)(T)))).$$

(These two things are not automatically equal as the maps are not all the same, however adding extra notation gets messy). Since $B_1 \cap B_2 = \emptyset$, the π's and the π^*'s commute so $S(T) = S'(T)$.

Now let $B_1 \subseteq B_2 \subseteq A$ as well as

$$S = \gamma_{B_2}^{A}(Q, \gamma_{B_1}^{B_2}(P, R)), \qquad S' = \gamma_{B_1}^{A}(\gamma_{B_2/B_1}^{A/B_1}(Q, P), R).$$

If $T \subseteq B_1 \subseteq B_2 \subseteq A$ then it is easy to see that $S(T) = R(T) = S'(T)$. If $T \subseteq B_2$ but $T \not\subseteq B_1$ then

$$S(T) = \gamma_{B_1}^{B_2}(P, R)(T) = \pi^* P(\pi(T))$$

$$S'(T) = \pi^* \gamma_{B_2/B_1}^{A/B_2}(Q, P)(\pi(T)) = \pi^* P(\pi(T)).$$

Finally, if $T \not\subset B_2$ then $S(T) = \pi^* Q(\pi(T))$ and

$$\begin{aligned} S'(T) &= \pi^* \gamma_{B_2/B_1}^{A/B_1}(Q, P)(\pi(T)) \\ &= \pi^* \pi^* Q(\pi(\pi(T))) \end{aligned}$$

But since $B_1 \subseteq B_2$, $\pi \circ \pi$ and $\pi^* \circ \pi^*$ above are equal to π and π^* in the expression for $S(T)$. ☐

As we did for $\bar{F}_N$, we can consider a reduced version of $\bar{\mathcal{F}}_N$ by simply setting $\bar{\mathcal{F}}_N(\emptyset) = \emptyset$. Again, we let $\bar{\mathcal{F}}_N$ be the reduced operad and $\bar{\mathcal{F}}_N^+$ be the non-reduced operad with $\bar{\mathcal{F}}_N(\emptyset) = \{*\}$.

Lemma 4.5.13. *The collection $\mathcal{T} : \bar{\mathcal{F}}_N \to$ FTrees given by taking critical trees is a map of operads in posets.*

Proof. By lemma 4.5.5 we already know that each map $\mathcal{T}_A : \bar{\mathcal{F}}_N(A) \to \mathrm{FTrees}(A)$ is a map of posets. Let $Q \in \bar{\mathcal{F}}_N(A/B)$, $P \in \bar{\mathcal{F}}_N(B)$ and $S = \gamma_B^A(Q, P)$. Then we need to check that

$$\mathcal{T}(S) = \mathcal{T}(P) \cup \pi^{-1}\mathcal{T}(Q)$$

where $\pi^{-1}\mathcal{T}(Q) = \{\pi^{-1}(T) \mid T \in \mathcal{T}(Q)\}$. Let $T \in \mathcal{T}(P)$ and $T \subset U \subseteq A$. If $U \subseteq B$ then $S(U) = P(U)$ and $P(U)|_T = (T^2)^N$ since $T \in \mathcal{T}(P)$. If $U \not\subseteq B$ then $S(U)|_B = (B^2)^N$ which implies that $S(U)|_T = (T^2)^N$ since $T \subseteq B$. Therefore $T \in \mathcal{T}(S)$.

Now let $T \in \pi^{-1}\mathcal{T}(Q)$ and $T \subset U \subseteq A$. Then $S(U) = \pi^* Q(\pi(U))$. Since $\pi(T) \in \mathcal{T}(Q)$ it implies that $Q(\pi(U))|_{\pi(T)} = (\pi(T)^2)^N$ so $\pi^* Q(\pi(U))|_T = (T^2)^N$ meaning $T \in \mathcal{T}(S)$.

For the converse, take $T \in \mathcal{T}(S)$ which means that by definition $S(U)|_T = (T^2)^N$ for all $T \subset U \subseteq A$. If $T \subseteq B$ then in particular $P(U)|_T = S(U)|_T = (T^2)^N$ for all $T \subset U \subseteq B$ so $T \in \mathcal{T}(P)$. If $T \not\subseteq B$ then $U \not\subseteq B$ either. Then

$$\pi^* Q(\pi(U))|_T = S(U)|_T = (T^2)^N$$

and therefore $Q(\pi(U))|_{\pi(T)} = (\pi(T)^2)^N$ so $T \in \pi^{-1}\mathcal{T}(Q)$. ☐

Corollary 4.5.14. *γ_B^A can be described as the grafting of trees with vertices decorated by $\mathcal{F}_N$.*

Proof. We only need to show that the composition maps γ_B^A preserve the decorations which we described in lemma 4.5.10. Let $\theta : \bar{\mathcal{F}}_N(A; = \mathcal{T}) \to \prod_{T \in \mathcal{T}'} \mathcal{F}_N(\delta T)$ be that bijection. We then claim that

$$\theta(S)(T) = \begin{cases} \theta(P(T)) & T \subseteq B \\ \theta(Q(\pi(T))) & \text{otherwise} \end{cases}.$$

If $T \subseteq B$ there is nothing to check. Therefore if $T \not\subseteq B$ then $S(T) = \pi^*(Q(\pi(T)))$. Then

$$\begin{aligned} (U, V) \in \theta(S)(T)_i \quad & \Leftrightarrow \quad (u, v) \in S(T)_i \\ & \Leftrightarrow \quad (u, v) \in \pi^*(Q(\pi(T)))_i \\ & \Leftrightarrow \quad (\pi(u), \pi(v)) \in Q(\pi(T))_i \\ & \Leftrightarrow \quad (\pi(u), \pi(v)) \in \theta(Q(\pi(T))) \end{aligned}$$

as required. □

Corollary 4.5.15. *$\bar{\mathcal{F}}_N$ is a well-labelled operad for any N.*

Proof. This is a direct consequence of the fact that the operad composition can be described as the grafting of trees and that

$$\bar{\mathcal{F}}_N(A; = \mathcal{T}) \cong \prod_{T \in \mathcal{T}'} \mathcal{F}_N(\delta T).$$

I.e. we will have an isomorphism $\tilde{f} : \Phi\mathcal{F}_N \to \bar{\mathcal{F}}_N$. □

Remark 4.5.16. As we could for $\bar{F}_N$, we could use the viewpoint of proposition 1.3.15 to prove corollary 4.5.15.

Lemma 4.5.17. $\bar{\mathcal{F}}_N(A; \supseteq \mathcal{T}) \cong \prod_{T \in \mathcal{T}'} \bar{\mathcal{F}}_N(\delta T)$.

Proof. Let $P = (P_T)_{T \in \mathcal{T}'} \in \prod_{T \in \mathcal{T}'} \bar{\mathcal{F}}_N(\delta T)$. We can then construct an element $Q \in \bar{\mathcal{F}}_N(A; \supseteq \mathcal{T})$ by applying a composition of operad composition maps $\gamma_{\mathcal{T}}$ which acts according to the tree $\mathcal{T}$. This is injective since $\bar{\mathcal{F}}_N$ is well-labelled. To see that it is surjective, notice that

$$\bar{\mathcal{F}}_N(A; \supseteq \mathcal{T}) = \coprod_{\mathcal{T} \subseteq \tilde{\mathcal{T}}} \bar{\mathcal{F}}_N(A; = \mathcal{T}) \cong \coprod_{\mathcal{T} \subseteq \tilde{\mathcal{T}}} \prod_{T \in \tilde{\mathcal{T}}} \mathcal{F}_N(\delta T).$$

Therefore, let $Q = (Q(T)) \in \prod_{T \in \tilde{\mathcal{T}}} \mathcal{F}_N(\delta T)$ and for $T \in \mathcal{T}$ define

$$\tilde{\mathcal{T}}_T = \{U \in \tilde{\mathcal{T}} \mid U \subseteq T, U \cap V \neq \emptyset \Rightarrow V \subseteq U \text{ for } V \in \delta T\}.$$

It is then not hard to show that

$$\left(\gamma_{\tilde{\mathcal{T}}_T}((Q(U))_{U \in \tilde{\mathcal{T}}_T}\right)_{T \in \mathcal{T}'}$$

is a preimage of Q, and so our map is surjective. □

We can now start to compare the operads $\bar{\mathcal{F}}_N$, $|\bar{\mathcal{F}}_N|$ and $\bar{F}_N$ with the aim of showing that $|\bar{\mathcal{F}}_N|$ is a combinatorial model for $\bar{F}_N$.

Definition 4.5.18. Define $\mu : \bar{F}_N(A) \to \bar{\mathcal{F}}_N(A)$ to simply be the product

$$\mu(x) = (\mu(x_B))_{B\subseteq A}.$$

To see that this is well-defined first note that $\mu(x_B)|_C = (C^2)^N$ if and only if $x_B|_C$ is constant. Then, if $x_B|_C \sim x_C$, we have $x_C(a)_i \leq x_C(b)_i$ if and only if $x_B(a)_i \leq x_B(b)_i$, and similarly $x_C(a) = x_C(b)$ if and only if $x_B(a) = x_B(b)$. Therefore $\mu(x_B)|_C = \mu(x_C)$.

Lemma 4.5.19. *$\mathcal{T}(\mu(x)) = \mathcal{T}(x)$.*

Proof. $T \in \mathcal{T}(\mu(x))$ if and only if for all $T \subset U \subseteq A$ we have $\mu(x)(U)|_T = (T^2)^N$. But in turn, this happens if and only if $x_U|_T \sim 0$ for all $T \subset U \subseteq A$ which means $T \in \mathcal{T}(x)$ if and only if $T \in \mathcal{T}(\mu(x))$. □

Proposition 4.5.20. *μ is a map of operads.*

Proof. Let $\gamma^A_B : \bar{F}_N(A/B) \times \bar{F}_N(B) \to \bar{F}_N(A)$ be the operad composition for the Fulton-Macpherson operads as before. Note that we can use proposition 1.1.6 to define it this way. Then to avoid confusion, denote by $\Gamma^A_B : \bar{\mathcal{F}}_N(A/B)\times\bar{\mathcal{F}}_N(B) \to \bar{\mathcal{F}}_N(A)$ the operad composition for the Fulton-Macpherson posets. Let $(x, y) \in \bar{F}_N(A/B) \times \bar{F}_N(B)$. We then need to show that $\mu(\gamma^A_B(x, y)) = \Gamma^A_B(\mu(x), \mu(y))$. Firstly, if $T \subseteq B$ then $\gamma^A_B(x, y)_T = y_T$ and so the left hand side is equal to $\mu(y_T)$. Similarly, $\Gamma^A_B(\mu(x), \mu(y))(T) = \mu(y)(T) = \mu(y_T)$ as required.

Now let $T \nsubseteq B$. Then $\gamma^A_B(x, y) = \pi^* x_{\pi(T)}$ and so the left hand side is equal to $\mu(\pi^* x_{\pi(T)})$. Similarly $\Gamma^A_B(\mu(x), \mu(y)) = \pi^*(\mu(x)(\pi(T))) = \pi^*(\mu(x_{\pi(T)}))$. Therefore we just need to show that μ commutes with the induced maps

$$\pi^* : S(W_N(A/B)) \to S(W_N(A))$$

and

$$\pi^* : \mathrm{SCP}_N(A/B) \to \mathrm{SCP}_N(A).$$

Firstly, $(a, b) \in \mu(\pi^* x_{\pi(T)})_i$ if and only if $\pi^* x_{\pi(T)}(a)_i \leq \pi^* x_{\pi(T)}(b)_i$ and $\pi^* x_{\pi(T)}(a)_j = \pi^* x_{\pi(T)}(b)_j$ for all $j < i$. This in turn happens if and only if $x_{\pi(T)}(\pi(a))_i \leq x_{\pi(T)}(\pi(b))_i$ and $x_{\pi(T)}(\pi(a))_j = x_{\pi(T)}(\pi(b))_j$ for all $j < i$. This happens if and only if $(\pi(a), \pi(b)) \in \mu(x_{\pi(T)})_i$ and by definition this happens if and only if $(a, b) \in \pi^*(\mu(x_{\pi(T)}))_i$ as required. □

We want to be able to use corollary 4.2.8 to compare $\bar{F}_N$ and $|\bar{\mathcal{F}}_N|$. To do this we need to show that certain subspaces of $\bar{F}_N(A)$ are contractible. We first prove some technical lemmas that we will use to do this. For $Q \in \bar{\mathcal{F}}_N(A)$ let

$$V(Q) = \mu^{-1}(U(Q)) = \{x \in \bar{F}_N(A) \mid \mu(x) \succeq Q\}.$$

Lemma 4.5.21. *Let $Q \in \bar{\mathcal{F}}_N(A)$ and $x \in \bar{F}_N(A)$. Then $x \in V(Q)$ if and only if $a <_{Q(T)_i} b \Rightarrow (x_T(a)_i < x_T(b)_i$ or $x_T(a)_j \neq x_T(b)_j$ for some $j < i)$.*

Proof. First assume that $a <_{Q(T)_i} b \Rightarrow (x_T(a)_i < x_T(b)_i$ or $x_T(a)_j \neq x_T(b)_j$ for some $j < i)$. We then need to show that $\mu(x)(T)_i \subseteq Q(T)_i$ for all $T \subseteq A$ and $i = 0, \ldots, N-1$. Start with $i = 0$. Then $(a, b) \in \mu(x)(T)_0$ if and only if $x_T(a)_0 \leq x_T(b)_0$. If $(a, b) \notin Q(T)_0$ then $(b, a) \in Q(T)_0$ by the totality of $Q(T)_0$. But then $b <_{Q(T)_0} a$ implies that $x_T(b)_0 < x_T(a)_i$ which is a contradiction. So $(a, b) \in Q(T)_0$.

Now assume that $\mu(x)(T)_j \subseteq Q(T)_j$ for all $j < i$. In particular, $a \equiv_{\mu(x)(T)_j} b \Rightarrow a \equiv_{Q(T)_j} b$. Now $(a, b) \in \mu(x)(T)_i$ if and only if $x_T(a)_j = x_T(b)_j$ for all $j < i$ and $x_T(a)_i \leq x_T(b)_i$.

Again, assuming that $(a,b) \notin Q(T)_i$ leads to the contradiction that $x_T(b)_i < x_T(a)_i$ or that $x_T(a)_j \neq x_T(b)_j$ for some $j < i$. So $(a,b) \in Q(T)_i$ as required.

Conversely, if we assume that $x \in V(Q)$ then $\mu(x)(T)_i \subseteq Q(T)_i$ for all $T \subseteq A$ and $i = 0, \ldots, N-1$. Therefore, if $a <_{Q(T)_i} b$ then $(a,b) \in Q(T)_i$ and $(b,a) \notin Q(T)_i$. In particular this means that $(b,a) \notin \mu(x)(T)_i$ also. Therefore we must have that $x_T(a)_i < x_T(b)_i$ or $x_T(a)_j \neq x_T(b)_j$ for some $j < i$ as required. $\square$

Lemma 4.5.22. *If $x \in V(Q)$ then $\mathcal{T}(x) \subseteq \mathcal{T}(Q)$.*

Proof. This is an easy consequence of lemma 4.5.5 and lemma 4.5.19. $\square$

Therefore $V(Q) \subseteq \bar{F}_N(A; \subseteq \mathcal{T}(Q))$ and so it can be identified with a subspace of $\bar{F}_N(A;\mathcal{T}(Q))$ via the projection map $\tau : \bar{F}_N(A) \to \bar{F}_N(A;\mathcal{T}(Q))$ from definition 2.2.13. We will use this definition for $V(Q)$ from now on.

Lemma 4.5.23. *$V(Q) \subseteq \bar{F}_N(A;\mathcal{T}(Q))$ is equal to the set of $x \in \bar{F}_N(A;\mathcal{T}(Q))$ such that $\mu(x_T) \succeq Q(T)$ for all $T \in \mathcal{T}(Q)'$.*

Proof. Suppose we have $x \in \bar{F}_N(A;\mathcal{T}(Q))$ such that $\mu(x_T) \succeq Q(T)$ for all $T \in \mathcal{T}(Q)'$. By recalling the proof of proposition 2.2.26, we need to check that $x_{\mathcal{T}(Q)(B)}|_B \not\sim 0$ for all $B \subseteq A$. Let $T = \mathcal{T}(Q)(B)$. If $x_T|_B \sim 0$ then $\mu(x_T)|_B = (B^2)^N$. This in turn implies that $Q(T)|_B = (B^2)^N$ which is a contradiction since $Q(T)|_B = Q(B) \in \mathrm{SCP}_N(B)$. The converse statement is clear. $\square$

Definition 4.5.24. Let $\widetilde{F}_N(A;\mathcal{T}(Q))$ be the set of $x = (x_T) \in \prod_{T\in\mathcal{T}(Q)'} \mathrm{Map}(T,\mathbb{R}^N)$ such that each x_T is non-constant and satisfying the usual coherence condition

$$x_T|_U = \lambda_{TU} x_U + v_{TU}$$

for some $\lambda_{TU} \geq 0$ and $v_{TU} \in \mathbb{R}^N$ representing a constant map. There is an evident projection map $\widetilde{F}_N(A;\mathcal{T}(Q)) \twoheadrightarrow \bar{F}_N(A;\mathcal{T}(Q))$. Let $\widetilde{V}(Q)$ be the preimage of $V(Q)$.

Remark 4.5.25. It is easy to see that $\widetilde{F}(A;\mathcal{T}(Q))$ is the product of $\bar{F}_N(A;\mathcal{T}(Q))$ with several copies of $(0,\infty)\times\mathbb{R}^N$. Since this is contractible, this means that $\widetilde{F}(A;\mathcal{T}(Q))$ and $\bar{F}_N(A;\mathcal{T}(Q))$ are homotopy equivalent. Similarly, $\widetilde{V}(Q)$ is the product of $V(Q)$ with the same contractible space and so the projection $\widetilde{V}(Q) \to V(Q)$ is a homotopy equivalence also.

Definition 4.5.26. Let $x \in \mathrm{Map}(A,\mathbb{R}^N)$. We then define $\mathcal{O}(x) \in \mathbb{R}^N$ by setting

$$\mathcal{O}(x)_i = min\{x(a)_i \mid a \in A\}.$$

Similarly, we define $\nu(x) \in \mathbb{R}_{\geq 0}$ by setting

$$\nu(x) = max\{x(a)_i - x(b)_i \mid a, b \in A, 0 \leq i \leq N-1\}.$$

We then define a subset of $\widetilde{V}(Q)$ by setting

$$V_1(Q) = \{x \in \widetilde{V}(Q) \mid \mathcal{O}(x_T) = 0 \text{ and } \nu(x_T) = 1 \text{ for all } T \in \mathcal{T}(Q)'\}.$$

Finally, for $U \subseteq T$ we denote by $\lambda_{TU}(x) \in \mathbb{R}_{\geq 0}$ the value that makes $x_T|_U - \lambda_{TU}(x)x_U$ a constant map. If $x \in V_1$ then it is easy to see that $\lambda_{TU}(x) = \nu(x_T|_U)$.

Definition 4.5.27. We define a particular element $z \in \widetilde{V}(Q)$ by setting $z = (\sigma(Q(T)))_{T \in \mathcal{T}(Q)'}$ where σ is as in definition 4.3.11. We extend each z_T to all of A by setting $z_T(a)_i = 0$ for $a \in A \setminus T$. Notice that by lemma 4.3.13, $\mu(z) = Q$ and so $\mathcal{T}(z) = \mathcal{T}(Q)$.

Definition 4.5.28. For $\epsilon \in \mathrm{Map}(\mathcal{T}(Q)', \mathbb{R}_{\geq 0})$, $x \in V_1(Q)$ and $T \in \mathcal{T}(Q)'$ we define a map $\phi(\epsilon, x)_T : T \to \mathbb{R}^N$ by

$$\phi(\epsilon, x)_T = x_T + \sum_{U \subseteq T} \lambda_{TU}(x) \epsilon_U z_U|_T.$$

One should notice that $\phi(0, x) = x$. Also, if the numbers ϵ_T are large and increase rapidly with $|T|$, then $\epsilon_T^{-1} \phi(\epsilon, x)_T$ is close to z_T.

Lemma 4.5.29. *Let $U \subseteq T$ in $\mathcal{T}(Q)'$. Then*

$$\phi(\epsilon, x)_T|_U = \lambda_{TU}(x) \phi(\epsilon, x)_U + v$$

as a map $U \to \mathbb{R}^N$ where v is a constant map. In particular, this implies that the tuple $\phi(\epsilon, x) = (\phi(\epsilon, x)_T)$ is in $\widetilde{F}_N(A; \mathcal{T}(Q))$ as long as each $\phi(\epsilon, x)_T$ is non-constant.

Proof. Let $V \in \mathcal{T}(Q)'$ with $V \subseteq T$ but $V \not\subseteq U$. Then since $\mathcal{T}(Q)$ is a tree, either V is disjoint from U so $z_V|_U = 0$ or $U \subset V$ and so $z_V|_U$ is constant. In either case, the corresponding terms in $\phi(\epsilon, x)_T$ have no effect on our hypothesis since they will contribute to v and so we may ignore them. We have also seen that $x_T|_U = \lambda_{TU}(x) x_U$ since $x \in V_1(Q)$ and one may also check that for $V \subseteq U \subseteq T$ we have $\lambda_{TV}(x) = \lambda_{TU}(x) \lambda_{UV}(x)$. Therefore

$$\begin{aligned}
\phi(\epsilon, x)_T|_U &= x_T|_U + \sum_{V \subseteq U} \lambda_{TV}(x) \epsilon_V z_V|_U + \text{ constant} \\
&= \lambda_{TU}(x) x_U + \sum_{V \subseteq U} \lambda_{TU}(x) \lambda_{UV}(x) \epsilon_V z_V|_U + \text{ constant} \\
&= \lambda_{TU}(x) \phi(\epsilon, x)_U + v.
\end{aligned}$$

□

Definition 4.5.30. Let $\epsilon \in \mathrm{Map}(\mathcal{T}(Q)', \mathbb{R}^N)$. We say that ϵ is admissible if for all $T \in \mathcal{T}(Q)'$ we have either

(1) $\epsilon_U = 0$ for all $U \subset T$, or

(2) $\epsilon_T > 1 + \sum_{U \subset T} |U| \epsilon_U$.

Lemma 4.5.31. *If ϵ is admissible then $\phi(\epsilon, x) \in \widetilde{V}(Q)$.*

Proof. We need to show that $\mu(\phi(\epsilon, x)_T) \succeq Q(T)$ for all $T \in \mathcal{T}(Q)'$. To do this we shall apply lemma 4.5.21. Also we shall decompose $\phi(\epsilon, x)_T$ as

$$\phi(\epsilon, x)_T = x_T + \epsilon_T z_T + \sum_{U \subset T} \lambda_{TU}(x) \epsilon_U z_U|_T$$

which we can do since $\lambda_{TT}(x) = 1$. Assume that we have $a <_{Q(T)_i} b$ and consider $\phi(\epsilon, x)_T(b) - \phi(\epsilon, x)_T(a)$.

First we consider case (1) in definition 4.5.30. Then we immediately have

$$\sum_{U \subset T} \lambda_{TU}(x) \epsilon_U (z_U(b) - z_U(a)) = 0$$

since $\epsilon_U = 0$. By the definition of z, we see that $\epsilon_T(z_T(b) - z_T(a))_j = 0$ for $j < i$ and is greater than or equal to 0 for $j = i$. Similarly, since $x_T \in V_1(Q)$ we have that either $(x_T(b) - x_T(a))_i > 0$ or $(x_T(b) - x_T(a))_j \neq 0$ for some $j < i$ as required.

Now consider case (2). Since $x \in V_1(Q)$ we have $0 \leq x_T(c)_i \leq 1$ for all $c \in T$ and so $(x_T(b) - x_T(a))_i \geq -1$. By considering the definition of z, we see that $(z_T(b) - z_T(a))_i$ is a positive integer and therefore $\epsilon_T(z_T(b) - z_T(a))_i \geq \epsilon_T$. Furthermore $0 \leq z_U(c)_i \leq |U|$ for all $c \in T$ and so $(z_U(b) - z_U(a))_i \geq -|U|$ for all $U \subset T$. Finally, since $x \in V_1(Q)$ we have $\lambda_{TU}(x) \leq 1$ for all $U \subseteq T$. Putting this all together

$$\begin{aligned}
(\phi(\epsilon, x)_T(b) - \phi(\epsilon, x)_T(a))_i &= (x_T(b) - x_T(a))_i + \epsilon_T(z_T(b) - z_T(a))_i + \\
&\qquad \sum_{U \subset T} \lambda_{TU}(x)\epsilon_U(z_U(b) - z_U(a))_i \\
&\geq -1 + \epsilon_T + \sum_{U \subset T} 1 \cdot \epsilon_U \cdot -|U| \\
&> 0
\end{aligned}$$

since ϵ is admissible. $\square$

Proposition 4.5.32. *$\widetilde{V}(Q)$ is a contractible space.*

Proof. Consider the composition of maps

$$V_1(Q) \xrightarrow{\iota} \widetilde{V}(Q) \xrightarrow{proj} V(Q)$$

where ι is the inclusion. It is easy to check that this is a homeomorphism $V_1(Q) \to V(Q)$ and so ι is a homotopy equivalence. Therefore if we show that ι is homotopic to a constant map, we can conclude that $\widetilde{V}(Q)$ is contractible.

To do this, we will produce a family of admissibles in $\mathrm{Map}(\mathcal{T}(Q)', \mathbb{R}_{\geq 0})$, parameterised by $[0, \infty)$, that have the asymptotics described at the end of definition 4.5.28. Firstly, for $T \in \mathcal{T}(Q)'$ define natural numbers $\alpha_T \in \mathbb{N}$ recursively by setting $\alpha_T = |T|(1 + \sum_{U \subset T} \alpha_U)$. Then for $t \in [0, \infty)$ we define $\epsilon(t) \in \mathrm{Map}(\mathcal{T}(Q)', \mathbb{R}_{\geq 0})$ by setting

$$\epsilon(t)_T = \begin{cases} \alpha_T^{t-|T^c|} - 1 & \text{if } t \geq |T^c| \\ 0 & \text{if } t \leq |T^c| \end{cases},$$

where T^c is equal to the complement of T in A. This is continuous as a map $\epsilon : [0, \infty) \to \mathrm{Map}(\mathcal{T}(Q'), \mathbb{R}_{\geq 0})$ since if $t = |T^c|$ then $\alpha_T^{t-|T^c|} - 1 = 0$.

We first show that $\epsilon(t)$ is admissible for all $t \in [0, \infty)$. Suppose we have $U \subset T$ such that $\epsilon(t)_U > 0$. In particular this means that we must have $t > |U^c| \geq |T^c| + 1$ and so $t - |T^c| > 1$. Therefore

$$\alpha_T^{t-|T^c|} \geq |T|^{t-|T^c|} \left(1 + \sum_{U \subset T} \alpha_U^{t-|T^c|} \right) \geq |T|^{t-|T^c|} + \sum_{U \subset T} |U| \alpha_U^{t-|T^c|}.$$

Now taking the components of the right hand side in turn, first we notice that because $|T| \geq 2$, $|T|^{t-|T^c|} > 2$. Then we also note that $t - |T^c| > t - |U^c|$ and so $\alpha_U^{t-|T^c|} > \alpha_U^{t-|U^c|} - 1 = \epsilon(t)_U$. Putting this together implies that

$$\epsilon(t)_T > 1 + \sum_{U \subset T} |U| \epsilon(t)_U$$

as required.

Now we define a homotopy $h : [0, \infty] \times V_1(Q) \to \widetilde{V}(Q)$ by setting $h(\infty, x) = z$ and for $t \in [0, \infty)$,

$$h(t, x)_T = (1 + \epsilon(t)_T)^{-1} \cdot \phi(\epsilon(t), x)_T.$$

From the definitions above, $\epsilon(0)$ is equal to 0 meaning $h(0, x) = x$ and therefore $h(0, -)$ is the inclusion map $V_1(Q) \xrightarrow{\iota} \widetilde{V}(Q)$. Because $\epsilon(t)$ is admissible, by lemma 4.5.31, $\phi(\epsilon(t), x) \in \widetilde{V}(Q)$ and therefore so is any scalar multiple. Finally we just need to show that h is continuous. This is clear except at $t = \infty$. However, it is clear that as $t \to \infty$ we have $\epsilon(t)_T \to \infty$ for all $T \in \mathcal{T}(Q)'$. It is also easy to see that

$$lim_{t\to\infty} \frac{\epsilon(t)_T}{\epsilon(t)_U} = lim_{t\to\infty} \frac{(|T|(1 + \sum_{U \subset T} \alpha_U))^{t-|T^c|} - 1}{\alpha_U^{t-|U^c|} - 1} = \infty.$$

Therefore, as $t \to \infty$ we have $h(t, x)_T$ tends to

$$lim_{t\to\infty}(1 + \epsilon(t)_T)^{-1} \cdot x_T + \frac{\epsilon(t)_T}{1 + \epsilon(t)_T} z_T + \sum_{U \subset T} \frac{\lambda_{TU}(x)\epsilon(t)_U}{1 + \epsilon(t)_T} z_U|_T = z_T$$

as required. ☐

Theorem 4.5.33. *$|\bar{\mathcal{F}}_N|$ and $\bar{F}_N$ are isomorphic as operads in the homotopy category. In particular, $\mu : \bar{F}_N \to \bar{\mathcal{F}}_N$ induces a collection of maps $\tilde{\mu} : \bar{F}_N \to |\bar{\mathcal{F}}_N|$ which is a level-wise homotopy equivalence and a map of operads up to homotopy.*

Proof. First we know that $\mu : \bar{F}_N \to \bar{\mathcal{F}}_N$ is a map of operads by proposition 4.5.20. We also know that $\bar{F}_N(A)$ has the homotopy type of a CW-complex since it is a compact real semi-algebraic set, [13]. Now let $Q \in \mathrm{SCP}_N(A)$ and consider $\mu^{-1}(U(Q)) \subseteq S(W_N(A))$. This is an open set because it is the preimage of $U_Q \subseteq |\,\mathrm{SCP}_N(A)|$ under the map $\bar{\mu} : S(W_N(A)) \to |SCP_N(A)|$ where

$$U_Q = \{t \in |\,\mathrm{SCP}_N(A)| \mid \mathrm{supp}(t) \cap U(Q) \neq \emptyset\}$$

which is an open set. Therefore if $Q = (Q(B)) \in \bar{\mathcal{F}}_N(A)$ then

$$\mu^{-1}(Q) = \prod_{B \subseteq A} \mu^{-1}(U(Q(B))) \cap \bar{F}_N(A)$$

is also an open set. Finally, by proposition 4.5.32 and remark 4.5.25, $\mu^{-1}(U(Q))$ is also contractible. Therefore we can apply corollary 4.2.8 to show that there exists an isomorphism of operads $\tilde{\mu} : \bar{F}_N \to |\bar{\mathcal{F}}_N|$ in the homotopy category. ☐

4.6 Further Combinatorial Operads

In this final section, we compare the Fulton-Macpherson posets with other well-known operads defined in a combinatorial way. In particular, we will define the Smith operads and show that they are weakly equivalent to the Fulton-Macpherson posets.

Definition 4.6.1. We define the functor E : SET $\to$ sSET from the category of sets to simplicial sets by

$$(EX)_m = \mathrm{Map}(\mathbf{m}, X)$$

where $\mathbf{m} = \{0, \ldots, m\}$, a slight change from our previous definition of $\mathbf{m}$.

Remark 4.6.2. The following are well-known:

- $|EX|$ is a contractible space.
- For any sets X and Y we have $E(X \times Y) = EX \times EY$.

Therefore, we can apply E to an operad $\mathcal{P}$ in SET to get an operad $E\mathcal{P}$ in sSET. In particular, if $E\mathcal{P}(A)$ has free Σ_A action for every finite set A then $E\mathcal{P}$ will be an E_∞ operad.

Definition 4.6.3. Consider the operad Ass from definition 1.2.3. If we apply E to this operad then we acquire the operad $E\,\mathrm{Ass}$ known as the Barratt-Eccles operad. It is known that this is an E_∞ operad, [2].

Definition 4.6.4. For $N \in \mathbb{N}$, we define suboperads of $E\,\mathrm{Ass}$ denoted $\mathcal{S}_N$, known as the Smith operads [26]. We describe the simplicial sets $\mathcal{S}_N(A)$ without proving they form a suboperad. If $|A| \leq 1$ then $E\,\mathrm{Ass}(A)$ is just a point and we set $\mathcal{S}_N(A) = E\,\mathrm{Ass}(A)$. If $|A| = 2$ then we define $\mathcal{S}_N(A)$ to be the $(N-1)$-skeleton of $E\,\mathrm{Ass}(A)$. Finally, let $|A| > 2$. For $B \subseteq A$ we have a restriction map $\rho^A_B : \mathrm{Ass}(A) \to \mathrm{Ass}(B)$ which induces a map $\rho^A_B : E\,\mathrm{Ass}(A) \to E\,\mathrm{Ass}(B)$. We define

$$\mathcal{S}_N(A) = \bigcap_{|B|=2} (\rho^A_B)^{-1}\mathcal{S}_N(B).$$

Lemma 4.6.5. *Let $|A| = 2$. Then $|\mathcal{S}_N(A)|$ is homeomorphic to the sphere S^{N-1}. In particular it is exactly the well known CW-structure given by having two k-cells for each $k < N$.*

Proof. First we describe the $(N-1)$-skeleton of $E\,\mathrm{Ass}(A)$ more explicitly. In the case $|A| = 2$, $\mathrm{Ass}(A)$ only has 2 elements and is isomorphic to A by identifying $a \in A$ with the ordering such that a is minimal. In this way, for $k < N$ we can identify the non-degenerate k-simplices as the lists of length $k+1$, namely $(a, b, a, b, \ldots)$ and $(b, a, b, a, \ldots)$.

To see that this forms the claimed CW-structure, take some non-degenerate k-simplex. Then this only has two non-degenerate faces, corresponding to removing either the first or the last element in the list. Removing any other element results in a repeat in the list. However this is in fact all of the non-degenerate $k-1$-simplices and via an inductive argument we can deduce that these form the sphere S^{k-1}. Therefore the resulting space is two k-simplices identified along their boundary which is of course a sphere. $\square$

Now we show that we can map from the operads $\bar{\mathcal{F}}_N$ to $\mathcal{S}_N$ in an explicit way.

Definition 4.6.6. Let $Q \in \bar{\mathcal{F}}_N(A)$. We define a relation $\tau_A(A) \in A^2$ by setting

- $(a, a) \in \tau_A(Q)$ for all $a \in A$.
- For $a, b \in A$ with $a \neq b$ we have $(a, b) \in \tau_A(Q)$ if and only if there exists $i \in \{0, \ldots, N-1\}$ such that $a <_{Q(\{a,b\})_i} b$.

Lemma 4.6.7. *$\tau_A(Q)$ is a total order for all $Q \in \bar{\mathcal{F}}_N(A)$.*

Proof. The only real work is in showing that $\tau_A(Q)$ is transitive. Assume we have $a <_{\tau_A(Q)} b$ and $b <_{\tau_A(Q)} c$ and that these occur because $a <_{Q(\{a,b\})_i} b$ and $b <_{Q(\{b,c\})_j} c$. Now consider $Q(\{a, b, c\})$. If this is in $\mathrm{ICP}_N(\{a, b, c\})$ then by coherence

$$Q(\{a, b, c\})|_{\{a,b\}} = Q(\{a, b\}),$$

$$Q(\{a,b,c\})|_{\{b,c\}} = Q(\{b,c\}),$$
$$Q(\{a,b,c\})|_{\{a,c\}} = Q(\{a,c\}).$$

First assume that $i \leq j$. This implies that

$$a <_{Q(\{a,b,c\})_i} b \leq_{Q(\{a,b,c)_i} c$$

and so $a <_{Q(\{a,b,c\})_i} c$ by the transitivity of $Q(\{a,b,c\})_i$. Therefore $a <_{Q(\{a,c\})_i} c$ and so $a <_{\tau_A(Q)} c$. The case for $j \leq i$ is similar.

Now assume that $Q(\{a,b,c\}) \notin \mathrm{ICP}_N(\{a,b,c\})$. Since $Q(\{a,b,c\}) \neq (\{a,b,c\}^2)^N$, we must have that $Q(\{a,b,c\})$ is constant on exactly one two-element subset of $\{a,b,c\}$. If $Q(\{a,b,c\})|_{\{a,b\}} = (\{a,b\}^2)^N$ then

$$a \equiv_{Q(\{a,b,c\})_j} b <_{Q(\{a,b,c\})_j} c.$$

Therefore, $a <_{Q(\{a,c\})_j} c$ and so $a <_{\tau_A(Q)} c$. The case when $Q(\{a,b,c\})|_{\{b,c\}} = (\{b,c\}^2)^N$ is similar. Finally, if $Q(\{a,b,c\})|_{\{a,c\}} = (\{a,c\}^2)^N$, by coherence we would require

$$c \equiv_{Q(\{a,b,c\})_i} a <_{Q(\{a,b,c\})_i} b$$

$$b <_{Q(\{a,b,c\})_j} c \equiv_{Q(\{a,b,c\})_j} a$$

which cannot happen if one checks the cases $i < j$, $j < i$ and $i = j$. $\square$

Lemma 4.6.8. *The collection of maps $\tau = \{\tau_A\}$ is a map of operads $\tau : \bar{\mathcal{F}}_N \to \mathrm{Ass}$.*

Proof. We need to show that for any $B \subseteq A$ the diagram

$$\begin{tikzcd}
\bar{\mathcal{F}}_N(A/B) \times \bar{\mathcal{F}}_N(B) \arrow[r, "\gamma^A_B"] \arrow[d, "\tau_{A/B}\times\tau_B"'] & \bar{\mathcal{F}}_N(A) \arrow[d, "\tau_A"] \\
\mathrm{Ass}(A/B) \times \mathrm{Ass}(B) \arrow[r, "\gamma^A_B"] & \mathrm{Ass}(A)
\end{tikzcd}$$

commutes. Let $Q \in \bar{\mathcal{F}}_N(A/B)$, $P \in \bar{\mathcal{F}}_N(B)$ and $S = \gamma^A_B(Q,P)$. Also, let $R_{A/B} = \tau_{A/B}(Q)$, $R_B = \tau_B(P)$ and $T = \gamma^A_B(R_{A/B}, R_B)$. We need to show that $\tau_A(S) = T$. First take $a, a' \in B$ with $a \neq a'$. Then $(a,a') \in \tau_A(S)$ if and only if $a <_{P(\{a,a'\})_i} a'$ for some i. Similarly, $(a,a') \in T$ if and only if $a <_{R_B} a'$ which in turn happens if and only if $a <_{P(\{a,a'\})_i} a'$ for some i. Now take $a \neq a'$ with $\{a,a'\} \nsubseteq B$. Then $(a,a') \in \tau_A(S)$ if and only if $\pi(a) <_{Q(\pi\{a,a'\})_i} \pi(a')$ for some i, where as usual $\pi : A \to A/B$ is the projection. Similarly, $(a,a') \in T$ if and only if $\pi(a) <_{R_{A/B}} \pi(a')$ which in turn happens if and only if $\pi(a) <_{Q(\pi\{a,a'\})_i} \pi(a')$ for some i. Therefore $\tau_A(S) = T$ as required. $\square$

Corollary 4.6.9. *This induces a map of operads $\tau : \bar{\mathcal{F}}_N \to E\,\mathrm{Ass}$ in* sSET.

Proof. Any partially ordered set P can be thought of as a simplicial set by setting P_m equal to the set of chains in P of length $m+1$. Then $(\tau_A)_m : (\bar{\mathcal{F}}_N(A))_m \to (E\,\mathrm{Ass}(A))_m$ is defined by

$$(\tau_A)_m(Q_0 \prec \cdots \prec Q_m)(i) = \tau_A(Q_i)$$

for $i \in \mathbf{m}$. Lemma 4.6.8 can then be used to show that this is a map of operads. $\square$

Proposition 4.6.10. *The image of $\tau : \bar{\mathcal{F}}_N \to E\,\mathrm{Ass}$ lands in $\mathcal{S}_N$.*

Proof. Using lemma 4.6.5, when $|B| = 2$, we see that $\mathcal{S}_N(B)$ identifies with lists in B^m generated by elements in $(E\,\mathrm{Ass}(B))_m$ such that there are at most $N-1$ swaps between the two elements of B. Consider a chain $Q_0 \prec \dots Q_d$ in $\bar{\mathcal{F}}_N(A)$ for any A. In particular take some $B \subseteq A$ with $|B| = 2$ and consider the chain $Q_0(B) \preceq \dots \preceq Q_d(B)$. Since the dimension of $\mathrm{SCP}_N(B)$ is equal to $N-1$, this chain has at most N unique elements. Therefore, if we apply τ_A to the elements of the original chain, there can be at most $N-1$ swaps between the induced ordering on the elements of B. Therefore $\tau_A(Q_0 \prec \dots Q_d) \in \mathcal{S}_N$ as required. ☐

We next prove the following intermediate result which in fact compares $|\bar{\mathcal{F}}_1|$ to another combinatorial operad, the Stasheff operad.

Proposition 4.6.11. *$|\bar{\mathcal{F}}_1|$ and K are isomorphic as operads where K is the Stasheff operad.*

Proof. First fix a finite set A. We then define a map $\theta_A : \bar{\mathcal{F}}_1(A) \to K(A)$ by setting

$$\theta_A(Q) = (t, \tau_A(Q))$$

where $\tau_A : \bar{\mathcal{F}}_1(A) \to \mathrm{Ass}(A)$ is as in definition 4.6.6 and $t : \mathcal{J}(A, \tau_A(Q)) \to [0,1]$ such that

$$t(J) = \begin{cases} 1 & \text{if } J \in \mathcal{T}(Q) \\ 0 & \text{otherwise} \end{cases}.$$

We then let $|\theta_A| : |\bar{\mathcal{F}}_1(A)| \to K(A)$ simply be the realisation of this map. We first need to show that this map is a homeomorphism. It is a simple check to see that $|\bar{\mathcal{F}}_1(A)|$ is the disjoint union of $\tau_A^{-1}(R)$ for $R \in \mathrm{Ass}(A)$. Therefore we might as well fix some ordering $R \in \mathrm{Ass}(A)$ and then show that $|\bar{\mathcal{F}}_N(A)| \cap \tau_A^{-1}(R)$ is isomorphic to $K(A,R)$. Fix some Stasheff tree $\mathcal{T}$ and consider the subset $X_{\mathcal{T}} \subseteq K(A,R)$ defined by

$$X_{\mathcal{T}} = \{t \in K(A,R) \mid \mathrm{supp}(t) \subseteq \mathcal{T}\}.$$

This space can be identified with $\mathrm{Map}(\mathcal{T}' \setminus \{A\}, [0,1]) \cong [0,1]^{\mathcal{T}' \setminus \{A\}}$. Now for the same $\mathcal{T}$, consider $\bar{\mathcal{F}}_1(A; \subseteq \mathcal{T}) \cap \tau_A^{-1}(R)$. Because we have fixed the ordering R, any element in this set is in fact completely determined by its critical tree and so

$$\bar{\mathcal{F}}_1(A; \subseteq \mathcal{T}) \cap \tau_A^{-1}(R) \cong \{\mathcal{S} \subseteq \mathcal{T} \mid \mathcal{S} \text{ is Stasheff}\} \cong \mathrm{Map}(\mathcal{T}' \setminus \{A\}, \{0,1\}).$$

The final set is isomorphic to the set of subsets of $\mathcal{T}'$. In turn,

$$|\bar{\mathcal{F}}_1(A; \subseteq \mathcal{T}) \cap \tau_A^{-1}(R)| \cong |\,\mathrm{Map}(\mathcal{T}' \setminus \{A\}, \{0,1\})| \cong [0,1]^{\mathcal{T}' \setminus \{A\}} \cong X_{\mathcal{T}}$$

and it is an easy check to see that the composition of these maps is in fact $|\theta_A|$. Since our choice of tree was arbitrary, this in fact shows that $|\theta_A|$ is a homeomorphism for all of $|\bar{\mathcal{F}}_1(A)|$ since it is a homeomorphism on each of a collection of closed subsets with disjoint interiors, and the images of the boundaries agree.

Now we check that $|\theta| = \{|\theta_A|\}$ is a map of operads. First notice that because $\tau_A : \bar{\mathcal{F}}_1 \to \mathrm{Ass}$ is a map of operads, we may ignore the ordering term since the projection $K \to \mathrm{Ass}$ is also a map of operads. For the first term, notice that because $K \cong WK$, its operad composition can be described as the grafting of trees. This is also true for $\bar{\mathcal{F}}_1$ by corollary 4.5.14 and so it is a simple check to see that $\theta = \{\theta_A\}$ is a map of operads. One then just needs to expand the definitions to see that the realisation $|\theta|$ is also a map of operads. We will not spell out the details however. ☐

Remark 4.6.12. Recall by theorem 2.4.13 that $\bar{F}_1$ is isomorphic to the Stasheff operad K. Therefore 4.6.11 also demonstrates an isomorphism of operads between $|\bar{\mathcal{F}}_1|$ and $\bar{F}_1$.

Corollary 4.6.13. *If $N = 1$ then $\tau_A : \bar{\mathcal{F}}_N(A) \to \mathcal{S}_N(A)$ induces a homotopy equivalence $|\tau_A| : |\bar{\mathcal{F}}_N(A)| \to |\mathcal{S}_N(A)|$.*

Proof. First it is not hard to see that $|\mathcal{S}_1(A)|$ is simply equal to the 0-skeleton of $E\,\mathrm{Ass}(A)$ which is the set $\mathrm{Ass}(A)$ and so $|\mathcal{S}_1(A)|$ is the associative operad. Since K is an A_∞ operad and $|\bar{\mathcal{F}}_1|$ is isomorphic to K in a way compatible with τ, we can deduce that $|\tau_A| : |\bar{\mathcal{F}}_1(A)| \to |\mathcal{S}_1(A)|$ is a homotopy equivalence as required. □

Lemma 4.6.14. *Let $|A| = 2$. Then $|\tau_A| : |\bar{\mathcal{F}}_N(A)| \to |\mathcal{S}_N(A)|$ is a homotopy equivalence.*

Proof. Label A by $A = \{a, b\}$. We have an alternative CW-structure on S^{N-1} to that defined by $|\mathcal{S}_N(A)|$. This is defined by having one 0-cell and one $(N-1)$-cell such that the boundary is identified to the 0-cell. We can form a homotopy equivalence $h : |\mathcal{S}_N(A)| \to S^{N-1}$ by mapping the $(N-1)$-simplex labelled by $(a, b, a, \dots)$ to the unique $(N-1)$-cell in S^{N-1} and everything else to the 0-cell.

Now consider the composite $h \circ |\tau_A| : |\bar{\mathcal{F}}_N(A)| \to S^{N-1}$. In this simple case where $|A| = 2$ we can label the $(N-1)$-simplices in $|\bar{\mathcal{F}}_N(A)|$ as elements in $\mathrm{Map}(\mathbf{N\text{-}1}, A)$. Indeed the $(N-1)$-simplices are labelled by chains $Q_0 \prec \dots \prec Q_{N-1}$ such that Q_i is separated in its i^{th} preorder and this separated preorder is in fact a total order on A. The image of $i \in \mathbf{N\text{-}1}$ corresponds to the least element in this ordering on A. This means that $|\tau_A|$ maps the simplex with image $(a, b, a, \dots)$ to the corresponding simplex in $|\mathcal{S}_N(A)|$ and similarly for $(b, a, b, \dots)$. Every other $(N-1)$-simplex in $|\bar{\mathcal{F}}_N(A)|$ is mapped to a degenerate simplex in $|\mathcal{S}_N(A)|$. Therefore the map $h \circ |\tau_A|$ maps the simplex $(a, b, a, \dots)$ in $|\bar{\mathcal{F}}_N(A)|$ to the unique $(N-1)$-cell in S^{N-1} and everything else to the basepoint. It is easy to see that this is also a homotopy equivalence. Therefore by the two out of three property, $|\tau_A|$ is a homotopy equivalence as well. □

Theorem 4.6.15. *$|\tau_A|$ is a weak equivalence for $N \neq 2$. Therefore $\tau : \bar{\mathcal{F}}_N \to \mathcal{S}_N$ induces a weak equivalence of operads in spaces.*

Proof. By corollary 4.6.13, we only need to concentrate on the case $N > 2$. Firstly, the work in [14] shows that $|\mathcal{S}_N(A)|$ is homotopy equivalent to $\mathrm{Inj}(A, \mathbb{R}^N) \simeq |\bar{\mathcal{F}}_N(A)|$ by demonstrating an isomorphism in homology. To do this, the author defines generators in $H^*(|\mathcal{S}_N(A)|)$ by using restriction maps $\rho^A_B : \mathcal{S}_N(A) \to \mathcal{S}_N(B)$, (where $|B| = 2$), which after realisation, one can compare with the sphere S^{N-1}. Notice that this is entirely analogous to our survey in section 2.1. It is easy to see that the following diagram commutes

$$\begin{tikzcd}
\bar{\mathcal{F}}_N(A) \arrow[r, "\tau_A"] \arrow[d, "\rho^A_B"'] & \mathcal{S}_N(A) \arrow[d, "\rho^A_B"] \\
\bar{\mathcal{F}}_N(B) \arrow[r, "\tau_B"] & \mathcal{S}_N(B)
\end{tikzcd}.$$

Then, by lemma 4.6.14, we will have that $\tau_A^* : H^*(|\mathcal{S}_N(A)|) \to H^*(|\bar{\mathcal{F}}_N(A)|)$ preserves generators and so is an isomorphism. It is also a homology isomorphism since everything here is free and finitely generated. By lemma 4.4.1, $|\bar{\mathcal{F}}_N(A)|$ will be simply connected. Therefore it is a consequence of the Hurewicz theorem that in fact $|\tau_A|$ is a weak equivalence as required. □

Remark 4.6.16. I expect that we can make the same conclusion as theorem 4.6.15 for the case $N = 2$. However it is not immediately clear how one should approach this.

4.7 The Flaws of $\bar{\mathcal{F}}_N(A)$

Initially in our research, the poset $\mathrm{ICP}_N(A)$ was the first to appear and its purpose was to construct a spine for the manifold $\bar{F}_N(A)$. The work in section 4.3 demonstrates that $|\mathrm{ICP}_N(A)|$ is indeed a spine for $\bar{F}_N(A)$. As we discussed in remark 2.4.15, this should be a useful step towards creating explicit isomorphisms $W\bar{F}_N \to \bar{F}_N$. The next step was to define SCP_N and then create a combinatorial model for $\bar{F}_N$, which we did by defining $\bar{\mathcal{F}}_N$ in a completely analogous way. As we demonstrated in lemma 4.5.7, there is an obvious way to embed the spine $\mathrm{ICP}_N(A)$ into $\bar{\mathcal{F}}_N(A)$ and the stratification by trees makes this analogous to the interior $F_N(A)$ of $\bar{F}_N(A)$. It was expected that the dimension of $|\bar{\mathcal{F}}_N(A)|$ would be the same as the dimension of the manifold $\bar{F}_N(A)$. Considering all of the other evidence, (same stratification by trees, homotopy equivalent spaces, analogous operad structure), it would have been safe to conjecture that in fact $|\bar{\mathcal{F}}_N|$ and $\bar{F}_N$ were isomorphic as operads. Therefore a collaring of one would define a collaring on the other, paving the way for our desired explicit isomorphism $W\bar{F}_N \to \bar{F}_N$. However, this is not the case as we discuss in this section. We do believe though that some refinement of our poset will be appropriate.

Definition 4.7.1. Let $Q \in \bar{\mathcal{F}}_N(A)$, $\mathcal{T} = \mathcal{T}(Q)$ and recall ξ and ξ^* from definition 4.3.24 and extend this in the obvious way to $\mathrm{SCP}_N(A)$. We then define a function

$$\rho : \bar{\mathcal{F}}_N(A) \to \mathbb{N}_0$$

$$\rho(Q) = \sum_{T \in \mathcal{T}'} \left(\left(\sum_{i=0}^{N-1} \xi^*(Q(T))_i \right) - 1 \right).$$

We now conjecture the following:

Conjecture 4.7.2. *There exists a partial order $\trianglelefteq$ on the set $\bar{\mathcal{F}}_N(A)$ such that*

- *Whenever $Q \lhd Q'$ we have $Q \prec Q'$ so the geometric realisation $|(\bar{\mathcal{F}}_N(A), \trianglelefteq)|$ is a subcomplex of $|(\bar{\mathcal{F}}_N(A), \prec)|$.*
- *The inclusion $|(\bar{\mathcal{F}}_N(A), \trianglelefteq)| \to |(\bar{\mathcal{F}}_N(A), \prec)|$ is a homotopy equivalence, so $|(\bar{\mathcal{F}}_N(A), \trianglelefteq)|$ is homotopy equivalent to $\bar{F}_N(A)$.*
- *The operad structure maps are full embeddings of posets with respect to $\trianglelefteq$.*
- *The map ρ from definition 4.7.1 is a rank function for $\trianglelefteq$.*

We also conjecture the strengthening of item 2 that in fact $|(\bar{\mathcal{F}}_N, \trianglelefteq)|$ and $\bar{F}_N$ are isomorphic as operads.

We now discuss why $\bar{\mathcal{F}}_N(A)$ with its defined ordering fails to satisfy this conjecture, but do point out cases where it is sufficient.

Remark 4.7.3. Recall that by lemma 4.5.10 we have

$$\bar{\mathcal{F}}_N(A; = \mathcal{T}) \cong \prod_{T \in \mathcal{T}'} \mathcal{F}_N(\delta T) \cong \prod_{T \in \mathcal{T}'} \mathrm{ICP}_N(\delta T).$$

Since geometric realisation preserves products, we then have

$$|\bar{\mathcal{F}}_N(A; = \mathcal{T})| \cong \prod_{T\in\mathcal{T}'} |\operatorname{ICP}_N(\delta T)|$$

and the right hand side will therefore have dimension $\sum_{T\in\mathcal{T}'}(N-1)(|\delta T|-1)$. Analogously, we also have $\bar{F}_N(A; = \mathcal{T}) \cong \prod_{T\in\mathcal{T}'} F_N(\delta T)$ and each $F_N(\delta T)$ has homological dimension $(N-1)(|\delta T|-1)$. Therefore our original poset structure seems correct here. With this in mind, it seems likely that any solution to 4.7.2 should be a refinement of the original poset structure.

The next example shows that in general, $|\bar{\mathcal{F}}_N(A)|$ will not have the same dimension as $\bar{F}_N(A)$ and so their respective operads cannot possibly be isomorphic.

Example 4.7.4. Let $A = \{a, b, c, d\}$ and $N = 3$. This means that $dim(\bar{F}_N(A)) = 8$ in this case but we can construct a chain of length 10 in $\bar{\mathcal{F}}_N(A)$. Indeed, consider $Q \in \bar{\mathcal{F}}_N(A)$ such that $\mathcal{T}(Q) = \mathcal{C}_A \cup \{\{a, b\}, \{c, d\}\}$ with

$$Q(A) = (A^2; A^2; a \equiv b < c \equiv d)$$

$$Q(\{a, b\}) = (a < b; 1_{\{a,b\}}; 1_{\{a,b\}})$$
$$Q(\{c, d\}) = (c < d; 1_{\{c,d\}}; 1_{\{c,d\}}).$$

Then consider $Q' \in \bar{\mathcal{F}}_N(A)$ with $\mathcal{T}(Q) = \mathcal{C}_A$ and $Q'(A) = (a \equiv c < b \equiv d; a \equiv c,\ b \equiv d; a < c,\ b < d)$. One can check that $Q \prec Q'$ in $\bar{\mathcal{F}}_N(A)$. However, we can also create a chain $C(Q)$ of length 5 in $\bar{\mathcal{F}}_N(A)$ such that Q is the maximal element. We can do this by increasing the index in which a and b are equivalent in $Q(\{a, b\})$ and the index in which c and d are equivalent in $Q(\{c, d\})$, one step at a time. Similarly, we can create a chain $C(Q')$ of length 5 in $\bar{\mathcal{F}}_N(A)$ such that Q' is the minimal element. This is because $Q'(A)$ is separated and so we can consider it as an element of $\operatorname{ICP}_N(A)$ where it has rank equal to 4. By transitivity, this means that $C(Q) \cup C(Q')$ is itself a chain and it contains 10 elements. Therefore the dimension of $|\bar{\mathcal{F}}_N(A)|$ is at least 9.

For other posets that we have considered in this thesis, we have calculated the dimension of the realisation by using a rank function. However, our next example shows that $\bar{\mathcal{F}}_N(A)$ does not have a rank function in general, let alone the one in definition 4.7.1.

Example 4.7.5. Again let $A = \{a, b, c, d\}$ and $N = 3$. Also let $Q(1) = Q$ and $Q(3) = Q'$ from example 4.7.4. We also define $Q(2) \in \bar{\mathcal{F}}_N(A)$ such that $\mathcal{T}(Q(2)) = \mathcal{C}_A \cup \{\{c, d\}\}$ and

$$Q(2)(A) = (a < b \equiv c \equiv d; b \equiv c \equiv d; b < c \equiv d)$$

$$Q(2)(\{c, d\}) = (c < d; 1_{\{c,d\}}; 1_{\{c,d\}}).$$

Finally, we define $Q(4), Q(5) \in \bar{\mathcal{F}}_N(A)$ such that $\mathcal{T}(Q(4)) = \mathcal{T}(Q(5)) = \mathcal{C}_A$ and

$$Q(4)(A) = (a \equiv c < b \equiv d; a < c,\ b \equiv d; b < d)$$

$$Q(5)(A) = (a < c < b \equiv d; b \equiv d; b < d).$$

One can check that $Q(1) \prec Q(3) \prec Q(4) \prec Q(5)$ and so if ρ is a rank function on $\bar{\mathcal{F}}_N(A)$ then $\rho(Q(5)) \geq \rho(Q(1)) + 3$. One can also check that $Q(1) \prec Q(2) \prec Q(5)$. However we claim that this chain cannot be refined. Indeed, first assume that we have $Q(1) \preceq P \preceq Q(2)$.

This means that $\mathcal{T}(P) = \mathcal{T}(Q(1))$ or $\mathcal{T}(P) = \mathcal{T}(Q(2))$. In both cases, first notice that $Q(1)(\{a,b\}) = Q(2)(\{a,b\})$ and $Q(1)(\{c,d\}) = Q(2)(\{c,d\})$ and so we must have

$$P(\{a,b\}) = Q(1)(\{a,b\}) = Q(2)(\{a,b\})$$

$$P(\{c,d\}) = Q(1)(\{c,d\}) = Q(2)(\{c,d\}).$$

Then in the first case, we are forced to have $a \equiv_{P(A)_i} b$ for all i and therefore it is easy to see that $P(A) = Q(1)(A)$ and so in fact $P = Q(1)$. In the second case, we must have $P(A)|_{\{a,b\}} = P(\{a,b\})$ and so in particular $a <_{P(A)_0} b$. But then $Q(1)(\{b,c,d\}) = Q(2)(\{b,c,d\})$ and so one can deduce that these two facts imply $P(A) = Q(2)(A)$ and so in fact $P = Q(2)$.

Now assume that we have $Q(2) \preceq P \preceq Q(5)$. Notice that $Q(2)(\{a,b,d\}) = Q(5)(\{a,b,d\})$ and $Q(2)(\{c,d\}) = Q(5)(\{c,d\})$ and so P must agree on these two sets also. First assume that $\mathcal{T}(P) = \mathcal{T}(Q(2))$. Then we must have $c \equiv_{P(A)_i} d$ for all i and so it is easy to see that $P(A) = Q(2)(A)$ and so in fact $P = Q(2)$. On the other hand, if $\mathcal{T}(P) = \mathcal{T}(Q(5))$ then we must have $P(A)|_{\{c,d\}} = P(\{c,d\})$ and so in particular $c <_{P(A)_0} d$. But since $Q(2)(\{a,b,d\}) = Q(5)(\{a,b,d\})$, one can then deduce that $P(A) = Q(5)(A)$ and so $P = Q(5)$. Because of this lack of refinement, this means that we would then have $\rho(Q(5)) = \rho(Q(1)) + 2$ but this is a contradiction.

Despite the counterexamples above, we do have positive results in low-dimensional cases.

Remark 4.7.6. As we have already seen in remark 4.6.12, when $N = 1$, the operads $|\bar{\mathcal{F}}_N|$ and $\bar{F}_N$ are isomorphic. Therefore $\bar{\mathcal{F}}_1(A)$ will obviously satisfy all of the hypotheses of conjecture 4.7.2. Similarly, if we restrict attention to $|A| = 2$ then $|\bar{\mathcal{F}}_N(A)|$ will be isomorphic to $\bar{F}_N(A)$. This is because in this case it is easy to see that $\bar{\mathcal{F}}_N(A) \cong \mathrm{SCP}_N(A)$. Then we see that

$$|\bar{\mathcal{F}}_N(A)| \cong |\,\mathrm{SCP}_N(A)| \cong S(W_N(A)) \cong \bar{F}_N(A).$$

We can also make statements about the dimension of $|\bar{\mathcal{F}}_N(A)|$ when $|A| = 3$.

Lemma 4.7.7. *Let $A = \{a,b,c\}$. Then ρ in definition* 4.7.1 *is a rank function for $\bar{\mathcal{F}}_N(A)$ for any N.*

Proof. Let $Q \prec Q'$ in $\bar{\mathcal{F}}_N(A)$ which implies that $\mathcal{T}(Q') \subseteq \mathcal{T}(Q)$. If $\mathcal{T}(Q) = \mathcal{T}(Q')$ then all of the required properties for ρ can be deduced from the properties of ξ^*. Therefore assume that $\mathcal{T}(Q') \subset \mathcal{T}(Q)$. In particular this implies that $\mathcal{T}(Q') = \mathcal{C}_A$ and we may assume without loss of generality that $\mathcal{T}(Q) = \mathcal{C}_A \cup \{\{b,c\}\}$. We claim that

$$\sum_{i=0}^{N-1} \xi^*(Q'(A))_i \geq \sum_{i=0}^{N-1} \left(\xi^*(Q(A))_i + \xi^*(Q(\{b,c\}))_i\right)$$

which will prove that $\rho(Q) < \rho(Q')$. Because $\mathcal{T}(Q'(A)) = \mathcal{C}_A$ we have that $Q'(A)|_{\{b,c\}} = Q'(\{b,c\})$ and therefore

$$\xi^*(Q'(A))_i \geq \xi^*(Q'(\{b,c\}))_i \geq \xi^*(Q(\{b,c\}))_i.$$

If $\xi^*(Q'(A))_i = \xi^*(Q'(\{b,c\}))_i$ then this implies that either $a \equiv_{Q'(A)_i} b$ or $a \equiv_{Q'(A)_i} c$. But since $b \equiv_{Q(A)_i} c$ this implies that $Q(A)_i = A^2$ and so $\xi^*(Q(A)_i) = 0$. Since $\xi^*(Q(A))_i$ can only be equal to 0 or 1, this verifies the inequality above.

Now assume that there does not exist $Q'' \in \bar{\mathcal{F}}_N(A)$ such that $Q \prec Q'' \prec Q$. Firstly this implies that we must have $Q(\{b,c\}) = Q'(\{b,c\})$. If not then we could form an element Q'' such that $Q''(A) = Q(A)$ and $Q''(\{b,c\}) = Q'(\{b,c\})$. Therefore $\xi^*(Q(\{b,c\}))_i = \xi^*(Q'(\{b,c\}))_i$. Next, notice that $\xi^*(Q'(A))_i \leq \xi^*(Q'(\{b,c\}))_i + 1$. We must also have $\xi^*(Q'(A))_i = \xi^*(Q'(\{b,c\}))_i + 1$ implies that $\xi^*(Q(A))_i = 1$. If not, then there exists an i such that $a \not\equiv_{Q'(A)_i} b$ and $a \not\equiv_{Q'(A)_i} c$ but $Q(A)_i = A^2$. Therefore we may construct Q'' so that $a \equiv_{Q''(A)_i} b$ and $Q''(A)_j|_{\{a,b\}} = Q(A)|_{\{a,b\}}$ for $j < i$. Combining this with the analysis above we see that we must have

$$\sum_{i=0}^{N-1} \xi^*(Q'(A))_i = \sum_{i=0}^{N-1} \left(\xi^*(Q(A))_i + \xi^*(Q(\{b,c\}))_i\right)$$

and so $\rho(Q') = \rho(Q) + 1$ as required. □

Proposition 4.7.8. *The simplicial complex $|\bar{\mathcal{F}}_N(A)|$ has dimension equal to $N(|A|-1)-1$ when $|A| = 3$.*

Proof. We aim to apply lemma 4.1.8. Firstly the minimal rank that an element can take is 0. This occurs when we have an element with binary critical tree, decorated by elements with signature $(0,\dots,0,1)$. The maximal rank is $N(|A|-1)-1$ which corresponds to an element that has the corolla as critical tree and is decorated with an element of signature $(|A|-1,\dots,|A|-1)$.

To finish the proof, we claim that if we take an element $Q \in \bar{\mathcal{F}}_N(A)$ such that $\rho(Q) \neq 0$, then we can produce an element $Q' \in \bar{\mathcal{F}}_N(A)$ such that $Q' \prec Q$ and $\rho(Q') = \rho(Q) - 1$. If $\mathcal{T}(Q) \neq \mathcal{C}_A$ then this is clear as we may use the properties of ξ^* and that $\bar{\mathcal{F}}_N(A; = \mathcal{T}) \cong \prod_{T\in\mathcal{T}'} \mathrm{ICP}_N(\delta T)$ by lemma 4.5.10. If $\mathcal{T}(Q) = \mathcal{C}_A$ but $\rho(Q) > 1$ then in particular the rank of Q as an element in $\mathrm{ICP}_N(A)$ is greater than 0. Therefore we can use proposition 4.3.27 to produce a lesser element in $\mathrm{ICP}_N(A)$ which then embeds in $\bar{\mathcal{F}}_N(A)$. If $\rho(Q) = 1$ then this means $Q(A)_i = A^2$ for $i < N-1$ and $Q(A)_{N-1}$ is a total order on A. Assume without loss of generality that c is the greatest element in this total order. We then construct Q' by setting

$$Q'(A) = (A^2, \dots A^2, a \equiv b < c)$$

$$Q'(\{a,b\}) = (\{a,b\}^2, \dots \{a,b\}^2, Q(A)|_{\{a,b\}}).$$

One can easily check that $Q' \prec Q$ and that $\rho(Q) = 0$ as required. □

Remark 4.7.9. If we have a poset structure $\trianglelefteq$ on $\bar{\mathcal{F}}_N(A)$ such that ρ is a rank function in general, then this should hopefully demonstrate that $|\bar{\mathcal{F}}_N(A)|$ has the same dimension as $\bar{F}_N(A)$. Indeed, it is reasonably easy to see, as was the case in lemma 4.7.8, that the minimal rank an element can take is 0. Again, this will occur when we have an element with binary critical tree, decorated by elements with signature $(0,\dots,0,1)$. The maximal rank is $N(|A|-1)-1$ which corresponds to an element that has the corolla as critical tree and is decorated with an element of signature $(|A|-1,\dots,|A|-1)$. We would then expect to be able to use similar techniques to create a chain of length $N(|A|-1)$ to then be able to conclude that the dimension of $|(\bar{\mathcal{F}}_N(A), \trianglelefteq)|$ is $N(|A|-1)-1$.

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