An Approximation Method for the Finite-Time Optimal Control for Nonlinear Systems

by

K. Dinesh

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ABSTRACT

In this thesis, we shall consider nonlinear systems of the form

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t)$$

together with a standard quadratic cost functional and replace the system by a sequence of time-varying approximations for which an optimal control problem can be solved explicitly. The basic system is, $\dot{x}(t) = A(x)x + B(x)u$, which can be approximated by a sequence of systems of the form $\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t)$. We then show that the sequence converges. Although it may not converge to a global optimal control of the nonlinear systems, we also consider an approximation sequence for the equation given by the necessary condition of the maximum principle and we shall see that first methods gives solutions very close to optimal solution in many cases. We shall also extend the result to partially observable nonlinear stochastic systems which can be written in the above form with additive Gaussian white noise. Some examples are shown to illustrate this theory.

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Chapter 1

Introduction and Basic Results

1.1 Introduction

Engineering is concerned with understanding and controlling the materials and forces of nature for the benefit of human kind. Control system engineers are concerned with understanding and controlling parts of their environment, often called systems, in order to provide useful economic products for society. In order to control a physical system it is necessary to obtain a good mathematical model of the system. Usually, all physical systems are inherently nonlinear. Thus, all control systems are nonlinear to a certain extent. Nonlinear control systems can be described by nonlinear differential equations. However, if the operating range of the control system is small, and if the involved nonlinearities are smooth, then the control system may be reasonably approximated by a linearized system, whose dynamics are described by a set of linear differential equations.

Control system design has been studied from the classical systems theory point of view and gradually evolved to meet more complex problems by using modern systems technology. Classical control design is generally a trial and error process in which various methods of analysis are used iteratively to determine the design parameters of an acceptable system. Acceptable is generally defined in terms of time and frequency domain criteria such as rise time, settling time, peak over shoot, gain and phase margin, and bandwidth. However, complex multiple-input, multiple-output systems are required to meet the demands of modern technology. For example, the design of a spacecraft attitude control system that minimizes fuel expenditure is not amenable to solution by classical methods. A new and direct approach to the synthesis of these complex system, called optimal control theory, has been made feasible by the development of the digital computer.

The objective of optimal control theory is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize some performance criterion.

Furthermore, new theories of optimal control have been developed by L.S. Pontryagin in former Soviet Union and R. Bellman in the United states. As a result the control engineer may have a new mathematical tool for the analysis and design of a control system in the time domain.

For the time domain analysis, the optimal control law of linear quadratic optimal control problems can be obtained by solving the matrix Riccati equation, or by determining the transition matrix of the Hamiltonian system [4, 35, 38]. However, for a general nonlinear optimal control problem, it is not possible to obtain the exact feedback control solution analytically. Never the less suboptimal feedback control can be obtained by using either the power series expansion method [40, 26, 27, 28, 42, 55]) or neighbouring optimal control method ([18, 37, 44]). In addition, the optimal feedback control can be obtained using Bellman's [13] dynamic programming method, but this method suffers from the curse of dimensionality.

Since general nonlinear systems are extremely difficult to deal with and most results which can be obtained are of an approximate nature, many authors have considered bilinear systems [1, 10, 5, 17, 30]. It is in a sense, a simple nonlinear system for which one can write explicit formulae for calculation of the kernels. However, it has some drawbacks; one has to compute a large number of terms in order to get a reasonable approximation. Control of processes governed by bilinear systems, including optimal control, has been studied previously by numerous authors (see, eg [1, 10, 5, 30]).

Hofer and Tibken [30] proposed a sequence of approximations to finite time bilinear optimal control problems in which they calculated the gain matrix iteratively from the Riccati differential equation. This method has the advantage that one can use linear optimal control law for each sequence of approximations; however, a disadvantage for on-line process control is the computational overhead. Aganovic and Gajic [1] introduced a new approach that instead of calculating the gain matrix iteratively from Riccati differential equation, computed the time-varying Lyapunov equation for each sequence of approximations.

Banks and MaCaffrey [8] applied the same idea (a sequence of approximations) to nonlinear parabolic systems. In this research a sequence of approximations to nonlinear optimal control problems has been proposed.

The thesis is organized as follows: Basic results of linear control theory, linear optimal filtering and functional analysis needed in the work are presented in Chapter 1.

In Chapter 2, a sequence of approximations to a nonlinear quadratic optimal control problem is introduced and shown to converge under certain conditions.

The type of system which is considered is a nonlinear one of the form $\dot{x}(t) = A(x)x(t) + B(x)u(t)$ which can be approximated by a sequence of systems of the form $\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t)$.

In Chapter 3, a sequence of approximations is introduced in order to find a global optimal control for nonlinear systems, using Pontryagin's minimum principle. This will replace the nonlinear systems by a sequence of two-point boundary value problems which is shown to converge under certain conditions.

In Chapter 4, a sequence of linear filtering equations is applied to nonlinear stochastic systems in order to estimate the state, which is corrupted by state excitation noise and measurement errors.

Practical applications of the theory developed in the previous chapters will be given in Chapter 5 and Chapter 6, using models of an inverted pendulum, an F-8 fighter aircraft and the Van der Pol oscillator.

Finally, in Chapter 7, conclusions and topics of further research are suggested.

1.2 Linear Optimal Control Theory

In this section, some of the results concerning linear control theory, namely, linear quadratic regulator problems and their application to deterministic and stochastic systems, are reviewed.

The linear regulator problem is a classical problem of optimal control theory and may be treated as a continuous or discrete problem. In the interest of brevity the remarks will be confined to the continuous version of the problem. The proof of these results and further details can be found in [4, 14, 16, 32, 35, 38].

A linear time-varying system of the form:

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \tag{1.1}$$

is considered, where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, A and B are $n \times n$, $n \times m$ dimensional matrices respectively. It is assumed that the complete state $x(t)$ can be accurately measured at all times and is available for feedback. A stabilizable time-invariant linear system can always be stabilized by a linear feedback law. Since the closed-loop poles can be located anywhere in the complex plane, the systems can be stabilized. By choosing the closed-loop poles far to the left in the complex plane, the convergence to the zero state can be made arbitrarily fast. However, increasing the closed-loop poles too far to the left requires a large control input. In any practical problems the control input cannot be increased beyond some value. These considerations lead quite naturally to formulation of an optimization problem, where one takes into account both the speed of convergence of the state to zero and the magnitude of the input amplitudes. To introduce this optimization problem for the linear systems (1.1), a quadratic cost functional (J) is formed:

$$\min_u J = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_0^{t_f} \left\{x^T(t)Q(t)x(t) + u^T(t)R(t)x(t)\right\} dt, \tag{1.2}$$

where F and Q are symmetric positive semi - definite matrices, R is symmetric positive definite, and t_f is some fixed time. As this optimization problem is a regulator problem, only the transient situation is considered where an arbitrary initial state must be reduced to the zero state. The problem formulation does not include a disturbance or a reference variable that should be tracked. These analyses lead to the following theorem:

THEOREM 1.2.1 *The optimal input for the deterministic optimal linear regulator* *is given by a linear control law:*

$$u(t) = -\Gamma(t)x(t),$$

where $\Gamma(t) = R^{-1}B^T(t)P(t)$,

which minimize the cost functional in equation (1.2). The symmetric positive semi - definite matrix P(t) satisfies the matrix Riccati equation:

$$-\dot{P}(t) = Q(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t),$$

with the terminal condition

$$P(t_f) = F.$$

REMARK 1.2.1 The weighting factor F penalizes the system at the specified terminal time t_f. The weighting factors Q and R apply penalties for trajectories of the state $x(t)$ and the control $u(t)$ over the time span $[t_0, t_f]$. It can be seen that if Q is large and R small, then optimal system will apply a large control effort and force the system to zero rapidly in order to avoid a large penalty due to large Q. On the other hand, if Q is small and R large, the control effort will be smaller, and the system will converge to zero slowly. Clearly, a wide variety of situations can be accommodated within this formulation.

1.2.1 Pontryagin's Maximum Principle

In order to find the optimal control $u(t)$ which minimizes the cost functional J subject to the constraints given by equation (1.1), the Pontryagin maximum principle is used. This principle provides the necessary conditions for optimality, resulting in a set of adjoint equations that may be treated as two-point boundary value problems. The proof of Pontryagin's principle is given by Pontryagin *et al* [45] and Fleming and Rishel [25]; it may be written in the following form.

THEOREM 1.2.2 *Pontryagin's Maximum Principle*
Consider the problem of minimizing the cost functional

$$J = \theta(x(t_f), t_f) + \int_{t_0}^{t_f} \phi(x(t), u(t), t)dt$$

subject to the constraint

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0$$

and assume that (t_0, x_0) *is fixed. Suppose, moreover, that* t_f *is free and that* $x(t_f)$ *is constrained to the terminal manifold defined by the equation*

$$N(x(t_f), t_f) = 0$$

and finally that u *is constrained to belong to some closed set* $\Omega \subseteq \mathcal{R}^m$. *Then necessary conditions for a minimum of* J *at* x^*, u^* *are given by:*

$$H(x^*(t), u^*(t), \lambda(t), t) \leq H(x(t), u(t), \lambda(t), t), \quad \forall\ t,\ u,\ x$$

and

$$\begin{array}{rclrcl} \frac{\partial H}{\partial \lambda} & = & \dot{x}, & \lambda & = & \frac{\partial \theta}{\partial x} + \left(\frac{\partial N^T}{\partial x}\right)\nu \\ \frac{\partial H}{\partial x} & = & -\dot{\lambda}, & -H & = & \frac{\partial \theta}{\partial t} + \left(\frac{\partial N^T}{\partial t}\right)\nu \end{array} \left.\vphantom{\begin{array}{c} \frac{\partial H}{\partial \lambda} \\ \frac{\partial H}{\partial x} \end{array}}\right\} at\ t\ =\ t_f$$

where H *denotes the Hamiltonian defined by*

$$H = \phi + \lambda^T f$$

and λ, ν are Lagrange multipliers.

REMARK 1.2.2 To find the optimal control, H is minimized among the admissible controls with values in Ω; strictly speaking one has therefore stated the 'minimum' principle. Lagrange multipliers λ are often called the costate (adjoint or dual) variables and the differential equation of λ is called the costate or adjoint equation.

EXAMPLE 1.2.1 For the linear regulator problem the Hamiltonian function H is given by:

$$H = \frac{1}{2}\left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right) + \lambda^T(t)\left(A(t)x(t) + B(t)u(t)\right)$$

Applying the minimum principle one can obtain the following conditions:

$$\begin{array}{lllllll}
\frac{\partial H}{\partial \lambda} & = & \dot{x} & = & A(t)x(t) & + & B(t)u(t) \\
\frac{\partial H}{\partial x} & = & -\dot{\lambda} & = & Q(t)x & + & A^T(t)\lambda(t) \\
\frac{\partial H}{\partial u} & = & 0 & \Rightarrow & Ru & + & B^T\lambda = 0
\end{array}$$

$$u(t) = -R^{-1}(t)B^T(t)\lambda(t)$$

For $\lambda = Px$ with $\lambda(t_f) = Fx(t_f)$, one can obtain the matrix Riccati differential equation

$$-\dot{P}(t) = Q(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t).$$

1.3 Linear Optimal Filtering

In the previous section it is assumed that the complete state vector can be measured accurately for the regulator problem. This assumption is often unrealistic. The most frequent situation is that for a given system

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$$

only certain combinations of the state, denoted by y, can be measured:

$$y(t) = C(t)x(t).$$

The quantity y, which is assumed to be an m- dimensional vector, with m usually less than the dimension n of state x, will be referred to as the observed or measured variable. Estimating the state vector of a linear system on the basis of observations of the output variables is treated as a vector random process due to measurement errors and stochastic disturbance of the variables.

Theoretically, it is an estimator for what is called the linear quadratic Gaussian problem, which is the problem of estimating the instantaneous "state" of a linear dynamic system perturbed by Gaussian white noise - by using measurements linearly related to the state, but corrupted by Gaussian white noise. Mathematically, the filter problem is treated by using probability theory and the theory of stochastic process in order to find optimal state vector for linear dynamical systems (linear optimal or Kalman-Bucy filter).

A historical account of the derivation of the so-called Kalman-Bucy filter is given by Brammer and Siffling [14]. The Kalman-Bucy filter can be regarded in discrete or continuous time and here the continuous version of the problem is considered. The proof of this results and further details can be found in [4, 14, 16, 19, 25].

1.3.1 Formulation of the Filter Problem

A linear system is considered in which the state vector $x(t)$ is stochastically disturbed and the output measurements $y(t)$ contain stochastic errors. The system is described by:

$$\dot{x} = A(t)x(t) + v(t)$$

$$y(t) = C(t)x(t) + w(t)$$

The state vector $x(t)$ to be estimated is n-dimensional. Let its initial value $x(t_0)$ be a random vector with zero expectation and given covariance matrix $\tilde{P}(t_0) = \tilde{P}_0$:

$$E\left\{[x(t_0) - E\{x(t_0)\}][x(t_0) - E\{x(t_0)\}]^T\right\} = \tilde{P}_0 \tag{1.3}$$

The measurement variable $y(t)$ has m components and the matrices A and C are known continuous functions of time t.

The disturbance $v(t) \in \mathcal{R}^n$ and the measurement error $w(t) \in \mathcal{R}^m$ are white noise with given covariance matrices:

$$\begin{aligned} E\{v(t)\} &\equiv 0 \quad E\{v(t)v^T(\tau)\} &= \tilde{Q}(t)\delta(t-\tau) \\ E\{w(t)\} &\equiv 0 \quad E\{w(t)w^T(\tau)\} &= \tilde{R}(t)\delta(t-\tau) \end{aligned} \tag{1.4}$$

where $\delta(t-\tau)$ is the Dirac δ function. $\tilde{Q}$ and $\tilde{R}$ are symmetric, $\tilde{Q}$ is positive semi definite, but $\tilde{R}$ must be positive definite.

Let the initial state, the disturbance process and the measurement process be mutually uncorrelated:

$$\begin{aligned} E\{x(t_0)v^T(t)\} &\equiv 0 \\ E\{x(t_0)w^T(t)\} &\equiv 0 \\ E\{v(t)w^T(\tau)\} &\equiv 0 \end{aligned} \tag{1.5}$$

A *linear unbiased* estimate of $x(t)$ based on the sample function $\mathrm{y}(\tau), \mathrm{t_0} \leq \tau \leq \mathrm{t}$ is formed. This estimate is denoted by $\hat{x}(t)$. The estimation error is defined as:

$$\tilde{x}(t) = x(t) - \hat{x}(t).$$

For unbiased estimation $E\{\tilde{x}(t)\} = 0$, its covariance matrix is defined by

$$\tilde{P}(t) = E\{\tilde{x}(t)\tilde{x}^T(t)\}.$$

The estimate must be optimal in sense that the components of the estimation error have *minimum variance.* This is equivalent to the requirement

$$\text{trace } \tilde{\mathrm{P}}(\mathrm{t}) \to \text{minimum}$$

The optimal covariance matrix $\tilde{P}(t)$ is found using the following theorem.

THEOREM 1.3.1 *Kalman-Bucy filter*

(i) The linear unbiased minimum variance estimate for the state $x(t)$ of the continuous-time stochastically disturbed system

$$\dot{x}(t) = A(t)x(t) + v(t), \quad E\{x(t_0)\} = 0$$

$$y(t) = C(t)x(t) + w(t)$$

for which the sample function $y(\tau)$, $t_0 \leq \tau \leq t$, *and a priori knowledge according to (1.3)-(1.5) are available, is given by the differential equation:*

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)\{y(t) - C(t)\hat{x}(t)\}, \quad \hat{x}(t_0) = 0. \tag{1.6}$$

(ii) The gain matrix is given by

$$K(t) = \tilde{P}(t)C^T(t)\tilde{R}^{-1}(t). \tag{1.7}$$

(iii) The covariance matrix of the estimation error obeys the matrix Riccati differential equation

$$\begin{aligned} \dot{\tilde{P}}(t) &= \tilde{Q}(t) + A(t)\tilde{P}(t) + \tilde{P}(t)A^T(t) - \tilde{P}(t)C^T(t)\tilde{R}^{-1}(t)C(t)\tilde{P}, \quad (1.8) \\ \tilde{P}(t_0) &= \tilde{P}_0. \end{aligned}$$

$\tilde{P}(t)$ *is positive semidefinite and symmetric matrix.*

1.3.2 Extended Kalman Filter

The Extended Kalman filter is similar to a linearized Kalman filter except that the linearization takes place about the filter's estimated trajectory rather than a the pre-computed trajectory.

Consider the equation

$$\dot{x}(t) = f(x, t) + v(t),$$

together with the measurement

$$y(t) = h(x,t) + w(t),$$

and suppose that $E\{v(t)v^T(t)\} = \tilde{Q}(t)$ and $E\{w(t)w^T(t)\} = \tilde{R}(t)$. Expand f and h in Taylor series to first order. Then,

$$\begin{aligned} f(x,t) &= f(x_1,t) + A(x_1,t)(x-x_1) \quad \text{and} \\ h(x,t) &= h(x_1,t) + C(x_1,t)(x-x_1) \end{aligned}$$

for some nominal trajectory $x_1(t)$. Then if

$$\begin{aligned} r(t) &\triangleq f(x_1,t) - A(x_1,t)x_1(t) \\ s(t) &\triangleq h(x_1,t) - C(x_1,t)x_1(t) \end{aligned}$$

the equations become

$$\begin{aligned} \dot{x}(t) &= A(x_1,t)x(t) + r(t) + v(t) \\ y(t) &= C(x_1,t)x(t) + s(t) + w(t). \end{aligned}$$

These equations are now linear and one may apply the Kalman filter equations to obtain the estimation equations:

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A(x_1,t)\hat{x}(t) + r(t) \\ &+ \tilde{P}(t)C^T(x_1,t)\tilde{R}^{-1}(t)\left[y(t) - s(t) - C(x_1,t)\hat{x}(t)\right] \\ \dot{\tilde{P}}(t) &= \tilde{Q}(t) + \tilde{P}(t)A^T(x_1,t) + A(x_1,t)\tilde{P}(t) \\ &- \tilde{P}(t)C^T(x_1,t)\tilde{R}^{-1}(t)C(x_1,t)\tilde{P}(t) \end{aligned}\right\} \tag{1.9}$$

The linearized estimation equations (1.9) with $x_1 = \hat{x}$ constitute the *extended Kalman* filter.

1.3.3 Separation Principle

Stochastic separation is of central importance for the optimal synthesis of linear stochastically disturbed control loops. This class of problem is largely solved by decomposition into deterministic linear control and stochastic linear filtering which both have a fully developed theory.

In continuous time, the linear stochastic control problem is formulated as follows:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + v(t), \\ y(t) &= C(t)x(t) + w(t), \end{aligned} \tag{1.10}$$

where $x(t_0)$ is normally distributed random vector with $E\{x(t_0)\} = \xi$, $v(t)$ and $w(t)$ are vector valued Gaussian random processes. Estimation error is defined by:

$$\tilde{x}(t) = x(t) - \hat{x}(t). \tag{1.11}$$

Then by substitution of (1.10) into (1.11), one may obtain

$$\dot{x}(t) = A(t)x(t) - B(t)\Gamma(t)\{x(t) - \tilde{x}(t)\} + v(t),$$

since $u(t) = -\Gamma(t)\hat{x}(t)$, $\Gamma(t) = R^{-1}(t)B^T(t)P(t)$ and optimal feedback gain $\Gamma(t)$ is obtained by solving the deterministic optimal control problem.

From equation (1.11) and (1.6) :

$$\dot{\tilde{x}}(t) = \{A(t) - K(t)C(t)\}\tilde{x}(t) + v(t) - K(t)w(t)$$

Writing these equations as a combined differential equation yields

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - B\Gamma & B\Gamma \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} v + \begin{bmatrix} 0 \\ -K \end{bmatrix} w \tag{1.12}$$

REMARK 1.3.1 Equation (1.12) is known as the *separation theorem or separation principle.* Here as well the dynamics of the error $\tilde{x}(t)$ is completely decoupled from the dynamics of the state $x(t)$: $\tilde{x}$ is influenced neither by x nor by Γ. The characteristic motions of x, conversely, are only determined by A, B and Γ, while $B\Gamma\tilde{x}$ and v merely act as external excitations. These considerations motivate the following theorem.

THEOREM 1.3.2 *(i) The linear unbiased minimum variance estimate for the state $x(t)$ of the system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + v(t) \quad E\{x(t_0)\} = \xi \\ y(t) &= C(t)x(t) + w(t) \end{aligned}$$

for which the sample function $y(\tau)$ and the input variable $u(\tau)$, $t_0 \le \tau \le t$, and a priori knowledge according to (1.3)-(1.5) are available, is determined by the differential equation

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)\{y(t) - C(t)\hat{x}(t)\} \quad \hat{x}(t_0) = \xi$$

(ii) the gain matrix $K(t)$ is given by equation (1.7) and the covariance matrix of the estimation error obeys the matrix Riccati differential equation (1.8).

1.4 Functional Analysis

In this section some elementary functional analysis which will be needed later is presented. This will include subjects like, normed vector space, Banach spaces and measure of matrix. More details of this material can be found in [31, 43].

1.4.1 Normed Vector Space

A vector space is a purely algebraic object, and if the processes of analysis are to be meaningful in it, a measure of distance must be supplied; the distance is known as normed in this context. Intuitively one expects distance to be a non-negative real number, to be symmetric, and to satisfy the triangle inequality. These consideration motive the following definition.

DEFINITION 1.4.1 Let $\mathcal{X}$ be a set. Associate with each pair of elements $f, g \in \mathcal{X}$ a non-negative number $d(f,g)$ such that for all $f, g, h \in \mathcal{X}$:

$$\begin{array}{llcll} (i) & d(f,g) & = & 0 & iff^{1}\ \ f=g \\ (ii) & d(f,g) & = & d(g,f) & (symmetry) \\ (iii) & d(f,g) & \leq & d(f,h)+d(h,g) & (triangle\ inequality) \end{array}$$

d is called a *metric* on $\mathcal{X}$, and $\mathcal{X}$ equipped with a metric is known as a *metric space.*

[1] if and only if

DEFINITION 1.4.2 Let $\mathcal{V}$ be a vector space, and suppose that to each element $f \in \mathcal{V}$ a non-negative number $\|f\|$ is assigned in such a way that for all $f, g \in \mathcal{V}$

$$\begin{array}{llll} (i) & \|f\| & = 0 & iff \;\; f = 0 \\ (ii) & \|\alpha f\| & = |\alpha|\|f\| & for\ any\ scalar\ \alpha \\ (iii) & \|f+g\| & \leq \|f\| + \|g\| & (the\ triangle\ inequality) \end{array}$$

The quantity $\|f\|$ is called the *norm* of f, and $\mathcal{V}$ is known as a *normed vector space.* A metric d is obviously obtained on setting $d(f,g) = \|f-g\|$, so all normed vector spaces are metric spaces. A vector space may often be equipped with more than one norm, the associated normed vector spaces are regarded as different unless the norms are same.

DEFINITION 1.4.3 Assume that a vector $f \in \mathcal{V}$ and a number r with $0 < r < \infty$ are given. The sets of points $\mathcal{S}(f,r) = \{g : \|f-g\| < r\}$ and $\tilde{\mathcal{S}}(f,r) = \{g : \|f-g\| \leq r\}$ are called respectively the *open* and *closed* balls with centre f and radius r.

DEFINITION 1.4.4 A subset $\mathcal{S}$ of $\mathcal{V}$ is said to be bounded iff it is contained in some ball (of finite radius). If $\mathcal{S}$ is bounded, its diameter is the diameter of the closed ball of smallest radius containing $\mathcal{S}$. The distance, written $dist(f,\mathcal{S})$, of a point f from $\mathcal{S}$ is the number $\inf_{g\in\mathcal{S}} \|f-g\|$.[2]

EXAMPLE 1.4.1 Let $\mathcal{V} = \mathcal{R}^n$. Define the Euclidean norm of $f = (f_1, \ldots, f_n)$ to be

$$\|f\|_2 = \left\{\sum_{i=1}^{n} f_i^2\right\}^{\frac{1}{2}}$$

[2]Inf and sup (abbreviations for infimum and supremum) denote the greatest lower bound and least upper bound respectively.

If $n = 3$ the norm is of course just the usual distance in $\mathcal{R}^3$.

DEFINITION 1.4.5 Let (f_n) be a sequence in $\mathcal{V}$. The sequence is said to *converge* iff there is a vector $f \in \mathcal{V}$ such that $\lim \|f_n - f\| = 0$. f is called the *limit* of the sequence, and it may be written as $f_n \to f$ or $\lim f_n = f$. The limit is unique, for if $f_n \to f$ and $f_n \to g$, by triangle inequality

$$\|f - g\| = \|f - f_n + f_n - g\| \leq \|f - f_n\| + \|f_n - g\|,$$

and the right-handed side tends to zero as $n \to \infty$, whence $f = g$.

DEFINITION 1.4.6 Let Ω be a subset of $\mathcal{R}^n$, and suppose that f is a complex valued function defined on Ω. Then f is said to be continuous at the point $x_0 \in \Omega$
(i) for each $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in \Omega$ and $|x - x_0| < \delta$;
(ii) for each sequence (x_n) in Ω with limit x_0, $\lim f(x_n) = f(x_0)$. f is said to be *continuous* iff it is continuous at every point of Ω;
(iii) f is said to be *uniformly continuous* on Ω iff for each $\varepsilon > 0$ there exit a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x, x_0 \in \Omega$ and $|x - x_0| < \delta$, i.e. ε, δ are independent of x_0.

DEFINITION 1.4.7 Let Ω be a subset of $\mathcal{R}^n$. The vector space of bounded continuous complex valued functions defined on Ω is denoted by $\mathcal{E}(\Omega)$.
(i) The space $\mathcal{E}(\Omega)$ may be normed in a number of ways. Set first

$$\|f\| = \sup_{x \in \Omega} |f(x)|$$

Then $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for each x, and it follows that $\|f + g\| \leq \|f\| + \|g\|$. Evidently therefore $\|\cdot\|$ is a norm on $\mathcal{E}(\Omega)$.

(ii) Let f denote $\mathcal{C}^m$ (m-dimensional complex space) valued function $f_1, \ldots, f_m$ where for each j, f_j is a bounded continuous complex valued function defined on a subset Ω of $\mathcal{R}^n$. The set of such functions with the laws of combination

$$\begin{aligned}(f+g)(x) &= (f_1(x)+g_1(x), \ldots, f_m(x)+g_m(x)), \\ (\alpha f)(x) &= (\alpha f_1(x), \ldots, \alpha f_m(x)) \quad (\alpha \in \mathcal{C}),\end{aligned}$$

is a vector space, which will be denoted by $\mathcal{E}(\Omega, \mathcal{C}^m)$; $\mathcal{E}(\Omega, \mathcal{R}^m)$ of course denotes the corresponding real space of $\mathcal{R}^m$ valued functions. The sup norm is defined by

$$\|f\| = \max_{1 \le j \le m} \sup_{x \in \Omega} |f_j(x)|$$

1.4.2 Banach Space

A set $\mathcal{S}$ in a normed vector space $\mathcal{V}$ is said to be complete iff each Cauchy sequence in $\mathcal{S}$ converges to a point of $\mathcal{S}$. $\mathcal{V}$ itself is known as a complete normed vector space or a *Banach space* $\mathcal{B}$ iff it is complete. The simplest Banach spaces are $\mathcal{R}^n$ and $\mathcal{C}^n$ - with any norm.

DEFINITION 1.4.8 Let $\mathcal{V}$ be a normed vector space. A sequence (f_n) in $\mathcal{V}$ is said to be *Cauchy* iff

$$\lim_{m,n \to \infty} \|f_n - f_m\| = 0,$$

that is, iff for each $\varepsilon > 0$ there is an n_0 such that $\|f_n - f_m\| < \varepsilon$ whenever $m, n > n_0$. It is an obvious consequence of the inequality

$$\|f_n - f_m\| = \|f_n - f + f - f_m\| \le \|f_n - f\| + \|f_m - f\|,$$

that if (f_n) is convergent it is Cauchy.

DEFINITION 1.4.9 *Lipschitz Condition*

Let $\mathcal{B}$ be Banach spaces and let $\mathcal{D}$ be a subset of $\mathcal{B}$. Operator $\mathcal{A}$ is said to satisfy a Lipschitz condition on $\mathcal{D}$ with Lipschitz constant q iff there is a $q < \infty$ such that

$$\|\mathcal{A}f - \mathcal{A}g\| \leq q\|f - g\| \quad (f, g \in \mathcal{D}).$$

In one dimension a function which satisfies a Lipschitz condition is absolutely continuous, and hence differential almost everywhere. It is convenient to have the following terminology available when $\mathcal{D}$ is unbounded.

(i) $\mathcal{A}$ will be said to satisfy a *Local Lipschitz condition* iff for each bounded $\mathcal{S} \subset \mathcal{D}$, $\mathcal{A}$ satisfies a Lipschitz condition on $\mathcal{S}$ with Lipschitz constant q_s (which may depend on $\mathcal{S}$).

(ii) $\mathcal{A}$ will be called a *contraction* iff it satisfies a Lipschitz condition with Lipschitz constant$q < 1$.

1.4.3 Measure of a matrix

The measure $\mu(A)$ of a matrix A is a mapping from $\mathcal{R}^{n \times n}$ into $\mathcal{R}$ [15, 21, 22], which is in some ways analogous to a norm but which leads to a sharper estimate of convergence than norms, principally because $\mu(A)$ may take negative values.

DEFINITION 1.4.10 Let A be a $n \times n$ matrix and $\|\cdot\|$ be an induced norm for matrices. The measure $\mu(A)$ of the matrix A is defined by

$$\mu(A) = \lim_{h \to 0^+} = \frac{\|I + hA\| - 1}{h}.$$

One can think of $\mu(A)$ as being the one-sided directional derivative of the mapping $\|\cdot\|$: $\mathcal{R}^{n \times n} \to \mathcal{R}_+$ at the point I (the $n \times n$ identity matrix), in the direction of A.

LEMMA 1.4.1 *Brauer [15]*
Every solution $x(t)$ of the linear system

$$\dot{x}(t) = A(t)x(t)$$

obeys

$$\|x(t)\| \leq \|x(t_0)\| \exp\left(\int_{t_0}^{t} \sigma(A(s))ds\right), \quad [t \geq t_0],$$

where $\sigma(A)$ denotes the largest eigenvalue of the symmetric matrix $\frac{1}{2}(A + A^T)$. Also, $\sigma(A)$ can be calculated from the relation

$$\sigma(A) = \sup_x \frac{x^T A x}{x^T x}.$$

1.4.4 Gronwall Inequality

One of the most useful results used in differential equations or control engineering is the Gronwall inequality. It is stated as follows:

THEOREM 1.4.1 *Gronwall Inequality*
Let K be a non-negative constant and let f and g be continuous non-negative functions on some interval $t_0 \leq t \leq t_f$ satisfying the inequality

$$f(t) \leq K + \int_{t_0}^{t} f(s)g(s)ds \quad \textit{for } t_0 \leq t \leq t_f. \qquad \text{Then}$$

$$f(t) \leq K \exp\left(\int_{t_0}^{t} g(s)ds\right) \quad \textit{for } t_0 \leq t \leq t_f.$$

Chapter 2

Optimal Control of Nonlinear Systems

2.1 Introduction

The optimal control of general nonlinear systems of the form

$$\dot{x} = f(x, u)$$

with the quadratic cost function

$$min\ J = x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^T Q x + u^T R u)dt$$

can be solved, in principle, by the use of Lie series and infinite dimensional systems theory [3, 6, 10]. However, the solution is complex and difficult to implement. For this reason we consider nonlinear systems of the form

$$\dot{x} = A(x)x + B(x)u$$

together with a quadratic cost and introduce a sequence of (time-varying) linear-quadratic approximations to the nonlinear problem. Similar methods have been applied to bilinear systems and distributed parameter systems [1, 8, 30], see also for a related freezing technique [9].

In section 2.2, the approximating sequence to the nonlinear systems is introduced, and in section 2.3 the proof of convergence of this sequence in an appropriate space is given.

2.2 The Approximating Sequence

Consider the nonlinear systems

$$\dot{x} = A(x)x + B(x)u, \; x(0) = x_0, \tag{2.1}$$

together with the quadratic cost function

$$J = x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^TQx + u^TRu)dt. \tag{2.2}$$

Here, x and u are vector functions with n, m components respectively. A is an $n \times n$ matrix function and B is an $n \times m$ matrix function; Q and F are real symmetric positive semi-definite matrices, R is a real symmetric positive definite matrix and t_f is some fixed time.

We introduce the following sequence of approximations to the problem of minimizing the cost (2.2) subject to the dynamics (2.1):

$$\begin{aligned} \dot{x}^{[0]} &= A(x_0)x^{[0]} + B(x_0)u^{[0]}, \; x^{[0]}(0) = x_0 \\ J^{[0]} &= x^{[0]^T}(t_f)Fx^{[0]}(t_f) + \int_0^{t_f} (x^{[0]^T}Qx^{[0]} + u^{[0]^T}Ru^{[0]})dt \end{aligned}$$

and for $k \geq 1$,

$$\begin{aligned}
\dot{x}^{[k]} &= A(x^{[k-1]}(t))x^{[k]} + B(x^{[k-1]}(t))u^{[k]}, \; x^{[k]}(0) = x_0 \\
J^{[k]} &= x^{[k]^T}(t_f)Fx^{[k]}(t_f) + \int_0^{t_f} (x^{[k]^T}Qx^{[k]} + u^{[k]^T}Ru^{[k]})dt
\end{aligned} \tag{2.3}$$

Since each approximating problem in (2.3) is linear (time-varying), quadratic we can write the optimal control in the form

$$u^{[k]} = -R^{-1}B^T(x^{[k-1]}(t))P^{[k]}x^{[k]}(t)$$

where $P^{[k]}$ is the solution of the usual Riccati equation

$$\begin{aligned}
\dot{P}^{[k]}(t) &= -Q - P^{[k]}A(x^{[k-1]}(t)) - A^T(x^{[k-1]}(t))P^{[k]} \\
&+ P^{[k]}B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))P^{[k]} \\
P^{[k]}(t_f) &= F
\end{aligned} \tag{2.4}$$

and the kth dynamical system becomes

$$\dot{x}^{[k]} = A(x^{[k-1]}(t))x^{[k]}(t) - B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))P^{[k]}x^{[k]}(t) \tag{2.5}$$

2.3 Proof of Convergence

In this section we shall prove that the sequence of coupled equations (2.4) and (2.5) converge under certain conditions on $A(x)$ and $B(x)$ and for small enough horizon time t_f (or small enough initial conditions x_0). To do this we first note that if $\Phi^{[i-1]}(t, t_0)$ denotes the transition matrix generated by $A(x^{[i-1]}(t))$ then [15], we have

$$\left\|\Phi^{[i-1]}(t, t_0)\right\| \leq exp\left[\int_{t_0}^{t} \mu(A(x^{[i-1]}(\tau))d\tau\right]$$

where $\mu(A)$ is the logarithmic norm of A. We next require an estimate for $\Phi^{[i-1]} - \Phi^{[i-2]}$.

LEMMA 2.3.1 *Suppose that $\mu(A(x)) \leq \mu$ for some constant μ and for all x that*

$$\|A(x) - A(y)\| \leq \alpha \|x - y\|, \ \forall \ x, \ y \in R^n$$

Then

$$\left\|\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)\right\| \leq \alpha e^{\mu(t-t_0)}(t - t_0) \sup_{s\in[t_0,t]} \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|$$

Proof: $\Phi^{[i-1]}$, $\Phi^{[i-2]}$ are solutions of the respective equations

$$\dot{z} = A(x^{[i-1]}(t))z, \ z(t_0) = I$$

$$\dot{w} = A(x^{[i-2]}(t))w, \ w(t_0) = I$$

Hence,

$$\frac{d}{dt}(z - w) = A(x^{[i-1]}(t))(z - w) + \left[A(x^{[i-1]}(t)) - A(x^{[i-2]}(t))\right] w$$

and so

$$z - w = \int_{t_0}^{t} \Phi^{[i-1]}(t, s) \left[A(x^{[i-1]}(s)) - A(x^{[i-2]}(s))\right] w(s) ds$$

i.e.

$$\begin{aligned} \|z - w\| \ &\leq \ \int_{t_0}^{t} exp\left(\int_{s}^{t} \mu(A(x^{[i-1]}(\tau)))d\tau\right) exp\left(\int_{t_0}^{s} \mu(A(x^{[i-2]}(\tau)))d\tau\right) \\ &\times \ \alpha \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\| ds \\ &\leq \ \alpha e^{\mu(t-t_0)}(t - t_0) \sup_{s\in[t_0,t]} \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|. \qquad \square \end{aligned}$$

We first consider the more general equation

$$\begin{aligned}\dot{x}^{[0]}(t) &= A(x_0)x^{[0]}(t) + C(x_0)x^{[0]}(t), \quad x^{[0]}(0) = x_0 \\ \dot{x}^{[i]}(t) &= A(x^{[i-1]})x^{[i]}(t) + C(x^{[i-1]})x^{[i]}(t), \quad x^{[i]}(0) = x_0, \quad i \geq 1 \qquad (2.6)\end{aligned}$$

In order to prove the sequence $x^{[i]}(t)$ converges to a solution, one need to show that $\|x^{[i]}(t) - x^{[i-1]}(t)\| \to 0$ as $i \to \infty$. It follows that

$$\begin{aligned}x^{[i]}(t) - x^{[i-1]}(t) &= \left[\Phi^{[i-1]}(t,0) - \Phi^{[i-2]}(t,0)\right] x_0 \\ &+ \int_0^t \Phi^{[i-1]}(t,s)C(x^{[i-1]}(s))\left[x^{[i]}(s) - x^{[i-1]}(s)\right] ds \\ &+ \int_0^t \Phi^{[i-1]}(t,s)\left[C(x^{[i-1]}(s)) - C(x^{[i-2]}(s))\right] x^{[i-1]}(s)ds \\ &+ \int_0^t \left[\Phi^{[i-1]}(t,s) - \Phi^{[i-2]}(t,s)\right] C(x^{[i-2]}(s))x^{[i-1]}(s)ds\end{aligned}$$

For uniform convergence, the following conditions are assumed.

$$\begin{array}{llcll} \overline{(A1)} & \mu(A(x)) & \leq & \mu & \forall\, x \in \mathcal{R}^n \\ \overline{(A2)} & \|A(x) - A(y)\| & \leq & \alpha\, \|x-y\| & \forall\, x,\ y \in \mathcal{R}^n \\ \overline{(A3)} & \|C(x) - C(y)\| & \leq & \beta\, \|x-y\| & \forall\, x,\ y \in \mathcal{R}^n \\ \overline{(A4)} & \|C(x)\| & \leq & \gamma & \forall\, x \in \mathcal{R}^n \end{array}$$

Also, $\xi^{[i]}(t)$ is defined as

$$\xi^{[i]}(t) = \sup_{s\in[0,t]} \|x^{[i]}(s) - x^{[i-1]}(s)\|$$

thus,

$$\begin{aligned}\xi^{[i]}(t) &\leq \alpha\, te^{\mu t}\xi^{[i-1]}(t)\|x_0\| \\ &+ \int_0^t e^{\mu(t-s)}\gamma\xi^{[i]}(s)ds + \int_0^t e^{\mu(t-s)}\beta\xi^{[i-1]}(s)e^{(\mu+\gamma)s}\|x_0\|ds \\ &+ \int_0^t \alpha te^{\mu(t-s)}\xi^{[i-1]}(s)\gamma e^{(\mu+\gamma)s}\|x_0\|ds\end{aligned}$$

since

$$\|x^{[i]}(t)\| \leq e^{(\mu+\gamma)t}\|x_0\|$$

from (2.6). Hence, if $t \in [0, T]$

$$\left(1 - \gamma \int_0^t e^{\mu(t-s)} ds\right) \xi^{[i]}(t) \leq \alpha \; T e^{\mu t} \xi^{[i-1]}(t)\|x_0\|$$
$$+\xi^{[i-1]}(t)\left\{\int_0^t e^{\mu(t-s)} \beta e^{(\mu+\gamma)s}\|x_0\| ds + \int_0^t \alpha T \; e^{\mu(t-s)} \gamma e^{(\mu+\gamma)s}\|x_0\| ds\right\}$$

The above equation may be written as

$$\xi^{[i]}(t) \leq \tilde{\lambda}(t)\xi^{[i-1]}(t)$$

where

$$\tilde{\lambda}(t) = \frac{\frac{\beta}{\gamma}\left[e^{(\mu+\gamma)t} - e^{\mu t}\right]\|x_0\| + \alpha T e^{(\mu+\gamma)t}\|x_0\|}{\left[1 + \frac{\gamma}{\mu}(1 - e^{\mu t})\right]}$$

and so, if $|\tilde{\lambda}(t)| < 1$ for some small x_0 and small $t \in [0, T]$ and $i \to \infty$ it follows

$$x^{[i]}(t) \to x(t) \text{ on } \mathrm{C}\ (\ [0, \mathrm{T}],\ \mathcal{R}^{\mathrm{n}}\ )$$

Moreover, it can easily be checked that

$$\sup_{t\in[0,T]} \|x(t)\| \leq \frac{1}{1-\nu}\xi^{[2]}(t) + \sup_{t\in[0,T]} \|x^{[1]}(t)\| \tag{2.7}$$

where

$$\nu = \sup_{t\in[0,T]} |\tilde{\lambda}(t)|$$

LEMMA 2.3.2 *Under the assumptions $\overline{A1} - \overline{A4}$ we have*

$$\|x^{[1]}(t)\| \leq e^{(\bar{\mu}+\gamma)t}\|x_0\|$$

and

$$\begin{aligned}\|x^{[2]}(t) - x^{[1]}(t)\| &\leq \left(e^{(\mu+\gamma)t} + e^{(\bar{\mu}+\gamma)t}\right)\|x_0\| \\ &\leq 2e^{(\mu+\gamma)t}\|x_0\|\end{aligned}$$

where $\bar{\mu} = \mu(A(x^{[0]}(t)))$. Since $\bar{\mu} \leq \mu$

Proof: This follows from Gronwall's lemma and inequality

$$\|x^{[2]}(t)\| \leq e^{(\mu+\gamma)t}\|x_0\|. \quad \square$$

COROLLARY 2.3.1 *Under the assumptions $\overline{A1} - \overline{A4}$ we have*

$$\sup_{t\in[0,T]} \|x(t)\| \leq \left(\frac{2}{1-\nu} + 1\right) e^{(\mu+\gamma)t}\|x_0\|$$

Proof: This follows directly from the above results. $\square$

REMARK 2.3.1 *If $\mu + \gamma < 0$, then from corollary (2.3.1), it follows*

$$\sup_{t\in[0,T]} \|x(t)\| \leq \left(\frac{2}{1-\nu} + 1\right)\|x_0\|.$$

Hence, although the conditions $\overline{A1} - \overline{A4}$ for all $x \in \mathcal{R}^n$, it is clear that only these conditions are required to hold in the ball

$$B_{x_0} = \left\{x : \|x\| \leq \left(\frac{2}{1-\nu} + 1\right)\|x_0\|\right\}.$$

This means that the results apply to polynomial systems which, of course, do not satisfy $\overline{A1} - \overline{A4}$ on the whole of $\mathcal{R}^n$.

In the case of the controlled sequence (2.5) we have

$$C(x^{[i-1]}(t)) = -B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t))P^{[i]}(t)$$

and so

$$\|C(x^{[i-1]}(t))\| \le \|B(x^{[i-1]}(t))\|^2 \; \|R^{-1}\| \; \|P^{[i]}(t)\|.$$

The following lemma provides a bound on $\|P^{[i]}(t)\|$ directly from (2.4) (and a bound on $\|P^{[i]}(t) - P^{[i-1]}(t)\|$ follows in a similar way).

LEMMA 2.3.3 *Let $\bar{P}^{[i]}(t) = P^{[i]}(t_f - t)$, satisfying the Riccati equation (2.4), and $\rho(t) = \|\bar{P}^{[i]}(t)\|$, such that $\bar{P}^{[i]}(0) = F$, then*

$$\dot{\rho}(t) \le c_1 + c_2\rho + c_3\rho^2, \quad \rho(0) = \|F\|$$

for some constants $c_1, c_2, c_3 > 0$. Hence, for some $\tau \in [0, t_f)$, let $\rho(t) \le \|F\| + \varepsilon$ for $t \in [\tau, t_f]$ for any $\varepsilon > 0$, and so on $[\tau, t_f]$,

$$\dot{\rho}(t) \le c_1 + (c_2 + c_3(\|F\| + \varepsilon))\,\rho$$

so that

$$\begin{aligned} \rho \quad &\le \quad e^{(c_2+c_3(\|F\|+\varepsilon))t}\|F\| \quad + \quad c_1 \int_0^t e^{(c_2+c_3(\|F\|+\varepsilon))(t-s)}ds \\ \|P^{[i]}(t)\| \quad &\le \quad e^{(c_2+c_3(\|F\|+\varepsilon))(t_f-t)}\|F\| \quad + \quad c_1 \int_t^{t_f} e^{(c_2+c_3(\|F\|+\varepsilon))(t_f-s)}ds \end{aligned}$$

and so for small enough t_f (or small c_2, c_3) it follows that $P^{[i]}(t)$ is bounded on $[\,0, t_f]$.

Proof: For any sequence $\bar{P}^{[i]}(t)$,

$$\begin{aligned}\dot{\bar{P}}^{[i]}(t) &= Q + \bar{P}^{[i]}(t)A(x^{[i-1]}(t)) + A^T(x^{[i-1]}(t))\bar{P}^{[i]}(t) \\ &- \bar{P}^{[i]}(t)B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t))\bar{P}^{[i]}(t)\end{aligned}$$

$$\bar{P}^{[i]}(0) = F$$

Integrate the above equation from $[0, t]$,

$$\begin{aligned}\bar{P}^{[i]}(t) &= \Phi^{[i-1]^T}(t,0)\bar{P}^{[i]}(0)\Phi^{[i-1]}(t,0) + \int_0^t \Phi^{[i-1]^T}(t,s)Q\Phi^{[i-1]}(t,s)ds \\ &+ \int_0^t \Phi^{[i-1]^T}(t,s)\bar{P}^{[i]}(s)B(x^{[i-1]}(s))R^{-1}B^T(x^{[i-1]}(s))\bar{P}^{[i]}(s)ds\end{aligned}$$

Taking the norm both side of the equation, we will obtain

$$\begin{aligned}\|\bar{P}^{[i]}(t)\| &\leq \|\Phi^{[i-1]}(t,0)\|^2\ \|\bar{P}^{[i]}(0)\| + \int_0^t \|\Phi^{[i-1]}(t,s)\|^2\ \|Q\|ds \\ &+ \int_0^t \|\Phi^{[i-1]}(t,s)\|^2\ \|\bar{P}^{[i]}(t)\|^2\ \|B(x^{[i-1]}(s))\|^2\ \|R^{-1}\|ds\end{aligned}$$

If we assume the condition $\|Q\| = q,\ \ \|R^{-1}\| = r$ and $\|B(x)\| \leq b,\ \forall\ x\ \in\ \mathcal{R}^n$, using the assumption $\overline{A1}$, we have

$$\rho(t) \leq e^{2\mu t}\ \|F\| + \int_0^t \left[q + rb^2\rho(t)^2\right] e^{2\mu(t-s)}ds.$$

Differentiate the above equation, then we have

$$\dot{\rho}(t) \leq 2\mu\rho(t) + q + rb^2\rho(t)^2.$$

Which may be written as

$$\dot{\rho}(t) \leq c_1 + c_2\rho + c_3\rho^2,$$

where $c_1 = q$, $c_2 = 2\mu$, and $c_3 = rb^2$. It is a nonlinear equation in $\rho(t)$. So if the solution $\rho(t) < \|F\| + \varepsilon$ for small t (or small c_2, c_3), then

$$
\begin{aligned}
\dot{\rho}(t) &\leq c_1 + (c_2 + c_3(\|F\| + \varepsilon))\,\rho \\
\rho(t) &\leq e^{(c_2+c_3(\|F\|+\varepsilon))t}\,\|F\| + c_1 \int_0^t e^{(c_2+c_3(\|F\|+\varepsilon))(t-s)} ds \\
\|P^{[i]}(t)\| &\leq e^{(c_2+c_3(\|F\|+\varepsilon))(t_f-t)}\|F\| + c_1 \int_t^{t_f} e^{(c_2+c_3(\|F\|+\varepsilon))(t_f-s)} ds \quad \square
\end{aligned}
$$

From lemma (2.3.3), it is shown that $P^{[i]}(t)$ is bounded by small interval time $t \in [0, t_f]$ (or small c_2, c_3). Hence, one should determine within this interval whether $P^{[i]}(t)$ is satisfying the Lipschitz condition. To this end, the following lemma explain that $P^{[i]}(t)$ is satisfying the Lipschitz condition on $[0, t_f]$.

LEMMA 2.3.4 *Let matrix $P^{[i]}(t)$ is satisfying the Riccati equation of (2.4). Such that $P^{[i]}(t) = Y^{[i]}(t)X^{[i]^{-1}}(t)$, $Y^{[i]}(t_f) = F$ and $X^{[i]}(t_f) = I$. Then,*

$$
\begin{aligned}
(i) \quad \dot{X}^{[i]}(t) &= A(x^{[i-1]}(t))X^{[i]}(t) - B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t))Y^{[i]}(t) \\
(ii) \quad \dot{Y}^{[i]}(t) &= -QX^{[i]}(t) - A^T(x^{[i-1]}(t))Y^{[i]}(t),
\end{aligned}
$$

(iii) $P^{[i]}(t)$ satisfies the Lipschitz condition; provided that $A(x^{[i-1]}(t))$ and $B(x^{[i-1]}(t))$ satisfies the Lipschitz condition.

Proof:

$$
\begin{aligned}
P^{[i]}(t) &= Y^{[i]}(t)X^{[i]^{-1}}(t),\ Y^{[i]}(t_f) = F,\ X^{[i]}(t_f) = I. \\
\dot{P}^{[i]}(t) &= \dot{Y}^{[i]}(t)X^{[i]^{-1}}(t) - Y^{[i]}(t)X^{[i]^{-1}}(t)\dot{X}^{[i]}(t)X^{[i]^{-1}}(t) \qquad (2.8)
\end{aligned}
$$

The equation (2.8) is satisfying Riccati equation (2.4). Hence,

$$
\begin{aligned}
&- (\dot{Y}^{[i]}(t)X^{[i]^{-1}}(t) - Y^{[i]}(t)X^{[i]^{-1}}(t)\dot{X}^{[i]}(t)X^{[i]^{-1}}(t) = Q + P^{[i]}(t)A(x^{[i-1]}(t)) \\
&+ A^T(x^{[i-1]}(t))P^{[i]}(t) - P^{[i]}(t)B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t))P^{[i]}(t)
\end{aligned}
$$

The above equation may be written as

$$-(\dot{Y}^{[i]} - Y^{[i]}X^{[i]^{-1}}\dot{X}^{[i]} = QX^{[i]} + Y^{[i]}X^{[i]^{-1}}A(x^{[i-1]}(t))X^{[i]^{-1}}$$
$$+A^T(x^{[i-1]}(t))Y^{[i]} - Y^{[i]}X^{[i]^{-1}}B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t))Y^{[i]}$$

The above equation is coupled with $\dot{X}^{[i]}(t)$ and $\dot{Y}^{[i]}(t)$. So if they are separated from the equation, thus

$$\begin{aligned}
\dot{X}^{[i]}(t) &= A(x^{[i-1]}(t))X^{[i]}(t) - B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t))Y^{[i]}(t),\\
\dot{Y}^{[i]}(t) &= -QX^{[i]}(t) - A^T(x^{[i-1]}(t))Y^{[i]}(t).
\end{aligned}$$

$X^{[i]}(t)$ and $Y^{[i]}(t)$ are differentiable, therefore, they are satisfying the Lipschitz condition. Now, we want to show that $X^{[i]^{-1}}(t)$ satisfies the Lipschitz condition. It follows

$$\begin{aligned}
\|X^{[i]^{-1}}(t_1) - X^{[i]^{-1}}(t_2)\| &= \left\|X^{[i]^{-1}}(t_1)\left(I - X^{[i]}(t_1)X^{[i]^{-1}}(t_2)\right)\right\|\\
&= \left\|X^{[i]^{-1}}(t_1)\left(X^{[i]}(t_2) - X^{[i]}(t_1)\right)X^{[i]^{-1}}(t_2)\right\|\\
&\leq \|X^{[i]^{-1}}(t_1)\|\ \|X^{[i]^{-1}}(t_2)\|\ \|X^{[i]}(t_1) - X^{[i]}(t_2)\|\\
&\leq \bar{\gamma}_1\ \|t_1 - t_2\| \qquad (2.9)
\end{aligned}$$

where $\bar{\gamma}_1$ $(= \gamma_1\|X^{[i]^{-1}}(t)\|^2)$ is the Lipschitz constant for $X^{[i]^{-1}}(t)$ and γ_1 is the Lipschitz constant for $X^{[i]}(t)$. From the equation (2.9), we can say $X^{[i]^{-1}}(t)$ also satisfies the Lipschitz condition. Since $Y^{[i]}(t)$ and $X^{[i]^{-1}}(t)$ satisfy the Lipschitz condition, $P^{[i]}(t)$ also satisfies this condition $\square$.

REMARK 2.3.2 *$P^{[i]}(t) = P^{[i]}(t, x^{[i-1]}(t))$, therefore*

$$\begin{aligned}
\|P^{[i]}(t) - P^{[i-1]}(t)\| &= \|P^{[i]}(t, x^{[i-1]}(t)) - P^{[i-1]}(t, x^{[i-2]}(t))\|\\
&\leq \gamma_2\|x^{[i-1]}(t) - x^{[i-2]}(t)\|
\end{aligned}$$

Hence, from the equation (2.6), if we take norm on $\|C(x^{[i-1]}(t)) - C(x^{[i-2]}(t))\|$, *thus*

$$\begin{aligned} \|C(x^{[i-1]}(t)) - C(x^{[i-2]}(t))\| &\leq \|R^{-1}\|\|B^T(x^{[i-1]}(t)) - B^T(x^{[i-2]}(t)\|\|P^{[i]}(t)\| \\ &+ \|R^{-1}\|\|B^T(x^{[i-2]}(t)\|\|P^{[i]}(t) - P^{[i-1]}(t)\|. \end{aligned}$$

If the sequence given by equation (2.5) is to converge we need a additional condition for B(x) in which it should satisfy the Lipschitz condition.

THEOREM 2.3.1 *If* t_f *(or* x_0*) is small enough, then under the conditions* $\overline{A1}-\overline{A2}$ *and*

$$\begin{array}{llll} \overline{(A3)'} & \|B(x) - B(y)\| & \leq \beta\, \|x-y\| & \forall\, x,\ y \in \mathcal{R}^n \\ \overline{(A4)'} & \|B(x)\| & \leq \gamma & \forall\, x \in \mathcal{R}^n \end{array}$$

approximation $x^{[i]}, u^{[i]}$ *converge (in* $C\ (\ 0, t_f; \mathcal{R}^n)\ )$ *to function* $x(t), u(t)$ *which minimize (2.2) over the set of feedback controls of the form* $-BR^{-1}B^TP(x)x$ □

Proof: The theorem is proved by lemmas (2.3.1-2.3.4).

2.4 Conclusions

In this chapter the optimization problem

$$min\ J = x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^TQx + u^TRu)dt$$

subject to the constraint

$$\dot{x} = A(x)x + B(x)u, \quad x(0) = x_0$$

has been considered. A new method which minimized (J) over the interval $[0, t_f]$, for some fixed t_f is proposed.

The idea was to transform the original nonlinear optimal control problem into a sequence of linear optimal control problems, so that for each sequence one may obtain a linear time-varying differential equation of the form

$$\dot{x}^{[i]} = A(x^{[i-1]}(t))x^{[i]} + B(x^{[i-1]}(t))u^{[i]}, \; x^{[i]}(0) = x_0$$

with cost functional

$$J^{[i]} = x^{[i]^T}(t_f)Fx^{[i]}(t_f) + \int_0^{t_f} (x^{[i]^T}Qx^{[i]} + u^{[i]^T}Ru^{[i]})dt$$

We have shown for small t_f (*or* x_0) the approximation of $x^{[i]}, u^{[i]}$ converged to the function $x(t), u(t)$.

The method used is promising and simple to implement. It consists of numerically solving the matrix Riccati differential equation and linear differential equation for each sequence. The results are very useful and some examples are given to illustrate the theory in chapter 5.

So far it has been shown that under some conditions, $x^{[i]}, u^{[i]}$ converge to $x(t), u(t)$ respectively and J is minimized in the interval $[0, t_f]$. Is this minimum a global minimum?

To this end, we will analyse the same methodology from different point of view, using Pontryagin's minimum principle. This yields the Hamiltonian which will provide the necessary conditions for optimal control.

This will be considered in the next chapter.

Chapter 3

Global Optimal Control of Nonlinear Systems

3.1 Introduction

In the previous chapter we studied nonlinear systems of the form

$$\dot{x} = A(x)x + B(x)u,$$

with a quadratic cost functional

$$J = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\{x^TQx + u^TRu\}dt.$$

In order to find the optimal control for such nonlinear systems, we introduced a sequence of linear quadratic approximations.

In this chapter, Pontryagin's minimum principle is used to find an optimal control for nonlinear systems. This method involves computing the Hamiltonian function which provides the necessary and sufficient conditions for optimality. From the necessary and sufficient conditions, one can form 2n nonlinear homogeneous differential equations. This leads to a nonlinear two point boundary value problem which has to be solved by iterative numerical techniques such as steepest descent, variation of extremals or quasi-linearization [35].

Here, a new approach is introduced that will transform the nonlinear two point boundary value problem into a sequence of linear two point boundary value problems. Since this is linear, the missing initial conditions to the sequence can be found by using the method of adjoints. Similar methods have been applied in order to obtain approximate solutions to the time-invariant Hamilton-Jacobi-Bellman equation [12], which arises in optimal control when plant is modeled by nonlinear dynamics, bilinear systems and distributed parameter systems; see [1, 8, 30].

This chapter is organized as follows: In section 3.2, the mathematical description of the problem is introduced. In section 3.3, adjoints method which will find the set of missing initial conditions for the linear two point boundary value problem is presented. The proof of convergence of the sequence will be discussed in section 3.4.

3.2 Problem formulation

In this section, we shall consider the optimization problem

$$min\ J = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\{x^TQx + u^TRu\}dt$$

with state equation

$$\dot{x} = A(x)x + B(x)u \quad x(0) = x_0.$$

Here, Q, R, and F are symmetric and positive definite and t_f is some fixed time. Also x and u are vector functions with n, m components respectively. A is an $n \times n$ matrix function, B is an $n \times m$ matrix function.

The Hamiltonian of the problem is

$$H = \frac{1}{2}\{x^T Q x + u^T R u\} + \lambda^T(t)\{A(x)x + B(x)u\}.$$

Therefore, the necessary conditions for optimality are:

$$\begin{aligned}
\tfrac{\partial H}{\partial \lambda} &= \dot{x} &&= A(x)x + B(x)u \\
\tfrac{\partial H}{\partial x} &= -\dot{\lambda} &&= Qx + \tfrac{\partial}{\partial x}\left(A(x)x\right)^T \lambda(t) + u^T \tfrac{\partial}{\partial x}\left(B(x)\right)^T \lambda(t) \\
\tfrac{\partial H}{\partial u} &= 0 &&= Ru + B^T(x)\lambda
\end{aligned}$$

so that

$$u = -R^{-1}B^T(x)\lambda(t)$$

From these equations, $2n$-dimensional coupled two-point boundary value problems are obtained:

$$\begin{aligned}
\dot{x} &= A(x)x - B(x)R^{-1}B^T(x)\lambda, \\
\dot{\lambda} &= -Qx + \bar{B}(x,\lambda)\lambda,
\end{aligned}$$

where

$$\bar{B}(x,\lambda) = -\frac{\partial}{\partial x}\left(A(x)x\right)^T - u^T\frac{\partial}{\partial x}B(x)^T.$$

We introduce the following sequence of approximations to the above equation:

$$\begin{aligned}
\dot{x}^{[0]} &= A(x_0)x^{[0]}(t) - B(x_0)R^{-1}B^T(x_0))\lambda^{[0]}(t), & x^{[0]}(t_0) &= x_0 \\
\dot{\lambda}^{[0]} &= -Qx^{[0]}(t) + \bar{B}(x_0, Fx_0)\lambda^{[0]}(t), & \lambda^{[0]}(t_f) &= Fx_0
\end{aligned}$$

and $k > 0$

$$\left.\begin{aligned}
\dot{x}^{[k]} &= A(x^{[k-1]}(t))x^{[k]}(t) - B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))\lambda^{[k]}(t), \\
x^{[k]}(t_0) &= x_0, \\
\dot{\lambda}^{[k]} &= -Qx^{[k]}(t) + \bar{B}(x^{[k-1]}(t), \lambda^{[k-1]}(t))\lambda^{[k]}(t), \\
\lambda^{[k]}(t_f) &= Fx^{[k-1]}(t_f).
\end{aligned}\right\} \quad (3.1)$$

Also, a sequence of approximations to the control input, which minimizes the cost function, is given by:

$$u^{[k]}(t) = -R^{-1}B(x^{[k-1]}(t))\lambda^{[k]}(t)$$

Let $z = [x, \lambda]^T$, then from equation (3.1) one can write:

$$\dot{z}^{[k]}(t) = \bar{A}(z^{[k-1]}(t))z^{[k]}(t), \quad z^{[k]}(t) = [x^{[k]}(t), \lambda^{[k]}(t)]^T$$

where

$$\bar{A}(z^{[k-1]}(t)) = \begin{bmatrix} A(x^{[k-1]}(t)) & -B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t)) \\ -Q & \bar{B}(x^{[k-1]}(t)), \lambda^{[k-1]}(t)) \end{bmatrix}$$

Hence $z^{[k]}(t)$ becomes set of $2n$ linear, homogeneous, differential equations for which n initial and n terminal conditions are known. Thus our initial problem has been transformed into linear two point boundary value problem. To determine unknown initial conditions the shooting method is used. This will be discussed in the following section.

3.3 Linear Two Point Boundary Value Problem

In this section one of the shooting methods, namely, the method of adjoints, is discussed. For the linear two point boundary value problem the method of adjoints finds the set of missing initial conditions in one pass through the process; for more details see [47].

Consider the set of $2n$ linear ordinary differential equations with variable coefficients

$$\dot{z}(t) = \bar{A}(t)z(t) \tag{3.2}$$

where

$$\begin{aligned}\bar{A}(t) &= 2n \times 2n \text{ matrix} \\ z(t) &= 2n \times 1 \text{ vector}\end{aligned}$$

The initial conditions are:

$$z_i(t_0) = c_i, \quad i = 1, 2, \ldots n; \tag{3.3}$$

and the terminal conditions are

$$z_{i_m}(t_f) = c_{i_m}, \quad m = 1, 2, \ldots (2n - n). \tag{3.4}$$

The adjoint system of equation (3.2) is defined as

$$\dot{y} = -\bar{A}^T(t)y$$

Hence

$$\sum_{i=1}^{2n}(y_i\dot{z}_i + \dot{y}_i z_i) = 0,$$

Above equation may also be written as

$$\sum_{i=1}^{2n}\frac{d}{dt}(y_i z_i) = 0$$

or as

$$\frac{d}{dt}\sum_{i=1}^{2n}(y_i z_i) = 0 \tag{3.5}$$

On integrating equation (3.5) over $[t_0, t_f]$, we have

$$\int_{t_0}^{t_f}\left\{\frac{d}{dt}\sum_{i=1}^{2n}(y_i z_i)\right\}dt = 0$$

or

$$\sum_{i=1}^{2n} y_i(t_f)z_i(t_f) - \sum_{i=1}^{2n} y_i(t_0)z_i(t_0) = 0 \tag{3.6}$$

Now, if we set terminal conditions for adjoint equation, that is

$$y_i^{(m)}(t_f) = \left.\begin{array}{cc} 1 & i = i_m \\ 0 & i \neq i_m \end{array}\right\} m = 1, 2, \ldots, (2n - n)$$

where the superscript m refers to the m th backward integration of the adjoint equations and i_m refers to subscript on the specified terminal conditions $y_{i_m}(t_f)$ in (3.4).

Since by (3.3) the initial conditions are given for $z_i(t_0), i = 1, 2, \ldots, n$, equation (3.6) can be written as

$$y_{i_m}^{(m)}(t_f)z_{i_m}(t_f) - \sum_{i=1}^{2n} y_i^{(m)}(t_0)z_i(t_0) = 0;$$

or

$$z_{i_m}(t_f) - \sum_{i=1}^{n} y_i^{(m)}(t_0)z_i(t_0) = \sum_{i=n+1}^{2n} y_i^{(m)}(t_0)z_i(t_0) \tag{3.7}$$

In matrix form we have, for (3.7),

$$\begin{bmatrix} z_{i_1}(t_f) - \sum_{i=1}^{n} y_i^{(1)}(t_0)z_i(t_0) \\ \vdots \\ z_{i_n}(t_f) - \sum_{i=1}^{n} y_i^{(n)}(t_0)z_i(t_0) \end{bmatrix} = \begin{bmatrix} y_{n+1}^{(1)}(t_0) & y_{n+2}^{(1)}(t_0) & \ldots & y_{2n}^{(1)}(t_0) \\ \vdots & \vdots & & \vdots \\ y_{n+1}^{(n)}(t_0) & y_{n+2}^{(n)}(t_0) & \ldots & y_{2n}^{(n)}(t_0) \end{bmatrix} \begin{bmatrix} z_{n+1}(t_0) \\ \vdots \\ z_{2n}(t_0) \end{bmatrix} \tag{3.8}$$

The set of missing initial conditions is found by solving (3.8):

$$\begin{bmatrix} z_{n+1}(t_0) \\ \vdots \\ z_{2n}(t_0) \end{bmatrix} = \begin{bmatrix} y_{n+1}^{(1)}(t_0) & y_{n+2}^{(1)}(t_0) & \ldots & y_{2n}^{(1)}(t_0) \\ \vdots & \vdots & & \vdots \\ y_{n+1}^{(n)}(t_0) & y_{n+2}^{(n)}(t_0) & \ldots & y_{2n}^{(n)}(t_0) \end{bmatrix}^{-1} \times \begin{bmatrix} z_{i_1}(t_f) - \sum_{i=1}^{n} y_i^{(1)}(t_0)z_i(t_0) \\ \vdots \\ z_{i_n}(t_f) - \sum_{i=1}^{n} y_i^{(n)}(t_0)z_i(t_0) \end{bmatrix} \tag{3.9}$$

provided, of course, the inverse exist.

In our case, we have

$$\dot{z}^{[k]}(t) = \bar{A}(z^{[k-1]}(t))z^{[k]}(t) \tag{3.10}$$

where

$$z^{[k]}(t) = [x^{[k]}(t), \lambda^{[k]}(t)]^T.$$

The initial conditions of $x^{[k]}(t_0)$ and terminal conditions of $\lambda^{[k]}(t_f)$ (see the equation (3.1)) are known. So one can find the missing initial conditions for each sequence of $\lambda^{[k]}$ using the method of adjoints.

Having determined all the initial conditions in (3.10), the differential equation becomes an initial value problem and can be solved numerically.

The sequence of approximate solutions $z^{[k]}(t)$ is developed by the process $\dot{z}^{[k]} = \bar{A}(z^{[k-1]})z^{[k]}$ which converges to the solution of the original problem under certain conditions. The convergence will be discussed in the following section.

3.4 Proof of Convergence

The equation (3.1) is integrated over $[t_0, t_f]$. Thus

$$\left.\begin{array}{rcl} x^{[k]}(t) & = & \Phi_x^{[k-1]}(t,t_0)x_0 + \int_{t_0}^{t} \Phi_x^{[k-1]}(t,\tau)\hat{B}(x^{[k-1]}(\tau))\lambda^{[k]}(\tau)d\tau \\ \lambda^{[k]}(t) & = & \Phi_\lambda^{[k-1]}(t_f - t + t_0, t_0)Fx^{[k-1]}(t_f) \\ & + & \int_t^{t_f} \Phi_\lambda^{[k-1]}(t_f - t + t_0, t_f + t_0 - \tau)Qx^{[k]}(t_f + t_0 - \tau)d\tau \end{array}\right\} \tag{3.11}$$

where $\hat{B}(x^{[k-1]}(t)) = -B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))$, $\Phi_x^{[k-1]}(t,t_0)$ is a transition matrix of $A(x^{[k-1]}(t)$ and $\Phi_\lambda^{[k-1]}(t_f - t + t_0, t_0)$ is a transition matrix of $\bar{B}(x^{[k-1]}(t), \lambda^{[k-1]}(t))$.

REMARK 3.4.1

- *From equation (3.11), one can say that $\lambda^{[k]}(t)$ and $x^{[k]}(t)$ is bounded by small t_f or x_0.*
- *Since each sequence consists of linear time-varying systems, one may solve a two point boundary value problem (bvp) for each k explicitly using the equation (3.9).*
- *Having found the missing initial conditions for each k for $\lambda^{[k]}(t)$, then this two point bvp becomes an initial value problem.*
- *Then in equation (3.10), all the initial conditions are known.*
- *Therefore, using the theorem developed in Chapter 2, we can show the equation (3.10), $\dot{z}^{[k]}(t) = \bar{A}(z^{[k-1]}(t))z^{[k]}(t)$, converges in $C(t_0, t_f; \mathcal{R}^n)$.*

THEOREM 3.4.1
Under the conditions $\overline{A1}, \overline{A2}, \overline{A3'}$, and $\overline{A4'}$ of theorem 2.3.1 (Chapter 2), the sequence of the systems given by (3.1) converges (in $C(t_0, t_f; \mathcal{R}^n)$ for sufficiently small t_f or x_0.

3.5 Conclusions

In this Chapter, a new method has been proposed in order to find a global optimal control for nonlinear systems of the form $\dot{x} = A(x)x + B(x)u$. This method first finds the Hamiltonian function and using the Pontryagin's minimum principle the necessary conditions for optimality are found. This provides a set of $2n$ first order nonlinear ordinary differential equations in which n initial and n terminal conditions are known. We have transformed the nonlinear two point boundary value problem into a sequence of linear two point boundary value problems. Since it is linear, the set of missing initial conditions can be found by using equation (3.9). It is shown here that this sequence converges (in $\mathcal{C}(t_0, t_f; \mathcal{R}^n)$ for sufficiently small t_f or x_0.

However, there are some disadvantages in this method where in practice one finds that sometimes, even though the adjoint equation is integrated backward with a set of n linear independent terminal vectors, the initial vectors are not numerically independent. In particular, this can happen when the matrix $A(t)$ in (3.1) has eigenvalues widely separated in value. As a result the inverse matrix in equation (3.9) becomes singular.

To illustrate this theory some examples are given in Chapter 5.

Chapter 4

Nonlinear Optimal Filtering

4.1 Introduction

In the previous chapters, it was assumed that all the states of the nonlinear systems of the form

$$\dot{x}(t) = A(x)x(t) + B(x)u(t) \tag{4.1}$$

are available for complete measurement. This assumption is often unrealistic and only certain nonlinear combinations of the state, denoted by y, can be measured:

$$y(t) = C(x)x(t) \tag{4.2}$$

The purpose of this chapter is to present a method of estimating the state vector, or finding approximations to the state vector, from the measured variable. For linear systems, this theory is well developed. The estimated state is then used to control the nonlinear system given by equation (4.1). The main contribution of this chapter is to derive an optimal control for a nonlinear stochastic system of the form 4.1 - 4.2, using the framework of linear filtering, separation principle and linear regulator control theory.

Usually, many dynamical systems contain noise and measurement errors which are considered as stochastic phenomena. Therefore, when considering the estimation of the state a certain degree of filtering of the noise is implicitly involved. In a sense, estimating the state of the systems is a filtering problem and choosing an optimum filter is essential.

The first landmark contribution to optimum filtering (or Kalman filtering) was made by Kalman [34], who formulated a recursive solution to the optimum linear filtering problem using a state-space model for dynamical system.

Kalman and Extended Kalman filters have been applied to satellite and flight navigation and mathematical modelling (see [19, 29, 49]).

This chapter is organized as follows: In section 4.2 a new method is proposed to obtain an optimum filter for a nonlinear stochastic system. In section 4.3, the proof of convergence of this method is given. Finally, the application of the filtering problem to control is introduced in section 4.4.

4.2 Proposed Solution

Consider the general nonlinear dynamical system whose state $x(t)$ evolves in time according to the following differential equation:

$$\dot{x} = A(x)x + v(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{4.3}$$

where

$x(t) \in \mathcal{R}^n$ is an n dimensional state vector,

$A(x)$ is an $n \times n$ real matrix function,

$x_0 \in \mathcal{R}^n$ is an n dimensional Gaussian random vector with $E\{x_0\} = \bar{x}_0$ and, $\mathrm{Cov}(x_0, x_0) \triangleq E\{[x_0 - \bar{x}_0][x_0 - \bar{x}_0]^T\} = \tilde{P}_0$,

$\tilde{P}_0$ is a positive semi-definite and symmetric matrix,

$v(t) \in \mathcal{R}^n$ is an n dimensional white noise with $E\{v(t)\} = 0 \;\forall\; t \geq t_0$ and, $E\{v(t)v^T(\tau)\} = \tilde{Q}(t)\delta(t-\tau)),\quad E\{x_0 v^T(t)\} = 0$,

$\tilde{Q}(t)$ is a positive semi-definite and symmetric matrix.

Also, consider the measurement process $y(t)$ to be given by

$$y(t) = C(x)x + w(t) \tag{4.4}$$

where

$y(t) \in \mathcal{R}^m$ is an m dimensional state vector,

$C(x)$ is an $m \times n$ real matrix function,

$w(t) \in \mathcal{R}^m$ is an m dimensional white noise with $E\{w(t)\} = 0 \;\forall\; t \geq t_0$ and, $E\{w(t)w^T(\tau)\} = \tilde{R}(t)\delta(t-\tau)),\quad E\{x_0 w^T(t)\} = 0$,

$\tilde{R}(t)$ is a positive definite and symmetric matrix.

A sequence of approximations is introduced to the equations 4.3 and 4.4 which has the following form:

$$\begin{aligned}
\dot{x}^{[0]}(t) &= A(\bar{x}_0)x^{[0]}(t) + v(t), \quad x^{[0]}(t_0) = \bar{x}_0 \\
y^{[0]}(t) &= C(\bar{x}_0)x^{[0]}(t) + w(t)
\end{aligned}$$

and for $k \geq 1$, k - iteration index

$$\left.\begin{array}{rcl} \dot{x}^{[k]}(t) & = & A(x^{[k-1]}(t))x^{[k]}(t) \;+\; v(t), \;\; x^{[k]}(t_0) = \bar{x}_0 \\ y^{[k]}(t) & = & C(x^{[k-1]}(t))x^{[k]}(t) \;+\; w(t) \end{array}\right\} \tag{4.5}$$

The estimate $\hat{x}^{[k]}(t)$ of $x^{[k]}(t)$ is given by a Kalman filter which has the following expression:

$$\left.\begin{array}{rcl} \dot{\hat{x}}^{[k]}(t) & = & A(\hat{x}^{[k-1]}(t))\hat{x}^{[k]}(t) + K^{[k]}(t)\left[y^{[k]}(t) - C(\hat{x}^{[k-1]}(t))\hat{x}^{[k]}(t)\right] \\ K^{[k]}(t) & = & \tilde{P}^{[k]}(t)C^T(\hat{x}^{[k-1]}(t))\tilde{R}^{-1}(t) \\ \dot{\tilde{P}}^{[k]}(t) & = & \tilde{Q}(t) + \tilde{P}^{[k]}A^T(\hat{x}^{[k-1]}(t)) + A(\hat{x}^{[k-1]}(t))\tilde{P}^{[k]}(t) \\ & - & \tilde{P}^{[k]}(t)C^T(\hat{x}^{[k-1]}(t))\tilde{R}^{-1}(t)C(\hat{x}^{[k-1]}(t))\tilde{P}^{[k]}(t) \end{array}\right\} \tag{4.6}$$

$$\tilde{P}^{[k]}(t_0) = \tilde{P}_0$$

$$\hat{x}^{[k]}(t_0) = \bar{x}_0$$

For k = 0, above equations become:

$$\begin{array}{rcl} \dot{\hat{x}}^{[0]}(t) & = & A(\bar{x}_0)\hat{x}^{[0]}(t) + K^{[0]}(t)\left[y^{[0]}(t) - C(\bar{x}_0)\hat{x}^{[0]}(t)\right] \\ K^{[0]}(t) & = & \tilde{P}^{[0]}(t)C^T(\bar{x}_0)\tilde{R}^{-1}(t) \\ \dot{\tilde{P}}^{[0]}(t) & = & \tilde{Q}(t) + \tilde{P}^{[0]}A^T(\bar{x}_0) + A(\bar{x}_0)\tilde{P}^{[0]}(t) \\ & - & \tilde{P}^{[0]}(t)C^T(\bar{x}_0)\tilde{R}^{-1}(t)C(\bar{x}_0)\tilde{P}^{[0]}(t) \\ \tilde{P}^{[0]}(t_0) & = & \tilde{P}_0 \\ \hat{x}^{[0]}(t_0) & = & \bar{x}_0 \end{array}$$

4.3 Proof of Convergence

In this section equations 4.5 and 4.6 are proved to converge under certain conditions on $A(x)$ and for small enough horizon time $T($ or small enough initial conditions x_0). To do this, the bound on $\|\Phi^{[i-1]}(t,t_0) - \Phi^{[i-2]}(t,t_0)\|$ has to be estimated. Therefore the following lemma is presented. This lemma is already presented in Chapter 2 (section 2.3) but for easy reference it is given again here.

LEMMA 4.3.1 *Suppose that $\mu(A(x)) \leq \mu$ for some constant μ and for all x that*

$$\|A(x) - A(y)\| \leq \alpha\|x-y\|, \ \forall \ x, \ y \in \mathcal{R}^n$$

Then

$$\left\|\Phi^{[i-1]}(t,t_0) - \Phi^{[i-2]}(t,t_0)\right\| \leq L \sup_{t_0 \leq t \leq T} \left\|x_t^{[i-1]} - x_t^{[i-2]}\right\|$$

where

$$L = \alpha e^{\mu(t-t_0)}(t-t_0).$$

Since white noise $v(t)$ is the derivative of the Wiener process W_t, $v(t)dt$ can be replaced by dW_t. Also, the state $x^{[i]}(t,\omega)$ is replaced by $x_t^{[i]}(\omega)$ where $\omega \in \Omega$ (probability space) is a random parameter appearing indirectly in equation 4.5. On integrating equation 4.5 over $[t_0, t]$ one can get following solution:

$$x_t^{[i]} = \Phi^{[i-1]}(t,t_0)x_0 + \int_{t_0}^{t} \Phi^{[i-1]}(t,s)dW_s \tag{4.7}$$

In order to prove the sequence $x_t^{[i]}$ converges to a solution one needs to show that $E\|x_t^{[i]} - x_t^{[i-1]}\|^2 \rightarrow 0$ as $i \rightarrow \infty$. First, note that

$$\begin{aligned} x_t^{[i]} - x_t^{[i-1]} &= \left[\Phi^{[i-1]}(t,t_0) - \Phi^{[i-2]}(t,t_0)\right] x_0 \\ &+ \int_{t_0}^{t} \left[\Phi^{[i-1]}(t,s) - \Phi^{[i-2]}(t,s)\right] dW_s \end{aligned}$$

The following conditions are assumed in order to show that $x_t^{[i]}$ converge uniformly:

$$\begin{array}{lll}
\overline{(A1)} & \mu(A(x)) & \leq \ \mu_0 \ \forall\, x \in \mathcal{R}^n \\
\overline{(A2)} & E\|x_0\|^2 & \leq \ c \\
\overline{(A3)} & \|\Phi^{[i-1]}(t,t_0)\| & \leq \ exp\left[\int_{t_0}^{t} \mu(A(x^{[i-1]}(\tau))d\tau\right]
\end{array}$$

Under these assumptions and by virtue of the inequality $\|a+b\|^2 \leq 2(\|a\|^2+\|b\|^2)$, it follows that

$$\begin{aligned}
E\|x_t^{[i]} - x_t^{[i-1]}\|^2 \ &\leq \ 2E\left\|\Phi^{[i-1]}(t,t_0) - \Phi^{[i-2]}(t,t_0)\right\|^2 c \\
&+ \ 2\int_{t_0}^{t} E\left\|\Phi^{[i-1]}(t,s) - \Phi^{[i-2]}(t,s)\right\|^2 ds
\end{aligned} \tag{4.8}$$

Using the lemma 4.3.1 and defining,

$$d_i = \sup_{t_0 \leq t \leq T} E\|x_t^{[i]} - x_t^{[i-1]}\|^2$$

it follows from equation 4.8 that

$$\begin{aligned}
d_i \ &\leq \ L_1 d_{i-1} \ \text{ where} \\
L_1 \ &= \ 2L^2[c + (T - t_0)]
\end{aligned}$$

By induction d_i satisfies

$$d_i \leq L_1^{i-1} d_1 \tag{4.9}$$

The value of d_1 can be calculated from the equation 4.7 and using the assumptions $\overline{A1} - \overline{A3}$, therefore

$$E\|x_t^{[i]}\|^2 \ \leq \ 2\|\Phi^{[i-1]}(t,t_0)\|^2 c + 2\int_{t_0}^{t} \|\Phi^{[i-1]}(t,s)\|^2 ds \ \ \leq L_2$$

$$L_2 = 2e^{2\mu(t-t_0)}c + \frac{1}{\mu}[e^{2\mu(t-t_0)} - 1].$$

Hence,

$$d_1 \leq 2E\|x_t^{[1]}\|^2 + 2E\|x_t^{[0]}\|^2$$
$$d_1 \leq 4L_2$$

therefore from equation 4.9

$$d_i \leq 4L_2 L_1^{[i-1]}.$$

So, if $|L_1| < 1$ for $t \in [t_0, T]$ and $i \to \infty$, it follows

$$\lim_{i\to\infty} E\|x_t^{[i]} - x_t^{[i-1]}\|^2 = 0, \qquad ac^* - \lim_{i\to\infty} x_t^{[i]} = x_t.$$

Remark 4.3.1 $\hat{x}_t^{[k]}$ converges to $E\{x_t^{[k]}|y_t^{[k]}[t_0,t]\}$ for each $k = 0, 1, \ldots, i$ by the Kalman filter algorithm on $(\mathcal{C}[t_0,T], \mathcal{R}^n)$. Since each sequence $x_t^{[k]}$ is linear time-varying stochastic process, state estimation can be done using a Kalman filter. However, $x_t^{[k]}$ converges to x_t as $i \to \infty$. Therefore, $\lim_{i\to\infty} \hat{x}_t^{[i]} = E\{x_t|y[t_0,t]\}$ on $(\mathcal{C}[t_0,T], \mathcal{R}^n)$. Since $x_t^{[k]}, \hat{x}_t^{[k]}$ converge, it is obvious that the sequences $K^{[k]}(t)$ and $\tilde{P}^{[k]}$ will converge to the solutions K(t) and $\tilde{P}(t)$, by methods similar to those presented in chapter 2.

Theorem 4.3.1
Using the conditions $\overline{A1} - \overline{A3}$, if T (or x_0) is small enough and small enough $\tilde{P}_0$, then the approximate sequences $\hat{x}_t^{[k]}, x_t^{[k]}$ converge on $(\mathcal{C}[t_0,T], \mathcal{R}^n)$ to function x_t defined by equation 4.5.

*almost certainly

4.4 Stochastic Optimal control

In this section, nonlinear stochastic systems of the form

$$\dot{x}(t) = A(x)x + B(x)u + v(t), \quad x(t_0) = x_0, \; E\{x_0\} = \bar{x}_0$$

together with the nonlinear measurement equation

$$y(t) = C(x)x + w(t).$$

are considered, where B is an $n \times p$ matrix, u is an $p \times 1$ matrix and A, C, v, w, and y have been defined in the previous section. Since these are nonlinear systems, there are no analytical solutions to the above equations. Therefore, before analyzing the nonlinear stochastic systems, linear stochastic systems are considered in order to solve the nonlinear problem.

Consider linear stochastic systems

$$\dot{x}(t) = A(t)x + B(t)u + v(t), \quad x(t_0) = x_0, \; E\{x_0\} = \xi$$

with the measurement equation

$$y(t) = C(t)x + w(t).$$

It is well known by Kalman Filtering that the observer to the system is given by

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)\left[y(t) - C(t)\hat{x}(t)\right], \quad \hat{x}(t_0) = \xi$$

The observation error is defined by $\tilde{x}(t)$, $\tilde{x} = x - \hat{x}$, where it has differential equation of the form

$$\dot{\tilde{x}}(t) = [A(t) - K(t)C(t)]\tilde{x}(t) + v(t) - K(t)w(t), \quad \tilde{x}(t_0) = x_0 - \xi.$$

The only difference in stochastic control systems is that the feedback control law is connected to the estimate of the state rather than state itself. Thus,

$$u(t) = -\Gamma\hat{x}(t), \quad \Gamma = R^{-1}(t)B^T(t)P(t),$$

where, P is obtained by solving the Riccati equation with $P(t_f) = F$. So, one may obtain combined differential equation of $x(t)$ and $\hat{x}(t)$ of the form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} A(t) - B(t)\Gamma(t) & B(t)\Gamma(t) \\ 0 & A(t) - K(t)C(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} v(t)$$
$$+ \begin{bmatrix} 0 \\ -K \end{bmatrix} w(t)$$

Using this linear stochastic system theory, a sequence of approximations is applied in order to tackle the nonlinear stochastic problem. This will be demonstrated in the following section.

4.4.1 The Approximating sequence

Consider the nonlinear system

$$\dot{x}(t) = A(x)x + B(x)u + v(t), \quad x(t_0) = x_0, \quad E\{x_0\} = \bar{x}_0$$

together with the nonlinear measurement equation

$$y(t) = C(x)x + w(t).$$

The control $u(t)$ is to be chosen so as to minimize the cost functional

$$min\ J(u) = E\left\{x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt\right\}.$$

A sequence of approximations to the above equations is introduced. Thus

$$\begin{aligned}
\dot{x}^{[k]}(t) &= A(\hat{x}^{[k-1]}(t))x^{[k]}(t) + B(\hat{x}^{[k-1]}(t))u^{[k]}(t) + v(t) \\
x^{[k]}(t_0) &= x_0,\ E\{x_0\} = \bar{x}_0 \\
u^{[k]}(t) &= -\Gamma^{[k]}(t)\hat{x}^{[k]}(t) \\
\Gamma^{[k]}(t) &= R^{-1}(t)B^T(\hat{x}^{[k-1]}(t))P^{[k]}(t) \\
-\dot{P}^{[k]}(t) &= Q + P^{[k]}(t)A(\hat{x}^{[k-1]}(t)) + A^T(\hat{x}^{[k-1]}(t))P^{[k]}(t) \\
&- P^{[k]}(t)B(\hat{x}^{[k-1]}(t))R^{-1}B^T(\hat{x}^{[k-1]}(t))P^{[k]}(t),\quad P^{[k]}(t_f) = F \\
y^{[k]}(t) &= C(\hat{x}^{[k-1]}(t))x^{[k]}(t) + w(t)
\end{aligned}$$

For $k = 0$, the above equations become

$$\begin{aligned}
\dot{x}^{[0]}(t) &= A(\bar{x}_0)x^{[0]}(t) + B(\bar{x}_0)u^{[0]}(t) + v(t) \\
x^{[0]}(t_0) &= x_0,\ E\{x_0\} = \bar{x}_0 \\
u^{[0]}(t) &= -\Gamma^{[0]}(t)\hat{x}^{[0]}(t) \\
\Gamma^{[0]}(t) &= R^{-1}(t)B^T(\bar{x}_0)P^{[0]}(t) \\
-\dot{P}^{[0]}(t) &= Q + P^{[0]}(t)A(\bar{x}_0) + A^T(\bar{x}_0)P^{[0]}(t) \\
&- P^{[0]}(t)B(\bar{x}_0)R^{-1}B^T(\bar{x}_0)P^{[0]}(t),\quad P^{[0]}(t_f) = F \\
y^{[0]}(t) &= C(\bar{x}_0))x^{[0]}(t) + w(t)
\end{aligned}$$

The estimate of $\hat{x}^{[k]}(t)$ of $x^{[k]}(t)$ is given by the Kalman filter algorithm which has the following expression:

$$\begin{aligned}
\dot{\hat{x}}^{[k]} &= A(\hat{x}^{[k-1]}(t))\hat{x}^{[k]}(t) + B(\hat{x}^{[k-1]}(t))u^{[k]}(t) \\
&+ K^{[k]}(t)\left[y^{[k]}(t) - C(\hat{x}^{[k-1]}(t))\hat{x}^{[k]}(t)\right], \quad \hat{x}(t_0) = \bar{x}_0 \\
K^{[k]}(t) &= \tilde{P}^{[k]}(t)C^T(\hat{x}^{[k-1]}(t))\tilde{R}^{-1}(t) \\
\dot{\tilde{P}}^{[k]}(t) &= \tilde{Q}(t) + \tilde{P}^{[k]}(t)A^T(\hat{x}^{[k-1]}(t)) + A(\hat{x}^{[k-1]}(t))\tilde{P}^{[k]}(t) \\
&- \tilde{P}^{[k]}(t)C^T(\hat{x}^{[k-1]}(t))\tilde{R}^{-1}(t)C(\hat{x}^{[k-1]}(t))\tilde{P}^{[k]}(t) \\
\tilde{P}^{[k]}(t) &= \tilde{P}_0.
\end{aligned}$$

For $k = 0$, the above equations become

$$\begin{aligned}
\dot{\hat{x}}^{[0]} &= A(\bar{x}_0)\hat{x}^{[0]}(t) + B(\bar{x}_0)u^{[0]}(t) + K^{[0]}(t)\left[y^{[0]}(t) - C(\bar{x}_0)\hat{x}^{[0]}(t)\right] \\
\hat{x}(t_0) &= \bar{x}_0 \\
K^{[0]}(t) &= \tilde{P}^{[0]}(t)C^T(\bar{x}_0)\tilde{R}^{-1}(t) \\
\dot{\tilde{P}}^{[0]}(t) &= \tilde{Q}(t) + \tilde{P}^{[0]}(t)A^T\bar{x}_0) + A(\bar{x}_0)\tilde{P}^{[0]}(t) \\
&- \tilde{P}^{[0]}(t)C^T(\bar{x}_0)\tilde{R}^{-1}(t)C(\bar{x}_0)\tilde{P}^{[0]}(t) \\
\tilde{P}^{[0]}(t) &= \tilde{P}_0
\end{aligned}$$

The sequence of observation errors $\tilde{x}^{[k]}(t)$ is defined by

$$\tilde{x}^{[k]}(t) = x^{[k]}(t) - \hat{x}^{[k]}(t)$$

and the combined differential equation of $x^{[k]}(t)$ and $\tilde{x}^{[k]}(t)$ is of the form:

$$\begin{bmatrix} \dot{x}^{[k]}(t) \\ \dot{\tilde{x}}^{[k]}(t) \end{bmatrix} =$$

$$\begin{bmatrix} A(\hat{x}^{[k-1]}(t)) - B(\hat{x}^{[k-1]}(t))\Gamma^{[k]}(t) & B(\hat{x}^{[k-1]}(t))\Gamma^{[k]}(t) \\ 0 & A(\hat{x}^{[k-1]}(t)) - K^{[k]}(t)C(\hat{x}^{[k-1]}(t)) \end{bmatrix}$$

$$\times \begin{bmatrix} x^{[k]}(t) \\ \tilde{x}^{[k]}(t) \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} v(t) + \begin{bmatrix} 0 \\ -K^{[k]} \end{bmatrix} w(t) \tag{4.10}$$

The advantage of the equation 4.10 is that the $\tilde{x}^{[k]}(t)$ is completely decoupled from the dynamics of the state $x^{[k]}(t)$. Therefore, $\tilde{x}^{[k]}(t)$ is influenced neither by $x^{[k]}(t)$ nor $\Gamma^{[k]}(t)$. The characteristic motions of $x^{[k]}(t)$, conversely, are only determined by $A(\hat{x}^{[k-1]}(t))$, $B(\hat{x}^{[k-1]}(t))$ and $\Gamma^{[k]}(t)$, while $B(\hat{x}^{[k-1]}(t))\Gamma^{[k]}(t)\tilde{x}^{[k]}(t)$ and v(t) merely act as external excitations. The algebraic separation principle is therefore also valid here.

REMARK 4.4.1 The equation 4.10 may be written as

$$\dot{X}^{[k]}(t) = \bar{A}(X^{[k-1]}(t))X^{[k]}(t) + V(t),$$

where $\forall\ X,\ V \in \mathcal{R}^{2n}$, $k = 0, 1, \ldots, i$ and $\bar{A} \in \mathcal{R}^{2n} \times \mathcal{R}^{2n}$. So, under the theorem 4.3.1 $X^{[i]}(t)$ will converge to X(t) as $i \to \infty$.

4.5 Conclusions

In this chapter, the problem of developing optimal filter for nonlinear system has been introduced using a new approach. Basically, it consists of a sequence of approximations to the nonlinear systems combined with optimal filtering based on the approximate systems. A numerical experiment to test the performance of the developed filter was conducted and the results will be shown in chapter 6.

Also, a technique for studying nonlinear stochastic control problems by replacing their defining equations by sequence of time-varying systems has been introduced. Each sequence may be approached by classical means. The sequences are shown to converge under mild assumptions.

A numerical example is shown to illustrate this theory in chapter 6.

Chapter 5

Practical Applications for Deterministic Systems

5.1 Introduction

In the previous chapters the control problem for nonlinear system in the form

$$\dot{x} = A(x)x + B(x)u \tag{5.1}$$

have been studied. A new algorithm to determine a suitable control input u that will stabilize the nonlinear systems within a finite time has been introduced. This control law is global optimum over the trajectories and minimizing the cost functional within a final time. That is minimizing the J, where

$$J = x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f} \left(x^T Qx + u^T Ru\right) dt.$$

The above equation can be replaced by a sequence of linear time varying systems of the form,

$$\begin{aligned}
\dot{x}^{[i]}(t) &= A\left(x^{[i-1]}(t)\right) x^{[i]}(t) + B\left(x^{[i-1]}(t)\right) u^{[i]}(t) \\
u^{[i]}(t) &= -R^{-1} B(x^{[i-1]}(t))^T P^{[i]}(t) x^{[i]}(t)
\end{aligned}$$

It has been shown that as $i \to \infty$, this equation is uniformly converging to (5.1).

That is,

$$\lim_{i\to\infty} x^{[i]}(t) = x(t)$$

$$\lim_{i\to\infty} u^{[i]}(t) = u(t)$$

The first approximation is calculated from the initial conditions. Second and subsequent approximations are based on the previous approximation. The iteration procedure will be stopped when the convergence criterion, $\|x^{[i]}(t) - x^{[i-1]}(t)\| < \epsilon$, is met; ϵ is a positive number and close to zero.

In this chapter, some example nonlinear systems will be used to demonstrate the practical aspects of the techniques introduced in this research. These systems are the inverted pendulum and F-8 fighter aircraft.

5.2 Inverted Pendulum

In this section the proposed control technique will be used to solve the control problem of the inverted pendulum on a moving cart. The objective of the problem is to keep the pole upright position; This has to be done only by movement of the cart along the horizontal axis.

Balancing an inverted pendulum using an automatic controller is a popular example in control engineering classes. It is simple enough for educational purpose, and at the same time demonstrate the usefulness of certain design techniques in modern control theory.

Different approaches have been used to solve this problem; Jorgensen [33] reduced the above nonlinear model of the system to a linear one. But this approach only work for small angles in the range of $(0 - 12^{\circ})$. This idea of using a linear model have been used by many authors; see([20]and [41]). Many other attempts have been carried out using the techniques of Neural Networks; see for an example([2]). There are other approaches like Fuzzy control systems, see ([51]), also applied to solve this problem. The method proposed here is able to solve the problem for large angles in the range $(0 - 60^{\circ})$.

5.2.1 Description of the Model

The system consists of a pendulum connected to a horizontal cart by a pin joint as shown in Figure 5.1. A control force u is applied to the cart. Let m the mass of the pendulum, M the mass of the cart, r the distance from the pivot to the center of mass of the pendulum, x the horizontal displacement of the cart and the θ the angle of inclination of the pendulum to the vertical direction. The equation of motion are (see [50])

$$(M+m)\ddot{x} + mr\ddot{\theta}\cos\theta - mr\dot{\theta}^2\sin\theta = u$$

$$\ddot{x}\cos\theta + r\ddot{\theta} - g\sin\theta = 0$$

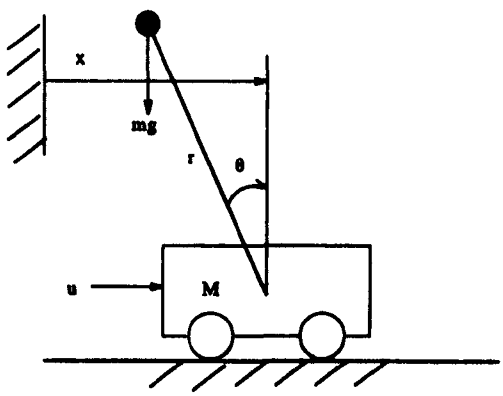


Figure 5.1: Inverted pendulum on a cart, including notation

If x_1, x_2, x_3 and x_4 represent state of the systems x, $\dot{x}$, θ and $\dot{\theta}$, then the state space form of the equations are:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{mrx_4^2 \sin x_3}{M + m\sin^2 x_3} - \frac{mg\sin 2x_3}{2(M + m\sin^2 x_3)} + \frac{u}{M + m\sin^2 x_3} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{(M+m)g\sin x_3}{r(M + m\sin^2 x_3)} - \frac{mx_4^2 \sin 2x_3}{2(M + m\sin^2 x_3)} - \frac{u\cos x_3}{r(M + m\sin^2 x_3)}
\end{aligned}$$

The above nonlinear equations representing the dynamics of the systems are rearranged to be of the form:

$$\dot{x} = A(x) + B(x)u$$

with

$$A(x) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_1(x) & a_2(x) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_3(x) & a_4(x) \end{vmatrix} \qquad and \quad B(x) = \begin{vmatrix} 0 \\ b_1(x) \\ 0 \\ b_2(x) \end{vmatrix} \tag{5.2}$$

where

$$\begin{aligned} a_1(x) &= \frac{-mg\sin 2x_3}{2x_3(M+m\sin^2 x_3)}, & a_2(x) &= \frac{mrx_4\sin x_3}{(M+m\sin^2 x_3)}, \\ a_3(x) &= \frac{(M+m)g\sin x_3}{rx_3(M+m\sin^2 x_3)}, & a_4(x) &= \frac{-mx_4\sin 2x_3}{2(M+m\sin^2 x_3)}, \\ b_1(x) &= \frac{1}{(M+m\sin^2 x_3)}, & b_2(x) &= \frac{-\cos x_3}{r(M+m\sin^2 x_3)}. \end{aligned}$$

5.2.2 Simulation Results: Method I, Sequence of Approximations

In this section numerical results are presented to demonstrate the performance of the method described in chapter 2. The method is basically a sequence of optimization policies.

The first approximation ($\dot{x}^{[0]} = A(x_0)x^{[0]} + B(x_0)u^{[0]}$) is calculated from the initial conditions of the nonlinear systems represented by state space form $\dot{x} = A(x)x + B(x)u$. In this case the matrices A and B are constant. After the first approximation the matrices A and B are time-varying. It has been shown that under certain conditions the nonlinear systems can be represented as systems of linear differential equations ($\dot{x}^{[n]} = A(x^{[n-1]})x^{[n]} + B(x^{[n-1]})u^{[n]}$) which will converge to the original nonlinear systems. The sequence of control laws ($u^{[n]} = -R^{-1}B^T(x^{[n-1]})P^{[n]}x^{[n]}$) is calculated by numerically integrating the matrix Riccati equation backwards in time for each iteration.

The values of the parameter g, m, M and r are given in table 5.1. By changing the values of the weighting matrices F, Q and R in the cost functional the shape of the response can be optimized. The design matrices F, Q and R have been chosen to obtain a good response. Their values are $F = I$, $Q = diag(1, 1, 100, 1)$ and $R = 1$. Initial conditions are x(0)=$x_0 = [0,\ 0,\ \theta,\ 0]$ and final optimization time t_f=10 seconds. The simulation results are presented in Figure 5.2 - 5.4 for

Parameter	Value
Acceleration due to gravity (g)	9.8 m/sec^2
Mass of the pendulum (m)	0.1 kg
Mass of the cart (M)	1.0 kg
Distance from pivot to center of mass of pendulum (r)	0.5 m

Table 5.1: Model Parameters

$\theta = 20°$, $30°$and $50°$. Simulation results demonstrate that the controller is able to balance the pendulum within 3 seconds. The dotted-dash line represents the first approximation, dashed line for second approximation and solid line represents the third approximation. It is evident that the sequence converges quickly and after the second iteration a good approximation tò the solution is reached.

5.2.3 Simulation Results: Method II, Sequence of Two Point Boundary Value Problems

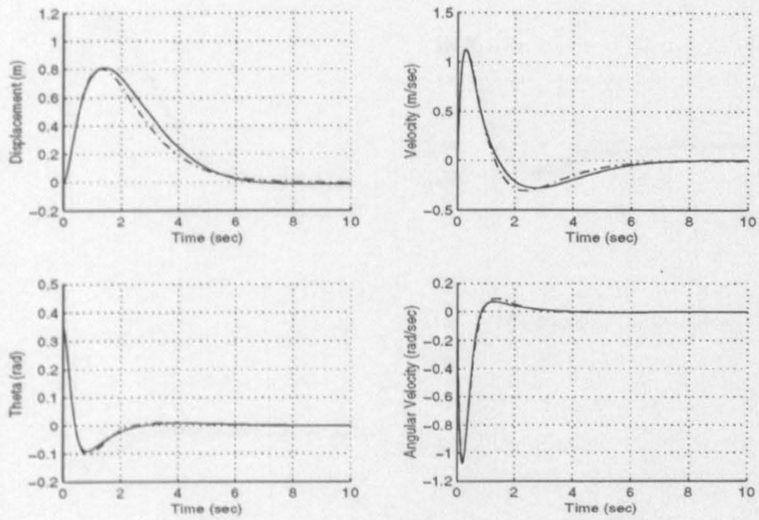
Here the algorithm developed in chapter 3, which is a sequence of two point boundary value problems, is applied to an inverted pendulum in order to illustrate this method.

The optimization problem,

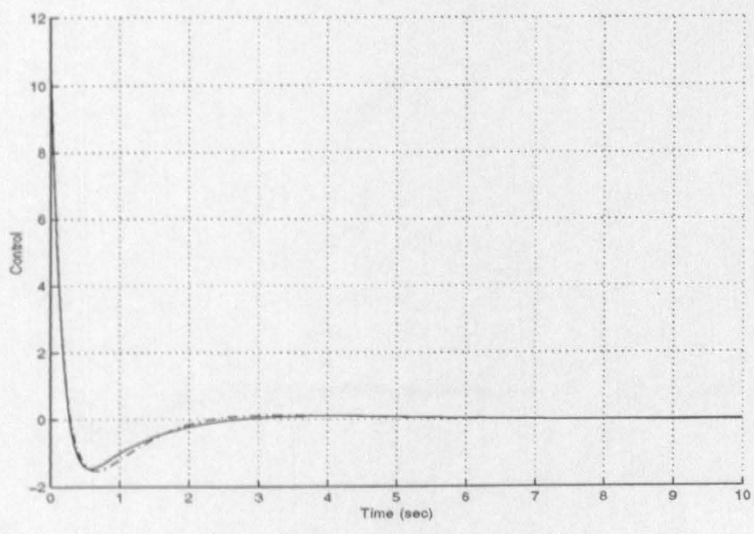
$$J = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\{x^TQx + u^TRu\}dt,$$

shall be considered, subject to the constraint

$$\dot{x} = A(x)x + B(x)u.$$

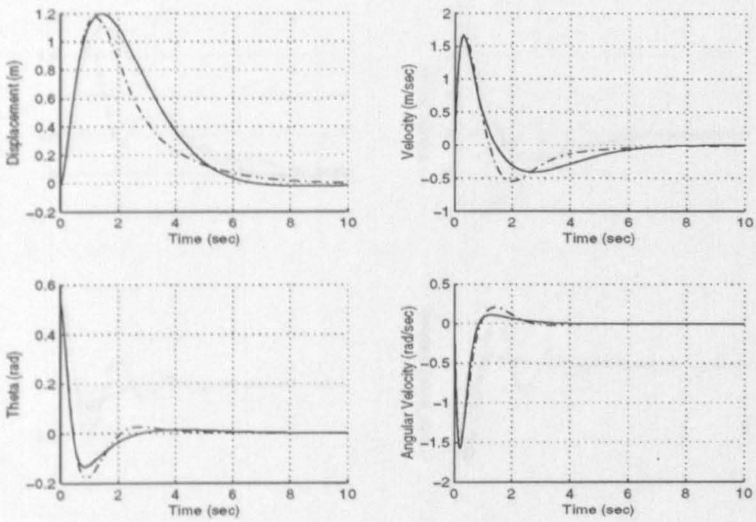


(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$

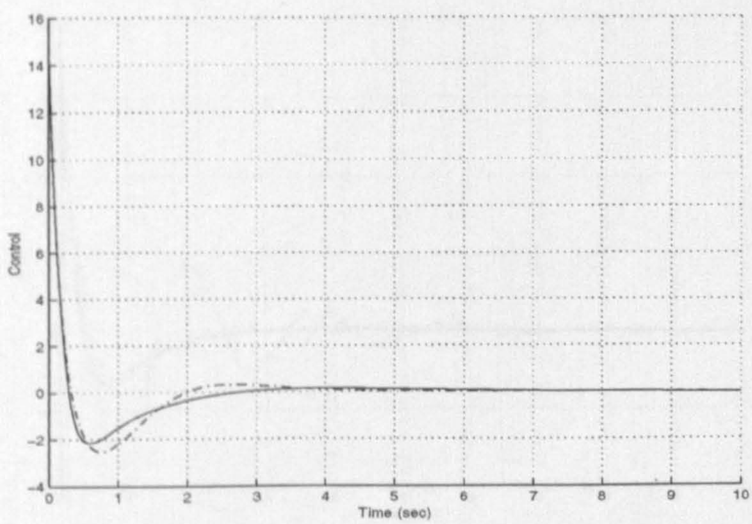


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 5.2: Response of the states of the inverted pendulum to the initial angle $\theta = 20°$, using the method(I), sequence of approximations.

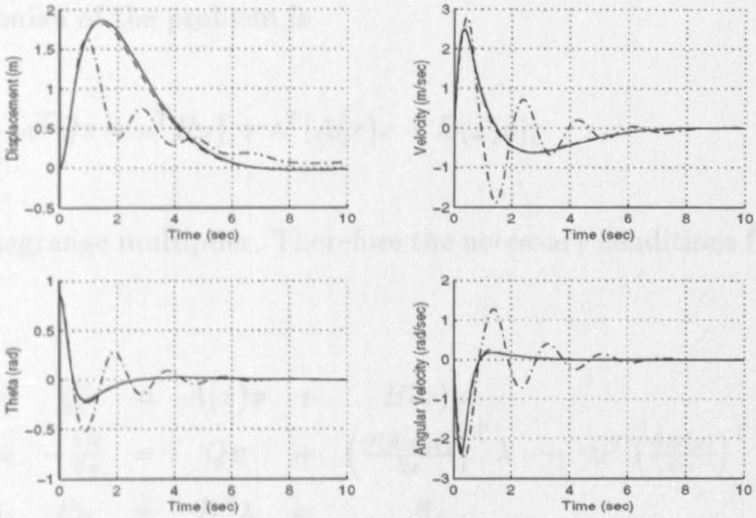


(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$

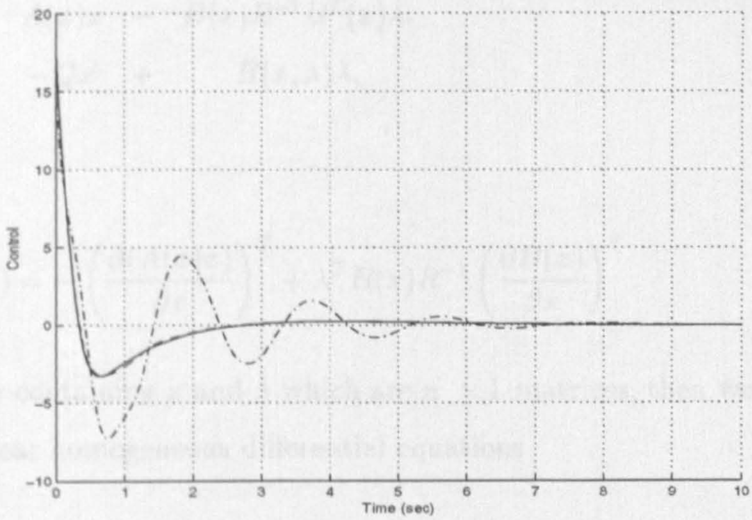


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 5.3: Response of the states of the inverted pendulum to the initial angle $\theta = 30°$, using the method(I), sequence of approximations.



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$



(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 5.4: Response of the states of the inverted pendulum to the initial angle $\theta = 50°$, using the method(I), sequence of approximations.

The Hamiltonian of the problem is

$$H = \frac{1}{2}\{x^T Q x + u^T R u\} + \lambda^T [A(x)x + B(x)u];$$

where, λ is Lagrange multiplier. Therefore the necessary conditions for optimality are:

$$\begin{array}{ccccccccc} \dot{x} & = & \frac{\partial H}{\partial \lambda} & = & A(x)x & + & B(x)u, & & \\ \dot{\lambda} & = & -\frac{\partial H}{\partial x} & = & Qx & + & \left(\frac{\partial (A(x)x)}{\partial x}\right)^T \lambda & + & u^T \left(\frac{\partial B(x)}{\partial x}\right)^T \lambda, \\ \frac{\partial H}{\partial u} & = & Ru & + & B^T \lambda & = & 0. & & \end{array}$$

If we rearrange above equation, it will be

$$\begin{array}{ccccc} \dot{x} & = & A(x)x & - & B(x)R^{-1}B^T(x)\lambda, \\ \dot{\lambda} & = & -Qx & + & \bar{B}(x,\lambda)\lambda, \end{array}$$

where

$$\bar{B}(x,\lambda) = -\left(\frac{\partial (A(x)x)}{\partial x}\right)^T + \lambda^T B(x) R^{-1} \left(\frac{\partial B(x)}{\partial x}\right)^T.$$

If z is vector containing x and λ which are $n \times 1$ matrices, then we have the set of $2n$ nonlinear homogeneous differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(x) & -B(x)R^{-1}B^T(x) \\ -Q & \bar{B}(x,\lambda) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \tag{5.3}$$

or we may write $\dot{z} = \bar{A}(z)z$. $\bar{A}$ is a 2n×2n matrix. For inverted pendulum, analytic functions $A(x)$ and $B(x)$ are defined in equation (5.2). The analytic function $\bar{B}(x,\lambda)$ is derived from differentiating $\frac{\partial (A(x)x)}{\partial x}$ and $\frac{\partial B(x)}{\partial x}$.

$\bar{B}(x,\lambda)$ has the form:

$$\bar{B}(x,\lambda) = -\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{a}_1(x) & \bar{a}_2(x) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \bar{a}_3(x) & \bar{a}_4(x) \end{bmatrix}^T + \lambda^T B R^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{b}_1(x) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{b}_2(x) & 0 \end{bmatrix}^T$$

where

$$\begin{aligned}
\bar{a}_1(x) &= \tfrac{mg}{2(M+m\sin^2 x_3)^2}\left\{m\sin^2 2x_3 - 2(M+m\sin^2 x_3)\cos 2x_3\right\} \\
&+ \tfrac{mrx_4^2}{(M+m\sin^2 x_3)^2}\left\{(M-m\sin^2 x_3)\cos x_3\right\}, \\
\bar{a}_2(x) &= \tfrac{2mrx_4 \sin x_3}{(M+m\sin^2 x_3)}, \\
\bar{a}_3(x) &= \tfrac{(M+m)(M-m\sin^2 x_3)g\cos x_3}{r(M+m\sin^2 x_3)^2} \\
&+ \tfrac{mx_4^2}{2(M+m\sin^2 x_3)^2}\left\{m\sin^2 2x_3 - 2(M+m\sin^2 x_3)\cos 2x_3\right\}, \\
\bar{a}_4(x) &= \tfrac{-mx_4 \sin 2x_3}{(M+m\sin^2 x_3)}, \\
\bar{b}_1(x) &= \tfrac{-m\sin 2x_3}{(M+m\sin^2 x_3)^2}, \\
\bar{b}_2(x) &= \tfrac{(M+m\sin^2 x_3)\sin x_3 + m\sin 2x_3 \cos x_3}{r(M+m\sin^2 x_3)^2}.
\end{aligned}$$

Sequence of approximations to equation (5.3) are introduced:

$$\begin{bmatrix} \dot{x}^{[i]}(t) \\ \dot{\lambda}^{[i]}(t) \end{bmatrix} = \begin{bmatrix} A(x^{[i-1]}(t)) & -B(x^{[i-1]}(t))R^{-1}B^T(x^{[i-1]}(t)) \\ -Q & \bar{B}(x^{[i-1]}(t), \lambda^{[i-1]}(t)) \end{bmatrix} \begin{bmatrix} x^{[i]} \\ \lambda^{[i]} \end{bmatrix}, \quad (5.4)$$

It is a $2n$ dimensional linear time-varying differential equation in which n initial conditions and n terminal conditions are known. The terminal conditions for each iteration $\lambda^{[i]}(t_f)$ is given by

$$\lambda^{[i]}(t_f) = Fx^{[i-1]}(t_f)$$

For the first iteration step $i = 0$, the value of $\lambda^{[0]}(t_f)$ is given by $\lambda^{[0]}(t_f) = Fx_0$. So the missing initial conditions for $\lambda^{[i]}(t_0)$ for each iteration can be found by using the method of adjoints. Having found the missing initial conditions the differential equations (5.4) will be integrated to find the response. This procedure will continue until the sequence of approximations converges.

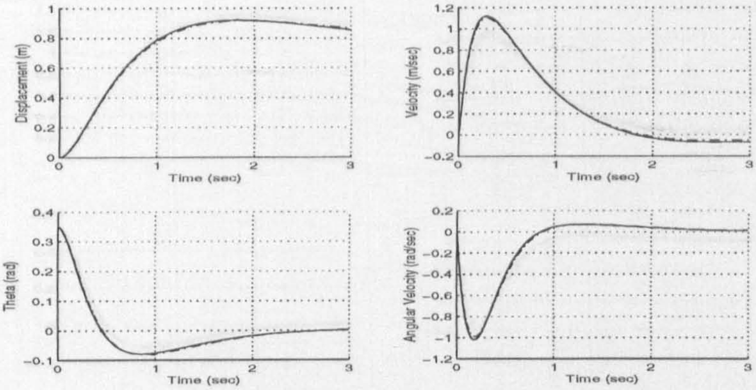
The same conditions which were used in section 5.2.2 are now applied to calculate the response of the inverted pendulum for method II. They are the weighting matrices F, Q and R, and their values are $F = I$, $Q = diag(1, 1, 100, 1)$ and $R = 1$. For different θ (20, 30, 50) the simulation results are plotted in Figures 5.5 - 5.7. As before the dotted-dash line represents the first approximation, dashed line for second approximation and solid line represents the third approximation.

5.2.4 Comparison of Method I and II for the Inverted Pendulum

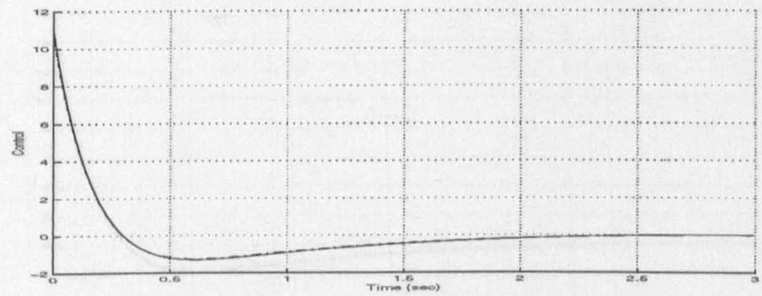
In this section it is shown that solving a sequence of Riccati equations (Method I) provide control policies which determine global optimum control for nonlinear systems. To this end, the second method, sequence of two point boundary value problems, is used.

The converged sequence is taken to calculate the cost functional for both methods under the same conditions. These conditions are weighting matrices F, Q, R and fixed horizon time. The cost functional for the inverted pendulum was calculated for 3 seconds for both methods.

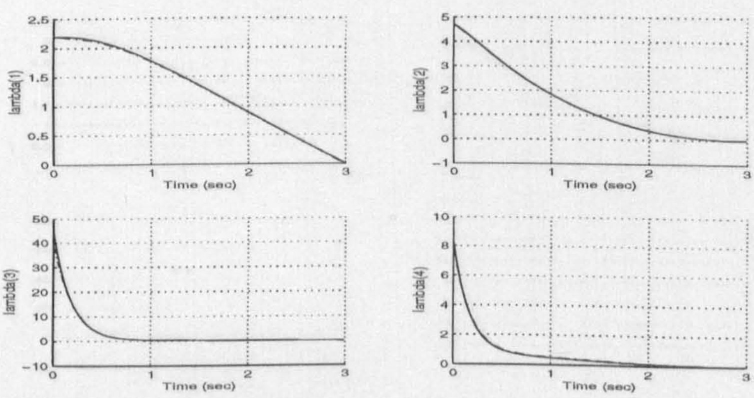
Table 5.2 shows values for the cost functional for different value of θ. From the values of the table 5.2 it is understood that both methods provide similar cost to the system. Therefore, intuitively, one can say the method I which is solving



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$

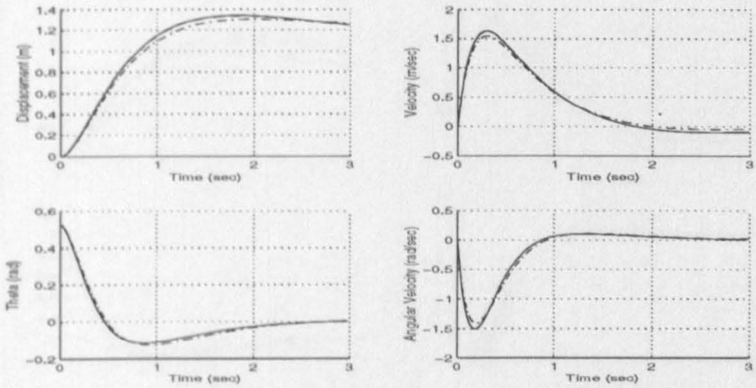


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

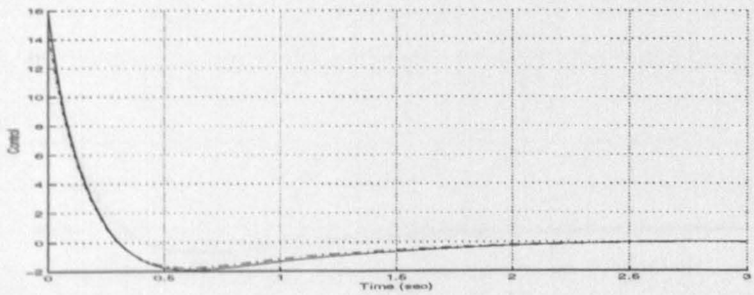


(c) Sequence of state $\lambda^{[0]}(t)$, $\lambda^{[1]}(t)$ and $\lambda^{[2]}(t)$

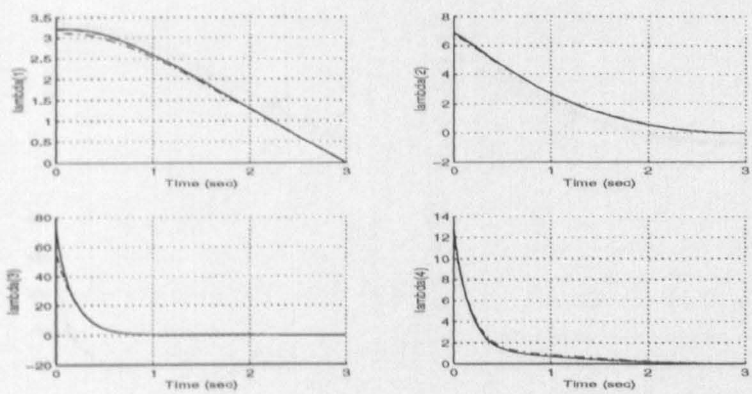
Figure 5.5: Response of the states of the inverted pendulum to the initial angle $\theta = 20°$, using the method(II), sequence of two point boundary value problems.



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$

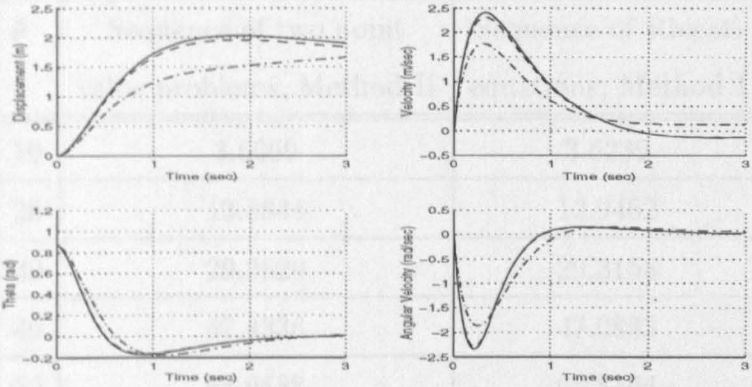


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

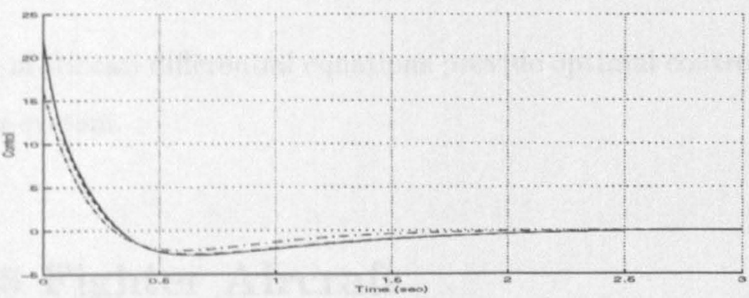


(c) Sequence of state $\lambda^{[0]}(t)$, $\lambda^{[1]}(t)$ and $\lambda^{[2]}(t)$

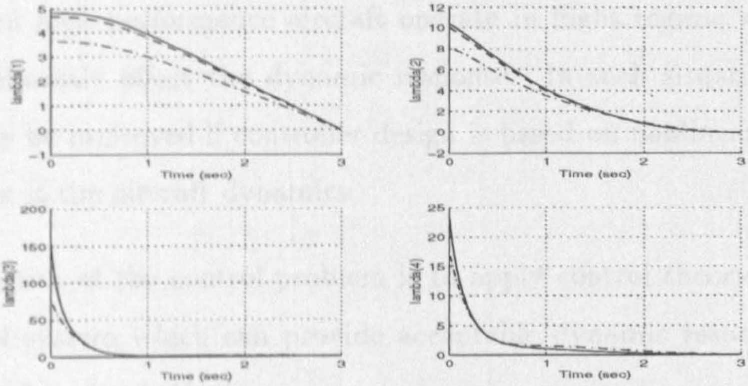
Figure 5.6: Response of the states of the inverted pendulum to the initial angle $\theta = 30°$, using the method(II), sequence of two point boundary value problems.



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$



(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$



(c) Sequence of state $\lambda^{[0]}(t)$, $\lambda^{[1]}(t)$ and $\lambda^{[2]}(t)$

Figure 5.7: Response of the states of the inverted pendulum to the initial angle $\theta = 50°$, using the method(II), sequence of two point boundary value problems.

θ	Sequence of two point value problems, Method II	Sequence of Riccati equations, Method I
10	3.6069	3.6239
20	13.8844	13.9462
30	29.2620	29.3158
40	47.4336	47.0833
50	64.9688	63.8404

Table 5.2: Cost of the Inverted Pendulum

the sequence of Riccati differential equations provide optimal control solutions to the nonlinear system.

5.3 F-8 Fighter Aircraft

As a second example, a nonlinear model of the F-8 fighter aircraft is considered here. Modern high performance aircraft operate in flight regime where nonlinearities significantly affect the dynamic response. In such situations dynamic response may be improved if controller design is based on nonlinear rather than linear models of the aircraft dynamics.

The objective of the control problem is to apply control theory to design of flight control system which can provide acceptable dynamic response over the entire range of angle of attack.

In an attempt to solve the control problem for the fighter aircraft, several techniques have been used. Starting with a linearized version of the nonlinear model of the aircraft, Garrard and Jordan [27] computed a linear control law using linear quadratic regulator theory. This linear control has been shown to work only for small angle of attack. In order to overcome this limitation of the linear control law which was based on a linear model, Garrard and Jordan introduced system nonlinearities in their model so that it is written in the form:

$$\dot{x} = Ax + Bu + \phi(x),$$

where $\phi(x)$ is an analytic vector function representing the system nonlinearities. Then feedback control for the above system is [27]:

$$u = -\frac{1}{2}R^{-1}B^T\frac{\partial v}{\partial x},$$

where $v(x)$ satisfies the Hamilton-Jacobi partial differential equation:

$$\frac{\partial v^T}{\partial x}Ax + \frac{\partial v^T}{\partial x}\phi(x) - \frac{1}{4}\frac{\partial v^T}{\partial x}BR^{-1}B^T\frac{\partial v^T}{\partial x} + x^TQx = 0, \quad v(0) = 0.$$

This equation cannot be solved analytically; perturbation procedures are used to obtain approximate solutions. Although by introducing this nonlinearities in their model, Garrard and Jordan [27] increased the flying range of the angle of attack for the aircraft. But this approach for the determination of the nonlinear control is very laborious and it becomes very complicated as one implements higher order control terms. It was found that as angle of attack approach 34.5°, it cannot recover from stall.

Another approach to the problem has been carried out by Desrochers and Al-Jaar [23]. In their method they design a controller for a reduced order or simplified nonlinear model of the nonlinear plant. Compared with that of [27] this technique is less complicated but on the other hand it was shown to work only for small angle of attack even less than that of the third order control introduced in [27].

To improve the aircraft performance a number of different approaches have been described recently using nonlinear control methods which are: Global Nonlinear Control Design and Extended Linear/Nonlinear Control Design. Global nonlinear design methods are used to design nonlinear feedback control laws that guarantee stability and acceptable responses of the closed-loop system over the entire state control space. In Banks and Mhana [9], it is shown that solving the infinite-time linear quadratic optimal control problem, pointwise on the state trajectory, results in global stability, near optimal quadratic control policy for nonlinear systems. Their method may be classified into this category and the application of this approach to aircraft control has been reported [9].

Extended linear control design methods are used for designing nonlinear control laws that ensure closed-loop stability and acceptable response in a region that consists of neighborhoods of all operating points. This method of designing on family of constant operating points, eg extended/pseudo-linearization [11, 46]. Application of the extended linear/nonlinear control approach to flight controller design for F-8 aircraft has been presented in Wang *et al.* [52, 53]. In their methods, at the high speed (243.84 m/s) trim condition, it was found that initial perturbation of angle of attack 33°, extended linear controller shows diverging response. Extended nonlinear flight controller captures the aircraft even up to an initial perturbation of angle of attack 37.5°.

The method which we proposed here is able to control the F-8 aircraft, at the high speed (243.84 m/s) trim condition, for larger angle of attack even up to 43°.

5.3.1 Mathematical Model

Under many circumstances the equation of motion of an aircraft naturally decouples into longitudinal modes and lateral modes. The longitudinal mode comprises the forward velocity, angle of attack (the angle the velocity vector makes with the wing), pitch rate and flight path angle (the angle the velocity vector makes with the horizontal). The lateral modes comprises the side velocity, the roll rate, and the yaw rate.

The longitudinal mode of the F-8 aircraft dynamics are considered here to demonstrate the theory introduced in this research. The equation representing the dynamics of the aircraft are [27]:

$$
\begin{aligned}
\dot{\alpha} &= \dot{\theta} - \alpha^2\dot{\theta} - 0.088\alpha\dot{\theta} - 0.877\alpha + 0.47\alpha^2 + 3.846\alpha^3 - 0.215\delta_\epsilon + 0.28\delta_\epsilon\alpha^2 \\
&+ 0.47\delta_\epsilon^2\alpha + 0.63\delta_\epsilon^3 - 0.019\theta^2 \\
\ddot{\theta} &= -0.396\dot{\theta} - 4.208\alpha - 0.47\alpha^2 - 3.564\alpha^3 - 20.967\delta_\epsilon + 6.265\delta_\epsilon\alpha^2 + 46\delta_\epsilon^2 \\
&+ 61.4\delta_\epsilon^3
\end{aligned}
$$

where α is the angle of attack in radians, θ the pitch angle in radians, $\dot{\theta}$ the pitch rate in radians per second and δ_ϵ the tail deflection angle. The above dynamical equation have been put in state space form where x_1, x_2, and x_3 represent the states α, θ, and $\dot{\theta}$ respectively and the control δ_ϵ by u. So that

$$
\begin{aligned}
\dot{x_1} &= -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 - x_1^2x_3 + 3.846x_1^3 \\
&- 0.215u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3 \\
\dot{x_2} &= x_3 \\
\dot{x_3} &= -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967u + 6.265x_1^2u \\
&+ 46u^2 + 61.4u^3
\end{aligned}
$$

Above equation may be written in the form :

$$\dot{x} = A(x)x + B(x)u$$

where

$$A(x) = \begin{bmatrix} -0.877 + 0.47x_1 - x_1x_3 + 3.846x_1^2 & -0.019x_2 & 1 - 0.088x_1 \\ 0 & 0 & 1 \\ -4.208 - 0.47x_1 - 3.564x_1^2 & 0 & -0.396 \end{bmatrix}$$

and

$$B(x) = \begin{bmatrix} -0.215 + 0.28x_1^2 \\ 0 \\ -20.967 + 6.265x_1^2 \end{bmatrix} \tag{5.5}$$

Terms involving nonlinearities in u are eliminated as the approach here can not account for nonlinear control terms. However, these terms have only a small effect on the dynamics as shown in [27].

5.3.2 Simulation Results: Method I, Sequence of Approximations

The response of the aircraft to the controller derived in chapter 2 was tested. The time response for several initial values of angle of attack $(35°, 43°)$ are shown in graphical form in Figure 5.8-5.9. In Figures, dotted-dashed line represents the first approximation, dashed line for second approximation and solid line represents the third approximation. After the second iteration a good approximation to the solution is reached. The weighting matrices F, Q and R have been chosen for obtaining a good response. Their values are: $F = I$, $Q = diag(50,1,1)$ and $R = 1$.

5.3.3 Simulation Results: Method II, Sequence of two point boundary value problems

Here the algorithm developed in chapter 3 is applied to the dynamics of the F-8 aircraft as a second example. To find a global minimum for the quadratic cost functional Hamilton's equation has been used.

The Hamiltonian of the problem is:

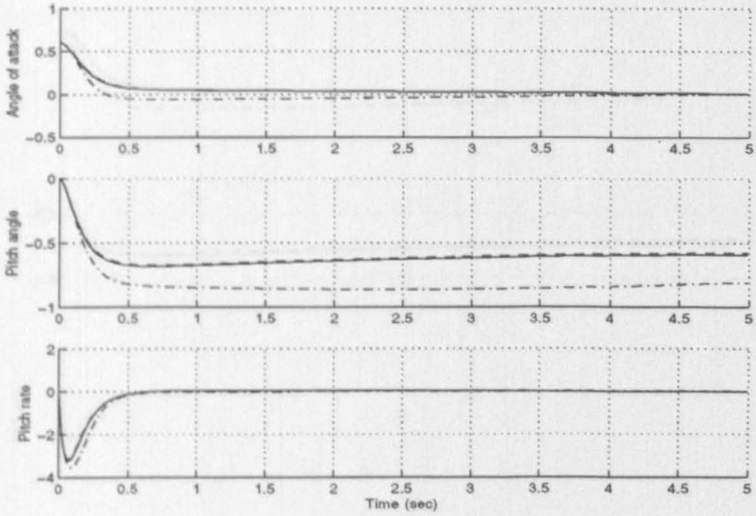
$$H = \frac{1}{2}\{x^T Q x + u^T R u\} + \lambda^T [A(x)x + B(x)u];$$

where, λ is the Lagrange multiplier. Therefore the necessary conditions for optimality are:

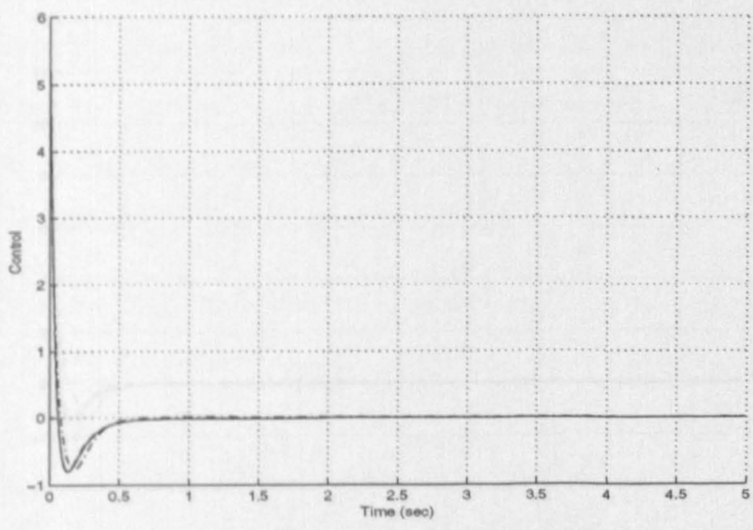
$$\begin{array}{rcccccccc} \dot{x} & = & \frac{\partial H}{\partial \lambda} & = & A(x)x & + & B(x)u, & & \\ \dot{\lambda} & = & -\frac{\partial H}{\partial x} & = & Qx & + & \left(\frac{\partial (A(x)x)}{\partial x}\right)^T \lambda & + & u^T \left(\frac{\partial B(x)}{\partial x}\right)^T \lambda, \\ \frac{\partial H}{\partial u} & = & Ru & + & B^T\lambda & = & 0. & & \end{array}$$

The above equations may be rearranged to form a set of nonlinear homogeneous differential equation as introduced in equation (5.3). For easy reference, this equation is given again here:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(x) & -B(x)R^{-1}B^T(x) \\ -Q & \bar{B}(x,\lambda) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \tag{5.6}$$

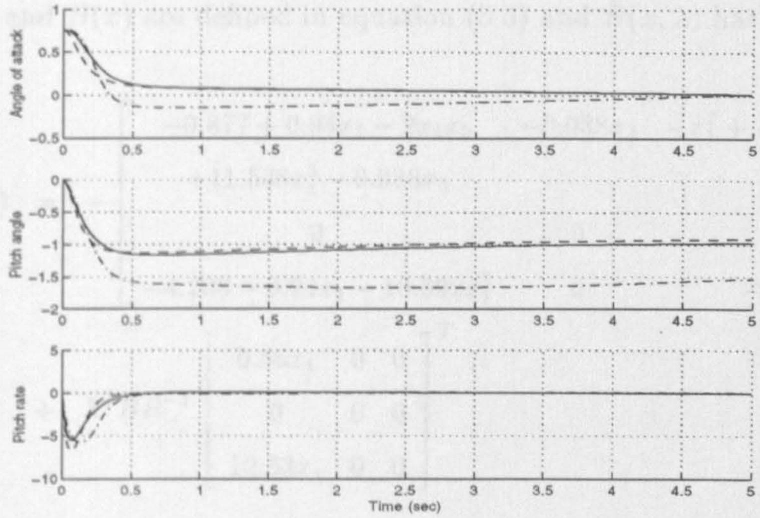


(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$

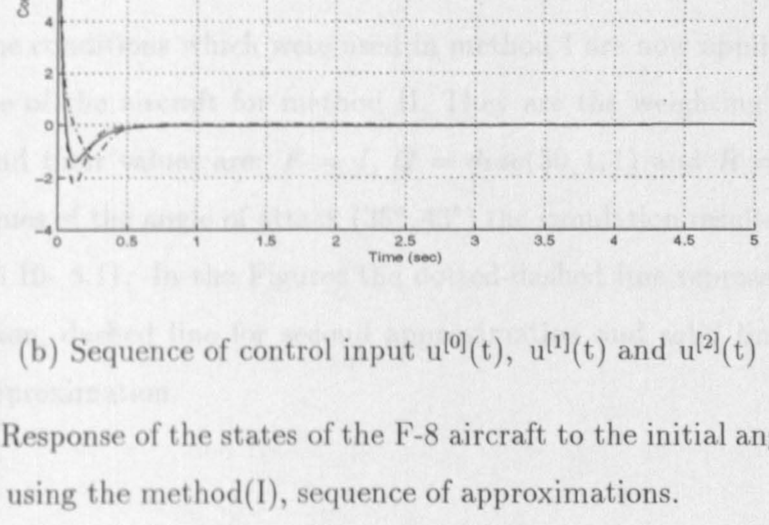


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 5.8: Response of the states of the F-8 aircraft to the initial angle of attack $\alpha(0) = 35°$, using the method(I), sequence of approximations.



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$



(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

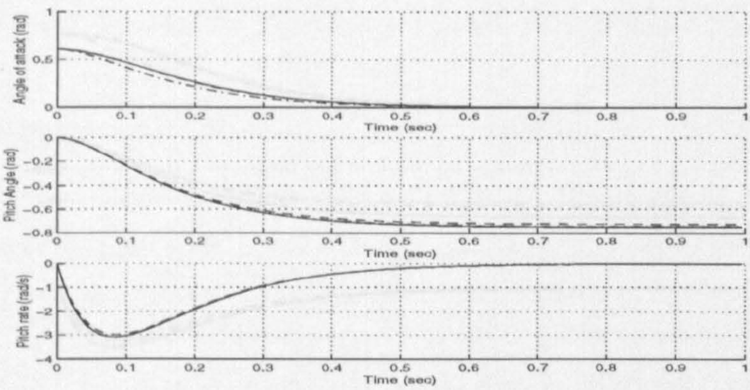
Figure 5.9: Response of the states of the F-8 aircraft to the initial angle of attack $\alpha(0) = 43°$, using the method(I), sequence of approximations.

where $A(x)$ and $B(x)$ are defined in equation (5.5) and $\bar{B}(x,\lambda)$ has the form:

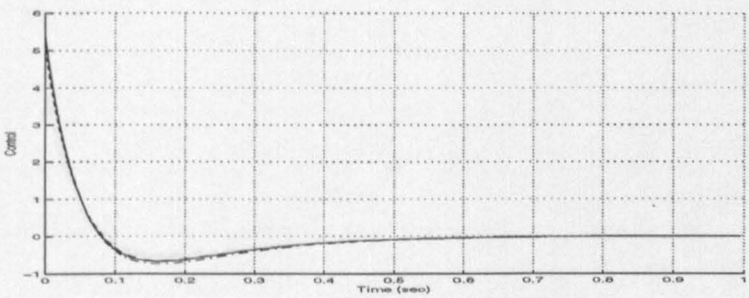
$$\bar{B}(x,\lambda) = -\begin{bmatrix} -0.877+0.94x_1-2x_1x_3 & -0.038x_2 & -x_1^2+1-0.088x_1 \\ +11.538x_1^2-0.088x_3 & & \\ 0 & 0 & 1 \\ -4.208-0.94x_1-10.692x_1^2 & 0 & -0.396 \end{bmatrix}^T$$
$$+\ \lambda^T B R^{-1}\begin{bmatrix} 0.56x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 12.53x_1 & 0 & 0 \end{bmatrix}^T.$$

A sequence of approximations is introduced into equation (5.6). Since we have known initial conditions for the state x and terminal conditions for λ, the original differential equation (5.6) is transformed into sequence of linear (time-varying) two point boundary value problems. The missing initial conditions may be found for $\lambda^{[i]}(t_0)$ using the method of adjoints. The terminal condition of $\lambda^{[i]}(t_f)$ for each iteration are approximated by $\lambda^{[i]}(t_f) = Fx^{[i-1]}(t_f)$. For the first iteration step $i=0$, the value of $\lambda^{[0]}(t_f)$ is given by $\lambda^{[0]}(t_f) = Fx_0$.

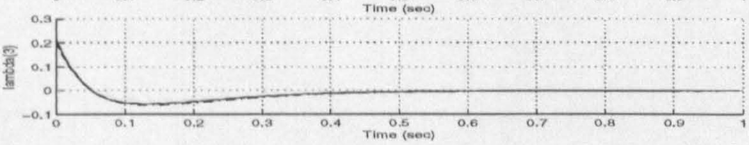
The same conditions which were used in method I are now applied to obtain the response of the aircraft for method II. They are the weighting matrices F, Q and R and their values are: $F = I$, $Q = diag(50,1,1)$ and $R = 1$. For the different values of the angle of attack $(35°, 43°)$ the simulation results are plotted in Figures 5.10- 5.11. In the Figures the dotted-dashed line represents the first approximation, dashed line for second approximation and solid line represents the third approximation.



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$

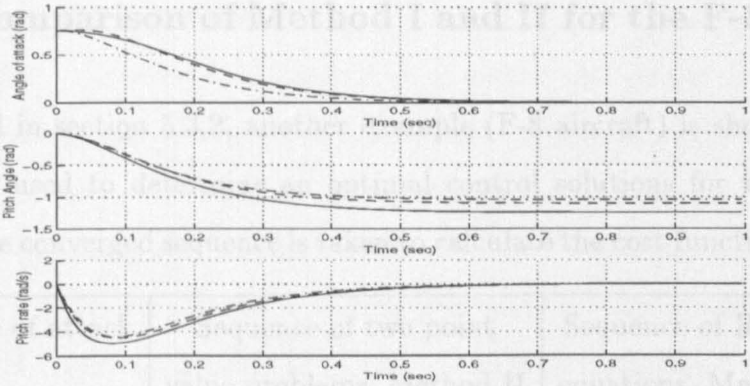


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

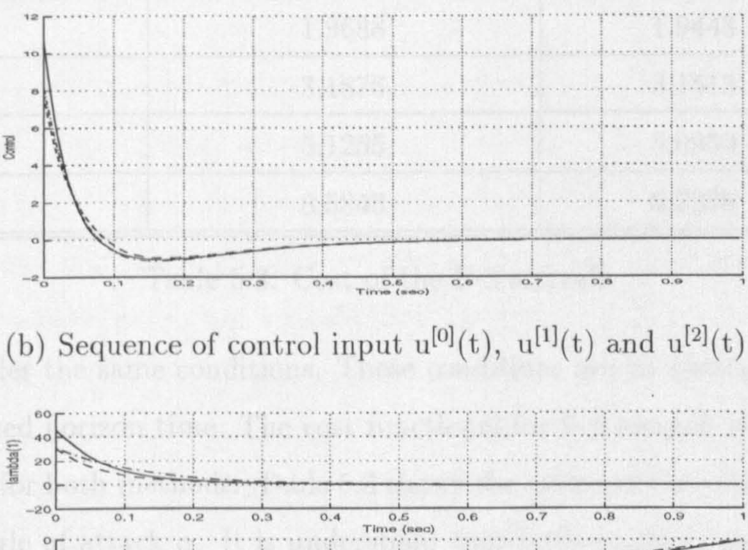


(c) Sequence of state $\lambda^{[0]}(t)$, $\lambda^{[1]}(t)$ and $\lambda^{[2]}(t)$

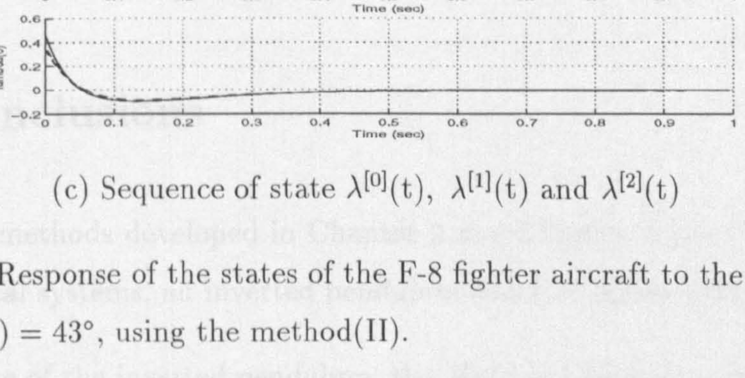
Figure 5.10: Response of the states of the F-8 fighter aircraft to the initial angle of attack $\alpha(0) = 35°$, using the method(II).



(a) Sequence of state $x^{[0]}(t)$, $x^{[1]}(t)$ and $x^{[2]}(t)$



(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$



(c) Sequence of state $\lambda^{[0]}(t)$, $\lambda^{[1]}(t)$ and $\lambda^{[2]}(t)$

Figure 5.11: Response of the states of the F-8 fighter aircraft to the initial angle of attack $\alpha(0) = 43°$, using the method(II).

5.3.4 Comparison of Method I and II for the F-8 Aircraft

As discussed in section 5.3.2, another example (F-8 aircraft) is shown in which Method I is used to determine an optimal control solutions for the nonlinear systems. The converged sequence is taken to calculate the cost functional for both methods under the same conditions. These conditions are weighting matrices F, Q, R and fixed horizon time. The cost functional for F-8 aircraft was calculated for 1 second for both methods. Table 5.3 shows the values of the cost for different values of angle of attack α. It is understood that both methods provide similar cost to the system.

Angle of attack α	Sequence of two point value problems, Method II	Sequence of Riccati equations, Method I
20	1.1580	1.1435
25	1.9688	1.9443
30	3.1876	3.1513
35	5.1255	5.0950
40	8.5848	8.7378

Table 5.3: Cost of the F-8 aircraft

5.4 Conclusions

Two control methods developed in Chapter 2 and Chapter 3 have been applied to two physical systems, an inverted pendulum and F-8 fighter aircraft.

In the case of the inverted pendulum, the Method I (sequence of approximations) and Method II (sequence of two point boundary value problems) provided a controller which is able to balance the pendulum provided that the initial angle is less than 60°. However, if the initial angle beyond 60° these methods would only work for very short time. For example, if the initial angle is 160°, these methods only converge for the first 0.5 seconds of the simulation. One may consider this 0.5 second as the horizon time used in the receding horizon philosophy where every 0.5 second interval is considered as a new optimization policy in which method I may be applied with new set of initial conditions, resulting in global (sub optimal) control for nonlinear systems. These ideas will be considered as future development.

The theory of Method II provides global optimal control for nonlinear systems but has some computational disadvantages. The set of initial vectors is determined from a set of linear independent terminal vectors by backward integration. However, due to numerical computation errors, the initial vectors may form a set of linear dependent vectors. As a result the inverse matrix at the initial time may become singular. This causes this method II to diverge very quickly. However, there are ways one can apply techniques such as orthonormalizing the matrix each time whenever these vectors become linear dependent. By applying these techniques to method II, the simulation can be run for long a period of time. These ideas will be considered for future developments of this work.

For F-8 fighter aircraft, both methods are able to recover from stall when the angle of attack is up to 43°.

Chapter 6

Practical Applications for Stochastic Systems

6.1 Introduction

In this chapter some applications of the theory developed in Chapter 4 are investigated.

For partially observable system with linear dynamics and quadratic integral cost criteria the separation principle is well known (see [54]). The important feature of the separation principle allows for solving the state estimation problem first, by explicitly characterizing the conditional mean of the state process given the measurements, and then solving the control problem which is completely observable and has a new state, the conditional mean (Kalman-filter estimator).

Using this linear filtering and regulator control theory a sequence of approximations to nonlinear filtering and nonlinear stochastic control problem is introduced and shown to converge under mild assumptions in Chapter 4. The converged sequence provides almost optimal control solutions for the nonlinear stochastic control problem.

For the nonlinear stochastic control problem, the traditional design method involves linearizing the nonlinear systems at the equilibrium point. This method has some drawbacks in that the performance may be poor when the system is operating far away from the equilibrium point.

The linearization and design of gain scheduled control of nonlinear systems have been discussed by many authors, see [36, 39, 48].

In this chapter nonlinear systems such as the Van der Pol oscillator and inverted pendulum have been chosen to examine the nonlinear filter problem and the nonlinear stochastic control problem. The results show that the proposed method provide a controller which is able to control the nonlinear stochastic system over large operating range and in the case of the inverted pendulum, the controller has been able to balance the pendulum over large angles in the range $0 - 60°$.

6.2 The Van der Pol Oscillator

The Van der Pol oscillator is characterized by the following differential equation [24]:

$$\ddot{x} - \epsilon\dot{x}(t)(1 - x^2(t)) + x(t) = 0$$

which describes a dynamical system with state-dependent damping coefficient $\epsilon(1 - x^2(t))$, ϵ is a positive parameter. The damping in the system goes from negative to zero to positive values as the value of $x^2(t)$ changes from less than to greater than unity. The oscillator's response is characterized by a limit cycle in the $x(t)$, $\dot{x}(t)$ plane (phase plane). The limit cycle approaches a circular shape as ϵ becomes very small, and it has a maximum value of $x(t)$ equal to 2.0 irrespective of ϵ. This type of oscillation occurs in the electronic tubes, which also exhibit what is known as thermal noise.

If x_1 and x_2 represent state of the system x and $\dot{x}$, then the state space form of the Van der Pol equation becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon(1 - x_1^2(t)) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Also suppose that the following measurement is taken:

$$y(t) = (1 + x_1^2)x_1(t) + w_1(t),$$

where v_1, v_2 and w_1 are white noise. Values for noise statistics are shown in the table 6.1. Also ϵ is taken to be 0.2. $\tilde{R}$ is a positive value, $\tilde{Q}$ is a (2×2) positive semi-definite matrix and $\tilde{P}_0$ is also a (2×2) positive semi-definite matrix.

Case	$\tilde{Q}_{11}$	$\tilde{Q}_{12}$	$\tilde{Q}_{21}$	$\tilde{Q}_{22}$	$\tilde{R}$	Figure	$\tilde{P}_0$
Vander Pol 1	0.5	0	0	0.5	4	6.1	0.1I
Vander Pol 2	5.0	0	0	5.0	10.0	6.2	0.1I

Table 6.1: Noise statistics

6.2.1 Numerical Experiment

In order to illustrate the algorithm developed in Chapter 4 (Bank-Dinesh approach), the nonlinear filtering is investigated using a Van der Pol oscillator.

In the figures, the following symbols are used:

$$\begin{aligned} \mathrm{EKF} &\equiv \text{Extened Kalman Filter algorithim} \\ \mathrm{B-D} &\equiv (\mathrm{Banks-Dinesh})\ \text{approach.} \end{aligned}$$

In B-D approach, the converged sequence has been taken to compare the response with EKF for the case Vander Pol 1 and Vander Pol 2 which are shown in the figure 6.1 and 6.2 respectively.

6.3 Stochastic Control

In this section a continuous time partially observed nonlinear system is considered as a regulator problem. For a system represented by differential equation

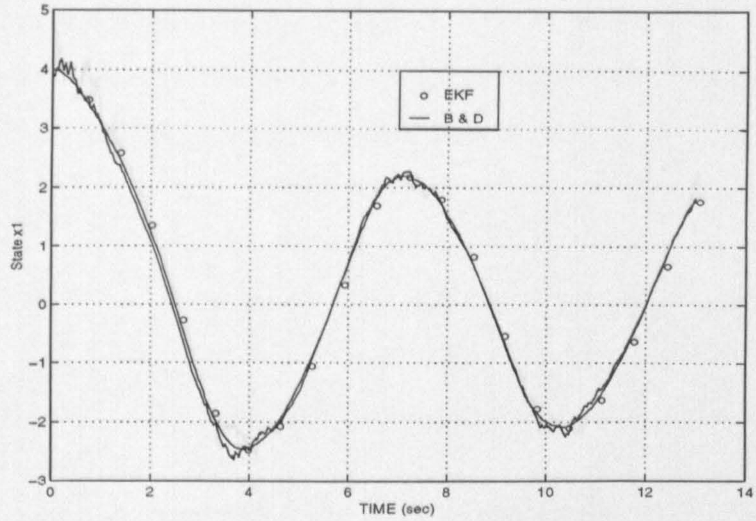
$$\dot{x} = A(x)x + B(x)u + v(t), \quad x(t_0) = x_0, \quad E\{x_0\} = \bar{x_0},$$

together with the nonlinear measurement equation given by

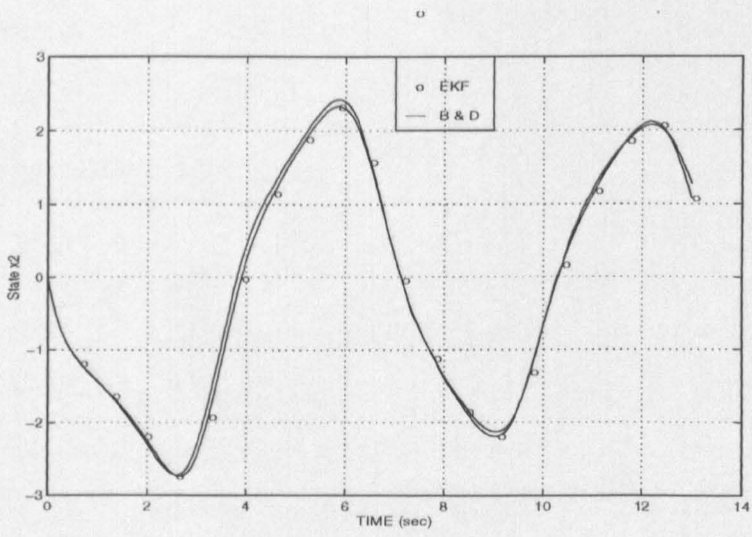
$$y(t) = C(x)x + w(t),$$

stochastic optimal control is found so as to minimize the cost

$$min\ J(u) = E\left\{x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f} \left(x^TQx + u^TRu\right)dt\right\}.$$

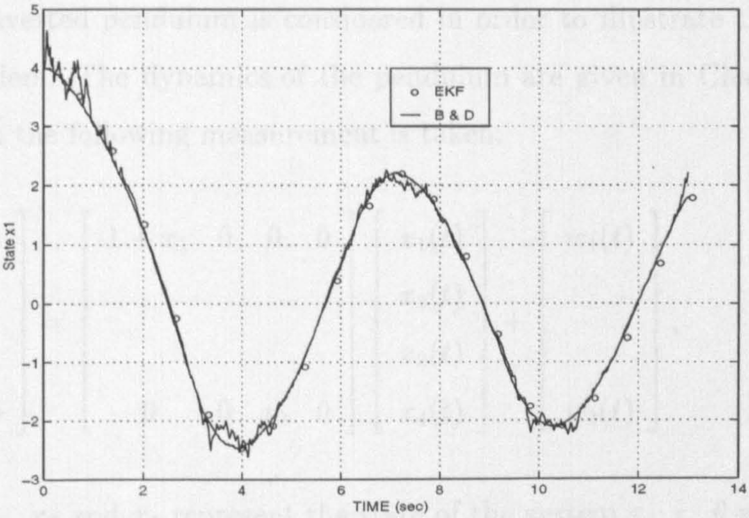


(a) First state and estimate by B-D, EKF

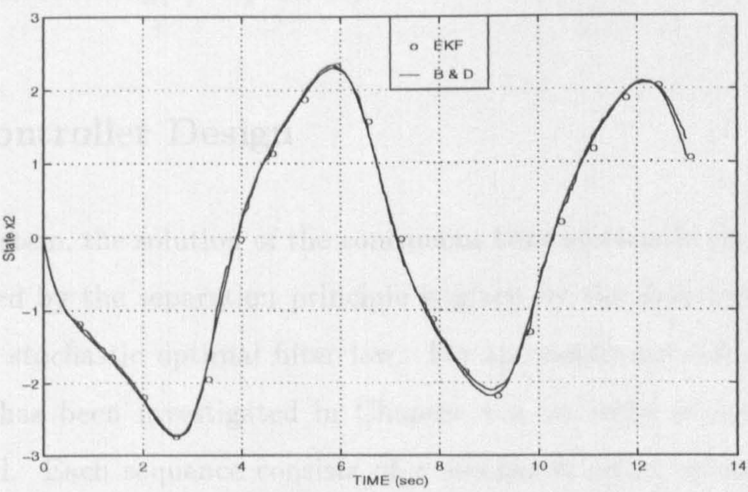


(b) Second state and estimate by B-D, EKF

Figure 6.1: Response of the states of the Van der Pol oscillator to the case Vander Pol 1.



(a) First state and estimate by B-D, EKF



(b) Second state and estimate by B-D, EKF

Figure 6.2: Response of the states of the Van der Pol oscillator to the case Vander Pol 2.

The noisy inverted pendulum is considered in order to illustrate the stochastic control problem. The dynamics of the pendulum are given in Chapter 5. Also suppose that the following measurement is taken:

$$\begin{bmatrix} y_1(t) \\ \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1+x_1 & 0 & 0 & 0 \\ \\ 0 & 0 & x_3 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ \\ w_2(t) \end{bmatrix},$$

where x_1, x_2, x_3 and x_4 represent the state of the system x, $\dot{x}$, θ and $\dot{\theta}$.

For the inverted pendulum, the stochastic control problem has been considered with the presence of small enough Gaussian white noise $v(t)$ and $w(t)$ in the state $x(t)$ and measurement $y(t)$ respectively.

6.3.1 Controller Design

For linear system, the solution of the continuous time stochastic control problem as determined by the separation principle is given by the deterministic control law and the stochastic optimal filter law. For the nonlinear stochastic control problem as has been investigated in Chapter 4 a sequence of approximations is introduced. Each sequence consists of a stochastic linear system, therefore, separation principle may be used to solve the control problem.

The design parameters are chosen as follows:
the deterministic control parameters $Q = diag(1,1,100,1)$, $R = 1$, $F = I$ and the filter parameters $\tilde{P}_0 = 0.01I$, $\tilde{R} = 2\ I$ and $\tilde{Q} = 10I$.

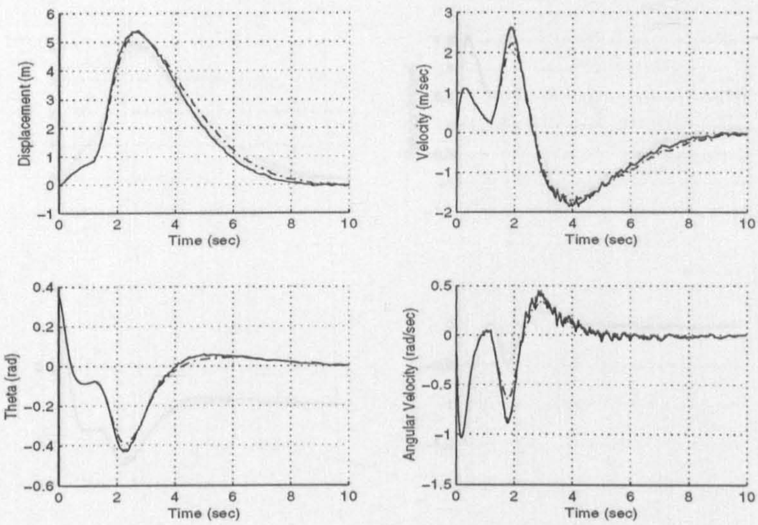
6.3.2 Simulation results

The simulation results are presented in Figures 6.3 - 6.5 for $\theta = 20°$, $30°$, and $50°$. The dotted line represents the first approximation, dashed line for second approximation and solid line represent the third approximation and the converged solution. The method is seen to converge rapidly to provide very effective control.

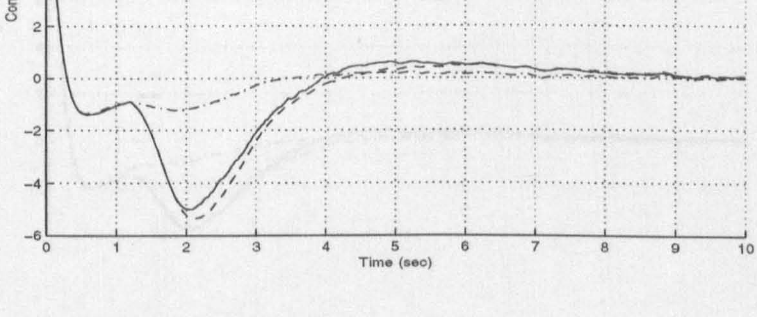
6.4 Conclusions

In this chapter a new technique for the control of partially observable stochastic nonlinear systems has been presented. For the sample problems considered, the procedure was seen to offer substantial improvement in systems performance as compared to linearization at equilibrium point. The resulting control is almost optimal.

Possible applications besides those already discussed might include, for example, optimal guidance and navigation policies for space and terrestrial vehicles and optimum closed loop process controllers.

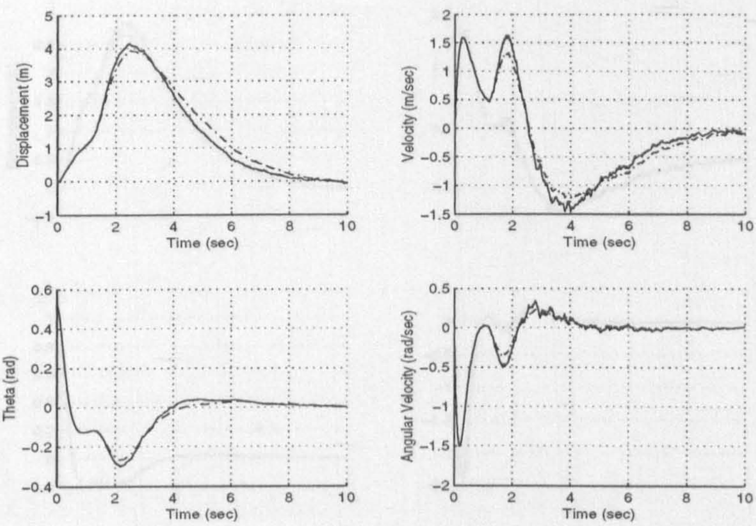


(a) Sequence of estimate state $\hat{x}^{[0]}(t)$, $\hat{x}^{[1]}(t)$ and $\hat{x}^{[2]}(t)$

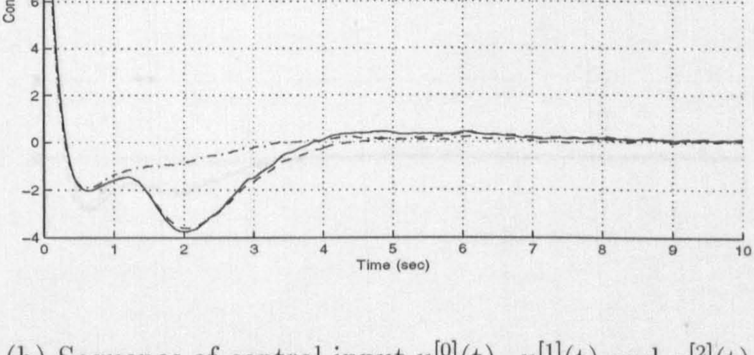


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 6.3: Response of the estimate states of the inverted pendulum to the initial angle $\theta = 20°$.

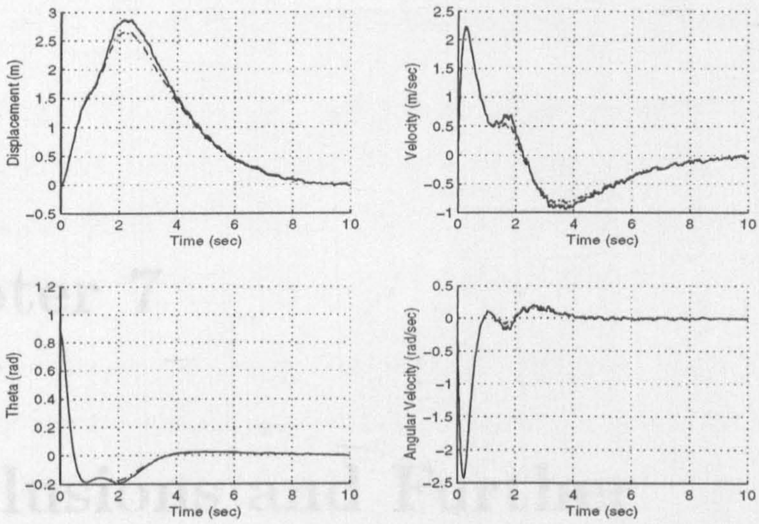


(a) Sequence of estimate state $\hat{x}^{[0]}(t)$, $\hat{x}^{[1]}(t)$ and $\hat{x}^{[2]}(t)$

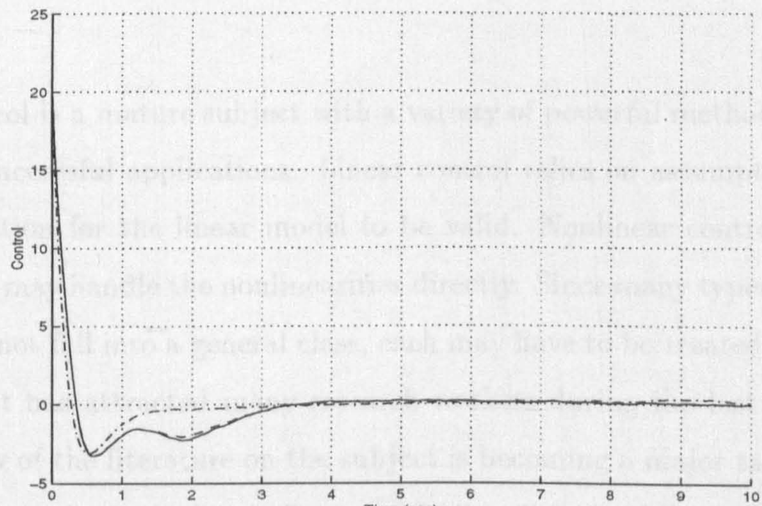


(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 6.4: Response of the estimate states of the inverted pendulum to the initial angle $\theta = 30°$.



(a) Sequence of estimate state $\hat{x}^{[0]}(t)$, $\hat{x}^{[1]}(t)$ and $\hat{x}^{[2]}(t)$



(b) Sequence of control input $u^{[0]}(t)$, $u^{[1]}(t)$ and $u^{[2]}(t)$

Figure 6.5: Response of the estimate states of the inverted pendulum to the initial angle $\theta = 50°$.

Chapter 7

Conclusions and Further Research

Linear control is a mature subject with a variety of powerful methods and a long history of successful applications. Linear control relies on assumptions of small range operation for the linear model to be valid. Nonlinear controllers, on the other hand, may handle the nonlinearities directly. Since many types of nonlinear systems do not fall into a general class, each may have to be treated individually. Therefore, it has attracted many research workers during the last thirty years. An overview of the literature on the subject is becoming a major task. However, the different mathematical technique used in the analysis of these systems can be recognized.

In this work we have considered the nonlinear system of the form

$$\begin{aligned} \dot{x}(t) &= A(x)x(t) \quad + B(x)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(x)x(t). \end{aligned}$$

It has been analyzed from the deterministic and stochastic points of view.

For the deterministic case, in chapter 2, a sequence of approximations to the nonlinear system has been introduced. It is shown that if t_f or(x_0) is small enough this sequence converges uniformly ($in\ \mathcal{C}[t_0, t_f], \mathcal{R}^n$) to function $x(t)$ and $u(t)$. The converged sequence is the solution to the nonlinear problem.

It should be noted that the method proposed here may easily be generalized to the case of nonlinear cost functional of the form

$$min\ J(u) = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left\{x^T(t)Q(x)x(t) + u^T(t)R(x)u(t)\right\}.$$

It would be another interesting area for future development.

In chapter 3, in order to verify that the proposed method in chapter 2 is an optimal control for a nonlinear system, the same problem has been considered with Pontryagin's minimum principle. It has provided a nonlinear two point boundary value problem (bvp). We have transformed the nonlinear two point bvp into a sequence of linear two point bvps and have shown it to converge ($in\ \mathcal{C}[t_0, t_f], \mathcal{R}^n$) for small t_f or(x_0). Since this method requires the determination of an inverse matrix, which may sometimes contain linearly dependent vectors due to numerical error, a singular matrix may result. In particular, this occurs when the matrix A has widely separated eigenvalues. Further research is required to ensure the linear independence of the vectors of the matrix obtained by using orthonormalization procedure.

In chapter 5, two examples are considered to demonstrate the controllers developed in chapter 2 and chapter 3. Having calculated the cost functional for the two methods under the same conditions, it is shown that both methods have similar costs for the nonlinear system. Therefore, the method developed in chapter 2 provides almost optimal control for a nonlinear system.

This method (in chapter 2) has many advantages:

- In fact, any nonlinear problem for which the corresponding problem for time-varying linear system can be solved is amenable to this method.
- This method may be applied using receding horizon philosophy for real time applications. This is another interesting topic for further research.
- It provides almost optimal control for nonlinear systems.
- Numerical experience shows that after a few iteration steps a good approximation to the solution is obtained.
- Using receding horizon philosophy, this method may be extended to obtain global control for a nonlinear system. This is another area for development.

In chapter 4, the nonlinear stochastic control problem has been considered. Using the same method, optimum filter and almost optimal control to the nonlinear stochastic system has been obtained.

In chapter 6, the performance of the filter and controller has been examined. The results show substantial improvement in systems performance.

Another promising area of research is the use of distributed parameter systems to extend the results obtained in this work. This can be done by writing the partial differential equations in the state-space form on some Hilbert space (see [7]).

Finally, note that the nonlinear system $\dot{x} = f(x, u)$ when written in the form $\dot{x} = A(x)x + B(x)u$ does not have a unique representation. Then there are many different matrix functions $A(x)$ and $B(x)$ which are possible. Future research may include the theories developed here can be applied to the system (inverted pendulum, F-8 aircraft) for different forms of $A(x)$ and $B(x)$, and whether all possible state-space forms are produced the same results.

Bibliography

[1] Z. Aganovic and Z. Gajic. The successive approximation procedure for finite time optimal control of bilinear systems. *IEEE Trans. on Automatic Control*, 39(9):1932–1935, 1994.

[2] C.W. Anderson. Learning to control an inverted pendulum using neural network. *IEEE Control Systems Magazine*, 9(3):31–37, 1989.

[3] S.P. Banks. On the optimal control of nonlinear systems. *Systems Control Letters*, 6:337–343, 1986.

[4] S.P. Banks. *Control Systems Engineering*. Prentice-Hall, London, 1988.

[5] S.P. Banks. Optimal control of distributed bilinear systems. *Systems Control Letters*, 19:629–636, 1988.

[6] S.P. Banks. Infinite-dimensional Carleman linearisation, the Lie series and optimal control of nonlinear PDE. *Int J. Systems Science*, 23:663–675, 1992.

[7] S.P. Banks and K. Dinesh. Approximate optimal control and stability of nonlinear finite and infinite dimensional systems. *Invited Paper for Annals of Operations Research*, submitted 1998.

[8] S.P. Banks and D. McCaffrey. Approximate optimal controllers for nonlinear parabolic systems. In *15th IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics*, volume 5, pages 185–190, Berlin, August 1997.

[9] S.P. Banks and K.J. Mhana. Optimal control and stabilization for nonlinear systems. *IMA Journal of Mathematical Control and Information*, 9:179–196, 1992.

[10] S.P. Banks and M.K. Yew. On a class of suboptimal controls for infinite dimensional bilinear systems. *Systems Control Letters*, 5:327–333, 1985.

[11] W.T. Baumann and W.J. Rugh. Feedback control of nonlinear systems by extended linearization. *IEEE trans.Automat.Control*, 31:40–46, 1986.

[12] R.W. Beard, G.N. Saridis, and J.T. Wen. Approximate solutions to the time-invarient Hamilton-Jacobi-Bellman equation. *J.Optimization Theory and Applications*, 96(3):589–626, 1998.

[13] R. Bellman. *Introduction to Mathematical Theory of Control Processes.* Academic Press, New York, 1971.

[14] K. Brammer and G. Siffling. *Kalman-Bucy Filter.* Artech House, Norwood, MA, 1989.

[15] F. BRAUER. PERTURBATIONS OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS.II. *J. of Mathematical Analysis and Applications*, 17:418–434, 1967.

[16] R.G. BROWN AND PATRICK.Y.C. HWANG. *Introduction to Random Signals and Applied Kalman Filtering.* JOHN WILEY, NEW YORK, 2ND EDITION, 1992.

[17] C. BRUNI, G. DIPILLO, AND G. KOCH. BILINEAR SYSTEMS: AN APPEALING CLASS OF "NEARLY LINEAR" SYSTEMS IN THEORY AND APPLICATIONS. *IEEE Trans. on Automatic Control*, 19(4):334–348, 1974.

[18] A.E. BRYSON AND Y.C. HO. *Applied Optimal Control.* HEMISPHERE PUBLISHING CORPORATION, WASHINGTON, DC, 1975.

[19] R.S. BUCY AND P.D. JOSEPH. *Filtering for Stochastic Processes with Applications to Guidance.* JOHN WILEY, NEW YORK, 1968.

[20] KA.C. CHEOK AND NAN.K LOH. A BALL-BALANCING DEMONSTRATION OF OPTIMAL AND DISTURBANCE-ACCOMMODATING CONTROL. *IEEE Control Systems Magazine*, 7(1):54–57, 1987.

[21] W.A. COPPEL. *Stability and Asymptotic Behavior of Differential Equations.* D.C.HEATH, BOSTON, 1965.

[22] C.A. DESOER AND H. HANEDA. THE MEASURE OF A MATRIX AS A TOOL TO ANALYZE COMPUTER ALGORITHIMS FOR CIRCUIT ANALYSIS. *IEEE Trans. on Circuit Theory*, 19(5):480–486, 1972.

[23] A.A. DESROCHERS AND R.Y. AL-JAAR. NONLINEAR MODEL SIMPLIFICATION IN FLIGHT CONTROL SYSTEM DESIGN. *J.Guidance*, 7:684–689, 1984.

[24] H.E. Emara-Shabaik and C.T. Leondes. Nonlinear filtering-the link between Kalman and extended Kalman filters. *Int.J. of Control*, 34(6):1207–1214, 1981.

[25] W.H. Fleming and R.W. Rishel. *Deterministic and Stochastic Optimal Control.* Springer-Verlag, New York, 1975.

[26] W.L. Garrard. Suboptimal feedback control for nonlinear systems. *Automatica*, 8:219–221, 1975.

[27] W.L. Garrard and J.M. Jordan. Design of nonlinear automatic flight control systems. *Automatica*, 13:497–505, 1977.

[28] S.T. Glad. Robustness of nonlinear state feedback-a survey. *Automatica*, 23:425–435, 1987.

[29] S. Haykin, P. Yee, and E. Derbez. Optimum nonlinear filtering. *IEEE Trans. on Signal Processing*, 45(11):2774–2786, 1997.

[30] E.P. Hofer and B. Tibken. An iterative method for the finite time bilinear quadratic control problem. *Journal of Optimization Theory and Applications*, 57(3):411–427, 1988.

[31] V. Hutson and J.S. Pym. *Applications of Functional Analysis and Operator Theory.* Academic Press, London, 1980.

[32] A. Isidori. *Nonlinear Control Systems,*. Springer Velag, new york, 2nd edition, 1989.

[33] V. Jorgensen. A ball-balancing system for demonstration of basic concepts in the state-space control theory. *Int. J.Elect.Enging Educ.*, 11:367–376, 1974.

[34] R. E. KALMAN. A NEW APPROACH TO LINEAR FILTERING AND PREDICTION PROBLEMS. *Trans.Amer.Soc.Mech.Eng., J.Basic Eng.,*, 82:35–45, 1960.

[35] D.E. KIRK. *Optimal Control Theory: An Introduction.* PRENTICE HALL, ENGLE WOOD CLIFFS, NEW JERSEY, 1992.

[36] A.J. KRENER. ON THE EQUIVALENCE OF CONTROL SYSTEMS AND LINEARIZATION OF NONLINEAR SYSTEMS. *SIAM J.Control*, 2:670–676, 1973.

[37] B. KUGELMANN AND H.J. PESCH. NEW GENERAL GUIDANCE METHOD IN CONSTRAINED OPTIMAL CONTROL. PART 1: NUMERICAL METHOD. *J.Optimization Theory and Application*, 67:421–435, 1990.

[38] H. KWAKERNAAK AND R. SIVAN. *Linear Optimal Control Systems.* JOHN WILEY, NEW YORK, 1972.

[39] D.A. LAWRENCE AND W.J. RUGH. GAIN SCHEDULING DYNAMIC LINEAR CONTROLLERS FOR A NONLINEAR PLANT. *Automatica*, 31:381–390, 1995.

[40] D.L. LUKES. OPTIMAL REGULATION OF NONLINEAR DYNAMICAL SYSTEMS. *SIAM Journal on Control*, 7:75–100, 1969.

[41] S. MORI, H. NISHIHARA, AND K. FURUTA. CONTROL OF UNSTABLE MECHANICAL SYSTEM: CONTROL OF PENDULUM. *Int. J. Control*, 23(5):673–692, 1976.

[42] Y. NISHIKAWA, N. SANNOMIYA, AND H. ITAKURA. A METHOD FOR SUBOPTIMAL DESIGN OF NONLINEAR FEEDBACK SYSTEMS. *Automatica*, 7:703–712, 1971.

[43] E. W. PACKEL. *Functional Analysis a Short Course.* INTEXT, NEW YORK, 1974.

[44] H.J. Pesch. Real-time computation of feedback controls for constrained optimal control problems, part 1: Neighboring extremals. *Optimal Control Applications and Methods*, 10:129–145, 1990.

[45] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko. *The Mathematical Theory of Optimal Processes.* Pergamon, 1964.

[46] C. Reboulet and C. Champetier. A new method for linearizing nonlinear systems: the pseudo-linearization. *Int.J.Control*, 40:631–638, 1984.

[47] S.M. Roberts and J.S Shipman. *Two point boundary value problems:Shooting Methods.* 31. Elsevier, New York, 1972.

[48] W.J. Rugh. Analytical framework for gain scheduling. *IEEE Control Systems Magazine*, 11(1):79–84, 1991.

[49] G. Shorshi and I.Y. Bar-Itzhack. Satellite autonomous navigation based on magnetic field measurements. *J. of Guidance, Control and Dynamics*, 18(4):843–850, 1995.

[50] J.E. Slotine and L.I Weiping. *Applied Nonlinear Control.* Prentice hall, Engle Wood Cliffs, New Jersey, 1991.

[51] Hua.O. Wang, K. Tanaka, and M.F. Griffin. An approach to fuzzy control of nonlinear systems: Stability and design issues. *IEEE Tran. on Fuzzy Systems*, 1(4):14–23, 1996.

[52] J. Wang and N. Sundararajan. A nonlinear flight controller design for aircraft. *Control Eng. Practice*, 3:813–825, 1995.

[53] J. Wang and N. Sundararajan. Extended nonlinear flight controller design for aircraft. *Automatica*, 8:1187–1193, 1996.

[54] W.M. Wonham. On the separation theorem of stochastic control. *SIAM J. on Control*, 6:312–326, 1968.

[55] T. Yoshida and K.A. Loparo. Quadratic regulatory theory for analytic nonlinear system with additive controls. *Automatica*, 25:531–544, 1989.