Learning through teaching: factors influencing teachers' mathematics knowledge

Rebecca Kay Warburton

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Dedication and Acknowledgements

I dedicate this thesis to my daughter, Esther, whose birth during the writing-up stage provided perspective. May you always ‘reach’ your goals.

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Abstract

Understanding mathematics teacher knowledge is an international endeavour, seen by researchers as a key part of improving pupil learning. Within the last few decades, several conceptions of teacher knowledge have been proposed within the literature including Mathematical Knowledge for Teaching (Ball and colleagues) and the Knowledge Quartet (Rowland and colleagues). However, multiple criticisms of these conceptions exist, prompting the introduction of a new approach to considering teacher knowledge within this thesis. Rather than seeking to categorise a knowledge unique to teaching different than the mathematical knowledge required for other professions, this research aims to examine how knowledge changes within the context of trainee secondary teachers in England. The poor mathematics results of school leavers in the UK as well as a shortage of mathematics teachers, has influenced government policies on teacher training. Bursaries differentiated by degree class and the introduction of government-sponsored 'subject knowledge enhancement' (SKE) courses (to graduates from numerate disciplines) attempt to increase the quality and supply of teachers. By examining how knowledge changes over a teacher training course, with emphasis on the divide between SKE and non-SKE course participants, it is proposed that further insights into the knowledge useful for teaching and how this knowledge needs to be organised can be gleaned. This mixed methods study employs questionnaires, interviews and observations to track the knowledge change of a sample of Postgraduate Certificate in Education (PGCE) students over their year-long course. Results of the current study suggest that changes in the quality rather than quantity of knowledge take place over a PGCE course, in other words, a change in the organisation of knowledge. In addressing the research questions, this study also: raises questions about what the Mathematical Knowledge for Teaching items measure; suggests potential changes to the Knowledge Quartet codes; evaluates the proposed alternative approach to knowledge; and, discusses implications for teacher training policy.

Table of Contents

List of Tables

List of Figures

List of Abbreviations

A-level	Advanced Level
ACME	Advisory Committee on Mathematics Education
ANOVA	Analysis of variance
AS level	Advanced Subsidiary level
BIDMAS	Brackets, Indices, Division, Multiplication, Addition, Subtraction
COACTIV	Cognitive Activation in the classroom (project)
CK	Content Knowledge (used for Shulman's content knowledge only)
CCK	Common Content Knowledge
DIF	Differential item functioning
EP	Exceptional Performance
GCSE	General Certificate of Secondary Education
IRT	Item Response Theory
ITT	Initial teacher training
IWB	Interactive white board
KCS	Knowledge of Content and Students
LMT	Learning Mathematics for Teaching (research project at University of Michigan)
LSIS	Learning and Skills Improvement Service
MKT	Mathematical Knowledge for Teaching
MQI	Mathematical quality of instruction
NCTL	National College of Teaching and Leadership
NCTM	National Council of Teachers of Mathematics
NQT	Newly Qualified Teacher
OECD	Organisation for Economic Co-operation and Development

Ofsted	The Office for Standards in Education, Children's Services and Skills
PCK	Pedagogical Content Knowledge
PGCE	Postgraduate Certificate in Education
PISA	Programme for International Student Assessment
PUFM	Profound understanding of fundamental mathematics
QCA	Qualifications and Curriculum Authority
QTS	Qualified Teacher Status
RQ(1-4)	Research Question (1-4)
SCAAT	School and College Achievement and Attainment Table (points)
SKE	Subject Knowledge Enhancement (course)
SCK	Specialised Content Knowledge
TDA	Training and Development Agency for Schools
TEDS-M	The Teacher Education and Development Study in Mathematics
UCAS	Universities and Colleges Admissions Service

1 Introduction

This chapter introduces the problem to be addressed within this thesis as well as providing useful background (contextual) information. General information about the study as a whole including: significance of the study, scope and boundaries, the language used, and structure of the thesis is also presented. Theoretical background is provided in Chapter 2.

1.1 The problem to be addressed

Research into content knowledge specifically for mathematics teaching has been conducted internationally (e.g. Delaney et al., 2008; Hill et al., 2004; Ma, 1999). However, despite teacher knowledge being a focus of mathematics education research within the last few decades (Hill et al., 2004), there is lack of consensus regarding the knowledge teachers require. For example, in the USA some teacher licensing tests involve solving middle-school level mathematical problems whereas others test ability to set questions and tasks for students (Hill et al., 2004). Not only does disagreement exist geographically, but also temporally. Shulman (1986) describes how the view of teacher knowledge has altered over time, with a changing interplay between content knowledge and knowledge of pedagogy.

Nevertheless, a key idea proposed within the literature is that mathematics teachers need to have knowledge which is generally different from other professionals who utilise mathematics, a knowledge which enables them to deal effectively with the issues of day-to-day practice (Stylianides and Stylianides, 2006). Indeed, several conceptions of knowledge for teaching mathematics have been proposed (and criticised) within the literature (see 2.1, p.14). This view is examined within the current research, joining with others in the view that the mathematics knowledge required by teachers can be discovered within the context of teaching practice (Steinbring, 1998; Ball and Bass, 2000; Davis and Simmt, 2006). Indeed, many researchers recognise that preparing to teach a concept often leads to a deeper understanding for the teacher (Steinbring, 1998; Ma, 1999; Leikin et al., 2000).

Despite teacher knowledge being a focus of mathematics education research within the last few decades (Hill et al., 2004), “... our understanding of what and how changes in teachers’ mathematical knowledge through teaching is relatively limited” (Leikin, 2005:1). Further, Stacey (2008) argues that research into secondary teachers’ mathematical knowledge should be a high priority since most research tends to focus on primary teachers.

Within the UK, the poor mathematics results of school-leavers is of growing concern to government and employers since: “demand for improved mathematical skills and understanding is growing” (ACME, 2011:1), yet a large proportion[1] of students fail GCSE Mathematics. Further, in the most recent Programme for International Student Assessment (PISA) survey, the UK ranked only 26 out of 34 participating countries in mathematics (OECD PISA, 2012). In order to address the issue, policy makers in England and elsewhere see defining and codifying the subject knowledge required for teaching as central to improving student learning (Askew, 2008).

Additionally, there is a shortage of mathematics teachers in the UK which has prompted government sponsorship of ‘subject knowledge enhancement’ (SKE) courses to graduates from numerate disciplines to increase the supply of mathematics teachers (TDA, 2010; McNutty, 2004). Additionally, financial incentives are offered to graduates to train in shortage subject areas including mathematics (Department for Education, 2010). These two backgrounds (those who have taken an SKE course and those who have not) of trainee teachers are taken into account for this research.

1.2 Towards a solution

Rather than seeking to categorise a knowledge unique to teaching, different from mathematical knowledge, this research aims to address gaps in the literature by examining how knowledge changes as trainee secondary mathematics teachers in England train and participate in teaching, with emphasis on the divide between SKE course participants and those deemed to already have sufficient subject knowledge.

1.3 Background information related to the context of the study

1.3.1 PGCE courses

A Postgraduate Certificate in Education (PGCE) course is one of several routes that leads to Qualified Teacher Status (QTS) - the accreditation required to teach in all state-maintained schools (except free schools and

[1] over a third did not achieve a pass at GCSE Mathematics this year (JCQ, 2015).

academies; Full Fact, 2015) within England and Wales. It can be taken as a one year full-time course or up to two years as a part-time course through a university or college in the UK (Department for Education, 2013d). Candidates must have a UK undergraduate degree (or recognised equivalent) and qualifications equivalent to a grade 'C' in GCSE English and Mathematics (Department for Education, 2013d) in order to apply to take a PGCE course. For a secondary mathematics PGCE course, the degree is usually required to be in mathematics or a mathematics related area such as engineering (see, for example, University of Manchester, 2015). However, institutions vary in their requirements as the precise level of mathematical knowledge required is determined somewhat subjectively. Indeed, those interviewing potential students determine whether the degree course taken by the student contained 'enough' mathematics. If the mathematical knowledge of the potential student is deemed to be lacking, there is the option of taking a SKE course to 'top-up' their mathematical knowledge before commencing the PGCE:

> Traditionally, having an A level or higher qualification in a specific or related subject has been considered the main barometer for determining the suitability of a candidate to teach at secondary level. Indeed many SKE tutors commented that they do not accept students without A levels or equivalent although many institutions place just as much importance on how prospective SKE students do at the interview. Many tutors acknowledged that there were other very important attributes, 'we look at the student as a whole, their experiences, profession and previous qualifications.' (Mathematics SKE Tutor). (Gibson et al., 2013:65).

The Department for Education website states:

> A PGCE course mainly focuses on developing your teaching skills, and not on the subject you intend to teach. For this reason, you are expected to have a good understanding of your chosen subject(s) – usually to degree level – before you start training. (Department for Education, 2013d:para.1).

Within England, all trainee teachers are required to pass a literacy and numeracy skills test (Department for Education, 2012b). After completing the course and skills tests, the candidate is then recommended for gaining QTS. They are then a Newly Qualified Teacher (NQT) required to take an induction period (three full terms of teaching in school) (Department for Education, 2013a).

There is no standardised PGCE curriculum which providers have to follow, but by the end of the course, PGCE students have to be able to demonstrate that they meet all of the QTS standards. PGCE courses usually combine teaching at the university or college by course tutors (mathematics teacher educators) and at least two placements in local secondary schools where trainees teach mathematics on a reduced timetable (UCAS, 2014).

1.3.2 PCGE course at Leeds

The PGCE course at the University of Leeds is a full-time, one year course which aims to enable students to teach the ages 11-18 (secondary school). It is typical of most PGCE course formats providing for some practical teaching experience alongside university-based training.

The practical teaching experience is in the form of two placements. The first placement (October – December) builds up to a 50% teaching timetable, taking classes mainly at KS3 but perhaps some A-level classes also. During the placement students are observed formally once a week (or more) by their school-based mentor (Meenan, 2012).

The second semester is similar to the first, except for a longer period (February – June) and with a greater teaching load (60% timetable). They are observed once a week by their school-based mentor and the second placement grades count towards their final marks (Meenan, 2012).

At Leeds, the focus of university teaching is on the “relationship between theory, research evidence and practice, and... the practical aspects of teaching and learning, such as assessment, differing learning styles, classroom organisation and behaviour management” (University of Leeds, 2011). Students are also encouraged to track their own mathematical knowledge through a subject knowledge audit throughout the course. Students’ final PGCE grades are based upon observations of their teaching practice placement as well as marks on three written assignments. All other tasks are marked as pass or fail for formative assessment and students retake them until they pass (Meenan, 2012).

1.3.3 Bursaries for teacher training

Degree class	**Bursary (£) 2012/13**
First	20,000
2:1	15,000
2:2	12,000

Table 1.1: Table showing differentiated bursaries available for teacher training in 2012/13 (Department for Education, 2013f)

Bursaries for teacher training have existed for many years but a new system of bursaries was introduced in 2011 with graduates with higher degree classes receiving larger financial incentives as an attempt to raise the quality of teachers (Department for Education, 2010). Existing bursaries of £9,000 were replaced by bursaries ranging from £12,000 to £20,000 depending on degree classification of the trainee (see **Table 1.1**, p.4). Further, from September 2012 funding for initial teacher training (ITT) was only available for those achieving at least a 2:2 class degree (Department for Education, 2010).

1.3.4 SKE courses

SKE courses were piloted for two years (2004/05 and 2005/06) before becoming a national programme (Stevenson, 2008).

A three year longitudinal evaluation of SKE courses was commissioned by the former Teaching Agency (Gibson et al., 2013). It aimed to evaluate the effectiveness of SKE courses for equipping trainees to meet QTS standards and become subject specialists in schools. It also compared SKE students with traditional entry (subject graduates) trainee teachers. Surveys and telephone interviews with SKE candidates, PGCE students and course tutors were carried out over the years 2009/10, 2010/11 and 2011/12. It included all subjects that offer SKE courses, including mathematics.

The report concluded that whilst “SKE courses provide trainees with a high level of subject knowledge and confidence… and a firm foundation and preparation for their PGCE training” (Gibson et al., 2013:11), the SKE students felt they had lower level subject knowledge than their peers who were subject graduates. Nevertheless, SKE students felt their knowledge may be more relevant to a school context than subject graduates (Gibson et al., 2013).

1.3.4.1 SKE course content

The ‘Evaluation of Subject Knowledge Enhancement Courses, Annual Report – 2011-12’ provides an overview of the nature of SKE courses:

> SKE courses:
>
> - Are intensive pre-ITT programmes specifically developed for graduates with some previous knowledge of the subject area who need to develop the depth of their understanding.
> - Are undertaken by individuals prior to proceeding onto a PGCE or for Qualified Teacher Status (QTS).
> - Enable those whose first degree was not focused on mathematics, physics or chemistry to train as a specialist teacher in one of these areas.
> - Focus on building and improving knowledge acquired by candidates at A-level, as part of a degree, or through occupational experience, in order to reach sufficient depth and breadth of subject knowledge appropriate for teaching secondary pupils.

- Have been designed to allow flexibility for both the course providers and trainees in terms of addressing individual needs in subject knowledge training.
- Are not intended to include significant in-school experience, but are designed instead to focus on subject knowledge.

(Gibson et al., 2013:25).

Thus, it is clear that SKE courses were designed to focus on subject knowledge. Despite this, responses from participants in the SKE course evaluation indicated that pedagogy was also taught on some SKE courses whether explicitly or implicitly. Indeed:

> …courses do not just help develop subject knowledge, but also motivate trainee teachers to think more broadly about teaching practices (e.g. developing group work techniques), as well as learning how to apply subject knowledge to practical activities/concepts (Gibson et al., 2013:40).

Most PGCE students and tutors interviewed for the SKE evaluation report: "…viewed the balance between subject knowledge and pedagogy of most SKE courses as being heavily weighted towards subject knowledge, with pedagogy being a much smaller aspect" (Gibson et al., 2013:12). However, just over a fifth of SKE students and PGCE students and a quarter of NQTs said that the balance of their SKE course was closer to 60/40 towards learning the subject (Gibson et al., 2013). Further, 12% of NQTs said their course had an equal balance of subject knowledge and pedagogy (Gibson et al., 2013). Thus, it seems courses varied as to the nature of the pedagogical instruction given. Some courses had specific sessions devoted to pedagogy or included school-based experience whilst other tutors felt good practice was modelled in the teaching of the subject matter, thus students implicitly gleaned teaching tips (Gibson et al., 2013).

As well as variations in the balance of subject knowledge taught, there were also differences in the level of this subject knowledge, with some interviewees stating that the course covered GCSE content but that they would not feel confident to teach A-level (Gibson et al., 2013).

As well as differences in the content and delivery of the SKE courses, the data collected for the SKE course evaluation showed there were also differences in assessment: "with some students taking exams at the end of each term to others just having unmarked continuous formative assessment methods [such as] portfolios, written assignments, practicals, coursework, reflective diaries, presentations, projects and examinations" (Gibson et al., 2013:70). Indeed, whilst SKE documentation, both official and across institutions, emphasises that teachers need to know 'mathematics in depth' (Adler et al., 2009), the content and assessment of SKE courses are not standardised, but set by each individual institution (TDA, 2011a) and vary widely across institutions. For example, London Metropolitan University assesses students: “Through the development of an ongoing subject audit… [to] build up a personalised subject knowledge folder” (London Metropolitan University, 2010). This self-reflective approach contrasts with the: “short tasks and tests relating to mathematical knowledge” (Birmingham City University, 2009) at Birmingham City University. This implies varying views as to what knowledge of ‘mathematics in depth’ consists.

There were also differences in contact time with some courses having a large proportion of online learning whereas other courses had more face-to-face learning (Gibson et al., 2013).

1.3.4.2 SKE course length

SKE courses vary from 2 weeks to 36 weeks in length in order that they can cater for candidates needs (Gibson et al., 2013).

The SKE course evaluation distinguishes between long courses and short SKE courses:

> Duration of long courses varies depending on the discipline and requirement but typically ranges from sixteen to 36 weeks. For people who have already applied for or started their ITT, but need to improve specific aspects of their subject knowledge, there are short SKE courses which can last from two to twelve weeks. (Gibson et al., 2013:25).

For the current research, graduates who had taken a long SKE course, as defined above, are referred to as ‘SKE students’. Students who had taken no SKE course or a short SKE course of two to four weeks in length were classified as ‘Non-SKE’ (there were no students in the sample involved in the current study who took courses between twelve and sixteen weeks). These groups were chosen since, as mentioned above, short courses are typically for refreshing specific aspects of knowledge. Indeed, most short course candidates involved in this research had mathematics degrees. Further, short courses are no longer funded by the government from 2013-14 but “incorporated as part of the normal refresher learning that forms part of an ITT course” (NCTL, 2013:2).

1.3.4.3 SKE candidates

Whilst SKE courses were intended to cater for students who are not deemed to have studied sufficient mathematics at degree level to qualify them for direct entry to a PGCE course, anecdotal evidence from one SKE course provider in England suggests that this is not always the case. Indeed, the course tutor informed me that their SKE students included: students that have 100% mathematics in their degree (asked to complete an SKE course for non-mathematics reasons), students that have more than 50% mathematics in their degree, and students that have no mathematics in their degree. Similarly, PGCE students at the same institution included some students who had less than 50% mathematics in their degree but who did not complete an SKE course.

1.3.4.4 SKE student numbers

According to Teaching Agency data (cited in Gibson et al., 2013), in 2009-10 there were 1,251 mathematics SKE students and in 2010-11 there were 1,082 mathematics SKE students. Given that there were 2,760 mathematics PGCE students recruited in 2010-11 (Teaching Agency, 2012), this represents over a third (39%) of trainee mathematics teachers.

1.3.4.5 Differences between SKE and Non-SKE PGCE students:

1.3.4.5.1 Level and type of knowledge and confidence

The SKE course evaluation report states that: “SKE courses provide trainees with a high level of subject knowledge and confidence in the subject and a firm foundation and preparation for their PGCE training” (Gibson et al., 2013:11). However, during the PGCE year, the SKE students reported that they felt their knowledge was at a lower level than their peers who had a degree in the subject (Gibson et al., 2013). Indeed, SKE students felt that those with a degree in the subject would have a better depth and breadth of knowledge with a better understanding of underlying principles and concepts, and therefore be better able to teach A-level, stretch the more able pupils, and answer pupils’ more complex questions. However, they also felt their knowledge may be surplus to the knowledge required at school level and that they may find it difficult to translate their higher-level knowledge into a form for pupils to understand (Gibson et al., 2013).

Further, SKE students felt that their own knowledge would be more up-to-date and relevant to the school curriculum than those who were graduates in the subject, which would enable them to: relate well to pupils; break down the subject for pupils; and understand the misconceptions and challenges that pupils face (Gibson et al., 2013).

1.3.4.5.2 Preparation for the PGCE

The SKE course evaluation report concludes that: “In terms of preparing students for the PGCE course, the SKE is considered to be ideal preparation” (Gibson et al., 2013:14). From the interviews conducted, there were many reasons provided by SKE students and tutors as to how the SKE course could prepare them for a PGCE:

1. SKE students were felt to be better prepared for what to expect on the PGCE in terms of its demands. Having spent time writing academic essays/assignments, putting together portfolios and receiving feedback, students were felt to be more prepared for academic work. Further, they could “take advantage of the summer studying, revising and organising resources” (Gibson et al., 2013:88).

2. SKE students were felt to be more familiar with the school curriculum and have it fresh in their minds.

3. Some SKE students were felt to have had experience of more up-to-date teaching methods and current teaching approaches as well as insight into school practices.

4. SKE students had the opportunity to build up a bank of resources

5. Through sharing ideas and experiences with other students, SKE students had the opportunity to build up a network of support.

6. If the SKE course is provided by the same institution as the PGCE course, SKE students would have had time to grow familiar with the institution and tutors and their style of working.

7. For SKE students who have been in previous employment, they were felt to have the advantage of ‘real life prior experience’ to draw upon.

8. Since SKE students recently learned and struggled with the subject, they were felt to be in a better position to empathise with pupils and provide better support.

9. Since some SKE students come into teaching as a career change, it was felt they may have thought more carefully about teaching as a vocation, whereas graduates may see teaching as the next logical step following their university studies. Thus SKE students may be more committed to teaching.

Nevertheless, the evaluation concludes that: “if there were any differences that they tended to even themselves out during the year and that there was no difference when it came to gaining employment” (Gibson et al., 2013:89).

1.4 Significance of the study

This research has the potential to make several contributions to the mathematics education research community and teacher training provision and policy in England.

Firstly, the mathematics education community could benefit as this study adds to the literature on secondary mathematics teacher knowledge. In particular, it broadens opportunities for researchers to study teacher knowledge in England through using the Mathematical Knowledge for Teaching (MKT) items in England. This is due to the fact that these items have not previously been used in England and this research therefore provides useful information about the transferability of the items. This study also adds to the international work of the Learning Mathematics for Teaching (LMT) research group (the authors of the items).

Secondly, this research adds to the literature by proposing a new approach to mathematics teacher knowledge which aims to overcome seven criticisms of existing conceptions of teacher knowledge identified within the literature review (Chapter 2). Instead of treating knowledge as a static checklist of things to know, the new approach focuses on knowledge change and multiple representations (geometric, algebraic and numeric) of knowledge. Further, this new approach incorporates a new way of representing knowledge change over time, namely, Knowledge Maps (see 2.3.2.2, p.46).

Thirdly, this research utilises the Knowledge Quartet observation schedule when analysing video recordings of PGCE students whilst teaching on their school placements. In doing so, this research highlights some potential issues with the schedule not previously discussed within the literature.

Further, this research has the potential to impact on future teacher training provision in England as it provides information about how knowledge changes over the course of a teacher training year and how this knowledge differs between SKE and Non-SKE PGCE students. Consequently, this research has the potential to impact teacher training policy in England as the existing government policies on funded SKE courses and differentiated training bursaries are taken into account and analysed for this study.

1.5 Scope and boundaries

This research focuses on trainee secondary mathematics teachers in England only. Primary teachers are not taken into account nor teachers in other countries. Qualified teachers are not focused on for this research.

There are several routes to becoming a teacher within England. For example taking an undergraduate degree in Education leading to QTS or ‘Teach First’ (Department for Education, 2014). This research only focuses upon full-time secondary mathematics PGCE courses.

The current research involves PGCE students from the academic year 2012/13, and SKE students from 2011/12 (covered by the SKE Evaluation report).

Recruitment data shows that 2,500 (rounded to the nearest 10) mathematics trainees were recruited in 2012/13 (Teaching Agency, 2012). This was 95% of the recruitment target of 2,635 (Teaching Agency, 2012). These figures were updated to 2,340 (89% of recruitment target) in the 2013-14 ITT number census (Department for Education, 2013g). Out of these, 2,030 were recruited from mainstream routes (Department for Education, 2013g). Out of the mainstream routes, 1,773 places were allocated for full-time PGCE courses across 65 institutions in December 2011 (TDA, 2011b). This was later updated to 1,761 places across 63 institutions in February 2012 (Department for Education, 2012a). The initial allocation was used for this research, that is, 1,773 students was the target population and all 65 institutions were invited to participate. The final number of secondary mathematics trainee teachers who took up places in 2012/13 (to the nearest 10) was 1,630 students (NCTL, 2013a).

When considering the mathematics National Curriculum to be taught by the PGCE students, only three aspects of it are addressed within this research, namely, number, geometry and algebra. Understanding a mathematical concept in terms of these three aspects is said to be indicative of a deep understanding of mathematics (discussed in 2.3.1, p.42). Further, these aspects are also chosen by other studies of teacher knowledge (e.g. TEDS-M discussed in 2.1.6, p.25). Furthermore, only Key Stage 3 and 4 are considered since A-level mathematics is not always taught by PGCE students (Meenan, 2012).

The sample of trainee teachers involved in this study is not a random one, thus caution needs to be exercised when generalising to the population.

Finally, there was no school pupil learning data collected since PGCE students only teach school pupils for part of the year and sometimes teaching is shared. As a result, claims cannot be made about pupil learning as a consequence of teacher knowledge.

1.6 Language of the thesis

Since this research analyses PGCE students who are in turn teaching school students, henceforth within this document, ‘students’ refers to the PGCE students and ‘pupils’ refers to the school children (see ‘List of Abbreviations’, p.xiii for further details on abbreviations adopted within this document).

1.7 Structure of the thesis

Chapter two is the main literature review for this research and is divided into three parts. Firstly, several existing conceptions of the knowledge required for teaching mathematics are detailed. This is followed by a critique of these ideas. Finally, a new approach to teacher knowledge is offered in response to the problems with existing conceptions.

Chapter three details and justifies the research questions arising from the literature review.

Chapter four describes and justifies the research methods used to address the research questions taking into consideration the underlying theoretical assumptions of the researcher which may impact upon the research design. The three main data collection methods used for this mixed methods research (questionnaires, interviews and observations) are then discussed in turn.

Chapter five presents the results from the three data collection methods. This chapter provides a foundation for Chapter six.

Chapter six ties together results from the three data collection methods in order to provide evidence in response to the research questions. It offers critical reflection and discussion of how the results compare with existing research.

Chapter seven comprises four main discussion points arising from this research. Each discussion point is based on a contribution to wider research or practice which is made as a result of conducting this study, but also includes consideration of how this study relates to existing literature, and implications of the findings.

Chapter eight presents overall conclusions of the study; limitations of the research methods and analyses; and suggests avenues for further research.

2 Literature Review

In order to analyse the knowledge change of mathematics PGCE students, a precise understanding of what is meant by 'knowledge' is required. Indeed: "…without some common understanding of what subject knowledge means and what it looks like in practice, there can be no coherent approach to... answering research questions about the role of teachers' mathematical knowledge in teaching" (Petrou and Goulding, 2011:9).

Shulman states:

> A conceptual analysis of knowledge for teachers would necessarily be based on a framework for classifying both the domains and categories of teacher knowledge, on the one hand, and the forms for representing that knowledge, on the other (1986:10).

By 'forms', Shulman refers to how knowledge categories are organised and stored in the mind (Shulman, 1986). Whilst I agree that both an understanding of the categories of knowledge under consideration as well as an understanding of the forms for representing that knowledge are required, for the current research, a means of measuring said knowledge is also desirable. However, as the literature review reveals, papers either aim to classify knowledge or aim to measure knowledge with the exception of Deborah Ball and colleagues who devote similar efforts to both categorising knowledge and developing measurement instruments. Part One (2.1, p.14) of this chapter (the literature review) is therefore structured as follows: beginning with Shulman's (1986) seminal paper, conceptions of knowledge for teaching by Leinhardt and Smith (1984), Prestage and Perks (2001), Ma (1999), Ball and colleagues' are examined, followed by the TEDS-M (Senk et al., 2008) and COACTIV (Krauss et al., 2008) research projects which focus on measuring knowledge. In particular (i) categories of knowledge, (ii) forms of knowledge, (iii) measuring knowledge, and (iv) representations of knowledge are discussed (where applicable). Finally, the Knowledge Quartet (Rowland and Turner, 2007) is presented which, in contrast to the other studies, considers knowledge in practice rather than categories and forms of knowledge. Following the literature review, **Table 2.1** (p.33) highlights the key points of each view of knowledge to assist comparison of the theories.

In Part Two (2.2, p.34) of this chapter, the literature reviewed in Part One is critiqued. In particular, Part Two shows that no single conception of teacher knowledge considers both categories and forms of knowledge whilst simultaneously offering a means to measure and represent that knowledge.

Further, many existing conceptions of knowledge treat content knowledge as dualistic, objective and static. A static conception of knowledge is not suitable for research looking at knowledge change. Therefore, in Part Three (2.3, p.42), a new conception of teacher knowledge is presented.

In conducting this literature review, it is acknowledged that there will inevitably be gaps as not all papers which involve teacher knowledge can be included due to time and space restrictions. I have endeavoured to include papers from the UK (the context of this research) which introduce categories of knowledge for teaching. Given the limited research projects which focus on measuring teacher knowledge on a large scale, projects from other countries are also included (Ball and colleagues in the USA, the COACTIV project in Germany and the TEDS-M international study).

2.1 Part One: Literature review

2.1.1 Shulman

2.1.1.1 (i) Categories of knowledge

Shulman's (1986) seminal paper reframes extant research into teachers' knowledge with a new focus on the role of content for teaching (Ball et al., 2008) which has since led to an explosion of studies within the field over the past two decades (Askew, 2008). Shulman (1986) describes how the view of teacher knowledge needed had previously altered over time, with a changing interplay between content knowledge and knowledge of pedagogy. Shulman's paper distinguishes between three categories of knowledge: subject matter content knowledge (CK), pedagogical content knowledge (PCK) and curriculum knowledge. CK "refers to the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986:9). PCK "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching"* (ibid.:9). Finally, 'curriculum knowledge' includes understanding of the way topics are organised within a school year and subsequent years, as well as knowledge of available materials and resources and how to use them.

CK, according to Shulman, includes an understanding of the structure of the subject matter beyond knowledge of basic facts and concepts of the domain. That is, knowledge of what is true and what constitutes 'truth' within the domain as well as an understanding of how concepts within the domain relate, and their relative importance. Thus, CK of a teacher is "at least equal to that of his or her lay colleague, the mere subject matter major" (ibid.:9).

However, “The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied” (ibid.:9).

2.1.1.2 (ii) Forms of knowledge

Prestage and Perks (2001) argue that research by Shulman does not consider in detail how subject knowledge is held within teachers’ minds. However, close reading of Shulman (1986) reveals that he does recognise the need to have more than categories of knowledge alone. Indeed, he presents three forms of teacher knowledge (in depth) which seem to have been overlooked within the literature. Hodgen also recognises this oversight:

> Whilst subsequent work has emphasised the aspects of Shulman's work that attempt to codify teacher knowledge, it is often overlooked that he did examine the forms of knowledge. This neglected area of Shulman's work related to the way teacher knowledge is 'held' and used in teaching (2011:35).

Shulman’s forms include: propositional knowledge, case knowledge, and strategic knowledge. ‘Propositional knowledge’ (rules or recommendations for teaching, for example ‘never smiling until Christmas’) is further divided into three categories:

1. Disciplined empirical/ philosophical enquiry (principles) – propositions derived from empirical research;
2. Practical experience (maxims) – practical propositions which are not supported by theory but from accumulated practice;
3. Moral or Ethical reasoning (norms) – propositions based on morals or ethics.

Shulman’s ‘case knowledge’ refers to knowledge of particular cases of specific teaching practices which have been experienced or read about. Unlike the other categories, ‘strategic knowledge’ is more ‘knowing’ or judgement than ‘knowledge’: it is a wisdom that transcends the usual propositional or case knowledge and can be demonstrated when two propositions clash (Shulman, 1986).

2.1.1.3 (iii) Representation of knowledge

N/A

2.1.1.4 (iv) Measuring knowledge

N/A

Content knowledge for mathematics teachers

Shulman's categories of knowledge refer to teaching in general without reference to a specific subject. However, PCK depends upon a specific subject to be taught. Indeed, Ball and colleagues criticise research on teacher knowledge for not linking PCK with a specific subject:

> …ironically, nearly one-third of the articles that cite pedagogical content knowledge do so without direct attention to a specific content area — the very emphasis of the notion — instead making general claims about teacher knowledge, teacher education, or policy (Ball et al., 2008:3).

A review of the literature which focuses specifically on mathematics knowledge for teaching are therefore presented. That is, work by Leinhardt and Smith (1984), Prestage and Perks (2001), Ma (1999), Ball and colleagues, followed by the TEDS-M (Senk et al., 2008) and COACTIV (Krauss et al., 2008) research projects and finally, the Knowledge Quartet (Rowland and Turner, 2007). The researchers either draw upon the work of Shulman or present alternative categories of knowledge.

2.1.2 Leinhardt and Smith

2.1.2.1 (i) Categories of knowledge

Leinhardt and Smith divide "the organization and content of subject matter knowledge used by expert arithmetic teachers" (1984:2) into two cognitive aspects: lesson structure which "includes the skills needed to plan and run a lesson smoothly" (ibid.:2) and subject matter. Subject matter knowledge includes "conceptual understanding, the particular algorithmic operations, the connection between different algorithmic procedures, the subset of the number system being drawn upon, understanding of classes of student errors, and curriculum presentation" (ibid.:2).

This understanding of subject matter knowledge is comparable to Shulman's. Whilst written before Shulman's paper, some aspects of their conception of subject matter knowledge include some aspects which Shulman labels as PCK, for example, knowledge of student errors. The view of mathematics that the authors take appears to be a fixed body of facts and concepts to be known which are outlined in a school curriculum. This can be seen in their view of how knowledge is held.

2.1.2.2 (ii) Forms of knowledge

Leinhardt and Smith describe content knowledge as being in both declarative (facts that are known about a domain) and procedural (the algorithms and heuristics that operate on those facts) forms.

2.1.2.3 (iii) Representation of knowledge

They further present a method for representing declarative knowledge: semantic networks. “A semantic network is a node-link structure in which concepts are represented as nodes that are linked together according to a defined set of relationships” (Leinhardt and Smith, 1984:9). Semantic networks are therefore comparable to concept maps. Indeed, concept maps “include concepts, usually enclosed in circles or boxes of some type, and relationships between concepts indicated by a connecting line linking two concepts. Words on the line... specify the relationship between the two concepts” (Novak and Canas, 2008:1).

2.1.2.4 (iv) Measuring knowledge

N/A

2.1.3 Prestage and Perks

2.1.3.1 (i) Categories of knowledge

A further categorisation is offered by Prestage and Perks (2001). They consider 'professional knowledge' which I understand to mean 'knowledge needed in the teaching profession'. They take 'subject knowledge' as a subset of professional knowledge and recognise that: “Existing research in the area of teachers' knowledge offers definitions of professional knowledge as well as explanations for the different forms of knowledge that a teacher holds” (Prestage and Perks, 2000:101). They thus define subject knowledge to be: “knowledge about the subject matter in mathematics, knowledge about its structure, the body of concepts, facts, skills and definitions, as well as methods of justification and proof" (ibid.:102).

Again, this view of subject matter is comparable to Shulman’s and could arguably be transferred to any school subject. It does not relate specifically to mathematics. However, the focus of the paper is on how the knowledge is held in the mind of the teacher rather than the content of that knowledge. Indeed, they hypothesise that subject matter knowledge is held in two forms: (1) learner-knowledge – “the knowledge needed to pass examinations, to find solutions to mathematical problems” (ibid.:102) and (2) teacher knowledge – “the knowledge needed to plan for others to come to learn the mathematics” (ibid.:102). They thus argue that subject knowledge auditing is necessary but not sufficient for trainee teachers.

2.1.3.2 (ii) Forms of knowledge

Prestage and Perks (2001) present three phases of knowledge in an attempt to address the lack of detailed forms of knowledge within the literature. Their three phases include:

(1) Professional traditions: knowledge brought to teaching practice from prior experiences of teacher training or their own learning. This knowledge can often go unquestioned.
(2) Practical wisdom: knowledge gained through experience in the classroom.
(3) Deliberate reflection: knowledge gained through reflection outside of the classroom.

Prestage and Perks use the term ‘phase’ to describe the above to suggest “the possibility of movement over time as a teacher's subject knowledge develops and transforms" (2001:204).

However, Prestage and Perks’ three phases seem to be limited to *sources* from which teachers attribute their knowledge acquisition rather than consideration of how knowledge is held.

2.1.3.3 (iii) Representation of knowledge

N/A

2.1.3.4 (iv) Measuring knowledge

N/A

2.1.4 Ma

2.1.4.1 (i) Categories of knowledge

More recently, Ma (1999) describes the depth, breadth and thoroughness of knowledge teachers need as ‘profound understanding of fundamental mathematics’ (PUFM). She argues that teachers need knowledge like a taxi driver’s knowledge of a city – flexible in order to get to different places in a variety of ways. This is similar to Skemp (1976) who distinguishes between ‘instrumental’ and ‘relational’ understanding. ‘Instrumental’ understanding is knowing ‘how’ to do something and can be seen as analogous to knowing one route for getting from A to B. Conversely, ‘relational’ understanding is knowing ‘how’ and ‘why’ and is similar to having a map in one’s head so that several routes are known between A and B and if one gets lost, one can still find their way.

This view of subject knowledge takes into consideration the subject of mathematics, recognising that due to the nature of mathematics, it requires a particular understanding of it in order to teach effectively. However, Ma's proposed construct of PUFM is not fully defined. Indeed, Askew (2008) states that Ma's work only partially describes of what PUFM consists.

2.1.4.2 (ii) Forms of knowledge and (iii) Representations of knowledge

Ma (1999) also considers how knowledge can be held. Indeed, she suspects that the teachers in her study possess: "differently structured bodies of mathematical knowledge" (*xx*). In addressing this, Ma (1999) introduces a way to represent this through 'knowledge packages' (**Figure 2.1**, p.19) – a collection of 'pieces' of knowledge (the ellipses in the diagram) which relate to a concept (the rectangle at the top) which can differ slightly from teacher to teacher in terms of the links (arrowed lines) and ellipses present.

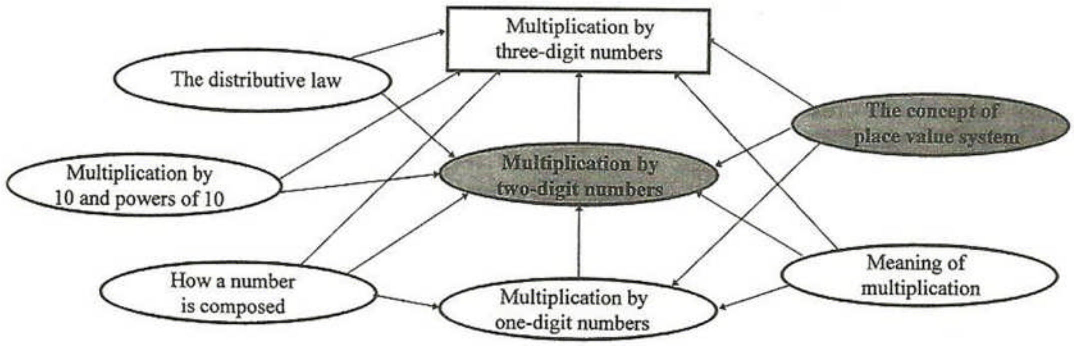

Figure 2.1: A knowledge package for multiplication by three-digit numbers (Ma 1999:47).

Ma recognises that: "Teachers with conceptual understanding and teachers with only procedural understanding... ha[ve] differently organized knowledge packages" (1999:22). Further, she recognises that conceptual understanding can differ in breadth and depth between teachers.

Ma's 'knowledge packages' are criticised by Ball and Bass (2000) as the term suggests knowledge which is not flexible enough to be used in practice.

2.1.4.3 (iv) Measuring knowledge

N/A

2.1.5 Ball and Colleagues

2.1.5.1 (i) Categories of knowledge

The Learning Mathematics for Teaching (LMT) group (Deborah Ball and colleagues) at the University of Michigan takes a 'bottom-up' approach to studying teacher knowledge by beginning with teaching practice. They ask:

“What do teachers do in teaching mathematics, and how does what they do demand mathematical reasoning, insight, understanding, and skill?” (Ball et al., 2008:4). In a sense, their approach is a ‘job analysis’ of the tasks of teaching. From their qualitative analysis of videos of teachers teaching, they define ‘mathematical knowledge for teaching’ (MKT) as “the *mathematical knowledge that teachers need to carry out their work as teachers of mathematics*” (ibid.:4). In particular, MKT builds upon Shulman’s original definitions, by further dividing his ‘subject matter content knowledge’ (CK) and ‘pedagogical content knowledge’ (PCK) categories with regard to mathematics. In their diagram (Figure 2.2, p.20) CK is divided into ‘common’ and ‘specialised’ content knowledge as well as ‘knowledge at the mathematical horizon’. ‘Common Content Knowledge’ (CCK) is mathematics knowledge which any well-educated adult should have. Indeed: “bankers, candy sellers, nurses, and other nonteachers are likely to hold such knowledge” (Hill and Ball, 2004:333) whereas ‘specialised content knowledge’ (SCK) is knowledge which is mathematical in nature and which is beyond that expected of a well-educated adult but not yet requiring any knowledge of students or of teaching. ‘Knowledge at the mathematical horizon’ is an understanding of the wider context of mathematical ideas that an aspect of the curriculum relates, thus allowing a teacher to know where a pupil is headed (Ball et al., 2009).

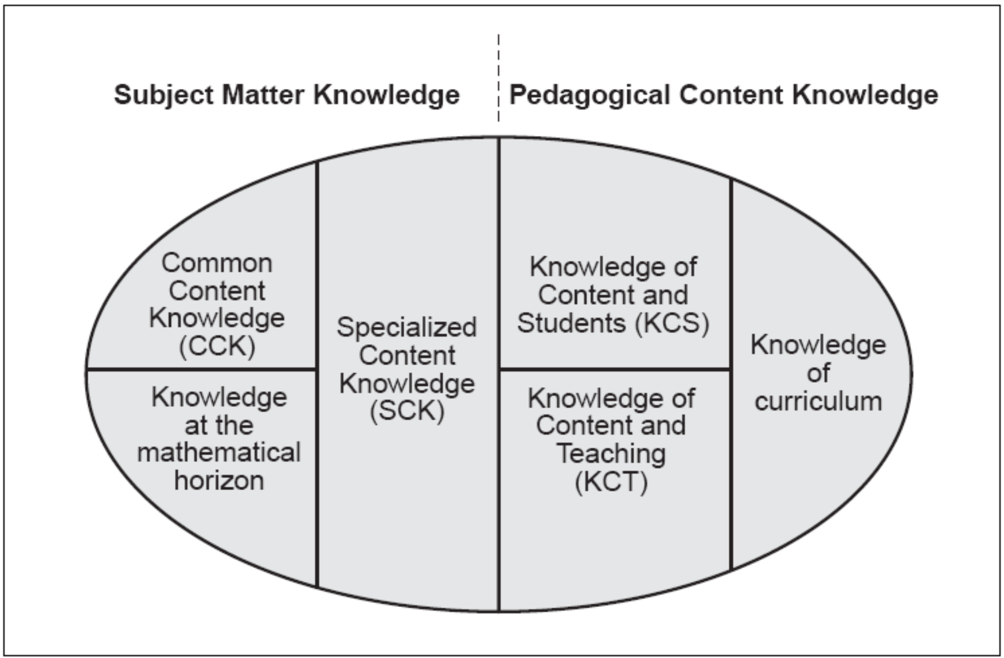

Figure 2.2: Domain map for 'mathematical knowledge for teaching' (Hill et al., 2008a:377).

MKT thus relates to mathematics subject knowledge but it is different in that it is knowledge needed by teachers to carry out the work of teaching. Ball et al. (2008) provide an example of the distinction between common and SCK by considering a subtraction problem:

$$\begin{array}{r} 307 \\ -\ \underline{168} \\ 139 \end{array}$$

They state that knowing how to solve the calculation is knowledge expected of an educated adult and is CCK. They argue that this is necessary but not sufficient for teaching. They provide the example of the following error:

$$\begin{array}{r} 307 \\ -\ \underline{168} \\ 261 \end{array}$$

They reason that a teacher not only needs to be able to identify the answer as being correct or not, but needs to figure out the source of the mathematical error (in this case the student has subtracted the smaller digit from the larger one in each column). Furthermore, a teacher needs to be able to do this quickly, often while a pupil is waiting for feedback. Ball et al. (2008) conclude:

> [An a]nalysis such as this [is] characteristic of the distinctive work teachers do and they require a kind of mathematical reasoning that most adults do not need to do on a regular basis. And although mathematicians engage in analyses of error, often of failed proofs, the analysis used to uncover a student error appears to be related to, but not the same as, other error analysis in the discipline. Further, there is no demand on mathematicians to conduct their work quickly as students wait for guidance (Ball et al., 2008:7).

In the example given, knowledge of students or of teaching *per se* is not necessarily required. Hence, Ball and colleagues distinguish SCK from PCK. Further, they "began to notice how rarely these mathematical demands were ones that could be addressed with mathematical knowledge learned in university mathematics courses" (ibid.:8). They further justify their view that MKT is different than graduates' mathematical knowledge by comparing the work of mathematics teachers to other professions which utilise mathematics:

> Accountants have to calculate and reconcile numbers and engineers have to mathematically model properties of materials, but neither group needs to explain why, when you multiply by ten, you "add a zero" (ibid.:11).

They argue that SCK is present in the mathematical tasks that teachers do. Such tasks include:

> Presenting mathematical ideas

> Responding to students' "why" questions
> Finding an example to make a specific mathematical point
> Recognizing what is involved in using a particular representation
> Linking representations to underlying ideas and to other representations
> Connecting a topic being taught to topics from prior or future years
> Explaining mathematical goals and purposes to parents
> Appraising and adapting the mathematical content of textbooks
> Modifying tasks to be either easier or harder
> Evaluating the plausibility of students' claims (often quickly)
> Giving or evaluating mathematical explanations
> Choosing and developing useable definitions
> Using mathematical notation and language and critiquing its use
> Asking productive mathematical questions
> Selecting representations for particular purposes
> Inspecting equivalencies
>
> (ibid.:10)

2.1.5.2 (ii) Forms of knowledge

Ball and colleagues consider how knowledge is held proposing that teachers need: "to unpack the elements of… mathematics to make its features apparent to students" (Ball et al., 2008:10). They term this unpacking of knowledge for teaching as 'decompression':

> …one needs to be able to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible. We refer to this as decompression. Paradoxically, most personal knowledge of subject matter, which is desirably and usefully compressed, can be ironically inadequate for teaching. In fact, mathematics is a discipline in which compression is central. Indeed, its polished, compressed form can obscure one's ability to discern how learners are thinking at the roots of that knowledge... Because teachers must be able to work with content for students in its growing, not finished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements… (Ball and Bass, 2000:98).

A 'decompression' of knowledge can be seen as the opposite to Ma's 'knowledge packages' since it is knowledge which is "pedagogically useful and ready, not bundled in advance..." (Ball and Bass, 2000:88).

As Hill and Ball (2004) put it:

> …**how** teachers hold knowledge may matter more than **how much** knowledge they hold. In other words, teaching quality might not relate so much to performance on standard tests of mathematics achievement as it does to whether teachers' knowledge is procedural or conceptual, whether it is connected to big ideas or isolated into small bits, or whether it is compressed or conceptually unpacked…(Hill and Ball, 2004:332)

2.1.5.3 (iii) Representing knowledge

N/A

2.1.5.4 (iv) Measuring knowledge

There is a pool of several hundred multiple-choice measures (LMT, 2007) developed by the LMT group to test for MKT. These began to be written in 2001 by mathematicians, mathematics educators, researchers and former teachers (Hill et al., 2005) and later piloted (Hill et al., 2004). The item writing process itself and subsequent factor analyses of the questions helped the LMT group to refine their ideas about the organisation and structure of MKT.

Some of the multiple-choice questions test CCK, others test SCK. However, the researchers recognise that it can be difficult to distinguish between the two. Nevertheless they provide some guidance:

> One way to illustrate this distinction is by imagining how someone who has not taught children but who is otherwise knowledgeable in mathematics might interpret and respond to these items. This test population would not find the items that tap ordinary subject matter knowledge difficult. By contrast, however, these mathematics experts might be surprised, slowed, or even halted by the mathematics-as-used-in-teaching-items; they would not have had access to or experience with opportunities to see, learn about, unpack and understand mathematics as it is used at the elementary level (Hill et al., 2004:12-13).

The MKT items are to be kept secure, but the LMT group have released a small number of them (see Ball and Hill, 2008) for distribution.

Appendix A (p.270) contains two of the released multiple-choice items. (A.1) can be considered a CCK item, whereas (A.2) can be argued to tap SCK. However, it should be noted that whilst some of the older MKT questions were categorised by the authors as assessing specific components of the MKT construct, this practice was discontinued. This was because it was found to be difficult to place questions into mutually exclusive categories, thus more recently developed questions are not classified (Blunk, 2013).

The MKT items are not criterion referenced but seek to enable teachers to be ordered relative to one another (Blömeke and Delaney, 2012).

The MKT measures have been extensively validated by the LMT group and other researchers. The measures have been analysed psychometrically (Hill et al., 2004), and teacher scores on the measures have been shown to correlate with their ‘mathematical quality of [classroom] instruction’ (MQI) (Hill et al., 2008b) as well as positively predicting pupil achievement (Ball et al., 2005). A selection of validation studies are outlined below.

Pre- and post-assessments using the MKT items on a sample of 398 teachers at the California Professional Development Institute show that teachers' MKT improved following a summer professional development workshop (Hill and Ball, 2004). The study also shows that the MKT items can be used at scale (with large numbers of teachers).

There are a series of papers looking at the contribution of MKT and curriculum materials (separately and together) to the MQI of a sample of teachers with a range of MKT as measured by the items. One such study provides evidence to support the hypothesis that MKT is a key contributor to MQI (Hill and Charalambous (2012). Indeed, although only a small sample of teachers was used, it was found that teachers scoring highly on the MKT items typically taught excellent lessons, whilst low scoring teachers often struggled. This pattern has been observed elsewhere. For example, Hill et al. (2005) found that scores on the MKT items significantly predicted pupil achievement gains.

In a study by Kersting et al. (2010), teachers were asked to analyse video clips of classroom situations:

> The teachers' written responses to each clip were scored along four dimensions, each representing important aspects of teachers' work: mathematical content (MC), student thinking (ST), suggestions for improvement (SI), and depth of interpretation (DI) (Kersting et al., 2010:574).

The authors found that "teacher scores on the [video analysis] were strongly related to their mathematics knowledge for teaching as measured by the MKT" (ibid.:177-178). Further, "Virtually all of the shared variance was explained by the MC Knowledge subscale" (ibid.:178). They also compared scores on the MKT items with scores of ST, SI and DI and found that the first two were not measured by the MKT items and a negligible amount of additional variance with the MKT items was explained by DI. The authors suggest that ST and SI might be closer to measuring Shulman's PCK rather than CK.

One reason Deborah Ball and colleagues decided to write their own items testing for MKT, was due to the lack of existing measures suitable for their purposes. Whilst many USA teacher licensing exams existed which could be used for large scale testing, the team felt these were incomplete since they tended to measure only mathematical subject knowledge rather than utilising this knowledge within teaching contexts (Hill and Ball, 2004). Further, the limited instruments measuring mathematics as used in teaching were not suitable for administering on a large scale. They state:

> The only existing measures of teachers' specialized knowledge for mathematics teaching consisted of interviews and open-ended written responses, or single multiple-choice items culled from early work by Ball, Post, and others contributing to the Teacher Education and Learning to Teach project (Kennedy, Ball, & McDiarmid, 1993). None of these measurement strategies was suitable… because they could not be implemented on a large scale, and little was known about their reliability (Hill and Ball, 2004:335).

2.1.6 The Teacher Education and Development Study in Mathematics (TEDS-M)

2.1.6.1 (i) Categories of knowledge

The Teacher Education and Development Study in Mathematics (TEDS-M) was an international study focusing on differences in teacher education programs between and within countries (Tatto et al., 2008), examining the nature and extent of knowledge for teaching among trainee mathematics teachers (Senk et al., 2008). The study sought to measure mathematical knowledge for teaching which was assumed to comprise of 'mathematical content knowledge' and 'mathematics pedagogical content knowledge' which are in turn subdivided (ibid.). 'Mathematical content knowledge' comprised: knowledge of mathematics (basic factual knowledge) including the structure of mathematics, different approaches and how they relate, and organisation of mathematics, whilst 'mathematics pedagogical content knowledge' included subject-related knowledge needed in order to teach mathematics. It was hypothesised to include: mathematics curricular knowledge, knowledge of planning mathematics, and knowledge of enacting mathematics (ibid.). The TEDS-M study also assumed that teacher knowledge is situated and applied.

2.1.6.2 (ii) Forms of knowledge

In terms of types of mathematics PCK, the TEDS-M study attempted to cover propositional and case-based knowledge of mathematics teachers, using Shulman's categories.

2.1.6.3 (iii) Representing knowledge

N/A

2.1.6.4 (iv) Measuring knowledge

Tatto and colleagues (2008) collected data at the end of teacher training in 16 countries around the world. Over 23,000 primary and lower secondary trainee teachers in total (from nationally representative samples) took a 60 minute written test under standardised, monitored conditions (ibid.).

By looking at national standards for teachers in the participating countries, the classroom situations that a teacher is expected to deal with were identified and questions within the TEDS-M survey were based on these standards. The mathematical content knowledge test comprised of tasks covering number, algebra and geometry and, to a lesser extent, data. Knowing, applying and reasoning were the three cognitive dimensions also covered. There was also a PCK test.

2.1.7 Cognitive Activation in the Classroom (COACTIV) Project

2.1.7.1 (i) Categories of knowledge

The test developed as part of the Cognitive Activation in the Classroom (COACTIV) Project (Krauss et al., 2008) was based on a conception of knowledge which includes 'content knowledge' ("deep understanding of the domain itself" (ibid.:1)) and pedagogical content knowledge ("how to make the subject comprehensible to others" ibid.:1) as theoretically separate. Pedagogical content knowledge is further subdivided into three sub-dimensions: selecting tasks, awareness of student misconceptions and difficulties and using representations/ analogies/ illustrations/ examples (ibid.). Content knowledge is not further divided but is based on Shulman's conception. Content knowledge is distinguished from the mathematical knowledge which all adults should have and university level mathematics which is not part of the school curriculum content (ibid.).

2.1.7.2 (ii) Forms of knowledge

N/A

2.1.7.3 (iii) Representing knowledge

N/A

2.1.7.4 (iv) Measuring knowledge

The COACTIV Project (Krauss et al., 2008) aimed to develop a reliable test of secondary mathematics teachers' professional knowledge. It surveyed mathematics teachers in Germany whose classes participated in the Programme for International Student Assessment (PISA) 2003/4.

The test included 13 items designed to test CK and 35 open-ended items designed to test PCK. According to Cole (2011), some of their items appear to asses Ball and colleagues' SCK.

2.1.8 The Knowledge Quartet

The Knowledge Quartet is different than other conceptions of mathematical knowledge for teaching as neither categories nor forms of knowledge are the focus. This section therefore deviates from the structure followed heretofore. Instead, the Knowledge Quartet is introduced first, followed by an explanation as to why this theory is seen to be different.

The Knowledge Quartet is a response to a concern that trainee Primary teachers' mathematics lessons (taught on their school placements) are often observed and discussed with their school-based mentors without a focus on the mathematical content of the lesson but rather general features of teaching such as behaviour management (Rowland and Turner, 2007). The Knowledge Quartet was subsequently developed to provide an empirically-based framework for reviewing trainee teachers' mathematics lessons with a focus on the mathematics content of the lesson including Shulman's (1986) 'subject matter knowledge' and PCK (Rowland and Turner, 2007).

The Knowledge Quartet is a collection of 21 codes (Appendix B, p.272) which emerged as a result of grounded theory analysis of video recordings of Primary Postgraduate Certificate in Education (PGCE) students over one academic year (ibid.), thus it can be considered a practice-based theory (Ball and Bass, 2003 cited in Weston et al, 2012). An association was recognised between the individual codes: "enabling [the authors] to group them (again by negotiation in the team) into four broad, superordinate categories" (Rowland and Turner, 2007:110) or 'dimensions' (Figure 2.3, p.28) in order to make using the codes more practical. Indeed, the authors: "did not want a 17-point tick-list (like an annual car safety check), but preferred a less complex, more readily-understood scheme which would serve to frame an in-depth discussion between teacher and observer" (ibid.:110). However, it is recognised that a teaching episode may be classified under two or more of these four categories (Rowland, 2012b).

The Knowledge Quartet is not 'owned' by Rowland and colleagues, but is an evolving set of codes, however, only three new codes have been added since 2002 so it is expected to be fairly comprehensive (Rowland, 2012b).

Whilst originally intended for analysing Primary school teachers, the Knowledge Quartet has been extended to include Secondary school mathematics teachers. Indeed, it was found that no codes needed to be added or removed when considering Secondary school teachers (Weston et al., 2012).

The four dimensions are as follows: foundation; transformation; connection; and contingency.

Foundation consists of knowledge, beliefs and understanding gained in preparation for teaching:

> The key components of this theoretical background are: knowledge and understanding of mathematics **per se** and knowledge of significant tracts of the literature on the teaching and learning of mathematics, together with beliefs concerning the nature of mathematical knowledge, the purposes of mathematics education, and the conditions under which pupils will best learn mathematics (Rowland et al., 2004:123).

The inclusion of beliefs within the Foundation dimension is different than Shulman's (1986) categorisation which does not consider beliefs.

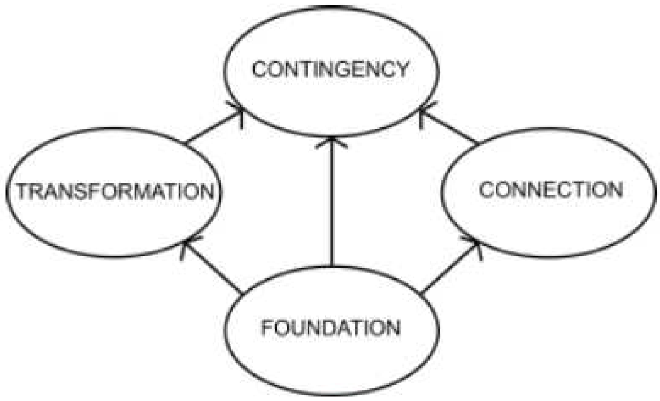

Figure 2.3: The relationships between the four dimensions that comprise the Knowledge Quartet (Rowland, 2012a).

Transformation involves the re-presentation of ideas to pupils "in the form of analogies, illustrations, examples, explanations and demonstrations" (Rowland et al., 2004:123) in order that pupils can understand. It concerns knowledge-in-action whilst planning to teach and in the act of teaching (ibid.:123). The decisions made by the teacher regarding how content is transformed is informed by foundation knowledge (Rowland and Turner, 2007).

Connection "binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content… [the authors'] conception of this coherence includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks" (Rowland et al., 2004:123). Connection: "includes the sequencing of topics of instruction within and between lessons, including ordering of tasks and exercises" (Rowland and Turner, 2007:114). The teacher thus needs to be aware of "structural connections within mathematics itself" (ibid.:114).

The final category, *contingency* is witnessed in the ability to react to unplanned occurrences in the classroom, for example, responding to children's ideas. "In commonplace language it is the ability to 'think on one's feet'" (Rowland et al., 2004:123).

The Knowledge Quartet team have worked with several groups of people to apply the Knowledge Quartet, including mathematics teacher educators and trainee teachers yet, through this process, the team noticed that others often conceptualise the dimensions of the Knowledge Quartet in different ways than originally intended (Weston et al., 2012). Thus, an online 'coding manual' (freely available online at www.knowledgequartet.org) which consists of a series of scenarios illustrating each dimension now exists to assist users of the Knowledge Quartet (Weston et al., 2012).

The Knowledge Quartet is therefore quite different than other conceptions of mathematical knowledge for teaching as the categories of knowledge are not the focus; instead how this knowledge can potentially be observed in practice is the motivation. Indeed:

> The Knowledge Quartet… is an empirically grounded theory of knowledge for teaching in which the distinction between different **kinds** of mathematical knowledge is of lesser significance than the classification of the **situations** in which mathematical knowledge surfaces in teaching… (Weston et al., 2012:179).

By doing this, Rowland and colleagues circumvent the issue of having a precise definition of mathematical content knowledge.

Similarly, forms of knowledge are not explicitly considered, with the exception of 'Foundation' knowledge. The knowledge in this category is said to be in 'propositional' form (Shulman, 1986). Despite this, the four dimensions of The Knowledge Quartet can be compared with Shulman's categories (c.f. **Table 2.1**, p.33).

2.1.9 Comparison table

The above information is summarised in **Table 2.1** (p.33) to aid comparison of the various conceptions of knowledge for teaching. The table has been adapted and extended from Cole's (2011) thesis, who provides a similar comparison of some of the studies addressed above. For those not included in her table, studies have been added, and the columns outlining the forms and representations of knowledge have been added to all studies as Cole does not address these aspects.

The table begins with Shulman's CK and PCK and the other theories are compared and contrasted with these categories in subsequent rows. Any aspects which are seen as distinct from these categories are listed under the ‘Distinct’ column. For each conception, the following are considered: whether it is grounded in teaching practice, grounded in mathematics, provides a means to measure knowledge, considers forms of knowledge, or provides a representation of knowledge.

	Content knowledge	**Pedagogical Content Knowledge**		**Distinct**	**Forms of knowledge?**	**Representation of knowledge?**	**Measurement?**	**Grounded in Teaching practice?**	**Grounded in mathematics?**
Shulman (1986)	Content knowledge	Curricular knowledge	Pedagogical Content Knowledge	General pedagogical knowledge	Case knowledge	No	No	No	No
	Amount and organization, per se, in the mind of the teacher	Full range of programs designed for teaching particular subjects	Particular form of content knowledge that embodies aspects of content germane to its teachability	Knowledge of educational contexts	Strategic knowledge				
	Understanding the structure of the subject matter in substantive and syntactic ways (Schwab, 1978)	Variety of instructional materials available in relation to those programs	Most useful forms of representations and most powerful analogies, illustrations and examples.	Knowledge of education ends, purposes, and values and their philosophical and historical grounds	Propositional knowledge – disciplined empirical/ philosophical enquiry				
	Knowledge of the what and why of the content	Set of indications and contraindications for the use of particular curriculum materials.	The ways of representing and formulating the subject that make it comprehensible to others	Knowledge of learners and their characteristics	Propositional knowledge: practical experience				
			Understanding what makes a particular topic easy or difficult.		Propositional knowledge: moral/ ethical reasoning				
			Student conceptions and preconceptions and misconceptions.						

	Content knowledge	Pedagogical Content Knowledge			Distinct	Forms of knowledge?	Representation of knowledge?	Measurement?	Grounded in Teaching practice?	Grounded in mathematics?
Leinhardt and Smith (1984)	Subject matter	n/a (before Shulman)			Lesson structure	Declarative (facts) and procedural (algorithms) forms.	Semantic networks	No	No	Yes
	“conceptual understanding, the particular algorithmic operations, the connection between different algorithmic procedures, the subset of the number system being drawn upon, understanding of classes of student errors, and curriculum presentation” (p.2)				“the skills needed to plan and run a lesson smoothly” (p.2)	“For example, basic multiplication facts can be stored in declarative form to be used procedurally when used to raise fractions to a common denominator”.	(for declarative knowledge). “a node-link structure in which concepts are represented as nodes that are linked together”			
Prestage and Perks (2001)	Subject matter					Yes	No/Yes	No	No	Yes
	“knowledge about the subject matter in mathematics, knowledge about its structure, the body of concepts, facts, skills and definitions, as well as methods of justification and proof." (p.102).					(1) learner-knowledge – “the knowledge needed to pass examinations, to find solutions to mathematical problems” (p.102) and (2) teacher knowledge – “the knowledge needed to plan for others to come to learn the mathematics” (p.102).	Model of knowledge forms rather than subject matter knowledge			
Ma (1999)	Profound understanding of fundamental mathematics - depth, breadth and thoroughness of mathematical content knowledge					‘Knowledge packages’	‘Knowledge packages’	No	No	Yes
TEDS-M (Senk et al, 2008)	Mathematical content knowledge	Curricular knowledge	Knowledge of planning for maths teaching	Knowledge of enacting maths		Propositional and case-based knowledge (following Shulman)	No	Yes	No	Yes
COACTIV (Blum and Krauss, 2006)		Tasks and multiple solutions	Misconceptions and difficulties	Explanations and representations		No	No	Yes	No	Yes

	Content knowledge			**Pedagogical Content Knowledge**			**Distinct**	**Forms of knowledge?**	**Representation of knowledge?**	**Measurement?**	**Grounded in Teaching practice?**	**Grounded in mathematics?**
LMT group - Ball et al	Common content knowledge	Specialised content knowledge	Knowledge at the horizon	Knowledge of curriculum	Knowledge of content and teaching	Knowledge of content and students		'decompressed' knowledge ready for use in the classroom	No	Yes	Yes	Yes
	The mathematical knowledge and skill used in settings other than teaching	The mathematical knowledge and skill unique to teaching	An awareness of how mathematical topics are related over the span of mathematics included in the curriculum .		Knowledge that combines knowing about teaching and knowledge about mathematics	Knowledge that combines knowing about students and knowledge about mathematics						
Knowledge Quartet (Rowland and Turner, 2007)	Foundation		Connections		Transformation		Contingency	No	No	No	Yes	Yes
	Knowledge and understanding of mathematics per se. Knowledge of significant tracts of literature and thinking which has resulted from systemic enquiry into teaching and learning. Espoused beliefs about maths.		The coherence of the planning or teaching displayed across an episode. Connected knowledge for teaching (Ball, 1990). Profound understanding of fundamental mathematics (Ma, 1999) Management of discourse. Sequencing of topics of instruction within and between lessons. Knowledge of structural connections within mathematics and awareness of the relative cognitive demands of different topics and tasks.		Knowledge-in-action as demonstrated both in planning to teach and in teaching itself. Based on Shulman's idea of teachers transforming content knowledge into "pedagogically powerful" ways.		Preparedness to deviate from initial teaching agenda. Ability to think on one's feet.					

Table 2.1: Table comparing selected theories of teacher knowledge with Shulman's categories (1986); adapted and extended from Cole (2011)

2.2 Part Two: Criticisms of Shulman's and other categories of knowledge

Within this section, the literature reviewed in Part One (2.1, p.14) is critiqued. It is shown that no single conception of teacher knowledge considers both categories and forms of knowledge whilst simultaneously offering a means to measure and represent that knowledge. In particular, criticisms of Shulman's categories, namely, problems with: dualism, objectivism, separating PCK from CK, problems with multiple representations and labels, problems with focusing on categories and the idea that PCK is a misguided explanation for results of empirical research are considered.

Table 2.1 (p.33) highlights how Shulman's categories of teacher knowledge revolutionised the way teacher knowledge was conceptualised within resulting literature. Indeed:

> Various attempts have been made to map out and identify the constituent parts of pedagogic content knowledge and to provide headings or schema under which it can be located and described for different subject areas (McNamara, 1991:119).

Whilst there are various alternative categorisation systems and labels of knowledge presented within the literature, there seems to be a general consensus that knowledge required *for teaching* mathematics is different than knowledge of mathematics. Indeed, according to Adler and colleagues:

> ...studies related to mathematical knowledge for teaching... do not cohere into a unified frame. However, they consistently reinforce Shulman's insight that knowing mathematics for oneself is not synonymous with enabling others to learn mathematics (Adler et al., 2009:2-3).

As Adler and Davis put it: "A new discourse is emerging, attempting to distinguish and mark out Mathematics for Teaching as a distinctive form of mathematical knowledge, produced in, and used for, the practice of teaching" (2006:272). However, this is an area of considerable debate. Indeed, Hodgen states: "… teacher knowledge in mathematics is an area of some controversy" (2007:2). Seven 'problems' with these categories are now discussed.

2.2.1 The problem of dualism

Tomlinson (1999) argues against 'dualistic' categories (in general) which present action as not being guided by deliberate thought, in other words a separation of body from mind. He argues that action as well as thought can be skilful. Further, Ellis (2007) argues that much of the literature on teacher knowledge treats subject knowledge as fixed, objective and universal: "while simultaneously theorizing separate categories of teacher knowledge as tacit and uncodifiable" (Ellis, 2007:449). In particular, Ellis (2007) criticises Shulman's categories:

> …in Shulman's typology, 'content knowledge'—framed by the problem of dualism—is just a given, something that is merely subject to the transformative action of PCK in classrooms. The point of origin (and the ownership) of subject knowledge is elsewhere and PCK is advanced as a theorization of the 'delivery' mechanism (Ellis, 2007:453).

Thus, the separation of CK from PCK can be seen as a dualistic distinction which reduces CK to an unproblematic given and which is acted upon and delivered by PCK within the classroom. Whilst this is arguably the reverse of Tomlinson's argument (knowledge is relegated whilst action in the classroom is elevated), separating PCK from CK can still be seen as a dualistic distinction.

2.2.2 The problem of objectivism

Ellis argues against research which: "relegate[s] subject knowledge to a prerequisite and unexamined category" and calls for researchers to: "[take] teachers' subject knowledge seriously" (2007:447), in other words "treating it as complex, dynamic and as situated as other categories of teachers' professional knowledge" (ibid.:447). Others (e.g. Banks et al., 2005) highlight this as a weakness of Shulman's categories, purporting that they ignore the dynamic nature of knowledge and instead treat it as a static entity.

Wilson et al. (1987) express a similar view:

> The shared assumption underlying… research is that a teacher's knowledge of the subject matter can be treated as a list-like collection of individual propositions readily sampled and measured by standardized tests. Thus researchers ask how much a teacher knows (how many such propositions) and not how that knowledge is organized, justified, or validated... [such research] has failed to provide insight into the character of the knowledge held by students and teachers and the ways in which that knowledge is developed, enriched and used in classrooms (Wilson et al., 1987:107).

Ellis (2007) asserts that an implication of an objectivist view of knowledge is that: "In policy terms, it becomes a commodity that can be counted ('audited'), 'boosted' or 'topped up' outside of practice—words and phrases used by the Training and Development Agency for Schools (TDA)" (Ellis, 2007:450).

For the current research, which seeks to analyse knowledge change, such a static conception of knowledge may not be useful.

2.2.3 The problem of separation: can CK be separated from PCK?

Whilst there are those who argue against Shulman’s categories of knowledge due to the way subject knowledge is conceptualised, others alternatively question whether separating CK from PCK is warranted from the outset.

Indeed, Shulman justifies the introduction of PCK as a separate category of knowledge particular to the work of teaching and separate from the knowledge of scholars and there are those who agree:

> …teachers may need to generate alternative approaches to the subject matter – analogies, illustrations, metaphors, examples – that take into consideration difference in student abilities, prior knowledge, and learning styles (Wilson et al., 1987:105).

However, the work of teachers and the work of scholars can be seen as similar for several reasons.

Firstly, McEwan and Bull (1991) claim that since scholars seek new knowledge in order to advance their discipline and teachers strive to advance knowledge growth within learners: “these two processes are formally identical… They are both arts of inquiry that "lead out into an expanding world of subject matter" (McEwan and Bull, 1991:331). Indeed, Ellis argues that “learning in a subject is also a process of being disciplined into the ways of thinking and feeling about subject concepts” (2007:450), thus the work of scholars and teachers can be seen as similar.

Secondly, scholarship and teaching can be said to have the common purpose of communicating ideas. Indeed, McEwan and Bull purport:

> …scholarship is no less pedagogic in its aims than teaching. Subject matter is always an expression of a desire to communicate ideas to others, whether they happen to be members of the scholarly community, newcomers to the field, or laypersons. Differences within the form and content of various expressions of subject matter reflect an under-standing of differences in the backgrounds of potential audiences and the circumstances of the subject matter's formulation (1991:331).

Finally, McEwan and Bull criticise Shulman’s categories, arguing that a distinction between CK and PCK is an “unnecessary and untenable complication” (1991:318) on epistemological grounds. Although they recognise that Shulman was not explicit about the underlying epistemology on which his categories are based, they argue that the language used by Shulman suggests an objectivist view (others have also labelled Shulman’s views as objectivist e.g. Banks et al. 2005). McEwan and Bull show that, from an objectivist perspective, separating content from pedagogy is untenable and instead purport the view that all knowledge is pedagogic:

"because all content knowledge, whether held by scholars or teachers, has a pedagogical dimension" (1991:318).

Taking an alternative view, McNamara asks: "whether in practice it is possible to make a clear distinction between subject knowledge and pedagogic content knowledge" (1991:119). He answers by citing empirical research which suggests it is not possible since both are "modified and influenced by practice"; teachers cannot articulate how they shape their practice and the effect of CK and PCK on practice will vary from teacher to teacher depending on their individual values and/ or training received. Thus, even if a separation of CK from PCK was justified, it could not be analysed in practice for research, thus the separation cannot be tested empirically. Moreover, from a measurement perspective, there are no empirical studies which show using quantitative methods (such as confirmatory factor analysis) that CK and PCK are separate constructs.

2.2.4 The problem of multiple representations

However, Shulman maintains that teachers need to know a variety of representations of subject matter, representations which are alternatives to scholars' representations (McEwan and Bull, 1991). Nevertheless, McEwan and Bull ask: "Why should scholars' representations be privileged in comparison with teachers' representations? And why should the teachability of representations be of no concern to scholars?" (1991:319). Going further, McNamara states:

> It is, nevertheless, a moot point as to whether or not representations are a particular characteristic of pedagogic content knowledge or, in effect, the very stuff of subject matter itself (1991:120).

Conversely, there are those who would separate the personal understanding of the teacher with the knowledge of how to represent those ideas for their pupils (e.g. Wilson et al., 1987). Indeed, in a study by Shulman and colleagues, Frank (a novice biology teacher) reflected on what is means to know biology for teaching:

> When you learn [biology] for teaching you have to know it a lot better I think… When you learn it to teach, you have to be able to handle… 150 different approaches to it because you have to be able to handle every different student's approach… They're going to ask you questions from different areas and you're going to have to be able to approach it from their mind-set. (Wilson et al., 1987:104).

Nevertheless, as I argue below (2.3.1, p.42), understanding further representations of a concept is key to a deeper understanding of said concept when learning mathematics, a discipline which: "involves the representation of ideas, namely the structuring of information in ways that permit problem solving and the manipulation of information (Putnam et al, 1990)." (McNamara, 1991:121). As McNamara puts it: "It is at least questionable whether and in what instances another layer of representation should be imposed upon subject knowledge which is itself a form of representation" (1991:122).

2.2.5 The problem of multiple labels

As the literature review in Part One (2.1, p.14) of this chapter reveals, there are many existing conceptions of teacher knowledge. The problem with this is that multiple terms are used potentially as different labels for the same ideas (Adler and Ball, 2009). This can cause disunity and confusion within the field.

2.2.6 The problem of focusing on categories

Another argument against Shulman's categories is whether such categories of knowledge are useful or even necessary. Indeed, Watson (2008) suggests that refining categories of knowledge types is not worthwhile as many people have developed into effective mathematics teachers without such categories.

On the other hand, Ball et al. (2008) maintain that precise definitions and categories of knowledge are necessary for research to build upon shared understanding and lead to progress within the field. They argue that lack of adequate definitions helps explain why research over the last few decades has failed to reform teacher education programs. Ball and Bass (2000) state that many people assume content knowledge is sufficient, or add knowledge of what pupils will learn in subsequent years to the list of required knowledge, yet Ball and her colleagues believe this perspective is too narrow. Hence, they argue for categories of knowledge which distinguish between content knowledge and content knowledge for teaching.

Alternatively, if time has been spent (unnecessarily?) refining definitions instead of analysing the differences between the use of mathematics in teaching *vis a vis* its use in other professions, then this may also explain the lack of progress within the field. In other words, there has been too much focus on the 'what' rather than the 'how'.

Watson proposes a further argument against classifications: that they give a disjointed view of knowledge and hide the underlying mathematical activity. Indeed, she states: "...I try to think about mathematical knowledge in teaching as a way of being and acting, avoiding categorisation and acquisition metaphors of knowledge" (2008:1). She proposes that instead of pedagogic categories, experience of learning mathematics is a more effective way to deepen and develop mathematical knowledge for teaching. She supports her claim by arguing that mathematical ways of thinking and ‘know-how’ allow analysis of teaching and learning situations, curriculum difficulties, and student errors. Further, she provides examples of teachers whose mathematical learning has informed their teaching practice. She states: “...the tasks of teaching can be seen as particular contextual applications of mathematical modes of enquiry" (ibid.:1). In which case, does teaching require additional mathematical *knowledge* or is the *context* of application simply different? If the latter, this would support Watson’s view that: "Any process of identifying types can go too far and losing [sic] overarching insight” (2008:1).

2.2.7 The problem with correlation studies

The introduction of PCK may have gained currency within the literature as a tenable explanation to empirical studies which suggest that teachers with greater mathematical content knowledge do not necessarily have more highly achieving pupils. Thus, if such correlation studies suggest that content knowledge does not predict pupil success, then this may have substantiated the belief that there must be another kind of knowledge required for effective teaching.

A few examples of such studies follow. Then it is argued that there is an alternative explanation to another type of knowledge.

Askew and colleagues, when examining primary school numeracy teachers, found that having a qualification in A-level mathematics or a mathematics degree was not necessarily “positively associated with larger pupil gains.” (Askew et al.,1997:66). Ma (1999) found that although Chinese pupils perform better than USA pupils in mathematics, the Chinese teachers usually have less years of formal education than USA teachers as they do not usually finish high school, yet USA teachers normally have at least a bachelor’s degree.

More recently, in his study of 21 secondary mathematics PGCE students in the academic year 2003-04, Tennant (2006) analysed the correlation between students’ degree classifications (converted into a five point scale and weighted by mathematical content) and final PGCE grades. He found a fairly strong negative correlation ($r = -0.52$), suggesting those who do well on a PGCE course tend to have worse first degree results with less mathematical content. However, looking at the previous 5 cohorts of students (92 students from 2000 to 2005), Tennant (2006) concluded that there was no correlation between students’ formal academic qualifications and successful performance on the PGCE course:

> Using all 92 students who have successfully completed the secondary mathematics PGCE course in the 5 academic years from 2000 to 2005, the unweighted correlation between degree result and success on the course was 0.11, the weighted correlation was -0.05, and the correlation between the weighting and success on the course was -0.16. (Tennant, 2006:50).

There are several limitations of this research, however, in particular, the small sample size, the fact that only students from one institution were included and obviously unsuccessful applicants to the PGCE course were not included (Tennant, 2006). Further, the grades awarded for PGCE courses are somewhat subjective although effort is made to standardise them (Tennant, 2006). Indeed, Stevenson (2008) highlights this as a key limitation of Tennant’s enquiry and adds that the level of measurement of the data is in fact ordinal rather than interval. Nevertheless, using ideas from Tennant’s work, Stevenson (2008) investigated the correlation between academic qualifications and PGCE grades at another institution (31 PGCE students in 2006-07) and also found no correlation. Additionally, she compared final subject knowledge grades and overall PGCE grades between students who had taken a ‘subject knowledge enhancement’ (SKE) course prior to their teacher training and the group as a whole. Although SKE students achieved lower scores on average than the group as whole, the difference was not statistically significant.

Conversely, in her review of the literature, Cole (2011) discusses educational production function studies which examine the relationship between teacher knowledge (often measured by number of mathematics courses taken) and pupil achievement. She concludes that such studies provide mixed, inconclusive findings, suggesting that: “The production function studies incorrectly assumed that the proxies [number of courses etc.] were accurate depictions of teacher knowledge” (Cole, 2011:26-28).

Wilson and colleagues explain the lack of consistency in studies that correlate teacher knowledge with pupil achievement: in such studies teacher knowledge has been measured by scores on standardised tests, which assumes subject matter knowledge is a list of propositions to be tested for: “Thus researchers ask how much a teacher knows (how many such propositions) and not how that knowledge is organized, justified, or validated” (Wilson et al., 1987:107).

A conclusion often made as a result of such correlation studies is that there must be another kind of mathematical knowledge needed for teaching in addition to knowledge of the subject to be taught. However, it is only more recently that the results of such studies have been reframed to consider not types or categories of knowledge but *how* content knowledge is held or organised within the teachers' minds. Indeed, "Without understanding more about *how* mathematical knowledge is brought to bear on the tasks of teaching, descriptions and audits of necessary knowledge are hypothetical" (Watson and Barton, 2011:68). In other words, if mathematical knowledge needs to be held in a different way for effective teaching then of course mathematical qualifications do not correlate with effective teaching. Indeed: "It is not just a question of what teachers know, but how they know it, how they are aware of it, how they use it and how they exemplify it" (Watson and Barton, 2011:67).

Following similar results in his study (no positive association between formal qualifications and pupil gains of primary numeracy teachers), Askew et al. (1997) state: "What would appear to matter… is not the level of formal qualification but the *nature of the knowledge about the subject* that teachers have" (1997:69). In considering Primary school teachers knowledge of numeracy, Bell (1991) also suggests that teachers do not necessarily need more mathematical knowledge, but better connected, deep, multi-faceted understanding.

Furthermore, as Kersting and her colleagues conclude: "Just having knowledge is not enough; it may be equally important that knowledge be organized and accessible in a flexible way" (2010:178). Indeed, "Teachers might know things in a theoretical context but be unable to activate and apply that knowledge in a real teaching situation" (Kersting et al., 2010:178).

In her study of trainee secondary teachers, Crisan (2012) concludes that teachers need to see mathematics as coherent and connected in order for them to make appropriate pedagogical decisions on the use of instructional resources. In the study, students revisited the square root concept and their prior understandings were challenged as they considered positive and negative roots and when and where these were appropriate.

Thus, researchers recognise that teachers need more than subject matter knowledge. In other words, becoming an effective teacher is not simply a case of acquiring more knowledge by taking more courses. Despite this, according to Wilson and colleagues:

> …educational researchers have failed to provide a theoretically or empirically based conceptualization of the professional knowledge base of teaching. Instead, research on teaching has concentrated on what effective teachers need to do, with the types of performance associated with their effectiveness. Researchers have been much less interested in what sorts of knowledge or understanding are needed to make desirable performance possible (Wilson et al., 1987:106).

Other attempts in the literature to describe the knowledge teachers require include: a 'mathematical sensibility' (Askew, 2008:22); 'connected' knowledge (Chinnappan and Lawson, 2005) or 'fluency' (Davis and Simmt, 2006; Chinnappan and Lawson, 2005). Further, Mason and Spence view knowledge not as "an approximation to some fixed or absolute state of justified true belief, but as at best a snapshot of a state of knowing that is in constant flux according to prevailing personal and social conditions" (1999:137). They define 'knowing-to' act in the moment as "a state of readiness as a result of what is being attended to. It consists of what is primed and ready to come to attention, and it excludes what is blocked or otherwise unavailable" (ibid.:151).

All these attempts seem related yet only seem to point research in the general direction rather than explicitly concretising what this fluid, connected knowledge or sensibility is or consists of. The current research attempts to understand how mathematical knowledge is held in the minds of teachers.

2.3 Part Three: Proposed approach to teacher knowledge

> ... we believe that (e)pistemological awareness is an important and informative part of the transparent research process that needs to be addressed and communicated to readers... Moreover, when authors make their (e)pistemological awareness and desired knowledge(s) within a particular research project unambiguous and explicit, this process of selfreflection can assist authors in selecting methods that instantiate and support their knowledge building... as well as choosing a theoretical perspective that is suited to the purposes of their research (Koro-Ljungberg et al., 2009:689).

2.3.1 Theoretical underpinning

As Part Two (2.2, p.34) discusses, there are seven potential issues with Shulman's categories of knowledge and those attempting to build on or refine his ideas. Indeed, teacher knowledge is an area of considerable debate (Hodgen, 2007). Further, there are objections to categories of teacher knowledge which treat content knowledge as fixed, objective and straightforward, perhaps unintentionally. It is my belief that a consideration of mathematical knowledge for teaching requires consideration of the ontology and the epistemology of mathematics. Indeed, according to Askew:

> If mathematical constructs are shaped through "analogies, metaphors, image, and logical constructs" then are these part of pedagogical content knowledge or part of subject knowledge? And does it matter? The answer comes down to a philosophical position on the epistemology of mathematics. If one believes that there are idealised mathematical forms that exist independently of representations, illustrations, examples and so forth, that there is a signifier/signified distinction (Walkerdine, 1988), then such 'unpacking' (or repackaging) is going to be seen as a pedagogic skill. If, on the other hand, one views mathematics as a 'language game' (Wittgenstein, 1953) only brought into being through representations, illustrations, examples and not existing outside these, then this is an aspect of subject knowledge as much as pedagogic (Askew, 2008:28-29).

For the current research, I present a view of the ontology and epistemology of mathematics which aims to overcome the seven limitations and criticisms of existing views (2.2, p.34) and which seeks to understand how knowledge for teaching mathematics is held.

This research takes the ontological view that mathematical concepts exist independently (are objective and fixed) but takes the epistemological view that accessing such concepts (by a person) cannot fully be achieved. However, a person can use approximations of the mathematical concepts in the form of semiotic representations. Indeed:

> Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments (microscopes, telescopes, measurement apparatus). The only way to have access to them and deal with them is using signs and semiotic representations (Duval, 2006:107).

By building up a repertoire of alternative representations of a concept, one comes closer to understanding the mathematical concept which itself transcends any single representation. Similarly, Wurnig and Townend (1997) recognise that a mathematical concept is not understood fully until different representations of the concept are understood:

> For many the development of a new area of mathematics for a student involves a trinity of the following areas of mathematics:
> numerical graphical algebraic
> and it is suggested that a concept is not completely understood until a student resides in the intersection of these three sets…
> (Wurnig and Townend, 1997:78).

Therefore, in contrast to Walkerdine's view (cited above by Askew), 'unpacking' mathematical concepts is not a pedagogic skill alone since anyone who wishes to understand/ use mathematical concepts (whether in the classroom or outside the classroom) must access them via representations. For example, the concept of multiplication is more than being able to execute the standard multiplication algorithm. Indeed: “Being able to calculate in multiple ways means that one has transcended the formality of an algorithm and reached the essence of the numerical operations - the underlying mathematical ideas and principles” (Ma, 1999:112). This resonates with Shulman’s ‘strategic knowledge’.

Akkoc and Ozmantar (2012) found that multiple representations can be utilised by teachers to construct a deeper understanding of mathematical concepts. Further, Wilson and colleagues (1987) encountered a multiplicity of representations in interviews and observations with teachers. However, their explanation for the necessity of teachers having a repertoire of representations was pedagogical and the extension of the teacher’s understanding was seen as a by-product:

> As students are multiple, so representations must be various. As multiplicity of connections renders understanding more durable and rich, so the range of variations produced by the transformations is argued, in principle, to be a virtue. Hence, teachers should possess a ‘representations repertoire’ for the subject matter they teach. And, as the representations repertoire grows, it may enrich or extend the teacher’s subject matter understanding **per se** (Wilson et al., 1987:113).

Wilson et al. (1987) consider the process by which a teacher takes subject matter and transforms it for teaching. They propose that a teacher originally has a particular preferred representation and then introduces further representations of the subject matter which can be developed by the teacher or student. They present a diagram of this process (**Figure 2.4**, p.45).

In their model, teachers begin with **comprehension** of a set of ideas. They provide the example of mathematics teachers understanding the relationship between fractions and decimals. **Transformation** then involves the four subprocesses shown in the diagram: *Critical interpretation* involves reviewing materials for instruction with reference to their current understanding of the topic; then in *representing* the subject matter, teachers draw on a range of “metaphors, analogies, illustrations, activities, assignments, and examples that teachers use to transform the content for instruction” (Wilson et al., 1987:120); these transformations are then *adapted* to the particular pupils to be taught in terms of their misconceptions, prior knowledge, ability, gender and motivations. These adaptations are then *tailored* to specific pupils in one class rather than the general pupil population. A plan for instruction is then developed as a result of these steps. **Instruction** is defined by Wilson and colleagues as the “observable performance of the teacher” (1987:120). **Evaluation** can take place both during and after instruction as they check for pupil understanding during teaching and from formal summative tests. As the teacher **reflects** on the teaching and learning that has taken place this can lead to **new comprehension** – an: ”understanding that has been enriched with increased awareness of the purposes of instruction, the subject matter of instruction, and the participants – teacher and [pupils]” (1987:120).

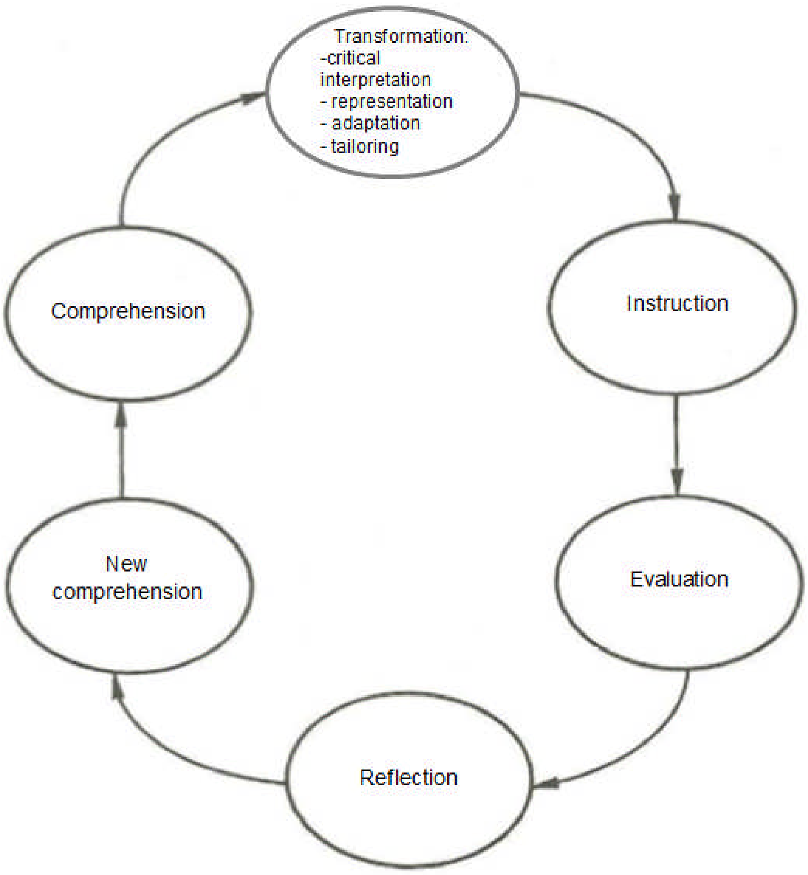

Figure 2.4: Model of pedagogical reasoning (Wilson et al, 1987).

Wilson et al. (1987) focus their research on subject matter knowledge and its role in teaching. They state that in addition to understanding the subject matter to be taught, teachers “must also possess a specialized understanding of the subject matter, one that permits them to foster understanding in most of their [pupils]” (1987:104). Thus, Wilson and colleagues see the process of acquiring multiple representations as unique to teaching. Alternatively, I take the view that such a process enables any person to understand mathematical concepts more deeply and this is not unique to the province of teaching. However, in practice, it is the nature of learning and teaching within schools which brings this to the attention of those researching education.

Having outlined the underlying ontological and epistemological assumptions for this research, a new approach to thinking about mathematical knowledge for teaching, including a way to represent that knowledge, is presented followed by discussion of how this new approach aims to overcome the criticisms of existing conceptions of knowledge which are highlighted in Part Two (2.2, p.34).

2.3.2 A new conception of teacher knowledge

2.3.2.1 Discrete/ continuous metaphor

As an extension of the ontological and epistemological assumptions which are described above, this research distinguishes between knowledge which is held in ‘discrete’ and ‘continuous’ forms. A ‘discrete’ form of knowledge can be learned once (unless forgotten) and can be used to solve mathematical tasks in a learned, fixed way (a procedural, propositional or an instrumental knowledge). In contrast, a ‘continuous’ form of knowledge is not necessarily learned only once but the knowledge can continuously be extended, developed and connected with other knowledge (a conceptual, strategic or relational knowledge). The process of acquiring further representations of a mathematical concept is a feature of building knowledge in a ‘continuous’ manner. As an example, the method of completing the square to solve a quadratic equation only needs to be learnt once (unless forgotten). Thus, one either possesses the knowledge to complete the square, or one does not at any given time. Further, knowledge of the procedure cannot be altered, deepened or extended; the process remains the same (though of course, one can become more adept or speedy at completing the square over time). Conversely, the knowledge needed to understand quadratic equations can be extended and deepened over time as one learns: further techniques for solving; connections between them; which technique would be most useful for a given equation; and, why the techniques work.

2.3.2.2 Representations of knowledge: Knowledge Maps

Drawing on the underlying ontological and epistemological viewpoints inspired by Duval’s (2006) ideas of representation and the discrete/ continuous metaphor, I introduce a representation of knowledge which I term a ‘Knowledge Map’. Knowledge Maps are a way of visually representing knowledge which can be drawn by a researcher for a teacher at cross-sections of specified time-periods such as a school year or professional development course (for the current research, over a teacher training course). Using some of the multiple representations or metaphors for multiplication which were listed by participants in a study by Davis and Simmt (2006), I present an example of a 'Knowledge Map' (**Figure 2.5**, p.47). Here, the central 'cloud' represents the concept of multiplication which itself cannot be perceived (grasped hold of). In contrast, the rectangles are discrete, concrete representations or methods of operationalising a concept – in this case multiplication – which are either known/ not known or can be executed or not at any given time (discrete form of knowledge). These representations can reside within different mathematical domains: algebraic, geometrical or numeric (I substitute the word 'graphical' provided by Wurnig and Townend (1997) in their quote above for 'geometric').

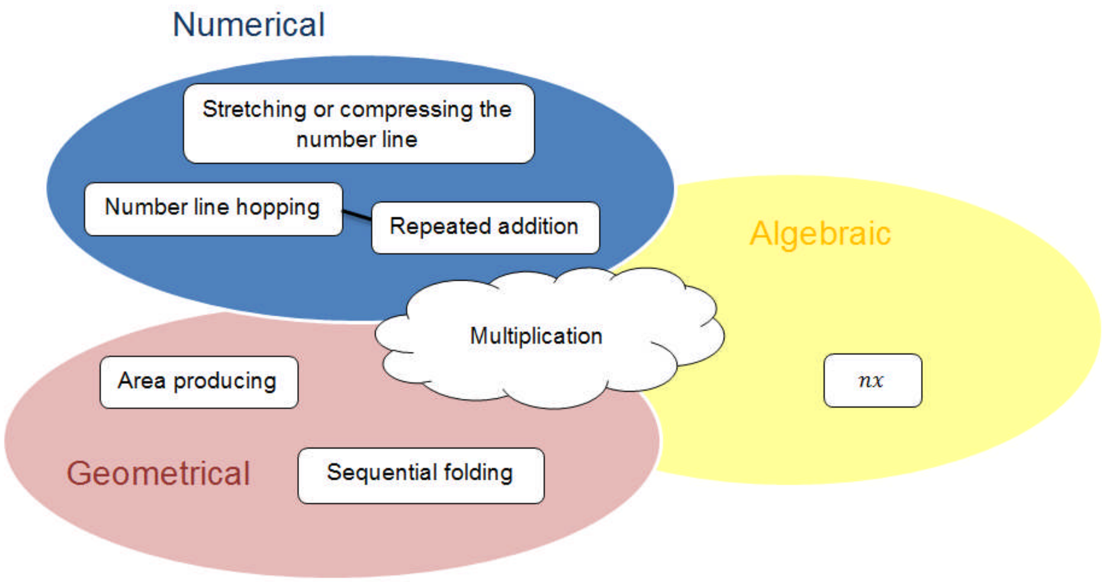

Figure 2.5: An example of a knowledge map for multiplication.

The changes between a set of Knowledge Maps through the addition of further links and domains shows how knowledge can be extended/ deepened over time (continuous form of knowledge). The addition of further domains will necessarily include the addition of further nodes, thus continuous knowledge is a superset of (includes) discrete knowledge.

Knowledge held in a continuous form can be developed over the course of teaching and can help explain how learning through teaching occurs. Additional representations can be learnt from text-books, pupils, professional development courses, colleagues and other resources teachers use and encounter during their teaching practice. Each representation adds to an increasingly complex set of connections and builds the teachers' knowledge of a mathematical concept which is otherwise not perceivable. Indeed: "access to the web of interconnections that constitute a concept is essential for teaching" (Davis and Simmt 2006:301). Moreover, such an understanding as represented above is not an isolated part of knowledge, but is connected to other concepts such as addition, division and square numbers.

Within the literature there are attempts to 'map-out' or represent how knowledge is organised within one's mind. For example, Ma (1999)'s 'knowledge packages'; and, use of concept maps (e.g., Chinnappan and Lawson, 2005). Unlike concept maps, the concept itself (multiplication) is not a node, but the multiple representations of the concept form a nucleus of nodes with multiple connections between them and other nuclei of nodes (concepts). A 'knowledge map' is thus not a 'concept map' since it is not composed of multiple "concepts... and relationships between concepts" (Novak and Canas, 2008:1) but multiple *representations* of a single concept and relationships between those representations. Thus a 'knowledge' map' can be seen as a finer-grained version of a concept map. Whilst concept maps are concerned with the organisation and hierarchical structure of concepts in one's mind (how a concept is related to other concepts) Knowledge Maps are concerned with the representations (discrete forms of knowledge) which, alongside connections between them, can be increased over time (continuous forms of knowledge). The connections between the nodes of discrete knowledge are complex and are thus only a few possible connections (lines) are drawn on the diagram. These lines represent an understanding of how representations relate to one another and act in a different way to the lines on a concept map. The links to the central construct in a concept map imply that the concept comprises of the surrounding nodes. However, I share the conclusion that Davis and Simmt reached upon completion of their study:

> By the end of the lengthy discussion of these representations of multiplication, there was consensus that the concept of multiplication was anything but transparent. In particular, it was underscored in the interaction that multiplication was not the sum of these interpretations. It was some sort of conceptual blend (2006:301).

A 'knowledge map' is thus not a 'concept map' since its focus is on *representations of* concepts rather than *relationships between* concepts.

Similarly, a 'knowledge map' is not a 'knowledge package' since the latter suggests pre-parcelled knowledge taken into the classroom. Rather, continuous knowledge is similar to a 'map' from which alternative routes can be highlighted/ discovered *during* the practice of teaching. Instead, like maps, they are periodically revised to show alternative routes (and even additional 'landmarks') which are discovered *during* the practice of teaching.

2.3.3 How the proposed approach addresses the problems of existing approaches

I claim that the discrete/ continuous metaphor and Knowledge Maps improve upon existing conceptions of knowledge. I will now outline how this new approach can be theoretically seen to overcome each of the seven limitations of existing conceptions (see 2.2, p.34). However, this will be returned to within the discussion chapter (7.3, p.230) in light of the data gathered and analysed for this research.

2.3.3.1 The problem of dualism

The discrete/ continuous distinction is not dualistic. Indeed, whilst knowledge held in discrete form is either known or not known (there is no in between), it can be held implicitly or explicitly (one can be aware of this knowledge or not) and it can either be a known fact or a known procedure (actions). A continuous form of knowledge, as a super-set of discrete knowledge, must therefore also have explicit or implicit knowledge contained within it and both knowledge of facts and techniques. Thus, the discrete/ continuous metaphor does not separate mind from body.

2.3.3.2 The problem of objectivism

As section 2.3.2 (p.46) states, this research assumes mathematical knowledge is fixed, independent and objective. However, unlike many other conceptions of knowledge for teaching, the epistemological view held is that coming to know this knowledge is not straightforward. Thus, this overcomes the objections which are discussed in Part Two (2.2, p.34) that existing categories of teacher knowledge treat subject knowledge as unproblematic and something that can be itemised on a list and audited. In contrast, continuous knowledge is seen as a complex process. Indeed, the discrete/ continuous metaphor emphasises the dynamic nature of knowledge by considering how it behaves over time. Moreover, a categorisation which focuses on how knowledge alters temporally is crucial for the current research which considers teachers' knowledge change since:
"Understanding developing knowledge is limited when measures of static knowledge are used" (Fennema and Franke, 1992:161).

Similarly, Knowledge Maps also aim to represent the dynamic nature of knowledge. Existing representations of knowledge such as 'knowledge packages' (Ma, 1999) are criticised as not being flexible enough: teachers need knowledge that is ready for use, not brought into the classroom pre-packaged (Ball and Bass, 2000). The Knowledge Maps also demonstrate how content knowledge is not an unproblematic given since, unlike concept maps, it is not individual mathematical concepts which form the nodes. Instead, concepts are not viewed as objects which can be simply listed on a map with links between them but as complex, idealistic entities which are comprised of (but which transcend) their various semiotic representations. The Knowledge Map of a given mathematical concept may therefore vary significantly between teachers or temporally for a given teacher.

2.3.3.3 The problem of separating PCK from CK

This research does not separate CK from PCK, thus learning and teaching are connected. Therefore, not only is gaining multiple representations seen to lead to an extension of the teacher's understanding of subject matter but by a teacher coming to know a mathematical concept through multiple representations, this is seen to contribute to more effective teaching.

2.3.3.4 The problem of multiple representations

As 2.3.3.3 (p.50) discusses, the current research treats the process of adding multiple representations to a teachers' repertoire as crucial for building the teacher's own understanding of the concept and helping their pupils to understand a concept. Thus, this research does not add another layer of representations onto a subject (mathematics) which is understood through representations as McNamara (1991) objects (see 2.2.4, p.37).

Whilst it can be seen that discrete and continuous knowledge is needed by all professions which require use of mathematics (because in order to understand mathematical concepts, one must build up representations of the concept) in practice, other professions may focus more on discrete forms of mathematical knowledge. For example, an accountant may only need to know and use one method for adding expenses (say). That one method may be sufficient to allow him/her to carry out his/her work, thus further methods are not required.

To extend the accounting example, the goal is to reach an answer (it does not necessarily matter which method is used) whereas, for teaching, the goal is to expound the method(s) of arriving at an answer and why such method(s) work. Indeed, Ma (1999) found that teachers who knew a range of methods - including non-standard ones - for subtraction with regrouping, were able to flexibly deal with non-standard methods proposed by pupils.

2.3.3.5 The problem of multiple labels

The discrete/ continuous metaphor (forms of mathematical content knowledge) resonates with and consolidates the plethora of labels defined by other researchers within the literature. There have been many attempts to label how knowledge can be held in the minds of teachers, yet the overarching distinction is between what I term ‘discrete’ and ‘continuous’ forms of knowledge. This distinction seems to emerge from the literature but is not termed as such.

Knowledge held in a discrete form is a procedural, propositional or an instrumental knowledge, whereas knowledge held in a continuous form is a conceptual, strategic or relational knowledge. Indeed, a ‘mathematical sensibility’ would be needed and can be developed as one learns a variety of methods and how to select the best or most useful method for solving a task; connections can be formed between topics and concepts; and ‘fluency’ is also required in selecting appropriate methods.

2.3.3.6 The problem of focusing on categories

The discrete/ continuous metaphor and ‘knowledge map’ representation does not focus on categorising different aspects of a teacher’s knowledge. In particular, it does not attempt to identify a different kind of mathematical knowledge necessary for teaching. Instead, this research focuses on how the knowledge teachers hold changes over time (in practice) with a view to understanding more about the organisation of mathematical knowledge for teaching.

2.3.3.7 The problem with correlation studies

In section 2.2.7 (p.39), the argument that a lack of correlation between teacher subject knowledge and pupil achievement must indicate that there is another type of knowledge needed by teachers is presented and is shown to be flawed. Indeed, some researchers have begun to recognise that the focus within the literature has been to identify *types* of knowledge rather than considering *how* subject knowledge needs to be held differently in order to teach others. The discrete/continuous metaphor along with Knowledge Maps seek to further understand the organisation of teachers’ knowledge and how it changes over time rather than identifying different knowledge types.

2.3.4 Conclusion

Part One (2.1, p.14) of this chapter shows that there are various categories within the literature for labelling teacher knowledge, with suggestions to how this knowledge can be held. However, the criticisms of these categories are outlined (in Part Two, p.34) and an alternative theory of knowledge for teaching mathematics (discrete/ continuous knowledge) is proposed (in Part Three, p.42) along with a means of representing such knowledge (Knowledge Maps) which recognises mathematical knowledge as dynamic, complex and interrelated with PCK.

According to Even:

> Teacher knowledge for teaching mathematics has received immense international attention in recent years. Yet, theoretical/conceptual perspectives to serve as the basis for analysing teacher knowledge and its relationship to the work of teaching are sparse (2009:146).

The current research aims to address this gap by proposing a theory of teacher knowledge with accompanying representation for analysing teacher knowledge in practice.

3 Research Questions

Chapter 1 (Introduction) highlights the issue of the poor mathematics results of school leavers within England and the subsequent research into teacher knowledge as a potential solution. The literature review (2.1, p.14) highlights various conceptions of teacher knowledge within the literature, many of which stem from Shulman's categories of knowledge. Several criticisms with these categorisations are identified, in particular, treating knowledge as fixed and static (2.2, p.34). Subsequently, section 2.3 (p.42) proposes an alternative ontological and epistemological view of mathematics and its relation to teaching and learning to be investigated in the current research within the context of teacher training in England. In particular, the proposed theoretical underpinning of the current research recognises the dynamic nature of knowledge. By examining how knowledge changes over a teaching training course it is anticipated that further insights into the knowledge useful for teaching and how this knowledge is organised can be gleaned.

The main aim of this research is to explore: how the mathematics-related knowledge[2] of trainee secondary teachers changes over time and whether there is a difference between Postgraduate Certificate in Education (PGCE) students who have taken a 'subject knowledge enhancement' (SKE) course and those who have not. In particular, the research questions to be addressed are as follows.

3.1 The Research Questions

When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course:

1. What is the nature of this knowledge?
2. (How) does this knowledge change?
3. Does mathematics-related knowledge differ between SKE and Non-SKE PGCE students?
4. What are some of the factors which have caused a change (if any)?

These research questions are explained and justified below.

[2] The term 'mathematics-related knowledge' is used here to denote knowledge which is related to mathematics as a subject (rather than general pedagogic knowledge) but since the form of this knowledge is to be explored within this study, I have avoided using the terms 'content knowledge' (Shulman, 1986) or Mathematical Knowledge for Teaching (Ball and colleagues).

3.2 Research Question 1 (RQ1)

The literature review shows a plethora of knowledge categories, many of which are theoretical rather than being grounded in teaching practice (see **Table 2.1**, p.33). However, there are exceptions, including work done by the Learning Mathematics for Teaching (LMT) group at the University of Michigan (2.1.5, p.19) and the Knowledge Quartet developed by Rowland and colleagues in the UK (2.1.8, p.27). Both these theories of knowledge for teaching stem from analysis of teachers' classroom practice. However, the research of the LMT group was conducted in the USA which raises questions regarding the transferability of the theory and multiple-choice Mathematical Knowledge for Teaching (MKT) items to England. Further, the Knowledge Quartet, although developed in the UK for secondary mathematics teachers, does not focus on categories of knowledge, only situations arising in teaching practice. Thus, for mathematics teachers in England, this gives rise to the first research question:

> When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: What is the nature of this knowledge?

For this study, the tools developed from these researchers, that is, the MKT multiple-choice items and the Knowledge Quartet observation analysis are used to explore mathematics teacher knowledge.

Firstly, the MKT items are used with students on PGCE courses in England, since these items have not been used in this context before, this has the potential to further the work of the LMT group and extend knowledge. If the items prove a useful measure in this context, then this study may provide a platform for further research involving these items within England. Moreover, if scores on these items relate to success on a PGCE course in England, then this may suggest that the construct of MKT (which the items are claimed to measure) is a useful view of the knowledge required to be a successful teacher in England.

In terms of the Knowledge Quartet, this framework is used to analyse observations of PGCE students teaching on their school placements as part of their PGCE course (for further details see 4.3.3.2, p.97).

From the literature review, it became clear that many existing categories of knowledge focus on labels for knowledge rather than how knowledge is held or organised within the mind (2.2.6, p.38). This research aims to address this gap through the introduction of a new understanding of teacher knowledge

(2.3.2, p.46). The utility of this new approach is explored whilst addressing the first research question.

3.3 Research Question 2 (RQ2)

A further issue the literature review highlights is the tendency of research to treat knowledge as static. Whilst there is some recognition that knowledge is dynamic, theories of knowledge with subsequent research methods for gathering research data and evidence for such theories are lacking.

For this research, a new theory of how knowledge may be held is proposed along with a way to represent and therefore analyse it in practice through Knowledge Maps (see 2.3.2, p.46). This approach recognises the dynamic nature of knowledge.

The second research question therefore stems from this gap within the literature:

> When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: (How) does this knowledge change?

Moreover, it relates to the first research question as the process of examining the knowledge change (if any) of trainee secondary mathematics teachers may demonstrate which elements of knowledge are useful to be retained for teaching practice and which elements ‘fall away’. By analysing the knowledge change of trainee rather than in-service teachers, potentially more information could be gleaned since trainees are at the beginning of their journey to become teachers and therefore may have the greatest changes (if any) to be made in terms of knowledge.

It is anticipated that through examining knowledge change, this may aid our understanding of the nature of knowledge for teaching. For example, if PGCE students’ mathematical knowledge became more connected, then it could be said that gaining connections is useful for teaching. On the other hand, if mathematics knowledge for teaching requires more than subject knowledge alone, then there may be no change in knowledge over the course of a PGCE.

3.4 Research Question 3 (RQ3)

Due to a shortage in mathematics teachers within England, the UK government have sponsored SKE courses to enable graduates without sufficient mathematical content within their degree to train to become teachers. There is a dearth of research into the effectiveness and implications of this policy. Available figures suggest over a third (39%) of mathematics PGCE students have taken an SKE course beforehand (see 1.3.4.4, p.8). For such a significant proportion of trainee teachers, it is surprising there is little research into these courses. Indeed, the literature search found only one report (Gibson et al. 2013) to evaluate the effectiveness of SKE courses in general (including mathematics). Although there was an element of the study which compared subject knowledge between SKE and Non-SKE students, this was limited to PGCE students' self-reported ratings on the level and confidence of subject knowledge they possessed and a single open-response survey question which asked for any thoughts on subject knowledge differences/ similarities between SKE and Non-SKE students. Thus, data were limited to students' perceptions about subject knowledge rather than seeking to obtain an understanding of the subject knowledge possessed by students (and how this was organised) directly.

There have been almost no data gathered on how SKE students compare with their Non-SKE peers in terms of completion rates, final PGCE grades and overall teaching quality. An exception to this is a small-scale study by Stevenson (2008) which was limited to one cohort of PGCE students on a single PGCE course in England. Stevenson found no significant difference in final PGCE course scores between SKE students in comparison to the group as a whole.

This research seeks to address these gaps in the literature through RQ3:

> When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: Does mathematics-related knowledge differ between SKE and Non-SKE PGCE students?

Unlike the studies by Gibson and colleagues and Stevenson, the current research seeks to understand the nature of content knowledge and whether any differences exist in terms of the nature and/ or organisation of this knowledge between SKE and Non-SKE PGCE students. More objective measures of knowledge are used to compare between students, namely, final PGCE results and MKT scores. Moreover, this study uses a larger sample of students from multiple institutions.

Looking at these differences (if any) may also help us to understand the nature of knowledge required to be an effective mathematics teacher (RQ1). For example, if SKE students tend to achieve higher final PGCE grades than their Non-SKE peers (say), this may suggest a mathematics degree is not necessary to become an effective secondary mathematics teacher.

3.5 Research Question 4 (RQ4)

Once an understanding of the nature of teacher knowledge and how it changes and differs between students has been gleaned, it is useful to know the factors which have led to knowledge change in order that future teachers can benefit from this knowledge and adapt their knowledge accordingly and as efficiently as possible. Therefore, the fourth and final research question facilitates the impact of this research on teacher training:

> When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: What are some of the factors which have caused a change (if any)?

These factors are anticipated to emerge as a result of following PGCE students on their PGCE course through observations and interviews.

The main aim of this research is expected to be met through answering the four research questions discussed above. The first three research questions are distinct but inter-related in that RQ2 and RQ3 also may provide evidence to address RQ1. In contrast, RQ4 reflects a different approach, that is, to enable the knowledge gained from RQ1 – RQ3 to be applied to teacher education programs and thus other future teachers.

4 Methodology

This chapter describes and justifies the research methods used to address the research questions (see Chapter 3). It begins with considering the underlying theoretical assumptions of the researcher which may impact upon the research design. The three main data collection methods (questionnaires, interviews and observations) are then discussed in turn.

4.1 Underlying assumptions and their impact on the research design

According to Romberg, research methods may depend on the researcher’s world view, “the time orientation of the questions being asked, whether the situation currently exists or not, the anticipated source of information, and product judgement” (1992:56). In terms of the researcher’s ‘world view’, such assumptions and/ or theoretical views can affect the research methods chosen (Shulman, 1988; Artigue et al., 2005) but may be taken for granted (Bikner-Ahsbahs, 2005). Indeed, researchers may be unaware of their underlying assumptions since: “...sometimes it can be challenging and/or impossible to label, conceptualize, articulate, and intentionally know what we know” (Koro-Ljungberg et al., 2009).

Nevertheless, it is important to try to understand such assumptions otherwise, if the researcher focuses purely on research methods and techniques, many researchers agree that this may cause limitations to the study. For example, Kilpatrick (1981) states: “As long as we ignore the theoretical contexts of our research... it will remain lifeless and ineffective." (1981:26). Further, according to Koro-Ljungberg et al. (2009), lack of epistemological reflection can lead to research designs which “...appear random, uninformed, inconsistent, unjustified, and/or poorly reported.” (2009:688). Finally, Popkewitz (1984) claims that the outcome of failing to situate research methods within a theoretical context is “knowledge that is often trivial…” (1984:ix).

Additionally, theoretical assumptions may not only affect the research methods selected for a study, but also the resulting theory emerging from the data analysis. Indeed: "The assumptions are so deeply rooted in the personal reality of the researcher that they become 'facts' that structure the perceptions of the theorist and shape his/her subsequent theorizing" (Popkewitz, 1984).

Within educational research in particular, Romberg (1992) claims that:

> ...different ideological perspectives are reflected in a group's assumptions about the knowledge that is to be taught, the work of students and how learning occurs, the work of teachers and professionalism, and the social organization and technology of the classroom and schools (1992:54).

The knowledge to be taught for this study is mathematics, thus, it follows that theoretical perspectives can be demonstrated through assumptions regarding the nature of mathematics and how mathematics is learned.

Romberg's (1992) categories of assumptions: *'Assumptions about…*	**Assumptions for the current research**
… the knowledge that is to be taught' (i.e. mathematics)	Mathematics is seen to exist independently but can only be accessed via semiotic representations.
…the work of students and how learning occurs'	Students learn mathematics through building up representations of the mathematical concepts. The more representations they have of a concept, the deeper the knowledge.
… the work of teachers and professionalism' for this research	Teachers have the task of helping students understand mathematical concepts by introducing them to multiple representations of the concepts. Teachers themselves need to know a variety of representations to aid their understanding of mathematical concepts and this is hypothesised to help them teach a variety of students.
… the social organization and technology of the classroom and schools' for this research	The classroom is constrained by external factors such as time pressures, exams and stipulated curricula. This means often only one representation can be covered within the lesson time. Often this single representation is the one stipulated by the curriculum and/ or tested in the examination(s).

Table 4.1: Table showing the assumptions made for the current research with respect to Romberg's (1992) four areas of assumptions

Section 2.3.1 (p.42) presents the assumptions held for this research. Since Romberg's work is widely regarded (NCTM, 2015) the assumptions are recapped here (**Table 4.1**, p.59) with respect to Romberg's four areas of assumptions stated above.

For this study, these underlying assumptions are considered when deciding upon the methodology to follow in order to have a consistent research design. However, it is recognised that there may be other assumptions held which are difficult to make explicit (c.f. Koro-Ljungberg et al., 2009).

4.2 Methodology

This section introduces mixed methods research and outlines some of the issues and controversies surrounding the use of mixed methods within social sciences research. The term 'mixed methods' is then defined and justified for the current study.

4.2.1 Debates regarding mixed methods

Historically, research has distinguished between qualitative and quantitative studies until relatively recently when mixed methods research is proposed as an alternative 'third approach' (McEvoy and Richards, 2006). This third approach has recently become more widespread, particularly within social sciences research (McEvoy and Richards, 2006). Indeed, the Handbook of Mixed Methods in Social & Behavioral Research was published in 2002 with a second edition in 2010 and the Journal of Mixed Methods Research was established in 2007. Within the field of mathematics education in particular, Hart and colleagues (2009) examined 710 research articles from six prominent English-language journals and classified 29% of them as mixed methods studies (Hart et al., 2009).

Despite the increased adoption of such methods, mixed methods research is still an area of ongoing discussion (Tashakkori and Creswell, 2007a). At one side of the argument, there are those researchers who see mixing methods as a "philosophical oxymoron" (Maxwell and Mittapalli, 2010:146) because it ignores the underlying ontological and epistemological views of quantitative and qualitative research. Indeed, as Guba states with regard to positivism and naturalism: "The one [paradigm] precludes the other just as surely as belief in a round world precludes belief in a flat one" (1987:31). At the other extreme, Gorard (2010) proposes that the term mixed methods is redundant since combining both qualitative and quantitative approaches is part of the 'natural' research process which all 'serious' researchers should undertake in order to have more complete and ethical research results. He further argues that there are no barriers to mixing methods, only the barriers researchers have created for themselves by choosing to divide methods into qualitative and quantitative types (an unnatural divide) - just one of many possible classifications.

Between mixed methods researchers themselves, there is also disagreement. In particular, there are many different uses of the term ‘mixed methods’. Indeed, Johnson, Onwuegbuzie and Turner (2007) examine 19 definitions of mixed methods provided by leaders in the field. Whilst many researchers agree that a mixed methods study involves both qualitative and quantitative elements, disagreements occur when considering how these strands are related (Tashakkori and Creswell, 2007a). Tashakkori and Creswell identify seven specific ways a study could be termed ‘mixed’:

> • two types of research questions (with qualitative and quantitative approaches),
> • the manner in which the research questions are developed (participatory vs. preplanned),
> • two types of sampling procedures (e.g., probability and purposive; see Teddlie & Yu, 2006),
> • two types of data collection procedures (e.g., focus groups and surveys),
> • two types of data (e.g., numerical and textual),
> • two types of data analysis (statistical and thematic), and
> • two types of conclusions (emic and etic representations, ‘‘objective’’ and ‘‘subjective,’’ etc.)
>
> (Tashakkori and Creswell, 2007b:4).

One way to group these various definitions is to distinguish between mixed methods as the collection and analysis of both qualitative and quantitative data types (methods focus) and mixed methods as incorporating both quantitative and qualitative approaches (methodology focus) to research (Tashakkori and Creswell, 2007a). Alternatively, Creswell and Tashakkori (2007) have identified four perspectives which researchers have taken on mixed methods:

> The first is a method perspective, in which scholars view mixed methods as focused on the process and outcomes of using both qualitative and quantitative methods and types of data. The second is a methodology perspective, in which writers discuss mixed methods as a distinct methodology that integrates aspects of the process of research such as worldview, questions, methods, and inferences or conclusions. The third is a paradigm perspective, in which researchers discuss an overarching worldview or several worldviews that provide a philosophical foundation for mixed methods research. The final and fourth perspective is the practice perspective, in which scholars view mixed methods research as a means or set of procedures to use as they conduct their research designs, whether these designs are survey research, ethnography, or others (Creswell and Tashakkori, 2007:303).

A further major issue of contention between researchers is the philosophical underpinnings of mixed methods research (Tashakkori and Creswell, 2007a). Opponents of mixed methods argue that research paradigms are incommensurable (Bergman, 2010). However, mixed methods researchers side-step this issue by proposing that multiple paradigms can be used within different stages of a research design or by searching for a single paradigm (Creswell, 2009). Mertens (2012) identifies three such paradigms: (1) dialectical pluralism in which post-positivist views are adopted in quantitative data collection, constructivist views in the qualitative data collection and both are combined throughout to gain deeper understandings from the emerging convergence and divergence of the two approaches; (2) pragmatism which allows for the selection of methods based on those felt to best address the research questions; (3) the transformative paradigm which pursues social justice and human rights and which could lead to the decision to use mixed methods if they are aligned with this ethical stance.

Mertens (2012) suggests that the multiple views regarding the role of paradigms in mixed methods research is due to multiple uses of the term paradigm. Bergman (2010) also recognises the different ways the term ‘paradigm’ is used and identifies two uses within the literature. Firstly, he states that most often researchers use the term ‘paradigm’ to differentiate between qualitative and quantitative research: “most authors in this vein even provide tables, which classify the differences between qualitative and quantitative methods on epistemological, ontological, and axiological grounds” (Bergman, 2010:174). A second use of the term ‘paradigm’ in the literature, which Bergman terms the ‘weaker form’ is equating it with a ‘worldview’, ‘approach’ or ‘framework’. Bergman argues that the first sense cannot be used when describing mixed methods research since:

> …qualitative and quantitative analysis techniques do not necessitate a particular view of the nature of reality, privilege a specific research theme and how to research it, or determine the truth value of data or the relationship between researchers and their research subject (Bergman, 2010:173).

Others such as Mertens (2012) and Gorard (2010) agree with this view. Bergman (2010) further argues that the Kuhnian argument that paradigms are incommensurable only holds in the strong form and in this sense mixed methods could never be proposed as an alternative/ new paradigm.

Recently, critical realism has been proposed as an underlying philosophy to mixed methods research. A critical realist stance features:

> an integration of a realist ontology (there is a real world that exists independently of our perceptions, theories, constructions) with a

> constructivist epistemology (our understanding of this world is inevitably a construction from our own perspectives and standpoint… (Maxwell and Mittapalli, 2010:146).

Thus, due to its ‘stratified ontology’ (Zachariadis et al., 2013) critical realism has been argued to: “[provide] a philosophical stance that is compatible with the essential methodological characteristics of *both* qualitative and quantitative research” (Maxwell and Mittipalli, 2010:147).

Given the multiple perspectives and definitions for mixed methods research, the next section outlines the view taken for the current study.

4.2.2 Mixed methods for this research

The current research takes aspects of both the ‘methodology’ and ‘paradigm’ perspectives towards mixed methods research out of the four perspectives proposed by Creswell and Tashakkori (2007) cited in the previous section (4.2.1, p.60). These are now discussed in order to define mixed methods for the current research.

4.2.2.1 Methodology perspective towards mixed methods

For this research, ‘mixed methods’ is seen as the combining of both qualitative and quantitative approaches throughout all aspects of the research design: the worldview, research questions, data collection methods, data analysis and conclusions.

The worldview: Section 2.3.1(p.42) explains that this research assumes a fixed objective reality (a view often associated with quantitative approaches) combined with the acknowledgement that individual understandings of the world exist (often associated with a constructivist epistemology and qualitative methods). This will be discussed further below (4.2.2.2, p.64).

The research questions: Tashakkori and Creswell (2007b) identify three possible formats of mixed methods research questions. In section 3.1 (p.53) the research questions can be seen as: “an overarching mixed (hybrid, integrated) research question, later broken down into separate quantitative and qualitative subquestions” (Tashakkori and Creswell, 2007b:208).

Data collection methods: As discussed below (4.3, p.65) this research utilises both quantitative and qualitative data collection methods in the form of questionnaires (featuring multiple-choice questions with one correct answer), semi-structured interviews and non-participant observations.

Data analysis: As discussed below, both quantitative analysis in the form of statistical and psychometric analysis (4.3.1.4- 4.3.1.5, pp.71-73) and qualitative analysis through the use of Knowledge Maps and the Knowledge Quartet observation schedule (4.3.2.5, p.89 and 4.3.3.2, p.97 respectively) are employed for this research.

Conclusions: This research draws together results from both the qualitative and quantitative data collection in addressing and drawing conclusions about the research questions, so much so that an additional results chapter (6, p.175) is written in order to effectively combine these elements.

4.2.2.2 Paradigm perspective towards mixed methods

The ontological and epistemological views taken for this research are expressed in section 2.3.1 (p.42) and section 4.2.2.1 (p.63) above and can be seen as similar to a critical realist perspective (defined in section 4.2.1, p.60), in that a realist ontology and constructivist epistemology are assumed.

Thus, this research can also be said to adopt a ‘paradigm’ perspective in that the underlying theoretical assumptions (which have similarities to a critical realist worldview) are seen as providing a philosophical foundation for mixed methods research (c.f. Maxwell and Mittapalli, 2010). Moreover, this critical realist type worldview is seen to apply to all aspects of the research design (see above) and I contend it can lead to a consistent research design (see 8.3, p.256, point 5).

I agree with those researchers who argue that critical realism can provide a philosophical foundation for mixed methods research (e.g. Maxwell and Mittapalli, 2010) since, in my view, critical realism can be seen to answer Guba’s remark above. That is, the world may objectively be round, but there are still individual people who believe that the world is flat (for example, members of The Flat Earth Society). In other words, there is a fixed external reality, but many individual constructions of said reality.

It is felt that not only is a mixed methods approach consistent with the underlying philosophical views of this research, but, it is felt that mixed methods can: “(a) provide better understanding, (b) provide a fuller picture and deeper understanding, and (c) enhance description and understanding” (Johnson et al., 2007:122). Further, I agree with Jennifer Mason’s view that: “social (and multi-dimensional) lives are lived, experienced and enacted simultaneously on macro and micro scales” (2006:12) and mixed methods can help to overcome the limitations of research which focuses on either macro or micro scales. Within the context of this research, the National picture of Postgraduate Certificate in Education (PGCE) courses and the effects of ‘subject knowledge enhancement’ (SKE) courses (macro scale) is important, but so too is the individual experiences and learning of individual PGCE students (micro scale). I feel achieving this balance is possible with a mixture of quantitative and qualitative research.

Therefore a combination of both qualitative and quantitative approaches (with no greater emphasis given to either approach) is used concurrently throughout this research study.

4.3 Research Methods

In order to understand more about the nature of mathematical knowledge for teaching (RQ1, p.54) existing conceptions of teacher knowledge identified in the literature review which are grounded in mathematics and teaching practice would need to be examined within the context of trainee secondary mathematics teachers in England. These theories include Mathematical Knowledge for Teaching (MKT) and the Knowledge Quartet. The same is the case for the proposed theory of knowledge.

Further, to address RQ2 (p.55), these three theories (two existing and one new) would need to be applied at the beginning and end of the PGCE course to see if a change had occurred. Additionally, the methods used would need to allow a comparison between SKE and Non-SKE PGCE students.

As justified below, data collection consisted of three main aspects in order to explore these three theories in the context of this study: questionnaires, interviews and observations. These are discussed in turn.

4.3.1 Questionnaires

The MKT measures were administered in a questionnaire[1] towards the beginning and end of the PGCE course. The questionnaire was in two parts as follows.

[1] ‘Questionnaire’ is used here instead of ‘test’ as the guidance for administering the items suggests respondents should be told that: “this is not an assessment of any individual teachers’ knowledge or skill” (Hill et al., n.d.-a). Further, the term ‘test’ has negative connotations. I therefore used the term ‘questionnaire’ with the respondents and continue to do so here.

4.3.1.1 Questionnaire: Part A

The first section of the questionnaire was designed to elicit biographical information (date of birth, gender), prior attainment (Mathematics GCSE and A-level grades; degree type and class), and whether or not an SKE course had been taken. The PGCE students were also informed that they would be contacted after their course to be invited to share their PGCE results.

This data was collected to enable correlations between prior attainment (GCSE, A-level mathematics and degree results) and PGCE grades to be calculated to see if these grades predict success on a PGCE course. In other words, to provide evidence (or otherwise) of the extent to which prior mathematical attainment (as a proxy measure for mathematical content knowledge) is needed for effective (as measured by success on a PGCE course) teaching (RQ1). Further, finding the correlation between degree classification (converted to a point scale) and PGCE course grades has the additional potential to corroborate or contradict smaller studies at single institutions (e.g. Tennant, 2006; Stevenson, 2008). Finally, data regarding whether the PGCE students have taken a SKE course or not is crucial for allowing comparison between SKE and Non-SKE PGCE students' data.

4.3.1.2 Questionnaire: Part B

The second section of the questionnaire comprised a purposive sample (see 4.3.1.6, p.73) of 18 multiple-choice items from the Learning Mathematics for Teaching Project (LMT) at the University of Michigan.

The difference between mean scores on the pre- and post-items could then be calculated using repeated measure t-tests and checked for statistical significance and effect size (RQ2). Further, this allows scores on multiple-choice items to be compared with degree classification scores to see if the former is a better predictor of PGCE course grades (RQ1). Indeed, the items are said to be well-suited for comparison with other constructs (LMT, n.d.). These comparisons may also provide evidence for or against the claims that there is a specialised MKT (RQ1).

The items can also be used "to make statements about how content knowledge differs among groups of teachers, or how a group of teachers performs at one or more time points" (LMT, n.d.). Moreover, the items are said to be well-suited for investigating "How knowledge of mathematics for teaching develops" (LMT, n.d.) particularly with programs that: "intend to broadly improve teachers' knowledge", have "pre-existing curricula", and have sufficient sample size (LMT, n.d.). Since the current study aims to make comparisons between two groups (SKE and Non-SKE) over time (at the beginning and end of a PGCE course) for a course which aims to improve teacher knowledge, the MKT items are potentially useful for this research.

However, there are three potential problems with using the MKT items for the current research. Firstly, this research involves trainee, not in-service, teachers. Secondly, this research involves a different context (England) than originally intended (the USA). Indeed, the LMT measures are practice-based in the sense that MKT is identified with reference to the practice of teaching and thus it can vary in places other than the USA (Blömeke and Delaney, 2012). Finally, this research involves secondary school trainee teachers, not elementary or middle school teachers as in the original study. These potential issues are described and addressed below.

(1) As mentioned above, the measures were designed for in-service teachers. However, they are not intended for highly knowledgeable teachers (LMT, n.d.) and Gleason (2010) established the reliability of using them for pre-service teachers instead of the originally intended subjects. However, as Gleason found that his item difficulty and marginal reliability information was different than the original study, he proposed that researchers run their own item response theory (IRT) r they investigated whether respobility information from the in-service teachers. This was done for the current study.

(2) Further, although these measures were developed and have primarily been used within the USA, they have been shown to be able to be adapted to different (non-U.S.) contexts in a number of other countries: Ireland (Delaney et al., 2008; Delaney, 2012), Norway (Fauskanger et al., 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012) and Korea (Kwon et al., 2012).

(3) Another potential issue with using the MKT items are that they are intended for elementary and middle school teachers, whereas the current research focuses on secondary mathematics teachers. However there is some overlap between the age groups of pupils as shown in Table 4.2 (p.68).

This suggests that some of the MKT items may be suitable for secondary teachers in England (Key Stage 3). Thus, item content was analysed and matched with topics within the secondary school curriculum to initially determine whether the items were potentially appropriate for use with secondary teachers in England. This process was not undertaken as a formal test of validity (this was tested later) but to gauge initial suitability. For example, if the question content was based on mathematics which the secondary teachers are not required to teach within England, they could be seen as having little use in this new context.

USA			England	
School	**Grade**	**Age**	**Year group**	**School**
Elementary school	Preschool	4-5	Reception	Primary School
	Kindergarten	5-6	Year 1	
	1st grade	6-7	Year 2	
	2nd grade	7-8	Year 3	
	3rd grade	8-9	Year 4	
	4th grade	9-10	Year 5	
	5th grade	10-11	Year 6	
	6th grade	11-12	Year 7	Secondary School
Middle School	7th grade	12-13	Year 8	
	8th grade	13-14	Year 9	
High School	9th grade	14-15	Year 10	
	10th grade	15-16	Year 11	

Table 4.2: Table showing the overlap between school systems in the USA and England by pupil ages

The MKT items are to be kept secure, but as an example of this process, one of the released items is shown below (**Figure 4.1**, p.69) along with the statement within the English National Curriculum which is felt to match with it (**Figure 4.2**, p.69).

A complete comparison of the mathematics curriculum in the USA and England is beyond the scope of this research. Although there are similarities as shown above, there may be differences which might influence the nature of MKT in the context of England (see 4.3.1.6, p.73 for details on how appropriate items were selected).

Nevertheless, the concept of MKT can be argued to be the same in the USA as in England since the tasks of teaching upon which the items are based are broadly similar in England (see 2.1.5.1, p.19).

If the above potential issues with using the MKT measures can be addressed as suggested and additionally they prove to validly measure MKT in England following psychometric analysis, then the MKT measures provide an opportunity to measure English secondary mathematics trainee teachers' MKT at scale.

A discussion of how the reliability and validity of using the MKT items within the context of the current research will be established follows, commencing with an overview of how this was done by other researchers within the contexts of Ireland (Delaney et al., 2008; Delaney, 2012), Norway (Fauskanger et al., 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012) and Korea (Kwon et al., 2012).

18. At the close of a lesson on reflection symmetry in polygons, Ms. White gave her students several problems to do. She collected their answers and read through them after class. For the problem below, several of her students answered that the figure has two lines of symmetry and several answered that it has four.

How many lines of symmetry does this figure have?

Which of the following is the <u>most</u> likely reason for these incorrect answers? (Circle ONE answer.)

a) Students were not taught the definition of reflection symmetry.

b) Students were not taught the definition of a parallelogram.

c) Students confused lines of symmetry with edges of the polygon.

d) Students confused lines of symmetry with rotating half the figure onto the other half.

Figure 4.1: Released MKT item example (Ball and Hill, 2008:15).

Level 5

When constructing models and drawing or using shapes, pupils measure and draw angles to the nearest degree and use language associated with angles. They know the angle sum of a triangle and that of angles at a point. They identify all the symmetries of 2D shapes. They convert one metric unit to another. They make sensible estimates of a range of measures in relation to everyday situations. They understand and use the formula for the area of a rectangle.

Figure 4.2: Level 5 descriptor for Key Stage 4 ‘Geometry and Measures’ strand of the Mathematics National Curriculum (QCA, 2010:32).

4.3.1.3 Validity and reliability of the questionnaires

Delaney et al. (2008) were the first to use the LMT items for use outside of the USA (in Ireland). Although English is the language used in both contexts, Delaney and colleagues still felt the need to adapt the items. Four categories of changes were recognised:

1. Changes related to the general cultural context;
2. Changes related to the school cultural context;
3. Changes related to mathematical substance;

4. Other changes.

Changes such as people’s names and differences in spelling between American and British English came under the first category. Changes related to the language and structure of school such as ‘the school district’s curriculum’ being changed to ‘the school curriculum’ came under the second category. Changes to units (of currency and measurements), school mathematical language, representations and anticipated pupil responses came under the third category. Other changes included the deletion of an answer choice and changes to the format and appearance of the question.

Researchers adapting the items for use in other contexts to date have used Delaney et al. (2008)’s categories as a guide, adding to them where necessary. For example, ‘changes related to the translation from American-English into Norwegian’ (see Fauskanger et al., 2012).

In order to determine whether the MKT items could be validly used for studying mathematical knowledge for teaching within Ireland, Delaney (2012) adopted a validation study by Schilling and Hill (2007) which was an application of Kane's approach (see 7.1.2.1, p.218 for details).

Within the other studies mentioned above (Delaney et al., 2008; Delaney, 2012; Fauskanger et al., 2012; Cole, 2012; Ng, 2012; Kwon et al., 2012), researchers took the original MKT items and adapted them either by translating the language or by making changes to the question wording and /or response options. The reliability of the measures used in the new contexts was generally established by using Cronbach’s alpha. Methods used to establish the validity of using the items in the new contexts included: comparing difficulty estimates of the items between the USA and the new country, calculating point-biserial correlations for each item, using Item Response Theories (one and two parameter depending upon the sample size of the study), and in the case of Korea (Kwon et al., 2012) correct response rates were also compared with the USA figures. In Ghana (Cole, 2012), and Ireland (Delaney, 2012), interviews were conducted with teachers to compare their verbal responses with those on the items.

Hambleton (2012) reviews the papers in which the MKT items are adapted for use in other countries. He is not in favour of comparing reliability scores between versions of the test because "the amount of test score variability in each group drives the value of the reliability estimates" (2012:451). Even if tests are well adapted, the reliability score could be poor and poorly adapted tests could have high reliability scores (Hambleton, 2012).

Conversely, Hambleton believes that plots of classical discrimination indices and comparison of test score reliabilities as used in the studies have some preliminary merit but have many possible interpretations, though he recognises the value of plots in spotting outliers. He states that comparing the rankings of items can be useful in highlighting items to be investigated further.

Moreover, Hambleton (2012) explains how comparing IRT statistics across groups can be problematic because each group is on a separate IRT scale. Further, choice of IRT model should be made after determining whether the chosen model fits the data.

Since for the current study the original MKT items will not be adapted at all and given the potential issues with the validation approaches used by other researchers, another method will be used for this research in order to establish the reliability and validity of using the MKT items in England, namely, Rasch analysis. This will now be discussed.

4.3.1.4 Rasch analysis of the questionnaire responses

Rasch analysis was developed by Georg Rasch, a Danish mathematician (see Rasch, 1960/1980). The approach compares test outcomes with a mathematical model which utilises the formal axioms that underpin measurement. Thus, the model shows what the expected responses should be if construct measurement is to be achieved (Tennant and Conaghan, 2007). The model is based on the idea that all responders are more likely to get easier questions correct and that responders with higher ability (on the single underlying trait to be measured by the items) are more likely to get all questions correct than those with lower ability (Bond and Fox, 2006).

Rasch analysis is a rigorous theory of measurement and allows several measurement issues relating to valid interval scaling to be addressed at the same time. Namely, internal construct validity (unidimensionality), the invariance of items and differential item functioning (DIF) (Tennant and Conaghan, 2007).

According to Tennant and Conaghan (2007) there are several situations when a Rasch analysis could be used. One of these is when a new scale is to be developed and items need to fit the model (that is they need to demonstrate unidimensionality, be free from DIF and fit the model expectations). This is the situation for this research. Although the items are not ‘new’ per se, the selection of a sample of the items and using them within a new context needs to provide responses which meet the requirements of the Rasch model.

A dichotomous Rasch model was used to analyse the psychometric properties of the sample of MKT items (18 items) on the pre-questionnaire (239 respondents). The following summarises the data collected.

The software used to conduct the Rasch analysis was RUMM2020 (Andrich et al., 2002). When testing items and individuals against the Rasch model there was no evidence of item or person misfit. One person was ‘extreme’ (that is they scored 0) which means this person’s ability could not be estimated. This is not considered a problem in the analysis.

It is important that the MKT items selected for this research appropriately target the students. That is, items should not be too easy or too hard because the Rasch estimates of item difficulty and/ or student ability will be poor and the reliability of the test will be low (Bond and Fox, 2006). The RUMM2020 software calculated the mean student location as 0.160 (Standard Deviation = 0.953), which indicates that the average student ability is only slightly higher than the average item difficulty (see **Figure 4.3**, p.73). The horizontal scale on **Figure 4.3** represents ability/ difficulty (conjointly). It can be seen that students (upper graph) tend to broadly overlap with the items (lower graph) suggesting items appropriately target the students. It could, however, be argued that there were not enough more difficult items.

There was no evidence of multi-dimensionality. This suggests a single underlying trait is being measured by the items (i.e. MKT) and that Rasch estimates are robust.

DIF can affect fit to the model. This occurs when sub-groups of respondents answer differently on particular items after having controlled for underlying ability. Hambleton (2012) states that 200 persons are required for carrying out DIF. This requirement was met.

The following sub-groups were checked for DIF: gender, institution (where PGCE course is being taken), whether a SKE course was taken or not, and mode of entry (whether online or paper and pencil test). Item 11b showed significant DIF for all of these, and 11a showed DIF for SKE only. This suggests Item 11 is biased towards certain groups of respondents and should be considered for removal from the questionnaire.

The results of the Rasch analysis suggest that the MKT items selected for this research can be appropriately used in England, with the possible exception of item 11.

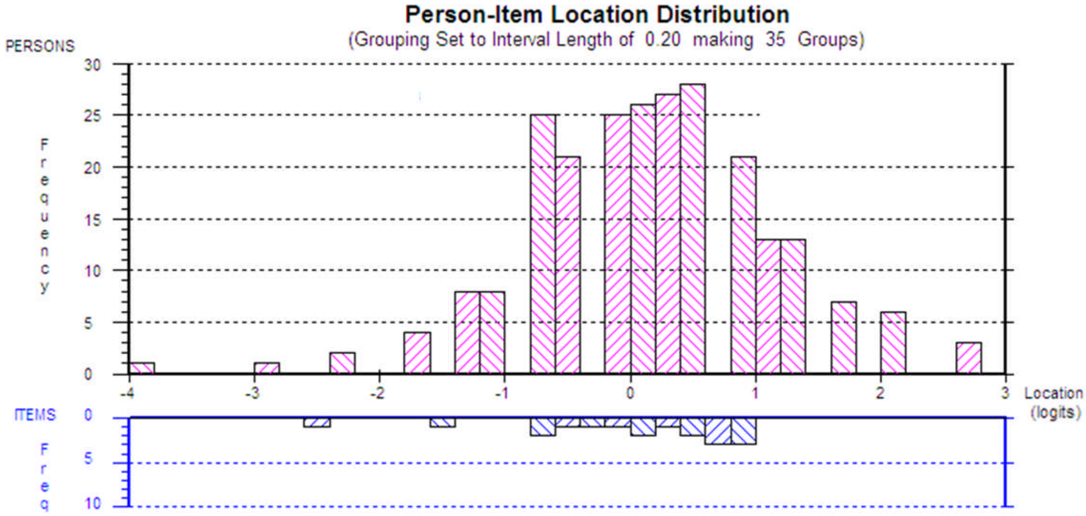

Figure 4.3: Distribution of person ability (upper graph) and item difficulty (lower graph).

4.3.1.5 Cronbach's alpha

As suggested by Gleason (2010) who also used the items with a sample of trainee teachers, the reliability (internal consistency) was calculated and found to be reasonable (0.67) as measured by Cronbach's alpha (18 items). The reliability of post-questionnaire items attempted by all respondents was also adequate (0.71) as measured by Cronbach's alpha.

4.3.1.6 Selecting items

As there are several hundred LMT items available for use, a sample needed to be selected for use for this research due to practical constraints on available survey time. The developers suggest that "between 12-25 items are required to achieve reliabilities of between .7 and .8 for each scale" (LMT, n.d.). This number of items was used as a guide (since reliabilities for pre-service teachers in England with purposively selected items may differ).

The LMT items are grouped into parallel forms (Form A and B) for different mathematical areas such as 'Geometry' and 'Number concepts and operations'. The forms available at the time of selection for this study are shown in Table 4.3 (p.74).

In existing studies which adapted the MKT items for use outside the USA, researchers took a complete form in one or two (three in the case of Norway) mathematical areas and adapted it/them for use in a specific context (country). The exception to this is Ireland (Delaney et al., 2008) where individual questions across topic areas were selected.

Age range	**Topic**	**Years (forms) available**
Elementary school	Patterns, Function and Algebra	2001, 2004, 2006
Elementary school	Geometry	2004
Elementary school	Number concepts and operations	2001, 2002, 2004, 2008
Elementary school	Place value	2006
Middle School	Patterns, Function and Algebra	2005, 2006
Middle School	Geometry	2005
Middle School	Number concepts and operations	2005, 2007
4-8	Rational numbers	2008
4-8	Geometry	2008
4-8	Proportional Reasoning	2008
4-8	Probability, data and statistics	2008

Table 4.3: Table showing the available MKT forms at the time of item selection

As mentioned above, Delaney et al. (2008) were the first to adapt the LMT items for use in another country and other researchers used Delaney et al. (2008)'s categories of changes as a guide when adapting items for use in other countries (see 4.3.1.3, p.69). Alternatively, for the current research, it was decided to select items from the pool of all possible items which did not require adaptation. This was decided for several reasons, which will now be discussed.

Firstly, adapting items may change their psychometric properties. This was emphasised by the authors in their guidance for using the items:

> Changing items – even by making small alterations in wording – means the psychometric properties no longer hold. These items have also been through extensive editing and revisions by mathematics educators and mathematicians. Even small wording changes (e.g., changing a preposition) can make the item mathematically ambiguous, and thus unusable in a measurement context (Hill et al., 2007b:2).

Indeed, when adapting items Kwon et al. (2012) found that in some cases adaptation proved to make the items more difficult for teachers and "introduced unanticipated validity issues" (2012:371).

Secondly, this research looks at teachers' knowledge in relation to geometry, number and algebra, thus items are needed which test all these areas. It was felt that including complete forms from all three areas would make the questionnaire too long. Nevertheless three forms were used in Norway, a total of 61 items. Since the LMT group recommend respondents take 1-2 minutes on each item, a two-hour questionnaire was felt to be impractical for use with trainee teachers who have a demanding year of training.

The categories developed by Delaney and colleagues were used to highlight questions which needed adapting. However, instead of adapting these items, if an item contained an element which required adaptation for use in England, it was eliminated from the item bank. Thus, a sample of the original items was used for this study with no adaptation. This process was iterated three times to ensure no items remained that contained any aspects unsuitable for use in England. Eliminated items required at least one of the changes suggested by Delaney et al. (2008). Some items contained all three of the first three categories of changes.

Out of the MKT forms available, those designed for elementary teachers and those on ‘Probability, data and statistics’ were disregarded as these areas are not the focus of the current research. This left forms for middle school teachers and grades 4-8. These equate to years 7-9 and 5-9 in the UK respectively, thus some of the questions could cover KS3 year groups of the National Curriculum in England. Nevertheless, it can be argued that they also cover KS4 as the National Curriculum levels are the same for KS3 and KS4 (Department for Education, 2013c). The remaining forms are shown in Table 4.4 (p.75).

Level	**Topic**
Middle School	Patterns, Function and Algebra
Middle School	Geometry
Middle School	Number concepts and operations
4-8	Rational numbers
4-8	Geometry
4-8	Proportional Reasoning

Table 4.4: Table showing the forms from which items were selected for the current research

Of the remaining 41 items, 10 were eliminated because they had no pre-existing psychometric information available. It was felt that since there was no statistical data for comparison, these items should not be used. A further 9 items were eliminated to reduce the number of items whilst maintaining as large a coverage of topics as possible. Items were plotted in a table of English National Curriculum levels and content areas. The range of National Curriculum topics and the corresponding levels that they are featured at (shown in grey) are shown in Table 4.5 (p.77). The numbers represent the number of MKT items which were felt to address the topic at that level.

Although the items were not originally designed to fit to any particular curriculum or grade system, I wanted to ensure that a range of National Curriculum topics were covered for this research.

The remaining items (22) were collated, renumbered and given to three academics to complete and comment on. All three had taught mathematics in school(s)/collage(s) for 17, 10 and 4 years respectively. One had been head of mathematics department for 12 years and been involved extensively in curriculum development across the secondary age and ability range. They were asked to answer the questions and consider their suitability for use in England. Two items caused particular concern. One was felt to be a topic not covered in secondary schools in England and the other discussed common misapplications of a theorem which was thought to depend upon the context/culture as a misapplication may not be common in all places. Further, the item contained a hand-drawn diagram which may cause problems when trying to reproduce the item (especially in an online format). It was decided to omit both these items. The remaining 20 item stems (made up of 34 individual questions) were still felt to be too many for PGCE students to complete in a reasonable time period and would negatively affect the response-rate. Further, LMT group recommend that between 18 and 25 items are used. Therefore, some further items were removed whilst considering the following:

(i) Enabling as large a spread of the National Curriculum topic areas to be covered as possible (see Table 4.6, p.78) and whilst maintaining roughly equal numbers of geometry, number and algebra questions (4, 5, and 3 respectively). Rows or columns in the above table with more than one item were considered and some were eliminated. For example, if two items were related to finding area of shapes, one was omitted.

(ii) The difficulties of the items. Estimates were provided for each item which indicated how difficult teachers in the USA had found them to be. A balance between easier and more difficult items was endeavoured to be struck.

(iii) The lengths of the items. Some items with several parts were removed to achieve a balance between shorter and longer items.

In total, 12 items (18 parts) were selected for use on the questionnaire (**Table 4.6**, p.78).

	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6	Level 7	Level 8	EP
Geometry									
2D and 3D shapes			1						
Angles						1			
Symmetry									
Perimeter and area and volumes				1	1	2	1		
Transformations									
Loci									
Congruence and similarity								1	
Sine, cosine and tangents									
Proof									
Measures									
Measuring and ordering objects									
Conversion of units									
Accuracy									
Number									
Counting, addition, subtraction					1				
Reading and writing numbers									
Place value									
Mental calculation									
Sequences									
Multiplication and division					2		1		
Decimal and negative numbers					1				
Fractions and percentages			1			3	1		
Ratio and proportion					2	1			
Powers and surds									
Accuracy (bounds)									
Algebra									
Formulae									
Co-ordinates									
nth term						1			
Equations						1	1	1	
Mappings, graphs, functions						2	1		1
Inequalities								1	
Expressions									
Brackets appropriately					1				

Table 4.5: Table showing MKT items (indicated by numbers) matched to the English National Curriculum levels and content areas (indicated by shaded fields)

Geometry	**Level 1**	**Level 2**	**Level 3**	**Level 4**	**Level 5**	**Level 6**	**Level 7**	**Level 8**	**EP**
2D and 3D shapes			1						
Angles									
Symmetry									
Perimeter and area and volumes				1		1	1		
Transformations									
Loci									
Congruence and similarity									
Sine, cosine and tangents									
Proof									
Number									
Counting, addition, subtraction					1				
Reading and writing numbers									
Place value									
Mental calculation									
Sequences									
Multiplication and division							1		
Decimal and negative numbers									
Fractions and percentages						1	1		
Ratio and proportion					1				
Powers and surds									
Accuracy (bounds)									
Algebra									
Formulae									
Co-ordinates									
nth term									
Equations							1		
Mappings, graphs, functions						1			1
Inequalities									
Expressions									
Brackets appropriately									

Table 4.6: Table showing the range of content areas and National Curriculum levels of the items selected for use (indicated by numbers) on the questionnaire

4.3.1.7 Administering the pre-questionnaire

As the MKT questions have been previously used in different contexts, several ways of administering the items exist: pencil and paper versions administered under examination conditions, paper and pencil versions sent through the mail, and online versions of the questions (Cole, 2011).

For this research, there were 65 institutions (universities/ collages) in

England allocated[2] by the Training and Development Agency for Schools (TDA) to run a PGCE secondary mathematics course (or equivalent) in the academic year 2012/13 (≈1,773 students, see 1.5, p.10). Due to the large numbers of students involved, an online version of the questionnaire was set up and additionally four of the larger courses (in terms of numbers of students) were visited in person by the principal researcher to administer a paper and pencil questionnaire towards the beginning of the course (including the University of Leeds). This was done in order to achieve a good response rate, since a pilot study of the online questionnaire yielded a very low response (less than 3%). As an incentive to participate, PGCE students could be entered into a prize draw to receive a £50 Amazon voucher.

On all four courses, the questionnaire was administered on a day when the students were scheduled to be together at their respective institution and was either in between two of their scheduled sessions or at the end of the day. Students had the option of participating in the questionnaire or withdrawing from the research.

In all cases I was introduced to the students, told them about the researcher focus and purposes and distributed information sheets about the research either before my visit or on the day. Students had the opportunity to ask questions.

Students were “clearly [told] that this [was] <u>not</u> an assessment of any individual teachers’ knowledge or skill” (Hill et al., 2007c), but was to enable me to see how their knowledge changed over their PGCE course. They were also reassured that the items were designed to be difficult and they were not expected to get all questions right. This was in accordance with guidance provided by the item authors:

> Explain that the assessment has, by design, some difficult items. We don’t expect every teacher to answer every item correctly – there is no evidence that if you can’t answer #4 correctly, for instance, that you are a poor teacher (Hill et al., 2007c).

Students were also informed that I would contact them after their course to invite them to share their PGCE result(s) with me if they were happy to do so.

The item authors “do not offer an opinion on whether to time-cap these assessments or not” (Hill et al., 2007c) but give the following advice: “Calculate about 2 minutes per problem stem included on the assessment,

[2] Those allocated small numbers of students (<10) may not have run their courses in 2012/13.

or alternatively, 1 minute per item" Hill et al. (2007c). Students were therefore instructed to spend no longer than 2 minutes on each multiple-choice question, yet no overall time limit was set.

The item authors recommended that the survey should stipulate that teachers work alone on the items. This was not written on the survey and students did not have to work in silence, but they were asked not to discuss the questions or answers as I was interested in their individual responses to the questions – but they could discuss other things whilst working on them. Nevertheless, the room usually fell into silence as the students became absorbed in the questions!

Most students returned the questionnaire back to me at the end, upon completion. Seven students could not stay but wanted to participate in the research so I allowed them to take the questionnaire away and return it later.

As it was not feasible to visit all institutions at the beginning of the academic year, all other institutions allocated to run a PGCE course that academic year were invited to take part via an online version of the pre-questionnaire.

4.3.1.8 The online version of the questionnaire

The online questionnaire (administered to all other institutions) was produced using Questionmark Perception software (Questionmark, 2008). After reviewing several software options, Questionmark Perception was the only one which would allow images and mathematical notation to be inserted within the question and thus maintain the formatting of the original questions. The importance of maintaining item formatting was stressed by the item authors.

> The [Microsoft] Word formatting on these items is unstable—margins and fonts can change, not only changing the "look" of the items but also potentially harming their content. When you print a proof of your form, check the items you have chosen carefully against ours (Hill et al., 2007c:1)

The online questionnaire was administered by contacting PGCE course tutors via email and asking them to forward a link to the questionnaire to their students, who could then choose whether or not to participate.

4.3.1.9 Post-questionnaires

The online and paper and pencil post-questionnaires were the same as the pre-questionnaires except students were not asked to give their prior mathematics attainment again and students' personal contact details were sought in order that students could be contacted after their course had finished (when their university emails addresses may no longer be valid). In addition, students were asked to state the length of the SKE course (had one been taken).

The questionnaires were administered in a similar way to the pre-questionnaires. Those institutions which were visited in person towards the beginning of the course were again visited towards the end, with the exception of one university where the PGCE course tutor handed out paper copies of the questionnaire to his students while they were in university for individual tutorials. The questionnaires were then posted back to me. Students who did not attend tutorials were sent the online version of the questionnaire.

For the online questionnaire, respondents were contacted directly by email and invited to participate in the post-questionnaire. PGCE course tutors were also sent an email with a link to forward to their students, should there be any who now wished to participate. Responses to the post-questionnaire were still considered useful even if the pre-questionnaire had not been completed, since PGCE results could still be compared with the questionnaire responses.

4.3.2 Interviews

In order to investigate how knowledge is held within the minds of PGCE students, a way of accessing knowledge is needed. Pre- and post-interviews were conducted alongside the questionnaires as they are said to be capable of helping researchers assess conceptual understanding (Heirdsfield, 2002) and to obtain information about processes – the 'how's and 'why's – as well as the end results (Ginsburg, 1981; Hunting, 1997). It is the understanding behind the answers given and the connections between knowledge (continuous knowledge) that needs to be elicited in order to have a more complete understanding of an individual's mathematical concept knowledge. Further, interviews are a technique used for similar studies within the literature. Indeed, Shulman and colleagues were also interested in how CK changed during teaching, and conducted research with 21 teachers of different disciplines including mathematics (Wilson et al., 1987). They experimented with ways to assess changing subject knowledge. The methods they employed included: free association (teachers had to tell everything they knew about key concepts or ideas within their subject matter), card sort tasks, and analysis of a piece of text (for maths teachers, this involved asking the teachers to critique a chapter from a textbook). This study was used to inform the current research.

4.3.2.1 Pre-interviews

The interviews for this research were based on Chinnappan and Lawson's (2005) study with some changes and informed by Shulman and colleagues' study (Wilson et al., 1987). In a study of teacher geometry knowledge, Chinnappan and Lawson (2005) conducted three interviews (audio and video taped) with two experienced teachers. The first interview involved the teachers freely recalling what they knew about 13 focus areas of geometry and how they would teach these topics to their students. The second interview involved think aloud problem-solving (Ericsson and Simon, 1993) on questions that required knowledge of the focus areas. The teachers were also asked about alternative solution methods and to comment on how their solution compared with solutions their pupils might give. The final interview enabled the interviewer to question the teachers on any aspects of their knowledge or relationships between knowledge that had not yet been discussed.

For the current study, only one interview (in two parts) was conducted which involved the trainee teachers freely recalling their knowledge on three focus areas (number, geometry and algebra) of the secondary curriculum, followed by interviewer prompts. Think aloud problem solving on three tasks was also undertaken on part two of the interview. The trainee teachers were not asked about how they would teach the topics to pupils or about how pupils would solve the problems since this was not the focus of this study.

Eight PGCE students at the University of Leeds (out of the twenty-four who responded to the pre-questionnaire) volunteered to be interviewed towards the beginning of their PGCE course. Three of these students subsequently dropped out of the course at various points in the academic year and were therefore removed from this study. The five remaining students are referred to as Students 1-5 (S1-5) within this document.

Interviews were conducted by me or one of my supervisors. This was to enable interviews to be conducted simultaneously with students during a narrow time-window when students would be on campus. All interviewers undertook training provided by me on how to conduct the interviews.

All pre-interviews were audio recorded in full (except one interview when the recorder was only switched on for the second part of the interview) and lasted between 23 minutes and 1 hour 14 minutes.

4.3.2.2 Interview: Part 1

The first part of the interview was semi-structured. Semi-structured interviews involve some pre-planned interview questions, whilst allowing for deviation dependent upon the subject's responses (Zazkis and Hazzan, 1999).

Specifically the interviewees were asked:

> "Tell me everything you know about 'squares'"
>
> "Tell me everything you know about 'rational numbers'"
>
> "Tell me everything you know about 'quadratic equations'"[3]

Further questioning was then dependent upon the students' responses in a 'clinical interview' style. A 'clinical interview' (Ginsburg, 1981) is a specific type of semi-structured interview and involves: "a flexible method of questioning" (Ginsburg, 1981:4) based on observation of the subject's responses (Piaget, 1929/ 2007). In general, clinical interview questions: are open-ended, maximise opportunity for discussion, and allow student and interviewer to reflect on responses (Hunting, 1997). By asking students to tell everything they know about a topic, it allows the student the freedom to discuss any aspects of their knowledge that comes to mind (free recall), whilst allowing further exploration of ideas. Indeed, Trochim (2006) recognises that issues can be explored in depth when follow-up questions are tailored to each interviewee. Further, Zazkis and Hazzan (1999) state:

> It is our impression that the mathematics education community does not need any further convincing. The value of clinical interviews as means to "enter the learners' mind" has been discovered by researchers and the appreciation for this method is growing (1999:430).

Further, Heirdsfield (2002) claims that researchers can assess conceptual understanding as well as the mathematical workings of interviewee's minds through listening to them within an interview setting.

For this research, a data collection method was needed which would enable the depth and breadth of knowledge to be accessed. From the above discussion, clinical interviews seemed to be a recognised, useful method. However, it was important that the interviews captured the student's own knowledge rather than knowledge that, due to the presence of an

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[3] These topics were chosen as they are key topics in the areas of number, geometry and algebra within the KS3 and KS4 curriculum and have connections to different areas of the curriculum. For example, rational numbers link to fractions and surds.

interviewer, they are led to discuss. Indeed, the interviewer “can inadvertently lead students to make correct responses” (Hunting, 1997:150) by suggesting answers “…so that there is only the illusion of discovery” (Ginsburg, 1981:6). This is particularly true for clinical interviews where the interviewer does not adhere to a list of pre-written questions.

In order to overcome potential researcher bias within the interviews, Hunting (1997) suggests that responses by the interviewer should be neutral with no variation in emphasis. However, he states that not only can it be difficult to be neutral with verbal cues, but nonverbal cues such as body language can be even more challenging to mask. Thus, a researcher could unintentionally bias the interview (see also Wrenn, 1939). As Tomlinson (1989) puts it: “to the extent that they define and pursue their own topic, [researchers] may miss the interviewee's construals and reactions, which they precisely wish to obtain” (1989:155). Conversely, by allowing the interviewee’s own perspectives and definitions to emerge, the researcher may “fail to do justice to their own research agenda” (ibid.:155). To address this issue, Tomlinson (1989) introduced ‘hierarchical focusing’ an approach which was taken for the current research (see Appendix C, p.273). This involves probing interviewees’ responses using ‘utterance linking’ (Tomlinson, 1989:170) (seeking elaboration on points raised by the interviewee using language used by the interviewee) to avoid putting words into the interviewee’s mouth. According to Tomlinson (1989), it is possible to know when the student’s knowledge of the topic (at that time) has been fully treated, that is, when the respondent can no longer provide further ideas but begins to repeat him/herself. When this occurred, the interviewer moved on to the next question. Thus, hierarchical focusing was expected to enable the students’ own knowledge to be accessed thoroughly whilst trying to minimise the influence of the researcher on their responses.

A related issue of any interview lies in the assumption that the respondent tells the truth. Indeed, there are several studies highlighted by Strang (1939) where interviewee’s comments did not correlate highly with actual measures. Such lack of correlation may not be caused by subjects purposefully lying, although there have been research studies were this is suspected (e.g. Flicker, 2004), but by: “Self-concern, inaccuracy of observation, memory, and judgment” (Strang, 1939:500). For this research, the interviewee’s desire to please or to achieve correct answers may lead to an inaccurate representation of their knowledge.

4.3.2.3 Interview: Part 2

For the second part of the interview, students were asked to solve three mathematics questions (Appendix D, p.274) while thinking aloud.

The first two questions were taken from the NRich website (http://nrich.maths.org) and are based on 'rational numbers' and 'squares' respectively. NRich offers mathematical problems designed to stimulate rich mathematical thinking based on National Curriculum content. Thus, they are felt to be useful for eliciting connections between mathematical knowledge.

The third question is adapted from Drijvers' assignment 9.2 (2003:219). This question was selected as it requires knowledge of a combination of ideas related to quadratic equations in order to solve it.

The three tasks are expected to elicit further connections between knowledge as answering the tasks may prompt knowledge which was not mentioned during the free-recall section of the interview (part one). In regard to their study, Chinnappan and Lawson stated:

> We can never be certain that we have tapped all that a teacher might have constructed and deconstructed about a specific topic. However, we argue that the use of the free-recall, problem solving and detailed probing activities did provide a good estimate of the functionally available knowledge of geometry and for teaching geometry (2005:205).

Thus, the free-recall combined with mathematical problems (with probes) is expected to also provide a good indication of the subjects' mathematical content knowledge in regard to specific topics in number, geometry and algebra.

Whilst solving the three mathematical tasks, subjects were asked to 'think aloud'. According to Ericsson and Simon 'think aloud' is the: "standard method for getting subjects to verbalize their thoughts concurrently" (1993:xiii). It involves the verbalisation of thoughts as they enter the subject's attention without requiring any description or explanation of their thoughts (ibid.). Verbalisation methods are "frequently used to gain information about… cognitive processes" (Ericsson and Simon, 1993:1). Indeed, several studies have used a student verbalisation approach (e.g. Bannert and Mengelkamp, 2008; Artzt and Armour-Thomas, 1992; Tanner and Jones, 1999).

Nevertheless, it is disputed whether verbalisation is a valid method of data collection as: (i) "many psychologists believe that people have no direct access to their mental processes" (Garofalo and Lester, 1985:166); (ii) cognitive processes may be affected by the act of verbalisation as subjects may focus on thoughts that they otherwise would not have heeded (Ericsson and Simon, 1993); (iii) a further problem is that verbalisation methods depend upon the participants being sufficiently verbal (Bannert and Mengelkamp, 2008).

However, there are arguments against these criticisms:

(i) Ericsson and Simon claim that thoughts, as the “end products of cognitive processes” (1993:xiii), can be verbalised as they consist of information retrieved, perceived or generated, yet, the processes of delivering the information cannot. Indeed, Tanner and Jones (1999) state that: “The significance of articulation or verbalisation in making processes explicitly available as *objects of thought* is confirmed by research” (1999:para.25, emphasis added). Therefore, if cognition involves both information and processes, this could be a cause of the disagreement as to whether mental processes can be verbalised/ accessed.

The other two criticisms can be said to only relate to specific levels as Ericsson and Simon (1993) present a three-level model of verbalisation approaches:

> ***Level 1: Talk aloud***
>
> For this method, the subject verbalises the contents of his or her short-term memory without explaining any processes (Bannert and Mengelkamp, 2008).
>
> ***Level 2: Think Aloud***
>
> In Level 2 verbalisations, unlike Level 1, thoughts which are not originally in verbal form have to be encoded so as to make them clear (Bannert and Mengelkamp, 2008). For example, context-bound phrases such as ‘substitute that into there’ may need to be adapted to ‘substitute the x value into the original equation’ which can lead to a longer time needed to complete the task.
>
> ***Level 3 Verbalisations***
>
> In contrast to the above methods, ‘Level 3’ verbalisations request the subject to give reasons and explanations for their thoughts and method choice. These verbalisations require more time due to the additional explanations required (Bannert and Mengelkamp, 2008).

(ii) According to Ericsson and Simon, Level 1 and 2 verbalisations do not affect the cognitive processes needed to perform the task as: “the sequence of thoughts is not changed by the added instruction to think aloud”

(1993:xiii). However, as Level 3 verbalisations: “[force] subjects to change their thought sequences in order to generate and verbalize overtly the information requested" (ibid.:xviii), this “will affect the primary task performance" (Bannert and Mengelkamp, 2008:44). Thus, the second criticism only seems to refer to Level 3 verbalisations.

(iii) Regarding the third criticism, Ericsson and Simon state that: "...verbal reporting is consistent with the structure of... normal cognitive processes..." (1993:xiii-xiv), explaining that it is natural for humans to think aloud when solving a problem. Further, even Level 3 explanations of thoughts are naturally practiced through explanations given to others as part of daily life. Moreover, they affirm that subjects quickly learn to verbalise their thoughts.

Nevertheless, Dominowski argues that because all levels of verbalisations used for research usually involves a researcher, this could modify the thoughts: “because of the presence of an observer" (1988:26). Whilst Ericsson and Simon agree that verbalising in social situations may cause different thoughts to be expressed than those generated by an individual subject, such explanations would be classed as Level 3 verbalisations and are not of the Level 1 and 2 types.

Thus, as ‘think aloud’ (Level 2 verbalisation) are used for this study, most criticisms do not apply to this level. Nevertheless, the problem of an interviewee not providing the desired verbalisations either due misunderstanding the think aloud process or due to becoming stuck on a task (and therefore having no thoughts to verbalise) still remains.

For the current research, this potential problem was addressed in two ways. Firstly, a practice problem was given so the students could practice thinking aloud while solving a mathematics task. It was not necessary for students to reach an answer to this question. In most cases students were stopped part way through once it was clear that students understood what was meant by thinking aloud or if an aspect of their verbalisation needed addressing. For example, on the post-interview one student had very long pauses (moments of silence) on the practice problem. The resulting feedback given by the interview was:

> So that was good, you didn’t try and explain what you were thinking. What I would say is if you are silent – and even if you are just thinking ‘I don’t know how to do this’ ‘where do I start?’ even if you’re just thinking that just verbalise that because I then know what’s going on in your mind... It was good when you said ‘oh I’ve not looked at anything like this for ages’, because that’s what you were thinking [Interviewer, S4 post-interview].

The second implemented method attempted to overcome the issue of a student becoming stuck on a task. If a student became stuck, the following procedure was implemented: “hints or heuristic suggestions [were provided] when blockage occur[red]. This often permits the child to demonstrate competencies that otherwise he or she would never "get to" during the problem solving, which adds to the information gained” (Goldin, 1997:57). However, as Goldin recognises, when specific suggestions are given, this limits the information that can be elicited about the respondent. However, it was felt that gaining some (prompted) information from the student was better than receiving no information due to lack of verbalisations. Further, these hints were designed to ‘give away’ as little as possible about the solution but instead sought to maintain the students’ engagement with the task by asking them questions rather than ‘telling’ them information. For example: ‘What are you tying to find out?’ and ‘What information do you know?’.

4.3.2.4 Post-interviews

The five remaining PGCE student volunteers were again interviewed during the final week of their course when they were already scheduled to be in the university for PGCE sessions.

The post-interview took the same format as the pre-interview with the same questions and mathematical tasks being posed. However, there were two differences. Firstly, all interviews were conducted by me. This was decided since conducting five interviews over the course of a week was feasible by one person, unlike the pre-interviews when eight were to be conducted during narrow time windows and I had to rely on the help of my supervisors.

The second difference was that students could use a LiveScribe smart-pen and proprietary (dot paper) notebook instead of an ordinary pen and paper during the interview. A LiveScribe pen is a digital pen with built-in audio recorder and camera which has the ability (when used with the special paper) to record both the marks made with the pen and any sounds/utterances made at the time. These recordings can then either be played back in real-time once uploaded to special software on a computer (with the notes/ marks made re-appearing in the order they were initially written on a virtual version of the paper page) or the audio can be replayed by ‘tapping’ any notes made on the paper with the pen. For example, if a diagram is ‘tapped’ with the pen, any sounds/utterances made at the time the diagram was drawn will be replayed via the pen’s speaker.

The use of a LiveScribe pen over an ordinary pen had several affordances. Firstly, Ericsson and Simon (1993) recognise that mathematical problem-solving can be disrupted the more verbalisations are made. Whilst Ericsson and Simon (1993)'s level 2 verbalisations were sought for this research, some interviewees tended to explain what they were writing down at times during the pre-interview. Hickman (2012) recognises the potential of the LiveScribe pen to reduce verbalisations in a think-aloud protocol and thus the impact on mathematical problem-solving. Whilst using the LiveScribe pen in the post-interviews, interviewees did not need to explain what they were writing down nor verbally label/ explain each part of their diagrams.

Secondly, the computer playback of notes/audio from the LiveScribe pen afforded easier transcription and analysis of the interviews later. In initial pilot work of Hickman's (2012) study, he found that participants in a stimulated-recall interview based on prior think-aloud problem solving with an audio recorder and ordinary pen led to an over-focus on attempts to link jottings with the recordings. For this research, this was true when trying to analyse the pre-interviews and this problem was eliminated with the LiveScribe pens in the post-interviews. Indeed, with the LiveScribe pens it does not matter if an interviewee crossed out some notes or skipped around the page(s) when making notes as the sequencing of their written working could be tracked alongside their spoken comments at that time.

Whilst there are other technologies available which allow audio and written notes to be recorded simultaneously (such as iPads), Weibel et al. (2011) recognise that LiveScribe pens: "enable users to exploit rich digital services whilst keeping the natural interaction common in traditional pen and paper interfaces" (2011:260). Indeed, the students required no training on how to write with the LiveScribe pen since it can also be used as an ordinary pen on ordinary paper, the only differences were the need to tap 'record' with the paper before starting. This was beneficial when working with PGCE students who had varying levels of experience with technology and whose mathematical knowledge was the main focus of the research.

4.3.2.5 Interview analysis

Following the interviews with the PGCE students, the interview audio recordings were fully transcribed. From the interview transcripts, discrete mathematical facts/statements were coded as 'discrete knowledge' (see Table 4.7, p.90) relating to one of the three interview focus topics: squares, rational numbers or quadratic equations.

Statement	Explanation
The opposite sides are parallel	
a square number's always going to be positive	
[quadratics] will have a squared part to it	
often it's a bell shaped curve when you do graph it	Does not matter if the statement is mathematically incorrect
S2: first you'd create your brackets, then.. I: so you're talking about factorising? S2: yeah factorising	This statement regarding solving a quadratic equation by factorising is embedded within the exchange with the interviewer, so the full exchange is coded.
let's have a look - how many lines of symmetry? [draws] it's got one, two... four lines of symmetry?	The first part is not needed (thought processes) only the resulting statement.
S2: erm obviously if you take a negative number, square it I: yeah S2: it's positive.. erm..	The statement need not be stated altogether – it could be separated by other comments by the student or interviewer.

Table 4.7: Table showing examples of discrete mathematical statements (green)

These statements were then tabulated (these will be referred to as 'Analysis Tables') with the discrete facts matched up between pre- and post-interviews. For example, a statement regarding the number of sides of a square on the pre-interview was paired with a similar statement on the post-interview (if such a statement existed). This enabled differences between the content covered between pre-and post-interviews to be seen at a glance. In other words, where a statement was made regarding a particular aspect of a topic on the pre-interview but not on the post-interview (or vice versa) the table would have a single statement on a row with no matching statement (see Table 4.8, p.91). The shading on the table is simply to help separate different aspects of the topic, for example, sides and angles of a square.

From the tables and coded transcripts, Knowledge Maps (see 2.3.2.2, p.46) were then drawn for each topic on each of the pre- and post-interviews for each PGCE student. Knowledge Maps are a visual representation of the tables but with only those statements which were not directly prompted by the interviewer appearing on the Knowledge Map. Thus, the Knowledge map can be seen to show the knowledge that was more immediately accessible for a student that did not require probing from the interviewer in order to access it. Including only unprompted responses meant a fairer comparison between interviewees (in terms of more accurately capturing their knowledge) since if one interviewer prompted responses on symmetry of squares for example, and another interviewer did not, it would not be fair to say that the student interviewed by the former interviewer had knowledge of symmetry whilst the later student did not.

Pre-Interview	**Post-Interview**
you could mean the geometrical shape	they're a 2 dimensional shape
got four angles or vertices	they have four equal angles
all 90 degrees.	all of 90 degrees
	they have four vertices
all the angles add up to 360 degrees	
four sided	
there's obviously two pairs of parallel sides	two sets of parallel lines
all the opposite sides are parallel	

Table 4.8: Table showing examples of paired mathematical statements within an Analysis Table

The Knowledge Maps were also able to show which statements made related to Geometric, Algebraic or Numeric representations by having separate coloured ellipses for each of these domains and placing the discrete nodes within an ellipse (or the intersection between two ellipses, if applicable). If the student saw connections between domains – for example, stating that square numbers (Numeric) are so called because they relate to the area of a square (Geometric), then the Knowledge Map would have an intersection of the Numeric and Geometric ellipses. Alternatively, ellipses would remain distinct if no connection was apparent. For example, regarding squares, one student commented that you could *either* mean square numbers (Numeric) *or* the square shape (Geometric). This suggests he saw the two areas as distinct. Only ellipses containing discrete nodes feature on the Knowledge Maps, so if, for example, a student only makes statements regarding a square shape, then only the Geometric ellipse will be on the Knowledge Map.

Where explicit connections between discrete statements are identified then connecting lines are drawn between the nodes.

Coding reliability:

The process of analysing transcripts to identify discrete mathematical statements to put into tabular form was conducted by two coders and responses compared. My principal supervisor was trained on which kinds of statements were to be highlighted through a training document which outlined the process with examples of mathematical statements and examples of statements that were not desired. The training document also included an interview transcript with highlighted statements with the resulting table as an example. I also met with my supervisor to discuss any questions relating to the training document and to give further information and training.

An interview transcript was selected at random (S4's pre-interview on squares). This transcript was printed with no highlighted sections. My supervisor was then left to highlight the discrete mathematical statements within the interview and these were compared with the statements already highlighted by myself. For this first trial, some questions were raised – whether to include statements which were prompted by the interviewer and whether to include statements which were mathematically incorrect. My supervisor had included all prompted statements but only some of the ones which were incorrect. Another issue raised was how much of the statement to highlight – in some cases the idea expressed was highlighted by both my supervisor and I but we each had variations on the exact words included in our highlighted section. It was clarified that both prompted and incorrect statements were to be included and slight variations in the exact words highlighted were not deemed to matter. Despite the questions raised, there was still 76% agreement between our highlighted statements. S4's post interview on squares was then coded and compared with 79% agreement. The differences came as a result of highlighting the same statements but in some cases I treated statements as two separate ones and my supervisor treated them as a single statement. For example, we both highlighted: "it's a four sided shape or quadrilateral" but I treated "it's a four sided shape" as a separate statement to "quadrilateral". In one case, we agreed that a statement could be treated as one or two statements depending upon the interpretation of the student's utterance. He said: "it's a type of rectangle er all sides have to be of equal length". This could either be taken to mean: 'a square is a type of rectangle with equal lengths' or 'a square is a type of rectangle' and 'a square has all lengths equal' as two separate ideas. Taking both interpretations into account the inter-coder reliability for the second transcript increases to 86%.

After these initial 'training' transcripts, two further transcripts were coded by my supervisor with an inter-coder reliability of 86% and 93% respectively.

4.3.3 Observations

As well as collecting data at the beginning and end of the course in an attempt to establish the changes that have taken place, suggesting reasons for these changes (RQ3) is also desirable, that is, understanding *how* knowledge has changed. In order to understand how knowledge grows in teaching, Shulman closely followed trainee teachers during their training year: “conducting regular interviews, asking them to read and comment on materials related to the subjects they teach, and observing their instruction after having engaged them in a planning interview” (1986:8). Data was also gathered on the training program itself. It was decided that a similar approach of observing students throughout their course would be taken for this study. That is, PGCE students were interviewed at intervals throughout the course, observed teaching on their school placements and all PGCE mathematics taught sessions at university were also observed.

Observations were chosen as a data collection method for several reasons. Firstly, Jersild and Meigs propose several instances where observations can be used:

> …a desire to probe aspects of behavior not accessible to the conventional paper and pencil, interview, or laboratory technics [sic]; a desire to obviate some of the subjective errors likely to enter into the customary rating procedures; an emphasis on the need for studying children in "natural" situations, and for studying the functioning child, including his social and emotional behavior, rather than to rely exclusively on static measurements of mental and physical growth (1939:472).

For the current research, observing student teachers in their ‘natural’ teaching (on school placements – see 1.3.2, p.4) and learning setting (PGCE course sessions at the University of Leeds) in order to study the non-static process of learning through teaching requires such a data collection method. Observations may be able to capture data which otherwise may not be collected by the questionnaire and interviews.

Other researchers note the usefulness of studying teachers in their ‘natural setting’ i.e. when teaching (e.g. Stylianides and Ball, 2004:31). A possible explanation for this is that “Teachers might know things in a theoretical context but be unable to activate and apply that knowledge in a real teaching situation” (Kersting et al., 2010:178). Moreover, "It can be argued on both empirical and philosophical grounds that what teachers learn is framed in the context in which that knowledge is acquired" (Cooney, 1999:168).

Jersild and Meigs (1939) identify three levels of control that an observer can have during data-collection: the observer could simply observe the situation without restricting any aspects; the observer could manipulate the situation in order to elicit certain behaviours; or the situation could be partially controlled by the observer by taking subjects to a specified room or confronting them with certain conditions or directions. Similarly, the researcher can either: observe from the outside as an unobtrusive onlooker (Biddle, 1967); or be a participant-observer, that is, the researcher immerses him/herself in the environment and interacts with the subjects (Trochim, 2006). Finally, Biddle (1967) indicates three angles an observer can take: the intent of behaviour, its objective characteristics, or its effects can be observed. Thus, although a seemingly objective occurrence such as a child hitting an older child (to use Biddle's example) can be observed: "The motive of the younger child is hostility, his action is aggression, but its effect is to create amusement in the older child" (ibid.:345).

4.3.3.1 PGCE placement observations

For the current research, a sub-sample of six PGCE students was initially selected to examine as volunteer case studies: three who had undertaken a mathematics enhancement course and three who had not. The sample was selected from those who had completed the pre-questionnaire and also participated in the pre-interviews. Six was determined as the number to be observed to allow for withdrawal of participants whilst being the maximum number that could reasonably be observed given time constraints. One student subsequently left the PGCE course and thus was withdrawn from the study. The remaining five students (S1-5) were observed whilst teaching lessons in their placement schools. The observed lessons were video-recorded (with permission).

Observations of placement lessons were non-participatory, with no aspects (intentionally) restricted or influenced by the researcher. The focus of the observations was to gather evidence on possible factors which led students' mathematics-related knowledge to change during the course. For example if a method of solution was proposed by a pupil in the class which was novel to the PGCE student, 'pupil methods' could be said to be a factor leading to teacher knowledge growth.

A limitation of observations is that observed behaviour may not be typical. Additionally, Herbert and Attridge highlight several possible occurrences which could alter 'normal' behaviour:

> …a fire drill, an accident, distracting repairs to equipment, a party? Did the teacher appear particularly uptight or the students preoccupied with the observer? Was information lost through faults in the observation equipment? (1975:14).

Furthermore, the presence of the observer may cause the observee to alter their behaviour. Indeed, "The presence of an observer might be expected to produce self-consciousness or other reactions that would distort the behavior which is being studied” (Jersild and Meigs, 1939:480). In my experience as a pupil, teachers purposefully behaved differently during Ofsted inspections.

This research attempted to overcome this by following the students over the nine month course since “many investigators claim that observer effect… wears off over time” (Herbert and Attridge, 1975:13). One way to overcome observer effect is for the observer to remain hidden through using a periscope, a one-way window, or a one-way mirror (Boyd and DeVault, 1966). However, this was felt to be impracticable for this research.

The PGCE students were interviewed immediately after the lessons (where possible) to discuss whether they had learnt any mathematics whilst preparing to teach that lesson or during the lesson (again in order to determine factors leading to knowledge change). For example, whilst preparing for the lesson, the student may have consulted a textbook which explained a topic in such a way as to cause them to make links with another area of mathematics). The post-lesson interviews also provided the opportunity to question PGCE students on any aspects of the lesson that required clarification or explanation. For example, if a particular mathematical example was provided in the lesson, the student could be asked whether that example was selected prior to the lesson or was selected ‘on the spot’.

Post-observation interviews were conducted immediately following the observations for several reasons. Firstly, the lesson would be fresh in both the researcher and students’ minds. Secondly, scheduling a time for a delayed post-interview would be difficult for students who already have a demanding year of teacher training. Thirdly, post-interviews seemed to fit in well during the wait for a meeting with students’ tutors and school based mentors (whilst mentors and tutors met to discuss the student prior to meeting with the student, the post-interview was conducted).

According to Rowland and Turner: "Contingent moments arise one after the other when teachers interact with a class… and their responses to these opportunities draw in various ways on their mathematical content knowledge" (2009:31). Thus, the observations may reveal aspects of mathematical content knowledge which could not be accessed via the questionnaire and interview, and the subsequent interviews can potentially allow the researcher’s interpretations of classroom situations to be verified (or otherwise).

Student	Placement 1 Observation date/ time (period) /class (set)	Placement 2 Observation date/ time (period) /class (set)
S1 (Non-SKE)	12/12/12 (Wednesday) 2pm-3pm (P5) Year 8 class (set 2/ 3)	22/04/2013 (Tuesday) 11:10-12:10 (P3) Year 10 class (set A2) 21/05/2013 (Tuesday) 11:10-12:10 (P3) Year 10 class (set A2)
S2 (SKE)	13/12/2012 (Thursday) 2pm-3pm (P5) Year 9 class (top set).	26/03/2013 (Tuesday) 11:15-12:15pm (P3) Year 9 top set 26/04/2013 11:15-12:15pm (P3) Year 10 class (set 8/10).
S3 (Non-SKE)	27/11/2012 13:00 - 13:55pm (P5) Year 10 class (set 5/9).	2/05/2013 (Thursday) 12-1pm (P3) Year 8 top set 08/05/2013 9:40-10:40am (P2) Year 8 top set (same class)
S4 (SKE)	05/12/2012 11:15am Year 9 class (set 2 /6).	19/04/2013 11:30-12:30pm Year 10 tops set but weak. 13/05/13 Same year 10 group.
S5 (Non-SKE)	29/11/2012 9:40-10:40am Year 10 class (top set) 10/12/2012 12-1pm. Year 10 class top set (same class).	25/03/2013 (Monday) 1:25-2:25pm (P4) Year 9 class (set 2 / 6). 17/04/2013 (Wednesday) 11:30-12:35 (P3) Year 8 class (set 3 of 3). *(written field notes only)*

Table 4.9: Table showing details of school placement observations

At least three observations were made throughout the year for each student (see **Table 4.9**, p.96). This tended to be one observation during the first placement (October – December 2012) and two observations during placement two (February-June 2013). Given the difficulty of obtaining permission from the schools to observe classes and arranging observations that were mutually convenient for all parties involved, the content of the lessons and the particular classes taught were not considered when arranging visits. Instead, the following factors were considered:

(a) Arranging observations to coincide with observations by course tutors where possible. This was for ethics reasons as it was felt it would cause the least disruption to the PGCE students who already have to be observed on both placements by tutors, and therefore it was not wished to put greater stress on the students by introducing extra observations for this research. Similarly, additional observations (especially with a video camera) could disrupt the school pupils in the classes being taught.

(b) Lessons were chosen at times in the day which would allow sufficient time to travel to the schools (which were located at various distances from the University of Leeds) via public transport.

(c) Having a break or free period after the lesson was also a factor in arranging the observations so that the students could be interviewed immediately after the lesson.

4.3.3.2 Observation analysis

Following the observations, The Knowledge Quartet (Rowland and Turner, 2007) was used to analyse video recordings of the school placement lessons. Full details of the Knowledge Quartet observation framework are in Part One of the Literature Review (2.1.8, p.27). Video recordings were taken as recommended by the Knowledge Quartet authors (Rowland, 2012b). The Knowledge Quartet is the only observation schedule known that: "provides a framework for analysis of the mathematical content knowledge that informs teacher insights when they are brought together in practice" (Turner and Rowland, 2011:196).

In particular, the ‘foundation’ element of the Quartet has the potential to provide insights into the trainees’ mathematical content knowledge upon which their lesson is based, and the ‘transformation’, ‘connection’ and ‘contingency’ elements of the Quartet (which focus on how trainees’ mathematical knowledge is used within a classroom situation) can enable additional insights into the trainees’ mathematical content knowledge to be gleaned.

I met with Tim Rowland at the University of Cambridge to receive training and guidance on using the Knowledge Quartet (Rowland, 2012b). The online Knowledge Quartet ‘coding manual’ (Weston et al., 2012) was also used to aid the coding of the video data.

4.3.3.3 PGCE session observations

In addition to observing students on their school placements, all taught mathematics education PGCE sessions at the University of Leeds were observed throughout the year. This consisted of 15 full day sessions on a Friday as well as 2 sessions on other weekdays. Sessions on a Monday morning were not observed as these were general sessions for all PGCE secondary students (Languages, Sciences, Arts etc.) and so did not have a mathematics focus.

The PGCE sessions were mainly pedagogical (for example sessions on Assessment for Learning, inclusion and classroom management) with some sessions devoted to subject knowledge (for example calculus and discrete/decision mathematics). Whilst the focus of this research is on mathematical knowledge rather than pedagogical knowledge all session were observed since during a pilot observation of a PGCE methods class with a non-mathematics focus it was surprising that mathematical topics were mentioned. Indeed, the session was focused on the services available by county-council mathematics advisors, yet the students were asked to analyse mathematics questions leading to a discussion of what triangle and perfect numbers were.

During observed sessions, field notes and audio recordings were simultaneously taken using a LiveScribe smart pen (see 4.3.2.4, p.88 for details).

I intended to act as a non-participant observer. However, sometimes the course tutors asked for my views on some aspects of mathematics or academic practice and sometimes I offered information which I felt would be useful to the students. In this respect I was not fully a non-participant as I was treated as an additional ‘advisory’ course tutor. It was not clear whether students saw me as a student (peer) or another member of staff.

5 Results

This chapter presents the results of the data collected. Results from each data collection method are presented in turn: questionnaires, interviews and observations. This provides a foundation for the next chapter where the results are tied together to show how they address each of the research questions.

5.1 Questionnaire results

5.1.1 Introduction

The questionnaires were conducted at the beginning and end of the Postgraduate Certificate in Education (PGCE) course and they collected biographical data as well as scores on the Mathematical Knowledge for Teaching (MKT) multiple-choice items. Descriptive statistics outlining the details/ characteristics of the respondents are first presented. Then, results regarding the prior attainment of students and their scores on the MKT items follow. How responses correlate with each other are considered throughout.

5.1.2 Descriptive statistics

Overall there were 329 respondents to the questionnaires, from 33 (out of 65) different institutions. Since the total number of secondary mathematics trainee teachers studying in 2012/13 was 1,630[6] students (NCTL, 2013a), this represents a sample size of approximately 20% of the population. However, not all these participants responded to both questionnaires, and may not have responded to all parts of the questionnaire(s). Out of these respondents, there were slightly more females (55.2%) than males (**Table 5.1**, p.100).

The questionnaire also collected data on whether or not the PGCE students had previously taken a ‘subject knowledge enhancement’ (SKE) course and the length of the course (if applicable). Since the lengths of SKE courses vary from two weeks to one academic year, the PGCE students were grouped according to the lengths of the courses rather than whether or not they had taken a course. As section 1.3.4.2 (p.7) explains, students who had taken no SKE course or a short SKE course of two to four weeks in length were classified as ‘Non-SKE’ and students who took a long SKE course of

[6] rounded to the nearest 10

five months or longer were classified as ‘SKE’ students. Since mean scores on both tests were very similar for those taking a short SKE course as for no SKE course this suggests this division was justified.

Out of those respondents who provided details on the length of their SKE course (if any), the majority were Non-SKE students (73.5%) with the remainder taking a (long) SKE course (see **Table 5.1**, p.100).

Male	**%**	**Female**	**%**	**Total Respondents**
137	44.8	169	55.2	329
SKE	**%**	**Non-SKE**	**%**	**Total Respondents**
73	26.5	202	73.5	275

Table 5.1: Table showing biographical details of questionnaire respondents

Students’ final PGCE results ($n = 138$) were collected after the course had ended. The distribution of these results is shown in **Table 5.2** (p.100).

PGCE grade	**Frequency**
Fail/ deferred	22
3 (Working towards good)	8
2 (Good)	53
1 (Outstanding)	55
Total	138

Table 5.2: Table showing the distribution of final PGCE grades

5.1.3 Prior mathematical attainment of respondents

The questionnaire requested data on PGCE students’ prior mathematical attainment, including GCSE, A-level and degree title and classification. These will be discussed in turn including the results of correlating prior attainment scores with final PGCE grades.

5.1.3.1 GCSE Mathematics (or equivalent)

Since respondents may have taken alternative qualifications to GCSE Mathematics such as Scottish O grades, O-levels, Irish Junior Certificates, or other qualifications from countries outside the UK, qualifications were converted to School and College Achievement and Attainment Table (SCAAT) points. “SCAAT points measure the achievement and attainment of pupils…across qualifications of different types” (LSIS, 2009:1), enabling qualifications to be compared. “SCAAT points are calculated by Ofqual, the qualifications regulator, and are applied to all eligible approved qualifications…” (LSIS, 2009:1). Table 5.3 (p.101) shows the distribution of SCAAT points which respondents achieved.

GCSE Grade	SCAAT points	Frequency
D	34	1
C	40	3
B	46	32
A	52	135
A*	58	119
	Total	290

Table 5.3: Table showing the distribution of SCAAT points for GCSE Mathematics (or equivalent)

There was a small, positive relationship between GCSE grades and PGCE grades ($r = 0.102,\ n = 136$) but this was not statistically significant ($p > 0.05$).

5.1.3.2 A-level Mathematics (or equivalent)

A-level Grade	SCAAT Points	Frequency
E	150	5
	160	1
	165	2
	170	1
D	180	18
	190	2
	200	1
C	210	37
	230	1
B	240	62
A	270	156
	Total	286

Table 5.4: Table showing the distribution of SCAAT points for A-level Mathematics (or equivalent)

286 students provided their A-level Mathematics grades (or equivalent). These qualifications were also converted to SCAAT points for ease of comparison (**Table 5.4**, p.101). There was a small negative correlation between A-Level grades and PGCE grades ($r = -0.123,\ n = 132$) but this was not statistically significant ($p > 0.05$).

5.1.3.3 A-level Further Mathematics (or equivalent)

A-level Further Mathematics is not always offered to students within Higher Education, or provision is only made for AS level Further Mathematics. Responses were therefore also converted to SCAAT points. Out of the 156 respondents who provided data on their Further Mathematics A-level qualification (or equivalent), 66% took AS or A-Level Further Mathematics to varying degrees of success (**Table 5.5**, p.102).

AS level Grade	A-level Grade	SCAAT points	Frequency
		0	53
	E	150	4
B		165	6
A	D	180	13
	C	210	9
	B	240	25
	A	270	46
		Total	156

Table 5.5: Table showing the distribution of SCAAT points for Further Mathematics A-level (or equivalent)

There was no statistically significant correlation ($p > 0.05$) between A-Level Further Mathematics grades and PGCE grades ($r = -0.08, n = 73$).

5.1.3.4 First degree classification and type

302 students across 21 institutions provided their prior degree data. There were 124 different degree titles. As well as various Mathematics and Statistics based degrees and degrees related to Economics, Finance and Engineering, there were also other titles such as Art and Archaeology, Building Surveying, Childhood and Youth studies, Film Studies and Sports Science. Degree classifications are shown in **Table 5.6** (p.102).

The table shows that 66% of respondents achieved a 2:1 or higher in their first degree. This is comparable with the National picture as 62% of all new entrants to all teacher training routes across all subjects in 2012/13 had a 2:1 or above (NCTL, 2013b).

Degree level	Frequency
First	71
2:1	127
2:2	94
Third	9
Other	1
Total	302

Table 5.6: Table showing the distribution of degree classifications

Following a similar methodology to Tennant (2006, see 2.2.7, p.39), the correlation between prior degree classification and final PGCE teaching scores ($n = 137$) was calculated as $r = 0.185$. This weak positive correlation was significant ($p = 0.03$) and is similar to Tennant's[7] ($r = 0.11$).

A linear regression established that prior degree classification could statistically significantly predict PGCE scores, $F(1, 135) = 4.804, p = 0.03$. However, degree classification only accounted for 3% of the explained variability in PGCE scores. The regression equation was: $PGCE\ grade = 1.04 + 0.254\ x\ (degree\ classification)$. However, this equation should be treated with caution as the assumption that residuals of the regression line are approximately normally distributed was violated.

The correlation between prior degree classification (weighted by mathematical content) and final PGCE teaching scores was ($r = -0.055$). Again this was very similar to Tennant's correlation ($r = -0.05$) and represents no significant correlation.

There was a weak negative correlation between the weightings of mathematical content and PGCE results ($r = -0.132$) suggesting having higher amounts of mathematical content on a first degree is associated with generally lower performance on a PGCE course. Again, this finding corroborates Tennant's study where the correlation was $r = -0.16$.

5.1.4 MKT (multiple-choice) items results

5.1.4.1 Descriptive statistics

There were 142 'genuine' respondents to Part B of both the pre- and post-questionnaire (MKT items). If a student had not responded to any of the multiple-choice questions or had answered the first few questions followed by no response to the rest of the questions, I did not treat these as genuine attempts to answer the questionnaire and excluded them from the analysis. Respondents were from 18 different institutions. Out of all respondents, there were slightly more females ($n = 72$, 51%) than males ($n = 70$, 49%) and there were 43 (30.3%) 'SKE' respondents and 99 (69.7%) 'Non-SKE' respondents.

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[7] Tennant's initial study was for 21 students in one academic year. However, he also combined these students with those from the previous four years to give 92 students in total:

> Using all 92 students who have successfully completed the secondary mathematics PGCE course in the 5 academic years from 2000 to 2005, the unweighted correlation between degree result and success on the course was 0.11, the weighted correlation was -0.05, and the correlation between the weighting and success on the course was -0.16 (Tennant, 2006:50).

Raw scores are not allowed to be disclosed due to the terms of use of the MKT items, but scores were approximately normally distributed for pre- and post-questionnaire and the questionnaire discriminated between students' knowledge well. In order to adhere to the terms of use, differences between mean scores are instead provided in the following sections.

5.1.4.2 Temporal changes

A repeated measures t-test showed that, on average, participants ($n = 142$) did significantly better on the post-questionnaire than the pre-questionnaire (the mean score increased by 1.31 marks), $t(141) = -5.498$, $p < 0.001$, $r = 0.420$ (medium effect size).

5.1.4.3 MKT scores as a predictor of PGCE success

Scores on the MKT questions on the pre- and post-questionnaire were also analysed to see whether they were better predictors of success on a PGCE course than degree results. However, scores on the pre-questionnaire MKT questions did not predict success on the PGCE course ($r = -0.076, n = 103$), neither did scores on the post-questionnaire ($r = -0.051, n = 105$). These correlations are not statistically significant.

5.1.4.4 Correlations with prior attainment scores

		Pre- MKT Qs score	**Post- MKT Qs score**
GCSE SCAAT points	Pearson Correlation	0.316	0.183
	p	<0.001	0.012
	N	226	189
A-level SCAAT points	Pearson Correlation	0.284	0.334
	p	<0.001	<0.001
	N	226	189
Further Mathematics A-level SCAAT points	Pearson Correlation	0.051	0.318
	p	0.603	0.001
	N	108	110

Table 5.7: Table showing correlations between prior mathematics qualifications and scores on the pre- and post-MKT questions

The correlations between prior mathematics attainment and scores on the MKT questions (pre- and post-) were calculated and are summarised in Table 5.7 (p.104). With the exception of Further Mathematics and pre-questionnaire MKT score, all correlations were significant at (at least) the 5% level. All correlations were positive and in some cases small to moderate.

The correlations between degree class and scores on the MKT questions were also calculated and also found to be small and positive: $r = 0.189, n = 236$ (pre-test), $r = 0.122, n = 196$ (post-test). The correlation between degree class and pre-questionnaire score is significant at the 0.01 level ($p = 0.003$). The post-questionnaire correlation is not significant.

5.1.4.5 Comparison between SKE and Non-SKE PGCE students

In order to determine whether a difference existed in MKT between SKE and Non-SKE students, a repeated measures ANOVA ($n = 141$) was used to analyse any changes in MKT scores (dependent variable) over time (between pre-and post- test; within subjects factor) and by group (Non-SKE and SKE; between-subjects factor). There was found to be a significant increase in overall MKT scores of 1.34 marks (7.4%) between the pre- and post-questionnaire ($F(1, 139) = 25.088, p < 0.001$ with a medium effect size: partial $\eta^2 = 0.153$). The between subjects (Non-SKE and SKE) effect was non-significant. This suggests that whilst time played a role in the increase of scores on the test, taking an SKE course did not.

The mean difference in MKT scores between the two groups (Non-SKE and SKE) was sustained over time (**Figure 5.1**, p.106). The mean difference in marks on the pre-questionnaire was just less than a third of a mark (0.31, 1.7%) and on the post-questionnaire it was just less than half a mark (0.47, 2.6%).

The results of the ANOVA suggest that whether a SKE course has been taken has no significant effect on MKT scores. However, since there was a sustained difference in mean scores between the two groups, this was further investigated by reiterating the ANOVA twice using sub-samples of the MKT items.

I attempted to classify the MKT items used for this research as assessing either ‘Common Content Knowledge’ (CCK) – mathematics knowledge which any well-educated adult should know - or other aspects of MKT (Other) beyond that expected of an educated adult but specific to the work of teaching (Ball et al., 2008). Some of the older MKT questions selected for use in this study had been previously categorised by the authors but this was discontinued as it was found to be difficult to place questions into mutually exclusive categories (Blunk, 2013). Thus, more recently developed questions are not classified. Whilst the following results provide some insight, they should be treated with caution given the difficulty of classifying questions as recognised by the authors.

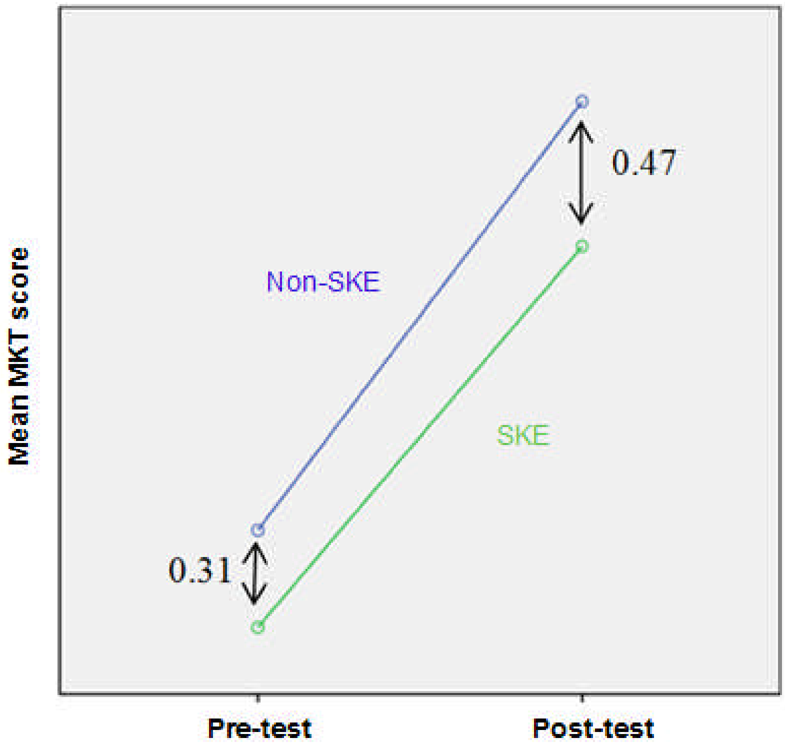

Figure 5.1: Graph showing differences in mean scores on MKT questions for the pre- and post-questionnaire for SKE and Non-SKE students (Warburton, 2014:347).

Most (13 items) were classified as requiring CCK in order to answer them correctly, with only 5 felt to require additional knowledge related to teaching (other aspects of MKT).

The repeated measures ANOVA was repeated first with scores on the CCK questions as the new dependent variable and then with Other MKT questions as the dependent variable.

For CCK questions there was a significant increase in overall scores of 1.01 marks ($F(1, 139) = 24.04$, $p < 0.001$, $\eta^2 = 0.147$), yet the other effects were non significant. The same was true for the Other questions, that is, there was a significant increase in overall scores of 0.32 marks ($F(1, 139) = 6.24$, $p = 0.014$, $\eta^2 = 0.043$) and all the other effects were non-significant. Although there was no significant effect of group on CCK and Other scores, **Figure 5.2** (p.107) shows that, on average, the SKE group performed worse on the CCK questions, but better on the Other questions than the Non-SKE group. These differences are small, but were sustained over time.

Additionally, similar to Stevenson (2008), final PGCE scores were compared between SKE and Non-SKE students. Whilst SKE students did better on average on the PGCE course ($M = 2.18$ points, $SE = 0.135$) than Non-SKE students ($M = 1.96$ points, $SE = 0.115$), this difference was not significant ($t = 1.184, p = 0.238, r = 0.102$).

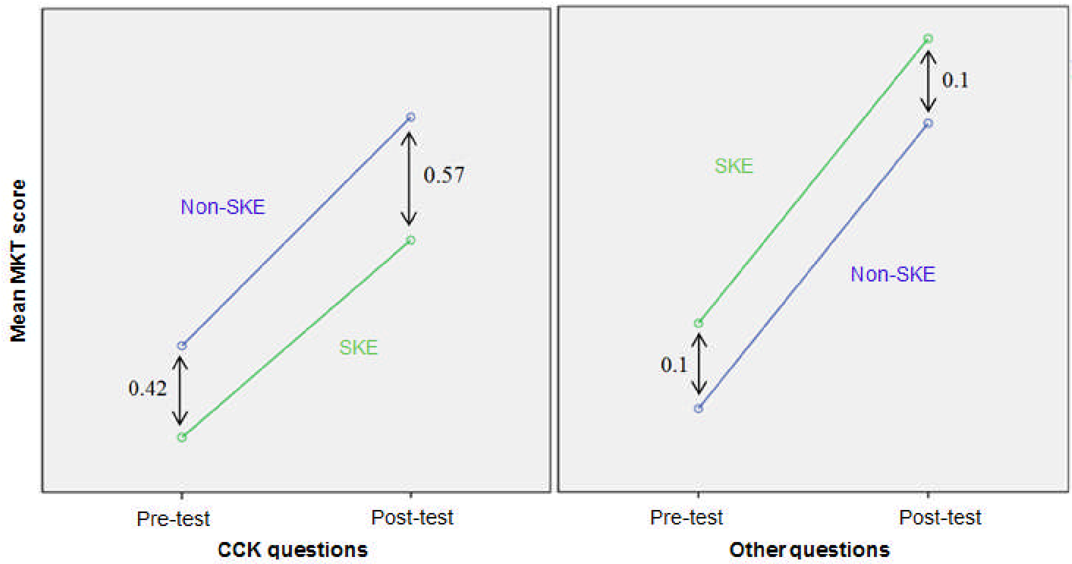

Figure 5.2: Graphs showing differences in mean scores on the CCK and Other questions on the pre- and post-tests by SKE group (Warburton, 2014:347).

5.2 Interview results

5.2.1 Introduction

The interviews were conducted towards the beginning and end of the PGCE course with five PGCE students studying at the University of Leeds. The interview format was the same for both pre- and post-interviews and consisted of asking students to share everything they knew about three topics: squares, rational numbers and quadratic equations. Further, students completed three mathematical problems based on these topics while talking aloud (see 4.3.2.3, p.85 for full details).

Following the interviews, Analysis Tables and Knowledge Maps (see 4.3.2.5, p.89) were created for each topic for each student on both the pre- and post-interviews. These were both used when analysing the interviews. Some example Analysis Tables[8] are provided in Appendix E (p.275) and all Knowledge Maps are included within this section.

To recap, the Knowledge Maps potentially include three domains: geometric, numeric and algebraic. These are represented as coloured ellipses on the Knowledge Maps, where applicable (Geometric = red, Numeric = blue, Algebraic = yellow). For example, when discussing ‘squares’ if a student

[8] Due to space restrictions not all tables could be included

referred to a square as having four sides, this indicates that one representation of the concept of ‘square’ is a geometric shape and a red ellipse will feature on the Knowledge Map with the discrete statement ‘a square has four sides’ as a node placed within it. If there are no references to ‘square’ residing in the numeric domain (for example, references to square numbers) or algebraic domain (for example references to x^2) , then the geometric (red) ellipse will be the only one to feature on the Knowledge Map drawn for that student (e.g. **Figure 5.9**, p.117). The ellipses can either appear as distinct or overlapping on a Knowledge Map depending whether links are identified between them by the student. For example, if a student explicitly refers to the connection between area of a square shape and square numbers then the geometric and numeric ellipses will be overlapping (e.g. **Figure 5.10**, p.117). Statements may either reside in the intersection of two ellipses or there may be a link (connecting line drawn) between two statements which each reside in different ellipses. Where explicit connections between discrete statements are identified *within* a domain then connecting lines are drawn between the nodes within an ellipse. In the case of rational numbers, only statements relating to the numeric domain were made by all students. However, whilst most referred to the fractional form of rational numbers, some did mention the decimal form. The Knowledge Maps for rational numbers therefore feature a division within the numeric ellipse to represent fractional and decimal forms of number (see for example, **Figure 5.12**, p.118).

In some cases, students made statements which were mathematically incorrect. Such statements were included on the Knowledge Maps but highlighted in red (e.g. **Figure 5.3**, p.110). Other statements made were partially correct, these were highlighted in yellow (e.g. **Figure 5.14**, p.120). Where statements of knowledge made in Part 1 of the interview were evidenced within Part 2 of the interviews (problem-solving), the corresponding nodes were highlighted in green. For example if a student stated that ‘all squares have sides of equal length’ and then used this fact to help him solve the squares task in Part 2, the node would be highlighted in green (e.g. **Figure 5.3**, p.110).

Student 1 is not included in this analysis since the pre-interview was not fully audio recorded, making comparison difficult.

In this section, results of the analysis for each student are discussed in turn for each of the three interview topics. Where direct quotes are provided, these have been modified - ‘erms’ have been removed and some instances of elision have been expanded (e.g. ‘whatcha’ → ‘what do you’) for clarity. Following the quote, an indication of whether the utterance is from the pre- or post-interview is provided in parentheses.

As the Knowledge Maps contain only unprompted responses from students these are used when comparing the content of knowledge between students (numbers of statements made, topics covered etc.). However, the Analysis Tables consider all responses and are used to compare quality and accuracy of responses between pre- and post-interview for each student individually regardless of whether these responses were prompted or not. For example, if a student demonstrated a misconception on the pre-interview which was also manifested on the post-interview. Whether the misconception was manifested as a result of prompting or not does not alter the claim that a misconception was sustained.

Following results from each student, a summary table is provided quantifying the Knowledge Map data for ease of comparison (**Table 5.9**, p.140). This is followed by a table summarising the results of the mathematical tasks in Part 2 of the interview (**Table 5.10**, p.141).

5.2.2 Student 2 (S2)

5.2.2.1 Squares

S2’s initial response to the question: “Can you tell me everything you know about squares?” was to state that: “you could mean the geometrical shape… but you can also mean the numerical power - squared” (pre). On the post-interview, S2 firstly described properties of a square shape and later stated: “you can have squared numbers” (post). This suggests that on both the pre- and post-interview, S2 distinguished between square numbers and square shapes as two separate ideas. This was represented on the Knowledge Maps as two detached circles (**Figure 5.3**, p.110 and **Figure 5.4**, p.111).

Comparing nodes and links on the Knowledge Maps, there was not much difference between pre- and post-interview (**Figure 5.3**, p.110 and **Figure 5.4**, p.111). There were 12 nodes with 4 links, and 13 nodes with 2 links on the pre- and post-interviews respectively.

In terms of differences in content (that was not explicitly prompted) between the pre- and post-interviews, S2 mentioned a few topics on the pre-interview that were not mentioned on the post-interview and vice versa. On the pre-interview (only), the sum of the angles of a square, the number of sides of a square and the fact that square numbers are always positive were mentioned. On the post-interview only square graph paper and the fact that squares are the bases of a squared-based pyramid were mentioned.

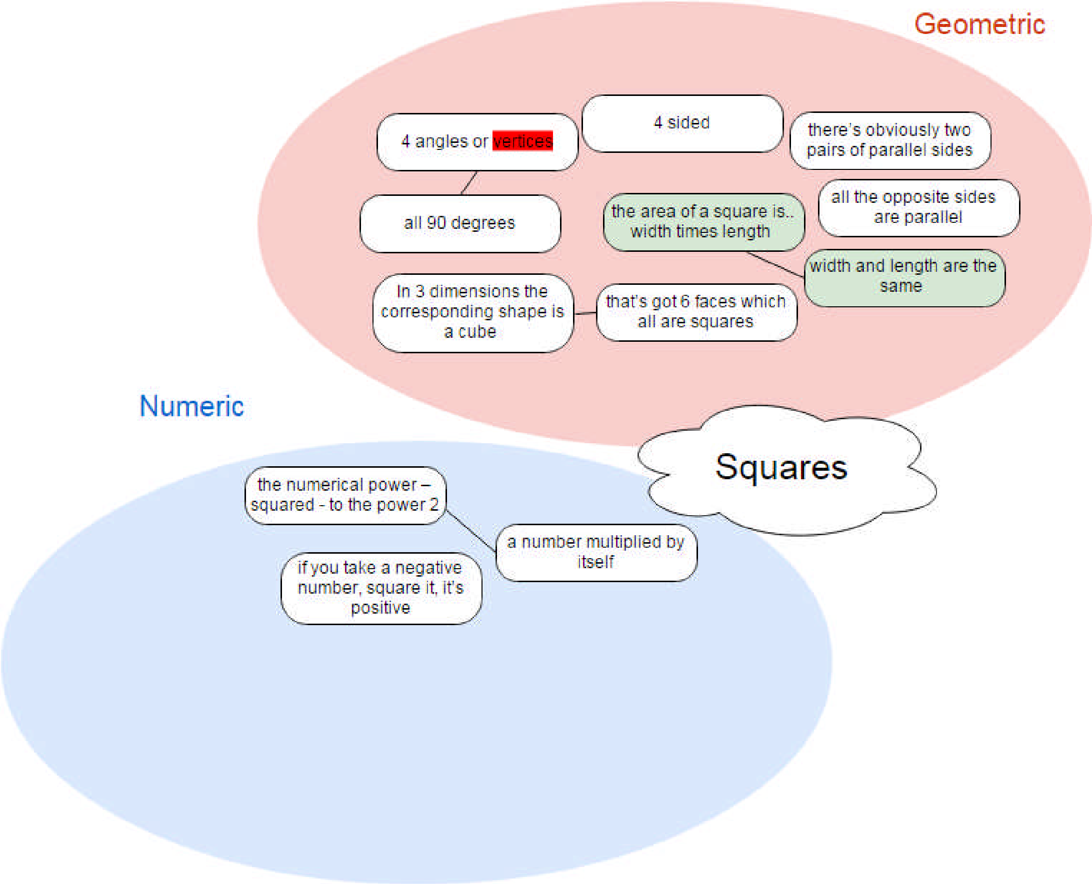

Figure 5.3: S2's pre-interview Knowledge Map for 'squares'.

Although areas of squares were mentioned on both interviews, in the pre-interview, this was only manifested during the problem-solving part of the interview.

Comparing the accuracy and quality of responses, there were some differences in the succinctness of statements and some misconceptions were manifested. On the post-interview, S2's comments seemed to be more succinct than in the pre-interview in many cases. For example:

> …all the opposite sides are parallel and there's obviously two pairs of parallel sides (pre)
>
> …two sets of parallel lines (post)
>
> …the area of a square is.. well width times length but they're the same (pre)
>
> …the area is.. equal to one of the lengths squared so multiplied by itself (post)

S2 demonstrated a possible misconception about squares in the pre-interview: "4 angles or vertices all 90 degrees" (pre). This suggests S2 thinks that vertices of a shape are the same as angles of a shape, which is not the case. Alternatively, he could have meant ‘four angles *and* vertices’. On the post-interview, angles and vertices were not equated, but were mentioned separately (at different times within the post-interview) as follows:

> …they have four equal angles, all of 90 degrees (post)
>
> …they have 4 vertices (post)

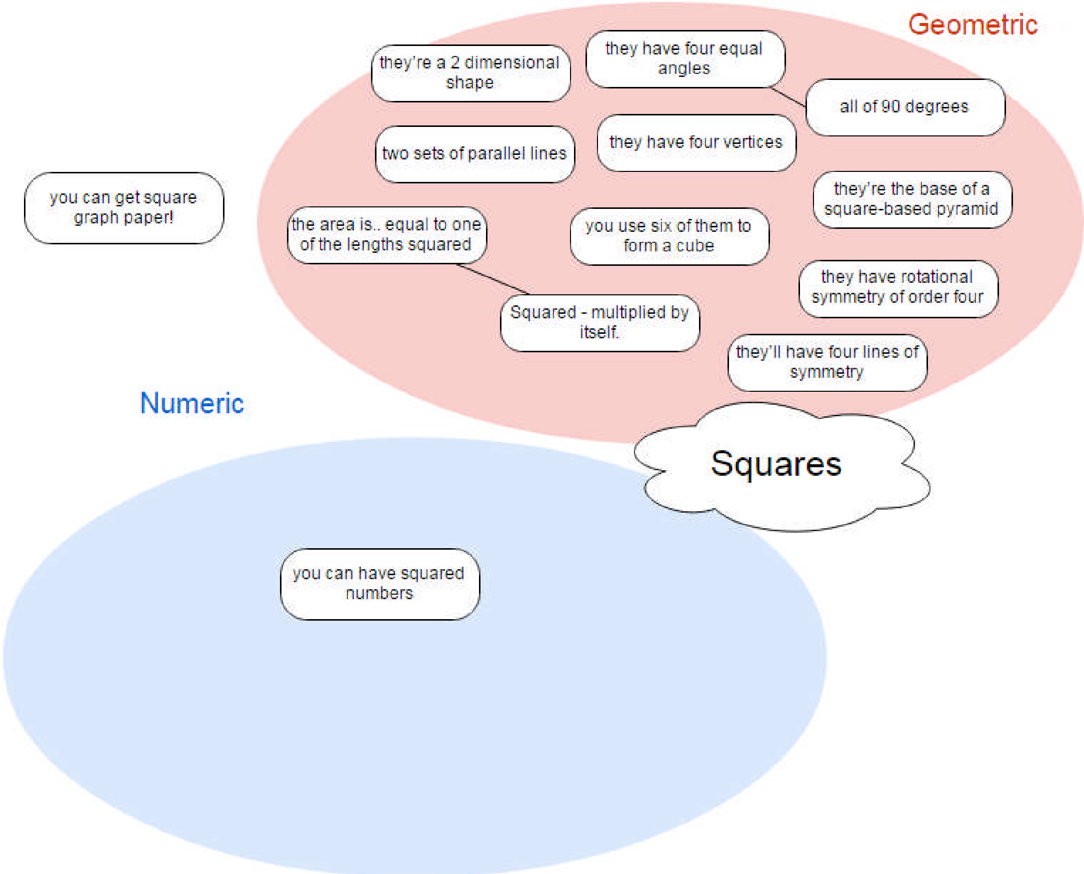

Figure 5.4: S2's post-interview Knowledge Map for ‘squares’.

5.2.2.2 Rational numbers

Responses on both the pre-and post-interviews were similar regarding rational numbers and limited compared to the other topics discussed. Only one and three nodes were on the pre- and post-interview Knowledge Maps respectively and these were all within the domain of fractional numbers. There was one link on the post-interview. Knowledge Maps were thus very similar for S2. He mentioned the same things about rational numbers (unprompted) on both interviews, down to the example of pi as an irrational number. However, on the pre-interview, this example was framed by his experience of learning at school. Conversely, pi as an example of a rational number was stated on the post-interview as a discrete fact.

A possible misconception was manifested on the pre-interview. He stated: “I think I’m right in linking surds to rational numbers – a surd is a type of rational number isn’t it?” (pre). Since he poses the question, we can infer he is not sure about this. Whilst it is true that taking the square root of a square number (e.g. 9) yields a rational number (i.e. an integer), this is not true for all surds (e.g. square root of 2).

On the post-interview only, S2 discusses how to add, subtract, divide and multiply rational numbers after the interviewer explicitly asked him about this. He was able to describe how to carry out the procedures for performing these operations with rational numbers but could not state why the procedure for dividing fractions worked beyond the statement: “I guess it’s something to do with it being the reciprocal… one being – No, I don’t [know]” (post).

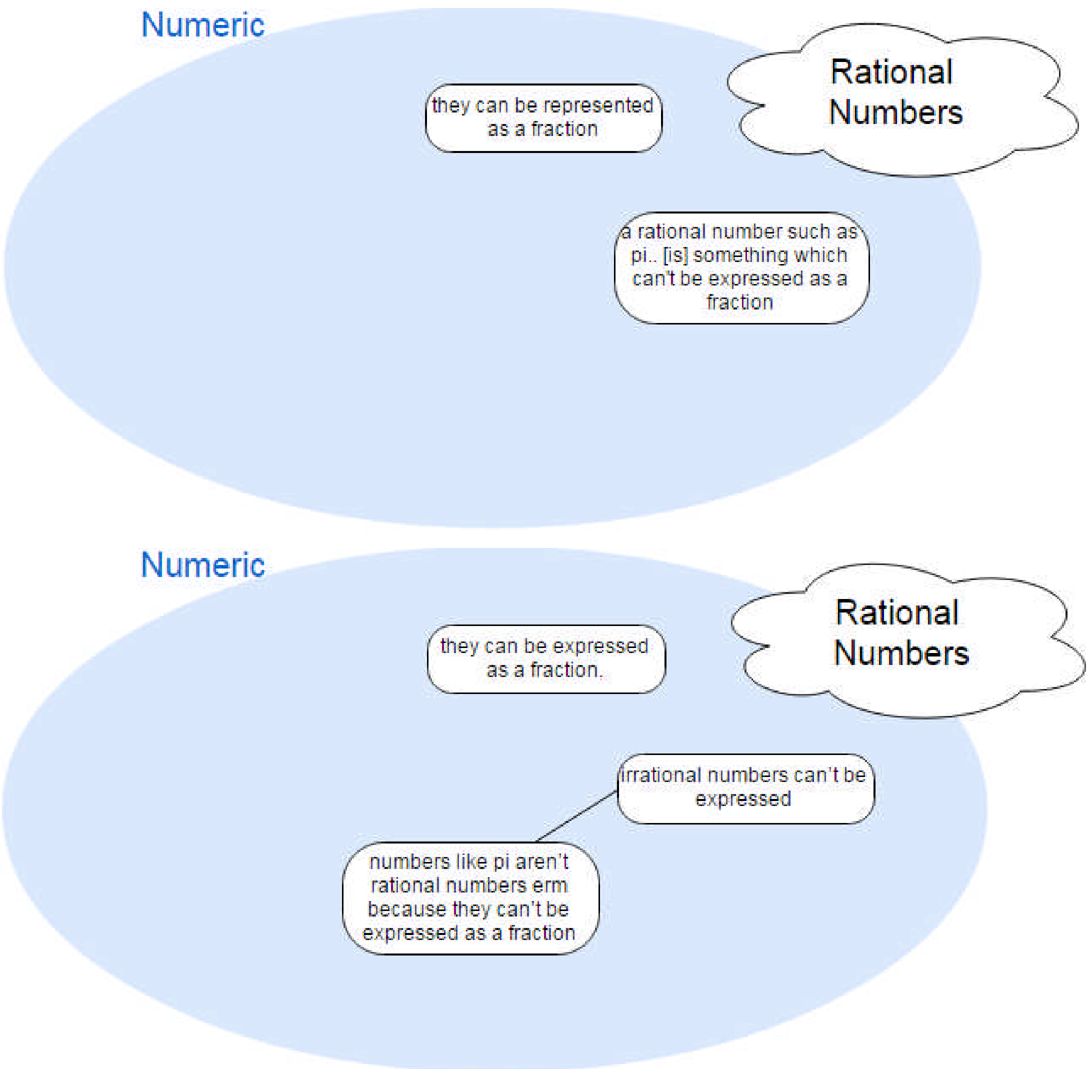

Figure 5.5: S2's pre- and post-interview Knowledge Maps for ‘rational numbers'.

5.2.2.3 Quadratic equations

The topic of quadratic equations elicited more responses for S2. There were nine nodes with four links and ten nodes with seven links on the pre- and post-interviews respectively (**Figure 5.6**, p.114 and **Figure 5.7**, p.114). The Knowledge Maps for the pre- and post-interviews both feature the algebraic and geometric (graphical) forms of quadratics with knowledge that these are linked demonstrated in both. For example, on the pre-interview, S2 states: “your constant shows where the graph crosses the x axis”. However, there are more links between the two on the post-interview as S2 explains how the discriminant corresponds to the general form of the graph:

> …you can determine how many roots... there are depending on the discriminant. So discriminant’s B squared minus 4ac. If that’s equal to zero, then there’s gonna be a repeated root. So the quadratic’s just gonna touch the x axes. If it’s greater than zero there’s gonna be two real roots or two distinct roots, so it’s going to cross the x-axes in two different locations. If it’s less than zero, we say it has imaginary roots – or no real roots, so it won’t cut that x-axis (post)

There are differences in content on the pre- and post-interview Knowledge Maps. On the pre-interview Knowledge Map only, S2 discusses more about the algebraic form of a quadratic equation:

> …they’ll always have a variable - most commonly if you look in a textbook it’ll be x more times than not… they’ll have a squared part to it, a linear part and then a.. constant… you don’t always need each part.. obviously you’re always gonna need the squared (pre)

On the post-interview only, he discusses calculus:

> …you can find the gradient of a quadratic by finding the first order derivative… That’d give you the solute- er the equation for the gradient and finding the second order – you can find whether it’s a maximum or a minimum (post)

Finally, he states that quadratics: “can be used to model events” on the post-interview, but is unable to provide any specific examples.

On both the pre- and post-interviews, S2 discusses how the algebraic form of a quadratic influences its graph. On the pre-interview, S2 incorrectly states that: “often it’s a bell shaped curve when you do graph it” (pre). Despite this statement, S2 draws a parabola as an example. Since a normal distribution tends to be described as a bell shaped curve, perhaps S2 is confusing the names of the two, but his sketch suggests he does understand what the general shape looks like. This statement is not made in the post-interview. On the contrary, S2 correctly sketches two parabola, stating that if the coefficient of x squared is positive, it will look like his first sketch (of an upward parabola), if negative, it will look like his second sketch (downward facing parabola). S2 goes into more detail about how the equation affects the graph on the pre-interview. Stating that the "x squared element of your equation is gonna show you the steepness of your curve" and that the "constant shows where the graph crosses the x axis". On both the pre- and post-interview, S2 states that he doesn't know exactly how the coefficient of the x term affects the graph.

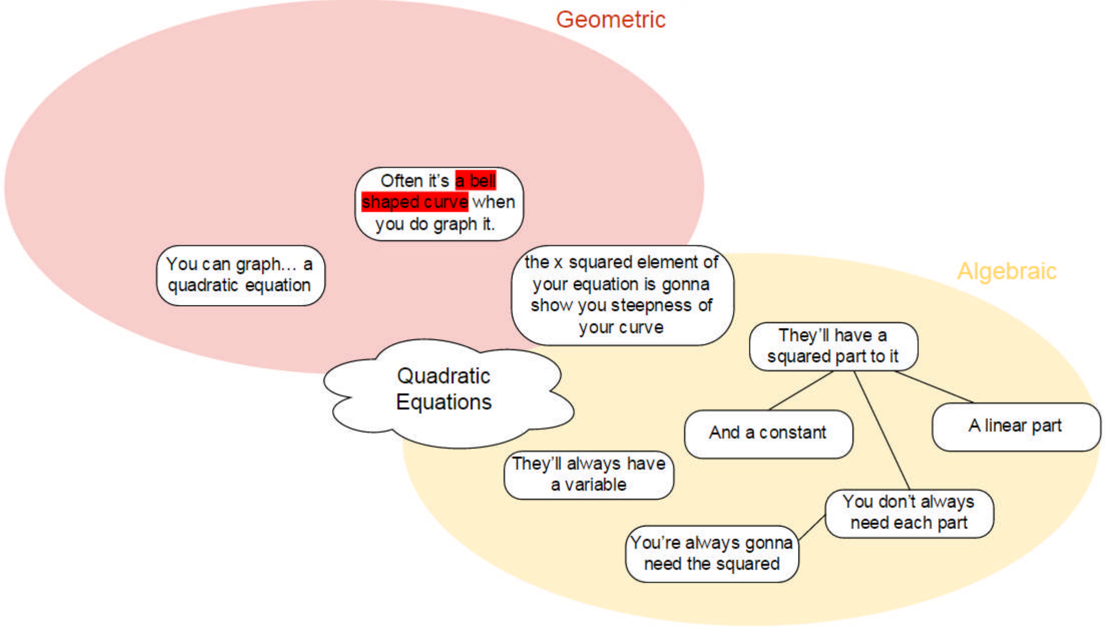

Figure 5.6: S2's pre-interview Knowledge Map for 'quadratic equations'.

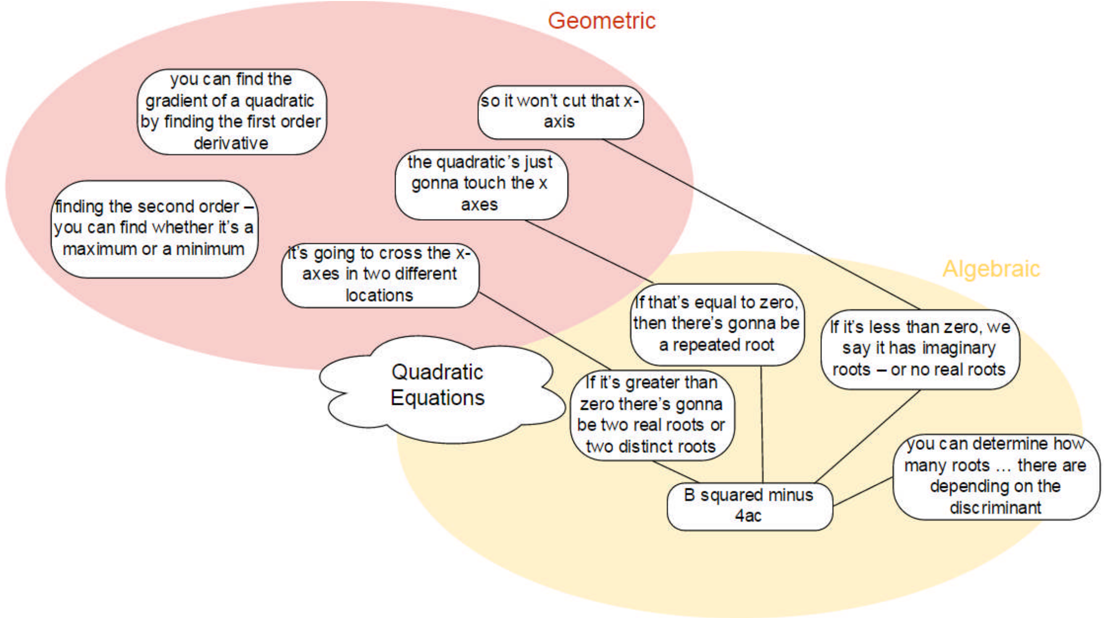

Figure 5.7: S2's post-interview Knowledge Map for 'quadratic equations'.

On both pre- and post-interview, S2 discussed the numbers of solutions of a quadratic equation and how to determine them using the discriminant. On the pre-interview, S2 states that there are two solutions to a quadratic equation but that “it could be… a repeated solution” or “you might not have any real solutions”. On the other hand, on the post-interview, S2 states that there are “zero, one or two” and then goes on to say there are only “two or zero… there’s never just one, ‘cause if it’s one, it’s repeated”. **Table 5.8** (p.115) contrasts the statements made in the pre- and post-interviews with respect to the discriminant and the roots of a quadratic equation:

Pre-interview	**Post-interview**
if the discriminant is.. less than zero – am I – if it’s less than zero then you’ve got no real equations	If it’s less than zero, we say it has imaginary roots – or no real roots
	so it won’t cut that x-axis.
if it’s equal to zero you’ve got a repeated equation	If that’s equal to zero, then there’s gonna be a repeated root
	So the quadratic’s just gonna touch the x axes
if it’s greater than zero, you’ve got two distinct solutions	If it’s greater than zero there’s gonna be two real roots or two distinct roots
	so it’s going to cross the x-axes in two different locations

Table 5.8: Table showing statements made regarding the roots of a quadratic equation made by Student 2 on the pre- and post-interviews

With the exception of S2 saying “equations” rather than “roots” or “solutions” on the pre-interview, which we could assume is a slip of the tongue, the statements made on pre- and post- are similar. However, on the post-interview, S2 also links the type of roots to what the graph of the corresponding quadratic function will look like. The table above suggests S2 understands the types and number of roots a quadratic equation has but the post-interview reveals some confusion on the actual number of roots (two) given the different types (repeated, imaginary etc.). Perhaps this confusion comes as a result of understanding the connection with the graph of a quadratic since the number of times the graph crosses the x-axis is one, none or two but this does not exactly correspond with the number of solutions (two).

Factorising as a method of solving a quadratic equation is also discussed on pre- and post- interviews. On the pre-interview all that is said is: “first you’d create your brackets", writes **Figure 5.8** (p.116) and then goes onto another method of solving.

$$x^2 + 2x + 4 = 0$$

$$(\quad x \quad) =$$

Figure 5.8: S2's example of factorising

On the post-interview, S2 goes into more detail, but this is rather procedural, that is, describing the process/ method of solving:

> …so factorising you're going to put double brackets… So out of that you can draw out your solutions so the idea is to make the left hand side of the equation equal to zero so that means that one of the brackets is gonna have to equal zero so for instance in our first bracket, if x is minus one, then that equals zero, so the whole of the equation would be equal to zero, similarly for the second part. X could equal minis 2 (post)

5.2.3 Student 3 (S3)

5.2.3.1 Squares

My first impression of comparing the pre- and post-interviews for S3 was that the responses on the post-interview seemed fuller. That is, S3 mentioned other related topics (other shapes) and other areas of mathematics (e.g. links between number and geometry) in the post-interview. In particular, S3 mentioned the relationship between the square, rhombus and parallelogram, and the connection between square numbers and finding areas of squares was made in the post-interview. The Knowledge Map for the post-interview thus includes an ellipse representing the geometry domain (square shape) and an ellipse representing the number domain (square numbers) with these ellipses intersecting (**Figure 5.10**, p.117). The number domain does not feature in the pre-interview Knowledge Map (**Figure 5.9**, p.117). There were, however, similar numbers of nodes and links between the pre- and post-interview Knowledge Maps: eight nodes with two links and nine nodes with three links respectively.

The only unprompted topic on the post-interview which was not on the pre-interview was the square's relation to cubes. Conversely, the fact that all squares are similar was only mentioned on the pre-interview.

During the pre-interview, S3 talked about pupils' sources of confusion regarding squares and the sequencing of squares within the curriculum. Although such Pedagogical Content Knowledge (PCK) and curricular knowledge is not the focus of this research, it is interesting that this was only present on the pre-interview (i.e. at the start of the PGCE course) and not at the end of a course which aims to prepare trainees with knowledge of teaching. However, this could be explained by S3 having a better idea of the foci of this research by the end of the course and thus by the post-interview, perhaps he knew this type of knowledge was not being sought.

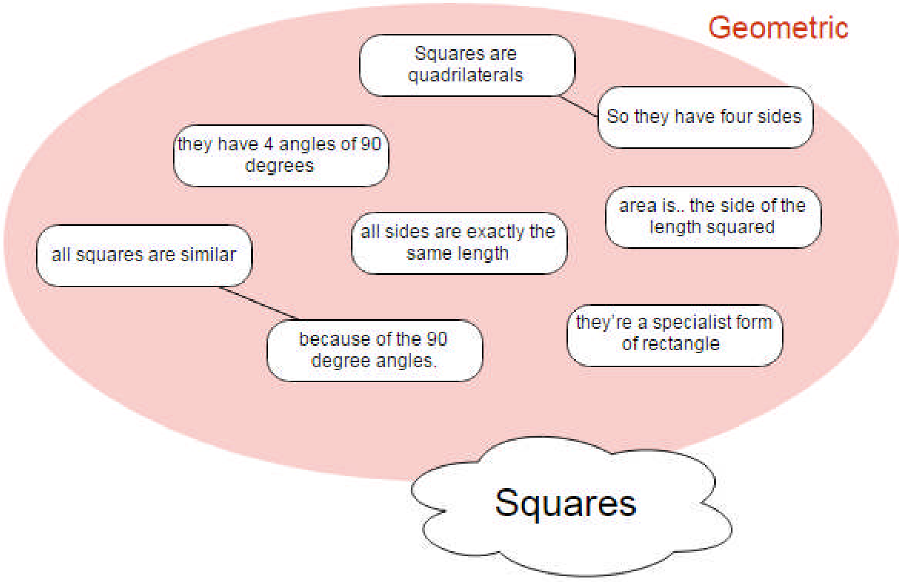

Figure 5.9: S3's pre-interview Knowledge Map for 'squares'.

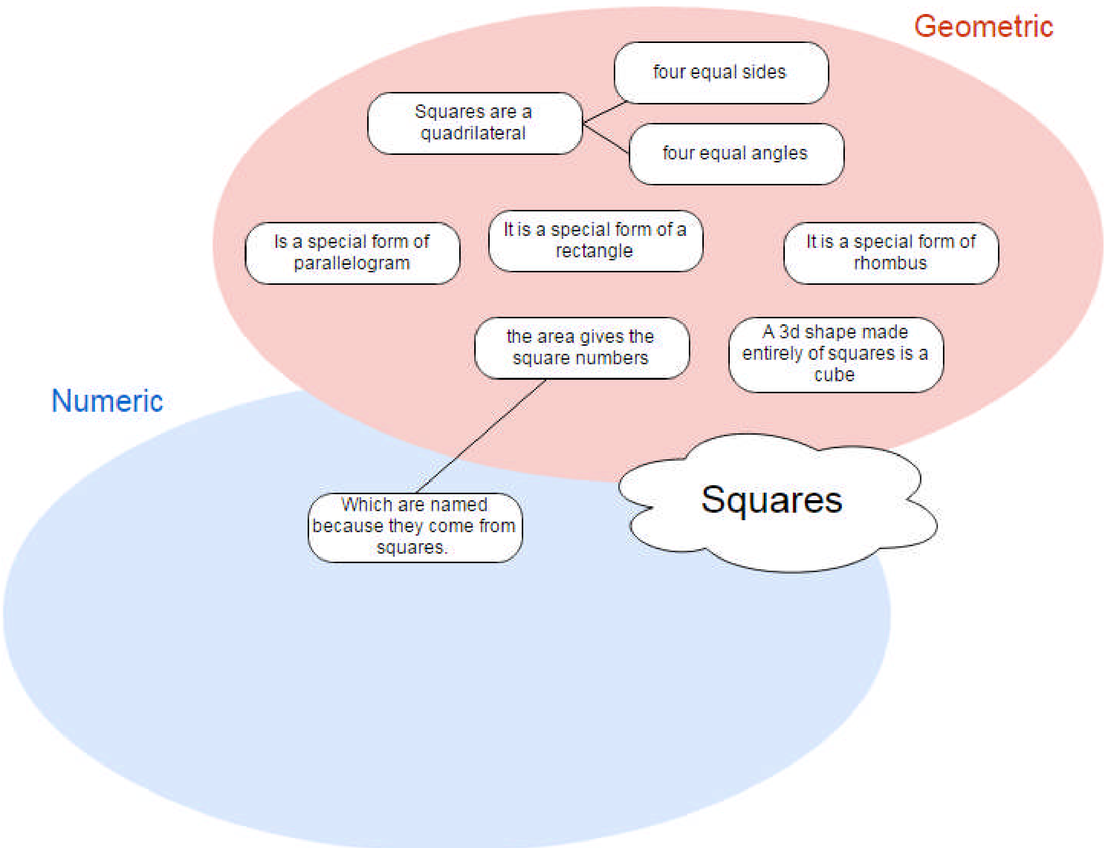

Figure 5.10: S3's post-interview Knowledge Map for 'squares'.

5.2.3.2 Rational numbers

There were some differences between S3's Knowledge Maps for rational numbers. On the post-interview (**Figure 5.12**, p.118), S3 included rational numbers as decimals – all terminating decimals and some non-terminating decimals - whereas, only the fractional form of rational numbers appeared on the pre-interview Knowledge Map (**Figure 5.11**, p.118). This was represented as two intersecting ellipses on the post-interview Knowledge Map. This was the main difference between the two Knowledge Maps as there were similar numbers of nodes and links: nine nodes with three links and seven nodes with four links respectively on the pre- and post-interview Knowledge Maps. There were differences between the content of the nodes, however.

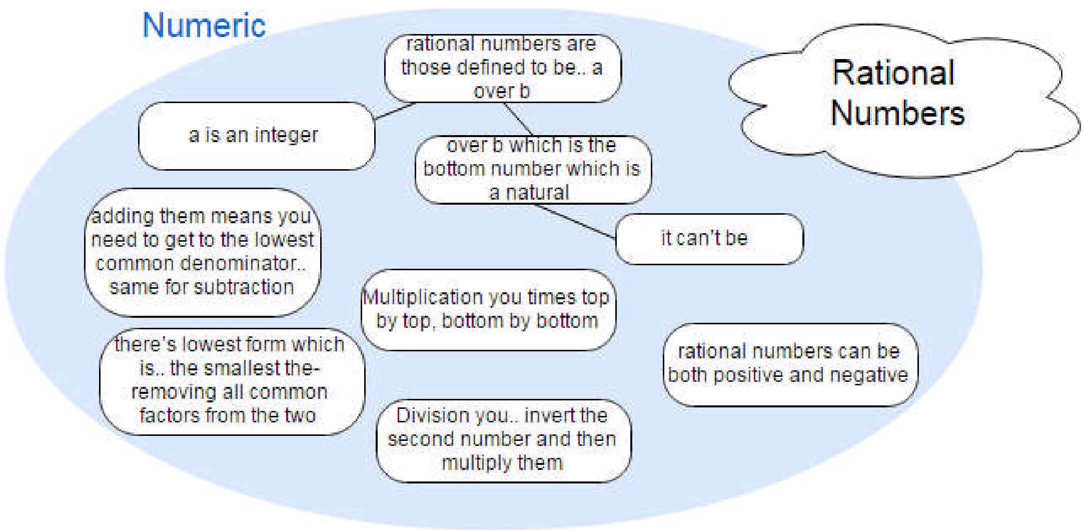

Figure 5.11: S3's pre-interview Knowledge Map for 'rational numbers'.

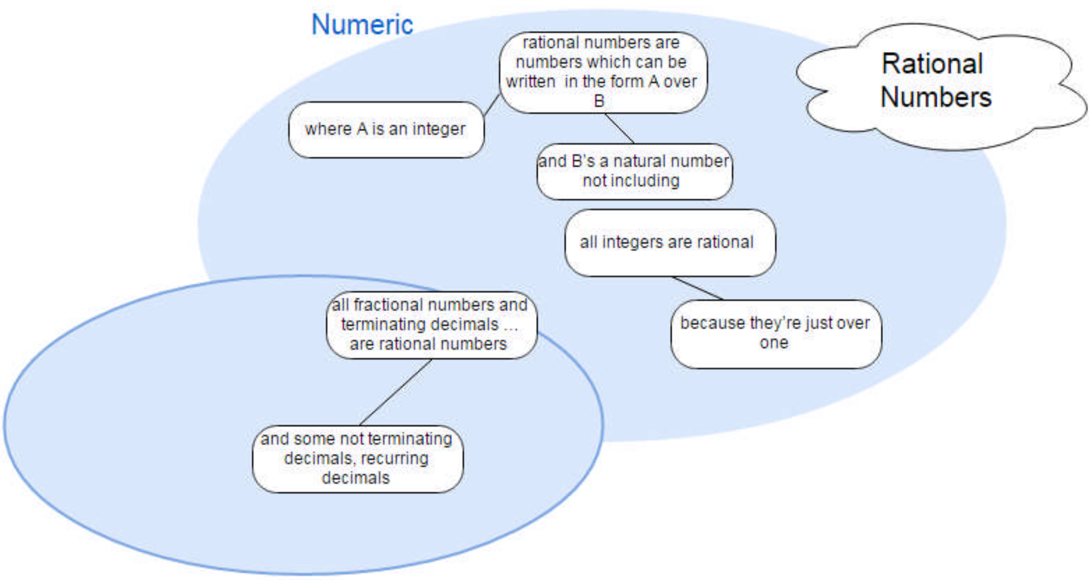

Figure 5.12: S3's post-interview Knowledge Map for 'rational numbers'.

Firstly, on the pre-interview only, S3 talked about the four operations on rational numbers and stated that “rational numbers can be both positive and negative”. On the pre-interview, when discussing the fractional form of rational numbers, S3 went on to explain what natural numbers and integers are (this was not recorded on the Knowledge Map since it was deemed not to refer to rational numbers):

> …‘a’ is an integer which is anything from minus infinity to infinity of like the countable numbers 0, 1, 2, 3, 4, minus 1, minus 2, minus 3, minus 4 (pre)

> …which is a natural number which is one, two, three, four because it can’t be zero (pre)

Finally, removing common factors in the numerator and denominator to get a fraction in lowest form was only mentioned on the pre-interview Knowledge Map.

There was only one statement made on the post-interview that was not on the pre-interview Knowledge Map in addition to the decimal representation of fractions as mentioned above:

> …all integers are rational they’re just over one (post)

As well as differences in content between the two interviews, there were also differences in the quality of statements made or the depth that S3 went into when making statements. Indeed, on the post-interview S3 explained about finding equivalent fractions in greater detail, explaining why it works:

> We find equivalent fractions by multiplying by one or different forms of one such as three over three, four over four, five over five which all cancel down to give one (post)

Further he stated: “standard addition has this because it’s over one” (post). Presumably, he is talking about adding whole numbers (which can be expressed as fractions over a denominator of one).

On the post-interview, S3 also went into greater depth about equivalent forms of rational numbers:

> …all rational numbers have a decimal equivalent and a fractional equivalent (pre)

> …all fractional numbers and… all terminating decimals… are rational numbers and some not terminating decimals, recurring decimals (post)

5.2.3.3 Quadratic equations

On the Knowledge Maps for both the pre (**Figure 5.13**, p.120) and post-interview (**Figure 5.14**, p.120), only the algebraic form of a quadratic equation is featured. Graphs of quadratic functions were only mentioned on the post-interview after this was explicitly prompted by the interviewer (and thus does not feature on the Knowledge Map):

> …you can graph them… so the graph of x squared is generally U shape… we have either, standard x squared – which is a U or minus x squared which is an N (post)

S3 went on to explain what the graph would look like if we have repeated roots, complex roots and two distinct, real roots.

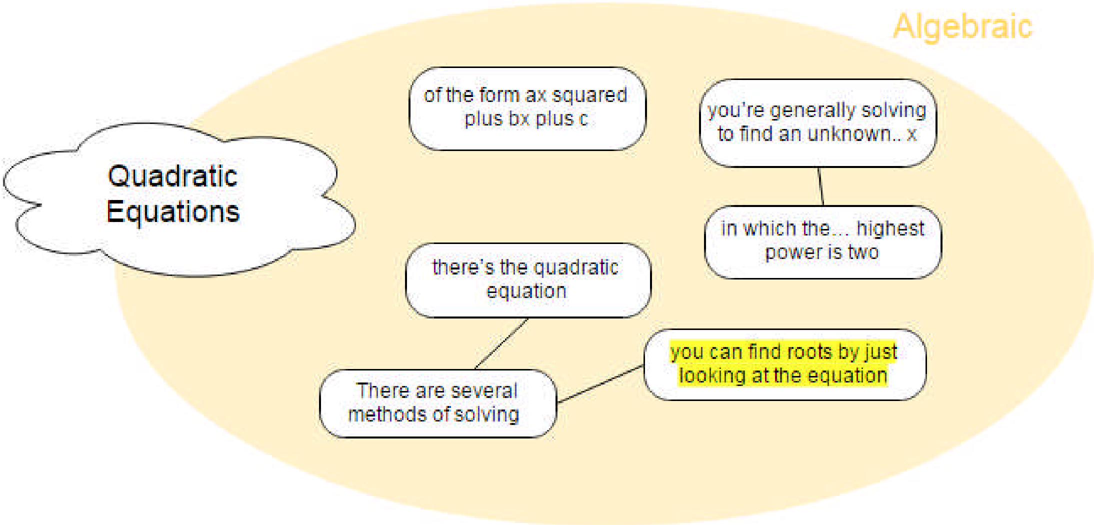

Figure 5.13: S3's pre-interview Knowledge Map for 'quadratic equations'.

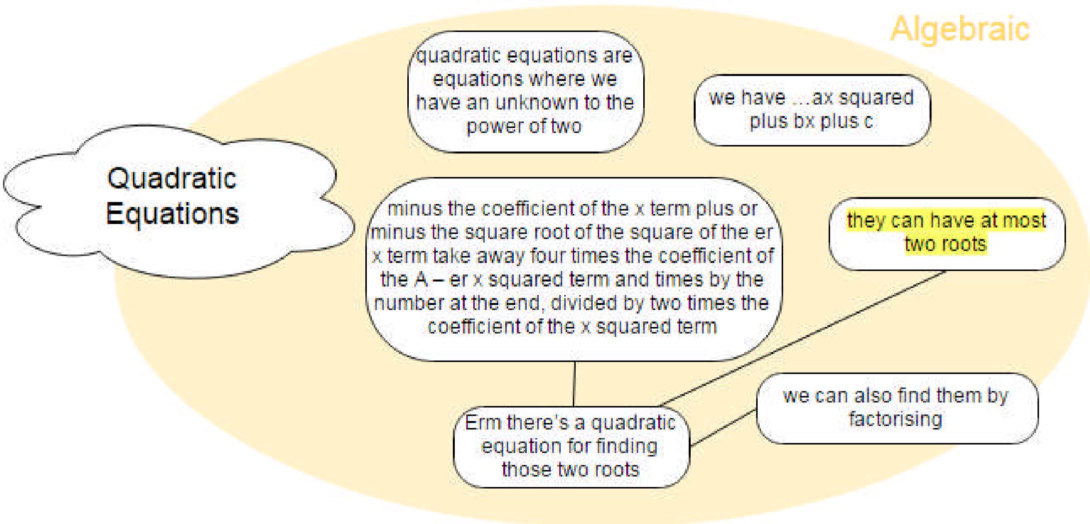

Figure 5.14: S3's post-interview Knowledge Map for 'quadratic equations'.

A possible reason why the geometric form (graphs) of quadratics was not mentioned without prompting is that S3 prefers to think algebraically rather than geometrically. This preference was manifested in comments during the problem-solving part of the post-interview as S3 said:

> …I hate geometry questions (post)

There are the same number of nodes and links on the pre- and post-interview Knowledge Map, namely six nodes and three links. Further, the content of these nodes is similar on both maps as the form of a quadratic equation was discussed and some of the methods for solving – though these methods differ as he mentions factorising on the post-interview Knowledge Map, but not on the pre-. He also stated what the quadratic formula is on the post-interview Knowledge Map and he discussed the number of roots of a quadratic. Two statements S3 made regarding roots seemed to contradict each other:

> …they can have at most two roots(post)

> …there's a quadratic equation for finding those two roots (post)

Therefore, S3 was questioned by the interviewer regarding the number of roots: "So you've said they can have at most two roots… and then you said the quadratic equation gives us… two… so what about less? How does it give it less?"

Within the dialogue of the interview, S3 concluded that there are two roots, though these could be imaginary or a repeated root. He further stated:

> …sometimes when you're teaching it in school there can be no roots but in... proper maths they do have roots they're just imaginary ones (post)

This suggests that S3 is aware that when teaching quadratics sometimes teachers say that there are no roots, because pupils have not yet met complex numbers. This is a possible explanation as to why the other PGCE students were confused about the numbers of roots in the interviews.

There were no responses made on the pre-interview that were not also made on the post-interview.

Despite similarities in the number of nodes and the content of these nodes, there was one difference in the depth of responses S3 gives between the pre- and post-interview. On the pre-interview, S3 states that: "There's the quadratic equation" for solving a quadratic. However on the post-interview he goes on to state what the quadratic formula is:

> …which is minus the coefficient of the x term plus or minus the square root of the square of the x term take away four times the coefficient of the A –x squared term and times by the number at the end, divided by two times the coefficient of the x squared term, which gives you both roots (post).

This explanation shows that S3's understanding of the quadratic formula is not tied to using specific values of coefficients (usually a, b and c), but instead knows which coefficients these terms relate to.

5.2.4 Student 4 (S4)

5.2.4.1 Squares

During the pre-interview with S4, he freely recalled some facts about squares and then commented that "off the top of [his] head" he couldn't think of anything further. The interviewer then prompted him quite heavily on most of the aspects on the interview hierarchy sheet, thus, the majority of his statements were heavily influenced by the interviewer. The Knowledge Maps are therefore useful for comparing the unprompted knowledge since prompting varied between pre- and post-interview.

Although the number of nodes and links are similar between the pre- (**Figure 5.15**, p.123) and post-interview (**Figure 5.16**, p.123) (nine nodes with two links and eight nodes with two links respectively), the post-interview discussed squares in terms of both geometry and number whereas the pre-interview Knowledge Map only shows statements made regarding geometry.

Unlike the other students, there were several misconceptions which were demonstrated in the pre-interview. Firstly, when considering whether the parallelogram related to a square, he concluded that it did not: "just trying to bang my head if there's any way it could be parallelogram but it's not" (pre). This was after discussing the relation of the rhombus and parallelogram to the square and following a comment from the interviewer that those two shapes were all he could think of that related. Thus, S4 may have been led to this conclusion due to assuming those two shapes were the only ones. Other misconceptions were as follows:

> …any number multiplied by itself is a square number (pre)
>
> …square numbers... are used in the Fibonacci sequence (pre)
>
> …if you multiply a square number by its square root you get a cube number... if you square a squared number you'll get a number to the power of four (pre)

The final statement above was perhaps not well expressed rather than being a misconception. The misconception that any number multiplied by itself is a square number was sustained in the post-interview: "a square number is a number multiplied by itself" (post). The other statements above were not commented on in the post-interview thus we cannot claim these misconceptions were either eliminated or sustained.

There were no statements on symmetry made in the post-interview.

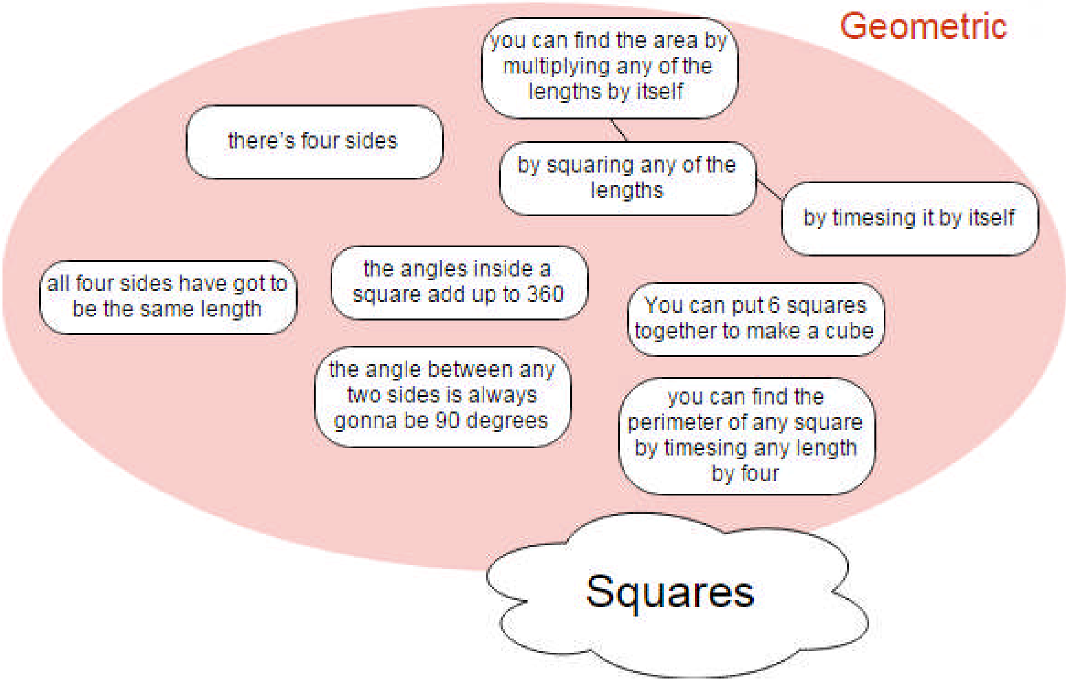

Figure 5.15: S4's pre-interview Knowledge Map for 'squares'.

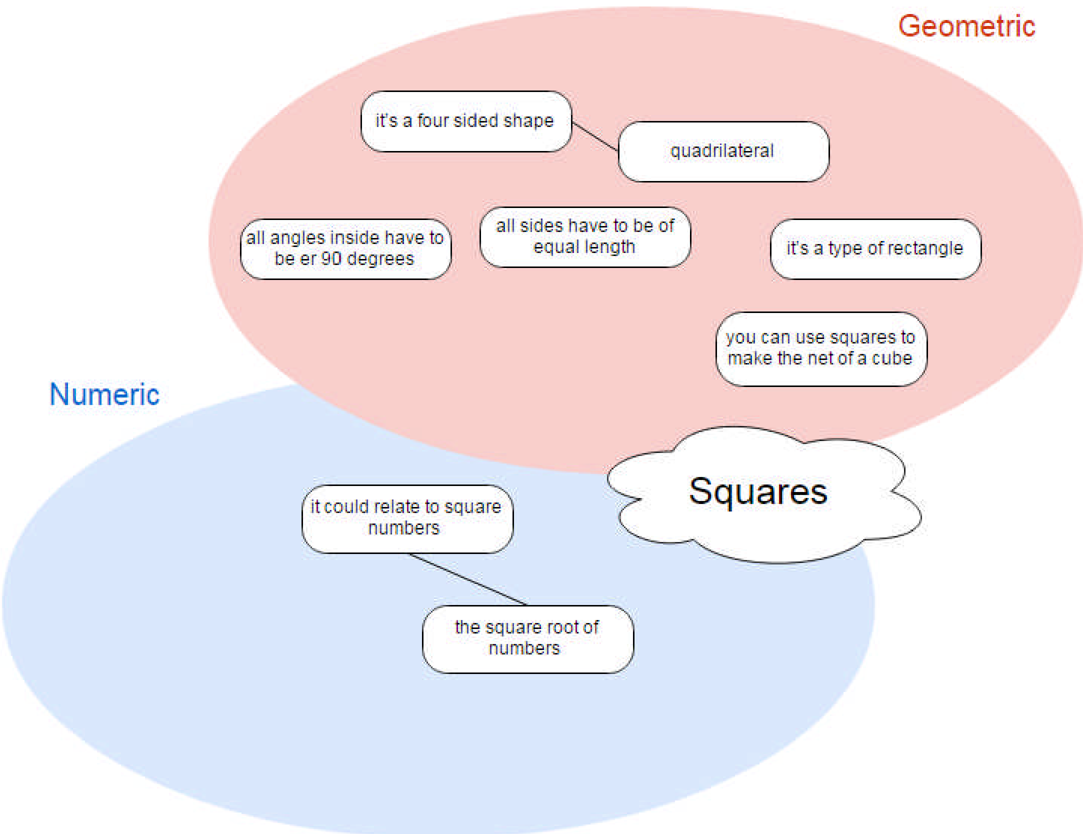

Figure 5.16: S4's post-interview Knowledge Map for 'squares'.

When asked about symmetry, S4 drew a diagram and counted the number of lines of symmetry. He also used the diagram to think about the order of rotational symmetry. He also struggled to explain what rotational symmetry meant, though it was apparent he did understand, only struggled to find the words to express his understanding:

> …rotational symmetry, let's have a think… so it could be rotated... It's four - four times and still give the same.. shape - think the same – trying to think - oh no it'd be – well it's ahh hard to explain [laughs].. it still being the same sort of positional shape if that makes any sense? (pre)

5.2.4.2 Rational numbers

On the pre-interview S4 could not remember what a rational number was: "I've forgotten rational numbers... that's embarrassing" (pre), so the interviewer provided the definition: "they're numbers of the form a over b where a and b are… integers". The pre-interview was heavily prompted and since Knowledge Maps only feature unprompted knowledge, the pre-interview Knowledge Map was blank/ could not be drawn. Further, any differences between comments made in the pre- and post-interview should be treated with caution as they may have been affected by interviewer prompts. The Knowledge Maps are therefore a better indication of the knowledge held by S4 (not requiring prompting) and are discussed first.

On the pre-interview, the following comment was made after a definition of rational number was provided by the interviewer:

> …any numerator that's an even number can… be made a rational number as long as it's divided by 2... (pre)

This suggests a bit of confusion about rational numbers as an even number is a rational number already. Perhaps he meant 'can be written in the form of a fraction'. This potential confusion about what a rational number is manifested itself later in the pre-interview as demonstrated in the following exchange between the interviewer (I) and S4 (S):

> I: What about if a number can't be written in this form? What do we say about it?
>
> S: that it's an irrational number
>
> I: and what's an irrational? Just one that can't be written in this form [laughs] so do you have an example?
>
> S: yep er so.. er.. well for example isn't one over three no–
>
> I: so one over three?
>
> S: yeah

I: isn't one over three-?

S: oh sorry erm.. oh so irra- are we thinking… d'ya know I still think I've got what rational numbers are wrong!" (pre)

Conversely, on the post-interview Knowledge Map (**Figure 5.17**, p.125), S4 did not require prompting in order to define a rational number and S4's responses suggested he understood rational numbers:

> …a rational number can be written as a fraction with two integers – an integer as the numerator and an integer as the denominator (post)

He further stated that "they come under real numbers as well". He also made reference to decimal forms of rational numbers – both recurring and non recurring:

> …you can get recurring rational numbers… so it can be written as a decimal that goes on forever… it could be a decimal that doesn't repeat as well (post)

S4 also discussed writing a fraction in its lowest form and stated that it could also be written as a whole number with a fraction if the fraction is top heavy. Being able to describe several different forms of rational numbers (fractions, mixed fractions, decimals) suggests he understands the concept of rational number well by the post-interview.

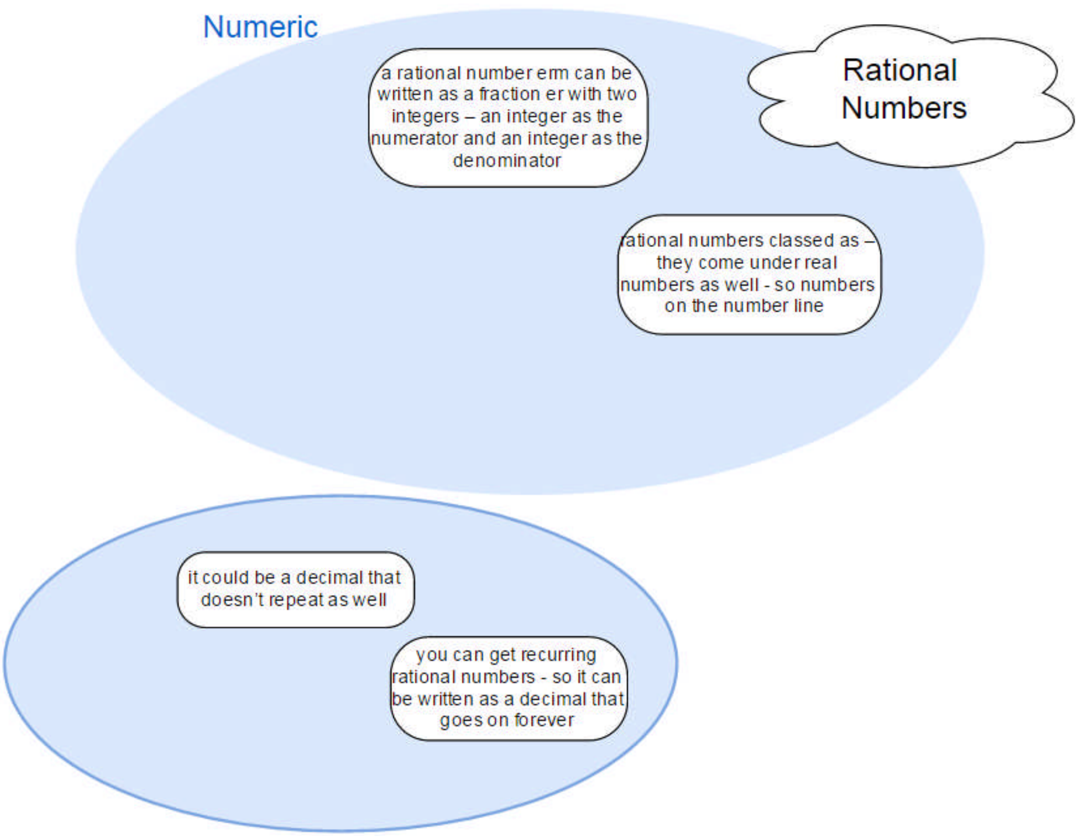

Figure 5.17: S4's post-interview Knowledge Map for 'rational numbers'.

Considering all responses on the interviews (prompted and unprompted), there were differences in the accuracy and depth of responses made. For example, on the pre-interview he stated: “denominator’s gotta be a positive number..” (pre) which is not strictly true, whereas on the post, he refines this: “so the denominator has to be – cannot be zero, obviously” (post).

Further, on the pre-interview he says: “they… always have a decimal representation” whereas on the post-interview he goes on to say what form this decimal representation takes: recurring decimal or “doesn’t repeat”. Thus, the depth of information given was greater on the post-interview in this case.

5.2.4.3 Quadratic equations

A noticeable difference between the Knowledge Maps of the pre- (**Figure 5.18**, p.127) and post-interview (**Figure 5.19**, p.128) is that on the pre-interview, the algebraic and geometric forms of a quadratic are treated separately, whereas not only are these two domains seen as linked in the post-interview Knowledge Map, but the number domain is also present as S4 considers the link to number, namely, sequences of number which are defined by a quadratic equation:

> Quadratic equations are linked to quadratic sequences where… the sequences grow at quadratic rates…. so for example… a quadratic sequence would be… when there’s a constant difference in the difference between the terms we’d get - you’d get them growing at a quadratic rate (post)

S4 was the only interviewee to refer to the number domain when considering quadratic equations.

Similarly there is a substantial difference in the number of discrete nodes between the pre- and post-interview Knowledge Maps, with the pre-interview having only 4 nodes and no links and the post-interview Knowledge Map having 12 nodes with 6 links across all three domains (number, geometry and algebra). However, the pre-interview with S4 was heavily prompted, which may account for the fewer number of unprompted nodes on the pre-interview Knowledge Map.

There were differences in the depth of the (prompted and unprompted) responses.

When discussing the form of a quadratic equation, a possible misconception was demonstrated on the post-interview. The form of a quadratic was stated on the pre-interview as follows:

> …so a quadratic equation can be written in the form ax squared plus bx plus c equals zero… it's always gonna have an x squared value (pre)

The post-interview 'definition' was not as full:

> …there's got to be at least one x squared for it to be a quadratic equation (post).

However, in the post-interview, S4 did go on to describe the general algebraic form of a quadratic equation but in the context of using the quadratic formula:

> …you can solve them using the quadratic equa- formula… when the equation takes this form... x squared plus bx plus c equals any value – it doesn't have to be zero, I've written zero down (post).

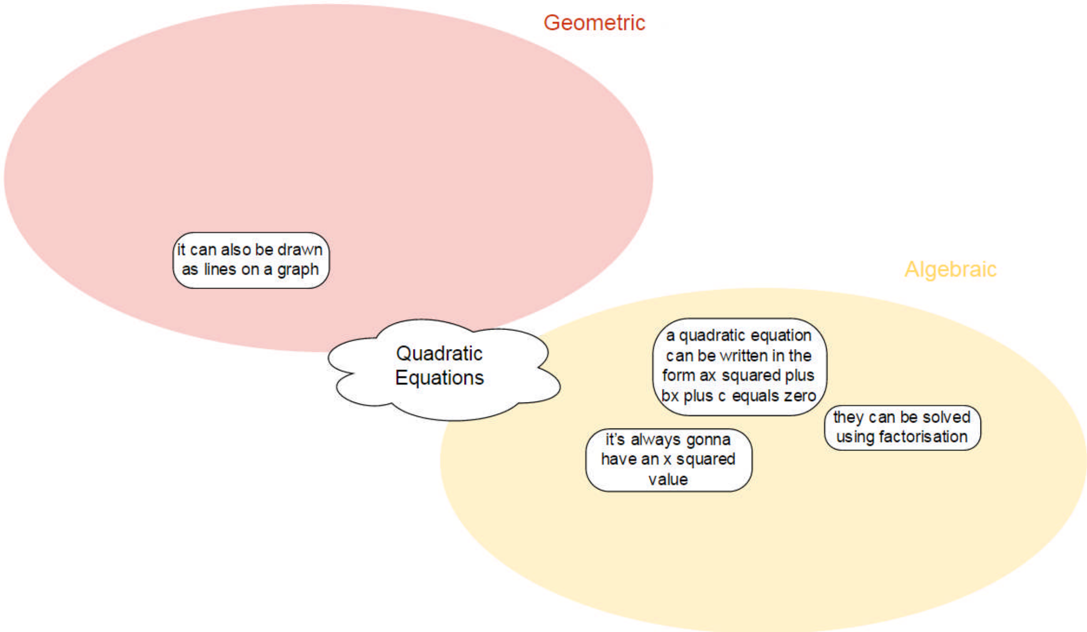

Figure 5.18: S4's pre-interview Knowledge Map for 'quadratic equations'.

This demonstrates a possible misconception since whilst it is true that a quadratic equation does not have to be written to equal zero in general, when using the quadratic formula it does. The quadratic formula was mentioned on both pre- and post-interview. S4 stated the formula on the post-interview but on the pre-interview the formula was only stated as he used it in solving the third task.

S4 discusses graphs of quadratic functions on both the pre- and post-interview:

> …it can also be drawn as lines on a graph (pre)
>
> The graph of a quadratic equation makes a… bowl shape (post)

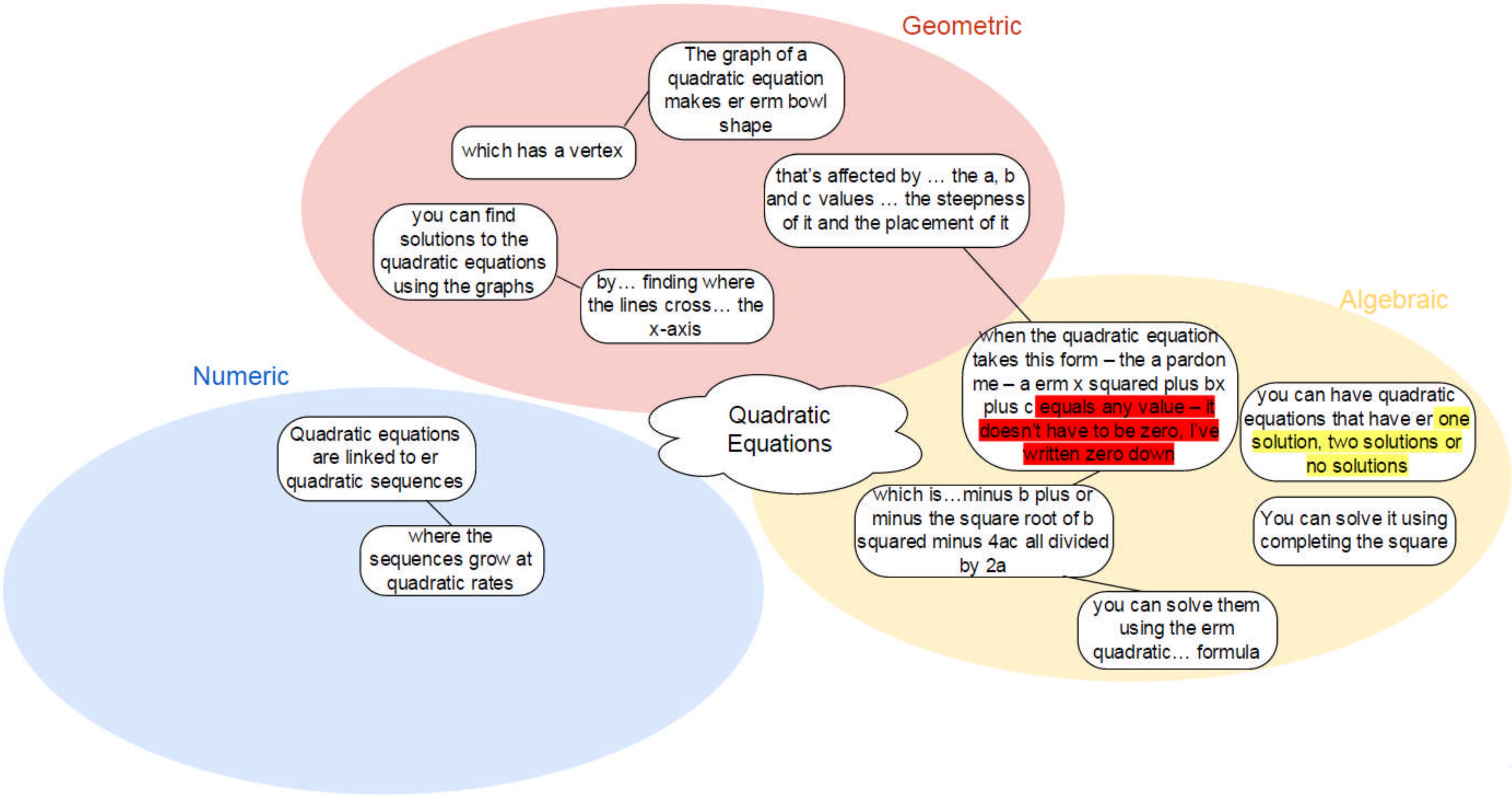

Figure 5.19: S4's post-interview Knowledge Map for 'quadratic equations'.

Thus, S4 is aware that a quadratic can be graphed, but the descriptions are not mathematically accurate. Indeed, when probed on the pre-interview by the interviewer as to the specific name of the shape of the graph, S4 eventually states: “parabula [sic]” (pre). The name is not mentioned on the post-interview to allow comparison.

S4 demonstrates his awareness of some links between the algebraic form of the equation and the corresponding graph on both pre- and post-interview:

> …so the C would move it ... up or down.. on the axi- on the.. y axis. (pre)

> …that’s affected by - the a, b and c values with how that shape’s – the steepness of it and the placement of it (post)

S4 states that there is either one, two or no solutions to a quadratic equation on both the pre- and post-interview. On the pre-interview, the interviewer told S4 that in the ‘one’ solution case, mathematicians said it was a repeated solution and when the graph doesn’t cross the x-axis: “in maths we say it does have a solution here but it’s a bit weird… it has a solution in another number field”. The interviewer then drew out the response of “complex number” from S4. On the post-interview, S4 goes into more detail about the solutions to a quadratic equation, linking this to the discriminant and also where the graph of the quadratic crosses the x-axis:

> …if the discriminant equals zero, then there’s only one solution – because you’re plussing or minusing that value. If it equals more than zero then you’ve got two solutions. If it equals less than zero - you can’t have it equal less than zero so there’d be no solutions ‘cause you can’t find the square root of a value less than zero unless you take into account complex numbers. And also you can use the graph as well because you can look at how – how many times the line meets or crosses the x-axis. So if the line of the graph – the quadratic equation doesn’t – doesn’t cross the x-axis then it has no solutions. If it just meets it once then it’s got one solution. If it crosses it twice, it’s got two solutions (post)

5.2.5 Student 5 (S5)

5.2.5.1 Squares

The Knowledge Maps for the pre- and post-interviews with S5 are similar in that both feature a consideration of squares in terms of shape and number and these domains are seen as linked (**Figure 5.20**, p.130 and **Figure 5.21**, p.130). There are also similar numbers of nodes and links: three nodes with two links and five nodes with two links on the pre- and post-Knowledge Maps respectively.

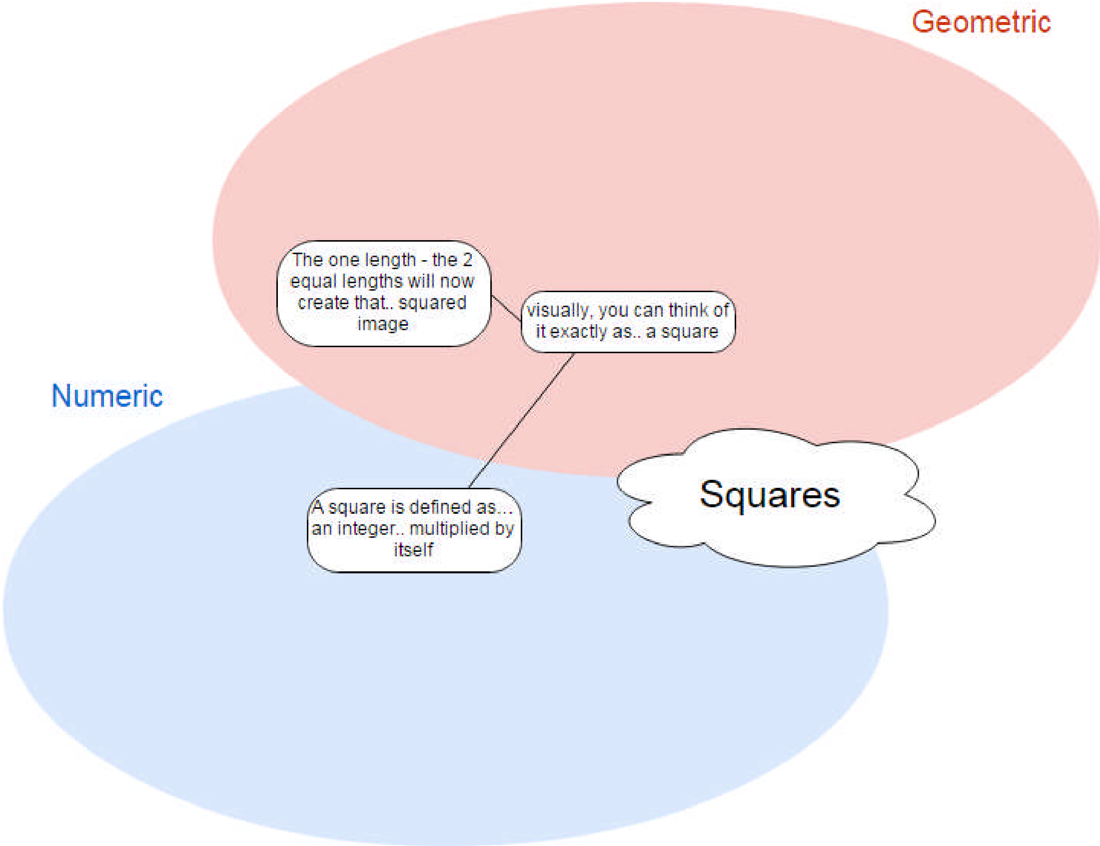

Figure 5.20: S5's pre-interview Knowledge Map for 'squares'.

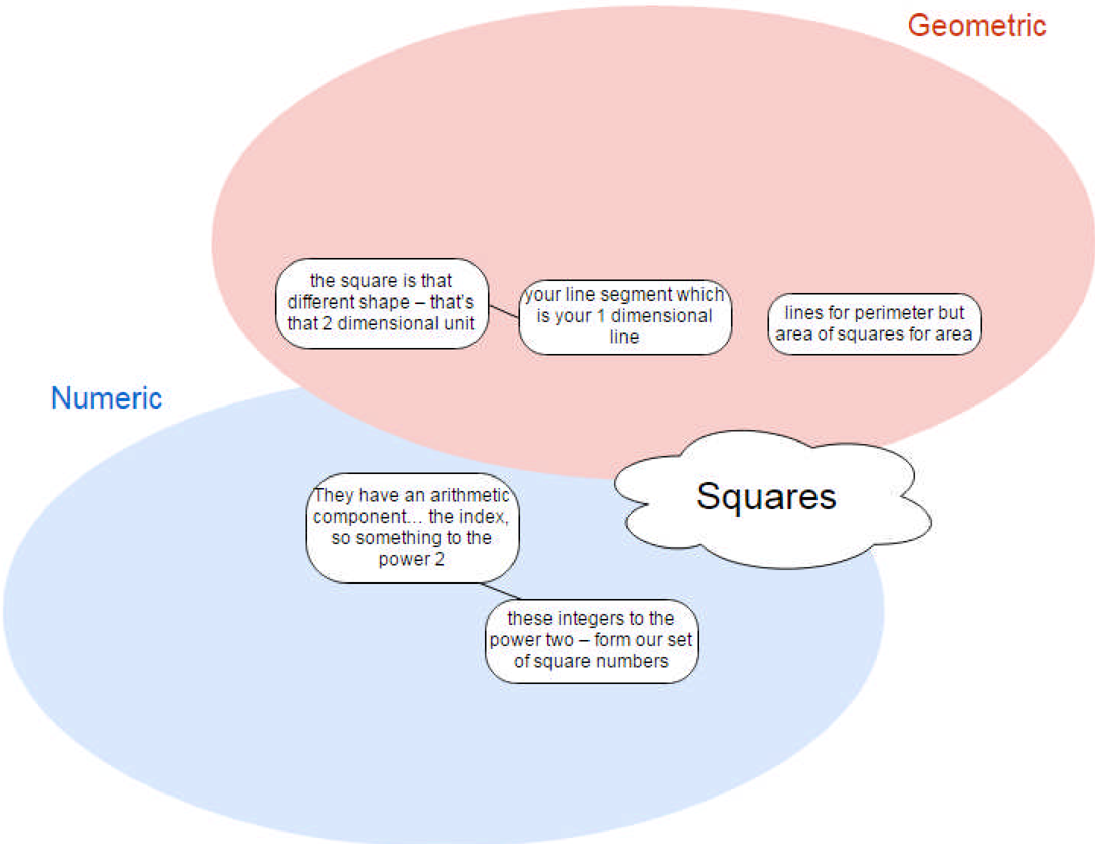

Figure 5.21: S5's post-interview Knowledge Map for 'squares'.

These Knowledge Maps are quite sparse compared to the other students' Knowledge Maps for squares but this is because it was difficult to pick out discrete statements made by S5 since his knowledge of squares was highly 'compressed' (Ball and Bass, 2000, see 2.1.5.2, p.22). That is, statements made were interlinked together in a complex way which was difficult to unpack. For example the following response:

> …so when thinking of a square vs something like a cube what we're considering is ultimately a 2-dimensional surface.. and to be honest I don't know if that surface changes outside of Euclidean geometry – that's that's where I'm comfortable. But the... I'm not sure if the square is the simplest representation but it would be the most.. common - yeah as opposed to a cube which would create a volume or a hypercube which would be a 4th dimensional (pre)

Here, the node 'a square is a 2-dimensional surface' would not quite capture all of what S5 is trying to convey since this statement not only has reference to higher dimensions (cubes, hypercube) but also alternative geometries.

The nodes on both Knowledge Maps relate to defining the concept of square rather than providing specific properties of the square shape or square numbers like the other students. In my view, this shows a deeper conceptual understanding of 'square', independent from properties of specific representations, for example, edges and angles as properties of the geometric representation of a square (we cannot talk about 'edges' when discussing square numbers). This creates a potential problem when drawing Knowledge Maps and will be discussed further in section 7.3 (p.230).

Comparing all S5's responses (prompted and unprompted) between the pre- and post-interviews, there was a broader knowledge of squares demonstrated in the pre-interview than the post-interview. For example, S5 considered non-Euclidean geometries and higher dimensions on the pre-interview (see quote above), whereas only Euclidean geometry is considered on the post-interview and dimensions are restricted to one and two dimensions:

> …the square is that different shape – that's that 2 dimensional unit, if you will versus just your line segment which is your 1 dimensional line (post).

When considering area of a square, S5 states on the pre-interview:

> …it will have both a finite area and a finite perimeter (pre)

The inclusion of the word 'finite' suggests knowledge of shapes with infinite areas such as the Sierpinski sponge. This word is omitted in the post-interview. However, the post-interview goes into greater depth regarding how the area of squares relates to finding areas of other shapes:

> …the area contained inside for cuboids - your sort of base times height – two opposing sides, will then give you the number of unit squares within. If you think of like having a sheet of grid paper and you're just counting the squares that it contains and it's the shapes – as their boundaries become more complicated, for example circles, you now have new calculations of area, but they're still in cm squared. So we're still – almost- thinking of how many of these little tiny squares can we fit inside the circle (post)

Perhaps this change in statements regarding area reflects the knowledge needed regarding area for teaching mathematics at secondary level, that is, students do not need to know about shapes with infinite area but they do have to calculate areas of different shapes, thus a knowledge of how areas relate may be useful.

A further difference between pre- and post-interview when considering area and perimeter of a square is that on the post-interview only, S5 converts the methods for finding them into algebraic form: "so area: x squared, perimeter: $4x$" (post).

On the post-interview, S5 mentions symmetrical properties of a square (without being explicitly prompted). These are not mentioned on the pre-interview. Again, symmetry is a topic to be covered on the school curriculum so is a possible explanation as to why this was mentioned more readily on the post-interview after teaching experience.

5.2.5.2 Rational numbers

Looking at the Knowledge Maps for the pre- (**Figure 5.22**, p.134) and post-interviews (**Figure 5.23**, p.134), there are a few differences. The main difference is that S5 refers to decimal representations on the post-interview but not the pre-interview. There are also differences in the number of nodes and links: three nodes with one link on the pre-interview and six nodes with three links on the post-interview.

Although S5 defines a rational number on both the pre- and post-interview, this is more clearly defined in the post-interview:

> …our n and m are both finite (pre)
>
> …integers N over M where M isn't zero (post)

On the pre-interview only he mentions that the set of rational numbers is smaller than the set of irrationals. On the post-interview only he says there are “limitlessly many” rational numbers and mentions mixed numbers “A N over M”.

Looking at all responses in the pre- and post-interviews (both prompted and unprompted), there were some things mentioned in the pre-interview only, but more things were mentioned in the post-interview only.

On the pre-interview, the names of the numerator and denominator as the top and bottom numbers in a fraction were explicitly provided and S5 went into depth about why we cannot have zero on the denominator of a fraction:

> …our top can be any but our denominator – our fraction breaks down if that’s zero because we don’t have a good system of saying how many zeroes go into a number that isn’t zero or any number basically. So we do we do want to avoid that case. It gives us something that we can’t represent (pre)

On the post-interview only, S5 explains how the decimal representation of number relates to fractional representations. No other student does this:

> Just beyond the decimal is sort of fractional place value so beyond one, there’s tens, hundreds, as we increment one way which we see in this sort of notation, we see how tenths, hundredths, thousands etc. (post)

He also makes the following statements on the post-interview only:

> You can’t take the square root of a negative number, that’s for complex numbers (post)

> you can’t simplify some of the surds like root two because it’s an irrational number (post)

S5 makes a conjecture within the post-interview:

> A rational number divided by any other rational number yields a rational number, whereas it’s not true for integers [pause] 2 divide by 2 is an integer, but divide by two is still rational, I think that’s right. Not certain of it though (post)

A misconception that there are more rational numbers than integers is made within the post-interview and he attempts to justify this with the following explanation, although he does express some doubt about his explanation:

> I think the proof is that you can map one over N for all N and so all of the… one over N from [inaud.] fits between… one and minus one, and then you have all of the other rationals outside of it, I think that’s the proof. Not a 100% sure about that one. It’s been a while (post)

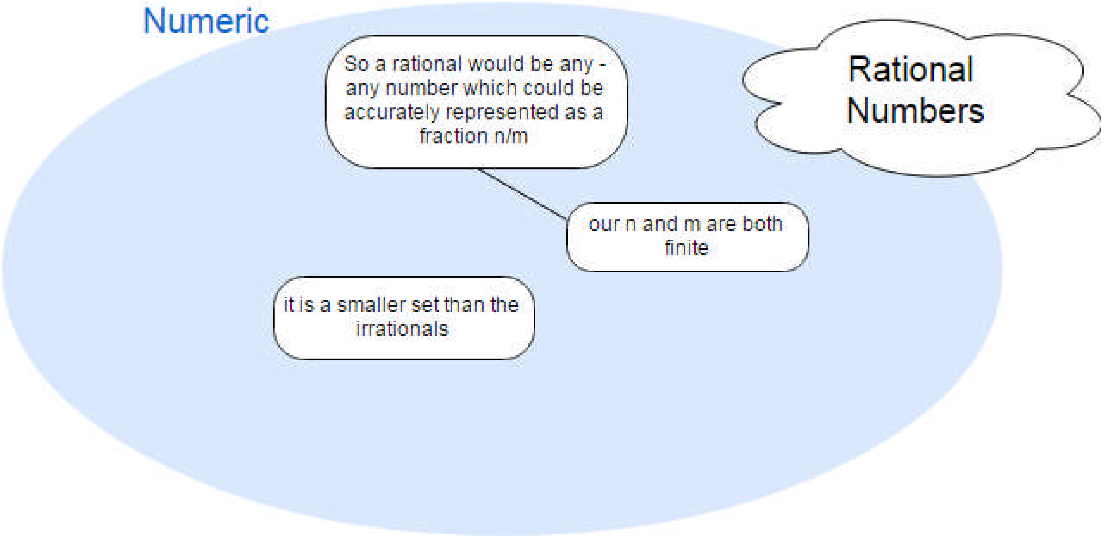

Figure 5.22: S5's pre-interview Knowledge Map for 'rational numbers'.

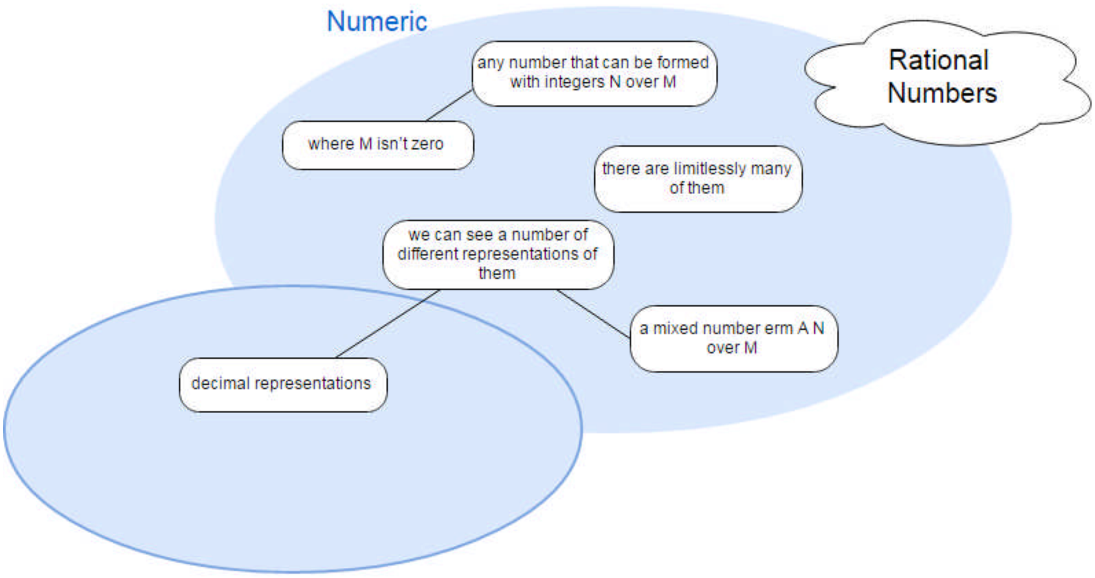

Figure 5.23: S5's post-interview Knowledge Map for 'rational numbers'.

5.2.5.3 Quadratic equations

A main difference between the pre- and post-interview Knowledge Maps is that the post-interview includes the geometric domain as S5 considers graphical representation of a quadratic equation. Clear links are also seen between this geometric domain and the algebraic, as S5 considers how the coefficients of the equation affect the shape of the graph. There are a greater number of nodes (13) on the post-interview than the pre-interview (10). There are also a greater number of links on the post- (8) than the pre-interview (5).

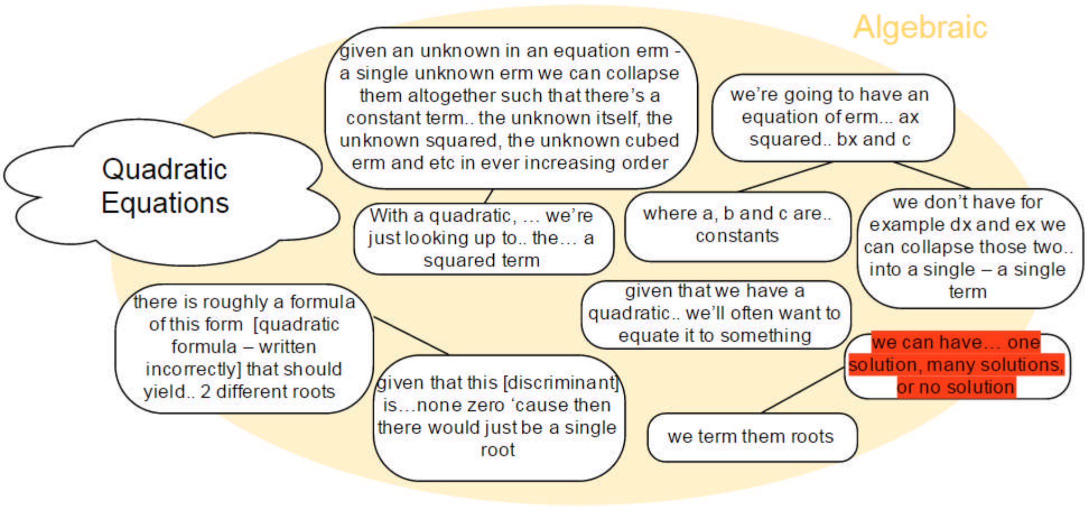

Figure 5.24: S5's pre-interview Knowledge Map for 'quadratic equations'.

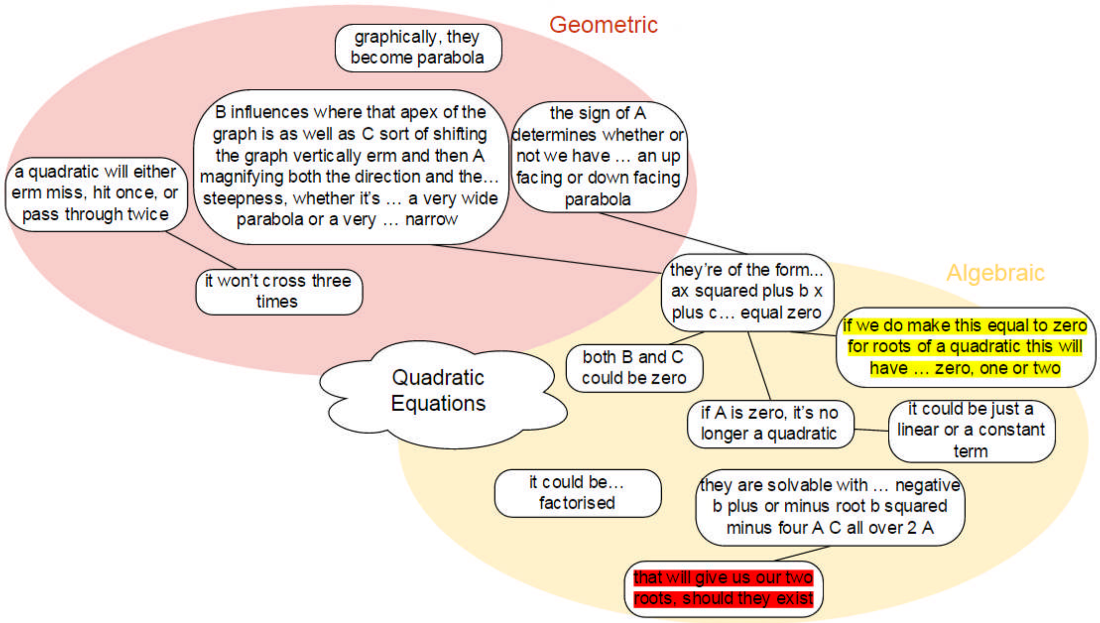

Figure 5.25: S5's post-interview Knowledge Map for 'quadratic equations'.

Considering all responses (prompted and unprompted) there were no statements made on the pre-interview that were not mentioned on the post-interview regarding quadratic equations. However, on the post-interview, the following additional ways of solving a quadratic equation were stated:

> …it could be I guess– factorised… there we go so that it goes… – no, this isn't right. Bx plus a, cx plus d er with no relation to the above a, b and c (post)
>
> …there's always by inspection if the numbers are easy (post)

On both the pre- and post-interview, S5 defines the algebraic form of a quadratic equation as: "ax squared plus bx plus c". However, on the pre-interview, this is preceded by an explanation of equations of higher order and then he states how quadratics fit into this overall picture:

> …equations can be collapsed into... I've forgotten the term for this.. I want to say orders, that given an unknown in an equation - a single unknown - we can collapse them altogether such that there's a constant term… the unknown itself, the unknown squared, the unknown cubed… etc. in ever increasing order. With a quadratic, I want to say we're just looking up to.. the - a squared term (pre)

He further explains how for general equations, we can treat two like terms:

> …for example dx and ex we can collapse those two.. into a single – a single term (pre).

This could be described as a 'top-down' approach by considering higher orders and then focusing in on quadratics specifically. On the other hand, the post-interview could be described as a 'bottom-up' approach to defining quadratics:

> …the key with a quadratic is this x squared term. So both B and C could be zero, but if A is zero, it's no longer a quadratic, it could be just a linear or a constant term (post)

Here, S5 considers just constant, linear and squared terms and selects from these the terms necessary for a quadratic (squared term).

On the post-interview only, he mentions equating the terms to zero:

> …and it's gotta be an equation [writes $y =$] not y equals, what am I doing? Erm but to make equal zero (post)

In the pre-interview, S5 does not mention what the terms are equal to, only that: "given that we have a quadratic.. we'll often want to equate it to something". On the post-interview, he links equating the equation to zero to finding solutions:

> …so if we do make this equal to zero for roots of a quadratic, this will have between zero and 2 solutions, so zero, one or two (post).

Solutions to a quadratic are also mentioned on the pre-interview, but there is some contradiction in the number of solutions:

> …when we have an equation like this with an unknown, we can have one solution, many solutions, or no solution.. and if I remember correctly, we term them roots.. so a quadratic would have roots that – again, zero, one or none – that it could be possible than in order to balance this equation there would 2 possible roots, 2 possible solutions (pre)

This confusion over number of roots is also manifested in the later responses regarding the quadratic formula and graphing a quadratic. S5 attempts to state the quadratic formula on both the pre- and post-interview:

> [writes **Figure 5.26**, p.137] that doesn't look right, it looks close, but it doesn't look right.... I have no confidence in this but there is

> roughly a formula of this form that should yield.. 2 different - 2 different roots given that this [discriminant] is I believe none zero because then there would just be a single root... but yes there is an equation for solving these (pre)

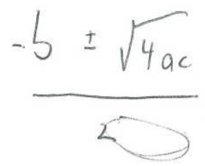

Figure 5.26: S5's attempt at writing the quadratic formula (pre-interview).

> …they are solvable with – I'm gonna mess it up- negative b plus or minus root b squared minus $4ac$ all over $2a$. 90% sure that that's right, I still have to look that up (post)

S5 correctly states the formula on the post-interview only, but he is not confident that he has stated it correctly. The interviewer tells him the correct form later on in the pre-interview in order to probe his understanding further.

On the pre-interview, again S5 considers the 'bigger picture' as he contemplates whether there is corresponding formula for solving equations of higher order:

> I can't remember if that is unique amongst the orders, I can't remember if there is a.. if there is a formula for third order of fourth order equations... I've not read that recently. No it's - no idea. (pre).

On the pre-interview, S5 discusses how the quadratic formula relates to the number of roots:

> if we have b squared equalling $4ac$ the… idea that we would get two roots we've now encountered a solution with a single root. (pre)

> Given that a is 0 we actually don't have a quadratic anymore.. so.. that seems to imply... no, nope that's that's what we can do... when $4ac$ exceeds b squared and we begin generating.. complex roots we no longer have a solution so now that's our zero… is that true?... or does that just mean they're complex roots? I don't know. I'm still uncertain of whether or not there's a no answer case. (pre)

> Well put it this way if we were limiting ourselves to real numbers then there is now definitely no answer… but whether or not there exist complex roots to quadratics, I don't know (pre)

On the post-interview, S5 also discusses how the quadratic formula can tell us the number of roots:

> …it can tell us when there's one root? Yes because it's going to resolve to the same root. Is that right? [pause] x squared has one root. A is one, B is zero, C is zero, yeah because the plus and minus will resolve to the same, same solution so it will tell us that we've only got one root. [pause]. Erm. I think that's true [pause]

not sure, not sure about that actually [long pause] er no, - we can't have sort of just one root, because that produces a linear equation, but we could have something that is the same root, like x plus 2 squared, so we only have one.. possible – or we only have one intercept on the x-axis but it still generates a quadratic. (post)

This confusion over the number of roots continues throughout discussion of graphs of quadratic functions on both the pre- and post-interview. Firstly, he states that the equations can be graphed:

…a second order equation... graphically might have.. an arc [draws **Figure 5.27**, p.138] if graphed against as.. as x changes.. the- the corresponding y (pre)

…graphically, they become parabola (post)

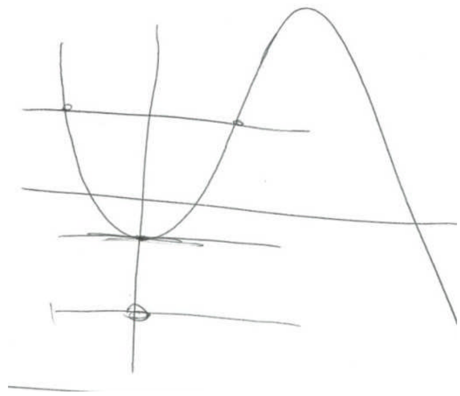

Figure 5.27: S5's sketch of a quadratic function (pre-interview).

On the pre-interview he then attempts to link the graph he has drawn (**Figure 5.27**, p.138) to the number of roots:

So maybe an example here, that if we were trying to find where this equation was... a certain y, there could be no answers but there is no intercept, that there is a single answer – a single point of intercept and then multiple answers... No I think I've got myself confused... That's just a squared term.... That's not really cor- I mean that's more a general.. I don't know. I don't know for this one (pre)

The interview (I) attempts to help:

I: but this does correspond with what you were saying about there's no roots, there's 1 or there's 2

S5: Well there's two, but I'm - to be honest I'm uncertain about that erm... I'm trying to think of - if there's a counter example, if ultimately, all of them have to cross all points or if there are actual... I don't think that's right, I don't think that's right at all. I do think you can have zero, one or none. But I don't - I can't think of the justification for that reasoning only that seems right to me. I don't have a better explanation than that (pre)

On the post-interview, S5 comments:

> …there's two but they're the same. The – because the root, my understanding of the root, is that it's basically the quadratic touching the x-axis, so the root is where our y equals zero, so the number of different solutions could either be zero, one or two, in a case like this one, we we only have one solution at x minus 2.. but in a graph like this one we might have x is two and x is three being our two different solutions (post)

The interview probes further on the post-interview to try to elicit his understanding of the number of roots:

> I: so you can have zero roots?
>
> S5: correct er what would be an example of that? That would be x squared plus four for which there is no – there's no rational x that allows us to have have this equation equal zero.
>
> I: so there are no rational roots?
>
> S5: yes [pause] er
>
> I: ok, so you can have- the number of different, rational roots can either be zero, one or two.
>
> S5: correct
>
> I: is that what you're saying?
>
> S5: yes, because now, now that that can of worms is opened I don't actually know how to deal with complex roots.
>
> I: oh ok, complex roots, right.
>
> S5: er I'm not even sure if that's a thing.
>
> I: ok
>
> S5: erm I get to rational space and I'm done. Irrationals at a pinch. Erm yeah, I'm not actually sure if you can this solvable with complex numbers – or, I imagine it's possible, I don't know how it's done (post)

On both pre- and post-interview, S5 seems unsure about complex numbers and perhaps this is the source of his confusion about the number of roots, since he understands there are cases with no real roots but that there has to be two solutions to a quadratic ("we can't have sort of just one root, because that produces a linear equation", post). This creates conflict for S5 which he cannot rectify without an understanding of complex number solutions.

Another difference between the pre- and post-interview is the depth he goes into when discussing the graph of the quadratic and how this is determined by the equation. On the pre-interview he only states that the equation can be graphed and what this will generally look like. On the other hand, S5 states on the post-interview:

> the sign of A determines whether or not we have sort of an up facing or down facing parabola… we can derive properties of the graph from these, B influences where that apex of the graph is as well as C sort of shifting the graph vertically and then A magnifying both the direction and the sort of steepness, whether it's sort of – a very wide parabola or a very shallow – mm narrow and wide, there we go, not narrow and shallow (post)

5.2.6 Summary table – Knowledge Maps

	Squares #nodes (#links)		**Rational numbers** #nodes (#links)		**Quadratic equations** #nodes (#links)	
	Pre	**Post**	**Pre**	**Post**	**Pre**	**Post**
Student 2 (SKE)	11 (4)	13 (2)	2 (0)	3 (1)	8 (4)	10 (7)
domains	G, N	G, N	N (fraction)	N (fraction)	A-G	A-G
Student 3 (Non-SKE)	8 (2)	9 (3)	9 (3)	7 (4)	5(3)	5 (3)
domains	G	G-N	N (fraction)	N (fraction–decimals)	A	A
Student 4 (SKE)	9 (2)	8 (2)	0 (0)	4 (0)	4 (0)	10 (6)
domains	G	G-N	-	N (fraction-decimals)	A, G	A-G, N
Student 5 (Non-SKE)	3 (2)	5 (2)	3 (1)	6 (3)	9 (5)	11 (8)
domains	G-N	G-N	N (fraction)	N (fraction–decimals)	A	A-G

Table 5.9: Table summarizing the numbers of (correct) nodes, links and domains present on the Knowledge Maps for each student

The summary table (**Table 5.9**, p.140) shows the numbers of (correct) nodes and links for each of the three topics for pre- and post-interviews. However, given the issues with drawing Knowledge Maps for highly compressed knowledge (as in the case of S5, see 5.2.5.1, p.129) these numbers should be treated with caution. That is, more nodes does not necessarily mean 'better or 'richer' knowledge, but knowledge which is 'decompressed' (Ball and Bass, 2000, see 2.1.5.2, p.22).

The summary table also shows the domains - number (N), geometry (G) and algebra (A) - featured on each map and whether these were linked (shown by a hyphen linking the two letters e.g. ‘A-G’ means the algebra and number domains featured on the map and were seen as linked, as opposed to ‘A, G’ which suggests they were distinct).

5.2.7 Problem solving

Part two of the interview involved solving three mathematical tasks related to the three topics discussed in part one, namely, squares, rational numbers and quadratic equations. **Table 5.10** (p.141) shows the marks for each student’s answers to the tasks. In the table, a tick refers to a correct answer, a cross refers to an incorrect answer or no correct answer achieved.

	Q1 pre	**Q1 post**	**Q2 pre**	**Q2 post**	**Q3 pre**	**Q3 post**	**Pre total**	**Post total**
S1	✓	x	✓	x	✓	x	3/3	0/3
S2	✓	x	✓	x	✓	✓	3/3	1/3
S3	✓	✓	x	x	x	x	1/3	1/3
S4	✓	x	✓	✓	x	✓	2/3	2/3
S5	x	x	✓	✓	✓	✓	2/3	2/3

Table 5.10: Table showing accuracy of responses to the mathematical interview tasks for each student

As can be seen from the table, three students got the same number of questions right on the pre- and post-interview tasks. Nevertheless, the correct answers were not always for the same tasks. For example, S4 got task 1 correct on the pre-interview but initially got it wrong on the post-interview. S2 got all of questions correct on the pre-interview but only one correct on the post-interview and S1 made mistakes on the post-interview meaning he got none of his previously correct answers right.

Thus, there does not seem to be a general pattern in responses to the interview tasks. There was not an increase in the number of correct answers over time. The reader is directed to section 6.2.2.3 (p.193) for details on further insights gained from the problem-solving part of the interview.

5.3 Observation results

5.3.1 Introduction

This section presents observation results from lessons taught by the PGCE students as part of their school placements. Three observations were made for each student throughout the PGCE year. See **Table 4.9** (p.96) for details on the context (times/dates and class) of the observed lessons. Results are presented for each student in turn. A lesson overview is presented for each observation, followed by a consideration of students’ mathematical knowledge as demonstrated in the lesson using the Knowledge Quartet (Rowland and Turner, 2007) to aid analysis of the video recordings (see Appendix B, p.272 for all codes).

I met with Tim Rowland at the University of Cambridge to receive further training and guidance on using the Knowledge Quartet (Rowland, 2012b). The following method (as used by Anne Thwaites) was outlined as one way to analyse observations using the Knowledge Quartet:

1. Write a detailed account of the lesson soon after the observation took place complete with times for each main ‘episode’ within the lesson.

2. Add to this detailed account any codes from the Knowledge Quartet which relate to the episodes. Not all episodes will have a code.

3. Select the more useful/ interesting episodes from the lesson description and write a more fluid description of what happened complete with some background/ context for the reader.

4. Add to this fluid account screen shots from the lesson and/ or transcript sections from the interviews where relevant.

5. Finally, cut down on the information and add references from other authors where applicable which may help to account for the behaviour observed.

The above method was adopted for this study. However, I have separated a general overview of the lesson from the Knowledge Quartet analysis. As in step 3 above, the more interesting episodes from the lesson are discussed with Knowledge Quartet codes indicated by single quotation marks and italicised (e.g. ‘*concentration on procedures’*). The frequencies of all codes given are featured in Appendix F (p.278).

5.3.2 Student 1 (S1)

5.3.2.1 Placement 1, Observation 1 (12/12/2012)

Lesson overview

The lesson objectives were to: "Simplify expressions with powers. Expand brackets with powers. Simplify and expand with powers”. There was a starter on the board for pupils to work on when they arrived with three questions involving simplifying, expanding and combining the two (simplifying and expanding) for linear expressions. Pupils who had finished were verbally instructed to think how they would define the terms ‘simplifying’ and 'expanding'. The solutions to the starter problems were demonstrated on the board with input from pupils.

Pupils were then asked to 'guess' what the answer to $d(4t+3)$ was (they had not seen an expression of this form before) and to write their 'guesses' on mini whiteboards. After discussing one pupil's (correct) answer to this question, S1 demonstrated the solution on the board whilst writing the pupil's answer in conventional form (with numbers before letters). The rest of the lesson follows this format, with other expressions provided on the board for pupils to expand, pupils then wrote their answers on whiteboards and S1 demonstrated the solutions on the board with pupil input. The questions developed to expanding brackets with negative numbers and finally introducing powers.

Knowledge Quartet Analysis

Throughout this lesson there was evidence of S1 '*concentrating on procedures*' as identified by the Knowledge Quartet framework. When talking the class through the method for expanding an expression, S1 explained:

> d times $4t$ is $d4t$ but that's – but we write it like this, we write the number first. So we get $4dt$ plus $4d$. So we multiply a d by $4t$, $4dt$, d times plus 3, plus $3d$.

Here, S1 tells the pupils what the procedure is for expanding this expression, whilst it is clear S1 understands the mathematical convention of writing the number first, he does not communicate this to the pupils. This can again be seen when a pupil states '$4sm$' as an answer and S1 writes '$4ms$' on the board. When this is questioned S1 states: "OK, so we've not necessarily clarified that, so 4 times m times s and four times s times m are equivalent." No explanation as to why this is the case is offered.

Other statements demonstrate S1's focus on conventions:

So let's just write this a bit neater, ok so we've got $d4t$ but we don't like to put numbers halfway through the expression.

> …there was a $4dt$ and a $3d$ so we can't combine them, we can't combine different letters and we can't combine different combinations of letters so if we have a dt we can't group that with t's.

> What's a d times a d? How can we write that even better?... what's being timesed by itself?... d, so we've got $4d$ squared, plus $3d$. That is the powers bit. If you've got a letter timesed by itself, it's that letter squared. So you've got $4d$ squared plus $3d$ there. You can write it out like this first if you want [indicates top line $4dd + 3d$] but want to make it nice and slick. Nice and efficient.

Here S1 hints at possible reasons for these seemingly arbitrary rules: to make it 'neater', 'efficient' and 'slick'. The following exchange demonstrates that S1 does understand the reasons why. A pupil asks if '$4dd + 3d$' could be written as $7d$ and S1 explains: "two D's multiplied together is different from one D. They're not like terms, they're not comparable, we can't add or subtract them".

However, S1 may have had difficulty 'unpacking' this knowledge to explain to the pupils. Conversely, S1 described a pupil's (correct) answer to the expansion (though it was written unconventionally) as "closer to the truth" rather than as being correct.

Additionally, when asking pupils to expand $-d(4e + 3)$, one student gives: $4 - de - 3d$ as the answer. S1 responds as follows:

> Now, the way that is written, it isn't correct. What you might have meant might have been correct. Let's go over what I mean by that. If we put a bracket here [writes $4(-d)e - 3d)$ on the board] four times minus d times e then subtract three times d then that would be correct but it's confusing. So I'm gonna go through an easier way of doing this. So [crosses out expression] what we're gonna think about is minus d times $4e$. It's gonna produce a minus, ok, because we've got something negative being multiplied by something positive. A minus times a positive, is a negative, so we've got a minus, and then we've got a d times a four times an e.

Here, S1 seems to understand the pupil's thinking behind his answer but again '*concentra[tes] on procedures'* and on conventions with only hints at reasons behind these.

One pupil's explanation of how he arrived at his answer for expanding $S(4 - 3m)$ reflects the '*concentra[tion] on procedures'* approach that S1 has taken:

> Pupil: I think I've found a way to do the thing – you swap the bracket and the four round and put the s where the four was so then it'd turn into $4s$ and then if you do it on the other side it'll turn it into the $3sm$.
>
> S1: OK, I can see what you're saying, formally, we're multiplying through. Formally we're thinking multiplying through by the s. But that one will work in these- what you've done. Let's see if it works a little later on.

Here, it seems the pupil sees the process of expanding brackets as simply swapping and moving the letters, numbers and brackets around rather than a process of multiplying through. S1 manages to evaluate the pupil's idea and it seems he knows that this unconventional method will work here but is either unsure if it will work in all cases or thinks that it will not but wants the pupil to discover this for himself.

The above analysis suggests that S1 had no misconceptions or lacked any understanding of *how* to expand and simplify expressions. Indeed, he demonstrated a sound knowledge of *procedure* and mathematical conventions. However, it seemed this knowledge was ‘taken for granted’ as the *reasons behind* the steps in the procedures and/ or conventions were not fully explained to the pupils.

For this lesson, the expressions offered showed ‘*decisions about sequencing*’ as identified by the Knowledge Quartet as they increased in difficulty in line with the National Curriculum levels. Indeed as S1 explained to the pupils: ‘the first question on the starter is level 4, simplifying, the second is level 5, expanding, and the third is level 6, putting it all together’. The expressions also progress from linear to quadratic and within these categories from all positive terms to multiplying with negative terms. S1 was also able to ‘*deviate from agenda’* and write other expressions on the spot for the pupils to expand upon seeing that the class were generally not ready to move onto the next more difficult expansion.

For the final part of the lesson S1 demonstrates his knowledge of his ‘*connection between [the] concept[s]*’ of a number squared and the area of a square shape. He asks the class: “…why do they call it squared? Why not two-ed? Why squared, are mathematicians just odd?”. When no pupils can offer the response he is looking for, he explains: “I like to think of it as the area of a square, ok so T times T would be the area of a T length square.”

5.3.2.2 Placement 2, Observation 1 (22/04/2013)

Lesson overview

The lesson was on Pascal’s Triangle which was not within a scheme of work at the school but S1 had previously taught this lesson as part of a job interview and decided to teach it. The lesson focused on one particular application of Pascal’s Triangle – calculating how many colours living organisms can see depending upon the number of cones they have in their eyes. Pascal’s Triangle emerges when tabulating the data. There was time left at the end of the lesson, so S1 discussed a further application of Pascal’s Triangle – expanding brackets. S1 asked the students to multiply out $(x+1)^2$, then $(x+1)^3$ and see if they could see a pattern. The method they had previously been taught (by another teacher) to expand brackets was the FOIL method (multiply **F**irst terms, **O**uter terms, **I**nner terms, then

Last terms together). During the lesson, the pupils found it difficult to expand $(x+1)^3$ using the FOIL method as it is not intended for higher terms. Only written notes were taken during this lesson as permission to record audio and video was not granted. Subsequently, it was difficult to apply the Knowledge Quartet in analysing the lesson due to insufficient data on episodes within the lesson. Instead, S1's reflections on the lesson during the post-observation interview are provided.

Post-Observation Interview Analysis

During the post-observation interview, S1 evaluated the methods for expanding brackets which were featured in the lesson: FOIL and the grid method (**Figure 5.28**, p.146).

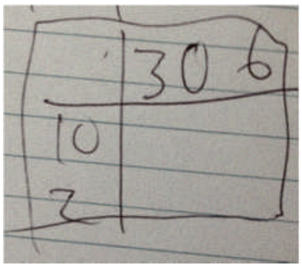

Figure 5.28: S1's example of the grid method for multiplying 36 by 12 (respective numbers are partitioned and multiplied separately before adding the four totals).

S1 is scathing of the FOIL method: "which is a terrible method, it's even worse than I thought it was because it ascribes meaning to something that's meaningless… it really makes no difference what order you do them in". Therefore, S1 is critical of a method which places emphasis on the order of multiplication when multiplication is commutative.

S1 prefers the grid method, which he explains as follows:

> …you can partition numbers the same way you partition an x bracket and that comes from the idea of a rectangle – finding the area of it – so you could have a length x and a length one, and then that could be length x and length one [draws diagram - Figure 5.29, p.146] – so that'd be a square – a special rectangle

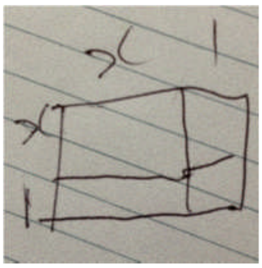

Figure 5.29: S1's diagram of how the grid method links to area of a rectangle (in this case a square).

Here, S1 '[*makes] connections between [the] concepts'* of multiplication and finding the area of a rectangle. He also makes connections between regular rectangles and squares.

Further, he praises the grid method because: "it links with the idea of area of a rectangle and they've already learnt it with multiplying numbers". Thus, S1 sees the value of an algorithm which has connections to other areas of the mathematics curriculum (geometry) and also which follows through from multiplying numbers to multiplying out two brackets to multiplying brackets with higher powers.

5.3.2.3 Placement 2, Observation 2 (21/05/2013)

Lesson overview

The main part of the lesson involved making a solar cooker from cardboard and aluminium foil with an exam style question on parabolas as a starter and plenary.

Knowledge Quartet Analysis

This lesson demonstrated S1's '*overt subject knowledge'* of parabolas:

> …last lesson we were having a look at parabolas – x squared graphs - and we came to the conclusion that any light travelling in vertically is gonna bounce off at the same angle to the tangent and hit the focus and it's all gonna just converge on that point. And this lesson, we're gonna use that information to make a solar cooker. It's more of a model of a solar cooker, it's not really big enough to actually cook things but we'll find out about it. And the idea is the cocktail skewer will be running through the focus of the parabola.

This suggests S1 is not only aware of the name of an x-squared graph but is aware of how properties and applications of the parabola relate to real-life situations.

Further, he demonstrates '*overt subject knowledge'* of the different methods for solving a quadratic equation and is able to evaluate the effectiveness of these methods for different situations:

> Now we could use factorising, completing the square or quadratic formula to get that [the solution] exactly – apart from factorising won't really work here because it's a decimal so it'd be harder to work out. But it's only asking you for a rough solution.

Here, because the numbers involved are decimals, S1 recognises that using the method of factorising would be more time consuming and the question only asks for an approximate solution, hence using the values generated from a sketch of the graph is sufficient. This suggests, S1 knows that sketching a graph will give an estimate of the solutions where it crosses the

x axis, whereas the quadratic formula, completing the square and factorising can give exact solutions.

S1's knowledge of quadratics also extends to him '*making connections between procedures*'. Indeed, he asks pupils who have finished the starter problem to: "Figure out how [the graph has] moved from an x squared graph". This extension task was most likely planned in advance since he later goes over the answer to this problem which is pre-written on the board:

> Right, if we have a look at how this graph has moved, it's moved two to the right [the equation $(x-2)^2 - 2$ appears above graph on the interactive white board (IWB)] and two down, and that's using completing the square which I think you've had a look at before so I'm just mentioning that, it's moved two to the right and two down, which I believe you had a look at with Mr X.

Here, S1 connects completing the square (algebraic domain) which they looked at in the past with another teacher to the graph of the equation (geometric domain) and also translations of graphs (geometric domain).

Further, for the plenary exam style question on parabolas, S1 again makes '*connections between procedures'*:

> …basically you're trying to solve where this graph is equal to nought. So where is this graph equal to nought? [pupil response] Right, when it's nought, when y is nought… Right so y is nought along this line, this graph crosses that line…. That's a good estimate for that [the solution]. If you use the quadratic formula, you'll get it exactly. Right so write down what you think it is from that.

Within this extract, S1 makes connections between ideas, firstly between solving simultaneous equations (algebraic domain) and the corresponding graphical representation (geometric domain) of this (where the two lines cross) and also between estimating the solution of a quadratic equation (algebraic domain) by using a sketch of the graph and the 'exact' solution when using the quadratic formula.

S1 also encourages his pupils to make connections between the algebraic and geometric domains of parabolas. Indeed, to the pupils who have finished making their solar cooker before others he asks them: "Let's imagine you wanted to put a grid on it, can you think of an equation where you would have a focus sitting right across the top, how can we have a focus sitting right across the top?".

However, S1 does '*concentrate on procedures'* when helping pupils answer the exam style questions. In order to plot the graph of $y = x^2 - 4x + 2$, S1 advises pupils to: "plug the value of x to find the y values" and his later explanation to the whole class also '*concentrates on procedures'* for answering the question: "If in the exam, you have an equation to work out and you've been given it on the form y equals letters, you can take different values of x, put them into the equation, get your values of y, plot your coordinates."

This lesson also demonstrates S1's knowledge of mathematics as a discipline with the implicit reference to mathematical models. Indeed, when calculating the potential energy of the solar cooker, S1 stated:

> A hundred and 57 point 5 watts of power… this is if it was perfectly efficient, if all the light bounced off the tin foil and hit the food, right and if none of the power was used heating the air and all that sort of business. This won't be completely realistic because we're not going to design something completely perfect…

This suggests S1 knows that they are working with a mathematical model which is a simplified version of reality, that is, the calculation does not take into account loss of energy from heating the air and assumes that the sun is shining directly onto the cooker and perfectly reflecting all the light. This episode could be coded as '*overt subject knowledge'*. However, it is felt that this code does not quite capture the knowledge of mathematics as a discipline which was demonstrated. This issue will be returned to in section 7.2 (p.226).

5.3.3 Student 2 (S2)

5.3.3.1 Placement 1, Observation 1(13/12/2012)

Lesson Overview

The lesson was on finding the volumes and surface areas of prisms and cylinders. The starter on the board for the pupils to work on when they arrived was to find the volume of a Toblerone chocolate bar (triangular prism) but to concentrate on the steps taken to arrive at the answer rather than the answer itself. These steps were then written on the board and pupils were asked which steps changed or remained the same for finding the volume of a cylinder. Pupils then worked on exam questions set on the board for finding volumes and marked their own work using mark schemes. The same procedure was repeated but for surface area of prisms and cylinders. As a plenary, pupils fed back how confident they were with the methods they had learnt.

Knowledge Quartet Analysis

In this lesson, S2 made explicit '*connections between procedures'* for finding the volume and surface area of prisms and cylinders. These procedures were written on the wipe-board as a method to follow and amended as necessary. For example, for volumes, only the second step was changed to ‘find the area of the circle’ instead of ‘find the area of the triangle’. Moreover, S2 demonstrated his understanding of the underlying reasons why the methods for finding volumes of prisms and cylinders are related. He explained:

> So with a prism we had a triangle and that triangle goes all the way through so no matter where you cut the prism, you’ll have that triangle on the front. So our cylinder is similar, no matter where you cut it…

S2 makes good ‘*choice(s) of examples’* within this lesson. Firstly, three different prisms were put on the board for pupils to calculate their respective volumes and these prisms were in different orientations to prompt pupils to think carefully about the correct measurements they needed to use, not simply the relative positions of the measurements given. Secondly, the use of the Toblerone bar is a useful example since it relates to a ‘real-life’ object and the box can be opened out flat to demonstrate surface area (though this was not done). Finally, when discussing the procedure for finding the surface area of a cylinder, one pupil states that the circumference of the circle needs to be found. When other pupils do not grasp why this is, S2 refers to a paper model he has of a cylinder:

> OK, PupilX is telling us that the circumference of the disk at the top – of the circle at the top – is the same as the unknown length of the rectangle, ok. So this is going to form a rectangle, we’ll see in a second. We know that the height is 2cm [cuts the cylinder model] This was folded around - the circumference is going all the way round the circle isn’t it?… OK, when we unfold it, it forms a rectangle [demonstrates]. So our middle piece of that cylinder is in fact a rectangle.

S2’s ‘*choice of representation’* (a paper model of a cylinder) is highly appropriate for cutting open and physically demonstrating the resulting rectangle shape. Having this model to hand suggests S2 had ‘*anticipated the complexity*’ of this idea and had prepared accordingly.

With regard to ‘*overt display of subject knowledge’* (or lack of) there were two mathematical ‘errors’ made which were highlighted by S2’s university tutor and school-based mentor in the post-lesson meeting. Firstly, during the final part of the lesson, pupils had the choice whether to practice more questions on surface area and volume or to rearrange the formula (displayed on the IWB) to make ‘surface area’ and ‘height’ the subjects respectively:

$$Volume = surface\ area\ x\ height$$

The above formula should state ‘cross-sectional area’ rather than ‘surface area’ – but this was an error rather than a misconception I believe.

Secondly, for the process of finding the surface area of a cylinder written on the board, “add them up” should only refer to adding the areas. The point was raised afterwards by S2’s tutor that pupils may think they need to add the circle circumference too.

5.3.3.2 Placement 2, Observation 1 (26/03/2013)

Lesson overview

The lesson, which was on the laws of indices, began with four pictures (including a superhero and Barrack Obama) on the IWB. The pupils had to say what links the pictures (power/s). Following this, pupils had to evaluate the answers to questions involving indices on the board. The main body of the lesson involved pupils walking round the room to look at five sets of coloured cards stuck on the walls. Each set contained expressions involving indices and the pupils had to: “spot the pattern, try and see if you can work out a rule, see what’s going on in each group, see if there’s a pattern there”. For example, $5^2 \times 5^3 = 5^5$ along with other similar equations would lead to the general rule $a^b \times a^c = a^{b+c}$. These rules were then restated on the board by the teacher with input from the pupils. S2 chose to write down what pupils said (even if incorrect) and allowed the other class members to correct the rule if needed. Clues to a sixth rule ($x^0 = 1$) were given on all five sets of cards and this was also discussed. During the second half of the lesson, pupils worked on text book exercises set by S2 using the rules they had written on the board whilst S2 helped individuals. For the plenary, S2 showed expressions on the IWB with several possible coloured answers. The pupils had to hold up the corresponding coloured card from their individual set of cards to indicate their answer to the question.

Knowledge Quartet Analysis

There are three instances within this lesson which demonstrate S2’s knowledge of different representations of numbers and this helped him to flexibly deal with pupils’ difficulties. These instances could be coded as ‘*choice of representation’* or alternatively, ‘*connections between concepts’* (to be discussed further in section 7.2, p.226).

Firstly, during the starter, the expression to be evaluated (0.5^2) causes difficulty for one pupil:

PupilC: I got minus 1

S2: minus 1

PupilC: I don't think it's right

S2: Anybody else get minus one? [no answer] Tell me what you did… Have a think, when we see squared, what does it tell us? PupilD?

PupilD: multiply by itself

S2: OK, so nought point five times nought point five? If it's easier we can think about it as fractions. If we thought about it as fractions what would it be?

Pupil responses: 'quarter' 'half'

S2: A half times a half?

[pupil response]

S2: zero point two five, ok

This demonstrates S2's knowledge of the connection between decimal and fractional form of rational numbers and he makes this connection for his pupils.

Secondly, when helping individuals with text book exercises, one pupil is stuck on $\frac{x^2}{x^2}$ as they know "it'd be x zero" but are not sure whether the final answer is zero or one. S2 demonstrates connection between representations by asking: "if you take something and divide it by itself, what do you get?" This re-presentation of the question helps the pupils to understand.

Thirdly, the final question in the plenary is a number raised to the power 0.5. Some pupils struggle with this and S2 demonstrated '*choice of representation*' with this question as he deliberately did not use the typical fractional form of one half so that pupils would think about the connections between decimals and fractions:

How else could we write a power of zero point five? What do our fractional powers tell us? … What root is it gonna be? If it's to the power a half?

Here S2 wants pupils to convert 0.5 to ½ in order to use the 'rule' written on the board for 'fractional powers'.

S2's flexibility with dealing with multiple representations and the connections between them is further demonstrated by making '*connections between procedures*'. Indeed, by providing an example of $4^{\frac{3}{2}}$, he asks the pupils which operation they would perform first – finding the square root or cubing 4 - and explains that both are equivalent but one may be easier to calculate. He also made use of brackets to indicate the different orders of calculation.

S2's foundation knowledge of mathematics as a discipline ('*overt display of subject knowledge*') is demonstrated through the following discussion which took place after one pupil correctly identifies the sixth rule: 'something to the power of zero is always one?'. S2 writes the following on the board in response:

$$x^0 = 1$$

$$0?$$

He asks: "what about zero to the power zero?" In '*response to [the] children's ideas*' of 'zero' and 'one' S2 replies:

> I'd say, it's zero but I think it's an issue that some people might disagree with alright? Some people might try and argue that it's one, but I'd say it's zero… Anything to the power zero is one, apart from – most probably – zero, because some people might argue, alright… We had a discussion about this at uni once and some people tried to argue that it was one. I'd say it's zero.

This demonstrates S2's foundation knowledge that there is an exception to the general rule when it comes to zero to the power zero which he is aware of as a result of discussions in a taught session on the PGCE course. However, it is not obvious that S2 fully understands why there is debate over the answer and that it is mathematicians who ultimately have this debate and set conventions rather than 'uni' students.

5.3.3.3 Placement 2, Observation 2 (26/04/2013)

Lesson overview

The lesson (on solving equations) began with a starter on the IWB with some expressions for the pupils to expand in their books. They had looked at expanding expressions a few weeks ago. After giving sufficient time to complete the three questions on the board, he asked individuals to talk through their answers and method of solution. Two further questions were then provided on the board as a means for pupils to determine which set of questions (given on worksheets) to focus on during the lesson so that pupils could work at their own level. S2 helps individuals with their worksheets. At intervals during the lesson two more sets of questions are put on the board for pupils to keep gauging their progress and to help them determine whether they are ready to move on to the next worksheet or need more practice. Those who had finished both sheets were given an extension task.

Knowledge Quartet Analysis

Throughout this lesson, S2 helps individual pupils to solve equations by '*concentrat[ing] on procedures'*. He gives instructions such as:

> …everything outside, we multiply by inside.
>
> Think about the steps. Get the letters on the left, numbers on the right.

These statements focus on the mechanical or algorithmic steps to reaching an answer. His assistance also takes the form of referring back to previous examples. This was also the case when going through the answers to the questions on the board with pupil input:

> S2: We're doing the opposite ok, so if we've got a minus five, we're gonna add on five. If we add a positive six, PupilI, how do we get rid of that? If we had positive 6, if that said $3f$ plus 6, how would we get rid of that plus 6?
>
> PupilJ: minus 6
>
> S2: minus 6 so we do the opposite
>
> Pupil: how come you do the opposite?
>
> S2: To get our letters by themselves, ok?

In this exchange, S2 focuses on the steps and 'rules' for solving the equation and the reasons provided for 'doing the opposite' are also arguably procedural rather than conceptual: "to get our letters by themselves". Further, one does not eliminate the terms in an equation but rearranges them into a form which will enable a solution to be seen, again demonstrating a focus on procedures rather than concepts.

Another example of '*concentrat[ion] on procedures'* is with pupils who have finished the worksheet and are given some further equations with more than one algebraic term as an extension. S2 explains:

> …when we're solving this equation [one they have been working on] what are we trying to get on our left?... The letters on the left, what do we try and get on our right?... Ok, what do you notice about these [the extension] questions? [pupil response: there's letters on both sides] what letters have we got on the right hand side? How many?... two. Ok. If we've got – what's the number before it?... That means we've got two of those x's. We've got $2x$ – plus $2x$ - on the right hand side, how do we move that over to the left?... Times it?... plus two x so we're gonna take two x off… We can only take x's off x's and numbers off numbers… Write down all the steps. We need to get the numbers to the right hand side.

Again rules such as ‘taking x’s off x’s’ are procedural in nature and do not relate to the underlying concepts. However, S2 makes ‘*connections with the procedure*’ learnt for solving linear equations with one x term, and extends this to equations with x terms on both sides of the equation. Linking these extension questions with the type the pupils are already familiar with is a positive thing.

There are some instances which hint that S2 understands the concepts involved. Firstly, whilst helping individuals on the worksheets, S2 demonstrates ‘*choice of representation’* as he re-presents part of a question ($12x = 0$) to a pupil: “if twelve lots of something is zero, what’s one lot?”. This re-phrasing of the algebraic equation into a word problem helps the pupil to complete the question. He does the same for several other pupils in the class with different (but similar) questions and in one case, he also refers to physical objects: “so $2x$ equals 8, what does one x equal?... That’s saying we’ve got two lots of something, I’ve got two bags, and the sum of those bags is eight, so if we had only one of the bags…”

With another individual he demonstrates ‘*connection between procedures*’:

> …right, I want you to write it as a fraction, ok? So write ten as a fraction. I’ll show you where we’re going. You’re telling me to divide by twenty so that’s ten divide by twenty, how do we cancel that fraction down? [listens] half? You’ve told me. Half…

Here he connects dividing ten by twenty with the fraction 10 over 20 which can be cancelled to give one half.

Perhaps S2’s decision to focus on the procedures and ‘rules’ is due to the ability of the class. Indeed, S2 commented that this was a low ability group who in the previous lesson struggled with the question: “If 2a = 6, what is a?”. He also indicated there were difficulties with behaviour and absences in the class.

5.3.4 Student 3 (S3)

5.3.4.1 Placement 1, Observation 1 (27/11/2012)

Lesson Overview

The focus of the lesson was ‘dividing mixed numbers’ and the starter involved the pupils answering 8 questions set on the IWB which involved converting mixed numbers to improper fractions. Afterwards, S3 reviewed the answers on the board before moving onto the main section of the lesson. He demonstrated how to divide mixed numbers on the board by first converting to improper fractions, then using the acronym LET (**L**eave the first number, **E**xchange the divide for a multiply and **T**urn the second fraction upside down).

Pupils then worked on questions individually in their books. After marking their own work, they could then select either a set of ‘red’ questions or ‘green’ questions from the board depending how confident they felt.

The final activity was working on questions from a grid which stated which level each question was at. The questions went up to a C grade – the target grade of the class.

Knowledge *Quartet Analysis*

S3 ‘*concentrate[d] on procedures’* when teaching pupils how to divide mixed numbers. Indeed, for converting a mixed number to a fraction, he instructed pupils to: “times the number by the bottom number and add to the top”. Further, when dividing the fractions, he said:

> Can anyone remember the word we used when we were dividing? PupilN? [PupilN: LET] LET. And what did LET mean? We leave the first one, [pupil: leave it, exchange, turn] thank you very much guys, can we not shout out. So we leave that – keep the 50 over 9, leave that alone. We exchange the divide sign by a times sign, and we turn this one over don’t we? ok? So, how do we multiply fractions, PupilO?... What do we multiply together? PupilP do you want to help him out? [PupilP: multiply both tops] multiply both tops.

Here the steps to solving the question are emphasised and reinforced by the mnemonic ‘LET’. This could be because this is the approach that the class are familiar with. Indeed, S3 commented in the post-observation interview that the LET technique was told to him by another teacher in the department.

The effects of ‘*concentrating on procedures’* is demonstrated by the confusion of some pupils when attempting to solve questions involving multiplying two fractions (only). Some pupils are not sure how to proceed with multiplying, despite this being one of the steps of dividing:

> S3: To do the divide ones you do the times ones, don’t you? So surely you can do times, times is easier than divide.
>
> Pupil: so don’t you have to swap it then?
>
> S3: [shakes head] Multiplying, you can just do it can’t you?

S3 later tries to ‘[*make] connections between procedures’*: “That’s why when we divide, we turn to multiply because we can do multiplying, ok?”. The pupils’ difficulty with the procedures is also demonstrated by one pupil using the method for adding two fractions when attempting to multiply two fractions.

The above utterances also show a lack of '*use of terminology'* as S3 refers to the numerator as 'top' and denominator as 'bottom'. However, S3 knows the correct mathematical names as in the post-observation interview he commented that he did not use mathematical language such as 'numerator' or 'denominator' because it confused the pupils. S3 said that the class did know the meanings of those words, but he did not want to 'put them off'.

'*Use of terminology'* can again be seen when going over the answers to the other questions. He states answers in the form: one number 'over' another number for every answer (for example six over five). Since this was the way the pupils gave the answers to S3, perhaps he chose to retain this phraseology rather than using alternative language such as 'six fifths'.

This lesson demonstrates that S3 is familiar with the procedure for dividing mixed fractions to the extent of knowing the individual steps involved and being able to set the first step (changing to improper fractions) as a starter activity. It is not clear from this lesson that S3 knows why the procedure works or whether he knows alternative representations of fractions (i.e. alternative language other than a number 'over' another number) but perhaps because of the low ability of this group, S3 chooses to focus on procedures and uses only one way to express fractions to avoid confusion – this is the reason given by S3 for not using the terms 'numerator' and 'denominator' but instead referring to 'top' and 'bottom' respectively.

5.3.4.2 Placement 2, Observation 1 (2/05/2013)

Lesson overview

The lesson started with a mental arithmetic test taken in silence. This section of the lesson was not video recorded, only audio recorded. Video recording commenced during the second part of the lesson after the test was over.

The second part of the lesson was on BIDMAS[1]. This acronym was written on the board and the pupils were asked to say what each letter stood for. S3 then demonstrated some examples on the board of using BIDMAS to get to a solution. Pupils then worked on questions from worksheets in their books

Knowledge Quartet Analysis

This lesson demonstrated S3's foundation knowledge of what BIDMAS

[1] An acronym to help remember the order of operation in a mathematical expression – Brackets, Indices, Division, Multiplication, Addition, Subtraction

stands for and how to apply it to mathematics questions (‘*overt display of subject knowledge*’). Further, it shows that S3 understands the underlying reasons behind BIDMAS. Indeed, this is shown in his response to a pupil’s question about $10 \times 4 + 3$:

> Pupil: when you do the times ten do you only do it to the four?
>
> S3: Yes, because… the numbers are either side of the times sign, aren’t they? So we do the – that’s why if we did three times four times ten it would be a different answer wouldn’t it? That’s why we’re looking at BIDMAS because we do the multiplication before we do the addition. OK. This is why we have brackets, so if wanted to do the addition first there, we would have put brackets round it, ok?

Here, S3 shows his understanding of why BIDMAS is needed and why brackets are used in mathematics. However, it is not clear whether he knows that BIDMAS is a convention rather than a mathematical truth/ absolute.

5.3.4.3 Placement 2, Observation 2 (08/05/2013)

Lesson Overview

The lesson was on estimating and S3 chose to use the convention of rounding to one significant figure as ‘the method’ to estimate. The starter therefore involved rounding numbers to one significant figure. Pupils then worked on questions individually. The second half of the lesson was on estimating square roots of numbers by looking at the square number above and below. This method was offered by a pupil and was the method S3 wanted to use.

Knowledge Quartet Analysis

S3’s focus on the convention of rounding to one significant figure in order to estimate was coded as ‘*adherence to textbook’*, though later it became apparent that this method was proposed by other teachers so ‘*adherence to peers’ techniques*’ (*not* a Knowledge Quartet code) would perhaps be more accurate. It is not clear why S3 chose to adhere to this technique when the following explanation to the class demonstrates that S3 does understand the underlying reasons for estimating (foundation knowledge) and when this method led to pupil confusion:

> So, if we have a difficult problem, it’s easier to estimate, ok. So we know if we work it out properly, our answer’s gonna be correct. So to estimate, we round our numbers to 1 significant figure. Ok, that’s why I had you do that for the starter. So in this one, we’re gonna estimate 4.99 times by 3.41. It’s not very nice to work out is it? So, first things first, 4.99 rounds to 5, what does 3.41 round to?... Three, ok. So what question am I actually going to be solving

> there? PupilG?... Five times three, fantastic, ok. Does somebody want to tell me what five times three is? PupilH?... 15, OK, so, does anyone have their calculator out? So PupilI here is gonna test us. What is 4.99 timesed by three point four one? So what's our actual answer? [Pupil: 17.01] So, our estimate here is fairly accurate, ok. So we knew that if we got the answer, 80 something when we… did it, that we'd done something wrong wouldn't we? OK, so we know our estimate was about 15.

Here, S3 explains how estimating can help one to know if an answer to a calculation is correct and contrasts an 'exact' answer on the calculator with an estimate to show how the estimate can be useful. However, S3's adherence to the method of rounding to one significant figure causes conflict with this understanding as demonstrated in the explanation to the class when discussing the problem 146.5 x 154.2 ÷ 2.41:

> So what is my first number going to round to? PupilJ? [PupilJ: is it 100 or is it 150?] Right, this is something that I was wanting to talk to you about, ok. You guys probably could estimate by rounding it up to 150 couldn't you? You guys can do your 15s [times table]. Normally when we teach it we'd say we estimate to one significant figure so 100, ok. You could say that you would estimate it to 150 because it would be more accurate. However, in all the Qs and answers I've put up, I've rounded it to one significant figure. But, in your exams, or in your SATs, if you round it to 150 and then get your estimate right, that will be ok. Alright. Just make sure that what you round it to, you can get it right. OK, so in this one, I'm going to round it to 100, ok.

Here S3 recognises that rounding to 150 would be more accurate but he adheres to the rule of 'rounding to one significant figure' despite this. This may cause confusion for the pupils especially the statement: "get your estimate right" since estimates are not right or wrong per se, only more or less accurate. However, reference to accuracy is made later on:

> …we've got a pretty accurate estimate here. So we knew that if you came out with something absolutely ludicrous like fifty thousand when you did that, it obviously wasn't going to be right.

Despite demonstrating estimating and then checking the actual answer to the question on a calculator twice in front of the class, some pupils wanted to use their calculators to simply work out the answers on the worksheet handed out rather than estimate the answer, suggesting they did not understand:

> S3: we're meant to be doing the estimates
>
> PupilC: yeah but I'm doing the estimates on my calculator
>
> S3: PupilC if you had your calculator when doing estimates, you wouldn't need to do an estimate!

In the post-observation interview, S3 reveals why he chose rounding to one significant figure as his approach to estimating:

> S3: I learnt that when you're estimating you're only supposed to go to one significant figure, I always just did it to whatever seemed appropriate, I didn't realise that there was a method that you're supposed to follow.
>
> Interviewer: so where does it state that?
>
> S3: it was just in teachers' things on the internet… it seemed to be the common – in all the lessons I looked at that other teachers had taught it always said to go to one significant figure.
>
> Interviewer: so on teachers' forums?
>
> S3: yeah
>
> Interviewer: does it say anything on the syllabus?
>
> S3: no it just said 'estimating' on the syllabus

Thus, S3 chose to use this method rather than do "whatever seemed appropriate".

Within the lesson, there was evidence that S3 '*concentrated on procedures'*. Indeed, when helping individual pupils with the starter questions, he said: "Well, what's the first significant figure? [pupil response] The tens, so we want tens. So we round to the nearest ten. What's the nearest ten from 13.5?". Here the procedure is to see what the first non-zero number is, and round according to the place value of that figure as can be seen in the following statement made by S3 when going over the answer to rounding 7.5 on the board:

> …not ten on this one… I expected that mistake, PupilE? [PupilE: 8] 8, ok so our first one that's non-zero is our units, ok? So to 1 significant figure that one is actually 8.

This also demonstrates S3's '*anticipation of complexity'* or 'decisions about sequencing' of his questions – he deliberately put this question after a few which involved rounding to the nearest ten and expected some pupils to continue rounding to the nearest ten even when this question involved only units. This was also the case with a later question:

> It's a tricky one this one… That one I put in there to try and catch people out.

This lesson also showed a progression in S3's explanation of what a significant figure is. Indeed, at the beginning of the lesson, S3's explanation consisted of providing examples of rounding to one significant figure and then the following statement: “there should only be one number that isn’t a zero in your answer, ok, if we’re going to one significant figure”.

However, by the post-observation interview, S3 gave the following ‘definition’: “it is the first non-zero term when reading from the left”. This ‘definition’ of significant figure is more succinct than the multiple examples S3 needed to give the pupils in order to ‘explain’ what it was. There are two explanations for this: either S3 knew this ‘definition’ all along and chose to use examples only when explaining to the pupils or, by teaching a lesson on significant figures and having to explain to others what one was, he may have formed an explanation which could be verbalised. Hence, the process of teaching caused him to transform his internal understanding of significant figures into a form which could be expressed to another.

5.3.5 Student 4 (S4)

5.3.5.1 Placement 1, Observation 1 (05/12/2012)

Lesson Overview

The lesson was on data handling. The scenario was as follows: “I’m going to be opening, a shoe factory, we’re really making shoes for people your age - so 13-14 year olds”. S4 then asks the class for their shoe sizes and divides the data into boys and girls sizes. S4 asks pupils (in groups) to first think about how these data can be used to help the shoe factory owner. Pupils ‘skip ahead’ and start calculating averages (mean, mode and median) and the range. The definitions of these are then written on the board and S4 asks different groups to calculate the answers for the shoe size data and to say what information it provides the factory. Finally, pupils are provided with some other data sets from which to calculate these values.

Knowledge Quartet Analysis

Within this lesson, there is an ‘*overt display of [S4’s] subject knowledge’* of how the mean, mode, median and range can be used to make decisions in real life situations. This knowledge is demonstrated through S4’s ‘*choice of examples*’ by using a shoe factory setting and through his ‘*awareness of purpose’*. Indeed, the purpose of the lesson is very much on helping pupils understand the meaning behind the values. For example, right at the beginning, S4 instructs pupils:

> I want you to think why would I want to use this data? What would this data be good for? So I’ve just got some numbers, why is it relevant to me in my shoe factory?… I want you to discuss how many ways you think we could use this data, and what can we

> learn from it, how's it going to be useful to me? Before we start working with it, what can we aim towards?

Thus, he first asks the pupils in groups to think about how this data can be used before they start calculating anything. This suggests that S4 wants the pupils to think about the reasons behind calculating statistics and what statistics can be used for rather than simply calculating averages. However, the pupils seem to get ahead of the lesson agenda as one group feeds back values for the median of boys' shoes. S4's '*awareness of purpose'* is demonstrated as he tries to steer the pupils' responses back to the underlying reasons: "so why would I need to know the median, why would I be looking to use this data?". Again this demonstrates S4's knowledge that averages provide specific information to help make decisions rather than being meaningless values to be calculated.

After letting the pupils work on some other data sets to find the mean, median, range and mode, S4 returns to the shoe factory values calculated and asks pupils about the information that these values provide. S4 again demonstrates '*overt subject knowledge'* of the meaning behind these values. Indeed, he states that the mean suggests: "On average… boys tend to have bigger feet than girls in our age group - in 13 to 14 year olds". He explains that the mode can inform the factory of which sizes to make more of and to explain the median, S4 provides an example on the board:

> …if I've got a data set and I've got a maximum of 12 shoe size and a minimum of two, and a median of ten, is there anything we can learn from that? Yeah? [Pupil: that there's more ten's elevens and twelves…] Yeah so we can use the median to work our how spread out our data are as well so that's something we can use the median for.

Finally, S4 explains that the range can tell us:

> …how many different sizes we're gonna need, for boys and girls. So what about the range of the boys compared to the range of the girls?... So a bigger range, yeah, so there's more variety of sizes for boys than there is for girls, that's one thing we can look at.

This lesson demonstrates S4's understanding of how the mean, median, mode and range can be used to inform decisions in a shoe factory. However, S4 experiences some difficulty when trying to explain what these values are to his pupils, particularly the mean:

> We call it, we call it the arithmetic mean and actually that is, like I say, it's like the average isn't it really? So the arithmetic average it's like the overall average for this set of data.

This confuses the terms ‘mean’, ‘average’ and ‘averages’. Although S4 knows how to calculate the mean, he has trouble teaching his pupils what the mean is. This is further demonstrated by the definitions of mean, median, mode and range which are written on the board for pupils to write into their books. S4 provides the following definitions:

> Mode – most common
>
> Range – breadth
>
> Median – middle value
>
> Mean – average value

Again, describing the ‘mean’ as the ‘average’ is confusing the everyday sense of the word ‘average’ (which involves summing the values and dividing by the number of values) with the mathematical/ statistical term ‘averages’ (which is an umbrella term for the mean, median and mode. Range, on the other hand, is a measure of spread). It is not clear that S4 realises that the ‘range’ is a measure of spread not an average.

Despite the confusion between the term ‘average’ used in the lesson, S4 does have a good understanding of these statistical methods. This is shown through his ‘*choice of examples’*. Indeed, the further data sets on the board were made up by him. He explained that his procedure for writing examples was to try and get a mix with some organised data and some not. He also wrote some examples with more than one mode and some with one mode and one example with the median value between two numbers (an even number of data points).

5.3.5.2 Placement 2, Observation 1(19/04/2013)

Lesson Overview

The lesson was a revision session on probability trees. Pupils were first asked to draw a probability tree for the outcomes of tossing three coins. Once pupils had drawn the probability tree, S4 asked them: “what are the probabilities of three heads?”, followed by two heads. The procedure for answering these questions using the probability trees was then explained by S4 on the board. For the remainder of the lesson, the pupils worked on further questions on probability trees in their books.

Knowledge Quartet Analysis

S4 ‘*concentrates on procedures’* when teaching the pupils how to calculate probabilities using probability trees. This is demonstrated by the following exchange:

S4: what are the probabilities of three heads?

Pupil: ‘is it likely?’

S4: We’re not looking for ‘likely’ we’re looking for numbers this time, remember when doing the probability scale we were looking at likely, like impossible. PupilC?

PupilC: do you have to measure…all the ones it could be and then work out what fraction of the ones are three heads from what the other outcome are?

S4: So, sorry, explain what you mean sorry, I didn’t quite get that. So you’re looking at?

PupilC: all the different outcomes and then see how likely it would be to get three heads from all the different outcomes

S4: I think you’re going down the right lines – how many ways can we get three heads following this tree?

The fact that S4 then goes on to explain to the pupils how to calculate the probability (‘*concentrating on procedures’*) suggests that he does not follow/ understand PupilC’s suggestion (which was to look at the sample space of results and see how many of these resulted in three heads – one out of eight):

So for each step as well when we’re working out the probability we need a head AND a head and a head and another head, so it’s AND every time. If you’re doing ‘and’ - when you’re looking at ‘and’ with probability - you always times the fractions together, you always times the different probabilities together so what we’ll have is we need it to be heads which we’ve got a fifty fifty chance of the first head [writes ½ with H over it] and then we need to times that, we need ‘and’ [writes x] we need another head [writes H, and ½] which’d be a fifty fifty chance, AND [writes x] from there [writes H] we also need our third head which has a fifty fifty chance [writes 1/2].

Here, the procedure of calculating the probability (multiplying) is the focus and S4 offers no explanation why we multiply, nor links it with PupilC’s suggestion.

For the next problem – finding the probability of tossing two heads - another pupil offers a suggestion for finding the answer which S4 does not seem to understand (lack of ‘*overt display of subject knowledge’*):

PupilI: is it three eighths?

S4: why is it three eighths?

PupilI: ‘cause there’s three ways to get two heads you just do it like that

> S4: One second PupilI people are talking over you… Go on, so you're saying there's three different ways of getting two heads, so we could get a head, tail heads, so that's one way we could do it, this route. So we could take head, then tail, then head [circles them on board] We could go head, head and tail so we could go that way, what's the last route we could go?
>
> PupilI: tail head head
>
> S4: yeah tail, head head [circles] and if take that route [top one], that gives us a probability of getting that route, we'd have a probability of one over eight, going that way, this way we'd have a probability of one over eight again – so we'd use the AND rule – timesing each step of the way – and then this third one, it'd also be one over eight. And like you said, because we've got three different ways of getting there, we could go that way, or we could go that way or we could go that way, we'd add these together [writes addition] to give us our answer of three over eight

For this example, adding up the number of ways does equal the same probability as calculating the 'AND' and 'OR' rules but in general, this method will not work unless the probabilities are the same on each 'branch'. It seems S4 does not see this link. Perhaps S4 wishes to stick with his procedure because his knowledge of this area of mathematics is limited. Indeed, in the post-observation interview, S4 admitted he did not know why you 'add for OR' and 'times for AND'. He said that's what he was taught in high school and in his degree (computer science) and A-level. He was never taught why it works.

S4 shows 'anticipation of complexity' in his 'choice of example' of tossing coins:

> …if we were rolling two dice, we'd have six different strands for the first dice, and then we'd have to branch off again and do another six, for each at the end of each branch if we're doing two different dice which'd be a lot more complicated – that's why I stuck with heads or tails.

Thus, selecting an event with only two possible outcomes at each stage/ toss keeps the probability tree simpler.

S4's knowledge of rational numbers is shown in this lesson by '*making connections between procedures'*. Indeed, when pupils struggle to multiply ½ three times, S4 rephrases the question, linking multiplying fractions with cutting an object thus re-presenting it in another format ('*choice of representation*'):

> …what's a half times a half?... So if I've got a half, and I cut it in half, what do I get?... quarters, and if I've got quarters and I cut them in half again… Yeah so our answer would be one eighth.

This demonstrates S4's knowledge of operations with rational numbers (fractions). He also '*deviates from agenda'* to recap multiplying fractions:

> Two times two is four- yeah just as a quick recap guys when you're timesing fractions together [writes 1/5 x] so say we had one over five times [3/6] three over six. You times the top numbers together, and then times the bottom numbers together to get your final answer…

5.3.5.3 Placement 2, Observation 2 (13/05/13)

Lesson Overview

For the starter for this lesson, pupils had to determine whether several statements were true or false. The focus of the lesson (unrelated to the starter) was index numbers[2] and S4 first reminded pupils how to find percentage increase and decrease and introduced them to crude rates. Index numbers were then introduced through an example of a school minibus decreasing in value over time. Then, chain index values and weighted index values were then demonstrated on the board with pupil input. For the plenary, S4 asked pupils to explain/recap what the procedures are for finding the index values discussed in the lesson.

Knowledge Quartet Analysis

One of the statements ('a square is a rectangle') in the starter activity caused confusion with one pupil and the following discussion ensued:

> Pupil: how is a square a rectangle?
>
> S4: a square is a type of rectangle so that's true
>
> Pupil: yeah but a square has got all the sides the same length, a rectangle hasn't
>
> S4: … so a rectangle can't be a square, but can a square still be a rectangle?
>
> Pupil: no
>
> S4: why not?
>
> Pupil: 'cause it can't
>
> S4: No?... What's the definition of a rectangle?... PupilC?
>
> [Inaud Pupil response]

---

[2] Index numbers are used in economics to show change over time. A base year is assigned an index number of 100. In the following year, an index number greater than 100 would indicate a rise (e.g. 105 indicates a 5% rise), whereas a number less than 100 would indicate a fall (e.g. 95 indicate a 5% fall).

> S4: it's a four sided shape, anything else?... so the actual – the correct answer for that guys is – a rectangle can't be a square, but a square can be a rectangle.

This demonstrates S4's understanding of the relationship between a square and a rectangle ('*connection between concepts'*). However, he does not dwell on explaining this fully to his pupils. In the post-observation interview, S4 comments: "To be honest really if I did that again I could drag out some actual definitions and try to enforce the fact that it is [square is a rectangle] and demonstrate how but as it was a starter I didn't want to spend too much time on it". Thus, S4 chose not to address this in the lesson due to time restrictions, further it was not related to the main purpose of the lesson.

S4's understanding of the concepts involved in the lesson is demonstrated through '*connections between procedures'*. Firstly, he makes connections between percentage increase and crude rates:

> So you'll have, say, the crude birth rate as an example, and you'd divide it – you'd have the number of births over the total population. So that's like your new value divided by the old one it's the lesser value over the total population but instead of timesing it by a hundred you times it by a thousand.

This also demonstrates S4's understanding of crude rates and how they are related to finding percentage change. Similarly, S4 demonstrates his understanding of index numbers by showing he understands the connections with percentage increase and decrease ('*connections between concepts*') and also how they apply to real life contexts:

> And index numbers are a bit similar as well, they're used in business because there's lots of things that change over time. Like I've put the cost of raw materials, the number of staff you've got on, value of assets and things like that… So right, because things are changing all the time in business, managers and directors and things want to monitor how they change over time and index numbers help us compare how things change – how different things change.

As his '*choice of example',* S4 uses the decreasing value of a school minibus to introduce index values. This is an appropriate example since: "Values of vehicles like cars and things really drop down really quickly".

S4 also tries to ensure that pupils fully understand index numbers by asking what the index value would be for the base year and also what an index number greater than 100 would mean. The pupils are able to respond correctly to these questions. S4 explains that they are correct because: "there's no time passed" and "it's really similar to percentage" respectively. Thus, he '*mak[es] connections between concepts'*.

S4 makes further '*decisions about sequencing'* by choosing to look at chain index numbers next which: "is when we're changing the base value every year. We're updating that year every time. So other than that, it is exactly the same". This is only a slight extension to index numbers so follows logically. He then chooses to go onto weighted index numbers. However, the following explanation indicates S4 is not as comfortable with the idea of weighted index numbers:

> We'll have one more things that might come up. It's called weighted index numbers. And that'll pop up – say a company has got values – say if they're spending money on something and it's made of different parts… So right, year 10, we might have an example for – where – I've got a company that wants to find the index numbers of its advertisement costs. So we're looking at the amount of costs it's – sorry, the amount it's spending on different parts of its advertising. For example, it might have newspaper adverts and TV adverts. But it might want to work them out together but it might consider one of them to be more worthwhile than the other, even though it costs more. So for example, it might think that newspaper adverts –even though they're cheaper – they are more important, and they might want to give that priority when working out the index number. So what we'd use, it's like a ratio we'd use. We'd set up a weighting for them both so it might do something like say 75 is the ratio for newspaper ads and then 25 as the value for TV ads, just so that there's a much bigger proportion of our index number going into the newspaper ads change rather than there is the TV ads. So then you would actually just use this formula to work it out. You'd work out all your index number separately so you'd… work out the index number for your newspaper value first, and then you'd work out the index value for your TV adverts. So you'd get two separate index numbers.

This explanation is not quite correct (lack of foundation knowledge). If the company wants to calculate its costs on advertising but it spends more money on newspaper ads than TV adverts, this needs to be reflected in the overall costs of advertising. The importance of the type of adverts does not play a role here. This also confuses the pupils as no one is able to tell him what a weighted index number is when he asks them during the plenary.

5.3.6 Student 5 (S5)

5.3.6.1 Placement 1, Observation 1 (29/11/2012)

Lesson Overview

The lesson objective was to find the volume of pyramids and cones. The lesson began with a starter on the board which asked pupils how much material would be needed to make the tent which was pictured (a triangular prism with two circular skylights). S5 went over the method and solution on the board with pupil input, then moved onto the main section of the lesson.

He asked pupils for their ideas of how to find the volume of cones and pyramids, starting by asking them why the two are related. Drawing upon pupil responses the formulae for finding the volumes were teased out. S5 demonstrated the solution to an example on the board, and pupils worked on the questions in their books. Another set of questions was then displayed on the board, differentiated by difficulty, for pupils to select the level they were comfortable with. The final part of the lesson involved pupils working on two problems involving finding the volume of a cone of vision and the virtual frustum (the three dimensional region which is visible on a computer screen – a truncated pyramid) respectively. These ‘choice of examples’ related volumes to real-life situations.

Knowledge Quartet Analysis

This lesson demonstrates S5’s foundational knowledge that a cone and a pyramid are related and this was key to the lesson which treated both as similar. Initially, S5 demonstrated ‘*use of terminology’* as he ‘translated’ the pupil responses to the question why a cone is also a pyramid (“because it’s got like one side at the bottom and the rest all go up to one point”) using more mathematical language:

> Right, we’ve got a flat base – and this is actually probably something to write down… we have an apex… Same thing with the cone.. base [points to] apex [points to].

Here, S5 ‘translated’ ‘side at the bottom’ to ‘base’ and ‘one point’ to ‘apex’.

S5’s ‘*response to pupils’ ideas’* on how to find the volume of these shapes was key to helping them understand the concepts behind the formula. He was supportive of all pupils’ ideas provided and drew out that the “height is definitely important” and “the base is a key point” from different pupils’ responses. One pupil then asked: “would it be the same to work out the area of a triangle?”. S5 ‘*respon[ded] to [this pupil’s] idea’* by building upon what was said:

> I really like that. I really like that. Because a triangle [draws square on IWB with triangle inside] – we just put it inside a square – we can figure out that it was just half

Building upon the pupil’s idea and the ‘*choice of representation’* by drawing a diagram of a triangle inside a square helped the pupils to ‘*make connections between concepts*’ as one pupil then asked: “would you put it inside a cube?”. It seems S5 had intended to bring out this connection as his next IWB slide had a cuboid and a cylinder on. He drew on the pupils’ prior knowledge of finding the volume of a cuboid and wrote one pupil’s response:

“width times height times depth” as $(w \times h) \times d$. This ‘*choice of representation*’ was deliberately chosen as S5 intended to ‘ma*ke connections between the procedures*’ of finding the area of a cuboid and finding the area of a cylinder as he later wrote: ‘Area(base) x h’ and explained that whatever the shape of the base, if one finds the area and times by the height, this will give the volume. S5 then made a pyramid and a cone appear within the cuboid and cylinder respectively on the IWB. Referring back to the square he drew with a triangle inside, the pupils were able to make connections. One pupil suggested that to find the volume: “would it just be the same but over two?”. S5 ‘*respon[ded] to [this] idea’* by drawing a cube and cutting it in half to demonstrate how half a cube would not be a pyramid. After further responses of ‘a quarter’ and ‘a third’ from the class, S5 revealed that the formula is a third of the area of the base times by the height.

S5 made a ‘*choice of representation’* to ‘make connections between procedures’ as he wrote the formula for finding the volume of a cylinder as: ‘area(base) times height divided by three’ and explained that this is the same as before.

S5’s ‘*choice of examples’* also tied in with S5’s aims of linking cones with pyramids and it seemed he wanted pupils to identify the base even when the solid was not orientated with the base at the bottom (he used a variety of orientations for his questions set on the board). Further, he set questions with the apex of the shape in different positions, explaining: “we’ve had our apex over the shape [gestures with hands]… doesn’t need to be”.

5.3.6.2 Placement 1, Observation 2 (10/12/2012).

Lesson Overview

The lesson was on Pythagoras’ Theorem applied to three dimensional problems. The lesson began with a starter on the board involving a length of cable running from a ship’s mast through a crossbar and down to the deck. The pupils had to use Pythagoras’ Theorem (2D) to calculate the length of the cable. For the main body of the lesson, there were lines in one, two and then three dimensions drawn on the IWB (with the exception of the 3D line) along with coordinate axes. He had the pupils come to measure with a tape measure the difference along the coordinate axes corresponding to the differences in each direction of the lines. For example, for the 2D line, pupils measured the change in the x direction and the change in the y direction and then to measure (but keep hidden) the actual distance of the line. The class were then instructed to use Pythagoras' Theorem (using the changes in direction measured) to say what the distance of the line was before this was checked against the tape-measured distance of the line. For the 3D line, S5 used a ribbon attached to the board at one end and held the other end away from the board so the pupils could measure the change in three directions (x, y, z) with the tape measure. After these demonstrations, S5 displayed the algebraic formulae for finding distances in one, two and three dimensions respectively on the board:

$$x$$

$$\sqrt{(x^2 + y^2)}$$

$$\sqrt{(x^2 + y^2 + z^2)}$$

[Equation 1]

Questions involving real-life scenarios (a plane landing and a mobile phone located at an unknown distance from a signal tower) were displayed on the board for the pupils to work on in pairs. A worksheet was also given out to complete when these problems had been solved. Pupils worked on these questions for the remainder of the lesson.

Knowledge Quartet Analysis

S5 demonstrated '*choice of representation'* and '*decisions about sequencing'* during the main body of the lesson with the progression from one to two and then three dimensions and the use of comparison between measuring the line and calculating it using Pythagoras' Theorem. Further, the ribbon worked well as a choice of representation since not only did S5 'not have 3D glasses for the board' but this enabled him to move the ribbon around to form two 2D triangles to demonstrate how Pythagoras' Theorem can be used within a 3D problem. S5's deep understanding of Pythagoras' Theorem and dimensions was demonstrated through these techniques.

S5 also '[*made] connections between procedures'* as he wrote faintly next to the first expression of Equation 1:

$$\sqrt{(x^2)}$$

This showed the link to the other expressions.

5.3.6.3 Placement 2, Observation 1 (25/03/2013)

Lesson Overview

The lesson was on loci. Upon entering the classroom, S5 had previously moved all the tables to the edges of the room, leaving a wide space in the middle. After introducing/ defining the word ‘loci’, the pupils had to position themselves around the classroom to form loci of points according to S5’s instructions. Such instructions included:

> we need a set of points that make a line that divides the room in half
>
> A locus of points that make a square
>
> we’re going to find a loci equidistant from the hat
>
> a locus that is equidistant from a line running through the centre of the room
>
> everyone to stand in a locus equidistant from the North Pole

From these examples, S5 drew out the names of the ‘shapes’ that had been created: parallel lines, a curve to match the curvature of the earth, and angle bisector respectively. The pupils then helped put the tables back and wrote definitions in their books of how they would explain the terms ‘locus’ and ‘equidistant’. S5 asked them to put an example of a locus under their explanations and selected ‘the path of a tennis ball’ as his example.

S5 then provided two questions on the board for pupils to discuss in groups and complete in their books. The answers were given at the end although some pupils were confused at first about the answer to the second question.

Knowledge Quartet Analysis

The lesson demonstrated S5’s ‘*choice of representation’* through the decision to have the pupils act as ‘points’ to form loci. However, the host teacher said that having the pupils being points on loci was her suggestion as she has done this in the past. Since S5 chose to use the host teacher’s idea of representation, perhaps this explains why the idea that the pupils were ‘points’ was not explicitly made to the pupils. Further, pupils struggled with forming some of the loci as the following example shows.

Once the pupils had formed the locus of a square shape, S5 introduced the term ‘equidistant’ as meaning ‘equal distance’ and asked:

> S5: what might be a condition that uses the word 'equidistant'?
>
> Pupil: would it be from like one side to the other side? [of the square]
>
> S5: One side from the other side?
>
> Pupil: [explains distance same from that corner to that corner]
>
> S5: So how how would you use the actual word ‘equidistant’? If I wanted to say that corner to that corner and that corner to that corner? [pupil doesn’t know] Can anyone help him? Because I

> think he's on the right track…. What about the distance from this side to this side - because we're standing in a square- and the distance between this side and this side – what's their relationship?
>
> Pupil: this side and this side would be equidistant
>
> S5: Just this side and this side?
>
> Pupil: and that side and that side
>
> S5: Yeah so there's two.

S5 left it at that. This showed his understanding of the properties of a square shape and '*decisions about sequencing'* to introduce the term equidistant when the pupils were standing in a square shape but perhaps does not reflect how the term is used in examination questions. The next instruction however was more typical of an exam question:

> I want everyone to stand – so we're going to find a loci equidistant from the hat – equidistant

The pupils then shuffled around still maintaining their square shape but moved so that they formed a square around the hat. In response to this S5 had another pupil use a tape measure ('*recognition of conceptual appropriateness*') to: "test whether they are accurate… are they actually equidistant?". A discussion then ensued about how many points the pupil needed to measure to check that they were all standing equidistant from the hat. Some pupils thought just one or two; some thought two to five, others thought 'the more the better'. The pupils did not seem to realise that all points needed to be equidistant and that the shape they would form would be a circle, not a square. S5 thus prompted them further ('*responding to [pupil] ideas'*):

> Because we started with a square, if everyone's the same distance, what shape are we trying to create? [some say square, some circle] Hmm both square and circle… How many people think square? [no-one raises their hand] How many people think circle? [everyone raises their hands] Nice, it is a circle. Equidistance. So, we know that our equidistant points they always have to be sort of the same distance from.

Additionally, when asked to present their definitions/ explanations of what 'locus' and 'equidistant' mean, some pupils were unable to do so. Similarly, when asked to put 'the path of a tennis ball' as an example of a locus under their explanations in their books, pupils struggled. As a '*choice' of example*' this was interesting as it is not one that features within the National Curriculum/ GCSE examination (Department for Education, 2013c) yet it demonstrates S5's understanding of the term 'locus' since a moving tennis ball could represent a single point which indeed satisfies the conditions of physics when thrown, thus S5 '*recognises the conceptual appropriateness'* of this example. However, the pupils struggled with this task suggesting S5 did not '*anticipate complexity'*.

At the end of the lesson S5 revealed the answer to the final question on the IWB (**Figure 5.30**, p.174), yet some pupils questioned it: "why wouldn't there be a point at the other side?" (referring to the curved section on the outside of the figure). S5 then demonstrated '*choice of representation'* by using his fingers to demonstrate the distance of two units on the two unit scale and then moving this distance marked with his fingers around the point in question to show how the locus remained 2 units from the line segments. He used the same technique in response to a question about the inner point.

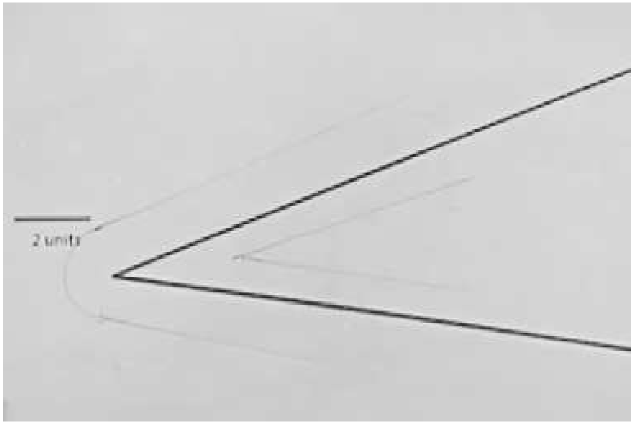

Figure 5.30: S5'S diagram on the IWB for placement 2, observation 1.

6 Addressing the Research Questions

Chapter 4 discusses how this research follows a mixed methods design in that both quantitative and qualitative methods are used to collect, analyse and address the research questions. This chapter ties together results from the three data collection methods (Chapter 5) in order to address the research questions (Chapter 3). Matters arising which require detailed discussion are returned to in Chapter 7. An evaluation of the research as a whole, including limitations of the study, are discussed in Chapter 8.

6.1 Research Question 1 (RQ1)

When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: what is the nature of this knowledge?

6.1.1 Introduction

This section discusses what results from this research suggest about the nature of mathematics-related knowledge of Postgraduate Certificate in Education (PGCE) course students. Specifically, correlations between degree results and scores on the Mathematical Knowledge for Teaching (MKT) items respectively with PGCE results are discussed to provide initial ideas about general constructs which may be associated with the mathematics-related knowledge required for teaching. Then, results from interviews and observations are discussed. Woven into the discussion are suggestions of how this research fits with other studies presented in the literature review (Chapter 2). The subsequent research questions (RQ2 and RQ3) also provide evidence to address this question by looking at how this knowledge changes and if there are differences between ‘subject knowledge enhancement’ (SKE) and Non-SKE students.

6.1.2 Findings

6.1.2.1 Mathematical content knowledge

In order to address this research question, prior mathematical attainment (GCSE Mathematics, A-level Mathematics and degree results) were taken as a proxy measure of mathematical content knowledge and scores on the MKT questions were used as a measure of MKT. How these measures correlated with final PGCE grades (a proxy measure of ‘teaching quality’) was then analysed in order to see if there was any relationship between these constructs. If one (or more) of these measures predicted success on a PGCE course, it could therefore be said to be a useful indicator of the knowledge needed by secondary mathematics teachers in England.

6.1.2.1.1 Prior attainment and success on the PGCE course

The results show no statistically significant correlation between GCSE Mathematics, A-level Mathematics or A-level Further Mathematics and final PGCE results respectively (5.1.3, p.100). Degree classification could statistically significantly predict PGCE success but the correlation was small ($r = 0.11$) and a linear regression analysis showed that degree classification only accounted for 3% of the explained variability in PGCE scores (5.1.3.4, p.102).

Intuitively, it makes sense that degree class plays a role (albeit a small one) in PGCE success since there is an academic (writing) element to the course (1.3.2, p.4). However, when the mathematical content of the degree was taken into account, there was no statistically significant correlation with final PGCE results ($r = -0.055$). This study, involving larger numbers of PGCE students across 18 institutions in England, corroborates the results of other studies (Tennant, 2006; Stevenson, 2008) suggesting that degree classification weighted by mathematical content does not substantially predict success on a PGCE course.

Thus, there is no evidence from this nor other studies that prior mathematical attainment (at GCSE, A-level or degree level) predicts success on a PGCE course. However, this does not mean that mathematical content knowledge is not important for PGCE course success. Indeed, minimum mathematical qualifications are a prerequisite for the PGCE course (1.3.1, p.2), so it may be that as long as these minimum requirements are met, success can be achieved. Further, the PGCE course is geared towards enabling graduates to teach mathematics rather than teaching mathematics to the graduates, who are assumed to already have such knowledge. Indeed, the PGCE course (at the University of Leeds) does not explicitly assess mathematical content knowledge (other institutions' assessments may differ) though it does encourage personal subject knowledge audits (see 1.3.2, p.4).

6.1.2.1.2 Mathematical knowledge for teaching (MKT) and success on the PGCE course

Results show (5.1.4.3, p.104) that scores on the MKT items do not predict PGCE success since there is no correlation between MKT scores on the pre- ($r = -0.076$) or post-questionnaire ($r = -0.051$) and PGCE grades respectively.

It was originally expected that scores on the MKT questions (which measure mathematical knowledge as used in teaching scenarios) would be a better predictor of PGCE success than prior mathematical attainment. However, this was not the case. This suggests that, within England, scores of MKT do not predict how effective a teacher is within the classroom (as measured by PGCE teaching score). This is in contrast to the findings of other studies which show that higher scores on the MKT questions predicted differences in the ‘mathematical quality of instruction’ (MQI) of teachers within the classroom (Hill et al., 2008b). There are several possible explanations for this.

Firstly, there are potential issues with the PGCE scores as the limitations section (8.2.1.1, p.251) discusses. These issues include lack of discrimination between scores and the potential for bias. This may have affected the correlation.

Secondly, whilst scoring highly on the items has been shown to relate to the MQI of teachers in the USA, it does not necessarily follow for teachers in England. Indeed, whilst Anderson-Levitt recognises that there are many commonalities between schooling and mathematics around the world, she infers that because there are some differences, the adapted USA tests cannot capture "the full range of mathematical knowledge for teaching on which teachers draw in the range of settings studied here [the five countries for which the items have been adapted]” (2012:447). The same may also be true for England.

Finally, the PGCE course may not focus on the specific aspects of the MKT construct which are tapped by the MKT items.

The results also show small to moderate ($r = 0.183$ to 0.334) correlations between GCSE, A-level and Further Mathematics A-level grades and MKT scores (5.1.4.4, p.104). This suggests prior mathematics attainment partially predicts success on the MKT items. This makes sense since the content of the items is based on mathematics at GCSE level (see 4.3.1.6, p.73). This corroborates the finding of Kersting and colleagues that: “Virtually all of the shared variance [of MKT scores] was explained by the MC [Mathematical Content] Knowledge subscale” (Kersting et al., 2010:178).

Additionally, there was also a small positive correlation between degree class (regardless of subject) and MKT score ($r = 0.189$). This suggests some relationship between the skills needed to successfully complete a degree and successfully respond to the MKT items.

6.1.3 Summary of findings

Results show that final PGCE grades are not significantly related to how well one performed at GCSE, A-level or degree level nor to the amount of mathematical content of prior degree. Further, PGCE success does not correlate with how well one performs on the MKT items.

Since RQ2, RQ3 and RQ4 also add to our understanding of the nature of knowledge for teaching (RQ1), the reader is directed to section 6.5 (p.213) which summarises results from all the research questions and which therefore provides additional insights for RQ1.

6.2 Research Question 2 (RQ2)

When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: (how) does this knowledge change over the course?

6.2.1 Introduction

This section examines the evidence gained from this study in addressing RQ2, namely, how the questionnaire, interview and observation data demonstrate changes in knowledge over the PGCE course. Each data collection method is discussed in turn, followed by a synthesis of the findings.

6.2.2 Findings

6.2.2.1 MKT score changes

In order to see whether the MKT of PGCE students changed over the PGCE course, pre- and post-questionnaire responses to the 18 MKT items were compared. The results (5.1.4.2, p.104) show a statistically significant increase of 1.31 marks (7.3%), on average, between the pre- and post-questionnaire and this represents a medium effect size. These results corroborate the findings of Hill and Ball (2004) who compared pre- and post-scores on a sample of MKT items with practicing teachers who had undergone summer professional development courses lasting between one and three weeks. They found an increase in scores equating to 2-3 items (4.1%) in the raw number correct (73 items). The increase of scores is arguably comparable between both studies. However, the PGCE course is considerably longer (approximately 9 months duration). This could be explained by the differing foci of the two courses. Indeed, “A PGCE course mainly focuses on developing [graduates’] teaching skills, and not on the subject [they] intend to teach” (Department for Education, 2013d:para.1). Conversely, the summer professional development courses were designed to teach mathematical content knowledge to in-service teachers:

> [the] morning session was heavily keyed to mathematics as it is used in elementary classrooms… The afternoon sessions… built on the morning’s mathematics by forging links to practice and state standards or providing teachers with activities for classroom use (Hill and Ball, 2004:334).

Thus, whilst the increase in scores over the PGCE is perhaps not as much as expected for a course of such length, the fact that the increase in scores was statistically significant is interesting given that the PGCE course is not intended to improve MKT specifically. Guidance on using the MKT items states that the MKT items are best used with:

> Programs that intend to broadly improve teachers’ knowledge in any of the content areas named above [Elementary number and operations; Elementary patterns, functions, and algebra; Geometry; Rational number; Proportional Reasoning; Data, probability, and statistics]; … Programs which target significant resources at improving teachers’ knowledge—not 2-3 hours of professional development, but sustained, serious mathematical work; Programs which work to improve teachers’ mathematical reasoning and analysis skills (LMT, n.d.).

Having observed all the mathematics PGCE sessions at the University of Leeds, I would not describe the course as explicitly meeting the above requirements since the focus is not on improving subject knowledge. Therefore, the PGCE course may be implicitly improving students’ MKT. This conclusion is supported by the juxtaposition of the finding for RQ2 (increase in MKT scores over the PGCE course) with the evidence for RQ1 which suggests MKT does not predict success on a PGCE course (6.1.2.1.2, p.176). Since MKT scores increase, but they do not predict success on the course, this suggests that the PGCE course implicitly equips students to do better on the MKT items whilst not explicitly taking MKT into account in the final assessments of the course. Since the PGCE course aims to prepare graduates to be teachers and the MKT items claim to test for the knowledge needed to be a successful teacher, it could be that the final PGCE grades do not capture all aspects (specifically those captured by the MKT items) of what is required to be a secondary mathematics teacher.

Alternatively, it could be that PGCE grades are not differentiating enough to provide an accurate reflection of the relationship between MKT and PGCE success. Indeed, PGCE scores only range from 1 to 4 with the majority of grades awarding being 1's and 2's (see 8.2.1.1, p.251). Moreover, as section 8.2.1.1 (p.251) discusses, there are further potential issues with the PGCE scores which may have effected the correlation between MKT scores and PGCE results.

6.2.2.2 Changes in Knowledge Maps

Scores on the MKT items were not the only aspect of knowledge to change over the PGCE course. Indeed, between the pre- and post-interviews there were changes in knowledge as evidenced by the Knowledge Maps. As students were asked about three mathematical topics in the interviews (squares, rational numbers and quadratic equations), changes over time are considered for each topic. For each topic (quantitative) changes in the numbers of nodes, links and domains are discussed as well as (qualitative) changes to the content of the nodes in terms of misconceptions and type of connections (discussed below) where applicable. This section refers back to the Knowledge Maps in section 5.2 (p.107) and the summary table (**Table 5.9**, p.140) which provides information on the numbers of nodes, links and domains for each of the students' Knowledge Maps.

In addressing this research question (RQ2), whilst it is differences between numbers of nodes, links and domains over time that is considered, the following discussion highlights that more nodes on the post-interview does not necessarily mean that more discrete mathematical facts (represented by the nodes) were learnt. Instead, knowledge may have become more accessible or have become 'decompressed' (Ball and Bass, 2000, see 2.1.5.2, p.22).

6.2.2.2.1 Squares

Nodes

Table 5.9 (p.140) shows that for the interview topic of squares, the change in the number of nodes between pre- and post-interview Knowledge Maps was small (one or two nodes) for all students. With the exception of S4, whose nodes decreased by one, nodes increased for all students.

The small change in numbers of nodes suggest that there was limited change in the number of discrete mathematical facts known by the students over the PGCE course. This is expected since, as discussed above, (6.1.2.1.1, p.176), the PGCE course is not intended to teach students mathematical content (minimum qualifications are required) but how to teach. Nevertheless, looking at the numbers of nodes alone is not sufficient to determine whether or not learning took place. The content of the nodes was also analysed, using the Analysis Tables (see Appendix E, p.275 for examples) for additional insights.

Comparing the content of the nodes between pre- and post-interview Knowledge Maps for all students reveals three types of changes to the students' knowledge: knowledge becoming more/ less accessible; knowledge being learnt or remembered; and knowledge incorporating familiarity with pupils and how they think. These are now discussed.

Firstly, there were some mathematical facts which became more (or less) accessible to the students over time. That is, some facts were stated as a result of interviewer prompting in the pre-interviews, but were freely recalled[11] by the post-interview (and vice versa). For example, S5 discussed area and perimeter of squares on the pre-interview after some prompting:

> I: could you tell me a bit about properties of the 2-dimensional shape of the square?
> S5: Properties of the 2-dimensional shape of the square.. erm..it's a erm - it will have both a finite area and a finite perimeter.. it will erm... [pause]
> I: and how would you calculate those?
> S5: Oh erm… the the for example.. the the height times the width would give the area, in this case they're the same, erm hence a square erm the perimeter would be 2 heights and 2 widths or 4 of one edge [pre]

Yet a node regarding area and perimeter featured on S5's post-interview Knowledge Map (**Figure 5.21**, p.130) only (thus was freely recalled). This suggests that although nodes for area and perimeter were only on the post-interview Knowledge Map, the knowledge was not 'new' but had become more accessible to S5 as he was able to freely recall these facts by the post-interview. A further example of this is found for S4. On the pre-interview, S4 was able to state some facts following interviewer prompting:

> I: …I'm thinking about.. shapes and and other names that you might use with the square and also shapes its related to. I'm thinking it's a regular...
> S4: erm... a regular... ooh God... er quad – quadrilater blah quad-ri-lat-eral
> I: regular quadrilateral ok

[11] Facts stated in response to the question 'Tell me everything you know about…' were classed as freely recalled as opposed to facts stated in response to more specific interviewer prompting.

> S4: and erm yeah it's also related to like say erm.. the rectangle… [pre]

By the post-interview he was also able to freely recall that a square was a quadrilateral and a type of rectangle (these nodes appear on only the post-interview Knowledge Map, **Figure 5.21**, p.130).

In contrast, there were instances where facts freely recalled on the pre-interview were not as immediately accessible (required prompting) on the post-interview. On the pre-interview Knowledge Map, S4 had nodes relating to area and perimeter of squares (**Figure 5.15**, p.123), whereas these aspects of square shapes were only mentioned when prompted on the post-interview:

> I: ok anything about…measurements?
> S4: oh right so.. yeah you could find the area of a square by erm– yeah so area of a square would be linked to square numbers as well so it'd be just the square of any – any length on the square. So it'd be that number times itself erm and obviously the perimeter would be four times any side – any side times four to find the perimeter [post]

Perhaps S4 did not come across the topics of area and perimeter during the PGCE course, so these were not as accessible to him as they had been.

Secondly, there were instances where knowledge was not discussed in the pre-interview but was during the post-interview, suggesting knowledge had been learnt or remembered over the PGCE course. However, it is difficult to determine whether remembering or learning had taken place. In some cases, it seems more likely that knowledge was remembered by the post-interview. For example, S2 talked about squares being the base of a square-based pyramid as well as the symmetrical properties of a square only on the post-interview (5.2.2.1, p.109). However, since these topics form part of the National Curriculum (Department for Education, 2013c) it is likely that S2 would have learnt them when he was a pupil at school yet did not remember them on the pre-interview. Since he discussed them on the post-interview, this suggests he may have revisited these topics during the PGCE course.

For other topics, which do not appear on the National Curriculum and are therefore unlikely to have been learnt by the students whilst they were pupils at school, it may be that they were learnt for the first time whilst on the PGCE course. However, it is difficult to say for certain. There are a few examples of this. S3 mentioned the relationship of squares to parallelograms and rhombuses only in the post-interview (5.2.3.1, p.116). Similarly, S3 appears to have learnt how the area of squares is related to square numbers as this was only discussed in the post-interview (5.2.3.1, p.116). S1's explanation of his experience of making this connection whilst on his PGCE school placement, see 5.3.2.1, p.142) supports this.

One final change in knowledge is the inclusion of considering pupils by the post-interview. Indeed, when discussing area and perimeter of squares on the post-interview, S5 states:

> there's a.. I say almost a measurement component – thinking about [pupils] trying to get them to differentiate between er perimeter and area [post]

This recognition of the difficulties pupils have with remembering which is area and which is perimeter was not mentioned on the pre-interview and is likely to have come as a result of direct experience with pupils on his school placement(s).

Domains

Table 5.9 (p.140) shows that for the interview topic of squares, the domains present on the Knowledge Maps either stayed the same (S2 and S5) or additional domains were added (S2 and S3) by the post-interview. With the exception of S2, all had linked number and geometry domains by the post-interview.

The change in domains between pre- and post-interviews to include both numeric and geometrical aspects of 'square' arguably marks a more substantial change than that of the number of nodes. Indeed, small fluctuations in the numbers of nodes could occur due to slight differences in which facts come to mind at the time of the interview. However, the addition of a new domain of knowledge to include numerical aspects of square as well as the connections between numerical and geometrical aspects can be seen to mark a substantial change in the way the student understands the concept. Indeed, "it is suggested that a concept is not completely understood until a student resides in the intersection of these three [numeric, geometric, algebraic] sets…" (Wurnig and Townend, 1997:78). Thus, the knowledge of squares demonstrated by two of the four students' post-interview Knowledge Maps can be described as richer with multiple representations (both geometric and numeric) as well as the connections between them.

Connections

The contrast between S3 and S5's interview progression is informative. Whilst S3 demonstrated a broader conception of squares by the post-interview (5.2.3.1, p.116) to include other mathematical shapes (rhombus, parallelogram) and other areas of mathematics (how area of a square relates to square numbers), the broader conception of squares demonstrated by S5 on the pre-interview 5.2.5.1, p.129) seemed to drop away by the post-interview. That is, S5 mentioned alternative geometries to Euclidean geometry, higher dimensions and an indication that he was aware of shapes with infinite areas on the pre-interview but these were not mentioned on the post-interview. One difference between the mathematical ideas mentioned is that S3's ideas are part of the school curriculum (Department for Education, 2013c) whereas, S5's ideas are beyond that required at compulsory education (Department for Education, 2013c). Perhaps exposure to the school curriculum on school-based placements has encouraged S3 to form links between the different areas of the curriculum and narrowed S5's focus to only those topics required by the syllabus. This fits with opinions expressed by students who participated in the Evaluation of SKE courses report:

> During the PGCE year, the SKE students considered their subject knowledge to be at a lower level (level 5) than traditional route teacher trainees (subject graduates) who were more likely to rate it at graduate/postgraduate level…They also felt however, that the knowledge that subject graduates might have could be less relevant to the school context (Gibson et al., 2013:11).
> Around a third of SKE students interviewed thought that those with a degree would have a more in-depth grasp of subject knowledge which may equip them to be more able to teach to A level; and also in terms of stretching children and potentially in answering more difficult complex questions – however some thought that some of this knowledge maybe 'surplus to what's required in schools' (Gibson et al., 2013:86).

Indeed, S5's mathematical knowledge was extensive, but beyond that required in order to teach at school level.

(De)compressed knowledge

S5's Knowledge Maps were more difficult to construct than for the other students since individual mathematical statements were difficult to pick out of S5's highly 'compressed' (Ball and Bass, 2000, see 2.1.5.2, p.22) utterances (5.2.5.1, p.129). That is, S5 referred to squares in higher dimensions and alternative geometries. Furthermore, S5's understanding of squares demonstrated a more general definition of the concept of 'squareness' rather than an understanding tied to specific properties of a square shape – for example, the number of sides a square (shape) has (5.2.5.1, p.129). Whilst S5's mathematical knowledge was at a high level, S5 indicated in a post-observation interview that he struggled to teach school pupils in lower ability groups (during a post-observation interview on placement 2). This corroborates the beliefs held by SKE students interviewed as part of the SKE evaluation report; they felt that: “those with a specialist degree had more subject knowledge…[Graduates] may be better equipped to teach to a higher level although may find it difficult to ‘dumb down’” (Gibson et al., 2013:86). Furthermore, this also relates to comments by Ball and colleagues mentioned in the Literature Review (2.1.5.2, p.22) that teachers need a ‘decompressed’ mathematical knowledge in order: “to unpack the elements of that mathematics to make its features apparent to [pupils]” (Ball et al., 2008:10).

Misconceptions

In terms of misconceptions, S2 had a misconception about squares which he freely recalled on the pre-interview (**Figure 5.3**, p.110) but not on the post-interview (**Figure 5.4**, p.111). Further, whilst S4 had no misconceptions on his Knowledge Maps (unprompted knowledge) for squares, he demonstrated some misconceptions within the pre-interview which resulted from prompts from the interviewer (and thus did not feature on the Knowledge Maps) and these were sustained on the post-interview.

6.2.2.2.2 Rational numbers

Nodes

For rational numbers there was greater variety between the change in numbers of nodes between the pre- and post-Knowledge Maps (**Table 5.9**, p.140). Indeed, S3’s nodes decreased by two whereas the other students increased their nodes anywhere between one and four.

Differing changes in the numbers of nodes suggest that some learning of mathematical facts in relation to rational numbers may have taken place for some of the students over the PGCE course. Indeed, comparing the content of the nodes between pre- and post-interview Knowledge Maps, the additional nodes tend to be additional facts about rational numbers rather than a decompression of facts or facts becoming more accessible. S4 provides a clear example of this. Indeed, S4’s pre-interview Knowledge Map was blank (the interviewer had to define a rational number for him). Moreover, even after the interviewer defined a rational number, S4 demonstrated confusion:

> I: and what’s an irrational? Just one that can’t be written in this form [laughs] So do you have an example?
> S4: yep er so.. er.. well for example isn’t one over three no–
> I: so one over three?
> S4: yeah
> I: isn’t one over three-?

> S4: oh sorry erm.. oh so irra- are we thinking erm.. d'ya know I still think I've got what rational numbers are wrongl...

However, by the post-interview, he was able to freely recall facts about rational numbers which populated the Knowledge Map (**Figure 5.17**, p.125) suggesting that he had learnt more about what a rational number was by the post-interview. This learning may have been prompted by the pre-interview.

Similar to the topic of squares, some changes were as a result of knowledge becoming more or less accessible rather than learning new knowledge. Indeed, on the pre-interview, S5 discussed other forms of rational numbers only when prompted:

> I: …can you tell me about other forms a rational number could take?
> S5: Other forms of rational numbers...erm... [pause] I suppose I think of... I think of rational numbers... I mean we usually think of not only the fraction but the fraction in- in least terms erm like 4/8ths we would consider as one half [pre]

By the post-interview, S5 freely recalled a variety of types of representation (**Figure 5.23**, p.134). This was also the case for S3 (**Figure 5.12**, p.118). This suggests this knowledge was more accessible by the post-interviews for both S3 and S5.

Prompted pre-interview statements	**Prompted post-interview statements**
Multiplication you times top by top, bottom by bottom	top by the top, bottom by bottom
Division you.. invert the second number and then multiply them	dividing fractional numbers, we leave the first one, change the divide sign to a times sign and flip the other one upside down so it becomes a times sum
adding them means you need to get to the lowest common denominator.. same for subtraction	if we're gonna add where the denominator's different we have to er equalise the denominators first, by finding equivalent fractions, which are fractions of the same size

Table 6.1 Section of the Analysis Table for S3's pre-and post-interviews showing statements related to operations on fractions

Conversely, for S3, there were a number of statements regarding operations on rational numbers in fractional form which were prompted on both the pre-and post-interviews. **Table 6.1** (p.186) shows a section of the Analysis Table for S3 comparing these statements.

Domains

Table 5.9 (p.140) shows that for the interview topic of rational numbers the number of links either stayed the same or increased by one or two. Further, with the exception of S4 whose pre-interview Knowledge Map was blank, all students started with only the fractional form of rational numbers on the pre-interview Knowledge Maps. By the post-interview, with the exception of S2 who maintained only the fractional form, all students had linked fractional and decimal forms of rational numbers.

Three out of the four students increased the number of forms of representations of the concept of rational numbers from considering only the fractional form at the start of the PGCE course to considering the decimal forms (including recurring and terminating decimals) as well as connections between the two (**Table 5.9**, p.140). This can be seen as a general deepening of knowledge over time (Wurnig and Townend, 1997). This is further demonstrated by increases in the depth or accuracy of students' interview responses. For example, S3 was able to provide an explanation as to how equivalent fractions work by the post-interview (by effectively multiplying by a fraction equal to one, see 5.2.3.2, p.118). Further, S5 was able to explain the link between decimal representation and fractional representation of rational numbers by discussing place value (5.2.5.2, p.132). Perhaps these increases in links between fractional and decimal forms of rational numbers can be explained by familiarity with the school curriculum which recognises this link (Department for Education, 2013b). S5's definition of rational numbers was also more precise by the post-interview as he specified that the numerator and denominator of a fraction were integers with the denominator non-zero (5.2.5.2, p.132).

The fact that S2's Knowledge Maps did not alter much between pre- and post-interview (**Figure 5.5**, p.112) does not necessarily have negative implications as his knowledge of rational numbers may have already been sufficient for teaching this topic within a school setting and thus he did not need to add to his knowledge over time but simply maintain it. Conversely, S2 only discussed the fractional form of rational numbers on both pre- and post-interviews, so perhaps he did not have to teach a lesson which required knowledge of rational numbers whilst on his school placements and was thus not exposed to this topic enough to form links between representations.

Misconceptions

There were no misconceptions on the pre- or post-interview Knowledge Maps for the topic of Rational Numbers. However, it worth noting that S4 had no freely recalled knowledge of rational numbers on the pre-interview.

6.2.2.2.3 Quadratic equations

Nodes

When comparing pre- and post-interview Knowledge Maps for the topic of quadratic equations, **Table 5.9** (p.140) shows that the numbers of nodes remained the same for S3, whilst S4's post-interview Knowledge Map changed substantially with an increase of six nodes. The other students' nodes both increased by two.

As for the other topics, there was evidence that knowledge was either learnt, forgotten or became more/ less accessible over time. Examples of each of these instances will now be discussed.

During the pre-interview, S2 was prompted to discuss the discriminant of the quadratic equation and the associated implications for the type of roots determined by the discriminant:

> I:... When you're solving the equation-
> S2: mm-hm - oh your aim is to find the value of x or values of x – you can have more than one erm solution
> I: yep
> S2: erm.. <u>I think</u>
> I: <u>how many</u> solutions?
> S2: two
> I: right
> S2: but obviously it could be.. a repeated solution
> I: right
> S2: erm... and you might not have any real solutions
> I: right
> S2: and to figure that out you'll look at the discriminant [pre]

Whereas, on the post-interview, this information was freely recalled (**Figure 5.7**,p.114). This suggests that knowledge about types of solutions to quadratic equations determined by the discriminant became more accessible to S2 over time.

Conversely, some facts were freely recalled on the pre-interview, but required prompting by the post-interview. Indeed, S4 mentioned that quadratic equations need a squared term on the pre-interview (**Figure 5.13**, p.120), yet required prompting from the interviewer on the post-interview:

> I: so what happens – you said it was over 2a [quadratic formula] so what happens if a is zero?

> S4: oh sorry, yeah – then it wouldn't be erm a quadratic equation because-
> I: right
> S4: yeah. That's make it – that'd mean it was a linear equation cos you wouldn't have any x squared values. It'd be zero timesed – [post]

There were also cases where knowledge was learnt or remembered. Indeed, on the post-interview, S2 linked the types of solutions shown by the discriminant with what the graph of the corresponding quadratic function would look like (**Figure 5.7**, p.114). These links were not discussed on the pre-interview at all and suggest these connections may have been learnt or remembered over the PGCE year. Similarly, S4 discussed quadratic rates on the post-interview only (**Figure 5.19**, p.128).

An explicit example of knowledge being remembered by the post-interview is the comparison of interviews with S3. During the pre-interview, he stated with regard to methods of solving quadratic equations:

> there are other methods of solving them [laughs] which I can't currently remember… there's a third method that I currently can't remember [pre]

Whereas, on the post-interview, he was able to identify several methods of solving, including 'factorising' (**Figure 5.14**, p.120) as a third method. There are six other similar examples for S4 and S5 where knowledge can be said to have become more accessible by the post-interview. These are not included due to space restrictions.

For S2, there were several nodes on the pre-interview interview Knowledge Map which were not on the post-interview Knowledge Map. Moreover, the content of the nodes was not discussed on the post-interview even with interviewer prompting. These nodes regarded the form of a quadratic equation:

> they'll always have a variable
> they'll have a squared part to it, a linear part and then a.. constant
> you don't always need each part.. erm obviously you're always gonna need the squared
> you can express them as an equation with an equals sign or you can translate them onto a graph – you can graph an equ- er sorry a quadratic equation (**Figure 5.6**, p.114)

This suggests that S2 may have forgotten this knowledge by the post-interview. Alternatively, this knowledge may have become 'compressed' (Ball and Bass, 2000, see 2.1.5.2, p.22). Indeed, S2 drew the graph of the quadratic function during the problem solving part of the post-interview, suggesting he was aware of the connection between graphs and quadratics but perhaps this knowledge became implicit. In other words, this knowledge was not stateable but was drawn upon in a problem-solving situation.

Domains

Table 5.9 (p.140) shows that the domains remained exactly the same for S2 and S3. On the other hand, S4's post-interview Knowledge Map changed substantially with the addition of a further domain (numeric) as well as a link between existing domains (algebraic, geometric). Moreover, S4 was the only student to refer to all three domains for any topic. The presence of all three domains for S4 suggests a rich understanding of quadratic equations which transcends any single form of representation (c.f. Akkoc and Ozmantar, 2012; Wurnig and Townend, 1997). S5 also gained the geometric domain by the post-interview and a connection between it and the existing algebraic domain.

A possible explanation for S3 only considering the algebraic form of a quadratic on both Knowledge Maps could be his preference for this representation over geometric representations. Indeed, S3 stated: "I hate geometry questions" (post).

(De)compressed knowledge

In the case of S5, there was evidence of a 'decompression' (Ball and Bass, 2000) of knowledge in his changing approach to defining quadratic equations. Indeed, on the pre-interview, S5 had a 'top-down' approach to defining quadratic equations by starting with polynomials of order n and reducing down to quadratics (see **Figure 5.24**, p.135), whereas, on the post-interview, he considered only the smallest terms needed for a quadratic equation (**Figure 5.25**, p.135). Since the National Curriculum, which progresses from linear equations to quadratic equations and then cubic equations later on (Department for Education, 2013c), this may explain S5's change of approach to a form more readily accessible by pupils who should have studied linear equations previously but not yet higher terms (Department for Education, 2013c). Additionally, this change in approach can be seen as a narrowing of S5's conception of quadratic equations. Indeed, S5 also considered whether there was a corresponding quadratic formula for equations of higher order on the pre-interview (5.2.4.3, p.126) which did not feature on the post-interview.

Misconceptions

The Knowledge Maps for the topic of quadratic equations feature more misconceptions (red and yellow highlighted nodes) than for the other topics. With the exception of S3, all students demonstrated some confusion over the number of roots of a quadratic equation on the post-interview. Whilst S4 and S5 also showed confusion over the numbers of roots on the pre-interview (**Figure 5.18**, p.127 and **Figure 5.24**, p.135), S2 seemed to understand the numbers of roots on the pre-interview, but then demonstrate confusion on the post-interview (**Figure 5.7**, p.114). For three of the students, they did not rectify misunderstandings over the number of roots of a quadratic equation over the PGCE course. Perhaps this can be explained by the incomplete treatment of quadratic equations on the compulsory school curriculum since complex roots are not covered within the National Curriculum Key Stages 1-4 (Department for Education, 2013c) potentially leading to the belief that there are no roots if the discriminant is less than zero.

Alternatively, the link between the graphical form of a quadratic and the equation could cause this confusion since the roots of a quadratic can be found by looking at where the graph crosses the x-axis and, for complex roots, the graph does not cross the x-axis, potentially leading to the conclusion that there are no roots. Again, these links are featured within the National Curriculum (Department for Education, 2013c). This idea is supported by S2's Knowledge Maps (**Figure 5.6**, p.114 and **Figure 5.7**, p.114) since the confusion over the numbers of roots occurs alongside S2's greater consideration of the links between the algebraic and graphical form of a quadratic. Similarly for S5, the link between the number of solutions, the graph of a quadratic and the quadratic formula (in particular the discriminant) seems to exacerbate his confusion over the numbers of roots on the post-interview (5.2.5.3, p.134). Whilst increased representations of a concept as well as increased connections between them are thought to enhance one's understanding of a mathematical concept, this example demonstrates that if there is a flaw in one form of the representations (in this case, not fully understanding that if there are *no real* roots there are *two complex* roots - in the algebraic domain) then this can lead to misconceptions within other domains (in this case geometric).

Conversely, if students become aware of these seeming contradictions in the numbers of roots within different representation domains, then this has the potential to highlight their misconceptions enabling them to address them and thus understand the concept more fully.

Alternatively, S3 was the only one to distinguish between ‘school maths’ and ‘proper maths’ (c.f. Chevallard’s ‘didactic transposition’, Bosch and Gascón, 2006) when discussing roots of a quadratic equation (5.2.3.3, p.119). His awareness of this difference perhaps explains why he was the only student not to be confused about the numbers of roots.

S3 and S5 both mentioned additional methods of solving quadratic equations from the pre- to the post-interview Knowledge Map (**Figure 5.14**, p.120 and **Figure 5.25**, p.135). For S5, this also included being able to state the quadratic formula correctly on the post-interview, whereas for the pre-interview his attempt was slightly wrong. The National Curriculum includes a number of methods for solving quadratics (Department for Education, 2013c) and is a possible explanation for this difference.

6.2.2.2.4 Summary of changes between pre- and post-interview Knowledge Maps

Overall, as evidenced by the Knowledge Maps, there were mixed changes in knowledge between the pre- and post-interviews which varied by student and the topic under consideration. This suggests that considering knowledge change is a complex process involving a variety of factors. It is not a case of knowledge increasing over time in a certain way such as learning more facts or gaining further representations of knowledge, but these changes occur in various combinations for each mathematical topic.

Further, changes in the quantity of nodes does not completely represent knowledge change. Indeed, the above discussion highlights that an increase in nodes could be because knowledge was learnt, became more accessible or became decompressed. Similarly, a node may not be present by the post-interview Knowledge Map and this could either be because knowledge was forgotten, became less accessible, or became implicit/ compressed.

In general, the Knowledge Maps confirm that learning took several forms – either through knowledge becoming more accessible, through learning new facts, by seeing the connections within mathematics or by gaining further representations of mathematical concepts.

The National Curriculum also seemed to play a role in shaping the post-interview Knowledge Maps in some cases. Indeed, the students’ knowledge seemed to converge towards the National Curriculum either from ‘above’ or ‘below’ (either focused down to school level or expanded up to school level).

6.2.2.3 Insights from Part Two of the interviews

Table 5.10 (p.141) shows the accuracy of responses to the mathematical tasks featured in part two of the pre- and post-interviews. The findings suggest that, as a result of the PGCE course, mathematical problem solving did not improve over time, that is, there was no improvement in the number of correct answers over time. Indeed, some tasks which were answered correctly at the beginning of the course, were answered incorrectly by the end of the course (e.g. S1). This finding may be expected since the PGCE course does not focus on improving graduates' mathematical problem-solving ability but on helping them to teach mathematics (see 1.3.1, p.2).

As discussed above (6.2.2.2, p.180), the problem-solving tasks were used to verify statements made during the free-recall part of the interviews. That is, if a statement made was also used to aid problem-solving, the corresponding node on the Knowledge Map was highlighted in green.

As discussed in the methodology chapter (4.3.2.3, p.85) the problem-solving tasks were used to ensure a good estimate of the knowledge held by the students was gained. Most of the knowledge students drew upon to solve the tasks was already discussed in part one of the interviews, suggesting that the free recall followed by interviewer prompts (part one of the interview) was effective in eliciting knowledge. However, there were some instances where facts were stated by students as they solved the tasks which were not previously mentioned. These instances, as well as possible explanations for them, will now be discussed.

Whilst working on the first interview task regarding squares, S2 used the fact that squares have sides of equal length on both the pre and post-interview:

> [side] CD is that same as [side] XD [pre]
> We know that [sides] A and B are gonna be the same [post]

S2 did not state that squares have equal sides during the first part of the interview, thus it could be that this fact was 'compressed' (Ball and Bass, 2000, see 2.1.5.2, p.22) or too obvious to state. Alternatively, given the other properties of squares that S2 mentioned, it was not necessary to state this fact to fully define a square (**Figure 5.3**, p.110).

S3 and S4 also referred to mathematical facts about squares during the problem-solving part of the interview which were not stated in the first part. Indeed, S3 said:

> We know that the angles are 45 degrees because it's half of them.. so that means that the top angle is.. 90 yeah cos it's a quarter - so it is a right angled triangle [pre]

The fact that cutting a square in half results in a right angled triangle was not stated by S3 during part one of either the pre- or post-interview. Similarly, S4 said:

> so the hypotenuse.. of the triangles that are made up when you cut it in half [pre]

This demonstrates that knowledge can be held by a student but can only be accessed given a stimulus such as a mathematical question which draws upon that knowledge.

For the topic of rational numbers, there was only one instance where a mathematical fact was used in the problem solving that wasn't stated by the student in part one of the interview. S2 used knowledge of fractional and decimal forms of rational numbers to convert a decimal to a fraction on both the pre- and post-interview:

> S2: it's saying that it does eighty three whole rotations around the circle and it's got point three three three...
> I: I think it means recurring
> S2: Oh is it? OK erm, so a third of a rotation- [pre]
> it's gone through eighty three whole rotations and a third of a rotation [post]

For the topic of quadratic equations, several students used facts related to differentiating quadratic equations during problem-solving which were not mentioned in part one of the interviews. Indeed, S3 discussed the vertex of a quadratic function on the post-interview problem-solving:

> the vertex is the.. either the maximum or minimum point of a quadratic graph
> because that's a positive, it'll be the minimum point
> The minimum point of all those graphs occurs when x is zero
> dy[by] dx equals $2x$ plus b, so the minimum point is when that – the gradient is zero [post]

Similarly, S5 stated on the post-interview task:

> its derivative equalling zero means it is the minima [post]

Furthermore, S4 stated:

> if I differentiated that'd give the area under it [pre]

The above examples all show that knowledge can be held and accessed in a problem-solving situation that may not be stated within a free-recall situation. In other words, some knowledge can be dependant upon some stimulus to be able to access it.

Conversely, there was one example of knowledge becoming stateable over time. Indeed, for the topic of quadratic equations, S2 used facts regarding minimum points of quadratic functions to aid him when working on task three on the pre-interview:

> Erm it's a.. minimum… Erm so the differential of the equation is gonna be equal to zero [pre]

However, by the post-interview, he was able to freely state facts relating to differentiating quadratic functions on the post-interview:

> and finding the second order – you can find whether it's a maximum or a minimum [post]

In summary, the problem-solving part of the interview shows that most of the knowledge used by students to solve the tasks was already stated during part one of the interview, suggesting the effectiveness of the interview structure (free recall followed by hierarchical focusing/ prompting) in eliciting a good estimate of the knowledge held by students for the three mathematical topics. However, there were some instances where additional knowledge was gleaned by the problem-solving tasks. As discussed above, this intimates that some knowledge may require a stimulus or specific context in order to be elicited.

6.2.2.4 Qualitative findings: observations

Five students on the PGCE course at the University of Leeds who participated in the interviews were also observed whilst teaching three lessons on their school placements (see 4.3.3.1, p.94 for full details). The Knowledge Quartet (Rowland and Turner, 2007, see 2.1.8, p.27) was used to analyse the video recordings of the lessons and resulting analyses are presented in section 5.3 (p.141). Since the Knowledge Quartet is a framework for reviewing trainee teachers' mathematics lessons with a focus on the mathematics content of the lesson (Rowland and Turner, 2007), it was intended to be used to highlight changes in knowledge (if any) over time for each student in order to address RQ2. However, issues arose when attempting to compare Knowledge Quartet codes over time.

Firstly, the numbers of each code cannot be used to make comparisons over time since codes can be used in both a positive or negative way. For example, an episode coded as 'identifying [pupil] errors' could be coded as such due to a lack of identifiying an error made by a pupil or because an error was identified.

Similarly, changes in the number of instances of a code over time may not be indicative of a change in knowledge but be a result of a number of factors such as the mathematical topic discussed, how familiar the pupils are with the topic, how participatory pupils are, or a number of other contextual factors. For example, S3's second observed lesson featured no 'responding to children's ideas' codes, whereas the third observation featured two instances. However, since the first half of the second lesson was a mental arithmetic test taken in silence, any discussion with pupils during this test would have been inappropriate.Thus, it cannot be concluded from the quantity of codes alone that S3 became better at responding to children's ideas over time. Similarly, less 'identifying pupil error' codes in an earlier lesson may be because less pupil errors were made during the lesson, rather than the teacher becoming more able to identify errors in a later lesson where more episodes were coded as such.

Due to these issues, identifying patterns over time was not straightforward. Indeed, no clear trends were identified.

Given the difficulties of using the individual Knowledge Quartet codes to compare knowledge over time and between students, each observed lesson was given a score for each of the four dimensions (Foundation, Transformation, Connection, Contingency, see 2.1.8, p.27) of the Knowledge Quartet under which the individual codes reside. These scores reflected the overall quality of the lesson in terms of these four areas using the coded episodes as a guide. Scores ranged from high to low and were grouped under five headings: high, medium-high, medium, medium-low, low. Scores are provided in **Table 6.2** (p.197). To achieve an indication of the reliability of these scores, a lesson transcript was selected at random and read by my supervisor. He underwent 60 minutes of coding training provided by me before independently scoring the lesson. An inter-coder reliability of 87.2% was achieved.

Table 6.2 (p.197) shows that Connection and Contingency scores were never higher than Foundation scores, whereas in one instance the Transformation score was higher than the Foundation score. Firstly, this suggests that good foundation knowledge is a pre-requisite for making and demonstrating connections in the classroom. The Knowledge Maps support this since the number of links between nodes are limited by the number of nodes. For example, if there are only two nodes on a Knowledge Map, there can only be a maximum of one link between them.

Secondly, the table suggests that ability and confidence to deal with contingent moments arising in the classroom (including unexpected pupil input) is low or low-medium when foundation knowledge of the topic taught is low-medium. However, if Foundation knowledge is high, this does not necessarily mean Contingency scores will be high. This may be because contingent moments do not always arise.

With the exception of one case, Transformation scores were also never higher than Foundation scores suggesting that teachers cannot help transform the subject matter knowledge into a form accessible for pupils beyond their own understanding of the subject. In the one case where a higher Transformation score than Foundation score was obtained, this was for S5's second observed lesson where although his knowledge of Pythagoras' Theorem was high, the score was reduced to 'medium-high' (**Table 6.2**, p.197) due to episodes of poor use of terminology:

> For this one are we working forwards or are we working backwards?
> …you've found the diagonal but for this one, it's backwards. So thinking about Pythagoras, how would you work backwards?

		Observation		
Student	**Dimension**	**O1**	**O2**	**O3**
S1	Foundation	High	High	N/A
	Transformation	Low	Medium	N/A
	Connection	Low-medium	High	N/A
	Contingency	Medium-high	Medium	N/A
S2	Foundation	High	Medium-high	Medium
	Transformation	Medium-high	Medium-high	Medium
	Connection	High	Medium-high	Low
	Contingency	High	Medium-high	Low
S3	Foundation	Medium	Medium-high	Medium
	Transformation	Low	Low-medium	Low-medium
	Connection	Low	Low	Medium
	Contingency	Low	Low	Low-medium
S4	Foundation	Medium-high	Low-medium	Medium-high
	Transformation	Medium-high	Low-medium	Medium
	Connection	Low-medium	Low-medium	Medium-high
	Contingency	Low	Low-medium	Low-medium
S5	Foundation	High	Medium-high	Medium-high
	Transformation	High	High	Low-medium
	Connection	High	Medium-high	Low-medium
	Contingency	High	Medium	Medium-high

Table 6.2: Table showing observed lesson scores for each dimension of the Knowledge Quartet

In these episodes, S5 is referring to rearranging Pythagoras' theorem to make one of the side lengths of the triangle the subject, but does not use this terminology. However, this phraseology is understood by the pupils who are subsequently able to progress with the problems set. Thus, S5 transforms the problems in such a way that the pupils can understand and progress but which is perhaps not mathematically accurate. Further, the way S5 made excellent '*choice(s) of representation'* with the use of a ribbon coming out from the board to show 3D Pythagoras' Theorem (see 5.3.6.2, p.170 for details) meant S5 scored a 'high' in the area of Transformation for this lesson.

These results imply that good subject knowledge is necessary but not sufficient for Transforming subject matter for pupils, demonstrating Connections within lessons, and being able to react confidently to Contingent moments arising in the classroom.

6.2.3 Summary of findings

Overall, there was an increase in scores on the MKT items over time. Further, there tended to be an increase in the number of nodes on the Knowledge Maps, though usually by only one node. However, both S4 and S5 increased the numbers of nodes for rational numbers and quadratic equations by between 3 and 8 nodes. Conversely, there were two decreases in the number of nodes: S3, rational numbers and S4, squares (by two and one node(s) respectively).

Since nodes represent discrete mathematical facts, it was not expected that there would be great changes in the numbers of nodes since the PGCE course is not intended to teach mathematical content knowledge – all students are required to have a minimum level of mathematical knowledge in order to commence the course.

For all students, the numbers of links on the Knowledge Maps either stayed the same or increased by as many as six connections (with the exception of S2 discussing squares where the number of links decreased). This increase in connections could be explained by the students revisiting mathematical topics with which they should already be familiar and being able to make further connections with other areas of mathematics which they are teaching to different classes (see 6.4.2.2.3, p.210, for more on this).

For all students, the mathematical domains on the Knowledge Maps either stayed the same or were added to and, in some cases, links were made between domains, that is, two or more domains were seen as related by the post-interview, rather than as separate and distinct representations. Again, being exposed to various representations of concepts, including alternative pupil methods could help explain this (see 6.4.2.1.4, p.208). Thus, changes over the course of the PGCE tended to be with regard to the organisation of knowledge including making connections between existing discrete knowledge and connections between alternative representations of the concepts, rather than gaining 'new' mathematical knowledge.

Most students improved in their explanations of mathematical concepts between the pre- and post-interview. This improvement took one of several forms: (i) explanations became more succinct (ii) explanations became fuller and went into greater detail (iii) explanations became more accurate. See RQ4 for possible factors leading to this change.

In terms of misconceptions, these were either eliminated or sustained by the post-interview depending upon the topic under consideration.

In general, changes in interview responses over time varied by the mathematical topic being discussed, that is, there were no general patterns over time but this varied by student and whether squares, rational numbers or quadratic equations were being discussed. This fits with findings of Ball and colleagues who found that MKT was: “at least partly domain specific, rather than simply related to a general factor such as overall intelligence, mathematics or teaching ability” (Hill et al., 2004:26). Nevertheless, the content of the National Curriculum seemed to be an explanatory factor in any changes in knowledge over the PGCE year as responses became more closely aligned with the content of the curriculum whether this involved adding topics or links present within the curriculum to the Knowledge Maps or removing superfluous elements of knowledge which are not required for mathematics at the secondary school level.

6.3 Research Question 3 (RQ3)

When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: Does it differ between SKE and Non-SKE PGCE students?

6.3.1 Introduction

This section compares the data (collected via questionnaires, interviews and observations) between students who had taken an SKE course prior to their PGCE and those who had not in order to determine if any differences between their knowledge exist. Each data collection method is taken in turn.

6.3.2 Findings

6.3.2.1 Questionnaires

Responses to the questionnaire were compared between SKE and Non-SKE students by means of a repeated measures ANOVA (5.1.4.5, p.105). Whilst, on average, scores on the MKT items were shown to improve over time, the between subjects effect (Non-SKE and SKE) was not significant, suggesting that whether or not a SKE course had been taken did not have an effect on the change in scores. However, there was a difference (albeit non significant) in scores which was sustained over time. Indeed, the Non-SKE group scored 1.7% and 2.6% more than the SKE group respectively on the pre- and post-questionnaire on average.

This finding corroborates views of the Learning Mathematics for Teaching (LMT) group who created the items. Indeed, they: “began to notice how rarely [the] mathematical demands [of teaching] were ones that could be addressed with mathematical knowledge learned in university mathematics courses” (Ball et al., 2008:8). This study supports this view as Non-SKE students did not perform significantly better than their SKE peers on the MKT items despite having studied mathematics at university level.

Similarly, the Rasch analysis also supports the above findings as there was only differential item functioning (DIF) for SKE for one item (see 4.3.1.4, p.71).

The MKT items were also separated into two groups, those which were felt to test for ‘Common Content Knowledge’ (CCK) and those testing Other aspects of MKT. Repeating the ANOVA with these groups it was found that SKE students performed slightly worse on average than traditional entry PGCE students on CCK items but slightly better on Other items (5.1.4.5, p.105). These differences were not significant, but were sustained over time. It is perhaps expected that graduates with high levels of mathematical content in their degrees (Non-SKE students) would perform better on CCK items than graduates with less mathematics in their degrees (SKE students), however, the non-significant difference between the two mean scores on both the pre- and post-questionnaire again supports other results in this study which suggest high levels of mathematical content within a degree are not necessary for successful completion of a PGCE course (6.1.2.1.1, p.176).

The Specialised Content Knowledge (SCK) items[12] are said to ‘surprise, slow or even halt’ those without teaching experience but who are knowledgeable in mathematics (Hill et al., 2004). The Non-SKE group, many of whom have degrees in mathematics but are not qualified teachers yet, can be said to fit this description, thus it may be expected that they perform less well on the Other items. However, it should arguably be the case with the SKE group as well since they have taken the SKE course to improve

[12] This label was given by the MKT item authors and the SCK items were included within the ‘Other items’ category used for the current research.

their content knowledge and like the Non-SKE group are not yet teachers. It may be worth exploring further the reasons why SKE students were not as ‘surprised, slowed or halted’ by the Other items than the Non-SKE students. One possible explanation is that the SKE course equips students with knowledge needed to answer these items. Indeed, Gibson et al (2013) found that many SKE courses include aspects of pedagogy (though the amount varies between courses). Thus, SKE students have potentially spent at least five months (the minimum length of SKE courses taken by students in the SKE group) learning mathematics with the intention of teaching as well as acquiring some knowledge of how to teach it, which perhaps explains higher mean scores on these items.

Additionally, when comparing final PGCE results, there was no significant difference between SKE and Non-SKE students (5.1.4.5, p.105). This corroborated results of Stevenson (2008) who also found no significant difference in a single cohort of students at one university on England.

In summary, no significant differences were found between SKE and Non-SKE PGCE students on the MKT items overall, on subsets of the MKT items or final PGCE scores.

6.3.2.2 Interviews

When comparing interview responses over time between SKE students (S2 and S4) and Non-SKE students (S3 and S5) there were no discernible patterns or features of their responses to enable a distinct difference in the knowledge held by these two groups to be highlighted. Indeed, **Table 5.9** (p.140) does not reveal any common differences in the numbers of nodes, domains and links between SKE and Non-SKE students. As for RQ2, responses varied by student and the mathematical topic under discussion rather than whether the student had taken an SKE course or not. However, there are a few incidents which could be explained by a SKE course not being taken immediately before commencing the PGCE course: not having the various methods of solving a quadratic equation immediately to hand upon commencement of the PGCE course and a difference in misconceptions. These will now be discussed.

Firstly, the Non-SKE students (S3 and S5) both mentioned two methods of solving quadratic equations in the pre-interview (**Figure 5.13**, p.120 and **Figure 5.24**, p.135), whereas, SKE students mentioned three (S2, **Figure 5.6**, p.114) or five (S4, **Figure 5.18**, p.127) methods. Taking a SKE course immediately prior to commencing the PGCE course may account for this difference since it may have been a few years since Non-SKE students had to solve quadratic equations, yet the SKE students may have covered them on their SKE course, making the methods more readily accessible to be recalled in the pre-interviews. This provides evidence for beliefs expressed by PGCE students interviewed as part of the SKE evaluation report; they thought: "former SKE students would be better prepared to do the PGCE course than traditional route students …. [one of the] main reasons [was] that they would be… more familiar with the curriculum…" (Gibson et al., 2013:14).

Secondly, with regard to misconceptions, the SKE students (S2 and S4) both had misconceptions about squares on the pre-interview. Indeed, S2 had a red highlighted node on the pre-interview Knowledge Map (**Figure 5.6**, p.114) and S4 had some misconceptions on the pre-interview which resulted from interviewer prompting (5.2.4.3, p.126). By the post-interview, S2 eliminated his misconception (**Figure 5.7**, p.114) but one of S4's misconceptions was sustained and the others cannot be commented on as they were not discussed to enable a comparison to be made (5.2.4.3, p.126).

The Non-SKE students (S3 and S5) did not demonstrate any misconceptions on the pre-interviews (for squares and rational numbers) (**Figure 5.9**, p.117, **Figure 5.11**, p.118, **Figure 5.20**, p.130, **Figure 5.22**, p.134). However, S5 had a misconception about the number of rational numbers in comparison to integers on the post-interview 5.2.5.2, p.132); this was not mentioned on the pre-interview to enable comparison.

For quadratic equations, the SKE students (S2 and S4) and Non-SKE student S5 all demonstrated misconceptions about the number of solutions to a quadratic equation either on the post-interview or both interviews (5.2.2.3, p.113, 5.2.4.3, p.126, 5.2.5.3, p.134). S3 was the only student to correctly identify and explain the number of solutions to a quadratic equation (5.2.3.3, p.119).

Of course having misconceptions and overcoming them can have positive implications for teachers. Indeed the SKE course evaluation report states that SKE students felt: "they would be better equipped than traditional route trainees to… understand misconceptions and the difficulties pupils faced often having experience of them on the SKE course." (Gibson et al., 2013:13). For this study, it was not only the SKE students who held misconceptions but the Non-SKE students also.

6.3.2.3 Observations

In order to address RQ3, the Knowledge Quartet analyses of the observed lessons taught by five PGCE students on their school placements were compared between those who had taken a SKE course and those who had not. However, there was only one possible difference found between the two groups and, interestingly, amended versions of the Knowledge Quartet codes were suggested for the episodes highlighting this difference. Thus, it could be said that the Knowledge Quartet analysis was not useful when comparing observations between two groups of students. This may be expected since the Knowledge Quartet was not originally intended for this purpose. The one difference identified will now be discussed.

The lesson observations reveal one potential difference between SKE and Non-SKE students, namely, knowledge of the conventions of the discipline of mathematics. Indeed, S1 (Non-SKE) demonstrated a sound understanding of mathematical conventions and mathematical models. Firstly, during the first observed lesson with S1, he made several statements showing he is aware of mathematical conventions when writing algebraic expressions:

> we write the number first
>
> so we've got D 4 T but we don't like to put numbers halfway through the expression
>
> …there was a 4DT and a 3D so we can't combine them, we can't combine different letters and we can't combine different combinations of letters so if we have a DT we can't group that with T's
>
> What's a D times a D? How can we write that even better?... what's being timesed by itself?... D, so we've got 4 D squared, plus 3D. That is the powers bit. If you've got a letter timesed by itself, it's that letter squared…You can write it out like this first if you want [indicates top line $4dd + 3d$] but want to make it nice and slick. Nice and efficient. [S1 P1 O1, see 5.3.2.1, p.142 for full details]

These instances were coded as '*concentrates on procedures'* but perhaps '*concentration on conventions'* (not a Knowledge Quartet code) would be a more accurate a description.

Secondly, S1's third observed lesson demonstrated S1's knowledge of the discipline of mathematics in terms of mathematical models. When referring to the potential energy of models of a solar cooker which pupils constructed during the lesson, S1 commented:

> A hundred and 57 point 5 watts of power… this is if it was perfectly efficient, if all the light bounced off the tin foil and hit the food, right and if none of the power was used heating the air and all that sort of business.

This suggests that S1 understands the limitations of simplified mathematical models (see 5.3.2.3, p.147 for full details). This episode was coded as '*overt subject knowledge*'. However, it was felt that '*overt discipline knowledge'* (not a Knowledge Quartet code) would perhaps better capture the knowledge of mathematics as a discipline which was portrayed by this episode.

S3 (Non-SKE) was also aware of conventions of mathematics as shown in S3's second observed lesson on BIDMAS. Indeed, S3 demonstrated knowledge of what BIDMAS stands for, how to apply it to a mathematics problem and underlying reasons behind BIDMAS (5.3.4.2, p.157). This was coded as '*overt subject knowledge'*. Further, in the post-interviews, S3 was the only student to recognise a difference between 'school maths' and 'proper maths' (5.2.3.3, p.119).

However, in some cases, both S1 and S3 tended to hold their knowledge of mathematics with its associated conventions implicitly, that is, they took it for granted so were not always able to explain this knowledge explicitly to their pupils. For example, during S1's first observed lesson, no explanations were offered about why the rules he stated for writing algebraic expressions held. Whilst it was clear S1 understood the rules, it seemed his knowledge was taken for granted and pupils had to adopt them without understanding why or where the rules came from (5.3.2.1, p.142). Conversely, S3 did make an attempt to explain the reasons behind the mathematical convention of BIDMAS:

> That's why we're looking at BIDMAS because we do the multiplication before we do the addition. OK. This is why we have brackets, so if wanted to do the addition first there, we would have put brackets round it, ok?

However, the fact that BIDMAS is a mathematical convention adopted to avoid confusion when reading and interpreting mathematical expressions was never explicitly stated in this lesson.

On the other hand, S2 (SKE) was not so knowledgeable about mathematics as a discipline as shown in his second observed lesson (see 5.3.3.2, p.151 for full details). Indeed, when discussing the answer to zero raised to the power zero, he stated:

> I'd say, it's zero but I think it's an issue that some people might disagree with alright, some people might try and argue that it's one, but I'd say it's zero… Anything to the power zero is one, apart

> from – most probably – zero, because some people might argue, alright… We had a discussion about this at uni once and some people tried to argue that it was one. I'd say it's zero

Whilst this statements shows that S2 is aware of the controversy surrounding the answer, it also shows his lack of understanding that mathematicians determine mathematical conventions rather than 'uni' students.

Knowledge of the discipline of mathematics was the only difference which tended to divide SKE and Non-SKE students, other instances varied by the individual student. Some examples follow:

Firstly, S1 and S3 were both Non-SKE students. However, S1 demonstrated the ability to evaluate mathematical methods, whereas, S3 did not. Indeed, S1 explained in depth the superiority of the 'grid method' over the 'FOIL method' (5.3.2.2, p.145) for expanding brackets (5.3.2.3, p.147), whereas, S3 used methods suggested by other teachers, namely, the LET method for dividing fractions (5.3.4.1, p.155) and rounding to one significant figure (as a technique when estimating, see 5.3.4.3, 158) despite these methods proving to be problematic in the classroom.

Secondly, both S1 (Non-SKE) and S2 (SKE) chose to focus on procedures rather than concepts when teaching lessons on expanding brackets (5.3.2.1, p.142 and 5.3.3.3, p.153). Conversely, S5's approach to teaching was decidedly conceptual which contrasted to the first lessons of the other students which featured '*concentration on procedures*'. In S5's first observed lesson not only did he demonstrate his understanding of the connection between procedures across different solid shapes and across dimensions, but he linked algebraic and geometric representations of shapes (5.3.6.1, p.168).

Finally, in contrast to some lessons where students were not keen to explore pupil's thinking (for example, S4's second lesson on probability trees, 5.3.5.2, p.163), S5's first lesson relied heavily on pupil input as he steered their ideas in the intended direction (5.3.6.1, p.168). S5's confidence with the subject matter was therefore demonstrated as he was not afraid to explore and respond meaningfully to pupils' ideas.

6.3.3 Summary of findings

In summary, the questionnaires and interviews reveal no differences between SKE and Non-SKE PGCE students in terms of scores on the MKT items, final PGCE grades, numbers of nodes, links and domains on the

Knowledge Maps. With regard to misconceptions demonstrated within the interviews, these tended to vary by mathematical topic.

The observations reveal only one difference between SKE and Non-SKE students, namely, understanding of mathematics as a discipline. Indeed, Non-SKE students (who had all taken a mathematics degree) seemed to better understand mathematical conventions and the nature of mathematical models.

6.4 Research Question 4 (RQ4)

When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course: What are some of the factors which have led to a change (if any)?

6.4.1 Introduction

The first three research questions seek to understand more about the nature of mathematical knowledge of trainee secondary mathematics teachers, how it changes over time, and how it differs between SKE and Non-SKE students. This final research question aims to understand some of the factors which have led to a change in knowledge over the course of the PGCE year.

Several factors and sources of the PGCE students' learning were identified during observations and post-observation interviews. These are discussed below. Where the students are quoted, the student (S), placement number (P) and observation number (O) are provided in square brackets e.g. '[S1 P1 O1]'.

6.4.2 Findings: Factors and sources of learning during the PGCE course

Factors for learning were categorised into three situations from which they arose. Firstly, sources the student consulted before the lesson in order to teach a topic which needed refreshing, further learning or clarification on the part of the PGCE student in order to teach it. Secondly, learning within the process of teaching itself, that is, during a lesson. Finally, learning from tutors or school based mentors on the PGCE course. These will be discussed in turn.

6.4.2.1 Sources consulted prior to teaching a lesson

6.4.2.1.1 Reading

Four of the five PGCE students observed for this research identified reading as a source of learning mathematics on the PGCE course.

S1 was able to make the connection between multiplication and area as a result of reading. Indeed, S1 explained:

> I read a book where Jo Boaler showed someone the rectangle and that was nice… I was shown grid method at school and then I read about it being in a rectangle I think and I'm not sure if I made the connection or if it was made for me [S1 P2 O1]

Thus reading research-based literature by Boaler helped S1 to make this connection.

S4 indicated that he also learnt mathematics by reading. In a post-observation interview, he explained:

> I've never actually done index numbers before, I don't think it was covered on the enhancement course or anything I'd done before so it was the first time I've come across it… with learning it myself I went through what the book said, I found things on the internet and I used them to piece together the worksheet and my examples and things like that [S4 P2 O2].

Here, S4 had to read the text book as well as sources online to learn the topic himself before he could teach it to others. S2 also said he had to learn a topic (fractional indices) before he could teach it:

> I: did you learn any maths while teaching or while preparing to teach that lesson [on indices]?
> S2: No because I've taught it before, but when I originally did, it became more clear because to be fair, I can't remember covering fractional indices at – obviously at A-level when – I was used to working with fractional indices with differentiation and integration and stuff but I had to go back and see what does fractional indices really mean, so yeah, not this time but when I did first teach this.
> I: so how did you find that out?
> S2: through text books [S2 P2 O1]

Thus, because S2 could not remember learning this topic before, he had to use text books to inform him about indices.

Finally, S5 said he had used the Oxford dictionary of mathematics and that led him onto further words to do with certain topics, thus he incidentally learnt mathematics whilst preparing to teach.

6.4.2.1.2 Other teachers

S3 was the only student who mentioned other teachers as the source of his learning on his PGCE course. These other teachers were either within the department at the placement school he was teaching at or on the Internet in teacher's forums. Indeed, S3 said the 'LET technique' (as a mnemonic to remember the steps when dividing fractions, see 5.3.4.1, p.155) was told to him by another teacher in the department. Further, ‘rounding to one significant figure’ as a strategy for estimating was “just in teachers’ things on the internet… in all the lessons I looked at that other teachers had taught” [S3 P1 O1]. Unfortunately, observation revealed that these techniques caused confusion for the pupils in the lessons. Indeed, the procedural approach of the LET method resulted in pupils being unable to proceed with multiplying fractions despite this being one of the steps when dividing fractions (5.3.4.1, p.155). Similarly, S3 and the pupils in class recognised that rounding to one significant figure was not effective in all cases as a more accurate estimate could be achieved when estimating the answers to calculations involving larger numbers such as 1609 x 2099 (5.3.4.1, p.155).

6.4.2.1.3 The internet

Both S1 and S5 mentioned the Internet as a source of learning about topics which they had either never been taught themselves or never had to teach. Indeed, S1 said he had learnt applications of Pascal’s Triangle and additional features as a result of researching on the internet following a request to teach a lesson on Pascal’s triangle for a job interview (post-observation interview, P2 O1). S5 said (in the post-observation interview for P1 O2) that he found out that 3D Pythagoras would not work in non-Euclidean space. He explained that in spherical geometry Pythagoras breaks down so its counterpart (3D) by extension would also not work. S5 said that he did not cover spherical geometry in his degree but it was mentioned so he knew it existed and he learnt a bit about spherical geometry by scanning Wikipedia.

S4 also said (in the post-observation interview for P2 O2) he used the internet as well as books when researching a topic (see section above) and to inform the selection of examples and questions:

> The examples I made up but I obviously used things I’d found on the net to guide it really… The [worksheet] questions are made up from past papers and other worksheets that I’d found online so I just pieced them together really… A lot of the examples I’d found were trying to link it with business stuff really so I just went with it [S4 P2 O2].

6.4.2.1.4 Personal investigation

Following a lesson on volumes of pyramids and cones (5.3.6.1, p.168), S5 explained during the post-observation interview that when preparing to teach this lesson he had asked himself ‘why is the volume of a pyramid and a cone a third that of a cuboid and cylinder respectively?’. In particular, he wondered why it was a third for a cone as well as a pyramid. Whilst he knew that one could derive the formulae by integration, he had to explore alternative explanations to satisfy his (pre-calculus) pupils. This led him to investigate the links between the volumes using a spreadsheet to calculate the volumes of solids with different dimensions. He had a column for radius, area of a circle and area of a square and then for volume of a cube and cylinder and then for square-based pyramid and cone. Looking at the values he realised that there was always the same difference (relationship) between them. This also worked for surface area. Thus, personal reflection and investigation led to S5 to learning connections between the volumes of a pyramid and cone.

6.4.2.2 Factors and sources of learning during teaching a lesson

6.4.2.2.1 Explaining to others

The act of explaining a mathematical concept to another person seemed to help some students to refine their explanations and to transform their implicit knowledge into a form which could be verbalised to their pupils. There were two examples of this.

Firstly, although S3 understood what a significant figure was, it was likely that he had not explained it to another person before. Indeed, during the course of a lesson on estimating, S3’s explanations became progressively more succinct (5.3.4.3, p.158) until he was able to define it as: “the first non-zero term when reading from the left” by the post-observation interview. This was more precise than the multiple examples S3 had to present to his pupils at the commencement of the lesson in an attempt to facilitate their understanding. Two possible explanations are: either S3 knew this ‘definition’ all along and chose to use examples only when explaining to the pupils, or, by teaching a lesson on significant figures and having to explain to someone else, he may have formed an explanation which could be verbalised. Hence the process of teaching caused him to transform his internal understanding of significant figures into a form which could be expressed to another. This is similar to George, an English teacher in a study by Wilson and colleagues (1987), who found it difficult to explain ‘theme’ to his pupils:

> Throughout his undergraduate education, he had been required to do thematic analyses in his literature courses. Yet he had never been required to define the concept of theme. Until he had to *teach* theme, his intuitive understanding of theme… had been sufficient. In thinking about communicating his understanding to his subjects, however, he struggled with explicating that intuition (Wilson et al, 1987:116).

A similar progression could also be seen with S4's explanations of averages to pupils in a lesson on data handling (5.3.5.1, p.161). Indeed, in the post-observation interview, S4 said he had learnt how to explain what mean, medium, mode and range were to his pupils when asked if he had learnt any mathematics whilst teaching this lesson.

Explanations becoming more succinct also occurred between pre- and post-interviews (5.2.2.1, p.109).

6.4.2.2.2 Using a method/ algorithm for a more advanced topic

In the post-observation interview for P2 O1, S1 explained how he had learnt that the FOIL method (5.3.2.2, p.145) did not work for higher powers within this lesson:

> I: You saw that when doing higher powers, FOIL didn't work
> S1: yeah I hadn't seen that before because I hadn't thought that far forward because I hadn't used it, but it's really a failed method which I have no interest in teaching – or promoting.

Here, S1 indicated that by using the FOIL method for a topic which appears later in the curriculum (involving higher powers), he saw that it no longer was useful.

6.4.2.2.3 Teacher reflection (prompted by pupil questioning and teaching multiple topics concurrently)

During the first lesson observation with S1, the final part of the lesson involved S1 making the link between area and square numbers for his pupils:

> Oh final bit - why do they call it squared? Why not two-ed? Why squared, are mathematicians just odd?... [Draws a square of side length t] there's a square, there's t there's t, what about that is t squared? Why squared, why not two-ed? It'd make more sense surely?... what is t times t? [shades in the square] what is t squared?… I like to think of it as the area of a square, ok so t times t would be the area of a t length square [S1 P1 O1].

In the post-observation interview, I asked S1 about this connection. He replied that the link came as a result of talking to a Year 11 pupil who was doing work on squares and who had asked S1 why it is called 'squared' (as opposed to 'two-ed'). S1 thought about this, and as he was also teaching area to another class at the time, this led to him making the link with squared numbers and area. Thus, several factors can be said to have played a role in S1's connection-making. Firstly, pupil questioning, and secondly, teaching different mathematical topics at different levels (different school year groups) whilst on the placements. In other words, because S1 was teaching a Year 11 class about area and also teaching a Year 8 class on powers, this aided

S1 in making this link. Whilst the learning of mathematics is arguably in a linear fashion, that is, progressing topic by topic, from easier concepts to more difficult ones, teaching mathematics can involve different mathematical topics at different levels within the same school day. This has the potential to lead to connections between topics to be learnt by the teacher.

6.4.2.3 Factors of learning relating to taught aspects of the PGCE course

6.4.2.3.1 Post-lesson meetings with tutor/ mentor

During the post-observation meeting with S4 and his school-based mentor and university tutor, S4 was told by his university tutor that the mean, median and mode are all ‘averages’ and that the range is a ‘measure of spread’. During the lesson, S4 had not made this distinction, in fact he had referred to them all as ‘averages’ (5.3.5.1, p.161). Thus, this was a learning opportunity provided by his university tutor.

6.4.2.3.2 PGCE taught sessions

The taught PGCE sessions at the University of Leeds fostered a culture of seeking alternative mathematical methods and encouraged students to explore alternatives with their pupils. Indeed, during eleven of the fifteen taught university sessions, there were at least one instance of tutors encouraging, promoting and highlighting alternative methods. Only a few of these many examples are provided below due to space limitations.

On the first day of the PGCE course (day 1, session 2) the course tutors told the PGCE students that, whilst teaching, they would encounter all sorts of mathematical methods not only from their pupils but also from their parents. Indeed, one tutor commented that he: “found 13 different methods in one class" for long multiplication.

On day 4, session 1, the PGCE students were asked to solve the following problem:

> Jo, Max and Jing shared 400 marbles amongst themselves. Jo received 28 marbles, Jing received seven times the total number of marbles Jo and Max received. How many more marbles did Jing received than Max?

After solving it, students were asked to write their different solution methods on the 4 boards in the classroom. Common across all the answers was the use of algebra. When asked by the tutor if they could do it without the use of algebra, the students could not respond. The guest tutor then drew a stick to represent 400 and split it into 8 parts (since Jing receives seven times the total of the others, so seven parts compared to the other single part – 8 parts in total). The tutor explained how each part was worth 50, since Jo had 28, Max must have 22 to make up to one part (50) and Jing has 7 x 50. This alternative representation led to a solution much quicker than the use of algebra.

On Day 4, session 1, there was a discussion about the different ways algebraic 'letters' could be used. Such uses include: abbreviations, computable numbers, constants (such as pi and e), specific unknown variables, generalised numbers (e.g. $a + b = b + a$), indeterminates in identities, parameters ($y = mx + c$), and variables. Thus, it was shown that mathematical objects can often take on different meanings depending on the context and that this is a potential source of confusion for pupils.

On Day 6, Session 2, the importance of connections within mathematics was explicitly focused upon. One slide of the PowerPoint Presentation used showed the following connections:

- Between 2x and 4x tables and doubling and halving
- Number bonds to 10 can help in a large range of calculations (many pupils survive by counting on and back on their fingers)
- Between multiplying by 3, 30 and 300
- Relationship between 4 operations (addition and subtraction, multiplication and division being inverses, multiplication – repeated addition, division – repeated subtraction)

The next slide suggested how visual representations can aid connection-making:

- Blank number lines for subtraction – adding on strategy
- Arrays to make the connection between multiplication and division
- Counting stick for multiplication tables, equivalence of fractions, decimals and percentages

Thus, this session taught the PGCE students about connections between not only mathematical concepts, but how these concepts can be represented to pupils to help them form these connections.

The above activities and tutor comments show how the PGCE course at the University of Leeds emphasised connections and alternative solution methods and representations. Further comments were also made such as making connections between different areas of the mathematics curriculum; that pupils may see alternative methods of completing a task and not to limit them; that "you need to see the topic in as many different ways as possible, and also you have to bear in mind, people learn in different ways"; and finally: "as a teacher you need to make those connections”.

6.4.3 Conclusion

In conclusion there are many sources which can account for the change in mathematical knowledge of the PGCE students involved in this research. These sources mainly stem from the necessity of having to teach a mathematical topic to others and either occur as a result of prior knowledge that a deeper understanding of the topic to be taught is needed or during the teaching process whereby it becomes clear that a topic implicitly understood requires some transformation or ‘decompression’ (Ball and Bass, 2000, see 2.1.5.2, p.22) in order to impart this understanding to others. Indeed, when preparing in advance to learn a topic for teaching, the following sources were used by students: other teachers, the internet, reading, and personal investigation. When learning occurred through the act of teaching, this was through repeated explanation to pupils (whereby the explanation became more refined), through using a mathematical method for a more advanced topic (and as a result, seeing the limitations of the method), or through pupil questions which prompted personal reflection combined with making connections between different topics as a result of teaching them concurrently.

Two further sources of learning through teaching were in post-lesson meetings with tutors/ mentors whereby incorrect mathematics was corrected and through taught PGCE sessions at university which created a culture of thinking about a variety of different mathematical methods and solutions.

6.5 Synthesis of findings

Since findings from RQ2, RQ3 and RQ4 also provide evidence to address RQ1, this section synthesises the findings from all RQs in order to show what this research contributes to our understanding of the nature of the mathematics-related knowledge held by trainee secondary mathematics teachers on a PGCE course.

The quantitative results which address RQ1 suggest that final PGCE grades are not significantly related to how well one performs at GCSE, A-level or degree level nor to the amount of mathematical content of said degree. A possible explanation is that the minimum requirements for the PGCE course (C grade in GCSE mathematics and an undergraduate degree; Department for Education, 2015) are sufficient to enable graduates to train to become a teacher without having to spend a great deal of time on the PGCE course learning mathematics. The Knowledge Map analysis corroborates this idea since it shows that the learning of new mathematical topics on the PGCE course was limited to a few cases.

Further, PGCE success does not correlate with how well a student performs on the MKT items. Thus, the knowledge required to be recommended for Qualified Teacher Status (QTS) in England (pass a PGCE course) is not directly related to prior mathematical attainment (GCSE, A-level or degree level mathematics as a proxy measure of mathematical content knowledge) and MKT.

However, whilst PGCE grades do not relate to scores on the MKT items, results show that MKT scores significantly improve over the PGCE course by 7.3% on average, which represents a medium effect size. Thus, some change in knowledge which is captured by the MKT items but which is not necessarily reflected within the final PGCE grade occurred over the PGCE year. Post-observation interviews provide some insight as they show that other kinds of mathematical learning such as making connections between existing mathematics concepts, learning applications of existing mathematical topics and learning alternative methods or representations of concepts take place on a PGCE course. Further, the Knowledge Map analyses suggest that knowledge became more (or less) accessible to some students over time or became compressed (or decompressed). Findings also suggest that in some cases knowledge became more closely aligned with the National Curriculum over time.

The above findings suggest a change in the quality of knowledge rather than knowledge quantity, or in other words, a change in the organisation (how it is held within the mind) of knowledge. Results from part two of the interviews substantiate this as they suggest that some knowledge can be held in such a way so as to only be made manifest with the presence of a stimulus (mathematical task) which draws upon that knowledge.

Additionally, when comparing the questionnaire, interview and observation data collected between SKE and Non-SKE PGCE students, there was only one main difference found between the two groups. That is, during classroom observations, Non-SKE students demonstrated greater understanding of mathematics as a discipline including conventions and knowledge of mathematical models.

7 Discussion

Four main areas meriting further discussion emerge from addressing the research questions for this study. These four 'discussion points' are attended to in this chapter. Each discussion point is based on a contribution which the current study makes to knowledge or practice. How the results relate to existing literature is also considered. The first discussion point concerns the Mathematical Knowledge for Teaching (MKT) items used for this study within a new (English) context as well as consideration of what these items measure. The second discussion point evaluates the Knowledge Quartet as used in this study whilst the third discussion point evaluates Knowledge Maps as introduced and used within this research. Finally, implications of the results for existing UK teacher training policies are discussed.

7.1 Discussion point one – MKT items

This section discusses the findings and implications of using a sample of MKT items for use with secondary mathematics trainee teachers in England. Firstly, methodological contributions of using the items within this new context are suggested. Then, contributions of the findings to the current understanding of what the MKT items measure are discussed.

7.1.1 Methodological contributions

The MKT items, a sample of which were used in this study, were developed in the USA and have since been adapted for use in a small number of other countries, namely, Ireland, Norway, Ghana, Indonesia, and Korea (see Ball, Blömeke, Delaney, & Kaiser, 2012). However, the current research marks the first instance of using the MKT items within England, furthering the work of the Learning Mathematics for Teaching (LMT) group at the University of Michigan (the item authors) by examining whether the construct and measures of MKT are applicable to an English context.

Specifically, psychometric analysis of responses to the MKT items selected for use in this study fit the Rasch model well (4.3.1.4, p.71). This has two methodological implications, namely, that the novel approach used to select items was appropriate and that items can be used with an alternative sample of respondents (secondary, pre-service teachers in England) than intended. These are now discussed in turn.

(i) An alternative approach to selecting items

The current research employed an approach different than researchers in other countries utilising the measures outside the USA in that the items selected for use were not adapted in any way. Instead, out of the pool of all items available at the time, only items which were felt to transfer well to England without any adaptation were selected (see 4.3.1.6, p.73) using Delaney et al. (2008)'s categories as a guide:

1. Changes related to the general cultural context;
2. Changes related to the school cultural context;
3. Changes related to mathematical substance;
4. Other changes.

These categories of changes were established by Delaney and colleagues as a means to classify changes required when adapting items for use in Ireland and have since been used by other researchers to help identify changes required for adaptation (e.g. Kwon et al., 2012; Ng, 2012). For the current study, instead of using them to adapt the items, the categories were used to identify items to be eliminated from use (see 4.3.1.6, p.73 for full details). Since responses to the resulting items fit the Rasch model well, selecting items by using Delaney and colleagues' categories in this way can be said to be of merit and marks a methodological contribution. Thus, this method could be adopted by other researchers wishing to select items for use in England or other English-speaking contexts.

(ii) An alternative sample of respondents – secondary, pre-service teachers in England

The MKT items were originally designed for in-service teachers in the USA. The current study used them for trainee teachers on a Postgraduate Certificate in Education (PGCE) course in England. In doing so, this study corroborates the findings of Gleason (2010) that the items can be reliably used with pre-service teachers. The reliability (internal consistency) of the 18 items selected was found to be reasonable on both the pre- and post-questionnaire (0.67 and 0.71 respectively) as measured by Cronbach's alpha (c.f. 4.3.1.5, p.73).

The MKT items were also intended for elementary and middle school teachers, whereas the current research focuses on secondary mathematics teachers. Section 4.3.1.2 (p.66) proposes that, theoretically, the items can be used with secondary mathematics teachers given the overlap between the ages of pupils in upper Elementary school and Middle school in the USA and lower Secondary school in England. Further, the mathematical content of the items was examined and items were selected which were felt to address content taught within secondary school mathematics lessons in England. The Rasch analysis indicates that using the items with this sample was appropriate. However, the mean student location calculated by the Rasch analysis software indicated that the average student ability was slightly higher than the average item difficulty (4.3.1.4, p.71). This is to be expected since the respondents for this study were secondary mathematics teachers rather than general elementary teachers.

A further inference of the MKT items performing well is that the sample selected for the current study can be successfully used in England. This provides a foundation for other researchers wishing to use these items within an English context.

7.1.2 What do the MKT items measure?

Whilst the psychometric analysis conducted for the current research suggests that the sample of MKT items perform well and can be used successfully within England, it cannot specify what the items measure.

The LMT group (MKT item authors) claim that the items measure their proposed construct of MKT (see 2.1.5.4, p.23 for full details) which involves a specialised form of mathematical knowledge unique to teaching. Nevertheless, Garner (2007) proposes that rather than measure MKT, the items instead measure a deep understanding of mathematics. I partially agree with Garner. Whilst I feel that the knowledge required for teaching mathematics is, as Garner states, not a different kind of mathematical knowledge but rather involves holding this knowledge in a different way (see 2.3.2, p.46), I feel Garner is not clear about what is meant by a ‘deep understanding’.

The LMT group used an adaptation of the argument-based approach to validation developed by Michael Kane (e.g. Kane, 1992) to validate the assumptions and inferences of responses to their MKT items. Garner (2007) used this same approach to show how a deep understanding of mathematics is an alternative explanation to what the items are claimed to measure (MKT). I also adopt the item authors’ approach to validation (explained below) to argue that the MKT items do not, in fact, measure MKT (a special form of knowledge unique to teaching) but that the items capture a ‘deep understanding’ of mathematical content knowledge involving ‘mathematical reasoning’. In doing so I draw on: (i) results from the MKT item authors themselves, (ii) results from the current study, and (iii) responses from other researchers (including Garner) commenting on the authors' validation approach.

Firstly, Kane's approach is outlined, followed by a description of the MKT item authors' adaptation of Kane's approach.

7.1.2.1 Kane's argument-based approach to validation

Kane's approach is based on the premise that it is the interpretation of test scores (inferences and decisions) that is validated, not the test itself nor scores on the test (Kane, 2001). Kane states that: "A test-score interpretation always involves an interpretive argument, with the test score as a premise and the statements and decisions involved in the interpretation as conclusions" (1992:527). In order to validate a particular test-score interpretation, one would provide appropriate evidence to support the plausibility of the associated interpretative argument (Kane, 1992).

In later work, Kane (2004) identifies two distinct stages of the approach:

> In the first stage, the proposed interpretation and uses of test scores are specified in terms of an interpretive argument leading from observed performances to conclusions and decisions. In the second stage, the plausibility of this argument is critically evaluated within a validity argument (Kane, 2004:136).

The reader is directed to Kane (2001) for full details on his validation approach.

7.1.2.2 Using Kane's argument approach for the MKT items

The LMT group adapt Kane's approach to validation to include three types of assumptions: elemental, structural and ecological (Schilling and Hill, 2007). Elemental assumptions and inferences: "[concern] the performance of specific test items, including consistency of items with the subjects' knowledge" (Schilling and Hill, 2007:71). Structural assumptions and inferences: "[concern] the internal structure of the test, including the consistency of the structure of the test with the structure of the test domain" (Schilling and Hill, 2007:71). Finally, ecological assumptions and inferences: "[concern] the external structure of the test, including the relationship of the test scales with external variables" (Schilling and Hill, 2007:71).

These three types of assumptions and inferences are examined by the LMT group with regard to the MKT items as follows:

> 1. Elemental assumption: The items reflect teachers' mathematical knowledge for teaching and not extraneous factors such as test taking strategies or idiosyncratic aspects of the items (e.g., flaws in items).

> A. Inference: Teachers' reasoning for a particular item will be consistent with the multiple choice answer they selected.
> 2. Structural assumption: The domain of mathematical knowledge for teaching can be distinguished by both subject matter area (e.g., number and operations, algebra) and the types of knowledge deployed by teachers. The latter types include the following: content knowledge (CK), which contains both common content knowledge (CCK), or knowledge that is common to many disciplines and the public at large and specialized content knowledge (SCK) or knowledge specific to the work of teaching; and knowledge of content and students (KCS), or knowledge concerning students' thinking around particular mathematical topics. Implications of this include:
> A. Inference: Items will reflect this organization with respect to both subject matter and types of knowledge in the sense that items reflecting the same subject matters and types of knowledge will have stronger inter-item correlations than items that differ in one or both of these categories. This will result in the appearance of multiple factors in an item factor analysis.
> B. Inference: Teachers can be reliably distinguished by unidimensional scores reflecting this organization by subject matter and types of knowledge. These scores are invariant with respect to different samples of items used to construct the scores.
> C. Inference: Teachers will tend to answer most problems (except those representing CCK) with knowledge specific to the work of teaching. Non-teachers will rely on test-taking skills, mathematical reasoning, or other means to answer these items.
> D. Inference: Teachers' reasoning for a particular item will reflect the type of reasoning (either CK or KCS) that the item was designed to reference.
> 3. Ecological assumption: The measures capture the content knowledge that teachers need to teach mathematics effectively to students.
> A. Inference: Higher scores on the scales derived from these measures are positively related to higher-quality mathematics instruction.
> B. Inference: Higher scores are positively related to improved student learning (Schilling and Hill, 2007:79-80)

Each of the above assumptions are now taken in turn and the LMT group's findings are re-examined in light of results from the current study as well as other researchers' views and findings. In doing so, it is argued that the evidence suggests that it is not a different kind of mathematical knowledge that is required for teaching (i.e. MKT) but a form of mathematical reasoning.

7.1.2.2.1 Elemental assumptions

Hill and colleagues address the elemental assumptions by examining whether a teacher's thinking was consistent with the multiple-choice answer chosen by using think-aloud interviews with a sample of teachers, non-teachers and professional mathematicians (Hill et al., 2007d). In particular they investigated whether respondents drew on the knowledge types intended to be accessed by the items or whether test-taking skills or other types of thinking or knowledge prevailed. Although the domain of MKT is subdivided into various categories of knowledge (**Figure 2.2**, p.20), in this analysis the group only focused on Content Knowledge (CK) items and items which were intended to measure Knowledge of Content and Students (KCS).

Hill and colleagues concluded that the examination provides evidence to support the elemental assumption (that the measures represent MKT) in the case of CK items but not the KCS items (Hill et al., 2007d). They found that mainly mathematical processes rather than test-taking skills or guessing were relied upon when answering the CK items. However, results from the KCS items were less conclusive. Indeed:

> While some teachers did use what we coded as KCS in their answers to these items, other teachers, non-teachers, and professional mathematicians also relied on mathematical reasoning in generating their answers (Hill et al., 2007d:92).

In addition, they found that test-taking skills occurred at a higher rate for KCS items.

In his commentary on the efforts of the LMT group to use his argument-based approach to validation with their MKT items, Kane (2007) states that mathematical knowledge would be expected to underlie responses to the KCS items. Moreover, Kersting and colleagues found that: “Virtually all of the shared variance [of MKT scores] was explained by the MC [Mathematical Content] Knowledge subscale” (Kersting et al., 2010:178). Given the prevalence of mathematical reasoning and processes used by respondents, Garner concludes: “Thus, success on the items was characterized by a deep understanding of mathematics, not a different kind of mathematics” (2007:171). Results from the current research corroborate the findings of Hill and colleagues and Kersting and colleagues as well as the views of Kane and Garner as small to moderate ($r = 0.183$ to 0.334) correlations between prior mathematical attainment (GCSE, A-level and Further Mathematics A-level grades) and MKT scores (5.1.4.4, p.104) were found. This suggests mathematical content knowledge plays a role in success on the items.

In addressing the elemental assumption, it is not conclusive that a specialised form of mathematical knowledge for teaching is required to respond to the MKT items, only that the kinds of thinking required involve ‘mathematical reasoning and processes’ by teachers, non-teachers and professional mathematicians and that test-taking skills were not wholly relied upon.

7.1.2.2.2 Structural assumptions

The think-aloud interviews with teachers, non-teachers and mathematicians used by Hill and colleagues to address the elemental assumptions as discussed above, were also used, in part, to address the structural assumptions. That is, they helped the item authors to determine whether the teachers drew on different types of teaching-specific knowledge to warrant a separation of the CK and KCS categories (Hill et al., 2007d).

The dimensionality of the test was also explored using exploratory factor analysis (Schilling, 2007). Hill and colleagues conclude that the evidence suggests that the MKT construct is composed of at least two sub-categories: CK and KCS (Hill et al., 2007d). However, multidimensionality was also found within the KCS items, suggesting that these items also test other types of thinking including mathematical reasoning. In examining the table of factor loadings provided by Schilling (2007), Garner states that: “There is not overwhelming evidence for two factors as opposed to three factors or one factor” (2007:172). Further, Schilling et al. (2007) state that a (unidimensional) Rasch model adequately fit the KCS items. Garner thus asks: “could it be that all items are essentially unidimensional?” (2007:172). The Rasch analysis conducted for the current research provides evidence that the sample of items used for this study may be unidimensional (4.3.1.4, p.71). However, the number of items is small ($n = 18$).

In responding to Garner’s remarks, Hill and colleagues suggest that they have evidence for the existence of a Specialised Content Knowledge (SCK) from other research which shows that professional mathematicians lacked flexibility with non-standard approaches and with decompressing their knowledge of mathematics (Hill et al., 2007d). As a result, they hypothesize that flexibility and decompression comprise SCK which they recognise as a slightly different conception of SCK than their original definition.

However, I contend that a lack of flexibility with non-standard approaches is not necessarily evidence of a lack of a different (specialised) form of knowledge but simply lack of practice with such approaches. Indeed, teachers gain experience with non-standard approaches during their teaching practice but this does not mean that they possess a different kind of knowledge, just greater experience in applying their mathematical content knowledge to understand non-standard methods.

With regard to the importance of teachers having a ‘decompressed’ (Ball and Bass, 2000) knowledge of mathematics, the current research provides some evidence to support this. S5 had a rich, conceptual understanding of squares which made the process of drawing discrete nodes on his pre-interview Knowledge Map difficult (5.2.5.1, p.129). Since it was difficult to separate (decompress) S5's knowledge into discrete nodes, S5's knowledge can be described as 'compressed'. Further, S5 indicated that he struggled when teaching low-ability groups (during a post-observation interview on P2) which exemplifies comments found in the 'subject knowledge enhancement' (SKE) evaluation report that: "those with a specialist degree had more subject knowledge… although may find it difficult to 'dumb down'" (Gibson et al., 2013:86). Moreover, S5 deferred successful completion of the PGCE course. This episode is an example of the potential difficulties a teacher can face when their knowledge is not sufficiently decompressed.

Nevertheless, a 'compressed' knowledge of mathematics is not necessarily a different kind of knowledge but can be seen as an alternative way of organising and holding mathematical content knowledge within one's mind.

Classroom observations from the current research reveal that connections between mathematical ideas allowed the PGCE students to be flexible when teaching pupils in the classroom. There were several instances of this. Firstly, during S2's second observed lesson (5.3.3.2, p.151), S2 used his knowledge of different representations of rational numbers to help transform a problem for pupils:

> OK, so nought point five times nought point five? If it's easier we can think about it as fractions. If we thought about it as fractions what would it be?

This enabled pupils to make the connection between decimal and fractional forms and subsequently complete the problem 0.5^2 (see 5.3.3.2, p.151 for full details). Similarly, he helped his pupils see this connection with the problem of a number raised to the power 0.5 within the same lesson (see 5.3.3.2, p.151).

Secondly, because of S2's understanding of the connection between fractions and division, he helped pupils to understand that because $\frac{x^2}{x^2}$ could be reinterpreted as dividing something by itself, the answer must be one (see 5.3.3.2, p.151) and he helped a pupil divide ten by twenty by writing it as a fraction and cancelling it (5.3.3.3, p.153).

Finally, S4 was also able to assist struggling pupils with the problem of multiplying ½ three times by using his knowledge of the conceptual connection between multiplying fractions and dividing an object into smaller sections (see 5.3.5.3, p.166):

> …what's a half times a half?... So if I've got a half, and I cut it in half, what do I get?... quarters, and if I've got quarters and I cut them in half again… Yeah so our answer would be one eighth

Here, presenting the question in an alternative format helped his pupils.

I argue that a connected understanding of mathematics enabled the PGCE students in these examples to be flexible in the classroom in transforming mathematics to help struggling pupils. However, I contend that a connected understanding of mathematics is key to understanding mathematical concepts themselves (2.3.1, p.42) and is thus not part of a specialised knowledge for teaching (although it is useful for teaching, precisely because pupils know and understand a variety of non-standard methods and representations).

In analysing the structural assumption, Schilling and colleagues also found that SCK items did not load on a separate factor to 'Common Content Knowledge' (CCK) items. However, they argue that this does not necessarily disprove their structural assumption that SCK should be distinguishable from CCK, rather that the: "inferences arising from assumptions can be replaced or modified so that they become consistent with both the assumptions and empirical data" (Schilling et al., 2007:122). They offer the suggestion that the sample of respondents used could have affected the factor analysis if, for example, a subgroup of teachers had the opportunity to learn about a specific trait tapped by the items on a professional development course. Indeed, they state: "we have subsequently found that SCK items are very inconsistent across different samples with respect to loading on a separate factor" (Schilling et al., 2007:123).

Conversely, the data fitting a Rasch (unidimensional) model appears to be consistent across samples (USA and English respondents). Researchers using the items in Ireland (Delaney, 2012), and Indonesia (Ng, 2012) also conducted Rasch analyses.

The above results suggest that the structural assumption (that the construct of MKT can be divided into CCK, SCK and KCS) is not warranted. It appears more likely that items are instead unidimensional.

7.1.2.2.3 Ecological assumption

To examine the ecological assumption, Hill et al. (2007a) investigated the relationship between teachers' performance on the MKT items and their ability in the classroom to provide 'mathematics-rich instruction' and the subsequent performance of the pupils being taught. They recorded mathematics lessons and analysed the videos by scoring the teachers' 'mathematical quality of instruction' (MQI).

Hill and colleagues found that there was a relationship between teachers' scores on the MKT items and their effectiveness in the classroom. They correlated MKT scores with scores of MQI and found significant strong positive correlations. Moreover, when comparing a teacher who achieved an average score on the MKT items and a teacher in the top quartile of scores that the higher-scoring teacher added an equivalent of 2-3 weeks instruction to her pupils' scores. Further:

> This effect occurred when controlling for relevant student and classroom characteristics, as well as critical teacher characteristics, such as years of experience, the average number of math methods and content courses, and certification status (Hill et al., 2007a:109).

In a more recent study, Hill and Charalambous (2012) found evidence to support the hypothesis that MKT is a key contributor to MQI. Indeed, although only a small sample of teachers was used, it was found that teachers scoring highly on the MKT items typically taught excellent lessons, whilst low scoring teachers often struggled to do so.

However, there are several criticisms of the 2007 study. Firstly, Kane (2007) argues that although positive correlations were found between scores on the MKT items with pupil change scores and MQI respectively, that this could be due to the inclusion of CCK items rather than the items testing components of knowledge specific to teaching. He points out that there was not a clear focus on the SCK items and the KCS items were not included in the analysis. Furthermore, Alonzo (2007) reasons that because only the CK items (which consist of CCK and SCK which was shown to align with the CCK factor) was shown to be correlated with MQI and gains in pupil achievement, that one could conclude only mathematics content knowledge is needed for teaching. In which case, "If CCK is what matters in teaching and learning, a straight test of elementary mathematics should suffice to measure knowledge for teaching" (Alonzo, 2007:136). Again, the result from the current research that a small to moderate correlation exists between prior mathematical attainment and scores on the MKT items (5.1.4.4, p.104) supports these comments.

Kane (2007) further criticises Hill and colleagues for controlling for teacher variables such as years of teaching experience, number of mathematics courses taken, etc. which may account for differences in MKT scores since these variables may in fact be important to MKT:

> Suppose, for example, that the specialized knowledge needed for teaching is acquired mainly through methods courses and experience (a plausible hypothesis), so that the scores on the SCK are highly correlated with these teacher variables. If so, then by controlling for these teacher variables, we make it very unlikely that the SCK scores will correlate with anything. If we want to know how much additional predictive power we get by introducing the SCK scores after the teacher variables have been included in a regression, including these other variables as covariates is reasonable. If we want to know how well the SCK test measures the kinds of knowledge needed in teaching, controlling for the teacher variables is not appropriate (Kane, 2007:185-6).

Several studies by the LMT group have shown that teacher scores on the LMT items have been shown to positively predict pupil achievement (Ball et al., 2005; Hill et al., 2005; Hill et al., 2007a). However, Engelhard and Sullivan point out that the content domains assessed for teachers were not matched with those of the pupils and: "Determining a relationship between the teacher's mathematical content knowledge and their [pupils'] content understanding may be best understood when the domains are matched" (2007:151).

Finally, Lawrenz and Toal (2007) argue that because the relationship between teaching practice and student achievement has not been established, and the LMT group sought to investigate this relationship, that the MKT items must be validated separately from pupil performance rather than claiming that the correlation found validates the measures. A related criticism is that the MKT measures: "…were not designed or validated to make statements about any individual teacher's level of mathematics knowledge" (LMT, n.d.). Thus, by analysing correlations between individual teachers' scores on the items and their classroom teaching, I contend that the LMT group are violating their own terms and conditions of use. I propose that, ideally, a further argument-based approach to validation should have been conducted for using the measures for this purpose prior to the correlation study. The current research also found that mean scores on a sample of MKT items increased over time for PGCE students (5.1.4.2, p.104), suggesting that as graduates train to become teachers they also become better equipped to respond to the MKT items, leading to a tentative conclusion that the items capture 'something' useful for teaching.

Furthermore, the increase in scores for the current study of 7.3% on average is arguably comparable in magnitude with an increase in scores of 4.1% of teachers on a summer professional development course (Hill and Ball (2004). However, whilst the professional development course was explicitly focused on improving teachers' MKT, the PGCE course was not (see

6.2.2.1, p.178 for a full discussion). If scores on the MKT items improve over time on a course not focusing on improving MKT but on training teachers, this provides evidence that the items capture something useful for teaching but that this ‘something’ is not necessarily the MKT construct.

From the evidence gained for the elemental and structural assumptions discussed above, this ‘something’ may be a form of mathematical content knowledge involving ‘mathematical reasoning’, rather than the MKT construct as defined by the item authors.

7.1.2.3 Concluding remarks

Rasch analysis is a rigorous theory of measurement, regarded by many as the gold standard against which item responses can be assessed (Kreiner, 2007). Since responses to the sample of items used for the current research fit the Rasch model well, this suggests that the MKT items are a good measure, yet the analysis cannot specify exactly *what* they measure. Results from empirical research indicate that scores on the MKT items significantly improve over a PGCE course (5.1.4.2, p.104) and correlate with the ‘mathematical quality of instruction’ of in-service teachers (Hill et al., 2008b; Hill et al., 2007a), thus the MKT items capture ‘something’ useful to teaching. However, the above discussion suggests that the items do not measure MKT, that is, a specialised form of mathematical knowledge unique to teaching, as theorised by the item authors, but instead a type of mathematical content knowledge involving ‘mathematical reasoning’.

7.2 Discussion point two – the Knowledge Quartet

This section (discussion point two) discusses three issues with applying the Knowledge Quartet to the current study. The Knowledge Quartet was developed to provide a framework for reviewing trainee teachers’ mathematics lessons with a focus on the mathematics content of the lesson (Rowland and Turner, 2007). This proved useful when analysing the lessons observed for the current research as it steered the coding to the mathematics of the lesson rather than other pedagogical features which were not the focus. Whilst the Knowledge Quartet was, for the most part, applied easily to the analysis of the classroom observations in that individual codes were easily matched to episodes within the lesson, three potential issues emerged. Indeed, issues with comparison and some instances where codes were not so easily identified despite reference to the online coding manual provided by the Knowledge Quartet authors. These three issues are now discussed in turn.

(i) Issues with temporal and participant comparison

For the current research, difficulties were encountered when attempting to compare lessons over time using Knowledge Quartet codes. Indeed, since the codes could be either for negative or positive episodes, the numbers of codes could not be compared over time, including changes over a series of lessons or within a single lesson. For example, S3's definition of 'rounding to one significant figure' progressed to become more succinct and accurate over the course of the lesson, but the Knowledge Quartet codes could not capture this (see 5.3.4.3, p.158). Similarly, codes could not be compared between students (6.3.2.3, p.203). Instead, this research took the approach of scoring each of the four dimensions of the Knowledge Quartet an overall score of high to low 6.2.2.4, p.195). It is recognised that the Knowledge Quartet was not designed for comparison over time, but since it was designed to enable trainee teachers to reflect on and improve their teaching, such a feature may potentially be useful.

(ii) Confusion over which code to select

Firstly, there were four instances where there was confusion over whether an episode was an example of '*making connections between procedures'*, '*making connections between concepts'* or '*choice of representations'*. However, this could be because of the way 'representations' are viewed for this research which may differ from the way representations are conceptualised within the Knowledge Quartet framework.

Three of these problematic instances are from S2's lessons, and one instance from S4's second observed lesson. I will now discuss these at length.

Toward the end of the first observation on placement two, pupils questioned whether the answer to $x^2 \div x^2$ is 1 or 0. S2 responded by asking the pupils: "Right, if we had, x squared divided by x squared" to which the pupils indicated the answer would be x^0. S2 then reframed the problem: "and if you take something and divide it by itself, what do you get?". This question helped the pupils to think about the problem, not in terms of rules of indices, but in terms of a division problem. Thus, S2 made a connection between the more familiar concept of division (covered since Key Stage 1 in the curriculum; Department for Education, 2013c) and rules of indices. This could therefore be coded as '*making connections between concepts'*.

Alternatively, it could be argued that since division was the concept involved all along, this is drawing attention to the division rule which still holds in this more abstract case, and is about translating the symbolic representation of the division problem and its representation in spoken terms: 'dividing a number by itself' – a 'conversion' between semiotic representation and natural language to use Duval's (2006) terminology. Thus, it could be coded as '*choice of representations'*.

The second episode is similar in nature to the first and occurred during the same lesson. Pupils were struggling with the question 0.5^2 and again, S2 broke down then reframed the problem:

> S2: have a think, when we see squared, what does it tell us? PupilD?
> PupilD: multiply by itself
> S2: OK, so nought point five times nought point five? If it's easier we can think about it as fractions. If we thought about it as fractions what would it be?
> PupilA (shouts out but is unheard): quarter
> S2: A half times a half?
> PupilB: nought point two five
> S2: zero point two five, ok

Coding this exchange is problematic since it involves making connections and representations but S2 does not make a '*connection between concepts'* he refers to alternative representations of the same concept (rational numbers), yet it is not quite a '*choice of representation'* since it occurs in a contingent moment rather than being deliberately planned in advance.

The third episode is again similar but occurred with the second observed lesson on placement 2. S2 instructed a pupil who was struggling with rearranging an equation in order to solve for x since they did not know the answer to ten divided by twenty. S2 instructed:

> …right, I want you to write it as a fraction, ok? So write ten as a fraction. I'll show you where we're going. You're telling me to divide by twenty so that's ten divide by twenty, how do we cancel that fraction down? [pupil responds] half? You've told me [S2 P2 O2].

Again S2 translated the division problem into another (fraction) representation in order to help the pupil answer the question. He explicitly made the connection between division and fractions for the pupil but through transforming the semiotic representations.

Finally, during S4's second observed lesson he assisted struggling pupils with the problem of multiplying ½ three times (see 5.3.5.2, p.163). He re-presented the problem as follows:

> …what's a half times a half?... So if I've got a half, and I cut it in half, what do I get?... quarters, and if I've got quarters and I cut them in half again… Yeah so our answer would be one eighth.

This episode was difficult to code because S4 '[*makes] connections between procedures*' by equating fraction multiplication with making a whole smaller but also uses a good '*choice of representation'* by transforming the numeric form of the question into a concrete form involving halves, quarters and eighths of an object.

These four episodes demonstrate how the 'transformation' and 'connection' dimensions of the Knowledge Quartet are not mutually exclusive when mathematical concepts are viewed as being accessed by various semiotic representations. This is because teaching mathematics is not seen as transforming content in order to be accessed by pupils, but rather making connections between semiotic representations in order to access the concept whether that be by a teacher, pupil or anyone else using mathematics.

(iii) Codes inadequately capturing the mathematical knowledge displayed

Section 6.3.2.3 (p.203) discusses the only difference between SKE and Non-SKE students which was highlighted by the data collected for the current research, namely, knowledge of the conventions of the discipline of mathematics. When coding teaching episodes related to mathematical conventions and models, it was found that existing Knowledge Quartet codes did not fully capture this aspect of the PGCE students' knowledge. Alternative codes are suggested in section 6.3.2.3 (p.203).

Similarly, during S3's third observed lesson, his adherence to a technique suggested by other teachers could be accurately coded as '*adherence to peers' techniques*' rather than '*adherence to textbook'* (see 5.3.4.3, p.158 for details).

A final example where existing Knowledge Quartet codes were felt to be inadequate was when S1 made use of applications to real life situations within his third observed lesson (5.3.2.3, p.147). The lesson focused on making a solar cooker as an application of parabolic curves. Whilst this was coded as '*overt subject knowledge'* and did indeed reflect S1's knowledge of parabola and quadratic equations, it was felt that this code did not fully capture this aspect (applications of mathematics) of subject knowledge.

7.3 Discussion point three - Knowledge Maps

This third discussion point concerns the approach to representing teachers' knowledge introduced within this thesis, namely, Knowledge Maps. Alongside the discrete/ continuous metaphor for knowledge, Knowledge Maps are proposed in response to criticisms (see 2.2, p.34) of existing conceptions of teacher knowledge. Following their use to analyse empirical data collected for the current research they are firstly evaluated in terms of how well they address the criticisms they are intended to overcome. Then, further contributions of Knowledge Maps to the literature on teacher knowledge are discussed. Finally, some limitations of Knowledge Maps are presented.

7.3.1 Evaluation of the theory of Knowledge Maps

Several existing conceptions of the mathematical knowledge required for teaching are presented in Chapter 2 (2.1, p.14). Seven problems with these conceptions are identified (see 2.2, p.34) and a new approach to conceptualising teacher knowledge is proposed (see 2.3, p.42) in order to address these problems. Section 2.3.3 (p.49) takes each problem in turn to argue how, theoretically, the new approach has the potential to overcome them. Each of the seven problems to existing conceptualisations of teacher knowledge are now revisited in light of evidence gained through data collection and analysis using the new approach.

7.3.1.1 The problem of dualism

In 2.2.2 (p.35) the separation of content knowledge (CK) from Pedagogical Content Knowledge (PCK) is argued to be a dualistic distinction which reduces CK (knowledge) to an unproblematic given whilst elevating the role of PCK (actions) within the classroom. Section 2.3.3.1 (p.49) argues that the discrete/ continuous distinction is not dualistic since both knowledge held in discrete or continuous form can either be a known fact (knowledge) or a known procedure (action). Further, theoretically, this section shows that the discrete/ continuous metaphor and therefore Knowledge Maps (as a representation of discrete/ continuous knowledge) do not separate thought from action. However, whilst using Knowledge Maps in practice for the current research, it was easier to include statements of knowledge as discrete nodes on the Knowledge Maps rather than statements of things the students could do, such as: 'the student can find the area of a square' or 'can graph a quadratic equation'. Indeed, there was a focus on students' utterances rather than their actions. This was the case even during the problem-solving part of the interview where students were required to ‘do’ mathematics. This is now discussed in more detail.

When analysing the interview transcripts, the Analysis Tables included statements made during both Part A (free recall) and Part B (problem solving) of the interview. However, there were further utterances made by students whilst problem-solving which could not be included within the Analysis Tables and therefore were not included in the Knowledge Maps. There was found to be three types of utterances which occurred during problem-solving and these were used in different ways when analysing the interviews.

Firstly, some utterances made during problem-solving were included as discrete nodes on the Knowledge Maps. For example, one student said whilst solving the second problem: “the area of a square is.. well width times length but they’re the same”. The corresponding statement: “width and length are the same on a square” was then included in the Analysis Table and consequently as a discrete node on the Knowledge Map.

Secondly, some utterances could be used to verify statements made within the free recall part of the interview. For example, when solving Task 1 (Appendix D, p.274), if a student recognised that the area of the blue square equalled the unknown side length squared, then this was used to verify that not only could the student state the mathematical fact (‘to find the area of a square, you square the length’), but could use it within a problem-solving situation. Such instances were marked on the Knowledge Maps by having nodes coloured green if the knowledge was also applied during problem-solving.

Lastly, some utterances made during problem-solving were not included in the Analysis Tables or resulting Knowledge Maps as these required some additional inference and it was felt that it was better to omit these than to speculate as to the knowledge which may or may not be held. For example, S2 remarked whilst attempting Task 1 that: “we know A and B are gonna be the same” when referring to the side lengths of the square in the problem. However, whilst it could be inferred that S2 knows ‘the sides of a square are equal’ it could be that this statement refers only to the specific example within the problem (as the side lengths looked the same length) and that S2 does not know this mathematical fact[13].

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[13] It is highly likely that S2 *does know* that a square has equal sides. However, this example demonstrates how making inferences was kept to a minimum for this research.

In conclusion, the Knowledge Maps drawn for this research tended to focus on statements of knowledge/ facts held by the students rather than the mathematics that they could perform. Thus, in practice, the Knowledge Maps can be described as dualistic.

7.3.1.2 The problem of objectivism

The argument that many existing conceptions of knowledge treat mathematical content knowledge as an objective list of things to be known rather than complex and dynamic as other forms of knowledge is presented in section 2.2.2 (p.35). Section 2.3.3.2 (p.49) asserts that the discrete/ continuous metaphor and Knowledge Maps do not treat knowledge as a static entity. Instead, concepts are complex idealistic forms which are accessed by various semiotic representations which vary both temporally and between teachers.

The Knowledge Maps drawn for this research do indeed distinguish between geometric, numeric and algebraic representations in which a PGCE student's[14] knowledge of a concept resides (e.g. **Figure 5.19**, p.128) and comparing Knowledge Maps for pre- and post-interviews shows that knowledge changes over time on the PGCE course (6.2.2.2, p.180). Therefore, in practice, Knowledge Maps do not treat content knowledge as static and fixed but a dynamic entity which alters over time. Alternatively, an individual Knowledge Map cannot show knowledge change in isolation, only when it is compared with another Knowledge Map.

7.3.1.3 The problem of separating PCK from CK

Section 2.2.3 (p.36) contends that separating CK from PCK is untenable since all knowledge has a pedagogic element to it. The new approach to knowledge taken for this research is unlike many existing conceptions of knowledge for teaching discussed in the Literature Review (2.1, p.14) in that it does not assume a qualitatively different type of knowledge separate from other ways of knowing and being able to use mathematics within other contexts. Instead, it proposes that learning mathematics is enhanced when multiple representations of the concept are added to one's repertoire and that this process is particularly useful when teaching since the various representations have the potential to convey a concept to a variety of pupils. Hence, the issue of whether a different type of knowledge exists is

[14] For this research PGCE students were involved but, arguably, Knowledge Maps may be drawn to represent anyone's mathematical knowledge who is capable of verbalising their knowledge in an interview setting.

somewhat circumvented for this study and thus overcomes this criticism (c.f. 2.3.3.3, p.50). This underlying theoretical viewpoint was not altered as a result of using Knowledge Maps in practice for the current research.

7.3.1.4 The problem of multiple representations

Section 2.2.4 (p.37) argues that existing conceptions of teacher knowledge which view teaching as re-presenting a subject for pupils are over-complicated for the subject of mathematics, since mathematics itself is comprised of representations. The proposed new approach to knowledge does not view teaching as the art of re-presenting the subject of mathematics to pupils. Instead, it takes the view that as the teacher gains deeper understanding of a mathematical concept through learning additional representations, this better equips the teacher to convey the concept to pupils (2.3.3.4, p.50). This view remains unchanged following data collection and analysis. Indeed, the Knowledge Maps confirmed that additional representations (domains) of concepts were gained over the course of the PGCE for at least one of the three topics for all students except S2 (**Table 5.9**, p.140), thus the hypothesis was not disproved. It is not expected that all students would gain additional representations for all three topics since they may not have experienced teaching all those specific topics on their school placements. It is recognised that further research is needed to explore whether further representations are gained after experience teaching them.

There were also cases where ‘new’ mathematical facts were learnt by the students (see 6.4.2.1.1, p.206) as indicated by the presence of additional nodes which contained facts not discussed on the pre-interview. It could be said that the only difference between teachers and pupils are where they lie on the continuum of the process of gaining further representations of mathematical concepts in order to understand them more deeply.

Nevertheless, using the Knowledge Maps to analyse the data collected for this research revealed an unanticipated occurrence, namely, it was found that when a misconception was present within one domain which was connected to another domain, confusion could be exacerbated. For example, S2 and S5 both had connected algebraic and geometric domains in their Knowledge Maps which seemed to intensify their confusion over the number of roots of a quadratic equation (see ‘Misconceptions’, p.190). This demonstrates the logical inverse of the hypothesis that greater (correct) representations lead to greater understanding, that is, incorrect, connected representations lead to greater confusion.

7.3.1.5 The problem of multiple labels

Section 2.2.5 (p.38) argues that the multiple categories and classifications of knowledge could cause confusion within the literature, especially when different names are used for the same ideas. In 2.3.3.5 (p.51), it is explained how the terms 'discrete' and 'continuous' emerge from a reading of the literature as they seem (to me) to capture the essence of what the various existing terms are attempting to convey. Thus, the discrete/ continuous metaphor attempts to unify the existing ideas. There is no empirical evidence for or against this theoretical argument.

7.3.1.6 The problem of focusing on categories

Section 2.2.6 (p.38) asserts that a focusing too much on developing categories of teacher knowledge within the literature is unnecessary since successful teachers can and have emerged without such categories, and it can also lead to a disjointed view of knowledge. Further, the discrete/continuous metaphor and 'Knowledge Map' representation do not focus on categorising different *aspects* of a teacher's knowledge (2.3.3.6, p.51) but instead seek to understand the *organisation* of teacher knowledge by observing how it behaves/changes over time.

Indeed, the data analysis from this study indicates that considering how 'accessible' and 'decompressed' knowledge is may be useful when examining teacher knowledge (see 0, p.234 for more on these concepts). 'Accessibility' and 'decompression' of knowledge concern the organisation of knowledge rather than knowledge categories.

7.3.1.7 The problem with correlation studies

Section 2.2.7 (p.39) proposes that because correlation studies had shown no relationship between the mathematical content knowledge of teachers and the subsequent performance of their pupils, researchers had sought out a different kind of knowledge to subject-matter knowledge to explain this result. Instead, section 2.3.3.7 (p.51) conjectures that rather than the *amount* or *type* of knowledge it is *how* mathematical knowledge is held which is important.

Knowledge Maps enabled this hypothesis to be examined for this research, and, although there were some mathematics topics which were learnt over the PGCE course (6.2.2.2.4, p.192), it was felt that the addition of further domains on the Knowledge Maps (representing alternative representations of knowledge) marked a more substantial change (6.2.3, p.198). Further, classroom observations and interviews reveal, respectively, that a connected understanding of mathematics enabled the PGCE students to help struggling pupils on their school placements (7.1.2.2.2, p.221) and, as in the case of S5, a 'compressed' knowledge of mathematics can lead to potential problems when teaching lower ability groups (7.1.2.2.2, p.221). These examples provide some evidence that the organisation of knowledge, rather than just the 'amount' of knowledge itself, has an impact on teaching and justifies further research into this area.

7.3.2 Contributions of Knowledge Maps

Having re-examined the seven problems with existing conceptions of teacher knowledge for the proposed conception of knowledge (7.3.1, p.230), it was found that some evidence exists to show that Knowledge Maps have the potential to overcome all but one of the problems (the problem of dualism, see 7.3.1.1, p.230), although further research is needed in some cases (the problem of multiple representations, see 7.3.1.4, p.233). The new approach can be seen to improve upon existing conceptions, providing a contribution to the literature on teacher knowledge. Moreover, two further contributions are made by Knowledge Maps, which are now discussed.

7.3.2.1 Contribution 1: 'decompressed' knowledge

An unanticipated result of drawing Knowledge Maps for this research is the contribution Knowledge Maps make to Ball and Bass' (2000) proposed theory that a 'decompressed knowledge' is required for teaching (see 2.1.5.2, p.22).

Indeed, whilst S5 had a high level understanding of mathematical concepts (see 5.2.5.1, p.129), it can be argued that his knowledge was not in a 'decompressed' form as described by Ball and Bass (2000). During the process of drawing Knowledge Maps to represent S5's knowledge, it was difficult to identify individual statements to form discrete nodes. For example, S5's response to: "Tell me everything you know about rational numbers" began as follows:

> When we think of numbers or quantity we think of them in sort of ever increasing denser and denser sets. Most people think of whole numbers –so what you can count on your fingers - and we usually progress to integers - the both positive and negative – but… quantas. And they're all unit distance between each other. The rationals take that a step further in that they can now express fractional distances and they begin filling in that density [S5 pre-interview].

This is a descriptive response which indicates how rational numbers fit within the 'bigger picture' of number in general and it relates rational numbers to the additional concepts of sets, the number line, and measurement. It contrasts with S4's concise definition which was more easily represented as nodes on a Knowledge Map (see **Figure 5.17**, p.125):

> …a rational number can be written as a fraction with two integers – an integer as the numerator and an integer as the denominator [S4, post-interview].

Similarly, for the topic of squares, S5's knowledge shared within the pre-interview was impressive (to the interviewer), yet his Knowledge Map was sparse (consisted of only a few nodes). This was due to the difficulty of translating his highly connected/ compressed knowledge into discrete nodes (see 5.2.5.1, p.129).

What is more, S5 indicated in a post-observation interview (Placement 2) that he struggled to teach lower ability pupils as he could not think how to simplify concepts in a way which would help them to understand. S5 also deferred completion of the PGCE course and this may have been due to these difficulties, although precise reasons are unknown.

The (single) case of S5's compressed knowledge and difficulty with unpacking his knowledge for teaching provides some evidence for Ball and Bass' ideas regarding 'decompression' of knowledge as important for teaching and also suggests the utility of Knowledge Maps in identifying knowledge which can be described as compressed. Indeed, it seems teachers need lots of nodes on their Knowledge Map to indicate that their knowledge of mathematical facts are at a suitable grain size (decompressed sufficiently) for expounding their knowledge to others, rather than being compressed as in the case of S5.

A potential application of Knowledge Maps is to use them towards the beginning of PGCE courses. Indeed, Knowledge Maps could be drawn for individual PGCE students and if, for a given student, drawing a Knowledge Map proves difficult, it could be an indication that their knowledge is too compressed as yet to allow them to expound their knowledge to learners. This could then be addressed within the course. However, as this was not an intended use of Knowledge Maps, further research would need to be conducted to validate the use of Knowledge Maps for this purpose.

7.3.2.2 Contribution 2: accessibility of knowledge

A second contribution is that Knowledge Maps provide an initial indication that the degree of 'accessibility' of mathematical knowledge may be important for mathematics teachers. The concept of 'accessibility' surfaced when considering how prompted knowledge should be treated during the

Knowledge Map analysis. Indeed, for the pre-interviews, some knowledge was prompted by an interviewer or diagram and the extent of interviewer prompting varied by interviewer (despite all interviewers receiving the same training - see 4.3.2.1, p.82). Thus, a policy was needed when drawing the Knowledge Maps to distinguish between knowledge which was freely recalled and prompted knowledge in order that a fair comparison of knowledge could be made. For example it would not be fair to assume that one student had knowledge of symmetry of squares and another did not if the former student was explicitly asked by the interviewer about symmetry and the latter student was not. It was decided that only the utterances made in response to the initial interview question ("Tell me everything you know about…") would be included in the Knowledge Maps, any further utterances made once the interviewer began further probing were not used to create nodes or links on the Knowledge Map but were used to inform other analyses of the data such as misconceptions and quality of explanations. However, it is recognised that differences in unprompted knowledge expressed by students could be a result of how talkative students are rather than differences in knowledge.

Similarly, some students had to write down a formula or draw a diagram in order to talk about their knowledge. For example, when asked about symmetry of a square, S4 drew a square with lines of symmetry on it before he could respond (5.2.4.1, p.122). This demonstrates a qualitatively different way of holding this knowledge – dependent upon a representation to derive the knowledge rather than 'knowing' the answer. Such knowledge (reliant upon a written representation) was also treated as 'prompted' knowledge and was not included on the Knowledge Maps.

This decision regarding prompted and unprompted knowledge led to the creation and use of Analysis Tables alongside the Knowledge Maps. Analysis Tables contained all utterances (both prompted and unprompted) which a student made during the pre- and post-interviews. If similar utterances were made on both the pre- and post- interview, they were paired in the Analysis Tables for ease of comparison (see 4.3.2.5, p.89 for full details). Using the Analysis Tables to compare knowledge over time (between pre- and post-interviews) led to several scenarios:

i. Knowledge was unprompted on both pre- and post-interview
ii. Knowledge was prompted on both pre- and post-interview
iii. Knowledge was unprompted on pre-interview but prompted on post-interview

iv. Knowledge was prompted on pre-interview but unprompted on post-interview

From the above scenarios (iii) and (iv) could be explained by ‘accessibility’ of knowledge. Indeed, if (iii) occurred, perhaps knowledge became less accessible, that is, knowledge which was once freely recalled, now required prompting. This may be due to the topic not being encountered on the PGCE course. On the other hand, if (iv) occurred, then it could be that knowledge became more accessible over time, that is, it was able to be freely recalled by the post-interview. Interestingly, two of the students used the phrase “off the top of my head” to refer to all knowledge they could freely recall. This provides a useful metaphor for accessibility with less accessible knowledge ‘buried away’ deep in the mind, requiring prompting in order to resurface it, and more accessible knowledge ‘on the top’ of one’s head ready and available to be freely recalled and used within the classroom. See ‘Nodes’ (p.180) for examples of knowledge becoming less/ more accessible.

Comparing knowledge between parts one (free recall) and two (problem-solving) of the interviews also supports the idea that some knowledge requires prompting in order to access it. Indeed, both S3 and S4 referred to mathematical facts about squares during the problem-solving part of the interview which were not stated in the first part (free recall). Whilst solving task 2 (Appendix D, p.274) S3 said:

> We know that the angles are 45 degrees because it’s half of them.. so that means that the top angle is.. 90 yeah because it’s a quarter - so it is a right angled triangle [pre]

This demonstrates that perhaps some knowledge may only be accessed following a prompt/ stimulus such as a mathematical question which draws upon that knowledge. In this case, the knowledge that cutting a square in half along the diagonal forms two right-angled triangles only emerged in response to task 1 where this knowledge helped S3 solve the task. See ‘Nodes’ (p.185 and p.188) for further examples.

Furthermore, there was one example where knowledge which once required a prompt/ stimulus was able to be freely recalled by the post-interview. Indeed, for the topic of quadratic equations on the pre-interview, S2 used facts regarding minimum points of quadratic functions to aid him when solving task three:

> Erm it’s a.. minimum… Erm so the differential of the equation is gonna be equal to zero [pre]

However, by the post-interview, he was able to freely state facts relating to differentiating quadratic functions:

> and finding the second order – you can find whether it's a maximum or a minimum [post]

This suggests that this knowledge may have become more accessible over time.

The above examples imply that knowledge may be held in one's mind to differing degrees of accessibility with some mathematical facts able to be accessed more readily, whereas other require some probing (in this case by an interviewer), a diagram or formula to help derive the knowledge, or a stimulus such as a mathematical task (c.f. 'knowing-to' act in the moment, Mason and Spence, 1999).

7.3.3 Limitations of Knowledge Maps

One limitation of Knowledge Maps is that changes in the number of nodes cannot be used independently in order to determine whether knowledge was learnt since an increase (or decrease) in nodes may be attributed to knowledge becoming accessible (less accessible) or decompressed (compressed) (see 6.2.2.2, p.180). Thus, an examination of the content of the nodes is also required.

Another limitation is that knowledge held by a student needs to be sufficiently 'unpacked' in order to draw a Knowledge Map (see 5.2.5.1, p.129). However, as the above discussion reveals (0, p.235) this can be seen as an affordance if a Knowledge Map is used to highlight students whose knowledge is not yet sufficiently decompressed.

A further limitation of the Knowledge Maps is that, when analysed on their own, they were insufficient to establish whether misconceptions were rectified over time or whether knowledge had been learnt or forgotten since Knowledge Maps only contain unprompted knowledge stated in the interviews. Indeed, the Analysis Tables were consulted to provide additional insight since the Tables contained all utterances (both prompted and unprompted) made by a student.

It should be noted that for this study, the Knowledge Maps were drawn by the researcher after the interviews and not by the students/ interviewees during the interview as in a study by Askew et al. (1997) where teachers drew concept maps within the interview. This obviously means that the interpretations of the researcher are inevitably incorporated into the Knowledge Maps and it is possible that a different representation would have been created should the students have been asked to draw Knowledge Maps of their own knowledge. However, it is felt that if this were the case, it would have taken too much time to be practicable in training students on how to draw Knowledge Maps and may also have prompted the students more than would be desirable into thinking about different geometric, numeric and algebraic representations, when this may not have been natural for them to think in these terms.

7.3.4 Conclusion

The Knowledge Map analyses conducted for this research confirm that knowledge change is a complex process involving a variety of factors. Indeed, results showed that changes in learning facts and/ or further representations of knowledge occurred in various combinations for different mathematical topics for each student (**Table 5.9**, p.140). Analyses also showed that change is difficult to track, particularly when change is viewed as a continuous process, since research studies can only collect data at intervals over time. Collecting data continuously via a method such as video capture would perhaps provide an unwieldy amount of information if recorded for longer than an hour or so.

Issues and problems with representing knowledge are not unique to this study. Indeed, when analysing expert and novices' knowledge of fractions through concept maps, Smith and colleagues found that: "Neither group's knowledge was simply composed but was instead structured as complex systems of related elements" (Smith et al., 1993:136). Thus, representing knowledge is anything but straightforward.

Furthermore, Shulman and colleagues were also interested in how CK changed during teaching and discussed the challenges of dealing with knowledge change in research:

> Although a portrait of the knowledge held by each teacher was an important starting point, the nature of our research, in tracing the growth of knowledge, makes the formulation of complete intellectual biographies an ongoing concern (Wilson et al., 1987:111).

Despite the challenges with any attempt to represent and track knowledge change over time, there was some evidence that Knowledge Maps may overcome the identified problems with existing conceptions of knowledge (except for the problem of dualism, 7.3.1.1, p.230). Further, they provide two key contributions to the literature. Firstly, in the case of S5, they evidenced Ball and Bass' (2000) ideas of 'decompression' of knowledge by demonstrating how compressed knowledge can lead to difficulties in teaching. Secondly, Knowledge Maps provide an initial indication that 'accessibility' of knowledge may be worth further exploration and research.

7.4 Discussion point four – implications for Government policies

This final discussion point concerns the contribution this study makes to evaluating existing teacher training policies and implications/ recommendations for current policy. As discussed in Chapter 1, the UK government have implemented two strategies in order to address the shortage of mathematics teachers within this country. Firstly, attractive bursaries are offered to high achieving graduates to train to be teachers (1.3.3, p.4). Secondly, they have introduced SKE courses (1.3.4, p.5). For ease of reference, these two strategies will be referred to as the 'Bursary Policy' and 'SKE Policy' respectively within this section. Results from the current research which relate to these policies will be discussed below.

7.4.1 Bursary policy

As mentioned in Chapter 1 (1.3.3, p.4), bursaries, differentiated by graduates' degree classifications are offered as follows: £20,000 to graduates with a first class degree, £15,000 to those holding a 2:1 and £12,000 to a 2:2, regardless of degree title.

Results from this research suggest that degree classification only accounted for 3% of the explained variability in PGCE scores (5.1.3.4, p.102). This corroborates results from another study which showed no correlation between degree classification and final PGCE score (Tennant, 2006). Thus, awarding £5,000 more to first class graduates than graduates with a 2:1 is not justified by empirical research and perhaps all graduates should be awarded the same bursary. Additionally, since the regression analysis conducted as part of this research suggests a linear relationship between degree class and PGCE scores (5.1.3.4, p.102), this raises the question of why increments in bursaries are not equal.

Nevertheless, the Department for Education (2013e) claim that those with higher degree class will be more likely to complete the course: "Degree class is also a good predictor of whether a trainee will complete their course and achieve QTS [Qualified Teacher Status]". However, for this research (n =244 PGCE students), there was no significant association between degree classification and whether or not the student completed the course ($p = 0.07,\ Fisher's\ exact\ test,\ Cramer's\ V = 0.155,$ small effect size; see Warburton, 2014). Indeed, out of those who did not complete the course, 43% achieved at least a 2:1.

However, the minimum requirement of achieving a grade C at GCSE Mathematics in order to take a PGCE course may be appropriate. Indeed, if those achieving a B grade or higher, say, at GCSE went on to achieve significantly higher PGCE scores, whilst those with a C grade tended to not complete the course or do significantly worse, then a possible implication may be to raise the minimum requirement to a B. Nevertheless, the results from the current research showed no correlation between GCSE nor A-level Mathematics and success on the PGCE course respectively (5.1.3, p.100).

In conclusion, offering differentiated bursaries by degree class may be a flawed policy since there is no evidence from this nor other studies conducted in England that higher degree classes lead to noticeably greater success on PGCE courses, nor increase the likelihood of course completion.

7.4.2 SKE policy

In 2010-11[15], 39% of PGCE mathematics students had taken a SKE course (Teaching Agency, 2012). Given that the implementation of SKE courses by the government effects such a large proportion of trainee teachers, it is surprising that there has been little existing research on the efficacy of SKE courses. The SKE evaluation report (Gibson et al., 2013) is the only known research which appraises SKE courses. However, it concerns SKE courses in general (not just mathematics) and relies on self-reports of students. In addition, there is a study which compares final PGCE results between SKE and Non-SKE students (Stevenson, 2008) but it was conducted with a single cohort of students at one university.

The current research is thus of significance as it adds to the limited existing studies on comparing SKE students with Non-SKE students and does so across several institutions using both quantitative and qualitative data.

The current study shows no significant effect of group (Non-SKE and SKE) on MKT score, suggesting that SKE and traditional entry PGCE students have similar levels of MKT both at the beginning and end of a PGCE course (5.1.4.5, p.105).

Further findings from this study corroborated the result of Stevenson (2008) that there is no significant difference in final PGCE scores between SKE and Non-SKE PGCE students (5.1.4.5, p.105).

Taken together, these results provide evidence that both SKE and Non-SKE students commence their PGCE course with similar levels of MKT and that

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[15] Figures for 2011-12 are not available.

PGCE courses help both groups to increase their scores on the MKT items over the course and achieve similar final PGCE scores. Since, the SKE evaluation course (Gibson et al., 2013) found that SKE students felt their subject knowledge was at a lower level than subject graduates yet more relevant to a school context, this provides a possible explanation as to why MKT scores were similar: SKE students may have a compensatory knowledge (as yet undefined) which allows them to achieve similar PGCE results and MKT results to their subject graduate peers despite their mathematical knowledge being at a lower level.

Since final PGCE scores and MKT scores were similar for both SKE and Non-SKE students, this provides evidence that the government strategy of offering SKE courses to graduates without mathematics degrees prior to teacher training is a good strategy to address the shortage of mathematics teachers whilst not compromising on teacher quality. Therefore, these results imply that SKE courses should be continued. However, further research is needed to investigate whether the SKE courses are the cause of this, in other words, it could be that non-mathematics graduates would have scored equally well on the MKT items and on the PGCE course without taking the SKE course. Additionally, further longitudinal research following the PGCE students into their teaching careers may provide useful insight into whether any differences between former SKE students and their Non-SKE peers become manifested over time during teaching practice.

Nevertheless, this research provides evidence that the SKE courses are not being implemented as intended by the government. Indeed, the intended purpose of SKE courses is to recruit mathematics and science teachers from the larger pool of graduates with sciences-related degrees since the limited numbers of graduates taking pure mathematics and sciences degrees would not provide sufficient teachers (Gibson et al., 2013). However, in practice, this is not the case. Indeed, the current research showed that some mathematics graduates completed an (albeit short) SKE course and some non-mathematics graduates did not take an SKE course. Furthermore (and surprisingly) some SKE students did not have degrees related to mathematics such as Economics, Finance and Engineering but had taken subjects in diverse areas such as Art and Archaeology, Building Surveying, Childhood and Youth studies, Film Studies and Sports Science (there were 124 different degree titles taken by the students involved in this research, see 5.1.3.4, p.102). This seems inconsistent with the government's intended participants of SKE courses.

7.4.3 Contradiction in policies

The Bursary Policy and SKE Policy are discussed and evaluated above with regard to results from the current research. However, this study also highlights a possible contradiction in these two government policies. This will now be discussed.

The Bursary Policy aims to attract high achieving graduates with the rationale that: “Teaching is increasingly a career for the most able graduates, and, for secondary teachers, those with excellent degree-level knowledge and enthusiasm for their specialist subject” (Department for Education, 2013e). Yet, graduates without a mathematics degree can take a SKE course which focus on mathematics content knowledge at a secondary school level (Gibson et al., 2013) rather than knowledge at a higher (degree) level. Indeed:

> The Government’s announcements about minimum qualifications required to receive a bursary and the more recent new Teaching Standards means that people now associate being an ideal trainee teacher candidate with possessing a high level of knowledge (to a graduate or equivalent level) in a strongly related subject. This is not always guaranteed in the case of those that have completed an SKE (Gibson et al., 2013:65).

The current research corroborates the statement: “This is not always guaranteed in the case of those that have completed an SKE”, especially since degrees not related to mathematics, such as Film Studies were taken by some SKE students (see 7.4.2, p.242).

Moreover, the Bursary Policy does not take into account degree titles. Hence, a Film Studies graduate who achieved a first class degree would be eligible for the full £20,000 training bursary whilst a mathematics degree graduate achieving a 2:1 would not. At face value, this contradicts the above statement that secondary teachers need “excellent degree-level knowledge and enthusiasm for their specialist subject”.

7.4.4 Implications/ recommendations for teacher training policy

The results from the current study highlight potential flaws in the Bursary Policy. That is, offering graduates bursaries dependent upon their degree class regardless of degree title does not seem financially beneficial since there seems to be (at best) a very small relationship between degree class and final PGCE scores and no relationship between degree class and course completion. Whilst the Bursary Policy may achieve the government’s purposes of attracting graduates into teaching, differentiating the bursaries by degree class does not seem appropriate. Thus, this thesis recommends revising the existing policy to offer teacher training bursaries at the same level to all graduates achieving a 2:2 or higher.

The current research also suggests that PGCE students deepen their knowledge of existing mathematical concepts on their PGCE course through: learning alternative methods (6.4.2.1.2, p.207), understanding how/why mathematical formulae work (6.4.2.1.4, p.208), learning limitations of mathematical methods in certain cases (6.4.2.2.2, p.210) and learning applications of concepts (6.4.2.1.3, p.208) as well as making connections between concepts (6.4.2.2.3, p.210). The process of revisiting mathematical concepts with the intention of teaching them to others seems to have the potential to lead to teacher learning. This corroborates other studies which show that opportunities to revisit mathematics have the potential to facilitate further mathematical learning (Pournara and Adler, 2014; Stacey, 2008). Perhaps a model similar to the two-year teacher training in France could be adopted whereby the first year: “provides students with an opportunity to reorganize their mathematical knowledge and to begin thinking about the teaching of mathematics” (Henry, 2000:272), the second year then focuses on pedagogy. Extending the teacher training period to two years would of course impact upon the numbers of trainees entering the teaching profession and would have financial implications (funding two years instead of one).

8 Conclusions

This chapter provides an overview of the current study before considering possible limitations of the study. Then, several potential avenues for further research are suggested.

8.1 Overview of the thesis

The poor mathematics results of school leavers in the UK has prompted government policies on teacher training in an attempt to raise the quality of teachers (Department for Education, 2010). Researchers have also focused on teacher knowledge as a key part of improving pupil learning. Indeed, within the last few decades, many researchers have attempted to define and codify the subject knowledge required for teaching (see Askew, 2008 and 2.1, p.14 for examples).

Within the UK, differentiated financial incentives are offered to graduates to train to be mathematics teachers (Department for Education, 2010) with high achieving graduates attracting more funding (Department for Education, 2013f). However, a shortage of mathematics teachers has also motivated government sponsorship of ‘subject knowledge enhancement’ (SKE) courses to graduates from numerate disciplines to increase the supply of mathematics teachers (TDA, 2010; McNutty, 2004).

Within the literature, several conceptions of knowledge for teaching mathematics are proposed (see 2.1, p.14). Shulman (1986) asserts that both domains and categories of knowledge as well as forms for representing that knowledge are necessary components of any conceptualisation of teacher knowledge. For the current research, a means to measure knowledge is also desirable and is added to Shulman’s list.

The literature review for the current study considers conceptions of knowledge for teaching by Shulman (1986), Leinhardt and Smith (1984), Prestage and Perks (2001), Ma (1999), Ball and colleagues, Senk and colleagues (2008), Krauss and colleagues (2008) and Rowland and Turner (2007). Chapter 2 discusses the following for each of these studies: (i) categories of knowledge, (ii) forms of knowledge, (iii) measuring knowledge, and (iv) representations of knowledge (where applicable). However, the literature review reveals (2.1.9, p.29) that papers either aim to classify knowledge or aim to measure knowledge with the exception of Deborah Ball and colleagues who devote similar efforts to both categorising knowledge and developing measurement instruments. As well as there being no single conception of teacher knowledge which considers both categories and forms of knowledge whilst simultaneously offering a means to measure and represent that knowledge, this thesis identifies seven criticisms of existing conceptions (2.2, p.34). Indeed, problems with: dualism, objectivism, separating Pedagogical Content Knowledge (PCK) from CK, problems with multiple representations and labels, problems with focusing on categories and the idea that PCK is a misguided explanation for results of empirical research are presented. Further, many existing conceptions of knowledge treat content knowledge as static and a static conception of knowledge is not suitable for research looking at knowledge change.

Given the problems and criticisms of existing conceptions of mathematics teacher knowledge, this thesis proposes a new approach. Indeed, rather than seeking to categorise aspects of knowledge for teaching, this research aims to examine how knowledge changes as trainee secondary mathematics teachers in England train and participate in teaching, with emphasis on the divide between SKE course participants and those deemed to already have sufficient subject knowledge.

This research takes the ontological view that mathematical concepts exist independently (are objective and fixed), whilst taking the epistemological view that accessing such concepts (by a person) cannot fully be achieved. Instead, one can use semiotic representations as approximations of the mathematical concepts and, by building a repertoire of alternative representations, one comes closer to understanding the mathematical concept which itself transcends any single representation (c.f. Duval, 2006). Drawing on these underlying ontological and epistemological viewpoints, this thesis introduces a representation of knowledge called ‘Knowledge Maps’ which can be drawn by a researcher to represent a teacher’s knowledge at cross-sections of specified time-periods.

By examining how knowledge changes over a teaching training course it is proposed that further insights into the knowledge useful for teaching and how this knowledge needs to be organised can be gleaned. This reflects the main aim of this research: to explore how the mathematics-related knowledge of trainee secondary teachers changes over time and whether there is a difference between Postgraduate Certificate in Education (PGCE) students who have taken an SKE course and those who have not. In particular, the research questions (3.1, p.53) for this study are as follows:

When considering the mathematics-related knowledge of trainee secondary mathematics teachers on a PGCE course:

1. What is the nature of this knowledge?
2. (How) does this knowledge change?
3. Does mathematics-related knowledge differ between SKE and Non-SKE PGCE students?
4. What are some of the factors which have caused a change (if any)?

In order to address these research questions whilst being consistent with underlying theoretical viewpoints, the current research employs mixed methods concurrently. For this research, ‘mixed methods’ is seen as the combining of both qualitative and quantitative approaches throughout all aspects of the research design: the worldview, research questions, data collection methods, data analysis and conclusions. Additionally, the underlying theoretical assumptions (which have similarities to a critical realist worldview) are seen as providing a philosophical foundation for mixed methods research.

In order to explore the nature of mathematical knowledge for teaching (RQ1), the two existing conceptions of teacher knowledge identified in the literature review as being grounded in mathematics and teaching practice (MKT and the Knowledge Quartet) are examined within the context of trainee secondary mathematics teachers in England alongside the proposed theory of knowledge. Data collection consisted of three main aspects in order to explore these three approaches in the context of this study: questionnaires, interviews and observations.

A sample of the Mathematical Knowledge for Teaching (MKT) measures (developed by Ball and colleagues in the USA) were administered in a questionnaire towards the beginning and end of the PGCE course, marking the first instance of these measures being used in England. The sample comprised of MKT items which were felt could be used in England without adaptation and which covered a range of mathematical topics on the secondary mathematics National Curriculum. All full-time secondary mathematics PGCE students in England were invited to respond to the questionnaire. Overall there were 329 respondents representing approximately 20% of the population.

PGCE students at the University of Leeds were invited to participate in semi-structured clinical interviews towards the beginning and end of the PGCE course. Five students were examined as volunteer case studies. They were asked to freely recall all they knew about three mathematical topics: squares, rational numbers and quadratic equations. They also solved three tasks relating to these topics whilst ‘thinking aloud’ (Ericsson and Simon, 1993). Knowledge Maps were used to represent and analyse the knowledge conveyed by the students during the interviews.

The five PGCE students were also observed whilst on their teaching practice placements. The Knowledge Quartet was used to analyse the classroom observations. All taught PGCE session at the University of Leeds were also observed.

Results of the current study show that whilst PGCE grades do not relate to scores on the MKT items, MKT scores improve by 7.3% over the PGCE course, which is a statistically significant improvement, representing a medium effect size. This suggests that the MKT items capture an aspect of knowledge useful for teaching which develops over the PGCE course but which is not reflected within the final PGCE grade. Post-observation interviews provide insight as they imply that other kinds of mathematical learning such as making connections between existing mathematics concepts, learning applications of existing mathematical topics and learning alternative methods or representations of concepts take place on a PGCE course. Further, the Knowledge Map analyses suggest that knowledge became more (or less) accessible to some students over time or became compressed (or decompressed). These findings suggest that changes in the quality rather than quantity of knowledge take place over a PGCE course. In other words, a change in the organisation of knowledge. Results from part two (problem-solving) of the interviews substantiate this as they imply that knowledge can be held in such a way so as to only be made manifest with the presence of a stimulus (mathematical task) which draws upon that knowledge.

Additionally, only one main difference was found between SKE and Non-SKE PGCE students when comparing the questionnaire, interview and observation data. That is, Non-SKE students demonstrated greater understanding of mathematics as a discipline (including conventions and knowledge of mathematical models) within the classroom.

Several potential factors for the change in PGCE students’ mathematical knowledge are identified by this thesis. These factors mainly stem from the necessity of having to teach a mathematical topic to others. When a student recognised they needed to learn a topic before they could teach it, the following sources were consulted: other teachers, the internet, reading, and personal investigation. When learning occurred through the act of teaching, this was through: repeated explanation to pupils (whereby the explanation became more refined), using a mathematical method for a more advanced topic (and as a result, seeing the limitations of the method), or through pupil questions which prompted personal reflection combined with making connections between different topics as a result of teaching them concurrently. Two further sources of learning through teaching were (i) in post-lesson meetings with tutors/ mentors whereby incorrect mathematics was corrected and (ii) through taught PGCE sessions at university which created a culture of thinking about a variety of different mathematical methods and solutions.

In addressing the research questions, this study also raises four points for further discussion which make wider contributions to existing literature or to practice (see Chapter 7). Firstly, this research marks the first instance of the MKT items (developed in the USA) being used in England. The novel approach to selecting a sample of items and their use within this context was found to be appropriate as suggested by the psychometric (Rasch) analysis. This provides a foundation for other researchers wishing to use these items within England. However, whilst the items performed well, this thesis argues that they do not measure a specialised form of mathematical knowledge for teaching (MKT) as intended by the item authors, but a form of mathematical content knowledge involving mathematical reasoning.

The second discussion point concerns the Knowledge Quartet, used for this research to analyse the school placement lessons taught by the PGCE students. It was found that whilst the Knowledge Quartet was a useful framework for steering the focus of the observations to the mathematical content, there were some issues with: (i) deciding which of several codes best captured an episode, and (ii) some codes inadequately capturing the mathematical knowledge demonstrated by the student.

The third discussion point evaluates the new approach to knowledge introduced within this thesis. Specifically, it highlights two contributions which Knowledge Maps make, namely, providing evidence for Ball and Bass' (2000) theory that a 'decompression' of knowledge is important for teaching and pointing to the idea of 'accessibility' of knowledge as meriting further research.

Finally, this thesis discusses current government teacher training policies in light of the findings of the current research. Whilst offering SKE courses to enable non mathematics graduates to train to be teachers (SKE policy) may be a good policy, results from this study suggest offering differentiated training bursaries (Bursary policy) may not be justified. Further, this thesis highlights a contradiction between the Bursary and SKE policies (7.4.3, p.244) and offers recommendations for future policy (7.4.4, p.244).

8.2 Limitations

This section highlights and discusses limitations of both the qualitative and quantitative data collection methods and analyses employed for this research.

8.2.1 Limitations of the quantitative methods and analyses employed

8.2.1.1 Limitations of data collection: PGCE scores

There may be issues with using final PGCE grades to calculate correlations for this research. Firstly, PGCE scores (which range from 1 to 4) are somewhat subjective as they are assigned by the PGCE students' course tutors and school-based mentors and are not standardised between institutions. In fact they may not even be standardised between multiple tutors and mentors on a single course. Further, anecdotal evidence from course tutors suggests that higher grades (1s and 2s) are more frequently awarded since lower grades reflect badly on the institution when Ofsted are inspecting. Thus, PGCE scores have the potential for bias.

On the other hand, Ofsted moderate grades by observing samples of students teaching. Thus, course tutors are also under pressure to mark students fairly and accurately. Nevertheless, the distribution of grades reflects a tendency to give higher grades (**Table 8.1**, p.252). Though this could be because of the high calibre of students rather than bias.

Alternatively, since students were invited to share their results with me, it could be the case that only those students achieving good grades wished to share their results, whereas those who failed the course or deferred completion of the course to the following year may have preferred not to say.

A further issue with PGCE grades is that those students who were failing the course may have left the course early which may also explain the lack of '4' grades in **Table 8.1** (p.252). This creates another potential problem with the correlations computed: those students who did not complete the PGCE course (effectively scored a '4') would not have completed the post-questionnaire and therefore would not have been included in the correlation between post-MKT scores and PGCE grades. The restricted range of PGCE grades may have therefore effected the correlations.

PGCE grade	Frequency	Valid %
1	55	39.6
2	53	38.1
3	8	5.8
Didn't complete the course	17	12.2
Deferred completion	6	3.6
Total	139	100.0
Missing data	167	
Total	306[16]	

Table 8.1: Table showing the distribution of respondents' PGCE grades

Unfortunately, data protection considerations prevented final PGCE grades from being collected from the institutions themselves. If this were possible, this would have resulted in a more accurate distribution of PGCE grades as well as fewer missing data.

Finally, using final PGCE teaching results as a proxy measure for 'teaching quality' has limitations. A better measure of 'teaching quality' perhaps would have been to observe the teachers teaching during their first (NQT) year of teaching as in Hill and colleagues' (2008b) study of 'mathematical quality of instruction'. However, time constraints would not permit this.

8.2.1.2 Limitations of data collection: degree classifications

When considering degree classifications, no allowance was made for the institution at which the degree was obtained (undergraduate institution data were not collected). it may be the case that a '2:1' at one institution is not equivalent to a '2:1' at another institution due to differing standards.

Another issue is that only first degree results were collected, no allowance was made for Masters degrees or PhDs. Thus, a graduate with a 2:2 in BSc Mathematics and an MSc in Mathematics or even a PhD in Mathematics could have been seen to have 'lower' knowledge than a graduate with a 'first' class in BSc Mathematics but no further qualifications.

[16] 306 students was the total number who responded in a meaningful way to at least one of the pre- and post-questionnaires out of approximately 1,630 full-time secondary mathematics PGCE students nationally.

8.2.1.3 Limitations of data collection: SKE courses

Although the different lengths of SKE courses were, to an extent, taken into account for this study, other differences between SKE courses were not. For example, there are differences in the level of mathematics knowledge covered on courses. Indeed, some focus on mathematics up to Key Stage 4, others include A-level topics (see 1.3.4.1, p.5). These differences in course content are a necessary part of the course since they aim to tailor to students' knowledge gaps. Additionally, there are differences in the balance between content knowledge and pedagogy on SKE courses (e.g. some SKE courses include placements in schools, others do not) as highlighted by the SKE evaluation report (Gibson et al., 2013). Thus, for example, whilst all students who took a six month course were grouped together for analysis purposes, it is difficult to establish whether these courses had similar content.

Furthermore, there are no standard ways of assessing which length of SKE course would best meet a student's needs. Policies may vary between institutions, thus a student put on a six month SKE course at one institution may have been placed on a nine month course if attending another institution.

8.2.1.4 Limitations of data analysis: estimating mathematical content of degrees

There are also limitations of taking degree title and classification as a proxy measure of mathematical content knowledge.

Firstly, the degree titles provided by the students in the questionnaire (124 different degree titles in total) were used to estimate the perceived mathematical content of the degree. This estimation was undertaken separately by four academics and the mean mathematical content score was calculated for each degree title. However, it could only be an 'educated guess' at best. This is because degree programs vary between and even within institutions depending upon the particular modules chosen by a student. Further, even if a complete course outline was provided, calculating what percentage of it counts as 'mathematical' is not clear cut. For example, in a Computing degree, a computer programming module could be said to be mathematical in nature whilst not being a 'traditional' mathematical discipline.

Secondly, "…it is not true that, for example, a 2:1 degree is somehow twice as valuable as a 3rd. Degree classification data is ordinal and does not truly meet the equal-interval criterion” (Stevenson, 2008:105). Yet, degree classification was treated as interval data when calculating correlations for this research to be consistent with other research (i.e. Tennant, 2006; Stevenson, 2008). Nevertheless, a Spearman's rank-order correlation provides similar results to the Pearson correlation (see 5.1.3.4, p.102), that is, a weak, positive correlation between 137 students' degree results and PGCE scores, which was statistically significant (r_s = .195, p = .022).

8.2.1.5 Limitations of data analysis: MKT items

The limitations of separating the MKT questions into items which test for ‘Common Content Knowledge’ (CCK) and Other items are discussed in Chapter 5 (5.1.4.5, p.105). In summary, it is difficult to separate items into mutually exclusive categories. Additionally, since the Rasch analysis suggests the items test for one underlying trait, it may not make sense to attempt to separate the questions into two groups testing different aspects of MKT.

8.2.2 Limitations of the qualitative methods employed

8.2.2.1 Limitations of data collection: interviews

A limitation of the interviews is that they were conducted with different interviewers (my supervisors and I) for the pre-interviews. This may have led to differences in interviewee responses by interviewer. That is, interviewees may have responded differently to me than they did to my supervisors since I may have been viewed as a peer due to also being a recent graduate of a similar age. Thus, interviewees may have felt more comfortable discussing mathematics with me than a university professor, say. Further, all post-interviews were conducted by me, thus there may have been a difference in the Knowledge Maps produced as a result of differences in interviewer between pre- and post- rather than differences in knowledge.

On the other hand, having the same interviewer or interview procedure does not necessarily mean that identical outcomes will be achieved. Indeed:

> Standardization of the interview procedure does not necessarily promote reliability and validity, because the highest authenticity is obtained when the approach is so skilfully [sic.] varied for each individual that he will make his most habitual, sincere, and accurate response (Strang, 1939:500).

This suggests, having different interviewers, if they each skilfully vary the interview to each student, has the potential to elicit responses from the students which are representative of their knowledge.

A further possible limitation of the interviews is within the wording of one of the questions. That is, by asking: “Tell me everything you know about quadratic equations”, it could be that students’ responses were focused on the algebraic representation due to use of the phrase ‘quadratic equations’ rather than ‘quadratics’ or ‘quadratic functions’. Different responses may have been elicited should these other phrases have been used. For a research study which is concerned with representations of mathematical concepts greater care should perhaps have been taken with this wording.

A limitation of using interviews to collect data on the factors leading to knowledge change on the PGCE course is that asking the students to talk about their mathematical learning has several potential reliability and validity issues. Firstly, students may feel embarrassed about a particular learning episode and not wish to share it (particularly if the mathematical knowledge involved was assumed to have been already known before the course commenced). Secondly, a student may simply have forgotten about a learning episode – particularly if it occurred a while before the interview. Thirdly, it is difficult to describe exactly how one has come to learn something, so even if students are aware that they did not know a particular mathematical topic before and now they do, they may not be sure how they came to know and therefore are unable to share in an interview setting how that learning came about.

8.2.2.2 Limitations of data collection: observations

This research could only observe students when undertaking aspects of their PGCE course such as attending their taught university sessions or on their placements. There may have been factors leading to knowledge change which occurred at students’ homes or other settings outside the course which were unable to be discovered during this research.

8.2.2.3 Limitations of observation data analysis: the Knowledge Quartet

The Knowledge Quartet – an existing observation schedule - was used to analyse data collected for this research in an attempt to standardise the analysis in accordance with other analyses of observations of trainee mathematics teachers and to focus attention on elements useful in addressing the research questions (mathematical content knowledge rather than pedagogy).

A limitation of the Knowledge Quartet is that since it is intended to be used to discuss how trainee teachers can improve their lessons (2.1.8, p.27) it could be described as a deficit model. This means some positive aspects of the students' lessons may have been overlooked. Nevertheless there were some instances of excellent teaching which were identified by using the Knowledge Quartet.

I was advised on the Knowledge Quartet training day (see 5.3.1, p.141) to seek second opinions on video clip analysis since others may have noticed different things. This is because a limitation of any observation analysis is that the researcher may sometimes: "project his own definitions on the behavior that he sees" (Jersild and Meigs, 1939:476). Seeking others' opinions was not done due to time constraints but may have been beneficial. Nevertheless, the authors stated that there is not a right or wrong answer when it comes to analysing lessons using the Knowledge Quartet, so the lack of seeking others' insights has no certain impact on the reliability of the analysis.

8.3 Further work

Five possible avenues for further research as appendages to this study are now suggested.

1. This research reports no difference between SKE and Non-SKE students' scores on the MKT items nor between final PGCE grades. However, data was only collected at the beginning and end of the PGCE course, thus no conclusions can be drawn about the success of SKE courses - it could be that SKE students would have performed no worse on the MKT items if they had not taken an SKE course but instead commenced the PGCE course directly. Further research could examine SKE students' scores on the MKT items prior to taking their SKE course, to see what effect (if any) the SKE course has.
2. Further research could be done to see if there is any relationship between scores on the MKT items and the 'mathematical quality of instruction' (MQI, see Hill et al., 2008b) of the teachers in England and also pupil achievement. This could be achieved by following the students involved in the current research over their first years of teaching, observing their teaching practice using the observation schedules set up by Ball and colleagues (Hill et al., 2008b) and also collecting test scores from the pupils they teach. Relationships have been found between scores on the MKT items with MQI and pupil achievement respectively in other countries (e.g. Delaney, 2012) as well as in the USA (Hill et al., 2008b) but this has not been done in the context of England. This could

potentially add to the work of Ball and colleagues as long as criticisms of existing studies are taken into account (see 2.2, p.34). Further, measures of MQI are potentially an improvement over using PGCE scores as a proxy measure for teaching effectiveness as was done in the current study.

3. This research corroborates other studies which suggest that when trainee teachers revisit mathematics they can achieve richer knowledge required for teaching (Stacey, 2008). However, little research exists into the form that this ‘revisiting’ should take, though Pournara and Adler (2014) provide initial ideas. Further work could explore how mathematics can be revisited within a teacher training course.
4. For this research it was difficult to draw Knowledge Maps for S5 who had a highly conceptual and compressed understanding of some mathematical concepts. S5 also indicated that he struggled when trying to teach lower ability groups on his school placements and deferred completion of the course. Further research could therefore be done to see if Knowledge Maps can be used as an initial indicator of whether a PGCE student has compressed mathematical knowledge towards the beginning of a PGCE course. If so, this could be focused on during the course as an attempt to avoid deferred completion.
5. Finally, further work could be done analysing the methodology of this thesis in light of McEvoy and Richards’ claim that: “adopting a critical realist perspective [to mixed methods research] may circumvent many of the problems associated with paradigm ‘switching’” (2006:66). The ontological and epistemological views taken for this research can be said to align with those of a critical realist approach (see 4.2.2.2, p.64) and it is felt that this thesis achieves consistency between (i) the research study, (ii) the view of mathematics, and (iii) the view of learning and teaching mathematics, since the same underlying approach has been adopted for each of these facets. There is evidence that this is not the case for other mixed methods research on mathematics teacher education. For example, in an analysis of three such papers, none explicitly revealed theoretical and epistemological justifications for utilising mixed methods research and tensions existed between the researcher’s perspectives on: research, mathematics and mathematics education (Pritchard, 2011). There is scope for further analysis of the current study to explore whether a critical realist perspective can help achieve greater consistency.

8.4 Concluding remarks

Parallels can be drawn between completing this doctorate, which consists of a thesis and oral examination, and the process of becoming a teacher.

Firstly, the process of writing this thesis is an example of continuously altering knowledge and understanding. Subsequent drafts of the thesis are analogous to a set of Knowledge Maps, representing current understanding at cross-sections of time. It is recognised that even the final version of the thesis is but one discrete representation which could be continuously refined and improved in the future.

Secondly, the final oral examination of the doctorate can be seen as a demonstration of the highest levels of subject matter competence through the ability to teach the subject through a brief oral exposition followed by discussion with examiners (Shulman, 1986).

As Shulman explains:

> To this day, the names we give our university degrees and the rituals we attach to them reflect those fundamental connections between knowing and teaching. For example, the highest degrees awarded in any university are those of “master” or “doctor”, which were traditionally interchangeable. Both words have the same definition; they mean “teacher”. … Thus, the highest university degree enabled its recipient to be called teacher (Shulman, 1986:6).

The underlying theoretical views of this thesis also recognise ‘fundamental connections between knowing and teaching’ or, in other words, between content knowledge and pedagogy. Indeed, results suggest that the process of teaching enables existing knowledge to be deepened and refined through gaining alternative representations of the subject matter. Thus, the teacher has the potential to learn through teaching as well as the pupils.

This thesis concludes with S5’s remarks about integers and rational numbers which can be seen to relate to the discrete/ continuous metaphor for knowledge introduced and used within this research. In other words, once discrete mathematical facts have been learnt, connections between those facts or alternative representations can then continuously be learnt as though ‘filling in the gaps’ between integers on a number line:

> integers - both positive and negative… they’re all unit distance between each other. The rationals take that a step further in that they can now express fractional distances and they begin filling in that density [S5, pre-interview].

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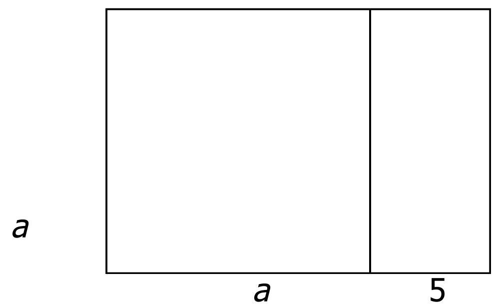
Appendix A

(A.1) Middle School Content Knowledge item (Ball and Hill, 2008)

20. Ms. Whitley was surprised when her students wrote many different expressions to represent the area of the figure below. She wanted to make sure that she did not mark as incorrect any that were actually right. For each of the following expressions, decide whether the expression <u>correctly represents</u> or <u>does not correctly represent</u> the area of the figure. (Mark REPRESENTS, DOES NOT REPRESENT, or I'M NOT SURE for each.)



	Correctly represents	Does not correctly represent	I'm not sure
a) $a^2 + 5$	1	2	3
b) $(a + 5)^2$	1	2	3
c) $a^2 + 5a$	1	2	3
d) $(a + 5)a$	1	2	3
e) $2a + 5$	1	2	3
f) $4a + 10$	1	2	3

(A.2) Elementary Content Knowledge item (Ball and Hill, 2008)

8. As Mr. Callahan was reviewing his students' work from the day's lesson on multiplication, he noticed that Todd had invented an algorithm that was different from the one taught in class. Todd's work looked like this:

$$
\begin{array}{r}
983 \\
\underline{\times\ 6} \\
488 \\
+\underline{5410} \\
5898
\end{array}
$$

What is Todd doing here? (Mark ONE answer.)

a) Todd is regrouping ("carrying") tens and ones, but his work does not record the regrouping.

b) Todd is using the traditional multiplication algorithm but working from left to right.

c) Todd has developed a method for keeping track of place value in the answer that is different from the conventional algorithm.

d) Todd is not doing anything systematic. He just got lucky – what he has done here will not work in most cases.

Appendix B

The Knowledge Quartet

FOUNDATION:

AP	awareness of purpose;
IE	identifying pupil errors;
OSK	overt subject knowledge;
TUP	theoretical underpinning of pedagogy;
UT	use of mathematical terminology;
ATB	adheres to textbook;
COP	concentrates on procedures.

TRANSFORMATION:

DT	teacher demonstration
UIM	use of instructional materials
CR	choice of representation;
CE	choice of examples;

CONNECTION:

MCP	making connections between procedures;
MCC	making connections between concepts;
AC	anticipation of complexity;
DS	decisions about sequencing;
RCA	recognition of conceptual appropriateness;

CONTINGENCY:

RCI	responding to children’s ideas;
UO	(use of opportunities);
DA	deviation from lesson agenda;
TI	teacher insight;
RAT	responding to the (un)availability of tools and resources.

Appendix C

Hierarchical Focusing as a Research Interview Strategy

"In general outline, the proposed strategy is as follows:

(1) Carry out and explicitly portray an analysis of the content and hierarchical structure of the domain in question as you, the researcher, construe it.

(2) Decide on your research focus: identify those aspects and elements of your topic domain whose construal you wish to elicit from interviewees.

(3) Visually portray a hierarchical agenda of questions to tap these aspects and elements in a way that allows gradual progression from open to closed framing, combining this as appropriate with contextual focussing. Include with this question hierarchy a skeleton of the same structure for use as a guide and record.

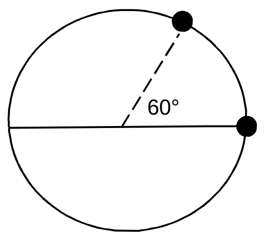
(4) Carry out the interview as open-endedly as possible, using the above strategies within a non-directive style of interaction so as to minimise researcher framing and influence. Tape-record the proceedings.

(5) Make a verbatim transcript and analyse the protocols, with use of the audiotape record where appropriate."

(Tomlinson, 1989: 162)

Appendix D

1. Imagine a point on a circle starts at (1,0), turns through 60 degrees anticlockwise around the circle and then stops.



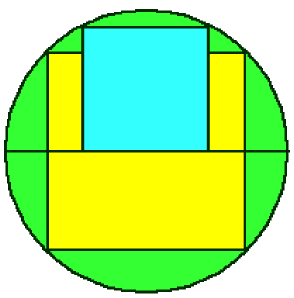
I was wondering, if the point hadn't stopped, and instead carried on until it had turned through 30,000 degrees, might it have finished the same distance above the horizontal axis? I took out my calculator and typed 30000 ÷ 360

The answer on the screen was 83.333333.

How can I use this to help me solve my problem?

2. A blue square of area 40 cm^2 is inscribed in a semicircle.

Find the area of the yellow square that is inscribed in a circle of the **same radius**.



3. Consider the family of functions: $y = x^2 + bx + 1,\ b \in \mathbb{R}$

Find the equation of the curve that goes through the vertex of every function in this family.

Appendix E

(E.1) Example Analysis Table (S3: Squares)

Pre-Interview	Post-Interview
Squares are quadrilaterals	Squares are a quadrilateral
they have four sides	four equal sides
all sides are exactly the same length	
they have 4 angles of 90 degrees	four equal angles
	all of the angles are 90 degrees
they're a specialist form of rectangle	It is a special form of a rectangle
	is a special form of parallelogram
	there's two pairs of parallel sides, which a square fits
	Rhombus
	a parallelogram is a quadrilateral
	A rectangle is a quadrilateral
	a rhombus is a parallelogram… all sides are the same length which also describes a square
all squares are similar simply because of the fact that they all have the 90 degree angles	
	the area gives the square numbers,
	which are named because they come from squares
area is.. the side of the length squared	the area of a square is two… sides multiplied together – because those are the same length
	then we get a square number from that
	the right-angled triangle comes from a square
	A square number is a number timesed by itself
	four squared is four times four which is 16
	Perimeter is the outside of the square
	four times by the length of the side
	a 3d shape made entirely of squares is a cube

(E.2) Example Analysis Table (S2: Rational Numbers)

Pre-Interview	Post-Interview
they can be represented as a fraction	they can be expressed as a fraction.
an irrational number such as pi.. something which can't be expressed as a fraction	irrational numbers can't be expressed
	numbers like pi aren't rational numbers because they can't be expressed as a fraction

	so if you're multiplying fractions you – it doesn't matter if the… denominators aren't equal you just multiply the top number, multiply the bottom number
	but for adding or subtracting fractions, you need to have them over the common denominator so you're probably going to need to manipulate them at some stage
	dividing fractions, again you don't need a common denominator, the easy way is to just flip the second fraction then multiply top by bottom
I think I'm right in linking surds to rational numbers – a surd is a type of rational number isn't it?	

(E.3) Example Analysis Table (S4: Quadratic Equations)

Pre-interview	**Post-interview**
so a quadratic equation can be written in the form ax squared plus bx plus c equals zero	
JM: can we say anything about A? S4: … it can't be zero	RW: … so what happens if a is zero? S4: … – then it wouldn't be a quadratic equation because-… that'd mean it was a linear equation 'cause you wouldn't have any x squared values.
it's always gonna have an x squared value	there's got to be at least one x squared for it to be a quadratic equation
they can be solved using factorisation	factorisation
it can also be drawn as lines on a graph	The graph of a quadratic equation makes …. bowl shape
so you could find– the.. x solutions are here and here [crosses on Figure 5]	you can find solutions to the quadratic equations using the graphs by finding where the lines… cross the x-axis
JM: what shape do we call that?.. it has a name that shape S4:... I wanna say arc, but I… don't think it is JM: it begins P, A S4: aww [pause] JM: pa- [Figure 6] S4: no it's gone.. parabula – cheers!	
so the C would move it ... up or down.. on the.. y axis.	that's affected by– the a, b and c values with how that shape's –the steepness of it and the placement of it.
you can use the quadratic formula to solve it.	you can solve them using the quadratic… formula
	which is minus b plus or minus the square root of b squared minus 4ac all divided by 2a when the quadratic equation takes this form – the a … x squared plus bx plus c equals any value – it doesn't have to be zero, I've written zero down

there's completing the square as another method of solving it	You can solve it using completing the square
	Trial and improvement
… no there's not always four so it could be a situation like this where.. there is no solution	you can have quadratic equations that have one solution, two solutions or no solutions
it's even possible it's possible to have just the one solution as well	if the discriminant equals zero, then there's only one solution – because you're plussing or minusing that value
but it's not possible to have more than two solutions	if it equals more than zero then you've got two solutions
it's a complex solution	you can't have it equal less than zero so there'd be no solutions 'cause you can't find the square root of a value less than zero unless you take into account complex numbers.
	also you can use the graph as well because you can look at how… many times the line meets or crosses the x-axis
	So if the line of the graph – the quadratic equation doesn't – doesn't cross the x-axis then it has no solutions. If it just meets it once then it's got one solution. If it crosses it twice, it's got two solutions
	Quadratic equations are linked to quadratic sequences where… the sequences grow at quadratic rates… – an example of a quadratic sequence would be… when there's a constant difference in the difference between the terms… you'd get them growing at a quadratic rate.
Polynomial equation is it?	
[the vertex is] the lowest point of every function	
if I differentiated that'd give the area under it	

Appendix F

Student	**Observation 1**		**Observation 2**		**Observation 3**	
S1	FOUNDATION		FOUNDATION		FOUNDATION	
	identifying errors	4	identifying errors	1		
	concentration on procedures	2	concentration on procedures	2		
	use of terminology	6	use of terminology	1		
			overt subject knowledge	2		
	TRANSFORMATION		TRANSFORMATION			
	choice of examples	3				
	CONNECTION		CONNECTION			
	decisions about sequencing	4				
	making connections between concepts	1	making connections between concepts	1		
			making connections between procedures	5		
	CONTINGENCY		CONTINGENCY			
	responding to children's ideas	7				
	deviation from agenda	3	deviation from agenda	1		

Student	**Observation 1**		**Observation 2**		**Observation 3**	
S2	FOUNDATION		FOUNDATION		FOUNDATION	
	awareness of purpose	2				
	use of terminology	1	Use of terminology	5	use of terminology	1
					identifying errors	1
					concentration on procedures	1 1
	TRANFORMATION		TRANSFORMATION		TRANSFORMATION	
	choice of representation	2	choice of representation	3	choice of representation	4
			choice of examples	1		
	CONNECTION		CONNECTION		CONNECTION	
	making connections between procedures	4	making connections between procedures	1	making connections between procedures.	2
	making connections between concepts	2	making connections between concepts/ procedures	2		
			decisions about sequencing	1		
					anticipation of complexity	1
	CONTINGENCY		CONTINGENCY		CONTINGENCY	
	responding to children’s ideas	8	responding to children’s ideas	3		

Student	**Observation 1**		**Observation 2**		**Observation 3**	
S3	FOUNDATION		FOUNDATION		FOUNDATION	
	concentration on procedures	6	concentration on procedures	3	concentration on procedures	9
	use of terminology	2	Use of terminology	1		
					adheres to textbook	2
					awareness of purpose	8
					overt subject knowledge	1
	TRANSFORMATION		TRANSFORMATION		TRANSFORMATION	
	Teacher demonstration	5	Demonstration	2	demonstration	2
	CONNECTION		CONNECTION		CONNECTION	
	decisions about sequencing	1			decisions about sequencing	3
	(lack of) anticipation of complexity	1			anticipation of complexity	2
	making connections between procedures	1				
	CONTINGENCY		CONTINGENCY		CONTINGENCY	
					responding to children’s ideas	2

Student	Observation 1		Observation 2		Observation 3	
S4	FOUNDATION		FOUNDATION		FOUNDATION	
	use of terminology	8	use of terminology	1	Use of terminology	3
	awareness of purpose	4	awareness of purpose	1		
	overt subject knowledge	3				
			concentration on procedures	3	concentration on procedures	2
	TRANSFORMATION		TRANSFORMATION		TRANSFORMATION	
			choice of examples	2	choice of examples	1
	CONNECTION		CONNECTION		CONNECTION	
	Anticipation of complexity	1	anticipation of complexity	1		
			decisions about sequencing	1	decisions about sequencing	1
					making connections between procedures/ decisions about sequencing	1
					making connections between procedures	2
					making connections between concepts	1
	CONTINGENCY		CONTINGENCY		CONTINGENCY	
	responding to children's ideas	3	responding to children's ideas	5	responding to children's ideas	2
			deviation from agenda	2		
			teacher insight/ realisation	3		

S5	FOUNDATION		FOUNDATION		FOUNDATION	
	use of terminology	2	use of terminology	2	use of terminology	4
	concentration on procedures	1				
	TRANSFORMATION		TRANSFORMATION		TRANSFORMATION	
	demonstration	1				
	choice of examples	5	choice of examples	4	choice of examples	3
	choice of representation	7	choice of representation	3	choice of representation	6
	CONNECTION		CONNECTION		CONNECTION	
	making connections between procedures	5	making connections between procedures	3		
			decisions about sequencing	2	decisions about sequencing	1
	Making connections between concepts	3				
					recognition of conceptual appropriateness	1
					anticipation of complexity	1
	CONTINGENCY		CONTINGENCY		CONTINGENCY	
	teacher insight/ realisation	2	teacher insight/ realisation	2		
	responding to children's ideas (or lack of!)	8	responding to children's ideas	2	responding to children's ideas	3
					deviation from agenda	1
					Responding to the availability of tools and resources	1