Analysis of Endogenous Asset Formation:

Production in General Equilibrium

with Incomplete Markets

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Contents

Chapter 1

Introduction

1.1 Motivation and the basic research problem

This thesis analyzes production in a general equilibrium model with incomplete markets (GEI). While most of the GEI literature derives results for asset structures exogenously specified, this thesis presents an early stage attempt on the study of endogenous asset formation in a simple two period general equilibrium model with incomplete markets. It focuses attention on the organization of production at first instance, because production is obviously a major source of asset endogeneity, since dividend payoffs, payoffs from bonds and any other financial assets like derivatives and options are all endogenous. Moreover, this thesis introduces a model of the firm in which the production sets available to firms are also endogenized. This allows economic relevant interpretations of the role firms play in terms of short and long run optimization behavior, and in terms of financial and real economic activities.

Ex ante, a study of endogenous asset structure formation is interesting for two main reasons. It allows to generalize the asset structure and (at minimum) to replicate the existing results in the literature, and to progress with the research program on production in general equilibrium with incomplete markets. The other reason, why this research could be interesting ex ante is that besides generalizing the asset structure, there exists the possibility to improve on existing economic results derived independently of the asset structure. This thesis provides a systematic criticism of the exogenous asset formation literature, and shows that the later ex ante consideration turns out to be the case. Among those equilibrium properties of principal interest to this thesis are:

- The endogenous asset formation model introduced in this thesis allows to analyze properties of economic equilibria for more general assets structures. In particular, we consider smooth and non-smooth real asset structures which are not independent of financial activities of the firms.

- The study of an endogenous asset formation model facilitates a reexamination of the results derived under exogenous asset structures. While the simplified exogenous asset structure of GEI models is sometimes regarded as sufficiently general [26], I examine properties of productive organization for its extension to an endogenous asset formation model.

- The endogenous asset model provides a more hospitable and realistic framework for studying the organization of production. This includes the modeling of the financing of the firm, a topic receiving seemingly less attention in contemporary literature.

The general equilibrium model with incomplete markets is a generalization of the Arrow-Debreu model where time and uncertainty enters the model in an essential way. Studying production in an incomplete markets framework is interesting, because the introduction of time and uncertainty into the analysis of economic general equilibrium brings many new phenomena into light which cannot be described by the model of the firm in the classical Arrow-Debreu framework. Those of foremost interest to this thesis include:

1. the interdependence of real and financial assets

2. the objective function of the firm

3. the efficient organization of production

4. the potential (ir)relevance of financial policy of the firm.

The interest of studying these properties comes at first instance from the contemporary view of the literature on what the goal of the firm should be (optimization problem) when markets are incomplete, namely the maximization of some sort of assigned utility to the firm's objective function. This thesis criticizes this approach to the study of the firm for the main reason of how to decide what utility to assign to it, and for the highly stylized model of the firm, in which financial assets essentially play no role. The classical model of production seems highly streamlined, and this thesis aims at improving on that. The primary aim is therefore, to establish a link between the real and financial sector (1), and to consider the role that financial assets play. With this in mind, what is the explicit derivation of the objective function of the firm (2) establishing this link? In particular, how is the efficient production boundary of each production set determined? The organization of production in (3) is another economic property of interest. The organizational productive inefficiency property of the classical GEI model of production is a further critique of the contemporary approach to the model of the firm. It is well known that the organization of production introduces an additional source of productive inefficiency in utility maximization models of the firm. We compare allocational efficiency of this model with the model introduced in this thesis. Finally, the study of the Modigliani and Miller theorem is motivated by the independence of the production set available to each firm from the firms' activities on capital markets (4). In the classical GEI model of production the financing of production is mostly not explicitly modeled. At best, firms issue stocks in order to finance production inputs at given production sets. However, the alternative interpretation of the classical model is also that production inputs are financed with the sell of total production output. From this view point, financial assets essentially play no role, and consequently the Modigliani and Miller theorem holds under standard assumptions.

Another important element of endogenous asset formation is the study of default. This research is still at its infancy, and only a few papers deal with this issue, despite its importance (Dubey, Geanokopolos and Shubik [45]). This paper determines default penalties for strategic default of individuals and shows that a GEI equilibrium with default exists. However, this research line is probably of less interest when considering default in an endogenous asset formation model. Firms with limited liability cannot be punished for defaulting. More important, the problem of the firm of acquiring cash through the stock market in order to build up production capacity will become even more difficult when a firm is expected to default. For simplicity, this thesis considers the case of no bankruptcy only, since much more work needs to be done on the way of endogenizing default. However, a first step already in progress in this direction is to endogenize the states of nature as a function of the firm's financial policy.

The economic interest in permitting incomplete markets derives from the introduction of a further element of realism into the analysis of economic equilibrium. The basic objective of the GEI theory is to expand the Arrow-Debreu model to a more general economic model with real and financial assets but with limited ability to trade into the future. For example, in the Arrow-Debreu model, since there is no need to transfer wealth between future uncertain states of the world, there is no trade on the stock market, hence no attention is paid to the different roles real and financial assets play. Thus, in the complete markets general equilibrium model, where markets function at their best, the role of the firm reduces to simple arithmetics. Essentially, the analysis of the firm in the Arrow-Debreu model becomes trivial, since many of the economic activities firms perform cannot be modeled. Among these are the financing of production through the stock market, the link between real and financial assets, and the endogenous asset structure formation. For these reasons I believe that the economic scenario, in which markets are incomplete is more realistic for a study of the firm.

A main criticism of the general equilibrium theory with incomplete markets concerns the exogenous specification of the asset structure. The current GEI theory heavily relies on results derived from an equilibrium analysis independent of the asset structure. This perhaps because the research program on endogenous asset formation is progressing relatively slowly. Problems associated with production in general equilibrium with incomplete markets impede the further development of this research program. These problems have been expressed in various ways. Here a few citations from the research frontier.

> *The firm fits into general equilibrium theory as a balloon fits into an envelope: flatted out! Try with a blown-up balloon: the envelope may tear, or fly away: at best, it will be hard to seal and impossible to mail.... Instead, burst the balloon flat, and everything becomes easy.* [20]

> *Evidently the question of what the goal of the firm should be with incomplete markets is widely thought to be one of the bugaboos of GEI analysis.* [2]

> *However, one reason for seeking more general asset structures is beyond the scope of the pure exchange model: it is that of being able to understand some of the phenomena linking financial markets and production.... I hope this is a motivation for moving towards more general asset structures. A case that is however very close to our framework is the one where the firm maximizes ex-post value of the output.* [8]

This thesis is an early attempt towards a study of endogenous asset formation. Despite the fact that production is a major source of asset endogeneity, there is only little significant progress observable in this direction. This not at least because there is no consensus in the literature about what the goal of the firm should be when markets are incomplete. The study of the objective function seems to be merely at a stand still since the papers by Drèze [21], and Grossman and Hart [28]. The present understanding of the objective function of the firm suggest assigning utilities to the firm, just as general equilibrium has always assigned utilities to consumers [26]. This approach is not satisfying. The immediate question about what utility to assign to which firm arises. It is possible to think of any utility, but they all have in common that they are decided by the modeler. This is precisely the starting point of this research program. This thesis deals at first instance with the problem of defining an objective function of the firm which is independent of any "average" utility assigned to it.

1.2 Definition of the problem and research questions

The aim of this thesis is to provide a simple two period general equilibrium model with an endogenized asset structure, and to study equilibrium properties of such a model as outlined in the previous section. This research is important because most of the general equilibrium literature on incomplete markets derives equilibrium properties for exogenously specified asset structures. This is a grave drawback of the GEI analysis as it implicitly assumes the equivalence of equilibrium properties of exogenous asset formation and endogenous asset formation models. This thesis aims at elaborating on this widely believed folklore.

Given the overall motivation of introducing an endogenized asset model, the more specific research question deals with defining an objective function of the firm. This problem receives much attention because production is a major source of asset endogeneity, since the production set, and payoffs from any financial securities are all endogenous. Introducing an objective function of the firm which is independent of extra information not contained in market prices would improve on models induced by Drèze [21], and Grossman and Hart [28]. These are the dominant GEI models on which most of the resent research on production in incomplete markets is based. Assigning utilities to firms is not unproblematic. The question of what utility to assign to which firm is not resolved at full satisfaction yet, and mostly specified by the modeler by assumption. This research line is motivated by the question: Do the preferences of shareholders, and the prices of shares on the stock market, influence the choices of firms among alternative state distributions of profits [20]?

This thesis therefore, is motivated by a different set of questions and asks: What role do financial assets play in an endogenous asset formation model, and what objective function of the firm would endogenously determine the firm's production set and dividend payoffs? Can the decentralization property of the Arrow-Debreu model be generalized to the case of incomplete markets? These questions lie at heart of my research program presented in this thesis.

1.3 Summary of main results

In this thesis, I consider a generalization of the asset structure beyond the smooth asset structure of the pure exchange general equilibrium model with incomplete markets. This involves establishing a link between financial assets and production in order to enhance the understanding of real world economic phenomena, where the real and financial sectors are not independent of each other. The main result on the asset structure introduced establishes a *class of regular endogenized smooth asset structures.* This result is very convenient and applies for stock market models, where stocks are the only financial assets considered. The expansion of this result to other financial assets, such as bonds, is primarily a matter of notation.[1]

An indispensable equilibrium property to be proved is the *existence of equilibria.* It is shown that for above mentioned regular asset structure, equilibria always exist. This result follows from the applications of Thom's parametric transversality theorem. Moreover, the presented proof of existence of equilibria is expanded to the case of convex, piece-wise linear production sets. The main proof of existence of equilibria is established under the unnecessary strong assumption of long run profits maximization for a reduced form model of the firm. This assumption is relaxed in subsequent chapters, and therefore, in order to guarantee existence of equilibria for the extensive form model of the firm an *equivalence result* between the reduced form model and the extensive form model is established.

Existence of equilibria for non-smooth production sets follows from the smooth existence proof case for a given *regularized endogenous asset structure.* For that, I first apply regularization theory from real analysis, in particular, I use the technique of convolution in order to smooth out the piece wise linear production manifold. It is shown that *the regularized production manifold approximates the convex hull of the piece-wise linear production set* sufficiently well.

While most of the general equilibrium with incomplete markets literature models incomplete markets by hypothesis, it is shown that *incomplete markets is a consequence of the idiosyncratic risk* present in this economy. This result contributes to the further understanding of why financial markets are incomplete. Moreover, most of the classical GEI takes the number of firms as exogenously given. This is at variance with the model presented here, where the number of

[1]However, some care is required in case of the possibility that firms can go bankrupt. This research is in progress, but not subject of this thesis.

firms (the degree of incompleteness) is endogenized. Hence, this thesis considers a model of *endogenously determined incomplete markets.*

I consider an economic scenario which is sufficiently interesting and rich in structure to study the organization of production in a general economic equilibrium. The main contribution of this thesis is the introduction of a model of the firm in a general equilibrium framework with incomplete markets, where the objective function of the firm is independent of any extra information provided by the group of owners of the firm. Essentially, this thesis rehabilitates the profit maximization objective of the firm of the Arrow-Debreu model for the more general case when time and uncertainty is explicitly modeled. This is at variance with the current GEI literature on the objective of the firm, which suggests to assign utilities to firms. Moreover, the sequential objective function of the firm introduced allows a *short and long run economic interpretation.* Firms build up production capacity in the long run, and given their installed production sets, choose a profit maximizing short run net activity. Over both periods firms choose real and financial quantities to maximize their profits, hence *the objective of each firm is long run profit maximization.*

The model introduced allows a *new formalization of ownership and control.* While in the classical GEI model shareholders directly impose control over the net activities of the firm, here, the owners control total production capacity available to a firm. This rehabilitates the idea that operational decisions are taken by managers acting in the interest of their stock holders when maximizing profits in a very traditional way.

A consequence of the model of production introduced is the preservation of the *decentralization theorem* of the Arrow-Debreu model when time and uncertainty enter the model in an essential way. This result improves on Drèze, and Grossman and Hart who were able to separate the activities of the firm from the activities of the consumers, once a present value vector determined by the consumers is assigned to the objective function of the firm.

While most of the GEI literature assumes that production is automatically financed, I explicitly model the *financing of the firm.* The long run financial problem of the firm is to issue stocks and to buy capital. The level of capital acquired determines the production set available to the firm in period two. The firm's short run financial activity is then to finance inputs of production with the revenue acquired by selling its output on the spot markets. This is at variance with classical GEI theory, where firms issue stocks in order to finance the inputs of production in period one, at fixed production capacity. In addition, the financing of production capacity determines the *size of the firm*, an important economic property largely ignored in the GEI literature.

The Modigliani and Miller theorem implicitly assumes that the firm's financial policy finances its production activity. Most proofs in GEI analysis adapt this assumption and replicate the validity of the theorem on the irrelevance of financial policies for the case of no bankruptcy. These literature includes papers by Stiglitz, Duffie, Shafer, and DeMarzo [[52],[53][14][23]]. I show that when the production set available to a firm is financed via financial markets the firm's *financial policy has generally real equilibrium effects*, depending on the firm's ability of acquiring capital.

A comparison of the model introduced in this thesis with the classical GEI production model suggests issues related to the *(in)efficient organization of production.* I show that the model introduced in this thesis is *productive superior* efficient relative to the models with utility maximizing firms. This result follows from the independence of the objective function of the firm from any present value vectors derived from the shareholders. *Efficiency properties are those of the classical pure exchange economies*. This suggests, at variance with centralized GEI models of production, that the *organization of production is efficient*.

Chapter 2

Literature Review

2.1 Related literature

This section aims at providing the broad context in which the literature on the specific phenomenon of production is embedded. The literature on general equilibrium under time and uncertainty dates back to the seminal paper by Arrow ([1][2]). This paper makes two fundamental contributions. It introduces a new approach to probability theory by introducing the idea that a random variable is a function defined on a set of states of nature. This leads to the equilibrium concept under time and uncertainty known as contingent market equilibrium. The second fundamental contribution of this paper is that it economizes on the number of contingent markets needed in order to obtain the same Pareto efficient allocation as in the contingent markets model by introducing financial markets. Arrow considered elementary securities which promise to deliver one unit of account if a particular state occurs and nothing otherwise.

The first to consider a set of markets with less financial securities than states of nature was Diamond [16]. This paper builds the cornerstone of the model of production in general equilibrium with incomplete markets. Diamond notices that with an incomplete financial markets system, Pareto efficiency does no more hold. Under the restrictive assumption of multiplicative uncertainty

Diamond showed constrained efficiency of equilibrium for the special case of a one good and single period model. In this model stock holders generally agree upon the production plan to be implemented by the firm.

Two main directions have emerged from Diamond's model where agents have limited possibilities to transfer wealth across time and states of nature. One branch of research deals with temporary equilibrium, where agents have exogenously given rules for forming expectations as functions of past and current variables. The other branch, and of primary interest to this thesis, is related to rational expectations [1] equilibrium, where agents correctly anticipate future variables. Either of these ideas closes the basic model of Diamond with an equilibrium concept. The later, formally introduced by Radner [46], who also realized problems associated with the objective function of the firm in establishing existence of equilibrium.

A paper which has much stimulated the literature on incomplete markets is due to Hart [31]. By means of examples he showed that equilibrium does not generally exist. This has lead to a huge literature on establishing existence of equilibrium. Two main branches can be classified. The first research line restricts attention to asset structures for which the payoff matrix cannot change rank. Cass [9] and Werner [56] considered a payoff structure in units of account while Geanokopolos and Polemarchakis [27] established existence for an asset structure with payoffs in a numeraire good. The other research line and of interest to this thesis shows that the set of economies for which equilibrium does not exist is of measure zero, implying that the probability of observing such an economy is very small and can therefore, be disregarded. The first proof in this direction is an extension of Balasko [4] to incomplete markets provided by Duffie and Shafer [11]. Other contributions to the study of existence of equilibria are Geanokopolos and Shafer [34], Husseini et al., [50], Hirsch et al., [37], Bottazzi [8].

[1]The idea of this concept was implicitly contained in Arrow's model

Beyond the one good model, generic equilibria are generally inefficient. This was first shown by Geanokopolos and Polemarchakis [27], Duffie and Shafer [23], and Geanokopolos et al. [33], each applying a different parameterization. The underlying idea of all proofs is simple. Since there is an incomplete income transfer space, agents' gradient vectors will generally not point in one direction.

Next section introduces the development of production within the context of this development of the general equilibrium literature with incomplete markets.

2.2 Literature on production

The benchmark model of the firm in general equilibrium under uncertainty is presented in the most elegant synthesis of Arrow and Debreu in the book called theory of value [12]. In this theory, the exogenously given primitive data of the economy consist of consumers described by preferences, and producers described by production sets. Agents optimize, have rational expectations, and markets clear. It is worth remarking here that producers are only and fully described by their exogenously given technology. In the context of a two period model, the exogenously given production sets range over both periods. The objective function of the firm is profit maximization. Each firm chooses a profit maximizing production plan at competitive prices. This corresponds to choosing inputs in period one with associated outputs in each state of the world in period two. For a complete set of contingent markets no problems arise regarding to what the goal of the firm should be, since at equilibrium all gradient vectors of the share holders point in the same direction.

New economic phenomena come into life once time and uncertainty enter the model in an essential way. These include problems related to the definition of an objection function of the firm at first instance. Others relate to the financing of the firm, a problem receiving seemingly less attention. The landmark paper studying specific problems of production economies is due to Diamond [16]. He restricts attention to a one good and single period model and shows that under the assumption of multiplicative uncertainty and objective of value maximization of the single firm equilibrium allocations are constrained efficient. A main property of this model is that the one period production set available to the firm is fully described by an exogenously given technology function. This function is independent of financial activities of the firm. The objective function of the firm is similar to the objective function of the firm in the Arrow-Debreu model with no stock market.

Beyond one good static models, Radner [46] formalized the general model of production and drew attention to problems associated with the objective function of the firm when spanning fails to hold. The question of what the goal of the firm should be in general equilibrium with incomplete markets has been at the center of the incomplete markets literature since Radner. Although unanimity of shareholder on the production plan of the firm was not explicitly stressed by Diamond, much of the subsequent work deals with finding conditions for which the market value objective of the firm generally holds. Ekern and Wilson [49] identified conditions for which unanimity of share holders on the production plan of the firm holds. These conditions amount to allow for production plans for each firm which are priced by the market. Radner[47] introduced a partial spanning assumption on production sets and shows that unanimity of shareholders can be achieved under this conditions. Other papers within the spanning literature are Grossman and Stiglitz [29], and Milne and Starrett [41]. In conclusion on the literature on market value maximization, a firm would have to know the effects of a given modification of a production plan on its market value. The literature shows that restrictive assumptions are needed in order to render this goal of the firm operational.

When spanning prevails, it was shown that share holders will unanimously agree on the firm's production plan. This however, is not more generally true in absence of partial spanning, as there is room for share holders to disagree about future values of risky investments. Drèze [21] was the first to recognize that stock prices do not always convey sufficient information to guide production decisions. He identified conditions for which unanimity of shareholders is established. One condition is that a collective group of new shareholders (after trade at the stock markets occurred) decide what production plan the firm should employ. This concept of the firm requires a second condition, which allows for side payments among share holders in order to achieve unanimous agreement on a production plan to be employed by the firm. Drèze was the first to introduce an objective function of the firm which is not independent of the preferences of the owners of the firm. Criteria for additional market information were derived from the Hicks Kaldor sum, which leads to a weighted shareholder criterion of the final group of shareholders. He established constrained efficiency for a single good model and introduced additional information from shareholders' preferences into the objective function of the firm. Grossman and Hart [28] criticized that production decisions are made by the final group of shareholders, since it would not allow to go beyond a two period model, because effects of production decisions on stock prices can be ignored in a model where final shareholders guide decisions on production plans. They therefore, introduced a decision criterion where production plans of the firms are guided by the initial shareholders (before trade at the stock market takes place). This required expanding the idea of competitive pricing to a framework of incomplete markets. They introduced the assumption of competitive price perceptions. The more recent literature on generalized production models maintains the centralized property of the objective function of the firm introduced by Drèze. For an example of this large literature see Duffie and Shafer [22], DeMarzo [14], Magill and Shafer [40], Geanokopolos et al. [33], and Magill and Quinzii [38].

Based on the concepts introduced by Drèze and Grossman and Hart are ideas related to voting mechanisms. This literature deals with economic problems like proxy fights or instability issues of the political economic for example. For a sample of voting applications of the these models see DeMarzo [15], Drèze [20] and Tvede and Cres [54], [55].

Other developments on the study of the objective function of the firm are related to the maximization of a function. For a sample of this research line see Dierker and Grodal [18], Bejan [7], and recent work by Magill and Quinzii [38].

The foundations for the market value maximization objective of the firm are provided by the Modigliani and Miller theorem [43] in a partial equilibrium set up. They showed that the value of the firm does not depend on how its production is financed. The first to prove the irrelevance of financial policy in general equilibrium model was Stiglitz [52],[53]. He noticed that in an incomplete markets framework, the theorem fails to hold for debt policies which may lead to bankruptcy, but do otherwise generally hold. More recent papers by Duffie and Shafer [23], and DeMarzo [14] replicate the validity of Modigliani and Miller's theorem for preference dependent objective functions of the firms, and interfirm security holdings.

Chapter 3

The Mathematical Model: Existence of Equilibrium

3.1 Introduction

This chapter establishes the corner stone of this thesis. It introduces the mathematical model, the main economic ideas, assumptions, and the mathematical notation. The quintessence of this chapter is the introduction of the model of the firm into an incomplete markets framework in an essential way that it eventually eliminates the present value problem of current production models with incomplete markets, where firms are utility maximizers. This amounts to assigning a sequential optimization structure to the firm, where the efficient sphere of the production set is not independent of the total number of financial assets issued by the firm. This naturally leads us into the world of endogenous asset formation and the study of economic phenomena of linking financial markets with production.

This chapter is unfortunately, as most of the general equilibrium literature on incomplete markets, unavoidably notational intensive. To keep potential confusion at a minimum, we introduce a long run profits maximization assumption. This assumption has the convenient advantage that it allows postponing a rigorous study of the precise nature of the objective function of the firm until chapter 5. Easy examples of the extensive form model of the firm however, can already be found in chapter 4.

The sine qua non of every model is then to prove existence of equilibrium. We establish generic existence for convex smooth, and convex piece-wise linear production sets. We also exhibit a class of endogenous asset structures for which equilibrium always exists. This class of smooth asset structures generalizes existing asset structures in two aspects. The efficient boundaries of the real assets structures are endogenously determined, and not independent of the firms' ability of acquiring cash through financial markets by issuing stocks. The other advantage of this asset structure introduced is that it allows interesting economic interpretations of the model of the firm, and therefore, to enhance the theory of the firm in general equilibrium with incomplete markets. Such are related to the long and short run optimization behavior of the firm, a new formalization of ownership and control, and the goal of the firm which is independent of extra information derived from any group of owners of the firm, for example. This list is not exhaustive, and further economic properties of the model of the firm will be introduced subsequently.

3.2 Assumptions, definitions and notation

We consider a two period $t \in T = \{0, 1\}$ model with technological uncertainty in period 1 represented by states of nature. An element in the set of mutually exclusive and exhaustive uncertain events is denoted $s \in \{1, ..., S\}$, where by convention $s = 0$ represents the certain event in period 0. Where no confusion of notation is expected, we sometimes denote S the set of all mutually exclusive uncertain events. We count in total $(S + 1)$ states of nature.

The economic agents are the $j \in \{1, ..., n\}$ producers and $i \in \{1, ..., m\}$ consumers which are characterized by sets of assumptions F and C below. There are $k \in \{1, ..., l\}$ physical commodities and $j \in \{1, ..., n\}$ financial assets, referred to as stocks. In fact, stocks are the only financial assets considered here. This allows for a sufficiently rich structure in order to introduce the benchmark model of production in its simplest form. Physical goods are traded on each of the $(S+1)$ spot markets. Producers issue stocks which are traded at $s = 0$, yielding a payoff in the next period at uncertain state $s \in \{1, ..., S\}$. The quantity of stocks issued by firm $j \in \{1, ..., n\}$ is denoted $z_j \in \mathbb{R}_-$, where $\hat{z} = (z_1, ..., z_n)^T$.

There are in total $l(S+1)$ physical goods available for consumption. The consumption bundle of agent $i \in \{1, ..., m\}$ is denoted by $x_i = (x_i(0), x_i(s), ..., x_i(S)) \in \mathbb{R}^{l(S+1)}_{++}$, with $x_i(s) = (x_i^1(s), ..., x_i^l(s)) \in \mathbb{R}^l_{++}$, and $\sum_{i=1}^m x_i = x$. The consumption space for each consumer $i \in \{1, ..., m\}$ is $X_i = \mathbb{R}^{l(S+1)}_{++}$, the strictly positive orthant. The associated price system is a collection of vectors represented by $p = (p(0), p(s), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$, with $p(s) = (p^1(s), ..., p^l(s)) \in \mathbb{R}^l_{++}$, the strictly positive orthant. Each consumer $i \in \{1, ..., m\}$ is endowed with initial resources $\omega_i \in \Omega$, where $\Omega = \mathbb{R}^{lT}_{++}$, and $\omega_i = (\omega_i(0), \omega_i(1))$ a collection of strictly positive vectors. Denote an initial resource vector at time period $t \in T = \{0, 1\}$, $\omega_i(t) = (\omega_i^1(t), ..., \omega_i^l(t)) \in \mathbb{R}^l_{++}$, and the sum of total initial resources, $\sum_{i=1}^m \omega_i = \omega$.

There is no aggregate risk in this economy. All risk in the economy is born by the producers. Hence, initial endowments do not depend on the state of nature. In total, there are n financial assets traded in period $t = 0$. Denote the quantity vector of stocks purchased by consumer $i \in \{1, ..., m\}$, $z_i = (z_i(1), ..., z_i(n)) \in \mathbb{R}^n_+$, a collection of quantities of stocks purchased from producers $j \in \{1, ..., n\}$, and denote $\sum_{i=1}^m z_i = z$, with associated stock price system $q = (q(1), ..., q(n)) \in \mathbb{R}^n_{++}$. Denote producer j's period $t = 0$ vector of capital purchase $y^j(0) \in \mathbb{R}^l_-$, and denote his period $t = 1$ state dependent net activity vector $y_j(s) = (y_j^1(s), ..., y_j^l(s)) \in \mathbb{R}^l$. Let $y_j(t = 1) = (y_j(s), ..., y_j(S)) \in \mathbb{R}^{lS}$ denote the collection of state dependent period t=1 net activity vectors. A period $t = 1$ input of production for every $s \in \{1, ..., S\}$ is by convention denoted $y_j^k(s) < 0$, and a production output in state $s \in \{1, ..., S\}$ satisfies $y_j^k(s) \geq 0$. For notational convenience, we treat quantity vectors as column vectors, and price vectors as row vectors, hence, we drop the notation for transposing vectors, whenever possible.

3.2.1 The model of the firm

Each firm $j \in \{1, ..., n\}$ issues stocks z_j at stock price q_j in period one in order to build up production capacity. A firm's total cash acquired via stock market determines the upper bound of the total value of production capacity it can install in the same period. Denote this liquidity constraint $q_j z_j = M_j$, where $M_j \in \mathbb{R}_+$ is a non-negative real number and $z_j \in \mathbb{R}_+$ a feasible financial policy of the firm $j \in \{1, ..., n\}$. M_j constraints the quantity of capital $y(0) \in \mathbb{R}_-^l$ a producer j can purchase at spot price system $p(0) \in \mathbb{R}_{++}^l$. The quantity of capital $y_j(0)$ purchased in period $t = 0$ determines a correspondence $\phi_j|_Z$. This correspondence defines the technology of the firm at feasible financial policy. For notational convenience let $y_j(0) := Z_j$[1] for every $j \in \{1, ..., n\}$. Hence, the production function of the firm, available to it in period $t = 1$ is not independent of the capital choices a firm takes in period $t = 0$. Let the production set available to each producer $j \in \{1, ..., n\}$ in period $t = 1$ be described by this technology, $\phi_j|_Z : \mathbb{R}_-^a \to \mathbb{R}_+^b$, a correspondence defined on the set of period $t = 1$ inputs, and denote it $Y_j|_z \subset \mathbb{R}^l$. Let S denote the set of all exogenously given states of nature. Then for each producer $j \in \{1, ..., n\}$ let the $t = 1$ one period production set be defined by a map $\Phi_j|_Z$ with domain $\mathbb{R}_-^a \times \mathbb{R}_{++}$ and range $\mathbb{R}_-^b$, and denote it $Y_j|_z(s) \subset \mathbb{R}^{lS}$, where $a + b = l$. In reality this correspondence is likely to map into $k \neq b$, as there is no reason to expect the same number of consumption goods l in each period. This restriction is purely for mathematical convenience, and changing dimension will not alter the analysis of this paper.

[1] I use this notational definition in order to economize on notation.

Denote the transformation map Φ for all producers $j \in \{1, ..., n\}$ and states of nature $s \in \{1, ..., S\}$, $\Phi_j|_Z : \mathbb{R}^a_- \times \mathbb{R}^S_{++} \to \mathbb{R}^b_+$.

Following paragraphs introduce and discuss the main assumptions underlying the model of the firm. These include: (i) an idiosyncratic risk assumption postulating that all risk present in the economy is born by the producers, (ii) a set of assumptions characterizing the endogenized production set available to each producer $j \in \{1, ..., n\}$, and (iii) for mathematical convenience an assumption on what the goal of the firm should be in order to derive a closed form equilibrium definition. This assumption is relaxed in subsequent chapters, where the sequential structure of the firm is studied in more detail.

We assume technological uncertainty which introduces idiosyncratic risk by stating that each producer $j \in \{1, ..., n\}$ has a production function $\Phi_j|_Z$ (at fixed $t = 0$ capital $y_j(0)$ defined on the set of $t = 1$ factors of production $\mathbb{R}^m_-$ and a set of random variables $s \in S$, each reflecting an exogenous realization in the set of finite states of nature $S = \mathbb{R}^S_{++}$. The production function itself is determined by the ability of the producer $j \in \{1, ..., n\}$ of accumulating capital in period $t = 1$. Thus, the level of production output $y^+_j(s)$ for a given technology $\Phi_j|_Z$, if state of nature $s \in \{1, ..., S\}$ occurs, is a function of the inputs y^-_j and state $s \in \{1, ..., S\}$ at fixed capital $Z_j = y_j(0)$. The boundary of the technology map is determined by the upper bound of the producer's total production capacity acquired by issuing stocks. For example $y^+_j(s) = \Phi_j|_Z(y^-, s)$. The main properties of this function are: non-decreasing, quasi-convexity, and differentiability, and for $y^-_j = 0$, $\Phi_j|_Z(0, s) = 0$. These are formally introduced in set of assumptions 3.1 (F) below.

It is now possible to expand Debreu's [12] assumptions on exogenously defined production sets to an economic setting, where period one production sets available to producers are endogenously determined by the firms' choice of production capacity in period $t = 0$. Denote a producer's financial policy $\tilde{Z}$[2],

[2]Note that $\tilde{Z}$ and Z represent a financial policy in the first case and a fixed level of capital in

here restricted to be a feasible quantity of stocks issued, determining the firms production capacity, $\Phi_j|_Z$, and let $\Phi_j|_Z \Rightarrow Y_j|_z$, where the capacity constraint $\sum_{i=1}^{m} \bar{z}_{ij} \leq z_j \leq 0$ is binding, and where $Y_j|_z(s)$ denotes the production set available to the firm $j \in \{1, ..., n\}$ in each state of the world $s \in \{1, ..., S\}$. Each producer $j \in \{1, ..., n\}$ is formally characterized by set of assumptions 3.1 (F). This set of assumptions determines the characterization of the short run production activities available to a firm.

Assumption 3.1 (F) *(i) For each $j \in \{1, ..., n\}$, $Y_j|_z \in \mathbb{R}^{lS}$ is closed, convex, and $(\omega + \sum_{j=1}^{n} Y_j|_z) \cap \mathbb{R}_+^{lS}$ compact for all $\omega_i \in \mathbb{R}_{++}^{lT}$. $0 \in Y_j|_z \Leftarrow Y_j|_z \supset \mathbb{R}_-^{lS}$. $Y_j|_z \cap \mathbb{R}_+^{lS} = \{0\}$. (ii) For each $j \in \{1, ..., n\}$ denote $\partial Y_j|_z \subset \mathbb{R}_+^{nS}$ a C^∞ manifold. (iii) For each $j \in \{1, ..., n\}$, transformation maps $\Phi|_Z(j)$ are non-linear representing decreasing returns to scale technology. (iv) For each $j \in \{1, ..., n\}$, endogenized production capacity is bounded above and is characterized by $z_j \in [\sum_{i=1}^{m} z_i(j), 0]$ in the closed interval of feasible financial policies.*

(i) The closedness assumption is introduced for its mathematical convenience. Convexity of the production set implies that no increasing returns to scale technologies are considered, describing the competitive economic environment. For example, it permits constant return to scale or decreasing return to scale technologies further specified by the assumption (iii) on the transformation maps $\Phi|_Z(j)$ for all $j \in \{1, ..., n\}$. It is assumed that the total production possibilities of the whole economy are bounded above. Finally a free disposal assumption is introduced, implying the possibility of inaction of the firm. A firm has always the choice of producing no outputs with zero inputs. It is assumed in (ii) that the efficient boundary of the production set is smooth. Here, C^∞ implies differentiability at any order required. The order depending on all transversality arguments employed. Assumption (iv) characterizes the bounds on the level of production capacity $y_j(0) \Rightarrow \Phi|_Z(j)$ accumulated at feasible financial policy.

the second case.

We now introduce the long run production sets. This requires to assume separability of the production sets over the two periods. Consider the the classical production set $Y_j \subset \mathbb{R}^{l(S+1)}$ as introduced in [12]. Assume that it satisfies separability:

Assumption 3.2 (FL) $(y(0),(1-\lambda)(1))y(1)+(\lambda(1)y'(1),...,(1-\lambda)(S)y(S)+\lambda(S)y'(S)) \in Y_j$ *for all* $y=(y(0),y(1),...,y(S))$, $y'=(y(0),y'(1),...,y'(S)) \in Y_j$ *and all* $\lambda(1),...,\lambda(S) \in [0,1]$.

Assumption 3.2 introduces separability of production across states of nature $s \in S$ in period $t=1$ given capital $y_j(0)$ in period $t=0$ for every $j \in \{1,...,n\}$. Then $Y_j|_z \in \mathbb{R}^{lS}$ introduced above is defined by: $Y_j|_z \in \mathbb{R}^{lS} = \{y(1),...,y(S) \in \mathbb{R}^{lS} | (y(0),y(1),...,y(S)) \in Y\}$. For the case that capital only is considered in period $t=0$ have a production set $Y_j \subset \mathbb{R}_- \times \mathbb{R}^{lS}$ rather than $Y_j \subset \mathbb{R}^{l(S+1)}$.

Proposition 3.1 *There exists* $Y_j(y(0),1),...,(y(0),S) \subset \mathbb{R}^l$ *such that* $Y_j|_z = Y_j(y(0),1)\times,...,\times Y_j(y(0),S)$.

An example of a production set Y_j requires to define a function ϕ_j such that $Y_j|_z$ results. Let there be two goods (capital) only, $l=2$. Then a required function can be expressed as $y^2 \leq F(y^1(0),y^2(0),y^1(1,s),s)$, where the first and second goods at date $t=0$ are the inputs of production (capital), the first good at $t=1$ is an input, the second good at $t=1$ is an output and the state $s \in S$ influences production to build the production set. In this example, clearly, the distribution of production varies across states of nature at date $t=1$.

Assumption 3.3 (P) *The objective of each producer* $j \in \{1,...,n\}$ *is to maximize its long run profits.*

Assumption 3.3 (P) of long run profits maximization is a convenient assumption which needs further explanation. The full elaboration of this assumption is subject of subsequent chapters. Here, it suffices to justify it for its mathematical and economical practicality, and its useful consequences. One idea underlying this assumption is to simplify the objective function of the firm such that it is mathematically more tractable for establishing existence of equilibrium. This assumption allows the introduction of a reduced form equilibrium for which existence of equilibria will be shown. The other idea is related to the simplification of the problem of linking the real with the financial sector. The assumption enables to think of the optimization structure of the firm as a sequential optimization problem, where in period two, given its production sets, each producer chooses a net activity plan in it such that it is profit maximizing. The assumption is convenient as we do not need to model the process of the firm of building up production capacity in period one by implicitly assuming that production capacity is given. This assumption leads us into a similar environment to the Arrow-Debreu model with private ownership firms, where the model of the firm is strongly simplified, and the objective of the firm of profit maximization well defined. Similar to the Arrow-Debreu model, where production capacity is exogenously determined, in the reduced form model firms take their productions sets as given.

Assumption 3.3 (P) is a reinterpretation of the profit maximization criterion in classical GEI models of production, where firms maximize profits by choosing net activity vectors over two period production sets. Here, it facilitates the introduction of a constraint sequential optimization structure, similar to the sequential optimization structure on the consumer side in classical GEI models. This sequential optimization structure has the convenient property of facilitating the introduction of one period production sets, similar to those of the standard Arrow-Debreu model, with the significant difference that a period two production set available to a firm is not independent of its capacity determined in period one. An immediate implication of assumption 3.3(P) is then, at variance with current models, that period one capital performs the role of determining the firms production sets, and therefore the firm's size. The level of capital a firm can buy depends on the firm's ability of acquiring financial liquidity on the stock market by issuing stocks. This sequential structure and the role financial assets play are non-trivial elements of the structure of the firm, as they eliminate the present value problem of the firm present in classical GEI models of production, where producers choose real quantities in period one, and share holders generally evaluate future income streams differently since gradient vectors generally point in different directions when incompleteness of financial markets is satisfied.

The algebraic form of the long run profit maximization assumption (3.3) is stated in equation (3.1).

$$(\overline{y}_j) \in argmax \left\{ \overline{p} \Box y_j \middle| \begin{array}{c} y_j(s) \in Y_j|_z(s) \\ \overline{qz}_j = \overline{p}(0) \centerdot \overline{y}_j(0) \end{array} \text{ for all } s \in S \right\}, \tag{3.1}$$

where $\Box$ is the box product, a state by state mathematical operation which is context dependent. Here, $\Box$ denotes the "s by s" inner product. This equation says that in period two, for a given state of nature $s \in \{1, ..., S\}$, and for given production set $Y_j|_z(s)$, each producer $j \in \{1, ..., n\}$ chooses a profit maximizing net activity plan $y_j(s)$ at given spot price system $p(s)$. This is essentially equivalent to the Arrow-Debreu model, or the expansion of it to a one period incomplete markets model as proposed by Diamond [16]. Let production capacity be characterized by the set of all feasible financial policies z_j in the interval $(\sum_{i=1}^m z_i(j), 0)$, and let $\sum_i^m \xi_i(j) = \sum_i^m z_i(j) - z_j \; j \in \{1, ..., n\}$. We then call this objective function a reduced form objective function, since financial policies and their role are not explicitly considered.

Notice here that all that information is available to the agents in period $t = 0$, after trade at the stock market has taken place. Therefore, in period $t = 0$, the producer's problem is to acquire capital by issuing stocks in order to build up production capacity. Once the level of capital is determined at the end of period $t = 0$, and all information available to the agents, there is no reason not to expect that the objective of long run profit maximization is not well defined.

Denote a long run equilibrium output vector associated with the production set boundary $\overline{y}_j \in \partial Y_{j,eff}|_z$. Denote the $t = 1$ maps implied by the long run profit maximization equation (3.1),

$$\pi_j : \mathbb{R}^l_{++} \times \mathbb{R}^l \to \mathbb{R}, \tag{3.2}$$

for all $s \in \{1, ..., S\}$, and $j \in \{1, ..., n\}$. For each state $s \in \{1, ..., S\}$ and all producers $j \in \{1, ..., n\}$ define the $(S \times n)$ total long run payoff matrix, a collection of n vectors denoted

$$\Pi(p_1, \Phi|_Z) = \begin{bmatrix} p(s) \cdot y_1(s) & ... & p(s) \cdot y_n(s) \\ ... & ... & ... \\ p(s) \cdot y_1(S) & ... & p(s) \cdot y_n(S) \end{bmatrix}. \tag{3.3}$$

$\Pi(p_1, \Phi|_Z)$ denotes the price and capacity dependent total payoff structure of the economy for equilibrium financial policies $\tilde{Z}$.

3.2.2 The consumers

Consumers play the same role in this production model as in the classical GEI model with production. They invest into firms because they want to transfer wealth between future uncertain states of nature, and to smooth out consumption across states of nature. Each consumer $i \in \{1, ..., m\}$ purchases stocks z_i

at stock price q in period one in return for a dividend stream in the next period. The consumer's optimization problem is to maximize utility subject to a sequence of $(S+1)$ budget constraints. Each consumer $i \in \{1, ..., m\}$ is characterized by set of assumptions C below. These are the standard assumptions for smooth economies introduced in Debreu [13].

Assumption 3.4 (C) *(i)* $u_i : \mathbb{R}^{l(S+1)}_{++} \to \mathbb{R}$ *is continuous on* $\mathbb{R}^{l(S+1)}_{++}$*, and* C^∞ *on* $\mathbb{R}^{l(S+1)}_{++}$*. (ii)* $U_i(x_i) = \left\{x'_i \in \mathbb{R}^{l(S+1)}_{++} : u_i(x'_i) \geq u_i(x_i)\right\} \subset \mathbb{R}^{l(S+1)}_{++}$*, for all* $x_i \in \mathbb{R}^{l(S+1)}_{++}$*. (iii) For each* $x_i \in \mathbb{R}^{l(S+1)}_{++}$*,* $Du_i(x_i) \in \mathbb{R}^{l(S+1)}_{++}$ *for all* $s \in S$*. (iv) For each* $x_i \in \mathbb{R}^{l(S+1)}_{++}$*,* $h^T D^2 u_i(x_i) h < 0,$ *for all nonzero hyperplane h such that* $(Du_i(x_i))^T h = 0.$

Assumptions (3.4) are introduced in order to obtain differentiable demand functions, and consequently differentiable equilibrium equations. Smoothness of the function u has the convenient property that we do not have to keep track of the order of differentiability of a C^k function for finite order of differentiation $k = 1, ..., K$. (ii) Smoothness on the non-negative orthant is introduced in order to avoid boundary problems when considering first or second order conditions, for example. The characterization of each consumer $i \in \{1, ..., m\}$ also introduces a (iii) strong monotonicity assumption for differentiable functions, and a family of local conditions implying strictly quasi-concavity of the utility function. This condition implies that locally, for each $x_i \in \mathbb{R}^{l(S+1)}_{++}$ the gradient vector $Du_i(x_i)$ changes direction for any small change dx on the indifference surface, so that the indifference surface is not locally flat.

Denote consumer i's sequence of $(S+1)$ budget constraints

$$B_{z_i} = \left\{ (x_i) \in \mathbb{R}^{l(S+1)}_{++} \middle| \begin{array}{c} p(0) \centerdot (x_i(0) - \omega_i(0)) = -qz_i \\ p(s) \cdot (x_i(s) - \omega_i(1)) = \Pi(p_1, \Phi|_Z)\theta_i(z_i) \end{array} \right\}, \tag{3.4}$$

where θ_i in (3.4), and (3.9) denote the endogenously determined ownership structure of consumer $i \in \{1, ..., m\}$, a $(n \times 1)$ vector defined by the mappings

$$\theta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+ \text{ for all } j \in \{1, ..., n\}, \tag{3.5}$$

where $z_i(j) \in R_+$ is a non-negative real number for every $j \in \{1, ..., n\}$. $\theta_{ij} = z_i(j)[\sum_i z_i(j)]^{-1}$ is the proportion of total payoff of financial asset $j \in \{1, ..., n\}$ hold by consumer $i \in \{1, ..., m\}$ after trade at the stock market took place in period one. For the moment, I assume that no discontinuities in θ_{ij} arise, and that no shareholder has market power. Hence, each consumer considers prices, production choices and also other's consumer choices to be fixed. This assumption allows to consider linear ownership structures. A convenient property of this assumption is that there arise nor problems in the definition of no arbitrage. Relaxing this assumption will enable to consider strategic interactions on the stock markets, a topic receiving seemingly less attention in the literature. In compressed notation, we write

$$B_{z_i} = \Big\{ (x_i) \in \mathbb{R}^{l(S+1)}_{++} \big| \; p(s) \cdot (x_i(s) - \omega_i(t)) \in \widehat{\Pi}[z_i|\theta_i(z_i)] \Big\}, \tag{3.6}$$

for all $s \in \{1, ..., S\}$, $t \in \{0, 1\}$, and consumers $i \in \{1, ..., m\}$, where

$$\widehat{\Pi}(p_1, \Phi|_Z) = \begin{bmatrix} -q_1 & \dots & -q_n \\ p(s) \centerdot y_1(s) & \dots & p(s) \centerdot y_n(s) \\ \dots & \dots & \dots \\ p(s) \centerdot y_1(S) & \dots & p(s) \centerdot y_n(S) \end{bmatrix} \tag{3.7}$$

represents the full payoff matrix of the economy of order $((S+1) \times n)$.

The sequential optimization problem of the consumer $i \in \{1, ..., m\}$ is to invest into firms in period one in order to smooth out future uncertain consumption and to optimize consumption of goods in every $(S + 1)$ spot market. For a given price system $p = (p(0), p(1), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$ of consumption goods and price system $q \in \mathbb{R}^n_{++}$ of financial assets (stocks), a consumer chooses bundles of consumption goods and quantities of stocks $(x, z)_i \in X_i \times \mathbb{R}^n_+$ such that $u_i(x_i; z_i)$ is maximized subject to the sequence of $(S+1)$ constraints in B_{z_i}. Algebraically, each $i \in \{1, ..., m\}$

$$(\overline{x}_i; \overline{z}_i) \in argmax\Big\{u_i(x_i; z_i) : z_i \in \mathbb{R}^n_+, x_i \in B_{z_i}\Big\}. \tag{3.8}$$

This optimization problem can be reformulated in a reduced form problem, where the reduced form budget set becomes

$$B_{\xi_i} = \left\{ (x_i) \in \mathbb{R}^{l(S+1)}_{++} \Big| \begin{array}{c} p(0) \cdot (x_i(0) - \omega_i(0)) = -q\xi_i \\ p(s) \cdot (x_i(s) - \omega_i(1)) = \Pi(p_1, \Phi|_Z)\theta_i(\xi_i) \end{array} \right\}, \tag{3.9}$$

where $\sum_i^m \xi_{ij} = \sum_i^m \overline{z}_i(j) - \overline{z}_j$ is the equation allowing to move between the two budget sets introduced, for all $j \in \{1, ..., n\}$, each consumer $i \in \{1, ..., m\}$ solves following problem:

$$(\overline{x}_i; \overline{\xi}_i) \in argmax\Big\{u_i(x_i; \xi_i) : \xi_i \in \mathbb{R}^n, x_i \in B_{\xi_i}\Big\}. \tag{3.10}$$

Recall that this is the reduced form maximization problem of the consumer. Here financial policies are exogenously modeled, and therefore, given by the firm. This simplifies the nature of the problem of interest. We relax this constraint in chapter 5.

3.2.3 Equilibrium equations

We introduce following prize normalization $\mathcal{S} = \{p \in R^{l(S+1)}_{++} : \|p\| = \Delta\}$ such that the Euclidean norm vector of the spot price system p is a strictly positive real number $\mathbb{R}_{++}$. A competitive equilibrium of the production economy defined by the initial resource vector $\omega \in \Omega$ is a price pair $(p, q) \in \mathcal{S} \times \mathbb{R}^n_{++}$ if equality between demand and supply of physical goods and financial assets is satisfied in all states of nature, $s = 0, 1, ..., S$. Its associated competitive equilibrium allocation is a collection of vectors $(x, y, \xi) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n} \times \mathbb{R}^{nm}$ of consumption, production and financial quantities. Market clearance conditions are determined by the aggregate excess demands for physical goods and for financial assets as expressed by the equilibrium equations:

$$\begin{array}{ll} \text{(i)} & \sum_{i=1}^{m}(\overline{x}_i(0) - \omega_i(0)) = \sum_{j=1}^{n} \overline{y}_j(0) \\ \text{(ii)} & \sum_{i=1}^{m}(\overline{x}_i - \omega_i(1)) = \sum_{s=1}^{S} \sum_{j=1}^{n} \overline{y}_j(s) \\ \text{(iii)} & \sum_{i=1}^{m}(\overline{z}_{ij}) = 0, \sum_{i=1}^{m} \theta(\overline{z}_i)_j = 1 \text{ for all } j \in \{1, ..., n\} \end{array} \tag{3.11}$$

In case of the reduced form equilibrium, have $\sum_{i=1}^{m} \overline{\xi}_i(j) = 0$, for all $j \in \{1, ..., n\}$, and $\sum_{i=1}^{m} \theta(\overline{\xi}_{ij}) = 1$ for all $j \in \{1, ..., n\}$ satisfied.

3.2.4 Definition of a stock market equilibrium

In a financial markets (stock market) general equilibrium with production, each producer $j \in \{1, ..., n\}$ optimizes a sequential optimization problem defined by equation (3.1) under assumption (3.3) of long run profit maximization. Every consumer $i \in \{1, ..., m\}$ optimizes the standard sequential optimization problem of the classical GEI model as in equation (3.10). Finally, all equilibrium conditions (3.11) are satisfied. It remains to formally introduce (in its reduced form) the definition of a long run profit maximization stock market equilibrium.

Definition 3.1 (FE) *A reduced form stock market equilibrium* $(\overline{p}, \overline{q})$ *with associated equilibrium allocations* $(\overline{x}, \overline{\xi})$, $(\overline{y})$ *for generic initial resources* $\omega \in \Omega$*, and each producer* $j \in \{1, ..., n\}$ *satisfying assumption 3.3* (P) *of maximizing long run profits satisfies:*

(i) $(\overline{x}_i; \overline{\xi}_i) \in argmax\left\{u_i(x_i; \xi_i) : \xi_i \in \mathbb{R}^n, x_i \in B_{\xi_i}\right\} \quad \forall i \in \{1, ..., m\}$

(ii) $(\overline{y}_j) \in argmax\left\{\overline{p}\Box y_j \middle| \begin{array}{c} y_j(s) \in Y_j|_z(s) \\ \overline{q}\overline{z}_j = \overline{p}(0) \centerdot \overline{y}_j(0) \end{array} \text{for all } s \in S\right\} \quad \forall\, j \in \{1, ..., n\}$

(iii) $\sum_{i=1}^m (\overline{x}_i - \omega_i) = \sum_j^n \overline{y}_j$

$\sum_i^m \xi_{ij} = \sum_i^m \overline{z}_{ij} - \overline{z}_j, \sum_i^m \theta(\overline{\xi}_{ij}) = 1, \forall j \in \{1, ..., n\}$

3.3 Existence of Equilibrium

This section of chapter 3 establishes the main existence result for the reduced form stock market general equilibrium model with long run profit maximizing firms introduced by definition (3.1). Section (3.4) introduces an extension of this result to piece-wise linear production manifolds. The novelty of this equilibrium concept is that production sets available to producers in period two are endogenized and not independent of the level of capital acquired via stock market in period one. The sequential objective function of the firm links the real with the financial sector of the economy. This information is implicitly contained in assumption 3.3(P).

We establish generic existence for smooth endogenized production manifolds for the equilibrium concept formally introduced in section (3.1). The relation between a reduced form and an extensive form equilibrium will formally be introduced in chapter 5. For the moment, it is sufficient to know that the later differs only in the sense that financial polices are explicitly modeled. The strategy of the proof is to show that a technical more tractable pseudo equilibrium with production exists, and that every pseudo equilibrium is also a stock market equilibrium with sequential structure of the firm. The precise relations between pseudo and (FE) equilibria are introduced in propositions (3.2), and (3.3). Subsection (3.3.1) establishes a class of smooth endogenized asset structures for which the existence theorem introduced in the same section guarantees existence.

Existence of pseudo equilibria for exchange economies with exogenous financial markets were established by Duffie, Shafer, Geanokopolos, Hirsh, Husseini, and others [[11],[34],[50],[37],[8]]. Geanokopolos et. al. [33] showed that pseudo equilibria exist for an economy with production for the case of exogenous financial markets and where the problem of the firm is to maximize the utility of the average share holder. We improve on this proof by showing that in a sequential incomplete markets model of the firm with decentralized decisions and objective function of the firm independent of the utility of share holders, pseudo equilibria with endogenously determined production sets exist.

Definition 3.2 *$if \nexists z \in \mathbb{R}^n_{++}$ $s.t.$ $\widehat{\Pi}(p_1, \Phi|_Z)[\sum_{i=1}^m \theta(z_i)_{s=1}^S] \geqslant 0,$ then $q \in \mathbb{R}^n_{++}$ is a no-arbitrage stock price relative to p_1.*

Lemma 3.1 $\exists \beta \in \mathbb{R}^S_{++}$ $s.t.$ $q = \sum_{s=1}^S \beta \Box \Pi(p_1, \Phi|_Z)[\sum_{i=1}^m \theta(z_i)_{s=1}^S]$.

Lemma (3.3) allows to rescale equilibrium prices without affecting equilibrium allocations, let $P_1 = \beta \Box p_1$. Note that this lemma is derived under the assumptions of linearity of the ownership structure and no discontinuities in it. The next step in deriving a pseudo equilibrium is to derive a normalized no arbitrage equilibrium definition [9]. Let $\beta \in \mathbb{R}^S_{++}$ be $(\frac{\lambda(s)}{\lambda})_{i=1}$, the gradient vector from the optimization problem of agent 1, called the Arrow-Debreu agent. The Walrasian budget set for the Arrow-Debreu agent is a sequence of constraints denoted

$$B_1 = \left\{ x_1 \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{c} P(0) \cdot (x_1(0) - \widetilde{\omega}_1(0)) = 0 \\ P(s) \cdot (x_1(s) - \omega_1(1)) = \sum_{j=1}^{n} \overline{\theta}_{1j} P(s) \cdot y_j(s) \end{array} \right\}, \tag{3.12}$$

for all $s \in \{1, ..., S\}$. Let $\widetilde{\omega}_1(0) = \sum_{j=1}^{n} \overline{\theta}_{1j} P(0) \cdot y_j(0)$.

For all consumers $i \geq 2$, the no arbitrage budget set consisting of a sequence of $(S+1)$ constraints is denoted

$$B_{i \geq 2} = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{c} P(0) \cdot (x_i - \widetilde{\omega}_i(0)) = 0 \\ P(s) \cdot (x_i(s) - \omega_i(1)) \in \langle \Pi(P_1, \Phi|_Z) \rangle \end{array} \right\}, \tag{3.13}$$

where $\langle \Pi(P_1, \Phi|_Z) \rangle$ denotes the span of the income transfer space of period $t = 1$. Let $\widetilde{\omega}_i(0) = \sum_{j=1}^{n} \overline{\theta}_{ij} P(0) \cdot y_j(0)$. Replace $\langle \Pi(P_1, \Phi|_Z) \rangle$ with L in $G^n(\mathbb{R})^S$ [3]

Lemma 3.2 *$G^n(\mathbb{R})^S$ is the Grassmann manifold with smooth $(S - n)$ dimensional*

[3] See i.e. Dieudonnè [19] for properties of the Grassmann manifold. See Duffie and Shafer for an exposition of the Grassmann manifold in economics [11].

structure, and L an n-dimensional affine subspace of $G^n(\mathbb{R})^S$.

Denote the pseudo opportunity set $B_i(P, L; \omega_i)$, for each $i \in \{2, ..., m\}$,

$$B_i = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{c} P \cdot (x_i - \widetilde{\omega}_i) = 0 \\ P(s) \cdot (x_i(s) - \omega_i(s)) \subset L \end{array} \right\}. \tag{3.14}$$

Let $\mathcal{S}' = \{p \in R^{l(S+1)}_{++} : \|p\| = \Delta\}$ be the set of normalized prices, and let $\Delta \in \mathbb{R}_{++}$ be a fixed strictly positive real number. This convenient normalization singles out the first good at the spot $s = 0$ as the numeraire. We introduce following definitions for the long run payoff maps associated with sets $\mathcal{S}$ and $\mathcal{S}'$ of normalized prices. This definition introduces the relation between β dependent payoff structures for a financial markets- and a pseudo equilibrium definition. The full payoff structure Π satisfying this definition is relabeled Γ.

Definition 3.3 *(i) For any* $p_1 \in \mathcal{S}$*, such that* $\pi : \mathcal{S} \times \mathbb{R}^l \to \mathcal{A}$ *, let* $\Gamma(P_1, \Phi|_Z) = \beta\square[(proj_\Delta(\frac{1}{\beta})^T \square P_1)\square y]$*, where T denotes the transpose,* $proj_\Delta = \Delta(\frac{z}{\|z\|})$*,* $\frac{1}{\beta} = (\frac{1}{\beta(s)}, ..., \frac{1}{\beta(S)}) \in \mathbb{R}^S_{++}$*, and* $\beta = (\beta(1), ..., \beta(S)) \in \mathbb{R}^S_{++}$*. (ii) For any* $p_1 \in \mathcal{S}'$*, such that* $\pi : \mathcal{S}' \times \mathbb{R}^l \to \mathcal{A}$*, let* $\Gamma(P_1, \Phi|_Z) = \beta\square[((\frac{1}{\beta})^T \square P_1) \cdot y]$*, where* $\mathcal{A}$ *is a set of* $(S \times n)$ *matrices A of order* $(S \times n)$*.*

Using the no arbitrage result of previous section (lemma (3.3)) and above definition (3.3) leads to the analytically more tractable concept of a pseudo stock market equilibrium for which we will establish existence. The main benefit of a pseudo equilibrium is that it allows to apply transversality arguments. This follows from the two consequences of the normalized gradient vector of the Arrow-Debreu agent. It gives his (i) standard GE demand functions satisfying boundary conditions, and (ii) it guarantees independency of aggregate demand functions, such that Walras' law applies [39].

Definition 3.4 *A pseudo stock market equilibrium* $(\bar{P}, \bar{L}) \in \mathcal{S}' \times G^n\mathbb{R}^S$ *with associates equilibrium allocations* $(\bar{x}, \bar{y}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+}$ *for generic initial resources* $\omega \in \Omega$*, satisfies:*

(i) $(\bar{x}_1) \arg\, max \left\{u_1(x_1)\ s.t.\ x_1 \in B_1(\bar{P}, \omega_1)\right\} \qquad i = 1$

(ii) $(\bar{x}_i) \arg\, max \left\{u_i(x_i)\ s.t.\ x_i \in B_i(\bar{P}, \bar{L}, \omega_i)\right\} \qquad \forall\, i \geq 2$

(iii) $\left\langle \Gamma(\bar{P}_1, \bar{\phi})\right\rangle \subset \bar{L}$*, proper if* $\left\langle \Gamma(\bar{P}_1, \bar{\phi})\right\rangle = \bar{L}$

(iv) $(\bar{y})_j \arg\max \left\{ \bar{p}(s) \cdot y_j(s) \middle| \ y_j(s) \in Y_j(s) \quad \forall s \in S \ \right\} \quad \forall\, j \in \{1, ..., n\}$

(v) $\bar{x}_1 + \sum_{i=2}^m \bar{x}_i = \sum_{i=1}^m \omega_i + \sum_{j=1}^n \bar{y}_j$

Lemma 3.3 *Under assumptions 3.4(C), demand mappings* $f_1(P, \omega_1)$ *and* $f_i(P, L, \omega_i)$ *for* $1 = 2, ..., m$*, from argmax (i) and (ii) are* C^∞*. Under assumptions 3.1(F), supply mappings* $g_j(P)$ *for* $j = 1, ..., n$*, from argmax (iv) are* C^∞*.*

A proof of this known result is omitted[4]. Smoothness of demand and supply functions follows from the setup of the model for smooth economies. Following results show the relation between pseudo and (FE) equilibria. They imply that, in order to prove existence of equilibrium it is sufficient to establish existence in the much easier case of a pseudo equilibrium, since every pseudo equilibrium is also a (FE) equilibrium. The advantage of a pseudo equilibrium is that the financial assets cancel out of the equations. This simplifies the existence proof. The propositions state that every pseudo equilibrium is a (FE) equilibrium, but the reverse is not always true. Therefore, establishing existence of equilibrium for the pseudo equilibrium case is sufficient in order to guarantee existence of

[4] A proof of this result for the case of an exchange economy can be found in Duffie and Shafer (1985), [11]. The expansion to production is obvious and follows from the set up

equilibrium of a reduced form (FE) equilibrium.

Proposition 3.2 *For every full rank stock market equilibrium with production* (FE)*,* $(\overline{p}, \overline{q})$*, with associated equilibrium allocations* $(\overline{x}, \overline{\xi}, \overline{y})$*, there exists a* $\beta \in \mathbb{R}^S_{++}$ *and a n-dimensional subspace* $L \in G^n(\mathbb{R}^S)$ *such that* $(\overline{P}, \overline{L})$ *is a pseudo equilibrium with associated equilibrium allocations* $(\overline{x}, \overline{y})$*.*

Proposition 3.3 *If* $(\overline{P}, \overline{L})$ *is a pseudo stock market equilibrium with associated equilibrium allocations* $(\overline{x}, \overline{y})$ *for every* $\beta \in \mathbb{R}^S_{++}$*, there exists a stock price system* $\overline{q} \in \mathbb{R}^n_{++}$ *and investment portfolios* $z = (z(1), ..., z(n)) \in \mathbb{R}^n_+$ *such that* $(\overline{p}, \overline{q})$ *with associated allocations* $(\overline{x}, \overline{\xi}, \overline{y})$ *is a* $(\overline{x}, \overline{y})$ *allocational equivalent stock market equilibrium (FE) with production.*

3.3.1 Regular endogenized payoff structure

Long run financial payoffs depend on the technology of the firm, which in turn depends on the production capacity installed via stock market, and on a set of regular prices. Hart illustrated by means of carefully chosen examples that equilibrium may not exist for some structures of the payoff matrix [31]. He showed that when price vectors are collinear the rank of the payoff matrix changes and that consequently equilibrium may fail to exist due to discontinuities of demand functions. We will exhibit a class of regular endogenous asset structures for smooth production economies for which equilibria will always exist. For this class of well behaved asset structures, generic existence of equilibrium is established by the application of Thom's parametric transversality theorem [32].

The class of endogenized asset structures considered, enables for interesting economic interpretations of economic phenomena beyond those of the pure exchange model, and allows enhancing the theory of the firm in a general equilibrium framework with endogenous incomplete markets, where production is not independent of the stock market.

Definition 3.5 *Define the rank dependent long run payoff maps* $\pi^{\rho} : \mathbb{R}^{l}_{++} \times \mathbb{R}^{l} \rightarrow \mathcal{A}^{\rho}$, *for* $0 \leq \rho \leq n$. *The set of reduced rank matrices* A^{ρ} *of order* $(S \times n)$ *with* $rank(A^{\rho}) = (n - \rho)$ *is denoted* $\mathcal{A}^{\rho}$, *and is of order* $(S \times n)$.

Proposition 3.4 *(i) for* $(1 \leq \rho < n)$, A^{ρ} *is a submanifold of* A *of codimension* $(S - n + \rho)\rho$. *(ii) for* $\rho = n$ *the set* $\mathcal{A}^{\rho}$ *is empty,* $\mathcal{A}^{\rho} = \{\varnothing\}$, *and (iii) for* $\rho = 0$, *the set of reduced rank matrices* $\mathcal{A}^{\rho}$ *is equivalent to the set of full rank matrices* $\mathcal{A}$, $\mathcal{A}^{\rho} = \mathcal{A}$.

Proposition (3.4) states rank properties of the income transfer space, the codomain of rank dependent payoff maps. For example, for any integers $(1 \leq \rho < n)$, A^{ρ}, the incomplete income transfer space is rank reduced. This properties are important when applying transversality arguments in the existence proof.

Theorem(3.1) below exhibits a regular asset structure $\mathcal{R}$ for the smooth production economy and shows that, for a map π to the ambient space $\mathcal{A}$ which is transverse to a submanifold $\mathcal{A}^{\rho}$ along all values of the domain of π, $\mathcal{R}$ is big in a topological sense. This follows from the transversality theorem for maps and submanifolds. Since $\mathcal{R}$ is open and dense, it follows that its complement, the set of critical values is closed and of measure zero. Denote the set satisfying $\Gamma \pitchfork \mathcal{A}^{\rho}$, $\mathcal{R}$, and its complement $\overline{\mathcal{R}}$.

Theorem 3.1 *(i)* $\pi_j \pitchfork \mathcal{A}^{\rho}_j$ *for integers* $(1 \leq \rho < n)_j$ *for all* $j = \{1, ..., n\}$. *(ii)* $\Gamma_j \pitchfork \mathcal{A}^{\rho}$ *for any* $\beta \in \mathbb{R}^{S}_{++}$ *and integers* $(1 \leq \rho < n)$ *for all* $j = \{1, ..., n\}$. *(iii)* $\mathcal{R} = \Gamma_j \pitchfork \mathcal{A}_j$ *is generic, since it is dense and open for all* $j = \{1, ..., n\}$.

The economic relevance of theorem (3.1) is that it exhibits a class of well defined smooth endogenized asset structures for production economies with production sets defined by set of assumptions 3.1 (F), for which for each $j \in$

$\{1, ..., n\}$ a sequential optimization structure of the firm applies. This result improves on Bottazzi [8] by generalizing the asset structure. It also improves on Duffie and Shafer [11], since the proposed asset structure is not more general but also independent of initial endowments and preferences.

For any gneneric production set structure satisfying theorem (3.1), equilibrium exists by the existence theorem below.

Definition 3.6 *Denote Ψ^ρ the vector bundle defined by (i) a basis $P^\rho = \{P \in \mathbb{R}^{l(S+1)}_{++} : rank(\Gamma(P_1, \Phi|_z)) = (n-1)\}$, and (ii) let the orthogonal income transfer space be denoted by $L^\perp \subset \langle \Gamma(P_1, \Phi|_z) \rangle^\perp$, then*

$$\Psi^\rho = \left\{ \begin{array}{c} (P, \langle \Gamma(P_1, \Phi|_z) \rangle^\perp, L^\perp) \in P^\rho \times G^{S-n+\rho}(\mathbb{R}^S) \times G^{S-n}(\mathbb{R}^S) \\ \textit{such that } L^\perp \subset \langle \Gamma(P_1, \Phi|_z) \rangle^\perp \end{array} \right\}.$$

We thus have defined a fiber bundle Ψ^ρ of codimension $l(S+1) - 1 - \rho^2$ containing the spot price system P and income transfer space $\langle \Gamma(.,.) \rangle$ consisting of a base vector P^ρ and fiber $G^{S-n}(\mathbb{R}^{S-n+\rho})$. We can now state the main result.

Theorem 3.2 *There exists a pseudo (FE) stock market equilibrium $(\overline{P}, \overline{L})$ with associated equilibrium allocations $(\overline{x}, \overline{y})$ for generic initial resources $\omega \in \Omega$. Moreover, by the relational propositions (3.2), and (3.3), a stock market equilibrium (FE) $(\overline{p}, \overline{q})$ with associated equilibrium allocations $(\overline{x}, \overline{\xi}, \overline{y})$ exists for generic initial resources $\omega \in \Omega$.*

Proof 3.1 (Theorem 3.2) *By proposition (3.4) and theorem(3.1), and using the definition of the vector bundle in (3.6), define a parameterized evaluation map Z^ρ on $\Psi^\rho \times \mathbb{R}^{l(S+1)m}_{++}$, where parameter space $\mathbb{R}^{l(S+1)m}_{++} = \Omega$ denotes the set of the economy's* *total initial endowments, such that Z^ρ maps into N, denoted*

$$Z^\rho : \Psi^\rho \times \mathbb{R}_{++}^{l(S+1)m} \to N.$$

For the normalized (Arrow-Debreu) agent have

$$Z_1^\rho : \Psi^\rho \times \mathbb{R}_{++}^{l(S+1)m} \to N. \tag{3.15}$$

This evaluation map is a submersion, since $D_{\omega_1} Z_1^\rho$ for all initial resources $\omega_1 \in \Omega$ is surjective everywhere. There exists for each $\omega_1 \in \Omega$

$$Z_{1,\omega_1 \in \Omega}^\rho : \Psi^\rho \to N \pitchfork_{\omega \in \Omega_\rho} \{0\}. \tag{3.16}$$

where $\{0\} \subset N$, and $\rho = 0$ satisfied. The dimension of the preimage of the evaluation map defined on the set $\{0\}$, $Z_{1,\omega_1 \in \Omega}^\rho(\{0\})$ is $l(S+1)-1$. by Thom's parametric transversality theorem[5], it follows that the subset $\Omega_\rho \cap \Omega$ is generic, since it is a dense and open set. Equilibria for this pseudo economy exist. By the equivalence propositions (3.2) and (3.3) know that full rank financial markets equilibria with endogenized smooth productions sets exist.

For all ρ satisfying $(1 \leq \rho \leq n)$ the preimage of the rank reduced evaluation map $Z_{1,\omega_1 \in \Omega}^\rho(\{0\})$ is $l(S+1)-1$ has dimension $l(S+1)-1-\rho^2$. By application of Thom's theorem this implies that for generic endowments $\omega \in \cap_\rho(\Omega_\rho)$ for all $\rho = 1, ..., n$ there is no reduced rank equilibrium, since for $Z_1^\rho(., \omega)$ the set of zeros is empty, $\{0\} = \emptyset$.

[5]See Thom R. (1954) "Quelques propriétés globales des variètès differentiables". Comm. Math. Helv. 28, 17-86.

3.4 Existence of equilibrium for endogenized convex piece-wise linear production sets

The linear activity model belongs to a large class of models with many important economic applications[6]. The Leontief Input-Output model is an example of such a linear activity model. The goal of this section therefore, is to establish generic existence of equilibrium for the linear activity model, a model with convex production sets and constant returns to scale technologies.

A linearity assumption on the transformation map ϕ_j is introduced by replacing the non-linearity assumption on the transformation map (iii) in set of assumptions 3.1(F) with assumption 3.5(L) below.

The main result of this section shows that, by similar arguments of the previous sections, equilibria exist for regularized production manifolds. This requires to firstly show that we can sufficiently well approximate the non-smooth production manifolds with smooth manifolds.

Assumption 3.5 (L) *The $t = 1$ transformation map $\phi_j|_z(s) : \mathbb{R}^a_- \to \mathbb{R}^b_+$ is piecewise linear for all $s \in \{1, ..., S\}$, and $j \in \{1, ..., n\}$.*

Geometrically, each endogenized period two production set $Y_j|_z$ is represented by a polyhedral cone, a set generated as a convex hull of a finite number of rays.

We apply techniques from regularization theory to production sets[7] in order to smooth out convex, piecewise linear production manifolds $\partial Y_j|_z$ by convolution, and show that these convolutes, denoted Φ_j, are compact and smooth manifolds approximating the piecewise linear production manifolds. Let the

[6] For details of such models see Gale (1960) for example [25].

[7] Similar to Chiappori and Rochet (Econometrica, 1987) who applied regularization theory to smooth out utilities.

state dependent convolute $\Phi_j(s)$ for producer $j \in \{1, ..., n\}$ be defined by equation

$$(\lambda_\sigma * \phi_j(y))(s)_j = \begin{cases} \int_{\mathbb{R}^m_-} (\lambda_\sigma(\zeta)\phi_j(y-\zeta)d\zeta)_j(s) & \text{for all } s \in S \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

for all $j \in \{1, ..., n\}$, where $y \in U_\sigma$, and $U_\sigma = \{y \in U : B(y, \sigma) \subset U\}$. Continuity of $\phi_j(s)$ implies the existence of a distance $\sigma = inf_t(\sigma_t)$, where $0 < \sigma < 1$. Associate with measure $\sigma \in [0, 1]$ the manifolds $\lambda_\sigma(j)$, for all $j \in \{1, ..., n\}$, defined by following equation.

$$\lambda_\sigma(y, s)_j = \frac{1}{\sigma}\lambda(\frac{y}{\sigma}(s)), \text{ for all } s \in S. \tag{3.18}$$

A convolution kernel $\lambda(s) \in L^1(\mathbb{R}^l_-)$ is a smooth, non-negative and symmetric manifold with mass equal to 1 and with compact support containing 0:

$$\lambda(y_0, s)_j = \begin{cases} \left(exp\left(\frac{-1}{1-\|y_0\|^2}\right) / \int_{\mathbb{R}^l_-} exp\left(\frac{-1}{1-\|y_0\|^2}\right) dy_0\right)(s) & \text{if } \|y_0\| < 1 \\ 0 & \text{if } \|y_0\| \geq 1 \end{cases}, \tag{3.19}$$

for all $s \in \{1, ..., S\}$, and for all $j \in \{1, ..., n\}$.

Proposition 3.5 *Each regularized production manifold $\partial\widetilde{Y}_j|_Z(s)$, defined by the convolute $\Phi_j(s)$, for all $s \in S$ and $j \in \{1, ..., n\}$ is C^∞ and compact.*

Proposition 3.6 *For every $j \in \{1, ..., n\}$ and C^∞ kernel λ, $\lambda*$ is bounded and con-*

verges to identity ϕ, it satisfies

$$|(\lambda_\sigma * \phi)_j(s) - \phi(s)|_j \leq \varepsilon(s)_j \textit{ for all } s \in S.$$

Theorem 3.3 *For every endogenously determined and regularized production manifold $\partial\widetilde{Y}_j|_Z$, $j \in \{1, ..., n\}$, there exists a pseudo (FE) stock market equilibrium $(\overline{P}, \overline{L})$ with associated equilibrium allocations $(\overline{x}, \overline{y})$ for generic initial resources $\omega \in \Omega$. Moreover, by the relational propositions (3.2), and (3.3), a stock market equilibrium (FE) $(\overline{p}, \overline{q})$ with associated equilibrium allocations $(\overline{x}, \overline{\xi}, \overline{y})$ exists for generic initial resources $\omega \in \Omega$.*

3.5 Conclusion

This chapter introduces the benchmark model of the firm in its reduced form. The reduced form equilibrium definition follows from a reinterpretation of the long run profits maximization assumption of the classical general equilibrium models of production with incomplete markets. This assumption has the convenient advantage that it enables to economize on mathematical notation, and therefore, to simplify the establishment of existence of equilibrium, since the precise nature of the structure of the firm needs not to be considered. The long run profits maximization assumption implicitly implies a sequential structure of the firm, where each firm issues stocks in period one in order to build up production capacity, and then, subject to given production sets and states of the world to choose a production plan in order to maximize its profits. Note that at this point in time, we model incomplete markets by hypothesis. This is improved on in chapter 5, where the extensive form model of the firm is introduced, and the optimal number of firms endogenously determined.

For the reduced form equilibrium concept introduced, existence of equilibria was proved for two asset structures: for convex production sets (i) and non-linear technologies, (decreasing returns to scale technologies), (ii) and for linear technologies (constant return to scale technologies). The later was shown by using techniques from regularization theory and smoothing out the piece-wise linear production manifolds by convolution. Existence of equilibrium is then shown for the approximated production set.

It remains to formally introduce an extensive form equilibrium which relaxes the long run profits assumption, and in which the role of financial assets are explicitly modeled. This is subject of chapter 4, where by means of examples the role of financial assets is illuminated, and more rigorously in chapter 5.

This chapter has also exhibited a class of endogenized smooth asset structures. This result is very convenient, then for every endogenized asset structure belonging to this general asset class, equilibrium always exists. Moreover, it enables to derive a theory of the firm in general equilibrium with incomplete markets, where the real sector is not independent of the financial sector. Consequently, interesting economic phenomena related to the firm can be studied. This result also provides a way of naturally introducing further financial assets, such as bonds for example, into the analysis of economic equilibrium with production.

The following chapter studies some of these properties by means of examples. In particular, the idea of chapter 4 is to contrast equilibrium properties of the model of production of the classical GEI model to properties of the model introduced in this thesis. Only the minimum structure is imposed on each example, such that each example highlights an economic interesting property in its simplest form. This allows to keep mathematical notation at a minimum.

Chapter 4

Examples: Equilibrium Properties Beyond Existence of Equilibrium

4.1 Introduction

This chapter intends to give some simple examples of properties of general equilibrium models with production beyond existence of equilibrium. These properties are mainly related to the objective function of the firm, and the role financial assets play in these models. The set of examples in part I aims at constructing a simple model and variations of it which is sufficiently rich in structure to reproduce some economic properties of a particular class of GEI models of production. These examples elaborate on the problems associated with the organization of production when firms maximize the utility of a group of shareholders or any other utility of a representative agent. The second set of examples in part II of this chapter studies the same equilibrium properties for a simplified version of the model introduced in chapter 3. For the purpose of illustration, we consider special cases only.

By means of examples, the main contribution of this chapter is to show that, in contrary to widely believed, the objective function of the firm can be viewed as independent of any form of utility of the owners or of the utility of a manager assigned to it. This in its simplest form establishes one of the main results of this thesis. An example of such an objective function is introduced in part II. This function turns out to have a nice property. It enables to prove another interesting result, namely, the generalization of the decentralization theorem of the Arrow-Debreu model to the case of incomplete markets. This result improves on Drèze [21] and Grossman and Hart [28] who where able to separate the activities of the agents only, but not the objective of the firm from the utility of the shareholders. The third main result suggests a reexamination of the Modigliani and Miller theorem in incomplete markets. It shows at a very preliminary level that real allocations are not independent of financial policies of the firm under standard GEI assumptions. This result follows from the objective function of the firm which links the real with the financial sphere through the way financial assets (stocks) are introduced. This result however, is incomplete at this stage and subject to further research. Finally, this chapter comments on the organization of production as an additional source of inefficiency. It shows that equilibrium allocations of the model introduced in chapter 3 are generally allocational superior efficient relative to any utility dependent production model. The degree of inefficiency depending on the present value vector assigned to the objective function of the firm.

Part I introduces a simple model with technological uncertainty. This models is designed in order to maintain the main properties of the classical GEI models. We use this model to reproduce some properties of production of centralized models. Part II studies production properties of the model introduced in chapter 3. We elaborate on the consequences of the two different ways production is organized in general equilibrium with incomplete markets. Despite the study of a very simple and highly stylized general equilibrium model in this chapter. We are able to make economic sense of this rather special case, and occasionally, where convenient, interpret this single agent model as an entrepreneurship model.

4.2 Examples: Part I

4.2.1 Introduction part I

Part I considers a set of examples replicating some equilibrium properties of centralized GEI models of production. These examples shall illustrate problems related to the organization of production when time and uncertainty are modeled explicitly. In order to illustrate the properties of interest, it is sufficient to consider a highly stylized 1 agent model. This model differs from current models in the following ways:

1. we consider technological uncertainty rather aggregate uncertainty,

2. (beyond the one agent model), initial ownership is not exogenously given, but modeled),

3. at variance with Magill and Quinzii [38], the firm finances production through the stock market, rather the bond market.

The main property of the classical GEI model of production that we maintain is the exogenously determined two period production set available to the producer. Given such a production set, the firm chooses inputs of production in period one with associated outputs in period two. This property is necessary in order to replicate the results known in the literature. Another property that we maintain is related to the objective function of the firm, where financial assets enter the objective function of the firm additively and independent of the production set. These two properties turn out to introduce a problem related to the question of how to model the firm.

4.2.2 Introduction to the centralized single agent reduced form model: Example (1)

The idea of considering a reduced form model is to introduce a simplified model where financial policies are not explicit modeled. Its counterpart, the extensive form model will be introduced later.

In the single agent reduced form model, the consumer performs the role as a consumer and as a producer. As a consumer the agent buys stocks z and receives a proportion of the real value of the firm $\theta(z) = 1$ (in this case) in the next period in return. As a producer the agent issues the quantity of stocks b (here b is not modeled explicitly) in order to finance a project. Notice the nature of the role financial assets play in this model. Namely, the firm issues stocks in order to finance factors of production in period one, taking its technology as exogenously given. This is at variance with the model introduced in the previews chapter. The return of financial investment the agent obtains as a consumer is denoted $R(\bar{y}, s)z$, and the dividend payoff the agent pays as a producer is denoted $R(\bar{y}, s)b$. The agent's $S + 1$ budget constraints are denoted

$$B_z = \left\{ \begin{array}{c} (x) \in \mathbb{R}^{S+1}_{++} : \\ p(0)x(0) = p(0)\omega(0) - \theta(\bar{z})p(0)y(0) - qz + qb \\ p(s)x(s) = p(s)\omega(1) + \theta(\bar{z})p(s)y(s) + R\left(\bar{y}, s\right) z - R\left(\bar{y}, s\right) b \end{array} \right\}, \tag{4.1}$$

where $R\left(\bar{y}, s\right) = \frac{D(\bar{y},s)}{\hat{b}}$ is the dividend payoff per stock issued[1]. Let $\xi = z - b$, then the agent's sequence of budget constraints can be rewritten as

[1] We sometimes abuse notation in this chapter. In particular, we do not write the dot product explicitly. This because, whenever possible, we think of the model as a single good model, and therefore, can omit it.

$$B_\xi = \left\{ (x) \in \mathbb{R}^{S+1}_{++} : \begin{array}{l} p(0)x(0) = p(0)\omega(0) - \theta(\bar{z})p(0)y(0) - q\xi \\ p(s)x(s) = p(s)\omega(1) + \theta(\bar{z})p(s)y(s) + R(\bar{y}, s)\,\xi \end{array} \right\}, \quad (4.2)$$

where $p(0)y(0)$ denotes the investment costs in period one associated with revenue $p(s)y(s)$ in each state of nature $s \in \{1, ..., S\}$ in period two. In this model the firm's production set is $Y = \mathbb{R}^{S+1}$ if only one good in each state of nature is considered (otherwise $\mathbb{R}^{l(S+1)}$). Note that a price normalization implies that $p(0) = 1$, and $p(s) = 1$ in every $s \in \{1, ..., S\}$. The production set is described by a function $\Phi : \mathbb{R}_- \to \mathbb{R}^S_+$, where $Y = \left\{ y \in \mathbb{R}^{S+1} : \Phi(y) \leq 0 \right\}$. Standard assumptions on technology sets apply. Ownership of the firm $\theta(.)$ is a function of quantity of stocks purchased as a consumer.

Definition 4.1 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, \bar{y})$ for generic initial resources $\omega \in \Omega$ if following conditions are satisfied:*

$$\begin{array}{ll} (i) & \left(\bar{x}; \bar{\xi}, \bar{y}\right) \in \arg\max \left\{ u(x) : x \in B_\xi \right\} \\ (ii) & \bar{\xi} = 0. \end{array} \quad (4.3)$$

Condition (ii) implies that the quantity of stocks that the consumer buys is equal to the quantity of stocks he issues as a producer. ξ denotes the net trade of stocks, where at equilibrium $\bar{\xi} = 0$ is satisfied. For the case that more than one consumption good is considered, $\bar{x}(0) = \omega(0) + \bar{y}(0)$, and $\bar{x}(s) = \omega(1) + \bar{y}(s)$ for all states of nature hold. The agent's optimization problem is to choose ξ and y such that utility of x is maximized.

Propositions (4.1), (4.2), and (4.3) state that in a single agent reduced form model, where economic activities are centralized, the utility maximization problem has a solution. The first two propositions show a first step towards modeling financial assets (on consumer side only), where ξ implying z and b implicitly contained in ξ, and for the case that the agent as a consumer takes financial policy of the firm b as given and chooses z to finance his preferred consumption bundle x. Proposition (4.3) shows the equivalence of these models.

Proposition 4.1 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $\left(\bar{x}, \bar{\xi}, \bar{y}\right)$ of the maximization problem (i), if and only if for generic initial resources $\omega \in \Omega$*

$$\bar{q} \text{ is a no-arbitrage price} \tag{4.4}$$

is satisfied.

Definition 4.2 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}, \bar{y})$ for for generic initial resources $\omega \in \Omega$ if following conditions are satisfied:*

$$\begin{array}{ll} \text{(i)} & (\bar{x}; \bar{z}, \bar{y}) \in \arg\max \{u(x) : (x; z, y) \in B_z\} \\ \text{(ii)} & \bar{z} + \hat{b} = 0 \end{array} \tag{4.5}$$

and $\bar{x}(0) = \omega(0) + \bar{y}(0)$, and $\bar{x}(s) = \omega(1) + \bar{y}(s)$ for all s hold for $l > 1$.

Proposition 4.2 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}, \bar{y})$ of the maximization problem (i), if and only if for generic initial resources $\omega \in \Omega$*

$$\bar{q} \text{ is a no-arbitrage price} \tag{4.6}$$

is satisfied.

Proposition 4.3 *The reduced form model (4.1) and the reduced form model (4.2) are equivalent if*

$$\bar{\xi} = \bar{z} - \hat{b} \tag{4.7}$$

The reduced form model (4.2) is equivalent to the reduced form model (4.1) if

$$\bar{z} - \hat{b} = \bar{\xi}. \tag{4.8}$$

4.2.3 The separated activities single agent reduced form model: Example (2)

This subsection expands the centralized reduced form model to an economic framework where decisions of the single agent are separated. This allows to introduce two separated optimizations problems, one for each role the agent plays. This example, although very simple, is non-trivial. It remarks on a fundamental issue regarding the literature on the objective function of the firm in general equilibrium with incomplete markets.

Suppose that the consumer assigns to the firm his own present value vector β. The objective of the agent as a producer is, given his own present value vector, to maximize the present value of streams of profits. This economic framework is sufficiently rich in structure in order to show the separation of activities of the agent as a consumer and as a producer. This is a variation of a contemporary result known in the GEI literature on centralized models of the firm.

Proposition 4.4 $(\bar{p}, \bar{q})$ *is a separated activities reduced form equilibrium with associated equilibrium allocations* $(\bar{x}, \bar{\xi}, (\bar{y}))$*, for generic initial resources* $\omega \in \Omega$*, if and only* *if for $\bar{\pi}$ assigned to the objective function of the firm it satisfies:*

$$
\begin{aligned}
&\textit{(i)} \quad \left(\bar{x}, \bar{\xi}\right) \arg\max\left\{u(\bar{x}) : \bar{x} \in B_{\xi}\right\} \\
&\textit{(ii)} \quad (\bar{y}) \in \arg\max\left\{\bar{\beta}\bar{p}y : y \in Y\right\} \\
&\textit{(iii)} \quad \bar{\xi} = 0.
\end{aligned}
\tag{4.9}
$$

Remark 1 *This result is sometimes referred to as the equivalence of the decentralization theorem of the Arrow-Debreu model for the case of incomplete markets [20]. This wording can be misleading. What the result actually does is separating the activities of the agent as a consumer from the activities as a producer. However, the objective function as a producer is not independent from the present value vector of the consumer. Consequently, as a producer, the agent maximizes a present value problem not independent of information contained in the utility of the consumer. This makes sense in this one agent set up if one is willing to think of this model as an entrepreneurship model. However, adding another agent to the model raises the question of what present value to assign to the single firm. What this result does not is decentralizing the objective function of the firm.*

Examples in part II will improve on this result and show that the model of the firm introduced in chapter 3 is independent of such additional information. This is interesting because it allows to improve on the decentralization property by decentralizing the objective function of the firm. Proposition (4.5) shows the inefficient organization of production of the reduced form model with separated activities of the agent. The level of inefficiency introduced into the model depends on the consumer's present value vector.

Proposition 4.5 *The organization of production is generally (in)efficient for any assigned present value β to the objective function.*

The model is production efficient, if there does not exist a production plan $\hat{y} \neq y$ in Y such that $u(\hat{x}) > u(x)$. Alternatively to above, to show that the organization of production of this centralized model introduces a source of inefficiency, it is sufficient to expand the model to two consumers. Then, need to assign some arbitrarily determined average β_i to the objective function of the firm. It is easy to see that for any different average or median present value vector assigned to the firm that net activities change accordingly, and consequently $u(\hat{x}) \neq u(x)$ for $\hat{y} \neq y$. Moreover, introducing another agent also introduces a new problem about what present value vector to assign to the firm.

4.2.4 Geometric first order conditions for the reduced form single agent model: Example (3)

For some cases, it may turn out convenient to have the geometric first order conditions and interpretation of equilibrium. Hence this application of convex sets analysis. Proposition (4.6) shows that the utility maximization problem of the reduced form model has a solution. Corollary (4.1) is the expansion of this result to the reduced form utility maximization problem of the centralized single agent model.

Proposition 4.6 *Let Y and Z be two nonempty convex sets. Then $(\bar{y}, \bar{z})$ is a solution of*

$$(\bar{x}; \bar{y}, \bar{z}) \in \arg\max \{u(y+z) : y \in Y,\ z \in Z\} \tag{4.10}$$

if and only if

$$\nabla u(\bar{y} + \bar{z}) \in N_Y(\bar{y}) \cap N_Z(\bar{z}). \tag{4.11}$$

Let $\tilde{Z}$ be a subset of the real line denoted by the interval $[0, Z]$.

Corollary 4.1 *Let Y and Ξ be two nonempty convex sets. Then $(\bar{y}, \bar{\xi})$ is a solution of*

$$\left(\bar{x}; \bar{y}, \bar{\xi}\right) \in \arg\max \{u(y + \xi) : y \in Y,\ \xi \in \Xi\} \tag{4.12}$$

if and only if

$$\nabla u(\bar{y} + \bar{\xi}) \in N_Y(\bar{y}) \cap N_\Xi(\bar{\xi}). \tag{4.13}$$

Let Ξ be a subset of the real line denoted by the interval $[0, \Xi]$.

The proof of these results are based upon the separation hyperplane theorem for convex sets [24]. Proposition (4.6) and corollary (4.1) are geometric reinterpretations of the reduced form model (4.2). This results say that it is necessary and sufficient to show that the gradient vector of the centralized optimization problem (utility maximization) must lie in the intersection of the convex cones in order to obtain a solution of the maximization problems.

We can further simplify the study of this income transfer model by considering following optimization problem for the agent. Since the choice of a portfolio ξ is equivalent to the choice of a vector of income transfers $\tau = \Pi\xi$, following result holds.

Proposition 4.7 *Let $\bar{\tau} = \Pi\bar{\xi}$, and $\bar{\tau} \in \langle\Pi\rangle$. Then $(\bar{\tau}, \bar{y})$ is a solution of*

$$(\bar{\tau}, \bar{y}) \in \arg\max \{u(\omega + y + \tau) : (\tau, y) \in \langle\Pi\rangle \times Y\} \tag{4.14}$$

if and only if

$$(\bar{\tau}, \bar{y}) \in \langle\Pi\rangle \times Y,\ \nabla u(\bar{x}) \in \langle\Pi\rangle^\perp \cap N_Y(\bar{y}) \tag{4.15}$$

where $\bar{x} = \omega + \bar{\tau} + \bar{y}$.

For $\bar{\beta}$ assigned to the objective function of the firm, the first order conditions of the optimization problem can be decomposed into two pairs of conditions:

$$\bar{\tau} \in \langle \Pi \rangle \,, \quad \bar{\beta} \in \langle \Pi \rangle^{\perp} \tag{4.16}$$

$$\bar{y} \in Y, \;\; \bar{\beta} \in N_Y(\bar{y}). \tag{4.17}$$

A proof of this result is omitted. It is a simple expansion of the proof of proposition (4.7) illustrated in the mathematical appendix. Proposition (4.8) replicates the separation result of the previous section for the geometrically reformulated model. The proof shows the geometric separation of activities of the agent as a consumer and as a producer.

Proposition 4.8 $(\bar{p}, \bar{q})$ *is an equilibrium of the geometrically reinterpreted reduced form equilibrium with associated equilibrium allocations* $(\bar{x}, \bar{\xi}, (\bar{y}))$ *and separated activities for generic initial resources* $\omega \in \Omega$*, if and only if for* $\bar{\beta}$ *assigned to the objective function of the firm it satisfies:*

$$\begin{array}{ll} (i) & \left(\bar{x}, \bar{\xi}\right) \in \arg\max \left\{u(\bar{x}) : \bar{x} \in B_{\xi}\right\} \\ (ii) & (\bar{y}) \in \arg\max \left\{\bar{\beta}\bar{p}y : y \in Y\right\} \\ (iii) & \bar{\xi} = 0. \end{array} \tag{4.18}$$

4.2.5 Single agent extensive form model: Example (4)

This subsection aims at reproducing in its simplest form the irrelevance of financial policy theorem of Modigliani and Miller [43]. The theorem states that whatever financial policy a firm chooses, consumers can always undo this, leaving effects on real allocations unchanged. The theorem implicitly assumes that an equilibrium production plan of the firm is financed by its policy.

The next result replicates the Modigliani and Miller theorem. For that, we add more structure to the model and introduce an extensive form model of the firm, where financial policies are explicitly modeled. The result shows that the firm's financial policy has no real effects. This result follows from the independence of the firm's production set from the actions of the firm in the financial sector, an assumption implicit in the theorem of the irrelevance of financial policies.

Denote the budget set of the consumer

$$B_z = \left\{(x) \in \mathbb{R}^{S+1}_{++} : px = p\omega + py + \Pi b + \Pi z\right\}, \tag{4.19}$$

where $\Pi = \begin{bmatrix} -q \\ \frac{D(1)}{b} \\ \vdots \\ \frac{D(S)}{b} \end{bmatrix}$

is the financial payoff matrix (vector, here). $\frac{D(s)}{b}$ denotes the payoff per stock issued in a particular state of nature. As a consumer, the agent takes (p, q, b, y) as given and chooses z which finances his most preferred consumption bundle x. As a producer he takes (p, q, x, z) and present value vector β as given and chooses b and y such that present value profits are maximized. This is formally introduced in following definition.

Definition 4.3 *$(\bar{p}, \bar{q})$ is an extensive form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}), (\bar{y}, \bar{b})$, for generic initial resources $\omega \in \Omega$, if following conditions are* *satisfied:*

$$\begin{array}{ll} \textit{(i)} & (\bar{x}, \bar{z}) \in \arg\max \{u(x) : x \in B_z\} \\ \textit{(ii)} & \left(\bar{y}, \bar{b}\right) \in \arg\max \left\{ \bar{\beta}\bar{p}y + \Pi b : \begin{array}{c} y \in Y \\ b \in \mathbb{R}_- \end{array} \right\} \\ \textit{(iii)} & \bar{z} - \bar{b} = 0. \end{array} \tag{4.20}$$

Proposition (4.9) asserts that the precise nature of the producer's financial policy has no real effects on equilibrium allocations, provided it finances the producer's production plan. The result follows from showing the equivalence between the extensive form and the reduced form model where financial policies are not explicitly modeled. Two properties of this model make the proof work. (i) as a consumer and as a producer the agent has access to the same market subspace $\langle \Pi \rangle$, and (ii) a no-arbitrage condition $\beta\Pi = 0$ holds. Hence, financial polices do not affect the budget set of the consumer, nor the present value of future streams of profits generated by the producer. As a consumer, the single agent can always undo the financial activities taken as a producer. The value of the firm depends only on the production plan chosen by the producer, and not on its financial policy.

Proposition 4.9 *If $(\bar{p}, \bar{q})$ is an extensive form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}), (\bar{y}, \bar{b})$, then $(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, \bar{y})$ for generic initial resources $\omega \in \Omega$ where*

$$\bar{\xi} = \bar{z} - \bar{b} \tag{4.21}$$

If $(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, \bar{y})$, then $(\bar{p}, \bar{q})$ is an extensive form equilibrium with associated equilibrium allocations

$(\bar{x}, \bar{z})$, $(\bar{y}, \bar{b})$ *for generic initial resources* $\omega \in \Omega$ *and for all* $(\bar{z}, \bar{b})$ *satisfying*

$$z - \bar{b} = \bar{\xi}. \tag{4.22}$$

Remark 2 *Recall that in this model production sets are exogenously given. This seems a strong assumption when considering the role of financial assets. Here, it is implicitly assumed that the firm's financial activity finances, given a fixed technology, the inputs of production. Next section elaborates on this restrictive assumption.*

4.3 Examples: Part II

4.3.1 Introduction part II

This set of examples considers variations of a simple single agent model with the main feature that the one period endogenized production set available to the firm in period two is not independent of its financial activities in period one. This model is a special case of the model introduced in chapter 3. In short, we consider variations of a special case of the endogenous asset formation model introduced in chapter 3 in its reduced form. The goal of this set of examples is to introduce an endogenous asset structure into the GEI model, and to show some properties of productive organization, and to contrast them with the results derived in Part I for a variation of the classical GEI model of production.

The financing of production in this model is at variance with the classical GEI model, where firms issue stocks to finance production inputs in period one. Here, we consider short and long run financing. In the long run, firms build up production capacity by issuing stocks in period one. This determines the production set available to the firm in period two. In the short run in period two, the firm finances production with the revenue generated by selling its output.

This is similar to the Arrow-Debreu model, once the production set is installed.

4.3.2 The endogenized production set single agent reduced form model: Example (5)

Consider the budget constraints of the agent as a consumer

$$
\begin{aligned}
p(0)x(0) &= p(0)\omega(0) - qz \\
p(s)x(s) &= p(s)\omega(1) + R(\bar{y}, s)z.
\end{aligned}
$$

In a one agent model the agent also performs the role of the producer, and therefore, adds following variables to his constraints

$$
\begin{aligned}
p(0)x(0) &= p(0)\omega(0) - qz + qb - p(0)\bar{k}(0) \\
p(s)x(s) &= p(s)\omega(1) + R(\bar{y}, s)z + p(s)y(0)
\end{aligned},
$$

where $\bar{k}(0)$ denotes the capital purchased. Let aside the modeling of financing production for a while, therefore, let $\xi = z - \hat{b}$, where $\hat{b}$ deotes a fixed level of capital at a feasible financial policy of the firm such that $\hat{b} \Rightarrow Y|_{\hat{b}}$. Here, take production set $Y|_{\hat{b}}$ as given. Then have following budget set

$$
B_\xi = \left\{ (x; y) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R}^{lS} : \begin{array}{l} p(0)x(0) = p(0)\omega(0) - q\xi - p(0)\bar{k}(0) \\ p(s)x(s) = p(s)\omega(1) + R(\bar{y}, s)\xi + p(s)y(0) \end{array} \right\}. \tag{4.23}
$$

The agent 's control problem is then to choose $(x; \xi, y)$ such that utility of consumption of goods is maximized. By reduced form, we mean a model where financial policies are not explicitly modeled and decisions of the agents not fully separated. We formally introduce this model via definition (4.4) below.

Definition 4.4 *A reduced form equilibrium* $(\bar{p}, \bar{q})$ *with associated equilibrium allocations* $\left(\bar{x}; \bar{\xi}, \bar{y}\right)$ *for generic initial resources* $\omega \in \Omega$ *satisfies:*

$$\begin{aligned}
&\textit{(i)} \quad \left(\bar{x}; \bar{\xi}, \bar{y}\right) \in \arg\max\left\{u(x) : x \in B_{\xi}\right\} \\
&\textit{(ii)} \quad \bar{\xi} = 0 \\
&\qquad\; \bar{x}(0) = \omega(0) + \bar{k}(0) \\
&\qquad\; \bar{x}(s) = \omega(1) + \bar{y}(s) \quad \textit{for all } s \in \{1, ..., S\}.
\end{aligned} \tag{4.24}$$

Proposition (4.10) is an interesting result. At first sight it seems to reproduce the result of propositions (4.4), and (4.8). This is commented on in remarks (3), and (4) below. However, remark (5) enables to interpret this result as not only separating the activities of the agent, but also separating the objective function of the firm from the utility of the owner of the firm. The result suggests that the classical GEI model of previous section is a special case of the model initially introduced in chapter 3.

Proposition 4.10 $(\bar{p}, \bar{q})$ *is a reduced form equilibrium with associated equilibrium allocations* $\left(\bar{x}; \bar{\xi}, \bar{y}\right)$ *for generic initial resources* $\omega \in \Omega$ *separating activities of the agent as a consumer and as a producer if assign the gradient vector* $\bar{\beta}$ *to the firm and following conditions are satisfied:*

$$\begin{aligned}
&\textit{(i)} \quad \left(\bar{x}; \bar{\xi}\right) \in \arg\max\left\{u(x) : x \in B_{\xi}\right\} \\
&\textit{(ii)} \quad (\bar{y}) \in \arg\max\left\{\bar{\beta}\bar{p}(s)y(s) : y \in B_{\xi}\right\} \\
&\textit{(iii)} \quad \bar{\xi} = 0 \\
&\qquad\; \bar{x}(0) = \omega(0) + \bar{k}(0) \\
&\qquad\; \bar{x}(s) = \omega(1) + \bar{y}(s) \qquad\qquad \textit{for all } s \in \{1, ..., S\}.
\end{aligned} \tag{4.25}$$

Remark 3 *Note that this separatation result is still dependent on the present value vec-*

tor of the consumer. Consequently, the objective function of the firm is not decentralized yet. Condition (iii) can be simplified if think of this model as a one good model.

Remark 4 *If one is willing to accept that the consumer assigns his own present value vector to the firm to evaluate future income streams, then this model reproduces the result from the literature (One can think of this model as an entrepreneurship model).*

Remark 5 *However, if one is willing to think that the single agent is perfectly able to separate his activity as a consumer and as a producer, then this model allows him as a producer not to attach the present value of the consumer to the objective function of the firm, since as a producer, he is not exposed to the no-arbitrage condition. This follows from the different role financial assets play. This gives following extension of the reduced form model introduced in chapter 3 presented in the next subsection.*

4.3.3 Decentralizing the objective function (by assumption of long run profit maximization): Example (6)

Consider the reduced form model introduced in part II. Assume that the producer maximizes long run profits. This means that at $t = 0$, ξ implicitly finances the production set available to a firm in $t = 1$. Denote the production set available to the firm $Y|_{\hat{b}}$ and assume that it exists. Long run profit maximization then implies that the producer chooses inputs of production, given production capacity, such that production of outputs maximizes his profits. The financing of production inputs in $t = 1$ is defined by the sell of production outputs. At $t = 1$ no other source of financing production is needed. The reduced form objective of long run maximization of profits is then to

$$(\bar{y}) \in \arg\max \left\{ \bar{p} \Box y : y(s) \in Y|_{\hat{b}}(s), \forall s \in S \right\}. \tag{4.26}$$

Proposition 4.11 $(\bar{p}, \bar{q})$ *is a reduced form long run profit maximizing equilibrium with decentralized objective function of the firm and with associated equilibrium allocations* $\left(\bar{x}, \bar{\xi}\right), (\bar{y})$ *for generic initial resources* $\omega \in \Omega$*, if following conditions are satisfied:*

$$
\begin{aligned}
&\textit{(i)} && \left(\bar{x}; \bar{\xi}\right) \in \arg\max\left\{u(x) : x \in B_{\xi}\right\} \\
&\textit{(ii)} && (\bar{y}) \in \arg\max\left\{\bar{p}\Box y : y(s) \in Y|_{\hat{b}}(s), \forall s \in S\right\} \\
&\textit{(iii)} && \bar{\xi} = 0 \\
& && \bar{x}(0) = \omega(0) + \bar{k}(0) \\
& && \bar{x}(s) = \omega(1) + \bar{y}(s) \textit{ for all } s \in \{1, ..., S\}.
\end{aligned}
$$

This result follows from remark (5). It shows the independence of the objective function from the present value vector of the agent as a consumer. The result is a consequence of the endogenous asset structure of the model, where the firm builds up production capacity by issuing stocks. This is in its simplest form one of the main results of this chapter. It generalizes, by means of a simple example, the decentralization property of the Arrow-Debreu model to the case where time and uncertainty explicitly enters the model in an essential way, and incomplete markets a consequence of idiosyncratic risk[2].

4.3.4 Productive efficiency of the reduced form model with decentralized objective function: Example (7)

Proposition 4.12 $(\bar{p}, \bar{q})$ *is a reduced form equilibrium with efficient organization of production and decentralized objective function of the firm (long run profit maximization) with associated equilibrium allocations* $(\bar{x}, \bar{\xi}), (\bar{y})$ *for generic initial resources* $\omega \in \Omega$.

[2] The later remark will be introduced in form of a result in the next chapter.

The proof of proposition (4.12) makes use of the fact that the objective function of the firm is independent of any assigned present value vectors of the consumer to it. In order to introduce productive inefficiency into the model it suffices, for example, to consider two consumers and a single firm. Then for any two different average gradient vectors assigned to the firm, profits change accordingly. The independence of the objective function from the utility of the consumer follows from the role financial assets play in this model. The problem of the firm in the reduced form model is essentially equivalent to the problem of the firm in the Arrow Debreu model, where one period production sets are taken as given. This is also the case there. This result implicitly states that any utility maximizing model of the firm in GEI introduces a further source of inefficiency due to the inefficient organization of production.

Remark 6 *This model has similar (in)efficiency properties of equilibrium as the classical GEI exchange model. The point here is that at variance with the classical GEI model of production the organization of production does not introduce a further source of inefficiency by attaching some β to the firm.*

Proposition 4.13 $(\bar{p}, \bar{q})$ *is a reduced form centralized financial markets equilibrium with equilibrium allocations $(\bar{x}, \bar{\xi}), (\bar{y})$ with inefficient organization of production of the firm for generic initial resources $\omega \in \Omega$, if and only if*

$$\bar{\beta}(s) \neq \overrightarrow{e} \text{ for every } s \in \{1, ..., S\} \tag{4.27}$$

is satisfied.

The condition $\bar{\beta}(s) \neq \overrightarrow{e}$ for every $s \in \{1, ..., S\}$ is generally satisfied for centralized general equilibrium models with incomplete markets. This follows from the no-arbitrage condition. $\overrightarrow{e}$ is a unit vector of appropriate dimension.

4.3.5 The extensive form model: Example (8)

We now introduce the extensive form model, where decisions are fully decentralized and financial policies explicitly modeled. Consider the consumer's constraints

$$p(0)x(0) = p(0)\omega(0) - qz$$
$$p(s)x(s) = p(s)\omega(1) + \theta(\bar{z})R(\bar{y}, s),$$

where qz is the value the consumer is willing to invest into the firm at expected return R. As a producer, the manager's job is to find b such that $qb = qz$. He then buys capital $k(0)$ such that income from selling stocks is equal to his expenditure on capital consumption, therefore, $qb = p(0)k(0)$. At $t = 0$, the producer's problem is to

$$\left(\bar{k}(0); \bar{b}\right) \in \arg\max \left\{\bar{q}b : \bar{q}\bar{z} \geq \bar{q}b = \bar{p}(0)k(0)\right\}, \tag{4.28}$$

where the level of capital, $\overline{k}(0)$, implies total production capacity available to the firm, a correspondence $\Phi|_{\hat{b}}$. This correspondence in turn determines the production set available to the firm, denoted $Y|_{\hat{b}}$. Given this production set, and the set of states of nature, the producer's $t = 1$ problem is to

$$\left(\bar{y}(s)\right) \in \arg\max \left\{\bar{p}(s)y(s) : y(s) \in Y|_{\hat{b}}(s), \forall s \in S\right\}. \tag{4.29}$$

Inputs of production are financed with sells from outputs. The level of revenue a firm can generate in each state $s \in \{1, ..., S\}$ depends on the available production set determined in the certain state of the world. The next result establishes the full version of the endogenous asset formation model of this thesis for the special case of a single agent model. It shows the independence of the objective function from any present value vector derived from the owners of the firm. It also establishes, through the objective function of the firm, the link between the real and financial sector. The generalization of this result is stated in chapter 5.

Proposition 4.14 $(\bar{p}, \bar{q})$ *is a decentralized objective function extensive form equilibrium with associated equilibrium allocations* $((\bar{x}, \bar{z}), (\bar{y}, \bar{b}))$ *for generic initial resources* $\omega \in \Omega$*, if for any feasible* $\hat{b} \leqslant \bar{z}$ *following conditions are satisfied:*

$$\begin{aligned}
&\textit{(i)} && (\bar{x}; \bar{z}) \in \arg\max \{u(x) : x \in B_z\} \\
&\textit{(ii)} && \underset{(\bar{y}, \hat{b}; (\bar{k}(0)))}{\arg\max} \left\{ \bar{q}b + \bar{p}\Box y : \begin{array}{c} \bar{q}\bar{z} \geq \bar{q}b = \bar{p}(0)k(0) \\ y(s) \in Y|_{\hat{b}}(s) \end{array} \quad s \in S \right\} \\
&\textit{(iii)} && \bar{z} - \hat{b} = 0 \quad \theta(\bar{z}) = 1 \\
& && \bar{x}(0) = \omega(0) + \bar{k}(0) \\
& && \bar{x}(s) = \omega(1) + \bar{y}(s) \textit{ for all } s \in \{1, .., S\}.
\end{aligned} \tag{4.30}$$

Next result is a first step towards a study of the Modigliani and Miller theorem in the endogenous asset formation model introduced in this thesis. It shows that the real and financial sectors are not independent of each other, and that consequently financial policies have real effects. This result follows from the way financial assets enter the model. In particular, the objective function of the firm links the real with the financial sphere. In the classical GEI model, firms issue stocks in order to finance production inputs in period one, here, firms issue stocks in order to buy capital and to build up their production set. Hence, real effects. Note that this result is established without considering other financial assets. The idea of the proof is to criticize the implicit assumption that financial policy is independent of the production set at first instance. More work needs to be done in order to proof the full version of the Modigliani and Miller theorem. This is work in progress.

Proposition 4.15 *(i) If* $((\bar{p}, \bar{q}), (\bar{x}, \bar{z}), (\bar{y}, \hat{b}))$ *is an extensive form equilibrium (EFE) with decentralized objective function for generic initial resources* $\omega \in \Omega$ *then* $((\bar{p}, \bar{q}),$ $(\bar{x}, \bar{\xi}), (\bar{y}))$ *is a reduced form equilibrium (RFE) with decentralized objective function for generic initial resources* $\omega \in \Omega$ *where*

$$\bar{\xi} = \bar{z} - \hat{b} \tag{4.31}$$

(ii) If $((\bar{p}, \bar{q}), (\bar{x}, \bar{\xi}), (\bar{y}))$ *is a (RFE) with decentralized objective function for generic initial resources* ω *then* $((\bar{p}, \bar{q}), (\bar{x}, \bar{z}), (\bar{y}, \hat{b}))$ *is a (RFE) with decentralized objective function for generic initial resources* $\omega \in \Omega$ *for any* $\hat{b} \leq \bar{z}$ *satisfying*

$$\bar{z} - \hat{b} = \bar{\xi}. \tag{4.32}$$

4.4 Geometric decentralization of activities and objective of the firm

This section considers a geometric approach to the study of the endogenous asset formation model. The result below separates the activities of the agent as a consumer and as a producer. Moreover, it separates the objective function of the firm from the present value vector derived from utility maximization.

Proposition 4.16 $(\bar{p}, \bar{q})$ *is a geometrically reinterpreted extensive form equilibrium* *with associated equilibrium allocations* $((\bar{x},\bar{z}),(\bar{y},\bar{b}))$ *for generic initial resources* $\omega \in \Omega$ *with decentralized objective function of the firm if for any feasible* $\hat{b} \leqslant \bar{z}$ *following conditions are satisfied:*

$$
\begin{array}{ll}
\textit{(i)} & (\bar{x};\bar{z}) \in \arg\max\{u(x) : x \in B_z\} \\
\textit{(ii)} & \underset{(\bar{y},\hat{b};(\bar{k}(0)))}{\arg\max}\left\{\bar{q}b + \bar{p}\Box y : \begin{array}{c} \bar{q}\bar{z} \geq \bar{q}b = \bar{p}(0)k(0) \\ y(s) \in Y|_{\hat{b}}(s) \end{array} \quad s \in S\right\} \\
\textit{(iii)} & \bar{z} - \hat{b} = 0 \quad \theta(\bar{z}) = 1 \\
& \bar{x}(0) = \omega(0) + \bar{k}(0) \\
& \bar{x}(s) = \omega(1) + \bar{y}(s) \text{ for all } s \in \{1,..,S\}.
\end{array}
\tag{4.33}
$$

Lemma 4.1 $\bar{x}|_{\bar{z}}$ *is a solution of*

$$max\{u(x;z) : x \in B\} \tag{4.34}$$

if and only if, $\bar{x}|_{\bar{z}} \in B$, *and*

$$\partial u(\bar{x}|_{\bar{z}}) \cap N_B(\bar{x}|_{\bar{z}}) \neq \{0\} \tag{4.35}$$

is satisfied.

Lemma 4.2 $\bar{y}|_{\bar{z}}$ *is a solution of*

$$max\{\Pi(p;z) : y \in Y|_{\bar{z}}\} \tag{4.36}$$

if and only if, $\bar{y}|_{\bar{z}} \in Y$, *and*

$$\partial u(\bar{y}) \cap N_Y(\bar{y}) \neq \{0\} \tag{4.37}$$

is satisfied.

Lemma 4.3 *Let $Y|_{\bar{\xi}}$ and Ξ be two nonempty convex sets. Then $(\bar{y}, \bar{\xi})$ is a geometric solution of the reduced form problem (4.4)*

$$\left(\bar{x}; \bar{y}, \bar{\xi}\right) \in arg\max\left\{u(y+\xi) : y \in Y|_{\bar{\xi}}\,,\ \xi \in \Xi\right\} \tag{4.38}$$

if and only if

$$\nabla u(\bar{y}+\bar{\xi}) \in N_{Y|_{\bar{\xi}}}(\bar{y}) \cap N_{\Xi}(\bar{\xi}). \tag{4.39}$$

Proof 4.1 (Proposition 4.16) *By lemma (4.3) $((\bar{p}, \bar{q}), (\bar{x}, \bar{\xi})))$ is a reduced form equilibrium satisfying (i) of the extensive form model with decentralized activities if and only if the geometric first order conditions of lemma (4.1) hold. The profit maximization problem (ii) of the extensive form model with decentralized activities $(\bar{y}, \hat{b})$ is satisfied if and only if the geometric first order condition of lemma (4.2) holds. Since using lemma (4.3) $(\bar{x}, \bar{z}), (\bar{y}, \hat{b})$ satisfies (i) of the (centralized) reduced form model if and only if both geometric first order conditions hold lemma (4.1), lemma (4.2), $((\bar{p}, \bar{q}), ((\bar{x}, \bar{z}), (\bar{y}, b)))$ is a geometric extensive form with decentralized activities equilibrium.*

4.5 Conclusion

This chapter first shows by means of simple examples some properties of classical GEI models with production for a simple model with technological uncertainty, and production inputs financed by the stock market. This class of models is referred to as centralized models, since the objective function of the firm is not independent of extra information provided by the stock holders. In particular, the examples elaborate on the dependency of the objective function of the firm on the utility of the (average) utility of the shareholders. This dependency is not unproblematic as the second set of examples shows, where a special case of the model of chapter 3 is considered. This model has enough structure to improve on economically interesting properties of the classical GEI model. For this model with endogenized production sets, we show that the objective function of the firm is independent of the utility of the stock holders. As a consequence, equilibrium properties change. For example: (i) we can decentralized the objective function of the firm, (ii) eliminate productive inefficiencies deriving from the organization of production, and (iii) we establish a link between the real sector and the financial sector of the economy. This result suggests a reexamination of the validity of the Modigliani and Miller theorem. (iv) Preliminary work on the Modigliani and Miller theorem suggests that it does not generally hold in the model introduced in this thesis.

Chapter 5

Decentralization of the Objective Function of the Firm

5.1 Introduction

This chapter formalizes some of the ideas introduced by means of simple examples in the previous chapter. The primary goal of this chapter is to elaborate on the long run profits maximization assumption introduced in chapter 3, and to formally introduce the objective function of the firm. For this purpose, we relax the long run profits maximization assumption of chapter 3, and define an extensive form equilibrium. This requires the firm to issue stocks in period one and to purchase capital. Total capital acquired determines the firm's production capacity described by a production function. This function in turn, describes the production set available to the firm in the next period in each state of the world for a particular feasible financial policy and given a set of states of the world. This economic intuition sketches the proof by construction of the first result of this chapter, -a decentralization theorem. This result formally introduces the objective function of the firm, and shows the independence of the objective function of the firm from any utilities of the owners of the firm. As a consequence of this independence of any extra information, firms do not play a

Nash equilibrium strategy but maximize their profits in a very traditional sense. This result rehabilitates the objective of the firm of the classical Arrow-Debreu model, where firms are profit maximizers.

Existence of equilibria for this model needs to be verified. For that, we show allocational equivalence between the reduced form- and the extensive form equilibrium. This equivalence result closes the missing gap in the existence proof of chapter 3, where existence of equilibrium was shown under the assumption of long run profits maximization. By the allocational equivalence result, we know that equilibria exist for the extensive form of the stock market model introduced in this chapter.

While much of the GEI literature models incomplete markets by hypothesis, we show that in our set up, incomplete markets is a consequence of the assumption of technological uncertainty introduced. This contributes to the literature, where predominantly the source of uncertainty considered is aggregate risk. The proof is based on the idea of market entrance, where firms enter the market as long they find positive long run profits opportunities. This result determines the optimal number of endogenous assets in the economy. This improves on GEI models with fixed number of financial assets.

The final part of this paper deals with an equivalence study between the class of centralized GEI models of production and the model introduced in this chapter. We reduce both models to exchange economies and compare equilibrium allocations of these pseudo exchange economies. It is shown that equilibrium allocations are generally different. This result follows from the properties of the model of the firm introduced, where the objective function of the firm is independent of any utility assigned to it. This result suggest that the way production is organized is non-trivial. Utility maximizing firms introduce a further source of inefficiency due to inefficient organization of production.

5.2 The objective function of the firm

The early research on the objective function of the firm in general equilibrium with incomplete markets is concentrated around following question: Do spot and equity prices provide firms with enough information to deduce what the appropriate objective function of the firm should be? The classical GEI with production literature replies no to this question. It then adds the suffix, not without further extra information ([39],[20],[26], and others). This additional information comes from the group of owners of the firm. We label the objective functions related to this research line, PO, and the corresponding economic model $\mathcal{E}_{PO}$[1]. This model of the firm is introduced in subsection (5.2.1)

We then ask a different question about what the objective function of the firm should be? What role do financial assets play in determining the production set of the firms, and what is the precise nature of the objective function in determining these sets? The objective function associated with this research line is labeled CO, and the corresponding model $\mathcal{E}_{CO}$. The introduction of this nomenclature allows us to use acronyms once we compare the models, and to economize on plain text. The model of the CO objective function firm is introduced in subsection (5.2.2).

5.2.1 Firms with (PO) objective functions

Let a firm in the standard GEI model maximize its present value of future income streams

$$\beta_j D_j = \sum\nolimits_{s=0}^{S} \beta_j(s) D_j(s), \text{ for all } j \in \{1, ..., n\} \tag{5.1}$$

[1]The labeling is arbitrary

satisfying the equation

$$\beta_j D_j = \beta_j \cdot (p \Box y_j)\,,\ \text{for all } j \in \{1, ..., n\} \tag{5.2}$$

for $\beta_j = (1, \beta_j(1), ..., \beta_j(S)) \in \mathbb{R}^{S+1}_{++}$, for all $j \in \{1, ..., n\}$. Let a firm's maximization problem be to

$$(\bar{y}_j) \underset{y_j \in Y_j}{\in} \arg\max \left\{ \bar{\beta}_j \cdot (\bar{p} \Box y) \right\},\ \text{for all } j \in \{1, ..., n\}, \tag{5.3}$$

where a net activity vector $y_j = (y_j(0), ..., y_j(S))$ is an element of the two period production set, $Y_j = \mathbb{R}^{l(S+1)}$. Let $\Pi = \begin{bmatrix} p(0) \cdot y_j(0) - q_j & ... & p(0) \cdot y_n(0) - q_n \\ p(1) \Box y_j(1) & ... & p(1) \Box y_n(1) \end{bmatrix}$ denote the full payoff matrix of order $((S+1) \times n)$. For β_j satisfying $\beta_j \Pi = 0 \Leftrightarrow \beta_j \in \langle \Pi \rangle^{\perp} \cap \mathbb{R}^{S+1}_{++}$, for all $j \in \{1, ..., n\}$, the study of the theory of the firm in incomplete markets reduces to the examination of how to determine β_j, the additional market information the firm needs to guarantee a well defined objective function. Since generally $\beta_i \neq \beta_j$, where $\beta_i \neq \beta_{i'} \Leftarrow \beta_i \Pi = 0$, and $S - n > 0$, for all $i \in \{1, ..., m\}$, criteria for additional market information can be derived from the Hicks Kaldor sum. Let

$$\bar{\beta}_j = \sum_{i=1}^{m} \bar{\theta}_{ij} \bar{\beta}_i,\ \text{for all } j \in \{1, ..., n\} \tag{5.4}$$

where θ_{ij} is the proportion of ownership of firm $j \in \{1, ..., n\}$ hold by individual $i \in \{1, ..., m\}$ after trade at the stock market occurred (Drèze criterion).

The other criterion

$$\bar{\beta}_j = \sum_{i=1}^{m} \bar{\xi}_{ij}\bar{\beta}_i, \text{ for all } j \in \{1, ..., n\} \tag{5.5}$$

represents a present value vector for firm $j \in \{1, ..., n\}$, where ξ_{ij} is the proportion of ownership of firm $j \in \{1, ..., n\}$ hold by individual $i \in \{1, ..., m\}$ before trade at the stock market takes place (Grossman and Hart criterion). Both criteria imply some notion of firms acting in the interest of shareholders (Ownership implies control). The additional information needed by firms, $\bar{\beta}_j$, is therefore, provided by the owners of the firm. The consequence of this derivation of the objective function of the firm is that, it comes at cost of centralizing decisions, and firms maximizing average utilities depending on the criterion utilized. The notion that ownership implies control is not independent of the choice of the criterion applied. In addition, it implies that share holders directly intervene into operational activities of the enterprise by directly controlling the net activity vector of the firm.

5.2.2 Firms with (CO) objective functions

We state a different research question about what the objective function of the firm should be, and ask: "What role do financial assets play (here stocks only) in determining the production set of the firm"? How is the production set of the firm determined, and what structure does the optimization problem of the firm have in order to determined this set? These questions are all related to the the problem of endogenizing asset structures, and the role of the objective function of the firm in this process.

Some of the examples presented in chapter 4 illustrate problems associated with the utility maximization approach introduced in the early 80's by Drèze

[21], Grossman Hart [28]. They suggest that the strategic game nature of the problem of the firm could be resolved by assigning utilities to firms. In order to derive a closed from solution of the model they assigned some average utility to the firms. The question of what utility to assign to the firm is to some extend still an open question. Many suggestions have been proposed. Beyond those of the early contributors are, the utility of the manager, the utility of the board of directors for example or the average utility of any other influential groups. What all these models have in common is that they assign a present value vector β_j, determined by those who control the firm, to the firm. One approach in resolving the problem of the firm is to define an objective function which is independent of this β_j. This is subject of this section of the thesis.

We therefore, construct an alternative theory of the firm, where each firm maximizes its long run profits by taking financial and real decisions sequentially, and independently of any utility assigned to it. The reduced form objective function is formally introduced in equ. (3.1) in chapter 3. In that model, the period two production set available to the firm in each state of nature is taken as given. Given that set, each firm chooses net activities maximizing long run profits. The assumption of long run profits maximization implicitly implies some short- and long run activity of the firm. The long run activity of the firm is to build up the production set available to it in period two. For that, each firm issues stocks in period one, buys capital, and builds up production capacity. The firms's short run activity in period two is then to chose inputs of production, given the production set available to it, such that profits are maximized. This suggests a sequential optimization structure of the firm where the efficient boundary of the real asset structure is not independent of of the choice of financial quantities chosen by the firm. Operational activities and decisions are left to the management of the firm.

The first result of this section formalizes this idea. It shows that the net activity of the firm in period two is independent of any extra information provided by the utility of shareholders, but not independent of the firm's level of capital acquired in period one. The independence of any utility assigned to the firm implies the decentralization property of the objective function. This result improves on Drèze [21]. Another economic implication of this result is that it allows a new interpretation of ownership and control. In the classical GEI model, ownership implies control over the net activity of a firm. This concept has various drawbacks. For example, stock holders would have to decide at the shareholders meeting on the future net activity of the firm. This is costly, requires managerial understanding and operational participation of the share holders, and perhaps requires some kind of voting process as decision mechanism. In the $\mathcal{E}_{CO}$ model, stockholders do not control the net activity of the firm, but control the total level of production capacity available to a profit maximizing firm.

Theorem 5.1 *For every producer* $j \in \{1, ..., n\}$*, the period two net activity vector* $y_j(s)$ *in available production set* $Y_j|_Z(s)$ *is independent of any present value vector* β_i *for all* $i \in \{1, ..., m\}$*, and* $s \in \{1, ..., S\}$*.*

Proof 5.1 *Using assumptions* $(3.1(F))$ *and* $(\mathbf{??}(T))$*, let* $q_j(i) = \sum_{s=1}^{S} \beta_i(s)\,[p(s) \cdot y_j(s)]$*, where* β_i *denotes* i*'s marginal evaluation of one additional unit of future income for* $\beta_i(s) \neq \beta_{i'}(s) \Leftarrow \beta_i\hat{\Pi} = 0$ *at* $S > n$*, for all* $i \in \{1, ..., m\}$ *, where*

$$\hat{\Pi}(p_1, q, y) = \begin{bmatrix} -q_1 & \cdots & -q_n \\ p(1) \cdot y_1(1) & \cdots & p(1) \cdot y_n(1) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p_1(S) \cdot y_n(S) \end{bmatrix}$$

represents the full payoff matrix of order $((S + 1) \times n)$ *.*

Denote $z_i(j)$ *the consumer's* $i \in \{1, ..., m\}$ *demand of quantity of stocks* $j \in \{1, ..., n\}$

evaluated at $q_j(i)$ *and* β_i.

Now, let

$$\sum\nolimits_{i=1}^{m} \bar{z}_i(j) = \bar{z}_j \textit{ for all } j \in \{1, ..., n\} \Rightarrow \bar{q}$$
$$\sum\nolimits_{i=1}^{m} (\bar{x}_i(0) - \tilde{\omega}_i(0)) = 0 \Rightarrow \bar{p}(0)$$
$$\sum\nolimits_{i=1}^{m} (\bar{x}_i(s) - \omega_i(s)) = \sum\nolimits_{j=1}^{n} \bar{y}_j(s), \;\; \forall s \in S \Rightarrow \bar{p}(s)$$

Consider $t = 0$ *optimization problem for* $i \in \{1, ..., m\}$ *and* $j \in \{1, ..., n\}$ *. For given equity prices system* $\bar{q}$ *and spot prices system* $\bar{p}(0)$*, let each consumer* $i \in \{1, ..., m\}$

$$\underset{x(0) \in B_{z_i}, z_i}{max} \; u_i(x_i; z_i) \Rightarrow \bar{x}_i(0), \bar{z}_i$$

and each producer $j \in \{1, ..., n\}$

$$\underset{z(j)}{max} \; \bar{q}z(j) : \bar{q}z(j) = \bar{q} \sum\nolimits_{i=1}^{m} \bar{z}_i(j) \Rightarrow \bar{z}(j).$$

Given maximum quantity of stocks producer $j \in \{1, ..., n\}$ *can sell on the stockmarket,* $\bar{z}(j)$*, the problem of the producer* $j \in \{1, ..., n\}$ *is then to purchase capital* $y(0)$*. The problem of produce* $j \in \{1, ..., n\}$ *is then to maximize the level of capitalat at given spot price system* $\bar{p}(0)$*, and given cash acquired by issuing stocks,* $\bar{q}\bar{z}(j) = \bar{M}_j$*. Let*

$$\underset{y(0)_j}{max} \; \bar{p}(0) y_j(0) : \bar{M}_j = \bar{p}(0) y_j(0) \Rightarrow \bar{y}_j(0).$$

Let the level of capital $\bar{y}_j(0)$*, at financial policy* $\bar{z}(j)$ *determines a correspondence* $\Phi|_z$ *. This correspondence maps* $\mathbb{R}^m_-$ *into* $\mathbb{R}^n_+$ *for every state of nature* $s \in \{1, ..., S\}$ *. This correspondence describes the production set available to the firm in period two, denoted* $Y_j \,|_{\bar{z}}(\, s)$ *for all* $s \in \{1, ..., S\}$*. Consider* $t = 1$ *optimization problem for* $i \in \{1, ..., m\}$ *and* $j \in \{1, ..., n\}$ *. For given* $\bar{p}(s)$*, each consumer* $i \in \{1, ..., m\}$

$$\underset{x(s) \in B_{z_i}}{max} \; u_i(x_i; \bar{z}_i) \Rightarrow \bar{x}_i(s),$$

and each producer $j \in \{1, ..., n\}$*, given his endogenized production set* $Y_j\,|_{\bar{z}}(s)$ *for all* $s \in \{1, ..., S\}$

$$\underset{y_j(s)}{max}\,\{\bar{p}(s) \cdot y_j(s) : y_j(s) \in Y_j\,|_{\bar{z}}\,(s)\} \Rightarrow \bar{y}_j(s).$$

We have constructed a sequential two argument linear objective function with endogenous asset structure

$$\underset{(\bar{z},\bar{y}(s),\bar{y}(0))_j}{\arg\max} \left\{ \bar{q}z_j + \bar{p}\Box y_j \;\middle|\; \begin{array}{l} y_j(s) \in Y_j\big|_z\,(s) \\ \bar{q}\sum_{i=1}^{m} \bar{z}_i(j) \geq \bar{q}z_j = \bar{p}(0)y_j(0) \quad \forall s \in S \end{array} \right\}, \tag{5.6}$$

which is independent of $\bar{\beta}_j$*, and consequently the choice of* y_j *is independent of all* $i \in \{1, ..., m\}$ *.*

The result shows, by construction, that the objective function of profit maximization is well defined, and independent of any assigned present value vector of a group of owners of the firm to it. This decentralizes the objective function. The result follows from the way financial assets are introduce into the model.

5.2.3 Extensive form equilibrium definition

We now relax the assumption of long run profit maximization in the equilibrium definition of chapter 3 and formally introduce the sequential model of the firm. Here, the firm's problem is to acquire capital via stock market in period one, and then, given production capacity and a set of states of nature, each firm faces a well defined profit maximization problem in the second period, similar to the Arrow-Debreu model. The main difference to the private ownership firm introduced by Debreu [12] is that in the (EFE) model, each firm takes financial and real quantity decisions sequentially, and production sets are endogenously determined by level of capital a firm can buy by issuing stocks.

Definition 5.1 (EFE) *An extensive form stock market equilibrium* $(\bar{p}, \bar{q})$ *with associated equilibrium allocations* $((\bar{x}, \bar{z}), (\bar{y}, \hat{z}))$ *for generic initial resources* $\omega \in \Omega$*, and each producer* $j \in \{1, ..., n\}$ *maximizing long run profits satisfies:*

$$\textit{(i)} \quad (\bar{x}_i, \bar{z}_i) \arg\max \left\{ u_i(x_i; z_i) : x_i \in B_{z_i},\ z_i \in \mathbb{R}^n_+ \right\}, \forall i \in \{1, ..., m\}$$

$$\textit{(ii)} \quad \underset{(\bar{y}(s), \bar{z}, \bar{y}(0))_j}{\arg\max} \left\{ \bar{q} z_j + p \square y_j : \begin{array}{c} \bar{q} \sum_{i=1}^m \bar{z}_i(j) \geq \bar{q} z_j = \bar{p}(0) y_j(0) \\ y_j(s) \in \left. Y_j \right|_{\hat{z}} (s)\ \forall s \in \{1, ..., S\} \end{array} \right\} \forall j$$

$$\textit{(iii)} \quad \sum_{j=1}^n \sum_{i=1}^m \bar{z}_i(j) = 0,\ \sum_{i=1}^m \theta_j(\bar{z}_i) = 1, \forall j \in \{1, ..., n\}$$

$$\bar{x}(0) = \omega(0) + \sum_{j=1}^n \bar{y}_j(0)$$

$$\bar{x}(s) = \omega(1) + \sum_{j=1}^n \bar{y}_j(s) \quad \forall\ s \in \{1, .., S\}.$$

Chapter 3 showed that equilibrium exists generically for this economic model under the assumption of long run profits maximization and reduced form model of the firm. It therefore, remains to be shown the equivalence between the reduced form equilibrium model of chapter 3 and the extensive form equilibrium model of the firm introduced in this chapter. This result is presented after we show that incomplete markets is a consequence of the assumption of technological uncertainty. The result (5.2) has an alternative interpretation. It is some preliminary version of the Modigliani and Miller theorem suggesting the potential irrelevance of the financial policies theorem [43], [14], [52], and others.

Proposition 5.1 $n < S \Longleftarrow \sum_j Y_j \,|_{\bar{z}}$

Proof 5.2 (Proposition 5.1) *Let* $S_j = 1$ *for every* $j \in \{1, ..., n\}$*, and* $\sum_j S_j = \mathcal{S}$*. Then long run profit prospects* $\pi(p) > 0$ *imply long run capacity adjustment and market entrance until* $n = S$*. Similar for negative long run prospects, the number of firms decreases until* $n = S$*, and* $\pi(p) = 0$ *satisfied. This violates assumption* (T)*. Let* $S > 1$ *for every* $j \in \{1, ..., n\}$*, and* $\sum_j S_j = \mathcal{S}$*. Then* $\pi(p) > 0$ *implies market entrance* *and the issue of new securities such that in the limit as* $\pi(p) \to 0$ *the number of firms increases until* $j \to n < S$ *by assumption* (T). *Similar for* $\pi(p) < 0$, *firms exit the market and as* $\pi(p) \to 0$ *the number of firms decreases until* $j \to n < S$ *by assumption* (T), *and* $\pi(p) = 0$ *satisfied.*

Theorem (5.2) establishes the missing gap in the existence proof in chapter 3, where existence of equilibrium was shown under assumption 3.3(P) of long run profits maximization for the reduced form model of the firm. Here, we show that every reduced form equilibrium is an extensive form equilibrium and vice versa. Hence, by the existence theorem of chapter 3, equilibria exist. The other interpretation of this result is that it suggests real allocational effects for different feasible financial policies of the firm, $z(j)$. Consequently, financial policies are non-neutral. This is the simplest version of the Modigliani and Miller theorem of irrelevance of financial policies. This result, however, is not complete at this stage, and a full version of the Modigliani and Miller theorem needs to be studied. This involves expanding the model of the firm to a more general endogenous asset structure formation model, including bonds, options,and other financial assets.

Theorem 5.2 *(i) If* $(\bar{p}, \bar{q})$ *is an extensive form equilibrium with associated equilibrium allocations* $((\bar{x}|_{\hat{z}}, \bar{z}), (\bar{y}|_{\hat{z}}, \hat{z}))$ *(DEFE) for generic initial resources* $\omega \in \Omega$, *then* $(\bar{p}, \bar{q})$, *is a reduced form equilibrium with associated equilibrium allocations* $((\bar{x}|_{\bar{\xi}}, \bar{\xi}), (\bar{y}|_{\bar{\xi}}))$ *(DRFE) for generic initial resources* $\omega \in \Omega$ *where*

$$\sum_{i=1}^{m} \bar{\xi}_{ij} = \sum_{i=1}^{m} \bar{z}_i(j) - \hat{z}_j \text{ for } j = 1, ..., n \tag{5.7}$$

(ii) If $(\bar{p}, \bar{q})$ *is a reduced form equilibrium with associated equilibrium allocations* $((\bar{x}|_{\bar{\xi}}, \bar{\xi}), (\bar{y}|_{\bar{\xi}}))$ *(DRFE) for generic initial resources* $\omega \in \Omega$, *then* $(\bar{p}, \bar{q})$, *is an extensive form equi-*

librium with associated equilibrium allocations $((\bar{x}|_{\hat{z}}, \bar{z}), (\bar{y}|_{\hat{z}}, \hat{z}))$ *(DEFE) for generic initial resources* $\omega \in \Omega$, *for any* $\hat{z}_j \leq \sum_{i=1}^{m} \bar{z}_i(j)$ *for* $j = 1, ..., n$ *satisfying*

$$\sum\nolimits_{i=1}^{m} \bar{z}_i(j) - \hat{z}_j = \sum\nolimits_{i=1}^{m} \bar{\xi}_{ij} \text{ for } j = 1, ..., n. \tag{5.8}$$

Lemma 5.1 $\bar{x}_i|_{\bar{\xi}}$ *is a solution of the reduced form problem*

$$max \left\{ u(x;\xi)_i : x_i|_{\bar{\xi}} \in B_{\xi_i} \right\} \tag{5.9}$$

if and only if, $\bar{x}_i|_{\bar{\xi}} \in B_{\xi_i}$, *and*

$$\partial u(\bar{x}_i|_{\bar{\xi}})_i \cap N_{B_{\xi_i}}(\bar{x}_i|_{\bar{\xi}}) \neq \{0\} \tag{5.10}$$

is satisfied.

Proof 5.3 (Lemma 5.1) *(i)* $\bar{x}_i|_{\bar{\xi}}$ *is a solution of utility max (RFE) if and only if* $\bar{x}_i|_{\bar{\xi}} \in B_{\xi_i}$ *and*

$$intU_{i,\bar{x}_i|_{\bar{\xi}}} \cap B_{\xi_i} = \varnothing.$$

By the separation theorem for convex sets (appendix), there exists $P = \beta_i p \in R_{++}^{l(S+1)}, P \neq 0$ *such that*

$$H_P^- = \left\{ x_i|_{\bar{\xi}} \in R_{++}^{l(S+1)} : P\, x_i|_{\bar{\xi}} \leq P\, x_i|'_{\bar{\xi}}, \forall\, x_i|_{\bar{\xi}} \in B_{\xi_i}, \forall\, x_i|'_{\bar{\xi}} \in intU_{i,\bar{x}_i|_{\bar{\xi}}} \right\}$$

since $\bar{x}_i|_{\bar{\xi}} \in B_{\xi_i}$,

$$H_P^- = \left\{ x_i|_{\bar{\xi}} \in R_{++}^{l(S+1)} : P\, \bar{x}_i|_{\bar{\xi}} \leq P\, x_i|'_{\bar{\xi}}, \forall\, x_i|'_{\bar{\xi}} \in intU_{i,\bar{x}_i|_{\bar{\xi}}} \right\}.$$

By continuity of utility, $intU_{i,\bar{x}_i|_{\bar{\xi}}} = U_{i,\bar{x}_i|_{\bar{\xi}}}$, *and by continuity of the scalar product,*

$$H_P^+ = \Big\{\forall\; x_i|_{\bar{\xi}}' \in U_{i,\bar{x}|_{\bar{\xi}}} : P\;\bar{x}_i|_{\bar{\xi}} \le P\;x_i|_{\bar{\xi}}', \forall\; x_i|_{\bar{\xi}}' \in U_{i,\bar{x}_i|_{\bar{\xi}}}\Big\} \quad \Leftrightarrow \quad P \in \partial u(\bar{x}_i|_{\bar{\xi}})_i$$
$$H_P^- = \Big\{\forall\; x_i|_{\bar{\xi}} \in B_{\xi_i} : P\;x_i|_{\bar{\xi}} \le P\;\bar{x}_i|_{\bar{\xi}},\Big\} \quad \Leftrightarrow \quad P \in N_{B_{\xi_i}}(\bar{x}_i|_{\bar{\xi}})_i$$

hence, there exists p such that $\partial u(\bar{x}_i|_{\bar{\xi}})_i \cap N_B(\bar{x}|_{\bar{\xi}})_i \neq \{0\}$ *is satisfied.*

(ii) Suppose that $\bar{x}_i|_{\bar{\xi}} \in B_{\xi_i}$*, and there exists* $P \in \partial u(\bar{x}_i|_{\bar{\xi}})_i \cap N_{B_{\xi_i}}(\bar{x}_i|_{\bar{\xi}})_i$, $P \neq 0$. *If* $\bar{x}_i|_{\bar{\xi}}$ *is not a solution of the (RFE) utility maximization problem, then there exists* $\bar{x}_i|_{\bar{\xi}}\prime \in intU_{i,\bar{x}_i|_{\bar{\xi}}} \cap B_{\xi_i}$. *Since* $P \in \partial u(\bar{x}_i|_{\bar{\xi}})_i$*, we have*

$$P\;x_i|_{\bar{\xi}}\prime > P\;\bar{x}_i|_{\bar{\xi}}$$

But since $P \in N_{B_{\xi_i}}(\bar{x}_i|_{\bar{\xi}})$ *and* $\bar{x}_i|_{\bar{\xi}}\prime \in B_{\xi_i}$*, it follows that* $P\;\bar{x}_i|_{\bar{\xi}}\prime \le P\;\bar{x}_i|_{\bar{\xi}}$ *which contradicts that* $\bar{x}_i|_{\bar{\xi}}\prime$ *is preferred to* $\bar{x}_i|_{\bar{\xi}}$.

Lemma 5.2 $\bar{y}_j|_{\bar{\xi}}$ *is a solution of*

$$max\Big\{\Pi(p;\xi)_j : \bar{y}_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}\Big\} \tag{5.11}$$

if and only if, $\bar{y}_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}$*, and*

$$\partial\Pi(\bar{y}|_{\bar{\xi}})_j \cap Y_j|_{\bar{\xi}}(\bar{y}_j|_{\bar{\xi}}) \neq \{0\} \tag{5.12}$$

is satisfied.

Proof 5.4 (Lemma 5.2) *(i)* $\bar{y}_j|_{\bar{\xi}}$ *is a solution of the (RFE) profit max problem if and only if* $\bar{y}_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}$ *and*

$$int\Pi_{\bar{y}_j} \cap Y_j|_{\bar{\xi}} = \varnothing.$$

By the separation theorem for convex sets (appendix), there exists $p \in \mathbb{R}^{lS}_{++}$, $p \neq 0$ *such that*

$$H_p^- = \left\{ y_j|_{\bar{\xi}} \in R^n : p\, y_j|_{\bar{\xi}} \leq p\, y_j|'_{\bar{\xi}}, \forall\, y_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}, \forall\, y_j|'_{\bar{\xi}} \in int\Pi_{j\,\bar{y}|_{\bar{\xi}}} \right\}$$

since $\bar{y}_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}$,

$$H_p^- = \left\{ y_j|_{\bar{\xi}} \in R^n : p\, \bar{y}|_{\bar{\xi}} \leq p\, y_j|'_{\bar{\xi}}, \forall\, y_j|'_{\bar{\xi}} \in int\Pi_{j\,\bar{y}|_{\bar{\xi}}} \right\}.$$

By continuity of Π_j, $int\Pi_{j,\bar{y}|_{\bar{\xi}}} = \Pi_{j,\bar{y}|_{\bar{\xi}}}$, *and by continuity of the scalar product,*

$$\begin{array}{lll} H_p^+ = \left\{ \forall\, y_j|'_{\bar{\xi}} \in \Pi_{j,y_j|_{\bar{\xi}}} : p\, \bar{y}_j|_{\bar{\xi}} \leq p\, y_j|'_{\bar{\xi}}, \forall\, y_j|'_{\bar{\xi}} \in \Pi_{j,\bar{y}_j|_{\bar{\xi}}} \right\} & \Leftrightarrow & p \in \partial\Pi(\bar{y}_j|_{\bar{\xi}})_j \\ H_p^- = \left\{ \forall\, y_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}} : p\, y_j|_{\bar{\xi}} \leq p\, \bar{y}_j|_{\bar{\xi}}, \right\} & \Leftrightarrow & p \in N_{Y_j|_{\bar{\xi}}}(\bar{y}_j|_{\bar{\xi}})_j \end{array}$$

hence, there exists p such that $\partial\Pi(\bar{y}_j|_{\bar{\xi}})_j \cap N_{j,Y_j|_{\bar{\xi}}}(\bar{y}_j|_{\bar{\xi}})_j \neq \{0\}$ *is satisfied.*

(ii) Suppose that $\bar{y}_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}$, *and there exists* $p \in \partial\Pi(\bar{y}_j|_{\bar{\xi}})_j \cap N_{Y_j|_{\bar{\xi}}}(\bar{y}_j|_{\bar{\xi}})_j, p \neq 0$. *If* $\bar{y}_j|_{\bar{\xi}}$ *is not a solution of the profit maximization problem (RFE), then there exists* $\bar{y}_j|_{\bar{\xi}'} \in int\Pi_{j,\bar{y}_j|_{\bar{\xi}}} \cap Y_j|_{\bar{\xi}}$. *Since* $p \in \partial\Pi(\bar{y}_j|_{\bar{\xi}})_j$, *we have*

$$p\, \bar{y}_j|_{\bar{\xi}'} > p\, \bar{y}|_{\bar{z}}$$

But since $p \in N_{Y_j|_{\bar{\xi}}}(\bar{y}_j|_{\bar{\xi}})$ *and* $\bar{y}_j|_{\bar{\xi}'} \in Y_j|_{\bar{\xi}}$, *it follows that* $p\, \bar{y}_j|_{\bar{\xi}'} \leq p\, \bar{y}_j|_{\bar{\xi}}$ *which contradicts that* $\bar{y}_j|_{\bar{\xi}'}$ *is preferred to* $\bar{y}_j|_{\bar{\xi}}$.

Proof 5.5 (Theorem 5.2) *Part (i). Let us first show that the reduced form equilibrium allocations* $((\bar{x}, \bar{\xi}), (\bar{y}))$ *satisfy the first order conditions (Lemma (5.1))* $\partial u(\bar{x}_i|_{\bar{\xi}})_i \cap N_{B_{\xi_i}}(\bar{x}_i|_{\bar{\xi}}) \neq \{0\}$ *and* $\bar{x}_i|_{\bar{\xi}} \in B_{\xi_i}$, *and (Lemma(5.2))* $\partial\Pi(\bar{y}|_{\bar{\xi}})_j \cap Y_j|_{\bar{\xi}}(\bar{y}_j|_{\bar{\xi}}) \neq \{0\}$ *and*

$\bar{y}_j|_{\bar{\xi}} \in Y_j|_{\bar{\xi}}$, *so that the conditions (i) and (ii) in the* (RFE) *are satisfied. The first order conditions for the consumer's problem in the* (EFE) *are*

$$p \cdot x_i|_{\hat{z}_j} = p \cdot \omega_i + \Pi(\bar{y}, \bar{p}) \begin{bmatrix} \bar{z}_i \\ \theta_j\left(\bar{z}_i\right) \end{bmatrix}, \text{and } \beta_i \Pi(\bar{y}, \bar{p}) = 0$$

and can be rewritten as

$$p \cdot \bar{x}_i|_{\bar{\xi}} = p \cdot \omega_i + \Pi(\bar{y}, \bar{p}) \sum\nolimits_{j=1}^{n} \xi_{ij}, \text{and } \beta_i \Pi(\bar{y}, \bar{p}) = 0$$

since $\bar{\xi}_i\big|_{\hat{z}} = \sum_{i=1}^{m} \bar{z}_i(j) + \hat{z}_j$, *for all* $j = 1, ..., n$, *so that (lemma (5.1)) above holds for any feasible* $\hat{z}_j \leqslant \sum_{i=1}^{m} \bar{z}_i(j)$ *for all* $j = 1, ..., n$. *The firm's problem in* (EFE) *is to*

$$\underset{\left(\bar{y}|_{\hat{z}}(s),(\hat{z};\bar{y}(0))\right)_j}{\arg\max} \left\{ \bar{q}z_j + \sum\nolimits_{s=1}^{S} \bar{p}(s)\ y_j|_{\hat{z}}(s) : \begin{array}{c} \bar{q} \sum\limits_{i=1}^{m} \bar{z}_i(j) \geq \bar{q}z_j = \bar{p}(0) y_j(0) \\ y_j|_{\hat{z}}(s) \in Y_j|_{\hat{z}}(s) \end{array} \quad s \in S \right\}.$$

For any feasible $\hat{z}_j \leqslant \sum_{i=1}^{m} \bar{z}_i(j)$ *the problem of the producer reduces to*

$$\underset{\left(\bar{y}|_{\bar{\xi}}(s)\right)_j}{\arg\max} \left\{ \sum\nolimits_{s=1}^{S} \bar{p}(s)\ y_j|_{\bar{\xi}}(s) : y_j|_{\bar{\xi}}(s) \in Y_j|_{\bar{\xi}}(s), s \in S \right\}$$

since feasible $\hat{z}_j \Rightarrow \Phi_j|_{\hat{z}}(s) \Longrightarrow Y_j|_{\hat{z}}(s)$ *for all* $s \in S$, *for which the first order conditions (lemma (5.2)) hold. The result follows ,since market clearing condition* $\bar{\xi}_i\big|_{\hat{z}_j} = \sum_{i=1}^{m} \bar{z}_i(j) - \hat{z}_j = 0$, *for all* $j = 1, ..., n$, *and* $\sum_{i=1}^{m} \bar{x}_i|_{\hat{z}_j}(0) = \sum_{i=1}^{m} \omega_i(0) + \sum_{j=1}^{n} \bar{y}_j(0)$, $\sum_{i=1}^{m} \bar{x}_i|_{\hat{z}_j}(s) = \sum_{i=1}^{m} \omega_i(1) + \sum_{j=1}^{n} \bar{y}_j|_{\hat{z}}(s)$ *for all* $s \in S$ *hold.*

Part (ii). If $((\bar{x}, \bar{\xi}), (\bar{y}))$ *is a* (RFE) *for implicit* $\hat{z}_j$, *for* $j = 1, ..., n$, *then the first order conditions are satisfied. This implies that for any feasible* $\hat{z}_j \leqslant \sum_{i=1}^{m} \bar{z}_i(j)$

$$\underset{\left(\bar{y}|_{\bar{\xi}}(s)\right)_j}{\arg\max} \left\{ \sum\nolimits_{s=1}^{S} \bar{p}(s)\ y_j|_{\bar{\xi}}(s) : y_j|_{\bar{\xi}}(s) \in Y_j|_{\bar{\xi}}(s),\ s \in S \right\} \tag{5.13}$$

expands to

$$\underset{\left(\bar{y}|_{\hat{z}}(s),(\hat{z};\bar{y}(0))\right)_j}{\arg\max} \left\{ \bar{q}z_j + \sum\nolimits_{s=1}^{S} \bar{p}(s)\, y_j|_{\hat{z}}(s) : \begin{array}{l} \bar{q}\sum_{i=1}^{m} \bar{z}_i(j) \geq \bar{q}\hat{z}_j = \bar{p}(0)y_j(0) \\ y_j|_{\hat{z}}(s) \in Y_j|_{\hat{z}}(s) \end{array} \quad s \in S \right\}, \tag{5.14}$$

for which the first order conditions are satisfied (Lemma (5.2)), hence $\bar{y}_j|_{\hat{z}}$ *is a solution of (ii) in* (EFE) *for feasible* $\hat{z}_j$*. Pick any feasible* $\hat{z}_j$ *and define*

$$\sum\nolimits_{i=1}^{m} z_i(j) - \hat{z}_j = \bar{\xi}_i, \text{ for all } j = 1, ..., n \tag{5.15}$$

such that $\Pi z_i + \Pi\hat{z}_j = \Pi\bar{\xi}_i$ *becomes* $\Pi(z_i - \hat{z}_j) = \Pi\bar{\xi}_i$*, then the first order conditions for the consumer of the* (EFE) *(Lemma (5.1)) are satisfied for* $(\bar{x}_i|_{\hat{z}_j}, z_i)$*.* $(\bar{x}_i|_{\hat{z}_j}, z_i)$ *is a solution of* (EFE) *(i) and* $(\bar{y}_j|_{\hat{z}}, \hat{z}_j)$ *is a solution of* (EFE) *(ii). The result follows from* $0 = \sum_{i=1}^{m} \xi_i = \sum_{i=1}^{m} z_i + \sum_{j=1}^{n} \hat{z}_j$ *and* $\sum_{i=1}^{m} \bar{x}_i|_{\hat{z}_j} \bar{(0)} = \sum_{i=1}^{m} \omega_i(0) + \sum_{j=1}^{n} \bar{y}_j(0)$, $\sum_{i=1}^{m} \bar{x}_i|_{\hat{z}_j}(s) = \sum_{i=1}^{m} \omega_i(1) + \sum_{j=1}^{n} \bar{y}_j|_{\hat{z}}(s)$ *for all* $s \in S$*.*

5.3 On the equivalence of $\mathcal{E}_E$, $\mathcal{E}_C$, and $\mathcal{E}_P$: The models

The second part of this chapter, sections (5.3) and (5.4), compares the model introduced in definition (5.1) with the classical GEI model of production. For this purpose, this section introduces assumptions, notation, and the asset structure of the classical GEI exchange-, and the classical GEI production model. Again, some nomenclature is unavoidable in order to economize on plain text when considering variations of the basic models, and when making comparisons between them.

The main conclusion of this part of chapter 5 suggests that the way the model of the firm is introduced into general equilibrium with incomplete markets has non-trivial equilibrium consequences. It shows that the organization of production in the decentralized objective function model is more efficient relative to any utility maximization objective function model. This result suggest the organization of production as a potential source of inefficiency. In order to derive this conclusion, we introduce the classical GEI exchange model, $\mathcal{E}_E$, as benchmark model. The idea is to reduce both production models $\mathcal{E}_P$ and $\mathcal{E}_C$ to an exchange model and to compare equilibrium allocations of these exchange economies with the classical GEI exchange model, $\mathcal{E}_E$.

Section (5.3) introduces the mathematical notation, the main assumptions, and the main properties of asset structures and budget sets of the models considered, $\mathcal{E}_E, \mathcal{E}_P, \mathcal{E}_C$, and variations of them. We first introduce the classical pure exchange GEI model, denoted $\mathcal{E}_E$, and then the production model $\mathcal{E}_P$. References to these models can be found at [26], [40].

Section (5.4) collects the results of this part of chapter 5. The main result of subsection (5.4.1) shows the allocational equivalence of the models, $\mathcal{E}_E \iff \mathcal{E}_C$. This result is a consequence of a series of equivalence results, where it is shown that $\mathcal{E}_E \iff \mathcal{E}_{FCE} \iff \mathcal{E}_{FC} \iff \mathcal{E}_C$. The economic relevance of this result follows from the equivalent welfare properties of the models. It implies the efficient organization of production. Subsection (5.4.2) studies a similar comparison between the pure exchange GEI model and the PO objective function model, $\mathcal{E}_P$. The main result of this section concludes that these two models are generally not allocational equivalent, $\mathcal{E}_E \not\Leftrightarrow \mathcal{E}_P$. This result follows by showing that following relations hold $\mathcal{E}_E \not\Leftrightarrow \mathcal{E}_{FPE} \iff \mathcal{E}_{FP} \iff \mathcal{E}_P$. The economic intuition of this result is that it suggest that the way production is organized in this model introduces a source of inefficiency, where the inefficiency comes form the present value vector, β_j, assigned to the firms. The final subsection (5.4.3) concludes that for fixed financial policies $Z_P = Z_C$, allocational equivalence between $\mathcal{E}_C$ and $\mathcal{E}_P$ does not hold, $\mathcal{E}_C \not\Leftrightarrow \mathcal{E}_P$. This result is a consequence of the main results of subsections (5.4.1) and (5.4.2). The degree of productive inefficiency introduced in PO objective function models depends on the weights assigned to the utility maximizing firms.

The final result in subsection (5.4.3) states that the CO objective function model maintains the standard efficiency properties of the pure exchange GEI model. This result is a consequence of the expansion of the decentralization theorem of the Arrow-Debreu model to incomplete markets introduced in the first part of this chapter. Consequently, the organization of productive activity of the CO objective function model does not introduce a further source of inefficiency. This improves on the classical GEI model of production in which the organization of production is generally inefficient [33].

5.3.1 The pure exchange economy, $\mathcal{E}_E$

The classical GEI pure exchange economy consists of an exogenous payoff structure. Denote a column vector of period two returns, measured in unit of account, $V_j = (V_j(1), ..., V_j(S))^T$ for all $j \in \{1, ..., n\}$. $V_j(s)$ denotes the unit payoff of an asset $j \in \{1, ..., n\}$ in uncertain state of the world $s \in \{1, ..., S\}$ in period two. The set of states of nature is exhaustive and elements are mutually exclusive. Denote the full matrix of payoffs of the pure exchange economy

$$\Pi_E(q, V) = \begin{bmatrix} -q_1 & ... & q_n \\ V_1(s) & ... & V_n(s) \\ \vdots & & \vdots \\ V_1(S) & ... & V_n(S) \end{bmatrix}, \tag{5.16}$$

where $q(j)$ is the price of financial asset $j \in \{1, ..., n\}$ in period one. Denote for each individual $i \in \{1, ..., \}$ a budget set

$$B_E(i) = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++}, z_i \in \mathbb{R}^n : p\Box\left(x_i - \omega_i\right) = \Pi z_i \right\}, \tag{5.17}$$

where $x_i \in \mathbb{R}^{l(S+1)}_{++}$ is a consumption bundle in the strictly positive orthant with associated spot price system $p \in \mathbb{R}^{l(S+1)}_{++}$. $z_i \in \mathbb{R}^n$ denotes a vector of financial securities in period one. Denote an initial endowments vector for consumer $i \in \{1, ..., m\}$, $\omega_i = (\omega_i(0), ..., \omega_i(S)) \in \Omega = \mathbb{R}^{l(S+1)}_{++}$. Let $Z_E := z_i(j)$ denote the number of the units of a financial asset $j \in \{1, ..., n\}$ an individual $i \in \{1, ..., m\}$ wants to trade at financial markets equilibrium in period $t = 0$, and let $z_i(j)$ denote the same number of units of a particular asset in period $t = 1$. The sequential optimization problem of the consumer $i \in \{1, ..., m\}$ is to invest into firms in period one in order to smooth out future uncertain consumption and to optimize consumption of goods in every $(S + 1)$ spot market. For a given price system $p = (p(0), p(1), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$ of consumption goods and price system $q \in \mathbb{R}^n_{++}$ of financial assets, a consumer chooses bundles of consumption goods and quantities of stocks $(x, z)_i \in X_i \times \mathbb{R}^n_{++}$ such that $u_i(x_i; z_i)$ is maximized subject to the sequence of $(S + 1)$ constraints in B_{z_i}. Formally

$$(x, z)_i \in \arg\max \{u_i(x_i) : x_i \in B_E(i), z_i \in \mathbb{R}^n\} \, \forall i \in \{1, ..., m\} \tag{5.18}$$

5.3.2 The economy with fixed production plans, $\mathcal{E}_{FC}$

The economy with fixed production plans is an intermediate model of production, where problems associated with the objective function are not explicitly considered. The relation between a net activity vector $y = (y_1, ..., y_n)$ of all producers $j \in \{1, ..., n\}$ and financial asset price vector $q = (q_1, ..., q_n)$ is obtained by solving for an equilibrium of the exchange economy $\mathcal{E}_{FCE}$ with fixed production plans y and running through all possible plans $y \in \Pi^n_{j=1} Y_j = \mathbb{R}^{l(S+1)2}$. Introduce virtual $t = 1$ endowments $\tilde{\omega}_i = (\omega_i(0), \tilde{\omega}_i(s)) \in \mathbb{R}^{l(S+1)}_{++}$, defined by

[2] Here, $\Pi^n_{j=1} Y_j$ is to be understood as the Cartesian product $Y_1 \times Y_2 \times, ..., \times Y_n$.

$$\tilde{\omega}_i(s) = \omega_i(s) + \sum\nolimits_{j=1}^{n} \theta_j\left(z_i\right) y_j \text{ for all } i \in \{1, ..., m\}\,, \tag{5.19}$$

where ownership structure in $t = 1$, θ_{ij} is a function of $t = 0$ portfolio trades $z_i(j)$, defined by

$$\theta_j\left(z_i\right) = z_i(j) / \sum\nolimits_{i=1}^{m} z_i(j) \text{ for all } j \in \{1, ..., n\}\,. \tag{5.20}$$

The budget set of an individual $i \in \{1, ..., m\}$ for the exchange economy with fixed production plans becomes

$$B_{FCE}(i) = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} p(0) \cdot (x_i(0) - \omega_i(0) = -qz_i \\ p(s)\Box(x_i(s) - \tilde{\omega}_i(s)) = V_j(s)z_i \end{array} \right\}, \tag{5.21}$$

where total payoff matrix, Π, is defined by

$$\Pi_{FCE}(p_1, q, V) = \begin{bmatrix} -q_1 & ... & q_n \\ V_1(s) & ... & V_n(s) \\ \vdots & & \vdots \\ V_1(S) & ... & V_n(S) \end{bmatrix}, \tag{5.22}$$

and where $V_j(s)_E = \frac{p(s) \cdot y_j(s)}{\sum_{i=1}^{m} z_i(j)} = V_j(s)_{FCE}$ denotes the period $t = 1$ payoff of asset $j \in \{1, ..., n\}$ in state $s \in \{1, ..., S\}$. The consumer's problem is to

$$(x, z)_i \in \arg\max \left\{u_i(x_i) : x_i \in B_{FCE}(i),\ z_i \in \mathbb{R}^n\right\} \forall i \in \{1, ..., m\} \tag{5.23}$$

5.3.3 The economy $\mathcal{E}_c$ with CO objective functions

The $\mathcal{E}_{CO}$ model is introduced in chapter 3 in its full length. For each $j \in \{1, ..., n\}$ there exists a $t = 1$ column vector of returns $V_j = (V_j(1), ..., V_j(S))^T$, where each payoff asset $V_j(s) = p(s) \cdot y_j(s)$, for $s \in \{1, ..., S\}$, denotes the revenue yield of an endogenized real asset $y_j(s) = (y_j^1(s), ..., y_j^l(s))^T \in \mathbb{R}^l$ at spot price $p(s) = \left(p^1(s), ..., p^l(s)\right) \in \mathbb{R}^l_{++}$. Denote the full matrix of payoffs

$$\Pi_C(q, p_1, y) = \begin{bmatrix} -q_1 & \dots & q_n \\ p(s) \cdot y_1(s) & \dots & p(s) \cdot y_n(s) \\ \vdots & & \vdots \\ p(s) \cdot y_1(S) & \dots & p(s) \cdot y_n(S) \end{bmatrix}. \tag{5.24}$$

Then, for every consumer $i \in \{1, ..., m\}$, denote the budget set

$$B_C(i) = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} p(0) \cdot (x_i(0) - \omega_i(0)) = -qz_i \\ p(s)\Box(x_i(s) - \omega_i(s)) = V(p_1, y)\theta(z_i) \end{array} \right\}. \tag{5.25}$$

The firm's problem is to maximizes long run profits,

$$\underset{(\bar{y})_j}{\in} \arg\max \left\{ \bar{p}(s)\Box y_j(s) \middle| \; y_j \in Y_j\big|_z \quad \forall s \in S \right\} \forall j \in \{1, ..., n\}. \tag{5.26}$$

Let for $Z_C := z_i(j)$ in $t = 0$ denote the demand of individual $i \in \{1, ..., m\}$ of financial asset $j \in \{1, ..., m\}$, issued by firm $j \in \{1, ..., m\}$, and let $t = 1$ ownership be a function of $t = 0$ demand for asset $z(j)$, defined by $\theta_{ij} = z_i(j) \left[\sum_{i=1}^{m} z_i(j)\right]^{-1}$ for all $j \in \{1, ..., m\}$. Let $\sum_{i=1}^{m} z_i(j) = z(j)$ be satisfied for all $j \in \{1, ..., m\}$. The consumer's problem is to

$$(x, z)_i \arg\max \left\{u_i(x_i) : x_i \in B_C(i),\ z_i \in \mathbb{R}_+^m\right\} \forall i \in \{1, ..., m\}. \tag{5.27}$$

5.3.4 The economy $\mathcal{E}_P$ with PO objective functions

This section introduces the classical GEI model with production. The main difference to chapter 3 is the interpretation of the long run profits assumption, leading to a different model of the firm, with three non-trivial consequences. (1) The objective function of the firm is indeterminate without any extra information provided by the owners of the firm. (2) The decentralization property of the standard Arrow-Debreu model fails to hold when markets are incomplete, and (3) for any $\beta_j \neq \hat{\beta}_j$ assigned to the objective function of the firm the organization of production is inefficient.

Assumption (P.2): Each firm $j \in \{1, ..., n\}$ maximizes long run profits.

The classical interpretation of this assumption is that each firm chooses inputs of production in period one with associates period two outputs. This structure of the firm follows from the introduction of two period productions sets. These sets are exogenously determined. The main drawback of this interpretation of assumption $(P.2)$ is that present-value prices are indeterminate for $S > n$, and consequently the objective of the firm indeterminate. This follows from the no-arbitrage condition $\beta_j \Pi = 0$ for all $j \in \{1, ..., n\}$. Drèze [21], and Grossman and Hart [28] solved this problem by assigning some average utility of the initial/final share holders to firms.

Assumptions (F.2): The standard assumptions on production sets introduced by Debreu [12] apply to production sets which are prolonged over two periods.

The sequential, one argument objective function of the firm is defined by

$$\left(\overline{y}_j\right) \underset{y_j \in Y_j}{\in} \arg\max \left\{\bar{\beta}_j \cdot (\bar{p}\Box y_j)\right\}, \text{ for all } j \in \{1, ..., n\}, \tag{5.28}$$

This equation is derived from the more general equation, where financial assets enter the objective function additively, and the no-arbitrage condition for each firm implying $\bar{\beta}_j \Pi = 0$ holds. Then

$$\left(\overline{y}_j, \overline{z}_j\right) \underset{y_j \in Y_j, z_j \in \mathbb{R}_-}{\in} \arg\max \left\{\bar{\beta}_j \cdot (\bar{p}\Box y_j) + \bar{\beta}_j \Pi\right\}, \text{ for all } j \in \{1, ..., n\}, \tag{5.29}$$

where the present-value vector $\bar{\beta}_j$ is derived from the utilities of the shareholders (Drèze, or Grossman and Hart criterion). For each $j \in \{1, ..., n\}$ there exists a $t = 0$ investment value vector $-q_j + p(0) \cdot y_j(0)$, and at $t = 1$ a column vector of returns $V_j = (V_j(1), ..., V_j(S))^T$, where each $V_j(s) = p(s) \cdot y_j(s)$, for $s \in \{1, ..., S\}$. Denote for each producer $j \in \{1, ..., n\}$ a net activity vector $y_j = \left(y_j^m(0) \times y_j^n(s)\right)_{s=1}^S$ in the two period production set $y_j \in Y_j = \mathbb{R}^{l(S+1)}$. Denote the full matrix of payoffs

$$\Pi_P(p,q,y) = \begin{bmatrix} -q_1 + p(0)\cdot y_1(0) & ... & q_n + p(0)\cdot y_n(0) \\ p(s)\cdot y_1(s) & ... & p(s)\cdot y_n(s) \\ \vdots & & \vdots \\ p(s)\cdot y_1(S) & ... & p(s)\cdot y_n(S) \end{bmatrix}. \tag{5.30}$$

A typical budgets set for consumer $i \in \{1, ..., m\}$ is denoted

$$B_P(i) = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++}, \ : \ \begin{array}{l} p(0)\cdot(x_i(0) - \omega_i(0)) + q\xi_i = [-q + p(0)\cdot y_j(0)]\,\theta_i \\ p(s)\Box(x_i(s) - \omega_i(s)) = [p(s)\cdot y_j(s)]\,\theta_i \end{array} \right\}. \tag{5.31}$$

Let for $Z_P := z_i(j) = (\theta_{ij} - \xi_{ij}) \sum_{i=1}^{m} z_i(j)$, denote the number of assets traded by consumer $i \in \{1, ..., m\}$, where $\xi_{ij} = \frac{z_i(j)}{\sum_{i=1}^{m} z_i(j)}$ denotes the initial ownership share, and $\theta_{ij} = \frac{\hat{z}_i(j)}{\sum_{i=1}^{I} \hat{z}_i(j)}$ the final ownership share after trade $z_i(j)$ at $t = 0$ took place. At $t = 0$, after trade on the stock market has taken place, individual $i \in \{1, ..., m\}$ holds a proportion θ_{ij} of $p\Box y_j$ for $s \in \{0, ..., S\}$. The consumer's problem is to choose optimal consumption and optimal ownership share θ_{ij} of $p\Box y_j$ such that for all $i \in \{1, ..., m\}$

$$(x, \theta_j)_i \in \arg\max \{u_i(x_i) : x_i \in B_P(i)\}. \tag{5.32}$$

5.3.5 The economy $\mathcal{E}_P$ with fixed production FP: $\mathcal{E}_{FP}$

The relation between the vector $y = (y_1, ..., y_n)$ and the vector $q = (q_1, ..., q_n)$ is obtained by solving for an equilibrium of the exchange economy $\mathcal{E}_{FPE}$ with fixed production plans y and running through all possible plans $y \in \Pi_{j=1}^{n} Y_j = \mathbb{R}^{l(S+1)n}$, given the PO objective function $\left\{\bar{\beta}_j \cdot (\bar{p}\Box y_j) : y_j \in Y_j\right\}$ for any definition of $\bar{\beta}_j$. Introduce virtual endowments $\tilde{\omega}_i = (\tilde{\omega}_i(0), \tilde{\omega}_i(s))_{s=1}^{S}$, where

$$\tilde{\omega}_i(0) = \omega_i(0) + \sum\nolimits_{j=1}^{n} \theta_{ji} y_j(0) \text{ for all } i \in \{1, ..., m\}, \tag{5.33}$$

$$\tilde{\omega}_i(s) = \omega_i(s) + \sum\nolimits_{j=1}^{n} \theta_{ji} y_j(s) \text{ for all } i \in \{1, ..., m\},\ s \in \{1, ..., S\}, \tag{5.34}$$

with net activity vectors $y_j|_{\bar{\beta}_j}$ not independent of the exogenously chosen criterion, $\bar{\beta}_j$. A net portfolio trades is denoted

$$z_i(j) = (\theta_{ij} - \xi_{ij}) \sum\nolimits_{i=1}^{m} z_i(j) \text{ for all } j \in \{1, ..., n\}. \tag{5.35}$$

The budget set of an individual $i \in \{1, ..., m\}$ for the induced exchange economy with fixed production plans is denoted

$$B_{FPE}(i) = \left\{ x_i \in \mathbb{R}_{++}^{l(S+1)},\ :\ \begin{array}{c} p(0) \cdot (x_i(0) - \tilde{\omega}_i(0)) = -q z_i \\ p(s)\Box(x_i(s) - \tilde{\omega}_i(s)) = V_j(s) z_i \end{array} \right\}, \tag{5.36}$$

where Π is defined as in the same way as in the model $\mathcal{E}_P$, but where $V_j(s)_{FP} = \frac{p(s) \cdot y_j(s)}{\sum_{i=1}^{I} z_i(j)} = V_j(s)_{FPE}$ denotes the payoff of asset $j \in \{1, ..., n\}$ in state $s \in \{1, ..., S\}$.

$$\Pi(p,q,y)_{FPE} = \begin{bmatrix} -q_1 + p(0)\cdot y_1(0) & \dots & q_n + p(0)\cdot y_n(0) \\ V_1(s) & \dots & V_n(s) \\ \vdots & & \vdots \\ V_1(S) & \dots & V_n(S) \end{bmatrix}. \quad (5.37)$$

5.4 On the equivalence of the $\mathcal{E}_E$, $\mathcal{E}_C$, and $\mathcal{E}_P$: (In)-efficient organization of production

This part of the chapter formally introduces the equilibrium definitions of the two models considered and their variations, and then presents a series of equivalence results. The main idea of this study is to show that both models can be reduced to an exchange economy. The two models are allocational equivalent if their derived exchange economies are allocational equivalent to the classical GEI exchange model. (i) We first show that $\mathcal{E}_E \iff \mathcal{E}_C$. This result is a consequence of a series of equivalence results, where it is shown that $\mathcal{E}_E \iff \mathcal{E}_{FCE} \iff \mathcal{E}_{FC} \iff \mathcal{E}_C$. (ii) We then show that, $\mathcal{E}_E \not\Leftrightarrow \mathcal{E}_P$. This result follows by showing that following relations hold: $\mathcal{E}_E \not\Leftrightarrow \mathcal{E}_{FPE} \iff \mathcal{E}_{FP} \iff \mathcal{E}_P$. (iii) It is then shown that for fixed financial policies $Z_P = Z_C$, allocational equivalence between $\mathcal{E}_C$ and $\mathcal{E}_P$ does not hold, $\mathcal{E}_C \not\Leftrightarrow \mathcal{E}_P$. This result is a consequence of the results introduced in subsections (5.4.1) and (5.4.2). It suggests that any utility dependent GEI model of production is organizational inefficient relative to the model introduced in this thesis. In other words, it implies organizational efficiency of the CO objective function model of the firm. (iv) The final result suggests that the $\mathcal{E}_C$ model maintains the standard inefficiency properties of the pure exchange GEI model, $\mathcal{E}_E$.

5.4.1 On the equivalence of $\mathcal{E}_E$, and $\mathcal{E}_C$

This section of the paper introduces the formal definitions of equilibrium of the pure exchange model and of all variations of the CO objective function model, $\mathcal{E}_C$. The results of this model are presented subsequently.

Definition 5.2 *A pure exchange financial markets (FM) equilibrium $\mathcal{E}_E$, is a pair of prices $(p,q) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R}^n_{++}$, and associated actions $(x,z) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{nm}$, for generic initial resources $\omega \in \Omega$, such that conditions (i)-(iii) are satisfied:*

$$\begin{array}{ll} (i) & (\bar{x}_i, \bar{z}_i) \in \arg\max \{u_i(x_i) : x_i \in B_E(i),\ z_i \in \mathbb{R}^n\} \quad \forall i \in \{1, ..., m\} \\ (ii) & \sum_{i=1}^m (\bar{x}_i - \omega_i) = 0 \\ (iii) & \sum_{i=1}^m \bar{z}_i = 0. \end{array} \tag{5.38}$$

Definition 5.3 *A CO objective function financial markets (FM) equilibrium with fixed production plans $y \in \Pi_{j=1}^n Y_j$, $\mathcal{E}_{FC}$, is an equilibrium of the induced exchange economy $\mathcal{E}_{FCE}$, consisting of a pair of prices $(p,q) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R}^n_{++}$ and associated equilibrium allocations $(x,z) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^n$, for generic initial resources $\omega \in \Omega$, such that conditions (i)-(iii) are satisfied:*

$$\begin{array}{ll} (i) & (\bar{x}, \bar{z})_i \in \arg\max \{u_i(x_i) : x_i \in B_F(i),\ z_i \in \mathbb{R}^n\}, \quad \forall i \in \{1, ..., m\} \\ (ii) & \sum_{i=1}^m (\bar{x}_i - \tilde{\omega}_i) = 0 \\ (iii) & \sum_{i=1}^m \bar{z}_i = 0. \end{array} \tag{5.39}$$

Definition 5.4 *A CO objective function financial markets (FM) equilibrium, $\mathcal{E}_C$, consists of pair of prices $(p,q) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R}^n_{++}$, and allocations $(x,z), (y,\hat{z}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}^{lSn} \times \mathbb{R}^n_-$, for generic initial resources $\omega \in \Omega$, such that conditions (i)-(iv) are* *satisfied:*

$$
\begin{array}{ll}
(i) & (\bar{x}_i, \bar{z}_i) \in \arg\max \left\{ u_i(x_i) : x_i \in B_C(i),\ z_i \in \mathbb{R}^n_+ \right\}, \forall i \in \{1, ..., m\} \\
(ii) & \underset{(\bar{y}(s), \bar{z}, \bar{y}(0))_j}{\arg\max} \left\{ \bar{q} z_j + \sum_{s=1}^{S} \bar{p}(s) y_j(s) : \begin{array}{l} \bar{q} \sum_{i=1}^{m} \bar{z}_i(j) \geq \bar{q} z_j = \bar{p}(0) y_j(0) \\ y_j(s) \in Y_j|_{\hat{z}}(s) \end{array} \right\} \\
& \forall j \in \{1, ..., n\} \\
(iii) & \sum_{i=1}^{m} (\bar{x}_i - \omega_i) = \sum_{j=1}^{n} \bar{y}_j \\
(iv) & \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{z}_i(j) = 0, \text{ and } \sum_{i=1}^{m} \theta(\bar{z}_i)_j = 1,\ \forall j \in \{1, ..., n\}.
\end{array}
\tag{5.40}
$$

Proposition (5.2) shows the allocational equivalence between the classical pure exchange GEI economy and the economy with fixed production plans. In such an economy it is implicitly assumed that each firm $j \in \{1, ..., n\}$ chooses the profit maximization net activity y_j automatically. The equivalence result essentially follows from the similar one period asset structure available to the agents in period two. The asset structure is a consequence of the interpretation of the long run profit maximization assumption.

Proposition 5.2 *(i) If (p, q) is a (FM) equilibrium of the exchange economy, $\mathcal{E}_E$, with associated equilibrium allocations (x, z) for generic initial resource $\omega \in \Omega$, and if*

$$\theta_j(z_i) = z_i(j) / \sum_{i=1}^{m} z_i(j) \text{ for all } j \in \{1, ..., n\} \tag{5.41}$$

$$\tilde{\omega}_i(s) = \omega_i(s) + \sum_{j=1}^{n} \theta_j(z_i)\, y_j(s), \text{ for all } i \in \{1, ..., n\} \tag{5.42}$$

$$y \in \Pi_{j=1}^{n} \partial Y_{j,eff} \tag{5.43}$$

are satisfied, then (p, q) is a (FM) equilibrium with fixed production plans, $\mathcal{E}_{FC}$, of the induced exchange economy, $\mathcal{E}_{FCE}$, with associated equilibrium allocations (x, y, z) for $\omega \in \Omega$. (ii) A (FM) equilibrium with fixed production plans $y \in \Pi_{j=1}^{n} Y_j$, $\mathcal{E}_{FC}$, of the induced exchange economy, $\mathcal{E}_{FCE}$, is also a (FM) equilibrium of the exchange economy $\mathcal{E}_E$.

Proposition (5.3) shows the allocational equivalence between the financial markets equilibrium with fixed production plans, $\mathcal{E}_{FC}$, and the CO objective function model, $\mathcal{E}_{C,}$ for the objective of the firm introduced in equation (3.1) for all $j \in \{1, ..., n\}$. The idea of the proof is to take an intermediate step, and to introduce an induced exchange economy, $\mathcal{E}_{FCE}$, where by definition $\mathcal{E}_{FCE} \Longrightarrow \mathcal{E}_{FC}$, and therefore, need to show that, $\mathcal{E}_{C} \Longleftrightarrow \mathcal{E}_{FCE}$. This equivalence then implies $\mathcal{E}_{C} \Longleftrightarrow \mathcal{E}_{FC}$.

Proposition 5.3 *(i) If (p, q) is a (FM) equilibrium with fixed production plans, $\mathcal{E}_{FC}$, with associated equilibrium allocations (x, y, z) for generic initial resources $\omega \in \Omega$, of the induced exchange economy, $\mathcal{E}_{FCE}$, and if*

$$\theta_j\left(z_i\right) = z_i(j)/\sum\nolimits_{i=1}^{m} z_i(j) \text{ for all } j \in \{1, ..., m\} \tag{5.44}$$

$$y \in \Pi_{j=1}^{n} \partial Y_{j,eff} = \sum\nolimits_{j=1}^{n} y_j \in \partial \left. Y_j\right|_{z,eff}, \text{ for all } j \in \{1, ..., n\} \tag{5.45}$$

are satisfied, then (p, q) is a (FM) equilibrium of the economy with CO objective functions, $\mathcal{E}_{C}$, with associated equilibrium allocations (x, y, z) for generic initial resources $\omega \in \Omega$. (ii) And if (p, q) is a (FM) equilibrium of the economy with CO objective functions, $\mathcal{E}_{C}$, with associated equilibrium allocations (x, y, z) for generic initial resources $\omega \in \Omega$, then (p, q) is a (FM) equilibrium with fixed production plans, $\mathcal{E}_{FC}$, with associated equilibrium allocations (x, y, z) for generic initial resources $\omega \in \Omega$, of the induced exchange economy, $\mathcal{E}_{FCE}$.

Proposition (5.2) and proposition (5.3) together imply the main result of this subsection. This result shows the allocational equivalence between the pure exchange GEI model and the CO objective function model. The economic intuition of this result suggests efficient productive organization. This is a consequence of the expansion of the decentralization theorem of the Arrow-Debreu model to incomplete markets proved in the first part of this chapter.

Theorem 5.3 *(i) If* (p, q) *is a* (FM) *equilibrium of the exchange economy,* $\mathcal{E}_E$*, with associated equilibrium allocations* (x, z) *for generic initial resources* $\omega \in \Omega$*, and if*

$$\theta_j\left(z_i\right) = z_i(j) / \sum\nolimits_{i=1}^{m} z_i(j) \text{ for all } j \in \{1, ..., n\} \tag{5.46}$$

$$y \in \Pi_{j=1}^{m} \partial Y_{j,eff} = y_j \in \partial \left. Y_j \right|_{z,eff}, \text{ for all } j \in \{1, ..., n\} \tag{5.47}$$

$$\tilde{\omega}_i(s) = \omega_i(s) + \sum\nolimits_{j=1}^{n} \theta_j\left(z_i\right) y_j, \text{ for all } i \in \{1, ..., m\} \tag{5.48}$$

are satisfied, then (p, q) *is a* (FM) *equilibrium of the economy with CO objective functions,* $\mathcal{E}_C$*, with associated equilibrium allocations* (x, y, z) *for generic initial resources* $\omega \in \Omega$*. (ii) A CO objective functions equilibrium* (p, q) *of* $\mathcal{E}_C$*, with associated equilibrium allocations* (x, y, z) *for generic initial resources* $\omega \in \Omega$*, is also an equilibrium* (p, q) *of the exchange economy,* $\mathcal{E}_E$*, with associated equilibrium allocations* (x, z) *for generic initial resources* $\omega \in \Omega$*, and conditions satisfied.*

5.4.2 On the equivalence of $\mathcal{E}_E$, and $\mathcal{E}_P$

This section introduces all equilibrium definitions and results of the PO objective function model, $\mathcal{E}_P$. Section (5.4.3) then states the two main welfare implications associated with the organization of production in classical general equilibrium models with incomplete markets.

Definition 5.5 *A PO objective function financial markets* (FM) *equilibrium with fixed production plans* $y \in \Pi_{j=1}^{n} Y_j$*,* $\mathcal{E}_{FP}$*, is an equilibrium of the induced exchange economy* $\mathcal{E}_{FPE}$*, consisting of a pair of prices* $(p, q) \in \mathbb{R}_{++}^{l(S+1)} \times \mathbb{R}_{++}^{n}$ *and associated equilibrium allocations* $(x, y, z) \in \mathbb{R}_{++}^{l(S+1)m} \times \mathbb{R}^{l(S+1)n} \times \mathbb{R}^{n}$*, for generic initial resources* $\omega \in \Omega$*,* *such that conditions (i)-(iii) are satisfied:*

$$\begin{array}{ll} (i) & (\bar{x},\bar{z})_i \in \arg\max\left\{u_i(x_i) : x_i \in B_{FP}(i),\ z_i \in \mathbb{R}^n\right\}, \quad \forall i \in \{1,...,m\} \\ (ii) & \sum_{i=1}^m (\bar{x}_i - \tilde{\omega}_i) = 0 \\ (iii) & \sum_{i=1}^m \bar{z}_i = 0. \end{array} \tag{5.49}$$

Definition 5.6 *A PO objective function financial markets (FM) equilibrium, $\mathcal{E}_P$, consists of a pair of prices $(p,q) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R}^n_{++}$, and associated allocations $(x,y,z) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n} \times \mathbb{R}^n$, for generic initial resources $\omega \in \Omega$, such that conditions (i)-(iv) are satisfied:*

$$\begin{array}{lll} (i) & \left(\bar{x}_i, \bar{\theta}_{ij,}\bar{\beta}_{ji}\right) \in \arg\max\left\{u_i(x_i) : x_i \in B_P(i)\right\} & \forall i \in \{1,...,m\} \\ (ii) & (\bar{y})_j \in \arg\max\left\{\bar{\beta}_j \cdot [\bar{p}(s)\Box y_j(s)] : y_j \in Y_j|_{\bar{\beta}_j}\right\} & \forall j \in \{1,...,n\} \\ (iii) & \sum_{i=1}^m (\bar{x}_i - \omega_i) = \sum_{j=1}^n \bar{y}_j & \\ (iv) & \sum_{i=1}^m \bar{\theta}_{ij} = 1,\ \forall j \in \{1,...,n\} & \end{array}, \tag{5.50}$$

where $\bar{\beta}_j = \sum_{i=1}^m \bar{\theta}_{ij}\bar{\beta}_{ji}$, for all $j \in \{1,...,n\}$. (Drèze or Grossman and Hart criterion)

Proposition (5.4) shows allocational non-equivalence between the classical pure exchange GEI economy and the PO objective function economy with fixed production plans. The economic intuition driving this result comes from the exogenously assigned present-value vectors to firms. Implicitly, proposition (5.4) further states that for different measurements of the present value vector $\bar{\beta}_j \neq \tilde{\beta}_j$ for all $j \in \{1,...,n\}$, corresponding net activity vectors of the firms are generally different, $\bar{y}_j|_{\bar{\beta}_j} \neq \tilde{y}_j|_{\tilde{\beta}_j}$. This in turn implies changes in affordable consumption sets for the consumers, and therefore, consumption bundles.

Proposition 5.4 *(i) If (p,q) is a (FM) equilibrium of the exchange economy, $\mathcal{E}_E$, with* *associated equilibrium allocations (x,z) for generic initial resources $\omega \in \Omega$, and if*

$$z_i(j) = (\theta_{ij} - \xi_{ij}) \sum_{i=1}^{m} z_i(j), \text{ for all } j \in \{1,...,n\} \tag{5.51}$$

$$\tilde{\omega}_i(0) = \omega_i(0) + \sum_{j=1}^{n} \theta_{ij} y_j(0), \text{ for all } i \in \{1,...,m\} \tag{5.52}$$

$$\tilde{\omega}_i(s) = \omega_i(s) + \sum_{j=1}^{n} \theta_{ij} y_j(s), \text{ for all } i \in \{1,...,m\} \tag{5.53}$$

are satisfied, then (p,q) is not an equivalent (FM) equilibrium with fixed production plans, $\mathcal{E}_{FP}$, of the induced exchange economy $\mathcal{E}_{FPE}$ with associated equilibrium allocations (x,y,z) for generic initial resources $\omega \in \Omega$. (ii) A (FM) equilibrium with fixed production plans, $\mathcal{E}_{FP}$, of the induced exchange economy, $\mathcal{E}_{FPE}$, is not an equivalent (FM) equilibrium of the exchange economy $\mathcal{E}_E$.

Proposition (5.5) shows the allocational equivalence of a PO objective function model with fixed production plans, $\mathcal{E}_{FP}$, with the PO objective function model, $\mathcal{E}_P$, if the objective of the firm introduced in equation (5.3) is satisfied for chosen criterion β_j for all $j \in \{1,...,n\}$. The idea of the proof is to take an intermediate step, and to introduce an induced exchange economy, $\mathcal{E}_{FPE}$, where by definition $\mathcal{E}_{FPE} \Longrightarrow \mathcal{E}_{FP}$, and therefore, need to show that, $\mathcal{E}_P \Longleftrightarrow \mathcal{E}_{FPE}$. This equivalence then implies $\mathcal{E}_P \Longleftrightarrow \mathcal{E}_{FP}$.

Proposition 5.5 *(i) If (p,q) is a (FM) equilibrium with fixed production plans $y \in \Pi_{j=1}^{n} Y_j = \mathbb{R}^{l(S+1)}$, $\mathcal{E}_{FP}$, with associated equilibrium allocations (x,y,z) for generic initial resources $\omega \in \Omega$, of the induced exchange economy $\mathcal{E}_{FPE}$, and if*

$$z_i(j) = (\theta_{ij} - \xi_{ij}) \sum_{i=1}^{m} z_i(j) \text{ for all } j \in \{1,...,n\} \tag{5.54}$$

$$y \in \Pi_{j=1}^{n} \partial Y_j = y_j \in Y_j|_{\bar{\beta}_j} \text{ for all } j \in \{1,...,n\} \tag{5.55}$$

aresatisfied, then (p,q) is a (FM) equilibrium of the PO objective functions economy $\mathcal{E}_P$ with associated equilibrium allocations (x,y,z) for generic initial resources $\omega \in \Omega$.

(ii) And every PO objective functions equilibrium, $\mathcal{E}_P$ is a (FM) equilibrium of the economy with fixed production plans, $\mathcal{E}_{FP}$, induced by the exchange economy $\mathcal{E}_{FPE}$.

Theorem (5.4) is the main result of this subsection. It is a consequence of proposition (5.4), and proposition (5.5). It states that the classical PO objective function model is not allocational equivalent to the classical pure exchange GEI model. This follows from the objective function of the firm, which is not independent of extra information not contained in market prices such as weighted present-value vectors. The result implicitly implies some degree of organizational productive inefficiency associated with different choices of present-value vectors $\bar{\beta}_j \neq \tilde{\beta}_j$ for all $j \in \{1, ..., n\}$.

Theorem 5.4 *(i) If (p, q) is a (FM) equilibrium of the exchange economy $\mathcal{E}_E$, with associated equilibrium allocations (x, z) for generic initial resources $\omega \in \Omega$, and if*

$$\theta_j (z_i) = z_i(j)/ \sum\nolimits_{i=1}^{m} z_i(j) \textit{ for all } j \in \{1, ..., n\} \tag{5.56}$$

$$y \in \Pi_{j=1}^{n} Y_j = y_{j,\bar{\beta}_j} \in Y_j|_{\bar{\beta}_j} \textit{, for all } j \in \{1, ..., n\} \tag{5.57}$$

$$\tilde{\omega}_i(0) = \omega_i(0) + \sum\nolimits_{j=1}^{n} \theta_{ij} y_j(0) \textit{, for all } i \in \{1, ..., m\} \tag{5.58}$$

$$\tilde{\omega}_i(s) = \omega_i(s) + \sum\nolimits_{j=1}^{n} \theta_{ij} y_j(s) \textit{, for all } i \in \{1, ..., m\} , s \in \{1, ..., S\} \tag{5.59}$$

are satisfied, then (p, q) is not an equivalent (FM) equilibrium of the economy with PO objective functions, $\mathcal{E}_P$, with associated equilibrium allocations (x, y, z) for generic initial resources $\omega \in \Omega$. (ii) A PO objective functions equilibrium, $\mathcal{E}_P$, for generic initial resources $\omega \in \Omega$, is not an equivalent (FM) equilibrium of the exchange economy, $\mathcal{E}_E$, and conditions satisfied.

5.4.3 (In)efficient productive organization

Theorem (5.5) and (5.6) introduce the main results of the analysis of this part of the paper. Theorem (5.5) shows the allocational non-equivalence between the model $\mathcal{E}_P$ and the model $\mathcal{E}_C$. This result follows from the inefficient productive organization of the $\mathcal{E}_P$ model, where each firm maximizes a weighted average utility exogenously assigned to it. At heart of this result is the interpretation of the long run profits maximization assumption, which suggests firms to choose inputs in period one with associated outputs in period two. This assumption is reinterpreted in the $\mathcal{E}_C$ model, where each firm issues stocks in period one, and then, after uncertainty has resolved, takes real decisions subject to installed production sets. This (endogenous asset formation) model of the firm does not only suggest efficient superior organization of economic activities relative to the private ownership model, but further allows a natural way of modeling the financing of production. Theorem (5.6) suggests production efficiency in the sense that no further source of inefficiency is introduced beyond those of the pure exchange GEI economy due to the organization of production. This results follows readily from the decentralization theorem of the objective function introduced earlier in this chapter.

Theorem 5.5 *(i) If (p,q), is a (FM) equilibrium of the economy $\mathcal{E}_C$, with associated equilibrium allocations (x,y,z) for generic initial resources $\omega \in \Omega$, and if conditions*

$$\sum_{i=1}^{m} \bar{\theta}_{ij} = 1 \Longleftrightarrow \sum_{i=1}^{m} \bar{z}_i = 0, \text{ for all } j \in \{1,...,n\} \tag{5.60}$$

$$\bar{\beta}_j = \sum_{i=1}^{m} \bar{\theta}_{ij}\bar{\beta}_i, \text{ for all } j \in \{1,...,n\} \tag{5.61}$$

$$(\bar{y}_j) \in \arg\max \left\{ \bar{\beta}_j \cdot (\bar{p}\Box y) : y_j \in Y_j|_{\bar{\beta}_j} \right\}, \text{ for all } j \in \{1,...,n\} \tag{5.62}$$

$$(\bar{z},\bar{y})_j \in \arg\max \left\{ \bar{p}(s)\Box y_j(s) \left| y_j \in Y_j|_z \right. \right\}, \text{ for all } j \in \{1,...,n\} \tag{5.63}$$

are satisfied, then (p,q) is not an equivalent (FM) equilibrium of the economy $\mathcal{E}_P$, with associated equilibrium allocations (x,y,z) for generic initial resources $\omega \in \Omega$. (ii) An equilibrium of the economy $\mathcal{E}_P$ is not an equivalent (FM) equilibrium of the economy $\mathcal{E}_C$ for conditions satisfied.

The idea of the proof is simple. For Π_C, Π_P and $B_C(i), B_P(i)$ for all $i \in \{1, ...m\}$, show that $B_C(i) \not\Leftrightarrow B_P(i) \Longrightarrow \max_{x_i, z_i} u_i(x_i)_C \not\Leftrightarrow \max_{x_i, \theta_i} u_i(x_i)_P$. Since $\langle \Pi_C \rangle \not\Leftrightarrow \langle \Pi_P \rangle \Longrightarrow B_C(i) \not\Leftrightarrow B_P(i)$, the result follows. The difference in the payoff span follows from the difference in net activity vectors associated with the objective functions, where $y_j|_{\bar{\beta}_j} \not\Leftrightarrow y_j|_{\hat{\beta}_j}$ for different definitions of $\bar{\beta}_j$. Consequently, for same $\bar{z}$, $y_j|_{\bar{z}} \not\Leftrightarrow y_j|_{\bar{\beta}_j}$ and $y_j|_{\hat{\beta}_j}$.

Theorem 5.6 *$\mathcal{E}_C$ is productive superior efficient relative to $\mathcal{E}_P$.*

Efficiency properties of $\mathcal{E}_C$ are those of the standard GEI exchange economy, $\mathcal{E}_E$.

5.5 Conclusion

This chapter has formalized some of the economic intuition derived in chapter 4 by means of simple examples. It formally introduces the full model of the firm for which under the assumption of long run profits maximization existence of equilibrium was shown in chapter 3. A formal study of the objective function of the firm shows that the objective function of the firm is well defined, and independent of extra information provided by the share holders. The extensive form definition of the model of the firm separates the objective of the firm from the objective of the stock holders, and enables a new formalization of the idea that ownership implies control. The equivalence of a reduced form and the extensive form equilibrium has another interesting implication. In its simplest form, it says that financial policy of the firm is non-neutral. This follows from the role financial assets (stocks) play.

The model considers idiosyncratic risk introduced by the assumption of technological uncertainty. It is show that incomplete markets is a consequence of the competitive nature of the model and this assumption. Consequently, the number of financial assets is endogenously determined. This result can be improved by considering endogenous default.

The second part of this chapter considers the (in)efficient organization of production. It essentially shows that the way production is organized in economic general equilibrium with incomplete markets is non-trivial. A main result suggests that utility maximizing firms introduce a further source of inefficiency in the economy. This is not the case for the profit maximization model introduced in this thesis.

Chapter 6

Conclusion

6.1 Concluding remarks and research outlook

A study of incomplete markets is a challenging project. Besides the study of economics, it requires some good working knowledge of some areas of mathematics. However, the fixed time constraint of a PhD course in economics does not facilitate a rigorous study of both fields. Therefore, I'll not be able to claim a complete set of results on the theory of the firm in incomplete markets as the constraints are those of an Economics PhD program. This is certainly not satisfying, given the high barriers to entry into this research field, and the mathematical sophistication. What I can claim however, is an alternative approach to the study of the firm in incomplete markets to the mainstream view of the role of the firm, and to enhance the theory of the firm in a model of production where the objective function of the firm is independent of the utilities of the owners of the firm. Some of the main economic contributions of this approach to the study of production are listed below.

This thesis is my first attempt towards a general equilibrium study of production when financial markets are incomplete. What the research suggests is that the way production is organized in a general equilibrium framework with incomplete markets is non-trivial. The main contribution of the paper is the model of the firm introduced in great detail in chapter 3, and in its extensive form in chapter 5. This model turns out to have a nice property, namely the equivalence of the decentralization property of the Arrow-Debreu model to incomplete markets. This result follows from the particular role financial assets play in this model.

The model of the firm is such that it establishes a link between the financial- and the real sector of the economy. In particular, each firm issues stocks, and purchases capital. The total level of capital a firm can purchase is bounded above by the total cash it is able to acquire by issuing stocks. Given total production capacity and states of nature, each firm has a well defined (endogenously determined) production set. Given that set, the activity of each firm is to choose profit maximizing real quantities at given prices. The objective of the firm of long run profit maximization is formally described by a two argument sequential optimization structure. The arguments are the financial and real quantities, and the sequential structure links the efficient boundary of the production set with the financial policy chosen by the firm.

This non-dichotomy of the financial-, and production sets suggests a reexamination of the Modigliani and Miller theorem. In its simplest form, chapter 4 and 5 show that every extensive form equilibrium can be transformed into a reduced form equilibrium for a fixed feasible financial policy of the firm. However, any small change in financial policy generates associated real effects. This result follows from the fact that the efficient boundary of the production set available to a firm is not independent of its capital accumulated in period one by issuing stocks. This result however, is only a partial study of the Modigliani and Miller theorem.

The full version of the Modigliani and Miller theorem still needs to be studied. This requires to expand the model and to include other financial assets, such as bonds for example. This expansion of the model is research in progress. There are many other practical directions to expand this model. These include going beyond two periods and to study short and/or long lived assets. One particular example could be to introduce this model of the firm into the overlapping generations model. Other interesting directions are related to the field of industrial organization, such as mergers and acquisitions, and strategic interactions. An obvious expansion of this model is to consider endogenous default.

From a mathematical perspective, there are still many more properties of the equilibrium manifold to be proved. These include showing connectedness and path connectedness of the equilibrium manifold, once it has been rigorously established as a smooth $\mathbb{R}^k$ submanifold of $\mathbb{R}^n$. There are some topological properties to be shown, including the ramifying covering over a set of parameters, the study of catastrophes, the characterization of the set of singularities, and many others. Beyond connectedness, it would be interesting to endow the equilibrium manifold with a Riemanian structure, and to study geodesics. One can then think, whether it is possible to define an economically meaningful geodesic. This is not possible in the standard Arrow-Debreu model as firstly remarked by Balasko [6]. However, the model presented here has more structure, and this question therefore, should be taken up again as such a definition is a very important step towards a study of economic policy. First attempts on this front are in progress.

I hope that this thesis pushes the boundaries of the general equilibrium literature by providing new insights into how to model profit maximizing firms in incomplete markets.

Appendix A

Mathematical Proofs

This section collects most of the proofs.

Proof A.1 (Proposition 3.1) *Consider the the classical production set $Y_j \subset \mathbb{R}^{l(S+1)}$ as introduced in [12]. By assumption 3.2 it follows separability of production across states of nature $s \in S$ in period $t = 1$ given capital $y_j(0)$ in period $t = 0$ for every $j \in \{1, ..., n\}$. Then in period $t = 1$ $Y_j|_z \in \mathbb{R}^{lS}$ becomes $Y_j|_z \in \mathbb{R}^{lS} = \{y(1), ..., y(S) \in \mathbb{R}^{lS} | (y(0), y(1), ..., y(S)) \in Y\}$. Clearly, the set $Y_{j1}(y(0))$ is the set of production possibilities at date t=1 given capital at t=0. See Chambers and Quiggin,[10].*

Proof A.2 (Lemma 3.3) *The proof is an immediate consequence of the separation theorem for $((S + 1) \times n)$ matrices in Gale (1960). This asserts that either there exists a portfolio $z \in \mathbb{R}^n_+$ such that $\widehat{\Pi}(p_1, \Phi|_Z)z \geq 0$, or there exists a present value vector $\beta \in \mathbb{R}^{S+1}_{++}$ such that $\beta\widehat{\Pi}(p_1, \Phi|_Z)z = 0$.*

Proof A.3 (Lemma 3.2) *Choose n linearly independent vectors v_j in L, for $j = 1, ..., n$, such that a matrix $A_{(S \times n)}$, a collection of n linearly independent vectors v_j is in the set $\mathcal{A}$ of $(S \times n)$ matrices. Then in the basis $S \in \mathbb{R}^S$, $L = span(A)$. Since $n < S$ by hypothesis, there are n linearly independent columns in A, $rank(A) = n$. Denote the set of permutations $\Sigma = \{1, ..., \sigma, ..., S\}$, and for each permutation $\sigma \in \Sigma$, denote the corresponding permutation matrix Ξ_σ of order $(S \times S)$. We premultiply A with Ξ_σ and* *denote it* $A_{\sigma(S\times n)} = \Xi_\sigma A$.

We need to show that this matrix is invertible and that it locally identifies L *in* $G^n(\mathbb{R}^S)$. *For mathematical convenience, we partition* $A_{\sigma(S\times n)} = \begin{pmatrix} A_{\sigma,1(n\times n)} \\ A_{\sigma,2(S-n\times n)} \end{pmatrix}_{(S\times n)}$.
By invertibility of $A_{\sigma,1(n\times n)}$, *since det* $\left(A_{\sigma,1(n\times n)}\right) \neq 0$ *as indicated by the rank*$\left(A_{\sigma,1(n\times n)}\right) = n$, *we obtain* $\begin{pmatrix} I_{(n\times n)} \\ A_{\sigma(S-n\times n)}A^{-1}_{\sigma,1(n\times n)} \end{pmatrix}$.
Let $U^n_\sigma = \left\{ L \in G^n(\mathbb{R}^S) : \begin{array}{|ll} (i) & \exists\, A_{\sigma(S-n\times n)}A^{-1}_{\sigma,1(n\times n)} \in \mathbb{R}^{(S-n)n} \\ (ii) & \begin{pmatrix} I_{(n\times n)} \\ A_{\sigma(S-n\times n)}A^{-1}_{\sigma,1(n\times n)} \end{pmatrix} \in L \end{array} \right\}$. *For each* $\sigma \in \Sigma$ *we define a homeomorphism* $\varphi^n_\sigma : U^n_\sigma \to \mathbb{R}^{(S-n)n}$ *by* $\begin{pmatrix} I_{(n\times n)} \\ \varphi^j_\sigma(L) \end{pmatrix} \in L$. *The collection of all charts* $\{U^n_\sigma, \varphi^n_\sigma\}^S_{\sigma=1}$ *defines a smooth, compact atlas without boundary for the Grassmanian manifold of dimension* $(S-n)n$. *This manifold is smooth since the overlap of all charts* $\sigma \neq \sigma' \in \Sigma$ *have a smooth coordinate changes* $\varphi^n_\sigma \circ \varphi^n_{\sigma'} : \varphi^n_{\sigma'}(U^n_\sigma \cap U^n_{\sigma'}) \to \varphi^n_\sigma(U^n_\sigma \cap U^n_{\sigma'})$. *The chart transformations are diffeomorphisms. The manifold has no boundary since the set* $\{U^n_\sigma\}_{\forall\, \sigma\in\Sigma}$ *defines an open cover on* $G^n(\mathbb{R}^S)$.

Proof A.4 (Proposition 3.2) ***Observation (1):*** *By lemma (3.3), there exists* $\beta \in \mathbb{R}^S_{++}$ *such that (FE) spot prices at* $\bar{p}$ *can be rescaled such that* $P = \beta\Box\bar{p}$, *then* $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$ *is a* $(\bar{x}, \bar{y}), (\bar{P}, L)$ *equilibrium. Since by definition* $\beta \in \mathbb{R}^S_{++}$ *is* $\left(\frac{\lambda^s}{\lambda^0}\right)_{i=1}$ *of agent 1 at* $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$, *agent 1's consumption bundle is* $\bar{x}_1$, *since* $\nabla u_1(x_1) = P$, *and* $P\Box(x - \omega) = 0$.
On the contrary, if have a $(\bar{x}, \bar{y}), (\bar{P}, L)$ *equilibrium, and* $\bar{z}_2, ..., \bar{z}_m$ *such that (i)* $\sum^m_{i=1}(\bar{x}_i - \omega_i) = \sum^n_{j=1}\bar{y}$, *(ii)* $\left\{\bar{x}_1 \in \mathbb{R}^{l(S+1)}_{++} : P(x - \omega) = 0\right\}$, *(iii)* $(\bar{x}, \bar{y}, \bar{z})$ *solves* $i \geq 2$ *maximization problem for constraints* $B^{FM}_{z_i}$. *Then by defining* $\bar{z}_1 = -\sum^m_{i=2}\bar{z}_i$, *every* $(\bar{x}, \bar{y}), (\bar{P}, L)$ *is a* $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$ *equilibrium.*

Remark: *Since agent 1 faces only the Arrow-Debreu constraints, his behavior is identical in both models.*

Observation (2): *Suppose* $(P, \tilde{L}) \in \Psi^{\rho}$ *are elements of the (FE) pseudo equilibrium manifold, and conditions (i)* $\begin{bmatrix} I & F \end{bmatrix} \hat{V}(P_1, \phi) = 0$, *and (ii)*

$$\begin{bmatrix} Q_{\rho \times (S-n)} \\ F^{Q}_{(n-\rho)\times(S-n)} \end{bmatrix}_{n\times(S-n)} - [L]_{n\times(S-n)} = 0 \text{ hold.}$$

Under these conditions, a consumption bundle $\bar{x}_i$ $(i \geq 2)$ *is feasible under the constraints (ii) in the* ψ *model if and only if* $\bar{x}_i$ $(\forall i)$ *is feasible under the constraints holding with equality in (i) in the (FE) model.*

The next step is then to show that $\tilde{L} = (L^{\perp} \subset \langle \Gamma^{\rho}(P_1, \phi)\rangle^{\perp})$ *exists. Recall* $\Psi^{\rho} = \left\{ \begin{array}{c} (P_1, \langle \Gamma^{\rho}(P_1, \phi)\rangle^{\perp}, L^{\perp}) \in P^{\rho} \times G^{S-n+\rho}(\mathbb{R}^S) \times G^{S-n}(\mathbb{R}^S) : \\ L^{\perp} \subset \langle \Gamma^{\rho}(P_1, \phi)\rangle^{\perp} \end{array} \right\}$. *Let* $e = \langle \Gamma^{\rho}(P_1, \phi)\rangle^{\perp}$ *and* $l = L^{\perp} \subset \langle \Gamma^{\rho}(P_1, \phi)\rangle^{\perp}$. *Relabel an element* $(\hat{P}, \hat{e}, \hat{l})$ *of* Ψ^{ρ} *in the orthogonal basis of* $\mathbb{R}^S$ *such that in the neighborhood of* $\hat{e}$*, the vector space* e *is spanned by the columns of a* $S \times (S - n + \rho)$ *matrix* $\begin{bmatrix} 1_{(S-n+\rho)\times(S-n+\rho)} \\ E_{(n-\rho)\times(S-n+\rho)} \end{bmatrix}$. *Similarly, in the neighborhood of* $\hat{l}$*, the vector space* l *in the same orthogonal basis of* $\mathbb{R}^S$ *is spanned by the columns of a* $S \times (S - n)$ *matrix* $\begin{bmatrix} 1_{(s-n)\times(s-n)} \\ L_{n\times(s-n)} \end{bmatrix}$. *We also rewrite the financial return matrix* $V(.,.)$ *in this basis, such that it becomes* $\hat{V}(P_1, \phi) = \begin{bmatrix} {}_{(n-\rho)\times(n-\rho)} \\ {}_{(S-n+\rho)\times(n-\rho)} \end{bmatrix}$, $S \times (n - \rho)$.

Condition (1): $e = (span(\Gamma^{\rho}(P_1, \phi))^{\perp}$.

Translate $\begin{bmatrix} I_{(S-n+\rho)\times(S-n+\rho)} \\ E_{(n-\rho)\times(S-n+\rho)} \end{bmatrix}_{S\times(S-n+\rho)} \rightarrow \begin{bmatrix} I & E \end{bmatrix}$ *then condition (1) becomes*

$$\begin{bmatrix} I & E \end{bmatrix} \hat{V}(P_1, \phi) = 0. \tag{A.1}$$

Condition (2): $l \in G^{S-n}(\mathbb{R}^S) \subset e \in G^{S-n+\rho}(\mathbb{R}^S)$.

Need to find a matrix Q such that $\begin{bmatrix} I \\ E \end{bmatrix} - \begin{bmatrix} I \\ L \end{bmatrix} = 0$. *We first partition* $\begin{bmatrix} I \\ E \end{bmatrix}$ *such that it becomes* $\begin{bmatrix} 1_{(S-n)\times(S-n)} & 0_{(S-n)\times\rho} \\ 0_{\rho\times(S-n)} & 1_{\rho\times\rho} \\ E'_{(n-\rho)\times(S-n)} & E''_{(n-\rho)\times\rho} \end{bmatrix}$, *then*

$\begin{bmatrix} 1_{(S-n)\times(S-n)} & 0_{(S-n)\times\rho} \\ 0_{\rho\times(S-n)} & 1_{\rho\times\rho}Q_{\rho\times(S-n)} \\ E'_{(n-\rho)\times(S-n)} & E''_{(n-\rho)\times\rho}Q_{\rho\times(S-n)} \end{bmatrix} = \begin{bmatrix} 1_{(S-n)\times(S-n)} \\ 1_{\rho\times\rho}Q_{\rho\times(S-n)} \\ E'_{(n-\rho)\times(S-n)} + E''_{(n-\rho)\times\rho}Q_{\rho\times(S-n)} \end{bmatrix}$

$= \begin{bmatrix} 1_{(S-n)\times(S-n)} \\ Q_{\rho\times(S-n)} \\ E^{Q}_{(n-\rho)\times(S-n)} \end{bmatrix}$. *Q is a* $(\rho \times S - n)$ *matrix. Condition (2) can then be written in terms of Q and E:*

$$\begin{bmatrix} Q_{\rho\times(S-n)} \\ E^{Q}_{(n-\rho)\times(S-n)} \end{bmatrix}_{n\times(S-n)} - [L]_{n\times(S-n)} = 0. \tag{A.2}$$

The final step is then to show that the pseudo equilibrium manifold Ψ^ρ*, parameterized by* P *and* Q *is locally identified by a diffeomorphism* $\Lambda(P, \tilde{L})$*, defined by* $(A.1) \times (A.1) \mapsto \Psi^\rho$*. The partial derivative* $D^{-1}_{P,Q}\Lambda(P, \tilde{L}(Q)$ *exists, moreover, the map is bijective.*

Proof A.5 (Proposition 3.3) *This part of the proof requires some definitions introduced in the main section (3.4). Using the definition of a pseudo equilibrium with production (3.6), let the value of the no-arbitrage stock price system* q *at* $t = 0$ *be defined by* $q = \sum_{s=1}^{S}(\Gamma(P_1, \overline{\phi}))\theta(z_i)_{s=1}^{S}]$*, let* $t = 1$ *spot price system* p_1 *be determined by* $\overline{p}_1 = proj((\frac{1}{\beta(s)})^T \Box P_1(s))$*, and let the* $z_1 = \sum_{i=2}^{m} z_i$*. Recall that there are no discontinuities in* $\theta(z_i)_{s=1}^{S}]$*. The equivalence of a pseudo equilibrium with production and a financial markets equilibrium with production then follows from similar arguments as in Magill and Shafer [39].*

Suppose $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ *is a* ψ *(FE) with production, and* $\bar{x}_1$ *solves (i) in* ψ*. Then agent* $1'$*s maximization problem, by observation (1) above, implies that* $\bar{x}_1$ *solves (i) in the (FE). Using observation (2) above, all other agents,* $i \geq 2$ *solve (ii) in the* ψ *(FE). Therefore, there exist investment portfolios* $\bar{z}_i$*, and asset price vectors* $\bar{q}_j = \sum_{s=1}^{S} \left(\Gamma^{\rho}(\bar{P}_1, \bar{\phi})\right)_j^s$*,* $j = 1, ..., n$*, and by rescaling* P_1 *by* $\beta \in \mathbb{R}^S_{++}$*, a spot price system* $\bar{p} = proj\left(\left(1, \frac{1}{\beta^s}\right)^T \Box \bar{P}\right)_{s=1}^{S}$ *such that* $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ *is a (FE).* $(\bar{x}, \bar{y})$ *is allocational equivalent.*

Proof A.6 (Proposition 3.4) *(i) We prove that the set* $\mathcal{A}^{\rho}_{(S \times n)}$ *of rank* $(n - \rho)$ *reduced matrices* $A^{\rho}_{(S \times n)}$*, for* $1 \leq \rho < n$*, is a submanifold of the smooth manifold defined by the full rank* n *matrices* $A_{(S \times n)}$ *in the set* $\mathcal{A}_{(S \times n)}$*. Since* $\mathcal{A}_{(S \times n)}$ *is homeomorphic to* $\mathbb{R}^{Sn}$*, for* $\mathcal{A}^{\rho}_{(S \times n)} \subset \mathcal{A}_{(S \times n)}$ *the reduced matrices manifold is shown to have codimension* $(S - n + \rho)\rho$*, for* $1 \leq \rho < n$*.*

Consider the open set U *of* $(S \times n)$ *matrices* $\tilde{a} = \left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array}\right]$ *of rank* $(\tilde{a}) = (n - \rho)$ *in the neighborhood of*
$a = \left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array}\right]$ *such that by invertibility of* $\bar{A}$ *in* a*, since*

$det\bar{A} \neq 0$, it remains invertible in $\tilde{a}$. Then $\tilde{a}$ has rank $(n-\rho)$ if and only if the last ρ columns of $\tilde{a}$ are spanned by the first $(n-\rho)$ columns in it. This implies that there exists a matrix $b_{(n-\rho)\times\rho}$ such that $\begin{bmatrix} \bar{B}_{(n-\rho)\times\rho} \\ \bar{D}_{(S-n+\rho)\times\rho} \end{bmatrix} = \begin{bmatrix} \bar{A}_{(n-\rho)\times(n-\rho)} \\ \bar{C}_{(S-n+\rho)\times(n-\rho)} \end{bmatrix} b_{(n-\rho)\times\rho} \Leftrightarrow$ $b_{(n-\rho)\times\rho} = \bar{A}^{-1}_{(n-\rho)\times(n-\rho)}\bar{B}_{(n-\rho)\times\rho}$*, and* $\bar{D}_{(S-n+\rho)\times\rho} = \bar{C}_{(S-n+\rho)\times(n-\rho)}\bar{A}^{-1}_{(n-\rho)\times(n-\rho)}\bar{B}_{(n-\rho)\times\rho}$.
Hence

$$U \cap \mathcal{A}^{\rho}_{(S\times n)} = \left\{ \tilde{a} = \left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array} \right] \in U : \bar{D} - \bar{C}\bar{A}\bar{B} = 0 \right\}.$$

Then, the map

$$U \to \mathbb{R}^{Sn} \simeq \mathbb{R}^{(n-\rho)(n-\rho)} \times \mathbb{R}^{(n-\rho)\rho} \times \mathbb{R}^{(S-n+\rho)(n-\rho)} \times \mathbb{R}^{(S-n+\rho)\rho},$$

$$\left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array} \right] \mapsto \left(\bar{A}, \bar{B}, \bar{C}, \bar{D} - \bar{C}\bar{A}\bar{B}\right),$$

is locally a diffeomorphism, a chart with the property that the set $U \cap \mathcal{A}^{\rho}_{(S\times n)}$ is the set of points such that the last $(S-n+\rho)\rho$ coordinates vanish, and therefore, the reduced matrices manifold $\mathcal{A}^{\rho}_{(S\times n)}$ has codimension $(S-n+\rho)\rho$. In the cases (ii) and (iii) we look at the corresponding elements in the set of reduced matrices $\mathcal{A}^{\rho}_{(S\times n)}$. In (ii), it is easy to see that there are no linear independent mappings, while in (iii) all mappings are linearly independent, and A is non-singular since A is of full rank.

Proof A.7 (Theorem 3.1) *(i) The linear map $D_y\pi_j$ is surjective everywhere in Y_j, and $Image(D_y\pi) + T_y(\mathcal{A}^{\rho}) = T_y(\mathcal{A})$ is satisfied for all $j \in \{1, ..., n\}$. (ii) The surjectivity of the push forward map for all $j \in \{1, ..., n\}$ does not change for any scaling $\beta \in \mathbb{R}^S_{++}$. (iii) Immediate consequence of the transversality theorem for maps to ambient manifolds and submanifolds, Hirsch [32]. Since each set $\pitchfork(\Gamma, \mathcal{A}; \mathcal{A}^{\rho})_j$ is residual for all $j \in \{1, ..., n\}$, their intersection is residual.*

Proof A.8 (Proposition 3.5) *For each $j \in \{1, ..., n\}$, denote the state dependent con-*

volute

$$\Phi(s)_j(\lambda_\sigma * \phi_j(y))_j(s) = \int_{\mathbb{R}^m_-}(\phi(y-\zeta)_j\lambda_\sigma(\zeta)d\zeta)_j(s) \tag{A.3}$$

Can restrict domain of integration to $Intsup(\lambda)$. See (Dieudonné [19]). Let $\lim_{p\to 0} y^p = -\infty$, and let $\lim_{p\to\infty} y^p = 0$. Denote $A = (\{-\infty, 0\})^m \subseteq \mathbb{R}^m_-$. For any $z \in \mathbb{R}^n_+$ there exists $y|_z \in A$. Denote the compact subset associated with any z, $A|_z$. $A|_z \subseteq A$. The the image of the continuous map $\Phi : A|_z \to \partial\widetilde{Y}|_z$ is compact by surjectivity of Φ.

Proof A.9 (Proposition 3.6) *Define for all $j \in \{1, ..., n\}$, $diam(\lambda(s))_j$ with $supp(\lambda(s))_j$ contained in the unit ball $\mathbb{R}^{m,}_-$. Let $\varepsilon_j(s) = (y(\phi, diam(\lambda)))_j(s)$. Now, for any C^∞ kernel $\lambda_j \in L^1(\mathbb{R}^m_-)$, since $\|\lambda_j\| \leq \|\lambda_j\|_{L^1} = 1$ can write for any map ϕ in $\mathbb{R}^{lS}$*

$$((\lambda * \phi_j - \phi)(y_0))_j(s) = \int_{\mathbb{R}^m_-}\left[(\phi(y_0-\zeta)_j - \phi(y_0))\,\lambda(\zeta)^{\frac{1}{2}}\,d\zeta\right]_j(s), \tag{A.4}$$

using Cauchy inequality

$$\begin{aligned} |(\lambda * \phi_j - \phi)(y_0)|_j^2(s) &\leq \left[\int_{\mathbb{R}^m_-}(\phi(y_0-\zeta)_j - \phi(y_0))\,\lambda(\zeta)^{\frac{1}{2}}\,d\zeta\right]_j^2(s) \\ &\leq \int_{\mathbb{R}^m_-}\left[|(\phi(y_0-\zeta)_j - \phi(y_0))|^2\,\lambda(\zeta)\,d\zeta\right]_j(s) \end{aligned} \tag{A.5}$$

by integration and using Fubini's theorem we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}^m_-}|\lambda * \phi_j - \phi)(y_0)|^2\,dy_0\right)_j(s) \\ &\leq \int_{\mathbb{R}^m_-}\int_{\mathbb{R}^m_-}\left\{\left[|(\phi(y_0-\zeta)_j - \phi(y_0))|^2\,\lambda(\zeta)\,d\zeta\right]dy_0 d\zeta\right\}_j(s) \\ &= \int_{\mathbb{R}^m_-}dy_0\int_{\mathbb{R}^m_-}\left[|(\phi(y_0-\zeta)_j - \phi(y_0))|^2\,\lambda(\zeta)\,d\zeta\right]_j(s) \\ &= \int_{\mathbb{R}^m_-}\left[\int_{\mathbb{R}^m_-}|(\phi(y_0-\zeta)_j - \phi(y_0))|^2\,\lambda(\zeta)\,dy_0\right]_j d\zeta(s) \\ &= \int_{\mathbb{R}^m_-}\left\{\left[\int_{\mathbb{R}^m_-}|(\phi(y_0-\zeta)_j - \phi(y_0))|^2\,dy_0\right]\lambda(\zeta)\,d\zeta\right\}_j(s) \end{aligned} \tag{A.6}$$

Since mass of λ is equal to one, and ζ ranges over its support,

$$\left(\int_{\mathbb{R}^m_-} |\lambda * \phi_j - \phi)(y_0)|^2 \, dy_0\right)_j (s) \quad \leq \quad \sup_{\|\zeta\| \leq \sigma} \left(\int_{\mathbb{R}^m_-} |(\phi(y_0 - \zeta)_j - \phi(y_0))|^2 \, dy_0\right)_j (s) \tag{A.7}$$

denote it $y(\phi, diam(\lambda))^2_j(s)$.

Remark 7 *The measurement error is then defined by an oscillation* $\sup_{\|\zeta\| \leq \sigma} \left(\int_{\mathbb{R}^m_-} |(\phi(y_0 - \zeta)_j - \phi(y_0))|^2 \, dy_0\right)^{\frac{1}{2}}_j (s)$.

Thus it follows that

$$\left(\int_{\mathbb{R}^m_-} |\lambda * \phi_j - \phi)(y_0)| \, dy_0\right)_j (s) \quad \leq \quad \sup_{\|\zeta\| \leq \sigma} \left(\int_{\mathbb{R}^m_-} |(\phi(y_0 - \zeta)_j - \phi(y_0))|^2 \, dy_0\right)^{\frac{1}{2}}_j (s) \tag{A.8}$$

Denote this oscillation $y(\phi, diam(\lambda))_j(s)$. It converges to zero when $diam(\lambda)$ converges to zero.

What remains to be shown is that $y(\phi, diam(\lambda))_j(s)$ is bounded above. Since mappings are smooth with compact support, the upper bound is obtained by

$$y(\phi, diam(\lambda)_j(s) \leq c \left(\sum_{k=1}^{m} \left|D^k \phi\,(y_0)\right|^2_j (s)\right)^{\frac{1}{2}} \tag{A.9}$$

where $c = k_1 \sigma$. k_1 is a constant of differentiation, and σ a distance.

Proof A.10 (Proposition 4.1) *Forming the Lagrangean*

$$\begin{aligned} L\left(\bar{x}, \bar{\xi}, \bar{y}, \bar{\lambda}, \bar{\mu}\right) &= u(x) - \lambda(0)\left[\bar{p}(0)x(0) - \bar{p}(0)\omega(0) + \bar{q}\xi - \theta(\bar{z})\bar{p}(0)y(0)\right] \\ &\quad - \sum\nolimits_{s=1}^{S} \lambda(s)\left[\bar{p}(s)x(s) - \bar{p}(s)\omega(1) + \theta(\bar{z})\bar{p}(s)y(s) + R(\bar{y}, s)\xi\right] \\ &\quad - \sum\nolimits_{s=0}^{S} \mu\,(s)\,\Phi(\bar{y}) \end{aligned} \tag{A.10}$$

The necessary and sufficient conditions for (x, ξ, y) to be a solution of L, are that there exists $\lambda \in \mathbb{R}^{S+1}_{++}$, and $\mu \in \mathbb{R}^{S+1}_{++}$ such that

$$\nabla L\left(\bar{x}, \bar{\xi}, \bar{y}, \bar{\lambda}, \bar{\mu}\right) \equiv 0$$

is satisfied. This is equivalent to

$$\begin{aligned}
\nabla u(\bar{x}) &= \bar{\lambda}\bar{p} \\
\bar{q} &= \left(\frac{\sum_{s=1}^{S} \bar{\lambda}\left(s\right)}{\bar{\lambda}(0)}\right)\bar{p}(s)\bar{y}(s) \\
\bar{\mu}\nabla\Phi(\bar{y}) &= \bar{\lambda}\bar{p} \\
\bar{p}\bar{x} - \bar{p}\omega &= \theta(\bar{z})\bar{p}\bar{y} + \Pi(\bar{y}, \bar{p})\bar{\xi} \\
\Phi(\bar{y}) &= 0
\end{aligned} \tag{A.11}$$

where $\Pi = \begin{bmatrix} -q \\ p(s)y(s) \\ \vdots \\ p(S)y(S) \end{bmatrix}$. *Let* $\bar{\beta} = \left(\frac{\sum_{s=1}^{S} \bar{\lambda}(s)}{\bar{\lambda}(0)}\right)$. *Then* $\bar{q} = \sum_{s=1}^{S} \bar{\beta}(s)\bar{p}(s)\bar{y}(s)$. *It follows from the first order conditions that*

$$\bar{\beta}\bar{p} = \frac{1}{\bar{\lambda}(0)}\nabla u(\bar{x}) = \frac{\bar{\mu}}{\bar{\lambda}(0)}\nabla\Phi(\bar{y}) \tag{A.12}$$

Proof A.11 (Proposition 4.2) *The necessary and sufficient conditions for (x, z, y) to be a solution of L, are that there exists $\lambda \in \mathbb{R}^{S+1}_{++}$, and $\mu \in \mathbb{R}^{S+1}_{++}$ such that*

$$\nabla L(\bar{x}, \bar{z}, \bar{y}, \bar{\lambda}, \bar{\mu}) \equiv 0$$

is satisfied. This is equivalent to

$$
\begin{aligned}
\nabla u(\bar{x}) &= \bar{\lambda}\bar{p} \\
\bar{q} &= \left(\frac{\sum_{s=1}^{S}\bar{\lambda}(s)}{\lambda(0)}\right)\bar{p}(s)\bar{y}(s) \\
\bar{\mu}\nabla\Phi(\bar{y}) &= \bar{\lambda}\bar{p} \\
\bar{p}\bar{x} - \bar{p}\omega &= \theta(\bar{z})\bar{p}\bar{y} + \Pi(\bar{p},\bar{y})\bar{z} \\
\Phi(\bar{y}) &= 0
\end{aligned}
\tag{A.13}
$$

Proof A.12 (Proposition 4.3) *From (1) have $\bar{\xi} = 0$, and from (2) have $\bar{z} + \hat{b} = 0$. The equivalence follows from $\bar{\xi} = \bar{z} + \hat{b} = 0$.*

Proof A.13 (Proposition 4.4) *Suppose that the agent assigns $\bar{\beta}$ to the producer. It remains to show that*

$$\max_{\bar{y}} \left\{\bar{\beta}\bar{p}y : y \in Y\right\} \tag{A.14}$$

is well defined. Since the first order conditions are such that there exists $\mu \in \mathbb{R}^{S+1}_{++}$. From

$$\nabla L(\bar{y}) \equiv 0 \tag{A.15}$$

have

$$\bar{\beta}\bar{p} = \bar{\nu}\nabla\Phi(\bar{y}) \tag{A.16}$$

it follows that

$$\bar{\beta}\bar{p} = \bar{\nu}\nabla\Phi(\bar{y}) \Longleftrightarrow \frac{\bar{\mu}}{\bar{\lambda}(0)}\nabla\Phi(\bar{y}) = \frac{1}{\bar{\lambda}(0)}\nabla u(\bar{x}) = \bar{\beta}\bar{p} \tag{A.17}$$

Proof A.14 (Proposition 4.5) *The source of inefficiency comes from the no-arbitrage condition, $\beta\Pi = 0$. This equation is indeterminate for the case that $S > n$. Therefore for any $\hat{\beta} \neq \beta$ assigned to the firm it follows that $\bar{y}|_{\hat{\beta}} \neq \bar{y}|_{\beta}$ in Y since*

$$\max_{\bar{y}|_{\hat{\beta}}} \left\{ \hat{\beta}\bar{p}y : y \in Y \right\} \quad \neq \quad \max_{\bar{y}|_{\beta}} \left\{ \beta\bar{p}y : y \in Y \right\} \tag{A.18}$$

Proof A.15 (Proposition 4.6) *Let $v : \mathbb{R}^{S+1} \times \mathbb{R}_{+} \rightarrow \mathbb{R}$ be defined by $v(y, z) = u(y + z)$. Then the two variable control problem above is equivalent to*

$$(\bar{y}, \bar{z}) \arg\max \left\{ v(y, z) : (y, z) \in Y \times Z \right\} \tag{A.19}$$

By application of the separation theorem for convex sets $(\bar{y}, \bar{z})$ is a solution of this control problem if and only if

$$(\nabla_y v(\bar{y}, \bar{z}), \nabla_z(\bar{y}, \bar{z})) \in N_{y \times Z}(\bar{y}, \bar{z}) \tag{A.20}$$

where ∇_y denotes the gradient of v with respect to y. From the definition of a normal cone it follows that

$$N_{y \times Z}(\bar{y}, \bar{z}) = N_y(\bar{y}) \times N_Z(\bar{z}) \tag{A.21}$$

and from the definition of the function v that

$$\nabla_y v(\bar{y}, \bar{z}) = \nabla_z v(\bar{y}, \bar{z}) = \nabla u(\bar{y} + \bar{z}) \tag{A.22}$$

so that $(\nabla_y v(\bar{y}, \bar{z}), \nabla_z(\bar{y}, \bar{z})) \in N_{Y \times Z}(\bar{y}, \bar{z})$ reduces to $\nabla u(\bar{y} + \bar{z}) \in N_Y(\bar{y}) \cap N_Z(\bar{z})$.

Proof A.16 (Corollary 4.1) *The proof is similar to the proof of proposition (4.6). It follows from $z \Rightarrow \xi$, and $Z \Rightarrow \Xi$.*

Proof A.17 (Proposition 4.7) *The proof is a slight modification of the proof of proposition (4.6).*

Proof A.18 (Proposition 4.8) $((\bar{p},\bar{q}),(\bar{x},\bar{\xi})))$ *satisfies (i) of the reduced form model with separated activities if and only if the geometric first order condition holds (4.16). The profit maximization problem (ii) of the reduced form model with separated activities* $(\bar{y})$ *is satisfied if and only if the first order condition above holds (4.17). Since* $(\bar{x},\bar{\xi},\bar{y})$ *satisfies (i) of the (centralized) reduced form model if and only if both geometric first order conditions hold*$(4.16, 4.17)$*,* $((\bar{p},\bar{q}),(\bar{x},\bar{\xi},(\bar{y})))$ *is a geometric reduced form with separated activities equilibrium if and only if assign* $\bar{\beta}$ *to the maximization problem of the firm.*

Proof A.19 (Proposition 4.9) *(1) show that* $(\bar{x},\bar{\xi},\bar{y})$ *satisfies the first order conditions so that (i) in definition of a reduced form equilibrium is satisfied. The first order conditions are*

$$\bar{p}\bar{x}-\bar{p}\bar{\omega}=\bar{p}\bar{y}+\Pi\bar{b}+\Pi\bar{z},\ \textit{and}\ \bar{\beta}\Pi=0 \tag{A.23}$$

which is equivalent to

$$\bar{p}\bar{x}-\bar{p}\bar{\omega}=\bar{p}\bar{y}+\Pi\bar{\xi},\ \textit{and}\ \bar{\beta}\Pi=0 \tag{A.24}$$

since $\bar{\xi}=\bar{z}+\bar{b}$ *holds, so that first order condition (above) holds. Next, show what the no-arbitrage condition implies for the firm for all* $(\bar{y},\bar{b})$*, the present value of the firm to the producer reduces to*

$$\bar{\beta}\bar{p}\bar{y}=\bar{\beta}\bar{p}\bar{y}+\bar{\beta}\Pi\bar{b}=\bar{\beta}\bar{p}\bar{y} \tag{A.25}$$

Thus the producer's problem in the extensive form equilibrium definition is equivalent to

$$(\bar{y}) \arg\max \left\{ \bar{\beta}\bar{p}y : y \in Y \right\} \tag{A.26}$$

for which the first order conditions are given (above). The last step is to recall that the market clearing condition $\bar{\xi} = \bar{z} + \bar{b} = 0$ holds, and from which the result follows.
(2) show that if $(\bar{x}, \bar{\xi}, \bar{y})$ is a solution to the reduced form problem, then the first order conditions(above) are satisfied. This implies that, for any $b \in \mathbb{R}$

$$\left(\bar{y}, \bar{b}\right) \arg\max \left\{ \bar{\beta}\bar{p}y + \Pi\bar{b} : (y; b) \in Y \times \mathbb{R} \right\} \tag{A.27}$$

since by no-arbitrage condition $\bar{\beta}\Pi = 0$. Therefore, can pick any $b \in \mathbb{R}$, and define

$$z = \bar{\xi} - \bar{b} \tag{A.28}$$

then the first order condition of extensive form equilibrium is satisfied by $(\bar{x}, z)$, and thus $(\bar{x}, z)$ is a solution of the extensive form equilibrium, since $(\bar{y}, \bar{b})$ is a solution of the extensive form equilibrium, and the result follows from $0 = \bar{\xi} = z + \bar{b}$.

Proof A.20 (Proposition 4.10) *Forming the Lagrangean*

$$\begin{aligned} L = \quad u(x) \quad & -\lambda(0) \left[\bar{p}(0)x(0) - \bar{p}(0)\omega(0) + \bar{q}\xi + \bar{p}(0)\bar{k}(0) \right] \\ & - \textstyle\sum_{s=1}^{S} \lambda(s) \left[\bar{p}(s)x(1) - \bar{p}(s)\omega(s) + R(\bar{y}, s)\xi + \bar{p}(s)y(s) \right] \\ & - \textstyle\sum_{s=1}^{S} \mu(s) \left[\Phi|_{\hat{b}} \left(\bar{y}(s) \right) \right] \end{aligned}$$

the first order conditions are necessary and sufficient for $(x; \xi, y)$ to be a solution of equilibrium definition (4.4) *if there exists $\lambda \in \mathbb{R}^{S+1}_{++}$ and $\mu \in \mathbb{R}^{S}_{++}$ such that*

$$\nabla L \left(\bar{x}, \bar{\xi}, \bar{y}, \bar{\lambda}, \bar{\mu} \right) \equiv 0$$

This is equivalent to

$$\nabla u(\bar{x}) = \bar{\lambda}\bar{p} \tag{A.29}$$

$$\bar{q} = \frac{\sum_{s=1}^{S} \bar{\lambda}(s)}{\bar{\lambda}(0)} R(\bar{y}, s) \tag{A.30}$$

$$\sum_{s=1}^{S} \bar{\mu}(s)\nabla \left.\Phi\right|_{Z}(\bar{y}(s)) = \sum_{s=1}^{S} \bar{\lambda}(s)\bar{p}(s) \tag{A.31}$$

$$\bar{p}\bar{x} = \bar{p}\omega + \Pi(\bar{y}, \bar{p})\bar{\xi} \tag{A.32}$$

$$\left.\Phi\right|_{\hat{b}}(\bar{y}(s) = 0 \tag{A.33}$$

Let $\bar{\beta}(s) = \left(\frac{\sum_{s=1}^{S} \bar{\lambda}(s)}{\bar{\lambda}(0)}\right)$, *then* $\bar{q} = \sum_{s=1}^{S} \bar{\beta}(s)R(\bar{y}, s)$ *and from* $(A.29)$ *and* $(A.31)$ *have*

$$\sum_{s=1}^{S} \bar{\lambda}(s)\bar{p}(s) = \nabla u(\bar{x}(s)) = \sum_{s=1}^{S} \bar{\mu}(s)\nabla \left.\Phi\right|_{\hat{b}}(\bar{y}(s))$$

using $\bar{\pi}$

$$\sum_{s=1}^{S} \bar{\beta}(s)\bar{p}(s) = \frac{1}{\bar{\lambda}(0)}\nabla u(\bar{x}(s)) = \sum_{s=1}^{S} \frac{\bar{\mu}(s)}{\bar{\lambda}(0)}\nabla \left.\Phi\right|_{Z}(\bar{y}(s)) \tag{A.34}$$

The first part of the proof shows that (4.4) has a solution and that (i) of (4.10) has a solution. It remains to show that part (ii) of (4.10) has a solution. Now, if assign $\bar{\beta}(s)$ *for each* $s \in S$ *to the optimization problem of the producer, and the producer takes* $\bar{\beta}$ *as given then*

$$(\bar{y}) \arg\max \left\{\sum_{s=1}^{S} \bar{\beta}(s)\bar{p}(s)y(s) : y(s) \in \left.Y\right|_{\hat{b}}\right\}. \tag{A.35}$$

This problem has a solution if there exists $\nu \in \mathbb{R}_{++}^{lS}$ *such that*

$$L(\bar{y}) \equiv 0 \tag{A.36}$$

This is equivalent to

$$\textstyle\sum_{s=1}^{S} \bar{\beta}(s)\bar{p}(s) = \sum_{s=1}^{S} \bar{\nu}(s)\nabla \left.\Phi\right|_{\hat{b}}(\bar{y}(s)) \tag{A.37}$$

The separation result follows from $(A.34)$ *and* $(A.37)$.

$$\sum_{s=1}^{S} \frac{\bar{\mu}(s)}{\bar{\lambda}(0)} \nabla \left. \Phi \right|_{\hat{b}} (\bar{y}(s)) = \sum_{s=1}^{S} \bar{\nu}(s) \nabla \left. \Phi \right|_{\hat{b}} (\bar{y}(s)). \tag{A.38}$$

Proof A.21 (Proposition 4.11) *The result follows from the first order conditions*

$$\sum_{s=1}^{S} \bar{\beta}(s)\bar{p}(s) = \frac{1}{\bar{\lambda}(0)} \nabla u(\bar{x}(s))$$
$$\bar{q} = \sum_{s=1}^{S} \bar{\beta}(s) R(\bar{y}, s)$$

and from

$$(\bar{y}) \arg\max \left\{ \sum_{s=1}^{S} \bar{p}(s) y(s) : y(s) \in \left. Y \right|_{\hat{b}} \right\}$$

This problem has a solution if there exists $\nu \in \mathbb{R}_{++}^{lS}$ *such that*

$$L(\bar{y}) \equiv 0$$

This is equivalent to

$$\sum_{s=1}^{S} \bar{p}(s) = \sum_{s=1}^{S} \bar{\nu}(s) \nabla \left. \Phi \right|_{\hat{b}} (\bar{y}(s)) \tag{A.39}$$

Equ. $(A.39)$ *is independent of the present value vector* $\bar{\beta}$ *of the consumer. This decentralizes the objective function of the firm.*

Proof A.22 (Proposition 4.12) *Consider a reduced form incomplete financial markets equilibrium with decentralized profit maximizing objective function* $(\bar{p}, \bar{q})$ *with associated equilibrium allocations* $(\bar{x}, \bar{\xi}), (\bar{y})$ *for an economy* $\omega \in \Omega$. *Let* (x, y) *not be a constraint productive efficient allocation at price system* $\bar{p}$ *and* $\bar{q}$, *and period one financial trade* $\bar{\xi} = \bar{z} + \hat{b}$ *for implicit feasible* $\hat{b}$. *Then, because* (x, y) *at* $\bar{\xi}$ *is a feasible competitive* *financial markets equilibrium with production allocation at $t = 1$, it satisfies*

$$\begin{aligned} \bar{x}(0) &= \omega(0) + \bar{k}(0) \\ \bar{x}(s) &= \omega(1) + \bar{y}(s) \\ \bar{\xi} &= 0, \end{aligned} \tag{A.40}$$

Because (x, y) is not efficient optimal, in the sense that period two allocations are not optimal, given financial constraint $\bar{\xi}$ implying production capacity $\bar{k}(0)$ and technology $\Phi|_{\hat{b}}$, which in turn implies the constraint production set available to the firm in $t = 1$, denoted $Y|_{\hat{b}}$, there must exist and alternative feasible allocation $(\hat{x}, \hat{y})$ within the constraint production set available to the producer $Y|_{\hat{b}}$ such that

$$u(\hat{x}(0), \hat{x}(s); \bar{\xi}) > u(x(0), x(s); \bar{\xi}), \textit{ for all } s \in \{1, .., S\} \tag{A.41}$$

We have that

$$\begin{aligned} \bar{p}(0) \cdot \hat{x}(0) &\geqslant \bar{p}(0) \cdot x(0) \\ \bar{p}(s) \cdot \hat{x}(s) &> \bar{p}(s) \cdot x(s), \textit{ for all } s \in \{1, .., S\} \end{aligned} \tag{A.42}$$

This implies that

$$\begin{aligned} \bar{p}(0) \cdot \hat{x}(0) &\geqslant \bar{p}(0) \cdot x(0) \\ \bar{p}(s) \cdot \hat{x}(s) &> \bar{p}(s) \cdot x(s), \textit{ for all } s \in \{1, .., S\} \end{aligned}$$

We then have for feasible $(\hat{x}, \hat{y})$ that

$$\begin{aligned} \bar{p}(0) \cdot \omega(0) + \bar{p}(0)\bar{k}(0) &\geqslant \bar{p}(0) \cdot \omega(0) + \bar{p}(0)\bar{k}(0) \\ \bar{p}(s) \cdot (\hat{y}(s) + \omega(1)) &> \bar{p}(s) \cdot (y(s) + \omega(1)), \end{aligned}$$

for all $s \in \{1, .., S\}$ *so that period two long run profits*

$$\bar{p}(s) \cdot \hat{y}(s) > \bar{p}(s) \cdot y(s). \tag{A.43}$$

However, this implies that $\bar{p} \cdot \hat{y} > \bar{p} \cdot y$ *, where* $\hat{y} \in Y|_{\hat{b}}$ *at equilibrium* $\bar{\xi}$*. This is a contradiction to the fact that* $y_j \in Y|_{\hat{b}}$ *is profit maximizing at price system* $\bar{p}$ *and* $\bar{\xi}$*.*

Proof A.23 (Proposition 4.13) *To see the inefficient organization of production of the first model (similar to the literature). Assign present value vector* $\bar{\beta}$ *to the firm, then for financial constraint* $\bar{\xi}$ *let the firm maximize its present value profits*

$$(\bar{y}) \arg\max \left\{ \sum_{s=1}^{S} \bar{\beta}(s)\bar{p}(s)y(s) : y(s) \in Y|_{\hat{b}} \right\} \tag{A.44}$$

Then $(A.44)$ *is equal to*

$$(\bar{y}) \arg\max \left\{ \sum_{s=1}^{S} \bar{p}(s)y(s) : y(s) \in Y|_{\hat{b}} \right\} \tag{A.45}$$

if and only if

$$\bar{\beta}(s) = e \text{ for every } s \in S \in \{1, .., S\} \tag{A.46}$$

where e *is an unit vector. This condition is generally not satisfied for a no-arbitrage equilibrium when* $S > n$*. For any* $\bar{\beta}(s)$

$$\vec{0} < \bar{\beta}(s) < e \tag{A.47}$$

the centralized model of production is less efficient than the fully decentralized objective function model since

$$\max_{\bar{y}} \Pi|_{\beta} (\bar{\beta}\bar{p}) < \max_{\bar{y}} \Pi(\bar{p}) \tag{A.48}$$

$y(s)|_{\bar{\beta}} \neq y(s) \in Y|_{\hat{b}}$.

Proof A.24 (Proposition 4.16) *The result follows from the first order conditions*

$$\sum_{s=1}^{S} \bar{\beta}(s)p(s) = \frac{1}{\bar{\lambda}(0)}\nabla u(\bar{x}(s))$$
$$\bar{q} = \sum_{s=1}^{S} \bar{\beta}(s)R(\bar{y}, s)$$

and from

$$(\bar{y}) \arg\max \left\{\sum_{s=1}^{S} \bar{p}(s)y(s) : y(s) \in Y|_{\hat{b}}\right\}$$

equivalent to

$$\underset{\left(\bar{y},\hat{b};(\bar{k}(0))\right)}{\arg\max} \left\{ \bar{q}b + \sum_{s=1}^{S} \bar{p}(s)y(s) : \begin{array}{l} \bar{q}\bar{z} \geq \bar{q}b = \bar{p}(0)k(0) \\ y(s) \in Y|_Z \end{array} \quad s \in S \right\} \tag{A.49}$$

for feasible $\hat{b}$. This problem has a solution for any feasible $\hat{b}$ equivalent to the solution in the (RFE), where there exists $\nu \in \mathbb{R}^{lS}_{++}$ such that

$$L(y) \equiv 0$$

This is equivalent to

$$\sum_{s=1}^{S} \bar{p}(s) = \sum_{s=1}^{S} \bar{\nu}(s)\nabla \left.\Phi\right|_{\hat{b}} (\bar{y}(s))$$

Equ. $(A.49)$ is independent of the present value $\beta(s)$ of the consumer for any feasible $\hat{b} \leq \bar{z}$. This decentralizes the objective function of the firm.

Proof A.25 (Proposition 4.15) *Part (i). Let us first show that $((\bar{x}, \bar{\xi}), (\bar{y}))$ satisfy the first order conditions (1) $\tau \in \langle \Pi(\bar{y}, \bar{p}) \rangle$, $\beta \in \langle \Pi(\bar{y}, \bar{p}) \rangle^{\perp}$, and (2) $y \in Y|_{\hat{b}}$, $p \in N_{Y|_Z}(y)$ so that conditions (i) and (ii) in (RFE) are satisfied, where $\tau = \Pi(\bar{y}, \bar{p})z$ is an income* *vector. The FOC's for the consumer's problem in (EFE) are*

$$px = p\omega + \Pi(\bar{y}, \bar{p}) \begin{bmatrix} \bar{z} \\ \theta(\bar{z}) \end{bmatrix}, \text{and } \beta\Pi(\bar{y}, \bar{p}) = 0 \tag{A.50}$$

and can be rewritten as

$$px = p\omega + \Pi(\bar{y}, \bar{p})\xi, \text{and } \beta\Pi(\bar{y}, \bar{p}) = 0$$

since $\bar{\xi}\big|_{\hat{b}} = \bar{z} + \hat{b}$, *so that (1) above holds for any feasible* $\hat{b} \leqslant \bar{z}$. *The firm's problem in (EFE)*

$$\underset{(\bar{y},(\hat{b};\bar{k}(0)))}{\arg\max} \left\{ \bar{q}b + \sum\nolimits_{s=1}^{S} \bar{p}(s)y(s) : \begin{array}{l} \bar{q}\bar{z} \geq \bar{q}b = \bar{p}(0)k(0) \\ y(s) \in Y|_{\hat{b}}(s) \end{array} \quad s \in S \right\}$$

for any feasible $\hat{b} \leqslant \bar{z}$ *reduces to*

$$\underset{(\bar{y}(s))}{\arg\max} \left\{ \sum\nolimits_{s=1}^{S} \bar{p}(s)y(s) : y(s) \in Y|_{\hat{b}}(s), s \in S \right\}$$

since feasible $b \Rightarrow \Phi|_{\hat{b}}(s) \implies Y|_{\hat{b}}$ *for which the first order condition (2) which is equivalent to* $\nu(s)\nabla\, \Phi|_{\hat{b}} = p(s)$ *above holds. The result follows ,since market clearing condition* $\bar{\xi}\big|_{\hat{b}} = \bar{z} + \hat{b} = 0$, *and* $\bar{x}(0) = \omega(0) + \bar{k}(0)$, $\bar{x}(s) = \omega(1) + \bar{y}(s)$ *for all* $s \in \{1, .., S\}$ *hold.*

Part (ii). If $((\bar{x}, \bar{\xi}), (\bar{y}))$ *is an (RFE) for implicit* $\hat{b}$, *then the first order conditions are satisfied. This implies that for any feasible* $\hat{b} \leq z$

$$\underset{(\bar{y}(s))}{\arg\max} \left\{ \sum\nolimits_{s=1}^{S} \bar{p}(s)y(s) : y(s) \in Y|_{\hat{b}}(s), s \in S \right\} \tag{A.51}$$

expands to

$$\underset{(\bar{y},(\hat{b};\bar{k}(0)))}{\arg\max}\left\{\bar{q}b+\sum\nolimits_{s=1}^{S}\bar{p}(s)y(s):\begin{array}{c}\bar{q}\bar{z}\geq\bar{q}b=\bar{p}(0)k(0)\\ y(s)\in Y|_{\hat{b}}(s)\end{array}\quad s\in S\right\} \tag{A.52}$$

for which the first order conditions are satisfied, hence $\bar{y}$ is a solution of (ii) in (EFE) for feasible $\hat{b}$. Pick any feasible $\hat{b}$ and define

$$z+\hat{b}=\bar{\xi} \tag{A.53}$$

such that $\Pi z+\Pi\hat{b}=\Pi\bar{\xi}$ becomes $\Pi(z+\hat{b})=\Pi\bar{\xi}$, then first order conditions for the consumer of the (EFE) $(A.50)$ is satisfied for $(\bar{x},z)$. $(\bar{x},z)$ is a solution of (EFE) (i) and $(\bar{y},\hat{b})$ is a solution of (EFE) (ii). The result follows from $0=\xi=z+\hat{b}$ and $\bar{x(0)}=\omega(0)+\bar{k}(0)$, $\bar{x}(s)=\omega(1)+\bar{y}(s)$ for all $s\in\{1,..,S\}$.

Proof A.26 (Lemma 4.1) *(i) $\bar{x}|_{\bar{z}}$ is a solution of utility max (4.1) if and only if $\bar{x}|_{\bar{z}}\in B$ and*

$$intU_{\bar{x}|_{\bar{z}}}\cap B=\varnothing.$$

By the separation theorem for convex sets (appendix), there exists $P=\beta_i p\in R^n, P\neq 0$ such that

$$H_P^-=\{x\in R^n:Px\leq Px',\forall x\in B,\forall x'\in intU_{\bar{x}}\}$$

since $\bar{x}|_{\bar{z}}\in B$,

$$H_P^-=\left\{x\in R^n:P\;\bar{x}|_{\bar{z}}\leq Px',\forall x'\in intU_{\bar{x}|_{\bar{z}}}\right\}.$$

By continuity of utility, $intU_{\bar{x}}=U_{\bar{x}}$, and by continuity of the scalar product,

$$\begin{array}{lll}H_P^+=\left\{\forall x'\in U_{\bar{x}|_{\bar{z}}}:P\bar{x}\leq Px',\forall x'\in U_{\bar{x}|_{\bar{z}}}\right\} & \Leftrightarrow & P\in\partial u(\bar{x}|_{\bar{z}})\\ H_P^-=\{\forall x\in B:Px\leq P\;\bar{x}|_{\bar{z}},\} & \Leftrightarrow & P\in N_B(\bar{x}|_{\bar{z}})\end{array}$$

hence, there exists p such that $\partial u(\bar{x}|_{\bar{z}}) \cap N_B(\bar{x}|_{\bar{z}}) \neq \{0\}$ *is satisfied.*

(ii) Suppose that $\bar{x}|_{\bar{z}} \in B$*, and there exists* $P \in \partial u(\bar{x}|_{\bar{z}}) \cap N_B(\bar{x}|_{\bar{z}}),\ P \neq 0.$ *If* $\bar{x}|_{\bar{z}}$ *is not a solution of the utility maximization problem (4.1) then there exists* $x' \in intU_{\bar{x}|_{\bar{z}}} \cap B$*. Since* $P \in \partial u(\bar{x}|_{\bar{z}})$*, we have*

$$Px\prime > P\,\bar{x}|_{\bar{z}}$$

But since $P \in N_B(\bar{x}|_{\bar{z}})$ *and* $x\prime \in B$*, it follows that* $Px\prime \leq P\,\bar{x}|_{\bar{z}}$ *which contradicts that* $x\prime$ *is preferred to* $\bar{x}|_{\bar{z}}$*.*

Proof A.27 (Lemma 4.2) *(i)* $\bar{y}|_{\bar{z}}$ *is a solution of profit max in (4.2) if and only if* $\bar{y}|_{\bar{z}} \in Y|_{\bar{z}}$ *and*

$$int\Pi_{\bar{y}} \cap Y|_{\bar{z}} = \varnothing.$$

By the separation theorem for convex sets (appendix), there exists $p \in R^n, p \neq 0$ *such that*

$$H_p^- = \left\{y \in R^n : py \leq py', \forall y \in Y, \forall y' \in int\Pi_{\bar{y}|_{\bar{z}}}\right\}$$

since $\bar{y}|_{\bar{z}} \in Y|_{\bar{z}}$ *,*

$$H_p^- = \left\{y \in R^n : p\,\bar{y}|_{\bar{z}} \leq py', \forall y' \in int\Pi_{\bar{y}|_{\bar{z}}}\right\}.$$

By continuity of utility, $int\Pi_{\bar{y}|_{\bar{z}}} = \Pi_{\bar{y}|_{\bar{z}}}$*, and by continuity of the scalar product,*

$$\begin{aligned} H_p^+ &= \left\{\forall y' \in U_{\bar{y}} : p\bar{y} \leq py', \forall y' \in \Pi_{\bar{y}|_{\bar{z}}}\right\} &\Leftrightarrow\quad & p \in \partial\Pi(\bar{y}|_{\bar{z}}) \\ H_p^- &= \{\forall y \in Y : py \leq p\,\bar{y}|_{\bar{z}}\,,\} &\Leftrightarrow\quad & p \in N_{Y|_{\bar{z}}}(\bar{y}|_{\bar{z}}) \end{aligned}$$

hence, there exists p such that $\partial u(\bar{y}|_{\bar{z}}) \cap N_{Y|_{\bar{z}}}(\bar{y}|_{\bar{z}}) \neq \{0\}$ *is satisfied.*

(ii) Suppose that $\bar{y}|_{\bar{z}} \in Y|_{\bar{z}}$, and there exists $p \in \partial\Pi(\bar{y}|_{\bar{z}}) \cap N_Y(\bar{y}|_{\bar{z}}), p \neq 0$. If $\bar{y}|_{\bar{z}}$ is not a solution of the profit maximization problem in (4.2), then there exists $y\prime \in int\Pi_{\bar{y}|_{\bar{z}}} \cap Y|_{\bar{z}}$. Since $p \in \partial\Pi(\bar{y}|_{\bar{z}})$, we have

$$py\prime > p\,\bar{y}|_{\bar{z}}$$

But since $p \in N_{Y|_{\bar{z}}}(\bar{y}|_{\bar{z}})$ and $y\prime \in Y|_{\bar{z}}$, it follows that $py\prime \leq p\,\bar{y}|_{\bar{z}}$ which contradicts that $y\prime$ is preferred to $\bar{y}|_{\bar{z}}$.

Proof A.28 (Lemma 4.3) *Let $v : \mathbb{R}^{S+1} \times \mathbb{R}_+ \to \mathbb{R}$ be defined by $v(y,\xi) = u(y+\xi)$. Then the two variable control problem above is equivalent to*

$$\left(\bar{y},\bar{\xi}\right) \arg\max \left\{ v(y,\xi) : (y,\xi) \in Y|_{\bar{\xi}} \times \Xi \right\} \tag{A.54}$$

By application of the separation theorem for convex sets $(\bar{y},\bar{\xi})$ is a solution of this control problem if and only if

$$\left(\nabla_y v(\bar{y},\bar{\xi}), \nabla_\xi(\bar{y},\bar{\xi})\right) \in N_{Y|_{\bar{\xi}}\times\Xi}(\bar{y},\bar{\xi}) \tag{A.55}$$

where ∇_y denotes the gradient of v with respect to y, and ∇_ξ denotes the gradient of v with respect to ξ. From the definition of a normal cone (appendix) it follows that

$$N_{Y|_{\bar{\xi}}\times\Xi}(\bar{y},\bar{\xi}) = N_{Y|_{\bar{\xi}}}(\bar{y}) \times N_\Xi(\bar{\xi}) \tag{A.56}$$

and from the definition of the function v that

$$\nabla_y v(\bar{y},\bar{\xi}) = \nabla_\xi v(\bar{y},\bar{\xi}) = \nabla u(\bar{y}+\bar{\xi}) \tag{A.57}$$

so that $\left(\nabla_y v(\bar{y},\bar{\xi}), \nabla_\xi(\bar{y},\bar{\xi})\right) \in N_{Y|_{\bar{\xi}}\times\Xi}(\bar{y},\bar{\xi})$ reduces to $\nabla u(\bar{y}+\bar{\xi}) \in N_{Y|_{\bar{\xi}}}(\bar{y}) \cap N_\Xi(\bar{\xi})$.

Proof A.29 (Proposition 5.2) *For $\Pi_E, \Pi_{FC}, \Pi_{FCE}$ and corresponding budget sets $B_E(i)$ and $B_{FCE}(i)$, show that, if conditions $(25-27)$ are satisfied, then this implies that (1)*

$$B_E(i) \Longleftrightarrow B_{FCE}(i), \text{ for all } i \in \{1, ..., m\} \tag{A.58}$$

The equivalence of the budget sets implies the equivalence of

$$\max_{(x,z)_i} u_i(x_i)_E \Leftrightarrow \max_{(x,z)_i} u_i(x_i)_{FCE}, \text{ for all } i \in \{1, ..., m\} \tag{A.59}$$

where $B_E \Longleftrightarrow B_{FCE} \Longleftarrow \langle \Pi_E \rangle = \langle \Pi_{FC} \rangle = \langle \Pi_{FCE} \rangle$ .We have $V_j(s)_E = \tilde{V}_j(s)_{FCE} = \frac{p(s) \cdot y_j(s)}{\sum_{i=1}^m z_i(j)}$, for all $j \in \{1, ..., n\}$. For $\tilde{\omega}_i(s)$, $\theta_j(z_i)$, and $y(s) \in \partial Y_{j,eff}$, have

$$\begin{aligned}
&p(s) \cdot (x_i(s) - \omega_i(s)) = V_j(s)_E z_i \\
&p(s) \cdot \left(x_i(s) - \left(\omega_i(s) + \textstyle\sum_{j=1}^n \theta_j(z_i)\, y_j(s)\right)\right) = \tfrac{p(s) \cdot y_j(s)}{\sum_{i=1}^m z_i(j)} z_i \\
&p(s) \cdot \left(x_i(s) - \left(\omega_i(s) + \textstyle\sum_{j=1}^n \theta_j(z_i)\, y_{j,eff}(s)\right)\right) = \tfrac{p(s) \cdot y_{j,eff}(s)}{\sum_{i=1}^m z_i(j)} z_i \\
&p(s) \cdot (x_i(s) - \tilde{\omega}_i(s)) = \tilde{V}_j(s)_{FCE} z_i
\end{aligned} \tag{A.60}$$

have that for all $i \in \{1, ..., m\}$

$$p(s) \cdot (x_i(s) - \omega_i(s)) = V_j(s)_E z_i = \tilde{V}_j(s)_{FCE} z_i = p(s) \cdot (x_i(s) - \tilde{\omega}_i(s)) \,. \tag{A.61}$$

For equilibrium $\bar{z}_i$, and $\bar{\pi}_i$ for all $i \in \{1, ..., m\}$, $\langle \Pi_E \rangle = \langle \Pi_{FCE} \rangle$

(2) Market-clearing conditions of the two models are equivalent,

$$\sum z_i = 0 \iff \begin{array}{c} \sum z_i = 0 \\ \sum_{i=1}^m \theta_j(z_i) = 1 \quad \forall j \in \{1, ..., n\} \end{array} \tag{A.62}$$

and

$$\begin{array}{c} \sum_{i=1}^{m}\left(x_i(0)-\omega_i(0)\right)=0 \\ \sum_{i=1}^{m}\left(x_i(s)-\omega_i(s)\right)=0 \end{array} \Longleftrightarrow \begin{array}{c} \sum_{i=1}^{m}\left(x_i(0)-\omega_i(0)\right)=0 \\ \sum_{i=1}^{m}\left(x_i(s)-\tilde{\omega}_i(s)\right)=0 \end{array}, \tag{A.63}$$

where at $t = 0$, $\sum_{i=1}^{m}\bar{z}_i(E) = \sum_{i=1}^{m}\bar{z}_i(FCE)$ *is obvious for same* $\bar{\beta}_i$, *for all* $i \in \{1,...,m\}$, *and at* $t = 1$ *(i)* $\theta_j(z_i) = z_i(j)/\sum_{i=1}^{m} z_i(j)$ *for all* $j \in \{1,...,n\}$, *is satisfied for (ii)* $\tilde{\omega}_i(s) = \omega_i(s) + \sum_{j=1}^{n}\theta_j(z_i)\,y_j$, *for all* $i \in \{1,...,m\}$, *and (iii)* $y \in \Pi_{j=1}^{n}\partial Y_{j,eff}$.

Proof A.30 (Proposition 5.3) *For* Π_{FCE}, Π_C, Π_{FC} B_{FCE}, *and* B_C. *(1) Have that*

$$B_{FCE} \Longleftrightarrow B_C \tag{A.64}$$

This implies that

$$\max_{(x,z)_i} u_i(x_i)_{FCE} \Leftrightarrow \max_{(x,z)_i} u_i(x_i)_C, \tag{A.65}$$

where $B_{FCE} \Longleftrightarrow B_C \Longleftarrow \langle\Pi_{FCE}\rangle = \langle\Pi_{FC}\rangle = \langle\Pi_C\rangle$. *We have that* $\tilde{V}_j(s)_{FCE,FC} = \frac{p(s)\cdot y_j(s)}{\sum_{i=1}^{m} z_i(j)}$, *for all* $j \in \{1,...,n\}$, *and* $\theta_j(z_i)$ *for all* $j \in \{1,...,n\}$, *then, at* $t = 1$ *have that for fixed* $\bar{z}_i, \bar{\beta}_i$ *for all* $i \in \{1,...,m\}$,

$$\begin{array}{c} p(s)\cdot\left(x_i(s)-\tilde{\omega}_i(s)\right)_{FCE} = p(s)\cdot\left(x_i(s)-\left(\omega_i(s)+\sum_{j=1}^{n}\theta_j(z_i)\,\bar{y}_j\right)\right) = \tilde{V}(s)_{FC,FCE}z_i = \\ \sum_{j=1}^{n}\frac{p(s)\cdot\bar{y}_j(s)}{\sum_{i=1}^{m} z_i(j)}z_i(j) = \sum_{j=1}^{n}\theta_j(z_i)\left[p(s)\cdot y_j(s)\right] = p(s)\cdot\left(x_i(s)-\omega_i(s)\right)_C \end{array} \tag{A.66}$$

for any $\bar{y} = (\bar{y}_1,...,\bar{y}_j,...,\bar{y}_n) \in \Pi_{j=1}^{n}\partial Y_{j,eff} = y_j \in \partial\, Y_j|_{\bar{z},eff}$, *for all* $j \in \{1,...,n\}$, *where for* $\bar{z}_i$

$$(\bar{z},\bar{y}(s),\bar{y}(0))_j \arg\max\left\{\bar{p}(s)\Box y_j(s)\;\middle|\; \begin{array}{l} y_j \in Y_j(s)|_{\bar{z}} \\ \bar{q}\sum_{i+1}^{m}\bar{z}_i(j) = \bar{q}\bar{z}(j) = \bar{p}(0)\cdot y_j(0) \quad \forall s \in S \end{array}\right\}. \tag{A.67}$$

We have that $\langle\Pi_C\rangle = \langle\Pi_{FCE}\rangle$ *is satisfied, and by definition (2),* $\langle\Pi_{FC}\rangle = \langle\Pi_{FCE}\rangle$.

(2) Market-clearing conditions of the two models are equivalent,

$$\sum_{i=1}^{m} z_i = 0 \iff \begin{array}{c} \sum_{j=1}^{n}\sum_{i=1}^{m} z_i(j) = 0 \\ \sum_{i=1}^{m} \theta_j\left(z_i\right) = 1 \qquad \forall j \in \{1,...,n\} \end{array} \tag{A.68}$$

and

$$\begin{array}{c} \sum_{i=1}^{m}\left(x_i(0)-\omega_i(0)\right)=0 \\ \sum_{i=1}^{m}\left(x_i(s)-\tilde{\omega}_i(s)\right)=0 \end{array} \iff \begin{array}{c} \sum_{i=1}^{m}\left(x_i(0)-\omega_i(0)\right)=0 \\ \sum_{i=1}^{m}\left(x_i(s)-\omega_i(s)\right)=\sum_{j=1}^{n} y_j(s) \end{array}, \tag{A.69}$$

where at $t=0$, (i) $\sum_{i=1}^{m} z_i(FCE)=0$, and $\sum_{i=1}^{m} z_i(C)_j = z(j)$ for all $j \in \{1,...,n\} \iff \sum_{j=1}^{n}\sum_{i=1}^{m} z_i(C)_j = 0$. At $t=1$ (ii) $\theta_j\left(z_i\right) = z_i(j)/\sum_{i=1}^{m} z_i(j)$ for all $j \in \{1,...,n\}$, and $\sum_{i=1}^{m}\theta_j\left(z_i\right)=1$ satisfied for all $j \in \{1,...,n\}$.(iii) $\sum_{i=1}^{m}\left(x_i(s)-\tilde{\omega}_i(s)\right) = \sum_{i=1}^{m}\left(x_i(s)-\left(\omega_i(s)+\sum_{j=1}^{n}\theta_j\left(z_i\right)y_j(s)\right)\right)$, consequently

$$\sum_{i=1}^{m}\left(x_i(s)-\left(\omega_i(s)+\sum_{j=1}^{n} y_j(s)\right)\right)=0 \iff \sum_{i=1}^{m}\left(x_i(s)-\omega_i(s)\right)=\sum_{j=1}^{n} y_j(s). \tag{A.70}$$

Proof A.31 (Theorem 5.3) *By definition of an induced exchange economy with fixed production plans $\mathcal{E}_{FC} \Longleftarrow \mathcal{E}_{FCE}$, and by proposition (5.2), $\mathcal{E}_{FCE} \iff \mathcal{E}_E$. By proposition (5.3), $\mathcal{E}_C \iff \mathcal{E}_{FCE}$. This proposition follows from $\mathcal{E}_C \iff \mathcal{E}_{FC}$, and where by definition $\mathcal{E}_{FC} \Longleftarrow \mathcal{E}_{FCE}$. We conclude that*

$$\mathcal{E}_C \iff \mathcal{E}_{FCE} \iff \mathcal{E}_E. \tag{A.71}$$

Proof A.32 (Proposition 5.4) *For $\Pi_E, \Pi_{FPE}, \Pi_{FCE}$, and $B_E(i)$, and $B_{FPE}(i)$ for all $i \in \{1,...,m\}$. Need to show that $\omega_{i,E} \neq \tilde{\omega}_{i,FPE} = \tilde{\omega}_{i,FP}$. Since $y \in \Pi_{j=1}^{n} Y_{j,FP} = y_j \in$*

$Y_j|_{\bar{\beta}_j}$ *but*

$$y_j \in Y_j|_{\bar{\beta}_j} \neq y_j \in \partial\, Y_j|_{\bar{z},eff} \Longrightarrow y \in \Pi_{j=1}^n Y_{j,FP} \neq y \in \Pi_{j=1}^n Y_{j,FC}. \tag{A.72}$$

for any definition of $\bar{\beta}_j$. *By proposition (1),* $\tilde{\omega}_{i,FPE} \neq \tilde{\omega}_{i,FCE} = \omega_{i,E}$, *for all* $i \in \{1,...,m\}$. *It follows that* $\langle \Pi_{FPE} \rangle \neq \langle \Pi_{FCE} \rangle = \langle \Pi_E \rangle$ *which implies that* $B_{FPE}(i) \neq B_E(i)$, *and* $\max\limits_{(x_i,\theta_i)} u_i(x_i)_{FPE} \neq \max\limits_{(x_i,z_i)} u_i(x_i)_E$. *We conclude that* $\mathcal{E}_E \not\Leftrightarrow \mathcal{E}_{FPE}$.

Proof A.33 (Proposition 5.5) *For* $\Pi_{FPE}, \Pi_{FP}, \Pi_P$, *and* $B_{FPE}(i)$, $B_P(i)$ *for all* $i \in \{1,...,m\}$.

$$B_{FPE}(i) \Longleftrightarrow B_P(i), \text{ for all } i \in \{1,...,m\}, \tag{A.73}$$

implies that

$$\max_{(x_i,\theta_i)} u_i(x_i)_{FP} \Longleftrightarrow \max_{(x_i,z_i)} u_i(x_i)_{FPE} \Longleftrightarrow \max_{(x_i,\theta_i)} u_i(x_i)_P, \text{ for all } i \in I, \tag{A.74}$$

where $z_i(j) = (\theta_{ij} - \xi_{ij}) \sum_{i=1}^m z_i(j)$ *for all* $j \in \{1,...,n\}$. $B_{FPE}(i) \Longleftrightarrow B_P(i)$, *for all* $i \in \{1,...,m\}$, *is implied by*

$$\langle \Pi_{FPE} \rangle = \langle \Pi_{FP} \rangle = \langle \Pi_P \rangle \tag{A.75}$$

where $y \in \Pi_{j=1}^n \partial Y_j = y_j \in Y_j|_{\bar{\beta}_j}$ *for all* $j \in \{1,...,n\}$ *is satisfied.*

$$\sum_{i=1}^m \bar{\theta}_{ij} \Longleftrightarrow \sum_{i=1}^m z_i(j) \tag{A.76}$$

where $z_i(j) = (\theta_{ij} - \xi_{ij}) \sum_{i=1}^m z_i(j)$ *for all* $j \in \{1,...,n\}$, *then*
$\sum_{i=1}^m z_i(j) = \sum_{i=1}^m (\theta_{ij} - \xi_{ij}) \sum_{i=1}^m z_i(j)$, *where* $\sum_{i=1}^m (\theta_{ij} - \xi_{ij}) = 0$.

$$\sum_{i=1}^m (x_i - \tilde{\omega}_i) = 0 \Longleftrightarrow \sum_{i=1}^m (x_i - \omega_i) = \sum_{j=1}^n y_j, \tag{A.77}$$

Where for $y \in \Pi_{j=1}^{n} \partial Y_j = y_j \in Y_j|_{\bar{\beta}_j}$ *for all* $j \in \{1, ..., n\}$,
$\sum_{i=1}^{m}(x_i - \tilde{\omega}_i) = |_{FPE} = \sum_{i=1}^{m}\left(x_i - \omega_i - \sum_{j=1}^{n}\theta_{ij}y_j\right)\Big|_{FP} = 0$
$\iff \sum_{i=1}^{m}(x_i - \omega_i) = \sum_{j=1}^{n} y_j\Big|_P$

Proof A.34 (Theorem 5.4) *By definition of an induced exchange economy with fixed production plans* $\mathcal{E}_{FP} \Longleftarrow \mathcal{E}_{FPE}$*, and by proposition (5.4)* $\mathcal{E}_{FPE} \not\Leftrightarrow \mathcal{E}_E$*. By proposition (5.5)* $\mathcal{E}_P \iff \mathcal{E}_{FPE}$*. This proposition follows from* $\mathcal{E}_P \iff \mathcal{E}_{FP}$*, and where by definition* $\mathcal{E}_{FP} \Longleftarrow \mathcal{E}_{FPE}$*. We conclude that*

$$\mathcal{E}_P \iff \mathcal{E}_{FPE} \not\Leftrightarrow \mathcal{E}_E. \tag{A.78}$$

Proof A.35 (Theorem 5.5) *For* $\Pi_C, \Pi_P, B_C(i), B_P(i)$*, and equilibrium* $\bar{z}$*, and* $\bar{\beta}_i$ *for all* $i \in \{1, ..., m\}$*. Then for* $\bar{\beta}_j = \sum_{i=1}^{m}\bar{\theta}_{ij}\bar{\beta}_i$*, for all* $j \in \{1, ..., n\}$ *have associated net activity vector* $y_j|_{\bar{\beta}_j}$ *. For any other definition of* $\hat{\beta}_j = \sum_{i=1}^{m}\hat{\xi}_{ij}\bar{\beta}_i$*, for all* $j \in \{1, ..., n\}$ *have associated net activity vector* $y_j|_{\hat{\beta}_j}$*, where obviously* $y_j|_{\bar{\beta}_j} \not\Leftrightarrow y_j|_{\hat{\beta}_j}$*, since* $\bar{\beta}_j \neq \hat{\beta}_j$*. For same equilbrium* $\bar{z} \Longrightarrow m(j) \in \mathbb{R}_+$*, for all* $j \in \{1, ..., n\}$*, have period two* $y_j|_{\bar{z}}$*. Then, the objective functions*

$$\bar{\beta}_j \cdot (\bar{p}\square y) \neq \hat{\beta}_j \cdot (\bar{p}\square y) \neq (\bar{p}\square y) \tag{A.79}$$

obviously imply that $y_j|_{\bar{z}} \neq y_j|_{\bar{\beta}_j} \neq y_j|_{\hat{\beta}_j}$ *.*
Consequently,

$$y_j|_{\bar{z}} \neq y_j|_{\bar{\beta}_j} \Longrightarrow \langle \Pi_C \rangle \not\Leftrightarrow \langle \Pi_P \rangle \tag{A.80}$$

and

$$\langle \Pi_C \rangle \not\Leftrightarrow \langle \Pi_P \rangle \Longrightarrow B_C \not\Leftrightarrow B_P. \tag{A.81}$$

Let $z_i = (\theta_{ij} - \zeta_{ij}) \sum_{i=1}^{m} z_i$*, and consumer* $i \in \{1, ..., n\}$ *choosing* $\theta_i \implies z_i$*, and vice versa* $z_i \implies \theta_i = \sum_{j=1}^{n} \frac{z_i(j)}{\sum_{i=1}^{m} z_i(j)}$*. Therefore, can rewrite utility maximization in* $\mathcal{E}_P$ *of* $\max\limits_{x_i,\theta_i} u_i(x_i)_P$ *into* $\max\limits_{x_i,z_i} u_i(x_i)_P$*. It follows*

$$\max_{x_i,z_i} u_i(x_i)_P \not\Leftrightarrow \max_{x_i,z_i} u_i(x_i)_C \tag{A.82}$$

since $B_C \not\Leftrightarrow B_P$*. We conclude that*

$$\mathcal{E}_P \not\Leftrightarrow \mathcal{E}_C \tag{A.83}$$

Proof A.36 (Theorem 5.6) *Follows from Theorem (5.3), and from the profit maximization Theorem (5.4) in chapter 5.*

Appendix B

Collection of Mathematical Theorems and Definitions

This section collects some theorems and mathematical concepts applied in this thesis. We state the results without proof. Most of the differential geometry material can be found in [32],[30],[17],[42],[19]. Other mathematical material on analysis can be found in [51], including calculus on manifolds . See [3] for functional analysis, and [48] for measure theory. Basic concepts in topology are found in [44], and [35], [36]. For concepts on convex sets see [24]. General equilibrium theory and mathematical concepts in general equilibrium such as the equilibrium manifold, can be found in [6],[5], [12], and [17].

Theorem B.1 (Separation Theorem for Convex Sets) *(i) If C and U are non-empty convex subsets of R^n with $C \cap U = \varnothing$, then there exists, $p \in R^n, p \neq 0$, such that*

$$\sup_{x \in C} px \leq \inf_{x' \in U} px' \tag{B.1}$$

(ii) if in addition C is compact, and U closed, then

$$\sup_{x \in C} px < \inf_{x' \in U} px' \tag{B.2}$$

Definition B.1 (Definition of a Convex Cone) *Let* $C \subset R^n$ *be a convex set. If* $\bar{x} \in C$, *then the normal cone to* C *at* $\bar{x}$ *is defined by*

$$N_C(\bar{x}) = \{P \in R^n : P\bar{x} \geq Px, \forall x \in C\}.$$

Definition B.2 (Abstract definition of a Manifold) *A manifold is a topological space which locally looks like Cartesian n-space* $\mathbb{R}^n$*; it is built up of pieces of* $\mathbb{R}^n$ *glued together by homeomorphisms. If these homeomorphisms are differentiable we obtain a differentiable manifold.*

Theorem B.2 (Transversality Theorem) *If the smooth map* $f : X \to Y$ *is transversal to a submanifold* $Z \subset Y$*, then the preimage* $f^{-1}(Z)$ *is a submanifold of X. Moreover, the codimension of* $f^{-1}(Z)$ *in X equals the codimension of Z in Y.*

Theorem B.3 (Parametric Transversality Theorem) *Let* V, M, N *be* C^r *manifolds without boundary and* $A \subset N$ *a* C^r *submanifold. Let* $F : V \to C^r(M, N)$ *satisfy the following conditions:*

(i) the evaluation map $F^{ev} : V \times M \to N$, $(v, x) \longmapsto F_v(x)$, *is* C^r;

(ii) F^{ev} *is transverse to A;*

(iii) $r >$*max*$\{0, \dim M + \dim A - \dim N\}$.

Then the set

$$\pitchfork (F; A) = \{v \in V : F_v \pitchfork A\} \tag{B.3}$$

is residual and therefore dense. If A *is closed in* N *and* F *is continuous for the strong topology on* $C^r(M, N)$ *then* $\pitchfork (F; A)$ *is also open.*

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