Embedding Theorem for the automorphism group of the α-enumeration degrees

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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Acknowledgements

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- Gerald Sacks and Chi Tat Chong for writing invaluable books on α-computability theory [23] [4] that convinced me and made it feasible within the scope of this thesis to work on the embedding theorem for the automorphism group of the α-enumeration degrees instead,

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- to my wife, Rheslyn, and my daughter Sophia for being by my side. Rheslyn's suggestions and feedback on formatting with LaTeX were also very helpful.

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Abstract

It is a theorem of classical Computability Theory that the automorphism group of the enumeration degrees $\mathcal{D}_e$ embeds into the automorphism group of the Turing degrees $\mathcal{D}_T$. This follows from the following three statements:

1. $\mathcal{D}_T$ embeds to $\mathcal{D}_e$,
2. $\mathcal{D}_T$ is an automorphism base for $\mathcal{D}_e$,
3. $\mathcal{D}_T$ is definable in $\mathcal{D}_e$.

The first statement is trivial. The second statement follows from the Selman's theorem: $A \leq_e B \iff \forall X \subseteq \omega[B \leq_e X \oplus \overline{X} \implies A \leq_e X \oplus \overline{X}]$. The third statement follows from the definability of a Kalimullin pair in the α-enumeration degrees $\mathcal{D}_e$ and the following theorem: an enumeration degree is total iff it is trivial or a join of a maximal Kalimullin pair.

Following an analogous pattern, this thesis aims to generalize the results above to the setting of α-Computability theory. The main result of this thesis is Embedding Theorem: the automorphism group of the α-enumeration degrees $\mathcal{D}_{\alpha e}$ embeds into the automorphism group of the α-degrees $\mathcal{D}_\alpha$ if α is an infinite regular cardinal and assuming the axiom of constructibility $V = L$. If α is a general admissible ordinal, weaker results are proved involving assumptions on the megaregularity.

In the proof of the definability of $\mathcal{D}_\alpha$ in $\mathcal{D}_{\alpha e}$ a helpful concept of α-rational numbers Q_α emerges as a generalization of the rational numbers $\mathbb{Q}$ and an analogue of hyperrationals. This is the most valuable theory development of this thesis with many potentially fruitful directions.

Contents

Chapter 1

Conventions

We state global conventions sometimes locally violated.

1.1 Notational conventions

Facts and proofs

Facts do not have proofs. If a proof for some other logical statement such as a proposition, lemma, theorem, corollary is not included, the statement will be marked with a box $\Box$ at the end.

Binding strength of arithmetical operators

$\Sigma_i i + x :\equiv (\Sigma_i i) + x$.

Binding strength of logical symbols

The equally strongest symbols are $\neg\ \forall\ \exists$. Then continued from the strongest to the weakest: $\wedge\ \vee\ \implies\ \iff$. Brackets override the binding strength. Thus for example $\forall x.\phi(x) \vee \psi \implies \chi \equiv ((\forall x.\phi(x)) \vee \psi) \implies \chi$.

Bounded quantifiers over several variables

A bounded quantifier over several variables bounds all of them. Thus $\forall a, b, c > 0.a + b + c = z$ abbreviates $\forall a > 0 \forall b > 0 \forall c > 0.a + b + c = z$ which is different from $\forall a \forall b \forall c > 0.a + b + c = z$ where only the third variable c is bounded.

Evaluation of the logic formulas

Brackets are used as a function $[] : \texttt{PROP} \to 2$ to give a valuation of a logic formula in $\texttt{PROP}$ with the valuation for the atoms as $[\texttt{false}] :\equiv 0$ and $[\texttt{true}] :\equiv 1$.

Ordinals

In the context of α-Computability Theory, the ordinal α always denotes an admissible ordinal, i.e. L_α satisfies the axioms of Kripke-Platek set theory. Similarly, the ordinal β denotes a limit ordinal.

1.2 Metatheory

Proofs in this thesis are carried out in ZFC - Zermelo-Fraenkel set theory with the axiom of choice. See appendix A.1 for the list of ZFC axioms.

Chapter 2

Thesis summary

The central focus of this thesis is to generalize the embedding theorem 2.1.17 that the automorphism group of the enumeration degrees embeds into the automorphism group of the Turing degrees to the setting of α-Computability Theory.

The enumeration reducibility and degrees are a major research area in Computability Theory with first investigations at least as early as 1961 [19] and 1974 [3].

A generalized notion of α-enumeration degrees in α-Computability Theory has been investigated by the author for the first time[1]. To generalize the major recent result meant on one hand the development of many notions and proving of many intermediate results. On the other hand this required a careful selection of the results and notions to be generalized.

2.1 Embedding Theorem in classical Computability Theory

We present some essential concepts first, including enumeration reducibility, semicomputability, Kalimullin pair and degree structures. Then we state and outline the proof of the embedding theorem in classical Computability Theory.

For a more detailed background in classical Computability Theory consult the Cooper's book Computability Theory [5] which includes the material on the following, but not limited to:

- Turing machines, computability, reducibility, degrees and jump,
- computably enumerable sets and degrees,
- Peano arithmetic, arithmetical hierarchy and sets,

[1] The only publication that the author is aware of and which treats α-enumeration reducibility briefly is [6]. However, α-enumeration degrees remain there untouched.

- the enumeration reducibility, degrees and jump,
- total sets and degrees,
- many-one reducibility and degrees,
- priority arguments.

2.1.1 Basic notions

In remark 2.1.1 below we recall the classes of the formulas in the arithmetical hierarchy. For a full definition, see [5].

Remark 2.1.1. (Arithmetical hierarchy)

- A formula ϕ belongs to the classes Σ_0 and Π_0 iff it is equivalent to a formula with bounded quantifiers only.
- A formula ϕ belongs to the class Σ_{n+1} iff it is equivalent to a formula of the form $\exists x_1 \exists x_2 \ldots \exists x_n \psi$ where ψ belongs to the class Π_n.
- A formula ϕ belongs to the class Π_{n+1} iff it is equivalent to a formula of the form $\forall x_1 \forall x_2 \ldots \forall x_n \psi$ where ψ belongs to the class Σ_n.
- A formula ϕ belongs to the class Δ_n iff ϕ belongs to both Σ_n and Π_n.

We use remark 2.1.1 further to define the definability classes over some set M.

Definition 2.1.2. (Definability classes over M)
Let $\mathcal{C}$ be a class of first-order formulas in the language of ZF. Specifically, let $\mathcal{C} \in \{\Sigma_n, \Pi_n, \Delta_n\}$ for some $n \in \mathbb{N}$. Let M be a set, e.g. the domain of the model of computation. Then $\mathcal{C}(M)$ denotes the subsets of M which are definable with some formula from $\mathcal{C}$ with parameters in M.

Note 2.1.3. For the subsequent concepts in classical Computability Theory, we use the least infinite ordinal ω in place of the natural numbers $\mathbb{N}$. This does not introduce any problems since $\omega = \mathbb{N}$ and has a benefit of an easier generalization of the notation to the context of α-Computability Theory introduced later.

Definition 2.1.4. (Computability and computable enumerability)

- The set $A \subseteq \omega$ is *computable* iff A is definable over ω with some Δ_1 formula, i.e. $A \in \Delta_1(\omega)$.

- The set $A \subseteq \omega$ is *computably enumerable* (c.e.) iff A is definable over ω with some Σ_1 formula, i.e. $A \in \Sigma_1(\omega)$.

Definition 2.1.5. (Canonical index of a finite set)
Let $D \subseteq \omega$ be a finite set. Then its *canonical index* n is defined as $n := \sum_{k \in D} 2^k$. Denote a finite set with the canonical index n as D_n. In other words, D_n is the set that contains an element $k < \omega$ iff the k^{th} digit of the binary expansion of n is 1.

Definition 2.1.6. (Reducibilities)

- The set $A \subseteq \omega$ is *enumerable* from $B \subseteq \omega$ denoted as $A \leq_e B$ iff there is a c.e. set W such that:
$$\forall a < \omega[a \in A \iff \exists n < \omega[\langle a, n \rangle \in W \land D_n \subseteq B]]$$
where D_n is the finite set given by the canonical index n and $\langle a, b \rangle$ is a natural number coding the pair (a, b).

- The set $A \subseteq \omega$ is *computable* from $B \subseteq \omega$ denoted as $A \leq_T B$ iff there is a Turing machine with an oracle B that computes the characteristic function of A.

A general r-reducibility such as the Turing or the enumeration reducibility gives rise to the ordered structure of the r-degrees as in definition 2.1.7 below.

Definition 2.1.7. (r-Degrees)
Let $\leq_r \in \{\leq_e, \leq_T\}$ be the enumeration or Turing reducibility. Then $\leq_r$ induces the *equivalence relation* $\equiv_r \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ as follows:
$$A \equiv_r B \iff A \leq_r B \land B \leq_r A$$
for any sets A and B which are subsets of ω. The *r-degree* of the set $A \subseteq \omega$ denoted as $\deg_r(A)$ is its equivalence class given by the equivalence relation $\equiv_r$. In notation,
$$\deg_r(A) := \{B \in \mathcal{P}(\omega) : A \equiv_r B\}.$$
The set of *r-degrees* denoted as $\mathcal{D}_r$ is the set of the partitions of $\mathcal{P}(\omega)$ partitioned by the equivalence relation $\equiv_r$ induced by the reducibility relation $\leq_r$. In notation,
$$\mathcal{D}_r := \{\deg_d(A) : A \subseteq \omega\}.$$
The reducibility relation $\leq_r$ induces the *order* $\leq$ on the set of r-degrees $\mathcal{D}_r$ as follows:
$$\forall A, B \in \mathcal{P}(\omega)[\deg_r(A) \leq \deg_r(B) \iff A \leq_r B].$$

Using the general definition definition 2.1.7, we can specifically define the structures of the enumeration and Turing degrees in definition 2.1.8.

Definition 2.1.8. (Degrees)

- The enumeration degrees are $\mathcal{D}_e \coloneqq \mathcal{P}(\omega)/\equiv_e$ where $\equiv_e$ is the equivalence relation induced by the enumeration reducibility $\leq_e$. In detail,
  $\mathcal{D}_e \coloneqq \{\deg_e(A) : A \subseteq \omega\} = \{\{B \in \mathcal{P}(\omega) : A \leq_e B \wedge B \leq_e A\} : A \subseteq \omega\}$.
- The Turing degrees are $\mathcal{D}_T \coloneqq \mathcal{P}(\omega)/\equiv_T$ where $\equiv_T$ is induced by $\leq_T$. In detail,
  $\mathcal{D}_T \coloneqq \{\deg_T(A) : A \subseteq \omega\} = \{\{B \in \mathcal{P}(\omega) : A \leq_T B \wedge B \leq_T A\} : A \subseteq \omega\}$.

Definition 2.1.9. (Computable join)
Let $A, B \subseteq \omega$. Then the computable join of A and B is defined as

$$A \oplus B \coloneqq \{2a : a \in A\} \cup \{2b + 1 : b \in B\}.$$

Definition 2.1.10. (Set complement)
Let $A \subseteq \omega$. Then the complement of A in ω is $\overline{A} \coloneqq \omega - A$.

Fact 2.1.11. [2] For any total functions $f, g : \omega \to \omega$ we have:

$$f \leq_e g \iff f \leq_T g.$$

Remark 2.1.12. (Correspondence between $\mathcal{D}_T$ and $\mathcal{D}_e$)

- Using fact 2.1.11 for any $A, B \subseteq \omega$ we have:
  $$A \leq_T B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$
- There is an embedding $\iota : \mathcal{D}_T \hookrightarrow \mathcal{D}_e$ from the Turing degrees into the enumeration degrees given by $A \mapsto A \oplus \overline{A}$.
- The set $\iota[\mathcal{D}_T] \subseteq \mathcal{D}_e$ is called *total enumeration degrees* and is denoted by $\mathcal{TOT}_e$.

Definition 2.1.13. (Semicomputability)
A set $A \subseteq \omega$ is *semicomputable* iff there is a computable function $s_A : \omega \times \omega \to \omega$ such that for all $x, y \in \omega$:

- $s_A(x, y) \in \{x, y\}$
- $x \in A \vee y \in A \implies s_A(x, y) \in A$.

Definition 2.1.14. (Kalimullin pair)

- Let $U \subseteq \omega$. The pair of sets $A, B \subseteq \omega$ is a U-*Kalimullin pair* denoted[3] as $\mathcal{K}_U(A, B)$ iff there is a set $W \leq_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

[2] [20] Corollary XXIV p153.

[3] Here the notation $\mathcal{K}_U(A, B)$ has two meanings: the first is the reference to the pair object (A, B), the second is the statement that this pair object (A, B) is a U-Kalimullin pair.

- The pair of sets $A, B \subseteq \omega$ is a *Kalimullin pair* denoted as $\mathcal{K}(A, B)$ iff there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

- A Kalimullin pair $\mathcal{K}(A, B)$ is *non-trivial* denoted as $\mathcal{K}_{\text{nt}}(A, B)$ iff both A and B are not c.e.

- A Kalimullin pair $\mathcal{K}(A, B)$ is *maximal* denoted as $\mathcal{K}_{\max}(A, B)$ iff for any Kalimullin pair $\mathcal{K}(C, D)$ if $A \leq_e C$ and $B \leq_e D$, then $A \equiv_e C$ and $B \equiv_e D$.

Example 2.1.15. Let $A \subseteq \omega$ be a semicomputable set. Then $\mathcal{K}(A, \overline{A})$.

Definition 2.1.16. (Kalimullin pair in the enumeration degrees $\mathcal{D}_e$)
Let $a, b, u \in \mathcal{D}_e$ be some enumeration degrees.

- Then the pair of the degrees a and b is a u-Kalimullin pair denoted as $\mathcal{K}_u(a, b)$ iff
$$\exists A \in a \exists B \in b \exists U \in u . \mathcal{K}_U(A, B).$$

- The pair of the degrees a and b is a Kalimullin pair denoted as $\mathcal{K}(a, b)$ iff
$$\exists A \in a \exists B \in b . \mathcal{K}(A, B).$$

2.1.2 Embedding Theorem

Induce $\leq$ by $\leq_e$ and $\leq_T$ on $\mathcal{D}_e$ and $\mathcal{D}_T$ respectively.

Theorem 2.1.17. (Embedding theorem [24][10][25][2] [4])
$$\exists \eta : \text{Aut}(\langle \mathcal{D}_e, \leq \rangle) \hookrightarrow \text{Aut}(\langle \mathcal{D}_T, \leq \rangle)$$

Proof. This follows from the following 3 statements:

- $\mathcal{D}_T$ degrees are embeddable in $\mathcal{D}_e$, i.e. $\exists \iota : \mathcal{D}_T \hookrightarrow \mathcal{D}_e$, see remark 2.1.12.

- $\mathcal{D}_T$ are an automorphism base for $\mathcal{D}_e$, i.e.
$$\forall f \in \text{Aut}(\mathcal{D}_e)[f_{|\iota(\mathcal{D}_T)} = 1_{\iota(\mathcal{D}_T)} \implies f = 1_{\mathcal{D}_e}]$$
This is implied by Selman's theorem 2.1.18 below.

- The total degrees $\mathcal{TOT}_e := \iota[\mathcal{D}_T]$ are definable in $\mathcal{D}_e$ (theorem 2.1.20).

Then $\eta(f) := \iota^{-1} \circ f \circ \iota$ is the required embedding. □

Theorem 2.1.18. (Selman's theorem[24])
$$A \leq_e B \iff \forall X \subseteq \omega[B \leq_e X \oplus \overline{X} \implies A \leq_e X \oplus \overline{X}].$$
□

[4] The last paper with which the proof of the Embedding Theorem was completed is [2]. See [2] p13.

Definability of the total degrees [10][25][2]

Theorem 2.1.19. (Definability of a Kalimullin pair [25])
The Kalimullin pair is definable in the enumeration degrees $\mathcal{D}_e$:

$$\forall a, b \in \mathcal{D}_e[\mathcal{K}(a, b) \iff \forall x \in \mathcal{D}_e.x = (a \vee x) \wedge (b \vee x)]$$

where $\vee$ denotes lattice join and $\wedge$ lattice meet. □

Theorem 2.1.20. (Definability of the total degrees [10] [25] [2] [5])
An enumeration degree total iff it is trivial or a join of a maximal Kalimullin pair, i.e.

$$\forall d \in \mathcal{D}_e[d \in \mathcal{TOT}_e \iff d = 0 \vee \exists a, b \in \mathcal{D}_e[(d = a \vee b) \wedge \mathcal{K}_{\max}(a, b)]].$$

Proof. $\Leftarrow$ direction follows from theorem 2.1.21. $\Rightarrow$ direction follows from theorem 2.1.22. □

Theorem 2.1.21. (Semicomputable cut existence [2])
Let $A, B \subseteq \omega$ and $\mathcal{K}_{\text{nt}}(A, B)$. Then there is semicomputable cut $C \subseteq \mathbb{Q}$ such that $A \leq_e C$ and $B \leq_e \overline{C}$. □

Theorem 2.1.22. (Maximal Kalimullin pair for a total set [10] [25])
Suppose that $D \subseteq \omega$ is total, i.e. $D \equiv_e D \oplus \overline{D}$ and that $D >_T \emptyset$. Then there are sets $A, B \subseteq \omega$ such that $D \equiv_e A \oplus B$ and $\mathcal{K}_{\max}(A, B)$. □

2.1.3 Dependency tree

We provide a dependency tree of the Embedding Theorem.

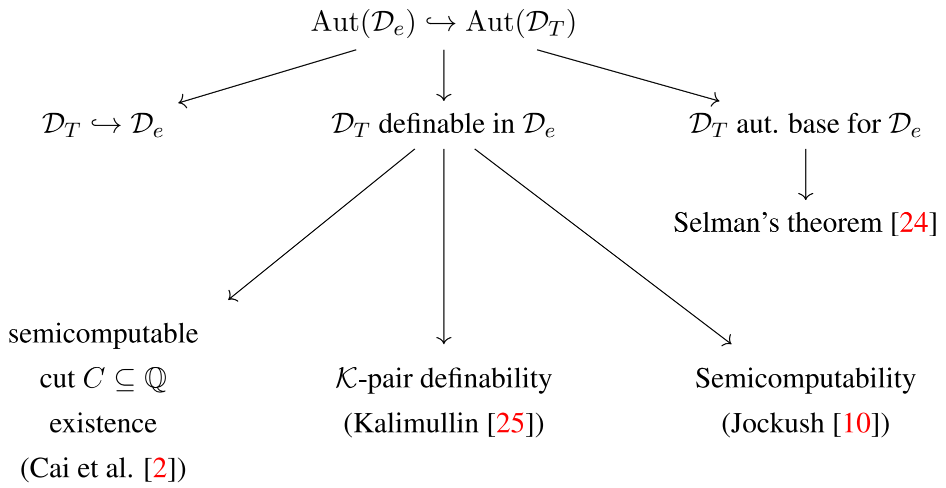
A node in the dependency tree is a statement or a mathematical area of development which *depends* upon all of its children as follows. If a node is a statement, then the proof of this statement requires the assumption of all the statements at the child nodes and the assumption of some of the statements achieved within all the mathematical areas of development at the child nodes. If a node is a mathematical area, then it to develop this area it is essential to require the assumption of all the statements at the child nodes and the assumption of some of the statements achieved within all the mathematical areas of development at the child nodes.

If a node and all its children are mathematical statements, this simplifies to saying that a node in the dependency tree is a statement implied by the conjunction of all its children.

For example, the root node is the Embedding Theorem stating $\text{Aut}(\mathcal{D}_e) \hookrightarrow \text{Aut}(\mathcal{D}_T)$ which follows from the three statements at the child nodes (namely

[5]The last paper with which the proof of the definability of the total degrees was completed is [2].

$\mathcal{D}_T \hookrightarrow \mathcal{D}_e$, $\mathcal{D}_T$ is definable in $\mathcal{D}_e$ and $\mathcal{D}_T$ is an automorphism base for $\mathcal{D}_e$) as shown in the proof of theorem 2.1.17.



2.2 Generalization to α-Computability Theory

We introduce intuitively α-Computability Theory, its methods and differences with classical Computability Theory. Then we present the main results of this thesis including the main result Embedding Theorem in α-Computability Theory. Finally, we provide a dependency tree of the Embedding Theorem which on one hand serves as a proof outline, on the other hand shows dependencies between the chapters of this thesis.

2.2.1 α-Computability Theory [23][4][17][6]

α-Computability Theory is the study of the definability theory over Gödel's L_α where α is an admissible ordinal, i.e. L_α satisfies the axioms of Kripke-Platek set theory. One can think of equivalent definitions on Turing machines with transfinite tape and time [12] [13] [14] [15] or on generalized register machines [16].

The α-degrees $\mathcal{D}_\alpha$ are the generalization of the Turing degrees. The α-enumeration degrees $\mathcal{D}_{\alpha e}$ are the generalization of the enumeration degrees. Note that ω-Computability Theory coincides with the classical Computability Theory. Similarly, the Turing degrees are the ω-degrees, the enumeration degrees are the ω-enumeration degrees.

In this summary we omit many basic definitions in α-Computability Theory. To get a glimpse of the next material, one should use the intuitions from ω-Computability Theory. For a proper introduction to α-Computability Theory, consult chapter 3.

2.2.2 Differences with classical Computability Theory and separation of notions

Limit stages

If $\alpha > \omega$, then limit stages of an algorithm for an extended Turing machine with the tape and time α have to be defined appropriately. Similarly, constructions and priority arguments at limit stages have to be specified.

Regularity and megaregularity

In α-Computability Theory one studies the definability properties of the subsets of α. In classical Computability Theory, the subsets of ω have nice properties - the central ones are regularity and megaregularity defined below.

Definition 2.2.1. (Regularity and megaregularity)

- A subset $A \subseteq \alpha$ is *regular* iff $\forall \gamma < \alpha . A \cap \gamma \in L_\alpha$.
- A subset $A \subseteq \alpha$ is *megaregular* iff for every function $f : \alpha \to \alpha$ which is Σ_1^0 definable over L_α with a parameter A we have:

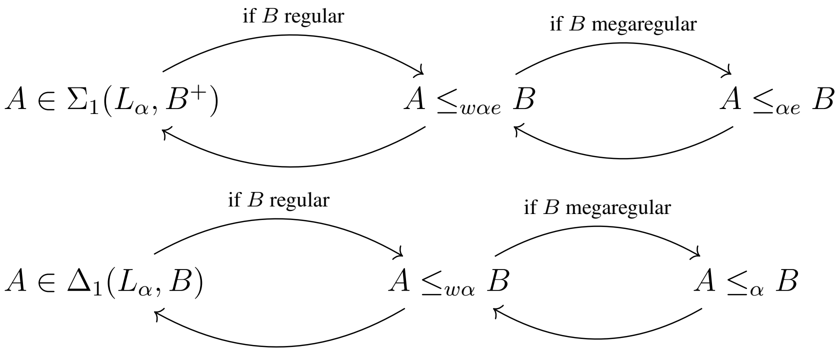
$$\forall K \in L_\alpha . f[K] \in L_\alpha .$$

Enumeration and Turing reducibility

Another major difference is that the enumeration reducibility in a generalized setting can be one of the three different reducibilities: Σ_1 definability with a positive parameter, weak α-enumeration reducibility and α-enumeration reducibility denoted as $A \in \Sigma_1(L_\alpha, B^+)$, $A \leq_{w\alpha e} B$, $A \leq_{\alpha e} B$ respectively for the parameters A and B.

Similarly the Turing reducibility corresponds to the following three notions: Δ_1 definability with a parameter, weak α-reducibility and α-reducibility, denoted as $A \in \Delta_1(L_\alpha, B)$, $A \leq_{w\alpha} B$, $A \leq_\alpha B$ respectively.

The exact relationship between the reducibilities is given below:



Projectum emerged

A well-known notion of a projectum in α-Computability Theory is a new concept relative to classical Computability Theory since in classical Computability Theory the notion of a projectum is trivial and thus not manifested.

Definition 2.2.2. (Projectum)[6]
$\alpha^* := \min\{\gamma < \alpha : \exists$ total injection $i : \alpha \rightarrowtail \gamma$ that is Σ_1 definable over $L_\alpha\}$

Example 2.2.3. $\alpha = \omega_1^{CK}$ is an admissible ordinal. Its projectum is $\alpha^* = \omega$.

Note that α many requirements can be given indices from α^* instead. Hence one uses projectum α^* to carry out a construction in only α^* many stages to satisfy α-many requirements. This technique enables one to bypass some of the difficulties caused by the lack of the megaregular and regular sets.

Summary

In summary the following are the major obstacles in generalization:

- Existence of limit stages in algorithms and constructions.
- Existence of non-regular and non-megaregular sets.
- Multiple generalized notions of reducibility.

To tackle these, one uses new notions such as projectum α^* of α.

2.3 Results

Let $\deg_{\alpha e}(Y)$ denote an α-enumeration degree that contains a set $Y \subseteq \alpha$. Let $K(U)$ denote an α-jump of U. The following are the major results presented in this thesis.

Theorem 4.3.6. (α-U-Kalimullin pair definability correspondence)
Assume $V = L$ and let α be an infinite regular cardinal. Let $A, B, U \subseteq \alpha$. Then

$$\mathcal{K}_U(A, B) \iff \forall X \subseteq \alpha. \deg_{\alpha e}(X \oplus U) = \deg_{\alpha e}(A \oplus X \oplus U) \wedge \deg_{\alpha e}(B \oplus X \oplus U).$$

□

Corollary 4.3.8. (Definability of an α-U-Kalimullin Pair)
Assume $V = L$ and let α be an infinite regular cardinal. Then

$$\forall a, b, u \in \mathcal{D}_{\alpha e}[\mathcal{K}_u(a, b) \iff \forall x \in \mathcal{D}_{\alpha e}.(a \vee x \vee u) \wedge (b \vee x \vee u) = x \vee u].$$

[6] Definition 6.1 in [1] on p174 or Theorem 1.20 in [4].

Proof. Follows from theorem 4.3.6. □

Corollary 4.4.2. (Maximal α-Kalimullin pair for a total set, see 2.1.22)
Assume $V = L$ and let α be an infinite regular cardinal. Then every nontrivial total degree is a join of a maximal $\mathcal{K}$-pair, i.e.

$$\forall a \in \mathcal{TOT}_{\alpha e} - \{0\} \exists b, c \in \mathcal{D}_{\alpha e}[(a = b \vee c) \wedge \mathcal{K}_{\max}(b, c)].$$

Proof. Follows from corollary 4.3.8 and the results on semicomputable sets in chapter 4. □

The proof of the next statement required a development of a new notion of α-rational numbers Q_α, see chapter 5.

Theorem 6.0.1. (Semicomputable Cut Existence Theorem, see 2.1.21)
Let A and B form a nontrivial α-Kalimullin pair, then there exists an α-semicomputable cut $C \subseteq Q_\alpha \cap L_\alpha$ such that $A \leq_{w\alpha e} C$ and $B \leq_{w\alpha e} \overline{C}$. □

Theorem 6.3.7. (Definability of total degrees)
Assume $V = L$ and let α be an infinite regular cardinal. A degree of $\mathcal{D}_{\alpha e}$ is total iff it is trivial or a join of a maximal $\mathcal{K}$-pair.

Proof. Follows from corollary 4.3.8, corollary 4.4.2 and theorem 6.0.1. □

Theorem 7.1.5. (Selman's theorem, see 7.1.5)
Let α be an admissible ordinal. Let $A, B \subseteq \alpha$ and let $A \oplus B \oplus K(U)$ be megaregular. Then

$$A \leq_{\alpha e} B \iff \forall X[X \equiv_{\alpha e} X \oplus \overline{X} \wedge B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}].$$

□

The main result of this thesis is the embedding theorem in α-Computability Theory.

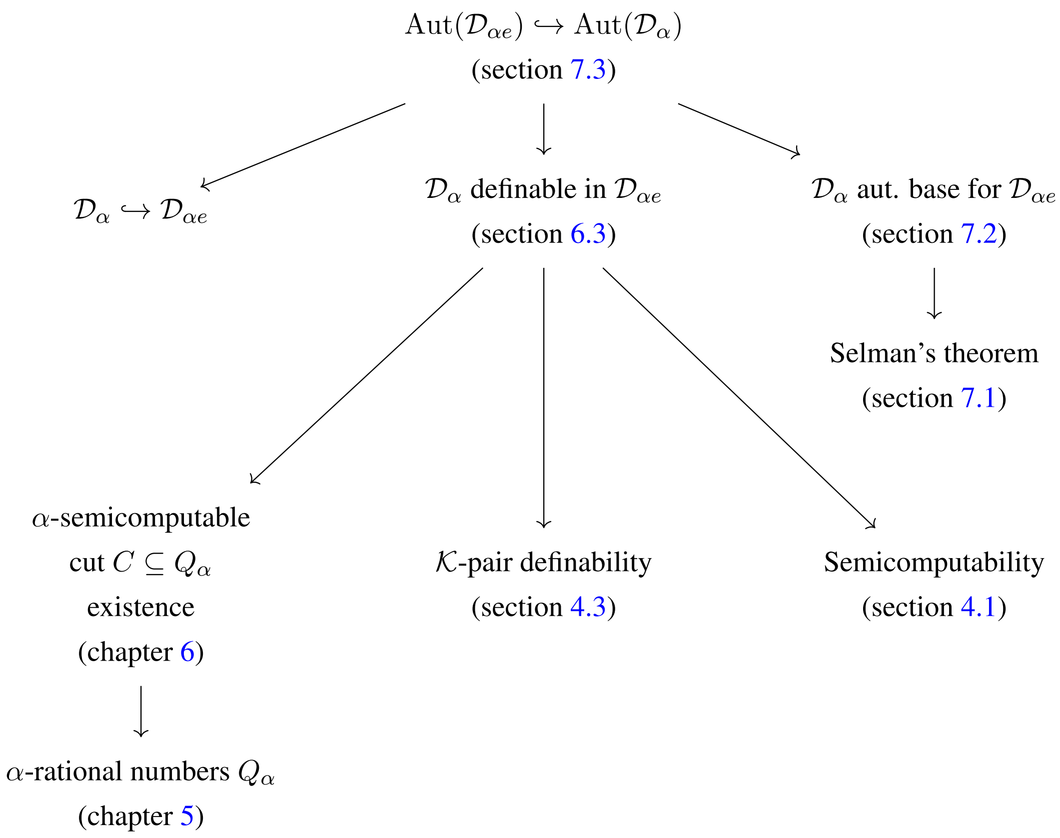
Theorem 7.3.1. (Embedding Theorem, see 2.1.17)
Assume $V = L$. Let α be an infinite regular cardinal. Then $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_\alpha)$.

Proof. Follows from theorem 6.3.7 and theorem 7.1.5. □

2.4 Dependency tree

We provide a dependency tree for the Embedding Theorem in α-Computability Theory.

A node in the dependency tree is a statement or a mathematical area of development which depends upon all of its children. If a node and all its children are mathematical statements, this simplifies to saying that a node in the dependency tree is a statement implied by the conjunction of all its children.



2.5 Established and new material

In section 2.3 we presented the major new results in this thesis. Here we summarize in more detail which results and notions in this thesis are new and which ones were introduced prior to this thesis. The greatest detail about the first appearance of the results themselves can be found in the text.

2.5.1 Classical Computability Theory

The new results in this thesis in α-Computability Theory are well-established in classical Computability Theory. This thesis does not introduce anything new in classical Computability Theory. Most of the used material from classical Computability Theory can be found in [5, 2, 10, 24, 25] .

2.5.2 Chapter 3: α-Computability Theory

Section 3.1: Set Theory

The definition of the ordinal sum 3.1.7 and its related result proposition 3.1.28 are new. Other material is well-established and most of it can be found in [1][4][8][23].

Section 3.2: Basic concepts in α-Computability Theory

This section contains introductory material about α-Computability all of which can be found in [4] and [23]. Some statements without the references (i.e. 3.2.11,3.2.12,3.2.13,3.2.14) are usually considered too trivial to be even stated explicitly outside of this thesis, yet are widely used implicitly.

Section 3.3: Higher Order Definability

Well-established material discussed in [23].

Section 3.4: Relativization

The parametrized definability or the definability with a parameter is very common and widely used in [1][4][23] for example. On the other hand, the restriction to a positive or a negative parameter seems to be introduced in this thesis for the first time, although it is a very natural direction to explore.

This section contains some new results introduced in this thesis, however, not significant as they are usually a straight-forward generalization, adaptation or consequence of other well-established statements with a general parameter or without it.

Section 3.5: α-computable index of a set

The material in this section is wisely used, but usually implicitly assumed in α-Computability Theory on the grounds of a generalized Church-Turing thesis. See the remark in [4] at the bottom of the page 7 for example.

Section 3.6: Projectum

A projectum is a well-established notion in α-Computability Theory, see [4][23]. The statements about the projectum in section 3.6 are well-known and often implicitly assumed in the literature (without a proof or a reference) as they are trivial to prove. Explicit proofs were given for the statements for which no reference was found. However, nothing in this section is new in this thesis.

Section 3.7.1: Reducibility

This section generalizes to α-Computability Theory the enumeration reducibility and its relationships with the Turing reducibility which are well-known in classical Computability Theory.

The weak enumeration operator and reducibility were introduced in [6] and easily seen equivalent definitions used in this thesis. Definition 3.7.2 and definition 3.7.3 are new in this thesis, but equivalent. The definition of the α-enumeration reducibility (definition 3.7.7) is different from the definitions in [6] and the relationships were not investigated as the focus of this thesis was different from the focus in [6].

Although this section contains new material, it is usually straight-forward to prove and see from the existing results in classical Computability Theory or the results in α-Computability Theory about the α-reducibility. The later is true by realizing that the original definition of the (weak) α-reducibility just copies the new definition of the (weak) α-enumeration reducibility twice - once for the positive part of the set and once for the positive part of the complement of the set. See fact 3.7.16 for a formal correspondence.

Section 3.8: Regularity

This section investigates the following notions of the regularity: regularity, quasiregularity, hyperregularity, megularity; and compares them with the notions of α-finiteness, α-computability and boundedness. Regularity and hyperregularity are well-established and investigated notions in α-Computability Theory, see [4][23].

Quasiregularity is a new notion which is slightly weaker than regularity. Megaregularity is a new notion which is slightly stronger than hyperregularity. The material which concerns quasiregularity and megaregularity in this thesis is new. The remaining material can be considered as well-known and is often assumed and used implicitly in the literature. Some explicit proofs are given in this thesis.

Section 3.9: Reducibilities and definability

This section explores mostly the relations between the Σ_1-definability with a positive parameter, weak α-enumeration reducibility and α-enumeration reducibility on general, regular and megaregular sets. An analogous exploration is present in [4] between the Δ_1-definability with a general parameter, weak α-reducibility and α-reducibility on general, regular and hyperregular sets, see [4] propositions 1.15, 1.30, 1.32. Hence the general ideas of making such comparisons between the notions of the definability is not new. However, the exact results concerned with the Σ_1-definability with a positive parameter and (weak) α-enumeration reducibility in this section are new. Their proof may be inspired by analogous results, but often differs as the definability with a positive parameter behaves quite differently from the definability with a general parameter.

Section 3.10: Degree Theory

The α-degrees are well-established and studied, see [4][23]. The α-enumeration degrees are newly introduced in this thesis. And so the results about them are new. Similarly, α-join operator is a new concept introduced in this thesis, but possibly general and useful enough to have been discovered somewhere before.

Section 3.11: Computability with infinite cardinal and assumption $V = L$

The well-established material of this section can be found in some standard texts on Set Theory that include material on regular cardinals, replacement axiom and constructive universe such as [21]. Some material in this section can be thought of as new, but it rather puts the well-established material from Set Theory into a perspective of α-Computability Theory through direct observations or simple proofs.

Section 3.12: α-enumeration jump

This section generalizes a well-known notion of the enumeration jump from classical Computability Theory (E.g. see [25].) unseen in α-Computability Theory before. Therefore all the material including definitions and results in this section is new.

Section 3.13: Simple construction

In this section a simple construction is presented which is new in this thesis.

2.5.3 Chapter 4: Kalimullin pair and semicomputability

All the material in this chapter except lemma 4.1.13 is new in α-Computability Theory and generalizes well-established material from classical Computability Theory. Section 4.1 generalizes the classical notion of semicomputability and some related results from [10]. Section 4.2 and section 4.4 generalize the results on the definability of a Kalimullin pair [25] and the results about maximal Kalimullin pairs [2] in classical Computability Theory.

2.5.4 Chapter 5: α-rational numbers Q_α

An idea to extend or generalize rational or real numbers is not new. Conway's field of the surreal numbers is a good example which extends the field of the real numbers into transfinite. However, the definition of the β-rational numbers Q_β is new and is well-suited for the application within this thesis. Just as the real numbers can be defined in many different ways (e.g. through Dedekind cuts, equivalence classes of Cauchy sequences or infinite decimal representations), and each is better suited for a different application, so also β-rational numbers and their definition can turn out to be more suitable in some applications than other well-established concepts.

Q_β is defined as a set of α-strings. An α-string is a new concept in this thesis which is a generalization of a string. A string is a well-known concept in the field of Computer Science. An α-rational is represented by a binary α-string. Binary representations are common in Computer Science. Transfinite binary representations are less common, but not new in this thesis, e.g. they are very useful (but usually implicit) in general priority arguments in α-Computability Theory.

As Q_β is a new concept in this thesis, also all results specific to it. The chapter uses a general well-known result from the model theory about the infinite countable dense orders (theorem 5.3.1).

2.5.5 Chapter 6: Semicomputable cut in Q_α

All the material in this chapter is new in α-Computability Theory and generalizes the labelling algorithm and the definability of the total enumeration degrees in the enumeration degrees from the classical Computability Theory [2]. The generalization is not straight-forward and it is claimed that the most creative and novel work of the thesis is present in this chapter.

2.5.6 Chapter 7: Embedding Theorem

All the material in this chapter is new in α-Computability Theory and generalized from classical Computability Theory. The Selman's Theorem is generalized from [24][30]. The Embedding Theorem is generalized from [2] p13.

Chapter 3

α-Computability Theory

α-Computability Theory is the study of the definability theory over Gödel's L_α where α is an admissible ordinal. In this thesis, α always denotes an admissible ordinal. One can think of equivalent definitions on Turing machines with a transfinite tape and time [12] [13] [14] [15] or on generalized register machines [16]. Recommended references for α-Computability Theory are [23], [4], [17] and [6]. Classical Computability Theory is α-Computability Theory restricted to $\alpha = \omega$.

In this chapter we introduce key notions and topics in α-Computability Theory relevant to this thesis including admissibility, α-finiteness, α-computability, α-computable enumerability, relativization, projectum, regularity, quasiregularity, hyperregularity and megaregularity, α-enumeration reducibility and α reducibility, degrees and an α-enumeration jump. We observe that α-Computability Theory behaves more like classical Computability Theory when α is an infinite regular cardinal. Finally, we perform a simple construction involving a pattern central to more complex arguments presented later in this thesis.

A lot of content in this chapter is essential to read the main results in chapter 4, chapter 5, chapter 6 and chapter 7. Many statements in this chapter are invoked in later proofs so frequently that their use is often implicit.

3.1 Set Theory

We mainly introduce Gödel's constructible hierarchy, Kripke-Platek set theory and admissible ordinals. Background on Set Theory can found in [8]. The material on admissible sets is mainly in [1].

Everything in this section is well-known except the notion of the ordinal sum (definition 3.1.7).

3.1.1 Ordinals and cardinals

Definition 3.1.1. (Ordinal and cardinal [8])

- The relation $< \subseteq X \times X$ is a *well-order* on X iff it is a total order and every non-empty subset of X has a least element wrt $<$.
- The set X is *transitive* iff $\forall Y \in X \forall Z \in Y.Z \in X$.
- An *ordinal* is a transitive set well-ordered by $\in$.
- A *cardinal* is an ordinal which is equal to its own cardinality.

Definition 3.1.2. (Cofinality of an ordinal [8])
The cofinality of an ordinal α is denoted as $\mathrm{cf}(\alpha)$ and defined as:

- $\mathrm{cf}(0) := 0$,
- $\mathrm{cf}(\gamma + 1) := 1$,
- $\mathrm{cf}(\delta) := \min\{\gamma \leq \delta : \exists f : \gamma \to \delta[\forall \epsilon < \delta \exists \beta < \gamma . f(\beta) > \epsilon]\}$ if $\lim(\delta)$.

Definition 3.1.3. (Regular and singular cardinal [8])
A *regular* cardinal is a cardinal that is equal to its own cofinality. Otherwise the cardinal is called *singular*.

Example 3.1.4. [8] $\aleph_0$ and $\aleph_{\gamma+1}$ are regular cardinals. $\aleph_{\omega_n}$ is a singular cardinal for $n \in \mathbb{N}$.

3.1.2 Ordinal arithmetic

Fact 3.1.5. [8] An ordinal α is a limit ordinal iff there exists an ordinal β s.t. $\alpha = \omega \cdot \beta$.

Theorem 3.1.6. (*Cantor Normal Form - CNF* [8])
For every positive ordinal α there exist unique positive integers $a_1, \ldots, a_k$ and ordinals $\alpha_1, \ldots, \alpha_k$ satisfying $\alpha_1 > \ldots > \alpha_k \geq 0$ s.t.

$$\alpha = \omega^{\alpha_1} \cdot a_1 + \ldots + \omega^{\alpha_k} \cdot a_k.$$

□

We define the ordinal sum. In α-Computability, the computations are performed within α time where α is an ordinal strong enough to be closed under the ordinal sum. This property is exploited in chapter 6.

Definition 3.1.7. (Ordinal sum[1])
For an ordinal γ we define the *ordinal sum* $\sum_{\beta<\gamma} \beta$ recursively:

[1] Introduce in this thesis.

- $\sum_{\beta<0}\beta := 0$,
- $\sum_{\beta<\gamma+1}\beta := \left(\sum_{\beta<\gamma}\beta\right) + \gamma$,
- $\sum_{\beta<\delta}\beta := \sup\{\sum_{\beta<\gamma}\beta : \gamma < \delta\}$ if $\lim(\delta)$.

Proposition 3.1.8. [2] Let $\alpha = \omega^{\alpha_1} \cdot a_1 + \ldots + \omega^{\alpha_k} \cdot a_k$ be an ordinal greater than ω expressed in CNF where $\alpha_1 > \ldots > \alpha_k \geq 0$. Then

i) $\forall\beta < \alpha(\beta + \alpha = \alpha) \iff \forall\beta, \gamma < \alpha(\beta + \gamma < \alpha) \iff \exists\delta(\alpha = \omega^\delta)$,

ii) $\forall\beta < \alpha(\beta \cdot \alpha = \alpha) \iff \forall\beta, \gamma < \alpha(\beta \cdot \gamma < \alpha) \iff \exists\epsilon\left(\alpha = \omega^{\omega^\epsilon}\right)$,

iii) $\sum_{\beta<\alpha}\beta = \alpha \iff \exists\delta(\lim(\delta) \wedge \alpha = \omega^\delta)$,

iv) $1 + \alpha = \alpha \iff \alpha_k \geq 1$,

v) $\omega \cdot \alpha = \alpha \iff \alpha_k \geq \omega$,

vi) $\lim(\delta) \implies \alpha \cdot \delta = \omega^{\alpha_1} \cdot \delta$.

□

3.1.3 Gödel's Constructible Universe

Definition 3.1.9. (Gödel's Constructible Universe [4][23])
Let `Ord` represent the class of all ordinals. *Gödel's constructible universe* is denoted by L and defined by transfinite recursion as follows:

- $L_0 := \emptyset$,
- $L_{\gamma+1} := \mathrm{Def}(L_\gamma) := \{x | x \subseteq L_\gamma$ and x is first-order definable over $L_\gamma\}$ for any $\gamma \in \mathtt{Ord}$,
- $L_\delta := \bigcup_{\gamma<\delta} L_\gamma$ for a limit ordinal $\delta \in \mathtt{Ord}$,
- $L := \bigcup_{\beta\in\mathtt{Ord}} L_\beta$.

3.1.4 Kripke-Platek set theory

Definition 3.1.10. (Kripke-Platek set theory axioms [1])

- Extensionality: $\forall a, b[a = b \iff \forall x[x \in a \iff x \in b]]$
- Empty set: $\exists a \forall x . x \notin a$

[2] Some of these facts can be found as exercises to the course Axiomatic Set Theory taught by Peter Holy [7].

- Pairing: $\forall a \forall b \exists c.c = \{a, b\}$
- Union: $\forall a \exists b \forall x[x \in b \iff \exists y[x \in y \wedge y \in a]]$
- Induction: For any formula $\phi(x)$ the following holds:
$$\forall x[\forall y \in x.\phi(y) \implies \phi(x)] \implies \forall x.\phi(x).$$
- Σ_0-separation: For any Σ_0 formula $\phi(x)$ the following holds:
$$\forall a \exists b \forall x[x \in b \iff x \in a \wedge \phi(x)].$$
- Σ_0-collection: For any Σ_0 formula $\phi(x, y)$ the following holds:
$$\forall u[\forall x \in u \exists y.\phi(x, y) \implies \exists z \forall x \in u \exists y \in z.\phi(x, y)].$$

3.1.5 Admissible ordinal

Definition 3.1.11. (Admissible ordinal [4][1])

- An ordinal α is Σ_n *admissible* iff α is a limit ordinal and L_α satisfies Σ_n-collection:
$\forall \phi(x, y) \in \Sigma_n(L_\alpha).L_\alpha \models \forall u[\forall x \in u \exists y.\phi(x, y) \implies \exists z \forall x \in u \exists y \in z.\phi(x, y)]$
where L_α is the α-th level of the Gödel's Constructible Hierarchy (definition 3.1.9).
- An ordinal α is *admissible* iff α is Σ_1 admissible.

Throughout the rest of the thesis, the ordinal α is always an admissible ordinal unless a weaker assumption is made explicitly.

Definition 3.1.12. (Stable ordinal [1])
An ordinal β is stable iff $L_\beta \prec_{\Sigma_1} L$, i.e. L_β is a Σ_1 elementary substructure of L.

Example 3.1.13. (Examples of admissible ordinals [4] [31])

- ω_1^{CK} - the Church-Kleene ordinal, the first non-computable ordinal,
- every stable ordinal α, e.g. δ_2^1 - the least ordinal which is not an order type of a Δ_2^1 subset of $\mathbb{N}$, 1st stable ordinal,
- every infinite cardinal in a transitive model of ZF [3].

Lemma 3.1.14. [4] Let α be an admissible ordinal. If $A \in \Delta_1(L_\alpha)$ and $A \subseteq K \in L_\alpha$, then $A \in \Delta_0(L_\alpha)$.

[3] [1] p53 Corollary 3.4.
[4] The relativization of lemma 3.1.14 is proposition 3.4.18.

Proof. This follows from the basic fact that KP $\vdash \Delta_1$-separation. □

Lemma 3.1.15. [1] Let α be an admissible ordinal. If $\delta \leq \alpha$, then $L_\delta \preceq_{\Sigma_0} L_\alpha$.

Proof. This follows from the fact that L_δ is transitive.

For a detailed proof of the relativized statement see proposition 3.4.21. □

Lemma 3.1.16. [4] Let α be an admissible ordinal. If $A \in \Delta_1(L_\alpha)$ and $A \subseteq K \in L_\alpha$, then $A \in L_\alpha$.

Proof. Assume $A \in \Delta_1(L_\alpha)$ and $A \subseteq K \in L_\alpha$. By lemma 3.1.14 $A \in \Delta_0(L_\alpha)$. Note that since the formula defining A is finite, it uses only a finite number of parameters from L_α. Thus there is some $\delta < \alpha$ s.t. all these L_α parameters are in L_δ. As $L_\delta \preceq_{\Sigma_0} L_\alpha$ by lemma 3.1.15 and $A \in \Delta_0(L_\alpha)$, so $A \in \Delta_0(L_\delta)$ and $A \in \text{Def}(L_\delta)$. Therefore $A \in L_\alpha$ as required. □

Definition 3.1.17. (Replacement axiom [8])
Let $\mathcal{C}$ be a class of formulas, e.g. Σ_n. Then L_α satisfies $\mathcal{C}$-replacement axiom iff we have the following: for any total function $f : \alpha \to \alpha$ definable with some formula in $\mathcal{C}$ and parameters from L_α and for any $K \in L_\alpha$ it is true that $f[K] \in L_\alpha$,

Lemma 3.1.18. If $L_\alpha \models \Sigma_1$-collection, then $L_\alpha \models \Sigma_1$-replacement.

Proof. Assume $L_\alpha \models \Sigma_1$-collection. Assume $f \in \Sigma_1(L_\alpha)$, $K \in L_\alpha$ and $f[K] \subseteq \hat{K} \in L_\alpha$. Note $f[K] \in \Delta_1(L_\alpha)$. Thus $f[K] \in L_\alpha$ by lemma 3.1.16. Therefore $L_\alpha \models \Sigma_1$-replacement as required. □

Proposition 3.1.19. (Equivalent notions of admissibility[5])
Let α be a limit ordinal. TFAE:

- α is admissible,
- L_α satisfies Σ_1-collection,
- L_α satisfies Σ_0-collection,
- $L_\alpha \models$ KP where KP is Kripke-Platek set theory,
- $L_\alpha \models \Sigma_1$-replacement for total functions: for any total $\Sigma_1(L_\alpha)$ definable function $f : \alpha \to \alpha$ and for any $K \in L_\alpha$ it is true that $f[K] \in L_\alpha$,
- $L_\alpha \models \Sigma_1$-replacement for partial function: for any partial $\Sigma_1(L_\alpha)$ definable function $f : \alpha \rightharpoonup \alpha$ and for any $K \in L_\alpha$: if $f[K]$ is defined (i.e. $\forall x \in K. f(x) \downarrow$), then $f[K] \in L_\alpha$.

[5]Uses [23] Chapter VII: Admissibility and Regularity, Section 1.1 and Proposition 1.5. More material on this can be also found in [1].

Proof. To prove the proposition note the following facts. By Proposition VII.1.5 of [23] we have: α is admissible iff $L_\alpha \models \Sigma_0$-Collection. Note that $L_\alpha \models \Sigma_1$-collection implies $L_\alpha \models \Sigma_1$-replacement by lemma 3.1.18.

If $L_\alpha \models \Sigma_1$-replacement, then imply that $L_\alpha \models \Sigma_0$-collection by constructing a total function $f : \alpha \to \alpha \in \Sigma_1(L_\alpha)$ from $\Sigma_0(L_\alpha)$ formula $\phi(x, y)$. □

Admissibility and cofinality

Definition 3.1.20. ($\Sigma_n(L_\alpha)$ cofinality[6])
Let $\rho \leq \alpha$. $\Sigma_n(L_\alpha)$ cofinality of ρ is defined as

$$\sigma_n\mathrm{cf}_\alpha(\rho) := \min\{\beta \leq \rho : \exists f : \beta \to \rho \in \Sigma_n(L_\alpha)[\forall\delta < \rho\exists\gamma < \beta.f(\gamma) > \delta]\}.$$

Abbreviate $\sigma_n\mathrm{cf}_\alpha(\alpha)$ as $\sigma_n\mathrm{cf}(\alpha)$.

The ordinal $\sigma_n\mathrm{cf}(\alpha)$ measures the extend to which α is not admissible:

Proposition 3.1.21. [7] α is Σ_n admissible iff $\sigma_n\mathrm{cf}(\alpha) = \alpha$.

Proof. α is Σ_n admissible iff $L_\alpha \models \Sigma_n$-collection iff $\sigma_n\mathrm{cf}(\alpha) = \alpha$. □

Corollary 3.1.22. [8] If α is not admissible, then $\sigma_0\mathrm{cf}(\alpha) < \alpha$.

Proof. Assume that α is not admissible. Then α is not Σ_0 admissible by proposition 3.1.19. Hence $\sigma_0\mathrm{cf}(\alpha) < \alpha$ by proposition 3.1.21. □

Computably inaccessible ordinal

Remark 3.1.23. A limit of admissible ordinals may not be an admissible ordinal. An ordinal which is admissible and a limit of admissible ordinals is called to be *computably inaccessible*.

Fact 3.1.24. [9] If $\alpha > \omega$ is admissible and $L_\alpha \models \Sigma_1$-separation, then α is computably inaccessible.

Proposition 3.1.25. [10] Let $\alpha > \omega$ be an admissible ordinal.

1. If $L_\alpha \models \Sigma_2$-replacement, then there exists $\mathcal{C} \subseteq \alpha$ s.t. $\forall\beta \in \mathcal{C}.L_\beta \models \Sigma_1$-replacement and $\alpha = \sup(\mathcal{C})$.

2. Therefore if an admissible ordinal α has a maximal admissible predecessor, then L_α cannot satisfy Σ_2-replacement.

[6] [4] Definition 1.24
[7] [4] 1.24
[8] This statement was discussed with Robert Lubarsky.
[9] Follows from [1] p175 Theorem 6.3. and p176 Theorem 6.8.
[10] The proof pointed out by Michael Rathjen.

Proof. As $L_\alpha \models \Sigma_2$-replacement, so $L_\alpha \models \Sigma_1$-separation. Hence α is computably inaccessible by fact 3.1.24. Therefore the statement 1 is true.

The statement 2 follows from the statement 1. □

Closure under ordinal operations

The strength of an admissible ordinal reflects in the strength of its closure under ordinal operations. Every admissible ordinal $\alpha > \omega$, is also an ϵ ordinal, i.e. $\omega^\alpha = \alpha$. Hence proposition 3.1.8 implies proposition 3.1.26 below.

Proposition 3.1.26. [11] Let α be an admissible ordinal and $\beta, \gamma < \alpha$, then

i) $\beta + \gamma < \alpha$,

ii) $\beta \cdot \gamma < \alpha$,

iii) $\beta^\gamma < \alpha$.

□

Proposition 3.1.27. [9] If α is an admissible ordinal, then $\alpha = \omega$ or $\omega \cdot \alpha = \alpha$.

Proof. For $\alpha = \omega$ both statements clearly hold. So suppose that $\alpha > \omega$. By proposition 3.1.26 an admissible ordinal is closed under ordinal multiplication. Hence by the proposition 3.1.8ii there exists an ordinal ϵ s.t. $\alpha = \omega^{\omega^\epsilon}$.

The proposition 3.1.27 is true since $\omega^\epsilon \geq \omega$ and hence by the proposition 3.1.8v we have $\omega \cdot \alpha = \alpha$. □

Proposition 3.1.28. [12] If α is an admissible ordinal, then $\sum_{\beta<\alpha} \beta = \alpha$.

Proof. Recall the definition of the ordinal sum $\sum_{\beta<\alpha} \beta$ for an ordinal α (definition 3.1.7). Let $\delta := \omega^\epsilon$, then δ is a limit ordinal and $\alpha = \omega^\delta$ and so by the proposition 3.1.8iii we have $\sum_{\beta<\alpha} \beta = \alpha$. □

3.2 Basic concepts in α-Computability Theory

The statements in this section are very common in α-Computability Theory and can be found or used in [4][23].

Assume that α is an admissible ordinal. In α-Computability Theory, sets in L_α play a similar role that finite sets play in the classical Computability Theory.

Definition 3.2.1. [4] A set $K \subseteq \alpha$ is α-*finite* iff $K \in L_\alpha$.

[11] [1] p274 Corollary 3.5.

[12] Introduced in this thesis.

Proposition 3.2.2. [4] There exists a $\Sigma_1(L_\alpha)$ definable bijection $b : \alpha \to L_\alpha$. □

Hence we can index α-finite sets with an index in α. Let K_γ denote an α-finite set $b(\gamma)$.

Proposition 3.2.3. [17][4] For every n, there is a $\Sigma_1(L_\alpha)$ definable bijection

$$p_n : \alpha \to \alpha \times \alpha \times \ldots \times \alpha \text{ (n-fold product).}$$

By proposition 3.2.3 we can also index pairs and other finite vectors from α^n by an index in α. Moreover, by proposition 3.2.3 one can consider a partial function $f : \alpha \times \alpha \times \ldots \times \alpha \rightharpoonup \alpha$ and its graph to be subsets of α.

Remark 3.2.4. [4] If K_1 and K_2 are α-finite subsets of α, then using the admissibility of α and proposition 3.2.3 the set $p_2^{-1}[K_1 \times K_2]$ is α-finite. Thus we can encode products of α-finite sets as α-finite subsets of α using p_n. This fact will be used implicitly.

Recall definition 2.1.2 that A is $\Sigma_1(L_\alpha)$ definable iff $A \in \Sigma_1(L_\alpha)$ iff A is definable with a Σ_1 formula with parameters in L_α.

Definition 3.2.5. (α-computability and computable enumerability [4])

- A function $f : \alpha \to \alpha$ is *α-computable* iff the graph of f is $\Sigma_1(L_\alpha)$ definable.
- A set $A \subseteq \alpha$ is *α-computably enumerable* (α-c.e.) iff $A \in \Sigma_1(L_\alpha)$.
- A set $A \subseteq \alpha$ is *α-computable* iff $A \in \Delta_1(L_\alpha)$ iff $A \in \Sigma_1(L_\alpha)$ and $\alpha - A \in \Sigma_1(L_\alpha)$.

Instead of using the definability over L_α, an alternative approach to α-Computability Theory studies computation on extended Turing machines.

Remark 3.2.6. (α-Computability on an extended Turing machine [14])
Let an α-machine be a Turing machine with time α and tape length α.

- $f : \alpha \to \alpha$ is α-computable iff f is α-machine computable,
- A is α-computable iff A is α-machine computable,
- A is α-c.e. iff A is α-machine c.e.

Proposition 3.2.7. [13] For every set $A \subseteq \alpha$ we have:

$$A \in L_\alpha \text{ iff } A \in \Delta_1(L_\alpha) \text{ and } A \text{ is bounded by some } \beta < \alpha.$$

[13] Proposition 1.12b in [4].

Proof. This follows from the fact that $L_\alpha \models \Delta_1$-separation.

For a detailed proof of a more general result, see lemma 3.4.13. □

Fact 3.2.8. [14] TFAE:

i) A is α-c.e.,

ii) $\exists f : \alpha \rightharpoonup \alpha$ s.t. $f \in \Sigma_1(L_\alpha) \wedge A = \text{dom}(f)$,

iii) $\exists f : \alpha \to \alpha$ s.t. $f \in \Sigma_1(L_\alpha) \wedge A = \text{rng}(f)$.

Theorem 3.2.9. (Uniformization Theorem[15])
Let $n \geq 1$. For each $\Sigma_n(L_\alpha)$ relation $R(x, y)$ there is a $\Sigma_n(L_\alpha)$ function f satisfying

$$\forall x < \alpha[\text{ if } \exists y < \alpha.L_\alpha \models R(x, y), \text{ then } L_\alpha \models R(x, f(x))].$$

Proposition 3.2.10. (α-finite union of α-finite sets[16])
α-finite union of α-finite sets is α-finite, i.e. if $K \in L_\gamma$, then $\bigcup_{\gamma \in K} K_\gamma \in L_\alpha$.

Proof. Follows from lemma 3.5.4 i). □

Proposition 3.2.11. Let $A, B \subseteq \alpha$. If A is unbounded and $B \in L_\alpha$, then $A - B \notin L_\alpha$.

Proof. As $B \in L_\alpha$, so B is bounded. Thus $A - B$ has to be unbounded and hence cannot be α-finite. □

Note 3.2.12. Suppose that $A \subseteq \alpha$ and $f : \alpha \to \alpha$ are α-computable. Is it true that $f[A]$ is also α-computable?

Not in general using fact 3.2.8 and the fact that there are α-computably enumerable sets which are not α-computable.

Fact 3.2.13. For any $A \subseteq \alpha$, it must hold $[A \notin L_\alpha \vee \overline{A} \notin L_\alpha]$.

Proposition 3.2.14. $\forall f \in \Sigma_1(L_\alpha)[\text{dom}(f) \in L_\alpha \implies f \in L_\alpha]$.

Proof. Define $g : x \mapsto \langle x, f(x) \rangle$. Note $g \in \Sigma_1(L_\alpha)$ since

$$z \in g \iff \exists x \exists y[z = \langle x, \langle x, y \rangle \rangle \wedge (x, y) \in f]$$

and $f \in \Sigma_1(L_\alpha)$. But then $f = g[\text{dom}(f)] \in L_\alpha$ as required since $\text{dom(f)} \in L_\alpha$, $g \in \Sigma_1(L_\alpha)$ and α is admissible. □

The computable join turns out to be very useful to encode within one set the information from two sets.

[14] From Proposition 1.12a in [4].
[15] Theorem 1.27 in [4].
[16] From [23] p162.

Definition 3.2.15. [17] The *computable join* of sets $A, B \subseteq \alpha$ denoted $A \oplus B$ is defined to be

$$A \oplus B := \{2a : a \in A\} \cup \{2b + 1 : b \in B\}.$$

The computable join satisfies the usual properties of the case $\alpha = \omega$.

3.3 Higher Order Definability

Recall that L_α is defined using the first order definability possibly transfinitely many times. The first order definability over such L_α relates to the higher order definability for a suitable α. Thus α-Computability Theory in one way is a study of the higher order definability too.

Let HYP denote the class of the hyperarithmetic sets. Recall Σ^1_n, Π^1_n, Δ^1_n are second order definable classes. Let $n : \mathcal{O} \rightharpoonup \omega_1^{CK}$ take a computable notation in Kleene's $\mathcal{O} \subseteq \omega$ to the ordinal represented by it. Then:

Theorem 3.3.1. (Correspondence with second-order definability[4][23])

i) $\forall A \subseteq \omega[A \in \mathrm{HYP} \iff A \in \Delta^1_1 \iff A \in L_{\omega_1^{CK}}]$,

ii) $\forall A \subseteq \omega_1^{CK}[A \in L_{\omega_1^{CK}} \iff n^{-1}[A] \in \Delta^1_1]$,

iii) $\forall A \subseteq \omega_1^{CK}[A \in \Sigma^0_1(L_{\omega_1^{CK}}) \iff n^{-1}[A] \in \Pi^1_1]$,

iv) $\forall A \subseteq \omega[A \in L_{\delta^1_2} \iff A \in \Delta^1_2]$,

v) $\forall A \subseteq \omega[A \in \Sigma^0_1(L_{\delta^1_2}) \Longleftarrow A \in \Sigma^1_2]$.

3.4 Relativization

We study the definability over L_α with a parameter $B \subseteq \alpha$. This can be thought of as an analogue of the oracle computation with the oracle B. Once we generalize the Turing and enumeration reducibilities later in section 3.7.1, we will compare them in section 3.9 with this parametrized definability of section 3.4 in a useful way.

The definability with a parameter is very common and widely used in [1][4][23] for example. Hence the results with a general parameter in this section are not new.

On the other hand, the restriction to a positive or a negative parameter seems to be introduced in this thesis for the first time, although it is a very natural direction to explore. Consequently, the results concerned with the definability with a positive parameter are assumed to be new in this thesis.

[17] From [4] p8.

3.4.1 Model with a parameter

We parametrize the arithmetical hierarchy and define a model L_α with a parameter $B \subseteq \alpha$.

Definition 3.4.1. (Arithmetical hierarchy with a parameter[18])
Let $B \subseteq \alpha$. We call the expression B^+ a *positive parameter* and the expression B^- a *negative parameter*. Let $\mathrm{QF}(\mathcal{A})$ denote the class of quantifier free formulas with parameters from $\mathcal{A}$. We define recursively $\mathrm{QF}(L_\alpha, \mathcal{B})$ - the class of quantifier free formulas with $\mathcal{B} \in \{B, B^+, B^-\}$ as a parameter as follows:

- if $K, M \in L_\alpha$, then $x_i = x_j$, $x_i = M$, $K = x_j$ and $K = M$ are in $\mathrm{QF}(L_\alpha)$,
- if $K \in L_\alpha$, then $x \in K$ is in $\mathrm{QF}(L_\alpha)$,
- if $\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha)$, then $\neg\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha)$,
- if $\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha)$, then $\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha, \mathcal{B})$ for $\mathcal{B} \in \{B, B^+, B^-\}$,
- $x \in B$ is in $\mathrm{QF}(L_\alpha, B^+)$,
- $\neg x \in B$ is in $\mathrm{QF}(L_\alpha, B^-)$,
- if $\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha, B^+)$ or in $\mathrm{QF}(L_\alpha, B^-)$, then $\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha, B)$,
- if $\phi(\overline{x})$ and $\psi(\overline{x})$ are both in $\mathrm{QF}(\mathcal{A})$, then both $\phi(\overline{x}) \wedge \psi(\overline{x})$ and $\phi(\overline{x}) \vee \psi(\overline{x})$ are in $\mathrm{QF}(\mathcal{A})$.

We define $\Delta_0(L_\alpha, \mathcal{B})$ - the class of formulas with bounded quantifiers with $\mathcal{B} \in \{B, B^+, B^-\}$ as a parameter as follows:

- if $\phi(\overline{x})$ is in $\mathrm{QF}(L_\alpha, \mathcal{B})$, then $\phi(\overline{x})$ is in $\Delta_0(L_\alpha, \mathcal{B})$,
- if $K \in L_\alpha$ and $\phi(\overline{x})$ is in $\Delta_0(L_\alpha, \mathcal{B})$, then both $\exists x_i \in K.\phi(\overline{x})$ and $\forall x_i \in K.\phi(\overline{x})$ are in $\Delta_0(L_\alpha, \mathcal{B})$.
- if $\phi(\overline{x})$ is in $\Delta_0(L_\alpha, B^+)$ or in $\Delta_0(L_\alpha, B^-)$, then $\phi(\overline{x})$ is in $\Delta_0(L_\alpha, B)$.

Let $\Sigma_0(L_\alpha, \mathcal{B}) \coloneqq \Pi_0(L_\alpha, \mathcal{B}) \coloneqq \Delta_0(L_\alpha, \mathcal{B})$. Then $\Sigma_1(L_\alpha, \mathcal{B})$ is the class of formulas with an existential quantifier and with $\mathcal{B}$ as a parameter. More precisely we define $\Sigma_{n+1}(L_\alpha, \mathcal{B})$ as follows:

- if $\phi(\overline{x})$ is in $\Pi_n(L_\alpha, \mathcal{B})$, then $\forall x_i.\phi(\overline{x})$ is in $\Pi_n(L_\alpha, \mathcal{B})$ and $\exists x_i.\phi(\overline{x})$ is in $\Sigma_{n+1}(L_\alpha, \mathcal{B})$,

[18] Introduced in this thesis.

- if $\phi(\overline{x})$ is in $\Sigma_n(L_\alpha, \mathcal{B})$, then $\exists x_i.\phi(\overline{x})$ is in $\Sigma_n(L_\alpha, \mathcal{B})$ and $\forall x_i.\phi(\overline{x})$ is in $\Pi_{n+1}(L_\alpha, \mathcal{B})$,
- if $\phi(\overline{x})$ is in $\Theta_n(L_\alpha, B^+)$ or in $\Theta_n(L_\alpha, B^-)$, then $\phi(\overline{x})$ is in $\Theta_n(L_\alpha, B)$ for $\Theta \in \{\Sigma, \Pi\}$,
- close the classes Σ_n, Π_n under the equivalence of the formulas.

Finally, define $\Delta_{n+1}(L_\alpha, \mathcal{B})$ as follows:

- $\Delta_n(L_\alpha, \mathcal{B}) \coloneqq \Sigma_n(L_\alpha, \mathcal{B}) \cap \Pi_n(L_\alpha, \mathcal{B})$,
- if $\phi(\overline{x})$ is in $\Delta_n(L_\alpha, B^+)$ or in $\Delta_n(L_\alpha, B^-)$, then $\phi(\overline{x})$ is in $\Delta_n(L_\alpha, B)$,
- close the classes Δ_n under the equivalence of the formulas.

Proposition 3.4.2. [19] Let f be a partial function. Let $A \subseteq \alpha$. Then

i) $\mathrm{dom}(f) \in \Pi^0_n(L_\alpha, A) \implies [f \in \Sigma^0_n(L_\alpha, A) \iff f \in \Delta^0_n(L_\alpha, A)]$,

ii) $\mathrm{dom}(f) \in \Delta^1_n \implies [f \in \Sigma^1_n \iff f \in \Pi^1_n \iff f \in \Delta^1_n]$.

Proof. As f is a function,

$$f(x) = y \iff \forall z[y = z \vee f(x) \neq z] \wedge x \in \mathrm{dom}(f).$$

□

Proposition 3.4.3. [20] The following are true:

i) if $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$, then $\phi(x, B) \in \Sigma_1(L_\alpha, B)$,

ii) If $\phi(x, B) \in \Sigma_1(L_\alpha, B)$, then $\phi(x, K) \in \Sigma_1(L_\alpha)$ for $K \in L_\alpha$.

Proof. The statement i is clearly true as the definability with the positive parameter B^+ is just the definability with the parameter B with some additional restrictions on what reference to B can be made.

To prove the statement ii, note that if $\phi(x, B) \in \Sigma_1(L_\alpha, B)$, then for any K we have $\phi(x, K) \in \Sigma_1(L_\alpha, K)$ by applying syntactic substitution of B with K. However, if in addition $K \in L_\alpha$, then $\phi(x, K) \in \Sigma_1(L_\alpha)$ as K is included as a parameter in L_α already. □

Proposition 3.4.4. (Properties of arithmetic definability[21])

i) $\Theta_n(L_\alpha, (B \oplus \overline{B})^+) = \Theta_n(L_\alpha, B)$ where $\Theta \in \{\Delta, \Sigma, \Pi\}$ and $n \in \mathbb{N}^+$,

[19] Adapted from proposition 1.7 in [23].
[20] Introduced in this thesis.
[21] Introduced in this thesis.

ii) $A \in \Delta_1(L_\alpha, B) \iff A \oplus \overline{A} \in \Sigma_1(L_\alpha, B) \iff A \oplus \overline{A} \in \Pi_1(L_\alpha, B)$.

Proof. i) We would like to show that for any formula ϕ and for any $n \in \mathbb{N}^+$ we have

$$\phi \in \Theta_n(L_\alpha, (B \oplus \overline{B})^+) \iff \phi \in \Theta_n(L_\alpha, B)$$

up to the equivalence of ϕ. By expressing the formula ϕ in a prenex form we reduce the problem to proving only the following two statements for any formula ϕ:

$$\phi \in \Sigma_1(L_\alpha, (B \oplus \overline{B})^+) \iff \phi \in \Sigma_1(L_\alpha, B), \quad (3.1)$$

$$\phi \in \Pi_1(L_\alpha, (B \oplus \overline{B})^+) \iff \phi \in \Pi_1(L_\alpha, B). \quad (3.2)$$

Assume $\phi \in \Sigma_1(L_\alpha, (B \oplus \overline{B})^+)$ and let $\phi \equiv \exists \overline{x} \phi'$ for $\phi' \in \Delta_0(L_\alpha, (B \oplus \overline{B})^+)$. Obtain the formula ϕ'' from the formula ϕ' by replacing every atom $x \in B \oplus \overline{B}$ for every variable x with the equivalent α-computable subformula $\exists y \leq x[(x = 2y \wedge y \in B) \vee (x = 2y + 1 \wedge y \notin \overline{B})]$. Observe $\phi' \equiv \phi''$. Note that $\phi'' \in \Delta_1(L_\alpha, B)$ and so $\phi \equiv \exists \overline{x} \phi'' \in \Sigma_1(L_\alpha, B)$. Hence $\phi \in \Sigma_1(L_\alpha, (B \oplus \overline{B})^+) \implies \phi \in \Sigma_1(L_\alpha, B)$ up to the equivalence of ϕ.

For the other direction, assume $\phi \in \Sigma_1(L_\alpha, B)$ and let $\phi \equiv \exists \overline{x} \phi'$ for $\phi' \in \Delta_0(L_\alpha, B)$. Use De Morgan's law to move the negations to the literals. To obtain the formula ϕ'' from ϕ', replace every literal $\neg x \in B$ with the subformula $2x + 1 \in B \oplus \overline{B}$ and afterwards every literal $x \in B$ with the subformula $2x \in B \oplus \overline{B}$. Observe $\phi' \equiv \phi''$. Note that $\phi'' \in \Delta_1(L_\alpha, (B \oplus \overline{B})^+)$ and so $\phi \equiv \exists \overline{x} \phi'' \in \Sigma_1(L_\alpha, (B \oplus \overline{B})^+)$. Hence $\phi \in \Sigma_1(L_\alpha, B) \implies \phi \in \Sigma_1(L_\alpha, (B \oplus \overline{B})^+)$ up to the equivalence of ϕ. Thus statement (3.1) is true. Statement (3.2) follows from a similar proof where Σ_1 and $\exists$ are replaced by Π_1 and $\forall$ respectively. Therefore $\Theta_n(L_\alpha, (B \oplus \overline{B})^+) = \Theta_n(L_\alpha, B)$ where $\Theta \in \{\Delta, \Sigma, \Pi\}$ and $n \in \mathbb{N}^+$ as required.

ii) Note $A \in \Delta_1(L_\alpha, B) \iff A \in \Sigma_1(L_\alpha, B) \wedge A \in \Pi_1(L_\alpha)$

$$\iff A \in \Sigma_1(L_\alpha, B) \wedge \overline{A} \in \Sigma_1(L_\alpha, B) \iff A \oplus \overline{A} \in \Sigma_1(L_\alpha, B).$$

Similarly, $A \in \Delta_1(L_\alpha, B) \iff A \in \Sigma_1(L_\alpha, B) \wedge A \in \Pi_1(L_\alpha)$

$$\iff \overline{A} \in \Pi_1(L_\alpha, B) \wedge A \in \Pi_1(L_\alpha, B) \iff A \oplus \overline{A} \in \Pi_1(L_\alpha, B).$$

□

Remark 3.4.5. (Model with a parameter[22])
If $\mathcal{B} \in \{B, B^-, B^+\}$ is a parameter in a formula $\phi(\overline{x}, B)$, we face a difficulty of interpreting the literal $x \in B$ (or $x \notin B$) in the model L_α if $B \notin L_\alpha$.

[22] This should not be confused with a relativized model $\langle L_\alpha, B\rangle$ in [4] p18 with the same notation.

For this purpose we define an extended model $\langle L_\alpha, \mathcal{B}\rangle$ which is a pair of L_α and a set $B \subseteq \alpha$. The language of the interpretable formulas over the model $\langle L_\alpha, \mathcal{B}\rangle$ is extended by a predicate B and has restrictions on the formulas depending on $\mathcal{B}$ to comply with definition 3.4.1.

Note the definability over L_α with a parameter $\mathcal{B}$ is equivalent to the definability over $\langle L_\alpha, \mathcal{B}\rangle$. Hence we use them interchangeably.

3.4.2 Bounded quantifier rearrangement

When having a formula with bounded quantifiers in front of the unbounded quantifiers, it is possible to rearrange it to an equivalent formula where the bounded quantifiers are behind the unbounded quantifiers. This rearrangement is the source of many statements in this subsection about the arithmetical hierarchy of the formulas with the unbounded quantifiers.

Proposition 3.4.6. [23] Assume that $\phi(x, y, z)$ is a $\Delta_0(L_\alpha)$-formula. Then the formula $\forall y \in K \exists z.\phi(x, y, z)$ is equivalent to some $\Sigma_1(L_\alpha)$ formula.

Proof. It is sufficient to prove the claim

$$\forall y \in K \exists z.\phi(x, y, z) \iff \exists H \forall y \in K \exists z \in H.\phi(x, y, z).$$

The $\Longleftarrow$ direction is clear. For the $\Longrightarrow$ direction assume $\forall y \in K \exists z.\phi(x, y, z)$. Define a partial function

$$f = \{(y, z) \in \alpha \times \alpha : y \in K \wedge \phi(x, y, z) \wedge \forall z' < z.\neg\phi(x, y, z')\}.$$

The function f is Δ_0 definable with K as a parameter from L_α. Thus f is Σ_1-definable over L_α. By the admissibility of α and the α-finiteness of K, the set $H := f[K]$ must by α-finite. Furthermore, note that $\forall y \in K \exists z \in H.\phi(x, y, z)$. Hence $\exists H \forall y \in K \exists z \in H.\phi(x, y, z)$ as required. □

We generalize proposition 3.4.6 to proposition 3.4.7 and proposition 3.4.9.

Proposition 3.4.7. [23] Let $\mathcal{B} \in \{B, B^+, B^-\}$. Assume $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_0$-collection. Assume that $\phi(x, y, z)$ is a $\Delta_0(L_\alpha, \mathcal{B})$ formula. Then the formula $\forall y \in K \exists z.\phi(x, y, z)$ is equivalent to some $\Sigma_1(L_\alpha, \mathcal{B})$ formula.

Proof. It is sufficient to prove the claim

$$\forall y \in K \exists z.\phi(x, y, z) \iff \exists H \forall y \in K \exists z \in H.\phi(x, y, z).$$

The $\Longleftarrow$ direction is clear. For the $\Longrightarrow$ direction assume $\forall y \in K \exists z.\phi(x, y, z)$. Note $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_0$-collection and $\phi(x, y, z) \in \Delta_0(L_\alpha, \mathcal{B})$. Therefore $\exists H \forall y \in K \exists z \in H.\phi(x, y, z)$ as required. □

[23] Introduced in this thesis.

Fact 3.4.8. Suppose that the collection axiom holds for the formula ϕ. Then:

i) $\forall y_1 \in K_1 \exists y_2.\phi(y_1, y_2) \iff \exists K_2 \forall y_1 \in K_1 \exists y_2 \in K_2.\phi(y_1, y_2)$

ii) $\exists y_1 \in K_1 \forall y_2.\neg\phi(y_1, y_2) \iff \forall K_2 \exists y_1 \in K_1 \forall y_2 \in K_2.\neg\phi(y_1, y_2)$

Proposition 3.4.9. (Bounded quantifier rearrangement[24])
Let $B \subseteq \alpha, n \in \mathbb{N}$ and $K \in L_\alpha$. Define $\Pi_{-1} := \emptyset$. Assume $\langle L_\alpha, B\rangle \models \Pi_{n-1}$-collection or $\langle L_\alpha, B\rangle \models \Delta_n$-collection. Then:

i) $\phi \in \Sigma_n(L_\alpha, B) \implies \exists \psi \in \Sigma_n(L_\alpha, B)[\psi \equiv \forall y \in K.\phi(y)]$

ii) $\phi \in \Pi_n(L_\alpha, B) \implies \exists \psi \in \Pi_n(L_\alpha, B)[\psi \equiv \exists y \in K.\phi(y)]$

Proof. First note that $\langle L_\alpha, B\rangle \models \Delta_n$-collection or $\langle L_\alpha, B\rangle \models \Pi_{n-1}$-collection implies $\langle L_\alpha, B\rangle \models \Pi_m$-collection for any $m < n$.

We prove the statements by the induction on $n \in \mathbb{N}$. The base case when $n = 0$ is trivial. For the inductive case, assume IH that the statements hold for some n. Let $\phi \in \Sigma_{n+1}(L_\alpha, B)$. Then $\phi \equiv \exists y'.\phi'(y, y')$ for some $\phi'(y, y') \in \Pi_n(L_\alpha, B)$. We have

$$\forall y \in K \exists y'.\phi'(y, y') \iff \exists K' \forall y \in K \exists y' \in K'.\phi'(y, y')$$

by $\Pi_n(L_\alpha, B)$-collection and fact 3.4.8i. By IH the formula $\exists y' \in K'.\phi'(y, y')$ is equivalent to some $\Pi_n(L_\alpha, B)$ formula. Hence $\exists K' \forall y \in K \exists y' \in K'.\phi'(y, y')$ is equivalent to some formula $\psi \in \Sigma_{n+1}(L_\alpha, B)$. Therefore using the equivalence,

$$\forall y \in K.\phi(y) \equiv \forall y \in K \exists y'.\phi'(y, y') \equiv \exists K' \forall y \in K \exists y' \in K'.\phi'(y, y') \equiv \psi$$

as required.

The second statement for the formula $\phi \in \Pi_{n+1}(L_\alpha, B)$ holds by applying the first statement on the formula $\neg\phi$ which is $\Sigma_{n+1}(L_\alpha, B)$ up to equivalence. This completes the induction. □

3.4.3 Transitivity for the arithmetical definability

If A is definable from the parameter B and B is definable from the parameter C, then A is definable from the parameter C. This subsection explorers the exact first-order definability class of A with the parameter C given the definability class of A with the parameter B and the definability class of B with the parameter C.

Proposition 3.4.10. (Transitivity for the arithmetical definability[25])
Assume $\langle L_\alpha, C\rangle \models \Delta_n$-collection. Then:

i) $\phi \in \Sigma_1(L_\alpha, B), B \in \Delta_n(L_\alpha, C) \implies \exists \psi[\phi \equiv \psi$ and $\psi \in \Sigma_n(L_\alpha, C)]$.

[24] Introduced in this thesis.
[25] No reference known. Possibly new in this thesis.

ii) $A \in \Sigma_1(L_\alpha, B), B \in \Delta_n(L_\alpha, C) \implies A \in \Sigma_n(L_\alpha, C)$

iii) $\phi \in \Pi_1(L_\alpha, B), B \in \Delta_n(L_\alpha, C) \implies \exists\psi[\phi \equiv \psi$ and $\psi \in \Pi_n(L_\alpha, C)]$.

iv) $A \in \Pi_1(L_\alpha, B), B \in \Delta_n(L_\alpha, C) \implies A \in \Pi_n(L_\alpha, C)$

v) $A \in \Delta_1(L_\alpha, B), B \in \Delta_n(L_\alpha, C) \implies A \in \Delta_n(L_\alpha, C)$

Proof. As $B \in \Delta_n(L_\alpha, C)$, so $x \in B \iff \theta(x, C)$ for some formula $\theta(x, C) \in \Delta_n(L_\alpha, C)$.

For the first statement, let $\phi(\overline{x}, B) \in \Sigma_1(L_\alpha, B)$ be a formula where $\overline{x}$ is a list of variables. WLOG let $\phi(\overline{x}, B)$ be in a prenex normal form, i.e. have all the negations, conjunctions and disjunctions in its quantifier-free subformula. Replace an atom $x' \in B$ in the formula $\phi(\overline{x}, B)$ by the subformula $\theta(x', C)$. Denote the new formula by $\psi(\overline{x}, C)$. We will prove by the structural induction on the formula that $\psi(\overline{x}, C) \in \Sigma_n(L_\alpha, C)$. Note that $\phi(\overline{x}, B)$ and $\psi(\overline{x}, C)$ are equivalent. This implies $\phi \equiv \psi$ and $\psi \in \Sigma_n(L_\alpha, C)$ as required.

For the second statement, let $\phi(x, B) \in \Sigma_1(L_\alpha, B)$ be a formula defining A. Note that $\phi(x, B)$ and $\psi(x, C)$ define the same set A. This implies $A \in \Sigma_n(L_\alpha, C)$ as required.

The third and the fourth statements follow by the duality from the first and the second respectively.

The statement v follows from the statements ii and iv.

Proof of $\psi(\overline{x}, C) \in \Delta_n(L_\alpha, C)$ if $\phi(\overline{x}, B) \in \mathrm{QF}(L_\alpha, B)$ by induction

- If $\phi(\overline{x}, B) \in \mathrm{QF}(L_\alpha)$, then $\psi(\overline{x}, C) \in \Delta_n(L_\alpha, C)$ trivially.

- If $\phi(\overline{x}, B) = x_i \in B$, then $\psi(\overline{x}, C) = \theta(x_i, C) \in \Delta_n(L_\alpha, C)$.

- If $\phi(\overline{x}, B) = \neg\phi'(\overline{x}, B)$, then by IH $\phi'(\overline{x}, B) \equiv \psi'(\overline{x}, C)$ for some formula $\psi'(\overline{x}, C) \in \Delta_n(L_\alpha, C)$. Thus $\psi(\overline{x}, C) = \neg\psi'(\overline{x}, C) \equiv \phi(\overline{x}, B)$ and $\psi(\overline{x}, C) \in \Delta_n(L_\alpha, C)$ as required.

- Let $\phi(\overline{x}, B) = \phi_0(\overline{x}, B)\blacklozenge\phi_1(\overline{x}, B)$ where $\blacklozenge \in \{\wedge, \vee\}$ and $\phi_i(\overline{x}, B) \in \Sigma_1(L_\alpha, B)$. Then $\psi(\overline{x}, C) = \psi_0(\overline{x}, C)\blacklozenge\psi_1(\overline{x}, C)$. By IH $\psi_i(\overline{x}, C) \in \Delta_n(L_\alpha, C)$. Hence $\psi(\overline{x}, C) \in \Delta_n(L_\alpha, C)$.

Proof of $\psi(\overline{x}, C) \in \Sigma_n(L_\alpha, C)$ by induction

- If $\phi(\overline{x}, B) \in \mathrm{QF}(L_\alpha, B)$, then $\psi(\overline{x}, C) \in \Delta_n(L_\alpha, C)$ by the argument above. Thus $\psi(\overline{x}, C) \in \Sigma_n(L_\alpha, C)$.

- Let $\phi(\overline{x}, B) = \forall y \in K.\phi'(\overline{x}, y, B)$. By IH $\phi'(\overline{x}, y, B) \equiv \psi'(\overline{x}, y, C)$ for some $\psi'(\overline{x}, y, C) \in \Sigma_n(L_\alpha, C)$. Then
$$\psi(\overline{x}, y, C) = \forall y \in K.\psi'(\overline{x}, y, C) \equiv \phi(\overline{x}, y, C).$$
Since $\langle L_\alpha, C\rangle \models \Delta_n$-collection, so $\psi(\overline{x}, C) \in \Sigma_n(L_\alpha, C)$ up to equivalence by proposition 3.4.9.

- If $\phi(\overline{x}, B) = \exists y.\phi'(\overline{x}, y, B)$ or $\phi(\overline{x}, B) = \exists y \in K.\phi'(\overline{x}, y, B)$, then $\phi'(\overline{x}, y, B) \equiv \psi'(\overline{x}, y, B)$ for some formula $\psi'(\overline{x}, B) \in \Sigma_n(L_\alpha, C)$ by IH. Thus $\psi(\overline{x}, C) \in \Sigma_n(L_\alpha, C)$ trivially as required.

As all induction steps are covered, this concludes the proof. □

Corollary 3.4.11. [26] Assume $\langle L_\alpha, C\rangle \models \Delta_n$-collection. If $B \in \Delta_n(L_\alpha, C)$, then $\langle L_\alpha, B\rangle \models \Delta_1$-collection.

Proof. Let $\phi \in \Delta_1(L_\alpha, B)$ and let $B \in \Delta_n(L_\alpha, C)$. Then by proposition 3.4.10v $\phi \in \Delta_n(L_\alpha, C)$ up to equivalence using $\langle L_\alpha, C\rangle \models \Delta_n$-collection. As $\langle L_\alpha, C\rangle \models \Delta_n$-collection and $\phi \in \Delta_n(L_\alpha, C)$ up to equivalence, so the collection holds for $\phi \in \Delta_1(L_\alpha, B)$. Therefore $\langle L_\alpha, B\rangle \models \Delta_1$-collection. □

3.4.4 Relativized Uniformization Theorem

We prove the relativization of theorem 3.2.9 for the case $n = 1$.

Lemma 3.4.12. [27] If $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement, then $\langle L_\alpha, B\rangle \models \Sigma_0$-collection.

Proof. Assume $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement. Let $\phi(x, y) \in \Sigma_0(L_\alpha, B)$. Define f by
$$(x, y) \in f \iff \phi(x, y) \wedge \forall y' < y.\neg\phi(x, y).$$
Then $f \in \Sigma_0(L_\alpha, B)$ up to equivalence trivially.

Let $K \in L_\alpha$ and assume $\forall x \in K \exists y.\phi(x, y)$. Then $f[K] \in L_\alpha$ as $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement. Furthermore, $\forall x \in K \exists y \in f[K].\phi(x, y)$. Therefore $\langle L_\alpha, B\rangle \models \Sigma_0$-collection as required. □

Lemma 3.4.13. [28] Let $A \subseteq \alpha$, $B \subseteq \alpha$, $\mathcal{B} \in \{B, B^+, B^-\}$ and $n \in \mathbb{N}$.

i) Assume that $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_n$-replacement. Then
$$A \in L_\alpha \text{ iff } A \oplus \overline{A} \in \Sigma_n(L_\alpha, \mathcal{B}) \text{ and } \exists\beta < \alpha.A \subseteq \beta.$$

[26] No reference known. Possibly new in this thesis.

[27] A straight-forward generalization of a well-known statement without a parameter, found in [23]VII for example.

[28] Introduced in this thesis.

ii) Assume that $\langle L_\alpha, B\rangle \models \Sigma_n$-replacement. Then

$$A \in L_\alpha \text{ iff } A \in \Delta_n(L_\alpha, B) \text{ and } \exists\beta < \alpha.A \subseteq \beta.$$

.

Proof. We prove the statement i. The direction $\implies$ is clear. For the other direction, assume that $A \oplus \overline{A} \in \Sigma_n(L_\alpha, \mathcal{B})$ and $A \subseteq \beta < \alpha$ for some β. WLOG let $A \neq \emptyset$ and let $a \in A$. Define a function $f : \alpha \to \alpha$ by

$$f(x) = y \iff (x \in A \wedge x = y) \vee (x \notin A \wedge y = a).$$

Since $A \oplus \overline{A} \in \Sigma_n(L_\alpha, \mathcal{B})$, the function f is $\Sigma_n(L_\alpha, \mathcal{B})$ definable. As $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_n$-replacement and β is α-finite, so we have that $A = f[\beta] \in L_\alpha$ as required.

The statement ii follows from the statement i since

$$A \oplus \overline{A} \in \Sigma_n(L_\alpha, B) \iff A \in \Delta_n(L_\alpha, B).$$

□

Lemma 3.4.14. [29] Assume $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement. Let $W \in \Sigma_1(L_\alpha, B)$. Then there is a function $\widehat{W} : \alpha \to L_\alpha \in \Sigma_1(L_\alpha, B)$ s.t.:

i) $\forall\gamma, \delta < \alpha[\gamma < \delta \implies \widehat{W}(\gamma) \subseteq \widehat{W}(\delta)]$.

ii) $W = \bigcup_{\gamma<\alpha} \widehat{W}(\gamma)$.

iii) $\forall x < \alpha[x \in W \iff \exists\gamma < \alpha.x \in \widehat{W}(\gamma)]$.

Proof. As $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement, so $\langle L_\alpha, B\rangle \models \Sigma_0$-collection by lemma 3.4.12. As $W \in \Sigma_1(L_\alpha, B)$, so there is a binary relation $P \in \Sigma_0(L_\alpha, B)$ s.t. $x \in W \iff \exists y < \alpha.P(x, y)$. Define

$$\widehat{W}(\gamma) := \pi_1[\{z \in p_2[\gamma] : P(z)\}] = \pi_1[p_2[\gamma] \cap P]$$

where $p_2 : \alpha \to \alpha \times \alpha$ is an α-computable bijection and $\pi_2 : \alpha \times \alpha \to \alpha$ is an α-computable projection. Thus $\widehat{W} : \alpha \to L_\alpha \in \Sigma_1(L_\alpha, B)$ by proposition 3.4.7 as $\langle L_\alpha, B\rangle \models \Sigma_0$-collection. The set $A := \{z \in p_2[\gamma] : P(z)\}$ is bounded as $A \subseteq p_2[\gamma] \in L_\alpha$. Since $P(z) \in \Sigma_0(L_\alpha, B)$, so $A \in \Sigma_0(L_\alpha, B)$. As $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement, so $A \in L_\alpha$ by lemma 3.4.13. Therefore $\widehat{W}(\gamma) := \pi_1[A] \in L_\alpha$ and so W is well-defined. Also observe that the function $\widehat{W} : \alpha \to L_\alpha$ satisfies the properties i-iii as required. □

Proposition 3.4.15. (Relativized Σ_1 Uniformization Theorem[30]) Assume $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement. Let $R \subseteq \alpha \times \alpha$ be a binary $\Sigma_1(L_\alpha, B)$-definable relation. Then there is a partial function $f \in \Sigma_1(L_\alpha, B)$ s.t. $\forall x \in \text{dom}(R).R(x, f(x))]$.

[29] Introduced in this thesis.
[30] Introduced in this thesis.

Proof. As $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement, so $\langle L_\alpha, B\rangle \models \Sigma_0$-collection by lemma 3.4.12. Using lemma 3.4.14, there is a $\Sigma_1(L_\alpha, B)$ function $\alpha \to L_\alpha, t \mapsto R_t$ s.t. $(x,y) \in R \iff \exists t < \alpha.(x,y) \in R_t$. Let $\pi_1 : \alpha \times \alpha \to \alpha$ be the α-computable projection to the first parameter. Define $f : \alpha \to \alpha$ by

$$(x,y) \in f \iff \exists t < \alpha[(x,y) \in R_t \wedge \forall s < t \forall z \in R_s.\pi_1 z \neq x].$$

Then $f \in \Sigma_1(L_\alpha, B)$ using proposition 3.4.7 and the fact that $\langle L_\alpha, B\rangle \models \Sigma_0$-collection. Also $\forall x \in \mathrm{dom}(R).R(x, f(x))$ as required. □

3.4.5 Axioms in a parameterized model

We investigate a relationship between separation, replacement and collection in a parameterized model $\langle L_\alpha, \mathcal{B}\rangle$ for $\mathcal{B} \in \{B, B^+, B^-\}$.

Separation

Remark 3.4.16. (Power of relative separation for bounded parameter)
The relative separation is too strong to be useful for bounded parameters. Let $B \subseteq \beta < \alpha$. Let $\mathcal{B} \in \{B, B^+\}$ and let $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_0$-separation, i.e. if $\phi(x) \in \Sigma_0(L_\alpha, \mathcal{B})$ and $K \in L_\alpha$, then $\{x \in K : \phi(x)\} \in L_\alpha$. If $\phi(x) :\equiv x \in B$, then $\phi(x) \in \Sigma_0(L_\alpha, \mathcal{B})$. Also $\beta \in L_\alpha$. Thus $B = \{x \in \beta : x \in B\} \in L_\alpha$. Therefore the definability over $\langle L_\alpha, \mathcal{B}\rangle$ is just the definability over L_α.

Proposition 3.4.17. [31] Let $n \in \mathbb{N}$ and $B \subseteq \alpha$. If $\langle L_\alpha, B\rangle \models \Sigma_n$-replacement, then $\langle L_\alpha, B\rangle \models \Delta_n$-separation.

Proof. Let $K \in L_\alpha$ and $\phi(x) \in \Delta_n(L_\alpha, B)$. Define $A := \{x \in K : \phi(x)\}$. Then clearly $A \in \Delta_n(L_\alpha, B)$. Also $A \subseteq K$. Therefore $A \in L_\alpha$ by lemma 3.4.13ii as required. □

Replacement and collection

We show that $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-replacement implies $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-collection. We outline a difficulty in stating $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-collection implies $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-replacement where $\mathcal{B}$ is a general parameter.

Proposition 3.4.18. [32] Let $\mathcal{B} \in \{B, B^-, B^+\}$. Assume $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_0$-collection. If $A \oplus \overline{A} \in \Sigma_1(L_\alpha, \mathcal{B})$ and $A \subseteq K \in L_\alpha$, then $A \in \Delta_0(L_\alpha, \mathcal{B})$.

Proof. Since $A \oplus \overline{A} \in \Sigma_1(L_\alpha, \mathcal{B})$, we have:

[31] No reference known. Possibly new in this thesis.
[32] New in this thesis.

- $x \in A \iff \exists \overline{y}.\phi(x, \overline{y})$ for some $\phi(x, \overline{y}) \in \Delta_0(L_\alpha, \mathcal{B})$,
- $x \notin A \iff \exists \overline{y}.\psi(x, \overline{y})$ for some $\psi(x, \overline{y}) \in \Delta_0(L_\alpha, \mathcal{B})$.

Recall $\veebar$ denotes exclusive logic or. As $A \subseteq K \in L_\alpha$, we have:

- $\forall x \in K \exists \overline{y}[\phi(x, \overline{y}) \veebar \psi(x, \overline{y})]$ where $\phi(x, \overline{y}) \veebar \psi(x, \overline{y}) \in \Sigma_0(L_\alpha, \mathcal{B})$.
- $\exists I \in L_\alpha \forall x \in K \exists \overline{y} \in I[\phi(x, \overline{y}) \veebar \psi(x, \overline{y})]$ since $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_0$-collection.

Hence we can define A with formulas with bounded quantifiers:

- $x \in A \iff x \in K \wedge \exists \overline{y} \in I.\phi(x, \overline{y})$,
- $x \notin A \iff x \notin K \vee \exists \overline{y} \in I.\psi(x, \overline{y})$.

Any of the two formulas implies $A \in \Delta_0(L_\alpha, \mathcal{B})$ as required. □

Proposition 3.4.19. [33] Let $n \in \mathbb{N}$. If $L_\alpha \models \Sigma_n$-replacement, then $L_\alpha \models \Sigma_n$-collection.

Proof. Assume $L_\alpha \models \Sigma_n$-replacement. Let $\phi(x, y) \in \Sigma_n(L_\alpha)$, $K \in L_\alpha$. Assume $\forall x \in K \exists y.\phi(x, y)$. By Uniformization Theorem 3.2.9, there is a function $f \in \Sigma_n(L_\alpha)$ s.t. $\forall x \in K.\phi(x, f(x))$. By the Σ_n-replacement, we have $\hat{K} := f[K] \in L_\alpha$. Furthermore, $\forall x \in K \exists y \in \hat{K}.\phi(x, y)$. Therefore $L_\alpha \models \Sigma_n$-collection. □

Proposition 3.4.20. [34] Let $n \in \{0, 1\}$. If $\langle L_\alpha, B\rangle \models \Sigma_n$-replacement, then $\langle L_\alpha, B\rangle \models \Sigma_n$-collection.

Proof. The case $n = 0$ is implied by lemma 3.4.12.

For the other case, assume $\langle L_\alpha, B\rangle \models \Sigma_n$-replacement. So $\langle L_\alpha, B\rangle \models \Sigma_0$-collection. Let $\phi(x, y) \in \Sigma_n(L_\alpha, B)$. As $\langle L_\alpha, B\rangle \models \Sigma_0$-collection and $\phi(x, y) \in \Sigma_n(L_\alpha, B)$, so there is $f \in \Sigma_n(L_\alpha, B)$ s.t. $\forall x[\exists y.\phi(x, y) \implies \phi(x, f(x))]$ using proposition 3.4.15.

Let $K \in L_\alpha$ and assume $\forall x \in K \exists y.\phi(x, y)$. Then $f[K] \in L_\alpha$ as $\langle L_\alpha, B\rangle \models \Sigma_n$-replacement. Furthermore, $\forall x \in K \exists y \in f[K].\phi(x, y)$. Therefore $\langle L_\alpha, B\rangle \models \Sigma_n$-collection. □

Proposition 3.4.21. [35] (Σ_0-elementary substructure of $\langle L_\alpha, \mathcal{B}\rangle$)
Let $\mathcal{B} \in \{B, B^-, B^+\}$. If $\delta \leq \alpha$, then $\langle L_\delta, \mathcal{B}\rangle \preceq_{\Sigma_0} \langle L_\alpha, \mathcal{B}\rangle$, i.e. $\langle L_\delta, \mathcal{B}\rangle$ is a Σ_0 elementary substructure of $\langle L_\alpha, \mathcal{B}\rangle$.

[33] A straight-forward consequence of Uniformization Theorem from [4].
[34] Introduced in this thesis.
[35] Introduced in this thesis.

Proof. By definition $\langle L_\delta, \mathcal{B}\rangle \preceq_{\Sigma_0} \langle L_\alpha, \mathcal{B}\rangle$ iff for every formula $\phi(\overline{x}) \in \Sigma_0(L_\delta, \mathcal{B})$ we have: $\langle L_\delta, \mathcal{B}\rangle \models \phi(\overline{x})$ iff $\langle L_\alpha, \mathcal{B}\rangle \models \phi(\overline{x})$. Let $K \in L_\delta$, then trivially for any $x < \delta$ we have: $\langle L_\delta, \mathcal{B}\rangle \models x \in K$ iff $\langle L_\alpha, \mathcal{B}\rangle \models x \in K$. Clearly $\langle L_\delta, \mathcal{B}\rangle \models x \in B$ iff $\langle L_\alpha, \mathcal{B}\rangle \models x \in B$. Also clearly, $\langle L_\delta, \mathcal{B}\rangle \models x = y$ iff $\langle L_\alpha, \mathcal{B}\rangle \models x = y$. Hence $\langle L_\delta, \mathcal{B}\rangle \models \phi(x)$ iff $\langle L_\alpha, \mathcal{B}\rangle \models \phi(x)$ for any atom $\phi(x) \in \Sigma_0(L_\delta)$. Inductive steps for $\wedge$, $\vee$, $\neg$ are clear. Inductive steps for the bounded quantifiers follow from the fact that each quantifier is bounded in some $K \in L_\delta$. Hence by the induction on the structure of the formula $\phi(\overline{x}) \in \Sigma_0(L_\delta)$ we conclude $\langle L_\delta, \mathcal{B}\rangle \models \phi(\overline{x})$ iff $\langle L_\alpha, \mathcal{B}\rangle \models \phi(\overline{x})$. Therefore $\langle L_\delta, \mathcal{B}\rangle \preceq_{\Sigma_0} \langle L_\alpha, \mathcal{B}\rangle$ as required. □

Remark 3.4.22. The definability with a predicate $x \in B$ is different from the definability with a set parameter B over L_δ if $B \not\subseteq L_\delta$. Whereas with the set parameter B we can access all the elements of B, possibly even the ones not in L_δ, when using the definability with the predicate $x \in B$, only the elements within the chosen model such as L_δ can be accessed. Therefore we cannot use proposition 3.4.18 and proposition 3.4.21 to generalize lemma 3.1.18 to prove that $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-collection implies $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-replacement.

3.5 α-computable index of a set

We study indexing functions and α-computable operations on such indices. The results of this section are important for later constructions where stage-dependent indexing and α-computability of the construction are important.

Lemma 3.5.1. (Pseudosupremum[36])
The function psup : $L_\alpha \to \alpha$ defined by

$$\texttt{psup}(K) = \begin{cases} 0 & K = \emptyset \\ \texttt{sup}(K) & K \neq \emptyset \end{cases}$$

is α-computable.

Proof. Note that

$$\begin{aligned} \texttt{psup}(K) = s \iff & [s = 0 \wedge K \subseteq \{0\} \vee \\ & s > 0 \wedge s \in K \wedge \forall x \in K.x \leq s \vee \\ & s > 0 \wedge s \notin K \wedge \forall x \in K.x < s \wedge \forall y < s \exists x \in K.y < x)] \end{aligned}$$

from which we deduce that the function psup is α-computable because the quantifiers inside the brackets are bounded. □

[36] Introduced in this thesis.

Proposition 3.5.2. [37] There exist α-computable functions $i : \alpha \times \alpha \to \alpha$ and $s_i : \alpha \to \alpha$ s.t.:

i) $\forall K \in L_\alpha \cap \mathcal{P}(\alpha)\, \exists \eta < \alpha \forall x < \alpha [x \in K \iff i(\eta, x) = 1]$,

ii) $\forall \eta < \alpha . i(\eta) := \{x < \alpha : i(\eta, x) = 1\} \in L_\alpha$,

iii) $\forall \eta < \alpha . s_i(\eta) = \begin{cases} 0 & i(\eta) = \emptyset \\ \mathtt{sup}(i(\eta)) & i(\eta) \neq \emptyset \end{cases}$

Proof. Using the bijection $b : \alpha \to L_\alpha \in \Sigma_1(L_\alpha)$ from proposition 3.2.2 and the function psup from lemma 3.5.1 respectively, define the functions i and s_i:

$$\begin{aligned} i(\eta, x) :=& [x \in b(\eta)], \\ s_i(\eta) :=& \{s < \alpha : \exists K \in L_\alpha [b(\eta) = K \wedge \mathtt{psup}(K) = s]\}. \end{aligned}$$

Clearly, i and s_i are both the required functions and i is α-computable. The function s_i is α-computable because the front quantifier is existential and functions b and psup are both α-computable. □

Therefore we can label α-finite subsets of α by indices $\eta < \alpha$. Let K_η denote an α-finite set $i(\eta)$ with an index η.

Lemma 3.5.3. [38] There exists an α-computable function $g : \alpha \times \alpha \times \alpha \to \alpha$ s.t.

$$\begin{aligned} D_\eta :=& \{x | g(\eta, x, 1) = 1\} \in L_\alpha, \\ E_\eta :=& \{x | g(\eta, x, 2) = 1\} \in L_\alpha \end{aligned}$$

and for every pair $(\hat{D}, \hat{E})$ of α-finite subsets of α there is an index $\eta < \alpha$ s.t. $D_\eta = \hat{D}$ and $E_\eta = \hat{E}$.

Therefore we can α-effectively number the pairs of the α-finite subsets of α by the indices of α.

Proof. Note that there are α-computable bijections $b : \alpha \to L_\alpha$ and $p_2 : \alpha \to \alpha \times \alpha$. Let π_1 and π_2 be the projections. Define $g(\eta, x, k) := [x \in b \circ \pi_k \circ p_2(\eta)]$. Then g is the required α-computable function. □

Lemma 3.5.4. [39] Let $i, j, k : \alpha \times \alpha \to \alpha$ be any α-computable numberings of α-finite subsets of α with their respective α-computable supremum functions $s_i, s_j, s_k : \alpha \to \alpha$ in a sense of proposition 3.5.2. Then

i) There is an α-computable function $u : \alpha \to \alpha$ s.t.

$$\forall \gamma < \alpha . \bigcup_{x \in j(\gamma)} i(x) = k(u(\gamma)).$$

[37] From proposition 1.7 in [4].

[38] Introduced in this thesis.

[39] Frequently used, yet usually implicitly assumed facts in α-Computability Theory.

ii) There is an α-computable function $v : \alpha \times \alpha \to \alpha$ s.t.

$$\forall \gamma, \delta < \alpha . k(v(\gamma, \delta)) = i(\gamma) \oplus j(\delta).$$

iii) There exist α-computable functions $i_{\pi_1}, i_{\pi_2} : \alpha \to \alpha$ s.t.

$$\forall l \in \{1, 2\} \forall \gamma < \alpha . k(i_{\pi_l}(\gamma)) = \{x_l : \langle x_1, x_2 \rangle \in i(\gamma)\}.$$

iv) There exists an α-computable function $i_{p_2} : \alpha \to \alpha$ s.t.

$$\forall \gamma < \alpha . k(i_{p_2}(\gamma)) = i(\gamma) \times j(\gamma).$$

v) There exists an α-computable function $w : \alpha \times \alpha \to \alpha$ s.t. if $\gamma, \delta < \alpha$, then

$$k(w(\gamma, \delta)) = \{\langle x, y \rangle : x \in j(\delta) \wedge y \in j(\gamma) \wedge y \in i(x)\}.$$

vi) There exists an α-computable function $t_{i,j} : \alpha \to \alpha$ s.t.

$$\forall \gamma < \alpha . i(\gamma) = j(t_{i,j}(\gamma)).$$

vii) Let $K(\gamma) := \bigcup_{x \in j(\gamma)} i(x)$. Then there exists an α-computable function $s_{i,j} : \alpha \to \alpha$ s.t.

$$\forall \gamma < \alpha . s_{i,j}(\gamma) = \begin{cases} 0 & K(\gamma) = \emptyset \\ \mathtt{sup}(K(\gamma)) & K(\gamma) \neq \emptyset \end{cases}.$$

Proof. To ensure that the following relations are functions, if there are multiple possible output values (resulting from the duplicate indices of α-finite subsets of α) in α, take the least one. To prove that the following formulas are $\Sigma_1(L_\alpha)$, proposition 3.4.6 is used to rearrange the bounded quantifiers from the outside to inside and remark 3.2.4 is used to encode the products.

i) Using the part vii) we have

$$u(\gamma) = \delta \iff \forall y \leq s_{i,j}(\gamma) \big[\exists x \leq s_j(\gamma) [j(\gamma, x) \cdot i(x, y) = 1] \iff k(\delta, y) = 1 \big]$$

which is $\Sigma_1(L_\alpha)$. To prove that the function $u : \alpha \to \alpha$ is also total, we need to prove that the α-finite union of α-finite sets is α-finite, i.e. if $K \in L_\alpha$, then $A := \bigcup_{\gamma \in K} K_\gamma \in L_\alpha$. By the part vii), A is bounded. Also $x \in A \iff \exists \gamma \in K . x \in K_\gamma$ and $x \in \overline{A} \iff \forall \gamma \in K . x \notin K_\gamma$. Since the quantifiers are bounded, so A is α-computable. Thus A is α-finite. Therefore $u : \alpha \to \alpha$ is a total α-computable function as required.

ii) Note that $v(\gamma, \delta) = \epsilon \iff$

$$\forall x \leq \max\{s_i(\gamma), s_j(\delta), s_k(\epsilon)\} [k(\epsilon, 2x) = i(\gamma, x) \wedge k(\epsilon, 2x + 1) = j(\delta, x)]$$

which is $\Sigma_1(L_\alpha)$ as required.

iii) Note $i_{\pi_l}(\gamma) = \delta \iff k(\delta) = \pi_l[i(\gamma)] \iff \forall x_l \leq s_k(\delta) [k(\delta, x_l) = 1 \iff \exists z \leq s_i(\gamma) . x_l = \pi_l z] \wedge \forall y \leq s_i(\gamma) [k(\delta, \pi_l y) = 1 \iff \exists z \leq s_i(\gamma) . \pi_l y = \pi_l z]$ which is $\Sigma_1(L_\alpha)$ as required.

iv) Note $i_{p_2}(\gamma) = \delta \iff \forall x < \alpha.k(\delta, x) = i(\gamma, \pi_1 x) \cdot j(\gamma, \pi_2 x) \iff$
$\forall x \leq s_k(\delta).k(\delta, x) = i(\gamma, \pi_1 x) \cdot j(\gamma, \pi_2 x) \wedge$
$\forall x_1 \leq s_i(\gamma) \forall x_2 \leq s_j(\gamma).k(\delta, \langle x_1, x_2 \rangle) = i(\gamma, x_1) \cdot j(\gamma, x_2)$ which is $\Sigma_1(L_\alpha)$ as required.

v) Note $w(\gamma, \delta) = \epsilon \iff$
$\forall x, y < \alpha.k(\epsilon, \langle x, y \rangle) = j(\delta, x) \cdot j(\gamma, y) \cdot i(x, y) \iff$
$\forall z \leq s_k(\epsilon).k(\epsilon, z) = j(\delta, \pi_1 z) \cdot j(\gamma, \pi_2 z) \cdot i(\pi_1 z, \pi_2 z) \wedge$
$\forall x \leq s_j(\delta) \forall y \leq \mathtt{max}\{s_j(\gamma), s_i(x)\}.k(\epsilon, \langle x, y \rangle) = j(\delta, x) \cdot j(\gamma, y) \cdot i(x, y)$ which is $\Sigma_1(L_\alpha)$ as required.

vi) Note $t_{i,j}(\gamma) = \delta \iff i(\gamma) = j(\delta) \iff$
$\exists s_\gamma[s_i(\gamma) = s_\gamma \wedge \forall x < s_\gamma.i(\gamma, x) = j(\delta, x)] \wedge$
$\exists s_\delta[s_j(\delta) = s_\delta \wedge \forall x < s_\delta.i(\gamma, x) = j(\delta, x)]$ which is $\Sigma_1(L_\alpha)$ as required.

vii) Using the function psup from lemma 3.5.1 we have $s_{i,j}(\gamma) = \mathtt{psup}(s_i[j(\gamma)])$ where $s_i[j(\gamma)]$ is an α-finite set by the admissibility of α, $j(\gamma)$ being an α-finite set and s_i being α-computable. Therefore $s_{i,j}$ is α-computable as required.

□

We can index α-c.e., α-computable sets by an index in α.

Note 3.5.5. (Index for α-c.e. set)
Note that every $\Sigma_1(L_\alpha)$ set is a domain of some function which is Σ_1-definable with a finite number of parameters (WLOG just one parameter $K \in L_\alpha$) over L_α with some first order formula ϕ. As the language (excluding the parameter from $K_\gamma \in L_\alpha$) where ϕ is defined is countable, we can encode the finite formula ϕ by some index $d < \omega$. The α-finite parameter K_γ of ϕ has index $\gamma < \alpha$. Hence we can encode the $\Sigma_1(L_\alpha)$ function by a pair $e := \langle d, \gamma \rangle < \alpha$. Note that the encoding/decoding of the index can be done in an α-computable way. Hence we can denote an α-c.e. set as W_e where e is its α-computable index.

Note 3.5.6. (Index for α-computable set)
Let A be an α-computable set. Then there exist $c, d < \alpha$ s.t. $A = W_c$ and $\alpha - A = W_d$ by note 3.5.5. Using this, we can assign an α-computable index for an α-computable set A as $e := \langle c, d \rangle$.

3.6 Projectum

The projectum α^* of α is defined below. We can index all α-c.e. sets with an index in α^*, see proposition 3.6.8. This turns out to be a useful property in constructions in α-Computability Theory as demonstrated in section 3.13.

We state an alternative definition of the projectum in definition 3.6.1 and then provide its equivalence with definition 2.2.2 in proposition 3.6.2.

Definition 3.6.1. (Projectum[40])
The Σ_1 *projectum* (projectum for short) of α is

$$\alpha^* := \min\{\gamma \leq \alpha : \exists A \subseteq \gamma[A \in \Sigma_1(L_\alpha) \wedge A \notin L_\alpha]\}.$$

Proposition 3.6.2. [41] The following ordinals are equal:

i) $\alpha^* := \min\{\gamma < \alpha : \exists A \subseteq \gamma[A \in \Sigma_1(L_\alpha) \wedge A \notin L_\alpha]\}$

ii) $\min\{\gamma \leq \alpha : \exists$ partial surjection $p_1 : \gamma \rightharpoonup \alpha \in \Sigma_1(L_\alpha)\}$

iii) $\min\{\gamma \leq \alpha : \exists$ total injection $i : \alpha \rightarrowtail \gamma \in \Sigma_1(L_\alpha)\}$.

□

Example 3.6.3. (Examples of a projectum)

- The projectum of an infinite regular cardinal α is $\alpha^* = \alpha$.
- The projectum of ω_1^{CK} is ω. [42]

Fact 3.6.4. (Admissibility of the projectum[43])
If α is admissible, then the projectum α^* of α is also admissible.

Proposition 3.6.5. [44] Suppose that $\alpha^* < \alpha$, then α is not Σ_2 admissible.

Proof. Let $\alpha^* < \alpha$ and $p_1 : \alpha^* \rightharpoonup \alpha$ be a partial surjection which is $\Sigma_1(L_\alpha)$ definable. Let $A := \mathrm{dom}(p_1)$. Extend p_1 to a total function $f : \alpha \to \alpha$ as follows:

$$f(x) = \begin{cases} p_1(x) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

As $A \in \Sigma_1(L_\alpha)$, so the total function $f \in \Sigma_2(L_\alpha)$. But $\alpha^* \in L_\alpha$ and $f[\alpha^*] = \alpha \notin L_\alpha$. Hence α is not Σ_2 admissible. □

Proposition 3.6.6. [45] Let $\alpha^* < \alpha$ and let $i : \alpha \to \alpha^*$ be a total α-computable injection. If $i[A]$ is α-computable, then $A \in L_\alpha$.

[40] Definition 1.19 in [4].
[41] Theorem 1.20 in [4].
[42] [4] p10 Remark 1.21a.
[43] [1] section V.7. p184 Corollary 7.13.
[44] Usually assumed implicitly. No reference known.
[45] Usually assumed implicitly. No reference known.

Proof. Let $p_1 : \alpha^* \rightharpoonup \alpha$ be an α-computable projection, i.e. $p_1 := i^{-1}$. Let $K := i[A]$. K is α-computable by the assumption. But K is also bounded by $\alpha^* < \alpha$. Thus $K \in L_\alpha$ by proposition 3.2.7.

Note that $p_1[K] = A$. Hence $A \in L_\alpha$ by the α-computability of p_1, α-finiteness of K and by the admissibility of α. □

Proposition 3.6.7. [46] Let $A \subseteq \delta < \alpha^*$ and $A \in \Sigma_1(L_\alpha)$. Then $A \in L_\alpha$. □

Proposition 3.6.8. (Indexing α-c.e. sets with a projectum[47])
We can index all α-c.e. sets just with indices from α^*.

Proof. Let W_d be an α-c.e. set with an index in α. Let $p_1 : \alpha^* \rightharpoonup \alpha$ be an α-computable projection (partial surjection). Let W^*_e denote an α-c.e. set with an index in α^*, i.e.

$$W^*_e := \{x < \alpha : \exists d < \alpha[p_1(e) = d \land x \in W_d]\}.$$

Note that W^*_e is $\Sigma_1(L_\alpha)$. As p_1 is a surjection, so every α-c.e. set W_d is represented by some set W^*_e where $p_1(e) = d$ as required. □

Lemma 3.6.9. [48] Suppose that $i : \alpha \to \alpha$ is an α-computable injection and $A \subseteq \alpha$. Then $A \equiv_{\alpha e} i[A]$.

Proof. Let $B := i[A]$. Then $A \leq_{\alpha e} B$ via $V := \{\langle\gamma, \delta\rangle : i[K_\gamma] = K_\delta\}$ where $V \in \Sigma_1(L_\alpha)$ since $i \in \Sigma_1(L_\alpha)$ and $i[K_\gamma] \in L_\alpha$ by the admissibility of α and α-computability of i. Similarly, $B \leq_{\alpha e} A$ via

$$W := \{\langle\gamma, \delta\rangle : K_\gamma \subseteq \text{rng}(i) \land i^{-1}[K_\gamma] = K_\delta\} \in \Sigma_1(L_\alpha).$$

Therefore $A \equiv_{\alpha e} i[A]$ as required. □

Proposition 3.6.10. [49] $\forall A \subseteq \alpha \exists B \subseteq \alpha^*. A \equiv_{\alpha e} B$.

Proof. Let $i : \alpha \to \alpha^*$ be the α-computable injection and define $B := i[A]$. Then $B \subseteq \alpha^*$ and $A \equiv_{\alpha e} B$ by lemma 3.6.9 as required. □

3.7 Reducibility

In this section the generalizations of the enumeration, Turing and many-one reducibilities in α-Computability Theory are introduced.

Each of the enumeration and Turing reducibility from the classical Computability Theory generalizes to several non-equivalent notions in α-Computability Theory. The correspondence between these different notions is introduced in section 3.9.

[46] From [23] p157.
[47] Usually assumed implicitly as in [4] theorem 3.5. p52. No reference known.
[48] Usually assumed implicitly. No reference known.
[49] Usually assumed implicitly. No reference known.

3.7.1 Enumeration reducibility

The generalization of the enumeration reducibility and operator to the context of α-Computability Theory is briefly investigated in [6] by Di Paola for the first time. The set A is *weakly α-enumeration reducible* to the set B denoted as $A \leq_{w\alpha e} B$ iff

$$\exists \epsilon < \alpha^* \forall x < \alpha [x \in A \iff \exists \eta [\langle x, \eta \rangle \in W^*_\epsilon \wedge K_\eta \subseteq B]] \tag{3.3}$$

where W^*_ϵ is an α-c.e. set with an index from the projectum

$$\alpha^* := \min\{\gamma \leq \alpha : \exists A \subseteq \gamma [A \in \Sigma_1(L_\alpha) \wedge A \notin L_\alpha]\}.$$

Similarly, the set A is *α-enumeration reducible* to the set B denoted as $A \leq_{\alpha e} B$ iff

$$\exists \epsilon < \alpha^* \forall \delta < \alpha [K_\delta \subseteq A \iff \exists \eta [\langle \delta, \eta \rangle \in W^*_\epsilon \wedge K_\eta \subseteq B]] \tag{3.4}$$

After these definitions, Di Paola studies the theory of α-enumeration operators which diverts from our goals. Hence unless stated otherwise, the definitions and results concerned with the enumeration reducibility and degrees in this thesis are new.

We provide three natural notions of the weak α-enumeration reducibility prove their equivalence in proposition 3.7.4. We also provide a suitable definition of the α-enumeration reducibility equivalent to 3.3.

Weak α-enumeration reducibility

Definition 3.7.1. (Weak α-enumeration reducibility[50])
The set A is *weak α-enumeration reducible* to B denoted as $A \leq_{w\alpha e} B$ iff there is a weak α-enumeration operator $\Phi^w \in \Sigma_1(L_\alpha)$ s.t. $A = \Phi^w(B)$ where

$$\Phi^w(B) := \{x < \alpha : \exists \delta < \alpha [\langle x, \delta \rangle \in \Phi^w \wedge K_\delta \subseteq B]\}.$$

Definition 3.7.2. (Very weak α-enumeration reducibility[51])
The set A is *very weak α-enumeration reducible* to B denoted as $A \leq_{vw\alpha e} B$ iff there is a very weak α-enumeration operator $\Phi^{vw} \in \Sigma_1(L_\alpha)$ s.t. $A = \Phi^{vw}(B)$ where

$$\Phi^{vw}(B) := \bigcup \{K_\gamma : \exists \delta < \alpha [\langle \gamma, \delta \rangle \in \Phi^{vw} \wedge K_\delta \subseteq B].$$

Definition 3.7.3. (Feeble α-enumeration reducibility[52])
The set A is *feeble α-enumeration reducible* to B denoted as $A \leq_{f\alpha e} B$ iff there

[50] Definition 3.7.1 is clearly equivalent to 3.3. The difference is that we do not require the α-c.e. set Φ^w to have an index from α^*.
[51] Introduced in this thesis.
[52] Introduced in this thesis.

is a feeble α-enumeration operator $\Phi^f \in \Sigma_1(L_\alpha)$ s.t. $A = \Phi^f(B)$ where

$$\Phi^f(B) := \{x < \alpha : \exists\gamma, \delta < \alpha[\langle\gamma, \delta\rangle \in \Phi^f \wedge x \in K_\gamma \wedge K_\delta \subseteq B]\}.$$

Proposition 3.7.4. (Equivalence of weak α-enumeration reducibilities[53])
The following three statements are equivalent for $A, B \subseteq \alpha$:

i) $A \leq_{w\alpha e} B$,

ii) $A \leq_{f\alpha e} B$,

iii) $A \leq_{vw\alpha e} B$.

In other words: $\leq_{w\alpha e} = \leq_{f\alpha e} = \leq_{vw\alpha e}$.

Proof. Note that

$$x \in \Phi^{vw}(B) \iff \exists\gamma, \delta < \alpha[\langle\gamma, \delta\rangle \in \Phi^{vw} \wedge x \in K_\gamma \wedge K_\delta \subseteq B].$$

So the definition of a very weak α-enumeration operator is equivalent to the definition of a feable α-enumeration operator, i.e. $\forall e < \alpha.\Phi_e^{vw}(B) = \Phi_e^f(B)$. Thus ii) iff iii). Next i) implies ii) since given an element $x < \alpha$, we can use the $\Sigma_1(L_\alpha)$ bijection $\alpha \to L_\alpha$ to retrieve the index γ of the α-finite set $K_\gamma = \{x\}$. To prove that ii) implies i), we assume that we have $A = \Phi^f(B)$. Define

$$\Phi^w := \{\langle x, \delta\rangle \in \alpha \times \alpha : \exists\gamma < \alpha[\langle\gamma, \delta\rangle \in \Phi^f \wedge x \in K_\gamma]\}.$$

As Φ^f is $\Sigma_1(L_\alpha)$, so is Φ^w. Then we have $x \in \Phi^w(B) \iff$
$\exists\delta < \alpha[\langle x, \delta\rangle \in \Phi^w \wedge K_\delta \subseteq B] \iff$
$\exists\gamma, \delta < \alpha[\langle\gamma, \delta\rangle \in \Phi^f \wedge x \in K_\gamma \wedge K_\delta \subseteq B] \iff x \in \Phi^f(B)$. So given Φ^f operator, we can construct Φ^w such that $\Phi^w(B) = \Phi^f(B)$. Hence ii) implies i). Therefore i) iff ii) iff iii) as required. □

The proposition above establishes the equivalence of the three notions defined. Therefore we will only talk about the weak α-enumeration reducibility and use any definition of the reducibility and its enumeration operator as convenient. We shorten *weak α-enumeration operator* to just *α-enumeration operator* and equip it with the index from α to obtain the following updated definitions widely used in this thesis.

Definition 3.7.5. (α-enumeration operator)
An *α-enumeration operator* is an α-c.e. set Φ. For a set $B \subseteq \alpha$, define

$$\Phi(B) := \{x \in \alpha : \exists\delta < \alpha(\langle x, \delta\rangle \in \Phi \wedge K_\delta \subseteq B)\}.$$

Let Φ_γ denote an α-c.e. set with an index γ.

Definition 3.7.6. (Weak α-enumeration reducibility)
A is *weakly α-enumeration reducible* to B denoted as $A \leq_{w\alpha e} B$ iff there exists an α-enumeration operator $\Phi \in \Sigma_1(L_\alpha)$ s.t. $A = \Phi(B)$.

[53] Introduced in this thesis.

α-enumeration reducibility

Definition 3.7.7. (α-enumeration reducibility[54])
The set A is α-enumeration reducible to B denoted as $A \leq_{\alpha e} B$ iff $\exists W \in \Sigma_1(L_\alpha)$ s.t.

$$\forall \gamma < \alpha[K_\gamma \subseteq A \iff \exists \delta < \alpha[\langle \gamma, \delta \rangle \in W \wedge K_\delta \subseteq B]].$$

Denote the fact that A reduces to B via W as $A = W(B)$.

Properties

Lemma 3.7.8. [55] $A \leq_{\alpha e} B \oplus C \wedge B \in \Sigma_1(L_\alpha) \implies A \leq_{\alpha e} C$

Proof. As $A \leq_{\alpha e} B \oplus C$, so there is some $W \in \Sigma_1(L_\alpha)$ s.t.

$$\forall \gamma < \alpha.[K_\gamma \subseteq A \iff \exists \delta[\langle \gamma, \delta \rangle \in W \wedge K_\delta \subseteq B \oplus C]].$$

We want to find $W_d \in \Sigma_1(L_\alpha)$ s.t.

$$\forall \gamma < \alpha.[K_\gamma \subseteq A \iff \exists \epsilon[\langle \gamma, \epsilon \rangle \in W_d \wedge K_\epsilon \subseteq C]].$$

Define

$$W_d := \{\langle \gamma, \epsilon \rangle : \exists \delta \exists K[\langle \gamma, \delta \rangle \in W \wedge K_\delta = K \oplus K_\epsilon \wedge \forall x \in K.x \in B]\}$$

which is $\Sigma_1(L_\alpha)$ by proposition 3.4.6 since $B \in \Sigma_1(L_\alpha)$ as required. □

Note 3.7.9. [56] If $A \leq_{\alpha e} \overline{A}$, is it true that $\overline{A} \leq_{\alpha e} A$?

No, consider the halting set H. Then $H \leq_e \overline{H}$, but $\overline{H} \not\leq_e H$.

Note 3.7.10. [57] Is it true that for any $A, B \subseteq \alpha$ we have: $A \leq_{\alpha e} B \iff \overline{A} \leq_{\alpha e} \overline{B}$?

No. For example, take $A = H$ and $B = \overline{H}$.

It is trivial to observe the following.

Fact 3.7.11. (Properties of an α-enumeration operator)
Let $A, B \subseteq \alpha$ and $\gamma < \alpha$. Then Φ_γ satisfies:

i) Closure under reducibility: $\Phi_\gamma(A) \leq_{w\alpha e} A$,

ii) Monotonicity: $A \subseteq B \implies \Phi_\gamma(A) \subseteq \Phi_\gamma(B)$,

iii) Witness property: if $x \in \Phi_\gamma(A)$, then $\exists K \subseteq A[K \in L_\alpha \wedge x \in \Phi_\gamma(K)]$.

Proposition 3.7.12. [58] If $A \leq_{w\alpha e} B$ and $B \leq_{\alpha e} C$, then $A \leq_{w\alpha e} C$.

[54] Definition 3.7.7 is clearly equivalent to 3.7.1. The difference is that we do not require the α-c.e. set W to have an index from α^*.
[55] Introduced in this thesis.
[56] Well-known from classical Computability Theory.
[57] Well-known from classical Computability Theory.
[58] Introduced in this thesis.

Proof. Assume $A \leq_{w\alpha e} B$. So there is an α-enumeration operator Φ s.t. $A = \Phi(B)$. In detail, for any $x < \alpha$ we have

$$x \in A \iff \exists \gamma < \alpha[\langle x, \gamma \rangle \in \Phi \wedge K_\gamma \subseteq B]. \tag{3.5}$$

Similarly, we assume $B \leq_{\alpha e} C$ and so there is $W \in \Sigma_1(L_\alpha)$ s.t. for any $\gamma < \alpha$ we have

$$K_\gamma \subseteq B \iff \exists \delta < \alpha[\langle \gamma, \delta \rangle \in W \wedge K_\delta \subseteq C]. \tag{3.6}$$

Putting statement (3.5) and statement (3.6) together we get

$$\begin{aligned} x \in A &\iff \exists \gamma < \alpha[\langle x, \gamma \rangle \in \Phi \wedge \exists \delta < \alpha[\langle \gamma, \delta \rangle \in W \wedge K_\delta \subseteq C]] \\ &\iff \exists \delta < \alpha[\langle x, \delta \rangle \in \widehat{\Phi} \wedge K_\delta \subseteq C] \\ &\iff x \in \widehat{\Phi}(C). \end{aligned}$$

where $\widehat{\Phi} := \{\langle x, \delta \rangle : \exists \gamma, \delta < \alpha[\langle x, \gamma \rangle \in \Phi \wedge \langle \gamma, \delta \rangle \in W]\} \in \Sigma_1(L_\alpha)$. Hence $A = \widehat{\Phi}(C)$ and so $A \leq_{w\alpha e} C$ as required. □

3.7.2 Total reducibility

The Turing reducibility can be thought of as a total reducibility wrt the enumeration reducibility. In this section we introduce total reducibilities in α-Computability Theory which are the generalizations of the Turing reducibility.

Definition 3.7.13. (Weak α-reducibility[59])
The set $A \subseteq \alpha$ is *weakly α-reducible* to the set $B \subseteq \alpha$ denoted as $A \leq_{w\alpha} B$ iff there is α-c.e. set $W \in \Sigma_1(L_\alpha)$ s.t. for any $x < \alpha$:

$$\begin{aligned} x \in A &\iff \exists \gamma < \alpha \exists \delta < \alpha[\langle x, \gamma, \delta, 1 \rangle \in W \wedge K_\gamma \subseteq B \wedge K_\delta \subseteq \overline{B}], \\ x \in \overline{A} &\iff \exists \gamma < \alpha \exists \delta < \alpha[\langle x, \gamma, \delta, 0 \rangle \in W \wedge K_\gamma \subseteq B \wedge K_\delta \subseteq \overline{B}]. \end{aligned}$$

Definition 3.7.14. (α-reducibility[60])
The set $A \subseteq \alpha$ is *α-reducible* to the set $B \subseteq \alpha$ denoted as $A \leq_\alpha B$ iff there is α-c.e. set $W \in \Sigma_1(L_\alpha)$ s.t. for any $\beta < \alpha$:

$$\begin{aligned} K_\beta \subseteq A &\iff \exists \gamma < \alpha \exists \delta < \alpha[\langle \beta, \gamma, \delta, 1 \rangle \in W \wedge K_\gamma \subseteq B \wedge K_\delta \subseteq \overline{B}], \\ K_\beta \subseteq \overline{A} &\iff \exists \gamma < \alpha \exists \delta < \alpha[\langle \beta, \gamma, \delta, 0 \rangle \in W \wedge K_\gamma \subseteq B \wedge K_\delta \subseteq \overline{B}]. \end{aligned}$$

3.7.3 Total and enumeration reducibilities

We state the correspondence between the total and enumeration reducibilities and their shared properties.

[59] [4] p6 definition 1.13.
[60] [4] p7 definition 1.14.

Correspondence

We state relations between the total and enumeration reducibilities which elucidate the use of the word *total* for total reducibilities.

Definition 3.7.15. (Total set[61])
A subset $A \subseteq \alpha$ is total iff $\overline{A} \leq_{\alpha e} A$ iff $A \equiv_{\alpha e} A \oplus \overline{A}$.

The following fact 3.7.16 provides alternative definitions of the total reducibilities $\leq_\alpha$ and $\leq_{w\alpha}$ in terms of the enumeration reducibilities on the total sets. Fact 3.7.16 follows from the definitions of the α-reducibilities (definition 3.7.13 and definition 3.7.14) and α-enumeration reducibilities (definition 3.7.6 and definition 3.7.7) provided in this section.

From definition 3.7.13 and definition 3.7.14 it is easy to see that $A \leq_\alpha B$ (or $A \leq_{w\alpha} B$) can be also interpreted as saying that the set A and its complement $\overline{A}$ can be (weakly) α-enumerated from the set B and its complement $\overline{B}$. We state this observation formally in the following fact.

Fact 3.7.16. (Total and enumeration reducibilities correspondence)

i) $\forall A, B \subseteq \alpha[A \leq_{w\alpha} B \iff A \oplus \overline{A} \leq_{w\alpha e} B \oplus \overline{B}]$.

ii) $\forall A, B \subseteq \alpha[A \leq_\alpha B \iff A \oplus \overline{A} \leq_{\alpha e} B \oplus \overline{B}]$.

Selman's Theorem gives a characterization of the α-enumeration reducibility for arbitrary sets in terms of the α-enumeration reducibility on the total sets.

Corollary 7.1.6. (Selman's theorem[62])
Assume $V = L$ and let α be an infinite regular cardinal. Then for all $A, B \subseteq \alpha$ we have

$$A \leq_{\alpha e} B \iff \forall X[X \equiv_{\alpha e} X \oplus \overline{X} \wedge B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}].$$

We defer the use of corollary 7.1.6 and its proof until chapter 7.

Shared properties

The following facts are easy to see for the (weak) α-reducibility and commonly implicitly assumed in α-Computability Theory. As the definitions of the enumeration reducibilities are simpler than those of the total reducibilities, the facts are even simpler to observe for the enumeration reducibilities.

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[61] A well-known definition in classical Computability Theory, see [2] Definition 1.4. The generalized version introduced in this thesis.

[62] Generalized to α-Computability Theory in this thesis.

Fact 3.7.17. (Properties of total and enumeration reducibilities)
Let $\leq_r \in \{\leq_{\alpha e}, \leq_\alpha\}$. Then

i) $\leq_r$ induces a partial order $\leq$ (i.e. $\leq$ is reflexive, antisymmetric, transitive) on the r-Degrees $\mathcal{D}_r$ which are the equivalence classes induced by $\leq_r$,

ii) In general weak reducibilities $\leq_{w\alpha e}$ and $\leq_{w\alpha}$ are not transitive [26],

iii) $\leq_r$ has a unique minimal degree denoted 0 containing $\emptyset$ and α,

iv) invariance under α-finite changes: if $A \leq_r B$, $A \triangle \hat{A} \in L_\alpha$, $B \triangle \hat{B} \in L_\alpha$, then $\hat{A} \leq_r \hat{B}$, where $\triangle$ denotes a set theoretic difference.

v) $A_0 \cup A_1 \leq_r A_0 \oplus A_1$,

vi) $A_0 \leq_r B \wedge A_1 \leq_r B \iff A_0 \oplus A_1 \leq_r B$,

vii) $\chi_A \equiv_\alpha A \oplus \overline{A}$ where χ_A is the characteristic function of A.

3.7.4 Many-one reducibility

We generalize many-one reducibility to the setting of α-Computability Theory.

Definition 3.7.18. (Many-one reducibility[63])
The set A is α-*many-one* reducible to the set B denoted as $A \leq_{\alpha m} B$ iff there exists a total α-computable function $f : \alpha \to \alpha$ satisfying

$$\forall x \in \alpha[x \in A \iff f(x) \in B].$$

It is easy to see the following.

Fact 3.7.19. (Many-one reducibility properties)

- $A \leq_{\alpha m} B \iff \overline{A} \leq_{\alpha m} \overline{B}$,
- If $A_1 \leq_{\alpha m} B_1$ and $A_2 \leq_{\alpha m} B_2$, then $A_1 \oplus A_2 \leq_{\alpha m} B_1 \oplus B_2$.

Proposition 3.7.20. [64] Assume $A \leq_{\alpha m} B$, then:

- $A \leq_{\alpha e} B$
- $\overline{A} \leq_{\alpha e} \overline{B}$

Proof. We prove the first statement. Assume that $A \leq_{\alpha m} B$ via the α-computable function $f : \alpha \to \alpha$. Hence $x \in A \iff f(x) \in B$ and so $K_\gamma \subseteq A \iff f[K_\gamma] \subseteq B$. Therefore $A \leq_{\alpha e} B$ via α-c.e. set

$$W := \{\langle \gamma, \delta \rangle : f[K_\gamma] = K_\delta\}.$$

The second statement follows from the first statement and fact 3.7.19. □

[63]Introduced in this thesis.
[64]Introduced in this thesis.

3.8 Regularity

A notion of a regularity measures how close a set behaves like a set in classical Computability Theory from the definability perspective. Many theorems and statements invoke assumptions on the regularity, e.g. Shore's Splitting Theorem 3.8.5.

We investigate different notions of regularity: regularity, quasiregularity, hyperregularity and megaregularity. We consider their closure under reducibilities and definability, degree invariance, relative strength and relation to totality.

3.8.1 Regularity and quasiregularity

We define regularity and quasiregularity. Regularity is a well-established notion in α-Computability Theory, see [23] or [4]. Quasiregularity was introduced in this thesis.

Definition 3.8.1. (Regularity and quasiregularity)

- A subset $A \subseteq \alpha$ is α-*regular* iff $\forall \gamma < \alpha . A \cap \gamma \in L_\alpha$.
- A subset $A \subseteq \alpha$ is α-*quasiregular* iff $\forall \gamma < \sup(A) . A \cap \gamma \in L_\alpha$.

If clear from the context, we just say *regular* and *quasiregular* instead of α-regular and α-quasiregular respectively.

The fact below follow directly from the definitions.

Fact 3.8.2. (Regularity and quasiregularity)

i) Every regular set is quasiregular.

ii) If $\sup(A) = \alpha$ and A is quasiregular, then A is regular.

Proposition 3.8.3. (Regularity closure under operations[65])

i) $\forall A \subseteq \alpha [A \text{ regular} \iff \overline{A} \text{ regular}]$,

ii) $\forall A, B \subseteq \alpha [A \text{ regular} \wedge B \text{ regular} \implies A \oplus B \text{ regular}]$,

iii) $\forall A, B \subseteq \alpha [A \text{ regular} \wedge B \text{ regular} \implies A \cup B \text{ regular}]$.

Proof. The statements follow from the following observations respectively:

i) If $A \cap \gamma \in L_\alpha$, then $\overline{A} \cap \gamma \in L_\alpha$.

ii) If $A \cap \gamma \in L_\alpha$ and $B \cap \gamma \in L_\alpha$, then $(A \oplus B) \cap \gamma \in L_\alpha$.

[65] Usually assumed implicitly. No reference known.

iii) If $A \cap \gamma \in L_\alpha$ and $B \cap \gamma \in L_\alpha$, then $(A \cup B) \cap \gamma \in L_\alpha$.

□

Theorem 3.8.4. (Sack's Theorem on regular set existence[66])
Let A be α-computably enumerable. Then there exists a regular, α-c.e. set B of the same α-degree as A.

Theorem 3.8.5. (*Shore's Splitting Theorem* [27])
Let C be α-c.e and regular. Let D be non-α-computable and α-c.e. Then there exist regular α-c.e. sets A and B s.t. $C = A \sqcup B$, $A \leq_\alpha C$, $B \leq_\alpha C$ and also $D \not\leq_\alpha A$ or $D \not\leq_\alpha B$. □

3.8.2 Megaregularity

We introduce in this thesis the notion of megaregularity. In remark 3.8.27 this will turn out to be much stronger than the notion of regularity.

Definition 3.8.6. (Megaregularity)
Let $B \subseteq \alpha$ and add B as a predicate to the language for the structure $\langle L_\alpha, B\rangle$.

Then B is *α-megaregular* iff the structure $\langle L_\alpha, B\rangle$ satisfies the axiom of $\Sigma_1(L_\alpha, B)$-replacement:

$$\forall f \in \Sigma_1(L_\alpha, B) \forall K \in L_\alpha . f[K] \in L_\alpha .$$

If the ordinal α is clear from the context, we just say *megaregular* instead of α-megaregular.

Remark 3.8.7. A person familiar with the notion of *hyperregularity* shall note that a set is megaregular iff it is regular and hyperregular (proposition 3.8.31).

Admissibility and megaregularity

Lemma 3.8.8. [67] If $f \in \Sigma_1(L_\alpha, A)$ and $K \in L_\alpha$, then $f[K] \in \Delta_1(L_\alpha, A)$.

Proof. As $f \in \Sigma_1(L_\alpha, A)$, so $f \in \Delta_1(L_\alpha, A)$ by proposition 3.4.2. Note that

$$y \in f[K] \iff \exists x \in K.(x, y) \in f.$$

So clearly $f[K] \in \Sigma_1(L_\alpha, A)$. As $f \in \Pi_1(L_\alpha, A)$, so $f[K] \in \Pi_1(L_\alpha, A)$ by proposition 3.4.9. As $f[K] \in \Sigma_1 \cap \Pi_1(L_\alpha, A)$, so $f[K] \in \Delta_1(L_\alpha, A)$ as required. □

Proposition 3.8.9. [68] If $L_\alpha \models \Sigma_n$-replacement and $A \in \Delta_n(L_\alpha)$, then A is megaregular.

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[66] Sacks [23], theorem 4.2.
[67] Generalized from [4] proposition 1.12b.
[68] Introduced in this thesis.

Proof. Assume $L_\alpha \models \Sigma_n$-replacement. Hence $L_\alpha \models \Sigma_n$-collection by proposition 3.4.19. Let $A \in \Delta_n(L_\alpha)$. Let $f \in \Sigma_1(L_\alpha, A)$ and $K \in L_\alpha$ be arbitrary. Then $f[K] \in \Delta_1(L_\alpha, A)$ by lemma 3.8.8. By $L_\alpha \models \Sigma_n$-collection and proposition 3.4.10v we have $f[K] \in \Delta_n(L_\alpha)$. Thus $f[K]$ has to be bounded since $\sigma_n \mathrm{cf}(\alpha) = \alpha$ by the Σ_n admissibility of α (proposition 3.1.21). As $f[K] \in \Delta_n(L_\alpha)$, $f[K]$ is bounded and $L_\alpha \models \Sigma_n$-replacement, so $f[K] \in L_\alpha$ by lemma 3.4.13ii. Therefore A is megaregular as required. □

Megaregularity closure

We prove that the megaregularity is closed downwards under the weak α-reducibility and thus α-degree invariant.

Lemma 3.8.10. [69] Assume $B \leq_{w\alpha e} C$ and $\langle L_\alpha, C^+\rangle \models \Sigma_0$-collection. Then:

i) $\phi \in \Sigma_1(L_\alpha, B^+) \implies \exists\psi[\phi \equiv \psi \text{ and } \psi \in \Sigma_1(L_\alpha, C^+)]$.

ii) $A \in \Sigma_1(L_\alpha, B^+) \implies A \in \Sigma_1(L_\alpha, C^+)$.

Proof. As $B \leq_{w\alpha e} C$, so

$$x \in B \iff \exists\delta < \alpha[\langle x, \delta\rangle \in W \wedge K_\delta \subseteq C] \iff \theta(x, C)$$

for some $W \in \Sigma_1(L_\alpha)$.

For the first statement, let $\phi(\overline{x}, B) \in \Sigma_1(L_\alpha, B^+)$ be a formula where $\overline{x}$ is a list of variables. Replace an atom $x' \in B$ in the formula $\phi(\overline{x}, B)$ by the expression $\theta(x', C)$. Denote the new formula by $\psi(\overline{x}, C)$. We will prove by the structural induction on the formula that $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C^+)$. Note that $\phi(\overline{x}, B)$ and $\psi(\overline{x}, C)$ are equivalent. This implies $\phi \equiv \psi$ and $\psi \in \Sigma_1(L_\alpha, C^+)$ as required.

For the second statement, let $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$ be a formula defining A. Note that $\phi(x, B)$ and $\psi(x, C)$ define the same set A. This implies $A \in \Sigma_1(L_\alpha, C^+)$ as required.

Proof of $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C^+)$ by induction

Note $\theta(x, C) \in \Sigma_1(L_\alpha, C^+)$. Hence express $\theta(x, C)$ as $\exists\overline{y}.R(x, \overline{y}, C)$ where $R(x, \overline{y}, C) \in \Delta_0(L_\alpha, C^+)$.

- If $\phi(\overline{x}, B) \in \mathrm{QF}(L_\alpha)$, then $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C^+)$ trivially.
- If $\phi(\overline{x}, B) = x_i \in B$, then $\psi(\overline{x}, C) = \theta(x_i, C) \in \Sigma_1(L_\alpha, C^+)$.

[69] Introduced in this thesis.

- Let $\phi(\overline{x}, B) = \phi_0(\overline{x}, B) \blacklozenge \phi_1(\overline{x}, B)$ where $\blacklozenge \in \{\wedge, \vee\}$ and $\phi_i(\overline{x}, B) \in \Sigma_1(L_\alpha, B^+)$. Then $\psi(\overline{x}, C) = \psi_0(\overline{x}, C) \blacklozenge \psi_1(\overline{x}, C)$. By IH $\psi_i(\overline{x}, C) \equiv \exists \overline{y_i}.R_i(\overline{x}, \overline{y_i}, C)$ where $R_i(\overline{x}, \overline{y_i}, C) \in \Delta_0(L_\alpha, C^+)$. Note that
$$\psi(\overline{x}, C) \equiv \exists \overline{y_0}, \overline{y_1}[R_0(\overline{x}, \overline{y_0}, C) \blacklozenge R_1(\overline{x}, \overline{y_1}, C)].$$
Hence $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C^+)$.

- Let $\phi(\overline{x}, B) = \forall y \in K.\phi'(\overline{x}, y, B)$. Then by IH we have
$$\psi(\overline{x}, C) = \forall y \in K \exists \overline{z} R'(\overline{x}, y, \overline{z}, C)$$
where $R'(\overline{x}, y, \overline{z}, C) \in \Delta_0(L_\alpha, C^+)$. Since $\langle L_\alpha, C^+ \rangle \models \Sigma_0$-collection, so $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C^+)$ by proposition 3.4.7.

- If $\phi(\overline{x}, B) = \exists y.\phi'(\overline{x}, y, B)$ or $\phi(\overline{x}, B) = \exists y \in K.\phi'(\overline{x}, y, B)$, then it can be verified easily using IH that also $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C^+)$.

As all induction steps are covered, this concludes the proof. □

Lemma 3.8.11. [70] Let $A \in \Sigma_1(L_\alpha, B)$, $B \leq_{w\alpha} C$ and $\langle L_\alpha, C \rangle \models \Sigma_0$-collection. Then $A \in \Sigma_1(L_\alpha, C)$.

Proof. Since $B \leq_{w\alpha} C$, so there are α-computably enumerable sets $W_0, W_1 \subseteq \alpha$ s.t.:
$$x \notin B \iff \exists \gamma, \delta < \alpha[\langle x, \gamma, \delta \rangle \in W_0 \wedge K_\gamma \subseteq C \wedge K_\delta \subseteq \overline{C}] \iff \theta_0(x, C),$$
$$x \in B \iff \exists \gamma, \delta < \alpha[\langle x, \gamma, \delta \rangle \in W_1 \wedge K_\gamma \subseteq C \wedge K_\delta \subseteq \overline{C}] \iff \theta_1(x, C),$$
where θ_0 and θ_1 are the abbreviations for the stated longer equivalent formulas. As $A \in \Sigma_1(L_\alpha, B)$, so $A = \{x < \alpha : \phi(x, B)\}$ where $\phi(x, B) \in \Sigma_1(L_\alpha, B)$. Using De Morgan's laws, WLOG let the negations in the formula $\phi(x, B)$ occur only at the level of literals. Construct a new formula $\psi(x, C)$ from the formula $\phi(x, B)$ by replacing the literals: $x \notin B$ by $\theta_0(x, C)$ and $x \in B$ by $\theta_1(x, C)$. Note $\theta_i(x, C) \in \Sigma_1(L_\alpha, C)$.

Note that $\phi(x, B)$ and $\psi(x, C)$ define the same set A. Moreover, one can prove by the induction on the structure of the formula with a similar proof as in the proof of lemma 3.8.10 that $\psi(x, C) \in \Sigma_1(L_\alpha, C)$. Here we use the assumption $\langle L_\alpha, C \rangle \models \Sigma_0$-collection in the following case. Suppose
$$\psi(\overline{x}, C) = \forall y \in K \exists \overline{z} R'(\overline{x}, y, \overline{z}, C)$$
where $R'(\overline{x}, y, \overline{z}, C) \in \Delta_0(L_\alpha, C)$ by IH and $K \in L_\alpha$. Since $\langle L_\alpha, C \rangle \models \Sigma_0$-collection, so $\psi(\overline{x}, C) \in \Sigma_1(L_\alpha, C)$ by proposition 3.4.7 as required.

Therefore $A \in \Sigma_1(L_\alpha, C)$ as required. □

[70] Introduced in this thesis.

Proposition 3.8.12. (Axiom closure under $\leq_{w\alpha e}$[71])
Assume that $A \leq_{w\alpha e} B$. Then:

i) If $\langle L_\alpha, B^+\rangle \models \Sigma_0$-collection $\wedge\ \Sigma_1$-replacement, then $\langle L_\alpha, A^+\rangle \models \Sigma_1$-replacement.

ii) If $\langle L_\alpha, B^+\rangle \models \Sigma_1$-collection, then $\langle L_\alpha, A^+\rangle \models \Sigma_1$-collection.

iii) If $\langle L_\alpha, B^+\rangle \models \Sigma_0$-collection $\wedge\ \Sigma_1$-separation, then $\langle L_\alpha, A^+\rangle \models \Sigma_1$-separation.

Proof. i) Let $f \in \Sigma_1(L_\alpha, A^+)$. Since $A \leq_{w\alpha e} B$ and $\langle L_\alpha, B^+\rangle \models \Sigma_0$-collection, so $f \in \Sigma_1(L_\alpha, B^+)$ by lemma 3.8.10ii. So if $K \in L_\alpha$, then $f[K] \in L_\alpha$ since $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement. Therefore $\langle L_\alpha, A^+\rangle \models \Sigma_1$-replacement.

ii) Let $\phi(x, y) \in \Sigma_1(L_\alpha, A^+)$. Since $A \leq_{w\alpha e} B$ and $\langle L_\alpha, B^+\rangle \models \Sigma_0$-collection, so $\phi(x, y) \in \Sigma_1(L_\alpha, B^+)$ up to equivalence by lemma 3.8.10i. So if $K \in L_\alpha$ and $\forall x \in K \exists y.\phi(x, y)$, then there is $\hat{K} \in L_\alpha$ s.t. $\forall x \in K \exists y \in \hat{K}.\phi(x, y)$ since $\langle L_\alpha, B^+\rangle \models \Sigma_1$-collection. Therefore $\langle L_\alpha, A^+\rangle \models \Sigma_1$-collection.

iii) Let $K \in L_\alpha$ and $\phi(x) \in \Sigma_1(L_\alpha, A^+)$. Define $\hat{K} := \{x \in K : \phi(x)\}$. We need to show that $\hat{K} \in L_\alpha$. Since $A \leq_{w\alpha e} B$ and $\langle L_\alpha, B^+\rangle \models \Sigma_0$-collection, so $\phi(x) \in \Sigma_1(L_\alpha, B^+)$ up to equivalence by lemma 3.8.10i. Thus $\hat{K} \in L_\alpha$ as required since $\langle L_\alpha, B^+\rangle \models \Sigma_1$-separation.

□

Proposition 3.8.13. (Megaregularity closure and degree invariance[72])

i) If $A \leq_{w\alpha} B$ or $A \leq_\alpha B$ and B is megaregular, then A is megaregular.

ii) If $A \equiv_{w\alpha} B$ or $A \equiv_\alpha B$, then [A is megaregular iff B is megaregular].

iii) If $A \in \Delta_1(L_\alpha)$, then A is megaregular.

Proof. Statements ii) and iii) follow from i). We prove i) as follows. A is megaregular iff every $\Sigma_1(L_\alpha, A)$ definable function f satisfies the replacement axiom.

Let $f \in \Sigma_1(L_\alpha, A)$. As $A \leq_{w\alpha} B$ and B is megaregular, so $f \in \Sigma_1(L_\alpha, B)$ by lemma 3.8.11. Hence f satisfies the replacement axiom as B is megaregular.

Therefore $\langle L_\alpha, A\rangle \models \Sigma_1$-replacement and so A must be megaregular. □

Often we will use proposition 3.8.13 implicitly.

[71] Introduced in this thesis.
[72] Introduced in this thesis.

3.8.3 Hyperregularity

We investigate two different notions of hyperregularity used in [23] and [4].

Definition 3.8.14. (Constructible hierarchy with a parameter[73])
Introduce to the language a predicate $x \in A$ to define the constructive hierarchy for a parameter $A \subseteq \alpha$:

- $L[A]_0 := \emptyset$,
- $L[A]_{\gamma+1} := \mathrm{Def}(L[A]_\gamma)$,
- $L[A]_\delta := \bigcup_{\gamma<\delta} L[A]_\gamma$ if $\lim(\delta)$,
- $L[A] := \bigcup_{\gamma\in\mathtt{Ord}} L[A]_\gamma$.

Lemma 3.8.15. [74] Assume $A \subseteq \alpha$. Then A is regular iff $L_\alpha = L[A]_\alpha$.

Proof. By induction $\forall\gamma.L[A]_\gamma = L[A \cap \gamma]_\gamma$. Hence A is regular iff $\forall\gamma < \alpha.L[A]_\gamma = L_\alpha$. □

Definition 3.8.16. (Sacks hyperregular[75])
A set $A \subseteq \alpha$ is Sacks hyperregular iff $\forall f \leq_{w\alpha} A \forall\gamma < \alpha \exists\delta < \alpha.f[\gamma] \subseteq \delta$.

Proposition 3.8.17. [76] If $A \in \Sigma_1(L_\alpha)$ and A is Sacks hyperregular, then A is regular. □

Proposition 3.8.18. [77] α is Σ_2-admissible iff $\forall A \in \Sigma_1(L_\alpha).A$ is Sacks hyperregular. □

It is easy to see that every α-computable set is Sacks hyperregular. By theorem 3.8.19 below the converse is not true which separates the notions of α-computability from the Sack's hyperregularity.

Theorem 3.8.19. [78] There exists a non-α-computable, Sacks hyperregular, α-c.e. set. □

Definition 3.8.20. (Chong hyperregular[79])
A set $A \subseteq \alpha$ is Chong hyperregular iff $L[A]_\alpha$ is an admissible structure - $L[A]_\alpha$ satisfies Σ_1-Collection, i.e. $\forall\phi(x, y) \in \Sigma_1(L[A]_\alpha, A)\forall K \in L[A]_\alpha$

$$\big[L[A]_\alpha \models \forall x \in K \exists y.\phi(x, y) \implies \exists\hat{K}\forall x \in K \exists y \in \hat{K}.\phi(x, y)\big].$$

[73] From [23] VII.3.5 Regularity p164.
[74] From [23] Proposition VII.3.6 p164.
[75] From [23] VI Hyperregularity and Priority p135.
[76] From [23] Prop VII.5.1. p167.
[77] From [23] Exercise VII.5.6. p174
[78] From [23] Theorem VII.5.3. p169
[79] From [4] Definition 1.32 p20

Proposition 3.8.21. [80] Assume $A \subseteq \alpha$ is regular. Then A is Sacks hyperregular iff A is Chong hyperregular.

Proof. First note that as A is regular, so $L[A]_\alpha = L_\alpha$ by lemma 3.8.15.

$\Rightarrow$: Assume that A is regular and Sacks hyperregular. Let $f \in \Sigma_1(L_\alpha, A)$, $K \in L_\alpha$ and $K \subseteq \mathrm{dom}(f)$. As A is regular, so $f \leq_{w\alpha} A$. As $K \in L_\alpha$, so $K \subseteq \gamma$ for some $\gamma < \alpha$. As A is Sacks hyperregular, so there is $\delta < \alpha$ s.t. $f[K] \subseteq f[\gamma] \subseteq \delta \in L_\alpha$.

$\Leftarrow$: Chong hyperregular implies Sacks hyperregular clearly.

□

3.8.4 Projectum and regularity

The projectum α^* can be thought of as a Σ_1-projectum. We generalize the Σ_1-projectum α^* to a Σ_n-projectum and relate a projectum with regularity.

Definition 3.8.22. (Σ_n-projectum[81])
The Σ_n *projectum* of α is

$$\sigma_n p(\alpha) := \min\{\gamma \leq \alpha : \exists A \subseteq \gamma[A \in \Sigma_n(L_\alpha) \wedge A \notin L_\alpha]\}.$$

Proposition 3.8.23. (Projectum and regularity[82])

- $A \subseteq \alpha^* \wedge A \in \Sigma_1(L_\alpha) \implies A$ quasiregular,
- $\alpha^* = \alpha \wedge A \in \Sigma_1(L_\alpha) \implies A$ regular.
- $A \subseteq \sigma_n p(\alpha) \wedge A \in \Sigma_n(L_\alpha) \implies A$ quasiregular,
- $\sigma_n p(\alpha) = \alpha \wedge A \in \Sigma_n(L_\alpha) \implies A$ regular.

Proof. All four statements are implied by the statement

$$A \subseteq \sigma_n p(\alpha) \wedge A \in \Sigma_n(L_\alpha) \implies \forall \beta < \sup(A). A \cap \beta \in L_\alpha$$

which is true. For suppose not, then

$$\exists \beta < \sup(A) \leq \sigma_n p(\alpha)[A \cap \beta \in \Sigma_n(L_\alpha) \wedge A \cap \beta \notin L_\alpha]$$

which is a contradiction to definition 3.8.22. □

[80] Introduced in this thesis.
[81] Definition 1.19 in [4].
[82] From [23] Proposition VII.2.1. p157.

3.8.5 Notions of regularity by strength

We inspect relation between α-finiteness, computability and different notions of regularity: quasiregularity, regularity, hyperregularity and megaregularity.

Proposition 3.8.24. [83] Every α-computable subset of α is regular.

Proof. Follows from proposition 3.2.7. □

Example 3.8.25. Let $\alpha = \omega_1^{CK}$. Then the set of the ordinal notations, Kleene's $\mathcal{O} \subseteq \omega$ is quasiregular, but not regular. As $\mathcal{O} \in \Sigma_1(L_\alpha)$, so by theorem 3.8.4 there is $B \subseteq \alpha$ s.t. $\mathcal{O} \equiv_\alpha B$ and B is regular.

Let $p_0 : \omega \rightharpoonup \alpha$ be a partial α-computable function taking a computable notation to its ordinal, i.e. p_0 is the α-computable projection constructed from $\mathcal{O}$ using proposition 3.6.2. Using $\mathcal{O}$ as an oracle, complete p_0 to a total surjective function $f : \alpha \to \alpha$. Then $f \in \Sigma_1(L_\alpha, \mathcal{O})$, $\omega \in L_\alpha$, but $f[\omega] \notin L_\alpha$. Hence neither $\mathcal{O}$ nor B is megaregular.

Example 3.8.26. [84] Assume $V = L$ and let α be an infinite regular cardinal. Then every subset of α is megaregular. But there are subsets of α which are not α-computable.

Remark 3.8.27. Hence using results from section 3.8 (statements 3.8.2, 3.8.24, 3.8.25, 3.8.26) we have the following *strict* separation of the notions where α-finiteness is the strongest condition and quasiregularity is the weakest:

α-finite $\implies$ α-computable $\implies$ megaregular $\implies$ regular $\implies$ quasiregular.

Proposition 3.8.28. (Computability, regularity, megaregularity[85])
Let $\mathcal{B} \in \{B, B^+\}$. Assume $\langle L_\alpha, \mathcal{B}\rangle \models \Sigma_1$-replacement. Then:

i) $A \oplus \overline{A} \in \Sigma_1(L_\alpha, \mathcal{B}) \implies A$ regular,

ii) $\neg A$ regular $\implies A \oplus \overline{A} \notin \Sigma_1(L_\alpha, \mathcal{B})$,

iii) If $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement and $A \oplus \overline{A} \leq_{\alpha e} B$, then A regular,

iv) If B megaregular, then B regular,

v) If B megaregular and $A \leq_\alpha B$, then A regular,

vi) $\forall \beta < \alpha. A \cap \beta \oplus \overline{A \cap \beta} \in \Sigma_1(L_\alpha, \mathcal{B}) \iff A$ regular.

[83] From Proposition 1.12b in [4].
[84] See corollary 3.11.4 for details.
[85] Introduced in this thesis. Some parts might have been used elsewhere implicitly, no reference known.

vii) If $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement and B total, then B regular,

viii) If $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement, B total and $A \in \Sigma_1(L_\alpha, B)$, then $A \in \Sigma_1(L_\alpha, B^+)$.

ix) If $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement and B total, then B megaregular,

Proof. The statements i-vi follow from lemma 3.4.13.

We show statement vii. Assume $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement and B total. So $B \oplus \overline{B} \leq_{\alpha e} B$. Clearly, $B \cap \beta \leq_\alpha B$ for any $\beta < \alpha$. Hence $B \cap \beta \oplus \overline{B \cap \beta} \leq_{\alpha e} B \oplus \overline{B} \leq_{\alpha e} B$. So $B \cap \beta \oplus \overline{B \cap \beta} \leq_{\alpha e} B$. Hence $B \cap \beta \oplus \overline{B \cap \beta} \in \Sigma_1(L_\alpha, B^+)$. As $\beta < \alpha$ was arbitrary, we have

$$\forall \beta < \alpha . B \cap \beta \oplus \overline{B \cap \beta} \in \Sigma_1(L_\alpha, B^+).$$

So B has to be regular using statement vi.

We show statement viii. Assume $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement and let B be total, i.e. $B \oplus \overline{B} \leq_{\alpha e} B$. Hence B is also regular by statement vii. Assume that $A \in \Sigma_1(L_\alpha, B)$. As B is regular, so $A \leq_{w\alpha e} B \oplus \overline{B}$. So $A \leq_{w\alpha e} B \oplus \overline{B} \leq_{\alpha e} B$. Hence $A \leq_{w\alpha e} B$ and so $A \in \Sigma_1(L_\alpha, B^+)$.

We show statement ix. Assume that $f \in \Sigma_1(L_\alpha, B)$. Then $f \in \Sigma_1(L_\alpha, B^+)$ by statement item viii. Thus f satisfies the replacement axiom as $\langle L_\alpha, B^+\rangle \models \Sigma_1$-replacement. Since we started with $f \in \Sigma_1(L_\alpha, B)$, so B is also megaregular as required. □

Since every megaregular set is also regular, we see directly from the definition of the regularity the following.

Fact 3.8.29. Let $A \subseteq \alpha$ be bounded. If A is regular or megaregular, then $A \in L_\alpha$.

Lemma 3.8.30. [86] A is regular and Sacks hyperregular iff

$$\forall \delta < \alpha \forall f [f \leq_{w\alpha} A \implies f \restriction \delta \in L_\alpha].$$

□

Proposition 3.8.31. [87] A is regular and Sacks hyperregular iff A is megaregular.

Proof. $\Rightarrow$: As A is Sacks hyperregular and regular, so

$$\forall \delta < \alpha \forall f [f \leq_{w\alpha} A \implies f \restriction \delta \in L_\alpha]$$

by lemma 3.8.30. Let $f \in \Sigma_1(L_\alpha, A)$ and $K \in L_\alpha$. As A is regular, so $f \leq_{w\alpha} A$. As $K \in L_\alpha$, so $K \subseteq \delta$ for some $\delta < \alpha$. Note that $f[K] = f \restriction \delta[K] \in L_\alpha$ since both $f \restriction \delta$ and K are α-finite. Therefore $\langle L_\alpha, A\rangle \models \Sigma_1$-replacement and so A is megaregular.

[86] From [23] Lemma VII.5.2 p168.
[87] Introduced in this thesis.

$\Leftarrow$: Assume that A is megaregular. Then A is regular by proposition 3.8.28iv. So $L[A]_\alpha = L_\alpha$ by lemma 3.8.15. Note $\langle L_\alpha, A\rangle \models \Sigma_1$-replacement. Thus $\langle L_\alpha, A\rangle \models \Sigma_1$-collection by proposition 3.4.20. So $\langle L[A]_\alpha, A\rangle \models \Sigma_1$-collection. Thus A is Chong hyperregular. As A is Chong hyperregular and regular, so A is Sacks hyperregular by proposition 3.8.21 as required. □

3.9 Reducibilities and definability

We state a correspondence between different notions of reducibility and definability. See section 3.9 for details.

We establish a correspondence between the weak α-enumeration reducibility $\leq_{w\alpha e}$, α-enumeration reducibility $\leq_{\alpha e}$ and Σ_1 definability with a positive parameter. This propagates to the correspondence between the α-reducibility, weak α-reducibility and Δ_1 definability with a parameter. At the end, we prove some results relating to the transitivity of the arithmetic definability.

3.9.1 Relation between $\leq_{\alpha e}$ and $\leq_{w\alpha e}$

We investigate the relation between the weak α-enumeration reducibility $\leq_{w\alpha e}$ and the α-enumeration reducibility $\leq_{\alpha e}$.

Lemma 3.9.1. [88] Let $B \subseteq \alpha$ and assume at least one of the following conditions:

i) $\langle L_\alpha, B\rangle \models \Sigma_1$-replacement.

ii) $\langle L_\alpha, B^+\rangle \models \Sigma_0$-collection $\wedge$ Σ_1-separation.

Then for any α-enumeration operator $\Phi \in \Sigma_1(L_\alpha)$ there exists set $W \in \Sigma_1(L_\alpha)$ s.t. $\Phi(B) = W(B)$, i.e. W satisfies

$$\forall \gamma < \alpha[K_\gamma \subseteq \Phi(B) \iff \exists \delta < \alpha[\langle \gamma, \delta\rangle \in W \wedge K_\delta \subseteq B]].$$

Proof. Construct W_ζ from an α-enumeration operator Φ_η. Define

$$W_\zeta := \{\langle \gamma, \delta\rangle \in \alpha \times \alpha : \exists K_\epsilon[(K_\delta = \bigcup_{\beta \in K_\epsilon} K_\beta) \wedge \forall x \in K_\gamma \exists \beta \in K_\epsilon.\langle x, \beta\rangle \in \Phi_\eta]\}.$$

The set W_ζ is $\Sigma_1(L_\alpha)$ by proposition 3.4.6 since it is defined using bounded quantifiers and α-c.e. set Φ_η. Note that the index ζ is uniformly α-computable from the index η.

Now let $\gamma < \alpha$ be arbitrary. By proposition 3.2.10 an α-finite union of α-finite sets is α-finite. Thus $K_\gamma \subseteq W_\zeta(B) \iff \exists \delta < \alpha[\langle \gamma, \delta\rangle \in W_\zeta \wedge K_\delta \subseteq B] \iff$ $\exists \delta < \alpha[[\exists K_\epsilon[(K_\delta = \bigcup_{\beta \in K_\epsilon} K_\beta) \wedge \forall x \in K_\gamma \exists \beta \in K_\epsilon.\langle x, \beta\rangle \in \Phi_\eta]] \wedge K_\delta \subseteq B] \iff$

[88] Introduced in this thesis.

$\exists K_\epsilon[\forall x \in K_\gamma \exists \beta \in K_\epsilon.[\langle x, \beta\rangle \in \Phi_\eta \wedge K_\beta \subseteq B] \wedge \forall \beta \in K_\epsilon.K_\beta \subseteq B] \iff$ (*)
$\forall x \in K_\gamma \exists \beta[\langle x, \beta\rangle \in \Phi_\eta \wedge K_\beta \subseteq B] \iff \forall x \in K_\gamma.x \in \Phi_\eta(B) \iff K_\gamma \subseteq \Phi_\eta(B)$.
Hence $W_\zeta(B) = \Phi_\eta(B)$ as required.

* Assume condition i). Define the relation

$$R(x, \beta) \iff \langle x, \beta\rangle \in \Phi \wedge K_\beta \subseteq B.$$

Note that $R \in \Sigma_1(L_\alpha, B)$. Using $\langle L_\alpha, B\rangle \models \Sigma_0$-replacement there is $f \in \Sigma_1(L_\alpha, B)$ s.t. $\forall x \in \mathrm{dom}(R).R(x, f(x))$ by proposition 3.4.15. The direction $\Leftarrow$ holds by taking $K_\epsilon := f[K_\gamma]$ which is α-finite using $\langle L_\alpha, B\rangle \models \Sigma_1$-replacement.

* Assume condition ii). Given

$$\forall x \in K_\gamma \exists \beta[\langle x, \beta\rangle \in \Phi_\eta \wedge K_\beta \subseteq B],$$

we have

$$\exists K'_\epsilon \forall x \in K_\gamma \exists \beta \in K'_\epsilon[\langle x, \beta\rangle \in \Phi_\eta \wedge K_\beta \subseteq B]$$

using $\Sigma_1(L_\alpha, B^+)$-collection. Then the direction $\Leftarrow$ holds by taking $K_\epsilon := \{\beta \in K'_\epsilon : K_\beta \subseteq B\} \in L_\alpha$ using $\Sigma_1(L_\alpha, B^+\}$-separation. □

Proposition 3.9.2. (Correspondence between $\leq_{w\alpha e}$ and $\leq_{\alpha e}$ reducibilities[89])

i) If $A \leq_{\alpha e} B$, then $A \leq_{w\alpha e} B$,

ii) If $A \leq_{w\alpha e} B$ and B is megaregular, then $A \leq_{\alpha e} B$.

Proof. i) is true by taking an α-enumeration operator Φ^w to be W from $\leq_{\alpha e}$.

To prove that ii), assume first $A \leq_{w\alpha e} B$. So $A \leq_{vw\alpha e} B$ by proposition 3.7.4. Define W from Φ^{vw} using lemma 3.9.1 and $\Sigma_1(L_\alpha, B)$-replacement true by the megaregularity of B. We have $A = W(B) = \Phi^{vw}(B)$ and

$$\forall \gamma < \alpha[K_\gamma \subseteq A \iff \exists \delta < \alpha[\langle \gamma, \delta\rangle \in W \wedge K_\delta \subseteq B]]$$

and so $A \leq_{\alpha e} B$. □

Corollary 3.9.3. [90] Let $e < \alpha$. Let $A, B \subseteq \alpha$ and let B be megaregular. Then $A \leq_{\alpha e} \Phi_e(B) \implies A \leq_{\alpha e} B$.

Proof. By fact 3.7.11i and proposition 3.9.2ii we have $A \leq_{\alpha e} \Phi_e(B) \leq_{\alpha e} B$. Then $A \leq_{\alpha e} B$ by the transitivity of $\leq_{\alpha e}$ as required. □

Corollary 3.9.4. (Correspondence between $\leq_{w\alpha}$ and $\leq_\alpha$ reducibilities)

i) $A \leq_\alpha B \implies A \leq_{w\alpha} B$,

ii) $A \leq_{w\alpha} B \implies A \leq_\alpha B$ if B is megaregular.

Proof. Follows from fact 3.7.16 and proposition 3.9.2. □

[89] Introduced in this thesis. Analogous to [4] Proposition 1.15 and Proposition 1.33.
[90] Introduced in this thesis.

3.9.2 Weak reducibilities and definability

We investigate the relationship between the weak α-enumeration reducibility and the Σ_1 definability with a positive parameter.

Definition 3.9.5. (Witness property[91])
We say that $\phi(x, B)$ has a *witness property* in a parameter B iff for an arbitrary $x < \alpha$: if $L_\alpha \models \phi(x, B)$, then there is $K \subseteq B$ s.t. $L_\alpha \models \phi(x, K)$ and $K \in L_\alpha$.

Lemma 3.9.6. (Bounded usage of a positive parameter[92])
Let $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$ and $A := \{x < \alpha : \langle L_\alpha, B^+\rangle \models \phi(x, B)\}$. Then

$$\forall x \in A \exists \delta < \alpha[\langle L_\alpha, B^+\rangle \models \phi(x, B \cap \delta)].$$

Moreover uniformly δ is $\Sigma_1(L_\alpha, B)$ definable from the formula ϕ.

Proof. Extend $\phi(x, B)$ to $\phi(\overline{x}, B)$ where $\overline{x}$ is a list of parameters in order to enable the proof by the structural induction. In the end, $\phi(x, B)$ can be thought of as $\phi(\overline{x}, B)$ with x as one free variable in $\overline{x}$ and other variables in the list $\overline{x}$ fixed. For every formula $\phi(\overline{x}, B)$ we construct a total α-computable bounding function $\delta_\phi(\overline{x})$ s.t. if $\langle L_\alpha, B^+\rangle \models \phi(\overline{x}, B)$, then also $\langle L_\alpha, B^+\rangle \models \phi(\overline{x}, B \cap \delta_\phi(\overline{x}))$.

- If $\phi(\overline{x}, B) \in \mathrm{QF}(L_\alpha)$, then $\delta_\phi(\overline{x}) := 0$.
- If $\phi(\overline{x}, B) = x_i \in B$, then $\delta_\phi(\overline{x}) := x_i$.
- If $\phi(\overline{x}, B) = \phi_1(\overline{x}, B)\blacklozenge\phi_2(\overline{x}, B)$ where $\blacklozenge \in \{\wedge, \vee\}$, then
$$\delta_\phi(\overline{x}) := \max(\delta_{\phi_1}(\overline{x}), \delta_{\phi_2}(\overline{x})).$$
- Let $\phi(\overline{x}, B) = \exists y.\psi(\overline{x}, y, B)$. If $\langle L_\alpha, B^+\rangle \models \phi(\overline{x}, B)$, then there is some $y < \alpha$ s.t. $\langle L_\alpha, B^+\rangle \models \psi(\overline{x}, y, B)$. Define
$$R := \{\langle \overline{x}, \delta_y\rangle : \delta_y = \delta_\psi(\overline{x}, y) \wedge y < \alpha \wedge \psi(\overline{x}, y, B)\}.$$
Note that $R \in \Sigma_1(L_\alpha, B)$. So by the relativized Uniformization Theorem 3.4.15 there is function $\delta_\phi \subseteq R$ which is $\Sigma_1(L_\alpha, B)$ definable and hence the function required.
- Let $\phi(\overline{x}, B) = \forall y \in K.\psi(\overline{x}, y, B)$ or $\phi(\overline{x}, B) = \exists y \in K.\psi(\overline{x}, y, B)$ for some $K \in L_\alpha$. Define $\delta_\phi(\overline{x}) := \sup(\delta_\psi[\{\overline{x}\} \times K])$. By the admissibility of α, the supremum is computed on an α-finite set, hence $\delta_\phi(\overline{x})$ is well-defined.

It is easy to verify assuming the induction hypothesis that the bounded functions constructed are $\Sigma_1(L_\alpha, B)$ definable, cover all the inductive

[91] Introduced in this thesis, but a general idea of a witness common in mathematics.
[92] Introduced in this thesis.

steps and satisfy the condition: if $\langle L_\alpha, B^+\rangle \models \phi(\overline{x}, B)$, then $\delta_\phi(\overline{x})\downarrow$ and $\langle L_\alpha, B^+\rangle \models \phi(\overline{x}, B \cap \delta_\phi(\overline{x}))$. Hence $\forall x \in A \exists \delta < \alpha[\langle L_\alpha, B^+\rangle \models \phi(x, B \cap \delta)]$ using the $\Sigma_1(L_\alpha, B)$ definable function δ_ϕ as required. □

Proposition 3.9.7. (Witness property[93])
Let $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$, $A := \{x < \alpha : \langle L_\alpha, B^+\rangle \models \phi(x, B)\}$ and B be a regular set. Then

$$\forall x \in A \exists K \in L_\alpha[L_\alpha \models \phi(x, K) \text{ and } K \subseteq B].$$

Proof. This follows from lemma 3.9.6 and the regularity of B as $K := B \cap \delta \in L_\alpha$ for $\delta < \alpha$. □

Definition 3.9.8. (Monotonicity of a formula[94])
A formula $\phi(x, B)$ is *monotone* in a parameter B iff for every $B, C \subseteq \alpha$:

$$\text{if } B \subseteq C, \text{ then } \{x < \alpha : L_\alpha \models \phi(x, B)\} \subseteq \{x < \alpha : L_\alpha \models \phi(x, C)\}.$$

Proposition 3.9.9. [95] Let $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$. Then $\phi(x, B)$ is monotone in a parameter B.

Proof. The proof follows from the structural induction on the formula $\phi(x, B)$, carried out in a similar way like in the proof of lemma 3.9.6. □

Proposition 3.9.10. (Σ_1 definability and $\leq_{w\alpha e}$ reducibility correspondence[96])

i) $A \leq_{w\alpha e} B \implies A \in \Sigma_1(L_\alpha, B^+)$,

ii) $A \in \Sigma_1(L_\alpha, B^+) \wedge B \text{ regular} \implies A \leq_{w\alpha e} B$.

Proof. i) Assume $A \leq_{w\alpha e} B$. So

$$A = \{x < \alpha : L_\alpha \models \phi(x, B)\}$$

where

$$\phi(x, B) \equiv \exists \gamma < \alpha[\langle x, \gamma\rangle \in \Phi \wedge K_\gamma \subseteq B]$$

for some weak α-enumeration operator $\Phi \in \Sigma_1(L_\alpha)$. Note that $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$ and $\phi(x, B)$ defines A. Hence $A \in \Sigma_1(L_\alpha, B^+)$.

ii) Assume $A \in \Sigma_1(L_\alpha, B^+)$. So

$$A = \{x < \alpha : L_\alpha \models \phi(x, B)\}$$

[93] Introduced in this thesis.
[94] Introduced in this thesis. The idea taken from the general monotonicity property of the enumeration operators in classical Computability Theory.
[95] Introduced in this thesis.
[96] Introduced in this thesis. Analogous to [4] Proposition 1.30.

for some $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$ where $\phi(x, B)$ has a witness property by the regularity of B and by proposition 3.9.7. Hence

$$\forall x < \alpha[x \in A \iff \exists \beta < \alpha.[\phi(x, K_\beta) \wedge K_\beta \subseteq B]].$$

So define a weak α-enumeration operator $\Phi := \{\langle x, \beta\rangle : \phi(x, K_\beta)\}$. As $\phi(x, B) \in \Sigma_1(L_\alpha, B^+)$, so $\phi(x, K_\beta) \in \Sigma_1(L_\alpha)$ by proposition 3.4.3. Note $A = \Phi(B)$. Hence $A \leq_{w\alpha e} B$.

□

Proposition 3.9.11. [97] The following are true:

i) $A \leq_{w\alpha} B \implies A \in \Delta_1(L_\alpha, B)$,

ii) $A \in \Delta_1(L_\alpha, B) \implies A \leq_{w\alpha} B$ if B is a regular set.

Proof. The statements follow from proposition 3.9.10 and fact 3.7.16. □

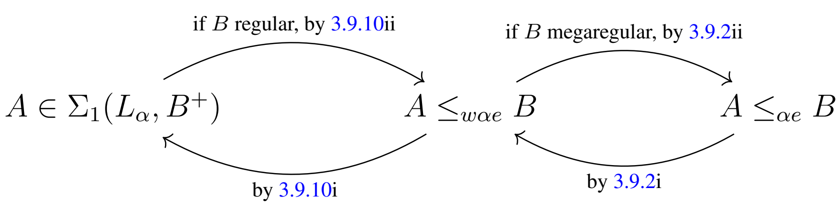
Proposition 3.9.12. [98] If $A \notin \Sigma_1(L_\alpha)$, $A \leq_{w\alpha e} C$, then $C \notin \Sigma_1(L_\alpha)$.

Proof. Assume $A \leq_{w\alpha e} C$ and $C \in \Sigma_1(L_\alpha)$. Then $A \in \Sigma_1(L_\alpha)$. □

3.9.3 Conclusions

We summarize the relations between the definability and reducibilities from this section in implication diagrams.

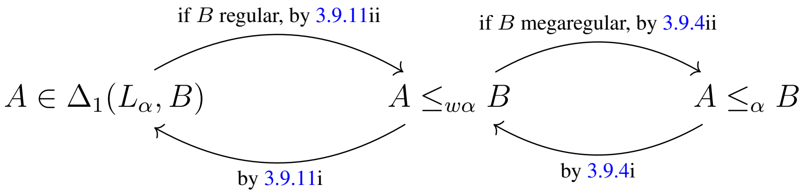
Proposition 3.9.13. (Σ_1 definability and α-enumeration reducibilities correspondence)
We have the following implication diagram:



□

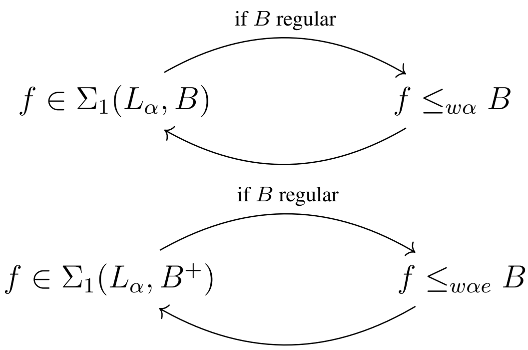
Proposition 3.9.14. (Δ_1 definability and α-reducibilities correspondence)
We have the following implication diagram:

[97] From Proposition 1.30 and Proposition 1.33 in [4].
[98] Introduced in this thesis.



□

Corollary 3.9.15. [99] We have the following implication diagrams:



Proof. Follows from proposition 3.4.2, proposition 3.9.13 and proposition 3.9.14.

□

3.10 Degree Theory

The generalizations of the Turing and the enumeration degrees are introduced briefly as the α degrees $\mathcal{D}_\alpha$ and the α-enumeration degrees $\mathcal{D}_{\alpha e}$ respectively.

Definition 3.10.1. (Degrees[100])
A degree structure is a set of equivalence classes induced by an equivalence relation induced by a reducibility relation. In particular:

- $\mathcal{D}_\alpha := \mathcal{P}(\alpha) / \equiv_\alpha$ is a set of α-*degrees*.
- $\mathcal{D}_{\alpha e} := \mathcal{P}(\alpha) / \equiv_{\alpha e}$ is a set of α-*enumeration* degrees.

Induce $\leq$ on $\mathcal{D}_\alpha$ and $\mathcal{D}_{\alpha e}$ by $\leq_\alpha$ and $\leq_{\alpha e}$ respectively.

3.10.1 Total degrees

Definition 3.10.2. (Total degrees[101])
A degree $d \in \mathcal{D}_{\alpha e}$ is *total* iff $\exists D \in d. D \equiv_{\alpha e} \overline{D}$. The set of the total degrees of $\mathcal{D}_{\alpha e}$ is denoted by $\mathcal{TOT}_{\alpha e}$.

[99] Adapted from the Corollary 1.31 in [4].
[100] The α-degrees are well-established, see [4][23]. The α-enumeration degrees are introduced in this thesis.
[101] The total α-enumeration degrees introduced and studied in this thesis.

Theorem 3.10.3. [102] Let χ_A be the characteristic function of A. Then

$$\forall A, B \subseteq \alpha[A \leq_\alpha B \iff \chi_A \leq_{\alpha e} \chi_B].$$

Corollary 3.10.4. (Degree embedding[103])
The map $\iota : \mathcal{D}_\alpha \to \mathcal{D}_{\alpha e}$ given by $\iota : \deg_\alpha(A) \mapsto \deg_{\alpha e}(A \oplus \overline{A})$ is an embedding whose image are the total degrees $\mathcal{TOT}_{\alpha e}$.

Proof. Use the fact $\chi_A \equiv_{\alpha e} A \oplus \overline{A}$ and theorem 3.10.3. □

From corollary 3.10.4 and definition 3.10.2 we can see directly the following fact.

Fact 3.10.5. (Equivalent definition of total degrees)
Let $\iota : \mathcal{D}_\alpha \hookrightarrow \mathcal{D}_{\alpha e}$ be the embedding from above. The total α-enumeration degrees $\mathcal{TOT}_{\alpha e}$ are the image of ι, i.e. $\mathcal{TOT}_{\alpha e} := \iota[\mathcal{D}_\alpha]$.

3.10.2 Unboundedness of $\mathcal{D}_{\alpha e}$

Proposition 3.10.6. (Set not α-enumerable from a given set[104])
Given a set $A \subseteq \alpha$, there is a set $C \subseteq \alpha$ s.t. C is not α-enumerable from A, i.e. $C \not\leq_{\alpha e} A$.

Proof. We must satisfy $\forall e < \alpha . C \neq \Phi_e(A)$. So define $C := \{e < \alpha : e \notin \Phi_e(A)\}$. Then $C \not\leq_{\alpha e} A$ as required. □

Corollary 3.10.7. (Unboundedness of α-enumeration degrees[105])
For every set $A \subseteq \alpha$, there is a set $B \subseteq \alpha$ s.t. $A <_{\alpha e} B$.

Proof. Let $C \subseteq \alpha$ s.t. $C \not\leq_{\alpha e} A$ using proposition 3.10.6. Then $A <_{\alpha e} B = A \oplus C$ as required. □

3.10.3 Properties of $\mathcal{D}_\alpha$ and $\mathcal{D}_{\alpha e}$

Proposition 3.10.8. (Properties of the degrees $\mathcal{D}_\alpha$ and $\mathcal{D}_{\alpha e}$[106])

- transitivity: $\forall a, b, c[a \leq b \wedge b \leq c \implies a \leq c]$,
- unboundedness: $\forall a \exists b . a < b$,
- cardinality of a degree: $\forall a . \# a = \# \alpha$.

[102] Theorem 2 in [6].
[103] Follows from Theorem 2 in [6]
[104] Introduced in this thesis.
[105] Introduced in this thesis.
[106] Introduced in this thesis for $\mathcal{D}_{\alpha e}$. Usually assumed implicitly for $\mathcal{D}_\alpha$.

- cardinality of the structure: $\#\mathcal{D}_\alpha = \#\mathcal{D}_{\alpha e} = \#\mathcal{P}(\alpha)$.

Proof. The transitivity is induced by the transitivity of $\leq_{\alpha e}$ and $\leq_\alpha$ (fact 3.7.17). The unboundedness of $\mathcal{D}_{\alpha e}$ follows from corollary 3.10.7 which also implies the unboundedness of $\mathcal{D}_\alpha$. Any two sets in the same degree are reducible to each other by some reduction procedure with an index from α and hence the cardinality of a degree is the cardinality of α and there have to be $\#\mathcal{P}(\alpha)$ many degrees. □

As an α-enumeration operator itself is an α-c.e. set, we have directly the following fact.

Fact 3.10.9. The least degree 0 of $\mathcal{D}_{\alpha e}$ is the set of $\Sigma_1(L_\alpha)$ subsets of α.

3.10.4 α-join operator

We introduce an α-join operator which is used to prove a degree theoretic statement corollary 3.10.12.

Definition 3.10.10. (α-join and α-join operator[107])
Let $A_\gamma \subseteq \alpha$ denote a set with an index $\gamma < \alpha$. For an index set $I \subseteq \alpha$ define the α-join of the set $\{A_\gamma\}_{\gamma \in I}$ to be

$$\bigoplus_{\gamma \in I} A_\gamma = \{i(\gamma, x) : \gamma \in I \wedge x \in A_\gamma\}$$

where $i : \alpha \times \alpha \to \alpha$ is an α-computable bijection. Call $\bigoplus_{\gamma \in I}$ an α-join operator.

Clearly, for $\forall \delta \in I[A_\delta \leq_{\alpha e} \bigoplus_{\gamma \in I} A_\gamma]$.

Proposition 3.10.11. [108] Let $\deg_{\alpha e}(A_0), \deg_{\alpha e}(A_1), \deg_{\alpha e}(A_2), \ldots$ be a β-sequence of strictly increasing α-enumeration degrees and $\beta \leq \alpha$. Then there is an α-enumeration degree $\deg_{\alpha e}(A)$ which is greater than any other degree in the sequence.

Proof. Let A be an α-join of all the degrees in the β-sequence. Then A is strictly greater than any other A_γ in the sequence as required since the sequence is strictly increasing. □

Corollary 3.10.12. [109] Any subset of α-enumeration degrees whose cardinality is at most $\#\alpha$ has an upper bound in $\mathcal{D}_{\alpha e}$. □

[107] Introduced in this thesis.
[108] Introduced in this thesis.
[109] Introduced in this thesis.

3.11 Computability with infinite cardinal and assumption $V = L$

Many notions in α-Computability Theory trivialize and simplify if we assume the axiom of constructibility $V = L$ and α is taken to be an infinite cardinal, even more when α is an infinite regular cardinal.

Throughout this section assume that $V = L$.

Fact 3.11.1. [110] Let α be an infinite cardinal. Let $A \subseteq \alpha$. Then:

i) A is regular,

ii) $A \in L_\alpha \iff \exists \beta < \alpha . A \subseteq \beta$.

3.11.1 Computability with infinite regular cardinal

Proposition 3.11.2. (α-finiteness of subsets of smaller cardinality[111])
Let α be an infinite regular cardinal. Let $A \subseteq \alpha$ and $\#A < \alpha$. Then $A \in L_\alpha$.

Proof. As $\#A < \alpha$, so A cannot be cofinal in α. Thus it has to be bounded by some $\beta < \alpha$. Hence $A \in L_\alpha$ by fact 3.11.1. □

Proposition 3.11.3. (Superadmissibility of infinite regular cardinal[112])
Let α be an infinite regular cardinal. Then L_α satisfies the full replacement axiom:

$$\forall f : \alpha \to \alpha \forall K \in L_\alpha . f[K] \in L_\alpha .$$

Proof. As α is an infinite regular cardinal, so if $K \in L_\alpha$, then $\#K < \alpha$. Also $\#f[K] \leq \#K$ by f being a single-valued function. Hence $f[K] \in L_\alpha$ by proposition 3.11.2. □

Corollary 3.11.4. [113] If α is an infinite regular cardinal, then every subset of α is megaregular.

Proof. Follows from proposition 3.11.3. □

Corollary 3.11.5. [114] If α is an infinite regular cardinal, then $\leq_{\alpha e} = \leq_{w\alpha e}$.

[110] Follows from [4]p5 part (d).
[111] Usually assumed implicitly in α-Computability Theory. Parts present in standard Set Theory texts, see [21].
[112] Usually assumed implicitly in α-Computability Theory. Parts present in standard Set Theory texts, see [21].
[113] Introduced in this thesis. A direct consequence of well-established facts.
[114] Introduced in this thesis. A direct consequence of well-established facts.

Proof. By proposition 3.9.2 if $A \leq_{\alpha e} B$, then $A \leq_{w\alpha e} B$. Also by proposition 3.9.2 if $A \leq_{w\alpha e} B$ and B is megaregular, then $A \leq_{\alpha e} B$. As α is an infinite regular cardinal, so B is megaregular by corollary 3.11.4. Therefore $\forall A, B \subseteq \alpha[A \leq_{w\alpha e} B \iff A \leq_{\alpha e} B]$ and so $\leq_{\alpha e}=\leq_{w\alpha e}$ as required. □

Remark 3.11.6. Fact 3.11.1 and corollary 3.11.4 imply that every subset of an infinite regular cardinal is regular and megaregular. This reveals that the computability on an infinite regular cardinal κ behaves similar to classical Computability Theory. A further comparison can be summarized as follows:

	classical CT	α-CT	κ-CT
computation domain	$\mathbb{N}$	L_α	L_κ
computable	$\Delta_1(\mathbb{N})$	$\Delta_1(L_\alpha)$	$\Delta_1(L_\kappa)$
c.e.	$\Sigma_1(\mathbb{N})$	$\Sigma_1(L_\alpha)$	$\Sigma_1(L_\kappa)$
finite	bounded	bounded and Δ_1	bounded
replacement strength	full replacement	Σ_1-replacement	full replacement

Proposition 3.11.7. [115] Assume that α is an infinite regular cardinal. Let $A := \bigcup_{\gamma \in I} K_\gamma$ where $\#I < \alpha$ and $I \subseteq \alpha$. Then A is α-finite.

Proof. If α is an infinite regular cardinal, then A is not cofinal in α as it is a union of $\#I$ subsets of α for $s < \alpha$ and each subset has a cardinality less than α. Hence $\#A < \alpha$ and so A is bounded in $\beta < \alpha$. Consequently $A = A \cap \beta \in L_\alpha$ by fact 3.11.1.

Alternatively, $\#I < \alpha$ and so $I \in L_\alpha$ by fact 3.11.1. By then A is an α-finite union of α-finite sets. Hence A is α-finite by proposition 3.2.10 as required. □

3.12 α-enumeration jump

We define a weak α-enumeration jump and an α-enumeration jump, then prove that they are equivalent under the weak α-enumeration reducibility. Thus if megaregular both jumps are in the same α-enumeration degree. We investigate several properties of an α-enumeration jump including totality, monotonicity and Σ_n-completeness of the n^{th} α-enumeration jump. The properties are established under assumptions such as Σ_n-replacement axiom or megaregularity. At the end we prove that any megaregular jump can be used equivalently as an oracle in constructions and priority arguments.

This section generalizes a well-known notion of the enumeration jump from classical Computability Theory unseen in α-Computability Theory before.

[115] Introduced in this thesis.

3.12.1 Jump definitions

We consider $K(A) := \{x < \alpha : x \in \Phi_x(A)\}$ as a weak α-jump of A and define its α-enumeration counterpart.

Definition 3.12.1. (Weak α-enumeration jump)
A *weak α-enumeration jump* of a set $A \subseteq \alpha$ is the set $J_{w\alpha e}(A)$ defined as follows:

- $K(A) := \{x < \alpha : x \in \Phi_x(A)\}$,
- $J_{w\alpha e}(A) := K(A) \oplus \overline{K(A)}$.

Definition 3.12.2. (α-enumeration jump)[116]
An *α-enumeration jump* of a set $A \subseteq \alpha$ is the set $J_{\alpha e}(A)$ defined as follows:

- $H^+(A) := \{\langle \gamma, \delta \rangle : K_\gamma \subseteq \Phi_\delta(A)\}$,
- $H^-(A) := \{\langle \gamma, \delta \rangle : K_\gamma \subseteq \overline{\Phi_\delta(A)}\}$,
- $J_{\alpha e}(A) := H^+(A) \oplus H^-(A)$.

The n^{th} α-enumeration jump of the set A is defined inductively as follows:

- $J^{(0)}_{\alpha e}(A) := A$
- $J^{(n+1)}_{\alpha e}(A) := J_{\alpha e}(J^{(n)}_{\alpha e}(A))$

Definition 3.12.3. (Halting set)
The halting set of $A \subseteq \alpha$ is defined as $H(A) := \{\langle x, y \rangle : x \in \Phi_y(A)\}$.

3.12.2 Equivalence of jump definitions

We investigate the relations and reductions between different jump definitions.

Lemma 3.12.4. $\exists e \forall x \forall y [x \in \Phi_y \iff \langle x, y \rangle \in \Phi_e]$.

Proof. The index e is defined by

$$\Phi_e := \{z : \exists x, y[z = \langle x, y \rangle \wedge x \in \Phi_y]\} \in \Sigma_1(L_\alpha)$$

as required. □

Lemma 3.12.5. There is an α-computable function $g : \alpha \times \alpha \to \alpha$ s.t.

$$\forall x, y[x \in \Phi_y \iff g(x, y) \in \Phi_{g(x,y)}].$$

[116] Generalized from [25].

Proof. The function g takes parameters x and y and produces the index $z = g(x, y)$ of an α-c.e. set Φ_z which behaves as follows. The enumeration of the α-c.e. set Φ_y is simulated. If x is enumerated by Φ_y, then Φ_z enumerates all of α, enumerating the least unenumerated element first, so $\Phi_z = \alpha$. If x is never enumerated by Φ_y, then Φ_z does not enumerate any elements and so $\Phi_z = \emptyset$. Note that g is α-computable and

$$x \in \Phi_y \iff \alpha = \Phi_z \iff g(x, y) \in \Phi_{g(x,y)}$$

as required. □

Proposition 3.12.6. (Reductions between jumps and halting sets)

1. $A \leq_{\alpha m} K(A) \leq_{\alpha m} H(A) \leq_{\alpha m} H^+(A)$,
2. $H^+(A) \leq_{w\alpha e} H(A) \leq_{\alpha m} K(A) \leq_{w\alpha e} A$ and $H(A) \leq_{w\alpha e} A$,
3. $\overline{A} \leq_{\alpha m} \overline{K(A)} \leq_{\alpha m} \overline{H(A)} \leq_{\alpha m} H^-(A)$,
4. $H^-(A) \leq_{w\alpha e} \overline{H(A)} \leq_{\alpha m} \overline{K(A)}$.

Proof. Note that A is α-many-one reducible to $K(A)$ as

$$a \in A \iff f(a) \in K(A)$$

where f is an α-computable function defined by

$$\Phi_{f(a)} := \{\langle y, \gamma\rangle : K_\gamma = \{a\} \wedge y \in \alpha\},$$

Thus $A \leq_{\alpha m} K(A)$. Next $K(A) \leq_{\alpha m} H(A)$ via $x \mapsto \langle x, x\rangle$. Finally, $H(A) \leq_{\alpha m} H^+(A)$ via

$$g = \{(\langle x, y\rangle, \langle \gamma, y\rangle) : K_\gamma = \{x\}\}.$$

Therefore $A \leq_{\alpha m} K(A) \leq_{\alpha m} H(A) \leq_{\alpha m} H^+(A)$ and the statement 1 holds.

To prove $H(A) \leq_{\alpha m} K(A)$, given $x, y < \alpha$, define

$$\Phi_z := \{\langle q, \delta\rangle : \langle x, \delta\rangle \in \Phi_y \wedge \gamma \in \alpha \wedge q \in \alpha\}.$$

Note that the index z is uniformly α-computable from the indices x, y and $\langle x, \delta\rangle \in \Phi_y \iff \langle z, \delta\rangle \in \Phi_z$, so $x \in \Phi_y(A) \iff z \in \Phi_z(A)$. Thus $\langle x, y\rangle \in H(A) \iff z(x, y) \in K(A)$. Hence $H(A) \leq_{\alpha m} K(A)$ as required.

To prove $H^+(A) \leq_{w\alpha e} H(A)$ notice that

$$\langle \gamma, \delta\rangle \in H^+(A) \iff \forall x \in K_\gamma[x \in \Phi_\delta(A)] \iff \forall x \in K_\gamma[\langle x, \delta\rangle \in H(A)]$$

and so $H^+(A) \leq_{w\alpha e} H(A)$ via

$$\Phi := \{\langle\langle \gamma, \delta\rangle, \epsilon\rangle \in \alpha : \forall x \in K_\gamma . \langle x, \delta\rangle \in K_\epsilon\} \in \Sigma_1(L_\alpha).$$

To prove $K(A) \leq_{w\alpha e} A$, let e be the index s.t. $\langle x, \delta\rangle \in \Phi_x \iff \langle x, \delta\rangle \in \Phi_e$ for any pair $\langle x, \delta\rangle$ using lemma 3.12.4. Then

$$x \in K(A) \iff x \in \Phi_x(A) \iff x \in \Phi_e(A).$$

Thus $K(A) = \Phi_e(A)$ and so $K(A) \leq_{w\alpha e} A$ via Φ_e.

Similarly, to prove $H(A) \leq_{w\alpha e} A$, let $\hat{e}$ be the index s.t.

$$\langle x, \delta \rangle \in \Phi_y \iff \langle \langle x, \delta \rangle, y \rangle \in \Phi_{\hat{e}}$$

for any $x, y, \delta < \alpha$ using lemma 3.12.4. Then

$$\langle x, y \rangle \in H(A) \iff x \in \Phi_y(A) \iff \langle x, y \rangle \in \Phi_{\hat{e}}(A).$$

Thus $H(A) = \Phi_{\hat{e}}(A)$ and so $H(A) \leq_{w\alpha e} A$ via $\Phi_{\hat{e}}$.

Therefore $H^+(A) \leq_{w\alpha e} H(A) \leq_{\alpha m} K(A) \leq_{w\alpha e} A$ and $H(A) \leq_{w\alpha e} A$, so the statement 2 holds.

As $A \leq_{\alpha m} K(A)$, so $\overline{A} \leq_{\alpha m} \overline{K(A)}$. As $K(A) \leq_{\alpha m} H(A)$, so $\overline{K(A)} \leq_{\alpha m} \overline{H(A)}$. Also $\overline{H(A)} \leq_{\alpha m} H^-(A)$ via g. Therefore $\overline{A} \leq_{\alpha m} \overline{K(A)} \leq_{\alpha m} \overline{H(A)} \leq_{\alpha m} H^-(A)$ and the statement 3 holds.

By symmetry we have $H^-(A) \leq_{w\alpha e} \overline{H(A)} \leq_{\alpha m} \overline{K(A)}$ and the statement 4 holds as required. □

Proposition 3.12.7. (Equivalence of the α-enumeration jump definitions) For any admissible ordinal α and $A \subseteq \alpha$ we have:

1. $J_{w\alpha e}(A) \leq_{\alpha m} H(A) \oplus \overline{H(A)} \leq_{\alpha m} J_{\alpha e}(A)$,
2. $J_{w\alpha e}(A) \equiv_{\alpha m} H(A) \oplus \overline{H(A)} \equiv_{w\alpha e} J_{\alpha e}(A)$ and $J_{w\alpha e}(A) \equiv_{w\alpha e} J_{\alpha e}(A)$,
3. $J_{w\alpha e}(A) \equiv_{\alpha e} H(A) \oplus \overline{H(A)} \equiv_{\alpha e} J_{\alpha e}(A)$ if $J_{w\alpha e}(A)$ is megaregular.

Proof. By proposition 3.12.6 we have $K(A) \leq_{\alpha m} H(A) \leq_{\alpha m} H^+(A)$ and $\overline{K(A)} \leq_{\alpha m} \overline{H(A)} \leq_{\alpha m} H^-(A)$. So $J_{w\alpha e}(A) \leq_{\alpha m} H(A) \oplus \overline{H(A)} \leq_{\alpha m} J_{\alpha e}(A)$ and the statement 1 holds.

By proposition 3.12.6 we have $H^+_{\alpha e}(A) \leq_{w\alpha e} H(A) \leq_{\alpha m} K(A)$ and $H^-_{\alpha e}(A) \leq_{w\alpha e} \overline{H(A)} \leq_{\alpha m} \overline{K(A)}$, so $J_{w\alpha e}(A) \geq_{\alpha m} H(A) \oplus \overline{H(A)} \geq_{w\alpha e} J_{\alpha e}(A)$ and hence $J_{w\alpha e}(A) \equiv_{\alpha m} H(A) \oplus \overline{H(A)} \equiv_{w\alpha e} J_{\alpha e}(A)$ and $J_{w\alpha e}(A) \equiv_{w\alpha e} J_{\alpha e}(A)$ using the statement 1.

The statement 3 follows from the statement 2 and the megaregularity of $J_{w\alpha e}(A)$. □

Therefore if $J_{w\alpha e}(A)$ is megaregular, we can use any of the definitions of an α-enumeration jump as convenient.

3.12.3 Totality

Clearly, by definition the degree of $J_{w\alpha e}(A)$ is a total degree. We investigate the totality of the α-enumeration jump $J_{\alpha e}(A)$.

Proposition 3.12.8. $J_{\alpha e}(A) \oplus \overline{J_{\alpha e}(A)} \leq_{w\alpha e} J_{\alpha e}(A)$.

Proof. Let $\triangle \in \{-,+\}$ and $\nabla \in \{-,+\} - \{\triangle\}$. Then $\overline{H^{\triangle}(A)} \leq_{w\alpha e} H^{\nabla}(A)$ via

$$\Phi = \{\langle\langle\gamma,\delta\rangle,\epsilon\rangle : \exists\beta < \alpha[K_\beta \neq \emptyset \wedge K_\beta \subseteq K_\gamma \wedge \{\langle\beta,\delta\rangle\} = K_\epsilon]\}$$

since

$$\langle\gamma,\delta\rangle \in \overline{H^{\triangle}(A)} \iff \exists\beta < \alpha[K_\beta \neq \emptyset \wedge K_\beta \subseteq K_\gamma \wedge \langle\beta,\delta\rangle \in H^{\nabla}(A)].$$

Thus

$$\overline{J_{\alpha e}(A)} = \overline{H^+(A) \oplus H^-(A)} = \overline{H^+(A)} \oplus \overline{H^-(A)} \leq_{w\alpha e} H^-(A) \oplus H^+(A) = J_{\alpha e}(A).$$

Therefore $J_{\alpha e}(A) \oplus \overline{J_{\alpha e}(A)} \leq_{w\alpha e} J_{\alpha e}(A)$ as required. □

The following definition will be useful in proposition 3.12.10 below.

Definition 3.12.9. (Pointclass union)
We define the *pointclass union* of the pointclasses θ_A and θ_B to be the pointclass θ by defining $\theta(D,C) \subseteq \mathcal{P}(D)$ from $\theta_A(D,C) \subseteq \mathcal{P}(D)$ and $\theta_B(D,C) \subseteq \mathcal{P}(D)$ for an arbitrary set of parameters D and a parameter $C \subseteq D$ as follows:

$$\theta_A \cup \theta_B(D,C) := \{E \subseteq D : \exists A, B \subseteq D$$
$$[E \in \Delta_0(D,C,B,A) \wedge A \in \theta_A(D,C) \wedge B \in \theta_B(D,C)]\}.$$

Proposition 3.12.10. (Definability class of the n^{th} α-enumeration jump)
Let $n \in \mathbb{N}$. Assume $\langle L_\alpha, A\rangle \models \Delta_n$-collection. Then:

1. $J^{(n)}_{\alpha e}(A) \in \Sigma_n \cup \Pi_n(L_\alpha, A)$
2. $J^{(n)}_{\alpha e}(A) \in \Delta_{n+1}(L_\alpha, A)$

Proof. The second statement follows from the first. We prove the first statement by the induction.

BC: If $n = 0$, then trivially $J^{(0)}_{\alpha e}(A) = A \in \mathrm{QF}(L_\alpha, A) \subset \Sigma_0 \cup \Pi_0(L_\alpha, A)$.

IC: Assume IH that $J^{(n)}_{\alpha e}(A) \in \Delta_{n+1}(L_\alpha, A)$. As $J^{(n)}_{\alpha e}(A) \in \Delta_{n+1}(L_\alpha, A)$ and $\langle L_\alpha, A\rangle \models \Delta_{n+1}$-collection, so $\langle L_\alpha, J^{(n)}_{\alpha e}(A)\rangle \models \Delta_1$-collection by corollary 3.4.11. By definition

$$J^{(n+1)}_{\alpha e}(A) = J_{\alpha e}(J^{(n)}_{\alpha e}(A)) = H^+(J^{(n)}_{\alpha e}(A)) \oplus H^-(J^{(n)}_{\alpha e}(A)).$$

Note that $H^+(J^{(n)}_{\alpha e}(A)) \in \Sigma_1(L_\alpha, J^{(n)}_{\alpha e}(A))$ and $H^-(J^{(n)}_{\alpha e}(A)) \in \Pi_1(L_\alpha, J^{(n)}_{\alpha e}(A))$ by proposition 3.4.9 using $\langle L_\alpha, J^{(n)}_{\alpha e}(A)\rangle \models \Delta_1$-collection. IH and $H^+(J^{(n)}_{\alpha e}(A)) \in \Sigma_1(L_\alpha, J^{(n)}_{\alpha e}(A))$ imply $H^+(J^{(n)}_{\alpha e}(A)) \in \Sigma_{n+1}(L_\alpha, A)$ by $\langle L_\alpha, A\rangle \models \Delta_{n+1}$-collection and proposition 3.4.10ii. $H^-(J^{(n)}_{\alpha e}(A)) \in \Pi_1(L_\alpha, J^{(n)}_{\alpha e}(A))$ and IH imply $H^-(J^{(n)}_{\alpha e}(A)) \in \Pi_{n+1}(L_\alpha, A)$ by $\langle L_\alpha, A\rangle \models \Delta_{n+1}$-collection and proposition 3.4.10iv. Hence $J^{(n+1)}_{\alpha e}(A) \in \Sigma_{n+1}(L_\alpha, A) \cup \Pi_{n+1}(L_\alpha, A)$ as required. □

Corollary 3.12.11. Let $n \in \mathbb{N}$. Assume $L_\alpha \models \Sigma_{n+1}$-replacement. Then $J^{(n)}_{\alpha e}(\emptyset)$ is megaregular.

Proof. As $L_\alpha \models \Sigma_{n+1}$-replacement, so $L_\alpha \models \Sigma_{n+1}$-collection by proposition 3.4.19. Thus $L_\alpha \models \Delta_n$-collection. Hence $J^{(n)}_{\alpha e}(\emptyset) \in \Delta_{n+1}(L_\alpha)$ by proposition 3.12.10. As $L_\alpha \models \Sigma_{n+1}$-replacement and $J^{(n)}_{\alpha e}(\emptyset) \in \Delta_{n+1}(L_\alpha)$, so $J^{(n)}_{\alpha e}(\emptyset)$ is megaregular by proposition 3.8.9. □

Corollary 3.12.12. Let $n \in \mathbb{N}$. Assume $L_\alpha \models \Sigma_{n+1}$-replacement. Then $J^{(n)}_{\alpha e}(\emptyset)$ is a total set.

Proof. This follows from corollary 3.12.11 and proposition 3.12.8. □

3.12.4 Monotonicity

Proposition 3.12.13. (Jump monotonicity)
Assume $A \leq_{\alpha e} B$. Then:

i) $H(A) \leq_{w\alpha e} H(B)$,

ii) $\overline{H(A)} \leq_{\alpha m} \overline{H(B)}$,

iii) $H(A) \oplus \overline{H(A)} \leq_{w\alpha e} H(B) \oplus \overline{H(B)}$,

iv) $J_{\alpha e}(B)$ is megaregular, then $J_{\alpha e}(A) \leq_{\alpha e} J_{\alpha e}(B)$.

Proof. The statement i: We have $H(A) \leq_{w\alpha e} A \leq_{\alpha e} B \leq_{\alpha e} H(B)$ using proposition 3.12.6. Thus $H(A) \leq_{w\alpha e} H(B)$.

The statement ii: As $K(A) \leq_{w\alpha e} A$ by proposition 3.12.6 and $A \leq_{\alpha e} B$, so $K(A) \leq_{w\alpha e} B$. As $K(A) \leq_{w\alpha e} B$, so $\exists y \forall x[x \in K(A) \iff \langle x, y\rangle \in H(B)]$. Hence $K(A) \leq_{\alpha m} H(B)$. So $\overline{K(A)} \leq_{\alpha m} \overline{H(B)}$. Thus $\overline{H(A)} \leq_{\alpha m} \overline{K(A)} \leq_{\alpha m} \overline{H(B)}$ using proposition 3.12.6. Hence $\overline{H(A)} \leq_{\alpha m} \overline{H(B)}$.

The statement iii follows from the statements i and ii.

The statement iv: If $J_{\alpha e}(B)$ is megaregular, then $J_{\alpha e}(B) \equiv_{\alpha e} H(B) \oplus \overline{H(B)}$ by proposition 3.12.7. By the statement iii and the megaregularity closure (proposition 3.8.13) we have

$$J_{\alpha e}(A) \equiv_{\alpha e} H(A) \oplus \overline{H(A)} \leq_{\alpha e} H(B) \oplus \overline{H(B)} \equiv_{\alpha e} J_{\alpha e}(B)$$

and so $J_{\alpha e}(A) \leq_{\alpha e} J_{\alpha e}(B)$ as required. □

3.12.5 Σ_n-completeness

We investigate the relativized Σ_1 completeness and non-relativized Σ_n-completeness of the α-enumeration jump.

Proposition 3.12.14. (Σ_1-completeness of a jump)
Let $A, W \subseteq \alpha$ be arbitrary. Then:

1. $W \leq_{w\alpha e} A \implies W \leq_\alpha J_{\alpha e}(A)$.
2. $\overline{W} \leq_{w\alpha e} A \implies W \leq_\alpha J_{\alpha e}(A)$.
3. A regular $\wedge W \in \Sigma_1(L_\alpha, A^+) \implies W \leq_\alpha J_{\alpha e}(A)$.
4. A regular $\wedge \overline{W} \in \Sigma_1(L_\alpha, A^+) \implies W \leq_\alpha J_{\alpha e}(A)$.

Proof. To prove the first statement, note $\Phi_\delta(A) = W$ for some $\delta < \alpha$ as $W \leq_{w\alpha e} A$. Thus

$$K_\gamma \subseteq W \iff K_\gamma \subseteq \Phi_\delta(A) \iff \langle \gamma, \delta \rangle \in H^+(A)$$

and so $W \leq_{\alpha e} H^+(A)$. Also

$$K_\gamma \subseteq \overline{W} \iff K_\gamma \subseteq \overline{\Phi_\delta(A)} \iff \langle \gamma, \delta \rangle \in H^-(A)$$

and so $\overline{W} \leq_{\alpha e} H^-(A)$. Hence

$$W \oplus \overline{W} \leq_{\alpha e} H^+(A) \oplus H^-(A) = J_{\alpha e}(A).$$

Therefore $W \leq_\alpha J_{\alpha e}(A)$ as required.

To imply the second statement, use the first statement as follows:

$$\overline{W} \leq_{w\alpha e} A \implies \overline{W} \leq_\alpha J_{\alpha e}(A) \implies W \leq_\alpha J_{\alpha e}(A).$$

The third and the fourth statement follow from the first the first and the second statements respectively and the regularity of A. □

Lemma 3.12.15. $\forall n \in \mathbb{N}[A \in \Sigma_{n+1}(L_\alpha) \implies \exists B \in \Pi_n(L_\alpha).A \in \Sigma_1(L_\alpha, B^+)]$.

Proof. By the definition of $\Sigma_{n+1}(L_\alpha)$ class we have that $x \in A \iff \exists y.(x, y) \in B$ for some $B \in \Pi_n(L_\alpha)$. Clearly, $A \in \Sigma_1(L_\alpha, B^+)$ as required. □

Proposition 3.12.16. (Σ_n-jump completeness)
Let $n \in \mathbb{N}$ and $A \subseteq \alpha$. Assume $L_\alpha \models \Sigma_{n+1}$-replacement. Then

$$A \in \Sigma_n(L_\alpha) \implies A \leq_\alpha J^{(n)}_{\alpha e}(\emptyset).$$

Proof. We prove the statement by induction. Note that $J^{(0)}_{\alpha e}(B) := B$. So the base case when $n = 0$ holds since $A \in \Sigma_0(L_\alpha) \implies A \leq_\alpha \emptyset$ trivially.

For the inductive case assume IH that $A \in \Sigma_n(L_\alpha) \implies A \leq_\alpha J^{(n)}_{\alpha e}(\emptyset)$ for any $A \subseteq \alpha$. We prove $\forall A \subseteq \alpha[A \in \Sigma_{n+1}(L_\alpha) \implies A \leq_\alpha J^{(n+1)}_{\alpha e}(\emptyset)]$. Let $A \in \Sigma_{n+1}(L_\alpha)$ be arbitrary. Then $\exists B \in \Pi_n(L_\alpha).A \in \Sigma_1(L_\alpha, B^+)$ by lemma 3.12.15. As $L_\alpha \models \Sigma_{n+1}$-replacement and $B \in \Delta_{n+1}(L_\alpha)$, so B is megaregular by proposition 3.8.9 and thus regular. As $A \in \Sigma_1(L_\alpha, B^+)$ and B is regular, so $A \leq_\alpha J_{\alpha e}(B)$ by proposition 3.12.14. As $\overline{B} \in \Sigma_n(L_\alpha)$, so $\overline{B} \leq_\alpha J^{(n)}_{\alpha e}(\emptyset)$ by IH. Thus $B \leq_\alpha J^{(n)}_{\alpha e}(\emptyset)$.

Note that $J^{(n)}_{\alpha e}(\emptyset)$ and $J^{(n+1)}_{\alpha e}(\emptyset)$ are megaregular by corollary 3.12.11 and $L_\alpha \models \Sigma_{n+2}$-replacement. As $J^{(n)}_{\alpha e}(\emptyset)$ is megaregular, so it is total by proposition 3.12.8. Using $B \leq_\alpha J^{(n)}_{\alpha e}(\emptyset)$ and the totality of $J^{(n)}_{\alpha e}(\emptyset)$, we have $B \oplus \overline{B} \leq_{\alpha e}$

$J_{\alpha e}^{(n)}(\emptyset)$. Hence $B \leq_{\alpha e} J_{\alpha e}^{(n)}(\emptyset)$. As $B \leq_{\alpha e} J_{\alpha e}^{(n)}(\emptyset)$ and $J_{\alpha e}^{(n+1)}(\emptyset) = J_{\alpha e}(J_{\alpha e}^{(n)}(\emptyset))$ is megaregular, so $J_{\alpha e}(B) \leq_{\alpha e} J_{\alpha e}(J_{\alpha e}^{(n)}(\emptyset)) = J_{\alpha e}^{(n+1)}(\emptyset)$ by proposition 3.12.13.

Since $J_{\alpha e}(B) \oplus \overline{J_{\alpha e}(B)} \leq_{w\alpha e} J_{\alpha e}(B)$ by proposition 3.12.8 and $J_{\alpha e}(B) \leq_{\alpha e} J_{\alpha e}^{(n+1)}(\emptyset)$, so $J_{\alpha e}(B) \oplus \overline{J_{\alpha e}(B)} \leq_{w\alpha e} J_{\alpha e}^{(n+1)}(\emptyset)$. As $J_{\alpha e}^{(n+1)}(\emptyset)$ is megaregular, so $J_{\alpha e}(B) \oplus \overline{J_{\alpha e}(B)} \leq_{\alpha e} J_{\alpha e}^{(n+1)}(\emptyset)$. As $J_{\alpha e}(B) \oplus \overline{J_{\alpha e}(B)}$ is total, so $J_{\alpha e}(B) \leq_{\alpha} J_{\alpha e}^{(n+1)}(\emptyset)$. Since $A \leq_{\alpha} J_{\alpha e}(B)$ and $J_{\alpha e}(B) \leq_{\alpha} J_{\alpha e}^{(n+1)}(\emptyset)$, so $A \leq_{\alpha} J_{\alpha e}^{(n+1)}(\emptyset)$.

Therefore

$$\forall A \subseteq \alpha[A \in \Sigma_{n+1}(L_\alpha) \implies A \leq_\alpha J_{\alpha e}^{(n+1)}(\emptyset)]$$

as required. □

3.12.6 Usage in oracle constructions

We may use the following proposition 3.12.17 implicitly in the oracle constructions and definability statements.

Proposition 3.12.17. Assume that $K(A)$ is megaregular, then

$$S \in \Delta_1(L_\alpha, J_{\alpha e}(A)) \implies S \in \Delta_1(L_\alpha, K(A)).$$

Proof. (Of proposition 3.12.17)

1. $J_{\alpha e}(A) \leq_{w\alpha e} H(A) \oplus \overline{H(A)} \leq_{w\alpha e} J_{w\alpha e}(A) = K(A) \oplus \overline{K(A)}$ by proposition 3.12.7.
2. $J_{\alpha e}(A) \oplus \overline{J_{\alpha e}(A)} \leq_{w\alpha e} J_{\alpha e}(A)$ by proposition 3.12.8.
3. $K(A)$ megaregular by assumption.
4. $J_{\alpha e}(A) \leq_{\alpha e} K(A) \oplus \overline{K(A)}$ by 1, 3.
5. $J_{\alpha e}(A) \oplus \overline{J_{\alpha e}(A)} \leq_{w\alpha e} K(A) \oplus \overline{K(A)}$ by 2, 4.
6. $J_{\alpha e}(A) \oplus \overline{J_{\alpha e}(A)} \leq_{\alpha e} K(A) \oplus \overline{K(A)}$ by 3, 5.
7. $J_{\alpha e}(A) \leq_{\alpha} K(A)$ by 6.
8. $J_{\alpha e}(A)$ megaregular by 3, 7.
9. $S \in \Delta_1(L_\alpha, J_{\alpha e}(A))$ by assumption.
10. $S \oplus \overline{S} \leq_{\alpha e} J_{\alpha e}(A) \oplus \overline{J_{\alpha e}(A)}$ by 8, 9.
11. $S \oplus \overline{S} \leq_{\alpha e} K(A) \oplus \overline{K(A)}$ by 10, 6.
12. $S \in \Delta_1(L_\alpha, K(A))$ by 11. □

3.13 Simple construction

We provide a simple construction that demonstrates the use of a projectum and an assumption on megaregularity in α-Computability Theory.

Proposition 3.13.1. [117] For a set $A \subseteq \alpha$ let $A \not\equiv_{w\alpha e} \emptyset$. Assume that the projectum of α is $\alpha^* = \omega$ or that $K(\emptyset) \oplus A$ is megaregular where $K(C) := \{x < \alpha : x \in \Phi_x(C)\}$ represents a weak α-jump[118] of C. Then there is $B \in \Sigma_1(L_\alpha)$ s.t. $A \not\leq_{w\alpha e} B$.

One could simply take B to be $\emptyset$ to prove the statement above. Instead we make a generic proof by constructing B from A given $A \not\equiv_{w\alpha e} \emptyset$. Later statements extend this proof by adding additional requirements, so it is important to present a simple construction first for better comprehension of later results.

Proof. (Of proposition 3.13.1)

Construction

To imply $A \not\leq_{w\alpha e} B$, we would like to construct B in such way so that for each $\gamma < \alpha$ we satisfy the requirement:

$$R_\gamma : A \neq \Phi_\gamma(B).$$

We construct B in α^* stages. At the stage $\gamma < \alpha^*$, given B_γ, we construct $B_{\gamma+1}$ from B_γ. In the end $B = \bigcup_{\gamma<\alpha^*} B_\gamma \subseteq \alpha$. To satisfy R_γ for every $\gamma < \alpha$ in only α^* many stages, we index the requirements and α-computably enumerable sets with the indices from the projectum α^* by using the partial α-computable surjection $p_1 : \alpha^* \rightharpoonup \alpha$ from proposition 3.6.2.

During the construction we prove by induction that at every stage $\gamma < \alpha^*$, the set B_γ is α-finite by proving $B_{\gamma+1} \in L_\alpha$ at the stage $\gamma < \alpha^*$ and assuming the following IH at that stage:

$$B_\gamma \in L_\alpha.$$

Using the α-finiteness of B_γ, we also prove that B_γ is uniformly α-computable from the index γ by the α-computable function $f : \alpha^* \to L_\alpha$, $f : \gamma \mapsto B_\gamma$. We use this function $f \in \Sigma_1(L_\alpha)$ to conclude that $B \in \Sigma_1(L_\alpha)$.

Let $B_0 := \emptyset$. Clearly, $B_0 \in L_\alpha$ and so the BC of the induction holds. Define $B_{\gamma+1}$ given B_γ at the stage γ below.

Stage γ: satisfy $A \neq \Phi_\gamma(B)$

Using the oracle $K(\emptyset)$ check if $\gamma \in \mathrm{dom}(p_1)$. If $\gamma \notin \mathrm{dom}(p_1)$, then set $B_{\gamma+1} := B_\gamma$ and proceed to the next stage $\gamma + 1$. Otherwise, proceed as follows.

As $A \not\equiv_{w\alpha e} \emptyset$, so $\Phi_s(\alpha) \neq A$. Hence one of the following must hold:

[117] Introduced in this thesis.

[118] See section 3.12 for more details.

- $\Phi_\gamma(\alpha) \subset A$: As $B \subseteq \alpha$, so $\Phi_\gamma(B) \subset A$ by the monotonicity of an α-enumeration operator Φ. Thus define $B_{\gamma+1} := B_\gamma$. The set $B_{\gamma+1}$ is α-finite as $B_\gamma \in L_\alpha$ by IH. Note that R_γ is satisfied by B as $\Phi_\gamma(B) \subset A$.

- $\exists x \in \Phi_\gamma(\alpha)$ s.t. $x \notin A$: Then take any $x < \alpha$ s.t. $x \in \Phi_\gamma(\alpha)$ and $x \notin A$. We must have some witness $K \subseteq \alpha$ in L_α s.t. $x \in \Phi_\gamma(K)$. Thus define $B_{\gamma+1} := B_\gamma \cup K$. The set $B_{\gamma+1}$ is α-finite as $B_\gamma \in L_\alpha$ by IH and $K \in L_\alpha$. Note that R_γ is satisfied by B as $B_{\gamma+1} \subseteq B$.

Limit construction

If $\alpha^* = \omega$, then this construction is not needed.

Otherwise, let $\delta < \alpha$ be a limit ordinal. Define $B_\delta := \bigcup_{\gamma<\delta} B_\gamma$. We prove B_δ is α-finite using the megaregularity of $K(\emptyset) \oplus A$.

Let $f : \alpha^* \to L_\alpha$ be the function $\gamma \mapsto B_\gamma$ which is defined as follows:

- $f(0) := \emptyset$,
- $f(\gamma+1) := \begin{cases} f(\gamma) & \text{if } \Phi_\gamma(\alpha) \subset A, \\ f(\gamma) \cup K_\delta & \text{if } \exists x[x \in \Phi_\gamma(\alpha) \wedge x \notin A] \text{ where} \\ & \delta := \mu\beta[x \in \Phi_\gamma(K_\beta) \wedge x \notin A]. \end{cases}$
- $f(\delta) = \bigcup_{\gamma<\delta} f(\gamma)$ if $\lim(\delta)$.

Note that $f \in \Sigma_1(L_\alpha, K(\emptyset) \oplus A)$ since f is $\Sigma_1(L_\alpha)$ definable with the oracles A and $K(\emptyset)$ as seen from its definition above.

Trivially $f(0) \in L_\alpha$ and if $f(\gamma) \in L_\alpha$, then $f(\gamma+1) \in L_\alpha$. Assume by the IH that f is total and well-defined on the domain δ where δ is a limit ordinal. Recall the α-computable bijection $b : \alpha \to L_\alpha$. Note that $I = b^{-1} \circ f[\delta] \in L_\alpha$ as $f \in \Sigma_1(L_\alpha, K(\emptyset) \oplus A)$, $\delta \in L_\alpha$ and $K(\emptyset) \oplus A$ is megaregular. Therefore $B_\delta = f(\delta) = \bigcup_{\beta \in I} K_\beta \in L_\alpha$ as required.

Conclusion

Note that B constructed in α^* stages satisfies $\forall \gamma < \alpha^*.A \neq \Phi_\gamma(B)$. So $A \not\leq_{w\alpha e} B$. Furthermore, note that $B \in \Sigma_1(L_\alpha)$ since $f \in \Sigma_1(L_\alpha)$ and

$$B = \{x < \alpha : \exists \gamma < \alpha^*.x \in f(\gamma)\}.$$

Hence $A \not\leq_{w\alpha e} B$ and $B \in \Sigma_1(L_\alpha)$ as required. □

Chapter 4

Kalimullin pair and semicomputability

A Kalimullin pair is an important relation in classical Computability Theory. The enumeration jump was defined in the enumeration degrees using a Kalimullin pair in [25]. Total degrees were defined in the enumeration degrees using a Kalimullin pair in [2].

There is a close connection between Kalimullin pairs and semicomputability. If A is semicomputable, then $\mathcal{K}(A, \overline{A})$, i.e. A and its complement $\overline{A}$ are a Kalimullin pair.

We define an α-Kalimullin pair and show that it is definable in the α-enumeration degrees $\mathcal{D}_{\alpha e}$ if $V = L$ and α is an infinite regular cardinal (corollary 4.3.8). We generalize some needed results on semicomputability by Jockusch [10] to conclude that every nontrivial total degree is a join of a maximal α-Kalimullin pair if $V = L$ and α is an infinite regular cardinal (corollary 4.4.2).

The work of this chapter is used in chapter 6 to prove that if $V = L$ and α is an infinite regular cardinal, then the total α-enumeration degrees $\mathcal{TOT}_{\alpha e}$ are definable in the α-enumeration degrees $\mathcal{D}_{\alpha e}$ (theorem 6.3.7):
an α-enumeration degree is total iff it is trivial or a join of a maximal α-Kalimullin pair.

4.1 Semicomputability

We lift some needed results of Jockusch [10] on semicomputable sets from the level ω to a level α.

4.1.1 Definition and closure

Definition 4.1.1. A set $A \subseteq \alpha$ is α-*semicomputable* iff there exists a total α-computable function $s_A : \alpha \times \alpha \to \alpha$ called a *selector function* satisfying:

i) $\forall x, y \in \alpha . s_A(x, y) \in \{x, y\}$,

ii) $\forall x, y \in \alpha[\{x, y\} \cap A \neq \emptyset \implies s_A(x, y) \in A]$.

Denote by $\mathrm{sc}(L_\alpha)$ the class of α-semicomputable sets.

We just say semicomputable instead of α-semicomputable if clear from the context.

Fact 4.1.2. (Semicomputability closure)

i) $A \in \mathrm{sc}(L_\alpha) \iff \overline{A} \in \mathrm{sc}(L_\alpha)$,

ii) $A \oplus B \in \mathrm{sc}(L_\alpha) \implies A \in \mathrm{sc}(L_\alpha) \wedge B \in \mathrm{sc}(L_\alpha)$.

4.1.2 Binary ordering

Definition 4.1.3. (Binary ordering)
Define $<_b \subseteq \mathcal{P}(\alpha) \times \mathcal{P}(\alpha)$ and $\leq_b \subseteq \mathcal{P}(\alpha) \times \mathcal{P}(\alpha)$ to be numerical orderings on the binary representation of the compared sets:

- $A <_b B :\iff \exists \beta \in \alpha[\beta \notin A \wedge \beta \in B \wedge A \cap \beta = B \cap \beta]$,
- $A \leq_b B :\iff A <_b B \vee A = B$.

Remark 4.1.4. The restrictions of the orderings $<_b$ and $\leq_b$ to α-finite sets are first-order definable and α-computable since an α-finite set is bounded.

Proposition 4.1.5. (Properties of binary ordering)
Let $\lhd \in \{<, \leq\}$, then:

i) $<_b$ is a strict total order,

ii) $\leq_b$ is a total order,

iii) $A \lhd_b B \iff \overline{B} \lhd_b \overline{A}$.

Proof. i) and ii) are trivial. Next we prove iii). Assume $A <_b B$. Then there is $\beta < \alpha$ s.t. $\beta \notin A$, $\beta \in B$, $A \cap \beta = B \cap \beta$. So $\beta \notin \overline{B}$, $\beta \in \overline{A}$, $\overline{A} \cap \beta = \overline{B} \cap \beta$. Thus $\overline{B} <_b \overline{A}$ and by symmetry $A <_b B \iff \overline{B} <_b \overline{A}$. Similarly, $A \leq_b B \iff \overline{B} \leq_b \overline{A}$. Therefore $A \lhd_b B \iff \overline{B} \lhd_b \overline{A}$ and iii) as required. $\square$

It is easy to see the following.

Fact 4.1.6. (Binary and subset ordering)

i) $A \subset B \implies A <_b B$,

ii) $A \subseteq B \implies A \leq_b B$,

iii) $A = B \iff A \equiv_b B$.

Note 4.1.7. If $A \leq_b C$ and $B \leq_b C$, is it true that $A \cup B \leq_b C$?

No. Consider $A = 011\ldots, B = 100\ldots, C = 110\ldots$. Then $A \cup B = 111\ldots$. Thus $A \leq_b C$ and $B \leq_b C$, but $\neg A \cup B \leq_b C$.

4.1.3 Left and right sets

Definition 4.1.8. Given a set A define $l_A := \{x \in \alpha : K_x \leq_b A\}, r_A := \overline{l_A}$.

Remark 4.1.9. If $A \notin L_\alpha$, then:

- $l_A = \{x < \alpha : K_x <_b A\}$ are α-finite sets *left* of A,
- $r_A = \{x < \alpha : A <_b K_x\}$ are α-finite sets *right* of A.

Proposition 4.1.10. (Properties of left/right α-finite sets)
Let $A \subseteq \alpha$ and $\beta, \gamma, \delta < \alpha$. Then:

i) $K \in L_\alpha \wedge K_\delta = \bigcup_{\gamma \in K} K_\gamma \wedge \delta \in l_A \implies K \subseteq l_A$,

ii) $\beta \in l_A \wedge \gamma \in r_A \wedge K_\beta \cap \delta = K_\gamma \cap \delta \implies K_\beta \cap \delta \subseteq A$.

Proof. i) Assume $K \in L_\alpha$, $K_\delta = \bigcup_{\gamma \in K} K_\gamma$, $\delta \in l_A$. So $K_\delta <_b A$.

Assume $\gamma \in K$. Then $K_\gamma \subseteq K_\delta <_b A$ by fact 4.1.6i. So $K_\gamma \leq_b K_\delta <_b A$. Hence $K_\gamma <_b A$ by the transitivity of $\leq_b$ (proposition 4.1.5). Therefore $\gamma \in l_A$ and so $K \subseteq l_A$ as required.

ii) Assume $\beta \in l_A$, $\gamma \in r_A$, $K_\beta \cap \delta = K_\gamma \cap \delta$. So $K_\beta <_b A$ and $A <_b K_\gamma$. As $K_\beta <_b A <_b K_\gamma$ and $K_\beta \cap \delta = K_\gamma \cap \delta$, so $K_\beta \cap \delta = A \cap \delta = K_\gamma \cap \delta$. Thus $K_\beta \cap \delta \subseteq A \cap \delta \subseteq A$ as required.

□

Proposition 4.1.11. For any $A \subseteq \alpha$ the sets l_A, r_A are α-semicomputable.

Proof. l_A is α-semicomputable since it has an α-computable selector function

$$s := \{(x, y, x) : K_x \leq_b K_y\} \cup \{(x, y, y) : K_x >_b K_y\}$$

by remark 4.1.4. □

Proposition 4.1.12. Let $A \subseteq \alpha$ be a quasiregular set, then $A \equiv_\alpha l_A \equiv_\alpha r_A$.

Proof. If $A \in \Delta_1(L_\alpha)$, then trivially $A \equiv_\alpha l_A \equiv_\alpha r_A$. Hence WLOG assume that $A \notin L_\alpha$ and use remark 4.1.9. Also WLOG $A \notin \Delta_1(L_\alpha)$ and so in the proof implicitly use the property

$$\forall x \in A \exists y, z[x < y < \alpha \wedge x < z < \alpha \wedge y \notin A \wedge z \in A].$$

Note that $\bigcup_{x \in K_\gamma} K_x \in L_\alpha$. Hence for any $\gamma < \alpha$ we have

$$K_\gamma \subseteq l_A \iff \exists \beta < \alpha[K_\beta <_b A \wedge \forall x \in K_\gamma . K_x <_b K_\beta].$$

Thus $l_A \leq_{\alpha e} A$ via

$$W := \{\langle \gamma, \delta \rangle : \exists \beta < \alpha[K_\delta = \{\beta\} \wedge \forall x \in K_\gamma . K_x <_b K_\beta]\} \in \Sigma_1(L_\alpha).$$

By symmetry $r_A \leq_{\alpha e} A$. Hence $l_A \oplus r_A \leq_{\alpha e} A$.

Let $\widehat{A}$ denote A or $\overline{A}$. Then $K_\gamma \subseteq \widehat{A} \iff$

$$\exists \beta_L, \beta_R < \alpha[\forall x \in K_\gamma \forall y \leq x[y \in K_{\beta_L} \iff y \in K_{\beta_R}] \wedge K_\gamma \subseteq \widehat{K_{\beta_L}} \wedge \beta_L \in l_A \wedge \beta_R \in r_A]$$

for any $\gamma < \alpha$ using the quasiregularity of A and proposition 4.1.10ii. Hence define

$$W := \{\langle \gamma, \delta \rangle : \exists \beta_L, \beta_R < \alpha\big[\forall x \in K_\gamma \forall y \leq x[y \in K_{\beta_L} \iff y \in K_{\beta_R}] \wedge \\ K_\gamma \subseteq \widehat{K_{\beta_L}} \wedge K_\delta = \{\beta_L\} \oplus \{\beta_R\}\big]\}.$$

Note that $W \in \Sigma_1(L_\alpha)$ and so $\widehat{A} \leq_{\alpha e} l_A \oplus r_A$ via W. Hence $A \oplus \overline{A} \leq_{\alpha e} l_A \oplus r_A$.

Therefore $A \oplus \overline{A} \equiv_{\alpha e} l_A \oplus r_A = l_A \oplus \overline{l_A} = r_A \oplus \overline{r_A}$ and so $A \equiv_\alpha l_A \equiv_\alpha r_A$ as required. □

4.1.4 Semicomputable set existence

Lemma 4.1.13. [1] If $A_0 \cap A_1 = \emptyset$ and $\forall i \in \{0,1\}. A_i \in \Sigma_1(L_\alpha, A_0 \sqcup A_1)$, then $A_0 \sqcup A_1 \equiv_\alpha A_0 \oplus A_1$.

Proof. We have $A_0 \sqcup A_1 \leq_\alpha A_0 \oplus A_1$ trivially. Let $i \in \{0,1\}$. For $A_0 \oplus A_1 \leq_\alpha A_0 \sqcup A_1$: $x \in A_i$ recognizable by $A_i \in \Sigma_1(L_\alpha, A_0 \sqcup A_1)$. Also $x \notin A_i$ is recognizable since $x \notin A_i \iff x \in A_{1-i} \vee x \notin A_0 \sqcup A_1$ by disjointness and both $x \in A_{1-i}, x \notin A_0 \sqcup A_1$ are recognizable from $A_0 \sqcup A_1$. Hence $A_0 \sqcup A_1 \equiv_\alpha A_0 \oplus A_1$. □

The lemma implies that if A_0, A_1 are disjoint α-incomparable α-computably enumerable sets, then $A_0 \sqcup A_1 \equiv_\alpha A_0 \oplus A_1$ [2].

Lemma 4.1.14. [3]

$$B \in \Sigma_1(L_\alpha) \wedge B >_\alpha 0 \implies \exists A[A \text{ regular } \wedge A \equiv_\alpha B \wedge l_A \notin \Pi_1(L_\alpha) \wedge l_A \notin \Sigma_1(L_\alpha)].$$

[1] From lemma 6 in [22] on p66.
[2] Proposition 3.3 in [29].
[3] Adapted from Lemma 5.5 in [10] for $\alpha = \omega$

Proof. By theorem 3.8.4 every $\Sigma_1(L_\alpha)$ set is α-equivalent to some regular set, so WLOG assume that B is regular. By Shore's Splitting Theorem 3.8.5, $\exists C_0, D_0 \in \Sigma_1(L_\alpha)[B = C_0 \sqcup D_0 \wedge C_0|_\alpha D_0]$ where $C_0|_\alpha D_0$ means that C_0 and D_0 are incomparable wrt α-reducibility. Using theorem 3.8.4 again, let C, D be α-c.e. regular sets s.t. $C \equiv_\alpha C_0$ and $D \equiv_\alpha D_0$. Define $A := C \oplus \overline{D}$.

Note $A = C \oplus \overline{D} \equiv_\alpha C_0 \oplus \overline{D_0}$. Hence $A \equiv_\alpha B$ by lemma 4.1.13 as required.

As D is regular, so $\overline{D}$ is regular. As C and $\overline{D}$ are regular, so $A = C \oplus \overline{D}$ is regular as required.

Next we prove $l_A \notin \Pi_1(L_\alpha) \wedge l_A \notin \Sigma_1(L_\alpha)$. For suppose to the contrary that $\neg(l_A \notin \Pi_1(L_\alpha) \wedge l_A \notin \Sigma_1(L_\alpha))$. Then $l_A \in \Sigma_1(L_\alpha) \vee l_A \in \Pi_1(L_\alpha)$.

- Case $l_A \in \Sigma_1(L_\alpha)$: Note that $\overline{D} \leq_{\alpha e} C \oplus \overline{C}$ via

$$\begin{aligned} W := \{\langle\gamma,\delta\rangle : \beta = \min\{\epsilon < \alpha : K_\gamma \cap \epsilon = K_\gamma\} \wedge \exists \zeta \in l_A \forall x < \beta[\\ (2x \in K_\delta \iff 2x+1 \notin K_\delta \iff 2x \in K_\zeta) \wedge \\ (x \in K_\gamma \implies 2x+1 \in K_\zeta) \wedge \\ (2x+1 \notin K_\zeta \implies x \in D)]\}. \end{aligned}$$

  The set W is α-c.e. since l_A and D are α-c.e. The condition $2x \in K_\delta \iff 2x+1 \notin K_\delta$ ensures that K_δ contains the initial segment $C \cap \beta$ of C. The conditions $2x \in K_\delta \iff 2x \in K_\zeta$ and $2x+1 \notin K_\zeta \implies x \in D$ ensure that K_ζ contains the initial segment $(C \cap \beta) \oplus (\overline{D} \cap \beta)$ of $C \oplus \overline{D}$. Finally, the condition $x \in K_\gamma \implies 2x+1 \in K_\zeta$ verifies that K_γ is a subset of $\overline{D}$, or more precisely a subset of its initial segment $\overline{D} \cap \beta$.

  As D is α-c.e., so this gives us $D \leq_\alpha C$ which is a contradiction to the case $l_A \in \Sigma_1(L_\alpha)$.

- Case $l_A \in \Pi_1(L_\alpha)$: Note that $r_A = \overline{l_A} \in \Sigma_1(L_\alpha)$. Hence similarly $C \leq_\alpha D$ using the fact that r_A and C are both α-c.e. by applying a symmetric argument to the one above. This is a contradiction to the case $l_A \in \Pi_1(L_\alpha)$.

So by the two cases

$$l_A \notin \Pi_1(L_\alpha) \wedge l_A \notin \Sigma_1(L_\alpha).$$

Therefore given $B >_\alpha 0$, there is a regular set A s.t.

$$A \equiv_\alpha B \wedge l_A \notin \Pi_1(L_\alpha) \wedge l_A \notin \Sigma_1(L_\alpha)$$

as required. □

Theorem 4.1.15. Let $B \subseteq \alpha$ be quasiregular and $B >_\alpha 0$. Then there exists an α-semicomputable set A s.t.

$$A \equiv_\alpha B \wedge A \notin \Sigma_1(L_\alpha) \wedge A \notin \Pi_1(L_\alpha).$$

Proof. If $\deg_\alpha(B)$ is α-c.e. degree, then WLOG let $B \in \Sigma_1(L_\alpha)$. Then by lemma 4.1.14 there is C s.t. C is quasiregular and

$$B \equiv_\alpha C \wedge L_C \notin \Sigma_1(L_\alpha) \wedge L_C \notin \Pi_1(L_\alpha).$$

By proposition 4.1.12 and quasiregularity of C we have that $C \equiv_\alpha L_C$ and so $B \equiv_\alpha L_C$. Hence $A := L_C$ is the required α-semicomputable set by proposition 4.1.11.

Otherwise $\deg_\alpha(B)$ is not an α-c.e. degree and so

$$\forall C \in \deg_\alpha(B)[C \notin \Sigma_1(L_\alpha) \wedge C \notin \Pi_1(L_\alpha)].$$

Note that $A := l_B \equiv_\alpha B$ by the quasiregularity of B and by proposition 4.1.12 and so $A \notin \Sigma_1(L_\alpha) \wedge A \notin \Pi_1(L_\alpha)$. Finally, A is α-semicomputable by proposition 4.1.11 as required. □

4.2 Kalimullin pair

We define an α-Kalimullin pair and establish some basic properties about it.

Definition 4.2.1. (Kalimullin pair[4])
Sets $A, B \subseteq \alpha$ are an α-*U-Kalimullin pair* denoted by $\mathcal{K}_U(A, B)$ iff

$$\exists W \leq_{\alpha e} U[A \times B \subseteq W \wedge \overline{A} \times \overline{B} \subseteq \overline{W}].$$

If clear, we omit the prefix α and say U-Kalimullin pair (or just U-$\mathcal{K}$-pair) and denote it by $\mathcal{K}_U(A, B)$. Similarly, if $U \in \Sigma_1(L_\alpha)$, then we say that A, B are a Kalimullin pair (or just $\mathcal{K}$-pair) and denote it as $\mathcal{K}(A, B)$.

The set W is called a *witness* to the U-Kalimullin pair.

Proposition 4.2.2. [5] If $A \leq_{\alpha e} U$, then $\forall B \subseteq \alpha . K_U(A, B)$.

Proof. Take the witness $W := A \times \alpha$. □

Proposition 4.2.3. If A is α-semicomputable, then $\mathcal{K}(A, \overline{A})$.

Proof. Define the witness $W \in \Sigma_1(L_\alpha)$ to the Kalimullin pair $\mathcal{K}(A, \overline{A})$ to be

$$W := \{\langle x, y\rangle \in \alpha : s_A(x, y) = x\}$$

where s_A is an α-computable selector function for an α-semicomputable set A. □

Definition 4.2.4. $A, B \subseteq \alpha$ are a *trivial* Kalimullin pair iff $\mathcal{K}(A, B)$ and $A \in \Sigma_1(L_\alpha) \vee B \in \Sigma_1(L_\alpha)$. If A, B are a not a trivial Kalimullin pair, they form a *nontrivial* Kalimullin pair, denoted by $\mathcal{K}_{\text{nt}}(A, B)$.

[4] Adapted from [25] Definition 2.1.
[5] Proposition 2.2 in [25] for $\alpha = \omega$.

Definition 4.2.5. (Maximal Kalimullin pair)
A Kalimullin pair $\mathcal{K}(A,B)$ is *maximal* denoted by $\mathcal{K}_{\max}(A,B)$ iff

$$\forall C, D[A \leq_{\alpha e} C \wedge B \leq_{\alpha e} D \wedge \mathcal{K}(C,D) \implies A \equiv_{\alpha e} C \wedge B \equiv_{\alpha e} D].$$

Remark 4.2.6. Note that in the definition of a maximal Kalimullin pair we use α-enumeration reducibility instead of a weak α-enumeration reducibility since we want that a maximal Kalimullin pair is definable (given that a Kalimullin pair is definable) in the structure $\langle \mathcal{D}_{\alpha e}, \leq \rangle$ where $\leq$ is induced by $\leq_{\alpha e}$.

Proposition 4.2.7. [6] Assume

$$A, B \subseteq \alpha \wedge A \notin \Sigma_1(L_\alpha) \wedge B \notin \Sigma_1(L_\alpha) \wedge \mathcal{K}(A,B)$$

where the witness of $\mathcal{K}(A,B)$ is W. Then

i) $A = \{a < \alpha : \exists b[b \notin B \wedge (a,b) \in W]\}$,

ii) $B = \{b < \alpha : \exists a[a \notin A \wedge (a,b) \in W]\}$.

Proof. i) Assume $A, B \subseteq \alpha$, $A \notin \Sigma_1(L_\alpha)$, $B \notin \Sigma_1(L_\alpha)$ and $\mathcal{K}(A,B)$. Define $A_2 := \{a : \exists b[b \notin B \wedge (a,b) \in W]\}$. We prove $A = A_2$ which implies part i.

- We prove that $A \subseteq A_2$. Let $a \in A$. We prove that $a \in A_2$. For suppose not, then $a \notin A_2$ and so $\forall b[\neg(b \notin B \wedge (a,b) \in W)]$. This implies $\forall b[b \in B \vee (a,b) \notin W]$. Hence $\forall b[(a,b) \in W \implies b \in B]$. As $\mathcal{K}(A,B)$, so $A \times B \subseteq W$ where W is the witness of the Kalimullin pair. Note that $B = \{b : \exists a \in \alpha.(a,b) \in W\}$ and so $B \leq_{\alpha e} W$. As $W \in \Sigma_1(L_\alpha)$ and $B \leq_{\alpha e} W$, so $B \in \Sigma_1(L_\alpha)$. But at the beginning we assumed that $B \notin \Sigma_1(L_\alpha)$. This is a contradiction. Hence $a \in A_2$ as needed and so $A \subseteq A_2$.
- We prove that $A_2 \subseteq A$. Assume $a \in A_2$. So there is some $b \notin B$ s.t. $(a,b) \in W$. We prove that $a \in A$. For suppose not, then $a \notin A$ and so $b \notin B \wedge a \notin A$. As $\mathcal{K}(A,B)$, so $\overline{A} \times \overline{B} \subseteq \overline{W}$. Hence $(a,b) \in \overline{W}$. As both $(a,b) \in W$ and $(a,b) \in \overline{W}$, this is a contradiction. Hence $a \in A$ as needed as so $A_2 \subseteq A$.

As both $A \subseteq A_2$ and $A_2 \subseteq A$, so $A = A_2$ and part i is true.

ii) The proof of part ii is symmetric.

□

Corollary 4.2.8. Assume

$$A, B \subseteq \alpha \wedge A \notin \Sigma_1(L_\alpha) \wedge B \notin \Sigma_1(L_\alpha) \wedge \mathcal{K}(A,B).$$

Then

[6] From [25] and proposition 1.8 in [2].

i) $A \leq_{w\alpha e} \overline{B}$ and $B \leq_{w\alpha e} \overline{A}$,

ii) $A \leq_{\alpha e} \overline{B}$ if B is megaregular, $B \leq_{\alpha e} \overline{A}$ if A is megaregular.

Proof. Follows from proposition 4.2.7. □

Lemma 4.2.9. (Kalimullin pair distributivity)
Suppose that $\bigwedge_{i\in 2} A_i \neq \emptyset$. Then

$$\bigwedge_{i\in 2} \mathcal{K}(A_i, B) \iff \mathcal{K}(\bigoplus_{i\in 2} A_i, B) \iff \mathcal{K}(\prod_{i\in 2} A_i, B)$$

Proof. Suppose $\bigwedge_{i\in 2} \mathcal{K}(A_i, B)$. For any $i \in 2$ let $A_i \times B \subseteq U_i \in \Sigma_1(L_\alpha)$ and $\overline{A}_i \times \overline{B} \subseteq \overline{U}_i$. Define the sets V, W, U_i^* as follows:

$$V := \{(2a + i, b) : (a, b) \in U_i, i \in 2\},$$
$$W := \{((a_0, a_1), b) : \forall i \in 2.(2a_i + i, b) \in V\},$$
$$U_i^* := \{(a_i, b) : \exists (a_0, a_1).((a_0, a_1), b) \in W\}.$$

The equivalences of lemma 4.2.9 follow from the following implications:

$$\bigwedge_{i\in 2} A_i \times B \subseteq U_i \in \Sigma_1(L_\alpha) \wedge \overline{A}_i \times \overline{B} \subseteq \overline{U}_i \implies$$
$$\Big(\bigoplus_{i\in 2} A_i\Big) \times B \subseteq V \in \Sigma_1(L_\alpha) \wedge \overline{\bigoplus_{i\in 2} A_i} \times \overline{B} \subseteq \overline{V} \implies$$
$$\Big(\prod_{i\in 2} A_i\Big) \times B \subseteq W \in \Sigma_1(L_\alpha) \wedge \overline{\prod_{i\in 2} A_i} \times \overline{B} \subseteq \overline{W} \implies (\text{by } \bigwedge_{i\in 2} A_i \neq \emptyset)$$
$$\bigwedge_{i\in 2} A_i \times B \subseteq U_i^* \in \Sigma_1(L_\alpha) \wedge \overline{A}_i \times \overline{B} \subseteq \overline{U}_i^*.$$

□

4.3 Definability of an α-Kalimullin pair

We prove that the set of the Kalimullin pairs is definable in $\mathcal{D}_{\alpha e}$ if $\alpha^* = \alpha$ or if $V = L$ and α is an infinite regular cardinal. For any $U \subseteq \alpha$ we prove that the set of the U-Kalimullin pairs is definable in $\mathcal{D}_{\alpha e}$ if $V = L$ and α is an infinite regular cardinal. The main part of the proof is the construction in section 4.3.1.

For this section let D_x, E_x be a pair of α-finite sets indexed by $x < \alpha$ according to lemma 3.5.3. For any $x < \alpha$ define

$$V_x := \{y < \alpha : D_x \subseteq D_y \wedge E_x \subseteq E_y\}$$

4.3.1 Key theorem and construction

Theorem 4.3.1. [7] Let $A, B, U \subseteq \alpha$. Let one of the conditions hold:

[7] Theorem 2.5 in [25] for $\alpha = \omega$.

i) the projectum of α is $\alpha^* = \omega$ and U is megaregular.

ii) $A \oplus B \oplus K(U)$ is megaregular.

Suppose $\neg\mathcal{K}_U(A, B)$. Then

$$\exists X, Y \subseteq \alpha[Y \leq_{\alpha e} X \oplus A \wedge Y \leq_{\alpha e} X \oplus B \wedge Y \not\leq_{w\alpha e} X \oplus U].$$

The following proof is a generalization of the proof for the case when $\alpha = \omega$ in [25].

Proof. This proof depends on lemmas 4.3.2 to 4.3.5 which are proved in section 4.3.2.

We perform a construction in α^* stages and define sets X, Y s.t. $\forall x < \alpha$:

$$x \in Y \iff x \in X \wedge D_x \subseteq A \iff x \in X \wedge E_x \subseteq B \tag{4.1}$$

which guarantees $Y \leq_{\alpha e} X \oplus A$ and $Y \leq_{\alpha e} X \oplus B$ by lemma 4.3.2 underneath.

Lemma 4.3.2. Let X, Y, A be any subsets of α. Assume that for any $x < \alpha$ we have

$$x \in Y \iff x \in X \wedge D_x \subseteq A$$

where D_x is an α-finite set with a uniformly α-computable index x. Then $Y \leq_{\alpha e} X \oplus A$. □

Index the requirements and α-enumeration operators by indices in α^* using proposition 3.6.8. Aim to meet for all $e < \alpha^*$ the requirements

$$R_e : Y \neq \Phi_e(X \oplus U).$$

Let $s < \alpha^*$ be a stage during the construction. We use the following sets to help to ensure the conditions stated just after them:

- X_s which is used to define the desired set X in the end as $X := \bigcup_{s<\alpha^*} X_s$.
- M_s which is used to put an extra condition (4.4) on X_s to ensure $Y \not\leq_{w\alpha e} X \oplus U$ (see Final verification).
- $N_s := \{e < \alpha : 2e < s\}$ which just stores the indices e of the enumeration operators used at the stage $2e + 2$ and together with I_s it is used to define M_{s+1} .
- I_s which is used to define M_{s+1} together with N_s and stores the indices $x \in M_s$ s.t. $D_x \subseteq A \wedge E_x \subseteq B \wedge x \notin \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U)$. This is used to ensure the condition 4.5 which is an essential property of the set M_s.
- $V_x := \{y < \alpha : D_x \subseteq D_y \wedge E_x \subseteq E_y\}$ which is also used in the definition of M_s, see 4.9.

At each stage $s < \alpha^*$ of the construction aim to define an α-finite set X_s and an α-computable set M_s so that for all $s < \alpha^*$ they satisfy:

$$X_s \subseteq X_{s+1} \tag{4.2}$$

$$M_{s+1} \subseteq M_s \tag{4.3}$$

$$X_{s+1} - X_s \subseteq M_{s+1} \tag{4.4}$$

$$\forall D, E \in L_\alpha[D \subseteq A \land E \subseteq B \implies \exists x \in M_s[D \subseteq D_x \subseteq A \land E \subseteq E_x \subseteq B]] \tag{4.5}$$

$$X_s \in L_\alpha \tag{4.6}$$

$$N_s \in L_\alpha \tag{4.7}$$

$$I_s \in L_\alpha \tag{4.8}$$

$$M_s := (\bigcap_{x \in I_s} V_x) - N_s = V_z - N_s \tag{4.9}$$

$$M_s \in \Delta_1(L_\alpha) \tag{4.10}$$

Pre-construction

Note that $N_s := \{e < \alpha : 2e < s\}$ and so it is α-finite as it is α-computable and bounded. Thus statement (4.7) is true.

Next we will use lemma 4.3.4 underneath.

Lemma 4.3.4. Let $I \in L_\alpha$. Then exists an index $z < \alpha$ which is uniformly α-computable from I s.t.

$$V_z = \bigcap_{x \in I} V_x.$$

□

By statement (4.9), the set M_s is defined at every stage $s < \alpha^*$ by the sets N_s and I_s. Since the set I_s is α-finite at the stage s by statement (4.8), so by lemma 4.3.4 there is an index z which is uniformly α-computable from I_s and $V_z = \bigcap_{x \in I_s} V_x$. Hence the equality

$$(\bigcap_{x \in I_s} V_x) - N_s = V_z - N_s$$

holds at every stage s where $I_s \in L_\alpha$. Consequently also the set V_z is α-computable at such stage s.

Since the set N_s is α-finite by statement (4.7) and V_z is α-computable at the stage s, so the set M_s has to be α-computable at the stage s, hence statement (4.10) holds.

When proving at the stage $s < \alpha^*$ that statement (4.5) holds, we use the fact that A and B are not α-finite by proposition 4.2.2 since $\neg\mathcal{K}_U(A, B)$. This given

α-finite sets D, E, enables us to find arbitrarily large α-finite supersets of D, E contained in A and B respectively.

Constructing X

The set X will be constructed in α^*-many stages.

- Stage $s = 0$. Set $X_0 := \emptyset$, $I_0 := \emptyset$. Observe statement (4.5) is true for $M_0 = \alpha$. Clearly, statements (4.6) to (4.8) are satisfied.

- Stage $s + 1 = 2e + 1$. By induction hypothesis let X_s, I_s be given and α-finite by statements (4.6) to (4.8). Define $X_{s+1} := X_s$, $I_{s+1} := I_s$. Trivially, statements (4.6) to (4.8) hold at the stage $s + 1$ by IH at the stage s.

  Note $M_{s+1} = M_s - \{e\}$ by statement (4.9). We claim that the set M_{s+1} satisfies statement (4.5). Let $D, E \in L_\alpha \wedge D \subseteq A \wedge E \subseteq B$. By IH on M_s there is $x \in M_s$ s.t.

  $$D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B.$$

  Note $D_x \in L_\alpha$, but by proposition 4.2.2 $A \notin L_\alpha$, hence $D_x \subset A$. Let $z \in A - D_x$. Then $\hat{D} := D_x \cup \{z\} \in L_\alpha$. By IH on M_s there is $y \in M_s$ s.t.

  $$\hat{D} \subseteq D_y \subseteq A \wedge E \subseteq E_y \subseteq B.$$

  If $x \neq e$, then $x \in M_{s+1} := M_s - \{e\}$. Otherwise $x = e \neq y$ and

  $$y \in M_{s+1} \wedge D \subseteq D_x \subset \hat{D} \subseteq D_y \subseteq A \wedge E \subseteq E_y \subseteq B.$$

  Therefore in any case the set M_{s+1} satisfies statement (4.5).

- Stage $s + 1 = 2e + 2$. Aim to find $x \in M_s$ s.t. one of the two following statements is true:

  1: $D_x \nsubseteq A \wedge E_x \nsubseteq B \wedge x \in \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U)$,

  2: $D_x \subseteq A \wedge E_x \subseteq B \wedge x \notin \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U)$.

  First we prove the existence of such $x \in M_s$. Assume that $\forall x \in M_s$ the statement 2 is false. Define

  $$W := \{\langle a, b\rangle : \exists x \in M_s[a \in D_x \wedge b \in E_x \wedge x \in \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U)]\}.$$

  Then $W \leq_{\alpha e} U$ by the megaregularity of U, lemma 4.3.3, statement (4.10) and statement (4.6) where lemma 4.3.3 is given below.

  Lemma 4.3.3. Assume $M_s \in \Sigma_1(L_\alpha)$ and $X_s \in L_\alpha$. Let

  $$W := \{\langle a, b\rangle : \exists x \in M_s[a \in D_x \wedge b \in E_x \wedge x \in \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U)]\}.$$

  Assume U is megaregular. Then $W \leq_{\alpha e} U$. □

We prove $A \times B \subseteq W$. Let $(a,b) \in A \times B$. By statement (4.5) for M_s it follows

$$\exists x \in M_s[a \in D_x \subseteq A \wedge b \in E_x \subseteq B].$$

Since statement 2 is false, we have

$$x \in \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U).$$

Thus $(a,b) \in W$. Since $\neg\mathcal{K}_U(A,B)$, there is a pair $(a,b) \in \overline{A} \times \overline{B}$ s.t. $(a,b) \in W$. Thus there is $x \in M_s$ s.t. $a \in D_x, b \in E_x$ and

$$x \in \Phi_e((X_s \cap (M_s \cap V_x)) \oplus U).$$

Hence $D_x \not\subseteq A$, $E_x \not\subseteq B$ and statement 1 is true for $x \in M_s$. Therefore there is $x \in M_s$ s.t. statement 1 or statement 2 is true. Choose such an element $x \in M_s$ using the oracle $A \oplus B \oplus K(U)$.

Case 1: If statement 1 is true for x, then

$$x \in \Phi_e((X_s \cup (M_s \cap V_x) \oplus U).$$

By fact 3.7.11ii and fact 3.7.11iii there is $F \subseteq X_s \cup (M_s \cap V_x)$ s.t.

$$F \in L_\alpha \wedge x \in \Phi_e(F \oplus U).$$

Thus define $X_{s+1} := X_s \cup F$, $I_{s+1} := I_s$. Note that $M_{s+1} := M_s$. The set F is α-finite, by IH X_s is α-finite and so the union $X_{s+1} = X_s \cup F$ is α-finite satisfying statement (4.6). Statement (4.8) is true by IH.

Case 2: Otherwise if statement 2 is true for x, then define $X_{s+1} := X_s \cup \{x\}$, $I_{s+1} := I_s \cup \{x\}$. Trivially, the sets X_{s+1}, I_{s+1} are α-finite using IH, hence satisfying statements (4.6) to (4.8). Note $M_{s+1} = M_s \cap V_x$ by statement (4.9). M_{s+1} satisfies statement (4.5): if $D \subseteq A, E \subseteq B, D \in L_\alpha, E \in L_\alpha$, then by the hypothesis on M_s, there is $y \in M_s$ s.t.:

$$D \cup D_x \subseteq D_y \subseteq A,$$
$$E \cup E_x \subseteq E_y \subseteq B.$$

Therefore

$$y \in M_s \cap V_x = M_{s+1}.$$

Note in both cases $X_{s+1} - X_s \subseteq M_{s+1}$, statement (4.4) being satisfied.

- Stage $s = 2e > 0$, $2e$ is a limit ordinal. If $\alpha^* = \omega$, then this stage does not arise. Hence assume that $A \oplus B \oplus K(U)$ is megaregular.

  Define $X_s := \bigcup_{r<s} X_r$, $I_s := \bigcup_{r<s} I_r$. We claim that these sets are α-finite.

Define a partial function $f : \alpha \rightharpoonup \alpha$ on the ordinals smaller than s by

$$f(r) := \{\gamma < \alpha : K_\gamma = X_r\}.$$

Note that by IH for all $r < s$, the set X_r is α-finite using statement (4.6). Also during the construction we only use the oracle $A \oplus B \oplus K(U)$. Thus the index $f(r)$ of an α-finite set X_r is also $A \oplus B \oplus K(U)$-computable. Consequently, the function f is $\Sigma_1(L_\alpha, A \oplus B \oplus K(U))$ definable. As $s < \alpha^*$, so s as a limit ordinal is an α-finite set. Therefore by the megaregularity of $A \oplus B \oplus K(U)$, the set $f[s]$ is also α-finite. But then $X_s = \bigcup_{\gamma \in f[s]} K_\gamma$ is α-finite by proposition 3.2.10. So statement (4.6) holds at the stage s as required. Applying similar reasoning, using the veracity of statement (4.8) for all $r < s$ by IH, we conclude that statement (4.8) holds at the stage s too.

Note $M_s := \bigcap_{r<s} M_r$ by statement (4.9). We prove that statement (4.5) holds at the stage s. Note that $M_s = V_z - N_s$ by statement (4.9) for some $z < \alpha$ satisfying both $D_z \subseteq A$ and $E_z \subseteq B$. Fix α-finite sets D and E s.t. $D \subseteq A$ and $E \subseteq B$. WLOG let $D_z \subseteq D$ and $E_z \subseteq E$. Define

$$Z := \{x < \alpha : D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B\}.$$

As $\neg\mathcal{K}(A, B)$ by the assumption, so $A \notin \Sigma_1(L_\alpha)$ and $B \notin \Sigma_1(L_\alpha)$ by proposition 4.2.2. Note that $A \oplus B$ is megaregular. Hence Z is unbounded by lemma 4.3.5 below.

Lemma 4.3.5. Let $D \subseteq A \subseteq \alpha$ and $E \subseteq B \subseteq \alpha$ satisfying $A, B \notin \Sigma_1(L_\alpha)$ and $D, E \in L_\alpha$. Define

$$Z := Z_{D,E} := \{x < \alpha : D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B\}.$$

Then:

i) $Z \equiv_{\alpha e} A \oplus B$,

ii) $\overline{Z} \leq_{w\alpha e} \overline{A} \oplus \overline{B}$,

iii) $Z \notin \Sigma_1(L_\alpha)$,

iv) Z is unbounded if $A \oplus B$ is megaregular.

□

On the other hand $N_s \subseteq s$. Thus $Z - N_s \neq \emptyset$. Note

$$\{x \in M_s : D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B\} =$$
$$\{x \in V_z - N_s : D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B\} =$$
$$Z - N_s \neq \emptyset.$$

Therefore

$\forall D, E \in L_\alpha\big[D \subseteq A \wedge E \subseteq B \implies \exists x \in M_s[D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B]\big]$

and so the statement statement (4.5) is satisfied at the limit stage s.

Finally, define $X := \bigcup_{s<\alpha^*} X_s$.

Defining Y

To define Y, first prove

$$\forall z \in X[D_z \subseteq A \iff E_z \subseteq B]:$$

Let $z \in X$. Then there is a stage $s+1 = 2e+2$ s.t. $z \in X_{s+1} - X_s$. In case 2 we have $D_z \subseteq A$ and $E_z \subseteq B$. In case 1 there is x s.t. $X_{s+1} - X_s \subseteq V_x$, $D_x \not\subseteq A$ and $E_x \not\subseteq B$. As $z \in X_{s+1} - X_s \subseteq V_x$, so $D_x \subseteq D_z$ and $E_x \subseteq E_z$. Thus $D_z \not\subseteq A$ and $E_z \not\subseteq B$. Define the set

$$Y := \{z \in X : D_z \subseteq A\} = \{z \in X : E_z \subseteq B\}.$$

Final verification

Note $Y \leq_{\alpha e} X \oplus A$ and $Y \leq_{\alpha e} X \oplus B$ as proved under statement (4.1).

We prove $Y \not\leq_{w\alpha e} X \oplus U$ by showing $Y \neq \Phi_e(X \oplus U)$ for an arbitrary $e < \alpha^*$. Consider a stage $s+1 = 2e+2$. In case 1 we have $X_{s+1} = X_s \cup F$ and there is x s.t. $x \in \Phi_e(F \oplus U)$, $D_x \not\subseteq A$ and $E_x \not\subseteq B$. Hence

$$x \in \Phi_e(X \oplus U) - Y.$$

In case 2 there is x s.t. $X_{s+1} = X_s \cup \{x\}$, $M_{s+1} = M_s \cap V_x$, $D_x \subseteq A$, $E_x \subseteq B$ and

$$x \notin \Phi_e((X_s \cup M_{s+1}) \oplus U).$$

Let $z \in X$. Then $\exists t. z \in X_{t+1} - X_t \subseteq M_{t+1}$ by statement (4.4). If $t \geq s$, then $z \in M_{s+1}$ by statement (4.3). If $t < s$, then $z \in X_s$ by statement (4.2). Hence $z \in X_s \cup M_{s+1}$ and thus $X \subseteq X_s \cup M_{s+1}$.

Hence $x \in Y - \Phi_e(X \oplus U)$ by fact 3.7.11ii. Therefore in both cases $Y \neq \Phi_e(X \oplus U)$ and so $Y \not\leq_{w\alpha e} X \oplus U$. □

4.3.2 Construction lemmas proved

We prove some lemmas used in the construction in section 4.3.1.

Lemma 4.3.2. Let X, Y, A be any subsets of α. Assume that for any $x < \alpha$ we have

$$x \in Y \iff x \in X \wedge D_x \subseteq A$$

where D_x is an α-finite set with a uniformly α-computable index x. Then $Y \leq_{\alpha e} X \oplus A$.

Proof. Recall $Y \leq_{\alpha e} X \oplus A \iff$

$$\exists W \in \Sigma_1(L_\alpha)\forall\gamma < \alpha[K_\gamma \subseteq Y \iff \exists\langle\gamma,\delta\rangle \in W.K_\delta \subseteq X \oplus A].$$

Note $K_\gamma \subseteq Y \iff \forall x \in K_\gamma.x \in Y \iff \forall x \in K_\gamma[x \in X \wedge D_x \subseteq A] \iff$

$$K_\gamma \subseteq X \wedge \bigcup_{x\in K_\gamma} D_x \subseteq A \iff \text{(By lemma 3.5.4i)}$$

$$K_\gamma \subseteq X \wedge K_{u(\gamma)} \subseteq A \iff \text{(By lemma 3.5.4ii)}$$

$$K_{v(\gamma,u(\gamma))} \subseteq X \oplus A.$$

Hence define

$$W := \{\langle\gamma,\delta\rangle < \alpha : \delta = v(\gamma,u(\gamma))\}.$$

As $u, v \in \Sigma_1(L_\alpha)$, so $W \in \Sigma_1(L_\alpha)$. Moreover,

$$K_\gamma \subseteq Y \iff \exists\langle\gamma,\delta\rangle \in W.K_\delta \subseteq X \oplus A.$$

Therefore $Y \leq_{\alpha e} X \oplus A$. □

Lemma 4.3.3. Assume $M_s \in \Sigma_1(L_\alpha)$ and $X_s \in L_\alpha$. Let

$$W := \{\langle a,b\rangle : \exists x \in M_s[a \in D_x \wedge b \in E_x \wedge x \in \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U)]\}.$$

Assume U is megaregular. Then $W \leq_{\alpha e} U$.

Proof. Let

$$S_e := \Phi_e((X_s \cup (M_s \cap V_x)) \oplus U).$$

We first prove $W \leq_{\alpha e} S_e$. Note $K_\gamma \subseteq W \iff \forall\langle a,b\rangle \in K_\gamma.\langle a,b\rangle \in W \iff$

$$\forall\langle a,b\rangle \in K_\gamma.\exists x \in M_s[a \in D_x \wedge b \in E_x \wedge x \in S_e] \iff$$

$$\forall\langle a,b\rangle \in K_\gamma.\exists x \in M_s[\langle a,b\rangle \in P_x \wedge x \in S_e]$$

where $i_P : \alpha \to \alpha \in \Sigma_1(L_\alpha)$ is a function of lemma 3.5.4iv and $P_x := K_{i_P(x)}$. Define ϕ and V:

$$\phi(\gamma,\delta) \iff \forall y \in K_\gamma \exists x \in K_\delta.y \in P_x.$$

$$V := \{\langle\gamma,\delta\rangle : K_\delta \subseteq M_s \wedge \phi(\gamma,\delta)\}.$$

Then continuing we have $K_\gamma \subseteq W \iff \forall y \in K_\gamma \exists x \in M_s[y \in P_x \wedge x \in S_e] \iff$

$$\exists\delta[K_\delta \subseteq M_s \wedge K_\delta \subseteq S_e \wedge \phi(\gamma,\delta)] \iff \exists\langle\gamma,\delta\rangle \in V.K_\delta \subseteq S_e.$$

where $K_\delta \in L_\alpha$ has to exist as an image of an α-computable function restricted to an $K_\gamma \in L_\alpha$ by the admissibility of α. Note

$$\phi(\gamma,\delta) \iff \exists H[H = w(\gamma,\delta) \wedge \forall y \in K_\gamma \exists x \in K_\delta.\langle x,y\rangle \in H]$$

where $w : \alpha \times \alpha \to \alpha \in \Sigma_1(L_\alpha)$ with

$$K_{w(\gamma,\delta)} := \{\langle x,y\rangle : x \in K_\delta \wedge y \in K_\gamma \wedge y \in P_x\}$$

is a function of lemma 3.5.4v. Hence $\phi(\gamma, \delta) \in \Sigma_1(L_\alpha)$. As $M_s \in \Delta_1(L_\alpha)$ by $M_s \in \Delta_1(L_\alpha)$, so $V \in \Sigma_1(L_\alpha)$. Therefore $W \leq_{w\alpha e} S_e$.

Note $V_x \in \Sigma_1(L_\alpha)$. By the assumptions $M_s \in \Sigma_1(L_\alpha)$ and $X_s \in L_s$ it is true that $M_s \in \Sigma_1(L_\alpha)$ and $X_s \in \Sigma_1(L_\alpha)$. Thus

$$(X_s \cup (M_s \cap V_x)) \in \Sigma_1(L_\alpha).$$

Hence

$$S_e \leq_{w\alpha e} (X_s \cup (M_s \cap V_x)) \oplus U \leq_{\alpha e} U$$

by fact 3.7.11i and lemma 3.7.8 respectively. Hence $S_e \leq_{w\alpha e} U$ by proposition 3.7.12.

As U is megaregular, so $S_e \leq_{\alpha e} U$ by proposition 3.9.2. Hence $W \leq_{w\alpha e} U$ by proposition 3.7.12. Finally, $W \leq_{\alpha e} U$ by the megaregularity of U again. □

Lemma 4.3.4. Let $I \in L_\alpha$. Then exists an index $z < \alpha$ which is uniformly α-computable from I s.t.

$$V_z = \bigcap_{x \in I} V_x.$$

Proof. Define f as follows

$$f(I) = z \iff D_z = \bigcup_{x \in I} D_x \wedge E_z = \bigcup_{x \in I} E_x.$$

By lemma 3.5.4i the function f is total and α-computable. Also

$$\bigcap_{x \in I} V_x = \{y < \alpha : \bigcup_{x \in I} D_x \subseteq D_y \wedge \bigcup_{x \in I} E_x \subseteq E_y\} = V_{f(I)} = V_z$$

as required. □

Lemma 4.3.5. Let $D \subseteq A \subseteq \alpha$ and $E \subseteq B \subseteq \alpha$ satisfying $A, B \notin \Sigma_1(L_\alpha)$ and $D, E \in L_\alpha$. Define

$$Z := Z_{D,E} := \{x < \alpha : D \subseteq D_x \subseteq A \wedge E \subseteq E_x \subseteq B\}.$$

Then:

i) $Z \equiv_{\alpha e} A \oplus B$,

ii) $\overline{Z} \leq_{w\alpha e} \overline{A} \oplus \overline{B}$,

iii) $Z \notin \Sigma_1(L_\alpha)$,

iv) Z is unbounded if $A \oplus B$ is megaregular.

Proof. i) First note that for all α-finite sets K_γ, K_δ there is some $x < \alpha$ s.t. $D_x = K_\gamma, E_x = K_\delta$. Hence if we require that D_x (or E_x) is fixed to some α-finite set $K \in L_\alpha$, still the remaining sets E_x (or D_x) include all α-finite sets. Note $A \leq_{\alpha e} Z$ via

$$W := \{\langle \gamma, \delta \rangle : \exists x < \alpha[D \cup K_\gamma \subseteq D_x \wedge K_\delta = \{x\}]\} \in \Sigma_1(L_\alpha).$$

Similarly, $B \leq_{\alpha e} Z$. Consequently, $A \oplus B \leq_{\alpha e} Z$. Define $I_{D,A}$ and $I_{E,B}$:

$$I_{D,A} := \{x < \alpha : D \subseteq D_x \subseteq A\},$$
$$I_{E,B} := \{x < \alpha : E \subseteq E_x \subseteq B\}.$$

Note $I_{D,A} \leq_{\alpha e} A$ via

$$W_A := \{\langle \gamma, \delta \rangle : \forall x \in K_\gamma . D \subseteq D_x \wedge \bigcup_{x \in K_\gamma} D_x = K_\delta\} \in \Sigma_1(L_\alpha).$$

Similarly $I_{E,B} \leq_{\alpha e} B$. Note that $Z = I_{D,A} \cap I_{E,B}$. Thus

$$Z \leq_{\alpha e} I_{D,A} \oplus I_{E,B} \leq_{\alpha e} A \oplus B.$$

Therefore $A \oplus B \equiv_{\alpha e} Z$.

ii) Note that $\overline{I_{D,A}} \leq_{w\alpha e} \overline{A}$ via

$\Phi^{vw} := \{\langle x, \delta \rangle : \exists y < \alpha [y \notin D_x \wedge y \in D \wedge K_\delta = \emptyset \vee y \in D_x \wedge K_\delta = \{y\}\}.$

Similarly, $\overline{I_{E,B}} \leq_{w\alpha e} \overline{B}$. Hence

$$\overline{Z} = \overline{I_{D,A}} \cup \overline{I_{E,B}} \leq_{w\alpha e} \overline{A} \oplus \overline{B}$$

as required.

iii) If $Z \in \Sigma_1(L_\alpha)$, then $Z \in \Sigma_1(L_\alpha)$ and $A \in \Sigma_1(L_\alpha)$, $B \in \Sigma_1(L_\alpha)$ which contradicts the assumption. Hence $Z \notin \Sigma_1(L_\alpha)$.

iv) From ii) and megaregularity of $A \oplus B$, we have $\overline{Z} \leq_{\alpha e} \overline{A} \oplus \overline{B}$. Note $\overline{A \oplus B} = \overline{A} \oplus \overline{B}$. Combining this with i) it yields $Z \leq_\alpha A \oplus B$. Hence $Z \in \Delta_1(L_\alpha, A, B)$. If Z was bounded, then by lemma 3.4.13 using the megaregularity of $A \oplus B$, Z is α-finite. This contradicts iii). Hence Z has to be unbounded.

□

4.3.3 Conclusions and definability

Theorem 4.3.6. [8] Let A, B, U be arbitrary subsets of α. The statements i) - iv) are equivalent. Moreover if $V = L$ and α is an infinite regular cardinal, then all the statements i) - v) are equivalent.

i) $\mathcal{K}_U(A, B)$, i.e. $\exists W \leq_{\alpha e} U[(A \times B \subseteq W) \wedge (\overline{A} \times \overline{B} \subseteq \overline{W})]$,

ii) $\exists f(x, y) \in \Delta_1(L_\alpha). \forall X \subseteq \alpha \forall x, y \in \alpha$

$[\Phi_x(A \oplus X) \cap \Phi_y(B \oplus X) \subseteq \Phi_{f(x,y)}(X \oplus U) \subseteq \Phi_x(A \oplus X) \cup \Phi_y(B \oplus X)]$,

iii) $\exists f(x, y) \in \Delta_1(L_\alpha) \forall x, y < \alpha [\Phi_x(A) = \Phi_y(B) \implies \Phi_{f(x,y)}(U) = \Phi_x(A)]$,

[8] From theorem 2.6 in [25] for $\mathcal{D}_e$. Thanks to Iskander Kalimullin for explaining the classical case $\alpha = \omega$, part ii) implies v).

iv) $\forall V_1, V_2 \subseteq \alpha\big[V_1 \leq_{\alpha e} A \wedge V_2 \leq_{\alpha e} B \implies \exists W \leq_{\alpha e} U[V_1 \cap V_2 \subseteq W \subseteq V_1 \cup V_2]\big]$,

v) $\forall X \subseteq \alpha[\deg_{\alpha e}(X \oplus U) = \deg_{\alpha e}(A \oplus X \oplus U) \wedge \deg_{\alpha e}(B \oplus X \oplus U)]$.

Proof. • First we prove that the statements i) to iv) are equivalent. The implications ii) $\implies$ iii), ii) $\implies$ iv), iv) $\implies$ i) are trivial. It remains to prove the implications i) $\implies$ ii) and iii) $\implies$ i).

- i) $\implies$ ii):

  Assume
  $$\exists W \leq_{\alpha e} U . A \times B \subseteq W \wedge \overline{A} \times \overline{B} \subseteq \overline{W}$$
  and let $W = \Phi(U)$ for some α-enumeration operator Φ.

  Define f s.t. for any $X \subseteq \alpha$, $x, y \in \alpha$:
  $$\Phi_{f(x,y)}(X \oplus V) := \{z \in \alpha : \exists D, E \in L_\alpha$$
  $$[z \in \Phi_x(D \oplus X) \cap \Phi_y(E \oplus X) \wedge D \times E \subseteq \Phi(V)]\}.$$

  Then f is α-computable and satisfies the condition ii).

- iii) $\implies$ i): Suppose that A and B satisfy the condition iii) with f being computable. Define a computable function g s.t. for every $Y \subseteq \alpha$ and $y < \alpha$:
  $$\Phi_{g(y)}(Y) = \begin{cases} \alpha & \text{if } y \in Y, \\ \emptyset & \text{otherwise.} \end{cases}$$
  Then A, B are a U-Kalimullin pair with a witness
  $$W = \{(m, n) : \Phi_{f(g(m), g(n))}(U) \neq \emptyset\}.$$

- Hence we proved that the statements i) to iv) are equivalent. Next we prove that the statements i) to v) are equivalent under the following assumption. Assume that $V = L$ and α is an infinite regular cardinal. Hence every subset of α is megaregular. The statement v) $\implies$ i) is the contrapositive of theorem 4.3.1 which follows from the assumption that both U and $A \oplus B \oplus K(\emptyset)$ are megaregular. Now we prove that ii) $\implies$ v).

- ii) $\implies$ v): Given ii) we would like to show that
  $$X \oplus U \equiv_{\alpha e} (A \oplus X \oplus U) \wedge (B \oplus X \oplus U).$$
  where $\wedge$ is the meet induced by $\leq_{\alpha e}$. Given some order relation $\leq$, recall the definition of its induced meet $\wedge$ that $C = A \wedge B$ iff
  $$(A \geq C) \wedge (B \geq C) \wedge \forall D[(A \geq D) \wedge (B \geq D) \implies C \geq D].$$

Trivially, $A \oplus X \oplus U \geq_{\alpha e} X \oplus U$ and $B \oplus X \oplus U \geq_{\alpha e} X \oplus U$. Let $D \subseteq \alpha$ be arbitrary and assume $A \oplus X \oplus U \geq_{\alpha e} D$ and $B \oplus X \oplus U \geq_{\alpha e} D$. Thus $\exists x, y < \alpha$ s.t. $\Phi_x(A \oplus X \oplus U) = D = \Phi_y(B \oplus X \oplus U)$. Thus using ii), we have $D \subseteq \Phi_{f(x,y)}((X \oplus U) \oplus U) \subseteq D$ and so $D = \Phi_{f(x,y)}((X \oplus U) \oplus U)$ and $X \oplus U \geq_{w\alpha e} D$. As $X \oplus U$ is megaregular, so $X \oplus U \geq_{\alpha e} D$. Therefore $X \oplus U \equiv_{\alpha e} (A \oplus X \oplus U) \wedge (B \oplus X \oplus U)$ as required.

- Hence i) $\iff$ ii) $\iff$ iii) $\iff$ iv) $\implies$ v) for any admissible α. Therefore i) $\iff$ ii) $\iff$ iii) $\iff$ iv) $\iff$ v) if $V = L$ and α is an infinite regular cardinal.

□

The statement i) iff v) establishes the definability of a U-Kalimullin pair.

Proposition 4.3.7. [9] Let $B \subseteq \alpha$. The set of all A s.t. $\mathcal{K}(A, B)$ is closed downwards under α-enumeration reducibility as well as closed under join.

Proof. Suppose $\mathcal{K}(A_0, B)$ and $A_1 \leq_{\alpha e} A_0$. Hence

$$\exists W_0 \in \Sigma_1(L_\alpha). A_0 \times B \subseteq W_0 \wedge \overline{A}_0 \times \overline{B} \subseteq \overline{W}_0.$$

Let $V_1 := A_1 \times \alpha$, $V_2 := \alpha \times B$. As $A_1 \leq_{\alpha e} A_0$, so $V_1 \leq_{\alpha e} A_0 \wedge V_2 \leq_{\alpha e} B$. Hence by theorem 4.3.6 (i implies iv), $\exists W_1 \in \Sigma_1(L_\alpha)$ s.t. $V_1 \cap V_2 \subseteq W_1 \subseteq V_1 \cup V_2$. Therefore $V_1 \cap V_2 = A_1 \times B \subseteq W_1$. Also

$$W_1 \subseteq V_1 \cup V_2 \iff \overline{V}_1 \cap \overline{V}_2 \subseteq \overline{W}_1$$

and so

$$\overline{V}_1 \cap \overline{V}_2 = (\overline{A}_1 \times \alpha) \cap (\alpha \times \overline{B}) = \overline{A}_1 \times \overline{B} \subseteq \overline{W}_1.$$

Hence $\mathcal{K}(A_1, B)$.

Let $\mathcal{K}(A_0, B) \wedge \mathcal{K}(A_1, B)$. If $A_i = \emptyset$ for $i \in 2$ then $A_0 \oplus A_1 \equiv_{\alpha e} A_{1-i}$ and so $\mathcal{K}(A_0 \oplus A_1, B)$. Otherwise $\mathcal{K}(A_0 \oplus A_1, B)$ by lemma 4.2.9. □

Corollary 4.3.8. (Definability of an U-Kalimullin Pair[10])
Assume $V = L$ and let α be an infinite regular cardinal. Then

$$\forall a, b, u \in \mathcal{D}_{\alpha e}[\mathcal{K}_u(a, b) \iff \forall x \in \mathcal{D}_{\alpha e}.(a \vee x \vee u) \wedge (b \vee x \vee u) = x \vee u].$$

Proof. Note that since α is an infinite regular cardinal, so $A \oplus B \oplus K(U)$ is megaregular. Thus the statement above follows from $(i \iff v)$ in theorem 4.3.6 and from the $\mathcal{K}$-pair being a degree theoretic property by its invariance under the αe-reducibility by proposition 4.3.7. □

[9] Proposition 1.7 in [2] for $\alpha = \omega$.
[10] The case for $\alpha = \omega$ proved in [25].

4.4 Maximal Kalimullin pair and total degrees

In this section we conclude that every nontrivial total degree is a join of a maximal Kalimullin-pair if $V = L$ and α is an infinite regular cardinal (corollary 4.4.2).

Proposition 4.4.1. (Maximality of semicomputable megaregular $\mathcal{K}$-pairs)[11]
Let $A \subseteq \alpha$ be megaregular. If $\mathcal{K}(A, \overline{A}) \wedge A \not\in \Sigma_1(L_\alpha) \wedge A \not\in \Pi_1(L_\alpha)$, then $\mathcal{K}_{\max}(A, \overline{A})$.

Proof. Suppose $\mathcal{K}(A, \overline{A})$ and $\mathcal{K}(C, D)$, $A \leq_{\alpha e} C$, $\overline{A} \leq_{\alpha e} D$. By proposition 4.3.7 $\mathcal{K}(A, D)$. By corollary 4.2.8 and the megaregularity of A we have $D \leq_{\alpha e} \overline{A}$. Similarly, $\mathcal{K}(C, \overline{A})$ and thus $C \leq_{\alpha e} \overline{\overline{A}} = A$ by corollary 4.2.8 and the megaregularity of A. □

Corollary 4.4.2. Assume $V = L$ and let α be an infinite regular cardinal. Then every nontrivial total degree is a join of a maximal $\mathcal{K}$-pair, i.e.

$$\forall a \in \mathcal{TOT}_{\alpha e} - \{0\} \exists b, c \in \mathcal{D}_{\alpha e}[(a = b \vee c) \wedge \mathcal{K}_{\max}(b, c)].$$

Proof. Since α is an infinite regular cardinal, thus the set of the (maximal) Kalimullin pairs is definable by corollary 4.3.8.

Suppose $a \in \mathcal{TOT}_{\alpha e} - \{0\}$. Note that a is a megaregular degree (at least one or equivalently every set in a is megaregular). Then by theorem 4.1.15, there is $A \subseteq \alpha$ s.t. A is α-semicomputable, $A \not\in \Sigma_1(L_\alpha)$, $\overline{A} \not\in \Sigma_1(L_\alpha)$ and $A \oplus \overline{A} \in a$ by the totality of a. As A is α-semicomputable, so $\mathcal{K}(A, \overline{A})$ by proposition 4.2.3. $\mathcal{K}(A, \overline{A})$ is nontrivial since $A \not\in \Sigma_1(L_\alpha)$ and $\overline{A} \not\in \Sigma_1(L_\alpha)$. Thus by proposition 4.4.1 and the megaregularity of A we have $\mathcal{K}_{\max}(A, \overline{A})$. □

By inspecting whether a degree which is not quasiregular could be a join of a maximal Kalimullin pair, one may establish the following:

Proposition 4.4.3. If $\deg_\alpha(B)$ is not a quasiregular degree, then there is C s.t. $0 <_\alpha C <_\alpha B$ and $\mathcal{K}_{\max}(C, \overline{C})$.

Proof. Since $\deg_\alpha(B)$ is not a quasiregular degree, then D is not quasiregular for any $D \equiv_\alpha B$. So B is not quasiregular.

Let $\beta < \alpha$ be the least ordinal s.t. $B \cap \beta \not\in L_\alpha$. Define $A := B \cap \beta$. Then $A \subset B$ by B not being quasiregular. By the minimality of β, the set A is quasiregular. A is bounded, but not α-finite, hence A cannot be α-computable. Thus $A >_\alpha \emptyset$. By theorem 4.1.15 there is α-semicomputable set C s.t. $A \equiv_\alpha C$, $C \not\in \Sigma_1(L_\alpha)$ and $C \not\in \Pi_1(L_\alpha)$. As C is α-semicomputable, so $\mathcal{K}(C, \overline{C})$. By proposition 4.4.1 we have that $\mathcal{K}_{\max}(C, \overline{C})$. □

[11]Generalized from Maximal $\mathcal{K}$-pairs in [2] for $\alpha = \omega$.

Chapter 5

α-rational numbers Q_α

In classical Computability Theory the definability of the total enumeration degrees was established by constructing a cut in the rational numbers $\mathbb{Q}$. We generalize this result in chapter 6, but for that we first need an analogue of rational numbers in α-Computability Theory.

Let α denote an admissible ordinal and let β denote a limit ordinal. A rational number can be represented by a bounded binary string. Thus the analogue, β-rational numbers Q_β, may be $\{\triangleleft, \triangleright\}^{<\beta}$, binary strings of order type less than β.

In this chapter we define β-rational numbers Q_β. We investigate Q_β briefly and Q_α with a little more detail covering the areas of representability, computability, dense total orders and analysis.

5.1 Basic concepts

Using strings we define β-rational numbers Q_β with its ordering and show how to represent them in L_β. We define a β-real number interval which is used later in section 5.4 to analyse Q_β further. We show that the ordering of Q_α is α-computable and an order type of an α-rational is uniformly α-computable.

5.1.1 Strings

We define strings of transfinite length and explain the notation involved.

Definition 5.1.1. (Language signature, character and string[1])
Let $\mathcal{L}$ be a set that contains the element λ.

- The set $\mathcal{L}$ is called the *language signature*.

[1]Strings are a common concept in the field of Computer Science. α-strings were introduced in this thesis.

- An element in $\mathcal{L}$ is called a *character*.
- The element λ is called the *empty character*.
- For an ordinal $\alpha \in \texttt{Ord}$ an *α-string* σ is a total function $\sigma : \alpha \to \mathcal{L}$ satisfying the following property:
$$\forall \gamma, \delta < \alpha[\gamma < \delta \wedge \sigma(\gamma) = \lambda \implies \sigma(\delta) = \lambda].$$
- If $\sigma(0) = \lambda$, then σ is called the *empty string* and denoted also by λ, i.e. $\sigma = \lambda$.

If the ordinal α is clear from the context, we just use the term *string* for an α-string.

Note 5.1.2. (Notation for strings)
We avoid complex definitions of the concepts and notation below and instead provide simple clarifying examples of α-strings for some ordinal α.

- abc is the *compact string notation* for the string
$$\{(0, a), (1, b), (2, c)\} \cup \{(\gamma, \lambda) \in \alpha \times \mathcal{L} : 2 < \gamma\}.$$
E.g. $abc\lambda = abc$. Here we make a convention to grow a string from the left to the right.
- $\sigma(\gamma)$ is the γ^{th} character of the string σ. E.g. $abc(0) = a$, $abc(1) = b$, $abc(2) = c$, $abc(3) = \lambda$.
- $\sigma[\gamma, \delta], \sigma[\gamma, \delta), \sigma(\gamma, \delta], \sigma(\gamma, \delta)$ denote *string intervals*. E.g. $abcd[1, 3] = bcd$, $abcd[1, 3) = bc$, $abcd[1, 5] = bcd\lambda\lambda = bcd = abcd[1, 3]$.
- $\sigma \cdot \tau$ denotes the *concatenation* of the strings σ and τ. E.g. if $\sigma = 000$ and $\tau = abc$, then $\sigma \cdot \tau = 000abc$.
- x^γ is the concatenation of γ many characters or strings x. E.g. $(bc)^3 = bcbcbc$, $0^4 \cdot a^2 = 0000aa$.
- $\sigma \upharpoonright \delta$ denotes the restriction of the string σ to the characters at the position less than δ, i.e.
$\forall c \in \mathcal{L} \forall \gamma < \alpha\big[c = (\sigma \upharpoonright \delta)(\gamma) \iff \gamma < \delta \wedge c = \sigma(\gamma) \vee \delta \leq \gamma \wedge c = \lambda\big].$
E.g. $(abcde) \upharpoonright 3 = abc$.
- The *order type* of the string σ denoted as $\mathrm{ot}(\sigma)$ is defined as
$$\mathrm{ot}(\sigma) := \min(\{\gamma \leq \alpha : \sigma(\gamma) = \lambda\}).$$
E.g. $\mathrm{ot}(\lambda) = 0, \mathrm{ot}(ab) = 2, \mathrm{ot}(abc \cdot d^\omega \cdot ad) = 3 + \omega + 2 = \omega + 2$.

- Let $S \subseteq \mathcal{L} - \{\lambda\}$ and $\beta < \texttt{Ord}$. Then $S^{<\beta}$ are the strings of an order type less than β that consist of the characters in S, i.e.

$$S^{<\beta} := \{\sigma : \beta \to S \cup \{\lambda\} | \sigma \text{ is a } \beta\text{-string}\}.$$

  E.g. $\{a, b\}^{<3} = \{\lambda, a, aa, ab, b, ba, bb\}$.

5.1.2 β-rationals Q_β with ordering

Definition 5.1.3. (β-rational numbers[2])

- Let β be a limit ordinal. Then *β-rational numbers* Q_β is a set of binary β-strings, i.e. $Q_\beta := \{\triangleleft, \triangleright\}^{<\beta}$.

- The ordering on Q_β is the total lexicographical ordering induced by the ordering on the substrings $\triangleleft, \lambda, \triangleright$ where $\triangleleft < \lambda < \triangleright$: for $\sigma, \tau \in Q_\beta$ define $<$ as following:

$$\sigma < \tau \iff \exists \gamma < \beta [\sigma \upharpoonright \gamma = \tau \upharpoonright \gamma \wedge (\sigma(\gamma), \tau(\gamma)) \in \{(\triangleleft, \lambda), (\triangleleft, \triangleright), (\lambda, \triangleright)\}].$$

Remark 5.1.4. (Order type of β-rational)
Recall note 5.1.2. As every β-rational is a β-string, the order type of the β-rational σ is the order type of the β-string σ. Specifically,

$$\text{ot}(\sigma) := \min\{\gamma < \beta : \sigma(\gamma) = \lambda\}.$$

Proposition 5.1.5. [3] Let β be a limit ordinal and let $\delta < \beta$ be 0 or a limit ordinal. Let $\pi \in Q_\beta$ be a β-string of order type δ. Let S be a set of β-rationals of order type in the interval $[\delta, \delta + \omega)$ for which π is a substring. In notation,

$$S := \{\rho \in Q_\beta : \pi \subseteq \rho \wedge \text{ot}(\rho) \in [\delta, \delta + \omega)\}.$$

Then

$$(S, <) \cong (Q_\omega, <).$$

Proof. Note

$$S = \{\pi \cdot \sigma \in Q_\beta : \text{ot}(\pi \cdot \sigma) \in [\delta, \delta + \omega)\} = \{\pi \cdot \sigma : \sigma \in Q_\omega\}.$$

□

5.1.3 β-real number unit interval I_β

Definition 5.1.6. (β-real number unit interval[4])
Let β be a limit ordinal. Then *β-real number unit interval* I_β is a set of binary strings of an order type less than or equal to β, i.e. $I_\beta := \{\triangleleft, \triangleright\}^{\leq\beta}$.

[2] Introduced in this thesis.
[3] Introduced in this thesis.
[4] Introduced in this thesis.

One could establish that the β-real numbers in the unit interval are a completion of the β-rational numbers in a natural sense.

Remark 5.1.7. (Greatest and the least element)
The β-real number unit interval I_β has the least element $\triangleleft^\beta$ and the greatest element $\triangleright^\beta$. Hence one should not think of I_β as being analogous to the generalization of all of $\mathbb{R}$, but only the interval $[0, 1] \subseteq \mathbb{R}$.

Similarly, the β-rational numbers Q_β are analogous to the generalization of the open unit interval $(0, 1) \subseteq \mathbb{Q}$.

However, the rational interval $((0, 1), <)$ and $(\mathbb{Q}, <)$ are models of the same theories and thus are indistinguishable. Hence, for the sake of the simplicity, we call Q_β as β-rational numbers.

One way to define β-real numbers R_β is as the set I_β without the greatest and the least element, i.e. $R_\beta \coloneqq I_\beta - \{\triangleleft^\beta, \triangleright^\beta\}$.

5.1.4 Representation of Q_α in L_α

Definition 5.1.8. (Binary representation of a subset by a string[5])
A *binary representation* of a subset $A \subseteq \beta$ for any ordinal β is a string σ of order type β consisting of symbols $0, 1$ satisfying the conditions:

- $\forall \gamma < \beta[\sigma(\gamma) = 0 \iff \gamma \notin A]$,
- $\forall \gamma < \beta[\sigma(\gamma) = 1 \iff \gamma \in A]$.

Example 5.1.9. The set $A = \{1, 2, 4\}$ is represented by a string $011010000\ldots$

Note 5.1.10. (Representation of a β-rational)
We *represent* every string $\sigma \in Q_\beta$ as the set $S \subseteq \beta$ defined as follows:

- $\triangleleft$ character in σ is represented by 01 string in S.
- $\triangleright$ character in σ is represented by 11 string in S.
- λ character (the termination of the string σ) is represented by 00 symbol in S. We conceive 00 substring in the binary representation of an β-rational as a termination marker since all the characters afterwards will be 0s in the representation.
- the set S is defined by its binary representation which is the concatenation of the representations of the characters of σ.

[5]Binary representations of various objects are very common in the field of Computer Science.

- If $\sigma \in Q_\beta$ is represented by $S \subseteq \beta$, then we may identify σ with S and thus have:

$$\sigma \in L_\beta \iff S \in L_\beta.$$

Remark 5.1.11. Every string $\sigma \in Q_\alpha$ is bounded. Therefore if we assume $V = L$ and α is an infinite regular cardinal, then $Q_\alpha \subseteq L_\alpha$.

5.2 Computability on Q_α

Proposition 5.2.1. [6] There exists an α-computable injection $i : Q_\alpha \cap L_\alpha \rightarrowtail \alpha$.

Proof. As $Q_\alpha \cap L_\alpha$ is an α-computable set and there exists an α-computable injection $L_\alpha \rightarrowtail \alpha$, so its restriction $Q_\alpha \cap L_\alpha \rightarrowtail \alpha$ has to be also an injection and α-computable. □

Proposition 5.2.2. [7] The ordering $<$ in $(Q_\alpha \cap L_\alpha, <)$ is α-computable.

Proof. The equality on Q_α is clearly α-computable. From definition 5.1.3 $<$ is $\Sigma_1(L_\alpha)$, by symmetry also $\Pi_1(L_\alpha)$ using the totality of $<$. Therefore $<$ is $\Delta_1(L_\alpha)$, thus α-computable. □

Proposition 5.2.3. (Computability of an order type[8])
The order type of a β rational ρ is uniformly $\Pi_0(L_\beta, \rho)$-definable.

Proof. Note that

$$\mathrm{ot}(\rho) = \delta \iff \forall \gamma \le \delta[\rho(\gamma) = \lhd \vee \rho(\gamma) = \rhd] \wedge \rho(\delta) = \lambda$$

which is Π_0 over L_β with ρ as a parameter. □

5.3 $(Q_\beta, <)$ as order

The classic result states that any infinite countable unbounded dense total orders are isomorphic, see theorem 5.3.1 below.

Theorem 5.3.1. [9] Suppose that $(A, <)$ and $(B, <)$ are both infinite countable unbounded dense total orders. Then $(A, <) \cong (B, <)$, i.e. $(A, <)$ is order isomorphic to $(B, <)$. □

We use theorem 5.3.1 to conclude corollary 5.3.3 that $(Q_\omega, <) \cong (\mathbb{Q}, <)$. This result provides us with more intuition about the β-rational numbers Q_β by inspecting the case when $\beta = \omega$.

[6] Introduced in this thesis.
[7] Introduced in this thesis.
[8] Introduced in this thesis.
[9] Follows from theorem 2.4.1. in [18] proved by Back and Forth method.

5.3.1 $(Q_\beta, <)$ as an unbounded dense total order

Proposition 5.3.2. [10] $(Q_\beta, <)$ is an unbounded dense total order.

Proof. Let σ, τ be β-rational numbers. If σ and τ are not equal, then let $p < \beta$ be the least position on which σ and τ disagree, i.e.

$$p := \min\{\gamma < \beta : \sigma(\gamma) \neq \tau(\gamma)\}.$$

If $\sigma(p) = \triangleleft$ or both $\sigma(p) = \lambda$ and $\tau(p) = \triangleright$, then $\sigma <_{Q_\beta} \tau$, otherwise $\tau <_{Q_\beta} \sigma$. Hence $(Q_\beta, <)$ is a total order.

Assume $\sigma < \tau$. Then $(\sigma(p), \tau(p)) \in \{(\triangleleft, \lambda), (\triangleleft, \triangleright), (\lambda, \triangleright)\}$. If $\sigma(p) = \triangleleft$, then define $\rho := \sigma \cdot \triangleright$. If $\sigma(p) = \lambda$, then define $\rho := \tau \cdot \triangleleft$. As $\sigma, \tau \in Q_\beta$, so the order type of σ and τ is bounded in β. Thus also the order type of ρ is bounded in β and so $\rho \in Q_\beta$. Note that $\sigma < \rho < \tau$ and so $(Q_\beta, <)$ is a dense order.

If $\sigma \in Q_\beta$, then $\sigma \cdot \triangleleft < \sigma < \sigma \cdot \triangleright$ and $\sigma \cdot \triangleleft, \sigma \cdot \triangleright \in Q_\beta$. Hence $(Q_\beta, <)$ is unbounded.

Therefore $(Q_\beta, <)$ is an unbounded dense total order. □

Corollary 5.3.3. [11] $(Q_\omega, <) \cong (\mathbb{Q}, <)$

Proof. The rational numbers $\mathbb{Q}$ are a countable infinite dense unbounded total order. Similarly, the ω-rationals Q_ω are an unbounded dense total order by proposition 5.3.2. Clearly, $\#Q_\omega = \aleph_0$. Therefore $(Q_\omega, <) \cong (\mathbb{Q}, <)$ by theorem 5.3.1 as required. □

5.4 Analysis on Q_β

We study density and weak forms of completeness of the β-rational numbers Q_β.

Proposition 5.4.1. (Infimum existence[12])
Every subset of Q_β has an infimum in I_β.

Proof. Let $S \subseteq Q_\beta$ and bounded below by $\tau \in Q_\beta$, i.e. $\forall \sigma \in S. \tau \leq \sigma$. We construct the infimum $\rho \in I_\beta$ of S in β stages. Start with the stage $\gamma = 0$. At the stage $\gamma < \beta$ assume that $\rho[0, \gamma)$ has been defined and define $\rho(\gamma)$ according to the following rules:

- If there is $\sigma \in S$ s.t. $\sigma[0, \gamma) = \rho[0, \gamma)$ and $\sigma(\gamma) = \triangleleft$, then set $\rho(\gamma) = \triangleleft$ and go to the stage $\gamma + 1$.

[10] Introduced in this thesis.
[11] Introduced in this thesis.
[12] Introduced in this thesis.

- If there is $\sigma \in S$ s.t. $\sigma[0,\gamma) = \rho[0,\gamma)$ and $\sigma(\gamma) = \lambda$, then set $\rho(\gamma) = \lambda$ and go to the stage $\gamma + 1$. Note that σ is the minimum of the set S and so in the end $\rho := \sigma$.

- Otherwise set $\rho(\gamma) = \triangleright$ and go to the stage $\gamma + 1$.

From the construction it is clear that ρ is equal or smaller than any other element of S. Also ρ is the maximal possible element. Hence ρ is the infimum of S as required. □

Proposition 5.4.2. [13] Every set bounded in order type $\delta < \beta$ is bounded by some β-rationals of the order type δ, i.e.

$$\forall S \subseteq Q_\beta \big[\exists \delta < \beta \forall \sigma \in S.\mathrm{ot}(\sigma) \leq \delta \implies \exists \tau_0, \tau_1 \in Q_\beta$$
$$[\mathrm{ot}(\tau_0) = \delta \wedge \mathrm{ot}(\tau_1) = \delta \wedge \forall \sigma \in S.\tau_0 \leq \sigma \leq \tau_1]\big].$$

Proof. Take $\tau_0 = \triangleleft^\delta$ and $\tau_1 = \triangleright^\delta$. Then $\forall \sigma \in S.\tau_0 \leq \sigma \leq \tau_1$ as required. □

Definition 5.4.3. (Set parameter infimum/supremum[14])
Let $(S, \leq)$ be an ordered set and let $A, B \subseteq S$.

- The A-*infimum* of the set B is defined as
$$\inf_A(B) := \max\{a \in A : \forall b \in B.a \leq b\}.$$

- The A-*supremum* of the set B is defined as
$$\sup_A(B) := \min\{a \in A : \forall b \in B.b \leq a\}.$$

Remark 5.4.4. In other words, A-infimum of the set B is the greatest lower bound of B which is in A. Similarly, A-supremum of the set B is the least upper bound of B which is in A.

Proposition 5.4.5. (Order type δ infimum existence and computability[15])
Let $S \subseteq Q_\beta$ be bounded in order type $\delta < \beta$. Define

$$Q := \{\rho \in Q_\beta : \mathrm{ot}(\rho) = \delta\}.$$

Then the Q-infimum of S exists and is uniformly $\Sigma_0(L_\alpha, S, \delta)$-definable.

Proof. Let υ be the infimum of the set S as constructed in proposition 5.4.1. Let γ be the order type of υ, i.e. $\gamma := \mathrm{ot}(\upsilon)$. Set $\rho_1 := \upsilon\overline{\triangleright}$, i.e. the string υ padded with the characters $\triangleright$ so that $\mathrm{ot}(\rho_1) = \delta$. Set $\rho_2 := \upsilon\triangleleft\overline{\triangleright}$ so that $\mathrm{ot}(\rho_2) = \delta$. Note that if $\gamma = \delta$, then $\rho_1 = \rho_2 = \upsilon$.

[13] Introduced in this thesis.
[14] Introduced in this thesis.
[15] Introduced in this thesis.

If $\forall \sigma \in S.\rho_1 \leq \sigma$, then set $\rho := \rho_1$, otherwise set $\rho := \rho_2$. Now by definition $\forall \sigma \in S.\rho \leq \sigma$. Also ρ is the maximum such β-rational number of the order type δ. As $\mathrm{ot}(\rho) = \delta$, so $\rho \in Q \subseteq Q_\beta$.

Hence ρ is the Q-infimum of the set S. Therefore the Q-infimum of the set S exists.

Next using proposition 5.2.3 we observe that $\inf_Q(S)$ is uniformly definable with bounded quantifiers from the parameters S and δ only since

$$\rho = \inf_Q(S) \iff \forall \sigma \in S.\rho \leq \sigma \wedge \forall \tau \leq \triangleright^\delta$$
$$[\mathrm{ot}(\tau) = \delta \wedge \forall \sigma \in S.\tau \leq \sigma \implies \tau \leq \sigma].$$

Therefore $\inf_Q(S)$ is uniformly $\Sigma_0(L_\alpha, S, \delta)$-definable as required. □

Proposition 5.4.6. [16] Let $S \subseteq Q_\alpha$ be bounded in order type $\delta < \alpha$ and let S be α-finite. Define

$$\hat{Q} := \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \delta\}.$$

Then the $\hat{Q}$-infimum of S denoted as $\inf_{\hat{Q}}(S)$ exists and is uniformly α-computable from S and δ.

Proof. Define

$$Q := \{\rho \in Q_\alpha : \mathrm{ot}(\rho) = \delta\}$$

By proposition 5.4.5 the α-rational $\rho := \inf_Q(S)$ exists and is uniformly $\Sigma_0(L_\alpha, S)$-definable. As S is α-finite, ρ is $\Sigma_0(L_\alpha)$-definable and hence α-computable. As $\mathrm{ot}(\rho) = \delta$, so ρ is also bounded. Hence ρ is α-finite by the admissibility of α. Thus $\rho \in \hat{Q}$. Given that $\rho \in \hat{Q}$ and $\hat{Q} \subseteq Q$, we have $\rho = \inf_Q(S) = \inf_{\hat{Q}}(S)$. Therefore $\inf_{\hat{Q}}(S)$ exists.

Using $\inf_Q(S) = \inf_{\hat{Q}}(S)$, the uniform α-computability of $\inf_{\hat{Q}}(S)$ from S and δ follows from the uniformly α-computability of $\inf_Q(S)$ from the parameters S and δ by proposition 5.4.5. □

Proposition 5.4.7. (Setwise density of Q_β for sets bounded in order type[17])
Let $A, B \subseteq Q_\beta$ and $\gamma < \beta$. Define

$$Q := \{\rho \in Q_\alpha : \mathrm{ot}(\rho) = \gamma\}.$$

Assume that

$$\forall \sigma \in A \forall \tau \in B.\sigma < \tau \wedge \mathrm{ot}[A \sqcup B] \subseteq \gamma.$$

Then

i) there is uniformly $\Sigma_0(L_\alpha, A)$-definable $\rho \in Q_\beta$ satisfying

$$\mathrm{ot}(\rho) = \gamma \wedge \forall \sigma \in A \forall \tau \in B.\sigma < \rho < \tau,$$

[16] Introduced in this thesis.
[17] Introduced in this thesis.

e.g. $\rho = \sup_Q(A)$,

ii) there is uniformly $\Sigma_0(L_\alpha, B)$-definable $\rho \in Q_\beta$ satisfying

$$\mathrm{ot}(\rho) = \gamma \wedge \forall \sigma \in A \forall \tau \in B.\sigma < \rho < \tau,$$

e.g. $\rho = \inf_Q(B)$.

Proof. Let $\rho := \inf_Q(B)$. Clearly, $\mathrm{ot}(\rho) = \gamma$. As

$$\forall \sigma \in A \forall \tau \in B.\sigma < \tau \wedge \mathrm{ot}[A \sqcup B] \subseteq \gamma,$$

so

$$\forall \sigma \in A \forall \tau \in B.\sigma < \rho < \tau].$$

Also ρ is uniformly $\Sigma_0(L_\alpha, B)$-definable by proposition 5.4.5. Therefore the statement ii holds as required.

The statement i is true by the dual proof of ii. □

5.5 Further directions

We investigated α-rational numbers Q_α as an α-computability theory analogue of the rational numbers $\mathbb{Q}$ in ω-computability theory. We proved the basic statements about Q_α sufficient for our purpose to construct an α-semicomputable cut. However, there are many further directions out of the scope of this thesis which may yield fruitful investigations:

- interactions between different notions of continuity on Q_α, e.g. ϵ-δ-continuity, α-sequential continuity, limit continuity, uniform continuity,
- relations between Q_α, hyperreals and surreals,
- generalizations of the real numbers, e.g. the completion of Q_α under α-Cauchy sequences, Dedekind cuts, subsets of α,
- continuity of a function $f : Q_\alpha \to Q_\alpha$ and the α-computability of its representation,
- α-metric spaces, α-Polish spaces and Higher Descriptive Set Theory.

Chapter 6

Semicomputable cut in Q_α

In this chapter we prove Semicomputable Cut Existence Theorem 6.0.1 below. This theorem is used in section 6.3 to obtain results about the definability of the total α-enumeration degrees $\mathcal{TOT}_{\alpha e}$ in the enumeration degrees $\mathcal{D}_{\alpha e}$.

Theorem 6.0.1. (Semicomputable Cut Existence Theorem)
Let A and B form a nontrivial $\mathcal{K}$-pair. Then there exists an α-semicomputable cut $C \subseteq Q_\alpha \cap L_\alpha$ s.t. $A \leq_{w\alpha e} C$ and $B \leq_{w\alpha e} \overline{C}$.

The proof of theorem 6.0.1 generalizes the proof of Theorem 2.3 in [2] for $\alpha = \omega$ to an admissible ordinal α. The main ingredient of the proof is the labelling algorithm provided in section 6.1 which involves a priority argument. In this thesis we mainly focus on the new parts arising from the generalization.

This chapter is organized as follows. Section 6.1 explains the labelling algorithm. The labelling algorithm outline is given in section 6.1.1 with some concepts presented intuitively. The formal definitions of the intuitive concepts are given in section 6.1.2. The rest of the section provides formal framework and steps of the labelling algorithm with some of its properties. Section 6.2 defines the cut C using the labelling algorithm and proves theorem 6.0.1. Section 6.3 concludes that the total α-enumeration degrees are definable in the α-enumeration degrees.

6.1 Labelling algorithm

In this section a labelling algorithm is used to construct an α-computable sequence of the labelling functions $q_s : \alpha_A \sqcup \alpha_B \to Q_\alpha$ for $s < \alpha$ to prove lemma 6.1.1 below. Later in section 6.2.4 this sequence of the labelling functions is used to define a semicomputable cut C in Q_α in order to prove Semicomputable Cut Existence Theorem 6.0.1.

Lemma 6.1.1. Let A and B form a nontrivial $\mathcal{K}$-pair, then there exist an α-computable sequence $q = \{q_s\}_{s<\alpha}$ of the α-computable partial labelling functions and an α-semicomputable cut $C \subseteq Q_\alpha \cap L_\alpha$ s.t.

$$A = A_C \coloneqq \{a \in \alpha_A : \exists s < \alpha . q_s(a) \in C\}$$
$$B = B_C \coloneqq \{b \in \alpha_B : \exists s < \alpha . q_s(b) \in \overline{C}\}.$$

6.1.1 Algorithm outline

Constructing the cut $C \subseteq Q_\alpha \cap L_\alpha$ from the witness W of $\mathcal{K}(A, B)$

We use the witness W to the $\mathcal{K}$-pair $\mathcal{K}(A, B)$ to construct the α-computable sequence q trying to satisfy two conditions for any $s < \alpha$:

$$(a, b) \in A \times B \implies q_s(a) < q_s(b), \tag{6.1}$$
$$(a, b) \in \overline{A} \times \overline{B} \implies q_s(b) < q_s(a). \tag{6.2}$$

Therefore at the beginning the elements from α_A are on the right of an α-rational line Q_α, the elements from α_B are on the left of the α-rational line Q_α. Hence if $\{a, b\} \subseteq \mathrm{dom}(q_s)$ and (a, b) has not entered the α-enumeration W at the stage s yet (i.e. $(a, b) \notin W_s$), then we have $q_s(b) < q_s(a)$. When (a, b) enters W, then we try to move a as much to the left and b as much to the right as possible so that $q_t(a) < q_t(b)$ at some stage $t > s$. This is not possible if there are some labels between b and a through which b and a cannot move as the conditions 6.1 and 6.2 would be violated. For example if $(a', b) \notin W_s$, $(a, b') \notin W_s$ and

$$q_s(b) < q_s(a') < q_s(b') < q_s(a),$$

then the label b cannot be moved to the right of the label a', similarly, the label a cannot be moved to the left of the label b'.

If it is not possible to have the label a to the left of the label b, then for such labels a and b we introduce a notion of a *dead zone* interval $[q_t(b), q_t(a)]$, see definition 6.1.13. We allow other labels to move out of the dead zone, but not inside it (unless the labels are of a higher priority).

To get rid of the dead zone intervals as much as possible, we have to introduce a *priority ordering* on the pairs $(a, b) \in \alpha_A \times \alpha_B$. If (a_1, b_1) is of a higher priority than (a_2, b_2), then it is more important to remove the dead zone $[q_t(b_1), q_t(a_1)]$ before removing the dead zone $[q_t(b_2), q_t(a_2)]$.

If $(a_1, b) \in W$, $(a_2, b) \notin W$ and $a_2 \in A$, then $a_1 \in A$. Therefore the main idea in defining the cut C to meet the conditions $A = A_C$ and $B = B_C$ is to place the labels $a_1 \in \alpha_A$ to the left of the labels $a_2 \in \alpha_A$ if a_1 label is witnessed by pairs in W of some fixed higher priority than the pairs in W witnessing a_2. The case of the labels $b \in \alpha_B$ is symmetric, where labels start on the left and with new witnessing pairs in W move towards right.

Moving the label a to the left of b for $(a, b) \in W$

Recall from the previous subsubsection that the goal of the labelling algorithm is to move labels in such a way that the conditions 6.1 and 6.2 are satisfied. Now we outline in a little more detail some related scenarios and steps of the labelling algorithm to resolve them.

Permanent dead zone Let $W = \{(a_0, b_0), (a_1, b_1), \ldots\}$ be a witness to $\mathcal{K}(A, B)$. Let q_0 be the initial labelling function, we have:

$$\ldots < q_0(b_2) < q_0(b_1) < q_0(b_0) < q_0(a_0) < q_0(a_1) < q_0(a_2) < \ldots$$

First (a_0, b_0) is enumerated and the labelling algorithm places a_0 to the left of b_0, i.e. $q_1(a_0) < q_1(b_0)$ using $W_1 := \{(a_0, b_0)\}$. Next (a_1, b_1) is enumerated and the labelling algorithm would like to place a_1 to the left of b_1. However, as $(a_0, b_1) \notin W_2$, $(a_1, b_0) \notin W_2$ where $W_2 := \{(a_0, b_0), (a_1, b_1)\}$, so we will instead have the labelling function q_2 s.t.

$$q_2(b_1) < q_2(a_0) < q_2(b_0) < q_2(a_1).$$

The label a_0 cannot be moved to the left of the label b_1 as $(a_0, b_1) \notin W_2$ and we need to satisfy the condition 6.2. Here the label b_1 is an *obstacle* for the label a_0. As a_1 and b_1 cannot satisfy the condition 6.1, so the interval $[q_2(b_1), q_2(a_1)]$ is a *dead zone*. If $(a_0, b_1) \notin W$, $(a_1, b_0) \notin W$, then a_1 will never be to the left of b_1 and such an interval $[q_2(b_1), q_2(a_1)]$ is called a *permanent dead zone*.

Strategy, its run and termination For each pair of the labels $(a, b) \in \alpha_A \times \alpha_B$ there is part of the labelling algorithm called *strategy* for the pair (a, b) which tries to move the labels a and b in such a way that the conditions 6.1 and 6.2 are satisfied.

When the strategy for the pair of labels (a, b) of the priority $p < \alpha$ runs at the stage $s < \alpha$, we call this a *strategy run* (s, p).

In lemma 6.2.6 we prove that such a strategy does not move labels forever, but eventually stops acting. Let $(a, b) \in W$. The termination of each strategy for some pair (a, b) is ensured by the following. If ever $q_s(a) < q_s(b)$ for some $s < \alpha$, then the condition 6.1 is satisfied and the strategy will never act again. Otherwise the interval $[q_s(b), q_s(a)]$ is a permanent dead zone. The labels inside the dead zone have to be of a higher priority (otherwise they would have to be moved out) and so using the induction hypothesis that the strategies of the higher priority have stopped acting we conclude that also the strategy for the pair (a, b) has to stop acting as it has nothing more to do.

Label clearing The goal of the strategy for the pair $(a, b) \in W$ is to ensure 6.1, i.e. $q_s(a) < q_s(b)$. If this is not possible, then the interval $[q_s(b), q_s(a)]$ is a dead zone. In order to satisfy 6.1 it is important to shrink the dead zone as much as possible. This may not be always possible with moving the labels a and b alone. Hence after the labels a and b have been moved as much as possible, other labels are attempted to be cleared out of the dead zone. This is called *label clearing* and we demonstrate its importance in the following scenario.

Consider we have the following ordering of the labels:

$$q_s(b_2) < q_s(a_3) < q_s(b_1) < q_s(a_0) < q_s(a_2) < q_s(b_0) < q_s(a_1),$$

the following memberships $(a_0, b_1) \notin W$, $(a_3, b_2) \notin W$, $(a_2, b_2) \notin W$, $(a_1, b_0) \notin W$, $(a_2, b_1) \in W$, $(a_1, b_1) \in W$, with the following priority ordering:

$$\mathtt{p}(a_1, b_1) > \mathtt{p}(a_0, b_2) > \mathtt{p}(a_2, b_1).$$

Then the strategy for the pair (a_1, b_1) would like to move the label a_1 to left of the label b_1, but it cannot do so as $(a_0, b_1) \notin W$ and $(a_1, b_0) \notin W$. However, we have $(a_2, b_1) \in W$ and so a_2 can be put to the left of b_1 and so the dead zone for the pair (a_1, b_1) can be shrunk by one label as desired. This is the label clearing of the strategy for the pair (a_1, b_1).

Note that the label clearing of the label a_2 is necessary as it cannot be done by the strategy (a_2, b_1) since $[q_s(b_2), q_s(a_0)]$ is a dead zone of the higher priority than the dead zone $[q_s(b_1), q_s(a_2)]$.

6.1.2 Formalizing the concepts

In subsection 6.1.1 we outlined the labelling algorithm and introduced intuitively some concepts as a label, labelling function, obstacle, dead zone, permanent dead zone, priority. In this subsection we formalize these and some more concepts including adjacent labels, left of an interval, connectedness since their precise meaning in the proof of Semicomputable Cut Existence Theorem 6.0.1 is crucial.

Labelling function, label, obstacle

Definition 6.1.2. (Labelling function and label)

- A *labelling function* is a total injection $\hat{q} : \alpha_A \sqcup \alpha_B \rightarrowtail Q_\alpha$ from two disjoint copies of α to α-rationals.
- If $c \in \alpha_A \sqcup \alpha_B$ and $\hat{q}(c) = \rho \in Q_\alpha$, then c is the *label* for ρ in $\hat{q}$.

Definition 6.1.3. (Labels inside/outside of the cut in Q_α)
Given a set $C \subseteq Q_\alpha$ and an α-sequence of partial labellings $q_s : \alpha_A \sqcup \alpha_B \rightarrowtail Q_\alpha$

for $s < \alpha$, define

$$A_C := \{a \in \alpha_A : \exists s < \alpha. q_s(a) \in C\},$$
$$B_C := \{b \in \alpha_B : \exists s < \alpha. q_s(b) \in \overline{C}\}.$$

Definition 6.1.4. (α-computable sequence of functions)
A sequence of partial functions $q_s : \alpha \rightharpoonup \alpha$ is α-computable iff the partial α-computable function $q : \alpha \times \alpha \rightharpoonup \alpha$ defined by $q = \lambda sc.q_s(c)$ is α-computable.

Recall definition 6.1.2 that a labelling function is total.

Definition 6.1.5. (Label obstacle)
Let $q_{s,p} : \alpha_A \sqcup \alpha_B \to Q_\alpha$ be an input labelling function of the labelling algorithm for the strategy run (s, p). A label b_o is a (left) *obstacle* for the label a (at the stage s and priority p) iff $q_{s,p}(b_o) < q_{s,p}(a)$ and $(a, b_o) \notin W_s$. Define the right obstacles for the labels in α_B symmetrically.

Definition 6.1.6. (Labelling function consistency)
A labelling function $\hat{q} : \alpha_A \sqcup \alpha_B \rightarrowtail Q_\alpha$ is *consistent* with respect to the set of pairs $W \subseteq \alpha_A \times \alpha_B$ iff

$$\forall (a, b) \in \alpha_A \times \alpha_B [\hat{q}(a) < \hat{q}(b) \implies (a, b) \in W].$$

Adjacency

The labelling algorithm has to be able to determine if two labels are next to each other, i.e. adjacent.

Definition 6.1.7. (Label adjacency)
Let $\hat{q} : \alpha_A \sqcup \alpha_B \rightarrowtail Q_\alpha$ be a labelling function and $c, d \in \alpha_A \sqcup \alpha_B$ be labels satisfying the condition $\hat{q}(c) < \hat{q}(d)$. We say that the two labels c and d are *adjacent* iff for every other label $e \in \alpha_A \sqcup \alpha_B$ either $\hat{q}(e) < \hat{q}(c)$ or $\hat{q}(d) < \hat{q}(e)$. We say that c is adjacent to d from the left and that d is adjacent to c from the right.

It is important to position and determine an exact position of a label wrt to some set of labels (which can be all the labels in some interval for example).

Definition 6.1.8. (Adjacency from the right/left)
Let $\hat{q} : \alpha_A \sqcup \alpha_B \rightarrowtail Q_\alpha$ be a labelling function and $d \in \alpha_A \sqcup \alpha_B$ be a label and $C \subseteq \alpha_A \sqcup \alpha_B$ be a set of labels. Then the label d is *adjacent* (from the right) to the labels C iff:

i) $\forall c \in C. \hat{q}(c) < \hat{q}(d)$,

ii) $\forall e \in \alpha_A \sqcup \alpha_B [\forall c \in C. \hat{q}(c) < \hat{q}(e) \wedge d \neq e \implies \hat{q}(d) < \hat{q}(e)]$.

We say that d is adjacent to c (from the right) iff d is adjacent to a set $\{c\}$. Define adjacency from the left symmetrically. If c and is adjacent to d from the left, then we say that the labels c and d are adjacent.

We have the following relation between definition 6.1.7 and definition 6.1.8.

Remark 6.1.9. Let $c, d \in \alpha_A \sqcup \alpha_B$ be labels. Then the label c is adjacent to the label d from the left/right iff the label c is adjacent to the set $\{d\}$ from the left/right respectively.

Priority

The priority ordering on the label pairs determines the order in which the strategies act and the order in which the dead zones are cleared out. The priority ordering prevents that a dead zone is cleared out of the labels and then new labels are added into it again. Ultimately, this guarantees that each strategy would stop acting eventually (lemma 6.2.6) which is essential to construct a static semicomputable cut to prove lemma 6.1.1.

Definition 6.1.10. (Priority on pairs)
The *priority* for a pair $(a, b) \in \alpha_A \times \alpha_B$ is given by an α-computable bijection $\alpha \times \alpha \to \alpha$. Denote this bijection as $\mathsf{p} : \alpha \times \alpha \to \alpha$.

Lemma 6.1.11. (Computability of priority)
There is an α-computable bijection $\mathsf{p} : \alpha \times \alpha \to \alpha$.

Proof. Take the inverse of the α-computable bijection $p_2 : \alpha \to \alpha \times \alpha$ in proposition 3.2.3. □

Priority in definition 6.1.10 inherits the ordering from the ordinal ordering. If $p_1 < p_2$, then the ordinal p_1 is smaller than the ordinal p_2 and p_2 is greater than p_1. For lower and higher priority terms, the meaning is reversed.

Definition 6.1.12. (Lower and higher priority)
Let $p_1 < p_2$. Then:

- p_1 is a *higher* priority than p_2.
- p_2 is a *lower* priority than p_1.

Dead zone

Definition 6.1.13. (Dead zone - DZ)
An interval $[q_s(b), q_s(a)]$ is a *dead zone* of a priority p iff $(a, b) \in W_s$, $q_s(b) < q_s(a)$ and $\mathsf{p}(a, b) = p$.

Definition 6.1.14. (Permanent dead zone - PDZ)
A dead zone $Z = [q_s(b), q_s(a)]$ is *permanent* iff

$$\forall t < \alpha[s < t \implies q_s^{-1} \upharpoonright Z = q_t^{-1} \upharpoonright Z].$$

Connectedness of α-rationals

α-rational numbers can be connected through some set of intervals such as dead zones. Connected dead zones of a higher priority are treated as one dead zone. Hence it is important for a labelling algorithm to know if two α-rationals are connected.

Definition 6.1.15. (Connectedness through the set of intervals)
Let $\rho, \sigma \in Q_\alpha$. Let $\mathcal{I}$ be *a*[1] set of intervals in Q_α. Denote ρ is connected to σ through $\mathcal{I}$ as $\texttt{conn}(\rho, \sigma, \mathcal{I})$. Define $\texttt{conn}(\rho, \sigma, \mathcal{I})$ inductively:

- $\exists I \in \mathcal{I}[\rho \in I \wedge \sigma \in I] \implies \texttt{conn}(\rho, \sigma, \mathcal{I})$
- $\texttt{conn}(\rho, \sigma, \mathcal{I}) \wedge \texttt{conn}(\sigma, \tau, \mathcal{I}) \implies \texttt{conn}(\rho, \tau, \mathcal{I})$

Lemma 6.1.16. Suppose that $\texttt{conn}(\rho, \sigma, \mathcal{I})$. Then there is $\mathcal{I}_f \subseteq \mathcal{I}$ s.t. $\texttt{conn}(\rho, \sigma, \mathcal{I}_f)$ and $\#\mathcal{I}_f < \aleph_0$.

Proof. Suppose that ρ and σ are connected through $\mathcal{I}$. Then there is a finite proof of this statement. As the proof is finite, so it refers to only finitely many intervals from $\mathcal{I}$. Let $\mathcal{I}_f$ be the set of these intervals. Then $\texttt{conn}(\rho, \sigma, \mathcal{I}_f)$ and $\#\mathcal{I}_f < \aleph_0$ as required. □

Lemma 6.1.17. Define the predicate $\texttt{fin}(K) :\iff \#K < \aleph_0$. Then $\texttt{fin}$ is α-computable on an α-finite domain.

Proof. Note

$$\texttt{fin}(K) \iff \exists n < \omega \exists f \in L_\alpha[\pi_1[f] = n \wedge \pi_2[f] = K]$$

where π_1, π_2 represent α-computable projections, $f : n \rightharpoonup K$ is a partial surjection. Clearly $\texttt{fin} : L_\alpha \to \{0, 1\} \in \Sigma_1(L_\alpha)$ as required. □

Lemma 6.1.18. The predicate $\texttt{conn}$ is α-computable on an α-finite domain, i.e.

$$\texttt{conn} : (Q_\alpha \cap L_\alpha) \times (Q_\alpha \cap L_\alpha) \times L_\alpha \to \{0, 1\}$$

is α-computable.

Proof. First observe that $\texttt{conn}(\rho, \sigma, \mathcal{I}_f)$ is clearly α-computable on the domain where $\mathcal{I}_f$ is finite, i.e. just try all the possible arrangements of the intervals in $\mathcal{I}_f$ in a finite time to test the connectedness of ρ and σ through $\mathcal{I}_f$. Recall

$$\texttt{fin}(K) \iff \#K < \aleph_0.$$

[1] $\mathcal{I}$ does not necessarily contain all the intervals in Q_α.

Using lemma 6.1.16 we have

$$\texttt{conn}(\rho,\sigma,\mathcal{I}) \iff \exists \mathcal{I}_f \in L_\alpha[\mathcal{I}_f \subseteq \mathcal{I} \wedge \texttt{fin}(\mathcal{I}_f) \wedge \texttt{conn}(\rho,\sigma,\mathcal{I}_f)].$$

Hence $\texttt{conn}(\rho,\sigma,\mathcal{I})$ is α-computable by the observation, α-finiteness of $\mathcal{I}$ and α-computability of $\texttt{fin}$ (lemma 6.1.17). □

The labelling function q_α and label connectedness

Remark 6.1.19. (Labels in PDZ and their labelling function q_α)
By definition 6.1.14, for every PDZ there is some stage $s < \alpha$ s.t. every label $c \in \alpha_A \sqcup \alpha_B$ in PDZ stops moving by that stage s, i.e. $\forall t < \alpha[s \leq t \implies q_s(c) = q_t(c)]$. Therefore we can define a partial labelling function $q_\alpha : \alpha_A \sqcup \alpha_B \rightharpoonup Q_\alpha$ on the domain of stationary labels by:

$$\forall c \in \alpha_A \sqcup \alpha_B \forall \rho \in Q_\alpha\big[(c,\rho) \in q_\alpha \iff \exists s < \alpha$$
$$[q_s(c) = \rho \wedge \forall t < \alpha\big(s \leq t \implies q_s(c) = q_t(c)\big)]\big].$$

The function q_α could be extended in a compatible way with the definition of a limit function in section 6.1.7, but its codomain and image would have to contain elements outside of the α-rationals Q_α.

Definition 6.1.20. (Connectedness of labels)
Let $c, d \in \alpha_A \sqcup \alpha_B$. The labels c, d are *connected* iff $q_\alpha(c)$ and $q_\alpha(d)$ are connected through PDZs according to definition 6.1.15.

6.1.3 The labelling function construction

A label a (or b) with a subscript if applicable is in α_A (or α_B). Index the enumeration of W as $W = \{(a_0, b_0), (a_1, b_1), \ldots\}$ and define

$$W_s := \{(a_t, b_t) : t < s\}.$$

We construct the labelling sequence $q : \alpha \times (\alpha_A \sqcup \alpha_B) \rightarrowtail Q_\alpha$ also denoted as $\{q_s\}_{s<\alpha}$ in α stages by constructing the labelling sequence $\{q_{s,p,u}\}_{0\leq s<\alpha, 0\leq p\leq s+1, -1\leq u\leq s+1}$ first in definition 6.1.24. The construction of $q_{s,p,u}$ depends on whether (s, p, u) is an initial, strategy, clearing or a limit triple as defined below in definition 6.1.21.

Using the triple indexing for the labelling function enables us to track the detailed changes of the labelling function which simplifies the proofs of the properties about the labelling function later. The first index s determines which α-finite subset of the set W is used by the labelling algorithm. At the stage $s = r + 1$ the set W_r is used. At the stage s where s is a limit ordinal, the set W_s is used. The second index p represents the priority of the pair (a, b) that is passed to the part of the labelling algorithm called the strategy run which attempts to move the label a

to the left of the label b. The clearing of the labels is executed after each strategy run. The third index u represents the stage within the algorithm for the clearing of the labels. At the stage u the pair (c, d) with the priority u is attempted to be cleared.

Definition 6.1.21. (Initial, strategy, clearing and limit triple)
Assume $0 \leq s < \alpha, 0 \leq p \leq s+1, -1 \leq u \leq s+1$. Then

- (s, p, u) is an *initial triple* iff $u = -1$.
- (s, p, u) is a *strategy triple* iff $u = 0$.
- (s, p, u) is a *clearing triple* iff $u > 0$ and $\neg\lim(u)$.
- (s, p, u) is a *limit triple* iff
  $\lim(u) \vee \lim(p) \wedge u = -1 \vee \lim(s) \wedge p = 0 \wedge u = -1$.

Note that the definition definition 6.1.21 allows for a triple to be both an initial and a limit triple.

Remark 6.1.22. (Equality relation on triples)
We extend the equality relation on triples by identifying

$$(s, p, s+1) = (s, p+1, -1),$$
$$(s, s+1, s+1) = (s+1, 0, -1).$$

In other words, $(s_1, p_1, u_1) = (s_2, p_2, u_2) \iff s_1 = s_2 \wedge p_1 = p_2 \wedge u_1 = u_2 \vee$

$$\exists s, p < \alpha[(s_1, p_1, u_1) = (s, p, s+1) \wedge (s_2, p_2, u_2) = (s, p+1, -1)] \vee$$
$$\exists s < \alpha[(s_1, p_1, u_1) = (s, s+1, s+1) \wedge (s_2, p_2, u_2) = (s+1, 0, -1)].$$

Remark 6.1.23. (Ordering on triples)
The ordering on the triples $(s, p, u) \in \alpha \times \alpha \times (\{-1\} \sqcup \alpha)$ is a usual alpha-numerical ordering: $(s_1, p_1, u_1) < (s_2, p_2, u_2) \iff$

$$s_1 < s_2 \vee s_1 = s_2 \wedge p_1 < p_2 \vee s_1 = s_2 \wedge p_1 = p_2 \wedge u_1 < u_2.$$

Definition 6.1.24. (Labelling sequence construction)
Construct the labelling sequence $\{q_{s,p,u}\}_{0 \leq s < \alpha, 0 \leq p \leq s+1, -1 \leq u \leq s+1}$ as follows:

- $q_{0,0,-1} := \emptyset$.
- $q_{s,p+1,-1} := q_{s,p,s+1}$.
- $q_{t,r,v} := \lim_{(s,p,u)<(t,r,v)}(q_{s,p,u})$ if (t, r, v) is a limit triple where $\lim_{(s,p,u)<(t,r,v)}(q_{s,p,u})$ is the limit function defined in section 6.1.7.
- $q_{s,p,0}$ is the output of the strategy run (s, p) on the input function $q_{s,p,-1}$ for a pair (a, b) of a priority p using W_{s+1}. See section 6.1.5 on a strategy run.

- $q_{s,p,u+1}$ where $-1 \leq u \leq s$ is the output of clearing labels of the pair $(c,d) \in \alpha_A \sqcup \alpha_B$ of the priority u from the the dead zone $[q_{s,p,u}(b), q_{s,p,u}(a)]$ where (a,b) is a label pair of a priority p. See section 6.1.6 on label clearing.

Using the sequence with triple indices define abridged sequences $\{q_{s,p}\}_{s<\alpha, p\leq s+1}$ and $\{q_s\}_{s<\alpha}$ by the assignment $q_s := q_{s,0} := q_{s,0,-1}$.

In the end, we have an α-computable sequence of α-computable partial labelling functions $q_s : \alpha_A \sqcup \alpha_B \to Q_\alpha$ for $s < \alpha$ as required. Note that if $t < \alpha$ is a limit ordinal, then $q_t = \lim_{s<t} q_s$. Also note that for any $s < \alpha$ the function q_s is constructed from W_s.

6.1.4 Label order type at the substage (s, p, u)

The strategy for the pair (a,b) moves the labels a and b first. Next if $[q_s(b), q_s(a)]$ is a dead zone, then it tries to move other labels out of this dead zone. Recall the order type $\mathrm{ot}(\rho)$ of an α-rational ρ is the order type of the binary string representing this α-rational. As ρ is an α-rational, so $\mathrm{ot}(\rho) < \alpha$. Every label moved has to be moved to an α-rational Q_α of a certain order type dependent on the stage s, priority p and the label itself. This is to guarantee that at every stage there is enough space for new adjacent labels which is made precise later, see proposition 6.2.3i.

Here we define an order type function $\mathtt{ot} : \alpha \times \alpha \times (\{-1\} \cup \alpha) \to \alpha$ which is used to specify a label order type during the strategy run in section 6.1.5 and during the label clearing in section 6.1.6. We also define a limit order type function $\mathtt{otLim} : \alpha \times \alpha \times (\{-1\} \cup \alpha) \to \alpha$ which is used to specify a label order type during the limit function construction in section 6.1.7.

Definition 6.1.25. (Order type functions)
The domain of the *order type function* $\mathtt{ot}$ is

$$\mathrm{dom}(\mathtt{ot}) := \{(s,p,u) \in \alpha \times \alpha \times (\{-1\} \cup \alpha) : 0 \leq s < \alpha, 0 \leq p \leq s, -1 \leq u \leq s\}.$$

The domain of the *limit order type function* $\mathtt{otLim}$ is

$$\mathrm{dom}(\mathtt{otLim}) := \{(s,p,u) \in \alpha \times \alpha \times (\{-1\} \cup \alpha) : (s,p,u) \text{ is a limit triple}\}.$$

Let $t, r, v < \alpha$ be limit ordinals. We define the order type function $\mathtt{ot} : \alpha \times \alpha \times (\{-1\} \cup \alpha) \to \alpha$ and the limit order type function $\mathtt{otLim} : \alpha \times \alpha \times (\{-1\} \cup \alpha) \to \alpha$ inductively:

- $\mathtt{ot}(0, 0, -1) := 0$,
- $\mathtt{ot}(s, p, u+1) := \mathtt{ot}(s, p, u) + 2$,

- $\texttt{ot}(s, p+1, -1) \coloneqq \texttt{ot}(s, p, s) + 2$,
- $\texttt{ot}(s+1, 0, -1) \coloneqq \texttt{ot}(s, s, s) + 2$,
- $\texttt{otLim}(s, p, v) \coloneqq \sup_{u<v} \texttt{ot}(s, p, u)$,
- $\texttt{otLim}(s, r, 0) \coloneqq \sup_{p<r} \texttt{ot}(s, p, 0)$,
- $\texttt{otLim}(t, 0, -1) \coloneqq \sup_{s<t} \texttt{ot}(s, 0, 0)$,
- $\texttt{ot}(s, p, v) \coloneqq \texttt{otLim}(s, p, v) \cdot 2$,
- $\texttt{ot}(s, r, 0) \coloneqq \texttt{otLim}(s, r, 0) \cdot 2$,
- $\texttt{ot}(t, 0, -1) \coloneqq \texttt{otLim}(t, 0, -1) \cdot 2$.

Lemma 6.1.26. (Properties of order type functions)

- $\texttt{ot}$ is well-defined on all triples.
- $\texttt{otLim}$ is well-defined on the limit triples.
- $\texttt{ot}$ is α-computable.
- $\texttt{otLim}$ is α-computable.

Proof. The proof is performed by the transfinite induction on the triple

$$\gamma = (s, p, u) \in \alpha \times \alpha \times (\{-1\} \cup \alpha)$$

where $p, u \leq s + 1$.

The base case is clear: $\texttt{ot}$ is well-defined and α-computable on the domain $\{(0, 0, -1)\}$.

For the inductive case when $\gamma + 1$ is not a limit triple, by IH assume that $\texttt{ot}$ is well-defined and α-computable on the domain $\gamma + 1$. Then clearly $\texttt{ot}(\gamma + 1) \coloneqq \texttt{ot}(\gamma) + 2$ is well-defined and α-computable uniformly from $\gamma + 1$. Hence $\texttt{ot}$ is well-defined and α-computable on the domain $\gamma + 2$.

For the inductive case when δ is a limit triple, by IH assume that $\texttt{ot}$ is well-defined and α-computable on the domain $D \coloneqq \delta$. Note $D \in L_\alpha$. Hence $K \coloneqq \texttt{ot}[D] \in L_\alpha$. Note that $\texttt{otLim}(\delta) \coloneqq \texttt{psup}(K)$ where $\texttt{psup}$ is α-computable by lemma 3.5.1. Also $\texttt{ot}(\delta) \coloneqq \texttt{otLim}(\delta) \cdot 2$. Hence $\texttt{otLim}(\delta)$ and $\texttt{ot}(\delta)$ are well-defined and uniformly α-computable from δ. Hence $\texttt{ot}$ is well-defined and α-computable on the domain $\delta + 1$, and $\texttt{otLim}$ is well-defined and α-computable on the domain $\{\gamma \leq \delta : \lim(\gamma)\}$.

Therefore the function $\texttt{ot}$ is well-defined and α-computable on the domain α, and the function $\texttt{otLim}$ is well-defined and α-computable on the domain $\{\delta < \alpha : \lim(\delta)\}$. □

6.1.5 Strategy for (a, b) of the priority p at the stage s

Strategy for (a, b) of the priority p executes a strategy run (s, p) trying to satisfy the condition

$$(a, b) \in W_{s+1} \implies q_{s+1}(a) < q_{s+1}(b).$$

Strategy run notation and order of execution

Definition 6.1.27. (Strategy run notation)
If a strategy runs at the stage $s < \alpha$ and for the pair (a, b) of an priority $p \leq s$, then denote this strategy run by the pair (s, p).

By the above the first strategy runs are the following:

$$(0, 0), (1, 0), (1, 1), (2, 0), (2, 1), (2, 2), (3, 0), \ldots$$

Definition 6.1.28. (Order of execution notation)
The function $\mathrm{oe}(s, p)$ denotes the *order of the execution* of the strategy run (s, p) and it is formally defined inductively as follows:

$$\begin{aligned}
\mathrm{dom}(\mathrm{oe}) &:= \{(s, p) : s \in \alpha, p \in s\} \\
\mathrm{oe}(0, 0) &:= 0 \\
\mathrm{oe}(s, p + 1) &:= \mathrm{oe}(s, p) + 1 \\
\mathrm{oe}(s, r) &:= \sup\{\mathrm{oe}(s, p) : p < r\} \text{ if } \lim(r) \\
\mathrm{oe}(s + 1, 0) &:= \mathrm{oe}(s, s) + 1 \\
\mathrm{oe}(t, 0) &:= \sup\{\mathrm{oe}(s, 0) : s < t\} \text{ if } \lim(t)
\end{aligned}$$

Example 6.1.29. For example $(0, 0)$ is executed first, so $\mathrm{oe}(0, 0) = 0$ (starting from 0). For others, $\mathrm{oe}(1, 0) = 1$, $\mathrm{oe}(1, 1) = 2$, ..., $\mathrm{oe}(n, p) = \frac{1}{2}n(n + 1) + p$, ..., $\mathrm{oe}(\omega, 0) = \omega$, etc.

Note that $\frac{1}{2}n(n + 1)$ is a triangular number.

Remark 6.1.30. Let δ be 0 or a limit ordinal. Let $n < \omega$ be a finite ordinal. Note that $\mathrm{oe}(\delta, 0) = \sum_{\beta<\delta} \beta$. More generally, the order of execution of a strategy run $(\delta + n, p)$ is

$$\mathrm{oe}(\delta + n, p) = (\sum_{\beta<\delta} \beta) + \delta \cdot n + n + p.$$

Strategy run (s, p) for the pair (a, b)

- Inputs:
  - $q_{\mathtt{in}} := q_{s,p,-1}$ (α-finite by IH, proposition 6.2.3iv)

  - W_{s+1} (α-finite as $W \in \Sigma_1(L_\alpha)$)
  - s
  - p

- Output: $q_{s,p,0} := q_{\mathtt{out}}$
- Goal: $(a, b) \in W_{s+1} \implies q_{\mathtt{out}}(a) < q_{\mathtt{out}}(b)$ (possibly unsatisfiable)

Given as input the partial labelling function $q_{\mathtt{in}}$, use the strategy underneath to modify $q_{\mathtt{in}}$. Once the strategy run completed, copy the updated $q_{\mathtt{in}}$ into $q_{\mathtt{out}}$.

Before the label clearing, the strategy for the pair (a, b) moves the labels a and b only and is allowed to label only two α-rationals of an order type θ and $\theta + 1$ where $\theta := \mathtt{ot}(s, p, -1)$ specified in definition 6.1.25.

If no rational is labelled by $a \in \alpha_A$, i.e. $q_{\mathtt{in}}(a) \uparrow$, then put the label a to the right of all the defined labels in $\mathrm{dom}(q_{\mathtt{in}})$, i.e. $q_{\mathtt{in}}(a) := \triangleright^\theta$. Similarly, if $q_{\mathtt{in}}(b) \uparrow$, then let $q_{\mathtt{in}}(b) := \triangleleft^\theta$ be left of all rationals in $\mathrm{dom}(q_{\mathtt{in}})$.

At the stage $s < \alpha$, if (a, b) has not entered W_{s+1}, do nothing. If the current labels of $q_{\mathtt{in}}$ satisfy the condition $q_{\mathtt{in}}(a) < q_{\mathtt{in}}(b)$, do nothing. Otherwise, try to move the label a to the left of the label b. This may not always be possible: if $b_o \in \alpha_B$, $(a, b_o) \notin W$ and $q_{\mathtt{in}}(b) < q_{\mathtt{in}}(b_o) < q_{\mathtt{in}}(a)$, then moving the label a left of the label b would cause $q_{\mathtt{in}}(a) < q_{\mathtt{in}}(b_o)$ which may prevent the conditions $A = A_C$ and $B = B_C$ from being satisfied.

Hence the label b_o is an obstacle for the label a to be moved left of the label b at the stage s and priority p, see definition 6.1.5. Define

$$B_o := \{b_o \in \alpha_B \cap \mathrm{dom}(q_{\mathtt{in}}) : q_{\mathtt{in}}(b_o) < q_{\mathtt{in}}(a) \wedge (a, b_o) \notin W_{s+1}\}$$

to be the set of the *obstacles* for the label a.

Hence try to move the label a to the left of the label b according to the following rules:

1. follow the rules below iff $q_{\mathtt{in}}(b) < q_{\mathtt{in}}(a)$.
2. if $B_o = \emptyset$, then put the label a to the left of all the labels in $q_{\mathtt{in}}$, i.e. $q_{\mathtt{in}}(a) := \triangleleft^{\theta+1}$.
3. if no label on the right of B_o is in a dead zone of a higher priority with some label in B_o, then place the label a adjacent to the set B_o from the right.
4. if $q_{\mathtt{in}}(a)$ is inside an interval protected by higher priority strategies and which contains some label from B_o, then do not move the label a.

5. otherwise do the following. First define the set Y.

   $Y \coloneqq \{\hat{a} \in \alpha_A : \exists \hat{b} \in \alpha_B.(\hat{a}, \hat{b}) \in S\} \cup \{\hat{b} \in \alpha_B : \exists \hat{a} \in \alpha_A.(\hat{a}, \hat{b}) \in S\}$ where

   $S \coloneqq \{(\hat{a}, \hat{b}) \in \alpha_A \times \alpha_B : (\hat{a}, \hat{b}) \in W_{s+1} \wedge \mathtt{p}(\hat{a}, \hat{b}) < p \wedge q_{\mathtt{in}}(\hat{b}) < q_{\mathtt{in}}(\hat{a}) \wedge \hat{b} \in B_o\}$

   and $\mathtt{p}(\hat{a}, \hat{b})$ is the priority of the pair $(\hat{a}, \hat{b})$. Note that the set S is the maximal set of pairs $(\hat{a}, \hat{b})$ such that the dead zone interval $[q_{\mathtt{in}}(\hat{b}), q_{\mathtt{in}}(\hat{a})]$ is protected by higher priority strategies and $\hat{b}$ is an obstacle for the label a. The set Y is the set which contains a label from $\alpha_A \sqcup \alpha_B$ iff it is in some pair in the set S. As Y is the main set of interest and S is an auxiliary maximal set, throughout the text we refer to Y as the *maximal set*. Now having defined the maximal set Y, in this step place the label a adjacent from the right to the right endpoint of the maximal set Y.

In a similar way move the label b as far right as possible.

Now copy the result $q_{\mathtt{in}}$ into the output labelling function $q_{\mathtt{out}}$. This completes the strategy run (s, p) for the pair (a, b).

Pseudocode

The function *strategy_run* takes as an input the labelling function $q_{\mathtt{in}} \coloneqq q_{s,p,-1}$ and the output is assigned to the function $q_{s,p,0}$.

function $strategy\_run(q_{\mathtt{in}}, W_{s+1}, s, p)$
Require: $q_{\mathtt{in}} \in L_\alpha$ by IH
    $\theta \coloneqq \mathtt{ot}(s, p, -1)$
    $(a, b) \coloneqq \mathtt{p}^{-1}(p)$
    if $a \notin \mathrm{dom}(q_{\mathtt{in}})$ **then**
        $q_{\mathtt{in}}(a) \coloneqq \triangleright^\theta$
    end if
    if $b \notin \mathrm{dom}(q_{\mathtt{in}})$ **then**
        $q_{\mathtt{in}}(b) \coloneqq \triangleleft^\theta$
    end if
    if $(a, b) \notin W_{s+1} \vee q_{\mathtt{in}}(a) < q_{\mathtt{in}}(b)$ **then**
        return $q_{\mathtt{in}}$
    end if
    $B_o \coloneqq \{b_o \in \alpha_B \cap \mathrm{dom}(q_{\mathtt{in}}) : q_{\mathtt{in}}(b_o) < q_{\mathtt{in}}(a) \wedge (a, b_o) \notin W_{s+1}\}$
    $q_{\mathtt{in}} \coloneqq move\_label\_a(q_{\mathtt{in}})$ ▷ Move the label a as much left as possible
    $q_{\mathtt{in}} \coloneqq move\_label\_b(q_{\mathtt{in}})$ ▷ Move the label b as much right as possible
    return $q_{\mathtt{in}}$
end function

function $move\_label\_a(q_{\mathtt{in}})$
  if $B_o = \emptyset$ **then**
    $q_{\mathtt{in}}(a) := \lhd^{\theta+1}$
    return $q_{\mathtt{in}}$
  end if
  $\hat{Q} := \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \theta + 1\}$
  if $\forall(\hat{a}, \hat{b}) \in W_{s+1}[\mathtt{p}(\hat{a}, \hat{b}) < p \implies (\neg\exists b_o \in B_o . q_{\mathtt{in}}(\hat{b}) \leq q_{\mathtt{in}}(b_o) < q_{\mathtt{in}}(\hat{a}))]$ **then**
    $q_{\mathtt{in}}(a) := \sup_{\hat{Q}}(B_o)$
    return $q_{\mathtt{in}}$
  end if
  $Z := \{d \in \mathrm{dom}(q_{\mathtt{in}}) : \exists(\hat{a}, \hat{b}) \in W_{s+1}[\mathtt{p}(\hat{a}, \hat{b}) < p \wedge q_{\mathtt{in}}(\hat{b}) \leq q_{\mathtt{in}}(d) \leq q_{\mathtt{in}}(\hat{a})]\}$
  if $a \in Z \wedge \exists b_o \in B_o \cap Z\big[\forall c \in \mathrm{dom}(q_{\mathtt{in}})[q_{\mathtt{in}}(b_o) < q_{\mathtt{in}}(c) < q_{\mathtt{in}}(a) \implies c \in Z]\big]$ **then**
    return $q_{\mathtt{in}}$
  end if
  $q_{\mathtt{in}}(a) := \sup_{\hat{Q}}(q_{\mathtt{in}}[Y])$
  return $q_{\mathtt{in}}$
end function

The function *move_label_b* is symmetric to the function *move_label_a* defined above. In particular, *move_label_b* uses the following sets:

$A_o := \{a_o \in \alpha_A \cap \mathrm{dom}(q_{\mathtt{in}}) : q_{\mathtt{in}}(b) < q_{\mathtt{in}}(a_o) \wedge (a_o, b) \notin W_{s+1}\},$
$Y_b := \{\hat{a} \in \alpha_A : \exists\hat{b} \in \alpha_B . (\hat{a}, \hat{b}) \in S_b\} \cup \{\hat{b} \in \alpha_B : \exists\hat{a} \in \alpha_A . (\hat{a}, \hat{b}) \in S_b\},$ where
$S_b := \{(\hat{a}, \hat{b}) \in \alpha_A \times \alpha_B : (\hat{a}, \hat{b}) \in W_{s+1} \wedge \mathtt{p}(\hat{a}, \hat{b}) < p \wedge q_{\mathtt{in}}(\hat{b}) < q_{\mathtt{in}}(\hat{a}) \wedge \hat{a} \in A_o\},$

where A_o is the set of the obstacles for the label b and Y_b is the set of the labels in the maximal set for the label b. The function *move_label_b* is defined as follows.

function $move\_label\_b(q_{\mathtt{in}})$
  if $A_o = \emptyset$ **then**
    $q_{\mathtt{in}}(b) := \rhd^{\theta+1}$
    return $q_{\mathtt{in}}$
  end if
  $\hat{Q} := \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \theta + 1\}$
  if $\forall(\hat{a}, \hat{b}) \in W_{s+1}[\mathtt{p}(\hat{a}, \hat{b}) < p \implies (\neg\exists a_o \in A_o . q_{\mathtt{in}}(\hat{b}) \leq q_{\mathtt{in}}(a_o) < q_{\mathtt{in}}(\hat{a}))]$ **then**
    $q_{\mathtt{in}}(b) := \sup_{\hat{Q}}(A_o)$
    return $q_{\mathtt{in}}$
  end if

$Z := \{d \in \mathrm{dom}(q_{\mathtt{in}}) : \exists(\hat{a},\hat{b}) \in W_{s+1}[\mathtt{p}(\hat{a},\hat{b}) < p \wedge q_{\mathtt{in}}(\hat{b}) \leq q_{\mathtt{in}}(d) \leq q_{\mathtt{in}}(\hat{a})]\}$

if $b \in Z \wedge \exists a_o \in A_o \cap Z\big[\forall c \in \mathrm{dom}(q_{\mathtt{in}})[q_{\mathtt{in}}(b) < q_{\mathtt{in}}(c) < q_{\mathtt{in}}(a_o) \implies c \in Z]\big]$ **then**

return $q_{\mathtt{in}}$

end if

$q_{\mathtt{in}}(b) := \sup_{\hat{Q}}(q_{\mathtt{in}}[Y_b])$

return $q_{\mathtt{in}}$

end function

Properties

We assume the hypothesis that the input function $q_{\mathtt{in}}$ is α-finite.

Lemma 6.1.31. The set of the obstacles B_o is α-finite and uniformly α-computable from the parameters $q_{\mathtt{in}}$, a and W_{s+1}.

Proof. The set B_o is α-finite as $q_{\mathtt{in}} \in L_\alpha, \mathrm{dom}(q_{\mathtt{in}}) \in L_\alpha$ by IH and $W_{s+1} \in L_\alpha$. By the definition of B_o, it is clearly uniformly α-computable from the parameters $q_{\mathtt{in}}$, a and W_{s+1} as required. ☐

Lemma 6.1.32. The set Z of the labels in the dead zones of a priority higher than p (or a set protected by strategies of a higher priority) in the function *move_label_a* is α-finite and uniformly α-computable from the parameters $q_{\mathtt{in}}$, p and W_{s+1}.

Proof. Z is α-finite as $\mathrm{dom}(q_{\mathtt{in}}) \in L_\alpha$ and $q_{\mathtt{in}} \in \Sigma_1(L_\alpha)$ by IH, $W_{s+1} \in L_\alpha$ and $\mathtt{p} \in \Sigma_1(L_\alpha)$. The uniform α-computability of Z is follows from its definition as required. ☐

Lemma 6.1.33. The set Y of the labels in the maximal set is α-finite and uniformly α-computable from the parameters $q_{\mathtt{in}}$, a, p and W_{s+1}.

Proof. Recall $Y := \pi_1[S] \cup \pi_2[S]$ where π_i is an α-computable projection and $S := \{(\hat{a},\hat{b}) \in \alpha_A \times \alpha_B : (\hat{a},\hat{b}) \in W_{s+1} \wedge \mathtt{p}(\hat{a},\hat{b}) < p \wedge q_{\mathtt{in}}(\hat{b}) < q_{\mathtt{in}}(\hat{a}) \wedge \hat{b} \in B_o\}$.

The set S is α-computable from the parameters $q_{\mathtt{in}}$, a, p and W_{s+1} using the uniform α-computability of the set B_o from the parameters $q_{\mathtt{in}}$, a and W_{s+1}. S is bounded as $S \subseteq W_{s+1}$. Hence S is α-finite. As π_i is α-computable, so $\pi_i[S] \in L_\alpha$, hence $Y \in L_\alpha$. The uniform α-computability of Y follows from the uniform α-computability of S. ☐

Lemma 6.1.34. The α-rationals $q_{\mathtt{out}}(a)$ and $q_{\mathtt{out}}(b)$ defined in the function *strategy_run* are computed in a uniform way, exist and are α-finite given that $q_{\mathtt{in}}$ is α-finite.

Proof. The statement is clear for the assignments on the lines 5, 8 and 21 as $\theta < \alpha$.

The assignments on the lines 26 and 33 use the function $\sup_Q(S)$. The function $\sup_Q(S)$ can compute using the same algorithm for any arguments Q and S where Q contains all the α-rationals of the same order type $\delta = \theta + 1$ and $\mathrm{ot}[S] \subseteq \delta$, see proposition 5.4.6. Hence an α-rational $\sup_Q(S)$ is computed in a uniform way.

Furthermore $S \in \{B_o, q_{\mathtt{in}}[Y]\}$ and both B_o and Y are α-computable uniformly from the parameters $q_{\mathtt{in}}$, a, p and W_{s+1}. Hence $q_{\mathtt{in}}(a) := \sup_{\hat{Q}}(S)$ is computed uniformly from $q_{\mathtt{in}}$, a, p, W_{s+1} and δ.

The α-finiteness of $q_{\mathtt{in}}(a) := \sup_{\hat{Q}}(S)$ where $S \in \{B_o, q_{\mathtt{in}}[Y]\}$ follows from the fact that the set $\hat{Q}$ contains only α-finite α-rationals.

By lemma 6.1.31, the set B_o is α-finite. By lemma 6.1.33, the set Y is α-finite. As $q_{\mathtt{in}}$ is also α-finite by the assumption, so $q_{\mathtt{in}}[Y]$ is α-finite. Hence S is α-finite and by proposition 5.4.6 the α-rational $q_{\mathtt{in}}(a) := \sup_{\hat{Q}}(S)$ exists, i.e. is well-(first-order)-defined. By duality $q_{\mathtt{in}}(b) := \inf_{\hat{Q}}(S_b)$ also exists for an appropriate S_b.

In the end $q_{\mathtt{in}}$ is copied to $q_{\mathtt{out}}$. Therefore $q_{\mathtt{out}}(a) \in L_\alpha$ and $q_{\mathtt{out}}(b) \in L_\alpha$ as required. □

Lemma 6.1.35. If the input function $q_{\mathtt{in}}$ is α-finite, then the output function

$$q_{\mathtt{out}} := \mathit{strategy\_run}(q_{\mathtt{in}}, W_{s+1}, s, p)$$

is also α-finite.

Proof. The function $q_{\mathtt{out}}$ is different from the α-finite function $q_{\mathtt{in}}$ on at most two labels in the label pair (a, b) of a priority p. By lemma 6.1.34 $q_{\mathtt{out}}(a)$ and $q_{\mathtt{out}}(b)$ have to be α-finite if changed. This change is α-finite and so $q_{\mathtt{out}}$ has to be α-finite too. □

Remark 6.1.36. (Object types[2])
Let M be the domain of some model. Then an element in M is a type-0 object. A function from M to M is a type-1 object. A function from M^M to M^M is a type-2 object (called a functional), etc.

In α-Computability Theory, the domain M of the model of the computation is α. For the purpose of this thesis we also consider a composition of a type-1 object with an α-computable bijection from/to α (e.g. $\alpha \to L_\alpha$, $\alpha \to \alpha \times \alpha$) to be a type-1 object. We do this since ultimately we are interested in assessing if a function is first-order definable over L_α and at what level of the arithmetical hierarchy.

[2] Consult a general book on Computability Theory, e.g. for a general idea see [5] Chapter 11, subsection The Scott model for lambda calculus.

Remark 6.1.37. (Type of the strategy run function)
We assume that the argument $q_{\mathtt{in}}$ to the function *label_clearing* which clears out the labels from the dead zone is always α-finite by IH, proposition 6.2.3iv. By lemma 6.1.35, the output of the function *label_clearing* has to be also α-finite. Therefore we can express the function with the type as

$$label\_clearing : L_\alpha \times L_\alpha \times \alpha \times \alpha \times \alpha \to L_\alpha.$$

Hence $label\_clearing$ is clearly a type-1 function.

Proposition 6.1.38. The procedure *move_label_a* is α-computable.

Proof. The procedure *move_label_a* is α-computable since:

- B_o is α-finite by lemma 6.1.31,
- W_{s+1} is α-finite,
- $\mathtt{p}$ is α-computable by lemma 6.1.11,
- Z is α-finite and uniformly α-computable by lemma 6.1.32,
- the instructions on the lines 26 and 33 are α-computable (by the uniformity of $\sup_Q(S)$ and S), an α-rational $q_{\mathtt{in}}(a)$ is well-defined and α-finite by lemma 6.1.34.

□

Proposition 6.1.39. The function *strategy_run* is α-computable.

Proof. By remark 6.1.37 the function *label_clearing* is a type-1 function, so it makes sense to talk about it being first-order definable over L_α. The function *strategy_run* is α-computable since:

- $q_{\mathtt{in}}$ is α-finite by IH proposition 6.2.3iv,
- $\mathtt{p}^{-1}$ is α-computable by lemma 6.1.11,
- $\mathtt{ot}$ is α-computable by lemma 6.1.26,
- B_o is α-finite and uniformly α-computable by lemma 6.1.31,
- the procedures *move_label_a* and *move_label_b* are α-computable by proposition 6.1.38 and its dual.

□

Next strategy runs

If $q_{\texttt{out}}(a) < q_{\texttt{out}}(b)$, then the strategy for (a, b) has satisfied its condition and never runs again. Otherwise it runs again later and prevents the strategies of lower priorities from placing labels in the interval $[q_{\texttt{out}}(b), q_{\texttt{out}}(a)]$ until the condition becomes satisfied (if that happens). We say the interval is marked as a *dead zone*.

6.1.6 Label clearing from DZ for a pair (a, b)

If $q_{\texttt{in}}(b) < q_{\texttt{in}}(a)$ and $(a, b) \in W$, then the interval $[q_{\texttt{in}}(b), q_{\texttt{in}}(a)]$ is a dead zone (DZ). See definition 6.1.13.

After the strategy run (s, p) for the pair (a, b) concludes, we would like to move labels out of the DZ interval $[q_{\texttt{in}}(b), q_{\texttt{in}}(a)]$ as much as possible. Let M be the set of labels in the maximum interval that contains the DZ $[q_{\texttt{in}}(b), q_{\texttt{in}}(a)]$ and other higher priority DZs connected to it.

See section 6.1.3 to recall that the sequence $q_{s,p,u}$ for $u \leq s + 1$ is used to construct $q_{s,p+1}$ after the strategy run (s, p) concludes and outputs $q_{s,p,-1}$. Also recall that $q_{s,p,v} := \lim_{u<v} q_{s,p,u}$.

Let $u < \alpha$ where $0 \leq u \leq s$. We construct the function $q_{s,p,u+1}$ given the function $q_{s,p,u}$. For every $u < s$ starting from 0 do the following in order: If $p = u$, then do nothing, i.e. $q_{s,p,u+1} := q_{s,p,u}$. Otherwise let $(c, d) \in \alpha_A \sqcup \alpha_B$ be a pair of the priority u, i.e. $\texttt{p}(c, d) = u$. Let η be the order type $\eta := \texttt{ot}(s, p, u)$. If $q_{s,p,u}(b) < q_{s,p,u}(c) < q_{s,p,u}(a)$ and it is consistent to move c left of M, then move c to the left of M to an α-rational ρ_c of an order type η, i.e. define $q_{s,p,u+1}(c) := \rho_c$. If $q_{s,p,u}(b) < q_{s,p,u}(d) < q_{s,p,u}(a)$ and it is consistent to move d right of M, then move d to the right of M to an α-rational ρ_d of an order type $\eta + 1$, i.e. define $q_{s,p,u+1}(d) := \rho_d$. If $q_{s,p,u+1}(e)$ has not been defined for a label $e \in \mathrm{dom}(q_{s,p,u})$, let $q_{s,p,u+1}(e) := q_{s,p,u}(e)$.

Pseudocode

The function *label_clearing* takes as an input the labelling function $q_{\texttt{in}} := q_{s,p,u}$ and the output is assigned to the function $q_{s,p,u+1}$.

function $label\_clearing(q_{\texttt{in}}, W_{s+1}, s, p, u)$
Require: $q_{\texttt{in}} \in L_\alpha$ by IH
    $(a, b) := \texttt{p}^{-1}(p)$
    if $q_{\texttt{in}}(a) < q_{\texttt{in}}(b) \vee (a, b) \notin W_{s+1} \vee p = u$ **then**
        return $q_{\texttt{in}}$
    end if
    $(c, d) := \texttt{p}^{-1}(u)$

$\eta \coloneqq \mathtt{ot}(s, p, u)$
if $q_{\mathtt{in}}(b) < q_{\mathtt{in}}(c) < q_{\mathtt{in}}(a) \wedge \forall \hat{b} \in \alpha_B \cap M.(c, \hat{b}) \in W_{s+1}$ **then**
$Q_c \coloneqq \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \eta\}$
$\rho_c \coloneqq \inf_{Q_c}(q_{\mathtt{in}}[M])$
$q_{\mathtt{in}}(c) \coloneqq \rho_c$
end if
if $q_{\mathtt{in}}(b) < q_{\mathtt{in}}(d) < q_{\mathtt{in}}(a) \wedge \forall \hat{a} \in \alpha_A \cap M.(\hat{a}, d) \in W_{s+1}$ **then**
$Q_d \coloneqq \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \eta + 1\}$
$\rho_d \coloneqq \sup_{Q_d}(q_{\mathtt{in}}[M])$
$q_{\mathtt{in}}(d) \coloneqq \rho_d$
end if
return $q_{\mathtt{in}}$
end function

Properties

We assume the hypothesis that the input function $q_{\mathtt{in}}$ is α-finite.

Lemma 6.1.40. The set M of the labels in the maximal interval extending the dead zone $[q_{\mathtt{in}}(b), q_{\mathtt{in}}(a)]$ is α-finite and uniformly α-computable from the parameters a, b and W_{s+1}.

Proof. Define

$$\mathcal{I} \coloneqq \{[q_{\mathtt{in}}(c), q_{\mathtt{in}}(d)] : (c, d) \in W_{s+1} \wedge \mathtt{p}(c, d) \leq \mathtt{p}(a, b)\}$$

to be the set of the DZs of the priority $\mathtt{p}(a, b)$ or higher. Clearly, $\mathcal{I}$ is uniformly α-computable from W_{s+1}, a, b as $\mathtt{p}$ is α-computable (lemma 6.1.11). The set $\mathcal{I}$ is α-finite as it is bounded up to α-computable encoding by W_{s+1} and $W_{s+1} \in L_\alpha$.

Recall definition 6.1.15: $\mathtt{conn}(\rho, \sigma, \mathcal{I})$ iff ρ and σ are connected through $\mathcal{I}$. So we have

$$M \coloneqq \{e \in \mathrm{dom}(q_{\mathtt{in}}) : \mathtt{conn}(q_{\mathtt{in}}(a), q_{\mathtt{in}}(e), \mathcal{I})\}.$$

The set M is uniformly α-computable from a, b and W_{s+1} since:

- $\mathcal{I}$ is α-finite,
- $\mathcal{I}$ is uniformly α-computable from a, b and W_{s+1},
- $\mathtt{conn}$ is α-computable on the domain where $\mathcal{I}$ is α-finite (lemma 6.1.18).

Finally M is α-finite since it is α-computable and bounded by $\mathrm{dom}(q_{\mathtt{in}}) \in L_\alpha$ assuming the hypothesis that $q_{\mathtt{in}}$ is α-finite.

Thus M is α-finite and uniformly α-computable from a, b and W_{s+1} as required. □

Lemma 6.1.41. The α-rationals ρ_c and ρ_d defined in the function *label_clearing* are computed in a uniform way, exist and are α-finite given that $q_{\texttt{in}}$ is α-finite.

Proof. The functions $\inf_Q(S)$ and $\sup_Q(S)$ can compute using the same algorithm for any arguments Q and S where Q contains all the α-rationals of the same order type δ and $\text{ot}[S] \subseteq \delta$, see proposition 5.4.6. Furthermore, $S = q_{\texttt{in}}[M]$ is uniformly α-computable since M is uniformly α-computable by lemma 6.1.40. Hence the α-rationals ρ_c and ρ_d are computed in a uniform way.

The α-finiteness of ρ_c and ρ_d follows from the fact that Q_c and Q_d contain only α-finite α-rationals.

By lemma 6.1.40, the set M is α-finite. Hence $q_{\texttt{in}}[M]$ has to be α-finite. By proposition 5.4.6 $\rho_c := \inf_{Q_c}(q_{\texttt{in}}[M])$ exists, i.e. is well-(first-order)-defined. By duality $\rho_d := \sup_{Q_d}(q_{\texttt{in}}[M])$ exists too.

Therefore $\rho_c \in L_\alpha$ and $\rho_d \in L_\alpha$ as required. □

Lemma 6.1.42. If the input function $q_{\texttt{in}}$ is α-finite, then the output function

$$q_{\texttt{out}} := \textit{label\_clearing}(q_{\texttt{in}}, W_{s+1}, s, p, u)$$

is also α-finite.

Proof. The function $q_{\texttt{out}}$ is different from the α-finite function $q_{\texttt{in}}$ on at most two labels in the label pair (c, d) of a priority u. By lemma 6.1.41 $q_{\texttt{out}}(c)$ and $q_{\texttt{out}}(d)$ have to be α-finite if changed. This change is α-finite and so $q_{\texttt{out}}$ has to be α-finite too. □

Remark 6.1.43. (Type of the label clearing function)
We assume that the argument $q_{\texttt{in}}$ to the function *label_clearing* is always α-finite. By lemma 6.1.42, the output of the function *label_clearing* has to be also α-finite. Therefore we can express the function with the type as

$$\textit{label\_clearing} : L_\alpha \times L_\alpha \times \alpha \times \alpha \times \alpha \to L_\alpha.$$

Hence *label_clearing* is clearly a type-1 function.

Proposition 6.1.44. The function *label_clearing* is α-computable.

Proof. By remark 6.1.43 the function *label_clearing* is a type-1 function, so it makes sense to talk about it being first-order definable over L_α. The function *label_clearing* is α-computable since:

- $q_{\texttt{in}}$ is α-finite by IH proposition 6.2.3iv,
- $\texttt{p}^{-1}$ is α-computable by lemma 6.1.11,
- $\texttt{ot}$ is α-computable by lemma 6.1.26,

- M is α-finite and uniformly α-computable by lemma 6.1.40,
- instructions on the lines 10 and 15 are α-computable (by the uniformity of $\inf_Q(S)$, $\sup_Q(S)$ and S) and the α-rationals ρ_c and ρ_d are well-defined and α-finite by lemma 6.1.41.

□

6.1.7 Limit labelling function

We define a function q_δ where $\delta = (t, r, v)$ is a limit triple.

The main idea in defining q_δ is that if some label $c \in \alpha_A \sqcup \alpha_B$ stops moving at some triple stage $\gamma < \delta$, then $q_\delta(c) := q_\gamma(c)$. Otherwise, the label c keeps moving and so it converges to some α-real point which is close to some α-rational ρ of an order type $\texttt{otLim}(\delta)$, see definition 6.1.25. However, there could be more labels that converge to the same α-real point. In such case, put the labels from α_A on the right of ρ and the labels from α_B on the left of ρ. This is to make sure that $q_\delta(b) < q_\delta(a)$ if still $(a, b) \notin W_h$ where $h = t$ if $\lim(t)$ and $h = t + 1$ otherwise. The idea is made precise and formal as follows.

Definition 6.1.45. (Limit of a labelling function)
Given a sequence of the labelling functions q_γ for a triple $\gamma < \delta$, we define the limit of this sequence as a function $q_\delta := \lim_{\gamma<\delta} q_\gamma$ as follows:

- Let
$$S(c) := \{q_\gamma(c) \in Q_\alpha : q_\gamma(c) \downarrow \wedge \gamma < \delta\} =$$
$$\{\sigma \in Q_\alpha : \exists\gamma < \delta \exists c \in \mathrm{dom}(q_\gamma).q_\gamma(c) = \sigma\}.$$
- Let
$$\hat{Q} := \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \texttt{otLim}(\delta)\}.$$
- $q_\delta(c) := q_\gamma(c)$ for $c \in \alpha_A \sqcup \alpha_B$ if
$$\exists\beta < \delta\forall\gamma[\beta \leq \gamma < \delta \implies q_\beta(c) = q_\gamma(c)].$$
Otherwise:
- $q_\delta(a) := \max\{\rho \in Q_\alpha : \mathrm{ot}(\rho) = \texttt{otLim}(\delta) \wedge \forall\gamma < \delta.\rho \leq q_\gamma(c)\} \cdot \triangleright^k = \sup_{\hat{Q}}(S(a)) \cdot \triangleright^k$ if $a \in \alpha_A$ where
$$k := \min\{k < t : \exists b \in \alpha_B.\mathtt{p}(a, b) = k\} =$$
$$\min\{k < t : \exists x \in W_h[\mathtt{p}(x) = k \wedge \pi_1(x) = a]\}$$
and π_1 is a projection to the first coordinate,
- $q_\delta(b) := \min\{\rho \in Q_\alpha : \mathrm{ot}(\rho) = \texttt{otLim}(\delta) \wedge \forall\gamma < \delta.q_\gamma(b) \leq \rho\} \cdot \triangleleft^l = \inf_{\hat{Q}}(S(b)) \cdot \triangleleft^l$ if $b \in \alpha_B$ where
$$l := \min\{l < t : \exists a \in \alpha_A.\mathtt{p}(a, b) = l\} =$$
$$\min\{l < t : \exists x \in W_h[\mathtt{p}(x) = l \wedge \pi_2(x) = b]\}$$

and π_2 is a projection to the second coordinate.

Properties

Lemma 6.1.46. (Uniform α-computability of the indices k and l)
Let $(a, b) = \mathtt{p}^{-1}(t)$.

- The index k is uniformly α-computable from a and W_h.
- The index l is uniformly α-computable from b and W_h.

Proof. The index k is uniformly α-computable from the parameters a and W_h as it is well-defined for a label a by the following formula:

$$\phi(a, k) := \exists x \in W_h[\mathtt{p}(x) = k \wedge \pi_1(x) = a] \wedge$$
$$\forall s < k \forall x \in W_h[\mathtt{p}(x) < k \implies \pi_1(x) \neq a]$$

which is $\Sigma_1(L_\alpha)$ as W_h is α-finite and both $\mathtt{p}$ and π_1 are α-computable.

The uniform α-computability of the index l follows by a dual argument. □

Lemma 6.1.47. Let γ be any triple. Let δ be a limit triple. Then:

i) $\mathrm{ot}[\mathrm{Im}(q_\gamma)] \subseteq \mathtt{ot}(\gamma)$,

ii) $\mathrm{ot}[\bigcup_{\gamma<\delta} \mathrm{Im}(q_\gamma)] \subseteq \mathtt{otLim}(\delta)$.

Proof. The statement i follows directly from the design of the labelling algorithm - to construct q_γ the algorithm places labels only on the α-rationals of an order type less than $\mathtt{ot}(\gamma)$. In particular, this is follows from the proof by induction using the following four observations:

- The strategy run (s, p) moves the labels to the order type
$$\mathtt{ot}(s, p, -1) + 1 < \mathtt{ot}(s, p, 0)$$
at most to construct the labelling function $q_{s,p,0}$: see the function *stragegy_run* lines 2, 5, 8; its subroutine *move_label_a* lines 21, 24, 26, 33 and its subroutine *move_label_b* lines 3, 6, 8, 15.

- The label clearing which constructs the function $q_{s,p,u+1}$ moves the labels to the order type $\mathtt{ot}(s, p, u) + 1 < \mathtt{ot}(s, p, u + 1)$ at most: see the function *label_clearing* lines 7, 9, 10, 11, 14, 15, 16.

- $\mathtt{otLim}(\delta) := \sup\{\mathtt{ot}(\gamma) : \gamma < \delta\}$ and $\mathtt{ot}(\delta) = \mathtt{otLim}(\delta) \cdot 2$ by definition 6.1.25.

- The limit labelling function q_δ moves the labels to the order type $\mathtt{ot}(\delta) = \mathtt{otLim}(\delta) \cdot 2$ at most by definition 6.1.45.

To see the statement ii, use the statement i and definition 6.1.25 to observe respectively that

$$\mathrm{ot}[\bigcup_{\gamma<\delta}\mathrm{Im}(q_\gamma)] \subseteq \sup\{\mathtt{ot}(\gamma) : \gamma < \delta\},$$

$$\mathtt{otLim}(\delta) := \sup\{\mathtt{ot}(\gamma) : \gamma < \delta\},$$

which implies $\mathrm{ot}[\bigcup_{\gamma<\delta}\mathrm{Im}(q_\gamma)] \subseteq \mathtt{otLim}(\delta)$ as required. □

Lemma 6.1.48. $S(c)$ is bounded in the order type:

$$\forall c \in \alpha_A \sqcup \alpha_B \forall\gamma < \delta[c \in \mathrm{dom}(q_\gamma) \implies \mathrm{ot}(q_\gamma(c)) < \mathtt{otLim}(\delta)].$$

Proof. Follows from lemma 6.1.47. □

For the remaining statements in this subsection we assume hypothesis 6.1.49 below. This is in fact the induction hypothesis for the proof of proposition 6.2.3v. The statements below are used to prove the inductive cases of the statements in proposition 6.2.3 assuming this IH.

Hypothesis 6.1.49. The function $q = \lambda\gamma c.q_\gamma(c)$ is a partial α-computable function with the domain $\{(\gamma, c) : \gamma < \delta \wedge c \in \mathrm{dom}(q_\gamma)\}$.

Fact 6.1.50. Hypothesis 6.1.49 implies that $S(c)$ is uniformly α-computable from the parameters c and δ.

Lemma 6.1.51. (Uniform α-computability of an α-rational $q_\delta(c)$)
Let $(a, b) = \mathtt{p}^{-1}(t)$. Assume hypothesis 6.1.49 about $q = \lambda\gamma c.q_\gamma(c)$. Then:

- $q_\delta(a)$ is α-finite and uniformly α-computable from δ, a and W_h.
- $q_\delta(b)$ is α-finite and uniformly α-computable from δ, b and W_h.

Proof. Note $q_\delta(a) := \inf_{\hat{Q}}(S(a)) \cdot \triangleright^k$. By lemma 6.1.48, the set $S(a)$ is bounded in order type, i.e. $\mathrm{ot}[S(a)] \subseteq \mathtt{otLim}(\delta)$. By hypothesis 6.1.49 and fact 6.1.50, $S(a)$ is α-computable. Thus $S(a)$ is α-finite. So the α-rational $\inf_{\hat{Q}}(S(a))$ exists and is uniformly α-computable from $S(a)$ and δ by proposition 5.4.6 and by the α-computability of $\mathtt{otLim}$ (lemma 6.1.26). The index k is uniformly α-computable from a and W_h by lemma 6.1.46. Hence $q_\delta(a)$ exists and is uniformly α-computable from $S(a)$, δ, a and W_h. But $S(a)$ is uniformly α-computable from a and δ by hypothesis 6.1.49 and fact 6.1.50. Hence $q_\delta(a)$ is uniformly α-computable from a, δ and W_h. Moreover $\inf_{\hat{Q}}(S) \in L_\alpha$ as $\hat{Q} \subseteq L_\alpha$. Hence as $k < \alpha$, so $q_\delta(a) \in Q_\alpha \cap L_\alpha$.

The α-finiteness and the uniform α-computability of $q_\delta(b)$ follows by a dual argument. □

Proposition 6.1.52. (α-computability of $q = \lambda\gamma c.q_\gamma(c)$ for $\gamma \leq \delta$)
Assume hypothesis 6.1.49 about $q = \lambda\gamma c.q_\gamma(c)$. Then q_δ is α-computable and q is α-computable on the domain where $\gamma \leq \delta$ and $c \in \mathrm{dom}(q_\gamma)$.

Proof. Define $\hat{Q} := \{\rho \in Q_\alpha \cap L_\alpha : \mathrm{ot}(\rho) = \mathtt{otLim}(\delta)\}$. Note that $S(c)$ is uniformly α-computable from the parameters c and δ by hypothesis 6.1.49 and fact 6.1.50. Also $S(c)$ is bounded as all α-rationals within are bounded in order type by $\mathtt{otLim}(\delta) < \alpha$ by lemma 6.1.48. Thus $S(c)$ is α-finite and we have a type-1 α-computable function $c \mapsto S(c)$ mapping a label $c \in \alpha_A \sqcup \alpha_B$ to its set $S(c)$ defined above. Similarly, we have α-computable functions $k : \alpha_A \cap \mathrm{dom}(q_\delta) \to h$ and $l : \alpha_B \cap \mathrm{dom}(q_\delta) \to h$ defined by the $\Sigma_1(L_\alpha)$ formula $\phi(a, k)$ and its dual above in lemma 6.1.46. Note that the function $\inf_Q : \alpha \to \alpha$ taking an index of an α-finite set and returning an index of an α-finite rational is α-computable by proposition 5.4.5. Dually, the function $\sup_Q : \alpha \to \alpha$ is also α-computable. Using these functions express the function $q_\delta = q_{t,r,v}$ as follows:

$$q_{t,r,v}(c) = \begin{cases} q_\beta(c) & \exists\beta < \delta\forall\gamma < \delta[\beta \leq \gamma \implies q_\beta(c) = q_\gamma(c)] \\ \inf_{\hat{Q}}(S(c)) \cdot \triangleright^{k(c)} & c \in \alpha_A \\ \sup_{\hat{Q}}(S(c)) \cdot \triangleleft^{l(c)} & c \in \alpha_B \end{cases}$$

From this definition we see that q_δ is α-computable by hypothesis 6.1.49 on q and the α-computability of the functions S, k, l, $\inf_Q$, $\sup_Q$ and $\mathtt{otLim}$ (lemma 6.1.26) whose index (or program) is α-computable from the triple $\delta = (t, r, v)$. Note that

$$\forall c \in \alpha_A \sqcup \alpha_B . q(t, r, v, c) = q_{t,r,v}(c).$$

Therefore q can compute the output for any input (γ, c) where $\gamma \leq \delta$ and $c \in \mathrm{dom}(q_\gamma)$ as required. □

6.2 Proof and verification

First we establish some properties and lemmas about the constructed labelling sequence $q = \{q_s\}_{s<\alpha}$. Next we define a semicomputable cut $C \subseteq Q_\alpha$ using the constructed sequence q. Finally we complete the proof of lemma 6.1.1 and Semicomputable Cut Existence Theorem 6.0.1 using such cut C.

6.2.1 Labelling function properties

Remark 6.2.1. (Domain of the labelling function)

- $\mathrm{dom}(q_s) = \{(a, b) \in \alpha_A \times \alpha_B : \mathtt{p}(a, b) < s\} = \mathtt{p}^{-1}[s]$,
- $\mathrm{dom}(q_{s,p}) = \{(a, b) \in \alpha_A \times \alpha_B : \mathtt{p}(a, b) < \max(s, p)\} = \mathtt{p}^{-1}(\max(s, p))$,

- $\mathrm{dom}(q_{s,p,u})$
  $= \{(a,b) \in \alpha_A \times \alpha_B : u = -1 \wedge \mathtt{p}(a,b) < s \vee u \geq 0 \wedge \mathtt{p}(a,b) < \max(s,p)\}$
  $= \{(a,b) \in \alpha_A \times \alpha_B : \mathtt{p}(a,b) < \max(s, p + \min(u,0))\}$
  $= \mathtt{p}^{-1}[\max(s, p + \min(u,0))]$.

Lemma 6.2.2. (α-finiteness of the domain of the labelling function)
The set $\mathrm{dom}(q_{s,p,u})$ is α-finite for any triple (s,p,u) for which $q_{s,p,u}$ is defined, i.e.

$$0 \leq s < \alpha, 0 \leq p \leq s+1, -1 \leq u \leq s+1.$$

Proof. The domain of $q_{s,p,u}$ is given explicitly in remark 6.2.1 above as $\mathrm{dom}(q_{s,p,u}) = \mathtt{p}^{-1}[t]$ for $t = \max(s, p+\min(u,0)) < \alpha$. As t is an α-finite set, α admissible and p α-computable, so $\mathrm{dom}(q_{s,p,u})$ has to be α-finite as required. □

Proposition 6.2.3. (Properties of the labelling function)
For any triple (s,p,u) for which $q_{s,p,u}$ is defined, the partial labelling function $q_{s,p,u} : \alpha_A \times \alpha_B \rightharpoonup Q_\alpha$ satisfies the following conditions:

i) $q_{s,p,u}$ is well-defined on its domain (explicitly given in remark 6.2.1),

ii) $\mathrm{Im}(q_{s,p,u}) \subseteq L_\alpha$,

iii) $q_{s,p,u}$ is a type-1 function,

iv) $q_{s,p,u} \in L_\alpha$,

v) The function

$$q : \alpha \times \alpha \times (\{-1\} \cup \alpha) \times (\alpha_A \sqcup \alpha_B) \rightharpoonup Q_\alpha$$

defined using the λ-term as $q := \lambda spuc.q_{s,p,u}(c)$ is α-computable.

Proof. (Of proposition 6.2.3)

$q_{s,p,u}$ as a type-1 function - property iii

In order to be able to talk about the α-computability and α-finiteness of the labelling function $q_{s,p,u}$, we need that it is a type-1 function, i.e. both its domain and codomain is a type-1 object, i.e. a subset of α up to the α-computable coding. The domain of $q_{s,p,u}$ is α-finite by lemma 6.2.2, so $\mathrm{dom}(q_{s,p,u}) \subseteq \alpha$ as required. The codomain of the function $q_{s,p,u}$ is α-rationals Q_α. Q_α coincides with L_α (which is just α up to α-computable coding) for an infinite regular cardinal α, but in other cases, $Q_\alpha \supset L_\alpha$. For example, Kleene's $\mathcal{O} \in Q_{\omega_1^{CK}}$, but $\mathcal{O} \notin L_{\omega_1^{CK}}$. However, the statement ii) states that $\mathrm{Im}(q_{s,p,u}) \subseteq L_\alpha$. Hence we can impose implicitly codomain $Q_\alpha \cap L_\alpha = L_\alpha$ on the labelling function $q_{s,p,u}$ to make it type-1. Therefore whenever the statement ii) holds for $q_{s,p,u}$, so does the statement iii).

Induction and base case - properties i, ii, iv, v

The rest of the proof is done by an induction on the triple (s, p, u). For the base case, when (s, p, u) is the first initial triple $(0, 0, -1)$, then $q_{0,0,-1} = \emptyset$. Hence trivially all conditions i) - iv) are satisfied.

For the inductive case, we have to prove that the properties hold at strategy, clearing and limit triples. We do not need to prove that the properties hold at initial triples because every initial triple is equivalent to clearing, limit or the first initial triple. See definition 6.1.21.

The function $q = \lambda spc.q_{s,p,u}(c)$ under the statement v) is α-computable as it is given by an algorithm consisting of the strategies (section 6.1.5), label clearing (section 6.1.6) and the limit function construction (section 6.1.7) which do not use any oracle, but only an α-finite part of the α-computably enumerable set W for particular values of s, p, u and c. We clarify by induction that q indeed computes the value $q_{s,p,u}(c)$ for any label $c \in \mathrm{dom}(q_{s,p,u})$ assuming that q computes the values for all predecessor triples $(s', p', u') < (s, p, u)$.

Inductive case for a strategy triple - properties i, ii

Assume that $(s, p, 0)$ is a strategy triple. The predecessor of $(s, p, 0)$ is the initial triple $(s, p, -1)$. By IH, let the statements i) and ii) hold for the labelling function $q_{s,p,-1}$. We have to show that they hold for the labelling function $q_{s,p,0}$. The labelling function $q_{s,p,0}$ is constructed by a strategy run (s, p) on the input function $q_{s,p,-1}$. This strategy run is associated with some pair of labels, say $(a, b) \in \alpha_A \times \alpha_B$. Only these two labels a and b can be moved by the strategy run (s, p).

Hence by IH $q_{s,p,0}$ is well-defined on $\mathrm{dom}(q_{s,p,0}) - \{a, b\}$ and $\mathrm{Im}(q_{s,p,0} - \{a, b\}) \subseteq L_\alpha$. Thus we need to show that $q_{s,p,0}$ is well-defined on a and b to conclude i) and that $q_{s,p,0}(a) \in L_\alpha$ and $q_{s,p,0}(b) \in L_\alpha$ to conclude ii). In other words, we have to show that for every strategy run for the pair (a, b) there exist α-rationals onto which the labels a and b can be placed and that these α-rationals are α-finite and satisfy the required conditions mentioned in section 6.1.5. By this follows from lemma 6.1.34 and IH property iv) that $q_{\mathtt{in}} = q_{s,p,-1} \in L_\alpha$.

Hence the conditions i) and ii) hold for the labelling function $q_{s,p,0}$.

Inductive case for a clearing triple - properties i, ii

Let $(s, p, u+1)$ be a clearing triple. Let $\mathsf{p}(c, d) = u$. By IH $q_{s,p,u}$ is well-defined on its domain and $\mathrm{Im}(q_{s,p,u}) \subseteq L_\alpha$. Note that $q_{s,p,u+1}$ can be different from $q_{s,p,u}$ on at most two labels c and d. In particular $q_{s,p,u+1}(c) \in \{q_{s,p,u}(c), \rho_c\}$ and $q_{s,p,u+1}(d) \in \{q_{s,p,u}(c), \rho_d\}$ where ρ_c and ρ_d are well-defined α-finite α-rationals by lemma 6.1.41 and α-finiteness of $q_{s,p,u}$ (IH property iv)). Hence the properties i) and ii) hold for the labelling function $q_{s,p,u+1}$ as required.

Inductive case for a strategy triple - property iv

Let $(s, p, 0)$ be a strategy triple. By IH $q_{s,p,-1}$ is α-finite. But $q_{s,p,0}$ is different from $q_{s,p,-1}$ on at most two labels a and b since every strategy run can move only two labels. Hence the strategy run (s, p) can make at most only an α-finite change to $q_{\mathtt{in}} = q_{s,p,-1}$ and so $q_{\mathtt{out}} = q_{s,p,0}$ has to be α-finite too. Hence the property iv) holds for the function $q_{s,p,0}$ as required.

Inductive case for a clearing triple - property iv

Let $(s, p, u + 1)$ be a clearing triple. By IH the function $q_{s,p,u}$ is α-finite. The function $q_{s,p,u+1}$ is different from $q_{s,p,u}$ on at most one label pair (the one of a priority u), see section 6.1.6. As this is an α-finite change, $q_{s,p,u+1}$ has to be α-finite too.

Inductive case for a strategy triple - property v

Let $(s, p, 0)$ be a strategy triple. By IH assume that $q = \lambda spuc.q_{s,p,u}(c)$ can compute the output for any input tuple (s', p', u', c) where $(s', p', u') < (s, p, 0)$ and $c \in \mathrm{dom}(q_{s',p',u'})$.

By the α-finiteness of $q_{\mathtt{in}} = q_{s,p,-1}$ (IH property iv)) and by proposition 6.1.39 all the instructions for a strategy run (s, p) for some pair $(a, b) \in \alpha_A \times \alpha_B$ are *uniformly* α-computable from $q_{\mathtt{in}}$, W_{s+1}, s and p. *Hence* q can compute the output for an input tuple $(s, p, 0, c)$ where $c \in \alpha_A \times \alpha_B$ is any label in the domain $\mathrm{dom}(q_{s,p,0})$. Therefore the property v) on q holds at the strategy triple $(s, p, 0)$ as required.

Inductive case for a clearing triple - property v

Let $(s, p, u + 1)$ be a clearing triple. By IH assume that $q = \lambda spuc.q_{s,p,u}(c)$ can compute the output for any input tuple (s', p', u', c) where $(s', p', u') < (s, p, u+1)$ and $c \in \mathrm{dom}(q_{s',p',u'})$.

By the α-finiteness of $q_{\mathtt{in}} = q_{s,p,u}$ (IH property iv)) and by proposition 6.1.44 all the instructions for label clearing are *uniformly* α-computable from $q_{\mathtt{in}}$, W_{s+1}, s, p and u. *Hence* q can compute the output for an input tuple $(s, p, u+1, c)$ where $c \in \alpha_A \times \alpha_B$ is any label in the domain $\mathrm{dom}(q_{s,p,u+1})$. Therefore the property v) on q holds at the clearing triple $(s, p, u + 1)$ as required.

Inductive case for a limit triple - properties i, ii, v

Let $\delta = (t, r, v)$ be a limit triple. Let

$$\begin{aligned} S(c) :=& \{q_\gamma(c) \in Q_\alpha : q_\gamma(c) \downarrow \wedge \gamma < \delta\} \\ =& \{\sigma \in Q_\alpha : \exists \gamma < \delta \exists c \in \mathrm{dom}(q_\gamma) . q_\gamma(c) = \sigma\}. \end{aligned}$$

By IH property v) the function $q = \lambda\gamma c.q_\gamma(c)$ is α-computable on the domain where $\gamma < \delta$ and $c \in \mathrm{dom}(q_\gamma)$. Hence hypothesis 6.1.49 is true.

By hypothesis 6.1.49 and lemma 6.1.51 $q_\delta(c)$ is α-finite and well-defined for any $c \in \{a, b\}$ where $\mathtt{p}(a, b) < t$. Therefore the labelling function $q_\delta : \alpha_A \sqcup \alpha_B \to Q_\alpha$ is well-defined on its domain and $\mathrm{Im}(q_\delta) \subseteq L_\alpha$. Hence i) and ii) hold for $q_\delta = q_{t,r,v}$ as required.

By hypothesis 6.1.49, IH property v) and proposition 6.1.52, the function $q = \lambda\gamma c.q_\gamma(c)$ is α-computable on the domain where $\gamma \leq \delta$ and $c \in \mathrm{dom}(q_\gamma)$. Therefore the property v) holds for q at the limit triple $\delta = (t, r, v)$ as required.

Inductive case for a limit triple - property iv

Let (t, r, v) be a limit triple. Note that $q_{t,r,v} = \lambda c.q(t, r, v, c)$. So the labelling function $q_{t,r,v}$ is α-computable since $\lambda c.q(t, r, c)$ is α-computable by the property v) proved above. Furthermore $\mathrm{dom}(q_{t,r,v}) \in L_\alpha$ by lemma 6.2.2. Hence $q_{t,r,v} \in L_\alpha$ by proposition 3.2.14 and thus the property iv) holds for the limit triple (t, r, v).

This completes the proof of proposition 6.2.3. □

Lemma 6.2.4. (Consistency of the labelling function)
For any triple γ, the labelling function q_γ is consistent:

$$\forall (a, b) \in \alpha_A \times \alpha_B [q_\gamma(a) < q_\gamma(b) \implies (a, b) \in W].$$

Proof. This follows directly from the labelling function construction in sections 6.1.5 to 6.1.7. □

Lemma 6.2.5. (Monotonicity of the label movement)

- $\forall s, t < \alpha \forall a \in \alpha_A [s \leq t \implies q_s(a) \geq q_t(a)]$
- $\forall s, t < \alpha \forall b \in \alpha_B [s \leq t \implies q_s(b) \leq q_t(b)]$

Proof. For the label $a \in \alpha_A$, if some strategy moves, then only more left on the α-rational line Q_α. For the limit construction, the label a is moved to a generalized infimum of the sequence $q_s(a)$. This is also to the left as required. □

6.2.2 Strategy termination

Lemma 6.2.6. (Strategy termination)
Every (a, b)-strategy stops acting within α-finite time.

Proof. If $(a, b) \notin W$, then the strategy never acts. If there ever exists a stage t s.t. $q_t(a) < q_t(b)$, then the strategy does not act again by lemma 6.2.5. Hence let s be large enough that $(a, b) \in W_s$ and by induction all strategies of a higher priority have stopped acting, yet $q_s(b) < q_s(a)$.

The only way that the (a, b)-strategy could act is that it would move either the label a or the label b. The cases for moving the label b are symmetric to moving the label a, hence let us consider only the case that the label a may be moved.

Let B_o be the set of obstacles for the label a. If the (a, b)-strategy acts and B_o ever gets empty, then it will move a to the left of all labels and afterwards $q_{s+1}(b) < q_{s+1}(a)$ and the strategy will never act again.

So suppose that $B_o \neq \emptyset$.

If $q_s(a)$ is inside an interval protected by higher priority dead zones, then it cannot be moved and will not be moved since the higher priority strategies have stopped acting.

If $q_s(a)$ is adjacent to the right endpoint of the maximal set Y that contains some label $b_o \in B_o$ and every other label in B_o that is right of b_o where the interval is protected by higher priority strategies, then $q_s(a)$ will not be moved again since the maximal set will not shrink as all higher priority strategies have stopped acting and nothing will be placed between the right endpoint and the label $q_s(a)$ since the (a, b)-strategy will protect the interval $[q_s(b), q_s(a)]$ from lower priority strategies placing the label inside.

Hence the only way that the (a, b)-strategy could act is that it would place a adjacent to B_o from the right. Therefore B_o, the set of obstacles for a, has to be shrinking and the label a advancing more towards left.

Define

$$B_{o'} := \{b_o \in \alpha_B : b_o \in \text{dom}(q_s) \wedge q_s(b_o) \in [q_s(b), q_s(a)] \wedge (a, b_o) \notin W_{s+1}\}$$

to be the set of the obstacles to a that are in the dead zone. Note that as $\text{dom}(q_s)$ is α-finite by lemma 6.2.2, so $B_{o'}$ is bounded by some $t < \alpha$, i.e. $B_{o'} \subseteq t \in \alpha_B$. Hence if the strategy (a, b) continues acting, $B_{o'}$ gets emptied by the stage t, then $q_t(a) < q_t(b)$ and the strategy does not act again.

Therefore by the induction on the priority of the pairs $(a, b) \in \alpha_A \times \alpha_B$ all the strategies terminate within an α-finite number of the steps as required. □

6.2.3 Dead Zone Lemma

Recall definition 6.1.20: two labels are connected iff they are in a connected union of PDZs.

Lemma 6.2.7. (Dead Zone Lemma)
Assume

$$A \times B \subseteq W \wedge \overline{A} \times \overline{B} \subseteq \overline{W} \wedge W \in \Sigma_1(L_\alpha).$$

Then

- If a_1 and b_1 are connected, then $a_1 \in A \iff b_1 \notin B$,
- If a_1 and a_2 are connected, then $a_1 \in A \iff a_2 \in A$,
- If b_1 and b_2 are connected, then $b_1 \notin B \iff b_2 \notin B$.

Proof. Let *HPPDZ* abbreviate *higher priority permanent dead zone* and HPPDZs its plural form.

Intersecting dead zones share a label, hence it is sufficient to prove the lemma under the assumptions that the labels a_1, a_2, b_1, b_2 are contained in the same PDZ since the conclusion follows by transitivity.

We prove the lemma by induction on the priority of a PDZ. Suppose that a_1, b_1 are in a PDZ $[q_\alpha(b), q_\alpha(a)]$ declared by the strategy (a, b). Hence $(a, b) \in W$. Redefine the sets

$$A_o := \{a_o \in \alpha_A : q_\alpha(b) < q_\alpha(a_o) < q_\alpha(a) \wedge (a_o, b) \notin W\},$$
$$B_o := \{b_o \in \alpha_B : q_\alpha(b) < q_\alpha(b_o) < q_\alpha(a) \wedge (a, b_o) \notin W\},$$

where A_o (B_o resp.) is the set of obstacles to b (to a resp.) that are in the PDZ $[q_\alpha(b), q_\alpha(a)]$. The interval $[q_\alpha(b), q_\alpha(a)]$ is a dead zone, hence both $A_o \neq \emptyset$ and $B_o \neq \emptyset$.

Let $\phi(a_1, b_2)$ denote the statement

$$a_1 \in A \iff b_2 \notin B \iff a \in A \iff b \notin B.$$

To prove Dead Zone Lemma we show for arbitrary labels $a_1 \in \alpha_A$ and $b_2 \in \alpha_B$ that if $\{q_\alpha(a_1), q_\alpha(b_2)\} \subseteq [q_\alpha(b), q_\alpha(a)]$, then $\phi(a_1, b_2)$.

Unlike for $\alpha = \omega$, in general the leftmost α_A-obstacle to b and the rightmost α_B-obstacle to a may not be defined, i.e. it may be the case that
$\min(q_\alpha[A_o]) \uparrow$ or $\max(q_\alpha[B_o]) \uparrow$. Hence we have to reason in general with quantifiers over the obstacle sets. There are 2 cases to consider:

Case 1: $\forall a_o \in A_o \forall b_o \in B_o . q_\alpha(b_o) < q_\alpha(a_o)$

There has to be a HPPDZ interval I_a (possibly an infinite union of HPPDZs) on the left of the label a. Let I_a denote the maximal possible such interval. There cannot be any label between I_a and a. Otherwise (a, b) strategy would have moved the label a to the left of such label not in I_a. The set I_a has to contain at least one B_o obstacle. Otherwise (a, b) strategy would have moved the label a to the right of B_o, i.e. between B_o and A_o.

Similarly, there is a maximal HPPDZ interval I_b on the immediate right of the label b and I_b contains some A_o obstacle.

Hence $I_a \cap I_b \neq \emptyset$ and the labels in the open interval $(q_\alpha(b), q_\alpha(a))$ are connected by HPPDZs. All such labels are more specifically in the interval $I = (I_a \cup I_b) \cap (q_\alpha(b), q_\alpha(a))$. Thus by IH for arbitrary labels a_1 and b_2 in I we have $a_1 \in A \iff b_2 \notin B$. Note that $A_o \cup B_o \subseteq I$. Let $a_o \in A_o \cap I$ and let $b_o \in B_o \cap I$. Then

$$a \in A \implies b_o \notin B \implies a_o \in A \implies b \notin B \implies a \in A$$

using the facts $(a, b_o) \notin W$, IH, $(a_o, b) \notin W$, $(a, b) \in W$ respectively. Thus

$$a \in A \iff b_o \notin B \iff a_o \in A \iff b \notin B$$

and so $\phi(a_o, b_o)$.

Let a_1 and b_2 be any labels in the PDZ $[q_\alpha(b), q_\alpha(a)]$. WLOG let $a_1 \neq a$ and $b_2 \neq b$. As $\{a_1, a_o, b_2, b_o\} \subseteq q_\alpha^{-1}[I]$ and I is a HPPDZ interval, so the labels a_1, a_o, b_2, b_o are connected and

$$a_1 \in A \iff b_o \notin B \iff a_o \in A \iff b_2 \notin B$$

by IH. Combining this with the statement $\phi(a_o, b_o)$ above we get

$$a \in A \iff b_2 \notin B \iff a_1 \in A \iff b \notin B$$

and so $\phi(a_1, b_2)$ as required.

Case 2: $\exists a_o \in A_o \exists b_o \in B_o . q_\alpha(a_o) < q_\alpha(b_o)$

Let $a_o \in A_o$ and $b_o \in B_o$ s.t. $q_\alpha(a_o) < q_\alpha(b_o)$. So $(a_o, b_o) \in W$. Note we have

$$q_\alpha(b) < q_\alpha(a_o) < q_\alpha(b_o) < q_\alpha(a).$$

Thus

$$a \in A \implies b_o \notin B \implies a_o \in A \implies b \notin B \implies a \in A$$

using $(a, b_o) \notin W$, $(a_o, b_o) \in W$, $(a_o, b) \notin W$ and $(a, b) \in W$ respectively. Hence the following statement $\phi(a_o, b_o)$ is true:

$$a \in A \iff b_o \notin B \iff a_o \in A \iff b \notin B \tag{6.3}$$

Next extend PDZ $[q_\alpha(b), q_\alpha(a)]$ to the maximum possible interval I by HPPDZs since we cannot move the labels to HPPDZs anyway. Then an arbitrary label $a_1 \in q_\alpha^{-1}[[q_\alpha(b), q_\alpha(a)]]$ cannot be moved out of I since there is its mutual obstacle $b_1 \in q_\alpha^{-1}[I]$ (i.e. $(a_1, b_1) \notin W$) that prevents a_1 being moved out of $[q_\alpha(b), q_\alpha(a)]$. Depending on where a_1 and b_1 are, we prove case by case $\phi(a_1, b_1)$.

Below follows the proof of $\phi(a_1, b_1)$ where a_1 is arbitrary and b_1 depends on a_1. To prove $\phi(a_1, b_2)$ for arbitrary labels a_1 and b_2, first apply a symmetric proof to prove $\phi(a_2, b_2)$ for an arbitrary label b_2 and a_2 dependent on b_2. Once we have $\phi(a_1, b_1)$ and $\phi(a_2, b_2)$ for arbitrary labels a_1 and b_2, we can conclude $\phi(a_1, b_2)$ as required.

Case 2.1: $q_\alpha[B_o] \cap [q_\alpha(a_1), q_\alpha(a)] = \emptyset$

The label a_1 has to be in a HPPDZ interval I_1. Otherwise the strategy (a, b) would have moved the label a to the left of the label a_1. Let I_1 be the maximum such possible interval. The label a cannot be put to the left of the interval I_1 as I_1 contains some obstacle $b_o \in B_o$. WLOG choose $a_o \in A_o$ and b_o so that they satisfy $q_\alpha(a_o) < q_\alpha(b_o)$. Also a_1 is not moved out of I_1, hence using the maximality of I_1 for the label a_1 there is some obstacle $b_1 \in I_1$, i.e. $(a_1, b_1) \notin W$ and $q_\alpha(b_1) < q_\alpha(a_1)$. But then b_o and a_1 are already connected through I_1. Hence by IH $b_o \notin B \iff a_1 \in A$. Using this and statement (6.3) we get

$$a \in A \iff b_1 \notin B \iff a_1 \in A \iff b \notin B$$

and so $\phi(a_1, b_1)$ as required.

Case 2.2: $q_\alpha[B_o] \cap [q_\alpha(a_1), q_\alpha(a)] \neq \emptyset$

There is $b_o \in B_o$ s.t. $q_\alpha(b_o) \in [q_\alpha(a_1), q_\alpha(a)]$. So $q_\alpha(a_1) < q_\alpha(b_o) < q_\alpha(a)$ and $(a_1, b_o) \in W$. WLOG choose $a_o \in A_o$ and b_o so that they satisfy $q_\alpha(a_o) < q_\alpha(b_o)$.

Case 2.2.1: $\exists a_o \in A_o . q_\alpha(a_o) < q_\alpha(b_1)$

WLOG let a_o satisfy the condition $q_\alpha(a_o) < q_\alpha(b_1)$. So $(a_o, b_1) \in W$. Note we have

$$q_\alpha(b) < q_\alpha(a_o) < q_\alpha(b_1) < q_\alpha(a_1) < q_\alpha(b_o) < q_\alpha(a).$$

Hence

$$a_1 \in A \implies b_1 \notin B \implies a_o \in A \implies b_o \notin B \implies a_1 \in A$$

using $(a_1, b_1) \notin W$, $(a_o, b_1) \in W$, statement (6.3) and $(a_1, b_o) \in W$ respectively. Thus

$$a_1 \in A \iff b_1 \notin B \iff a_o \in A \iff b_o \notin B.$$

Therefore by statement (6.3) again we conclude $\phi(a_1, b_1)$ as required.

Case 2.2.2: $\forall a_o \in A_o . q_\alpha(b_1) < q_\alpha(a_o)$

Note that

$$q_\alpha(b), q_\alpha(b_1) < q_\alpha(a_o) < q_\alpha(a_1) < q_\alpha(b_o) < q_\alpha(a).$$

We claim $b \notin B \iff b_1 \notin B$.

- Subcase $q_\alpha(b) < q_\alpha(b_1)$:
  Note

  $$q_\alpha(b) < q_\alpha(b_1) < q_\alpha(a_o) < q_\alpha(a_1) < q_\alpha(b_o) < q_\alpha(a).$$

  By symmetry of Case 2.1 we have $\phi(a_2, b_1)$ for the given b_1 and a_2 dependent upon it. Hence $b \notin B \iff b_1 \notin B$.

  Note that $\phi(a_1, b_1)$ does not follow immediately by symmetry since in Case 2.1 we show the existence of such b_1 satisfying the necessary conditions implying $\phi(a_1, b_1)$. But here we first start with a_1 and fix it. On the other hand, the symmetry proof starts from b_1 first.

- Subcase $q_\alpha(b) = q_\alpha(b_1)$:
  Note

  $$q_\alpha(b) = q_\alpha(b_1) < q_\alpha(a_o) < q_\alpha(a_1) < q_\alpha(b_o) < q_\alpha(a).$$

  So $b = b_1$ and $b \notin B \iff b_1 \notin B$.

- Subcase $q_\alpha(b) > q_\alpha(b_1)$:
  Note

  $$q_\alpha(b_1) < q_\alpha(b) < q_\alpha(a_o) < q_\alpha(a_1) < q_\alpha(b_o) < q_\alpha(a).$$

  As $b, b_1 \in q_\alpha^{-1}[I]$, the labels b and b_1 have to be connected by HPPDZs and so $b \notin B \iff b_1 \notin B$ by IH.

In all subcases above we have $b \notin B \iff b_1 \notin B$. Combining this with statement (6.3) we have $b_1 \notin B \iff b_o \notin B$. Notice $q_\alpha(a_1) < q_\alpha(b_o)$, so $(a_1, b_o) \in W$. Thus

$$a_1 \in A \implies b_1 \notin B \implies b_o \notin B \implies a_1 \in A$$

using $(a_1, b_1) \notin W$, $b_1 \notin B \iff b_o \notin B$ and $(a_1, b_o) \in W$ respectively. Hence

$$a_1 \in A \iff b_1 \notin B \iff b_o \notin B.$$

Using this and statement (6.3) we conclude $\phi(a_1, b_1)$ for Case 2.2.2 as required.

In all possible cases we have $\phi(a_1, b_2)$ as needed. This completes the proof of Dead Zone Lemma. □

6.2.4 Defining the cut

Definition 6.2.8. (Cut C in $Q_\alpha \cap L_\alpha$)
Define the cut C in the α-finite α-rationals and its complement[3] D by:

- $C := \{\rho \in Q_\alpha \cap L_\alpha : \exists b \in \overline{B} \exists s < \alpha[\rho \leq q_s(b) \downarrow$
  or $\{\rho, q_\alpha(b)\}$ is a subset of a PDZ $]\}$.

- $D := \{\rho \in Q_\alpha \cap L_\alpha : \exists a \in \overline{A} \exists s < \alpha[q_s(a) \downarrow \leq \rho$
  or $\{\rho, q_\alpha(a)\}$ is a subset of a PDZ $]\}$.

The following lemma establishes that C is a cut in $Q_\alpha \cap L_\alpha$.

Lemma 6.2.9. (Closure of C and D under $<_{Q_\alpha}$)

- C is downwards closed in $Q_\alpha \cap L_\alpha$:
$$\forall \sigma, \rho \in Q_\alpha \cap L_\alpha[\sigma < \rho \in C \implies \sigma \in C].$$

- D is upwards closed in $Q_\alpha \cap L_\alpha$:
$$\forall \sigma, \rho \in Q_\alpha \cap L_\alpha[\sigma > \rho \in D \implies \sigma \in D].$$

Proof. Let $\sigma, \rho \in Q_\alpha \cap L_\alpha$ and $\sigma < \rho \in C$. If $\exists b \in \overline{B} \exists s < \alpha.\rho \leq q_s(b) \downarrow$, then clearly $\sigma \leq q_s(b) \downarrow$ and so $\sigma \in C$. If $\exists b \in \overline{B}$ s.t. $\{\rho, q_\alpha(b)\}$ is a subset of a PDZ, then $\sigma < q_\alpha(b)$ or $q_\alpha(b) \leq \sigma < \rho$. If $\sigma < q_\alpha(b)$, then $\sigma \in C$. If $q_\alpha(b) \leq \sigma < \rho$, then σ is in the PDZ with $q_\alpha(b)$ and so $\{\sigma, q_\alpha(b)\}$ is a subset of the PDZ. Hence $\sigma \in C$. In all cases $\sigma \in C$. Therefore C is downwards closed.

The proof that D is upwards closed in $Q_\alpha \cap L_\alpha$ is symmetric. □

Lemma 6.2.10. $C \cap D = \emptyset$

Proof. Assume for a contradiction that $\exists \rho \in Q_\alpha.\rho \in C \cap D$.
As $\rho \in C$, so $\exists b \in \overline{B} \exists s < \alpha[\rho \leq q_s(b) \downarrow$ or $\{\rho, q_\alpha(b)\}$ is a subset of a PDZ].
As $\rho \in D$, so $\exists a \in \overline{A} \exists s < \alpha[q_s(a) \downarrow \leq \rho$ or $\{\rho, q_\alpha(a)\}$ is a subset of a PDZ].
WLOG let s be large enough so that $q_s(a) \downarrow \wedge q_s(b) \downarrow$. Since $a \in \overline{A} \wedge b \in \overline{B}$, we have $(a, b) \in \overline{W}$. Therefore $q_s(b) < q_s(a)$ by construction. There are four cases:

1. Case $q_s(a) \leq \rho \leq q_s(b)$: this is impossible since $q_s(b) < q_s(a)$.

2. Case $\{\rho, q_\alpha(b)\}$ is a subset of a PDZ and $q_s(a) \leq \rho$: since $(a, b) \notin W$ we have $q_\alpha(b) \leq q_\alpha(a) \leq q_s(a) \leq \rho$. Thus $q_\alpha(a)$ is in the same PDZ as $q_\alpha(b)$, so a and b are connected, but $a \notin A \wedge b \notin B$ which leads to the contradiction of lemma 6.2.7.

[3] For the proof, see proposition 6.2.15.

3. Case $\rho \leq q_s(b)$ and $\{\rho, q_\alpha(a)\}$ is a subset of a PDZ: this is symmetric to the case above.

4. Case $\{\rho, q_\alpha(a)\}$ is a subset of a PDZ and $\{\rho, q_\alpha(b)\}$ is a subset of a PDZ: thus $q_s(a)$ and $q_s(b)$ are connected and by lemma 6.2.7 $a \in A \iff b \notin B$ which leads to the contradiction.

□

Lemma 6.2.11. Assume $C \subseteq E \wedge D \subseteq \overline{E}$. Then $A = A_E \wedge B = B_E$.

Proof. We show

$$A = A_E \coloneqq \{a \in \alpha_A : \exists s < \alpha . q_s(a) \in E\}.$$

If $a \notin A$, then

$$\forall s < \alpha (q_s(a) \downarrow \implies q_s(a) \in D \subseteq \overline{E}),$$

and thus $a \notin A_E$.

If $a \in A$, then since $\mathcal{K}_{\text{nt}}(A, B)$, we have $\exists b \notin B.(a, b) \in W$ by proposition 4.2.7. By lemma 6.2.6, let s be a stage s.t. (a, b) strategy and all other strategies of a higher priority have stopped acting. Two cases are possible: $q_s(a) < q_s(b)$ or $[q_s(b), q_s(a)]$ is a PDZ. Each implies $q_s(a) \in C \subseteq E$ and thus $a \in A_E$. Therefore $A = A_E$ as required.

By a symmetric proof $B = B_E$. □

6.2.5 Proof completion

We complete the proof of Semicomputable Cut Existence Theorem 6.0.1.

Proof. (Of lemma 6.1.1) We use the labelling algorithm in section 6.1 to construct a labelling sequence

$$q = \{q_s : \alpha_A \sqcup \alpha_B \mapsto Q_\alpha \cap L_\alpha\}_{s<\alpha}.$$

This sequence is α-computable by proposition 6.2.3v. Using this sequence q, the cut C is defined in definition 6.2.8. Since the ordering on Q_α is α-computable, C is α-semicomputable. Taking $E \coloneqq C$ by lemma 6.2.10 we have $C \subseteq E \wedge D \subseteq \overline{E}$. Therefore by lemma 6.2.11 it follows that $A = A_C$ and $B = B_C$ which completes the proof of lemma 6.1.1. □

Lemma 6.2.12. $A_C \leq_{w\alpha e} C$ and $B_C \leq_{w\alpha e} \overline{C}$.

Proof. Note $A_C \leq_{\alpha e} C$ via

$$W \coloneqq \{\langle a, \delta \rangle : \exists s < \alpha . \{q_s(a)\} = K_\delta\}$$

which is $\Sigma_1(L_\alpha)$ as $q : \alpha \times (\alpha_A \sqcup \alpha_B) \to Q_\alpha$ is α-computable by proposition 6.2.3v. Similarly, $B_C \leq_{w\alpha e} \overline{C}$ as required. □

Proof. (Of theorem 6.0.1) Assume $\mathcal{K}_{\text{nt}}(A, B)$. Then there is a semicomputable cut $C \in Q_\alpha \cap L_\alpha$ by lemma 6.1.1 s.t. $A = A_C$ and $B = B_C$. Applying lemma 6.2.12 we get $A \leq_{w\alpha e} C$ and $B \leq_{w\alpha e} \overline{C}$ as required. □

6.2.6 More about the cut C

We establish some additional properties about the constructed semicomputable cut $C \subseteq Q_\alpha \cap L_\alpha$ which are not necessary for the proof of theorem 6.0.1.

Proposition 6.2.13. The cut $C \subseteq Q_\alpha \cap L_\alpha$ is a proper cut. In particular,

- $C \neq \emptyset$,
- $C \neq Q_\alpha \cap L_\alpha$.

Proof. Suppose that $C = \emptyset$ or that $C = Q_\alpha \cap L_\alpha$. Then $C \in \Delta_1(L_\alpha)$. Remember that by theorem 6.0.1 we have $A \leq_{w\alpha e} C$ and $B \leq_{w\alpha e} \overline{C}$. As $C \in \Delta_1(L_\alpha)$, so $A \in \Sigma_1(L_\alpha)$ and $B \in \Sigma_1(L_\alpha)$. But by the assumption in theorem 6.0.1, $\mathcal{K}(A, B)$ is nontrivial which is a contradiction. □

Lemma 6.2.14. $C \cup D = Q_\alpha \cap L_\alpha$

Proof. Assume not, then there is an α-finite α-rational $\rho \in Q_\alpha \cap L_\alpha - C \cup D$. Define $E := \{\pi \in Q_\alpha \cap L_\alpha : \pi < \rho\}$. Then $C \subseteq E$ and $D \subseteq \overline{E}$ and so $A = A_E$ by lemma 6.2.11. Hence $A = A_E \leq_{w\alpha e} E$ by lemma 6.2.12. But note that $E \in \Sigma_1(L_\alpha)$ and so $A \in \Sigma_1(L_\alpha)$. But $\mathcal{K}(A, B)$ is nontrivial which is a contradiction. □

Proposition 6.2.15. $D = \overline{C} := Q_\alpha \cap L_\alpha - C$

Proof. Remember definition 6.2.8 that $C \subseteq Q_\alpha \cap L_\alpha$ and $D \subseteq Q_\alpha \cap L_\alpha$. Hence $D = \overline{C}$ follows from lemma 6.2.10 stating $C \cap D = \emptyset$ and lemma 6.2.14 stating $C \cup D = Q_\alpha \cap L_\alpha$. □

6.3 $\mathcal{D}_\alpha$ definable in $\mathcal{D}_{\alpha e}$

We prove that the total degrees $\mathcal{TOT}_{\alpha e}$ are definable in the α-enumeration degrees $\mathcal{D}_{\alpha e}$ if $V = L$ and α is an infinite regular cardinal (theorem 6.3.7).

Lemma 6.3.1. The following are true about the function q_α:

- $\text{dom}(q_\alpha) \in \Sigma_2(L_\alpha)$,
- $q_\alpha \in \Sigma_2(L_\alpha)$.

Proof. Note that the labelling sequence $q : \alpha \times (\alpha_A \sqcup \alpha_B) \to Q_\alpha$ is α-computable by proposition 6.2.3v.

So $\mathrm{dom}(q_\alpha) \in \Sigma_2(L_\alpha)$ since

$$c \in \mathrm{dom}(q_\alpha) \iff \exists s < \alpha \forall t < \alpha[s \leq t \implies q_s(c) = q_t(c)].$$

And $q_\alpha \in \Sigma_2(L_\alpha)$ since

$$q_\alpha(c) = \rho \iff \exists s < \alpha \forall t < \alpha[s \leq t \implies \rho = q_t(c)].$$

□

Lemma 6.3.2. Let $\sigma, \rho \in Q_\alpha \cap L_\alpha$. Then the statement

"σ and ρ are together in a PDZ"

is $\Sigma_2(L_\alpha)$ definable.

Proof. Note $\{\sigma, \rho\} \subseteq$ a PDZ $\iff$
$\exists (a, b) \in W \exists t < \alpha[\forall u < \alpha(u \geq t \implies q_u(b) = q_t(b) \leq \sigma, \rho \leq q_u(a) = q_t(a))]$.
As the labelling sequence q is α-computable by proposition 6.2.3v, so by looking at the quantifier arrangement the statement has to be $\Sigma_2(L_\alpha)$ definable as required.

□

Recall section 3.12 that $J^{(n)}_{\alpha e}(A)$ denotes the n^{th} α-enumeration jump of A.

Lemma 6.3.3. (Definability of C and $\overline{C}$)
Assume $L_\alpha \models \Sigma_3$-replacement, then:

1. $C \in \Sigma_1(L_\alpha, B \oplus J^{(2)}_{\alpha e}(\emptyset))$,
2. $\overline{C} \in \Sigma_1(L_\alpha, A \oplus J^{(2)}_{\alpha e}(\emptyset))$,
3. $C \in \Delta_1(L_\alpha, A \oplus B \oplus J^{(2)}_{\alpha e}(\emptyset))$.

Proof. Note $C = \{\rho \in Q_\alpha \cap L_\alpha : \exists b \in \alpha_B \exists s < \alpha \exists \sigma \in Q_\alpha \cap L_\alpha$
$[b \in \overline{B} \wedge (\rho \leq q_s(b) \downarrow$ or $\sigma = q_\alpha(b) \wedge \{\rho, \sigma\}$ is a subset of a PDZ $)]\}$.

The statement "$\sigma = q_\alpha(b)$" is $\Sigma_2(L_\alpha)$ by lemma 6.3.1. The statement "$\{\rho, \sigma\}$ is a subset of a PDZ" is $\Sigma_2(L_\alpha)$ by lemma 6.3.2. The statement "$\rho \leq q_s(b) \downarrow$" is α-computable by the α-computability of the labelling sequence q, see proposition 6.2.3v. Hence the formula

"$\rho \leq q_s(b) \downarrow$ or $\sigma = q_\alpha(b) \wedge \{\rho, \sigma\}$ is a subset of a PDZ"

is $\Sigma_2(L_\alpha)$ definable. Using the Σ_n-completion of the n^{th} αe-jump (proposition 3.12.16) and $L_\alpha \models \Sigma_3$-replacement the set defined by such formula is α-reducible to the set $J^{(2)}_{\alpha e}(\emptyset)$. So clearly the set defined by the formula

"$b \in \overline{B} \wedge (\rho \leq q_s(b) \downarrow$ or $\sigma = q_\alpha(b) \wedge \{\rho, \sigma\}$ is a subset of a PDZ)" is α-reducible to the set $B \oplus J^{(2)}_{\alpha e}(\emptyset)$. Thus C is $\Sigma_1(L_\alpha, B \oplus J^{(2)}_{\alpha e}(\emptyset))$ definable and the statement 1 holds.

Similarly, $\overline{C}$ is $\Sigma_1(L_\alpha, A \oplus J^{(2)}_{\alpha e}(\emptyset))$ definable. Hence the statement 2 is true.

By the statements 1 and 2 we have

$$C \in \overline{C} \in \Sigma_1(L_\alpha, A \oplus B \oplus J^{(2)}_{\alpha e}(\emptyset)).$$

Therefore

$$C \in \Delta_1(L_\alpha, A \oplus B \oplus J^{(2)}_{\alpha e}(\emptyset))$$

and thus the statement 3 is true. □

Remark 6.3.4. The reason that the Σ_1 definition of C and $\overline{C}$ needs only one of the sets B and A respectively, not both, is corollary 4.2.8 which says $A \leq_{w\alpha e} \overline{B}$ and $B \leq_{w\alpha e} \overline{A}$ for a non-trivial Kalimullin pair $\mathcal{K}_{\text{nt}}(A, B)$.

Theorem 6.3.5. Suppose that A and B form a non-trivial maximal $\mathcal{K}$-pair, then there exists an α-semicomputable set C s.t. $C \equiv_{\alpha e} A$, $\overline{C} \equiv_{\alpha e} B$ and $\mathcal{K}(C, \overline{C})$. Furthermore, if $L_\alpha \models \Sigma_3$-replacement and $A \oplus B \oplus J^{(2)}_{\alpha e}(\emptyset)$ is megaregular, then $\deg_{\alpha e}(A \oplus B)$ is a total degree.

Proof. (Of theorem 6.3.5)
1. Assume $\mathcal{K}_{\max}(A, B)$.
2. Assume $\mathcal{K}_{\text{nt}}(A, B)$.
3. $\exists C \in \text{sc}(L_\alpha)[A \leq_{w\alpha e} C \wedge B \leq_{w\alpha e} \overline{C}]$ by 2 and theorem 6.0.1.
4. $\mathcal{K}(C, \overline{C})$ by 3.
5. $A \leq_{w\alpha e} C \wedge B \leq_{w\alpha e} \overline{C}$ by 3.
6. Assume $L_\alpha \models \Sigma_3$-replacement and $A \oplus B \oplus J^{(2)}_{\alpha e}(\emptyset)$ is megaregular.
7. C and $\overline{C}$ are megaregular by 6, lemma 6.3.3 and proposition 3.8.13.
8. $A \leq_{\alpha e} C \wedge B \leq_{\alpha e} \overline{C}$ by 5 and 7.
9. $A \equiv_{\alpha e} C \wedge B \equiv_{\alpha e} \overline{C}$ by 1, 4, 8.
10. Define $T := C \oplus \overline{C}$.
11. T is total and $T \in A \oplus B$ by 9, 10. □

Corollary 6.3.6. Assume $V = L$ and let α be an infinite regular cardinal. If a and b are α-enumeration degrees that form a non-trivial maximal $\mathcal{K}$-pair, then there exists an α-semicomputable set C s.t. $C \in a$, $\overline{C} \in b$ and $\mathcal{K}(C, \overline{C})$. Hence $a \oplus b$ is a total degree.

Proof. As α is an infinite regular cardinal, so $\mathcal{K}$-pair is definable in $\mathcal{D}_{\alpha e}$ by corollary 4.3.8. Thus assume $\mathcal{K}_{\max}(a, b) \wedge \mathcal{K}_{\text{nt}}(a, b)$ and let $A \in a$ and $B \in b$. We have $\mathcal{K}_{\max}(A, B) \wedge \mathcal{K}_{\text{nt}}(A, B)$. If α is an infinite regular cardinal, then every subset of α is megaregular. Hence $A \oplus B \oplus J^{(2)}_{\alpha e}(\emptyset)$ is megaregular. As α is an infinite regular cardinal, so $L_\alpha \models \Sigma_3$-replacement. Next apply theorem 6.3.5 to conclude $a \oplus b = \deg_{\alpha e}(A \oplus B)$ is total as required. □

Theorem 6.3.7. (Definability of total degrees)
Assume $V = L$. Let α be an infinite regular cardinal. A degree of $\mathcal{D}_{\alpha e}$ is total iff it is trivial or a join of a maximal $\mathcal{K}$-pair.

Proof. (Of theorem 6.3.7) Assume that α is an infinite regular cardinal. So $\mathcal{K}$-pair is definable in $\mathcal{D}_{\alpha e}$ by corollary 4.3.8. Thus the proposed definition makes a sense in $\mathcal{D}_{\alpha e}$.

The $\implies$ direction follows from corollary 4.4.2.

Clearly, the trivial α-enumeration degree $\deg_{\alpha e}(\emptyset)$ is total.

The rest of $\impliedby$ direction follows from corollary 6.3.6 implied by theorem 6.3.5 and theorem 6.0.1. □

Chapter 7

Embedding Theorem for $\mathrm{Aut}(\mathcal{D}_{\alpha e})$

In this chapter we first prove Selman's theorem 7.1.6 in α-Computability Theory assuming $V = L$ and α being an infinite regular cardinal. Finally, we use this theorem to prove the main result of this thesis, Embedding Theorem 7.3: the automorphism group of the α-enumeration degrees $\mathcal{D}_{\alpha e}$ embeds into the automorphism group of the α degrees $\mathcal{D}_{\alpha}$ assuming $V = L$ and α being an infinite regular cardinal.

7.1 Selman's theorem

Selman's theorem is generalized to the setting of α-Computability Theory from classical Computability Theory [24][30].

Definition 7.1.1. (Odd enumeration and α-finite part)

- Let $B \subseteq \alpha$. The total function $f : \alpha \to \alpha$ is an *odd enumeration* of B iff
$$f[\{2\gamma + 1 < \alpha : \gamma < \alpha\}] = B.$$
- B *odd* α*-finite part* is a function $\tau : [0, 2s) \to \alpha$ for $s < \alpha$ s.t.
$$\forall x < \alpha[2x + 1 \in \mathrm{dom}(\tau) \implies \tau(2x + 1) \in B].$$
- Let $|\tau|$ denote the order type of $\mathrm{dom}(\tau)$, i.e. $|\tau| := \mathrm{ot}(\mathrm{dom}(\tau))$.

If $\mathrm{dom}(\tau)$ is an initial segment of α, we have $|\tau| = \mathrm{dom}(\tau)$. If $\tau : [0, 2s) \to \alpha$ is a function, then $\mathrm{dom}(\tau) = 2s$ and so $|\tau| = 2s$. If τ is a B odd α-finite part, then there is an odd enumeration $f : \alpha \to \alpha$ of B s.t. $\tau \subseteq f$.

Lemma 7.1.2. Let $f : \alpha \to \alpha$ be an odd enumeration of B. Then $B \leq_{w\alpha e} f$.

Proof. We have $B \leq_{w\alpha e} f$ via
$$\Phi := \{\langle b, \delta\rangle : \exists\gamma < \alpha[2\gamma + 1 < \alpha \wedge K_\delta = \{(2\gamma + 1, b)\}]\}.$$
□

Lemma 7.1.3. Assume that $A \leq_{w\alpha e} f$. Then $f^{-1}[A] \leq_{w\alpha e} f$.

Proof. Note $f^{-1}[A] \coloneqq \{x < \alpha : f(x) \in A\}$. Let $A \leq_{w\alpha e} f$ via $\Psi \in \Sigma_1(L_\alpha)$. Then $f^{-1}[A] \leq_{w\alpha e} f$ via

$$\Phi \coloneqq \{\langle x, \delta\rangle : \exists y < \alpha[K_\delta = K_\epsilon \cup \{(x, y)\}\} \wedge \langle y, \epsilon\rangle \in \Psi]\} \in \Sigma_1(L_\alpha).$$

□

Proposition 7.1.4. [1] Let $A, B \subseteq \alpha$ and $A \not\leq_{w\alpha e} B$. Assume that $A \oplus B \oplus K(\emptyset)$ is megaregular. Then there exists an odd enumeration $f : \alpha \to \alpha$ of B s.t. $A \not\leq_{w\alpha e} f$ and $B \leq_{\alpha e} f$.

Proof. **Construction**

Since $B \equiv_{\alpha e} B \cup (\alpha - \sup(B))$, WLOG redefine B to be $B \cup (\alpha - \sup(B))$ so that it is unbounded. We construct a sequence of B odd α-finite parts in α many stages s.t.:

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_s \subseteq \ldots$$

In the end, the desired function $f : \alpha \to \alpha$ is defined as $f = \bigcup_{s<\alpha} \tau_s$.

Let $\tau_0 = \emptyset$. If s is a limit ordinal, then let $\tau_s \coloneqq \bigcup_{r<s} \tau_r$. Now assume that τ_s has been constructed, then at the stage s construct τ_{s+1}:

- Stage $s = 2e$:
  Set $\tau_{s+1} \coloneqq \tau_s \cdot 0 \cdot b$ where
  $$b = \mu y\{y < \alpha : y \in B \wedge y \notin \tau_s[\{2\gamma + 1 < |\tau_s| : \gamma < \alpha\}]\}$$
  and $0 \cdot b$ is the concatenation of 0 and b. E.g. if $\tau = a \cdot b$, then $\tau(0) = a, \tau(1) = b$. Note that b is well-defined since B is unbounded.

- Stage $s = 2e + 1$:
  Use e and τ_s to define the set C as
  $$C \coloneqq \{x < \alpha | \exists \rho \supseteq \tau_s[\rho \text{ is a } B \text{ odd } \alpha\text{-finite part and } x = \rho(|\tau_s|) \wedge |\tau_s| \in \Phi_e(\rho)]\}.$$
  Since $A \not\leq_{w\alpha e} B$ by assumption and $C \leq_{w\alpha e} B$, we have $C \neq A$. Thus we have two cases:

  - Case $\exists x < \alpha[x \in C \wedge x \notin A]$: Then let τ_{s+1} be the minimal ρ from C. Note $\tau_{s+1} = \tau_s \cdot x \cdot b \cdot \sigma$ for some $b \in B$ where σ is a B odd α-finite part.
  - Case $\exists x < \alpha[x \notin C \wedge x \in A]$: Then let $\tau_{s+1} \coloneqq \tau_s \cdot x \cdot b$ for some $b \in B$.

[1] Generalized from [30] for $\alpha = \omega$.

Verification

By the two cases above we have for all $e, s, x < \alpha$:

$$s = 2e + 1 \wedge x = \tau_{s+1}(|\tau_s|) \implies x \in C \wedge x \notin A \vee x \notin C \wedge x \in A \quad (7.1)$$

Note that $f = \bigcup_{s<\alpha} \tau_s$ is an odd enumeration of B. This is ensured by stages $s = 2e < \alpha$. If $b \in B$, then a pair $(2\gamma + 1, b)$ is added to f for some $2\gamma + 1 < \alpha$ at the stage $s = 2b < \alpha$ at latest.

Moreover, $A \not\leq_{w\alpha e} f$. For suppose not, then $A \leq_{w\alpha e} f$. Hence $f^{-1}[A] \leq_{w\alpha e} f$ by lemma 7.1.3 and so there is $e < \alpha$ s.t. $f^{-1}[A] = \Phi_e(f)$ and thus:

$$\forall l < \alpha[f(l) \in A \iff l \in f^{-1}[A] \iff l \in \Phi_e(f)] \quad (7.2)$$

Consider the stage $s = 2e + 1$. Let $l = |\tau_s|$. Note $\tau_s \subseteq f$.

- Case 1: $l \in f^{-1}[A] \implies f(l) \in A \implies l \in \Phi_e(f)$ using statement (7.2). By the witness property of an α-enumeration operator there exists B odd α-finite part ρ of f s.t. $\rho \supseteq \tau_s \wedge l \in \Phi_e(\rho) \wedge \rho(l) = f(l)$. So $f(l) \in C$. Hence $f(l) \in C \cap A$. Also by statement (7.1) we have

  $$(f(l) \in C \wedge f(l) \notin A) \vee (x \notin C \wedge f(l) \in A).$$

  This is a contradiction.

- Case 2: $l \notin f^{-1}[A] \implies f(l) \notin A \implies l \notin \Phi_e(f)$ using statement (7.2). By the monotonicity of an α-enumeration operator there is no B odd α-finite part ρ of f s.t. $\rho \supseteq \tau_s \wedge l \in \Phi_e(\rho)$. So $f(l) \notin C$. Hence $f(l) \notin A \wedge f(l) \notin C$. By statement (7.1) we have

  $$(f(l) \in C \wedge f(l) \notin A) \vee (x \notin C \wedge f(l) \in A).$$

  This is a contradiction.

Hence in any case $A \not\leq_{w\alpha e} f$ as required.

Let $g : \alpha \to \alpha$ be defined by $g : s \mapsto \gamma$ where $K_\gamma = \tau_s$. During the construction we use the oracle $A \oplus B \oplus K(\emptyset)$, hence $g \in \Sigma_1(L_\alpha, A \oplus B \oplus K(\emptyset))$. We show that g is well-defined and that τ_s is α-finite at a limit stage s. Let $s < \alpha$ be a limit stage and $g_{|s}$ be a restriction of g to the subdomain $s \subseteq \alpha$. Then $g[s] = I \in L_\alpha$ since $s \in L_\alpha$, $g_{|s} \in \Sigma_1(L_\alpha, A \oplus B \oplus K(\emptyset))$ and by the megaregularity of the oracle $A \oplus B \oplus K(\emptyset))$. Hence $\tau_s = \bigcup_{r<s} \tau_r = \bigcup_{\gamma \in I} K_\gamma$ is α-finite as required. Therefore $\forall s < \alpha . \tau_s \in L_\alpha$, as needed.

Note that $f \in \Delta_1(L_\alpha, A \oplus B \oplus K(\emptyset))$ since the construction of f uses the oracle $A \oplus B \oplus K(\emptyset)$. By that and the megaregularity of $A \oplus B \oplus K(\emptyset)$, also f must be megaregular. By lemma 7.1.2 we have $B \leq_{w\alpha e} f$. By the megaregularity of f, we have $B \leq_{\alpha e} f$ as required. □

Theorem 7.1.5. (Selman's theorem for admissible ordinals)
Let $A, B \subseteq \alpha$ and let $A \oplus B \oplus K(\emptyset)$ be megaregular. Then

$$A \leq_{\alpha e} B \iff \forall X[X \equiv_{\alpha e} X \oplus \overline{X} \wedge B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}].$$

Part of the following proof is adapted from the classical case present in [30].

Proof. $\Rightarrow$ direction is by the transitivity of $\leq_{\alpha e}$.

For the $\Leftarrow$ direction assume that

$$\forall X[X \equiv_{\alpha e} X \oplus \overline{X} \wedge B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}].$$

We want to show that $A \leq_{\alpha e} B$. Assume not, then $A \not\leq_{\alpha e} B$ and so $A \not\leq_{w\alpha e} B$ as $A \oplus B \oplus K(\emptyset)$ is megaregular. Then by proposition 7.1.4 and the megaregularity of $A \oplus B \oplus K(\emptyset)$ there exists a total function f s.t. $A \not\leq_{w\alpha e} f$, but $B \leq_{\alpha e} f$. But f is total and so $A \leq_{\alpha e} f$ which is a contradiction to the statement $A \not\leq_{w\alpha e} f$. Hence $A \leq_{\alpha e} B$ as required. □

Corollary 7.1.6. (Selman's theorem for regular cardinals)
Assume $V = L$. Let α be an infinite regular cardinal. Then for any $A, B \subseteq \alpha$ we have:

$$A \leq_{\alpha e} B \iff \forall X[X \equiv_{\alpha e} X \oplus \overline{X} \wedge B \leq_{\alpha e} X \oplus \overline{X} \implies A \leq_{\alpha e} X \oplus \overline{X}]$$

Proof. If α is a regular cardinal, then every subset of α is megaregular. Hence $A \oplus B \oplus K(\emptyset)$ is megaregular. The remaining proof of the corollary follows from theorem 7.1.5. □

7.2 $\mathcal{TOT}_{\alpha e}$ as an automorphism base for $\mathcal{D}_{\alpha e}$

We use Selman's theorem to conclude that $\mathcal{TOT}_{\alpha e}$ are an automorphism base for $\mathcal{D}_{\alpha e}$ under some assumptions.

Definition 7.2.1. (Automorphism base [11])
TFAE:

- The subset $B \subseteq \text{dom}(\mathcal{M})$ is an automorphism base of the model $\mathcal{M}$.
- $\forall f, g \in \text{Aut}(\mathcal{M})[f_{|B} = g_{|B} \implies f = g]$
- $\forall f \in \text{Aut}(\mathcal{M})[f_{|B} = 1_{|B} \implies f = 1]$

Theorem 7.2.2. [2] Assume $V = L$. Let α be an infinite regular cardinal. The total degrees $\mathcal{TOT}_{\alpha e}$ are an automorphism base for $\mathcal{D}_{\alpha e}$.

[2]For α-Computability Theory introduced in this thesis. A well-known result in classical Computability Theory concluded by the work in [24].

Proof. 0. Assume α is a regular cardinal.

1. $\forall a, b \in \mathcal{D}_{\alpha e}[a \leq b \iff \forall x \in \mathcal{TOT}_{\alpha e}(b \leq x \implies a \leq x)]$ by 0 and corollary 7.1.6.

2. $\forall a, b \in \mathcal{D}_{\alpha e}[a = b \iff \forall x \in \mathcal{TOT}_{\alpha e}(b \leq x \iff a \leq x)]$ by 1.

3. Assume $f \in \mathrm{Aut}(\mathcal{D}_{\alpha e})$.

4. $\forall a, b \in \mathcal{D}_{\alpha e}[a \leq b \iff f(a) \leq f(b)]$ by 3.

5. Assume $\forall x \in \mathcal{TOT}_{\alpha e}.f(x) = x$.

6. Assume $y \in \mathcal{D}_{\alpha e}$.

7. $\forall x \in \mathcal{TOT}_{\alpha e}.(f(y) \leq f(x) \iff y \leq x)$ by 4.

8. $\forall x \in \mathcal{TOT}_{\alpha e}.(f(y) \leq x \iff y \leq x)$ by 5, 7.

9. $f(y) = y$ by 2, 8.

10. $\forall y \in \mathcal{D}_{\alpha e}.f(y) = y$ by 6, 9.

11. $\mathcal{TOT}_{\alpha e}$ is an automorphism base for $\mathcal{D}_{\alpha e}$ by 3, 5, 10. □

7.3 Embedding Theorem

The main result of this thesis is:

Theorem 7.3.1. (*Embedding Theorem*[3])
Assume $V = L$. Let α be an infinite regular cardinal, then the automorphism group of the α-enumeration degrees is embeddable in the automorphism group of the total α-enumeration degrees: $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{TOT}_{\alpha e})$.

Proof. (Of theorem 7.3.1) The embedding theorem is implied by 3 statements:

1. There exists an embedding $\iota : \mathcal{D}_\alpha \hookrightarrow \mathcal{D}_{\alpha e}$ where $\mathrm{Im}(\iota) = \mathcal{TOT}_{\alpha e}$,

2. $\mathcal{TOT}_{\alpha e}$ are an automorphism base for $\mathcal{D}_{\alpha e}$,

3. $\mathcal{TOT}_{\alpha e}$ are definable in $\mathcal{D}_{\alpha e}$.

The first statement follows from theorem 3.10.4 where the embedding $\iota : \mathcal{TOT}_{\alpha e} \hookrightarrow \mathcal{D}_{\alpha e}$ is given by $\iota : \deg_\alpha(A) \mapsto \deg_{\alpha e}(A \oplus \overline{A})$. The second statement is theorem 7.2.2 which is the generalization of a result by Selman [24]. The third statement is theorem 6.3.7 which is the generalization of the definability of the total degrees in the enumeration degrees [2].

By the 3 statements we can define an injective group homomorphism $\eta : \mathrm{Aut}(D_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_\alpha), \eta : f \mapsto \iota^{-1} \circ f \circ \iota$. As $\mathrm{Im}(\iota) = \mathcal{TOT}_{\alpha e}$ and $\mathcal{TOT}_{\alpha e}$ definable and hence invariant under an automorphism f so $\mathrm{Im}(f \circ \iota) = \mathcal{TOT}_{\alpha e}$.

[3] For α-Computability Theory introduced in this thesis. A well-known result in classical Computability Theory concluded by the work in [2]. The proof of this result outlined in [2] p13.

Thus the composition of $\iota^{-1} : \mathcal{TOT}_{\alpha e} \to \mathcal{D}_\alpha$ (where ι injective) with $f \circ \iota : \mathcal{D}_\alpha \to \mathcal{D}_{\alpha e}$ is well-defined. Since

$$\eta(f) \circ \eta(g) = (\iota^{-1} \circ f \circ \iota) \circ (\iota^{-1} \circ g \circ \iota) = \iota^{-1} \circ f \circ g \circ \iota = \eta(f \circ g),$$

so η is a group homomorphism. If $\eta(f) = \eta(g)$ then $\forall a \in \mathcal{TOT}_{\alpha e}.f(a) = g(a)$, since $\mathcal{TOT}_{\alpha e}$ is an automorphism base for $\mathcal{D}_{\alpha e}$ so $f = g$ and hence η injective. This η is the required embedding. □

7.4 Further directions

Embedding theorem 7.3.1 establishes for a certain class of admissible ordinals α that $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_\alpha)$. One of the obstacles to prove the Embedding theorem for all admissible α is a difficulty in proving the definability of a Kalimullin pair and the total degrees in the enumeration degrees. We proved that a Kalimullin pair is definable in $\mathcal{D}_{\alpha e}$ if all subsets of α are megaregular (theorem 4.3.6). This condition seems to be necessary. The lack of the megaregularity of the sets is a consequence of the lack of the admissibility on L_α. This gives us the following intuition:

The less admissibility L_α has, the less definability $\mathcal{D}_{\alpha e}$ is expected to have.

Slaman and Woodin [28] proved that the Turing degrees have a trivial automorphism group iff they are biinterpretable with second order arithmetic. Also every automorphism in a structure has to preserve all definable relations. This gives us another intuition:

The less definability a structure has, the more non-trivial automorphism group it is expected to have.

Combining the two intuitions above into the third one we have:

The less admissibility L_α has, the more non-trivial automorphism group $\mathcal{D}_{\alpha e}$ is expected to have.

Chapter 8

Open problems

We present several open problems in Higher Computability Theory at a various level of difficulty, where the number of stars $\star$ marking the problem is proportional to the level of the difficulty guessed by the author. For comparison, Embedding Theorem 7.3.1, the main result of this thesis stands at the difficulty level of about three stars ($\star\star\star$).

8.1 α-Computability Theory

8.1.1 General

Does the following relativization of the Uniformization Theorem 3.2.9 hold?

Conjecture 8.1.1. ($\star$) (Relativized Uniformization Theorem)
Let $n \geq 1$ and $A \subseteq \alpha$. Then for each $\Sigma_n(L_\alpha, A)$ relation $R(x, y)$ there is a $\Sigma_n(L_\alpha, A)$ function f satisfying:

$$\forall x < \alpha[\text{if } \exists y < \alpha.\langle L_\alpha, A\rangle \models R(x, y), \text{then } \langle L_\alpha, A\rangle \models R(x, f(x))].$$

Note that conjecture 8.1.1 in the case of $n = 1$ was established by proposition 3.4.15. Likely, the cases $n > 1$ require additional assumptions on the structure $\langle L_\alpha, A\rangle$. Using a stronger Relativized Uniformization Theorem we could strengthen proposition 3.4.20.

Conjecture 8.1.2. ($\star$) Let $A_\delta := \{\gamma < \alpha : K_\gamma \subseteq K_\delta\}$. Then $A_\delta \in L_\alpha$.

Conjecture 8.1.2 could simplify many of the proofs in this thesis, especially the ones where indices of α-finite sets are used, e.g. section 3.5. Trivially, A_δ is α-computable. However, it does not seem to be clear that A_δ is also bounded. This seems to depend on the function used to index α-finite sets.

Fact 3.6.4 states that the projectum of an admissible ordinal is admissible. Proposition 3.1.21 gives a relation between the admissibility and the cofinality.

How can we relate the projectum and the cofinality? Can we do so in the following way?

Conjecture 8.1.3. ($\star\star$) (Projectum and cofinality)

1. If $\delta = \sigma_n p(\alpha)$, then there is a partial surjection $s : \delta \rightharpoonup \alpha \in \Sigma_n(L_\alpha)$ s.t.
$$\forall y < \alpha \exists x < \delta . y < s(x),$$
2. $\sigma_{n+1}\mathrm{cf}(\alpha) \leq \sigma_n p(\alpha)$.

Proof. (Of 2 assuming 1.)
Let $\delta = \sigma_n p(\alpha)$. So there is a partial surjection $s : \delta \rightharpoonup \alpha \in \Sigma_n(L_\alpha)$ s.t. $\forall y < \alpha \exists x < \delta . y < s(x)$. Define $s'(x) = s(\mu z[x \leq z \wedge s(z) \downarrow])$. Note that $s' : \delta \twoheadrightarrow \alpha$ is a total $\Sigma_{n+1}(L_\alpha)$ definable surjection and $\forall y < \alpha \exists x < \delta . y < s'(x)$. Therefore $\sigma_{n+1}\mathrm{cf}(\alpha) \leq \delta$ as required. □

We ask the next question in order to understand the relation between different levels of the definability.

Question 8.1.4. ($\star\star$) Suppose that α_1 and α_2 are admissible ordinals and that $\alpha_1 < \alpha_2$. Let $A, B \subseteq \alpha_1$. If $A \leq_{\alpha_1 e} B$, is it then true that $A \leq_{\alpha_2 e} B$?

8.1.2 Totality

Question 8.1.5. ($\star\star$) Recall section 3.12. If $J_{\alpha e}(A)$ is a megaregular set, then $J_{\alpha e}(A)$ is total. In general, is $J_{\alpha e}(A)$ a total set for any $A \subseteq \alpha$?

The next question asks if it is sufficient to define the totality with the weaker reducibility instead. If that is the case, then one could conclude the totality using the weaker reducibility without invoking the requirement of megaregularity.

Question 8.1.6. ($\star\star$) Assume that $A \oplus \overline{A} \leq_{w\alpha e} A$. Is it true that $A \oplus \overline{A} \leq_{\alpha e} A$?

8.1.3 Regularity

Proposition 3.8.13 already implies that megaregularity is closed under the Δ_1 definability. The next conjecture states that also regularity is closed under the Δ_1 definability.

Conjecture 8.1.7. ($\star$) If $A \in \Delta_1(L_\alpha, B)$ and B is regular, then A is regular.

Megaregularity seems as a too strong notion in the context of positive definability and enumeration reducibilities. Moreover, it does not even have the desired properties. The next question asks if there is a better notion in this context.

Question 8.1.8. ($\star\star$) (Natural notion of positive megaregularity)
Let α be a general admissible ordinal, not necessarily an infinite regular cardinal. Does there exists a relation $R \subseteq \mathcal{P}(\alpha)$ satisfying the following properties?

1. $\forall A \subseteq \alpha[$ if A megaregular, then $R(A) \wedge R(\overline{A})]$.
2. $\forall A, B \subseteq \alpha[A \leq_{w\alpha e} B \wedge R(B) \implies A \leq_{\alpha e} B]$.
3. $\forall A, B \subseteq \alpha[A \leq_{w\alpha e} B \wedge R(B) \implies R(A)]$.

8.1.4 Automorphisms

Megaregularity emerged as a very important notion. It would be good to know some of its degree theoretic properties. The negative answer to the next question would imply that it cannot be definable in a degree structure in question.

Question 8.1.9. ($\star\star$) Does an automorphism preserve megaregularity?

The next question seems a very natural one to ask after proving in this thesis that $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_\alpha)$ for an infinite regular cardinal assuming $V = L$.

Question 8.1.10. ($\star\star\star\star$) (Embedding Conjecture for admissible ordinals)
Let α be a general admissible ordinal. Is it true that $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_\alpha)$?

There is already a classical conjecture that the Turing degrees are rigid [28]. In remark 3.11.6 we saw that the computability on the infinite regular cardinals behaves much like the classical computability theory. Thus it may be natural to conjecture in general that the α-degrees $\mathcal{D}_\alpha$ are rigid if α is an infinite regular cardinal and $V = L$. Together with the main result of this thesis $\mathrm{Aut}(\mathcal{D}_{\alpha e}) \hookrightarrow \mathrm{Aut}(\mathcal{D}_\alpha)$ it can imply the following conjecture characterizing $\mathrm{Aut}(\mathcal{D}_{\alpha e})$.

Conjecture 8.1.11. ($\star\star\star\star$) If $V = L$ and α is an infinite regular cardinal, then the automorphism group of the α-enumeration degrees is trivial.

8.2 Transfer principle

Question 8.2.1. ($\star\star\star$) (Transfer principle for infinite regular cardinals)
Assume $V = L$. In remark 3.11.6 we observed that the α-Computability with infinite regular cardinals behaves similarly to the classical Computability Theory. Also all the main results of this thesis (section 2.3) are the generalizations of the classical results which hold for infinite regular cardinals α.

Is there some general transfer principle that would enable us to conclude that any result of a certain form which holds in classical Computability Theory is true also in α-Computability Theory if α is an infinite regular cardinal?

8.3 ϵ,ζ - Computability Theory

In remark 3.2.6 it was stated that Δ_1 definability over L_α behaves like the computability on the extended Turing machine with a tape of an order type α and the computational time α. Can we generalize this further?

Definition 8.3.1. For $A \in L$ define its powerset to be

$$\mathcal{P}(A) := \{B \in L : B \subseteq A\}.$$

Definition 8.3.2. (ϵ,ζ-Constructible Hierarchy)
Let δ, ϵ and ζ be limit ordinals or ∞ (i.e. `Ord`) and $\epsilon \geq \zeta$. Let $\gamma < \infty$. Define the ϵ,ζ-level of the constructible hierarchy recursively:

$$L_{\zeta,\zeta} := L_\zeta,$$

$$L_{\zeta+\gamma+1,\zeta} := \mathrm{Def}(L_{\zeta+\gamma,\zeta}) \cap \mathcal{P}(\zeta),$$

$$L_{\zeta+\delta,\zeta} := (\bigcup_{\gamma<\delta} L_{\zeta+\gamma,\zeta}) \cap \mathcal{P}(\zeta).$$

Definition 8.3.3. A function $f : \zeta \to \zeta$ is ϵ,ζ-computable iff f is $\Sigma_1(L_{\epsilon,\zeta})$-definable.

Question 8.3.4. ($\star\star$) Is a function $f : \zeta \to \zeta$ ϵ,ζ-computable iff it is a computable on a Turing machine with a tape of an order type ζ and time ϵ?

Appendix A

Axioms

A.1 Zermelo-Fraenkel set theory with choice

Definition A.1.1. (Zermelo-Fraenkel set theory with choice)
Zermelo-Frankel set theory with choice (ZFC) is the theory specified by the following axioms:

- extensionality: $\forall x \forall y[x = y \iff \forall z(z \in x \iff z \in y)]$,
- foundation: $\forall x[x \neq \emptyset \implies \exists y \in x.x \cap y = \emptyset]$,
- schema of separation: for any formula $\phi(a)$ in which y is not free we have
$$\forall x \exists y \forall z[z \in y \iff z \in x \wedge \phi(z)],$$
- pairing: $\forall x \forall y \exists z[x \in z \wedge y \in z]$,
- union: $\forall x \exists y \forall v \forall w[v \in w \wedge w \in x \implies v \in y]$,
- schema of replacement: for any formula $\phi(x, y)$ we have
$$\forall K[\forall x \in K \exists! y.\phi(x, y) \implies \exists I \forall x \in K \exists y \in I.\phi(x, y)],$$
- infinity: $\exists x[\emptyset \in x \wedge \forall y \in x.y \cup \{y\} \in x]$,
- powerset: $\forall x \exists y \forall z[z \subseteq x \implies z \in y]$,
- choice: $\forall x[\emptyset \notin x \implies \exists f : x \to \bigcup x(\forall y \in x.f(y) \in y)]$

Appendix B

Abbreviations

iff	if and only if
s.t.	such that
wrt	with respect to
BC	base case
IC	inductive case
IH	inductive hypothesis
TFAE	the following are equivalent
WLOG	without the loss of generality
QED	quod erat demonstrandum, i.e. that which was to be demonstrated
KP	Kripke-Platek set theory
AC	the axiom of choice
ZF	Zermelo-Fraenkel set theory
ZFC	Zermelo-Fraenkel set theory with AC
CK	Church-Kleene
CNF	Cantor normal form
c.e.	computably enumerable
$\mathcal{K}$-pair	Kalimullin pair
DZ	dead zone
PDZ	permanent dead zone
HPPDZ	higher priority permanent dead zone

Appendix C

Notation

C.1 General

$\mathbb{N}$	natural numbers, i.e. $\mathbb{N} \coloneqq \{0, 1, 2, \ldots\}$
$\mathbb{Q}$	rational numbers
$\sum$	sum
$\prod$	product
$\min(A)$	the minimum of the set A
$\max(A)$	the maximum of the set A
$\inf(A)$	the infimum of the set A
$\sup(A)$	the supremum of the set A
$x \coloneqq y$	assignment, x is defined to be y
$\equiv$	equivalence relation
$\cong$	isomorphism

C.2 Functions

$A \to B$	total function from A to B
$A \rightharpoonup B$	partial function from A to B
$A \hookrightarrow B$	embedding from A to B
$A \twoheadrightarrow B$	epimorphism from A to B, i.e. surjection
$A \rightarrowtail B$	monomorphism from A to B, i.e. injection
$f : x \mapsto y$	the function f maps x to y
$f[A]$	the image of the function f on the set A
$\mathrm{dom}(f)$	the domain of the function f, i.e. $\mathrm{dom}(f : A \to B) = A$
$\mathrm{dom}(R)$	the domain of the relation R, e.g. if $R \subseteq X \times Y$, then $\mathrm{dom}(R) \coloneqq \{x \in X : \exists y \in Y.(x, y) \in R\}$
$\mathrm{cod}(f)$	the codomain of the function f, i.e. $\mathrm{cod}(f : A \to B) = B$

$\mathrm{Im}(f)$	the image of the function f, i.e. $\mathrm{Im}(f) \coloneqq \{y \in \mathrm{cod}(f) : \exists x \in \mathrm{dom}(f).f(x) = y\}$
$\mathrm{rng}(f)$	the range of the function f, i.e. $\mathrm{rng}(f) \coloneqq \mathrm{Im}(f)$, e.g. $\mathrm{rng}(\mathrm{id} : \mathbb{Z} \to \mathbb{R}) = \mathbb{Z}$
f^{-1}	the inverse of the function f
$f \upharpoonright D$	the restriction of the function f to the domain D
$f(x) \downarrow$	the function f is defined on x, i.e. $x \in \mathrm{dom}(f)$

C.3 Logic

$\neg$	logic negation
$\wedge$	logic and / lattice meet
$\vee$	logic or / lattice join
$\veebar$	exclusive logic or, i.e. $A \veebar B \equiv A \wedge \neg B \vee \neg A \wedge B$
$\implies$	implies
$\iff$	if and only if
$\exists$	existential quantifier
$\forall$	universal quantifier
$\phi(\overline{x})$	formula ϕ with the list of free parameters $\overline{x}$
$\phi \equiv \psi$	the formula ϕ is equivalent to the formula ψ
$\mathcal{M} \models \phi$	the formula ϕ is true in the model $\mathcal{M}$
$T \models \phi$	the formula ϕ is true in every model of the theory T
$T \vdash \phi$	the formula ϕ is provable from the theory T
$\mathcal{M} \preceq_{\Sigma_n} \mathcal{N}$	$\mathcal{M}$ is a Σ_n elementary substructure (or submodel) of $\mathcal{N}$

C.4 Set Theory

$\emptyset$	the empty set
$\in$	set membership relation
$\cap$	set intersection
$\cup$	set union
$\sqcup$	disjoint set union
$\setminus$	set difference, i.e. $A \setminus B \coloneqq A - B$
$\triangle$	symmetric difference, i.e. $A \triangle B \coloneqq A \cup B - A \cap B$
$\subset$	strict subset relation
$\subseteq$	subset relation
$\overline{A}$	complement of the set A
$A \times B$	the Cartesian product of the set A and B

$\mathcal{P}(A)$	the powerset of the set A
$\#A$	the cardinality of the set A, i.e. $\#A = \lvert A \rvert$
ω	the first infinite ordinal
ω_1^{CK}	the Church-Kleene ordinal, i.e. the first uncomputable ordinal
$\mathrm{cf}(\alpha)$	the cofinality of an ordinal α
$\sigma_n\mathrm{cf}_\alpha(\delta)$	the $\Sigma_n(L_\alpha)$ cofinality of δ
$\sigma_n\mathrm{cf}(\alpha)$	the $\Sigma_n(L_\alpha)$ cofinality of α
α	admissible ordinal
α^*	the projectum of α
$\sigma_n p(\alpha)$	the Σ_n-projectum of α
$\lim(\beta)$	β is a limit ordinal
$\aleph_0$	the first infinite cardinal
$\aleph_\alpha$	$(1+\alpha)^{\text{th}}$ infinite cardinal
$\mathrm{Def}(\mathcal{M})$	the first-order definable sets over the model $\mathcal{M}$
L_γ	γ^{th} level of Gödel's constructible hierarchy
$L[A]_\gamma$	γ^{th} level of Gödel's constructible hierarchy with a parameter A
L	Gödel's constructible universe
V	von Neumann's universe
$\mathcal{O}$	Kleene's $\mathcal{O}$ - the set of ordinal notations in $\mathbb{N}$
	for computable ordinals
`Ord`	the class of all ordinals
∞	element of order type `Ord`
$[\gamma, \delta)$	half-closed, half-open ordinal number interval, i.e. $[\gamma, \delta) := \delta \setminus \gamma$
$[\gamma, \delta]$	closed ordinal number interval, i.e. $[\gamma, \delta] := (\delta \setminus \gamma) \cup \{\delta\}$

C.5 β-rational numbers and strings

Q_β	set of β-rational numbers
R_β	set of β-real numbers
I_β	the β-real number unit interval
λ	the empty string
$\sigma \cdot \tau$	the concatenation of the string σ with the string τ
$\lhd^\gamma$	the concatenation of γ zero characters
$\rho, \sigma, \tau, \upsilon$	strings bounded in β, β-rational numbers
$\mathrm{ot}(\sigma)$	order type of a string σ, e.g. $\mathrm{ot}(\lhd\rhd\lhd\lhd) = 4$, $\mathrm{ot}(\rhd^\omega\lhd\rhd) = \omega + 2$
$\sigma \restriction \gamma$	restriction of a string σ to the characters
	at the position less than γ, e.g. $\lhd\lhd\rhd\lhd \restriction 3 = \lhd\lhd\rhd$
$\sigma[\gamma]$	the character at the position γ of the string σ, e.g. $(\rhd\lhd)[1] = \lhd$
$\sigma[\gamma, \delta]$	the closed interval of the string σ from $\sigma[\gamma]$ to $\sigma[\delta]$,

	e.g. $(\lhd\rhd\lhd\lhd)[1,3] = \rhd\lhd\lhd$
$\inf_A(B)$	A-infimum of the set B, see 5.4.3
$\sup_A(B)$	A-supremum of the set B, see 5.4.3

C.6 α-Computability Theory

$\overline{A}$	complement of A, i.e. $\overline{A} := \alpha - A$
$A \oplus B$	α-computable join of sets A and B
π_l	projection, i.e. $\pi_l(\langle x_1, \ldots, x_n\rangle) = x_l$
λ	λ operator in α-calculus, e.g. $\lambda x.x = \mathrm{id} : x \mapsto x$
μ	μ operator, i.e. $\mu x[x \in A] = \min(A)$
Φ_e	(weak) α-enumeration operator
W_e	α-computably enumerable set
K_γ	β-finite set, i.e. $K_\gamma \in L_\beta$
$\leq_T$	Turing reducibility
$\leq_e$	enumeration reducibility
$\leq_{w\alpha}$	weak α-reducibility
$\leq_\alpha$	α-reducibility
$\leq_{w\alpha e}$	weak α-enumeration reducibility
$\leq_{\alpha e}$	α-enumeration reducibility
$A\|_r B$	sets A and B are incomparable wrt r-reducibility,
	i.e. $A \not\leq_r B$ and $A \not\geq_r B$
$\mathcal{D}_T = \mathcal{P}(\omega) / \equiv_T$	set of Turing degrees
$\mathcal{D}_e = \mathcal{P}(\omega) / \equiv_e$	set of the enumeration degrees
$\mathcal{TOT}_e$	set of the total enumeration degrees
$\mathcal{D}_\alpha = \mathcal{P}(\alpha) / \equiv_\alpha$	set of α-degrees
$\mathcal{D}_{\alpha e} = \mathcal{P}(\alpha) / \equiv_{\alpha e}$	set of the α-enumeration degrees
$\mathcal{TOT}_{\alpha e}$	set of the total α-enumeration degrees
$\deg_\alpha(A)$	α-degree of a set A
$\deg_{\alpha e}(A)$	α-enumeration degree of a set A
$K(A)$	weak α-jump of a set A
$J_{w\alpha e}(A)$	weak α-enumeration jump of a set A
$J_{\alpha e}(A)$	α-enumeration jump of a set A
$J_{\alpha e}^{(n)}(A)$	n^{th} α-enumeration jump of a set A
A^+	enumeration of a set A
A^-	enumeration of a set $\overline{A}$
$\mathrm{QF}(\mathcal{A})$	the class of quantifier free formulas
	with parameters from $\mathcal{A}$
$\Delta_n^0, \Sigma_n^0, \Pi_n^0$	definability classes of the arithmetical hierarchy

$\Delta_n, \Sigma_n, \Pi_n$	definability classes of the arithmetical hierarchy
$\Delta^1_n, \Sigma^1_n, \Pi^1_n$	definability classes of the analytical hierarchy
$\Sigma_1(L_\alpha)$	the class of Σ^0_1 formulas with parameters in L_α or sets definable with such formulas
$\Sigma_1(L_\alpha, A)$	the class of Σ^0_1 formulas over L_α with A as a param.
$\Sigma_1(L_\alpha, A^+)$	the class of Σ^0_1 formulas over L_α with A as a positive parameter
$\langle L_\alpha, \mathcal{B}\rangle$	extended model with a parameter $\mathcal{B} \in \{B, B^+, B^-\}$, see 3.4.5
$\mathrm{sc}(L_\alpha)$	the class of α-semicomputable sets
$\mathcal{K}_U(A, B)$	U-Kalimullin pair, U-$\mathcal{K}$-pair
$\mathcal{K}(A, B)$	Kalimullin pair, $\mathcal{K}$-pair
$\mathcal{K}_{\mathrm{nt}}(A, B)$	nontrivial Kalimullin pair
$\mathcal{K}_{\max}(A, B)$	maximal Kalimullin pair
`psup`	pseudosupremum function, see 3.5.1
`fin`	finiteness predicate, i.e. $\mathtt{fin}(A) \iff \#A < \aleph_0$

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