Diophantine approximation: the twisted, weighted and mixed theories

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Abstract

This PhD thesis consists of five papers dealing with problems in various branches of Diophantine approximation. The results obtained contribute to the theory of twisted, weighted, multiplicative and mixed approximation.

In Paper I a twisted analogue of the classical set of badly approximable linear forms is introduced. We prove that its intersection with any suitably regular fractal set is of maximal Hausdorff dimension. The linear form systems investigated are natural extensions of irrational rotations of the circle. Even in the latter one-dimensional case, the results obtained are new.

The main result of Paper II concerns a weighted version of the classical set of badly approximable pairs. We establish a new characterization of this set in terms of vectors that are well approximable in the twisted sense. This naturally generalizes a classical result of Kurzweil. In Paper II we also study the metrical theory associated with a weighted variant of the set introduced in Paper I. In particular, we provide a sufficient condition for this variant to have full Hausdorff dimension. This result is extended in Paper III to imply the stronger property of 'winning'.

Paper IV addresses various problems associated with the Mixed Littlewood Conjecture. Firstly, we solve a version of the conjecture for the case of one p-adic value and one pseudo-absolute value with bounded ratios. Secondly, we deduce the answer to a related metric question concerning numbers that are well approximable in the mixed multiplicative sense. This provides a mixed analogue to a classical theorem of Gallagher.

In Paper V we develop the metric theory associated with the mixed Schmidt Conjecture. In particular, a Khintchine-type criterion for the 'size' of the natural set of mixed well approximable numbers is established. As a consequence we obtain a combined mixed and weighted version of the classical Jarník-Besicovich Theorem.

Contents

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Author's declaration

I declare that all the material in this thesis is my own work unless stated otherwise. The thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Y. Bugeaud, S. Harrap, S. Kristensen & S. Velani, *On shrinking targets for* $\mathbb{Z}^m$ *actions on tori*, Mathematika **56** (2010), 193–202.

II S. Harrap, *Twisted inhomogeneous Diophantine approximation and badly approximable sets*, Acta Arith. **151** (2012), 55–82.

III S. Harrap & N. Moshchevitin, *A note on weighted badly approximable linear forms*, manuscript (2011).

IV S. Harrap & A. Haynes, *The mixed Littlewood conjecture for pseudo-absolute values*, submitted, 16 pages (2011).

V S. Harrap & T. Yusupova, *On a mixed Khintchine problem in Diophantine approximation*, manuscript (2011).

Chapter 1

Introduction

1.1 Classical Diophantine approximation

We begin with a brief description of the classical results within Diophantine approximation. Diophantus of Alexandria is widely believed to have been the first Greek mathematician to recognize the rationals as numbers. The branch of number theory that bears his name is the study of approximating irrational numbers with rationals. The smaller the distance between an irrational and the rational that we approximate it with, the better the approximation is considered to be. For example, two commonly used rational approximations for π are $22/7$ and $355/113$. We calculate that

$$\left|\pi - \frac{22}{7}\right| > \left|\pi - \frac{355}{113}\right|;$$

and so $\frac{355}{113}$ is the better of the two approximations. The question of whether a given rational approximation can always be improved is trivially answered upon recalling that the rationals form a dense subset of the real numbers. It follows immediately that for every irrational number x and every positive integer q there exists an integer p such that

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{2q}. \tag{1.1}$$

So, for any positive integer q, every irrational number can be approximated by some rational with an error of at most $1/(2q)$. It is in improving this statement that provides the motivation for more detailed study.

1.1.1 Dirichlet's theorem and its consequences

No discussion of Diophantine approximation would be complete without initial mention of the following fundamental result. As we will see later it has much wider consequences than one might at first expect.

Theorem 1.1 (Dirichlet 1842)**.** *For any real number x and any natural number N there exist integers p and q such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{qN} \qquad \textit{where} \qquad 1 \leq q \leq N.$$

The result was proven by Dirichlet [31] using his famous 'pigeonhole principle'. This principle essentially states that if N objects are placed in $(N-1)$ boxes then one box must contain at least two objects. Dirichlet's theorem shows that statement (1.1) can be significantly improved upon. Moreover, the following important consequence tells us about the 'rate' at which irrationals can be approximated by rationals (for more details see §6 of [5]).

Corollary 1.2. *For any real number x there exist infinitely many integers p and q (with $q > 0$) such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}. \tag{1.2}$$

For each real number x it is easy to find examples of rationals p/q for which inequality (1.2) holds. When x is rational the exercise is trivial. On the other hand, the *(simple) continued fraction expansion* of any irrational number x is given by

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where $a_0, a_1, a_2, \ldots$ are natural numbers. In abbreviated notation we write $x = [a_0; a_1, a_2, \ldots]$. The quantity a_k is called the *kth partial quotient of x* and the rational $p_k/q_k = [a_0; a_1, \ldots, a_k]$ is called the *kth convergent of x*. The convergents p_k/q_k of x satisfy (1.2) are in some sense the 'best' rational approximations for x. For more details, see [60].

It is natural to ask whether the right hand side of inequality (1.2) can in general be improved. Using the theory of continued fractions it was shown by Hurwitz [51] that it can. Moreover he found an 'optimal' constant associated with this rate of approximation. Modern proofs can be found in [24, 48].

Theorem 1.3 (Hurwitz 1891). *For any real number x there exist infinitely many integers p and q (with $q > 0$) such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{\sqrt{5}\, q^2}. \tag{1.3}$$

Furthermore, the constant $1/\sqrt{5}$ in the above inequality is best possible.

The constant is 'best possible' in the sense that when $x = (\sqrt{5}+1)/2$ is the golden ratio then for any $\epsilon > 0$ we have

$$\left| \frac{\sqrt{5}+1}{2} - \frac{p}{q} \right| > \frac{1}{(\sqrt{5}+\epsilon)\, q^2}.$$

In this way the golden ratio is considered to be 'difficult' to approximate by rationals; or 'badly' approximable. To be precise, we say a real number x is *badly approximable* if there exists a constant $c(x) > 0$ such that for all integers p and $q > 0$

$$\left| x - \frac{p}{q} \right| > \frac{c(x)}{q^2}.$$

In view of Hurwitz' theorem we necessarily have that $0 < c(x) \leq 1/\sqrt{5}$. Numbers which are not badly approximable will be referred to as *well approximable.*

One may notice that the set of badly approximable numbers is invariant under integer translation. In fact, this will be the case with most sets considered in this thesis. For that reason we will often restrict our attention to the unit interval $[0, 1)$ and it should be understood that no generality is lost in doing this. The set of badly approximable numbers lying in $[0, 1)$ will be denoted by **Bad**.

A beautiful property enjoyed by the badly approximable numbers is that their partial quotients are bounded. More precisely, an irrational $x = [a_0; a_1, a_2 \ldots]$ is in **Bad** if and only if there exists a constant $B \geq 1$ such that $a_k \leq B$ for every $k \in \mathbb{N}$. This connection is accentuated by the fact that the golden ratio φ has continued fraction expansion given by $\varphi = [1; 1, 1, \ldots]$.

The golden ratio is also an example of a quadratic irrational. All quadratic irrationals are badly approximable due to the fact that they have periodic continued fraction expansions. This means their partial quotients take on only finitely many values and are thus bounded. It is widely believed that the continued fraction expansion of any irrational algebraic number that is not quadratic contains arbitrarily large partial quotients; i.e., they are not badly approximable. On the other hand, the following remarkable theorem of Roth shows that irrational algebraic numbers are in general 'not too far away' from being badly approximable.

Theorem 1.4 (Roth 1955)**.** *For any irrational algebraic number x and any real $\tau > 1$ there exist only finitely many pairs of integers p and q (with $q > 0$) such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{\tau+1}}. \tag{1.4}$$

In contrast with the idea of badly approximable numbers, we can consider irrationals that are extremely well approximable by rationals. For any $\tau \geq 1$ let $W(\tau)$ be the set of real numbers $x \in [0, 1)$ for which (1.4) holds for infinitely many integers p and q (with $q > 0$). We refer to $W(\tau)$ as the set of *τ-approximable numbers.* Note that in view of Dirichlet's theorem we have $W(1) = [0, 1)$.

An irrational number x is said to be *very well approximable* if it is contained in $W(\tau)$ for some $\tau > 1$. We denote by VWA the set of very well approximable numbers in $[0, 1)$. Thus

$$VWA = \bigcup_{\tau>1} W(\tau).$$

Given $\tau > 1$, it is relatively straightforward to construct numbers in $W(\tau)$. However, Liouville [73] was the first to construct examples of numbers that lie in every $W(\tau)$ and the set of all numbers satisfying this property now bears his name. More precisely, we say an irrational x is a *Liouville number* if

$$x \in \bigcap_{\tau>1} W(\tau).$$

The sets **Bad** and VWA provide good points of reference as they represent two extremes of approximation. In the next subsection we ask the question of how large the sets **Bad** and VWA are. Equivalently, we ask how likely it is that a given real number is contained in one of these sets. It turns out that 'most' numbers fall somewhere in between the two concepts, although as we have seen both **Bad** and VWA are certainly non-empty.

1.1.2 Metric Diophantine approximation

As is usual we shall say that *almost no* points of some set $A \subset \mathbb{R}^k$ have a certain property if the points with the property form a subset of k-dimensional Lebesgue measure zero. Conversely, we say that *almost all* points have the property if almost no points do not have the property. The k-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^k$ will be denoted by $\lambda_k(A)$ or simply by $\lambda(A)$ when no confusion can occur. We say A is a *null* set if $\lambda(A) = 0$ and that A has *full* measure if it's complement is null. Lastly, we denote by $\|\,.\,\|$ the distance to the nearest integer; that is,

$$\|x\| \;=\; \min_{p \in \mathbb{Z}} |x - p\,|$$

for any real number x. This notation is useful as it allows us to forgo mention of the numerator p of any rational approximation p/q. For example, the existence of integers p, q (with $q > 0$) such that the inequality

$$\left| x - \frac{p}{q} \right| \;\leq\; \frac{C}{q^2}$$

holds is equivalent to the existence of a natural number q for which

$$\|qx\| \;\leq\; C/q. \tag{1.5}$$

We return to discussion of Hurwitz' theorem. With reference to this result, one can show that the constant $1/\sqrt{5}$ can be improved if we are to ignore the golden ratio and its equivalents. In particular, inequality (1.5) has infinitely many solutions for all x not equivalent to the golden ratio if we take $C = 1/2^{3/2}$. Furthermore, this constant is then optimal. The story does not stop here. If we also remove the possibility of irrationals equivalent to a solution of the equation $x^2 + 2x = 1$ then the constant may be reduced further. To be precise, the optimal constant associated with inequality (1.5) is then given by $C = 5/\sqrt{221}$.

This process of disregarding irrationals equivalent to the roots of certain integer polynomials can be repeated indefinitely. The sequence of associated optimal constants tends to $1/3$ (giving rise the the *Lagrange spectrum* [27]) and cannot be reduced further via the same method. Therefore, the inequality

$$\|qx\| \;\leq\; 1/(3q) \tag{1.6}$$

has infinitely many integer solutions q for all but a countable set of irrational x. In other words, inequality (1.6) has infinitely many solutions for almost all real numbers. It is this weakening from requiring that all points enjoy a certain property to requiring almost all points satisfy it that characterises the so-called *metric* theory of Diophantine approximation. To establish what happens when $C < 1/3$ we require some more sophisticated theorems.

We begin by generalising the notion of τ-approximable numbers. Let $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ be a non-negative, real-valued arithmetic function. Denote by $W(\psi)$ the set of real numbers $x \in [0, 1)$ that satisfy the inequality

$$\|qx\| \leq \psi(q) \tag{1.7}$$

for infinitely many natural numbers q. This notation should not be confused with that for the set $W(\tau)$, which will remain as shorthand for when $\psi : q \to q^{-\tau}$. We refer to $W(\psi)$ as the set of *ψ-approximable numbers* and the function ψ as an *approximating function.* Via inequality (1.7), each approximating function defines a closed neighbourhood, in this case the interval $[p/q - \psi(q)/q, p/q + \psi(q)/q]$, around any rational p/q. This neighbourhood consists of the set of real numbers x that are approximable by the rational to within an error of $\psi(q)/q$. In general we refer to the neighbourhoods as *error domains* associated with inequalities of the type (1.7).

The following groundbreaking theorem is fundamental to the metrical theory of Diophantine approximation. Modern proofs can be found in [24, 49, 94].

Theorem 1.5 (Khintchine 1924)**.** *For any approximating function ψ we have*

$$\lambda\left(W(\psi)\right) = \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} \psi(r) < \infty. \\ \\ 1, & \displaystyle\sum_{r=1}^{\infty} \psi(r) = \infty \ \textit{ and } \psi \textit{ is monotonic.} \end{cases}$$

It follows from Khintchine's theorem that **Bad** and VWA are of Lebesgue measure zero. Furthermore, it implies that for any $C > 0$ inequality (1.5) has infinitely many solutions for almost all x. Khintchine's theorem is very delicate. One way of demonstrating this is the following consequence. For almost every $x \in \mathbb{R}$ we have that

$$\inf_{q\in\mathbb{N}} q(\log q)(\log\log q)\,\|qx\| = 0,$$

whereas for any $\epsilon > 0$ and for almost every $x \in \mathbb{R}$,

$$\inf_{q\in\mathbb{N}} q(\log q)(\log\log q)^{1+\epsilon}\,\|qx\| > 0. \tag{1.8}$$

Infima type expressions of this form are commonplace in discussions of metric theorems. This particular statement should be compared with those later described in §1.3.1 and §2.4.

The 'convergence' part of Khintchine's theorem is a trivial consequence of the Borel-Cantelli Lemma from probability theory (see for example §1.2 of [49]). Regarding the 'divergence' part, in his original paper Khintchine [57] actually required that $q\psi(q)$ be decreasing. It was subsequently shown by others that this condition can be weakened to the assumption that $\psi(q)$ is decreasing. In their seminal paper [35], Duffin & Schaeffer produced a counterexample showing that this monotonicity assumption is absolutely necessary. More precisely, they constructed a non-monotonic approximating function ψ for which $\lambda(W(\psi)) = 0$ but $\sum_{r=1}^{\infty} \psi(r)$ diverges.

For any approximating function ψ consider the set

$$W'(\psi) := \{x \in [0,1) : |qx - p| < \psi(q) \text{ for i.m. } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with } (p,q) = 1\}.$$

Essentially, the coprimality restriction on p and q here ensures that the rational approximations p/q to x are in reduced form. It is clear that $W'(\psi) \subset W(\psi)$. A further consequence of the Borel-Cantelli Lemma is that

$$\lambda(W'(\psi)) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r}\,\psi(r) < \infty,$$

where φ denotes Euler's totient function. Inspired by this, Duffin & Schaeffer suggested an alternative statement to that of Khintchine free of any conditions on the function ψ.

Conjecture 1.6 (Duffin-Schaeffer 1941)**.** *For any approximating function ψ we have*

$$\lambda(W'(\psi)) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r}\,\psi(r) = \infty.$$

The Duffin-Schaeffer Conjecture represents one of the most profound unsolved problems in metric Diophantine approximation and has inspired a great deal of research into related problems. For various partial results see Chapter 2 of [49]. In [35], Duffin & Schaeffer were able to show that their conjecture is true in certain circumstances. Their result is utilised in its own right in Paper IV.

Theorem 1.7 (Duffin-Schaeffer 1941)**.** *Conjecture 1.6 is true if in addition we have*

$$\limsup_{N\to\infty} \left(\sum_{r=1}^{N} \frac{\varphi(r)}{r}\,\psi(r)\right) \left(\sum_{r=1}^{N} \psi(r)\right)^{-1} > 0. \tag{1.9}$$

In [26], Catlin provided a possible criterion for the full measure of $W(\psi)$ without imposing monotonicity or coprimality.

Conjecture 1.8 (Catlin 1976)**.** *For any approximating function ψ we have*

$$\lambda(W(\psi)) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty} \psi(r) \max_{t \geq 1} \frac{\psi(rt)}{rt} = \infty.$$

Catlin claimed that his conjecture was equivalent to that of Duffin & Schaeffer. However, a flaw in his proof was uncovered by Vaaler [96]. Whether or not the two conjectures are actually equivalent remains unknown.

It is no coincidence that in the metric theorems and conjectures described above the measures of the underlying sets $W(\psi)$ and $W'(\psi)$ are zero or one. This characteristic is related to the well known 'ergodicity' (or 'metrically transitive') property of sets invariant under translation by the rationals. Roughly speaking, if a Lebesgue measurable set A is invariant under rational translation then either A or its complement is of measure zero. The 'zero-one' laws associated with the sets $W(\psi)$ and $W'(\psi)$ were originally established by Cassels [22] and Gallagher [44] respectively. These, and more general zero-one laws, prove to be extremely useful for establishing the divergent part of Khintchine-type theorems. In particular, showing full measure is reduced to showing positive measure.

1.1.3 Hausdorff measure and dimension

Consider the sets of τ-approximable numbers for varying values of τ. As the rate of approximation increases we would intuitively expect the corresponding sets $W(\tau)$ to get smaller in size. For example, we would expect $W(3)$ to be a smaller set than $W(2)$. However, Lebesgue measure fails to distinguish between the two sets from a metric point of view. We have no way of differentiating between them without appealing to a finer method of measuring size. One such method is the concept of Hausdorff measure and dimension.

In what follows, a *dimension function* is a continuous and monotonic function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $f(0) = 0$ and A represents an arbitrary subset of $\mathbb{R}^k$. Let $\mathcal{B} = \{B_t\}_{t \in \mathbb{N}}$ be a countable collection of balls in $\mathbb{R}^k$ with diameters d_t. For any $\rho > 0$, we say $\mathcal{B}$ is a *ρ-cover for A* if $d_t \leq \rho$ for every $t \in \mathbb{N}$ and $A \subset \bigcup_{t \in \mathbb{N}} B_t$. Let

$$\mathcal{H}^f_\rho(A) := \inf \left\{ \sum_{t=1}^{\infty} f(d_t) : \mathcal{B} \text{ is a } \rho\text{-cover for } A \right\}.$$

It is easy to see that $\mathcal{H}^f_\rho(A)$ increases as ρ decreases and so it approaches a limit as $\rho \to 0$. This limit could be zero or infinity. The *Hausdorff f-measure* $\mathcal{H}^f(A)$ of A is defined by

$$\mathcal{H}^f(A) := \lim_{\rho\to 0} \mathcal{H}^f_\rho(A).$$

In the case that $f(r) = r^s$ for some $s \geq 0$, then $\mathcal{H}^f$ is denoted by $\mathcal{H}^s$ and is referred to as *s-dimensional Hausdorff measure.* When s is an integer, k say, then $\mathcal{H}^k(A)$ is a rescaling of k-dimensional Lebesgue measure. In particular, it can be shown that

$$\mathcal{H}^k(A) = c_k^{-1}\lambda_k(A),$$

where c_k is the 'volume' of the k-dimensional unit ball in the sense of Lebesgue. Thus, Hausdorff measure is naturally a refinement of Lebesgue measure.

For any subset A one can easily verify that there exists a unique critical value of s at which $\mathcal{H}^s(A)$ 'jumps' from infinity to zero. The value taken by s at this discontinuity is called the *Hausdorff dimension* of the set A and is denoted $\dim A$. More formally,

$$\dim A = \inf\left\{s > 0 \colon \mathcal{H}^s(A) = 0\right\}.$$

At the critical point $s = \dim A$ the quantity $\mathcal{H}^s(A)$ can be zero, infinite or positive and finite. We say a set $A \subset \mathbb{R}^k$ has *full* dimension if $\dim A = k$.

We now discuss the role of Hausdorff measure and dimension in the theory of Diophantine approximation. The first 'dimension' result was due to Jarník [53] who in 1928 proved that $\dim \mathbf{Bad} = 1$. Since $\mathbf{Bad}$ has Lebesgue measure zero it immediately follows that

$$\mathcal{H}^s\left(\mathbf{Bad}\right) = \begin{cases} 0, & s \geq 1. \\ \\ \infty, & 0 \leq s < 1. \end{cases}$$

The set VWA of very well approximable numbers also has full dimension. This is a consequence of the following classical result obtained by Jarník in [54] and independently by Besicovitch in [14].

Theorem 1.9 (Jarník-Besicovitch 1929/1934)**.** *For any real $\tau \geq 1$ we have*

$$\dim W(\tau) = \frac{2}{1+\tau}.$$

This theorem verifies our intuition regarding the sets $W(2)$ and $W(3)$. The set

$W(2)$ has dimension $2/3$ and the set $W(3)$ has dimension $1/2$ and so $W(3)$ is 'smaller' than $W(2)$. Moreover, it follows from the definition of Hausdorff dimension that

$$\mathcal{H}^s\left(W(\tau)\right) = \begin{cases} 0, & s > 2/(1+\tau). \\ \\ \infty, & 0 \leq s < 2/(1+\tau). \end{cases}$$

However, we are unable to determine directly from the Jarník-Besicovitch Theorem the s-dimensional Hausdorff measure of $W(\tau)$ at the critical exponent $s = 2/(1+\tau)$. The following result deals with this issue and is regarded as the natural generalisation of Khintchine's theorem to Hausdorff measures.

Theorem 1.10 (Jarník 1931)**.** *Let ψ be any approximating function and let f be a dimension function such that $r^{-1}f(r) \to \infty$ as $r \to 0$ and $r^{-1}f(r)$ is decreasing. Then,*

$$\mathcal{H}^f\left(W(\psi)\right) = \begin{cases} 0, & \sum_{r=1}^{\infty} r\, f\left(\frac{\psi(r)}{r}\right) < \infty. \\ \\ \infty, & \sum_{r=1}^{\infty} r\, f\left(\frac{\psi(r)}{r}\right) = \infty \ \textit{and } \psi \textit{ is monotonic.} \end{cases}$$

It is worth noting that when $\mathcal{H}^f$ is equivalent to Lebesgue measure Jarník's result does not apply. This is because the condition $r^{-1}f(r) \to \infty$ as $r \to 0$ excludes the possibility that $f(r) = r$. However, in this case Khintchine's theorem provides the relevant result. We remark that the monotonicity assumption in Jarník's theorem once more seems vital. In fact, very little is known when this restriction is not imposed. Hausdorff measure versions of both the Duffin-Schaeffer Conjecture and the Catlin Conjecture can be found in [7].

Jarník's theorem shows that for any $\tau > 1$ we have

$$\mathcal{H}^{2/(1+\tau)}(W(\tau)) = \infty.$$

However, it is much more powerful than this. For example, take the approximating functions given by

$$\psi_1(r) = \frac{1}{r^2} \quad \text{and} \quad \psi_2(r) = \frac{1}{r^2 \log r}. \tag{1.10}$$

Then, the Jarník-Besicovich Theorem implies that

$$\dim W(\psi_1) \;=\; \dim W(\psi_2) \;=\; \frac{2}{3}.$$

But, if f is the dimension function given by

$$f(r) = r^{2/3}\left(\log r^{-1}\right)^{-1}$$

then we have

$$\sum_{r=1}^{\infty} r f\left(\frac{\psi_1(r)}{r}\right) \;=\; \sum_{r=1}^{\infty}(r\log(r^3))^{-1} \;=\; \infty,$$

whilst

$$\begin{aligned}\sum_{r=1}^{\infty} r f\left(\frac{\psi_2(r)}{r}\right) \;&=\; \sum_{r=1}^{\infty}\left(r\log^{2/3} r\log(r^3\log r)\right)^{-1}\\ &\ll\; \sum_{r=1}^{\infty}(r\log^{5/3} r)^{-1} \;<\; \infty.\end{aligned}$$

It follows from Jarník's theorem that

$$\mathcal{H}^f(W(\psi_1)) = \infty \quad \text{whilst} \quad \mathcal{H}^f(W(\psi_2)) = 0.$$

Thus, we are able to distinguish between sets of the same Hausdorff dimension.

1.1.4 Linear forms approximation

We describe how the classical one-dimensional results of the preceding sections can be generalised to higher dimensions. Throughout, for any vector $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we let

$$\|\mathbf{x}\| := \max_{1\le j\le k} \|x_j\| \qquad \text{and} \qquad |\mathbf{x}| := \max_{1\le j\le k} |x_j|\,.$$

For any integers $n \geq 1$ and $m \geq 1$ let $x_{ji}\,(1 \leq j \leq n, 1 \leq i \leq m)$ be real numbers. For any integer vector $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{Z}^m$ there is a related system $(L_j)_{1\le j\le n}$ of n homogeneous linear forms in m variables given by

$$L_j(\mathbf{q}) \;=\; q_1x_{j1} + \cdots + q_mx_{jm} \qquad\qquad (1 \leq j \leq n).$$

This system can be written more concisely as $\mathcal{L}\mathbf{q}$, where the $n \times m$ matrix

$$\mathcal{L} := (x_{ji}) = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1m} & \cdots & x_{nm} \end{pmatrix}$$

may be regarded as a point in $\mathbb{R}^{nm}$. The set of $n \times m$ real matrices will be denoted by $\mathrm{Mat}_{n\times m}(\mathbb{R})$. However, as the sets and arguments we shall consider are invariant under translation by integer vectors these matrices will often be identified with points in the unit cube $[0,1)^{nm} \subset \mathbb{R}^{nm}$, which will be denoted by $\mathbb{I}^{nm}$. As a result we may use the phrases "matrix in $\mathrm{Mat}_{n\times m}(\mathbb{I})$" and "point in $\mathbb{I}^{nm}$" interchangeably and without confusion.

In general we shall be concerned with minimising the quantity $\|\mathcal{L}\mathbf{q}\|$. When $n \geq 2$ and $m = 1$ this equates to approximating points in $\mathbb{I}^n$ by rational vectors $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})$. In this case the matrix $\mathcal{L}$ takes the form of a vector $\mathbf{x} \in \mathbb{I}^n$ and we shall denote it as such. This is the theory of *simultaneous* Diophantine approximation. On the other hand, when $n = 1$ and $m \geq 2$ the associated problems are referred to as *dual* approximation.

To begin, the following famous result of Minkowski allows us to deduce a multi-dimensional version of Dirichlet's theorem.

Theorem 1.11 (Minkowski's Linear Forms Theorem 1891)**.** *For any square matrix $\mathcal{L} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ there is a non-zero integer vector $\mathbf{q} \in \mathbb{Z}^n_{\neq\mathbf{0}}$ such that*

$$|L_1(\mathbf{q})| \leq c_1 \qquad \text{and} \qquad |L_j(\mathbf{q})| < c_j \quad (2 \leq j \leq n)$$

provided that $c_1 \cdots c_n \geq |\det \mathcal{L}|$.

Corollary 1.12. *For any point $\mathcal{L} \in \mathbb{I}^{nm}$ and any real $N > 1$ there exists a non-zero integer vector $\mathbf{q} \in \mathbb{Z}^m_{\neq\mathbf{0}}$ such that*

$$\|\mathcal{L}\mathbf{q}\| \leq N^{-m/n}, \qquad |\mathbf{q}| \leq N.$$

Corollary 1.13. *For any point $\mathcal{L} \in \mathbb{I}^{nm}$ there exist infinitely many non-zero integer vectors $\mathbf{q} \in \mathbb{Z}^m_{\neq\mathbf{0}}$ such that*

$$\|\mathcal{L}\mathbf{q}\| \leq |\mathbf{q}|^{-m/n}. \tag{1.11}$$

Cassel's book [24] contains a short proof of Corollary 1.12 using the Linear Forms Theorem, but for an independent proof in the same mould as Dirichlet's original box-counting arguement see [48, 92]. The exponent 'm/n' in inequality (1.11) can be interpreted as a normalization of the error domains.

For any approximating function ψ let

$$W(\psi, n, m) = \left\{ \mathcal{L} \in \mathbb{I}^{nm} : \|\mathcal{L}\mathbf{q}\| \leq \psi(|\mathbf{q}|) \text{ for inf. many } \mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}} \right\}.$$

Note that $W(\psi, 1, 1) = W(\psi)$ and so $W(\psi, n, m)$ represents a higher dimensional analogue of the ψ-approximable numbers. The following theorem, attributed to Groshev [47], provides a complete multidimensional analogue of Khintchine's classical result. For obvious reasons it is often referred to as the 'Khintchine-Groshev Theorem'. A modern proof can be found in [94].

Theorem 1.14 (Groshev 1938). *For any approximating function ψ we have*

$$\lambda_{nm}\left(W(\psi, n, m)\right) = \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} r^{m-1}\psi^n(r) < \infty. \\ \\ 1, & \displaystyle\sum_{r=1}^{\infty} r^{m-1}\psi^n(r) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

The one-dimensional counterexample of Duffin & Schaeffer ensures that the monotonicity condition imposed on ψ cannot be removed in general. However, Groshev's theorem can be improved upon when $nm > 1$. Firstly, it can be deduced from a theorem of Schmidt [88] (or Sprindzuk [94]) that the monotonicity assumption is unnecessary when $m \geq 3$. Secondly, it is a consequence of the 1962 result of Gallagher [46] that the same is true when we have $n \geq 2$ and $m = 1$. Suprisingly, it was not until very recently that the problematic '$m = 2$' case was resolved, leading to the following complete statement [12].

Theorem 1.15 (Beresnevich-Velani 2010). *When $nm > 1$ the monotonicity condition in Groshev's Theorem can be dropped.*

We now shift our attention to the notion of badly approximable points. When $nm > 1$ there is no known optimal constant corresponding to inequality (1.11) in the sense of Hurwitz' theorem. That said, various bounds for such constants are known (see Chapter II of [92] for example). In particular, it does make sense to consider badly approximable points in higher dimensions. The set of *badly approximable linear* *forms* is defined by

$$\mathbf{Bad}(n,m) \;=\; \left\{ \mathcal{L} \in \mathbb{I}^{nm} : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq \mathbf{0}}} |\mathbf{q}|^{m/n} \|\mathcal{L}\mathbf{q}\| > 0 \right\}.$$

When $m = n = 1$ it is readily seen that $\mathbf{Bad}(n,m)$ reduces to $\mathbf{Bad}$. An immediate consequence of Groshev's theorem is that $\mathbf{Bad}(n,m)$ is of Lebesgue measure zero.

In 1954, Davenport [28] proved that $\mathbf{Bad}(2,1)$ has continuum many elements. A year later, Cassels [23] showed the same for the set $\mathbf{Bad}(n,1)$. A simpler proof of this can be found in Davenport's follow-up paper [29]. The analogous statement concerning $\mathbf{Bad}(n,m)$ was emphatically proven by Schmidt [91] in 1969 when he established that it has full Hausdorff dimension nm. To do this Schmidt utilised certain infinite topological 'games' and a 'transference' theorem.

1.1.5 Schmidt games and winning sets

The concept of solving mathematical problems by the means of topological games is often attributed to Mazur in the early 1930s. One such game was described by Banach in 1935 and this soon became known as the Banach-Mazur game. In 1966, Schmidt [90] introduced a generalization of the Banach-Mazur game for usage in number theory. As an application he used his game to reprove Jarník's result that $\dim \mathbf{Bad} = 1$. It was with these ideas that he was later able to prove the multidimensional analogue relating to the set $\mathbf{Bad}(n,m)$. For completeness, we include a brief account of Schmidt (α,β) games here.

Let (X,d) be a complete metric space and let $S \subset X$ be a given set. For reasons that will soon become apparent we refer to S as the *target set*. In what follows $B(c,r)$ will denote a closed ball with centre $c \in X$ and radius $r > 0$. Suppose that $0 < \alpha < 1$ and $0 < \beta < 1$ and consider the following game involving players A and B.

Player B starts the game by choosing a closed ball $B_0 = B(b_0, r)$ in X for some $b_0 \in X$ and some $r > 0$. Player A must then choose a point $a_0 \in X$ such that $A_0 = B(a_0, \alpha r) \subset B_0$. The game progresses with player B specifying a ball $B_1 = B(b_1, (\alpha\beta)r) \subset A_0$ and then player A a ball $A_1 = B(a_1, (\alpha^2\beta)r) \subset B_1$. Continuing in this fashion the players pick a nested sequence of non-empty closed balls $B_0 \subset A_0 \subset B_1 \subset A_1 \subset \cdots \subset B_t \subset A_t \subset \cdots$ and the diameters of the balls tend to zero as $t \to \infty$. As this sequence is in a complete metric space, the intersection of these balls is a single point $x \in X$. We declare player A the winner if $x \in S$, and player B the winner otherwise.

Each player employs a strategy for his/her choices of centres of balls as a consequence of his/her opponent's previous choices. If for certain α and β player A can choose a strategy to win the game regardless of how player B plays, we say that the target set S is an (α, β)-*winning* set (on X). If α is such that S is an (α, β)-winning set (on X) for every $0 < \beta < 1$, we say that S is an α-*winning* set (on X). Finally, we simply say S is *winning* (on X) if it is α-winning for some $\alpha > 0$.

We now mention two important properties of winning sets. Firstly, the intersection of countably many α-winning sets is again α-winning. Secondly, an α-winning set S on $\mathbb{R}^{nm}$ has full Hausdorff dimension; i.e., $\dim S = nm$. It is these properties that make the game such a powerful tool, a tool that we utilise in Paper III.

One might have expected that player B could always win the game if we took S to be the Lebesgue null set $\mathbf{Bad}(n, m)$. However, Schmidt was able to show that the opposite is true using a transference theorem of Mahler; he showed that set of badly approximable linear forms is winning.

1.1.6 Transference theorems

Statements which allow information about some Diophantine problem relating to one set of linear forms to be deduced from information concerning another set of linear forms are known as *transference theorems*. The first theorems of this kind in Diophantine approximation came to light from the work of Khintchine [58, 59] and were later extended by Dyson [36] and Jarník [56]. Here, we present an updated version of a more general transference result of Mahler [75] taken from the Appendix of [3].

First, recall that $\mathcal{L}$ denotes the $n \times m$ real matrix corresponding to the n linear forms in m variables given by

$$L_j(\mathbf{q}) = \sum_{i=1}^{m} q_i x_{ji} \qquad (1 \le j \le n).$$

We will denote by $\mathcal{M}$ the $m \times n$ real matrix corresponding to the m linear forms in n variables given by

$$M_i(\mathbf{u}) = \sum_{j=1}^{n} u_j x_{ji} \qquad (1 \le i \le m).$$

In other words, $\mathcal{M} = \mathcal{L}^T \in \mathrm{Mat}_{m\times n}(\mathbb{I})$.

Theorem 1.16 (Mahler 1939). *Suppose there are integers vectors $\mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}}$ such that*

$$\|L_j(\mathbf{q})\| \;\leq\; c_j, \qquad |q_i| \;\leq\; N_i,$$

for some positive constants $c_j > 0$ and $N_i > 0$ satisfying

$$\max_{1 \leq i \leq m} \left\{ d_i := (\ell - 1) N_i^{-1} d^{1/(\ell-1)} \right\} \;<\; 1,$$

where

$$d := \prod_{1 \leq j \leq n} c_j \prod_{1 \leq i \leq m} N_i \qquad \text{and} \qquad \ell := n + m.$$

Then, there are integer vectors $\mathbf{u} \in \mathbb{Z}^n_{\neq \mathbf{0}}$ such that

$$\|M_i(\mathbf{u})\| \;\leq\; d_i, \qquad |\mathbf{u}_j| \;\leq\; U_j,$$

where

$$U_j := (\ell - 1) c_j^{-1} d^{1/(\ell-1)}.$$

It is readily verified that a consequence of Theorem 1.16 is the following statement concerning the set of badly approximable linear forms.

Corollary 1.17. *For any point $\mathcal{L} \in \mathbb{I}^{nm}$ we have*

$$\mathcal{L} \;\in\; \mathbf{Bad}(n, m) \quad \Longleftrightarrow \quad \mathcal{M} \;\in\; \mathbf{Bad}(m, n).$$

We now turn our attention to well approximable sets. Recently, Beresnevich & Velani [10] introduced the *Mass Transference Principle*. In short, this principle allows us to transfer Lebesgue measure theoretic statements for 'limsup' sets to Hausdorff measure theoretic statements.

Recall that the *limit superior* of a sequence of balls B_t ($t \in \mathbb{N}$) in $\mathbb{R}^k$ is defined by

$$\limsup_{t \to \infty} B_t := \bigcap_{s=1}^{\infty} \bigcup_{t=s}^{\infty} B_t.$$

Any set that can be written in this form for some sequence of balls will be referred to as a *limsup* set. For example, the set $W(\psi, n, m)$ can be written as

$$W(\psi, n, m) = \bigcap_{N=1}^{\infty} \bigcup_{|\mathbf{q}|=N}^{\infty} B_{\psi(|\mathbf{q}|)}(\mathbf{q}),$$

where $B_r(\mathbf{q}) = \{\mathcal{L} \in \mathbb{I}^{nm} : \|\mathcal{L}\mathbf{q}\| \leq r\}$. Given a dimension function f and a ball

$B = B(c, r)$ in $\mathbb{R}^k$, define the ball

$$B^f := B(c, f(r)^{1/k}).$$

When $f(r) = r^s$ for some $s > 0$ we write B^s in place of B^f.

Theorem 1.18 (Mass Transference Principle 2006)**.** *Let $\{B_t\}_{t\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ whose radii $r(B_t) \to 0$ as $t \to \infty$. Let f be a dimension function such that $r^{-k}f(r)$ is monotonic and suppose that for any ball B in $\mathbb{R}^k$ we have*

$$\mathcal{H}^k \left(B \cap \limsup_{t\to\infty} B_t^f \right) = \mathcal{H}^k(B).$$

Then, for any ball B in $\mathbb{R}^k$

$$\mathcal{H}^f \left(B \cap \limsup_{t\to\infty} B_t^k \right) = \mathcal{H}^f(B).$$

One of the most remarkable consequences of this principle is that Khintchine's theorem implies Jarník's theorem. This suggests that the Lebesgue theory of limsup sets underpins the Hausdorff theory, which is rather surprising as Hausdorff measure is a refinement of Lebesgue measure. The Mass Transference Principle also enables us to deduce the Jarník-Besicovitch Theorem from Dirichlet's theorem.

A 'slicing' technique introduced in [11] allows the Mass Transference Principle to be generalised. The resulting multidimensional theorem is concerned with limsup sets arising from neighbourhoods of hyperplanes in $\mathbb{R}^k$ rather than simply balls. As a consequence we can transfer Lebesgue measure statements for systems of linear forms to Hausdorff measure statements. In particular, a linear forms version of Jarník's theorem can be deduced directly from Theorems 1.14 & 1.15, reproducing an earlier result of Dickinson & Velani [30] proven via classical methods.

Theorem 1.19 (Dickinson-Velani 1997)**.** *Let $nm > 1$ and let ψ be a decreasing approximating function. Then, for any dimension function f such that $r^{-n(m-1)}f(r)$ is increasing and $r^{-nm}f(r)$ is decreasing we have*

$$\mathcal{H}^f\left(W(\psi, n, m)\right) = \begin{cases} 0, & \sum_{r=1}^{\infty} f\left(\frac{\psi(r)}{r}\right) \psi^{-n(m-1)}(r)\, r^{nm+m-1} < \infty. \\ \\ \mathcal{H}^f(\mathbb{I}^{nm}), & \sum_{r=1}^{\infty} f\left(\frac{\psi(r)}{r}\right) \psi^{-n(m-1)}(r)\, r^{nm+m-1} = \infty. \end{cases}$$

This theorem essentially implies the following result of Dodson [32], which is general multidimensional version of the Jarník-Besicovitch Theorem involving the lower order of an approximating function ψ. The *lower order (at infinity)* $\lambda(g)$ of a function $g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is defined by

$$\lambda(g) = \liminf_{r\to\infty} \frac{\log g(r)}{\log r}. \tag{1.12}$$

Note that $\lambda(g)$ is strictly positive if g is increasing.

Theorem 1.20 (Dodson 1997)**.** *Let $nm > 1$ and for any decreasing approximating function ψ let λ be the lower order at infinity of $1/\psi$. Then,*

$$\dim W(\psi, n, m) = \begin{cases} n(m-1) + \frac{n+m}{\lambda+1}, & \lambda > \frac{m}{n}. \\ \\ nm, & \lambda \leq \frac{m}{n}. \end{cases}$$

This result shows that the 'size' of $W(\psi, n, m)$ decreases as the speed of approximation governed by ψ increases. Dodson's theorem is often considered to be the Hausdorff dimension version of Groshev's theorem with the 'volume' sum in the latter replaced by the lower order. These results provide us with a complete picture of the standard metric theory associated with the sets introduced in previous sections.

1.1.7 Absolutely friendly measures

One can study metric Diophantine approximation with respect to more general measures than those of Lebesgue and Hausdorff. The problem of approximating by rationals the points of some compact subset $K \subset \mathbb{R}^k$ has received much recent attention. In order to make progress one needs to be very careful in defining the measure supported on K. It turns out that it must be 'well behaved' in some sense.

In their 2004 paper [63], Kleinbock, Lindenstrauss & Weiss introduced the notion of 'friendly' measures, a class of measures adhering to certain rigid geometrical conditions. A more restrictive subclass was investigated by Pollington & Velani in [85] and it is with these 'absolutely friendly' measures that this thesis is concerned.

The following properties will be enforced on any locally finite Borel measure μ supported on a compact subset $K \subset \mathbb{R}^k$. Throughout, $B(c, r)$ will denote the closed ball in $\mathbb{R}^k$ with centre $c \in K$ and radius $r > 0$. Suppose D is a strictly positive real number. A measure μ is called *D-Federer* (or *D-doubling*) if there exists $r_0 > 0$ such that for any $c \in K$ and any positive $r < r_0$ we have

$$\mu(B(c,2r)) \ \leq \ D\mu(B(c,r)).$$

We say that μ is *Federer* (or *doubling*) if it is D-Federer for some $D > 0$.

Next, let $\mathcal{S}$ denote a generic $(k-1)$-dimensional hyperplane in $\mathbb{R}^k$ and for any $\epsilon > 0$ let $\mathcal{S}^{(\epsilon)}$ denote the ϵ-neighbourhood of $\mathcal{S}$. To be precise,

$$\mathcal{S}^{(\epsilon)} := \left\{ \mathbf{x} \in \mathbb{R}^k : d_{\mathcal{S}}(\mathbf{x}) < \epsilon \right\},$$

where $d_{\mathcal{S}}(\mathbf{x})$ denotes the Euclidean distance from $\mathbf{x} \in \mathbb{R}^k$ to $\mathcal{S}$. Suppose $C, \eta > 0$ are real numbers. Then a measure μ is said to be (C,η)-*absolutely decaying* if there exists $r_0 > 0$ such that for any hyperplane $\mathcal{S}$, any $\epsilon > 0$, any $c \in K$ and all positive $r < r_0$ we have

$$\mu(B(c,r) \cap \mathcal{S}^{(\epsilon)}) \leq C \left(\frac{\epsilon}{r}\right)^{\eta} \mu(B(c,r)).$$

We say μ is *absolutely decaying* if it is (C,η)-absolutely decaying for some $C, \eta > 0$. If a measure is Federer and absolutely decaying then it is said to be *absolutely friendly* (or *absolutely η-friendly* if the exponent requires emphasis) as defined in [85].

In addition, the measure μ is δ-*Ahlfors regular* if there exist strictly positive constants δ and r_0 such that for $c \in K$ and $r < r_0$

$$ar^{\delta} \leq \mu(B(c,r)) \leq br^{\delta} \ ,$$

where $0 < a \leq 1 \leq b$ are constants independent of the ball. It is easily verified that if μ is δ-Ahlfors regular then

$$\dim K = \delta. \tag{1.13}$$

In the one-dimensional situation (i.e., $K \subset \mathbb{R}$), one can check that the Ahlfors regular property implies the absolutely friendly property. However, this is not true in general for $\mathbb{R}^k$.

We remark that k-dimensional Lebesgue measure on $\mathbb{I}^k$ is 2^k-Federer, $(1,1)$-absolutely decaying and k-Ahlfors regular. In addition, if (1.13) holds, then the restriction of δ-dimensional Hausdorff measure to K is absolutely friendly. Further examples of absolutely friendly measures can be found in [65].

Establishing metric Diophantine results in the context of these general measures is difficult in general. Even proving an analogue of the convergence part of Khintchine's theorem requires new ideas. In [85], Pollington & Velani deduced that if μ is an absolutely η-friendly measure supported on a compact subset K of $\mathbb{R}^n$ then for any approximating function ψ we have

$$\mu(W(\psi,n,1)\cap K)=0 \quad \text{if} \quad \sum_{r=1}^{\infty} r^{\eta\frac{n+1}{n}-\eta-1}\psi^{\eta}(r) < \infty.$$

In all likelihood this result is not best possible. Moreover, a divergence type statement for absolutely friendly measures currently seems out of reach.

Whilst general metric results concerning well approximable points seem illusive, a great deal of progress has been made concerning badly approximable points in compact sets. The first result of this kind can be found in [65], where Kleinbock & Weiss proved that $\dim(\mathbf{Bad}(n,1)\cap K)=\dim K$ whenever K supports an Ahlfors regular absolutely friendly measure. This result was independently reproduced by Kristensen, Thorn & Velani [67] whilst developing a very general framework for establishing dimension statements. The ideas of [67] are applicable to a large class of badly approximable sets and are utilised in both Papers I & II. The result of Kleinbock & Weiss was strengthened by Fishman in a recent paper [42].

Theorem 1.21 (Fishman 2009)**.** *Let K be a compact subset of $\mathbb{I}^{nm}$ supporting an absolutely friendly finite Borel measure μ. Then, $\mathbf{Bad}(n,m)\cap K$ is a winning set on K.*

A large class of sets K for which Theorem 1.21 is applicable arise naturally as attractors of an irreducible finite family of contracting similarity maps of $\mathbb{R}^{nm}$ satisfying the open set condition (see [40, 52]). Examples include fractals such as the middle-third Cantor set, the von Koch curve and the Sierpinski gasket, to name but a few. It was shown in [41] that for such sets K the winning property implies full dimension for $\mathbf{Bad}(n,m)\cap K$.

The intersection of the set of very well approximable numbers with compact sets, including fractals, has also received much recent attention. For example, Weiss [98] demonstrated in 2001 that almost no points in the middle-third Cantor set C (with respect to Hausdorff measure on C) are very well approximable. Further discussion of problems of this type can be found in [64, 71] and the references therein.

1.1.8 Diophantine approximation with weights

Another way of generalising the classical problems of Diophantine approximation is via the concept of 'weighting'. This roughly corresponds to considering 'rectangular' error domains, rather than 'square' ones, in the problems previously discussed. To some extent the concept is motivated by another consequence of Minkowski's Linear Forms Theorem.

Corollary 1.22. *Fore any real n-tuple $\mathbf{k} = \{k_1, \ldots, k_n\}$ satisfying*

$$k_j \; > \; 0 \quad (1 \leq j \leq n) \qquad \text{and} \qquad \sum_{j=1}^{n} k_j \; = \; 1 \tag{1.14}$$

and any vector $\mathbf{x} \in \mathbb{R}^n$ there exist infinitely many natural numbers q such that

$$\max_{1 \leq j \leq n} \left(\|qx_j\|^{1/k_j} \right) \leq \frac{1}{q}. \tag{1.15}$$

This result demonstrates that if we attach a 'weight' $1/k_j$ to each component $\|qx_j\|$ we are still able to deduce an analogue of Dirichlet's theorem. For obvious reasons, the study of problems of this type is called *weighted* Diophantine approximation (or Diophantine approximation *with weights*). We remark that in the case $n = 2$ it is usual to put $k_1 = i$ and $k_2 = j$.

For any approximating function ψ and any n-tuple $\mathbf{k}$ define the set of $(\mathbf{k}, \psi)$-*approximable vectors* by

$$W(\mathbf{k}, \psi) := \left\{ \mathbf{x} \in \mathbb{I}^n : \max_{1 \leq j \leq n} \left(\|qx_j\|^{1/k_j} \right) \leq \psi(q) \;\; \text{for inf. many } q \in \mathbb{N} \right\}.$$

It is clear that when $k_1 = \cdots = k_n = 1/n$ this set reduces to $W(\psi, n, 1)$. In [59], Khintchine proved the following weighted version of his classical one-dimensional theorem.

Theorem 1.23 (Khintchine, 1926)**.** *For any n-tuple of reals $\mathbf{k}$ satisfying (1.14) and any approximating function ψ we have*

$$\lambda_n \left(W(\mathbf{k}, \psi) \right) \quad = \quad \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} \psi(r) \; < \; \infty. \\[2em] 1, & \displaystyle\sum_{r=1}^{\infty} \psi(r) \; = \; \infty \;\; \text{and } \psi \text{ is monotonic.} \end{cases}$$

It follows from a result of Harman [49, Theorem 3.8] that the monotonicity assumption on ψ can be removed if $n \geq 2$. Other problems relating to the set $W(\mathbf{k}, \psi)$ have received very little attention. Indeed, it was not until the early 1980s that interest in weighted problems resurfaced through the study of badly approximable points. Schmidt [93] noticed that the classical methods of Davenport for **Bad**$(1/2,\, 1/2)$ can be transferred to the weighted setting. He showed that for any real pair (i, j) satisfying

$$i, j > 0 \quad \text{and} \quad i + j = 1 \tag{1.16}$$

the set **Bad**(i, j) of (i, j)-*badly approximable vectors* defined by

$$\mathbf{Bad}(i, j) \;=\; \left\{ \mathbf{x} \in \mathbb{I}^2 : \inf_{q \in \mathbb{N}} q \cdot \max \left\{ \|qx_1\|^{1/i},\, \|qx_2\|^{1/j} \right\} > 0 \right\}$$

is uncountable. In the same paper, he made a famous conjecture concerning the intersection of these sets.

Conjecture 1.24 (Schmidt 1983)**.**

$$\mathbf{Bad}(2/3, 1/3) \;\cap\; \mathbf{Bad}(1/3, 2/3) \;\neq\; \emptyset.$$

Building on Davenport's work, Pollington & Velani [84] showed in 2000 that for every choice of reals i, j satisfying (1.16) we have

$$\dim\left(\mathbf{Bad}(i, j) \cap \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)\right) = \dim\left(\mathbb{I}^2\right) = 2. \tag{1.17}$$

In 2011, Badziahin, Pollington & Velani [3] settled the Schmidt Conjecture emphatically.

Theorem 1.25 (Badziahin-Pollington-Velani 2011)**.** *Let (i_t, j_t) be a countable number of pairs of real numbers each satisfying (1.16). Also, suppose that*

$$\liminf_{t \to \infty} \min\{i_t, j_t\} \;>\; 0. \tag{1.18}$$

Then,

$$\dim\left(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \right) \;=\; 2.$$

Note that if the number of pairs (i_t, j_t) is finite then (1.18) is trivially satisfied.

1.2 The inhomogeneous theory

1.2.1 Classical inhomogeneous approximation

We return to the fundamental problem of approximating a real number x by rationals p/q. The problem of minimising the values taken by $\|qx\|$ essentially equates to the question of how often the quantity $qx \pmod 1$ approaches the origin. However, the role that the origin plays seems uncertain. *Inhomogeneous* Diophantine approximation deals with the more general question of how often the sequence $qx \pmod 1$ approaches some fixed point α in the unit interval.

To begin, the following result of Cassels' shows that an inhomogeneous analogue of Dirichlet's theorem does not hold in general.

Theorem 1.26 ([24], Chapter III, Theorem III). *Let ψ be an approximating function for which $\psi(q) \to 0$ as $q \to \infty$. Then, there exists a real α and an irrational x such that the system*

$$\|qx - \alpha\| \leq \psi(N), \qquad |q| \leq N$$

has no solutions $q \in \mathbb{Z}_{\neq 0}$ for infinitely many $N \in \mathbb{N}$.

Corollary 1.27. *There exists a real number α and an irrational x such that for infinitely many $N \in \mathbb{N}$ the inequality*

$$\|qx - \alpha\| \leq \frac{1}{N}$$

has no integer solutions q with $|q| \leq N$.

Despite this setback, the following observation was made by Khintchine. For any irrational x and real α there exist infinitely many natural numbers q such that

$$\|qx - \alpha\| \leq \frac{1+\epsilon}{\sqrt{5}q}, \tag{1.19}$$

where $\epsilon > 0$ is an arbitrary constant. The inequality is 'optimal' in the sense that it cannot be improved upon in the case that $\alpha = sx + t$ for some $s, t \in \mathbb{Z}$. The underlying reason for this is that when $\alpha = sx + t$ the behaviour of the quantity $\|qx - \alpha\| = \|(q-s)\alpha\|$ is essentially homogeneous. To some extent this explains the similarity with Hurwitz' theorem. On the other hand, the following 'optimal' statement was deduced by Minkowski [80] when this case is ruled out.

Theorem 1.28 (Minkowski 1901). *For any real number α and any irrational x such that $\alpha \neq sx + t$ for any $s, t \in \mathbb{Z}$ there exist infinitely many non-zero integers q such* *that*

$$\|qx-\alpha\| \;\leq\; \frac{1}{4\,|q|}.$$

Furthermore, the constant $1/4$ *in the above inequality is best possible.*

The constant $1/4$ is not only best possible, but 'tight' in the following sense. For any $\epsilon>0$ there exist an irrational x and a real α not of the form $sx+t$ such that

$$|q|\,\|qx-\alpha\| > 1/4-\epsilon$$

for all non-zero integers q. Moreover, $\liminf_{|q|\to\infty}|q|\,\|qx-\alpha\| = 1/4$.

In terms of well approximable sets associated with inhomogeneous Diophantine approximation there are two obvious ways to proceed. Firstly, one could think of the the real number α (or more generally the real vector $\boldsymbol{\alpha}\in\mathbb{I}^n$) as fixed and consider the set

$$W^{\boldsymbol{\alpha}}(\psi,n,m) \;=\; \left\{\mathcal{L}\in\mathbb{I}^{nm} : \|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| \leq \psi(|\mathbf{q}|) \text{ for inf. many } \mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}\right\}.$$

A very general result of Schmidt [89] gives rise to the following statement.

Theorem 1.29 (Schmidt 1964)**.** *Let* $nm>1$ *with* $m\neq 2$*. Then for any approximating function* ψ *and any real vector* $\boldsymbol{\alpha}\in\mathbb{I}^n$ *we have*

$$\lambda_{nm}\left(W^{\boldsymbol{\alpha}}(\psi,n,m)\right) \;=\; \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} r^{m-1}\psi^n(r) \;<\; \infty. \\[2ex] 1, & \displaystyle\sum_{r=1}^{\infty} r^{m-1}\psi^n(r) \;=\; \infty. \end{cases}$$

The case '$n=m=1$' is excluded as the statement would then be false due to Duffin & Schaeffer's counterexample. The point is that no monotonicity condition is enforced on the function ψ, so when $\boldsymbol{\alpha}=\mathbf{0}$ the result coincides with Theorem 1.15. For obvious reasons, Schmidt's theorem is often referred to as a 'singly metric' result.

The second scenario one can consider concerns the 'doubly metric' problems arising when the inhomogeneous part $\boldsymbol{\alpha}$ is viewed as a variable in its own right. Consider the set

$$W_*(\psi,n,m) \;=\; \left\{(\mathcal{L},\boldsymbol{\alpha})\in\mathbb{I}^{nm}\times\mathbb{I}^n : \|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| \leq \psi(|\mathbf{q}|) \text{ for i. m. } \mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}\right\}.$$

An inhomogeneous version of Groshev's theorem concerning $W_*(\psi,n,m)$ is much easier to prove than in the singly metric case. The following theorem, originally proved by Cassels [24], also follows from Schmidt's general statement in [89].

Theorem 1.30 (Cassels 1957)**.** *For any approximating function ψ we have*

$$\lambda_{nm+n}\left(W_*(\psi,n,m)\right) \;=\; \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} r^{m-1}\psi^n(r) \;<\; \infty. \\[2ex] 1, & \displaystyle\sum_{r=1}^{\infty} r^{m-1}\psi^n(r) \;=\; \infty. \end{cases}$$

Cassels' result is strikingly similar to Theorem 1.29, but is slightly more general in the sense that the desired Diophantine property holds only for almost all pairs $(\mathcal{L},\boldsymbol{\alpha})$. For this reason it is considered a doubly metric statement. Note that even in the one-dimensional case no monotonicity restriction is enforced on ψ.

Hausdorff dimension results for the sets $W_*(\psi,n,m)$ and $W^{\boldsymbol{\alpha}}(\psi,n,m)$ were established by Dodson [33] and Levesley [70] respectively in the late 1990s. They represent the natural inhomogeneous analogues of Theorem 1.20. Inhomogeneous versions of Jarník's theorem can be found in Bugeaud's paper [18].

Of great recent interest and of relevance to this thesis is the concept of inhomogeneous badly approximable points. In view of the previous discussion, there are two formulations to be considered. Firstly, the set of *badly approximable affine forms* is defined by

$$\mathbf{Bad}_*(n,m) \;=\; \left\{(\mathcal{L},\boldsymbol{\alpha})\in\mathbb{I}^{nm}\times\mathbb{I}^n : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}} |\mathbf{q}|^{m/n}\,\|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| > 0\right\}.$$

And secondly, for any fixed vector $\boldsymbol{\alpha}$ in $\mathbb{I}^n$ one defines the set of *badly approximable inhomogeneous forms (with respect to the vector $\boldsymbol{\alpha}$)* by

$$\mathbf{Bad}^{\boldsymbol{\alpha}}(n,m) \;=\; \left\{\mathcal{L}\in\mathbb{I}^{nm} : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}} |\mathbf{q}|^{m/n}\,\|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| > 0\right\}.$$

Both sets are of Lebesgue measure zero due to the theorems of Cassels and Schmidt respectively.

In the landmark paper [62], Kleinbock used ideas and techniques from the theory of dynamical systems to prove that $\mathbf{Bad}_*(n,m)$ is of full Hausdorff dimension. Essentially, his method is based on a deep connection between badly approximable systems of linear forms and certain orbits of lattices in Euclidean space. In the same paper, Kleinbock noted that his method yields that the set of vectors $\boldsymbol{\alpha}$ for which

$\mathbf{Bad}^{\boldsymbol{\alpha}}(n,m)$ has full dimension is itself of full dimension. Inspired by this, he conjectured that both $\mathbf{Bad}_*(n,m)$ and $\mathbf{Bad}^{\boldsymbol{\alpha}}(n,m)$ are winning sets on $\mathbb{I}^{nm}$. A general result implying that the latter set is indeed winning was proven by Einsiedler & Tseng [39] in 2011.

Theorem 1.31 (Einsiedler-Tseng 2011)**.** *Let K be a closed subset of $\mathbb{I}^{nm}$ supporting an absolutely friendly finite Borel measure μ. Then, for any $\boldsymbol{\alpha} \in \mathbb{I}^n$ the set $\mathbf{Bad}^{\boldsymbol{\alpha}}(n,m) \cap K$ is a winning set on K.*

The related conjecture corresponding to the set $\mathbf{Bad}_*(n,m)$ remains open, although Einsiedler & Tseng did outline a possible method for obtaining a proof.

Conjecture 1.32 (Kleinbock 1999)**.** *Let K be a closed subset of $\mathbb{I}^{nm} \times \mathbb{I}^n$ supporting an absolutely decaying finite Borel measure μ. Then, the set $\mathbf{Bad}_*(n,m) \cap K$ is a winning set on K.*

1.2.2 Twisted approximation

Consider initially a rotation of the unit circle through an angle x. Identifying the circle with the unit interval $[0,1)$ and the base point of the iteration with the origin, we are considering the numbers $0, x, 2x, \ldots$ modulo one. If x is rational, the rotation is periodic. On the other hand, a trivial consequence of Minkowski's theorem is that the sequence $\{qx\}_{q\in\mathbb{N}}$ modulo one is dense in the unit interval for any irrational x. Furthermore, a celebrated result of Weyl [99] states that the sequence is uniformly distributed in $[0,1)$ for any irrational x. In view of Weyl's result, the sequence $\{qx\}_{q\in\mathbb{N}}$ modulo one must visit any fixed set in $[0,1)$ of positive measure infinitely often for almost every x. The 'shrinking target problem' introduced in [50] formulates the natural question of what happens if the target set – a set of positive measure – is allowed to shrink with time. For example and more precisely, is there an optimal 'shrinking rate' for which the sequence $\{qx\}_{q\in\mathbb{N}}$ modulo one visits the shrinking target infinitely often? In the specific case of irrational rotations of the circle, the shrinking target sets correspond to subintervals of $[0,1)$ whose lengths decay according to some specified approximating function ψ. In other words, the problem translates to considering the set of $\alpha \in [0,1)$ satisfying the familiar inequality

$$\|qx - \alpha\| \leq \psi(q).$$

In practice, we consider a more general problem than that described above. For each point $\mathcal{L} \in \mathbb{I}^{nm}$ define the set

$$W_{\mathcal{L}}(\psi, n, m) := \left\{ \boldsymbol{\alpha} \in \mathbb{I}^n : \|\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}\| \leq \psi(|\mathbf{q}|) \text{ for inf. many } \mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}} \right\}.$$

To avoid the degenerate situation that $\left\{\mathcal{L}\mathbf{q} : \mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}}\right\}$ is restricted to at most a countable collection of parallel, positively separated, hyperplanes in $\mathbb{I}^n$ we assume throughout that the associated group $G = \mathcal{L}^T\mathbb{Z}^n + \mathbb{Z}^m$ has rank $n + m$. In one-dimension this corresponds to the condition that x is irrational and in the simultaneous case that $1, x_1, \ldots, x_n$ are linearly independent over the rationals. For this reason we say that a point $\mathcal{L} \in \mathbb{I}^{nm}$ is *irrational* if G has rank $n + m$ and *rational* otherwise.

The metrical theory associated with the set $W_{\mathcal{L}}(\psi, n, m)$ was investigated in a groundbreaking paper by Kurzweil [69] in 1955. In what follows we say a decreasing approximating function ψ is *divergent* if $\sum_{r \in \mathbb{N}} r^{m-1}\psi^n(r) = \infty$.

Theorem 1.33 (Kurzweil 1955)**.** *Let ψ be a fixed decreasing approximating function. Then, for almost all irrational points $\mathcal{L} \in \mathbb{I}^{nm}$ we have*

$$\lambda_{nm}\left(W_{\mathcal{L}}(\psi, n, m)\right) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty} r^{m-1}\psi^n(r) \;=\; \infty. \tag{1.20}$$

Furthermore, the points $\mathcal{L} \in \mathbb{I}^{nm}$ for which (1.20) holds for every decreasing approximating function are precisely those in $\mathbf{Bad}(n, m)$.

One might have expected that no matter what the choice of irrational $\mathcal{L}$ or decreasing approximating function ψ we would be able to conclude that the set $W_{\mathcal{L}}(\psi, n, m)$ has full measure if ψ is divergent. But, Kurzweil's result demonstrates that for every point $\mathcal{L} \in \mathbb{I}^{nm} \setminus \mathbf{Bad}(n, m)$ there exists a divergent decreasing approximating function ψ for which the full measure conclusion fails to hold. On the other hand, by once more appealing to the Borel-Cantelli Lemma it is easy to show that for *any* irrational $\mathcal{L}$ and *every* approximating function ψ we have

$$\lambda_{nm}\left(W_{\mathcal{L}}(\psi, n, m)\right) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} r^{m-1}\psi^n(r) \;<\; \infty.$$

This subtle distinction is what makes the metrical theory in the twisted setting more delicate, and sophisticated, than its standard homogeneous counterpart.

Over the last few years, there has been much activity in investigating problems of this type. For example, when $\psi(q) := q^{-v}$ for some $v > 1$, Bugeaud [17] and independently Schmeling & Trubetskoy [87] obtained the Hausdorff dimension of the set $\mathbf{W}_{\mathcal{L}}(\psi, 1, 1)$. Of particular relevance is a result of Kim [61] stating that for any irrational x the set of real α for which

$$\liminf_{q \to \infty} q \, \|qx - \alpha\| = 0 \tag{1.21}$$

has full Lebesgue measure. Rather surprisingly, Beresnevich, Bernik, Dodson & Velani [7] were able to use the Mass Transference Principle to show that this result and indeed the dimension result of Bugeaud and Schmeling & Trubetskoy are consequences of the fact that for any irrational x and any real α the inequality (1.19) has infinitely many solutions.

Kim's paper inspired activity concerning the complementary measure zero set associated with (1.21). In Paper I it is established that the higher dimensional analogue

$$\mathbf{Bad}_{\mathcal{L}}(n, m) = \left\{ \boldsymbol{\alpha} \in \mathbb{I}^n : \inf_{\mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}}} |\mathbf{q}|^{m/n} \, \|\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}\| > 0 \right\}$$

has full Hausdorff dimension. This is proven as a consequence of a more general result concerning the intersection of $\mathbf{Bad}_{\mathcal{L}}(n, m)$ with suitably regular compact subsets of $\mathbb{I}^n$.

The assertions of Paper I motivated the work of Tseng [95], who showed that the one dimensional set $\mathbf{Bad}_{\mathcal{L}}(1, 1)$ is winning for every $\mathcal{L} \in \mathbb{I}$. The analogous statement for $\mathbf{Bad}_{\mathcal{L}}(n, m)$ was later proven by Moshchevitin [81]. Even these recent improvements have since been built upon, culminating in a 'complete' statement established by Einsiedler & Tseng [39] in 2011. Their result was independently obtained by Broderick, Fishman & Kleinbock [16], also in 2011.

Theorem 1.34 ([16, 39], 2011)**.** *Let K be a closed subset of $\mathbb{I}^{nm}$ supporting an absolutely decaying locally finite Borel measure μ. Then, for any $\mathcal{L} \in \mathbb{I}^{nm}$ the set $\mathbf{Bad}_{\mathcal{L}}(n, m) \cap K$ is a winning set on K.*

It is intriguing that for the twisted badly approximable set $\mathbf{Bad}_{\mathcal{L}}(n, m)$ the measure μ supported on K is only required to be absolutely decaying to conclude the winning property for $\mathbf{Bad}_{\mathcal{L}}(n, m) \cap K$. This should be compared with the classical inhomogeneous setting of Theorem 1.31, where at the moment we must assume the stronger condition that μ is absolutely friendly to reach the same conclusion for the set $\mathbf{Bad}_{*}(n, m)$.

1.3 Multiplicative Diophantine approximation

One of the most important unsolved problems in Diophantine approximation, and indeed number theory in general, is due to Littlewood [74].

Conjecture 1.35 (Littlewood 1930s). *For every $x_1, x_2 \in \mathbb{R}$,*

$$\liminf_{q\to\infty} q\,\|qx_1\|\,\|qx_2\| \;=\; 0. \tag{1.22}$$

Questions relating to this conjecture have been the subject of much concerted effort in recent years. Loosely speaking, the supremum norm of the classical problems has been replaced by the geometric mean in this *multiplicative* setting. The first significant contribution toward Conjecture 1.35 was made by Cassels & Swinnerton-Dyer [25] who showed that (1.22) holds when x_1 and x_2 are chosen from the same cubic field.

The Littlewood Conjecture has come to light recently because of its spectacular connection to 'measure rigidity' problems concerning the space of unimodular lattices (see [76], for example). This connection was exploited to devastating effect by Einsiedler, Katok & Lindenstrauss [37] in 2006. They proved that the set of pairs $(x_1, x_2) \in \mathbb{R}^2$ which do not satisfy (1.22) has Hausdorff dimension zero.

We remark that nothing seems to be gained by adding an extra real variable to the Littlewood Conjecture. The statement that

$$\liminf_{q\to\infty} q\,\|qx_1\|\,\|qx_2\|\,\|qx_3\| \;=\; 0$$

for all $x_1, x_2, x_3 \in \mathbb{R}$ is weaker than Conjecture 1.35 since $\|qx_3\| \leq 1/2$. However, the problem does not seem to be any easier to solve using current methods.

Conjecture 1.35 can be reformulated to read that the set

$$\mathbf{Bad}_L := \left\{\mathbf{x} \in \mathbb{I}^2 : \inf_{q\in\mathbb{N}} q\,\|qx_1\|\,\|qx_2\| > 0\right\}$$

is empty. Whilst $\mathbf{Bad}_L$ would seem to be the natural multiplicative analogue of **Bad**, the assertion of the Littlewood Conjecture suggests it might be a 'bad' choice. In [4], the larger sets

$$\mathbf{Mad}^\lambda := \left\{\mathbf{x} \in \mathbb{I}^2 : \inf_{q\in\mathbb{N}} q\,\log^\lambda q\;\|qx_1\|\,\|qx_2\| > 0\right\},$$

for $\lambda \geq 0$, were introduced by Badziahin & Velani. They argued that $\mathbf{Mad}^1$ should be considered the 'true' set of multiplicatively badly approximable numbers. Moreover, they conjectured the following.

Conjecture 1.36 (Badziahin-Velani 2010)**.**

$$\begin{aligned} \mathbf{Mad}^{\lambda} &= \emptyset \quad \textit{for any } \lambda < 1. \\ \dim \mathbf{Mad}^{\lambda} &= 2 \quad \textit{for any } \lambda \geq 1. \end{aligned}$$

Despite the fledgling nature of this conjecture, progress towards solving it has already been made. In the follow-up paper [1], Badziahin proved a result implying that the second part of Conjecture 1.36 is true when $\lambda > 1$. Knowledge had previously been limited to a result of Moshchevitin & Bugeaud [21], who showed that the set $\mathbf{Mad}^2$ enjoys full Hausdorff dimension.

On a different note, one can consider vectors that are well approximable in a multiplicative sense. For any approximating function ψ define the set

$$M(\psi, n) := \left\{ \mathbf{x} \in \mathbb{I}^n : \prod_{j=1}^{n} \|qx_j\| \leq \psi(q) \text{ for inf. many } q \in \mathbb{N} \right\}.$$

When $n = 1$ this set coincides with $W(\psi, 1, 1)$. The following Khintchine-type result can be deduced from a much more general theorem of Gallagher [45] and provides the Lebesgue metric theory associated with $M(\psi, n)$.

Theorem 1.37 (Gallagher 1962)**.** *For any approximating function ψ,*

$$\lambda_n\left(M(\psi, n)\right) = \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} \psi(r) \log^{n-1} r < \infty. \\ \\ 1, & \displaystyle\sum_{r=1}^{\infty} \psi(r) \log^{n-1} r = \infty \textit{ and } \psi \textit{ is monotonic.} \end{cases}$$

A consequence of Gallagher's theorem is that the Lebesgue measure of the set $\mathbf{Mad}^{\lambda}$ is zero if $\lambda \leq 2$ and full otherwise. It is not known whether the monotonicity condition imposed in the theorem is necessary when $n \geq 2$. The fact that we are able to remove monotonicity in the classical statements relies heavily on the error domains defined by the function ψ being convex. However, when $n \geq 2$, the error domains associated with the set $M(\psi, n)$ are 'hyperbolic' in shape. It should be stressed that the reliance on convexity also extends to classical proofs of zero-one laws.

In [9], Beresnevich, Haynes & Velani introduced a multiplicative version of the Duffin-Schaeffer Conjecture. Despite the difficulties relating to the lack of convexity, they were able to establish a complete multiplicative analogue of the Duffin-Schaeffer Theorem. In particular, their result provides a sufficient criterion for the full measure of the set

$$M'(\psi, n) := \left\{ \mathbf{x} \in \mathbb{I}^n : \prod_{j=1}^{n} |qx_j - p| \leq \psi(q) \text{ for i.m. } p \in \mathbb{Z}, q \in \mathbb{N} \text{ with } (p,q) = 1 \right\}.$$

In addition, a 'cross fibering principle' developed in [9] enabled the authors to establish zero-one laws for the sets $M(\psi, n)$ and $M'(\psi, n)$.

1.3.1 The 'mixed' problems

In 2004, de Mathan and Teulié [79] proposed a problem closely related to the Littlewood Conjecture. Let $\mathcal{D} = \{n_k\}_{k \geq 0}$ be an increasing sequence of positive integers with $n_0 = 1$ and $n_k | n_{k+1}$ for all k. We refer to such a sequence as a *pseudo-absolute value sequence*, or more simply as a *$\mathcal{D}$-adic sequence*. The *$\mathcal{D}$-adic pseudo-absolute value* $|\cdot|_{\mathcal{D}} : \mathbb{N} \to \{n_k^{-1} : k \in \mathbb{N}\}$ is then defined by

$$|q|_{\mathcal{D}} = \inf\{n_k^{-1} : q \in n_k \mathbb{Z}\}.$$

In the case when $\mathcal{D} = \{a^k\}_{k=0}^{\infty}$ for some integer $a \geq 2$ we write $|\cdot|_{\mathcal{D}} = |\cdot|_a$. If p is a prime then $|\cdot|_p$ is the usual $p-$adic absolute value. Finally, when the quotients of consecutive elements of the sequence $\mathcal{D}$ are bounded we say $\mathcal{D}$ has *bounded ratios.* That is, $\mathcal{D}$ has bounded ratios if there exists a constant $M \geq 2$ such that $n_k/n_{k-1} \leq M$ for every $k \in \mathbb{N}$.

The following problem was proposed in [79] and is often referred to as the de Mathan-Teulié Conjecture.

Conjecture 1.38 (Mixed Littlewood Conjecture 2004)**.** *For any pseudo-absolute value sequence $\mathcal{D}$ and for every $x \in \mathbb{R}$ we have*

$$\liminf_{q \to \infty} q \, |q|_{\mathcal{D}} \, \|qx\| \; = \; 0. \tag{1.23}$$

This conjecture bears more than a superficial resemblance to the Littlewood Conjecture. If $\mathcal{D} = \{n_k\}$ is a pseudo-absolute value sequence then the numbers $|q|_{\mathcal{D}}$ can be thought of as an approximation to the values of $\|qx_2\|$, where $x_2 \in \mathbb{R}$ is the real number with simple continued fraction expansion

$$x_2 = [0;\ n_2/n_1,\ n_3/n_2, \dots].$$

In the case that $|\cdot|_{\mathcal{D}} = |\cdot|_a$ for some integer $a \geq 2$ the Mixed Littlewood Conjecture has a dynamical formulation in terms of the action of a certain diagonal group on a quotient space of

$$\mathrm{SL}_2(\mathbb{R}) \times \prod_i \mathrm{SL}_2(\mathbb{Q}_{p_i}),$$

where $\{p_i\}$ is the collection of primes dividing a. By employing 'measure rigidity' results in this setting Einsiedler & Kleinbock [38] proved that when $|\cdot|_{\mathcal{D}} = |\cdot|_a$ the set of $x \in \mathbb{R}$ which do not satisfy (1.23) has Hausdorff dimension zero. Their result is in direct analogy with that of Einsiedler, Katok & Lindenstrauss concerning the set of exceptions to the Littlewood Conjecture.

The subject of the Mixed Littlewood Conjecture with more than one pseudo-absolute value has also been a topic of recent interest. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two pseudo-absolute value sequences then it is conjectured that for any $x \in \mathbb{R}$,

$$\liminf_{q \to \infty} q|q|_{\mathcal{D}_1}|q|_{\mathcal{D}_2}\|qx\| = 0. \tag{1.24}$$

We say that any collection of integers $a_1, \dots, a_s$ are *multiplicatively independent* if the numbers $\log a_1, \dots, \log a_s$ are linearly independent over the rationals. It is shown in [38] that the Furstenberg Orbit Closure Theorem (see Theorem IV.1 of [43]) implies that (1.24) is true for all $x \in \mathbb{R}$ whenever $\mathcal{D}_1 = \{a^k\}$ and $\mathcal{D}_2 = \{b^k\}$ for two multiplicatively independent integers a and b. In other words, the Mixed Littlewood Conjecture with two or more distinct p-adic values is true. This statement was recently strengthened by Bourgain, Lindenstrauss, Michel & Venkatesh [15] who proved that there is a constant $\kappa > 0$ such that for all $x \in \mathbb{R}$,

$$\liminf_{q \to \infty} q(\log\log\log q)^{\kappa}|q|_a|q|_b\|qx\| = 0. \tag{1.25}$$

These results provide a contrast to the classical setting of the Littlewood Conjecture, where nothing seems to be gained by adding more real variables. The results rely on understanding the dynamics of semigroups of toral endomorphisms associated with the sequences $\mathcal{D}_1$ and $\mathcal{D}_2$. In simple terms, each of these semigroups Σ takes the form of a countable set of positive integers. When a and b are multiplicatively independent the set $\Sigma_{a,b} = \left\{a^l b^k\right\}_{l,k \geq 0}$ forms a non-lacunary semigroup. That is, a semigroup which cannot be generated by one element. Under these conditions, the techniques of [38] and [15] are applicable. However, the methods do not readily extend to the case of lacunary semigroups.

The related metric question of how fast the infimum in (1.24) tends to zero was tackled by Bugeaud, Haynes & Velani in [19]. They established a mixed analogue of Gallagher's theorem. The following improvement is proved in §4.1 of [9].

Theorem 1.39 ([9], 2010)**.** *For $n \geq 1$ choose any two integers $s, t \geq 0$ such that $n = s+t$. Let $p_1, \ldots, p_s$ be distinct prime numbers and let ψ be a decreasing approximating function. Then, for almost every $(x_1, \ldots, x_t) \in \mathbb{R}^t$ the inequality*

$$|q|_{p_1} \cdots |q|_{p_s} \, \|qx_1\| \cdots \|qx_t\| \;\leq\; \psi(q)$$

has infinitely (resp. finitely) many solutions $q \in \mathbb{N}$ if the sum

$$\sum_{r=1}^{\infty} \psi(r) \log^{n-1} r$$

diverges (resp. converges).

Chapter 2

Summary of papers

We now summarize the results and methods used in each of the papers on which this thesis is based. Complete reproductions of the papers are included as an appendix to the thesis.

2.1 Paper I

The fundamental motivation for Paper I is the result of Kim described in §1.2.2. That is, for any irrational x the set of real α for which

$$\liminf_{q\to\infty} q\,\|qx-\alpha\| = 0 \tag{2.1}$$

has full Lebesgue measure. In Paper I the complementary measure zero set associated with (2.1) is considered; namely the set

$$\mathbf{Bad}_x := \left\{\alpha \in \mathbb{I} : \inf_{q\in\mathbb{Z}_{\neq 0}} |q|\,\|qx-\alpha\| > 0\right\}.$$

Essentially, Kim's result comes about upon considering problems associated with a rotation of the unit circle through an angle x. In the paper we consider more general actions than circle rotations and, as far as we know, were the first to consider the general twisted badly approximable set

$$\mathbf{Bad}_{\mathcal{L}}(n,m) = \left\{\boldsymbol{\alpha} \in \mathbb{I}^n : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}} |\mathbf{q}|^{m/n}\,\|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| > 0\right\}.$$

When there is no need to refer to specific values of n and m in Paper I, the set $\mathbf{Bad}_{\mathcal{L}}(n,m)$ is simply denoted $\mathbf{Bad}_{\mathcal{L}}$ for conciseness.

The underlying goal of the paper is to show that no matter which point $\mathcal{L} \in \mathbb{I}^{nm}$ we choose, the set $\mathbf{Bad}_{\mathcal{L}}(n,m)$ is of maximal Hausdorff dimension.

Theorem 2.1. *For any $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$,*

$$\dim \mathbf{Bad}_{\mathcal{L}}(n,m) \;=\; n\,.$$

In terms of the more familiar one-dimensional setting, the theorem reads as follows.

Corollary 2.2. *For any $x \in \mathbb{R}$,*

$$\dim \mathbf{Bad}_x = 1\,.$$

Note that if x is rational, the set $\mathbf{Bad}_x$ contains all points in the unit interval bounded away from a finite set of rationals. Thus, for rational x not only is $\mathbf{Bad}_x$ of full dimension but it is of full Lebesgue measure. In higher dimensions, similar phenomena occur in which the finite set of points is replaced by a finite set of affine subspaces. To be precise, when $\mathcal{L}$ is rational (in the sense defined in §1.2.2) the orbit $\{\mathcal{L}\mathbb{Z}^m\}$ fails to be dense in $\mathbb{I}^n$ and thus $\mathbf{Bad}_{\mathcal{L}}(n,m)$ is of full Lebesgue measure.

Inspired by the works of Kleinbock & Weiss [65] and Kristensen, Thorn & Velani [67], Theorem 2.1 is deduced as a consequence of a general statement concerning the intersection of $\mathbf{Bad}_{\mathcal{L}}(n,m)$ with certain compact subsets of $\mathbb{R}^n$.

Theorem 2.3. *Let $K \subseteq \mathbb{I}^n$ be a compact set supporting an absolutely decaying, δ-Ahlfors regular measure μ and assume that $\delta > n-1$. Then, for any $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$,*

$$\dim(\mathbf{Bad}_{\mathcal{L}}(n,m) \cap K) = \delta.$$

Although Theorem 2.3 constitutes the main result of the paper, an 'auxiliary' result is also proven. We include it here for the simple fact that it is new and of independent interest. In short, it strengthens and generalises a theorem of Pollington [82] and de Mathan [77, 78] that answers a question of Erdős. A sequence $\{\mathbf{y}_i\} := \{\mathbf{y}_i := (y_{1,i},\ldots,y_{n,i}) \in \mathbb{Z}^n_{\neq \mathbf{0}}\}$ is said to be *lacunary* if there exits a constant $\lambda > 1$ such that

$$|\mathbf{y}_{i+1}| \;\geq\; \lambda\,|\mathbf{y}_i| \qquad \forall\, i \in \mathbb{N}\,.$$

Given a sequence $\{\mathbf{y}_i\}$ in $\mathbb{Z}^n$, let

$$\mathbf{Bad}_{\{\mathbf{y}_i\}} := \left\{\mathbf{x} \in \mathbb{I}^n : \inf_{i\in\mathbb{N}} \|\mathbf{y}_i \cdot \mathbf{x}\| > 0\right\}\,.$$

Theorem 2.4. *Let $\{\mathbf{y}_i\}$ be a lacunary sequence in $\mathbb{Z}^n$. Furthermore, let $K \subseteq \mathbb{I}^n$ be a compact set which supports an absolutely η-decaying, δ-Ahlfors regular measure μ such that $\delta > n-1$. Then*

$$\dim(\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K) = \delta.$$

On setting $n = 1$, $K = \mathbb{I}$ and μ to be one-dimensional Lebesgue measure, Theorem 2.4 corresponds to the result of Pollington and de Mathan referred to above. Note that Moshchevitin [81, Lemma 1] has since shown that the set $\mathbf{Bad}_{\{\mathbf{y}_i\}}$ is winning for any lacunary sequence $\{\mathbf{y}_i\}$. However, the problem of strengthening his statement to the full generality of Theorem 2.4 is still open at the time of writing.

It is in proving Theorem 2.4 that we use the general framework developed by Kristensen, Thorn & Velani in [67]. Their machinery was designed for establishing dimension statements for a large class of badly approximable sets. Essentially, our proof is an application of the framework and reduces to showing that the conditions of the main theorem in [67] are satisfied. A simplification of this main theorem, geared towards the particular application we have in mind, is presented in §3 of Paper I. For brevity we refrain from restating it here.

To prove Theorem 2.3 we utilise the existence of 'special' sequences which for the most part are constructed in [20]. In §2 of [20], it is shown that associated with each irrational matrix $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ there exists a sequence of integer vectors $\mathbf{y}_i \in \mathbb{Z}^n$ satisfying the following properties:

(i) $1 = |\mathbf{y}_1| < |\mathbf{y}_2| < |\mathbf{y}_3| < \dots$,

(ii) $\|\mathcal{L}^T\mathbf{y}_1\| > \|\mathcal{L}^T\mathbf{y}_2\| > \|\mathcal{L}^T\mathbf{y}_3\| > \dots$,

(iii) For all non-zero $\mathbf{y} \in \mathbb{Z}^n$ with $|\mathbf{y}| < |\mathbf{y}_{i+1}|$ we have that $\|\mathcal{L}^T\mathbf{y}\| \geq \|\mathcal{L}^T\mathbf{y}_i\|$.

Such a sequence $\{\mathbf{y}_i\}$ is referred to as a *sequence of best approximations to $\mathcal{L}$*. In the one-dimensional case ($n = m = 1$), when $\mathcal{L}$ is an irrational number x, the sequence of best approximations is precisely the sequence of denominators associated with the convergents of the continued fraction representing x. It is also shown in [20] that one can find a lacunary subsequence of $\{\mathbf{y}_i\}$ which is not 'too sparse'. Applying Theorem 2.4 to this subsequence and then using methods similar to those found in §6 of Chapter V in Cassels' book [24] we obtain a proof of Theorem 2.3.

It is worth noting that since publication Theorem 2.3 has been extended in many ways. As discussed earlier, improved statements can now be found in [16, 39, 81, 95].

2.2 Paper II

In short, the intentions of Paper II are to establish versions of both Kurzweil's theorem and Theorem 2.1 within the weighted setting. At the heart of these extensions is a natural generalization of the set $\mathbf{Bad}(i,j)$. For any n-tuple of real numbers $\mathbf{k} = \{k_1, \ldots, k_n\}$ such that

$$k_j > 0 \quad (1 \leq j \leq n) \quad \text{and} \quad \sum_{j=1}^{n} k_j = 1, \tag{2.2}$$

we define

$$\mathbf{Bad}(\mathbf{k}) = \left\{ \mathbf{x} \in \mathbb{I}^n : \inf_{q \in \mathbb{N}} q \cdot \max_{1 \leq j \leq n} \left(\|qx_j\|^{1/k_j} \right) > 0 \right\}.$$

We refer to $\mathbf{Bad}(\mathbf{k})$ as the set of $\mathbf{k}$-*badly approximable vectors*. Of particular interest is the relationship of $\mathbf{Bad}(\mathbf{k})$ with the following 'well approximable' twisted sets. For any approximating function ψ, any irrational vector $\mathbf{x} \in \mathbb{I}^n$ and any n-tuple $\mathbf{k}$ satisfying (2.2) let

$$W_{\mathbf{x}}(\mathbf{k}, \psi) := \left\{ \boldsymbol{\alpha} \in \mathbb{I}^n : \max_{1 \leq j \leq n} \left(\|qx_j - \alpha_j\|^{1/k_j} \right) \leq \psi(q) \ \text{ for inf. many } q \in \mathbb{N} \right\}.$$

The main result of Paper II relies on an understanding of the metric theory surrounding $W_{\mathbf{x}}(\mathbf{k}, \psi)$. By utilising the Borel-Cantelli Lemma it is easy to show that for every n-tuple $\mathbf{k}$ satisfying (2.2), any irrational vector $\mathbf{x} \in \mathbb{I}^n$ and every approximating function ψ we have

$$\lambda_n \left(W_{\mathbf{x}}(\mathbf{k}, \psi) \right) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \psi(r) < \infty.$$

On the other hand, the set of irrational vectors for which we obtain a set of full measure is dependent on the choice of approximating function.

Theorem 2.5. *Let ψ be a fixed decreasing approximating function. Then, for almost all irrational vectors $\mathbf{x} \in \mathbb{I}^n$*

$$\lambda_n \left(W_{\mathbf{x}}(\mathbf{k}, \psi) \right) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty} \psi(r) = \infty.$$

This result follows immediately from a more general statement that can be found in the Appendix of Paper II. Effectively, Cassels' proof of Theorem 1.30 is merged with that of Gallagher's general metric result from [45] to provide a more general doubly metric theorem. It should also be compared with the results of Schmidt [89] and Sprindzuk [94].

In what follows, a decreasing approximating function for which the sum $\sum_{r=1}^{\infty}\psi(r)$ diverges will be referred to as *divergent* and the set of all divergent decreasing approximating functions will be denoted D. Recognising the similarities between Theorem 2.5 and Kurzweil's result, the paper asks whether there exist irrational vectors $\mathbf{x}$ such that a set of full measure is obtained regardless of the choice of divergent approximating function. In other words, consider the set

$$V(\mathbf{k},\psi) := \{\mathbf{x}\in\mathbb{I}^n : \lambda_n\left(W_{\mathbf{x}}(\mathbf{k},\psi)\right) = 1\}.$$

Note that Theorem 2.5 is equivalent to the statement "$\lambda_n\left(V(\mathbf{k},\psi)\right) = 1$ for every $\psi\in D$". Then, we wish to characterise the intersection

$$\bigcap_{\psi\in D} V(\mathbf{k},\psi). \tag{2.3}$$

It is certainly not obvious that this intersection is non-empty in general. Kurzweil's theorem is precisely the statement that

$$\bigcap_{\psi\in D} V(\mathbf{n^{-1}},\psi) \;=\; \mathbf{Bad}(\mathbf{n^{-1}}),$$

where $\mathbf{n^{-1}} = (n^{-1},\ldots,n^{-1})\in\mathbb{I}^n$. With this in mind, the following result represents the main theorem of Paper II and generalises Kurzweil's theorem from the classical to the weighted setting.

Theorem 2.6. *For every n-tuple $\mathbf{k}$ of real numbers satisfying (2.2) we have*

$$\bigcap_{\psi\in D} V(\mathbf{k},\psi) \;=\; \mathbf{Bad}(\mathbf{k}).$$

With reference to §1.1.8, Theorem 1.23 and the natural generalisation of statement (1.17) now immediately imply that the intersection on the LHS above is of Lebesgue measure zero and full Hausdorff dimension respectively. We remark that for ease of notation only the case '$n = 2$' is proved in the paper. However, the arguments can easily be extended to a general n.

The proof of Theorem 2.6 owes much to Kurzweil's original techniques and takes the form of two inclusion lemmas. Firstly, it is shown that if $\mathbf{x}\notin\mathbf{Bad}(\mathbf{k})$ then $\mathbf{x}\notin\bigcap_{\psi\in D}V(\mathbf{k},\psi)$. In particular, for every such $\mathbf{x}$ a function $\psi_0 := \psi_0(\mathbf{x})\in D$ is constructed in such a way that

$$\lambda_n\left(W_{\mathbf{x}}(\mathbf{k},\psi_0)\right) = 0.$$

The proof of the second inclusion is much more tricky as it involves proving that for any $\mathbf{x} \in \mathbf{Bad}(\mathbf{k})$ and every divergent approximating function ψ we have

$$\lambda_n\left(W_{\mathbf{x}}(\mathbf{k},\psi)\right) = 1. \tag{2.4}$$

The hard work is done in showing that the LHS of (2.4) is strictly positive for some refinement of the function ψ. Essentially, this involves transferring Kurzweil's original methods to the weighted setting. Finally, by constructing other approximating functions from ψ we are able to directly apply a powerful lemma presented in Kurzweil's paper to show that (2.4) always holds.

Paper II also offers a supplementary result concerning the natural weighted analogue of the simultaneous twisted set $\mathbf{Bad}_{\mathcal{L}}(n,1)$. To the best of our knowledge such an analogue has not been studied before and is defined as follows. For any n-tuple $\mathbf{k}$ of real numbers satisfying (2.2) and any $\mathbf{x} \in \mathbb{I}^n$ let

$$\mathbf{Bad}_{\mathbf{x}}(\mathbf{k}) \;=\; \left\{\boldsymbol{\alpha} \in \mathbb{I}^n : \inf_{q\in\mathbb{Z}_{\neq 0}} |q| \cdot \max_{1\le j\le n}\left(\|qx_j - \alpha_j\|^{1/k_j}\right) > 0\right\}.$$

Whilst a complete weighted version of Theorem 2.1 still seems out of reach, Paper II does make a contribution towards determining the Hausdorff dimension of $\mathbf{Bad}_{\mathbf{x}}(\mathbf{k})$.

Theorem 2.7. *For any n-tuple $\mathbf{k}$ of real numbers satisfying (2.2) and any vector $\mathbf{x} \in \mathbf{Bad}(\mathbf{k})$,*

$$\dim \mathbf{Bad}_{\mathbf{x}}(\mathbf{k}) \;=\; n.$$

The proof of this theorem once again makes use of the general framework developed by Kristensen, Thorn & Velani. An account of this framework slightly different to that used in Paper I is given in §5 of Paper II. In all likelihood the above result is true without the assumption on $\mathbf{x}$ and to that end the following conjecture is made.

Conjecture 2.8. *For any n-tuple $\mathbf{k}$ of real numbers satisfying (2.2) and any irrational vector $\mathbf{x} \in \mathbb{I}^n$,*

$$\dim \mathbf{Bad}_{\mathbf{x}}(\mathbf{k}) \;=\; n.$$

It seems that the ideas of Paper I are not extendible to the full weighted setting of Conjecture 2.8; a new approach may be required. Note that Theorem 2.7, together with (1.17) trivially implies that the conjecture is true for a set of irrational vectors $\mathbf{x}$ of full Hausdorff dimension.

2.3 Paper III

Paper III takes the form of a short note whose intention is to generalise Theorem 2.7. For any n-tuple $\mathbf{k}$ of real numbers satisfying (2.2) let

$$\mathbf{Bad}(\mathbf{k},n,m) = \left\{\mathcal{L}\in\mathbb{I}^{nm} : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}} |\mathbf{q}|^m \cdot \max_{1\leq j\leq n}\left(\|L_j(\mathbf{q})\|^{1/k_j}\right) > 0\right\}$$

denote the linear forms version of the set $\mathbf{Bad}(\mathbf{k})$ discussed above. For completeness, we mention that $\mathbf{Bad}(\mathbf{k},n,m)$ was also shown to be winning by Kleinbock & Weiss in [66]. For any matrix $\mathcal{L}\in \mathrm{Mat}_{n\times m}(\mathbb{R})$ let

$$\mathbf{Bad}_{\mathcal{L}}(\mathbf{k},n,m) = \left\{\boldsymbol{\alpha}\in\mathbb{I}^{n} : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}} |\mathbf{q}|^m \cdot \max_{1\leq j\leq n}\left(\|L_j(\mathbf{q})-\alpha_j\|^{1/k_j}\right) > 0\right\}.$$

The main result of the Paper III is the following improvement of Theorem 2.7.

Theorem 2.9. *For any n-tuple $\mathbf{k}$ of real numbers satisfying (2.2) and any point $\mathcal{L}\in\mathbf{Bad}(\mathbf{k},n,m)$ the set $\mathbf{Bad}_{\mathcal{L}}(\mathbf{k},n,m)$ is winning.*

The proof of this statement uses similar ideas to those of Paper I. This time we construct a weighted varient $\{\mathbf{w}_i\}$ of the sequence of best approximations and, using the properties of $\mathbf{Bad}(\mathbf{k},n,m)$, show that one can find a suitable subsequence of $\{\mathbf{w}_i\}$ whose supremum norms are lacunary. Once more it seems difficult to remove the restriction on the choice of matrix $\mathcal{L}$ using current methods. Indeed, the problem seems to boil down to the fact that the supremum norm $|\,.\,|$ on $\mathbb{R}^n$ is not equivalent to the weighted norm $|\,.\,|_{\mathbf{k}}$ defined by $|\mathbf{x}|_{\mathbf{k}} = \max_{1\leq j\leq n}(|x_j|^{1/k_j})$. This lack of equivalence can be bypassed under the assumption that $\mathcal{L}\in\mathbf{Bad}(\mathbf{k},n,m)$ but in general it poses too big an obstacle to clear. Certainly we can always construct a subsequence of $\{\mathbf{w}_i\}$ whose weighted norms are lacunary but this does not imply there exists a subsequence whose supremum norms are lacunary and the former is not a strong enough property with which to prove winning in the context of the devices we use. The proof is completed using Moshchevitin's previously mentioned result concerning $\mathbf{Bad}_{\{\mathbf{w}_i\}}$ and arguments similar to those used in Chapter V of [24].

2.4 Paper IV

Paper IV is concerned with problems surrounding the Mixed Littlewood Conjecture. In particular, the main result makes a contribution towards solving the conjecture in the case of more than one pseudo-absolute value.

Recall that Einsiedler & Kleinbock [38] have proven that if p_1 and p_2 are two distinct primes then for any $x \in \mathbb{R}$

$$\liminf_{q \to \infty} q|q|_{p_1}|q|_{p_2}\|qx\| = 0.$$

It was mentioned in their paper that the dynamical machinery they used does not readily extend to the case of more general pseudo-absolute values. However, the main result of Paper IV demonstrates how recent measure rigidity theorems can be combined with bounds for linear forms in logarithms to obtain more general results.

Theorem 2.10. *Suppose that $a \geq 2$ is an integer and that $\mathcal{D} = \{n_k\}$ is a pseudo-absolute value sequence all of whose elements are divisible by finitely many fixed primes coprime to a. If there is a $\delta \geq 0$ with*

$$\log n_k \leq k^\delta \quad \textit{for all} \quad k \geq 2, \tag{2.5}$$

then for any $x \in \mathbb{R}$ we have that

$$\inf_{q \in \mathbb{N}} q|q|_a|q|_{\mathcal{D}}\|qx\| = 0. \tag{2.6}$$

Our proof is inspired in part by Furstenberg's original proof of his Orbit Closure Theorem [43], and by the ideas used by Bourgain, Lindenstrauss, Michel & Venkatesh in [15]. Of huge significance to us is the intrinsic relationship between entropy and dimension (a thorough account of various forms of entropy complete with definitions is given in §2 of Paper IV). In the proof we combine this relationship with a consequence of a measure rigidity theorem of Lindenstrauss [72]. This allows us to reduce the proof to showing that the closure of the set $\left\{a^l n_k x\right\}_{l,k\geq 0}$ is of strictly positive dimension. Despite the fact that $\left\{a^l n_k\right\}_{l,k\geq 0}$ is not in general a semigroup, a lower bound for linear forms in logarithms due to Baker & Wüstholz [6] aids us in accomplishing this goal.

Of particular interest is the case in which consecutive elements of the sequence $\mathcal{D}$ have bounded ratios. Here, Theorem 2.10 gives a quite satisfactory answer to the problem at hand.

Corollary 2.11. *Suppose that $a \geq 2$ is an integer and that $\mathcal{D}$ is a pseudo-absolute value sequence with bounded ratios, all of whose elements are coprime to a. Then for any $x \in \mathbb{R}$ we have that*

$$\inf_{q \in \mathbb{N}} q|q|_a|q|_{\mathcal{D}}\|qx\| = 0.$$

After establishing Theorem 2.10 the paper turns to the problem of determining the almost everywhere behaviour of the quantity $q\,|q|_{\mathcal{D}}\,\|qx\|$. In particular it is shown that the statement of Theorem 1.39 can be extended to general pseudo-absolute values $|\cdot|_{\mathcal{D}}$. In short, we prove an analogue of Gallagher's theorem pertaining to a mixed variant of the set $M(\psi, n)$. The quality of approximation obtained will necessarily depend on the rate at which the sequence $\mathcal{D}$ grows. For this reason, define the quantity

$$\mathcal{M}(N) = \max\left\{k : n_k \leq N\right\}.$$

Theorem 2.12. *Suppose that ψ is a decreasing approximating function and that $\mathcal{D} = \{n_k\}$ is a pseudo-absolute value sequence satisfying*

$$\sum_{k=1}^{\mathcal{M}(N)} \frac{\varphi(n_k)}{n_k} \gg \mathcal{M}(N) \text{ for all } N \in \mathbb{N}. \tag{2.7}$$

Then for almost all $x \in \mathbb{R}$ the inequality

$$|q|_{\mathcal{D}}\,\|qx\| \leq \psi(q) \tag{2.8}$$

has infinitely (resp. finitely) many solutions $q \in \mathbb{N}$ if the sum

$$\sum_{r=1}^{\infty} \mathcal{M}(r)\psi(r) \tag{2.9}$$

diverges (resp. converges).

The proof of this result is a direct application of the Duffin-Schaeffer Theorem. With reference to this theorem, most of the work required to establish Theorem 2.12 is in showing that condition (1.9) holds for some suitably chosen function ψ. We note that when (2.9) converges the inequality (2.8) always has finitely many solutions, regardless of whether or not (2.7) is satisfied. When $|\cdot|_D = |\cdot|_p$ for some prime p we have that $\mathcal{M}(N) \asymp \log N$, and Theorem 2.12 reduces in this case to the previously mentioned result from [19].

To see what Theorem 2.12 means in terms of the infima type expressions that occur in the Mixed Littlewood Conjecture consider the following. If $\mathcal{D}$ satisfies (2.7) then for almost every $x \in \mathbb{R}$ we have that

$$\inf_{q\in\mathbb{N}} q\mathcal{M}(q)(\log q)(\log\log q)\,|q|_{\mathcal{D}}\,\|qx\| \;=\; 0,$$

while on the other hand for any $\epsilon > 0$ and for almost every $x \in \mathbb{R}$,

$$\inf_{q\in\mathbb{N}} q\mathcal{M}(q)(\log q)(\log\log q)^{1+\epsilon}\,|q|_{\mathcal{D}}\,\|qx\| \;>\; 0.$$

This should be compared with statements (1.8) and (1.25).

The hypothesis on $\mathcal{D}$ in Theorem 2.12 is not that restrictive in practice. Although it is possible to choose $\mathcal{D}$ so that (2.7) does not hold, any reasonably chosen pseudo-absolute value sequence should satisfy the condition. In particular if $\mathcal{D}$ has bounded ratios or even if the elements of $\mathcal{D}$ are divisible only by some finite collection of primes then it is easy to check that (2.7) is satisfied. For the interested reader it is indicated in §6 of Paper IV how one can construct a sequence $\mathcal{D}$ for which (2.7) fails.

2.5 Paper V

Paper V develops the metric theory associated with a mixed version of the Schmidt Conjecture. Previous study of mixed problems in the simultaneous setting had been limited to the paper [2], where Badziahin, Levesley & Velani derived a mixed analogue to Theorem 1.25. In Paper V a metrical theorem is established concerning the mixed and weighted simultaneous set

$$W_{\mathcal{D}}(i,j,\psi) := \left\{x \in \mathbb{I} : \max\left\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\right\} \le \psi(q) \ \text{ for inf. many } q \in \mathbb{N}\right\},$$

defined for any two real numbers i, j satisfying

$$i,\, j \;>\; 0 \quad \text{and} \quad i + j = 1. \tag{2.10}$$

In Paper IV it was demonstrated that when $\mathcal{D}$ has bounded ratios the Lebesgue measure of the sets $M(\psi, 2)$ and its natural mixed counterpart depend on the asymptotic behaviour of the same sum. In Paper V we show that the sets $W(i, j, \psi)$ and $W_{\mathcal{D}}(i, j, \psi)$ enjoy a similar property. In doing so, we provide a complete mixed and weighted analogue of Khintchine's theorem.

Theorem 2.13. *For any pair of reals i, j satisfying (2.10), any decreasing approximating function ψ and any $\mathcal{D}$-adic sequence with bounded ratios we have*

$$\lambda\left(W_{\mathcal{D}}(i,j,\psi)\right) \quad = \quad \begin{cases} 0, & \sum_{r\in\mathbb{N}} \psi(r) \ < \ \infty. \\ \\ 1, & \sum_{r\in\mathbb{N}} \psi(r) \ = \ \infty. \end{cases}$$

We remark that obtaining an equivalent statement to that of Theorem 2.13 for pseudo-absolute value sequences with non-bounded ratios, whilst desirable, would require more than trivial improvements over the techniques presented. In addition, it is worth emphasising that the degenerate cases '$i = 0$' and '$j = 0$' are not considered here. On employing the convention that $x^{1/y} = 0$ when $y = 0$ for all real x, it is easily verified that in the former case Theorem 2.13 reduces to the classical one-dimensional result of Khintchine, whilst in the latter case the measure of the corresponding set $W_{\mathcal{D}}(1,0,\psi)$ trivially fulfils a 'zero-one' law. Indeed,

$$W_{\mathcal{D}}(1,0,\psi) \quad = \quad \begin{cases} \mathbb{I}, & \psi(q) > |q|_{\mathcal{D}} \text{ for infinitely many } q \in \mathbb{N}. \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is proven in Paper V that the monotonicity assumption imposed on the function ψ in Theorem 2.13 is absolutely necessary. Furthermore, the 'most natural' mixed analogue of the Duffin-Schaeffer Conjecture is shown to be false. To be precise, the following statement is proven. For notational purposes, let

$$\mathcal{A} := \mathcal{A}(\mathcal{D},\psi,i) := \left\{ r \in \mathbb{N} : \, |r|_{\mathcal{D}} < \psi^{i}(r) \right\}.$$

Theorem 2.14. *For any pair of reals i, j satisfying (2.10) and any bounded $\mathcal{D}$-adic sequence there exists an approximating function $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ for which*

$$\lambda(W_{\mathcal{D}}(i,j,\Phi)) = 0 \quad \textit{but} \quad \sum_{r\in\mathcal{A}} \frac{\varphi(r)}{r}\, \Phi^{j}(r) = \infty.$$

For the most part the proof of this result uses ideas akin to the original arguments of Duffin & Schaeffer.

Theorem 2.13 is proven as a consequence of a more general Hausdorff measure result established in the paper.

Theorem 2.15. *Fix any pair of reals i, j satisfying (2.10), any $\mathcal{D}$-adic sequence with bounded ratios and any real $s \in (i, 1]$. Then, for any approximating function ψ for which $r^{1-s}\psi^{i+js}(r)$ is decreasing we have*

$$\mathcal{H}^s\left(W_{\mathcal{D}}(i,j,\psi)\right) = \begin{cases} 0, & \sum_{r\in\mathbb{N}} r^{1-s}\,\psi^{i+js}(r) < \infty. \\ \\ \mathcal{H}^s([0,1)), & \sum_{r\in\mathbb{N}} r^{1-s}\,\psi^{i+js}(r) = \infty \ \text{ and } \psi \text{ is monotonic.} \end{cases}$$

It should be mentioned that we do not claim the conditions imposed in Theorem 2.15 are optimum. In fact, we suspect that the assumption that $r^{1-s}\psi^{i+js}(r)$ is decreasing may be unnecessary. Our method is based on the notion of 'ubiquity', a fundamental tool for establishing measure theoretic statements.

The concept of ubiquitous systems was first introduced by Dodson, Rynne & Vickers in [34] as a method of determining lower bounds for the Hausdorff dimension of limsup sets. Recently, this idea was developed by Beresnevich, Dickinson & Velani in [8] to provide a very general framework for establishing the Hausdorff measure of a large class of limsup sets. A simplified account of ubiquity, tailored to our needs, is presented in §4 of the paper.

Another consequence of Theorem 2.15 is the following statement.

Corollary 2.16. *Choose any pair of reals i, j satisfying (2.10), any $\mathcal{D}$-adic sequence with bounded ratios and any decreasing approximating function ψ. Then, if there exists a real number τ such that*

$$\tau = \lim_{r\to\infty} \frac{-\log\psi(r)}{\log r} < \frac{1}{i}$$

we have

$$\dim\left(W_{\mathcal{D}}(i,j,\psi)\right) = \frac{2-i\tau}{1+j\tau}.$$

This result provides a complete analogue to the Jarník-Besicovich Theorem, which corresponds to the case when $i = 0$ and $j = 1$. We remark that when $\psi(q) = q^{-1/i}$ the set $W_{\mathcal{D}}(i,j,\psi)$ is empty.

Appendix

Papers

This appendix contains full reproductions of the papers on which this thesis is based. Each paper has self-contained section and equation numbering along with its own list of references.

Paper I

ON SHRINKING TARGETS FOR $\mathbb{Z}^m$ ACTIONS ON TORI

YANN BUGEAUD, STEPHEN HARRAP, SIMON KRISTENSEN
AND SANJU VELANI

Abstract. Let $\mathcal{L}$ be an $n \times m$ matrix with real entries. Consider the set $\mathbf{Bad}_{\mathcal{L}}$ of $\boldsymbol{\alpha} \in [0,1)^n$ for which there exists a constant $c(\boldsymbol{\alpha}) > 0$ such that for any $\mathbf{q} \in \mathbb{Z}^m$ the distance between $\boldsymbol{\alpha}$ and the point $\{\mathcal{L}\mathbf{q}\}$ is at least $c(\boldsymbol{\alpha})\,|\mathbf{q}|^{-m/n}$. It is shown that the intersection of $\mathbf{Bad}_{\mathcal{L}}$ with any suitably regular fractal set is of maximal Hausdorff dimension. The linear form systems investigated in this paper are natural extensions of irrational rotations of the circle. Even in the latter one-dimensional case, the results obtained are new.

1. Introduction

Consider initially a rotation of the unit circle through an angle x. Identifying the circle with the unit interval $[0,1)$ and the base point of the iteration with the origin, we are considering the numbers $0, \{x\}, \{2x\}, \ldots$ where $\{\,.\,\}$ denotes the fractional part. If x is rational, the rotation is periodic. On the other hand, it is a classical result of Weyl [24] that any irrational rotation of the circle is ergodic. In other words, $\{qx\}_{q\in\mathbb{N}}$ is equidistributed for irrational x.

Almost every orbit of an ergodic transformation visits any fixed set of positive measure infinitely often. The 'shrinking target problem' introduced in [9] formulates the natural question of what happens if the target set – the set of positive measure – is allowed to shrink with time. For example and more precisely, is there an optimal 'shrinking rate' for which almost every orbit visits the shrinking target infinitely often? In the specific case of irrational rotations of the circle, the shrinking target sets correspond to subintervals of $[0,1)$ whose lengths decay according to some specified function ψ. In other words, the problem translates to considering inequalities of the type

$$\|qx - \alpha\| < \psi(q), \tag{1.1}$$

where $\alpha \in [0,1)$ and $\|\,.\,\|$ denotes the distance to the nearest integer. The following statement dates back to Khintchine [10] and gives the 'optimal' choice of ψ in the non-trivial case that x is irrational and $\alpha \neq sx + t$ for any integers s and t. The inequality

$$\|qx - \alpha\| < \frac{C(x)}{q} \tag{1.2}$$

is satisfied for infinitely many integers q with $C(x) := \frac{1}{4}\sqrt{1-4\lambda(x)^2}$ – the quantity $\lambda(x) := \liminf_{q\to\infty} q\,\|qx\|$ is the Markoff constant of x. Note that $\lambda(x)$ is strictly positive whenever x is badly approximable by rationals. Thus, the above statement strengthens a result of Minkowski [18]; namely that (1.2) has infinitely many solutions with $C(x) = \frac{1}{4}$. In the trivial case that x is irrational and $\alpha = sx+t$ for some integers s and t, the classical theorem of Hurwitz implies that the inequality

$$\|qx-\alpha\| < \frac{1+\epsilon}{\sqrt{5}q} \qquad (\epsilon > 0) \tag{1.3}$$

is satisfied for infinitely many integers q. Since (1.3) is weaker than (1.2), it follows that for any irrational x and any α the inequality (1.3) has infinitely many solutions. We now describe a metrical statement in which the right hand side of (1.3) and indeed (1.2) can be significantly improved – at a cost!

Kurzweil [14] showed that, for any non-increasing function $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ such that $\sum \psi(r) = \infty$ and for almost every irrational x, the set of α for which (1.1) has infinitely many solutions is of full Lebesgue measure. This cannot be improved upon in the sense that there exist irrational x and a function ψ for which $\sum \psi(r) = \infty$, but the 'full measure' conclusion fails to hold. Hence, the 'almost every' aspect of Kurzweil's result does not extend to all irrationals x without modification – the divergent sum condition is not enough.

Over the last few years, there has been much activity in investigating the shrinking target problem associated with irrational rotations of the circle. For example, when $\psi(q) := q^{-v}$ $(v > 1)$, Bugeaud [3] and independently Schmeling & Trubetskoy [21] have obtained the Hausdorff dimension of the set of α for which inequality (1.1) has infinitely many solutions. Fayad [8], A.-H. Fan & J. Wu [7], Kim [11] and Tseng [22, 23] have built upon the work of Kurzweil in various directions. In particular, Kim has proved that for any irrational x, the set of α for which

$$\liminf_{q\to\infty} q\,\|qx-\alpha\| = 0 \tag{1.4}$$

has full measure. Rather surprisingly, Beresnevich, Bernik, Dodson & Velani [1] have shown that this result and indeed the dimension result of Bugeaud and Schmeling & Trubetskoy are consequences of the fact that for any irrational x and any α the inequality (1.3) has infinitely many solutions.

The result of Kim is the underlying motivation for our work. In this paper we investigate the complementary measure zero set associated with (1.4); namely

$$\mathbf{Bad}_x := \left\{\alpha \in [0,1) : \exists\, c(\alpha) > 0 \text{ s.t. } \|qx-\alpha\| \geq \frac{c(\alpha)}{q}\ \forall\, q \in \mathbb{N}\right\}. \tag{1.5}$$

In fact, we will be concerned with more general actions than rotations of the circle. Broadly speaking, there are two natural ways to generalise circle rotations. One option is to increase the dimension of the torus; i.e. to consider the sequence $\{q\mathbf{x}\}$ in $[0,1)^n$ where $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$. The other option is to increase the dimension of the group acting on the torus; i.e. to consider the sequence $\{\mathbf{x} \cdot \mathbf{q}\}$ where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\mathbf{q} = (q_1, \ldots, q_m)^T \in \mathbb{Z}^m$.

It is possible to consider both the above mentioned options at the same time by introducing a $\mathbb{Z}^m$ action on the n-torus by $n \times m$ matrices. Indeed, we may consider the points $\{\mathcal{L}\mathbf{q}\} \in [0,1)^n$ where $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ is fixed and $\mathbf{q}$ runs over $\mathbb{Z}^m$. In this case, the natural analogue of $\mathbf{Bad}_x$ is the set

$$\mathbf{Bad}_{\mathcal{L}} := \left\{ \boldsymbol{\alpha} \in [0,1)^n : \exists\, c(\boldsymbol{\alpha}) > 0 \text{ s.t. } \|\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}\| \geq \frac{c(\boldsymbol{\alpha})}{|\mathbf{q}|^{m/n}} \;\forall\, \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} \right\}.$$

Here and throughout, for a vector $\mathbf{x}$ in $\mathbb{R}^k$ we will denote by $|\mathbf{x}|$ the maximum of the absolute values of the coordinates of $\mathbf{x}$; i.e. the infinity norm of $\mathbf{x}$. Also, $\|\mathbf{x}\| := \min_{\mathbf{y}\in\mathbb{Z}^n} |\mathbf{x} - \mathbf{y}|$.

The underlying goal of this paper is to show that no matter which of the $\mathbb{Z}^m$ actions defined above we choose, the set $\mathbf{Bad}_{\mathcal{L}}$ is of maximal Hausdorff dimension.

Theorem 1.1. *For any $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$,*

$$\dim \mathbf{Bad}_{\mathcal{L}} \;=\; n\,.$$

In terms of the more familiar setting of irrational rotations of the circle, the theorem reads as follows.

Corollary 1.2. *For any $x \in \mathbb{R}$,*

$$\dim \mathbf{Bad}_x = 1\,.$$

Note that if x is rational, the set $\mathbf{Bad}_x$ is easily seen to contain all points in the unit interval bounded away from a finite set of points. Thus, for rational x not only is $\mathbf{Bad}_x$ of full dimension but it is of full Lebesgue measure. In higher dimensions, similar phenomena occur in which the finite set of points is replaced by a finite set of affine subspaces. The reader is referred to [5] and §5 below for further details.

Inspired by the works of Kleinbock & Weiss [12] and Kristensen, Thorn & Velani [13], we shall deduce Theorem 1.1 as a simple consequence of a general statement concerning the intersection of $\mathbf{Bad}_{\mathcal{L}}$ with compact subsets of $\mathbb{R}^n$. The latter includes exotic fractal sets such as the Sierpinski gasket and the van Koch curve.

2. The setup and main result

Let (X, d) be a metric space and (Ω, d) be a compact subspace of X which supports a non-atomic finite measure μ. Throughout, $B(c, r)$ will denote a closed ball in X with centre c and radius r. The measure μ is said to be *δ-Ahlfors regular* if there exist strictly positive constants δ and r_0 such that for $c \in \Omega$ and $r < r_0$

$$ar^\delta \leq \mu(B(c, r)) \leq br^\delta \ ,$$

where $0 < a \leq 1 \leq b$ are constants independent of the ball. It is easily verified that if μ is δ-Ahlfors regular then the Hausdorff dimension of Ω is δ; i.e.

$$\dim \Omega = \delta \ . \tag{2.1}$$

For further details including the definition of Hausdorff dimension the reader is referred to [17].

In the above, take $X = \mathbb{R}^n$ and let $\mathcal{S}$ denote a generic $(n-1)$-dimensional hyperplane. For $\epsilon > 0$, let $\mathcal{S}^{(\epsilon)}$ denote the ϵ-neighbourhood of $\mathcal{S}$. The measure μ is said to be *absolutely η-decaying* if there exist strictly positive constants C, η and r_0 such that for any hyperplane $\mathcal{S}$, any $\epsilon > 0$, any $c \in \Omega$ and any $r < r_0$,

$$\mu(B(c, r) \cap \mathcal{S}^{(\epsilon)}) \leq C \left(\frac{\epsilon}{r}\right)^\eta \mu(B(c, r)).$$

It is worth mentioning that if μ is δ-Ahlfors regular and absolutely η-decaying, then μ is an absolutely friendly measure as defined in [20].

Armed with the notions of Ahlfors regular and absolutely decaying, we are in the position to state our main result.

Theorem 2.1. *Let $K \subseteq [0, 1]^n$ be a compact set which supports an absolutely η-decaying, δ-Ahlfors regular measure μ such that $\delta > n - 1$. Then, for any $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$,*

$$\dim(\mathbf{Bad}_{\mathcal{L}} \cap K) = \delta.$$

In view of (2.1), the theorem can be interpreted as stating that within K the set $\mathbf{Bad}_{\mathcal{L}}$ is of maximal dimension. It is easily seen that Theorem 1.1 is a consequence of Theorem 2.1 – simply take $K = [0, 1]^n$ and μ to be n-dimensional Lebesgue measure. Trivially, n-dimensional Lebesgue measure is n-Ahlfors regular and absolutely 1-decaying. More exotically, the natural measures associated with self-similar sets in $\mathbb{R}^n$ satisfying the open set condition are absolutely η-decaying and δ-Ahlfors regular

– see [12, 20]. Thus, Theorem 2.1 is applicable to these sets which in general are of fractal nature.

Although Theorem 2.1 constitutes our main result, we state an 'auxiliary' result in this section for the simple fact that it is new and of independent interest. In short, it strengthens and generalises a theorem of Pollington [19] and de Mathan [15, 16] that answers a question of Erdős. A sequence $\{\mathbf{y}_i\} := \{\mathbf{y}_i := (y_{1,i}, \ldots, y_{n,i})^T \in \mathbb{Z}^n \setminus \{\mathbf{0}\}\ \}$ is said to be *lacunary* if there exits a constant $\lambda > 1$ such that

$$|\mathbf{y}_{i+1}| \ \geq \ \lambda\, |\mathbf{y}_i| \qquad \forall\, i \in \mathbb{N}\ .$$

Given a sequence $\{\mathbf{y}_i\}$ in $\mathbb{Z}^n$, let

$$\mathbf{Bad}_{\{\mathbf{y}_i\}} := \{\mathbf{x} \in [0,1]^n : \exists\; c(\mathbf{x}) > 0 \text{ s.t. } \|\mathbf{y}_i \cdot \mathbf{x}\| \geq c(\mathbf{x})\ \forall\ i \in \mathbb{N}\}\ .$$

Theorem 2.2. *Let $\{\mathbf{y}_i\}$ be a lacunary sequence in $\mathbb{Z}^n$. Furthermore, let $K \subseteq [0,1]^n$ be a compact set which supports an absolutely η-decaying, δ-Ahlfors regular measure μ such that $\delta > n-1$. Then*

$$\dim(\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K) = \delta.$$

On setting $n = 1$, $K = [0,1]$ and μ to be one-dimensional Lebesgue measure, Theorem 2.2 corresponds to the theorem of Pollington and de Mathan referred to above.

3. Preliminaries for Theorem 2.2

The proof of Theorem 2.2 makes use of the general framework developed in [13] for establishing dimension statements for a large class of badly approximable sets. In this section we provide a simplification of the framework that is geared towards the particular application we have in mind. In turn, this will avoid excessive referencing to the conditions imposed in [13] and thereby improve the clarity of our exposition.

As in §2, let (X, d) be a metric space and (Ω, d) be a compact subspace of X which supports a non-atomic finite measure μ. Let $\mathcal{R} := \{R_a \in X : a \in J\}$ be a family of subsets R_a of X indexed by an infinite countable set J. The sets R_a will be referred to as the *resonant sets*. Next, let $\beta : J \to \mathbb{R}_{>0} : a \mapsto \beta_a$ be a positive function on J such that the number of $a \in J$ with β_a bounded above is finite. Thus, β_a tends to infinity as a runs through J. We are now in the position to define the badly approximable set

$$\mathbf{Bad}(\mathcal{R}, \beta) := \left\{ x \in \Omega : \exists\; c(x) > 0 \text{ s.t. } d(x, R_a) \geq \frac{c(x)}{\beta_a}\ \forall\ a \in J \right\}\ ,$$

where $d(x, R_a) := \inf_{r \in R_a} d(x, r)$. Loosely speaking, $\mathbf{Bad}(\mathcal{R}, \beta)$ consists of points in Ω that 'stay clear' of the family $\mathcal{R}$ of resonant sets by a factor governed by β.

The goal is to determine conditions under which $\dim \mathbf{Bad}(\mathcal{R}, \beta) = \dim \Omega$; that is to say that the set of badly approximable points in Ω is of maximal dimension. With this in mind, we begin with some useful notation. For any fixed integer $k > 1$ and any integer $t \geq 1$, let $B_t := \{x \in \Omega : d(c, x) \leq 1/k^t\}$ denote a generic closed ball in Ω of radius $1/k^t$ with centre c in Ω. For any $\theta \in \mathbb{R}_{>0}$, let $\theta B_t := \{x \in \Omega : d(c, x) \leq \theta/k^t\}$ denote the ball B_t scaled by θ. Finally, let $J(t) := \{a \in J : k^{t-1} \leq \beta_a < k^t\}$. The following statement is a simple consequence of combining Theorem 1 and Lemma 7 of [13] and realises the above mentioned goal.

Theorem KTV. *Let (X, d) be a metric space and (Ω, d) be a compact subspace of X which supports of a δ-Ahlfors regular measure μ. Let k be sufficiently large. Then for any $\theta \in \mathbb{R}_{>0}$, any $t \geq 1$ and any ball B_t there exists a collection $\mathcal{C}(\theta B_t)$ of disjoint balls $2\theta B_{t+1}$ contained within θB_t such that $\#\mathcal{C}(\theta B_t) \geq \kappa_1 k^\delta$. In addition, suppose for some $\theta \in \mathbb{R}_{>0}$ we also have that*

$$\#\left\{2\theta B_{t+1} \subset \mathcal{C}(\theta B_t) : \min_{a \in J(t+1)} d(c, R_a) \leq 2\theta k^{-(t+1)}\right\} \leq \kappa_2 k^\delta , \tag{3.1}$$

where $0 < \kappa_2 < \kappa_1$ are absolutely constants independent of k and t. Furthermore, suppose

$$\dim \left(\cup_{a \in J} R_a\right) < \delta . \tag{3.2}$$

Then

$$\dim \mathbf{Bad}(\mathcal{R}, \beta) = \delta .$$

Note that the theorem together with (2.1) implies that $\dim \mathbf{Bad}(\mathcal{R}, \beta) = \dim \Omega$.

4. Proof of Theorem 2.2

We are given a lacunary sequence $\{\mathbf{y}_i\}$. For each index $i \in \mathbb{N}$ and any integer p, consider the hyperplane $\mathcal{S}_{p,i} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{y}_i \cdot \mathbf{x} = p\}$. It is easily verified that $\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K$ is equivalent to the set of $\mathbf{x}$ in K for which there exists a constant $c(\mathbf{x}) > 0$ such that $\mathbf{x}$ avoids the $c(\mathbf{x})/|\mathbf{y}_i|_2$–neighbourhood of $\mathcal{S}_{p,i}$ for every choice of i and p; that is

$$\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K = \left\{\mathbf{x} \in K : \exists\, c(\mathbf{x}) > 0 \text{ s.t. } \min_{\mathbf{y} \in \mathcal{S}_{p,i}} |\mathbf{x} - \mathbf{y}|_2 \geq \frac{c(\mathbf{x})}{|\mathbf{y}_i|_2} \ \forall\ (p, i) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

Here $|\,.\,|_2$ is the standard Euclidean norm in $\mathbb{R}^n$. With reference to §3, set

$$X := \mathbb{R}^n\,, \quad \Omega := K\,, \quad d := |\,.\,|_2\,, \quad J := \{(p,i) \in \mathbb{Z}\times\mathbb{N}\}\,,$$
$$a := (p,i) \in J\,, \quad R_a := \mathcal{S}_{p,i} \quad \text{and} \quad \beta_{\rm a} := |\mathbf{y}_{\rm i}|_2\,.$$

It follows that

$$\mathbf{Bad}(\mathcal{R},\beta) = \mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K\,.$$

The upshot of this is that the proof of Theorem 2.2 is reduced to showing that the conditions of Theorem KTV are satisfied.

For $k > 1$ and $t \geq 1$, let B_t be a generic closed ball of radius k^{-t} and centre in K. For k sufficiently large and any $\theta \in \mathbb{R}_{>0}$, Theorem KTV guarantees the existence of a collection $\mathcal{C}(\theta B_t)$ of disjoint balls $2\theta B_{t+1}$ contained within θB_t such that

$$\#\mathcal{C}(\theta B_t) \geq \kappa_1\, k^{\delta}\,.$$

The positive constant κ_1 is independent of k and t. We now endeavor to show that the additional condition (3.1) on the collection $\mathcal{C}(\theta B_t)$ is satisfied. To this end, set $\theta := (2k)^{-1}$ and proceed as follows. Fix $t \geq 1$ and assume that there exists an index i such that

$$k^t \leq |\mathbf{y}_i|_2 < k^{t+1}\,. \tag{4.1}$$

If this is not the case, the left hand side of (3.1) is zero and the additional condition is trivially satisfied. Associated with the index i is the family of hyperplanes $\{\mathcal{S}_{p,i} : p \in \mathbb{Z}\}$. The distance between any two such hyperplanes is at least $|\mathbf{y}_i|_2^{-1} > k^{-(t+1)}$. The diameter of the ball θB_t is $k^{-(t+1)}$. Thus, for any element of the sequence $\{\mathbf{y}_i\}$ satisfying (4.1) there is at most one hyperplane passing through θB_t. Assume, the hyperplane $\mathcal{S}_{p,i}$ passes through θB_t and consider the counting function

$$\omega(t,p,i) := \#\left\{2\theta B_{t+1} \subset \mathcal{C}(\theta B_t) : 2\theta B_{t+1} \cap \mathcal{S}_{p,i} \neq \emptyset\right\}\,.$$

The balls $2\theta B_{t+1}$ are disjoint and each is of diameter $4\theta k^{-(t+1)}$. Thus, on setting $\epsilon := 8\theta k^{-(t+1)}$ it follows that

$$\begin{aligned}\omega(t,p,i) \;&\leq\; \#\left\{2\theta B_{t+1} \subset \mathcal{C}(\theta B_t) : 2\theta B_{t+1} \subset \mathcal{S}_{p,i}^{(\epsilon)}\right\} \\ &\leq\; \frac{\mu(\theta B_t \cap \mathcal{S}_{p,i}^{(\epsilon)})}{\mu(2\theta B_{t+1})}\,.\end{aligned}$$

On making use of the fact that μ is absolutely η-decaying and δ-Ahlfors regular, it is readily verified that

$$\omega(t,p,i) \;\leq\; \kappa\, k^{\delta-\eta}\,.$$

The absolute constant κ is dependent only on η and δ. Next, let $\upsilon(t,\{\mathbf{y}_i\})$ denote the number of elements of the sequence $\{\mathbf{y}_i\}$ satisfying (4.1). Since $\{\mathbf{y}_i\}$ is lacunary, we find that for k sufficiently large

$$\upsilon(t,\{\mathbf{y}_i\}) \leq 1 + \log(\sqrt{n}\,k)/\log\lambda \;<\; \frac{\kappa_1}{2\,\kappa}k^{\eta}\,.$$

Here, $\lambda > 1$ is the lacunarity constant and we have used the fact that $|\mathbf{y}| \leq |\mathbf{y}|_2 \leq \sqrt{n}\;|\mathbf{y}|$ for $\mathbf{y} \in \mathbb{Z}^n$. On combining the above upper bound estimates, we have that

$$\begin{aligned}\text{l.h.s. of (3.1)} \;\;&<\;\; \upsilon(t,\{\mathbf{y}_i\}) \;\times\; \omega(t,p,i) \\ &\leq\;\; \frac{\kappa_1}{2\kappa}k^{\eta} \;\times\; \kappa k^{\delta-\eta} \;=\; \tfrac{1}{2}\,\kappa_1 k^{\delta}\;.\end{aligned}$$

Thus, with $\theta := (2k)^{-1}$ the collection $\mathcal{C}(\theta B_t)$ satisfies (3.1). Finally, note that the collection $\{\mathcal{S}_{p,i} : (p,i) \in \mathbb{Z}\times\mathbb{N}\}$ of hyperplanes (resonant sets) is countable and so

$$\dim\left(\cup\,\mathcal{S}_{p,i}\right) \;=\; n-1\;.$$

We are given that $\delta > n-1$ and so (3.2) is trivially satisfied. Thus, the conditions of Theorem KTV are satisfied and Theorem 2.2 follows.

5. Preliminaries for Theorem 2.1

The proof of Theorem 2.1 makes use of the existence of 'special' sequences which for the most part are constructed in [5]. Throughout, $\mathrm{Mat}^*_{n\times m}(\mathbb{R})$ will denote the collection of matrices $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ for which the associated group $G := \mathcal{L}^T\mathbb{Z}^n + \mathbb{Z}^m$ has rank $n+m$. In Section 2 of [5], it is shown that associated with each matrix $\mathcal{L} \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ there exists a sequence $\{\mathbf{y}_i\}$ of integer vectors $\mathbf{y}_i = (y_{1,i},\ldots,y_{n,i})^T \in \mathbb{Z}^n$ satisfying the following properties:

(i) $1 = |\mathbf{y}_1| < |\mathbf{y}_2| < |\mathbf{y}_3| < \ldots$,

(ii) $\|\mathcal{L}^T\mathbf{y}_1\| > \|\mathcal{L}^T\mathbf{y}_2\| > \|\mathcal{L}^T\mathbf{y}_3\| > \ldots$,

(iii) For all non-zero $\mathbf{y} \in \mathbb{Z}^n$ with $|\mathbf{y}| < |\mathbf{y}_{i+1}|$ we have that $\|\mathcal{L}^T\mathbf{y}\| \geq \|\mathcal{L}^T\mathbf{y}_i\|$.

Such a sequence $\{\mathbf{y}_i\}$ is referred to as a *sequence of best approximations* to $\mathcal{L}$. In the one-dimensional case ($n = m = 1$), when $\mathcal{L}$ is an irrational number x, the sequence of best approximations is precisely the sequence of denominators associated with the convergents of the continued fraction representing x.

Let $\{\mathbf{y}_i\}$ be a sequence of best approximations to a matrix $\mathcal{L} \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$. A further property enjoyed by $\{\mathbf{y}_i\}$, is that

$$\|\mathcal{L}^T\mathbf{y}_i\| \leq |\mathbf{y}_{i+1}|^{-m/n} \quad \forall\; i \in \mathbb{N}. \tag{5.1}$$

This property is easily deduced via Dirichlet's box principle – see Section 2 of [5] for the details.

The following result, which is taken from Section 4 of [5], enables us to extract a lacunary subsequence from a given sequence of best approximations. This will allow us to utilise Theorem 2.2 in the course of establishing Theorem 2.1.

Lemma BL. *Let* $\mathcal{L} \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ *and let* $\{\mathbf{y}_i\}$ *be a sequence of best approximations to* $\mathcal{L}$*. Then, there exists an increasing function* $\phi : \mathbb{N} \to \mathbb{N}$ *such that* $\phi(1) = 1$ *and for* $i \geq 2$

$$\left|\mathbf{y}_{\phi(i)}\right| \geq \sqrt{9n}\,\left|\mathbf{y}_{\phi(i-1)}\right| \quad \textit{and} \quad \left|\mathbf{y}_{\phi(i-1)+1}\right| \geq \frac{\left|\mathbf{y}_{\phi(i)}\right|}{9n}\ . \tag{5.2}$$

It is clear that the sequence $\left\{\mathbf{y}_{\phi(i)}\right\}$ is lacunary and that it also satisfies (5.1); i.e.

$$\left\|\mathcal{L}^T\mathbf{y}_{\phi(i)}\right\| \leq \left|\mathbf{y}_{\phi(i)+1}\right|^{-m/n} \quad \forall\, i \in \mathbb{N}. \tag{5.3}$$

The next inequality follows directly from the definition of the norms involved. For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$, we have that

$$\|\mathbf{x}\cdot\mathbf{y}\| \,<\, k\,|\mathbf{x}|\,\|\mathbf{y}\|\ . \tag{5.4}$$

We end this section with a short discussion that allows us to assume that $\mathcal{L} \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ when proving Theorem 2.1. With this in mind, suppose $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ and that the rank of the associated group $G := \mathcal{L}^T\mathbb{Z}^n + \mathbb{Z}^m$ is strictly less than $n+m$. Then, it is easily verified that $\{\mathcal{L}\mathbf{q} : \mathbf{q} \in \mathbb{Z}^m\}$ is restricted to at most a countable family of positively separated, parallel hyperplanes in $\mathbb{R}^n$. Let S denote the set of these hyperplanes. Then,

$$K \setminus S = \mathbf{Bad}_{\mathcal{L}} \cap K\ .$$

We are given that $\delta > n-1$ which together with (2.1) implies that $\dim K$ is strictly greater than $\dim S$. Thus, $\dim(K\setminus S) = \dim K$ and the statement of Theorem 2.1 follows for any $\mathcal{L} \notin \mathrm{Mat}^*_{n\times m}(\mathbb{R})$.

6. Proof of Theorem 2.1

Without loss of generality, assume that $\mathcal{L} \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ and let $\{\mathbf{y}_i\}$ be a sequence of best approximations to $\mathcal{L}$. In view of Lemma BL, there exists a lacunary subsequence $\left\{\mathbf{y}_{\phi(i)}\right\}$ of the sequence of best approximations. For any $c > 0$, let

$$\mathbf{B}_{\left\{\mathbf{y}_{\phi(i)}\right\}}(c) := \left\{\boldsymbol{\alpha} \in K\colon\ \left\|\mathbf{y}_{\phi(i)}\cdot\boldsymbol{\alpha}\right\| \geq c\ \forall\, i \in \mathbb{N}\right\}\ .$$

It is readily verified that $\mathbf{Bad}_{\{\mathbf{y}_{\phi(i)}\}} \cap K = \bigcup_{c>0} \mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c)$ and that

$$\dim \mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \;\to\; \dim\left(\mathbf{Bad}_{\{\mathbf{y}_{\phi(i)}\}} \cap K\right) \quad \text{as} \quad c \to 0\,.$$

For c sufficiently small, suppose for the moment that

$$\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \;\subseteq\; \mathbf{Bad}_{\mathcal{L}} \cap K\;. \tag{6.1}$$

On utilising Theorem 2.2, it follows that

$$\dim\left(\mathbf{Bad}_{\mathcal{L}} \cap K\right) \;\geq\; \dim \mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \;\to\; \delta \quad \text{as} \quad c \to 0\,.$$

The upshot of this is that $\dim(\mathbf{Bad}_{\mathcal{L}} \cap K) \geq \delta$. For the complementary upper bound statement, trivially

$$\dim\left(\mathbf{Bad}_{\mathcal{L}} \cap K\right) \;\leq\; \dim K \;\stackrel{(2.1)}{=}\; \delta\,.$$

This completes the proof of Theorem 2.1 modulo the inclusion (6.1).

To establish (6.1), fix a point $\boldsymbol{\alpha}$ in $\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c)$ and let $\mathbf{q}$ be any non-zero integer vector. For c sufficiently small, there exists an index $i \in \mathbb{N}$ such that

$$\left|\mathbf{y}_{\phi(i)}\right| \;\leq\; 9n\left(\frac{2m}{c}\right)^{m/n} |\mathbf{q}|^{m/n} \;<\; \left|\mathbf{y}_{\phi(i+1)}\right|. \tag{6.2}$$

The existence of such an index is guaranteed by the first of the inequalities in (5.2) as long as c is sufficiently small. By the definition of $\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c)$ and the trivial equality

$$\mathbf{y}_{\phi(i)} \cdot \boldsymbol{\alpha} \;=\; \mathbf{q} \cdot \mathcal{L}^T \mathbf{y}_{\phi(i)} - \mathbf{y}_{\phi(i)} \cdot (\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}),$$

we immediately have that

$$0 < c \leq \left\|\mathbf{y}_{\phi(i)} \cdot \boldsymbol{\alpha}\right\| = \left\|\mathbf{q} \cdot \mathcal{L}^T \mathbf{y}_{\phi(i)} - \mathbf{y}_{\phi(i)} \cdot (\mathcal{L}\mathbf{q} - \boldsymbol{\alpha})\right\|\;. \tag{6.3}$$

On applying the triangle inequality and making use of (5.4), it follows that

$$c \leq m\,|\mathbf{q}|\left\|\mathcal{L}^T \mathbf{y}_{\phi(i)}\right\| + n\left|\mathbf{y}_{\phi(i)}\right|\|\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}\|\,. \tag{6.4}$$

However,

$$m\,|\mathbf{q}|\left\|\mathcal{L}^T \mathbf{y}_{\phi(i)}\right\| \;\stackrel{(5.3)}{\leq}\; m\,|\mathbf{q}|\left|\mathbf{y}_{\phi(i)+1}\right|^{-n/m} \;\stackrel{(6.2)}{\leq}\; \frac{m}{(9n)^{n/m}\frac{2m}{c}}\left(\frac{\left|\mathbf{y}_{\phi(i+1)}\right|}{\left|\mathbf{y}_{\phi(i)+1}\right|}\right)^{n/m} \;\stackrel{(5.2)}{\leq}\; \frac{c}{2}$$

and

$$n\left|\mathbf{y}_{\phi(i)}\right|\|\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}\| \;\stackrel{(6.2)}{\leq}\; 9n^2\left(\frac{2m}{c}\right)^{m/n} |\mathbf{q}|^{m/n}\,\|\mathcal{L}\mathbf{q} - \boldsymbol{\alpha}\|\;,$$

which together with (6.4) yields that

$$\|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| > \frac{c^{m/n+1}}{9n^2(2m)^{m/n}}\,|\mathbf{q}|^{-m/n}\,.$$

In other words, for any c sufficiently small

$$\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \subseteq \left\{\boldsymbol{\alpha}\in K : \exists\; c(\boldsymbol{\alpha})>0 \text{ s.t. } \|\mathcal{L}\mathbf{q}-\boldsymbol{\alpha}\| \geq \frac{c(\boldsymbol{\alpha})}{|\mathbf{q}|^{m/n}} \;\forall\; \mathbf{q}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}\right\}.$$

The right hand side is $\mathbf{Bad}_{\mathcal{L}}\cap K$ and this establishes (6.1) which in turn completes the proof of Theorem 2.1.

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Paper II

TWISTED INHOMOGENEOUS DIOPHANTINE APPROXIMATION AND BADLY APPROXIMABLE SETS

STEPHEN HARRAP

ABSTRACT. For any real pair $i, j \geq 0$ with $i + j = 1$ let $\mathbf{Bad}(i, j)$ denote the set of (i, j)-badly approximable pairs. That is, $\mathbf{Bad}(i, j)$ consists of irrational vectors $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$ for which there exists a positive constant $c(\mathbf{x})$ such that

$$\max\left\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\right\} > c(\mathbf{x})/q \qquad \forall\, q \in \mathbb{N}.$$

A new characterization of $\mathbf{Bad}(i, j)$ in terms of 'well-approximable' vectors in the area of 'twisted' inhomogeneous Diophantine approximation is established. In addition, it is shown that $\mathbf{Bad}_{\mathbf{x}}(i, j)$, the 'twisted' inhomogeneous analogue of $\mathbf{Bad}(i, j)$, has full Hausdorff dimension 2 when $\mathbf{x}$ is chosen from $\mathbf{Bad}(i, j)$. The main results naturally generalise the $i = j = 1/2$ work of Kurzweil.

1. INTRODUCTION

1.1. **Background – the homogeneous theory.** A classical result of Dirichlet states that for any real number x there exist infinitely many natural numbers q such that

$$\|qx\| \leq \frac{1}{q}, \tag{1.1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. This result can easily be generalised to higher dimensions. In particular, the following 'weighted' simultaneous version is valid. Choose any positive real numbers i and j satisfying

$$i, j \geq 0 \quad \text{and} \quad i + j = 1. \tag{1.2}$$

Then, for any vector $\mathbf{x} \in \mathbb{R}^2$ there exist infinitely many natural numbers q such that

$$\max\left\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\right\} \leq \frac{1}{q}. \tag{1.3}$$

Without loss of generality, if $i = 0$ we employ the convention that $\|x\|^{1/i} = 0$ and so the above statement reduces to Dirichlet's original result. It is natural to ask whether the right hand side of inequality (1.3) can in general be tightened. That is, can $1/q$ be replaced by c/q for some absolute constant $c \in (0, 1)$ whilst still allowing (1.3) to hold infinitely often for all real vectors $\mathbf{x}$? It is still an open problem as to whether there exists an 'optimal' constant in this sense. On the other hand, in the one-dimensional setting of statement (1.1) such an 'optimal' constant (namely $1/\sqrt{5}$) was found by Hurwitz [12].

The above discussion motivates the study of real vectors $\mathbf{x}$ for which the right hand side of (1.3) cannot be improved by an arbitrary positive constant. Throughout, we will impose the following natural restriction on these vectors. We say $\mathbf{x} := (x_1, x_2)$ is *irrational* (abbreviated *irr.*) if its components x_i together with 1 are linearly independent over the rationals.

Definition 1.1. An irrational vector $\mathbf{x}$ is (i,j)-*badly approximable* if there exists a constant $c(\mathbf{x}) > 0$ such that

$$\max\left\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\right\} \; > \; \frac{c(\mathbf{x})}{q} \qquad \forall\, q \in \mathbb{N}.$$

The set of all such vectors will be denoted $\mathbf{Bad}(i,j)$.

We remark that the results of this paper (for $i, j > 0$) remain true when $\mathbf{x}$ is not assumed to be irrational in the above and later definitions. However, we choose to avoid this degenerate case to simplify our arguments.

One may notice that the set $\mathbf{Bad}(i,j)$ is invariant under translation by integer vectors. In fact, this will be the case with most sets considered in this paper. For that reason we will often restrict our attention to the unit square $[0,1)^2$ (or the unit n-cube when in higher dimensions) and it should be understood that no generality is lost in doing this. For example, if $i = 0$ the set $\mathbf{Bad}(0,1)$ will be identified with $[0,1) \times \mathbf{Bad}$, where $\mathbf{Bad}$ is the standard one dimensional set of badly approximable numbers. In other words, $\mathbf{Bad}(0,1)$ consists of vectors $\mathbf{x}$ with $x_1 \in [0,1)$ and

$$x_2 \in \mathbf{Bad} := \left\{x \in [0,1) : \exists\, c(x) > 0 \text{ s.t. } \|qx\| > \frac{c(x)}{q} \quad \forall\, q \in \mathbb{N}\right\}.$$

Definition 1.2. A mapping $\psi : \mathbb{N} \to \mathbb{R}$ is an *approximating function* if ψ is strictly positive.

Definition 1.3. For any approximating function ψ, define $W(i,j,\psi)$ to be the set of vectors $\mathbf{x} \in [0,1)^2$ such that the inequality

$$\max\left\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\right\} \; \leq \; \psi(q)$$

holds for infinitely many natural numbers q.

Application of the following classical theorem of Khintchine [15] yields that for every pair of reals i, j satisfying (1.2) the set $\mathbf{Bad}(i,j)$ is of two-dimensional Lebesgue measure zero. Throughout, Lebesgue measure will be denoted λ.

Khintchine's Theorem (1926). *For any pair of reals i, j satisfying (1.2) and any approximating function ψ we have*

$$\lambda\left(W(i,j,\psi)\right) = \begin{cases} 0, & \sum_{r=1}^{\infty} \psi(r) \; < \; \infty. \\ \\ 1, & \sum_{r=1}^{\infty} \psi(r) \; = \; \infty \;\; \textit{and } \psi \textit{ is monotonic.} \end{cases}$$

We remark that the monotonicity restriction imposed on the function ψ can be relaxed due to a result of Harman (see Theorem 3.8 of [11]).

The question of whether each null set $\mathbf{Bad}(i,j)$ is non-empty was formally[1] answered by Pollington & Velani [20] who showed that for every choice of reals i, j satisfying (1.2) we have

$$\dim\left(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1,0) \cap \mathbf{Bad}(0,1)\right) = \dim\left([0,1)^2\right) = 2. \tag{1.4}$$

Here, and throughout, 'dim' denotes standard Hausdorff dimension. With this result in mind, the aim of this paper is to obtain an expression for $\mathbf{Bad}(i,j)$ in terms of 'well-approximable' vectors in the area of 'twisted' inhomogeneous Diophantine approximation.

1.2. **Background – the 'twisted' theory.** Another result of Khintchine states that for any irrational x and any real α there exist infinitely many natural numbers q such that

$$\|qx - \alpha\| \leq \frac{1+\epsilon}{\sqrt{5}q}, \tag{1.5}$$

where $\epsilon > 0$ is an arbitrary constant. The inequality is 'optimal' and differs from Hurwitz's homogeneous '$\alpha = 0$' theorem by only the constant ϵ. When certain restrictions are placed on the choice of α, a tighter 'optimal' inequality was found to hold by Minkowski [21]. The right hand side of (1.5) can be replaced with $1/(4q)$ if it is assumed that α is not of the form $\alpha = sx + t$ for some integers s and t. Both of these statements imply that the sequence $\{qx\}_{q\in\mathbb{N}}$ modulo one is dense in the unit interval for any irrational x. Moreover, Kronecker's Theorem (see [17]) implies that the sequence $\{q\mathbf{x}\}_{q\in\mathbb{Z}}$ modulo one is dense in $[0,1)^2$ for any irrational vector $\mathbf{x}$. Further still, a celebrated result of Weyl [29] states that the sequence is uniformly distributed in $[0,1)^2$ for any irrational vector $\mathbf{x}$.

[1] The arguments used by Davenport in [7] to show that $\mathbf{Bad}(1/2,1/2)$ is uncountable can easily be adapted to show that $\mathbf{Bad}(i,j)$ is uncountable for every choice of reals i,j satisfying (1.2).

Naturally, this leads to the concept of approximating real vectors $\boldsymbol{\alpha}$ in $[0,1)^2$ by the sequence $\{q\mathbf{x}\}_{q\in\mathbb{N}}$ modulo one with a prescribed rate of accuracy. For obvious reasons we call this approach 'twisted' Diophantine approximation.

Definition 1.4. Fix an approximating function ψ, any irrational vector $\mathbf{x}$ and a pair or reals (i,j) satisfying (1.2). Then $W_{\mathbf{x}}(i,j,\psi)$ will denote the set of vectors $\boldsymbol{\alpha} := (\alpha_1,\alpha_2) \in [0,1)^2$ such that the inequality

$$\max\left\{\|qx_1-\alpha_1\|^{1/i}, \|qx_2-\alpha_2\|^{1/j}\right\} \leq \psi(|q|)$$

holds for infinitely many non-zero integers q.

Establishing a Khintchine-type result for the Lebesgue measure of $W_{\mathbf{x}}(i,j,\psi)$ is more difficult than in the homogeneous case. That said, by utilising the Borel-Cantelli lemma from probability theory it is easy to show that for every i, j satisfying (1.2), any irrational $\mathbf{x}$ and every approximating function ψ we have

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty}\psi(r) \, < \, \infty.$$

One might therefore expect that no matter what the choice of reals i, j, irrational $\mathbf{x}$ or approximating function ψ we should be able to conclude that $\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) = 1$ if the above sum diverges. However, the following statement suggests that once the reals i, j have been fixed the set of irrational vectors for which we do obtain a set of full measure is dependent on the choice of approximating function. This subtle distinction makes the metrical theory in the 'twisted' setting more delicate, and sophisticated, than its standard homogeneous counterpart.

Theorem 1.5 (Twisted Khintchine-type Theorem)**.** *Let ψ be a fixed monotonic approximating function. Then, for λ-almost all irrational vectors $\mathbf{x} \in [0,1)^2$*

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty}\psi(r) \, = \, \infty.$$

This result is a consequence of a more general result that can be found in the Appendix. In what follows we say a function ψ is *divergent* if $\sum_{r=1}^{\infty}\psi(r) = \infty$. The set of all divergent approximating functions will be denoted by D.

Definition 1.6. Fix a pair of reals i, j satisfying (1.2). Then, for each $\psi \in D$ let

$$V(i,j,\psi) := \{\text{irr. } \mathbf{x} : \lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) = 1\}.$$

Note that Theorem 1.5 is equivalent to the statement "$\lambda\left(V(i,j,\psi)\right) = 1$ for each $\psi \in D$". In view of Theorem 1.5 we ask whether there exist irrational vectors

$\mathbf{x}$ such that a set of full measure is obtained regardless of the choice of divergent approximating function. In other words, we wish to characterise the set

$$\bigcap_{\psi \in D} V(i, j, \psi).$$

It is certainly not obvious as to whether the intersection is non-empty. Previous activity has been restricted to the classical $i = j = 1/2$ case where elements of $\mathbf{Bad}\,(1/2, 1/2)$ are commonly referred to as *simultaneously badly approximable pairs.* The most notable breakthrough was made by Kurzweil [18], who proved the following remarkable result.

Kurzweil's Theorem (1955).

$$\bigcap_{\psi \in D} V(1/2,\ 1/2,\ \psi) = \mathbf{Bad}(1/2,\ 1/2).$$

The work of Kim [14] in a similar vein inspired activity concerning real vectors that are badly approximable in the twisted sense.

Definition 1.7. Fix an irrational vector $\mathbf{x} \in [0, 1)^2$ and two real numbers i and j satisfying (1.2). Define $\mathbf{Bad}_{\mathbf{x}}(i, j)$ as the set of vectors $\boldsymbol{\alpha} \in [0, 1)^2$ for which there exists a constant $c(\boldsymbol{\alpha}) > 0$ such that

$$\max\left\{\|qx_1 - \alpha_1\|^{1/i}, \|qx_2 - \alpha_2\|^{1/j}\right\} > \frac{c(\boldsymbol{\alpha})}{|q|} \qquad \text{for all } q \in \mathbb{Z}_{\neq 0}.$$

The set $\mathbf{Bad}_{\mathbf{x}}(i, j)$ represents the natural twisted analogue of $\mathbf{Bad}(i, j)$. Previous results are once again limited to the classical $i = j = 1/2$ setting. In particular, Bugeaud et al [4] proved the following result (see also [22, 28]).

Theorem BHKV (2010). *For any irrational* $\mathbf{x} \in [0, 1)^2$,

$$\dim\left(\mathbf{Bad}_{\mathbf{x}}\left(\frac{1}{2}, \frac{1}{2}\right)\right) = 2.$$

At the time of writing there were no known results concerning the Hausdorff dimension of $\mathbf{Bad}_{\mathbf{x}}(i, j)$ for a general pair i and j.

2. The main results

2.1. **Statements of results.** The following statement represents our main theorem and generalises Kurzweil's theorem from the classical '1/2–1/2' to the full weighted setting.

Theorem 2.1. *For every pair of reals i and j satisfying (1.2),*

$$\bigcap_{\psi \in D} V(i,j,\psi) = \mathbf{Bad}(i,j).$$

In view of Khintchine's theorem and statement (1.4), Theorem 2.1 immediately implies that the intersection on the LHS above is of two-dimensional Lebesgue measure zero and of maximal Hausdorff dimension.

Our next result makes a contribution towards determining the Hausdorff dimension of $\mathbf{Bad}_{\mathbf{x}}(i,j)$.

Theorem 2.2. *For any real i and j satisfying (1.2) and any $\mathbf{x} \in \mathbf{Bad}(i,j)$,*

$$\dim \left(\mathbf{Bad}_{\mathbf{x}}(i,j)\right) \;=\; 2.$$

The proof of this theorem makes use of a general framework developed by Kristensen, Thorn & Velani [16]. This framework was designed for establishing dimension results for large classes of badly approximable sets and the above statement constitutes one further application. In all likelihood Theorem 2.2 is true without the assumption on $\mathbf{x}$.

Conjecture 2.3. *For any real i and j satisfying (1.2) and any irrational vector $\mathbf{x} \in [0,1)^2$,*

$$\dim \left(\mathbf{Bad}_{\mathbf{x}}(i,j)\right) \;=\; 2.$$

It seems that the ideas of [4], which also make use of the framework in [16], cannot be extended to the full weighted setting of Conjecture 2.3; a new approach may be required. We remark that Theorem 2.2, together with (1.4), trivially implies that the conjecture is true for a set of irrational vectors $\mathbf{x}$ of full dimension.

2.2. **Higher dimensions.** We describe the n-dimensional generalisation of the sets $\mathbf{Bad}(i,j)$ and $V(i,j,\psi)$ along with the higher dimensional analogue of the statements in §2.1. Throughout, λ_n will denote standard n-dimensional Lebesgue measure.

Fix any n-tuple of reals $\mathbf{k} := k_1, \ldots, k_n \geq 0$ such that $\sum_{j=1}^{n} k_j = 1$. We naturally define $\mathbf{Bad}(\mathbf{k})$ to be the set of vectors $\mathbf{x} := (x_1, \ldots, x_n) \in [0,1)^n$ for which there exists a constant $c(\mathbf{x}) > 0$ such that

$$\max\left\{ \|qx_1\|^{1/k_1}, \ldots, \|qx_n\|^{1/k_n} \right\} > \frac{c(\mathbf{x})}{q} \qquad \forall\, q \in \mathbb{N}.$$

For any approximating function ψ and any irrational vector $\mathbf{x} \in [0,1)^n$, we denote by $W_{\mathbf{x}}(i,j,\psi)$ the set of vectors $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in [0,1)^n$ such that

$$\max\left\{\|qx_1 - \alpha_1\|^{1/k_1}, \dots, \|qx_n - \alpha_n\|^{1/k_n}\right\} \le \psi(|q|)$$

for infinitely many non-zero integers q. Also, let

$$V(\mathbf{k}, \psi) := \{\mathbf{x} \in [0,1)^n : \lambda_n\left(W_{\mathbf{x}}(\mathbf{k}, \psi)\right) = 1\}.$$

The proof of Theorem 2.1 can be extended in the obvious way, with no new ideas or difficulties, allowing us to establish the following statement. For every real n-tuple $\mathbf{k}$ such that $k_1, \dots, k_n \ge 0$ and $\sum_{j=1}^n k_j = 1$ we have

$$\bigcap_{\psi \in D} V(\mathbf{k}, \psi) = \mathbf{Bad}(\mathbf{k}). \tag{2.1}$$

Khintchine's theorem and statement (1.4) can also be generalised and yield that the above intersection is of n-dimensional Lebesgue measure zero and of full Hausdorff dimension n. In proving statement (2.1) the notation gets rather awkward and so for the sake of clarity we will prove the '$n = 2$' case only.

3. Multiplicative Diophantine Approximation

This section comprises of a brief discussion of related problems in the area of multiplicative Diophantine approximation, where loosely speaking the supremum norm is replaced by the geometric mean. For example, one could consider the set of vectors that are 'well approximable' in a multiplicative sense.

Definition 3.1. Let ψ be any approximating function. Then, define

$$M(\psi) := \left\{\mathbf{x} \in [0,1)^2 : \|qx_1\|\,\|qx_2\| \le \psi(q) \text{ for inf. many } q \in \mathbb{N}\right\}.$$

The relevant measure-theoretic result concerning $M(\psi)$ was found by Gallagher [10] who proved a theorem implying the following.

Gallagher's Theorem (1962). *For any approximating function ψ,*

$$\lambda\left(M(\psi)\right) = \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} \psi(r)\log(r) < \infty. \\[2ex] 1, & \displaystyle\sum_{r=1}^{\infty} \psi(r)\log(r) = \infty \textit{ and } \psi \textit{ is monotonic.} \end{cases}$$

It is natural to develop a twisted theory for the multiplicative setup.

Definition 3.2. Fix any approximating function ψ and any irrational vector $\mathbf{x}$ in $[0,1)^2$. Then, define

$$M_{\mathbf{x}}(\psi) := \left\{\boldsymbol{\alpha} \in [0,1)^2 : \|qx_1 - \alpha_1\| \, \|qx_2 - \alpha_1\| \leq \psi(|q|) \ \text{ for inf. } q \in \mathbb{Z}_{\neq 0}\right\}.$$

The following statement is a consequence of Theorem 6.1 (see the Appendix).

Theorem 3.3. *Fix any approximating function ψ. Then for λ-almost all irrational vectors $\mathbf{x} \in [0,1)^2$ we have*

$$\lambda\left(M_{\mathbf{x}}(\psi)\right) = \begin{cases} 0, & \sum_{r=1}^{\infty} \psi(r)\log(r) \ < \ \infty. \\ \\ 1, & \sum_{r=1}^{\infty} \psi(r)\log(r) \ = \ \infty \ \ \textit{and } \psi \textit{ is monotonic.} \end{cases}$$

Once more one could ask whether there exist irrational vectors $\mathbf{x}$ such that a set of full measure is obtained irrespective of the choice of approximating function. Accordingly, let D_M denote the set of approximating functions for which $\sum_{r=1}^{\infty}\psi(r)\log(r)$ diverges and define

$$V_M(\psi) := \{\text{irr. } \mathbf{x} : \lambda\left(M_{\mathbf{x}}(\psi)\right) = 1\}.$$

Consider the intersection

$$\bigcap_{\psi \in D_M} V_M(\psi). \tag{3.1}$$

In view of Theorem 2.1, one might expect that (3.1) is equivalent to a multiplicative analogue of the set of badly approximable pairs. However, quite how such an analogue should be defined is up for debate.

One could argue that a valid choice for a set of *multiplicatively badly approximable numbers* might be

$$\mathbf{Bad}_L := \left\{\mathbf{x} \in [0,1)^2 : \exists\, c(\mathbf{x}) > 0 \text{ s.t. } \|qx_1\| \, \|qx_2\| > \frac{c(\mathbf{x})}{q} \quad \forall\, q \in \mathbb{N}\right\}.$$

The famous Littlewood conjecture states that the set $\mathbf{Bad}_L$ is empty. For recent developments and background concerning the Littlewood conjecture see [9], [19] and the references therein.

Another candidate for the multiplicatively badly approximable numbers is the larger set

$$\mathbf{Mad}^1 := \left\{\mathbf{x} \in [0,1)^2 : \exists\, c(\mathbf{x}) > 0 \text{ s.t. } \|qx_1\| \, \|qx_2\| > \frac{c(\mathbf{x})}{q\log q} \quad \forall\, q \in \mathbb{N}\right\},$$

recently introduced in [1]. Hence, the following question arises:

$$\textit{Can } \bigcap_{\psi\in D_M} V_M(\psi) \textit{ be characterized as } \mathbf{Bad}_L \textit{ or } \mathbf{Mad}^1?$$

Even establishing that $\mathbf{Bad}_L \subseteq \bigcap_{\psi\in D_M} V_M(\psi)$ seems non-trivial.

4. Proof of Theorem 2.1

4.1. **Proof of Theorem 2.1 (Part 1).** If either $i = 0$ or $j = 0$ the theorem simplifies to the classical one-dimensional version of Kurzweil's theorem corresponding to the set **Bad**. Therefore, we can and will assume hereafter that $i, j > 0$. The proof of Theorem 2.1 takes the form of two inclusion propositions, the first of which is proved in this section.

Proposition 4.1. *For every real $i, j > 0$ such that $i + j = 1$,*

$$\bigcap_{\psi\in D} V(i,j,\psi) \subseteq \mathbf{Bad}(i,j).$$

Proof. We will show that if $\mathbf{x} \notin \mathbf{Bad}(i,j)$ then $\mathbf{x} \notin \bigcap_{\psi\in D} V(i,j,\psi)$. In particular, we will prove that for every such $\mathbf{x}$ there exists an approximating function $\psi_0 \in D$ for which

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi_0)\right) = 0. \tag{4.1}$$

That is, the set of points $\boldsymbol{\alpha} := (\alpha_1, \alpha_2) \in [0,1)^2$ that satisfy the inequality

$$\max\left\{\|qx_1 - \alpha_1\|^{1/i}, \|qx_2 - \alpha_2\|^{1/j}\right\} \le \psi_0(|q|)$$

for infinitely many non-zero integers q has Lebesgue measure zero.

If $\mathbf{x} \notin \mathbf{Bad}(i,j)$ then by definition there exists a sequence $\{q_k\}_{k\in\mathbb{N}}$ of non-zero integers such that

$$\max\left\{\|q_kx_1\|^{1/i}, \|q_kx_2\|^{1/j}\right\} < \frac{c_k}{|q_k|}, \qquad |q_k| < |q_{k+1}| \quad \forall\ k\in\mathbb{N}, \tag{4.2}$$

where $c_k > 0$ and $c_k \to 0$ as $k \to \infty$. We may assume that

$$1 > c_k > 2^{3/(2\min\{i,j\})} c_{k+1} \qquad \forall\ k\in\mathbb{N}. \tag{4.3}$$

If this were not the case then we could simply work with a suitable subsequence of $\{q_k\}$. It may also be assumed that the values $(c_k)^{-1/3}$ are positive integers for every natural number k. These assumptions guarantee that for every $k \in \mathbb{N}$

$$(c_k)^{-\frac{1}{3}} \ge 2. \tag{4.4}$$

For each $k \geq 1$, let $n_k := |q_k|\,(c_k)^{-1/3}$ and let $n_0 := 0$. In view of (4.4), the sequence $\{n_k\}_{k\in\mathbb{N}}$ is strictly increasing and takes positive integers values. Next, for each natural number r define

$$\psi_0(r) := \begin{cases} 1, & r \leq n_1. \\ |q_{k+1}|^{-1}\,(c_{k+1})^{\frac{1}{3}}\,, & n_k < r \leq n_{k+1} \qquad \text{for every } k \geq 1. \end{cases}$$

It is clear that ψ_0 is an approximating function. To show $\psi_0 \in D$, note that

$$\begin{aligned}
\sum_{r=1}^{\infty} \psi_0(r) &> \sum_{k=1}^{\infty} \sum_{r=n_k+1}^{n_{k+1}} \psi_0(r) \\
&= \sum_{k=1}^{\infty} (n_{k+1} - (n_k+1) + 1)\,\psi_0(n_{k+1}) \\
&= \sum_{k=1}^{\infty} \left(|q_{k+1}|\,(c_{k+1})^{-\frac{1}{3}} - |q_k|\,(c_k)^{-\frac{1}{3}}\right) |q_{k+1}|^{-1}\,(c_{k+1})^{\frac{1}{3}} \\
&= \sum_{k=1}^{\infty} \left(1 - \frac{|q_k|}{|q_{k+1}|}\left(\frac{c_{k+1}}{c_k}\right)^{\frac{1}{3}}\right) \\
&> \sum_{k=1}^{\infty} \left(1 - \left(\frac{c_{k+1}}{c_k}\right)^{\frac{1}{3}}\right) \qquad \text{(since } |q_k| < |q_{k+1}|\text{)} \\
&\overset{(4.3)}{>} \sum_{k=1}^{\infty} \left(1 - 2^{-1/(2\min\{i,j\})}\right) \\
&\geq \sum_{k=1}^{\infty} \frac{1}{2} \;=\; \infty,
\end{aligned}$$

as required.

Finally, we endeavour to show (4.1) holds for ψ_0. To that end, for each non-zero integer q let

$$\mathbf{R}_{\psi_o}(q) := \left\{\boldsymbol{\alpha} \in [0,1)^2 : \max\left\{\|qx_1 - \alpha_1\|^{1/i}, \|qx_2 - \alpha_2\|^{1/j}\right\} \leq \psi_0(|q|)\right\}$$

denote the closed rectangular region in the plane centred at the point $q\mathbf{x}$ (mod 1) of side lengths $2\psi_0^i(|q|)$ and $2\psi_0^j(|q|)$ respectively. All closed rectangular regions of this type will be simply referred to as 'rectangles' and all points within any rectangle will tacitly be considered modulo one. It is clear that

$$\begin{aligned}
W_{\mathbf{x}}(i,j,\psi_0) &= \left\{\boldsymbol{\alpha} \in [0,1)^2 : \boldsymbol{\alpha} \in \mathbf{R}_{\psi_o}(q) \text{ for inf. many } q \in \mathbb{Z}_{\neq 0}\right\} \\
&= \left\{\, \boldsymbol{\alpha} \in [0,1)^2 : \boldsymbol{\alpha} \in \bigcup_{|q|=n_{k-1}+1}^{n_k} \mathbf{R}_{\psi_o}(q) \text{ for inf. many } k \in \mathbb{N} \,\right\}. \qquad (4.5)
\end{aligned}$$

In view of the Borel-Cantelli lemma, to show that equation (4.1) holds it is enough to show that

$$\sum_{k=1}^{\infty} \lambda\left(\bigcup_{|q|=n_{k-1}+1}^{n_k} \mathbf{R}_{\psi_o}(q)\right) < \infty. \tag{4.6}$$

We estimate the LHS of (4.6) by estimating the measure of each union of rectangles of the form

$$\mathbf{R}^*_{\psi_o}(k) := \bigcup_{|q|=n_{k-1}+1}^{n_k} \mathbf{R}_{\psi_o}(q), \qquad \text{for } k \in \mathbb{N}.$$

We refer to a union of rectangles of this type as a 'collection'. For each k, the collection $\mathbf{R}^*_{\psi_o}(k)$ consists of $2(n_k - n_{k-1})$ rectangles in $[0,1)^2$ each centred at some point $q\mathbf{x}$ for which $n_{k-1} < |q| \leq n_k$. By definition, every rectangle in a collection has the same area.

To estimate the measure of $\mathbf{R}^*_{\psi_o}(k)$ we will cover it with a collection of larger rectangles whose areas will in some sense increase at a 'controllable' rate. For each $k \in \mathbb{N}$ let

$$\begin{aligned}\mathbf{S}^*_{\psi_o}(k) : \;=\; \bigcup_{|q|=1}^{|q_k|} \Bigg\{ \boldsymbol{\alpha} \in [0,1)^2 : \|qx_1 - \alpha_1\| \leq \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k) \\ \text{and } \; \|qx_2 - \alpha_2\| \leq \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k) \Bigg\}.\end{aligned}$$

To clarify, each collection $\mathbf{S}^*_{\psi_o}(k)$ consists of $2\,|q_k|$ rectangles in $[0,1)^2$, one centred at each point $q\mathbf{x}$ with $1 \leq |q| < |q_k|$. The side lengths of each of these rectangles are

$$2\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k)\right) \quad \text{and} \quad 2\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k)\right)$$

respectively. An upper bound for the Lebesgue measure of $\mathbf{S}^*_{\psi_o}(k)$ can be easily deduced. We have

$$\lambda\left(\mathbf{S}^*_{\psi_o}(k)\right) \leq 2^3\,|q_k| \left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k)\right)\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k)\right) \tag{4.7}$$

for every $k \geq 1$.

We wish to show that $\mathbf{S}^*_{\psi_o}(k)$ covers $\mathbf{R}^*_{\psi_o}(k)$. Since the rectangles of $\mathbf{S}^*_{\psi_o}(k)$ are larger than those of $\mathbf{R}^*_{\psi_o}(k)$, any rectangle of $\mathbf{R}^*_{\psi_o}(k)$ centred at a point $q'\mathbf{x}$ with $n_{k-1} < |q'| \leq |q_k|$ will automatically be contained in the corresponding rectangle of $\mathbf{S}^*_{\psi_o}(k)$. Therefore, it will suffice to check that any rectangle of $\mathbf{R}^*_{\psi_o}(k)$ centred at a point $q'\mathbf{x}$ with $|q_k| < |q'| \leq n_k$ is covered by some rectangle of $\mathbf{S}^*_{\psi_o}(k)$. By construction, we have $|q_k| < n_k$ and so rectangles of this type are present in every collection $\mathbf{R}^*_{\psi_o}(k)$.

For each of the integers q' with $|q_k| < |q'| \leq n_k$ we can find a natural number m such that $|q' - mq_k| \leq |q_k|$. This implies there must exist a rectangle in $\mathbf{S}^*_{\psi_o}(k)$ that is centred at the point $(q' - mq_k)\mathbf{x}$. Now, m can always be chosen in a way such that $|mq_k| < |q'|$. It follows that

$$|m| < \frac{|q'|}{|q_k|} \leq \frac{n_k}{|q_k|}. \tag{4.8}$$

Consider the distance between the points $q'\mathbf{x}$ and $(q' - mq_k)\mathbf{x}$. We have

$$\begin{aligned} \|q'x_1 - (q' - mq_k)x_1\| \;&= \|-mq_kx_1\| &&\leq |m|\,\|q_kx_1\| \\ &&&\overset{(4.2)}{<} |m| \left(\frac{c_k}{|q_k|}\right)^i \\ &&&\overset{(4.8)}{<} \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i. \end{aligned}$$

Similarly,

$$\|q'x_2 - (q' - mq_k)x_2\| < \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j.$$

Combining these two inequalities yields that any rectangle of $\mathbf{R}^*_{\psi_o}(k)$ centred at a point $q'\mathbf{x}$ with $|q_k| < |q'| \leq n_k$ is contained in a rectangle in $\mathbf{S}^*_{\psi_o}(k)$ centred at $(q' - mq_k)\mathbf{x}$. This shows that $\mathbf{S}^*_{\psi_o}(k)$ is a cover for $\mathbf{R}^*_{\psi_o}(k)$ and so

$$\sum_{k=1}^{\infty} \lambda\left(\mathbf{R}^*_{\psi_o}(k)\right) \quad \leq \quad \sum_{k=1}^{\infty} \lambda\left(\mathbf{S}^*_{\psi_o}(k)\right).$$

Estimate (4.7) yields that the RHS is bounded above by

$$\begin{aligned} &\sum_{k=1}^{\infty} 8\,|q_k| \left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k)\right)\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k)\right) \\ =\;& \sum_{k=1}^{\infty} 8\,|q_k| \left((c_k)^{-\frac{1}{3}}(c_k)^i\,|q_k|^{-i} + |q_k|^{-i}(c_k)^{\frac{i}{3}}\right) \\ &\qquad \times \left((c_k)^{-\frac{1}{3}}(c_k)^j\,|q_k|^{-j} + |q_k|^{-j}(c_k)^{\frac{j}{3}}\right) \\ =\;& 8\sum_{k=1}^{\infty} |q_k|\,|q_k|^{-i-j}\left((c_k)^{i-\frac{1}{3}} + (c_k)^{\frac{i}{3}}\right)\left((c_k)^{j-\frac{1}{3}} + (c_k)^{\frac{j}{3}}\right). \end{aligned}$$

However, we have that $i+j=1$ and so this reduces to

$$\begin{aligned} & 8\sum_{k=1}^{\infty}\left((c_k)^{i+j-\frac{2}{3}}+(c_k)^{\frac{i+j}{3}}+(c_k)^{\frac{i}{3}+j-\frac{1}{3}}+(c_k)^{i+\frac{j}{3}-\frac{1}{3}}\right) \\ = \;& 8\sum_{k=1}^{\infty}\left(2\,(c_k)^{\frac{1}{3}}+(c_k)^{\frac{2i}{3}}+(c_k)^{\frac{2j}{3}}\right) \\ \le \;& 8\sum_{k=1}^{\infty}4\,(c_k)^{2\min\{i,j\}/3}. \end{aligned}$$

By assumption (4.3) this is strictly less that

$$32\sum_{k=1}^{\infty}(c_1)^{2\min\{i,j\}/3}\,2^{-(k-1)} \;=\; 64\,(c_1)^{2\min\{i,j\}/3} \quad < \quad \infty,$$

as required. □

4.2. **Proof of Theorem 2.1 (Part 2).** In this section we prove the complementary inclusion to that of Proposition 4.1.

Proposition 4.2. *For every real $i,j>0$ such that $i+j=1$,*

$$\mathbf{Bad}(i,j)\subseteq\bigcap_{\psi\in D}V(i,j,\psi).$$

Proof. We are required to show that if $\mathbf{x}\in\mathbf{Bad}(i,j)$ then for every divergent approximating function ψ we have that

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right)=1.$$

To do this we first prove the intermediary result that for every $\mathbf{x}\in\mathbf{Bad}(i,j)$ we have

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right)>0 \tag{4.9}$$

for every $\psi\in D$.

Fix $\mathbf{x}\in\mathbf{Bad}(i,j)$. By definition there exists a constant $c(\mathbf{x})>0$ such that for all natural numbers q

$$\max\left\{\|qx_1\|^{1/i},\|qx_2\|^{1/j}\right\}>\frac{c(\mathbf{x})}{q}.$$

Choose any function $\psi\in D$. To ensure that certain technical conditions required later in the proof are met we will work with a refinement of ψ. Let

$$a^*:=2^{-1/\max\{i,j\}} \qquad \text{and} \qquad a_*:=2^{-1/\min\{i,j\}},$$

then for each $r \in \mathbb{N}$ let

$$\psi_1(r) := \min\left\{\psi(r), \quad \frac{a^*}{2}, \quad \frac{a_*\, c(\mathbf{x})}{2\,|r|}\right\}.$$

Choose any integer k such that

$$k > 4, \tag{4.10}$$

and for each natural number r define

$$\psi_2(r) := \begin{cases} \psi_1(k), & r \leq k. \\ \psi_1(k^{t+1}), & k^t < r \leq k^{t+1} \qquad \text{for each } t \in \mathbb{N}. \end{cases}$$

It is easy to see that for each $r \in \mathbb{N}$

$$\psi_2(r) \leq \psi_1(r) \leq \psi(r) \tag{4.11}$$

and that $\psi_1 \in D$. It is also clear that ψ_2 is decreasing and strictly positive. Furthermore,

$$\begin{aligned} \sum_{r=1}^{\infty} \psi_2(r) &\geq \sum_{t=1}^{\infty} \sum_{r=k^t+1}^{k^{t+1}} \psi_2(r) \\ &= \sum_{t=1}^{\infty} \left(k^{t+1} - k^t\right) \psi_2(k^{t+1}) \\ &= \frac{1}{k} \sum_{t=1}^{\infty} \left(k^{t+2} - k^{t+1}\right) \psi_1(k^{t+1}) \\ &\geq \frac{1}{k} \sum_{t=1}^{\infty} \sum_{r=k^{t+1}+1}^{k^{t+2}} \psi_1(r) \\ &= \frac{1}{k} \sum_{r=k^2+1}^{\infty} \psi_1(r) \quad = \quad \infty, \end{aligned}$$

and so ψ_2 too is a divergent approximating function.

Inequality (4.11) and the characterisation of $W_{\mathbf{x}}(i,j,\psi)$ given by (4.5) guarantee that the following statement is sufficient to prove that (4.9) holds for every approximating function ψ. For every integer $r \geq 1$ we have

$$\lambda\left(\bigcup_{|q|=r+1}^{\infty} \mathbf{R}_{\psi_2}(q)\right) \geq a_*\, c(\mathbf{x})/8. \tag{4.12}$$

To prove this statement we show that there cannot exist a natural number t_0 such that (4.12) fails to hold when $r = k^{t_0}$. Assume that such a t_0 exists and consider the collection of rectangles given by

$$\mathbf{R}_t := \mathbf{R}(\psi_2, t) := \bigcup_{|q|=k^{t_o}+1}^{k^t} \mathbf{R}_{\psi_2}(q) \qquad \text{for } t = t_0+1,\ t_0+2,\ldots.$$

We will demonstrate that the measure of the collection $\mathbf{R}_t$ is unbounded as t increases. This is a contradiction as each collection $\mathbf{R}_t$ is contained in $[0,1)^2$.

By construction each collection $\mathbf{R}_{t+1}$ is obtained from $\mathbf{R}_t$ by adding $2(k^{t+1}-k^t)$ new rectangles. These new rectangles are centred at the points $q\mathbf{x}$ for which $k^t < |q| \le k^{t+1}$. Therefore, we may estimate $\lambda(\mathbf{R}_{t+1} \setminus \mathbf{R}_t)$ by finding an upper bound for the number of the new rectangles that intersect any existing rectangle of $\mathbf{R}_t$. In practice, we find an upper bound to the cardinality of the set $\mathbf{J}_{t+1} \cap \mathbf{2R}_t$, where $\mathbf{J}_{t+1}$ denotes the set of points $q\mathbf{x}$ for which $k^t < |q| \le k^{t+1}$ and

$$\mathbf{2R}_t := \bigcup_{|q|=k^{t_o}+1}^{k^t} \mathbf{R}_{2\psi_2}(q) \qquad \text{for } t = t_0+1, t_0+2,\ldots.$$

This will suffice as ψ_2 is non-increasing. Before proceeding we first notice that, since the vector $\mathbf{x}$ was chosen from $\mathbf{Bad}(i,j)$, if $q\mathbf{x}$ and $q'\mathbf{x}$ are members of $\mathbf{J}_{t+1}$ then

$$\max\left\{\|qx_1 - q'x_1\|^{1/i}, \|qx_2 - q'x_2\|^{1/j}\right\} \ge \frac{c(\mathbf{x})}{|q-q'|} \ge \frac{c(\mathbf{x})}{2k^{t+1}}, \tag{4.13}$$

providing that the integers q and q' are distinct.

The collection $\mathbf{2R}_t$ can be partitioned into two exhaustive subcollections (which we will assume without loss of generality are non-empty). Recalling that $a_* := 2^{-1/\min\{i,j\}}$, define

$$\mathbf{2R}_t^{(1)} := \bigcup \mathbf{R}_{2\psi_2}(q),$$

where the union runs over all non-zero q with $k^{t_0} < |q| \le k^t$ such that

$$2\psi_2(|q|) < \frac{a_*\, c(\mathbf{x})}{2k^{t+1}}.$$

In turn, let

$$\mathbf{2R}_t^{(2)} := \bigcup \mathbf{R}_{2\psi_2}(q),$$

where this time the union runs over q with $k^{t_0} < |q| \le k^t$ such that

$$2\psi_2(|q|) \ge \frac{a_*\, c(\mathbf{x})}{2k^{t+1}}.$$

The intersections $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(1)}$ and $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}$ will be dealt with independently.

The subcollection $\mathbf{2R}_t^{(1)}$ consists of rectangles of side lengths

$$2(2\psi_2(|q|))^i \quad \text{and} \quad 2(2\psi_2(|q|))^j$$

respectively and we have

$$2\,(2\psi_2(|q|))^i \;<\; \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^i \qquad \text{and} \qquad 2\,(2\psi_2(|q|))^j \;<\; \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

This follows upon noticing that $\max\{a_*^i, a_*^j\} = 1/2$. Thus, statement (4.13) implies at most one element of $\mathbf{J}_{t+1}$ can lie in each rectangle of $\mathbf{2R}_t^{(1)}$ and so $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(1)}$ contains at most $2(k^t - k^{t_0}) < 2k^t$ elements.

Estimating the cardinality of $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}$ requires more work and we argue as follows. If a point $\boldsymbol{\alpha}_0$ lies in the subcollection $\mathbf{2R}_t^{(2)}$ then it must lie in a rectangle of the form $\mathbf{R}_{2\psi_2}(q_0) \subseteq \mathbf{2R}_t^{(2)}$ for some integer q_0 with $k^{t_0} < |q_0| \le k^t$. This rectangle has respective side lengths $2(2\psi_2(|q_0|))^i$ and $2(2\psi_2(|q_0|))^j$ and by definition we have

$$2\,(2\psi_2(|q_0|))^i \;\ge\; 2\left(\frac{a_*\,c(\mathbf{x})}{2k^{t+1}}\right)^i \qquad \text{and} \qquad 2\,(2\psi_2(|q_0|))^j \;\ge\; 2\left(\frac{a_*\,c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

Hence, there must exist a point $\mathbf{y}(\boldsymbol{\alpha}_0) \in \mathbf{R}_{2\psi_2}(q_0)$ such that $\boldsymbol{\alpha}_0$ is contained in a subrectangle of $\mathbf{R}_{2\psi_2}(q_0)$ centred at $\mathbf{y}(\boldsymbol{\alpha}_0)$. Call this subrectangle $\mathbf{S}(\boldsymbol{\alpha}_0)$. By definition, $\mathbf{S}(\boldsymbol{\alpha}_0)$ has side lengths $(a_*\,c(\mathbf{x})/2k^{t+1})^i$ and $(a_*\,c(\mathbf{x})/2k^{t+1})^j$. The fact that $\max\{a_*^i, a_*^j\} = 1/2$, twinned with equation (4.13), once more guarantees that only one point of $\mathbf{J}_{t+1}$ may lie in any subrectangle of this type. Moreover, any two such subrectangles containing respective points $q\mathbf{x}$ and $q'\mathbf{x}$, both in $\mathbf{J}_{t+1}$, must be disjoint. Thus, the cardinality of $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}$ cannot exceed $\lambda(\mathbf{2R}_t^{(2)})/\lambda\left(\mathbf{S}(\boldsymbol{\alpha}_0)\right)$. We estimate the size of $\lambda(\mathbf{2R}_t^{(2)})$ by utilising the following lemma.

Lemma 4.3. *For every $t = t_0 + 1, t_0 + 2, \ldots,$*

$$\lambda\left(\boldsymbol{2R}_t\right) \;\le\; 2\lambda\left(\boldsymbol{R}_t\right).$$

Proof of Lemma 4.3. For $s \in \mathbb{N}$, let

$$\mathbf{R}^s := \bigcup_{|q|=k^{t_0}+1}^{k^{t_0}+s} \mathbf{R}_{\psi_2}(q) \qquad \text{and} \qquad \mathbf{2R}^s := \bigcup_{|q|=k^{t_0}+1}^{k^{t_0}+s} \mathbf{R}_{2\psi_2}(q).$$

To prove Lemma 4.3 it suffices to show that $\lambda\left(\mathbf{2R}^s\right) \le 2\lambda\left(\mathbf{R}^s\right)$ for all s. We proceed by induction. If $s = 1$, then

$$\lambda(\mathbf{R}^1) \;=\; 2\psi_2^i(k^{t_0}+1)\cdot 2\psi_2^j(k^{t_0}+1) \;=\; 4\psi_2(k^{t_0}+1).$$

Further,

$$\lambda(\mathbf{2R}^1) \;=\; 2(2\psi_2(k^{t_0}+1))^i \cdot 2(2\psi_2(k^{t_0}+1))^j \;=\; 2\cdot 4\psi_2(k^{t_0}+1) \;=\; 2\lambda(\mathbf{R}^1)$$

and the statement holds.

Next, assume the hypothesis holds when $s = s'$ and define a transformation T on the torus $[0,1)^2$ by

$$T(\boldsymbol{\alpha}) := \left(2^i\alpha_1,\ 2^j\alpha_2\right) \qquad \forall\ \boldsymbol{\alpha} \in [0,1)^2.$$

For any subset $A \subseteq [0,1)^2$, we denote by $T(A)$ the set of points $T(\boldsymbol{\alpha})$ for which $\boldsymbol{\alpha} \in A$. Let $A^{s'+1} := \mathbf{R}^{s'+1} \setminus \mathbf{R}^{s'}$, then, since ψ_2 does not exceed $a^*(i,j)/2$, we have

$$\lambda(T(A^{s'+1})) = 2^i \cdot 2^j \cdot \lambda(A^{s'+1}) = 2\lambda(A^{s'+1}). \tag{4.14}$$

It is also clear that

$$\mathbf{2R}^{s'+1} = \mathbf{2R}^{s'} \cup T(A^{s'+1}),$$

from which it follows that

$$\begin{aligned}
\lambda(\mathbf{2R}^{s'+1}) &= \lambda(\mathbf{2R}^{s'} \cup T(A^{s'+1})) \\
&\le \lambda(\mathbf{2R}^{s'}) + \lambda(T(A^{s'+1})) \\
&\le 2\lambda(\mathbf{R}^{s'}) + 2\lambda(A^{s'+1}) && \text{(by assumption and (4.14) resp.)} \\
&= 2\lambda(\mathbf{R}^{s'} \cup A^{s'+1}) && \text{(since } \mathbf{R}^{s'} \text{ and } A^{s'+1} \text{ are disjoint)} \\
&= 2\lambda(\mathbf{R}^{s'+1}).
\end{aligned}$$

$\square$

We return to our calculation. The assumption that statement (4.12) is false now implies that

$$\lambda(\mathbf{2R}_t^{(2)}) \le \lambda(\mathbf{2R}_t) \le 2\lambda(\mathbf{R}_t) < a_*\, c(\mathbf{x})/4.$$

Thus,

$$\#(\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}) \quad \le \quad \frac{\lambda(\mathbf{2R}_t^{(2)})}{\lambda(\mathbf{S}(\boldsymbol{\alpha}_0))} \quad < \quad \frac{a_*\, c(\mathbf{x})}{4\,(a_*\, c(\mathbf{x})/2k^{t+1})^{i+j}} \quad = \quad \frac{k^{t+1}}{2}$$

and we have found our second upper bound.

Recalling our intention to estimate $\lambda\left(\mathbf{R}_{t+1} \setminus \mathbf{R}_t\right)$, we can now write an upper bound for the number of rectangles added to $\mathbf{R}_t$ to make $\mathbf{R}_{t+1}$ that intersect existing rectangles of $\mathbf{R}_t$. Indeed,

$$\#\left(\mathbf{J}_{t+1} \cap \mathbf{2R}_t\right) \quad \le \quad 2k^t + k^{t+1}/2. \tag{4.15}$$

This follows upon noticing that

$$\mathbf{J}_{t+1} \cap \mathbf{2R}_t \quad = \quad (\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(1)}) \cup (\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}).$$

To complete our argument we require one final piece of notation. Let

$$\mathbf{L}_{t+1} := \{q \in \mathbb{Z}_{\neq 0} : q\mathbf{x} \in \mathbf{J}_{t+1}, q\mathbf{x} \notin \mathbf{2R}_t\}.$$

The integers $q \in \mathbf{L}_{t+1}$ each correspond to a rectangle of $\mathbf{R}_{t+1}$ that does not intersect any rectangle of $\mathbf{R}_t$. So, by (4.15)

$$\begin{aligned}
\#(\mathbf{L}_{t+1}) &\geq 2(k^{t+1} - k^t) - (2k^t + k^{t+1}/2) \\
&= (2 - 4/k - 1/2)k^{t+1} \\
&\overset{(4.10)}{>} (2 - 1 - 1/2)k^{t+1} \\
&= k^{t+1}/2. \qquad (4.16)
\end{aligned}$$

We now estimate $\lambda(\mathbf{R}_{t+1} \setminus \mathbf{R}_t)$ by considering the inclusion

$$\mathbf{R}_{t+1} \setminus \mathbf{R}_t \supset \bigcup_{q \in \mathbf{L}_{t+1}} \mathbf{R}_{\psi_2}(q). \qquad (4.17)$$

The rectangles $\mathbf{R}_{\psi_2}(q)$ in the above union have side lengths $2\psi_2^i(|q|)$ and $2\psi_2^j(|q|)$ respectively. Further, if $q, q' \in \mathbf{L}_{t+1}$ then $k^t < |q|, |q'| \leq k^{t+1}$ and so

$$\max\left\{\|qx_1 - q'x_1\|^{1/i}, \|qx_2 - q'x_2\|^{1/j}\right\} \overset{(4.13)}{\geq} \frac{c(\mathbf{x})}{2k^{t+1}}. \qquad (4.18)$$

Recall that by definition ψ_2 is constant on each $\mathbf{L}_{t+1}$, taking the value $\psi_2(k^{t+1})$. Also, recall that

$$\psi_2(r) \leq \frac{a_*\, c(\mathbf{x})}{2\,|r|}.$$

Therefore, we have

$$2\psi_2^i(|q|) = 2\psi_2^i(k^{t+1}) < \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^i$$

and

$$2\psi_2^j(|q|) = 2\psi_2^j(k^{t+1}) < \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

Combining these inequalities with statement (4.18) yields that the rectangles $\mathbf{R}_{\psi_2}(q)$ on the RHS of (4.17) are disjoint. Hence,

$$\lambda(\mathbf{R}_{t+1} \setminus \mathbf{R}_t) \geq \sum_{q \in \mathbf{L}_{t+1}} \lambda(R_{\psi_2}(q)) = 2^2 \sum_{q \in \mathbf{L}_{t+1}} \psi_2(|q|) > 2k^{t+1}\psi_2(k^{t+1})$$

by estimate (4.16). Moreover, this quantity clearly exceeds

$$2(k^{t+1} - k^t)\psi_1(k^{t+1}) = \sum_{|q|=k^t+1}^{k^{t+1}} \psi_1(k^{t+1}) = \sum_{|q|=k^t+1}^{k^{t+1}} \psi_1(|q|).$$

Finally, ψ_1 is divergent, whence $\sum_{t>t_0} \lambda\left(\mathbf{R}_{t+1} \setminus \mathbf{R}_t\right) = \infty$. Since $R_t \subseteq R_{t+1}$ for any $t > t_0$, this implies that $\lambda(R_t) \to \infty$ as $t \to \infty$. However, each set R_t is contained in $[0,1)^2$ and so a contradiction is reached. This means the assumption that (4.12) fails for some $r = k^{t_0}$ is indeed false, and consequently

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) > 0$$

for every $\psi \in D$ as desired.

To complete the proof of Proposition 4.2 we must now show if $\mathbf{x} \in \mathbf{Bad}(i,j)$ then

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) = 1$$

for every $\psi \in D$. Our method will be through the application of two lemmas, the first of which is due to Kurzweil ([18, Lemma 13]).

Lemma 4.4 (Kurzweil). *Let U and V be subsets of $[0,1)^2$. If $\lambda(U) > 0$ and V is dense in $[0,1)^2$ then $\lambda(U \oplus V) = 1$, where $U \oplus V := \{u + v \pmod 1 : \ u \in U, v \in V\}$.*

Lemma 4.5. *For every $\psi \in D$ and for every natural number s we have*

$$\sum_{r=1}^{\infty} \psi(sr) = \infty.$$

Proof of Lemma 4.5. Suppose $s \geq 1$ and for ease of notation set $\psi(0) := \psi(1)$. Consider the s-subseries $\sum_{r=0}^{\infty} \psi(sr+k)$ for each $k = 0, \ldots, s-1$. Every term $\psi(r')$, $r' \in \mathbb{N}$, appears exactly once in exactly one s-subseries. If every s-subseries had a finite sum then the original series $\sum_{r=1}^{\infty} \psi(r)$ would also have a finite sum (precisely equal to the sum of the sums of the s-subseries). Since the original series does not have a finite sum, at least one of the s-subseries must diverge, say $\sum_{r=0}^{\infty} \psi(sr+k_0) = \infty$. Since ψ is decreasing $\psi(sr) \geq \psi(sr+k_0)$ and so $\sum_{r=0}^{\infty} \psi(sr) = \infty$. $\square$

Returning to the proof of Proposition 4.2, fix a divergent approximating function ψ and a vector $\mathbf{x} \in \mathbf{Bad}(i,j)$. Once again, we will refine ψ before proceeding. Firstly, we will construct a function $\psi_3 \in D$ such that

$$\lim_{r \to \infty} \left(\frac{\psi_3(r)}{\psi(r)} \right) = 0. \tag{4.19}$$

Let $r_0 = 0$ and choose $r_1 \geq 1$ such that the inequality $\sum_{r=1}^{r_1} \psi(r) \geq 1$ holds. Then in general construct inductively a strictly increasing sequence $\{r_k\}_{k=0}^{\infty}$ such that for each k

$$\sum_{r=r_{k-1}+1}^{r_k} \psi(r) \geq k. \tag{4.20}$$

This is always possible since $\sum_{r=1}^{\infty}\psi(r)$ diverges, so the partial sums from any starting point must tend to infinity. Next, define $c_r := 1/\sqrt{k}$ if $r_{k-1} < r \le r_k$ and $\psi_3(r) := c_r\psi(r)$. Equation (4.19) therefore holds as $\psi_3(r)/\psi(r) = c_r$ tends to zero. Both ψ and $\{c_r\}$ are strictly positive and decreasing, hence ψ_3 is strictly positive and decreasing. Also, by construction, inequality (4.20) guarantees that

$$\sum_{r=r_{k-1}+1}^{r_k} \psi_3(r) \;=\; \frac{1}{k}\sum_{r=r_{k-1}+1}^{r_k} \psi(r) \;\ge\; 1,$$

and so

$$\sum_{r=1}^{r_k} \psi_3(r) \ge k.$$

This shows that the sum of ψ_3 diverges and we have verified that $\psi_3 \in D$.

By Lemma 4.5,

$$\sum_{r=1}^{\infty} \psi_3(sr) = \infty,$$

for every natural number s. Consequently, there must exist a strictly increasing sequence of natural numbers $\{s_r\}_{r\in\mathbb{N}}$ with $s_r \to \infty$ as $r \to \infty$ such that

$$\sum_{r=1}^{\infty} \psi_3(s_r \cdot r) = \infty.$$

Accordingly, we define $\psi_4(r) := \psi_3(s_r \cdot r)$. For any fixed non-zero integer q' we have that

$$\lim_{|q|\to\infty}\left(\frac{\psi_4(|q|)}{\psi(|q+q'|)}\right) = 0. \tag{4.21}$$

It is also clear that ψ_4 is a divergent approximating function and therefore we know by the intermediary result (4.9) that

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi_4)\right) \;>\; 0. \tag{4.22}$$

In addition, if we choose some vector $\mathbf{y}$ such that

$$\mathbf{y} \;\in\; W_{\mathbf{x}}(i,j,\psi_4) \;\stackrel{(4.5)}{=}\; \bigcap_{k=1}^{\infty}\bigcup_{|q|=k}^{\infty} \mathbf{R}_{\psi_4}(q),$$

then for every natural number k there are infinitely many integers q with $|q| \ge k$ such that $\mathbf{y} \in \mathbf{R}_{\psi_4}(q)$. It follows that $\mathbf{y} + q'\mathbf{x}$ is a member of the set of $\boldsymbol{\alpha} \in [0,1)^2$ for which

$$\max\left\{\|(q+q')x_1 - \alpha_1\|^{1/i},\, \|(q+q')x_2 - \alpha_2\|^{1/j}\right\} \;\le\; \psi_4(|q|)$$

for infinitely many integers q satisfying $|q| \geq k$. For large enough k, equation (4.21) implies that for each q with $|q| \geq k$ the set of $\boldsymbol{\alpha}$ defined above is contained in the rectangle $\mathbf{R}_\psi(q+q')$. It follows that $\mathbf{y}+q'\mathbf{x}$ is contained in infinitely many rectangles of the form $\mathbf{R}_\psi(q)$; i.e.,

$$\mathbf{y} + q'\mathbf{x} \in \bigcap_{k=1}^{\infty} \bigcup_{|q|=k}^{\infty} \mathbf{R}_\psi(q) \;=\; W_{\mathbf{x}}(i,j,\psi) \tag{4.23}$$

for every natural number q'.

With reference to Lemma 4.4, set

$$U := W_{\mathbf{x}}(i,j,\psi_4) \quad \text{and} \quad V := \{q\mathbf{x} : q \in \mathbb{Z}_{\neq 0}\}.$$

By equation (4.22) we have $\lambda(U) > 0$ and, as mentioned in §1.2, Kronecker's Theorem implies that V is dense in $[0,1)^2$ if $\mathbf{x}$ is irrational. Hence, Lemma 4.4 implies that $\lambda(U \oplus V) = 1$, from which equation (4.23) gives

$$\lambda\left(W_{\mathbf{x}}(i,j,\psi)\right) = 1$$

and the proof of Proposition 4.2, and indeed that of Theorem 2.1, is complete. □

5. Proof of Theorem 2.2

The proof of Theorem 2.2 makes use of the framework developed in [16]. This framework was specifically designed to provide dimension results for a broad range of badly approximable sets. In this section we show that $\mathbf{Bad}_{\mathbf{x}}(i,j)$ falls into this category when $\mathbf{x}$ is chosen from $\mathbf{Bad}(i,j)$. First, we provide a simplification of the framework tailored to our needs.

Let $\mathcal{R} := \{R_a \subset \mathbb{R}^2 : a \in J\}$ be a family of subsets R_a of $\mathbb{R}^2$ indexed by an infinite countable set J. We will refer to the sets R_a as *resonant sets*. Furthermore, it will be assumed that each resonant set takes the form of a Cartesian product; i.e., that each set R_a can be split into the images $R_{a,s} \subset \mathbb{R}$, $s = 1,2$, of its two projection maps along the two coordinate axis. Next, let $\beta : J \to \mathbb{R}_{>0} : a \mapsto \beta_a$ be a positive function on J such that the number of $a \in J$ with β_a bounded above is finite. Thus, as a runs through J the function β_a tends to infinity. Also, for $s = 1,2$, let $\rho_s : \mathbb{R}_{>0} \to \mathbb{R}_{>0} : r \mapsto \rho_s(r)$ be any real, positive, decreasing function such that $\rho_s(r) \to 0$ as $r \to \infty$. We assume that either $\rho_1(r) \geq \rho_2(r)$ or $\rho_2(r) \geq \rho_1(r)$ for large enough r. Finally, for each resonant set R_a define a rectangular neighbourhood $\mathcal{F}_a(\rho_1,\rho_2)$ by

$$\mathcal{F}_a(\rho_1,\rho_2) := \left\{\mathbf{x} \in \mathbb{R}^2 : |x_s - R_{a,s}| \leq \rho_s\left(\beta_a\right) \text{ for } s = 1,2\right\},$$

where $|x_s - R_{a,s}| := \inf_{a\in\mathcal{R}_{a,s}} |x_s - a|$.

We now introduce the general badly approximable set to which the results of [16] relate. Define $\mathbf{Bad}(\mathcal{R},\beta,\rho_1,\rho_2)$ to be the set of $\mathbf{x}\in[0,1)^2$ for which there exists a constant $c(\mathbf{x})>0$ such that

$$\mathbf{x}\notin c(\mathbf{x})\mathcal{F}_a(\rho_1,\rho_2) \qquad \forall\, a\in J.$$

That is, $\mathbf{x}\in\mathbf{Bad}(\mathcal{R},\beta,\rho_1,\rho_2)$ if there exists a constant $c(\mathbf{x})>0$ such that for all $a\in J$

$$|x_s - R_{a,s}| \ \geq\ c(\mathbf{x})\rho_s\left(\beta_a\right) \qquad (s=1,2).$$

The aim of the framework is to determine conditions under which the general set $\mathbf{Bad}(\mathcal{R},\beta,\rho_1,\rho_2)$ has full Hausdorff dimension. With this in mind, we begin with some useful notation. For any fixed integers $k>1$ and $t\geq 1$, define

$$F_t := \left\{\mathbf{x}\in[0,1)^2 : |x_s - c_s| \leq \rho_t(k^t) \text{ for each } s=1,2\right\}$$

to be the generic closed rectangle in $[0,1)^2$ with centre $\mathbf{c}:=(c_1,c_2)$ and of side lengths given by $2\rho_1(k^t)$ and $2\rho_2(k^t)$ respectively. Next, for any $\theta\in\mathbb{R}_{>0}$, let

$$\theta F_t := \left\{\mathbf{x}\in[0,1)^2 : |x_s - c_s| \leq \theta\rho_s(k^t) \text{ for each } s=1,2\right\}$$

denote the rectangle F_t scaled by θ. Finally, let

$$J(t) := \left\{a\in J : k^{t-1}\leq\beta_a<k^t\right\}.$$

The following statement is a simplification of Theorem 2 of [16], made possible by the properties of two-dimensional Lebesgue measure λ.

Theorem KTV (2006). *Let k be sufficiently large. Suppose there exists some $\theta\in\mathbb{R}_{>0}$ such that for any $t\geq 1$ and any rectangle F_t there exists a collection $\mathcal{C}(\theta F_t)$ of disjoint rectangles $2\theta F_{t+1}$ contained within θF_t such that*

$$\#\mathcal{C}(\theta F_t) \geq \kappa_1 \frac{\lambda\left(\theta F_t\right)}{\lambda\left(\theta F_{t+1}\right)} \tag{5.1}$$

and

$$\begin{aligned}\#\left\{2\theta F_{t+1}\subset\mathcal{C}(\theta F_t) : R_a\cap 2\theta F_{t+1}\neq\emptyset \text{ for some } a\in J(t+1)\right\}&\\ \leq\kappa_2\frac{\lambda\left(\theta F_t\right)}{\lambda\left(\theta F_{t+1}\right)},&\end{aligned} \tag{5.2}$$

where $0<\kappa_2<\kappa_1$ are absolute constants independent of k and t. Suppose that

$$\dim\left(\cup_{a\in J}R_a\right) < 2, \tag{5.3}$$

then

$$\dim\left(\mathbf{Bad}(\mathcal{R},\beta,\rho_1,\rho_2)\right) = 2.$$

We can now prove Theorem 2.2.

Proof of Theorem 2.2. Fix two positive reals i, j with $i + j = 1$ and some $\mathbf{x} \in \mathbf{Bad}(i, j)$. It is once more assumed that $i, j > 0$, for in this case the theorem would otherwise follow immediately from Corollary 1 of [4]. With reference to the above framework, set

$$\begin{aligned} &J := \{q \in \mathbb{Z}_{\neq 0}\}, \quad a := q \in J, \quad R_a := R_q = \left\{q\mathbf{x} + \mathbf{p} : \mathbf{p} \in \mathbb{Z}^2\right\} \\ &\beta_a := \beta_q = |q|, \quad \rho_1(r) := 1/r^i \quad \text{and} \quad \rho_2(r) := 1/r^j. \end{aligned}$$

By design we then have

$$\mathbf{Bad}(\mathcal{R},\beta,\rho_1,\rho_2) = \mathbf{Bad}_{\mathbf{x}}(i,j)$$

and so the proof is reduced to showing that the conditions of Theorem KTV are satisfied.

For $k > 1$ and $t \geq 1$, let F_t be a generic closed rectangle with centre in $[0, 1)^2$ and of side lengths $2k^{-ti}$ and $2k^{-tj}$ respectively . For k sufficiently large and any $\theta \in \mathbb{R}_{>0}$ it is clear that there exists a collection $\mathcal{C}(\theta B_t)$ of closed rectangles $2\theta F_{t+1}$ within θF_t each of side lengths $4\theta k^{-(t+1)i}$ and $4\theta k^{-(t+1)j}$ respectively. Moreover, the number of rectangles in this collection exceeds

$$\left\lfloor \frac{2\theta k^{-ti}}{4\theta k^{-(t+1)i}} \right\rfloor \times \left\lfloor \frac{2\theta k^{-tj}}{4\theta k^{-(t+1)j}} \right\rfloor .$$

Here, the notation $\lfloor\,.\,\rfloor$ denotes the integer part. For large enough k the above is strictly positive and is bounded below by

$$\begin{aligned} \frac{1}{2}\left(\frac{2\theta k^{-ti}}{4\theta k^{-(t+1)i}}\right) \times \frac{1}{2}\left(\frac{2\theta k^{-tj}}{4\theta k^{-(t+1)j}}\right) &= \frac{1}{16}\left(\frac{4\theta^2 k^{-t(i+j)}}{4\theta^2 k^{-(t+1)(i+j)}}\right) \\ &= \frac{1}{16}\frac{\lambda(\theta F_t)}{\lambda(\theta F_{t+1})}. \end{aligned}$$

Hence, inequality (5.1) holds with $\kappa_1 := 1/16$.

We endeavour to show that the additional condition (5.2) on the collection $\mathcal{C}(\theta F_t)$ is satisfied. To this end, we fix $t \geq 1$ and proceed as follows. Choose two members of distinct moduli from the set $J(t + 1)$; i.e., choose two integers q and q' such that

$$k^t \leq |q'| < |q| < k^{t+1}. \tag{5.4}$$

Associated with the integers q and q' are the resonant sets R_q and $R_{q'}$, whose elements take the form $q\mathbf{x}+\mathbf{p}$ and $q'\mathbf{x}+\mathbf{p}'$ respectively (for some $\mathbf{p},\mathbf{p}' \in \mathbb{Z}^2$). Consider the minimum distance between a point in R_q and one in $R_{q'}$. For $s=1,2$,

$$\begin{aligned} |(qx_s+p_s)-(q'x_s+p'_s)| &= |(q-q')x_s+p_s-p'_s| \\ &\geq \|(q-q')x_s\|. \end{aligned}$$

Since $\mathbf{x} \in \mathbf{Bad}(i,j)$ either

$$\|(q-q')x_1\| \quad \geq \quad \left(\frac{c(\mathbf{x})}{|q-q'|}\right)^i \quad \overset{(5.4)}{>} \quad \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^i$$

or

$$\|(q-q')x_2\| \quad \geq \quad \left(\frac{c(\mathbf{x})}{|q-q'|}\right)^j \quad \overset{(5.4)}{>} \quad \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

Therefore, if we set

$$\theta := \frac{1}{2}\min\left\{\left(\frac{c(\mathbf{x})}{2k}\right)^i, \left(\frac{c(\mathbf{x})}{2k}\right)^j\right\}$$

then the rectangle θF_t has respective side lengths

$$2\theta k^{-ti} \quad = \quad \min\left\{\left(\frac{c(\mathbf{x})}{2k}\right)^i, \left(\frac{c(\mathbf{x})}{2k}\right)^j\right\} k^{-ti} \quad \leq \quad \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^i$$

and

$$2\theta k^{-tj} \quad = \quad \min\left\{\left(\frac{c(\mathbf{x})}{2k}\right)^i, \left(\frac{c(\mathbf{x})}{2k}\right)^j\right\} k^{-tj} \quad \leq \quad \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

So, for any two integers q, q' of distinct moduli in $J(t+1)$, if a member of R_q lies in θF_t then no members of $R_{q'}$ may lie in θF_t. Only one point of R_q may lie in θF_t (since $\lambda(\theta F_t) < 1$) and so only two points over all possible resident sets may lie in any rectangle θF_t; those corresponding to q and $-q$. Hence,

$$\#\left\{2\theta F_{t+1} \subset \mathcal{C}(\theta F_t) : R_q \cap 2\theta F_{t+1} \neq \emptyset \text{ for some } q \in J(t+1)\right\} \leq 2,$$

which for large enough k is certainly less than

$$\frac{k}{32} \quad = \quad \frac{1}{32}\frac{\lambda(\theta F_t)}{\lambda(\theta F_{t+1})}.$$

So, with θ as defined above and with $\kappa_2 := 1/32 < \kappa_1$, the collection $\mathcal{C}(\theta F_t)$ satisfies inequality (5.2).

Finally, note that the family $\mathcal{R}$ of resonant sets takes the form of a countable number of countable sets and so

$$\dim\left(\cup_{q\in J} R_q\right) = 0$$

and inequality (5.3) trivially holds. Thus, the conditions of Theorem KTV are satisfied and the theorem follows. $\square$

6. Appendix

We conclude the paper by proving a general result implying Theorems 1.5 & 3.3 as stated in the main body of the paper. The result is an extension of Cassels' inhomogeneous Khintchine-type theorem [6, Chapter VII, Theorem II]. The proof is a modification of Cassels' original argument and also borrows ideas from the work of Gallagher. It should also be compared with result of Schmidt [24] and Sprindzuk [27].

Theorem 6.1. *For any sequence $\{A_q\}_{q\in\mathbb{N}}$ of measurable subsets of $[0,1)^n$ let A denote the set of all pairs $(\mathbf{x},\boldsymbol{\alpha})\in[0,1)^n\times[0,1)^n$ for which there exists infinitely many $q\in\mathbb{N}$ and $\mathbf{p}\in\mathbb{Z}^n$ such that*

$$q\mathbf{x}-\boldsymbol{\alpha}-\mathbf{p}\in A_q. \tag{6.1}$$

Then,

$$\lambda_{2n}(A):=\begin{cases}0, & \sum_{r=1}^{\infty}\lambda_n(A_r)<\infty.\\ 1, & \sum_{r=1}^{\infty}\lambda_n(A_r)=\infty.\end{cases}$$

Proof. We begin by considering the case in which the sum $\sum_{r=1}^{\infty}\lambda_n(A_r)$ converges. Fix $\boldsymbol{\alpha}\in[0,1)^n$. For each natural number q a vector $\mathbf{x}$ satisfying (6.1) uniquely determines the integral vector $\mathbf{p}$ in such a way that $|\mathbf{p}|<q$. Therefore, the measure of the set of all $\mathbf{x}\in[0,1)^n$ that satisfy (6.1) for each q is given by

$$\lambda_n\left(\bigcup_{\mathbf{p}\in[0,\,q)^n}\frac{(A_q\oplus\boldsymbol{\alpha})\oplus\mathbf{p}}{q}\right)=\sum_{\mathbf{p}\in[0,\,q)^n}\lambda_n\left(\frac{(A_q\oplus\boldsymbol{\alpha})\oplus\mathbf{p}}{q}\right),$$

since the union is disjoint. The dilation property of λ_n yields that this is equivalent to

$$q^{-n}\sum_{\mathbf{p}\in[0,\,q)^n}\lambda_n\left((A_q\oplus\boldsymbol{\alpha})\oplus\mathbf{p}\right)=q^{-n}q^n\cdot\lambda_n\left(A_q\oplus\boldsymbol{\alpha}\right)=\lambda_n\left(A_q\right),$$

by the translational invariance of λ_n. Now, if $\sum_{r=1}^{\infty}\lambda_n(A_r)<\infty$, then for any $\epsilon>0$ the set of vectors satisfying (6.1) for any $q\geq Q$ has measure at most $\sum_{q\geq Q}\lambda_n(A_q)<\epsilon$ for large enough Q. In particular, the set of $\mathbf{x}$ with infinitely many solutions to (6.1) has measure at most ϵ. This completes the proof of the convergence case.

Let us now assume that the sum $\sum_{r=1}^{\infty} \lambda_n(A_r)$ diverges. Define the function $a_q : \mathbb{R}^n \to \mathbb{R}$ for each natural number q as follows. Let

$$a_q(\mathbf{x}) := \begin{cases} 1, & \exists\, \mathbf{p} \in \mathbb{Z}^n \text{ s.t. } \mathbf{x} - \mathbf{p} \in A_q. \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that each a_q is measurable since it is equivalent to the characteristic function of a countable union of measurable sets in $\mathbb{R}^n$. Next, for every natural number Q define the function $\mathcal{A}_Q : [0,1)^n \times [0,1)^n \to \mathbb{R}$ by

$$\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\alpha}) := \sum_{q \le Q} a_q(q\mathbf{x} - \boldsymbol{\alpha}).$$

We wish to verify that $\mathcal{A}_Q$ is measurable. To that end, we introduce the following lemma, which is a generalisation of a well known result in measure theory and follows via simple modification of the classical proof (see for example [26, Chapter 2, Proposition 3.9]).

Lemma 6.2. *If f is a measurable function on $\mathbb{R}^n$ then it follows that the function $F_q(\mathbf{x}, \boldsymbol{\alpha}) := f(q\mathbf{x} - \boldsymbol{\alpha})$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$ for every natural number q.*

Since a_q is finite valued (and finite sums of finite valued measurable functions are measurable functions) Lemma 6.2 implies that $\mathcal{A}_Q$ is indeed measurable on $[0,1)^n \times [0,1)^n$. Furthermore, by construction, it is apparent that $\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\alpha})$ is simply the number of natural q with $q \le Q$ such that

$$q\mathbf{x} - \boldsymbol{\alpha} - \mathbf{p} \in A_q \qquad \text{for some } \mathbf{p} \in \mathbb{Z}^n.$$

Hence, to complete the proof of Theorem 6.1 it suffices to show $\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\alpha}) \to \infty$ almost everywhere as $Q \to \infty$. We will hereafter consider $\mathcal{A}_Q$ as a random variable in a probability space with probability measure λ_n.

For any positive measurable function $f : [0,1)^n \times [0,1)^n \to \mathbb{R}^n_{\ge 0}$ we denote the *expectation* of f by

$$E(f) := \int_{[0,1)^n} \int_{[0,1)^n} f(\mathbf{x}, \boldsymbol{\alpha})\, d\mathbf{x}\, d\boldsymbol{\alpha}.$$

If the *variance* $V(f) := E(f^2) - E(f)^2$ of f is finite then the famous Paley-Zygmund inequality (see for example [13, Ineq. II, p.8]) states that

$$\lambda_n\left(\{(\mathbf{x}, \boldsymbol{\alpha}) : f(\mathbf{x}, \boldsymbol{\alpha}) \ge \epsilon E(f)\}\right) \;\ge\; (1-\epsilon)^2 \frac{(E(f))^2}{E(f^2)},$$

for any sufficiently small $\epsilon > 0$. We will use this inequality to reach our desired conclusion.

Before applying the Paley-Zygmund inequality to $\mathcal{A}_Q$ we must show that $V(\mathcal{A}_Q)$ is finite. It suffices to show that both $E(\mathcal{A}_Q)$ and $E((\mathcal{A}_Q)^2)$ are finite. To do this we require the following lemma [6, Chapter VII, Lemma 3].

Lemma 6.3 (Cassels)**.** *Let a be a measurable function of period one of the variable $\mathbf{x} \in \mathbb{R}^n$. Then,*

$$\int_{[0,1)^n} a(q\mathbf{x} + \boldsymbol{\alpha})\, d\mathbf{x} \;=\; \int_{[0,1)^n} a(\mathbf{x})\, d\mathbf{x},$$

for any vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ and any integer $q \neq 0$.

We note that a_q is of period one and so

$$\begin{aligned}
E(\mathcal{A}_Q) &= \int_{[0,1)^n}\int_{[0,1)^n} \mathcal{A}_Q(\mathbf{x},\boldsymbol{\alpha})\, d\mathbf{x}\, d\boldsymbol{\alpha} \\
&= \sum_{q\le Q}\int_{[0,1)^n}\int_{[0,1)^n} a_q(q\mathbf{x}-\boldsymbol{\alpha})\, d\mathbf{x}\, d\boldsymbol{\alpha} \\
&\overset{\text{Lem. 6.3}}{=} \sum_{q\le Q}\int_{[0,1)^n}\int_{[0,1)^n} a_q(\mathbf{x})\, d\mathbf{x}\, d\boldsymbol{\alpha} \\
&= \sum_{q\le Q}\int_{[0,1)^n}\int_{[0,1)^n} \chi_{A_q}(\mathbf{x})\, d\mathbf{x}\, d\boldsymbol{\alpha} \\
&= \sum_{q\le Q}\lambda_n(A_q),
\end{aligned} \tag{6.2}$$

which is indeed finite. Further,

$$\begin{aligned}
E((\mathcal{A}_Q)^2) &= \int_{[0,1)^n}\int_{[0,1)^n} (\mathcal{A}_Q(\mathbf{x},\boldsymbol{\alpha}))^2\, d\mathbf{x}\, d\boldsymbol{\alpha} \\
&= \sum_{q,r\le Q}\int_{[0,1)^n}\int_{[0,1)^n} a_q(q\mathbf{x}-\boldsymbol{\alpha})a_r(r\mathbf{x}-\boldsymbol{\alpha})\, d\mathbf{x}\, d\boldsymbol{\alpha} \\
&= \sum_{q,r\le Q}\int_{[0,1)^n}\int_{[0,1)^n} a_{r-s}(-\boldsymbol{\alpha}')a_r(s\mathbf{x}'-\boldsymbol{\alpha}')\, d\mathbf{x}'\, d\boldsymbol{\alpha}',
\end{aligned}$$

via the change of variables $\mathbf{x}' := \mathbf{x}$, $\boldsymbol{\alpha}' := \boldsymbol{\alpha} - q\mathbf{x}$ and $s := r - q$. Here, the range of $\mathbf{x}'$ and $\boldsymbol{\alpha}'$ can both be taken as $[0,1)^n$ since the function a_q is periodic. Let

$$\mathcal{A}^{(r,s)}(\mathbf{x}',\boldsymbol{\alpha}') := \int_{[0,1)^n}\int_{[0,1)^n} a_{r-s}(-\boldsymbol{\alpha}')a_r(s\mathbf{x}'-\boldsymbol{\alpha}')\, d\mathbf{x}'\, d\boldsymbol{\alpha}'.$$

Then, if $r = q$ then $s = 0$ and we have

$$\begin{aligned}
\mathcal{A}^{(r,s)}(\mathbf{x}',\boldsymbol{\alpha}') &= \int_{[0,1)^n}\int_{[0,1)^n} (a_q(-\boldsymbol{\alpha}'))^2 \, d\mathbf{x}'\, d\boldsymbol{\alpha}' \\
&= \int_{[0,1)^n}\int_{[0,1)^n} a_q(-\boldsymbol{\alpha}')\, d\mathbf{x}'\, d\boldsymbol{\alpha}' \\
&= \lambda_n(A_q).
\end{aligned}$$

However, if $r \neq q$ then $s \neq 0$ and we get

$$\begin{aligned}
\mathcal{A}^{(r,s)}(\mathbf{x}',\boldsymbol{\alpha}') &= \int_{[0,1)^n} a_{r-s}(-\boldsymbol{\alpha}')d\mathbf{x}' \int_{[0,1)^n}\int_{[0,1)^n} a_r(s\mathbf{x}'-\boldsymbol{\alpha}')\, d\mathbf{x}'\, d\boldsymbol{\alpha}' \\
&\overset{\text{Lem. 6.3}}{=} \lambda_n(A_{r-s})\int_{[0,1)^n}\int_{[0,1)^n} a_r(\mathbf{x}')\, d\mathbf{x}'\, d\boldsymbol{\alpha}' \\
&= \lambda_n(A_q)\lambda_n(A_r).
\end{aligned}$$

These equivalences yield that

$$\begin{aligned}
E((\mathcal{A}_Q)^2) &= \sum_{q,r\leq Q} \mathcal{A}^{(r,s)}(\mathbf{x}',\boldsymbol{\alpha}') \\
&= \sum_{q\leq Q}\lambda_n(A_q) + \sum_{\substack{q,r\leq Q:\\ q\neq r}} \lambda_n(A_q)\lambda_n(A_r) \\
&\leq \sum_{q\leq Q}\lambda_n(A_q) + \left(\sum_{q\leq Q}\lambda_n(A_q)\right)^2 \\
&\leq (1-\epsilon)^{-2}\left(\sum_{q\leq Q}\lambda_n(A_q)\right)^2 \\
&= (1-\epsilon)^{-2}\left(E(\mathcal{A}_Q)\right)^2,
\end{aligned}$$

for any sufficiently small $\epsilon > 0$ and large enough Q (because $\sum_{q\leq Q}\lambda_n(A_q) \to \infty$ as $Q \to \infty$ by assumption). Note that the final bound is finite as required.

In view of the Paley-Zygmund inequality we have that

$$\lambda_n\left(\left\{(\mathbf{x},\boldsymbol{\alpha}) : \mathcal{A}_Q(\mathbf{x},\boldsymbol{\alpha}) \geq \epsilon\sum_{q\leq Q}\lambda_n(A_q)\right\}\right) \;\geq\; (1-\epsilon)^4 \;\geq\; 1-4\epsilon.$$

Finally, since $\mathcal{A}_Q$ increases monotonically with Q, we have that $\mathcal{A}_Q(\mathbf{x},\boldsymbol{\alpha}) \to \infty$ in $[0,1)^n \times [0,1)^n$ except on a set of measure at most 4ϵ. This completes the proof as the choice of ϵ is arbitrary. $\square$

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Paper III

A NOTE ON WEIGHTED BADLY APPROXIMABLE LINEAR FORMS

STEPHEN HARRAP AND NIKOLAY MOSHCHEVITIN

Abstract. We prove a result in the area of twisted Diophantine approximation related to the theory of Schmidt games.

1. Introduction

In 2007, Kim [9] proved that for any irrational x the set of real $\alpha \in [0,1)$ for which

$$\liminf_{q\to\infty} q\,\|qx-\alpha\| \;=\; 0 \qquad (1.1)$$

has full Lebesgue measure. Here and throughout, $\|\,.\,\|$ denotes the distance to the nearest integer. This inspired investigation into the complementary null set

$$\mathbf{Bad}_x \;=\; \left\{\alpha \in [0,1) : \inf_{q\in\mathbb{N}} q\,\|qx-\alpha\| > 0\right\},$$

often referred to as the set of *twisted badly approximable numbers*. In 2010 it was shown by Bugeaud, the first author, Kristensen & Velani [2] that this set, and indeed its natural generalisation to n linear forms in m variables, is of full Hausdorff dimension. Moreover, Tseng [19] proved shortly after that $\mathbf{Bad}_x$ enjoys the stronger property of being winning (in the sense of Schmidt[1]) for all real numbers x.

In this note we consider the following generalisation of the set $\mathbf{Bad}_x$, which incorporates the idea of 'weighting' each component of approximation. This roughly corresponds to badly approximable points avoiding 'rectangular' neighbourhoods of rational vectors rather than avoiding 'square' ones. Let x_{ji} $(1 \le i \le m, 1 \le j \le n)$ be real numbers and let

$$L_j(\mathbf{q}) \;=\; \sum_{i=1}^{m} q_i x_{ji} \qquad\qquad (1 \le j \le n)$$

be the related system of n homogeneous linear forms in the variables $q_1, \ldots, q_m$. Denote by $\mathcal{L}$ the $n \times m$ real matrix corresponding to the real numbers x_{ji} and by $\mathrm{Mat}_{n\times m}(\mathbb{R})$ the set of all such matrices. Then, for any n-tuple of real numbers

[1]We refer the reader to [16] and [17] for all necessary definitions and results on winning sets.

$\mathbf{k} = \{k_1, \ldots, k_n\}$ such that

$$k_j \; > \; 0 \quad (1 \leq j \leq n) \quad \text{and} \quad \sum_{j=1}^{n} k_j \; = \; 1, \tag{1.2}$$

we consider the set

$$\mathbf{Bad}_{\mathcal{L}}(\mathbf{k}, n, m) \; = \; \left\{ \boldsymbol{\alpha} \in [0,1)^n : \inf_{\mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}}} \max_{1 \leq j \leq n} \left(|\mathbf{q}|^{mk_j} \left\| L_j(\mathbf{q}) - \alpha_j \right\| \right) > 0 \right\}.$$

Here, $|\mathbf{x}|$ denotes the maximum of the absolute values $|x_j|$ for any $\mathbf{x} \in \mathbb{R}^k$. For brevity, we will simply write $\mathbf{Bad}_{\mathcal{L}}(n,m)$ in the classical case '$k_1 = \cdots = k_n = 1/n$'.

Recently, Einsiedler & Tseng [6] extended the results of [2] and [19] to show, amongst other related results, that the set $\mathbf{Bad}_{\mathcal{L}}(n,m)$ is winning for any matrix $\mathcal{L} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ (see also [10] and [14]). However, their result does not extend to the weighted setting.

Schmidt was the first to consider a weighted version of the badly approximable numbers. In [18], he introduced sets of the form

$$\mathbf{Bad}(i,j) \; = \; \left\{ (x_1, x_2) \in [0,1)^2 : \inf_{q \in \mathbb{Z}_{\neq 0}} \max \left\{ |q|^i \left\| q x_1 \right\|, \, |q|^j \left\| q x_2 \right\| \right\} > 0 \right\},$$

for real numbers $i, j > 0$ satisfying $i + j = 1$. Whilst a metric theorem of Khintchine [8] implies that these sets are of Lebesgue measure zero, Schmidt noted that each set is certainly non-empty. Much later, building on the earlier work of Davenport [5], it was proven by Pollington & Velani [15] that the sets $\mathbf{Bad}(i,j)$ are always of full Hausdorff dimension. Subsequently, Badziahin, Pollington & Velani [1] have solved a famous conjecture made by Schmidt in [18] concerning the intersection of any two of the sets.

Inspired by these developments, and those of [2], the following statement was proven in [7].

Theorem A (2011). *For any real $i, j > 0$ satisfying $i+j = 1$ and any $\mathbf{x} \in \mathbf{Bad}(i,j)$, the twisted inhomogeneous set*

$$\mathbf{Bad}_{\mathbf{x}}(i,j) \; = \; \left\{ (\alpha_1, \alpha_2) \in [0,1)^2 : \inf_{q \in \mathbb{Z}_{\neq 0}} \max \left\{ |q|^i \left\| q x_1 - \alpha_1 \right\|, \, |q|^j \left\| q x_2 - \alpha_2 \right\| \right\} > 0 \right\}$$

is of full Hausdorff dimension.

The conclusion is non-trivial as the sets $\mathbf{Bad}_{\mathbf{x}}(i,j)$ are also of Lebesgue measure zero. The purpose of this note is to extend this result to the full linear forms setting and to strengthen the statement from full Hausdorff dimension to winning. To do this we are required to define one final badly approximable set, the natural generalisation of $\mathbf{Bad}(i,j)$. For any n-tuple of real numbers $\mathbf{k}$ satisfying (1.2) let

$$\mathbf{Bad}(\mathbf{k},n,m) \;=\; \left\{\mathcal{L}\in \mathrm{Mat}_{n\times m}(\mathbb{R}) : \inf_{\mathbf{q}\in\mathbb{Z}^m_{\neq\mathbf{0}}} \max_{1\leq j\leq n} \left(|\mathbf{q}|^{mk_j}\,\|L_j(\mathbf{q})\|\right) > 0\right\}.$$

This set is also known to have zero Lebesgue measure and full Hausdorff dimension [11].

1.1. **Statement of Results.** In this note we prove the following strengthening of Theorem A.

Theorem 1.1. *For any n-tuple $\mathbf{k}$ satisfying (1.2) and any matrix $\mathcal{L}\in\mathbf{Bad}(\mathbf{k},n,m)$ the set $\mathbf{Bad}_{\mathcal{L}}(\mathbf{k},n,m)$ is $1/2$ winning.*

We prove Theorem 1.1 by adapting the proof of Theorem X (Chapter 5) of Cassels' book [4]. In short, his theorem implies that the set $\mathbf{Bad}_{\mathcal{L}}(n,m)$ is non-empty. We note that removing the assumption that $\mathcal{L}\in\mathbf{Bad}(\mathbf{k},n,m)$, whilst desirable, does not seem possible using the methods presented here. Indeed, a complete weighted analogue to the main theorem of [2] currently seems out of reach. For completion, we mention the following trivial consequence of Theorem 1.1 in the more familiar two dimensional setting.

Corollary 1.2. *For any real numbers $i,j>0$ satisfying $i+j=1$ and any vector $\mathbf{x}\in\mathbf{Bad}(i,j)$ the set $\mathbf{Bad}_{\mathbf{x}}(i,j)$ is $1/2$ winning.*

2. Proof of Theorem 1.1

For simplicity we will assume throughout that the group $G=\mathcal{L}^T\mathbb{Z}^n+\mathbb{Z}^m$ has rank $n+m$. This is because Kronecker's Theorem (see [12]) then asserts that the dual subgroup $\Gamma=\mathcal{L}\mathbb{Z}^m+\mathbb{Z}^n$ is dense in $\mathbb{R}^n$. In the degenerate case when the rank of G is strictly less than $n+m$ it is easily verified that $\{\mathcal{L}\mathbf{q} : \mathbf{q}\in\mathbb{Z}^m\}$ is restricted to at most a countable collection H of parallel, positively separated, hyperplanes in $\mathbb{R}^n$. We therefore have $\mathbb{R}^n\setminus H=\mathbf{Bad}_{\mathcal{L}}(\mathbf{k},n,m)$, from which it is easily deduced that $\mathbf{Bad}_{\mathcal{L}}(\mathbf{k},n,m)$ is winning.

In what follows

$$M_i(\mathbf{u}) \;=\; \sum_{j=1}^{n} u_j x_{ji} \qquad\qquad (1\leq i\leq m)$$

denotes the transposed set of m homogeneous linear forms in the variables $u_1,\ldots,u_n$ corresponding to the matrix $\mathcal{M}=\mathcal{L}^T$ (the dual forms to L_j). Choose a matrix $\mathcal{L}\in\mathbf{Bad}(\mathbf{k},n,m)$ and assume without loss of generality that we have $k_1=\max_{1\leq j\leq n}k_j$.

We begin by utilising the following lemma, which allows us to switch between the matrices in $\mathbf{Bad}(\mathbf{k}, n, m)$ and the related 'dual' set. The lemma is a consequence of a general transference theorem which can be found in Chapter V of Cassels' book [4].

Lemma 2.1. *Let* $\mathbf{Bad}^*(\mathbf{k}, m, n)$ *be the set of matrices* $\mathcal{M} \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ *such that*

$$\inf_{\mathbf{u}\in\mathbb{Z}^n_{\neq\mathbf{0}}} \max_{1\leq i\leq m} \left(\max_{1\leq j\leq n} \left(|\alpha_i|^{1/(mk_j)} \right) \|M_i(\mathbf{u})\| \right) > 0.$$

Then,

$$\mathcal{L} \in \mathbf{Bad}(\mathbf{k}, n, m) \quad \Longleftrightarrow \quad \mathcal{M} \in \mathbf{Bad}^*(\mathbf{k}, m, n).$$

For any $T \geq 1$ and any $(n+1)$ strictly positive real numbers $\beta_1, \ldots, \beta_{n+1}$ let

$$\Pi_T(\beta_1, \ldots, \beta_{n+1}) = \Big\{ (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^m : |u_j| \leq \beta_j T^{mk_j} \ \ (1 \leq j \leq n) \\ \text{and } \max_{1\leq i\leq m} |M_i(\mathbf{u}) - v_i| \leq \beta_{n+1} T^{-1} \Big\}.$$

For ease of notation we will hereafter consider sets of this type as genuine subsets of $\mathbb{R}^{n+m}$, the origin of which will be denoted $\mathbf{0}$. Now, since $\mathcal{L} \in \mathbf{Bad}(\mathbf{k}, n, m)$, Lemma 2.1 immediately implies there exists a constant $\gamma = \gamma(\mathcal{L}) \in (0, 1)$ such that

$$\Pi_T(1, \ldots, 1, \gamma) \cap \mathbb{Z}^{n+m} = \{\mathbf{0}\}.$$

However, the set $\Pi_T(\gamma^{-m}, 1 \ldots, 1, \gamma)$ is a convex, symmetric, closed, bounded region in space whose volume is given by

$$2\gamma^{-m} T^{mk_1} \cdot \prod_{j=2}^{n} 2\, T^{mk_j} \cdot 2^m \gamma^m T^{-1} = 2^{n+m}.$$

Therefore, by Minkowski's Convex Body Theorem (see Appendix B of [4]) we have that

$$\Pi_T(\gamma^{-m}, 1 \ldots, 1, \gamma) \cap \mathbb{Z}^{n+m} \neq \{\mathbf{0}\}.$$

This means for any $T \geq 1$ there exists at least one integer vector $\mathbf{z} = (\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^{n+m}$ such that

$$\mathbf{z} \in \Pi_T(\gamma^{-m}, 1 \ldots, 1, \gamma) \setminus \Pi_T(1, \ldots, 1, \gamma).$$

Choose such an integer vector with the smallest possible first coordinate $u_1 \geq 1$ for which $\max_{1\leq i\leq m} |M_i(\mathbf{u}) - v_i|$ attains its minimal value. Denote this vector

$$\mathbf{z}(T) = (\mathbf{u}(T), \mathbf{v}(T)) = (u_1(T), \ldots u_n(T), v_1(T), \ldots, v_m(T)),$$

and by

$$\phi(T) = \max_{1\leq i\leq m} \|M_i(\mathbf{u}(T))\| = \max_{1\leq i\leq m} |M_i(\mathbf{u}(T)) - v_i(T)|$$

the minimal value taken. Note that the rank assumption imposed on $\mathcal{L}$ ensures that $\mathbf{z}(T)$ always exists and is unique up to sign change (for similar constructions, see [13] or Section 2 of [3]).

The following set of inequalities will be useful. Since $\mathbf{z}(T) \in \Pi_T(\gamma^{-m}, 1\ldots, 1, \gamma)$ we have

$$|u_1(T)| \;\leq\; \gamma^{-m}\, T^{mk_1}, \qquad\qquad |u_j(T)| \;\leq\; T^{mk_j} \quad (2 \leq j \leq n) \tag{2.1}$$

and also

$$\phi(T) \;\leq\; \gamma\, T^{-1}. \tag{2.2}$$

Morover, since $\mathbf{z}(T) \notin \Pi_T(1, \ldots, 1, \gamma)$ we know

$$|u_1(T)| \;>\; T^{mk_1} \quad \text{and so} \quad \max_{1\leq j\leq n} \left(|u_j(T)|^{1/(mk_j)}\right) \;=\; |u_1(T)|^{1/(mk_1)}. \tag{2.3}$$

Recalling that $\mathcal{M} \in \mathbf{Bad}^*(\mathbf{k}, m, n)$, we therefore have

$$\phi(T) \;\geq\; \gamma \left(\max_{1\leq j\leq n} \left(|u_j(T)|^{1/(mk_j)}\right)\right)^{-1} \;=\; \gamma\, |u_1(T)|^{-1/(mk_1)} \;\geq\; \gamma^{1+1/k_1}\, T^{-1}. \tag{2.4}$$

Next, we prove a lemma regarding the rate of growth of a suitable sequence of the Euclidean norms of the integer vectors $\mathbf{u}(T)$ (c.f. [13, Theorem 1.2]). Put $R := \lceil \gamma^{-1/k_1} \rceil + 1$ and define $T_r = R^r$ (for $r = 0, 1, \ldots$). For notational convenience let $\mathbf{z}_r = (\mathbf{u}_r, \mathbf{v}_r) = \mathbf{z}(T_r)$ and $\phi_r = \phi(T_r)$. Inequality (2.4) yields that ϕ_r is strictly decreasing as

$$\begin{aligned} \phi_r \;\;\geq\;\; \gamma^{1+1/k_1}\, T_r^{-1} \;\;=\;\; \gamma^{1+1/k_1}\, R\, T_{r+1}^{-1} \;\;&\geq\;\; \gamma^{1+1/k_1} \left(\gamma^{-1/k_1} + 1\right) T_{r+1}^{-1} \\ &>\;\; \gamma\, T_{r+1}^{-1} \\ &\geq\;\; \phi_{r+1}. \end{aligned}$$

The final inequality follows from (2.2), which also implies

$$\phi_r \;\;\leq\;\; \gamma\, R\, T_{r+1}^{-1}. \tag{2.5}$$

This will be utilised later, as will the observation that $\phi_r \to 0$ as $r \to \infty$.

Lemma 2.2. *The sequence of vectors $\{\mathbf{u}_r\}_{r=0}^{\infty}$ can be partitioned into finitely many subsequences in such a way that the Euclidean norms of the vectors of each subsequence form a lacunary sequence.*

Proof. Consider the Euclidean norm $|\,.\,|_e$ of each integer vector $\mathbf{u}_r$. From (2.3) we have

$$\begin{aligned} T_r^{2mk_1} \;<\; |u_1(T_r)|^2 \;&\leq\; |\mathbf{u}_r|_e^2 \qquad (2.6)\\ &\overset{(2.1)}{\leq}\; \gamma^{-2m}\, T_r^{2mk_1} + \sum_{j=2}^{n} T_r^{2mk_j} \\ &<\; \gamma^{-2m} \sum_{j=1}^{n} T_r^{2mk_j} \\ &\leq\; \gamma^{-2m} n\, T_r^{2mk_1}, \qquad (2.7) \end{aligned}$$

since we are assuming $k_1 = \max_{1\leq j\leq n} k_j$. Now, choose any natural number t such that $R^{tmk_1} \geq 2n^{1/2}\gamma^{-m}$. Then,

$$|\mathbf{u}_{r+t}|_e \;\overset{(2.6)}{>}\; T_{r+t}^{mk_1} \;=\; R^{tmk_1}\, T_r^{mk_1} \;\geq\; 2n^{1/2}\gamma^{-m}\, T_r^{mk_1} \;\overset{(2.7)}{>}\; 2\,|\mathbf{u}_r|_e\,.$$

So, the sequence $\{\mathbf{u}_r\}_{r=0}^{\infty}$ can be partitioned into a finite collection of subsequences $\{\mathbf{u}_{t_0+tr}\}_{r=0}^{\infty}$ such that each subsequence is 2-lacunary; that is

$$\left|\mathbf{u}_{t_0+t(r+1)}\right|_e \;\;\geq\;\; 2\,|\mathbf{u}_{t_0+tr}|_e \quad \forall\, r.$$

$\square$

This lemma allows us to use the following powerful result, which is taken from [14].

Lemma 2.3. *If a sequence $\{\mathbf{w}_r\}_{r=0}^{\infty}$ of non-zero integral vectors is such that the corresponding sequence of Euclidean norms is lacunary then the set*

$$\left\{\boldsymbol{\alpha} \in [0,1)^n : \inf_r \|\mathbf{w}_r \cdot \boldsymbol{\alpha}\| > 0\right\}$$

is 1/2 *winning.*

Corollary 2.4. *The set*

$$\mathbf{Bad}_{\{\mathbf{u}_r\}} \;=\; \left\{\boldsymbol{\alpha} \in [0,1)^n : \inf_r \|\mathbf{u}_r \cdot \boldsymbol{\alpha}\| > 0\right\}$$

is 1/2 *winning.*

We remark that the set $\mathbf{Bad}_{\{\mathbf{u}_r\}}$ was shown in [2] to have full Hausdorff dimension for any sequence $\{\mathbf{u}_r\}_{r=0}^{\infty}$ of non-zero integral vectors whose Euclidean norms form a lacunary sequence. Corollary 2.4 follows from a result of Schmidt [16] stating that countable intersections of α-winning sets are also α-winning and the observation that

$$\mathbf{Bad}_{\{\mathbf{u}_r\}} \;=\; \bigcap_{t_0=0}^{t-1} \left\{\boldsymbol{\alpha} \in \mathbb{R}^n : \inf_r \|\mathbf{u}_{t_0+tr} \cdot \boldsymbol{\alpha}\| > 0\right\}.$$

We are now ready to prove Theorem 1.1. Choose $\boldsymbol{\alpha} \in \mathbf{Bad}_{\{\mathbf{u}_r\}}$ and assume

$$\inf_r \|\mathbf{u}_r \cdot \boldsymbol{\alpha}\| \;\geq\; \epsilon \;>\; 0.$$

For any $\mathbf{q} \in \mathbb{Z}^m_{\neq \mathbf{0}}$, the trivial equality

$$\mathbf{u}_r \cdot \boldsymbol{\alpha} \;=\; \sum_{i=1}^{m} q_i M_i(\mathbf{u}_r) \;-\; \sum_{j=1}^{n} (L_j(\mathbf{q}) - \alpha_j) u_j(T_r),$$

in conjunction with the triangle inequality yields that

$$\begin{aligned} 0 < \epsilon < \|\mathbf{u}_r \cdot \boldsymbol{\alpha}\| \;&\leq\; m \max_{1\leq i\leq m} \left(\|M_i(\mathbf{u}_r)\| \, |q_i|\right) \;+\; n \max_{1\leq j\leq n} \left(\|(L_j(\mathbf{q}) - \alpha_j)\| \, |u_j(T_r)|\right) \\ &\leq\; m\phi_r \, |\mathbf{q}| \;+\; n \max_{1\leq j\leq n} \left(\|(L_j(\mathbf{q}) - \alpha_j)\| \, |u_j(T_r)|\right). \end{aligned} \tag{2.8}$$

Here, we have employed the fact that $\|a\mathbf{z}\| \leq |a| \, \|\mathbf{z}\|$ for all $a \in \mathbb{R}$ and all $\mathbf{z} \in \mathbb{R}^k$.

Since ϕ_r is strictly decreasing and $\phi_r \to 0$ as $r \to \infty$ we are free to choose r in such a way that

$$\phi_r \;<\; \frac{\epsilon}{2m\,|\mathbf{q}|} \;\leq \phi_{r-1}, \tag{2.9}$$

whereby inequality (2.8) yields

$$\max_{1\leq j\leq n} \left(\|(L_j(\mathbf{q}) - \alpha_j)\| \, |u_j(T_r)|\right) \;\geq\; \epsilon/2n.$$

Finally, notice that combining (2.5) with (2.9) implies

$$T_r \;\leq\; 2m\epsilon^{-1}\gamma R \, |\mathbf{q}| \, ,$$

and so we have

$$|u_1(T_r)| \;\overset{(2.1)}{\leq}\; \gamma^{-m} \, T_r^{mk_1} \;\leq\; (2mR)^{mk_1} \gamma^{m(k_1-1)} \epsilon^{-mk_1} \, |\mathbf{q}| \, ,$$

and similarly (for $2 \leq j \leq n$)

$$|u_j(T_r)| \;\leq\; (2mR\gamma)^{mk_1} \epsilon^{-mk_1} \, |\mathbf{q}| \, .$$

Therefore,

$$\max_{1\leq j\leq n} \left(\|L_j(\mathbf{q}) - \alpha_j\| \, |\mathbf{q}|^{mk_j}\right) \;\geq\; \kappa,$$

for some constant $\kappa > 0$. Since the choice of vector $\mathbf{q}$ was arbitrary we have shown that $\boldsymbol{\alpha} \in \mathbf{Bad}_{\mathcal{L}}(\mathbf{k}, n, m)$, and in particular that $\mathbf{Bad}_{\{\mathbf{u}_r\}} \subseteq \mathbf{Bad}_{\mathcal{L}}(\mathbf{k}, n, m)$. In view of Corollary 2.4, the desired conclusion easily follows.

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Paper IV

THE MIXED LITTLEWOOD CONJECTURE FOR PSEUDO-ABSOLUTE VALUES

STEPHEN HARRAP AND ALAN HAYNES

Abstract. In this paper we study the Mixed Littlewood Conjecture with pseudo-absolute values. We show that if p is a prime and $\mathcal{D}$ is a pseudo-absolute value sequence satisfying mild conditions then

$$\inf_{q\in\mathbb{N}} q|q|_p|q|_{\mathcal{D}}\|qx\| = 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

Our proof relies on a measure rigidity theorem due to Lindenstrauss and lower bounds for linear forms in logarithms due to Baker and Wüstholz. We also deduce the answer to the related metric question of how fast the infimum above tends to zero, for almost every x.

1. Introduction

For $x \in \mathbb{R}$ let $\|x\|$ denote the distance from x to the nearest integer. The Littlewood Conjecture is the assertion that for every $x_1, x_2 \in \mathbb{R}$,

$$\inf_{q\in\mathbb{N}} q\,\|qx_1\|\,\|qx_2\| \;=\; 0. \tag{1.1}$$

This conjecture has come to light recently because of its connection to measure rigidity problems for diagonal actions on the space of unimodular lattices. This connection was exploited by Einsiedler, Katok, and Lindenstrauss [10] to show that the set of pairs $(x_1, x_2) \in \mathbb{R}^2$ which do not satisfy (1.1) has Hausdorff dimension zero.

More recently de Mathan and Teulié [17] have proposed a problem which is closely related to the Littlewood Conjecture. Let $\mathcal{D} = \{n_k\}_{k\geq 0}$ be an increasing sequence of positive integers with $n_0 = 1$ and $n_k|n_{k+1}$ for all k. We refer to such a sequence as a *pseudo-absolute value sequence*, and we define the *$\mathcal{D}$-adic pseudo-absolute value* $|\cdot|_{\mathcal{D}} : \mathbb{N} \to \{n_k^{-1} : k \geq 0\}$ by

$$|q|_{\mathcal{D}} = \min\{n_k^{-1} : q \in n_k\mathbb{Z}\}.$$

In the case when $\mathcal{D} = \{a^k\}_{k=0}^{\infty}$ for some integer $a \geq 2$ we also write $|\cdot|_{\mathcal{D}} = |\cdot|_a$. If p is a prime then $|\cdot|_p$ is the usual $p-$adic absolute value.

The de Mathan and Teulié Conjecture, which we will refer to as the Mixed Littlewood Conjecture, is the assertion that for any pseudo-absolute value $|\cdot|_{\mathcal{D}}$ and for every $x \in \mathbb{R}$,

$$\inf_{q\in\mathbb{N}} q\,|q|_{\mathcal{D}}\,\|qx\| \;=\; 0. \tag{1.2}$$

The distribution of values of the quantities $|q|_{\mathcal{D}}$ mimics the distribution of values of $\|qx_2\|$, for suitably chosen x_2. In the case when $\mathcal{D} = |\cdot|_a$ for some integer $a \geq 2$ the Mixed Littlewood Conjecture also has a dynamical formulation in terms of the action of a certain diagonal group on a quotient space of

$$\mathrm{SL}_2(\mathbb{R}) \times \prod_i \mathrm{SL}_2(\mathbb{Q}_{p_i}),$$

where $\{p_i\}$ is the collection of primes dividing a. By employing measure rigidity results in this setting Einsiedler and Kleinbock [11] proved that when $|\cdot|_{\mathcal{D}} = |\cdot|_a$ the set of $x \in \mathbb{R}$ which do not satisfy (1.2) has Hausdorff dimension zero.

The case of the Mixed Littlewood Conjecture with more than one $p-$adic or pseudo-absolute value has also been a topic of recent interest. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two pseudo-absolute value sequences then it is reasonable to conjecture that for any $x \in \mathbb{R}$,

$$\inf_{q\in\mathbb{N}} q|q|_{\mathcal{D}_1}|q|_{\mathcal{D}_2}\|qx\| = 0. \tag{1.3}$$

It is shown in [11] that the Furstenberg Orbit Closure Theorem [12, Theorem IV.1] implies that (1.3) is true whenever $\mathcal{D}_1 = \{a^k\}$ and $\mathcal{D}_2 = \{b^k\}$ for two multiplicatively independent integers a and b. This result was strengthened by Bourgain, Lindenstrauss, Michel, and Venkatesh [5] who proved a result which implies (see [7, Section 4.6]) that there is a constant $\kappa > 0$ such that for all $x \in \mathbb{R}$,

$$\inf_{q\in\mathbb{N}} q(\log\log\log q)^{\kappa}|q|_a|q|_b\|qx\| = 0.$$

These results rely on understanding the dynamics of semigroups of toral endomorphisms. They provide a contrast to the situation of the original Littlewood Conjecture, where nothing seems to be gained by adding more real variables.

It was pointed out by Einsiedler and Kleinbock in [11] that the dynamical machinery used to study these problems does not readily extend to the case of more general pseudo-absolute values. Our first result in this paper demonstrates how recent measure rigidity theorems can be combined with bounds for linear forms in logarithms to obtain more general results.

Theorem 1.1. *Suppose that $a \geq 2$ is an integer and that $\mathcal{D} = \{n_k\}$ is a pseudo-absolute value sequence all of whose elements are divisible by finitely many fixed primes coprime to a. If there is a $\delta \geq 0$ with*

$$\log n_k \leq k^{\delta} \quad \textit{for all} \quad k \geq 2, \tag{1.4}$$

then for any $x \in \mathbb{R}$ we have that

$$\inf_{q\in\mathbb{N}} q|q|_a|q|_{\mathcal{D}}\|qx\| = 0. \tag{1.5}$$

Our proof of this theorem is inspired in part by Furstenberg's original proof of his Orbit Closure Theorem [12], and by the ideas used by Bourgain, Lindenstrauss, Michel, and Venkatesh in [5]. Of particular interest is the case when consecutive elements of the sequence $\mathcal{D}$ have bounded ratios (cf. [1, 6, 11, 16, 17]), and we will say that $\mathcal{D}$ and $|\cdot|_{\mathcal{D}}$ have bounded ratios in this case. This roughly corresponds in the original Littlewood Conjecture to having

$$\inf_{q\in\mathbb{N}} q\|qx_2\| > 0,$$

which is indeed the only interesting case of that conjecture anyway. For the bounded ratios case our theorem gives a quite satisfactory answer to the problem at hand.

Corollary 1.1. *Suppose that $a \geq 2$ is an integer and that $\mathcal{D}$ is a pseudo-absolute value sequence with bounded ratios, all of whose elements are coprime to a. Then for any $x \in \mathbb{R}$ we have that*

$$\inf_{q\in\mathbb{N}} q|q|_a|q|_{\mathcal{D}}\|qx\| = 0.$$

After establishing Theorem 1.1 we will turn to the problem of determining the almost everywhere behaviour of the quantities on the left hand side of (1.2). The analogue of this problem for the Littlewood Conjecture was established by Gallagher [13] in the 1960's. He proved that if $\psi : \mathbb{N} \to \mathbb{R}$ is any non-negative decreasing function for which

$$\sum_{r\in\mathbb{N}} \log(r)\psi(r) = \infty \tag{1.6}$$

then for almost every $(x_1, x_2) \in \mathbb{R}^2$

$$\|qx_1\|\,\|qx_2\| \;\leq\; \psi(q) \text{ for infinitely many } q \in \mathbb{N}. \tag{1.7}$$

For example this shows that for almost every $(x_1, x_2) \in \mathbb{R}^2$ we can improve (1.1) to

$$\inf_{q\in\mathbb{N}} q(\log q)^2(\log\log q)\,\|qx_1\|\,\|qx_2\| \;=\; 0.$$

Although Gallagher's method does not readily apply to the mixed problems that we are considering, it has recently been shown using other techniques [7] that if p is a prime, if ψ is as above, and if (1.6) holds then for almost every $x \in \mathbb{R}$,

$$|q|_p\,\|qx\| \;\leq\; \psi(q) \text{ for infinitely many } q \in \mathbb{N}.$$

Here we will show how this result can be extended to non $p-$adic pseudo-absolute values $|\cdot|_{\mathcal{D}}$. The quality of approximation that we obtain will necessarily depend on the rate at which the sequence $\mathcal{D}$ grows. For this reason, given a pseudo-absolute value sequence $\mathcal{D}$ we define $\mathcal{M} : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by

$$\mathcal{M}(N) = \max\{k : n_k \leq N\}.$$

Theorem 1.2. *Suppose that $\psi : \mathbb{N} \to \mathbb{R}$ is non-negative and decreasing and that $\mathcal{D} = \{n_k\}$ is a pseudo-absolute value sequence satisfying*

$$\sum_{k=1}^{\mathcal{M}(N)} \frac{\varphi(n_k)}{n_k} \gg \mathcal{M}(N) \text{ for all } N \in \mathbb{N}, \tag{1.8}$$

where φ denotes the Euler phi function. Then for almost all $x \in \mathbb{R}$ the inequality

$$|q|_{\mathcal{D}} \, \|qx\| \leq \psi(q) \tag{1.9}$$

has infinitely (resp. finitely) many solutions $q \in \mathbb{N}$ if the sum

$$\sum_{r=1}^{\infty} \mathcal{M}(r)\psi(r) \tag{1.10}$$

diverges (resp. converges).

We also note that when (1.10) converges the inequality (1.9) always has finitely many solutions, regardless of whether or not (1.8) is satisfied.

When $|\cdot|_D = |\cdot|_p$ for some prime p we have that $\mathcal{M}(N) \asymp \log N$, and Theorem 1.2 reduces in this case to the previously mentioned result from [7]. To see what Theorem 1.2 means in terms of the infima type expressions that occur in the Mixed Littlewood Conjecture, if $\mathcal{D}$ satisfies (1.8) then for almost every $x \in \mathbb{R}$ we have that

$$\inf_{q \to \infty} q\mathcal{M}(q)(\log q)(\log\log q) \, |q|_{\mathcal{D}} \, \|qx\| \; = \; 0,$$

while on the other hand for any $\epsilon > 0$ and for almost every $x \in \mathbb{R}$,

$$\inf_{q \to \infty} q\mathcal{M}(q)(\log q)(\log\log q)^{1+\epsilon} \, |q|_{\mathcal{D}} \, \|qx\| \; > \; 0.$$

Furthermore the hypothesis on $\mathcal{D}$ in Theorem 1.2 is not that restrictive in practice. Although it is possible to choose $\mathcal{D}$ so that (1.8) does not hold, any reasonably chosen pseudo-absolute value sequence should satisfy the condition. In particular if $\mathcal{D}$ has bounded ratios or even if the elements of $\mathcal{D}$ are divisible only by some finite collection of primes then it is easy to check that (1.8) is satisfied. For the interested reader we will indicate in Section 6 how one can construct a sequence $\mathcal{D}$ for which (1.8) fails.

2. Preliminaries for the proof of Theorem 1.1

2.1. **Invariant measures for continuous transformations.** Suppose X is a compact metric space and let $\mathcal{B}$ denote the σ-algebra of Borel subsets of X. Let $\mathfrak{M} = \mathfrak{M}(X)$ be the set of all probability measures on $(X, \mathcal{B})$, and if $T : X \to X$ is a continuous map let $\mathfrak{M}_T = \mathfrak{M}_T(X)$ be the subset of $T-$invariant measures in $\mathfrak{M}$. In other words

$$\mathfrak{M}_T = \{\mu \in \mathfrak{M} \ : \ \mu(B) = \mu(T^{-1}B) \text{ for all } B \in \mathcal{B}\}.$$

The set $\mathfrak{M}$ has a natural topology coming from the Riesz Representation Theorem, and we refer to this as the weak* topology on $\mathfrak{M}$. The following basic lemma summarizes some of the important properties of this topology (see [19, Theorems 6.4, 6.5, 6.10] for proofs).

Lemma 2.1. *If X is a compact metric space then we have that:*

(i) The space $\mathfrak{M}$ is compact and metrizable in the weak* topology,
(ii) The set $\mathfrak{M}_T$ is a non-empty, closed, convex subset of $\mathfrak{M}$, and
(iii) The extreme points of $\mathfrak{M}_T$ are exactly the measures $\mu \in \mathfrak{M}$ for which T is an ergodic measure preserving transformation of (X, μ).

Let $\mathfrak{E}_T = \mathfrak{E}_T(X)$ be the subset of extreme points of $\mathfrak{M}_T(X)$. Since $\mathfrak{M}$ is metrizable and $\mathfrak{M}_T$ is compact and convex, by the Choquet Representation Theorem [18, Chapter 3] for any $\mu \in \mathfrak{M}_T$ there is a probability measure λ supported on $\mathfrak{E}_T$ with the property that

$$\mu = \int_{\mathfrak{E}_T} m \ d\lambda(m). \tag{2.1}$$

This is the *ergodic decomposition of $\mu \in \mathfrak{M}_T$*.

2.2. **Entropy and dimension.** Suppose that X is a compact metric space with metric d and that $T : X \to X$ is continuous. For $n \in \mathbb{N}$ and $\epsilon > 0$ we say that a subset $A \subseteq X$ is *(n, ϵ)-separated with respect to T* if for any $\alpha, \beta \in A, \alpha \neq \beta$, we have that

$$\max_{0 \le i \le n-1} d(T^i\alpha, T^i\beta) \ge \epsilon.$$

Let $s_n(T, \epsilon)$ be the largest cardinality of an $(n, \epsilon)-$separated subset of X with respect to T. The *topological entropy of T* is defined as

$$h_{\text{top}}(T) = \lim_{\epsilon \to 0} \ \limsup_{n \to \infty} \frac{\log s_n(T, \epsilon)}{n}.$$

Next if $\mu \in \mathfrak{M}_T$ and $\mathcal{P} \subseteq \mathcal{B}$ is a finite partition of X we set

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P),$$

and we let

$$h_\mu(T,\mathcal{P}) = \lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}\right),$$

where

$$\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P} = \left\{\bigcap_{i=0}^{n-1} T^{-i}P_i : P_0,\ldots,P_{n-1}\in\mathcal{P}\right\}.$$

The *metric entropy of T with respect to μ* is defined as

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T,\mathcal{P}),$$

where the supremum is taken over all finite partitions $\mathcal{P} \subseteq \mathcal{B}$. When there is no confusion we may also refer to $h_\mu(T)$ as the entropy of μ.

The map from $\mathfrak{M}_T$ to $[0,\infty)$ which sends μ to h_μ is affine [19, Theorem 8.1]. Also the topological and metric entropies associated to a continuous transformation are connected by the formula

$$h_{\mathrm{top}}(T) = \sup\{h_\mu(T) : \mu\in\mathfrak{M}_T\}, \tag{2.2}$$

which is known as the variational principle [19, Theorem 8.6].

A concept which is somewhat related to topological entropy is the notion of the upper box dimension of a subset $A \subseteq X$. For $\epsilon > 0$ we say that $B \subseteq A$ is $\epsilon-$separated if for any $\alpha,\beta \in B, \alpha\neq\beta$, we have that $d(\alpha,\beta)\geq\epsilon$. Let $s(A,\epsilon)$ be the largest cardinality of an $\epsilon-$separated subset of A. The *upper box dimension of A* is defined as

$$\overline{\dim}(A) = \limsup_{\epsilon\to 0}\frac{\log s(A,\epsilon)}{\log(1/\epsilon)}.$$

First we establish an elementary fact. Here and in what follows we are working in the metric space $(\mathbb{R}/\mathbb{Z}, \|\cdot\|)$.

Lemma 2.2. *For any $A \subseteq \mathbb{R}/\mathbb{Z}$ and $\epsilon > 0$ we have that*

$$s(A-A,\epsilon) \leq 2s(A,\epsilon)^2,$$

where $A - A = \{\alpha-\beta : \alpha,\beta\in A\}$.

Proof. Given $\epsilon > 0$ let $\{\alpha_1, \dots, \alpha_k\}$ be an $\epsilon-$separated subset of A with maximum cardinality. Then we have that

$$A \subseteq \bigcup_{1 \le i \le k} B(\alpha_i, \epsilon),$$

where $B(\alpha_i, \epsilon)$ denotes the open ball of radius ϵ centred at α_i. This gives that

$$A - A \subseteq \bigcup_{1 \le i,j \le k} (B(\alpha_i, \epsilon) - B(\alpha_j, \epsilon)) = \bigcup_{1 \le i,j \le k} B(\alpha_i - \alpha_j, 2\epsilon),$$

and therefore

$$s(A - A, \epsilon) \le \sum_{1 \le i,j \le k} s(B(\alpha_i - \alpha_j, 2\epsilon), \epsilon) = 2s(A, \epsilon)^2.$$

□

In our proof of Theorem 1.1 we will link upper box dimension and entropy using following lemma.

Lemma 2.3. *Suppose that $a \in \mathbb{N}, a \ge 2$, and let $T_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the map $T_a(\alpha) = a\alpha$. If $A \subseteq \mathbb{R}/\mathbb{Z}$ is a closed set satisfying $T_a(A) \subseteq A$ then for any $\epsilon > 0$ there exists a measure $\mu \in \mathfrak{M}_{T_a}(A)$ with*

$$h_\mu(T_a) \ge \overline{\dim}(A) \cdot \log a - \epsilon.$$

In particular if $\overline{\dim}(A) > 0$ then there is a measure $\mu \in \mathfrak{E}_{T_a}(A)$ with positive entropy.

Proof. Let $d = \overline{\dim}(A)$ and assume without loss of generality that $d > 0$. Choose $\{\epsilon_m\} \subseteq \mathbb{R}$, decreasing to 0, with

$$d = \lim_{m \to \infty} \frac{\log s(A, \epsilon_m)}{\log(1/\epsilon_m)}.$$

Then for any $0 < \delta < d$ there is an integer m_0 with

$$s(A, \epsilon_m) \ge (1/\epsilon_m)^{d-\delta} \text{ for all } m \ge m_0.$$

Now for the moment fix a δ and an $m \ge m_0$ and let n be the integer which satisfies $a^{-n} \le \epsilon_m < a^{-n+1}$. Then if $\{\alpha_1, \dots, \alpha_k\}$ is an ϵ_m-separated subset of A of maximum cardinality we have that $k \ge a^{(n-1)(d-\delta)}$ and that

$$\|\alpha_i - \alpha_j\| \ge a^{-n} \text{ for all } 1 \le i < j \le k.$$

It is not difficult to check that the latter condition implies that for any $i \ne j$ we can find an integer $0 \le \ell < n$ with

$$\|a^\ell \alpha_i - a^\ell \alpha_j\| \ge 1/2a.$$

In other words the set $\{\alpha_1, \ldots, \alpha_k\}$ is $(n, 1/2a)-$separated with respect to T_a. This gives that

$$\frac{\log s_n(T_a|_A, 1/2a)}{n} \geq (d-\delta)\log a - \frac{(d-\delta)\log a}{n}.$$

Now our choice for n must tend to infinity with m and this gives that

$$\limsup_{n\to\infty} \frac{\log s_n(T_a|_A, 1/2a)}{n} \geq (d-\delta)\log a.$$

Finally by letting δ tend to zero we obtain that

$$h_{\text{top}}(T_a|_A) \geq \overline{\dim}(A) \cdot \log a.$$

The first claim of the lemma then follows from the variational principle (2.2). For the second claim let μ be any measure in $\mathfrak{M}_{T_a}(A)$ with positive entropy. Using the ergodic decomposition (2.1) and the fact that entropy is affine we have that

$$h_\mu(T_a) = \int_{\mathfrak{E}_{T_a}} h_m(T_a) d\lambda(m).$$

Since this integral is positive there must be a collection of ergodic measures, of positive measure with respect to λ, which have positive entropy. This finishes the proof of the lemma. $\square$

2.3. **Diophantine approximation.** For each $c > 0$ we define $\mathrm{Bad}(c) \subseteq \mathbb{R}$ to be the collection of real numbers x which satisfy

$$\inf_{q\in\mathbb{N}} q\|qx\| \geq c.$$

We say that a real number x is *badly approximable* if $x \in \cup_{c>0}\mathrm{Bad}(c)$, and we say that x is *well approximable* otherwise. The sets $\mathrm{Bad}(c)$ are invariant under integer translation and so we also think of them, as well as the sets of badly and well approximable numbers, as subsets of $\mathbb{R}/\mathbb{Z}$.

From the classical theory of continued fractions it has long been known that almost every x, with respect to Lebesgue measure, is well approximable [3, 4]. Recently Einsiedler, Fishman, and Shapira, using a measure rigidity theorem due to Lindenstrauss [15], have shown that we may draw the same conclusion with Lebesgue measure replaced by any times-a invariant measure with positive entropy.

Theorem 2.1. [9, Theorem 1.4] *Suppose $a \in \mathbb{N}$ and let $T_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the map $T_a(\alpha) = a\alpha$. If $\mu \in \mathfrak{E}_{T_a}$ has positive entropy then $\mu-$almost every $x \in \mathbb{R}/\mathbb{Z}$ is well approximable.*

Finally we say that $a_1, \ldots, a_s \in \mathbb{N}$ are *multiplicatively independent* if the real numbers $\log a_1, \ldots, \log a_s$ are linearly independent over $\mathbb{Q}$. We will use the following deep theorem of Baker and Wüstholz on lower bounds for linear forms in logarithms.

Theorem 2.2. [2] *Suppose that $a_1, \ldots, a_s \in \mathbb{N}$ are multiplicatively independent. Then there exists a constant $\kappa > 0$, which depends only on $a_1, \ldots, a_s$, such that for any $b_1, \ldots, b_s \in \mathbb{Z}$, not all 0, we have that*

$$\left|\sum_{r=1}^{s} b_r \log a_r\right| \geq \left(3 \cdot \max_{1\leq r\leq s} |b_r|\right)^{-\kappa}.$$

3. Proof of Theorem 1.1

Let $\Sigma_a = \{a^\ell\}_{\ell\geq 0}$ and let $\Sigma_a\mathcal{D} = \{a^\ell n_k\}_{\ell,k\geq 0}$. For $\alpha \in \mathbb{R}$ let $A(x) \subseteq \mathbb{R}/\mathbb{Z}$ denote the closure of the set $(\Sigma_a\mathcal{D})x = \{a^\ell n_k x\}_{\ell,k\geq 0} \subseteq \mathbb{R}/\mathbb{Z}$. If $x \in \mathbb{Q}$ then (1.5) is trivially satisfied, so for the remainder of the proof we will assume that $x \notin \mathbb{Q}$.

Now suppose that for some $x \in \mathbb{R}$ there were a constant $c > 0$ such that

$$\inf_{q\in\mathbb{N}} q|q|_a|q|_{\mathcal{D}}\|qx\| > c.$$

Then for any $\ell, k \geq 0$ we would have that

$$\inf_{q\in\mathbb{N}} q\left\|q(a^\ell n_k x)\right\| \geq \inf_{q\in\mathbb{N}} \left(a^\ell n_k q\right)\left|a^\ell n_k q\right|_a \left|a^\ell n_k q\right|_{\mathcal{D}} \left\|a^\ell n_k q x\right\| > c.$$

In other words we would have that $(\Sigma_a\mathcal{D})x \subseteq \mathrm{Bad}(c)$, which would then imply that the set $A(x)$ does not contain any well approximable points. Therefore in order to prove Theorem 1.1 we just have to show that for any $x \in \mathbb{R}\setminus\mathbb{Q}$ the set $A(x)$ contains a well approximable point.

First we will show that we can always find long sequences of integers in $\Sigma_a\mathcal{D}$ with ratios close to 1 (see equation (3.5) below). Let $\{p_1 < \cdots < p_s\}$ be the collection of prime numbers which divide the elements of $\mathcal{D}$, and for each $k \geq 0$ write

$$n_k = p_1^{b_k^{(1)}} \cdots p_s^{b_k^{(s)}}.$$

Hypothesis (1.4) guarantees that for any $k \geq 2$,

$$\max_{1\leq r\leq s} b_k^{(r)} \leq 2k^\delta. \tag{3.1}$$

Now for each $\ell \in \mathbb{N}$ let $\sigma_\ell \in \mathbb{Z}$ and $\tau_\ell \in [0, 1)$ be selected so that $\sigma_\ell \geq 0$ and

$$\sum_{r=1}^{s} b_\ell^{(r)} \frac{\log p_r}{\log a} = \sigma_\ell + \tau_\ell.$$

Note that this is the same as writing n_ℓ in the form $a^{\sigma_\ell+\tau_\ell}$, and doing this makes it technically easier to compare the ratios of these numbers. Let M be the smallest integer greater than $2\log a$. Then given $k\ge 2$, one of the intervals $[m/M,(m+1)/M), 0\le m<M$, contains at least k/M of the numbers $\{\tau_\ell\}_{1\le\ell\le k}$. Label the numbers which fall in this interval as $\tau_{\ell_1}<\cdots<\tau_{\ell_{k'}}$.

Next set $\sigma'=\max_{1\le i\le k'}\sigma_{\ell_i}$ and for each $1\le i\le k'$ let

$$t_i=a^{\sigma'-\sigma_{\ell_i}}n_{\ell_i}\in\Sigma_a\mathcal{D}.$$

Then for any $1\le i<j\le k'$ we have that

$$\begin{aligned}\log\left(\frac{t_j}{t_i}\right)&=\sum_{r=1}^{s}\left(b_{\ell_j}^{(r)}-b_{\ell_i}^{(r)}\right)\log p_r+\left(\sigma_{\ell_i}-\sigma_{\ell_j}\right)\log a\\&=\log a\left(\tau_{\ell_j}-\tau_{\ell_i}\right),\end{aligned}\tag{3.2}$$

and this shows that

$$0<\log\left(\frac{t_j}{t_i}\right)<\frac{\log a}{M}.\tag{3.3}$$

Next using (3.1) we have that

$$|\sigma_{\ell_j}-\sigma_{\ell_i}|\le\frac{s\log p_s}{\log a}\cdot\max_{1\le r\le s}\left(1+\left|b_{\ell_j}^{(r)}-b_{\ell_i}^{(r)}\right|\right)\le\left(\frac{4s\log p_s}{\log a}\right)k^{\delta},$$

and so by applying Theorem 2.2 to (3.2) we deduce that there are constants $C,\kappa>0$, which depend only on $p_1,\ldots,p_s$, and a, such that

$$\log\left(\frac{t_j}{t_i}\right)\ge\frac{C}{k^{\delta\kappa}}.\tag{3.4}$$

To avoid technicalities from here on we will assume that $k\ge\max\{2M,2C^{1/(\delta\kappa)}\}$. Combining (3.3) and (3.4) with the inequalities

$$1+\alpha\le e^{\alpha}\le 1+2\alpha,\ 0\le\alpha\le 1,$$

we have that

$$1+\frac{C}{k^{\delta\kappa}}\ \le\ \frac{t_j}{t_i}<\ 1+\frac{2\log a}{M},\quad\text{for all}\ \ 1\le i<j\le k'.\tag{3.5}$$

Next we claim that we can always find a number $\gamma\in[0,1)$ satisfying

$$\gamma\in\left[\frac{1}{t_1a^2},\frac{1}{t_1a}\right)\cap(\Sigma_a-\Sigma_a)\,x.$$

To see why this is true notice that since $x\notin\mathbb{Q}$ the point 0 is an accumulation point of $(\Sigma_a-\Sigma_a)x=\Sigma_ax-\Sigma_ax$. Also this set is symmetric about 0, so it contains infinitely many points which lie in the interval $(0,1/t_1a^2)$. If β is one of these points then we can find an integer $b \in \mathbb{N}$ with $a^b\beta \in [1/t_1a^2, 1/t_1a)$, and our claim is verified by taking $\gamma = a^b\beta$.

With γ as above, for any $q \in \mathbb{N}$ and for any $1 \le i \le k'$ we have from (3.5) that

$$\frac{1}{a^2} \le t_i\gamma \le t_1\gamma\left(1+\frac{2\log a}{M}\right) < \frac{2}{a} \le 1.$$

Furthermore if $i < k'$ then from the lower bound in (3.5) we obtain

$$t_{i+1}\gamma - t_i\gamma \ge \frac{t_i\gamma C}{k^{\delta\kappa}} \ge \frac{C}{a^2k^{\delta\kappa}}.$$

Thus for each $q \in \mathbb{N}$ we have that

$$s\left(A(x)-A(x),\frac{C}{a^2k^{\delta\kappa}}\right) \ge k' \ge \frac{k}{M}.$$

Then by Lemma 2.2 we have

$$s\left(A(x),\frac{C}{a^2k^{\delta\kappa}}\right) \ge \left(\frac{k}{2M}\right)^{1/2}, \tag{3.6}$$

and this gives that

$$\overline{\dim}(A(x)) \ge \limsup_{k\to\infty}\frac{\log\left(\left(\frac{k}{2M}\right)^{1/2}\right)}{\log\left(\frac{2a^2k^{\delta\kappa}}{C}\right)} = \frac{1}{2\delta\kappa} > 0.$$

Finally Lemma 2.3 ensures that there is an ergodic times-a invariant measure μ, supported on $A(x)$, which has positive entropy. By Theorem 2.1 we have that $\mu-$almost every point is well approximable, but since $\mu(\mathbb{R}/\mathbb{Z}\setminus A(x)) = 0$ this implies that $A(x)$ contains a well-approximable point. This finishes the proof of the theorem.

4. PRELIMINARIES FOR THE PROOF OF THEOREM 1.2

Let $\Psi : \mathbb{N} \to \mathbb{R}$ be any non-negative function and for each $q \in \mathbb{N}$ define $A_q = A_q(\Psi) \subseteq \mathbb{R}/\mathbb{Z}$ by

$$A_q(\Psi) = \{x \in \mathbb{R}/\mathbb{Z} : \|qx\| \le \Psi(q)\}.$$

Then define $A(\Psi) \subseteq \mathbb{R}/\mathbb{Z}$ by

$$A(\Psi) = \limsup_{q\to\infty} A_q(\Psi) = \{x \in \mathbb{R}/\mathbb{Z} : x \in A_q \text{ for infinitely many } q\}.$$

In our problem we are interested in the case when $\Psi(q) = |q|_{\mathcal{D}}^{-1}\psi(q)$, for a pseudo-absolute value $|\cdot|_{\mathcal{D}}$ and a non-negative monotonic function $\psi : \mathbb{N} \to \mathbb{R}$. If λ denotes Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ then we would like to show for this choice of Ψ that $\lambda(A(\Psi)) = 1$ depending on whether or not the sum (1.10) diverges. First of all we demonstrate that the divergence or convergence of the sum in question is equivalent to the divergence or convergence of the measures of the corresponding sets A_q. Here and in what follows we write $d_k = n_k/n_{k-1}$ for each $n_k \in \mathcal{D}, k \geq 1$.

Lemma 4.1. *If $\mathcal{D}$ is any pseudo-absolute value sequence then for $N \in \mathbb{N}$ we have that*

$$\sum_{r=1}^{N} \frac{1}{|r|_{\mathcal{D}}} \overset{(i)}{\asymp} N\mathcal{M}(N) \overset{(ii)}{\asymp} \sum_{r=1}^{N} \mathcal{M}(r).$$

Consequently if $\psi : \mathbb{N} \to \mathbb{R}$ is any non-negative decreasing function then

$$\sum_{r=1}^{\infty} \lambda\left(A_r\left(\frac{\psi}{|\cdot|_{\mathcal{D}}}\right)\right) = \infty \iff \sum_{r=1}^{\infty} \mathcal{M}(r)\psi(r) = \infty. \tag{4.1}$$

Proof. For the proof of (i) we have that

$$\begin{aligned}
\sum_{r=1}^{N} \frac{1}{|r|_{\mathcal{D}}} &= \sum_{k=0}^{\mathcal{M}(N)} n_k \sum_{\substack{r=1 \\ n_k | r,\ n_{k+1} \nmid r}}^{N} 1 \\
&= \sum_{k=0}^{\mathcal{M}(N)} n_k \sum_{\substack{n \leq N/n_k \\ d_{k+1} \nmid n}} 1 \\
&= \sum_{k=0}^{\mathcal{M}(N)} n_k \left(\left(1 - \frac{1}{d_{k+1}}\right) \frac{N}{n_k} + O(1) \right) \\
&= N \sum_{k=0}^{\mathcal{M}(N)} \left(1 - \frac{1}{d_{k+1}}\right) + O\left(\sum_{k=0}^{\mathcal{M}(N)} n_k \right).
\end{aligned} \tag{4.2}$$

Now notice that $1/2 \leq (1 - 1/d_{k+1}) < 1$ for all k and that

$$\sum_{k=0}^{\mathcal{M}(N)} n_k \leq \sum_{k=0}^{\mathcal{M}(N)} \frac{n_{\mathcal{M}(N)}}{2^{\mathcal{M}(N)-k}} \leq 2n_{\mathcal{M}(N)} \leq 2N. \tag{4.3}$$

As claimed this shows that (4.2) is bounded above and below by universal constants times $N\mathcal{M}(N)$.

For (ii) we have that

$$\begin{aligned}
\sum_{r=1}^{N} \mathcal{M}(r) &= \sum_{k=0}^{\mathcal{M}(N)-1} k(n_{k+1} - n_k) + \mathcal{M}(N)(N - n_{\mathcal{M}(N)} + 1) \\
&= (N+1)\mathcal{M}(N) - \sum_{k=0}^{\mathcal{M}(N)} n_k
\end{aligned}$$

The latter quantity is clearly less than $2N\mathcal{M}(N)$, and by (4.3) it is also greater than a constant times $N\mathcal{M}(N)$.

Finally for the proof of (4.1), first of all suppose that $\psi(m_i)/|m_i|_{\mathcal{D}} \geq 1/2$ for some infinite increasing sequence of integers $\{m_i\}_{i\in\mathbb{N}}$. Then for each i we have that $A_{m_i} = \mathbb{R}/\mathbb{Z}$ so that the left hand side of (4.1) surely diverges. On the other hand using (ii) we have that

$$\sum_{r=1}^{m_i} \mathcal{M}(r)\psi(r) \geq \psi(m_i)\sum_{r=1}^{m_i}\mathcal{M}(r) \gg |m_i|_{\mathcal{D}}\left(m_i\mathcal{M}(m_i)\right) \geq \mathcal{M}(m_i),$$

and this tends to infinity with i.

Now for the other case assume that there is an $r_0 \in \mathbb{N}$ such that $\psi(r)/|r|_{\mathcal{D}} < 1/2$ for all $r \geq r_0$. In this case we have that

$$\lambda\left(A_r\left(\frac{\psi}{|\cdot|_{\mathcal{D}}}\right)\right) = \frac{2\psi(r)}{|r|_{\mathcal{D}}} \quad \text{for all } r \geq r_0. \tag{4.4}$$

Now by the monotonicity of ψ together with (i) and (ii) we obtain

$$\begin{aligned}
\sum_{r=r_0}^{N}\frac{\psi(r)}{|r|_{\mathcal{D}}} &= \sum_{r=r_0}^{N}(\psi(r)-\psi(r+1))\sum_{m=r_0}^{r}\frac{1}{|m|_{\mathcal{D}}} + \psi(N+1)\sum_{m=r_0}^{N}\frac{1}{|m|_{\mathcal{D}}}\\
&\asymp \sum_{r=r_0}^{N}(\psi(r)-\psi(r+1))\sum_{m=r_0}^{r}\mathcal{M}(m) + \psi(N+1)\sum_{m=r_0}^{N}\mathcal{M}(m)\\
&= \sum_{r=r_0}^{N}\mathcal{M}(r)\psi(r),
\end{aligned}$$

and this together with (4.4) finishes the proof of the lemma. □

For any Ψ as above if

$$\sum_{r\in\mathbb{N}}\lambda(A_r(\Psi)) < \infty$$

then by the Borel-Cantelli Lemma we have that $\lambda(A(\Psi)) = 0$. One half of Theorem 1.2 follows from this observation together with (4.1). Unfortunately the converse of the Borel-Cantelli Lemma is not true in general for the sets $A_r(\Psi)$. In other words there are examples of functions Ψ for which

$$\sum_{r\in\mathbb{N}}\lambda(A_r(\Psi)) = \infty$$

and yet $\lambda(A(\Psi)) = 0$. Duffin and Schaeffer observed this in [8] and they also showed in the same paper that under certain conditions this type of anomalous behaviour can be avoided.

Theorem 4.1. [8] *If $\Psi : \mathbb{N} \to \mathbb{R}$ is a non-negative function which satisfies*

$$\sum_{r \in \mathbb{N}} \Psi(r) = \infty \tag{4.5}$$

and

$$\limsup_{N \to \infty} \left(\sum_{r=1}^{N} \frac{\varphi(r)\Psi(r)}{r} \right) \left(\sum_{r=1}^{N} \Psi(r) \right)^{-1} > 0 \tag{4.6}$$

then $\lambda(A(\Psi)) = 1$.

5. Proof of Theorem 1.2

If (1.10) converges then as previously remarked the result of Theorem 1.2 follows from the Borel-Cantelli Lemma. Therefore we assume that (1.10) diverges. We set $\Psi(q) = \psi(q)/|q|_{\mathcal{D}}$ and we assume without loss of generality that $\Psi(q) < 1/2$ for all but finitely many q (otherwise the conclusion of the theorem is trivial). Then by (4.1) and (4.4) we know that (4.5) is satisfied, so in order to prove Theorem 1.2 it is sufficient to show that (4.6) also holds.

First of all we show that there is a universal constant $C > 0$ such that

$$\sum_{\substack{r=1 \\ d \nmid r}}^{N} \frac{\varphi(r)}{r} \geq CN \quad \text{for any} \quad d, N \in \mathbb{N} \quad \text{with} \quad d \geq 2. \tag{5.1}$$

To verify this we have that

$$\begin{aligned} \sum_{\substack{r=1 \\ d \nmid r}}^{N} \frac{\varphi(r)}{r} &= \sum_{r=1}^{N} \frac{\varphi(r)}{r} - \sum_{\substack{r=1 \\ d | r}}^{N} \frac{\varphi(r)}{r} \\ &= \sum_{d=1}^{N} \frac{\mu(d)}{d} \sum_{1 \leq r \leq N/d} 1 - \sum_{\substack{r=1 \\ d | r}}^{N} \frac{\varphi(r)}{r}, \end{aligned}$$

where $\mu : \mathbb{N} \to \{\pm 1, 0\}$ is the Möbius function. For the first sum in this expression we use the fact that

$$\begin{aligned} \sum_{d=1}^{N} \frac{\mu(d)}{d} \sum_{1 \leq r \leq N/d} 1 &= N \sum_{d=1}^{N} \frac{\mu(d)}{d^2} - \sum_{d=1}^{N} \left\{ \frac{N}{d} \right\} \frac{\mu(d)}{d} \\ &= \frac{6N}{\pi^2} - N \sum_{d=N+1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d=1}^{N} \left\{ \frac{N}{d} \right\} \frac{\mu(d)}{d} \\ &\geq \frac{6N}{\pi^2} - C_1 \log(N+1), \end{aligned}$$

for some universal constant $C_1 > 0$. For the second sum we simply use the inequality

$$\sum_{\substack{r=1\\ d|r}}^{N} \frac{\varphi(r)}{r} \leq \frac{N}{d}.$$

Together these estimates give

$$\sum_{\substack{r=1\\ d\nmid r}}^{N} \frac{\varphi(r)}{r} \geq \left(\frac{6}{\pi^2} - \frac{1}{d}\right) N - C_1 \log(N+1).$$

Now since $d \geq 2$ we have $6/\pi^2 - 1/d > 0$ and therefore there exists an $N_0 \in \mathbb{N}$ such that

$$\left(\frac{6}{\pi^2} - \frac{1}{d}\right) N \geq 2C_1 \log(N+1) \quad \text{for all } N \geq N_0,$$

which means that

$$\sum_{\substack{r=1\\ d\nmid r}}^{N} \frac{\varphi(r)}{r} \geq \frac{1}{2}\left(\frac{6}{\pi^2} - \frac{1}{2}\right) N \quad \text{for all } N \geq N_0.$$

To take care of the smaller values of N we choose $C_2 > 0$ so that

$$\sum_{\substack{r=1\\ d\nmid r}}^{N} \frac{\varphi(r)}{r} \geq C_2 N \qquad \text{for all } N < N_0, d \leq N_0. \tag{5.2}$$

This is clearly possible since the summand is always positive and the range of values for both N and d is finite. However if (5.2) holds for all $d \leq N_0$ then it also holds for all $d > N_0$, since the left hand side only depends on $N < N_0$ in those cases. This establishes (5.1) with

$$C = \min\left\{C_2, \frac{1}{2}\left(\frac{6}{\pi^2} - \frac{1}{2}\right)\right\}.$$

For the final part of the proof we have that

$$\sum_{r=1}^{N} \frac{\varphi(r)\psi(r)}{r\,|r|_{\mathcal{D}}} = \sum_{r=1}^{N} (\psi(r) - \psi(r+1)) \sum_{m=1}^{r} \frac{\varphi(m)}{m\,|m|_{\mathcal{D}}} + \psi(N+1) \sum_{m=1}^{N} \frac{\varphi(m)}{m\,|m|_{\mathcal{D}}}. \tag{5.3}$$

We estimate the inner sums here by

$$\begin{aligned}
\sum_{m=1}^{r} \frac{\varphi(m)}{m\,|m|_{\mathcal{D}}} &= \sum_{k=1}^{\mathcal{M}(r)} \sum_{\substack{m=1 \\ n_k|m,\ n_{k+1}\nmid m}}^{r} \frac{\varphi(m)}{m\,|m|_{\mathcal{D}}} \\
&= \sum_{k=1}^{\mathcal{M}(r)} \sum_{\substack{1\le m\le r/n_k \\ d_{k+1}\nmid m}} \frac{\varphi(n_k m)}{m} \\
&\geq \sum_{k=1}^{\mathcal{M}(r)} \varphi(n_k) \sum_{\substack{1\le m\le r/n_k \\ d_{k+1}\nmid m}} \frac{\varphi(m)}{m} \\
&\geq \frac{Cr}{2} \sum_{k=1}^{\mathcal{M}(r)} \frac{\varphi(n_k)}{n_k}.
\end{aligned}$$

By hypothesis (1.8) the last sum here is $\gg \mathcal{M}(r)$ and so by inequality (i) in Lemma 4.1 we have that

$$\sum_{m=1}^{r} \frac{\varphi(m)}{m\,|m|_{\mathcal{D}}} \gg \sum_{m=1}^{r} \frac{1}{|m|_{\mathcal{D}}}.$$

This together with (5.3) and the monotonicity of ψ gives

$$\begin{aligned}
\sum_{r=1}^{N} \frac{\varphi(r)\psi(r)}{r\,|r|_{\mathcal{D}}} &\gg \sum_{r=1}^{N} (\psi(r)-\psi(r+1)) \sum_{m=1}^{r} \frac{1}{|m|_{\mathcal{D}}} + \psi(N+1) \sum_{m=1}^{N} \frac{1}{|m|_{\mathcal{D}}} \\
&= \sum_{r=1}^{N} \frac{\psi(r)}{|r|_{\mathcal{D}}}.
\end{aligned}$$

This shows that (4.6) is satisfied and we conclude our proof by applying Theorem 4.1.

6. Concluding remarks

We mentioned in the introduction that hypothesis (1.8) in Theorem 1.2 is not particularly restrictive. However there are sequences $\mathcal{D}$ for which it fails to hold. To see how one might construct such a sequence, for each $k \geq 0$ let $A_k = 2^{k^2}$ and set

$$n_k = \prod_{p \leq A_k} p,$$

where the product is over prime numbers. Then by one of Mertens' Theorems [14, §22.7] we have that

$$\frac{\varphi(n_k)}{n_k} = \prod_{p\leq A_k} \left(1 - \frac{1}{p}\right) \ll \frac{1}{\log A_k},$$

and this implies that

$$\sum_{k=1}^{\infty} \frac{\varphi(n_k)}{n_k} < \infty.$$

It is clear in this example that if $\mathcal{D} = \{n_k\}$ then (1.8) is not satisfied.

It would be interesting to determine whether or not hypothesis (1.8) can be removed from the proof of Theorem 1.2. Indeed another interesting question is to determine whether hypothesis (1.4) can be removed from the proof of Theorem 1.1. Both of these problems seem to require more than trivial improvements over the techniques which we have presented.

Finally we remark that the ideas in our proof of Theorem 1.2 can be extended to prove metric results about approximations involving more than one pseudo-absolute value. In particular given two pseudo-absolute value sequences $\mathcal{D}_1$ and $\mathcal{D}_2$ and a monotonic function $\psi : \mathbb{N} \to \mathbb{R}$ we could give conditions on $\mathcal{D}_1, \mathcal{D}_2$, and ψ which would guarantee that the inequality

$$|q|_{\mathcal{D}_1}|q|_{\mathcal{D}_2}\|qx\| \leq \psi(q) \tag{6.1}$$

has infinitely many solutions $q \in \mathbb{N}$ for almost every $x \in \mathbb{R}$. However the conditions would depend very much on how the sequences $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect. For example if $\mathcal{D}_1 = \{2^k\}$ and $\mathcal{D}_2 = \{3^k\}$ then by [7, Theorem 1], inequality (6.1) has infinitely many solutions for almost every x if and only if

$$\sum_{r \in \mathbb{N}} (\log r)^2 \psi(r) = \infty.$$

However if $\mathcal{D}_1 = \mathcal{D}_2 = \{2^k\}$ then by [7, Theorem 2], the inequality has infinitely many solutions for almost every α if and only if

$$\sum_{r \in \mathbb{N}} r\psi(r) = \infty.$$

This shows that there are two extremes of the problem, and most sequences behave in a way that falls between these two extremes. It doesn't seem readily obvious how to find a nice, tractable divergence condition which will apply in the most general case of metric problems involving more than one pseudo-absolute value. In the case of two pseudo-absolute values this might not be too difficult, but for more than two the problem seems to get complicated quickly.

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Paper V

ON A MIXED KHINTCHINE PROBLEM IN DIOPHANTINE APPROXIMATION

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Abstract. Let $\mathcal{D} = \{n_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of natural numbers such that the ratio of any two consecutive elements is bounded and let (i, j) be a pair of strictly positive real numbers with $i+j = 1$. For any decreasing non-negative arithmetic function ψ consider the set of $x \in [0, 1)$ such that

$$\max\left\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\right\} \leq \psi(q)$$

for infinitely many $q \in \mathbb{N}$. We prove that the Lebesgue measure of this set is zero or one depending on whether the sum $\sum_{r=1}^{\infty} \psi(r)$ converges or diverges respectively. This provides a complete mixed analogue of a classical theorem of Khintchine. In turn, we find the Hausdorff dimension of this set in the case that $\psi(q) = q^{-\tau}$ for some $\tau \geq 1$. This extends the classical Jarník-Besicovich theorem to the mixed and weighted setting.

1. Introduction

Choose any positive real numbers i and j satisfying

$$i, j > 0 \quad \text{and} \quad i + j = 1 \tag{1.1}$$

and let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be any non-negative arithmetic function. For reasons that will become apparent, we refer to ψ as an *approximating function*. Consider the set $W(i, j, \psi)$ of real vectors $\mathbf{x} = (x_1, x_2) \in [0, 1)^2$ for which the system of inequalities

$$\left|x_1 - \frac{p_1}{q}\right| \leq \frac{\psi^i(q)}{q}, \qquad \left|x_2 - \frac{p_2}{q}\right| \leq \frac{\psi^j(q)}{q} \tag{1.2}$$

is satisfied by infinitely many $p_1, p_2 \in \mathbb{Z}$ and $q \in \mathbb{N}$. Clearly, this set depends heavily on the choice of function ψ. Essentially, if $\mathbf{x} \in W(i, j, \psi)$ we are saying that $\mathbf{x}$ can be approximated by rational points $(p_1/q, p_2/q)$ at a 'rate' described by the approximating function ψ. The exponents i and j act as 'weights', perturbing this rate of approximation across each component of $\mathbf{x}$.

Throughout, n-dimensional Lebesgue measure will be denoted λ_n. In 1926, Khintchine [15] proved the following remarkable statement concerning the two-dimensional Lebesgue measure of the set $W(i, j, \psi)$. For any pair of reals i, j satisfying (1.1) and any approximating function ψ we have

$$\lambda_2\left(W(i,j,\psi)\right) \quad = \quad \begin{cases} 0, & \sum_{r=1}^{\infty} \psi(r) \ < \ \infty. \\ \\ 1, & \sum_{r=1}^{\infty} \psi(r) \ = \ \infty \ \text{ and } \psi \text{ is monotonic.} \end{cases}$$

In fact, the monotonicity restriction imposed on ψ in the 'divergent' part of Khintchine's theorem can be relaxed. This follows from a result of Harman (see Theorem 3.8 of [16]).

One can consider the following multiplicative variant of the set $W(i,j,\psi)$. Let

$$M(\psi) := \left\{\mathbf{x} \in [0,1)^2 : \ \|qx_1\| \, \|qx_2\| \ \leq \ \psi(q) \ \text{ for inf. many } q \in \mathbb{N}\right\},$$

where $\| \, . \, \|$ denotes the distance to the nearest integer. A measure theoretic statement concerning the set $M(\psi)$ was found by Gallagher [13] in 1962. For any approximating function ψ

$$\lambda_2\left(M(\psi)\right) \quad = \quad \begin{cases} 0, & \sum_{r=1}^{\infty} \psi(r)\log(r) \ < \ \infty. \\ \\ 1, & \sum_{r=1}^{\infty} \psi(r)\log(r) \ = \ \infty \ \text{ and } \psi \text{ is monotonic.} \end{cases}$$

It is an open question as to whether the monotonicity assumption on ψ can be safely removed in this setting. For recent progress, see [4].

In 2004, de Mathan & Teulié [20] introduced a related setup realised by retaining the condition that $\|qx_1\|$ is small but replacing the condition on $\|qx_2\|$ with a condition of divisibility. To elaborate we require some notation. A sequence $\mathcal{D} := \{n_k\}_{k=0}^{\infty}$ of positive integers is said to be a *pseudo-absolute value sequence*, or simply a *$\mathcal{D}$-adic sequence*, if it is strictly increasing with $n_0 = 1$ and $n_k | n_{k+1}$ for all k. We say a pseudo-absolute value sequence has *bounded ratios* if the quotients n_{k+1}/n_k do not exceed some universal constant. The *$\mathcal{D}$-adic pseudo-absolute value* $| \, . \, |_{\mathcal{D}} : \mathbb{N} \to \{1/n_k : \ k \in \mathbb{N}\}$ is then defined by

$$|q|_{\mathcal{D}} := 1/n_{\omega_{\mathcal{D}}(q)} = \inf\{1/n_m : q \in n_m\mathbb{Z}\}.$$

When $\{n_{k+1}/n_k\}_{k=0}^{\infty}$ is the constant sequence equal to a prime number p, the pseudo absolute value $| \cdot |_{\mathcal{D}}$ is the usual p-adic absolute value $| \cdot |_p$.

Within this setup, one can define a 'mixed' version of the set $M(\psi)$. Let

$$M_{\mathcal{D}}(\psi) := \{x \in [0,1) : \, |q|_{\mathcal{D}} \, \|qx\| \, \leq \, \psi(q) \text{ for inf. many } q \in \mathbb{N}\}.$$

Recently, in [17], the following analogue of Gallagher's theorem was established concerning the set $M_{\mathcal{D}}(\psi)$. For any approximating function ψ and any $\mathcal{D}$-adic sequence with bounded ratios we have

$$\lambda_1\left(M_{\mathcal{D}}(\psi)\right) \quad = \quad \begin{cases} 0, & \displaystyle\sum_{r=1}^{\infty} \psi(r) \log(r) \, < \, \infty. \\ \\ 1, & \displaystyle\sum_{r=1}^{\infty} \psi(r) \log(r) \, = \, \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Again, it is currently unknown whether the monotonicity assumption is necessary.

Somewhat surprisingly the metric theory relating to a mixed analogue of $W(i,j,\psi)$ has not yet been explored. The intentions of this paper are to do exactly that. In particular, a metric theorem is established concerning the one-dimensional set

$$W_{\mathcal{D}}(i,j,\psi) := \left\{x \in [0,1) : \, \max\left\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\right\} \leq \psi(q) \text{ for inf. many } q \in \mathbb{N}\right\}.$$

As we have seen, for each approximating function ψ the Lebesgue measure of the sets $M(\psi)$ and $M_{\mathcal{D}}(\psi)$ depend on the asymptotic behaviour of the same sum (assuming that $\mathcal{D}$ has bounded ratios). We show that the sets $W(i,j,\psi)$ and $W_{\mathcal{D}}(i,j,\psi)$ enjoy a similar property.

For the case when $\psi(q) = 1/q$ and $\mathcal{D}$ has bounded ratios, the 'badly approximable' complement of the set $W_{\mathcal{D}}(i,j,\psi)$ was studied in [1]. This seems to constitute all previous knowledge of mixed problems in Diophantine approximation outside the multiplicative setting.

2. Statement of Results

For notational purposes, let $\mathcal{A} := \mathcal{A}(\mathcal{D},\psi,i) := \{r \in \mathbb{N} : \, |r|_{\mathcal{D}} < \psi^i(r)\}$. The main result of this paper is the following analogue to Khintchine's theorem.

Theorem 2.1. *For any pair of reals i, j satisfying (1.1), any decreasing approximating function ψ and any $\mathcal{D}$-adic sequence with bounded ratios we have*

$$\lambda_1\left(W_{\mathcal{D}}(i,j,\psi)\right) \quad = \quad \begin{cases} 0, & \sum_{r\in\mathbb{N}} \psi(r) \ < \ \infty. \\ & \\ 1, & \sum_{r\in\mathbb{N}} \psi(r) \ = \ \infty. \end{cases}$$

We remark that one is free replace the volume sum $\sum_{r\in\mathbb{N}}\psi(r)$ with the sum $\sum_{r\in\mathcal{A}}\psi^j(r)$ in the statement of Theorem 2.1. This might be expected as the problem can be restated as one of Diophantine approximation with restricted denominator; namely, we can write

$$W_{\mathcal{D}}(i,j,\psi) = \left\{x \in [0,1) : \|qx\| < \psi^j(q) \quad \text{for inf. many } q \in \mathcal{A}\right\}.$$

The two sums in question are equivalent under the assumption that ψ is monotonic. However, the sum $\sum_{r\in\mathcal{A}}\psi^j(r)$ is in many ways the 'genuine' volume sum. Indeed, the monotonicity assumption can be dropped in the convergence case of Theorem 2.1 when this sum is considered. That said, to bring the similarity between Theorem 2.1 and Khintchine's theorem to the forefront we present the statement as above.

In section §3 we demonstrate that the monotonicity assumption imposed on the function ψ is indeed necessary, and further that the natural mixed analogue of the Duffin-Schaeffer Conjecture (see §3.2) does not hold in the mixed setting.

It is worth emphasising that the degenerate cases '$i = 0$' and '$j = 0$' are not considered in this paper. On employing the convention that $x^{1/y} = 0$ when $y = 0$ for all real x, it is easily verified that in the former case Theorem 2.1 reduces to a classical one-dimensional result of Khintchine (see §3.1), whilst in the latter case the measure of the corresponding set $W_{\mathcal{D}}(1,0,\psi)$ trivially fulfils a 'zero-one' law. Indeed,

$$W_{\mathcal{D}}(1,0,\psi) \quad = \quad \begin{cases} [0,1), & \psi(q) > |q|_{\mathcal{D}} \text{ for infinitely many } q \in \mathbb{N}. \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, we remark that obtaining an equivalent statement to that of Theorem 2.1 for pseudo-absolute value sequences with non-bounded ratios, whilst desirable, would require more than trivial improvements over the techniques which we have presented.

Theorem 2.1 is a consequence of the following more general Hausdorff measure result. Throughout, $\mathcal{H}^s$ denotes standard s-dimensional Hausdorff measure and 'dim' represents Hausdorff dimension. Recall that when $s = 1$ Hausdorff measure is comparable with one-dimensional Lebesgue measure.

Theorem 2.2. *Fix any pair of reals i, j satisfying (1.1), any $\mathcal{D}$-adic sequence with bounded ratios and any real $s \in (i, 1]$. Then, for any approximating function ψ for which $r^{1-s}\psi^{i+js}(r)$ is decreasing we have*

$$\mathcal{H}^s\left(W_{\mathcal{D}}(i,j,\psi)\right) \;=\; \begin{cases} 0, & \sum_{r\in\mathbb{N}} r^{1-s}\,\psi^{i+js}(r) \;<\; \infty. \\ \\ \mathcal{H}^s([0,1)), & \sum_{r\in\mathbb{N}} r^{1-s}\,\psi^{i+js}(r) \;=\infty \;\; \text{and } \psi \text{ is monotonic.} \end{cases}$$

It should be mentioned that we do not claim the conditions imposed in Theorem 2.2 are optimal. In fact, we suspect that the assumption that $r^{1-s}\psi^{i+js}(r)$ is decreasing may be unnecessary. A further consequence of Theorem 2.2 is the following statement.

Corollary 2.3. *Choose any pair of reals i, j satisfying (1.1), any $\mathcal{D}$-adic sequence with bounded ratios and any decreasing approximating function ψ. Then, if there exists a real number τ such that*

$$\tau \;=\; \lim \frac{-\log\psi(r)}{\log r} \;<\; \frac{1}{i}$$

we have

$$\dim\left(W_{\mathcal{D}}(i,j,\psi)\right) \;=\; \frac{2-i\tau}{1+j\tau}.$$

This generalises a classical theorem of Jarník [19] and Besicovich [8], which corresponds to the case when $i=0$ and $j=1$. We remark that when $\psi(q)=q^{-1/i}$ the set $W_{\mathcal{D}}(i,j,\psi)$ is empty.

3. Removing monotonicity

3.1. **The work of Duffin and Schaeffer.** For any approximating function ψ let

$$W(\psi) := \{x \in [0,1) : \|qx\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$$

denote the standard set of ψ-approximable numbers. A one-dimensional version of Khintchine's theorem states that the Lebesgue measure of $W(\psi)$ is zero or one depending upon whether the sum $\sum_{r=1}^{\infty}\psi(r)$ converges or diverges respectively. Once more, a monotonicity assumption is imposed on ψ in the divergent case.

In their seminal paper [12], Duffin & Schaeffer produced a counterexample showing that the monotonicity assumption is absolutely necessary. In particular, they constructed a general approximating function ϕ for which $\lambda_1(W(\phi)) = 0$ but the sum $\sum_{r=1}^{\infty}\phi(r)$ diverges. However, they did conjecture that under certain stronger conditions the assumption can be dropped. For any approximating function ψ define the set $W'(\psi)$ by

$$W'(\psi) := \{x \in [0,1) : |qx - p| < \psi(q) \text{ for inf. many } (p,q) \in \mathbb{N} \times \mathbb{N} \text{ with } (p,q) = 1\}.$$

The set differs from $W(\psi)$ by only the coprimality restriction on p and q. This restriction ensures that the rational approximations p/q to x are in reduced form.

Duffin-Schaeffer Conjecture (1941). *For any approximating function ψ we have*

$$\lambda_1(W'(\psi)) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r}\, \psi(r) = \infty,$$

where φ denotes Euler's totient function.

It is clear that $W'(\psi) \subset W(\psi)$, which, in view of Khintchine's theorem, implies that the complementary statement

$$\lambda_1(W'(\psi)) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r}\, \psi(r) < \infty$$

holds for every approximating function ψ. The Duffin-Schaeffer Conjecture represents one of the most profound unsolved problems in metric Diophantine approximation. For a thorough account including recent progress made concerning the conjecture see §2 of [16].

3.2. **The mixed setting.** One might hope to prove similar results to those of Duffin and Schaeffer within the mixed simultaneous setting. Indeed, in §3.3 we demonstrate that the monotonicity assumption in Theorem 2.1 is also absolutely necessary by constructing a counterexample of our own. More to the point, our example will show that the 'natural' mixed analogue of the Duffin-Schaeffer Conjecture does not hold.

First, let us discuss what a mixed analogue of the Duffin-Schaeffer Conjecture might be. Let $W'_{\mathcal{D}}(i,j,\psi)$ denote the set of points $x \in [0,1)$ for which the conditions

$$\max\left\{ |q|_{\mathcal{D}}^{1/i}, |qx-p|^{1/j} \right\} \le \psi(q), \qquad (p,q) = 1,$$

hold for infinitely many natural numbers q. One might naively propose that the analogue would read " for any pair of reals i, j satisfying (1.1), any approximating function ψ and any $\mathcal{D}$-adic sequence with bounded ratios we have

$$\lambda_1(W'_{\mathcal{D}}(i,j,\psi)) = 1 \quad \text{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r}\, \psi(r) = \infty."$$

However, it is not difficult to see this is false; for example, take

$$\phi(q) = \begin{cases} 1/2, & (n_k, q) = 1 \text{ for all } k \in \mathbb{N}, \\ \\ 0, & \text{otherwise,} \end{cases}$$

and we have that $W'_{\mathcal{D}}(i, j, \phi)$ is empty but the corresponding volume sum diverges.

A more astute, and natural, proposal for a mixed Duffin-Schaeffer Conjecture is that we have

$$\lambda_1(W'_{\mathcal{D}}(i, j, \psi)) = 1 \quad \text{if} \quad \sum_{r \in \mathcal{A}} \frac{\varphi(r)}{r} \psi^j(r) = \infty.$$

The example given above certainly does not contradict this statement as the set $\mathcal{A}(\phi)$ is empty. That said, we will prove that this natural proposal is also false. To be precise, we prove the following.

Theorem 3.1. *For any pair of reals i, j satisfying (1.1) and any $\mathcal{D}$-adic sequence there exists an approximating function $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ for which*

$$\lambda_1(W_{\mathcal{D}}(i, j, \Phi)) = 0 \quad \textit{but} \quad \sum_{r \in \mathcal{A}} \frac{\varphi(r)}{r} \Phi^j(r) = \infty.$$

Note that since

$$W'_{\mathcal{D}}(i, j, \Phi) \subset W_{\mathcal{D}}(i, j, \Phi) \quad \text{and} \quad \sum_{r \in \mathcal{A}} \frac{\varphi(r)}{r} \Phi^j(r) \;\leq\; \sum_{r \in \mathcal{A}} \Phi^j(r),$$

the example constructed in Theorem 3.1 both proves the necessity of the monotonicity assumption in Theorem 2.1 and disproves our natural proposal for a mixed Duffin-Schaeffer Conjecture.

3.3. **Proof of Theorem 3.1.** First we show for large R and small $\epsilon > 0$ that there exists an approximating function ϕ such that

$$\sum_{r \in \mathcal{A}(\phi)} \frac{\varphi(r)}{r} \phi^j(r) \;>\; 1, \qquad \phi(r) = 0 \quad \text{when} \quad r \leq R,$$

but the set of $x \in (0, 1)$ such that

$$\|qx\| \;<\; \phi^j(q) \qquad \text{for some } q \in \mathcal{A}(\phi), \tag{3.1}$$

has Lebesgue measure strictly less than ϵ.

Let α be a positive number strictly smaller than both $(\epsilon/2)^{1/j}$ and $(1/2)^{1/i}$ and choose primes $p_1, p_2, \ldots, p_s$ with $p_t > R$ $(t = 1, \ldots, s)$ for some natural number s to be specified later. Since $\mathcal{D}$ has bounded ratios we can choose the primes p_t in such a way that each is coprime to every integer n_k. Next, let

$$K := K(s,\alpha) = \min\{k \in \mathbb{N} : n_k \geq p_1 \cdots p_s/\alpha\}.$$

Finally, upon setting

$$N := n_K p_1 \cdots p_s$$

define

$$\phi(q) : \;\; = \;\; \begin{cases} q\alpha/N, & n_K \mid q, \quad q \mid N, \quad q \neq n_K. \\ \\ 0, & \text{otherwise.} \end{cases}$$

We claim that ϕ satisfies the desired properties. Let A_q denote the set in $(0,1)$ consisting of the $q-1$ open intervals of length $2\phi^j(q)/q$ with centres at the rationals p/q $(p = 1, \ldots, q-1)$ and the open intervals $(0, \phi^j(q)/q)$ and $(1 - \phi^j(q)/q, 1)$. The upper bound for α guarantees that these intervals are disjoint and so the Lebesgue measure of A_q is given by $2\phi^j(q) = 2q^j\alpha^j/N^j$. Furthermore, we have

$$A_N = \bigcup_{\substack{q \mid N: \\ n_K \mid q \\ q \neq n_K}} A_q$$

and for all q in this union

$$|q|_{\mathcal{D}} \;\leq\; \frac{1}{n_K} \;=\; \frac{p_1 \cdots p_s}{N} \;\leq\; \frac{n_K \alpha}{N} \;<\; \frac{q\alpha}{N} \;=\; \phi(q) \;<\; \phi^i(q);$$

i.e., $q \in \mathcal{A}(\phi)$. Hence, property (3.1) will be satisfied by irrational $x \in (0,1)$ if and only if $x \in A_N$. However, $\lambda_1(A_N) = 2\alpha^j < \epsilon$.

All that remains is to show

$$\sum_{r \in \mathcal{A}(\phi)} \frac{\varphi(r)}{r} \phi^j(r) \quad > \quad 1.$$

Via the change of variables $\ell := rn_K^{-1}, M := Nn_K^{-1}$ we have

$$\sum_{r \in \mathcal{A}(\phi)} \frac{\varphi(r)}{r} \phi^j(r) \quad = \quad \frac{\alpha^j}{N^j} \sum_{\substack{q \mid N: \\ n_K \mid q \\ q \neq n_K}} \frac{\varphi(q)}{q^{1-j}} \quad = \quad \frac{\alpha^j}{M^j} \frac{\varphi(n_K)}{n_K} \sum_{\substack{\ell > 1: \\ r \mid M}} \frac{\varphi(\ell)}{\ell^{1-j}},$$

since n_K and all divisors of M are pairwise coprime. It is readily verified that the function

$$f(n) := \quad \sum_{d \mid n} \frac{\varphi(d)}{d^{1-j}}$$

is multiplicative. Therefore,

$$f(n) \;=\; \prod_{t=1}^{m}\left(1+\frac{\varphi(q_t)}{q_t^{1-j}}+\frac{\varphi(q_t^2)}{q_t^{2(1-j)}}+\cdots+\frac{\varphi(q_t^{\alpha_t})}{q_t^{\alpha_t(1-j)}}\right),$$

where $n=q_1^{\alpha_1}\cdots q_m^{\alpha_m}$ is the unique prime factorization of n (see [18] for example). Also, it follows from the assumption that $\mathcal{D}$ has bounded ratios that the quantity $\varphi(n_K)/n_K$ is bounded below by some positive constant, $\kappa>0$ say, which depends only upon $\mathcal{D}$. Now, choose s large enough so that

$$\prod_{t=1}^{s}\left(1+1/p_t^{j}-1/p_t\right) \quad > \quad 1\,+\,1/\alpha^j\kappa. \tag{3.2}$$

This is always possible because $0<j<1$ and so the above product diverges when extended over all primes. Then, since $M=p_1\cdots p_s$ we have

$$\begin{aligned}\sum_{r\in\mathcal{A}(\phi)}\frac{\varphi(r)}{r}\,\phi^j(r) \quad &= \quad \frac{\alpha^j}{M^j}\frac{\varphi(n_K)}{n_K}\left(\prod_{t=1}^{s}\left(1+\varphi(p_t)/p_t^{1-j}\right)-1\right)\\ &\geq \quad \frac{\alpha^j\kappa}{M^j}\left(\prod_{t=1}^{s}\left(1+(p_t-1)/p_t^{1-j}\right)-1\right)\\ &> \quad \alpha^j\kappa\left(\prod_{t=1}^{s}\left(1+1/p_t^{j}-1/p_t\right)-1\right)\\ &\overset{(3.2)}{>} \quad 1,\end{aligned}$$

as required. Note that this argument is not applicable to the Duffin-Schaeffer Conjecture itself. This is because when '$j=1$' we cannot choose s in such a way that (3.2) holds since then the product on the LHS of (3.2) reduces to the trivial one.

We are now in a position to construct our counterexample. Let ϕ_1 satisfy the above properties with $R=R_1:=1$ and $\epsilon=\epsilon_1=2^{-2}$. Then for some R_2 we have $\phi_1(q)=0$ for all $q\geq R_2$. Let ϕ_2 satisfy the above properties with $R=R_2$ and $\epsilon=\epsilon_2=2^{-2}$. Continue to choose numbers R_t and construct functions ψ_t satisfying the above properties with $R=R_t$ and $\epsilon=\epsilon_t=2^{-t}$. Then, define

$$\Phi(q):\;=\;\begin{cases}\phi_1(q), & q<R_2.\\[2ex] \phi_t(q), & R_t\leq q<R_{t+1},\quad t\in\mathbb{N}.\end{cases}$$

Then, it is clear that

$$\sum_{r\in\mathcal{A}(\Phi)}\frac{\varphi(r)}{r}\,\Phi^j(r) \quad = \quad \infty,$$

but for $x \in (0,1)$ the system

$$\|qx\| < \Phi^j(q), \qquad q \in \mathcal{A}(\Phi), \qquad q > R_t$$

can be satisfied only if x belongs to a set of measure at most

$$\sum_{r=t}^{\infty} 2^{-r} = 2^{-t+1},$$

as required.

4. Ubiquitous Systems

Ubiquity is a fundamental tool for establishing measure theoretic statements. We will utilise this notion in proving Theorem 2.2. This section comprises of a brief description of a restricted form of ubiquity tailored to our needs.

The concept of ubiquitous systems was first introduced by Dodson, Rynne & Vickers in [11] as a method of determining lower bounds for the Hausdorff dimension of limsup sets. Recently, this idea was developed by Beresnevich, Dickinson & Velani in [2] to provide a very general framework for establishing the Hausdorff measure of a large class of limsup sets. A simplified account of ubiquity is presented in [7].

4.1. **The ubiquity setup.** Let (Ω, d) be a compact metric space endowed with a non-atomic probability measure μ and assume that any open subset of Ω is μ-measurable. Throughout, $B(c,r)$ will denote a ball in Ω centred at a point c and of radius $r > 0$. The following regularity condition will be imposed on the measure of balls: There exist positive constants a, b, δ and r_0 such that for any $c \in \Omega$ and $r \leq r_0$

$$ar^{\delta} \leq \mu(B(c,r)) \leq br^{\delta}.$$

If this power law holds then μ is referred to as *δ-Ahlfors regular*. It is easy to see that if μ is δ-Ahlfors regular then $\dim \Omega = \delta$ and that μ is comparable to δ-dimensional Hausdorff measure $\mathcal{H}^{\delta}$.

Let $\mathcal{R} = \{R_a \in \Omega : a \in J\}$ be a collection of points R_a in Ω indexed by some infinite, countable set J. The points R_a are referred to as the *resonant points*. Next, let $\beta : J \to \mathbb{R}_{>0} : a \mapsto \beta_a$ be a positive function defined on J for which the number of $a \in J$ with β_a bounded above is always finite. Finally, given an approximating function Ψ define

$$\Lambda(\Psi) := \{x \in \Omega : x \in B(R_a, \Psi(\beta_a)) \text{ for infinitely many } a \in J\}.$$

It is the measure of this set in which we are interested.

To demonstrate the 'limsup' nature of $\Lambda(\Psi)$ first choose any two positive increasing sequences $l := \{l_k\}$ and $u := \{u_k\}$ such that $l_k < u_k$ and $\lim_{k\to\infty} l_k = \infty$. These sequences will be referred to as the *lower* and *upper* sequences respectively. For $k \in \mathbb{N}$ let

$$\Lambda_l^u(\Psi, k) := \bigcup_{a \in J_l^u(k)} B(R_a, \Psi(\beta_a)),$$

where $J_l^u(k) := \{a \in J : l_k < \beta_a \le u_k\}$. Then, it is easily seen that

$$\Lambda(\Psi) = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \Lambda_l^u(\Psi, k).$$

We can now define what it means to be a ubiquitous system. Let $\rho : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be any function with $\rho(r) \to 0$ as $r \to \infty$ and let

$$\Delta_l^u(\rho, k) := \bigcup_{a \in J_l^u(k)} B(R_a, \rho(u_k)).$$

Definition (Local m-ubiquity) Let $B = B(c, r)$ be an arbitrary ball in Ω of radius $r \le r_0$. Suppose there exists a function ρ, sequences l and u and an absolute constant $\kappa > 0$ such that

$$\mu(B \cap \Delta_l^u(\rho, k)) \ \ge\ \kappa\mu(B) \qquad \forall\, k \ge k_0(B). \tag{4.1}$$

Then the pair $(\mathcal{R}, \beta)$ is said to be a *local μ-ubiquitous system relative to* (ρ, l, u).

The function ρ will be referred to as the *ubiquitous function.* Also, as is noted in [2], the appearance of the lower sequence l is in the above definition is irrelevant. Indeed, to establish inequality (4.1) it suffices to show

$$\mu\left(B \cap \bigcup\ _{a \in J^u(k)} B(R_a, \rho(u_k))\right) \ \ge\ \kappa\mu(B) \qquad \forall\, k \ge k_0(B), \tag{4.2}$$

where $J^u(k) := \{a \in J : \beta_a \le u_k\}$.

Finally, we will say a function h is *u-regular* if there exists a strictly positive constant $\lambda < 1$, which may depend on u, such that for k sufficiently large

$$h(u_{k+1}) \ \le\ \lambda h(u_k).$$

We now present the main results associated with ubiquitous systems tailored to our needs. The first theorem (see [2, Corollary 2]) concerns the μ-measure of the limsup set $\Lambda(\Psi)$ and corresponds to the '$s = 1'$ case of Theorem 2.1. The second (see [3, Theorem 10]) deals with the s-dimensional Hausdorff measure $\mathcal{H}^s$ of $\Lambda(\Psi)$ for $0 < s < 1$. Due to the nature of the framework it is necessary to deal with the two scenarios separately.

Theorem BDV1 (2006). *Let (Ω, d) be a compact metric space equipped with a δ-Ahlfors regular measure μ. Suppose that $(\mathcal{R}, \beta)$ is a local μ-ubiquitous system relative to (ρ, l, u) and that Ψ is an approximation function. Furthermore, suppose that either Ψ or ρ is u-regular and that*

$$\sum_{k=1}^{\infty} \left(\frac{\Psi(u_k)}{\rho(u_k)} \right)^{\delta} \quad = \quad \infty.$$

Then,

$$\mu\left(\Lambda(\Psi)\right) \quad = \quad 1.$$

Theorem BDV2 (2006). *Let (Ω, d) be a compact metric space equipped with a δ-Ahlfors regular measure μ. Suppose that $(\mathcal{R}, \beta)$ is a local μ-ubiquitous system relative to (ρ, l, u) and that Ψ is an approximation function. Furthermore, suppose that $0 < s < \delta$. Let g be the positive function given by $g(r) := \Psi^s \rho^{-\delta}$ and let $G := \limsup_{k \to \infty} g(u_k)$.*

(i) *Suppose that $G = 0$ and Ψ is u-regular. Then,*

$$\mathcal{H}^s(\Lambda(\Psi)) = \infty \qquad \textit{if} \qquad \sum_{k=1}^{\infty} g(u_k) = \infty.$$

(ii) *Suppose that $0 < G < \infty$. Then, $\mathcal{H}^s(\Lambda(\Psi)) = \infty$.*

Before proceeding, we mention a generalisation of the Cauchy condensation test, attributed to Oscar Schlömilch, which can be found in [9, Theorem 2.4]. We will appeal to this result multiple times in our proofs.

Schlömilch's Theorem (Late 19th Century). *Let $\sum_{r=0}^{\infty} a_r$ be an infinite real series whose terms are positive and decreasing and let $m_0 < m_1 < \cdots$ be a strictly increasing sequence of positive integers for which there exists a constant $M > 0$ such that*

$$\frac{m_{k+1} - m_k}{m_k - m_{k-1}} \leq M \quad \textit{for every } k \in \mathbb{N}. \tag{4.3}$$

Then the series $\sum_{r=0}^{\infty} a_r$ converges if and only if the series $\sum_{k=0}^{\infty} (m_{k+1} - m_k) a_{m_k}$ converges.

Note that when $m_k = n_k$, condition (4.3) is satisfied for some $M > 0$ if and only if the sequence $\mathcal{D}$ has bounded ratios.

5. Proof of Theorem 2.2

For the divergence part of Theorem 2.2 we will appeal to the ubiquity framework described in the previous section. The convergence part follows by well-known arguments stemming from the Borel-Cantelli Lemma. For completeness we include a short proof here. For each s with $i < s \leq 1$ let $\mathcal{H}^s$ denote s-dimensional Hausdorff measure and assume that the sum $\sum_{r\in\mathbb{N}} r^{1-s}\,\psi^{i+js}(r)$ converges. The case '$s = 1$' corresponds to the setting of Theorem 2.1, where $\mathcal{H}^1$ is comparable to one-dimensional Lebesgue measure. Define the quantity $m := m(k)$ for each $k \in \mathbb{N}$ as the unique natural number for which

$$\frac{1}{n_m} \quad < \quad \psi^i(n_k) \quad \leq \quad \frac{1}{n_{m-1}}. \tag{5.1}$$

This is always possible since ψ is decreasing and the elements of $\mathcal{D}$ are increasing. Since the pseudo absolute value is discrete, for each $k \in \mathbb{N}$ we have

$$\begin{aligned}
\#\left\{n_k < q \leq n_{k+1} : |q|_{\mathcal{D}} < \psi^i(q)\right\} \quad &\leq \quad \#\left\{n_k < q \leq n_{k+1} : |q|_{\mathcal{D}} < \psi^i(n_k)\right\} \\
&= \quad \#\left\{n_k < q \leq n_{k+1} : |q|_{\mathcal{D}} \leq \frac{1}{n_m}\right\} \\
&= \quad \#\left\{n_k < q \leq n_{k+1} : n_m |\, q\right\} \\
&= \quad \frac{n_{k+1} - n_k}{n_m} \\
&\overset{(5.1)}{<} \quad (n_{k+1} - n_k)\psi^i(n_k)
\end{aligned} \tag{5.2}$$

Next, choose any natural number q for which $|q|_{\mathcal{D}} < \psi^i(q)$. Then the set of real numbers $x \in (0,1)$ satisfying

$$\max\left\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\right\} \quad < \quad \psi(q) \tag{5.3}$$

is covered by the $q-1$ open intervals of length $2\psi^j(q)/q$ with centres at the rationals p/q $(p = 1, \ldots, q-1)$ and the open intervals $(0, \psi^j(q)/q)$ and $(1 - \psi^j(q)/q, 1)$. Thus, the Hausdorff measure $\mathcal{H}^s$ of this set is at most $2^s q^{1-s}\psi^{js}(q)$.

For any $\epsilon > 0$ and any sufficiently large integer k_0, the set of $x \in (0,1)$ satisfying inequality (5.3) for some $q > n_{k_0}$ has Hausdorff measure at most

$$\begin{aligned}
2^s \sum_{\substack{q > n_{k_0}: \\ |q|_{\mathcal{D}} < \psi^i(q)}} q^{1-s}\psi^{js}(q) \quad &\leq \quad 2^s M^{1-s} \sum_{k=k_0}^{\infty} \sum_{\substack{n_k < q \leq n_{k+1}: \\ |q|_{\mathcal{D}} < \psi^i(q)}} n_k^{1-s}\psi^{js}(n_k) \\
&\overset{(5.2)}{<} \quad 2^s M^{1-s} \sum_{k=k_0}^{\infty} (n_{k+1} - n_k) n_k^{1-s}\psi^{i+js}(n_k).
\end{aligned}$$

However, the function $r^{1-s}\psi^{i+js}(r)$ is assumed decreasing and $\mathcal{D}$ is assumed to have bounded ratios and so we may apply Schlömilch's theorem. This allows us to make the final sum as small as we like, smaller than some $\epsilon > 0$ say. In particular, the set of x satisfying inequality (5.3) for infinitely many q has Hausdorff measure measure at most ϵ and our proof is complete.

We now demonstrate how the ubiquity framework can be applied to the set $W_{\mathcal{D}}(i,j,\psi)$. Firstly, choose a natural number c. It is then easy to see that $W_{\mathcal{D}}(i,j,\psi)$ can be expressed in the form $\Lambda(\Psi)$ with

$$\Omega := [0,1], \quad \Psi(r) := \psi^j(r)/r, \quad J := \left\{(p,q) \in \mathbb{N}\times\mathbb{N} : 0 \le p \le q,\ |q|_{\mathcal{D}} < \psi^i(q)\right\},$$

$$a := (p,q) \in J, \quad \beta_a := q, \quad R_a := p/q, \quad u_k := l_{k+1} := n_{ck}, \quad \mu := \lambda_1, \quad \delta := 1,$$

$$J_l^u(k) := \left\{(p,q) \in J : n_{c(k-1)} < q \le n_{ck}\right\}, \quad \Lambda_l^u(\Psi,k) := \bigcup_{(p,q)\in J_l^u(k)} B(p/q, \psi^j(q)/q),$$

so that

$$W_{\mathcal{D}}(i,j,\psi) \ = \ \limsup_{k\to\infty} \Lambda_l^u(\Psi,k).$$

It is clear that λ_1 is δ-Ahlfors regular and that the metric d is in this case simply the standard Euclidean metric $d(x,y) := |x-y|$. The reason for the presence of the constant c will be described later.

We would like to show that this system is locally λ_1-ubiquitous relative to (ρ,l,u), for l and u as chosen above and some real positive function ρ satisfying with $\rho(r) \to 0$ as $r \to \infty$. After some thought it becomes apparent that an appropriate choice of ubiquitous function might be $\rho(q) := \gamma/q^2\psi^i(q)$ for some constant $\gamma > 0$. For then, the sum

$$\sum_{k=1}^{\infty} \left(\frac{\Psi(u_k)}{\rho(u_k)}\right)^{\delta} \ = \ \sum_{k=1}^{\infty} \frac{n_{ck}^2\psi^i(n_{ck})\psi^j(n_{ck})}{\gamma\, n_{ck}} \ = \ \frac{1}{\gamma}\sum_{k=1}^{\infty} n_{ck}\psi(n_{ck}),$$

diverges if and only if the sum $\sum_{r=1}^{\infty}\psi(r)$ diverges by the result of Schlömilch.

Next, we point out an important observation. When $\sum_{r\in\mathbb{N}} r^{1-s}\psi^{i+js}(r) = \infty$ and $s \in (i,1]$ we may assume that

$$\psi^i(r) \ > \ 1/r \qquad \text{for all } r \in \mathbb{N}. \tag{5.4}$$

To see this, let $\{r_k\}$ be the increasing sequence of integers for which $\psi^i(r_k) \le 1/r_k$. Then, for $s \in (i,1]$ we have

$$\sum_{k\in\mathbb{N}} r_k^{1-s}\psi^{i+js}(r_k) \ \le \ \sum_{k\in\mathbb{N}} r_k^{-(1+j/i)s} < \infty \quad \text{and} \quad \sum_{r\in\mathbb{N}\setminus\{r_k\}} r^{1-s}\psi^{i+js}(r) = \infty.$$

But, for each $k \in \mathbb{N}$ we have

$$\psi^i(r_k) \leq \frac{1}{r_k} \leq |r_k|_{\mathcal{D}}$$

and so $r_k \notin \mathcal{A}$. The upshot is that we may choose $J \subset \mathbb{N} \times (\mathbb{N} \setminus \{r_k\})$ in the ubiquity setup and neither the set $W_{\mathcal{D}}(i, j, \psi)$ nor the divergence of the corresponding volume sum is affected by the removal of the integers r_k.

Note that the above observation implies that $\rho(r) \to 0$ as $r \to \infty$. Furthermore, assume that the ratios of consecutive elements of $\mathcal{D}$ are bounded by $M \geq 2$; i.e., $n_{k+1}/n_k \leq M$ for all $k \in \mathbb{N}$. Then the monotonicity of ψ immediately implies that

$$\frac{\psi^j(n_{c(k+1)})}{n_{c(k+1)}} \leq \frac{\psi^j(n_{ck})}{n_{c(k+1)}} \leq \frac{\psi^j(n_{ck})}{M^c n_{ck}}$$

and so Ψ is trivially u-regular. Therefore, to prove the divergent part of Theorem 2.2 it suffices to show the following holds.

Proposition 5.1. *Let $\rho(q) := \gamma/q^2\psi^i(q)$. Then, the system defined above is a locally λ_1-ubiquitous relative to the triple $(\rho, n_{c(k-1)}, n_{ck})$ for some $c \in \mathbb{N}$ and some $\gamma > 0$ to be specified later.*

We begin by generalising the sequence specified in (5.1). Fix $k \in \mathbb{N}$, then for every natural number c define $m_k := m_k(c)$ as the unique natural number for which

$$\frac{1}{n_{cm_k}} \quad < \quad \psi^i(n_{ck}) \quad \leq \quad \frac{1}{n_{c(m_k-1)}}. \tag{5.5}$$

Again, this is always possible since ψ is decreasing and the elements of $\mathcal{D}$ are increasing. To prove Proposition 5.1 we will require the following consequence of a classical theorem of Dirichlet.

Proposition 5.2. *Fix $c \in \mathbb{N}$. Then, for every $x \in \mathbb{R}$ and every $k \in \mathbb{N}$ there exists $p/q \in \mathbb{Q}$ with $n_{cm_k} \leq q \leq n_{ck}$ such that*

$$\left|x - \frac{p}{q}\right| \quad < \quad \frac{n_{cm_k}}{qn_{ck}} \qquad \textit{and} \qquad |q|_{\mathcal{D}} \quad < \quad \frac{1}{n_{cm_k}}. \tag{5.6}$$

Proof of Proposition 5.2. Dirichlet's theorem states that for all $x' \in \mathbb{R}$ and for all $N \in \mathbb{N}$ there exists $p/q' \in \mathbb{Q}$ with $q' \leq N$ such that

$$|x' - p/q'| \quad < \quad 1/q'N.$$

Let $N := n_{ck}/n_{cm_k}$. Observation (5.4) guarantees that $N \geq 1$. Next, set $x := x'n_{cm_k}$ and $q = n_{cm_k}q'$. Then, for all $x \in \mathbb{R}$ we have

$$\left| xn_{cm_k} - \frac{pn_{cm_k}}{q} \right| \quad < \quad \frac{n_{cm_k}^2}{qn_{ck}}$$

whereby upon division by n_{cm_k} the desired inequality is reached. Furthermore, $n_{cm_k} \leq q \leq n_{cm_k}n_{ck}/n_{cm_k} = n_{ck}$ and $|q|_{\mathcal{D}} \leq 1/n_{cm_k}$ as required. $\square$

In what follows $K(c,k)$ will denote the set of integers q with $q \leq n_{ck}$ for which $|q|_{\mathcal{D}} < 1/n_{cm_k}$, whereas $K^*(c,k)$ will denote those integers q with $n_{c(k-1)} < q \leq n_{ck}$ for which $|q|_{\mathcal{D}} < 1/n_{cm_k}$. Finally, as proposed, set $\rho(r) := \gamma/r^2\psi^i(q)$ for some $\gamma > 0$.

In view of statement (4.2), to prove Proposition 5.1 we now need only show there exists an absolute constant $\kappa > 0$ such that

$$\lambda_1\left(I \cap \bigcup_{\substack{q \leq n_{ck}: \\ |q|_{\mathcal{D}} < \psi^i(q)}} \bigcap_{p=0}^{q-1} B\left(\frac{p}{q}, \rho(n_{ck})\right)\right) \geq \kappa\lambda_1(I) \tag{5.7}$$

for all every interval $I = [a,b] \subset [0,1)$ and for all k sufficiently large.

Assumption the ratios of consecutive elements of $\mathcal{D}$ are bounded above by some integer M. Upon setting $\gamma := M^{2c}$, it is easily verified that the LHS of (5.7) is bounded below by

$$\lambda_1\left(I \cap \bigcup_{K^*(c,k)} \bigcap_{p=0}^{q-1} B\left(\frac{p}{q}, \frac{n_{cm_k}}{q\, n_{ck}}\right)\right). \tag{5.8}$$

To see this simply note that for $n_{c(k-1)} < q \leq n_{ck}$ we have

$$n_{ck} \; < \; q \prod_{t=c(k-1)+1}^{ck} \frac{n_t}{n_{t-1}} \; \leq \; qM^{ck-(c(k-1)+1)+1} \; = \; qM^c$$

and by definition

$$n_{cm_k} \; = \; n_{c(m_k-1)} \prod_{s=c(m_k-1)+1}^{cm_k} \frac{n_s}{n_{s-1}} \; \leq \; n_{c(m_k-1)}M^c \; < \; \psi^{-i}(n_{ck})M^c.$$

Proposition 5.2 now implies that $(5.8) = \lambda_1(I) - \lambda_1(\mathcal{J})$ where

$$\mathcal{J} := \bigcup_{K(c,k-1)} \bigcap_{p=0}^{q-1} B\left(\frac{p}{q}, \frac{n_{cm_k}}{q\, n_{ck}}\right).$$

However, for each q with $n_{c(k-1)} < q \leq n_{ck}$ there are at most $\lambda_1(I)q + 3$ possible choices for p and so

$$\begin{aligned}\lambda_1(\mathcal{J}) &\leq 2\sum_{K(c,k-1)} \frac{n_{cm_k}}{qn_{ck}}(\lambda_1(I)q+3) \\ &= 2\lambda_1(I)\frac{n_{cm_k}}{n_{ck}}\sum_{K(c,k-1)} 1 \quad + \quad \frac{6n_{cm_k}}{n_{ck}}\sum_{K(c,k-1)}\frac{1}{q}.\end{aligned}$$

Notice that $\#(K(c,k)) = n_{ck}/n_{cm_k}$ and similarly $\#(K^*(c,k)) = (n_{ck} - n_{c(k-1)})/n_{cm_k}$. Therefore,

$$\begin{aligned}\frac{6n_{cm_k}}{n_{ck}}\sum_{K(c,k-1)}\frac{1}{q} &\leq \frac{6n_{cm_k}}{n_{ck}}\sum_{t=1}^{k-1}\sum_{K^*(c,t)}\frac{1}{q} \\ &\leq \frac{6n_{cm_k}}{n_{ck}}\sum_{t=1}^{k-1}\frac{(n_{ct}-n_{c(t-1)})}{n_{ct}n_{cm_k}} \\ &< \frac{6(M-1)(k-1)}{n_{ck}} \\ &< \frac{\lambda_1(I)}{4},\end{aligned}$$

for large enough k. Moreover,

$$\begin{aligned}2\lambda_1(I)\frac{n_{cm_k}}{n_{ck}}\sum_{K(c,k-1)} 1 &\leq 2\lambda_1(I)\frac{n_{c(k-1)}}{n_{ck}} \\ &\leq 2\lambda_1(I)\left(\prod_{t=c(k-1)+1}^{ck}\frac{n_t}{n_{t-1}}\right)^{-1} \\ &\leq 2\lambda_1(I)2^{-ck-(c(k-1)+1)+1} \\ &= 2^{1-c}\lambda_1(I).\end{aligned}$$

It follows that for $c \geq 2$ and for k large enough we have $\lambda_1(\mathcal{J}) \leq 3\lambda_1(I)/4$, and inequality (5.7) indeed holds with $\kappa := 1/4$.

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