Lagrangian structures and multidimensional consistency

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The candidate confirms that the work submitted is her own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

- Chapters 2 and 5 are based on Lobb, S.B. and F.W. Nijhoff. Lagrangian multiforms and multidimensional consistency. *Journal of Physics A: Mathematical and Theoretical*, **42** (2009) 454013.
- Chapter 3 is based on Lobb, S.B. and F.W. Nijhoff. Lagrangian multiform structure for the lattice Gel'fand-Dikii hierarchy. *Journal of Physics A: Mathematical and Theoretical*, **43** (2010) 072003.
- Chapter 4 is based on Lobb, S.B., F.W. Nijhoff and G.R.W. Quispel. Lagrangian multiform structure for the lattice KP system. *Journal of Physics A: Mathematical and Theoretical*, **42** (2009) 472002.

The contribution of the candidate to the results in all of the above jointly-authored publications was to provide the proofs and carry out all explicit computations; the concepts were developed in discussion with the other authors.

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Abstract

The conventional point of view is that the Lagrangian is a scalar object (or equivalently a volume form), which through the Euler-Lagrange equations provides us with one single equation (i.e., one per component of the dependent variable). Multidimensional consistency is a key integrability property of certain discrete systems; it implies that we are dealing with infinite hierarchies of compatible equations rather than with one single equation. Requiring the property of multidimensional consistency to be reflected also in the Lagrangian formulation of such systems, we arrive naturally at the construction of Lagrangian multiforms, i.e., Lagrangians which are the components of a form and satisfy a closure relation.

We demonstrate that the Lagrangians of many important examples fit into this framework: the so-called ABS list of systems on quad graphs, which includes the discrete Korteweg-de Vries equation; the Gel'fand-Dikii hierarchy, which includes the discrete Boussinesq equation; and the bilinear discrete Kadomtsev-Petviashvili equation.

On the basis of this we propose a new variational principle for integrable systems which brings in the geometry of the space of independent variables, and from this principle we can then derive any equation in the hierarchy. We also extend the notion of Lagrangian forms, and the corresponding new variational principle, to continuous systems, using the example of the generating partial differential equation for the Korteweg-de Vries hierarchy.

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Chapter 1

Introduction

This chapter gives an introduction to the areas of study relevant to the thesis; it gives an overview of the main results obtained in the past by others in the field, and provides some motivation and context for the new material in the following chapters. All of the results found during the course of my PhD studies lie in the realm of integrable systems (in the main, discrete integrable systems), so this is where we begin, followed by the theory of variational principles. We mention briefly in what context Lagrangian forms have appeared previously, and the attempts to establish a discrete variational complex, before giving a description of the contents of each chapter in the thesis.

1.1 Discrete integrable systems

In recent years there has been a growing interest in the integrability of discrete systems (systems with the independent variables taking discrete values) defined on two- or multidimensional lattices. Perhaps initially the study of such systems was motivated by the search for accurate approximations to continuous systems. However, it is becoming more and more the accepted point of view that lattice systems are important in their own right from a theoretical perspective, and, in fact, are thought to be richer and more generic than their continuous counterparts. We obtain continuous equations via continuum limits from discrete equations, and so the former may be regarded as a degeneration of the latter [93]. Indeed, taking derivatives we start with a difference operation, for example

$$\Delta f(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon}, \tag{1.1}$$

and then take a limit $\epsilon \to 0$ to obtain the derivative

$$\frac{df}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}, \tag{1.2}$$

suggesting that the original difference operation is the more fundamental operation. However, the theory of difference equations is still lagging behind that of differential equations, despite efforts at the beginning of the 20th century by mathematicians such as Nörlund[97] and Birkhoff[13]. One reason for this is that difference equations are essentially non-local, making their study more difficult. However, they have a wide variety of applications and so it is highly desirable that the theory is developed further.

1.1.1 Applications of discrete systems

Discrete systems are important from a practical perspective, since they appear in many real-life situations, in the areas of economics, financial mathematics, biology; they are vital to numerical analysis and hence any branch of science where computer calculations are involved, and of course it is possible that on a very small scale time and space may be discrete rather than continuous. There is even a growing interest in ultra-discrete systems (where not only the independent variables, but also the dependent variables, are discrete) which began with the study of cellular automata in the 1940s, and continued in the integrable case with, for example, the work of Ablowitz et al. [32, 4], Tokihiro et al. [115, 74], and Joshi and Lafortune [47, 48], although we will not be considering ultra-discrete systems here.

Discrete systems have been proposed in physics to model the fundamental interactions on the scale of the Planck constant where space and time themselves can be thought of as being discrete [59, 114], a possibility on which many famous physicists, including several Nobel laureates, have speculated. Gerard 't Hooft (who won the Nobel prize for physics in 1999), wrote in [114] that through investigating the quantization of black holes, he was led to the "suspicion that space at the Planck scale is discrete," since "the finiteness of the entropy of a black hole implies that the number of bits of information that can be stored there is finite and determined by the area of its horizon."

T.D. Lee, another winner of the Nobel prize in physics (this time in 1957) wrote a paper with the title *Can Time Be a Discrete Dynamical Variable?* [59] which led to several further publications by Lee and others on the formulation of fundamental physics in terms of difference equations [60, 61]. There he examined the possibility that time is discrete; note that in the relativistic case if either space or time is discrete, then both must be, since due to the required Lorentz invariance space and time have to be treated symmetrically. The usual continuous theory of mechanics then appears as an approximation.

1.1.2 Integrability of discrete systems

Many people have different views on what the definition of integrability should be, and already in the theory of continuous systems the notion of integrability is multifaceted; there was a book published recently entitled *What is Integrability?*[129]. As a working definition people often take the existence of a Lax pair[58], the existence of an infinite sequence of conservation laws[77], that a system is amenable to the inverse scattering transform method[33], or the definition due to Liouville of the existence of a sufficient number of independent invariants which are in involution[8]. Algorithms have been developed to test for integrability of a given equation, such as those in [76].

In the discrete realm, one single definition of integrability is even more elusive. There have been attempts to detect integrability, initially for maps[122], using the techniques of singularity confinement[38], algebraic entropy[12], assorted algebraic and arithmetic approaches (see [37] for a good review), and multidimensional consistency [91, 92, 14]; here follows a brief outline of some of these methods.

- *Singularity confinement* can be considered to be the discrete analogue of the Painlevé property; it is the analysis of the singularity structure of an equation. Essentially the Painlevé property for ordinary differential equations (ODEs) is the “absence of movable critical points in the general solution,” the theory being that “the solution of an ODE cannot escape the structure of singularities of the ODE. Such a structure can be studied from the equation itself, without any a priori knowledge of the solution, providing a deep insight into the possibility or not of performing the explicit integration” [22]. Singularity confinement is the requirement that movable singularities of mappings are cancelled out after a finite number of iterations [38].

- The *algebraic entropy* of an equation is a number defined by the growth of the degrees of the iterates of a map, and works well as a detector of integrability: integrable systems have a vanishing entropy, non-integrable systems have a non-vanishing entropy[12]. It is interesting to note that in a given dimension there is a minimum for the value of the entropy, i.e. there exists an interval around zero in which the entropy cannot assume values. The implication of this is that one cannot get arbitrarily close to integrability.

- A possible difference analogue of the Painlevé property is that an equation should admit sufficiently many finite-order meromorphic solutions. It was suggested by Ablowitz et al in [1] that “the integrability of many difference equations is related to the structure of their solutions at infinity in the complex plane and that Nevanlinna theory provides many of the concepts necessary to detect integrability in a large class of equations.” (Nevanlinna theory is a branch of complex analysis which deals with the value distribution theory of holomorphic functions in one variable).

- *Diophantine integrability*[40] requires that the logarithmic height of iterates of the discrete equation in an appropriate number field grow no faster than polynomially. It may be equivalent to the algebraic entropy approach, but it is easier to check numerically for a large number of iterates.

- *Multidimensional consistency* was proposed independently by Nijhoff and Walker[91, 92, 123], and by Bobenko and Suris[14], in the early years of this decade (although it was already implicit in some pre-2000 work, c.f. e.g. [95, 82, 84, 26]). In brief, it is the property that an equation can be embedded in a higher dimensional lattice, imposing copies of the equation (with appropriate lattice parameters) simultaneously in all possible lattice directions, and no inconsistency or multivaluedness occurs when the dependent variables are evaluated at any lattice site. This is analogous to the existence of commuting flows in the continuous realm, and in some cases provides us with an algorithmic way in which to construct Lax pairs. The definition will be made more explicit in Chapter 2.

The work in this thesis is concerned with the Lagrangian structure of systems which have the property of multidimensional consistency; we regard it as the key property or criterion of integrability for discrete systems.

1.1.3 Examples of integrable discrete systems

The study of integrable lattice systems is fairly new, and the earliest examples appear in the mid 1970s and early 1980s, when the research was focused on discretizing known continuous soliton systems [2, 3, 44, 45, 23, 82, 106]. Most of the known examples are 2-dimensional equations, probably the most famous being the *lattice potential Korteweg-de Vries* (KdV) equation, which was first presented in [82] in the form

$$(p - q + u_{n,m+1} - u_{n+1,m})(p + q - u_{n+1,m+1} + u_{n,m}) = p^2 - q^2, \tag{1.3}$$

where $u_{n,m}$ is the dependent variable evaluated at the lattice site (n, m), and $p, q \in \mathbb{C}$ are parameters. It was derived in [82] via the direct linearization method, and actually already appeared in numerical methods under the guise of the ϵ-algorithm [124], which is an acceleration method for slowly converging sequences. The lattice potential KdV equation provides an excellent example of a 2-dimensional lattice system; it is the simplest known such equation which still demonstrates all the richness of properties characteristic of these systems. Also 2-dimensional lattice systems themselves are very easy to visualize; they are obviously more general than 1-dimensional systems, and yet we only need to go to 3 dimensions in order to easily see the mechanism used to verify multidimensional consistency (or, equivalently in this case, consistency-around-a-cube).

The paper [87] contains a nice overview of other examples of discrete integrable lattice systems, we present some of them here.

- There are other integrable lattice systems in the KdV family; there is a discrete version of the *modified KdV equation*, related to (1.3) by a Miura transformation, having the form

  $$p(v_{n,m}v_{n,m+1} - v_{n+1,m}v_{n+1,m+1}) = q(v_{n,m}v_{n+1,m} - v_{n,m+1}v_{n+1,m+1}), \tag{1.4}$$

  where $v_{n,m}$ is the dependent variable.

- Also in the list of those systems of KdV type, there is a lattice Schwarzian KdV equation

  $$\frac{(z_{n,m} - z_{n,m+1})(z_{n+1,m} - z_{n+1,m+1})}{(z_{n,m} - z_{n+1,m})(z_{n,m+1} - z_{n+1,m+1})} = \frac{p^2}{q^2}, \tag{1.5}$$

  where $z_{n,m}$ is the dependent variable. Equation (1.5) is invariant under

Möbius transformations, i.e. under transformations of the form

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}. \tag{1.6}$$

- A famous example outside of the KdV family is the lattice analogue of the *sine-Gordon equation*, which was first presented by Hirota in [44] and as such is sometimes called the *Hirota equation.* This is

$$\sin(\theta_{n,m}+\theta_{n+1,m}+\theta_{n,m+1}+\theta_{n+1,m+1}) = \frac{p}{q}\sin(\theta_{n,m}-\theta_{n+1,m}-\theta_{n,m+1}+\theta_{n+1,m+1}), \tag{1.7}$$

  where $\theta_{n,m}$ is the dependent variable.

- Using the property of multidimensional consistency (again, this will be explained in depth in Chapter 2), a classification of two-dimensional scalar integrable lattice systems was given, in the affine linear case, by Adler, Bobenko and Suris [5, 6] (resulting in what is hereafter referred to as the ABS list). In addition to the known examples of lattice systems of KdV type, cf. [44, 45, 82, 106], this provided us with some new examples of integrable scalar lattice equations. More details will appear in the introduction to Chapter 2, the chapter concerned with 2-dimensional lattice systems.

- The example of the *lattice Gel'fand-Dikii hierarchy* comprises an infinite hierarchy of discrete integrable systems, where each equation higher up in the hierarchy has a higher number of components. At the lower end appear the well-known equations of the lattice potential KdV (which has only one component, and can be considered as a somewhat degenerate case here), and lattice Boussinesq equations; higher members of the hierarchy are coupled systems of partial difference equations.

- The *lattice Boussinesq equation* deserves special mention, as it has attracted much interest lately, for example with the construction of multisoliton solutions in [43], and with regard to the pentagram map in [103]. It first appeared in [86] along with the rest of the lattice Gel'fand-Dikii hierarchy; it can be written either as a coupled system of 3 equations, or as a scalar 9-point equation

$$\begin{aligned} \frac{p^3-q^3}{p-q+u_{n+1,m+1}-u_{n+2,m}} - \frac{p^3-q^3}{p-q+u_{n,m+2}-u_{n+1,m+1}} \\ +(p+2q)(u_{n+2,m+1}+u_{n,m+1})-(2p+q)(u_{n+1,m+2}+u_{n+1,m}) \\ +(p-q+u_{n+1,m+2}-u_{n+2,m+1})u_{n+2,m+2}+(p-q+u_{n,m+1}-u_{n+1,m})u_{n,m} \\ +u_{n+1,m}u_{n+2,m+1}-u_{n,m+1}u_{n+1,m+2}=0. \end{aligned} \tag{1.8}$$

It was shown in [123] to be consistent on a 27-point cube.

- Once we step out of the realm of 2-dimensional integrable lattice systems, in order to generalize to higher dimensional systems, we have at present far fewer examples to work with. However, there are still very important and well-known cases here. Most of the known examples of integrable 3-dimensional lattice systems are discrete equations of Kadomtsev-Petviashvili (KP) type [84]. The so-called *lattice KP equation* itself can be written

$$\frac{p-r+u_{n,m+1,k+1}-u_{n+1,m+1,k}}{p-r+u_{n,m,k+1}-u_{n+1,m,k}} = \frac{q-r+u_{n+1,m,k+1}-u_{n+1,m+1,k}}{q-r+u_{n,m,k+1}-u_{n,m+1,k}}. \tag{1.9}$$

- There is a *lattice modified KP equation* [84] with the form

$$\begin{aligned} p\left(\frac{v_{n,m,k+1}}{v_{n+1,m,k+1}}-\frac{v_{n,m+1,k}}{v_{n+1,m+1,k}}\right)+q\left(\frac{v_{n+1,m,k}}{v_{n+1,m+1,k}}-\frac{v_{n,m,k+1}}{v_{n,m+1,k+1}}\right) \\ +r\left(\frac{v_{n,m+1,k}}{v_{n,m+1,k+1}}-\frac{v_{n+1,m,k}}{v_{n+1,m,k+1}}\right) = 0, \end{aligned} \tag{1.10}$$

where $v_{n,m,k}$ is the dependent variable and p, q, r are lattice parameters.

- There is also a *lattice Schwarzian KP equation* [83, 27] with the form

$$\frac{(z_{n+1,m+1,k}-z_{n,m+1,k})(z_{n+1,m,k+1}-z_{n+1,m,k})(z_{n,m+1,k+1}-z_{n,m,k+1})}{(z_{n+1,m+1,k}-z_{n+1,m,k})(z_{n+1,m,k+1}-z_{n,m,k+1})(z_{n,m+1,k+1}-z_{n,m+1,k})}=1. \tag{1.11}$$

  where $z_{n,m,k}$ is the dependent variable.

- The *bilinear discrete KP equation*, which we will be dealing with later, is

$$A_{jk}\tau_i\tau_{jk}+A_{ki}\tau_j\tau_{ki}+A_{ij}\tau_k\tau_{ij}=0, \tag{1.12}$$

  where $\tau_{i,j,k}$ is the dependent variable and $A_{ij}=-A_{ji}$ are constants. This is also known as the *Hirota-Miwa* [78] equation.

1.1.4 Discrete systems from continuous systems (and vice versa)

We mentioned earlier that continuous equations may arise by taking limits of discrete equations. On the other hand, it is often possible to pass from a continuous equation to a discrete equation via *Bäcklund transformations*, which map solutions of an equation into new solutions. In this way, many new discrete integrable equations have been derived; indeed the most famous examples are known to be the Bäcklund transformations of famous continuous integrable systems.

In contrast to the relatively new study of discrete integrable systems, the theory of continuous integrable systems has been researched since the 19th century. One of the most famous examples of integrable equations is the KdV equation, which was derived in 1895 in a study of shallow water waves by Diederik Korteweg and Gustav de Vries [54] and takes the form

$$u_t=u_{xxx}+6uu_x, \tag{1.13}$$

where u is the dependent variable and x,t are the independent variables corresponding to space and time respectively. It was shown that this equation possesses an exact *solitary wave* solution, or *soliton*, a wave that maintains its shape while it travels at constant speed. In an oft-repeated story this phenomenon had been observed in 1834 by John Scott Russell [110] in the Union Canal in Scotland, where he rode on horseback alongside such a wave.

The KdV equation possesses much of the paraphernalia associated with integrable systems, such as a *Lax pair*[58], and an infinite sequence of conservation laws[77]. A Lax pair is simply an over-determined system of linear equations, from which the nonlinear equation can be derived through the compatibility conditions. The Lax pair for the KdV equation is

$$\psi_{xx} + u\psi = \lambda\psi, \tag{1.14a}$$

$$\psi_t = 4\psi_{xxx} + 6u\psi_x + 3u_x\psi. \tag{1.14b}$$

Here, ψ is some function of x and t, and λ is an additional (*spectral*) parameter. If these two linear equations are to be compatible, we require $(\psi_{xx})_t = (\psi_t)_{xx}$, and this leads to the conclusion that either $\psi \equiv 0$, or that u solves the KdV equation (1.13).

The Lax pair is an essential tool in finding exact solutions of a nonlinear partial differential equation via the *inverse scattering transform method*, which first appeared in a famous paper in 1967 by Gardner, Greene, Kruskal and Miura [33]. Details of this method can be found in many texts; since it is not relevant to this thesis it suffices here to say that the existence of a Lax pair is an important indicator of integrability.

If we introduce the variable $v \equiv \partial_x \ln \psi$, then (1.14a) becomes

$$u = \lambda - v_x - v^2, \tag{1.15}$$

and on substituting this into (1.14b) and differentiating with respect to x, we arrive at the equation

$$v_t = v_{xxx} - 6v^2 v_x + 6\lambda v_x, \tag{1.16}$$

which, when $\lambda = 0$, is known as the *modified KdV equation*. Note that it is invariant under the transformation $v \mapsto -v$. Making this transformation does not affect (1.16) at all, however it does change (1.15), changing u to something new, for instance we may call it $\tilde{u}$, where

$$\tilde{u} = \lambda + v_x - v^2. \tag{1.17}$$

Thus we have mapped a solution u to the KdV equation (1.13) to another solution $\tilde{u}$ of the same KdV equation, with the aid of the solutions v of the modified KdV equation (1.16).

From (1.15) and (1.17) we get

$$\tilde{u} + u = 2(\lambda - v^2), \tag{1.18a}$$

and

$$\tilde{u} - u = 2v_x, \tag{1.18b}$$

and if we introduce the new variable w for which $w_x = u$, we arrive at a relation which is the x-dependent part of the Bäcklund transformation

$$(\tilde{w} + w)_x = 2\lambda - \frac{1}{2}(\tilde{w} - w)^2. \tag{1.19}$$

Note that w also satisfies the *potential KdV equation*

$$w_t = w_{xxx} + 3w_x^2. \tag{1.20}$$

Here is the point at which we make the shift to discrete systems. We now consider the transformation $w \mapsto \tilde{w}$ to be a discrete shift in the variable w, and we can also consider another analogous shift $w \mapsto \hat{w}$ given by

$$(\hat{w} + w)_x = 2\mu - \frac{1}{2}(\hat{w} - w)^2. \tag{1.21}$$

Because of the permutability property of the Bäcklund transformations, under certain conditions it doesn't matter in which order we compose them; we may apply the " ˜ "-shift first, and then the " ˆ "-shift, or the other way around, we will arrive at the same result, so that that $\tilde{\hat{w}} = \hat{\tilde{w}}$. This is a highly non-trivial property, with a proof relying on the spectral properties lying behind the equation; it can be expressed using the diagram in Figure 1.1.

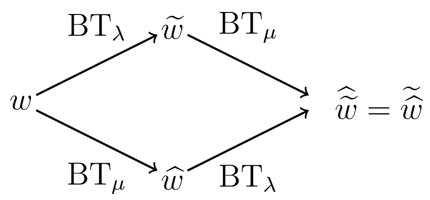

Figure 1.1: Permutability property of Bäcklund transformations.

The exact statement about the permutability of Bäcklund transformations is as follows: The Bäcklund transformations (1.19) and (1.21) for different parameters λ and μ generate solutions (with a suitable choice of integration constants) for which we have the commutation diagram of Fig. 1.1.

The proof of this is computational. In addition to (1.19) and (1.21) we have also the relations

$$(\tilde{\hat{w}} + \hat{w})_x = 2\lambda - \frac{1}{2}(\tilde{\hat{w}} - \hat{w})^2, \tag{1.22}$$

$$(\hat{\tilde{w}} + \tilde{w})_x = 2\mu - \frac{1}{2}(\hat{\tilde{w}} - \tilde{w})^2. \tag{1.23}$$

Taking the upper route of Fig. 1.1, we can solve $\tilde{w}$ in terms of w and $\hat{\tilde{w}}$:

$$\tilde{w} = \frac{1}{2}(\hat{\tilde{w}} + w) + \frac{(\hat{\tilde{w}} - w)_x + 2(\lambda - \mu)}{\hat{\tilde{w}} - w}, \tag{1.24}$$

and inserting this back into (1.19) we get

$$\begin{aligned}\lambda + \mu &= (\hat{\tilde{w}} + w)_x + \partial_x^2 \ln(\hat{\tilde{w}} - w) + \frac{1}{2}\big(\partial_x \ln(\hat{\tilde{w}} - w)\big)^2 \\ &\quad + \frac{1}{8}(\hat{\tilde{w}} - w)^2 + 2\left(\frac{\lambda - \mu}{\hat{\tilde{w}} - w}\right)^2. \end{aligned} \tag{1.25}$$

The relation (1.25) is clearly symmetric with respect to the interchange of λ and μ (and we can derive a similar relation taking the lower route of Fig. 1.1). Thus, with appropriate choice of integration constants, starting from an arbitrary "seed" w we may find solutions which are symmetric under interchange of λ and μ, and hence where $\hat{\tilde{w}} = \tilde{\hat{w}}$.

The permutability property allows us to eliminate the derivatives in x to derive from (1.19) and (1.21) a purely discrete equation

$$(w - \hat{\tilde{w}})(\tilde{w} - \hat{w}) + 4(\lambda - \mu) = 0. \tag{1.26}$$

This equation is the *discrete potential KdV equation.* Thus we have travelled from a fully continuous integrable system through to a fully discrete system. This discrete system is also integrable, in the sense that it is multidimensionally consistent, possesses a Lax pair, and soliton solutions [9, 10, 94].

It is easy to see from Figure 1.1 that we can iterate the Bäcklund transformations to construct an entire lattice, thanks to the permutability property; this lattice will have the form of Figure 1.2.

Finally, the discrete potential KdV equation (1.26) is truly the discrete analogue of the potential KdV equation (1.20) by the fact that in taking continuum limits of (1.26), we arrive once more at the fully continuous equation (1.20). Thus we are able to pass from the fully continuous equation (1.20) via Bäcklund transformations to the fully discrete equation (1.26), and then from there back to the fully continuous equation (1.20) via continuum limits. This demonstrates the intimate connections between continuous and discrete integrable systems.

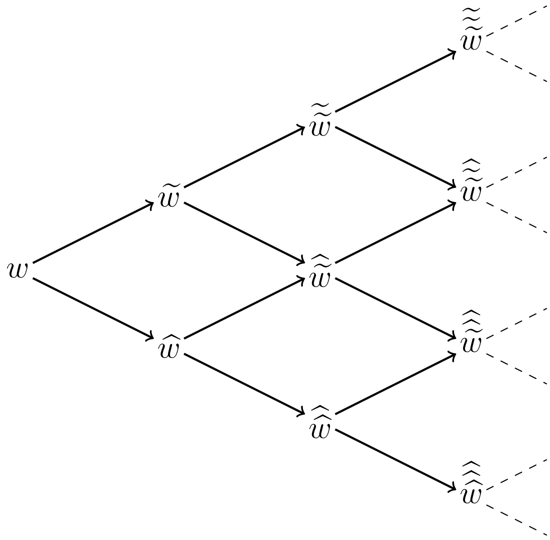

Figure 1.2: Lattice generated by Bäcklund transformations.

1.2 History of variational principles

The theory of variational principles has its origins in the 17th century, when Leibniz proposed the idea of the *vis viva* to describe the action of a force[62]. This *living force* is a scalar quantity closely related to the kinetic energy of a particle; together with another scalar quantity the *work function* (which can be thought of as the *potential energy*), it is enough to determine completely the motion of a particle or system. This was in contrast to the Newtonian approach to mechanics, which relies on finding all the forces acting on every particle at each instant.

Instrumental in developing the theory were Euler and Lagrange, who advocated a principle of least action. This principle can be illustrated in the following way. Suppose a particle is at a point P_1 at an initial time t_1 with a given velocity, and it reaches a point P_2 at time t_2. So far we do not know what path this particle may have followed to travel between the two points, but we can make a guess. Of course, in all probability this guess will turn out to be wrong, but that is irrelevant; we will be able to correct this path in order to discover that which the particle actually follows.

So we have a particle travelling between the points (P_1, t_1) and (P_2, t_2)

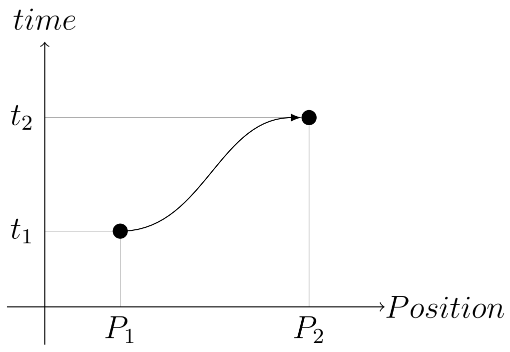

Figure 1.3: Tentative path from (P_1, t_1) to (P_2, t_2).

along some arbitrary curve, as in Figure 1.3. The constraint we put on this is that the sum of the kinetic and potential energies is constant, always equal to the value it had at (P_1, t_1). This fixes the velocity of the particle at each point in the path, and thus enables us to calculate the time at which the particle will pass any given point in the path. We can then work out the time-integral of the vis viva (the vis viva is in fact simply twice the kinetic energy) between P_1 and P_2; this integral is what is called the *action*.

$$\mathit{Action} = \int_{t_1}^{t_2} \mathit{vis\ viva}\ \ \mathrm{d}t. \tag{2.1}$$

For each path that we choose, this action will take a different value, and the principle states that it is the path for which the action assumes a minimum value which is the one we want; it happens to be the one the particle follows in reality.

As an example of how this works, suppose we have an action to be minimized of the form

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}; t)\ \mathrm{d}t, \tag{2.2}$$

where $\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$. Varying the path slightly (keeping the end points fixed), for small ϵ we get

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \delta\mathcal{L}(x,\dot{x};t)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \mathcal{L}(x+\epsilon\phi,\dot{x}+\epsilon\dot{\phi};t)-\mathcal{L}(x,\dot{x};t)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \epsilon\left(\frac{\partial\mathcal{L}}{\partial x}\phi+\frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{\phi}\right)\mathrm{d}t+\mathrm{O}(\epsilon^2). \end{aligned}\tag{2.3}$$

Since we want $\delta S = 0$, to first order in ϵ we must have

$$0=\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial x}\phi+\frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{\phi}\right)\mathrm{d}t,\tag{2.4}$$

which on integrating by parts is

$$0=\int_{t_1}^{t_2}\frac{\partial\mathcal{L}}{\partial x}\phi\,\mathrm{d}t+\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\phi\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)\phi\,\mathrm{d}t.\tag{2.5}$$

The second term will disappear since the variation ϕ is zero at the end-points t_1 and t_2, leaving us with

$$0=\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)\right)\phi\,\mathrm{d}t,\tag{2.6}$$

and so, since ϕ is arbitrary,

$$\frac{\partial\mathcal{L}}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)=0.\tag{2.7}$$

The equation (2.7) is often called the *Euler-Lagrange equation*, and is the equation of motion of the particle.

A great advantage of the use of these variational methods in mechanics is its independence of the choice of coordinate systems, of particular importance with regard to theories such as that of general relativity. A possible disadvantage is that all forces must be derivable from a scalar quantity (the work function), and so it cannot deal for example with frictional forces, at least not until the microscopic forces are taken into account. It is however a much less cumbersome, and more beautiful, description than the Newtonian approach.

1.2.1 Hamiltonian versus Lagrangian formulation

Conventionally the Hamiltonian has been the central object in (continuous) integrable systems [29]. Of course, it is often possible to pass between Lagrangian and Hamiltonian theories via Legendre transforms, although this is in many (non-Newtonian) cases not a trivial matter. Nevertheless, most integrable partial differential equations seem to admit a Lagrangian description; in fact, a universal Lagrangian structure for integrable systems admitting a Lax pair was formulated by Zakharov and Mikhailov [128]. In the discrete case one can argue that the Lagrangian is the more fundamental object; attempting to pass from the Lagrangian to the Hamiltonian description does not furnish us with an invariant Hamiltonian, and so there seems to be little point in doing so.

We adhere to Dirac's opinion [24], that the Lagrangian formulation for classical dynamics, rather than the Hamiltonian, is the more fundamental perspective. The action functional is a relativistic invariant, and so the Lagrangian method can easily be expressed in relativistic form, whereas the Hamiltonian is one component of an energy-momentum tensor, and so a particular time variable is singled out. Dirac sought a Lagrangian formulation for quantum mechanics, and in doing so noted that "*we must try to take over the ideas of the classical Lagrangian theory, not the equations of the classical Lagrangian theory*" [24] mainly due to the fact that the Euler-Lagrange equations involve partial derivatives of the Lagrangian with respect to the coordinates and velocities, and it is difficult to assign a meaning to these derivatives in the quantum realm. Indeed, this was at least in part our motivation for studying the Lagrangian structure of integrable systems, it seems that especially in the case of discrete systems the most fruitful path to quantization is via the Lagrangian.

1.2.2 Discrete variational principles

A discrete calculus of variations was first developed outside the scope of integrable systems in the 1970s by Cadzow[17], Logan[68] and Maeda[69]. Cadzow's original motivation was the use of the digital computer in modern systems and the solution of control problems, and it became clear that the formulation of a discrete calculus of variations was important for numerical methods, in optimization and engineering problems. In the continuous case the central object is the *action*, which is the time integral of a scalar quantity, called the *Lagrangian* after one of the founders of the calculus of variations, Joseph Louis Lagrange (in mechanics this is the difference between the kinetic and potential energies). Now in the discrete realm instead of an integral we have a sum over the independent variable(s).

Cadzow derived the discrete Euler equation in [17] for a 1-dimensional system with independent variable n and dependent variable r_n; his approach was then put into a more formal language by Logan[68], which made it clearer that the equations are derived via a variational approach. The aim was to find a sequence $\{\bar{r}_M, \bar{r}_{M+1}, \ldots \bar{r}_{N+1}\}$ which would extremize the functional

$$J[\{r_n\}] = \sum_{n=M}^{N} F(n, r_{n+1}, r_n). \tag{2.8}$$

A necessary condition for this is that $\delta J[\{r_n\}] = 0$, i.e.,

$$\begin{aligned} 0 &= \delta \sum_{n=M}^{N} F(n, r_{n+1}, r_n) \\ &= \sum_{n=M}^{N} \left\{ \frac{\partial}{\partial r_{n+1}} F(n, r_{n+1}, r_n) \delta r_{n+1} + \frac{\partial}{\partial r_n} F(n, r_{n+1}, r_n) \delta r_n \right\} \\ &= \frac{\partial}{\partial r_M} F(M, r_{M+1}, r_M) \delta r_M + \frac{\partial}{\partial r_{N+1}} F(N, r_{N+1}, r_N) \delta r_{N+1} \\ &\quad + \sum_{n=M+1}^{N} \left\{ \frac{\partial}{\partial r_n} F(n-1, r_n, r_{n-1}) + \frac{\partial}{\partial r_n} F(n, r_{n+1}, r_n) \right\} \delta r_n. \end{aligned} \tag{2.9}$$

So, defining

$$F_1(x, y, z) = \frac{\partial}{\partial y} F(x, y, z) \;\text{ and }\; F_2(x, y, z) = \frac{\partial}{\partial z} F(x, y, z), \tag{2.10}$$

Cadzow calls the following the *discrete Euler equation*

$$F_1(n-1, r_n, r_{n-1}) + F_2(n, r_{n+1}, r_n) = 0, \; n = M+1, \ldots, N. \tag{2.11}$$

It should be noted that this holds away from the boundary, since it is valid for $M + 1 \leq n \leq N$; at the boundary points we should have instead

$$F_2(M, r_{M+1}, r_M) = 0, \quad \text{and} \quad F_1(N, r_{N+1}, r_N) = 0. \tag{2.12}$$

The later paper by Logan [68] also showed how to obtain conservation theorems, or first integrals of the discrete Euler equation, from a direct study of the invariance properties of the discrete Lagrangian $F(n, r_{n+1}, r_n)$. This is in effect the discrete analogue of the well-known Noether's theorem in continuous theory (Noether's theorem first appeared in [96]; it states that to any differentiable symmetry of the action of a physical system, there corresponds a conservation law).

Maeda was also instrumental in the development of discrete variational principles, he sought to find a natural discretization of classical mechanics, initially by examining the role of the Poisson bracket in the Hamiltonian theory [69], subsequently studying the Lagrangian formulation of discrete systems [71].

Much progress has been made since: Lagrangian structures have been established for several discrete integrable systems such as Lagrangian mappings [121, 122, 80]. Furthermore, Lagrangians and/or actions have been constructed for integrable two-dimensional lattice equations, cf. [20, 86, 5]. Discrete Lagrangian systems on arbitrary graphs were proposed in [98], and a discrete variational complex was set up in [46].

The usual point of view is that the Lagrangian is a scalar object (or equivalently a volume form), which through the Euler-Lagrange equations provides us with one single equation (i.e. one per component of the dependent variable). In contrast, we take the point of view that in the case of an integrable system, where due to the multidimensional consistency several equations can be imposed simultaneously on one and the same dependent variable, the Lagrangian should reflect this property; it should be an extended object capable of producing a multitude of consistent equations from a variational principle. Thus we propose in this thesis an action in which the key ingredient is a Lagrangian 2-form (in the case of integrable discrete equations in two independent variables) or, more generally, a multiform (in the case of a larger number of independent variables).

Although the notion of a Lagrangian multiform is not new, and goes back to Cartan and Lepage [21, 63], cf. also [51] for a review, even in those theories the role of the Lagrangian is that of a volume form producing the equations of motion in a conventional way.

There have been attempts made to construct discrete analogues of differential forms, e.g. by Hydon and Mansfield in [46, 73]. They developed a formalism of variational complexes for discrete systems, with the purpose of obtaining conservation laws of arbitrary partial difference equations (a differential-difference calculus appeared earlier in [56]). Their approach is different to the one we take; Hydon and Mansfield start from a general principle and then seek examples on which to apply it, whilst we start with the examples and develop a general principle on the basis of these examples. It is not obvious how the abstract notation of [46, 73] should be applied in concrete computations, and we found that for our purposes it was even unnecessary, as the Lagrangian forms enter in an entirely natural way.

1.3 Outline of thesis

Chapter 1 is the introductory chapter, consisting of an overview of the area of integrable systems and discrete integrable systems in particular, outlining some key results and trends in the field.

Chapter 2 deals chiefly with 2-dimensional discrete integrable systems. There is a more in-depth introduction to equations on quadrilateral graphs, including the recent classification results of Adler, Bobenko and Suris [5]. We give a Lagrangian formulation of these equations, show that the Lagrangians satisfy a closure relation, and propose a novel variational principle on that basis. The Lagrangians now appear in the guise of Lagrangian forms, as opposed to scalar objects.

Chapter 3 is concerned with an example, or hierarchy of examples, of a 2-dimensional multicomponent system, the *Gel'fand-Dikii hierarchy*. We present a universal Lagrangian structure for this hierarchy, demonstrate that it satisfies a closure relation and hence fits in with our new variational principle.

Chapter 4 contains a higher dimensional system, the 3-dimensional *bilinear lattice Kadomtsev-Petvishvili equation*. Again we present a Lagrangian structure for this system, demonstrate that it obeys a higher dimensional closure relation, and is compatible with the new variational principle.

Chapter 5 deals with the continuous analogue of the theory of Lagrangian forms and the new variational principle, as illustrated by the examples of the linear and full nonlinear generating partial differential equation for the Korteweg-de Vries hierarchy, and other generating partial differential equations.

Chapter 6 is a concluding chapter which contains some discussion, a brief account of two lines of recent development which could not be incorporated into the thesis, speculations on the results and future directions of research.

Chapter 2

Lagrangian 2-forms

2.1 2-dimensional lattice systems

A large and important class of discrete systems is that of equations on quad graphs, i.e., planar graphs with quadrilateral faces. These are equations on 2-dimensional lattices, which link points defined on the vertices of an elementary plaquette (a quadrilateral of minimal size, see Figure 2.2). Probably the best-known example of such equations is the lattice potential KdV equation, already mentioned in the Introduction, which first appeared in [44].

2.1.1 Notation

To fix the notation we will use throughout this chapter, let n_1, n_2 be the independent variables which constitute the coordinates in 2-dimensional space, $u = u(n_1, n_2)$ be the dependent variable, and α_1, α_2 be lattice parameters corresponding to the lattice directions n_1, n_2 respectively (these lattice parameters can be thought of as measures for the grid size). Shifts in the dependent variable u will be denoted by subscripts, so that $u_1 = u(n_1 + 1, n_2)$ and $u_2 = u(n_1, n_2 + 1)$, backwards shifts will be shown as $u_{-1} = u(n_1 - 1, u_2)$, and shifts in 2 lattice directions will be written as $u_{1,2} = u(n_1 + 1, u_2 + 1)$. This is all illustrated in Figure 2.1.

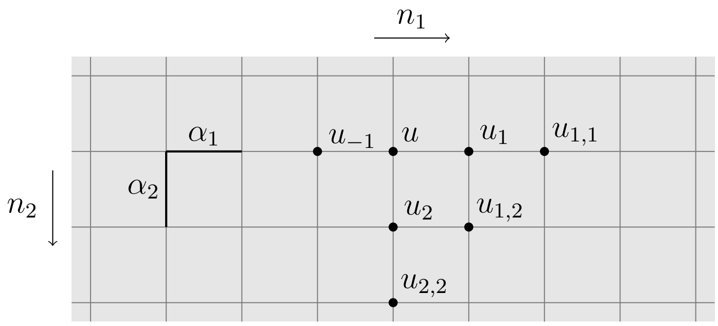

Figure 2.1: 2-d lattice.

In this notation, equations on quad graphs will have the form

$$Q(u, u_1, u_2, u_{1,2}; \alpha_1, \alpha_2) = 0, \tag{1.1}$$

for some function Q, so that they involve the 4 points around an elementary plaquette as shown in Figure 2.2.

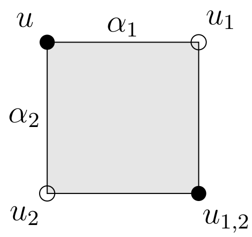

Figure 2.2: Elementary plaquette

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2.1.2 Multidimensional consistency

Of these equations, we are interested in those that are integrable, a notion which, as we have discussed, for discrete systems is even more difficult to define than for continuous systems. However, since its introduction in [91, 14], *multidimensional consistency* has come to be regarded as one of the hallmarks of integrability for discrete systems. To repeat, it is the property that several copies of an equation may be imposed simultaneously on a higher dimensional lattice, and no inconsistency or multivaluedness occurs in the evaluation of the dependent variables at any lattice site.

For equations of the form $Q = 0$ where Q is *multilinear*, i.e., Q is linear in each of the arguments u and its shifts $u_1, u_2, u_{1,2}$ (although not necessarily linear in the lattice parameters α_1, α_2), we may solve the equation uniquely for any argument. Then there is a simple test of multidimensional consistency, which in this case is equivalent to *consistency around a cube*, or 3-dimensional consistency. Start by introducing a third lattice direction associated with the independent variable n_3 and lattice parameter α_3, so we now consider the dependent variable u to depend on n_1, n_2 and n_3. Impose copies of the equation (1.1) on each elementary plaquette in the 3-dimensional lattice, so that in addition to (1.1) we have

$$Q(u, u_2, u_3, u_{2,3}; \alpha_2, \alpha_3) = 0, \tag{1.2}$$

and

$$Q(u, u_3, u_1, u_{3,1}; \alpha_3, \alpha_1) = 0, \tag{1.3}$$

and all shifted copies of the equations (1.1), (1.2) and (1.3)

$$Q(u_3, u_{31}, u_{23}, u_{31,23}; \alpha_1, \alpha_2) = 0, \tag{1.4}$$

$$Q(u_1, u_{12}, u_{31}, u_{12,31}; \alpha_2, \alpha_3) = 0, \tag{1.5}$$

$$Q(u_2, u_{23}, u_{12}, u_{23,12}; \alpha_3, \alpha_1) = 0. \tag{1.6}$$

Now, given initial data of u, u_1, u_2, u_3, there are in principle 3 different ways to compute $u_{1,2,3}$ depending on the order in which we apply the equations (1.1), (1.2) and (1.3), i.e., depending on which route around the elementary cube shown in Figure 2.3 we take.

We say that the equation (1.1) is consistent around a cube if the value of $u_{1,2,3}$ is independent of the way in which it is computed.

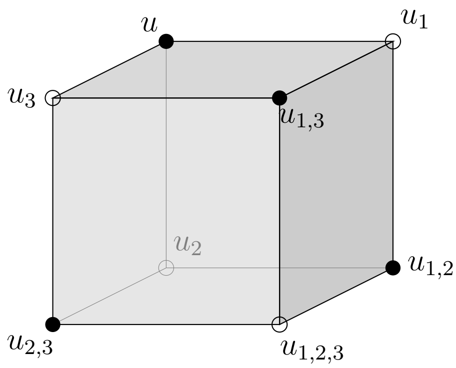

Figure 2.3: Elementary cube.

As an example, take the discrete (potential) KdV equation:

$$(u - u_{1,2})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0, \tag{1.7}$$

which we can rearrange to get an expression for $u_{1,2}$

$$u_{1,2} = \frac{u(u_1 - u_2) - \alpha_1 + \alpha_2}{u_1 - u_2}. \tag{1.8a}$$

In the other pairs of lattice directions, we have also

$$u_{2,3} = \frac{u(u_2 - u_3) - \alpha_2 + \alpha_3}{u_2 - u_3}, \tag{1.8b}$$

and

$$u_{3,1} = \frac{u(u_3 - u_1) - \alpha_3 + \alpha_1}{u_3 - u_1}. \tag{1.8c}$$

Shifting (1.8a) in the third direction to get $u_{1,2,3}$ and then substituting in the values of $u_{2,3}$ and $u_{3,1}$ from (1.8b) and (1.8c) respectively gives

$$\begin{aligned} u_{1,2,3} &= \frac{u_3(u_{3,1} - u_{2,3}) - \alpha_1 + \alpha_2}{u_{3,1} - u_{2,3}} \\ &= -\frac{(\alpha_1 - \alpha_2)u_1u_2 + (\alpha_2 - \alpha_3)u_2u_3 + (\alpha_3 - \alpha_1)u_3u_1}{(\alpha_1 - \alpha_2)u_3 + (\alpha_2 - \alpha_3)u_1 + (\alpha_3 - \alpha_1)u_2}, \end{aligned} \tag{1.9}$$

which is clearly invariant under cyclic permutation. Thus we would get the same expression for $u_{1,2,3}$ if we started with (1.8b) shifted in the first direction, or (1.8c) shifted in the second direction, and so there is no multivaluedness arising when we evaluate $u_{1,2,3}$. Hence the discrete KdV equation (1.7) is consistent around a cube.

Note that for the discrete KdV equation $u_{1,2,3}$ depends on u_1, u_2 and u_3 but is independent of u, i.e., the equation possesses the *tetrahedron property*, so-called because, as can be seen from Figure 2.3, joining the white vertices $u_{1,2,3}, u_1, u_2, u_3$ makes a tetrahedron shape. A similar relation exists between the black vertices of the cube.

2.1.3 ABS list

The equation (1.7) also possesses all the symmetries of the square, i.e., D_4 symmetries. This can be seen from the fact that the equation is clearly unchanged under the transformation $u \leftrightarrow u_1$, $u_2 \leftrightarrow u_{1,2}$, and is also unchanged under the transformation $u_1 \leftrightarrow u_2$, $\alpha_1 \leftrightarrow \alpha_2$, see Figure 2.4.

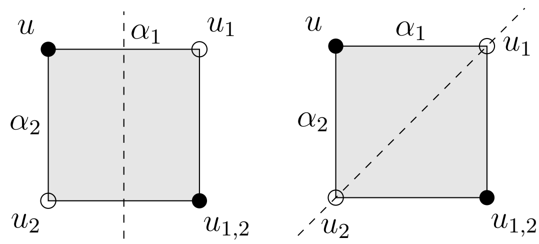

Figure 2.4: D_4 symmetries.

In [5], the classification problem of quadrilateral lattice equations of the form (1.1) was considered, where Q has the following properties:

1. Multilinearity.

2. D_4 symmetry.

3. Multidimensional consistency.

4. Tetrahedron property.

The result of the classification study of [5] was a list of 9 equations, up to Möbius transformations, labelled H1-H3, Q1-Q4 and A1-A2. Some of these were already well-known, e.g. H1 is the discrete KdV equation, and H3 is the discrete modified KdV equation, but the classification also produced some new equations. We will refer to the list of equations resulting from the classification as the *ABS list*. The tetrahedron property was replaced in a later paper [6] by certain non-degeneracy conditions; the classification there led solely to the list Q1-Q4. The original list is as follows:

H1:

$$(u - u_{1,2})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0;$$

H2:

$$(u - u_{1,2})(u_1 - u_2) - (\alpha_1 - \alpha_2)(u + u_1 + u_2 + u_{1,2}) - \alpha_1^2 + \alpha_2^2 = 0; \tag{1.10}$$

H3:

$$\alpha_1(uu_1 + u_2u_{1,2}) - \alpha_2(uu_2 + u_1u_{1,2}) + \delta(\alpha_1^2 - \alpha_2^2) = 0; \tag{1.11}$$

Q1:

$$\alpha_1(u - u_2)(u_1 - u_{1,2}) - \alpha_2(u - u_1)(u_2 - u_{1,2}) + \delta^2\alpha_1\alpha_2(\alpha_1 - \alpha_2) = 0; \tag{1.12}$$

Q2:

$$\begin{aligned} \alpha_1(u - u_2)(u_1 - u_{1,2}) - \alpha_2(u - u_1)(u_2 - u_{1,2}) & \\ +\alpha_1\alpha_2(\alpha_1 - \alpha_2)(u + u_1 + u_2 + u_{1,2}) & \\ -\alpha_1\alpha_2(\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2) &= 0; \end{aligned} \tag{1.13}$$

Q3:

$$
\begin{aligned}
(\alpha_2^2-\alpha_1^2)(uu_{1,2}+u_1u_2)+\alpha_2(\alpha_1^2-1)(uu_1+u_2u_{1,2})& \\
-\alpha_1(\alpha_2^2-1)(uu_2+u_1u_{1,2})& \\
-\delta^2(\alpha_1^2-\alpha_2^2)(\alpha_1^2-1)(\alpha_2^2-1)/(4\alpha_1\alpha_2) &= 0;
\end{aligned}
$$

Q4:

$$
\begin{aligned}
a_0uu_1u_2u_{1,2}+a_1(uu_1u_2+u_1u_2u_{1,2}+uu_2u_{1,2}+uu_1u_{1,2})& \\
+a_2(uu_{1,2}+u_1u_2)+a_2'(uu_1+u_2u_{1,2})& \\
+a_2''(uu_2+u_1u_{1,2})+a_3(u+u_1+u_2+u_{1,2})+a_4 &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
a_0=a+b,\ a_1=-\alpha_2a-\alpha_1b,\ a_2=\alpha_2^2a+\alpha_1^2b,& \\
a_2'=\frac{ab(a+b)}{2(\alpha_1-\alpha_2)}+\alpha_2^2a-(2\alpha_1^2-\frac{g_2}{4})b,& \\
a_2''=\frac{ab(a+b)}{2(\alpha_2-\alpha_1)}+\alpha_1^2b-(2\alpha_2^2-\frac{g_2}{4})a,& \\
a_3=\frac{g_3}{2}a_0-\frac{g_2}{4}a_1,\ a_4=\left(\frac{g_2}{4}\right)^2a_0-g_3a_1&
\end{aligned}
$$

with $a^2=r(\alpha_1)$, $b^2=r(\alpha_2)$ and $r(x)=4x^3-g^2x-g_3$;

A1:

$$
\alpha_1(u+u_2)(u_1+u_{1,2})-\alpha_2(u+u_1)(u_2+u_{1,2})-\delta^2\alpha_1\alpha_2(\alpha_1-\alpha_2)=0;
$$

A2:

$$
\begin{aligned}
(\alpha_2^2-\alpha_1^2)(uu_1u_2u_{1,2}+1)+\alpha_2(\alpha_1^2-1)(uu_2+u_1u_{1,2})& \\
-\alpha_1(\alpha_2^2-1)(uu_1+u_2u_{1,2}) &= 0.
\end{aligned}
$$

In H3, Q1, Q3 and A1, δ is an extra arbitrary parameter.

2.2 Lagrangians for 2-dimensional lattice systems

It was shown in [5] that all these equations admit an action principle, which is based on the *3-leg form* of the quadrilateral equation, a construct introduced in [14]. The 3-leg form is a way of writing the equation in the following way

$$\psi(u, u_1; \alpha_1) - \psi(u, u_2; \alpha_2) = \phi(u, u_{1,2}; \alpha_1, \alpha_2) \tag{2.1}$$

so that there are two "short" legs (u, u_1) and (u, u_2), and one "long" leg $(u, u_{1,2})$.

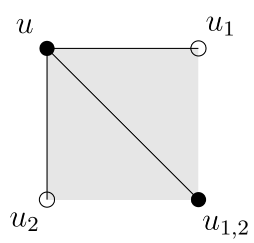

Figure 2.5: 3 legs.

Remark: In the generic case, the D_4 symmetry is not always manifest on the level of the 3-leg form, e.g. in the case of Q4 where the 3-leg form lives on the level of the uniformizing variables of the relevant elliptic curve. It is through the connection with the affine linear form of the equations that the symmetry under reversal of the shifts becomes apparent.

The actions in [5] were obtained by integrating the terms in the 3-leg form to create functions $L(x, x_1; \alpha_1)$ and $\Lambda(x, x_{1,2}; \alpha_1, \alpha_2)$, both of which are symmetric with respect to the interchange of the first two arguments, and x is related to u by the point transformations

- $u = x$ for H1, H2, Q1 and A1,
- $u = x^2$ for Q2,
- $u = e^{2x}$ for H3, $Q3|_{\delta=0}$ and A2,

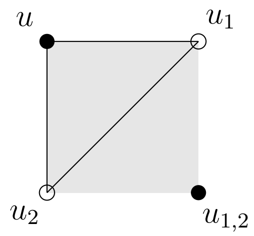

Figure 2.6: Alternative 3 legs for Lagrangian.

- $u = \cosh 2x$ for $Q3|_{\delta\neq 0}$, and
- $u = \wp(x)$, where $\wp$ is the Weierstrass elliptic function, for Q4.

The functions L and Λ are defined as follows

$$\psi(u, u_1; \alpha_1) = \psi(f(x), f(x_1); \alpha_1) = \frac{\partial}{\partial x} L(x, x_1; \alpha_1), \tag{2.2a}$$

$$\phi(u, u_{1,2}; \alpha_1, \alpha_2) = \phi(f(x), f(x_{1,2}); \alpha_1, \alpha_2) = \frac{\partial}{\partial x} \Lambda(x, x_{1,2}; \alpha_1, \alpha_2). \tag{2.2b}$$

Then if E_1 denotes the set of edges in the n_1-direction, i.e., all those which have the label α_1 associated with them, E_2 denotes the set of edges in the n_2-direction, i.e., all those which have the label α_2 associated with them, and E_3 denotes the set of diagonals in the direction $x \leftrightarrow x_{1,2}$, the action is defined by

$$S = \sum_{(x,x_1)\in E_1} L(x, x_1; \alpha_1) - \sum_{(x,x_2)\in E_2} L(x, x_2; \alpha_2) - \sum_{(x,x_{1,2})\in E_3} \Lambda(x, x_{1,2}; \alpha_1, \alpha_2). \tag{2.3}$$

By summing this in a different way, one can infer 4-point Lagrangians, however, for our purpose it is more useful to identify 3-point Lagrangians $\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2)$, defined on the 3 'legs' as in Figure 2.6, which are anti-symmetric with respect to the interchange of lattice directions. These Lagrangians are obtained from the 4-point Lagrangians by a reflection in the diagonal term, made possible by the symmetries of the equation. In terms of these new 3-point Lagrangians the action will take the form

$$S = \sum_{n_1,n_2\in\mathbb{Z}} \mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2), \tag{2.4}$$

and in this specific form Lagrangians $\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2)$ of all ABS equations can be established. In some cases, namely lattice equations "of KdV type" (i.e., lattice equations equivalent to the KdV, modified KdV, or Schwarzian KdV equations) a Lagrangian description had been previously established [20, 86].

The discrete Euler-Lagrange equations arising from the variational principle that $\delta S = 0$, under local variations $\delta u(n_1, n_2)$ of the dependent variable, are given by

$$\frac{\partial}{\partial u}\Big(\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2) + \mathcal{L}(u_{-1}, u, u_{-1,2}; \alpha_1, \alpha_2) + \mathcal{L}(u_{-2}, u_{1,-2}, u; \alpha_1, \alpha_2)\Big) = 0. \tag{2.5}$$

This can easily be seen as follows:

$$\begin{aligned} 0 &= \delta S \\ &= \sum_{n_1, n_2 \in \mathbb{Z}} \Big\{ \frac{\partial}{\partial u}\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2)\delta u + \frac{\partial}{\partial u_1}\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2)\delta u_1 \\ &\qquad + \frac{\partial}{\partial u_2}\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2)\delta u_2 \Big\} \\ &= \sum_{n_1, n_2 \in \mathbb{Z}} \Big\{ \frac{\partial}{\partial u}\mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2) + \frac{\partial}{\partial u}\mathcal{L}(u_{-1}, u, u_{-1,2}; \alpha_1, \alpha_2) \\ &\qquad + \frac{\partial}{\partial u}\mathcal{L}(u_{-2}, u_{1,-2}, u; \alpha_1, \alpha_2) \Big\} \delta u, \end{aligned} \tag{2.6}$$

which implies (2.5).

Below we list specific examples of ABS lattice equations together with their 3-point Lagrangians. Although similar formulae can be established for the remaining cases in the ABS list, we will restrict ourselves here to these particular examples. It should be noted that the discrete Euler-Lagrange equations (2.5) do not give the quadrilateral lattice equations themselves, but rather a discrete derivative of the original equation which is defined on 7 points of the lattice (lattice equations on 7-point stencils have attracted a considerable amount of interest in recent years, cf. e.g. [81]). The Euler-Lagrange equation actually results in a compound of two copies of the 3-leg form of the original equation: one reflected in the n_1-direction and one reflected in the n_2-direction. For example, the discrete Euler-Lagrange equation for H1 is

$$u_1 - u_{-2} - \frac{\alpha_1 - \alpha_2}{u - u_{1,-2}} + u_{-1} - u_2 - \frac{\alpha_1 - \alpha_2}{u - u_{-1,2}} = 0, \tag{2.7}$$

which contains two copies of (1.7) defined on the points shown in Figure 2.7. Note

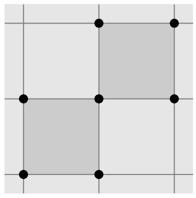

Figure 2.7: 7-point stencil.

that this is entirely analogous to the continuous case, where the Lagrangian for the continuous KdV equation produces through the Euler-Lagrange equations not the KdV equation itself, but a derivative.

The situation is similar for all of the Lagrangians of the systems given below; through the Euler-Lagrange equations we get a 7-point equation defined on the points shown in Figure 2.7, consisting of two copies of the original 4-point equation.

2.2.1 H1

This is the discrete potential KdV equation, one of the most fundamental examples in discrete integrable systems. The Lagrangian was first given in [20].
The original equation is

$$(u - u_{1,2})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0; \tag{2.8a}$$

written in 3-leg form this is

$$(u + u_1) - (u + u_2) = \frac{\alpha_1 - \alpha_2}{u - u_{1,2}}, \tag{2.8b}$$

and it possesses the Lagrangian

$$\mathcal{L} = (u_1 - u_2)u - (\alpha_1 - \alpha_2) \ln(u_1 - u_2). \tag{2.8c}$$

2.2.2 H2

The original equation is

$$(u - u_{1,2})(u_1 - u_2) - (\alpha_1 - \alpha_2)(u + u_1 + u_2 + u_{1,2}) - \alpha_1^2 + \alpha_2^2 = 0; \tag{2.9a}$$

written in multiplicative 3-leg form this is

$$\frac{u+u_1+\alpha_1}{u+u_2+\alpha_2}=\frac{u-u_{1,2}+\alpha_1-\alpha_2}{u-u_{1,2}-\alpha_1+\alpha_2}, \tag{2.9b}$$

where we may take logarithms to obtain the additive 3-leg form of (2.1) (this is also the case with many of the following equations). It possesses the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\;& (u+u_1+\alpha_1)\ln(u+u_1+\alpha_1)-(u+u_2+\alpha_2)\ln(u+u_2+\alpha_2)\\ &-(u_1-u_2+\alpha_1-\alpha_2)\ln(u_1-u_2+\alpha_1-\alpha_2)\\ &+(u_1-u_2-\alpha_1+\alpha_2)\ln(u_1-u_2-\alpha_1+\alpha_2).\end{aligned} \tag{2.9c}$$

2.2.3 H3

This is also known as the discrete modified (potential) KdV equation.

The original equation is

$$\alpha_1(uu_1+u_2u_{1,2})-\alpha_2(uu_2+u_1u_{1,2})+\delta(\alpha_1^2-\alpha_2^2)=0; \tag{2.10a}$$

written in 3-leg form this is

$$\frac{e^{2x+2x_1}+\delta e^{2a_1}}{e^{2x+2x_2}+\delta e^{2a_2}}=\frac{\sinh(x-x_{1,2}-a_1+a_2)}{\sinh(x-x_{1,2}+a_1-a_2)}, \tag{2.10b}$$

where $u=e^{2x}$ and $\alpha_1=e^{2a_1}$, and it possesses the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\;& -\mathrm{Li}_2\left(\frac{uu_1}{-\alpha_1\delta}\right)+\mathrm{Li}_2\left(\frac{uu_2}{-\alpha_2\delta}\right)+\mathrm{Li}_2\left(\frac{\alpha_2u_1}{\alpha_1u_2}\right)-\mathrm{Li}_2\left(\frac{\alpha_1u_1}{\alpha_2u_2}\right)\\ &+\ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(u)+\ln(\alpha_2^2)\ln\left(\frac{u_1}{u_2}\right),\end{aligned} \tag{2.10c}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function

$$\mathrm{Li}_2(z)=-\int_0^z\frac{\ln(1-z)}{z}dz. \tag{2.10d}$$

2.2.4 $\mathbf{Q1}|_{\delta=0}$

The original equation is

$$\alpha_1(u-u_2)(u_1-u_{1,2})-\alpha_2(u-u_1)(u_2-u_{1,2})=0; \tag{2.11a}$$

written in 3-leg form this is

$$\frac{\alpha_1}{u-u_1}-\frac{\alpha_2}{u-u_2}=\frac{\alpha_1-\alpha_2}{u-u_{1,2}}, \tag{2.11b}$$

and it possesses the Lagrangian

$$\mathcal{L}=\alpha_1\ln(u-u_1)-\alpha_2\ln(u-u_2)-(\alpha_1-\alpha_2)\ln(u_1-u_2). \tag{2.11c}$$

2.2.5 $\mathbf{Q1}|_{\delta\neq 0}$

The original equation is

$$\alpha_1(u-u_2)(u_1-u_{1,2})-\alpha_2(u-u_1)(u_2-u_{1,2})+\delta^2\alpha_1\alpha_2(\alpha_1-\alpha_2)=0; \tag{2.12a}$$

written in 3-leg form this is

$$\left(\frac{u-u_1+\alpha_1\delta}{u-u_1-\alpha_1\delta}\right)\left(\frac{u-u_2-\alpha_2\delta}{u-u_2+\alpha_2\delta}\right)=\left(\frac{u-u_{1,2}+\alpha_1\delta-\alpha_2\delta}{u-u_{1,2}-\alpha_1\delta+\alpha_2\delta}\right), \tag{2.12b}$$

and it possesses the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\;& (u-u_1+\alpha_1\delta)\ln(u-u_1+\alpha_1\delta)-(u-u_1-\alpha_1\delta)\ln(u-u_1-\alpha_1\delta)\\ &-(u-u_2+\alpha_2\delta)\ln(u-u_2+\alpha_2\delta)+(u-u_2-\alpha_2\delta)\ln(u-u_2-\alpha_2\delta)\\ &-(u_1-u_2+\alpha_1\delta-\alpha_2\delta)\ln(u_1-u_2+\alpha_1\delta-\alpha_2\delta)\\ &+(u_1-u_2-\alpha_1\delta+\alpha_2\delta)\ln(u_1-u_2-\alpha_1\delta+\alpha_2\delta).\end{aligned} \tag{2.12c}$$

2.2.6 Q3$|_{\delta=0}$

Written in a slightly different form, this equation is known as the Homotopy equation, and appears in [82]. The original equation is

$$(\alpha_2^2-\alpha_1^2)(uu_{1,2}+u_1u_2)+\alpha_2(\alpha_1^2-1)(uu_1+u_2u_{1,2})-\alpha_1(\alpha_2^2-1)(uu_2+u_1u_{1,2})=0; \tag{2.13a}$$

written in 3-leg form this is

$$\left(\frac{\sinh(x-x_1+a_1)}{\sinh(x-x_1-a_1)}\right)\left(\frac{\sinh(x-x_2-a_2)}{\sinh(x-x_2+a_2)}\right)=\left(\frac{\sinh(x-x_{1,2}+a_1-a_2)}{\sinh(x-x_{1,2}-a_1+a_2)}\right), \tag{2.13b}$$

where $u=e^{2x}$ and $\alpha_1=e^{2a_1}$, and it possesses the Lagrangian

$$\begin{aligned}\mathcal{L} &= -\mathrm{Li}_2\left(\frac{\alpha_1 u}{u_1}\right)+\mathrm{Li}_2\left(\frac{u}{\alpha_1 u_1}\right)+\mathrm{Li}_2\left(\frac{\alpha_2 u}{u_2}\right)-\mathrm{Li}_2\left(\frac{u}{\alpha_2 u_2}\right)\\ &\quad +\mathrm{Li}_2\left(\frac{\alpha_1 u_1}{\alpha_2 u_2}\right)-\mathrm{Li}_2\left(\frac{\alpha_2 u_1}{\alpha_1 u_2}\right)+\ln(\alpha_1^2)\ln\left(\frac{\alpha_2 u_1}{\alpha_1 u_2}\right).\end{aligned} \tag{2.13c}$$

2.2.7 A1

The original equation is

$$\alpha_1(u+u_2)(u_1+u_{1,2})-\alpha_2(u+u_1)(u_2+u_{1,2})-\delta^2\alpha_1\alpha_2(\alpha_1-\alpha_2)=0; \tag{2.14a}$$

written in 3-leg form this is

$$\left(\frac{u+u_1+\alpha_1\delta}{u+u_1-\alpha_1\delta}\right)\left(\frac{u+u_2-\alpha_2\delta}{u+u_2+\alpha_2\delta}\right)=\left(\frac{u-u_{1,2}+\alpha_1\delta-\alpha_2\delta}{u-u_{1,2}-\alpha_1\delta+\alpha_2\delta}\right), \tag{2.14b}$$

and it possesses the Lagrangian

$$\begin{aligned}\mathcal{L} &= (u+u_1+\alpha_1\delta)\ln(u+u_1+\alpha_1\delta)-(u+u_1-\alpha_1\delta)\ln(u+u_1-\alpha_1\delta)\\ &\quad -(u+u_2+\alpha_2\delta)\ln(u+u_2+\alpha_2\delta)+(u+u_2-\alpha_2\delta)\ln(u+u_2-\alpha_2\delta)\\ &\quad -(u_2-u_1+\alpha_1\delta-\alpha_2\delta)\ln(u_2-u_1+\alpha_1\delta-\alpha_2\delta)\\ &\quad +(u_2-u_1-\alpha_1\delta+\alpha_2\delta)\ln(u_2-u_1-\alpha_1\delta+\alpha_2\delta).\end{aligned} \tag{2.14c}$$

2.2.8 A2

The original equation is

$$(\alpha_2^2-\alpha_1^2)(uu_1u_2u_{1,2}+1)+\alpha_2(\alpha_1^2-1)(uu_2+u_1u_{1,2})-\alpha_1(\alpha_2^2-1)(uu_1+u_2u_{1,2})=0; \tag{2.15a}$$

written in 3-leg form this is

$$\left(\frac{\sinh(x+x_1+a_1)}{\sinh(x+x_1-a_1)}\right)\left(\frac{\sinh(x+x_2-a_2)}{\sinh(x+x_2+a_2)}\right)=\left(\frac{\sinh(x-x_{1,2}+a_1-a_2)}{\sinh(x-x_{1,2}-a_1+a_2)}\right), \tag{2.15b}$$

where $u=e^{2x}$ and $\alpha_1=e^{2a_1}$, and it possesses the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; & -\mathrm{Li}_2(\alpha_1uu_1)+\mathrm{Li}_2\left(\frac{uu_1}{\alpha_1}\right)+\mathrm{Li}_2(\alpha_2uu_2)-\mathrm{Li}_2\left(\frac{uu_2}{\alpha_2}\right)\\ & +\mathrm{Li}_2\left(\frac{\alpha_1u_2}{\alpha_2u_1}\right)-\mathrm{Li}_2\left(\frac{\alpha_2u_2}{\alpha_1u_1}\right)+\ln(\alpha_1^2)\ln\left(\frac{\alpha_2u_2}{\alpha_1u_1}\right).\end{aligned} \tag{2.15c}$$

For the Lagrangians of the cases given in the above list, we will next establish an important new property.

2.3 Closure relation

The main observation central to the thesis is that all these lattice systems, together with their 3-point Lagrangians as given in the previous section, possess a remarkable property which we refer to as the *closure relation*, when we embed both the equation and the Lagrangian in a 3-dimensional lattice. In order to formulate this property we introduce the notation of the difference operator Δ_i which acts on functions $f=f(u)$ of $u=u(n_1,n_2,n_3)$ by the formula $\Delta_if(u)=f(u_i)-f(u)$, and on a function $g=g(u,u_j,u_k)$ of u and its shifts by the formula $\Delta_ig(u,u_j,u_k)=g(u_i,u_{i,j},u_{i,k})-g(u,u_j,u_k)$, in which, as before, the suffix i denotes a shift in the direction associated with the variable n_i. The following statement holds true.

Proposition:

All the 3-point Lagrangians given in the ABS list (2.8a)-(2.15c) *when embedded in a three-dimensional lattice, satisfy the following relation on solutions of the quadri-*

lateral lattice system:

$$\Delta_1\mathcal{L}(u,u_2,u_3;\alpha_2,\alpha_3)+\Delta_2\mathcal{L}(u,u_3,u_1;\alpha_3,\alpha_1)+\Delta_3\mathcal{L}(u,u_1,u_2;\alpha_1,\alpha_2)=0. \quad (3.1)$$

The relation (3.1) we refer to as the *closure relation.*

Proof: This can be established in most cases by explicit computation, and has been directly verified for H1-H3, $Q1|_{\delta=0}$, $Q1|_{\delta\neq 0}$, $Q3|_{\delta=0}$, A1 and A2. Below we demonstrate the computations for these cases. For H1, the computation is relatively straightforward, relying merely on manipulation of logarithmic terms, but cases such as that of H3 are somewhat more involved and rely on a number of identities for the dilogarithm function Li_2, see Appendix A for more on the dilogarithm function.

2.3.1 Total derivatives

Before we embark on explicit computations, note that we are free to add total derivative terms to the Lagrangian, provided these terms are also antisymmetric with respect to the interchange of the lattice directions n_1 and n_2. To see this, suppose we have a total derivative term $F(u,u_1,u_2;\alpha_1,\alpha_2)$. By definition, the Euler-Lagrange equations will be identically zero, so that

$$\frac{\partial}{\partial u}F(u,u_1,u_2;\alpha_1,\alpha_2)+\frac{\partial}{\partial u}F(u_{-1},u,u_{-1,2};\alpha_1,\alpha_2)+\frac{\partial}{\partial u}F(u_{-2},u_{1,-2},u;\alpha_1,\alpha_2)=0. \quad (3.2)$$

The last 2 terms do not depend on u_1 or u_2, so $\frac{\partial}{\partial u}F(u,u_1,u_2;\alpha_1,\alpha_2)$ cannot contain either u_1 or u_2. We must have

$$\frac{\partial}{\partial u}F(u,u_1,u_2;\alpha_1,\alpha_2)=f(u;\alpha_1,\alpha_2), \quad (3.3)$$

for some function f. Similarly

$$\frac{\partial}{\partial u}F(u_{-1},u,u_{-1,2};\alpha_1,\alpha_2)=g(u;\alpha_1,\alpha_2), \quad (3.4)$$

for some function g, and

$$\frac{\partial}{\partial u}F(u_{-2},u_{1,-2},u;\alpha_1,\alpha_2)=h(u;\alpha_1,\alpha_2), \quad (3.5)$$

for some function h. From (3.2) we can see that

$$f(u;\alpha_1,\alpha_2)+g(u;\alpha_1,\alpha_2)+h(u;\alpha_1,\alpha_2)=0, \tag{3.6}$$

and so

$$f(u;\alpha_1,\alpha_2)=-g(u;\alpha_1,\alpha_2)-h(u;\alpha_1,\alpha_2). \tag{3.7}$$

Substituting this for f and shifting equations (3.4) and (3.5) in the n_1 and n_2 directions respectively, we have

$$\begin{aligned}
\frac{\partial}{\partial u}F(u,u_1,u_2;\alpha_1,\alpha_2) &= -g(u;\alpha_1,\alpha_2)-h(u;\alpha_1,\alpha_2), && (3.8)\\
\frac{\partial}{\partial u_1}F(u,u_1,u_2;\alpha_1,\alpha_2) &= g(u_1;\alpha_1,\alpha_2), && (3.9)\\
\frac{\partial}{\partial u_2}F(u,u_1,u_2;\alpha_1,\alpha_2) &= h(u_2;\alpha_1,\alpha_2). && (3.10)
\end{aligned}$$

Now suppose we have functions $G(u;\alpha_1,\alpha_2)$ and $H(u;\alpha_1,\alpha_2)$ such that

$$\frac{\partial}{\partial u}G(u;\alpha_1,\alpha_2)=g(u;\alpha_1,\alpha_2), \text{ and } \frac{\partial}{\partial u}H(u;\alpha_1,\alpha_2)=h(u;\alpha_1,\alpha_2), \tag{3.11}$$

then up to a constant term we will have

$$F(u,u_1,u_2;\alpha_1,\alpha_2)=-G(u;\alpha_1,\alpha_2)-H(u;\alpha_1,\alpha_2)+G(u_1;\alpha_1,\alpha_2)+H(u_2;\alpha_1,\alpha_2). \tag{3.12}$$

For the closure relation to hold, we require that

$$\begin{aligned}
0 &= \Delta_1F(u,u_2,u_3;\alpha_2,\alpha_3)+\Delta_2F(u,u_3,u_1;\alpha_3,\alpha_1)+\Delta_3F(u,u_1,u_2;\alpha_1,\alpha_2)\\
&= -G(u_1;\alpha_2,\alpha_3)-H(u_1;\alpha_2,\alpha_3)+G(u_{1,2};\alpha_2,\alpha_3)+H(u_{3,1};\alpha_2,\alpha_3)\\
&\quad +G(u;\alpha_2,\alpha_3)+H(u;\alpha_2,\alpha_3)-G(u_2;\alpha_2,\alpha_3)-H(u_3;\alpha_2,\alpha_3)\\
&\quad -G(u_2;\alpha_3,\alpha_1)-H(u_2;\alpha_3,\alpha_1)+G(u_{2,3};\alpha_3,\alpha_1)+H(u_{1,2};\alpha_3,\alpha_1)\\
&\quad +G(u;\alpha_3,\alpha_1)+H(u;\alpha_3,\alpha_1)-G(u_3;\alpha_3,\alpha_1)-H(u_1;\alpha_3,\alpha_1)\\
&\quad -G(u_3;\alpha_1,\alpha_2)-H(u_3;\alpha_1,\alpha_2)+G(u_{3,1};\alpha_1,\alpha_2)+H(u_{2,3};\alpha_1,\alpha_2)\\
&\quad +G(u;\alpha_1,\alpha_2)+H(u;\alpha_1,\alpha_2)-G(u_1;\alpha_1,\alpha_2)-H(u_2;\alpha_1,\alpha_2), && (3.13)
\end{aligned}$$

which implies that

$$\begin{aligned}
0 \;=\;& G(u;\alpha_1,\alpha_2)+G(u;\alpha_2,\alpha_3)+G(u;\alpha_3,\alpha_1)\\
&+H(u;\alpha_1,\alpha_2)+H(u;\alpha_2,\alpha_3)+H(u;\alpha_3,\alpha_1)\\
&-G(u_1;\alpha_2,\alpha_3)-G(u_1;\alpha_1,\alpha_2)-H(u_1;\alpha_2,\alpha_3)-H(u_1;\alpha_3,\alpha_1)\\
&-G(u_2;\alpha_3,\alpha_1)-G(u_2;\alpha_2,\alpha_3)-H(u_2;\alpha_1,\alpha_2)-H(u_2;\alpha_3,\alpha_1)\\
&-G(u_3;\alpha_1,\alpha_2)-G(u_3;\alpha_3,\alpha_1)-H(u_3;\alpha_1,\alpha_2)-H(u_3;\alpha_2,\alpha_3)\\
&+G(u_{1,2};\alpha_2,\alpha_3)+G(u_{2,3};\alpha_3,\alpha_1)+G(u_{3,1};\alpha_1,\alpha_2)\\
&+H(u_{1,2};\alpha_3,\alpha_1)+H(u_{2,3};\alpha_1,\alpha_2)+H(u_{3,1};\alpha_2,\alpha_3). \qquad (3.14)
\end{aligned}$$

So we must have

$$G(u_{1,2};\alpha_2,\alpha_3)+H(u_{1,2};\alpha_3,\alpha_1)=k(\alpha_2,\alpha_3,\alpha_1), \qquad (3.15)$$

for some function k. Thus F must have the form

$$\begin{aligned}
F(u,u_1,u_2;\alpha_1,\alpha_2) &= -G(u;\alpha_1,\alpha_2)-k(\alpha_3,\alpha_1,\alpha_2)+G(u;\alpha_3,\alpha_1)\\
&\quad +G(u_1;\alpha_1,\alpha_2)+k(\alpha_3,\alpha_1,\alpha_2)-G(u_2;\alpha_3,\alpha_1)\\
&= -G(u;\alpha_1,\alpha_2)+G(u;\alpha_3,\alpha_1)+G(u_1;\alpha_1,\alpha_2)\\
&\quad -G(u_2;\alpha_3,\alpha_1). \qquad (3.16)
\end{aligned}$$

Now, $F(u,u_1,u_2;\alpha_1,\alpha_2)$ does not depend on α_3, so

$$G(u;\alpha_3,\alpha_1)=P(u;\alpha_1)+q(\alpha_3,\alpha_1), \qquad (3.17)$$

for some functions P and q, and F becomes

$$F(u,u_1,u_2;\alpha_1,\alpha_2)=-P(u;\alpha_2)+P(u;\alpha_1)+P(u_1;\alpha_2)-P(u_2;\alpha_1), \qquad (3.18)$$

so it is some antisymmetric function. Hence, if F is a total derivative of the form (3.18), it will obey the closure relation.

2.3.2 H1

To illustrate the proposition in the simplest case, we perform the following computation. By explicit form of the Lagrangians we have

$$
\begin{aligned}
& \Delta_1\mathcal{L}(u,u_2,u_3;\alpha_2,\alpha_3)+\Delta_2\mathcal{L}(u,u_3,u_1;\alpha_3,\alpha_1)+\Delta_3\mathcal{L}(u,u_1,u_2;\alpha_1,\alpha_2)\\
= \; & (u_{1,2}-u_{1,3})u_1-(\alpha_2-\alpha_3)\ln(u_{1,2}-u_{1,3})-(u_2-u_3)u\\
& +(\alpha_2-\alpha_3)\ln(u_2-u_3)+(u_{2,3}-u_{1,2})u_2-(\alpha_3-\alpha_1)\ln(u_{2,3}-u_{1,2})\\
& -(u_3-u_1)u+(\alpha_3-\alpha_1)\ln(u_3-u_1)+(u_{1,3}-u_{2,3})u_3\\
& -(\alpha_1-\alpha_2)\ln(u_{1,3}-u_{2,3})-(u_1-u_2)u+(\alpha_1-\alpha_2)\ln(u_1-u_2). \qquad (3.19)
\end{aligned}
$$

Noting that the differences between the double-shifted terms have the form

$$
\begin{aligned}
u_{1,2}-u_{1,3} &= -\frac{(\alpha_2-\alpha_3)u_1+(\alpha_3-\alpha_1)u_2+(\alpha_1-\alpha_2)u_3}{(u_1-u_2)(u_2-u_3)(u_3-u_1)}(u_2-u_3)\\
&= A_{1,2,3}(u_2-u_3), \qquad (3.20)
\end{aligned}
$$

where $A_{1,2,3}$ is invariant under permutations of the indices, the expression (3.19) reduces to

$$
\begin{aligned}
& A_{1,2,3}(u_2-u_3)u_1-(\alpha_2-\alpha_3)\ln\big(A_{1,2,3}(u_2-u_3)\big)-(u_2-u_3)u\\
& +(\alpha_2-\alpha_3)\ln(u_2-u_3)+A_{1,2,3}(u_3-u_1)u_2-(\alpha_3-\alpha_1)\ln\big(A_{1,2,3}(u_3-u_1)\big)\\
& -(u_3-u_1)u+(\alpha_3-\alpha_1)\ln(u_3-u_1)+A_{1,2,3}(u_1-u_2)u_3\\
& -(\alpha_1-\alpha_2)\ln\big(A_{1,2,3}(u_1-u_2)\big)-(u_1-u_2)u+(\alpha_1-\alpha_2)\ln(u_1-u_2)\\
= \; & 0, \qquad (3.21)
\end{aligned}
$$

where we have tried to organize the succession of terms to make it manifest which groupings of terms cancel out against each other.

2.3.3 H2

By explicit form of the Lagrangians we have that

$$
\begin{aligned}
\Gamma \quad &\equiv \quad \Delta_1\mathcal{L}(u,u_2,u_3;\alpha_2,\alpha_3)+\Delta_2\mathcal{L}(u,u_3,u_1;\alpha_3,\alpha_1)+\Delta_3\mathcal{L}(u,u_1,u_2;\alpha_1,\alpha_2)\\
&= \quad (u_3+u_{31}+\alpha_1)\ln(u_3+u_{31}+\alpha_1)-(u_3+u_{23}+\alpha_2)\ln(u_3+u_{23}+\alpha_2)\\
&\quad -(u_{31}-u_{23}+\alpha_1-\alpha_2)\ln(u_{31}+u_{23}+\alpha_1-\alpha_2)\\
&\quad +(u_{31}-u_{23}-\alpha_1+\alpha_2)\ln(u_{31}+u_{23}-\alpha_1+\alpha_2)\\
&\quad -(u+u_1+\alpha_1)\ln(u+u_1+\alpha_1)+(u+u_2+\alpha_2)\ln(u+u_2+\alpha_2)\\
&\quad +(u_1-u_2+\alpha_1-\alpha_2)\ln(u_1-u_2+\alpha_1-\alpha_2)\\
&\quad -(u_1-u_2-\alpha_1+\alpha_2)\ln(u_1-u_2-\alpha_1+\alpha_2)\\
&\quad +(u_1+u_{12}+\alpha_2)\ln(u_1+u_{12}+\alpha_2)-(u_1+u_{31}+\alpha_3)\ln(u_1+u_{31}+\alpha_3)\\
&\quad -(u_{12}-u_{31}+\alpha_2-\alpha_3)\ln(u_{12}+u_{31}+\alpha_2-\alpha_3)\\
&\quad +(u_{12}-u_{31}-\alpha_2+\alpha_3)\ln(u_{12}+u_{31}-\alpha_2+\alpha_3)\\
&\quad -(u+u_2+\alpha_2)\ln(u+u_2+\alpha_2)+(u+u_3+\alpha_3)\ln(u+u_3+\alpha_3)\\
&\quad +(u_2-u_3+\alpha_2-\alpha_3)\ln(u_2-u_3+\alpha_2-\alpha_3)\\
&\quad -(u_2-u_3-\alpha_2+\alpha_3)\ln(u_2-u_3-\alpha_2+\alpha_3)\\
&\quad +(u_2+u_{23}+\alpha_3)\ln(u_2+u_{23}+\alpha_3)-(u_2+u_{12}+\alpha_1)\ln(u_2+u_{12}+\alpha_1)\\
&\quad -(u_{23}-u_{12}+\alpha_3-\alpha_1)\ln(u_{23}+u_{12}+\alpha_3-\alpha_1)\\
&\quad +(u_{23}-u_{12}-\alpha_3+\alpha_1)\ln(u_{23}+u_{12}-\alpha_3+\alpha_1)\\
&\quad -(u+u_3+\alpha_3)\ln(u+u_3+\alpha_3)+(u+u_1+\alpha_1)\ln(u+u_1+\alpha_1)\\
&\quad +(u_3-u_1+\alpha_3-\alpha_1)\ln(u_3-u_1+\alpha_3-\alpha_1)\\
&\quad -(u_3-u_1-\alpha_3+\alpha_1)\ln(u_3-u_1-\alpha_3+\alpha_1). \qquad (3.22)
\end{aligned}
$$

On rearranging, this is

$$\begin{aligned}
\Gamma \;=\;& u_{12}\ln\left(\frac{(u_1+u_{12}+\alpha_2)(u_{12}-u_{31}-\alpha_2+\alpha_3)(u_{23}-u_{12}+\alpha_3-\alpha_1)}{(u_2+u_{12}+\alpha_1)(u_{12}-u_{31}+\alpha_2-\alpha_3)(u_{23}-u_{12}-\alpha_3+\alpha_1)}\right)\\
&+u_{23}\ln\left(\frac{(u_2+u_{23}+\alpha_3)(u_{23}-u_{12}-\alpha_3+\alpha_1)(u_{31}-u_{23}+\alpha_1-\alpha_2)}{(u_3+u_{23}+\alpha_2)(u_{23}-u_{12}+\alpha_3-\alpha_1)(u_{31}-u_{23}-\alpha_1+\alpha_2)}\right)\\
&+u_{31}\ln\left(\frac{(u_3+u_{31}+\alpha_1)(u_{31}-u_{23}-\alpha_1+\alpha_2)(u_{12}-u_{31}+\alpha_2-\alpha_3)}{(u_1+u_{31}+\alpha_3)(u_{31}-u_{23}+\alpha_1-\alpha_2)(u_{12}-u_{31}-\alpha_2+\alpha_3)}\right)\\
&+u_1\ln\left(\frac{(u_1+u_{12}+\alpha_2)(u_3-u_1-\alpha_3+\alpha_1)(u_1-u_2+\alpha_1-\alpha_2)}{(u_1+u_{31}+\alpha_3)(u_3-u_1+\alpha_3-\alpha_1)(u_1-u_2-\alpha_1+\alpha_2)}\right)\\
&+u_2\ln\left(\frac{(u_2+u_{23}+\alpha_3)(u_1-u_2-\alpha_1+\alpha_2)(u_2-u_3+\alpha_2-\alpha_3)}{(u_2+u_{12}+\alpha_1)(u_1-u_2+\alpha_1-\alpha_2)(u_2-u_3-\alpha_2+\alpha_3)}\right)\\
&+u_3\ln\left(\frac{(u_3+u_{31}+\alpha_1)(u_2-u_3-\alpha_2+\alpha_3)(u_3-u_1+\alpha_3-\alpha_1)}{(u_3+u_{23}+\alpha_2)(u_2-u_3+\alpha_2-\alpha_3)(u_3-u_1-\alpha_3+\alpha_1)}\right)\\
&+\alpha_1\ln\left(\frac{(u_3+u_{31}+\alpha_1)(u_{23}-u_{12}+\alpha_3-\alpha_1)(u_{23}-u_{12}-\alpha_3+\alpha_1)}{(u_2+u_{12}+\alpha_1)(u_3-u_1+\alpha_3-\alpha_1)(u_3-u_1-\alpha_3+\alpha_1)}\right.\\
&\qquad\left.\cdot\frac{(u_1-u_2+\alpha_1-\alpha_2)(u_1-u_2-\alpha_1+\alpha_2)}{(u_{31}-u_{23}+\alpha_1-\alpha_2)(u_{31}-u_{23}-\alpha_1+\alpha_2)}\right)\\
&+\alpha_2\ln\left(\frac{(u_1+u_{12}+\alpha_2)(u_{31}-u_{23}+\alpha_1-\alpha_2)(u_{31}-u_{23}-\alpha_1+\alpha_2)}{(u_3+u_{23}+\alpha_2)(u_1-u_2+\alpha_1-\alpha_2)(u_1-u_2-\alpha_1+\alpha_2)}\right.\\
&\qquad\left.\cdot\frac{(u_2-u_3+\alpha_2-\alpha_3)(u_2-u_3-\alpha_2+\alpha_3)}{(u_{12}-u_{31}+\alpha_2-\alpha_3)(u_{12}-u_{31}-\alpha_2+\alpha_3)}\right)\\
&+\alpha_3\ln\left(\frac{(u_2+u_{23}+\alpha_3)(u_{12}-u_{31}+\alpha_2-\alpha_3)(u_{12}-u_{31}-\alpha_2+\alpha_3)}{(u_1+u_{31}+\alpha_3)(u_2-u_3+\alpha_2-\alpha_3)(u_2-u_3-\alpha_2+\alpha_3)}\right.\\
&\qquad\left.\cdot\frac{(u_3-u_1+\alpha_3-\alpha_1)(u_3-u_1-\alpha_3+\alpha_1)}{(u_{23}-u_{12}+\alpha_3-\alpha_1)(u_{23}-u_{12}-\alpha_3+\alpha_1)}\right). \qquad (3.23)
\end{aligned}$$

Then, using the 3-leg form of equation, we get

$$\begin{aligned}
\Gamma \;=\;& u_{12}\ln\left(\frac{u_{12}+u_{123}+\alpha_3}{u_{12}+u_{123}+\alpha_3}\right)+u_{23}\ln\left(\frac{u_{23}+u_{123}+\alpha_1}{u_{23}+u_{123}+\alpha_1}\right)\\
&+u_{31}\ln\left(\frac{u_{31}+u_{123}+\alpha_2}{u_{31}+u_{123}+\alpha_2}\right)+u_1\ln\left(\frac{u+u_1+\alpha_1}{u+u_1+\alpha_1}\right)+u_2\ln\left(\frac{u+u_2+\alpha_2}{u+u_2+\alpha_2}\right)\\
&+u_3\ln\left(\frac{u+u_3+\alpha_3}{u+u_3+\alpha_3}\right)+\alpha_1\ln(1)+\alpha_2\ln(1)+\alpha_3\ln(1). \qquad (3.24)
\end{aligned}$$

Up to constant imaginary terms resulting from the multivaluedness of the logarithm function, which can be chosen in such a way that they vanish, this is zero.

2.3.4 H3

As mentioned earlier, the dilogarithm function is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-z)}{z} dz. \tag{3.25}$$

The functional relations involving the dilogarithm that are useful for the following computations are given below. Proofs of these identities, and some discussion of the dilogarithm function, appear in Appendix A. The pivotal functional relation is the five-term identity

$$\begin{aligned} \mathrm{Li}_2(s) + \mathrm{Li}_2(t) - \mathrm{Li}_2(st) &= \mathrm{Li}_2\left(\frac{s-st}{1-st}\right) + \mathrm{Li}_2\left(\frac{t-st}{1-st}\right) \\ &\quad + \ln\left(\frac{1-s}{1-st}\right) \ln\left(\frac{1-t}{1-st}\right). \end{aligned} \tag{3.26}$$

An additional two identities needed are the following, both valid for all real x.

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{1}{2}\big(\ln(-x)\big)^2 - \frac{\pi^2}{6}, \tag{3.27}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\big(\ln(1-x)\big)^2. \tag{3.28}$$

Equation (3.27) holds regardless of whether the arguments are positive or negative. Equations (3.26), (3.28) require additional imaginary terms depending on the sign of the arguments; however, these cancel out in the course of the closure relation calculations.

The Lagrangian for H3 is

$$\begin{aligned} \mathcal{L}_{\alpha_1\alpha_2} &\equiv \mathcal{L}(u, u_1, u_2; \alpha_1, \alpha_2) \\ &= -\mathrm{Li}_2\left(\frac{uu_1}{-\alpha_1}\right) + \mathrm{Li}_2\left(\frac{uu_2}{-\alpha_2}\right) + \mathrm{Li}_2\left(\frac{\alpha_2 u_1}{\alpha_1 u_2}\right) - \mathrm{Li}_2\left(\frac{\alpha_1 u_1}{\alpha_2 u_2}\right) \\ &\quad + \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right) \ln(u) + \ln(\alpha_2^2) \ln\left(\frac{u_1}{u_2}\right). \end{aligned} \tag{3.29}$$

We make a change of variables similar to those that appear in the 3 leg form of H3. This will make the computations simpler and easier to follow. With the abbreviations

$$A = \frac{uu_1}{-\alpha_1}, \quad B = \frac{uu_2}{-\alpha_2}, \quad C = \frac{uu_3}{-\alpha_3}, \tag{3.30}$$

the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\alpha_1\alpha_2} &= -\mathrm{Li}_2(A) + \mathrm{Li}_2(B) + \mathrm{Li}_2\left(\frac{A}{B}\right) - \mathrm{Li}_2\left(\frac{\alpha_1^2 A}{\alpha_2^2 B}\right) \\ &\quad + \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(u) + \ln(\alpha_2^2)\ln\left(\frac{u_1}{u_2}\right),\end{aligned} \tag{3.31}$$

whilst the equations of evolution, written in the variables A, B, C, are as follows:

$$\frac{\alpha_1^2}{\alpha_2^2}\frac{1-A}{1-B} = \frac{1-B_1}{1-A_2}, \tag{3.32a}$$

$$\frac{\alpha_2^2}{\alpha_3^2}\frac{1-B}{1-C} = \frac{1-C_2}{1-B_3}, \tag{3.32b}$$

$$\frac{\alpha_3^2}{\alpha_1^2}\frac{1-C}{1-A} = \frac{1-A_3}{1-C_1}, \tag{3.32c}$$

where for example A_1 denotes A shifted in the n_1-direction. The definitions of A, B, C give the relations

$$\frac{\alpha_1^2 A}{\alpha_2^2 B} = \frac{B_1}{A_2}, \quad \frac{\alpha_2^2 B}{\alpha_3^2 C} = \frac{C_2}{B_3}, \quad \frac{\alpha_3^2 C}{\alpha_1^2 A} = \frac{A_3}{C_1}, \tag{3.33}$$

which, together with (3.32a)-(3.32c), give expressions for A_2, B_1, etc explicitly in terms of A, B, C. To write these in a simple way, define the function $H_{A,B} \equiv H(A, B; \alpha_1, \alpha_2)$ to be

$$H_{A,B} = \frac{\alpha_2^2(1-B) - \alpha_1^2(1-A)}{A-B}, \tag{3.34}$$

leading to the following

$$\begin{aligned}A_3 &= \frac{C}{\alpha_1^2}H_{C,A}, \quad B_1 = \frac{A}{\alpha_2^2}H_{A,B}, \quad C_2 = \frac{B}{\alpha_3^2}H_{B,C}, \\ A_2 &= \frac{B}{\alpha_1^2}H_{A,B}, \quad B_3 = \frac{C}{\alpha_2^2}H_{B,C}, \quad C_1 = \frac{A}{\alpha_3^2}H_{C,A}.\end{aligned} \tag{3.35}$$

Defining the quantity Γ as below

$$\Gamma \equiv \Delta_3\mathcal{L}_{\alpha_1\alpha_2} + \Delta_1\mathcal{L}_{\alpha_2\alpha_3} + \Delta_2\mathcal{L}_{\alpha_3\alpha_1}, \tag{3.36}$$

we may now write both the Lagrangians and their shifted versions in terms of A, B and C, which leads to

$$
\begin{aligned}
\Gamma \quad = \quad & \boxed{\mathrm{Li}_2\left(\frac{B}{\alpha_1^2}H_{A,B}\right) + \mathrm{Li}_2\left(\frac{\alpha_1^2 A}{\alpha_2^2 B}\right) - \mathrm{Li}_2\left(\frac{A}{\alpha_2^2}H_{A,B}\right)} \\
& \boxed{+\mathrm{Li}_2\left(\frac{C}{\alpha_2^2}H_{B,C}\right) + \mathrm{Li}_2\left(\frac{\alpha_2^2 B}{\alpha_3^2 C}\right) - \mathrm{Li}_2\left(\frac{B}{\alpha_3^2}H_{B,C}\right)} \\
& \boxed{+\mathrm{Li}_2\left(\frac{A}{\alpha_3^2}H_{C,A}\right) + \mathrm{Li}_2\left(\frac{\alpha_3^2 C}{\alpha_1^2 A}\right) - \mathrm{Li}_2\left(\frac{C}{\alpha_1^2}H_{C,A}\right)} \\
& \boxed{+\mathrm{Li}_2\left(\frac{\alpha_1^2 H_{B,C}}{\alpha_3^2 H_{A,B}}\right) + \mathrm{Li}_2\left(\frac{\alpha_3^2 H_{A,B}}{\alpha_2^2 H_{C,A}}\right) \boxed{+ \mathrm{Li}_2\left(\frac{\alpha_2^2 H_{C,A}}{\alpha_1^2 H_{B,C}}\right)}} \\
& \boxed{-\mathrm{Li}_2\left(\frac{H_{B,C}}{H_{A,B}}\right) - \mathrm{Li}_2\left(\frac{H_{A,B}}{H_{C,A}}\right) \boxed{- \mathrm{Li}_2\left(\frac{H_{C,A}}{H_{B,C}}\right)}} \\
& \boxed{-\mathrm{Li}_2\left(\frac{A}{B}\right) - \mathrm{Li}_2\left(\frac{B}{C}\right) \boxed{- \mathrm{Li}_2\left(\frac{C}{A}\right)}} \\
& + \ln\left(\frac{\alpha_3^2}{\alpha_1^2}\right)\ln(H_{A,B}) + \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(H_{B,C}) + \ln\left(\frac{\alpha_2^2}{\alpha_3^2}\right)\ln(H_{C,A}) \\
& - \ln\left(\frac{\alpha_3^2}{\alpha_1^2}\right)\ln(A) - \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(B) - \ln\left(\frac{\alpha_2^2}{\alpha_3^2}\right)\ln(C) \\
& - \ln(\alpha_1^2)\ln(\alpha_2^2) - \ln(\alpha_2^2)\ln(\alpha_3^2) - \ln(\alpha_3^2)\ln(\alpha_1^2) \\
& +(\ln(\alpha_1^2))^2 + (\ln(\alpha_2^2))^2 + (\ln(\alpha_3^2))^2, \qquad (3.37)
\end{aligned}
$$

where we have we rearranged the terms in a way that suggests which dilogarithm identities to use and where. Applying the dilogarithm identity (3.27) to the terms in the dashed-line boxes, the argument of the dilogarithm functions can be inverted. This enables us to use identity (3.26) on the terms grouped in the solid-line boxes, using the definition of $H_{A,B}$ to simplify the outcome. We will gather all the logarithm terms together at the end.

$$
\begin{aligned}
\Gamma \;=\;& +\mathrm{Li}_2\left(\frac{(A-B)H_{A,B}}{\alpha_1^2(A-1)}\right) + \mathrm{Li}_2\left(\frac{A(B-1)}{B(A-1)}\right)\\
&+\mathrm{Li}_2\left(\frac{(B-C)H_{B,C}}{\alpha_2^2(B-1)}\right) + \mathrm{Li}_2\left(\frac{B(C-1)}{C(B-1)}\right)\\
&+\mathrm{Li}_2\left(\frac{(C-A)H_{C,A}}{\alpha_3^2(C-1)}\right) + \mathrm{Li}_2\left(\frac{C(A-1)}{A(C-1)}\right)\\
&+\mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)(B-C)H_{B,C}}{\alpha_3^2(C-1)(B-A)H_{A,B}}\right) + \mathrm{Li}_2\left(\frac{(C-A)(B-1)}{(B-A)(C-1)}\right)\\
&-\mathrm{Li}_2\left(\frac{(B-C)H_{B,C}}{(B-A)H_{A,B}}\right) - \mathrm{Li}_2\left(\frac{C-A}{B-A}\right)\\
&-\mathrm{Li}_2\left(\frac{A(B-C)}{B(A-C)}\right) - \mathrm{Li}_2\left(\frac{A-B}{A-C}\right)\\
&+\ln\left(\frac{\alpha_2^2(B-1)}{\alpha_1^2(A-1)}\right)\ln\left(\frac{A-B}{B(A-1)}\right) + \ln\left(\frac{\alpha_3^2(C-1)}{\alpha_2^2(B-1)}\right)\ln\left(\frac{B-C}{C(B-1)}\right)\\
&+\ln\left(\frac{\alpha_1^2(A-1)}{\alpha_3^2(C-1)}\right)\ln\left(\frac{C-A}{A(C-1)}\right)\\
&+\ln\left(\frac{\alpha_2^2(B-1)(C-A)H_{C,A}}{\alpha_3^2(C-1)(B-A)H_{A,B}}\right)\ln\left(\frac{(A-1)(B-C)}{(C-1)(B-A)}\right)\\
&-\ln\left(\frac{(C-A)H_{C,A}}{(B-A)H_{A,B}}\right)\ln\left(\frac{B-C}{B-A}\right) - \ln\left(\frac{C(A-B)}{B(A-C)}\right)\ln\left(\frac{B-C}{A-C}\right)\\
&-\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2 H_{B,C}}{\alpha_2^2 H_{C,A}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{H_{B,C}}{H_{C,A}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{A}{C}\right)\right)^2\\
&+\frac{\pi^2}{6} + \ln\left(\frac{\alpha_3^2}{\alpha_1^2}\right)\ln(H_{A,B}) + \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(H_{B,C}) + \ln\left(\frac{\alpha_2^2}{\alpha_3^2}\right)\ln(H_{C,A})\\
&-\ln\left(\frac{\alpha_3^2}{\alpha_1^2}\right)\ln(A) - \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(B) - \ln\left(\frac{\alpha_2^2}{\alpha_3^2}\right)\ln(C)\\
&-\ln(\alpha_1^2)\ln(\alpha_2^2) - \ln(\alpha_2^2)\ln(\alpha_3^2) - \ln(\alpha_3^2)\ln(\alpha_1^2)\\
&+(\ln(\alpha_1^2))^2 + (\ln(\alpha_2^2))^2 + (\ln(\alpha_3^2))^2. \qquad (3.38)
\end{aligned}
$$

Again, using identity (3.27) on the terms in the dashed-line boxes, and subsequently identity (3.26) on the terms grouped in the solid-line boxes, we obtain

$$\begin{aligned}
\Gamma \quad = \quad & \mathrm{Li}_2\left(\frac{(C-B)H_{B,C}}{\alpha_3^2(C-1)}\right) + \mathrm{Li}_2\left(\frac{(A-C)H_{C,A}}{\alpha_1^2(A-1)}\right) \\
& + \mathrm{Li}_2\left(\frac{(B-C)H_{B,C}}{\alpha_2^2(B-1)}\right) + \mathrm{Li}_2\left(\frac{(C-A)H_{C,A}}{\alpha_3^2(C-1)}\right) \\
& + \ln\left(\frac{\alpha_2^2(B-1)}{\alpha_1^2(A-1)}\right)\ln\left(\frac{A-B}{B(A-1)}\right) + \ln\left(\frac{\alpha_3^2(C-1)}{\alpha_2^2(B-1)}\right)\ln\left(\frac{B-C}{C(B-1)}\right) \\
& + \ln\left(\frac{\alpha_1^2(A-1)}{\alpha_3^2(C-1)}\right)\ln\left(\frac{C-A}{A(C-1)}\right) \\
& + \ln\left(\frac{\alpha_2^2(B-1)(C-A)H_{C,A}}{\alpha_3^2(C-1)(B-A)H_{A,B}}\right)\ln\left(\frac{(A-1)(B-C)}{(C-1)(B-A)}\right) \\
& - \ln\left(\frac{(C-A)H_{C,A}}{(B-A)H_{A,B}}\right)\ln\left(\frac{B-C}{B-A}\right) - \ln\left(\frac{C(A-B)}{B(A-C)}\right)\ln\left(\frac{B-C}{A-C}\right) \\
& + \ln\left(\frac{C(A-B)}{B(A-C)}\right)\ln\left(\frac{(A-1)(B-C)}{(B-1)(A-C)}\right) \\
& + \ln\left(\frac{\alpha_3^2(C-1)}{\alpha_1^2(A-1)}\right)\ln\left(\frac{(C-A)H_{C,A}}{(B-A)H_{A,B}}\right) \\
& - \frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2 H_{B,C}}{\alpha_2^2 H_{C,A}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{H_{B,C}}{H_{C,A}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{A}{C}\right)\right)^2 \\
& - \frac{1}{2}\left(\ln\left(-\frac{A(C-1)}{C(A-1)}\right)\right)^2 - \frac{1}{2}\left(\ln\left(-\frac{(B-A)(C-1)}{(C-A)(B-1)}\right)\right)^2 \\
& + \frac{1}{2}\left(\ln\left(-\frac{A-C}{A-B}\right)\right)^2 + \ln\left(\frac{\alpha_3^2}{\alpha_1^2}\right)\ln(H_{A,B}) + \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(H_{B,C}) \\
& + \ln\left(\frac{\alpha_2^2}{\alpha_3^2}\right)\ln(H_{C,A}) - \ln\left(\frac{\alpha_3^2}{\alpha_1^2}\right)\ln(A) - \ln\left(\frac{\alpha_1^2}{\alpha_2^2}\right)\ln(B) - \ln\left(\frac{\alpha_2^2}{\alpha_3^2}\right)\ln(C) \\
& - \ln(\alpha_1^2)\ln(\alpha_2^2) - \ln(\alpha_2^2)\ln(\alpha_3^2) - \ln(\alpha_3^2)\ln(\alpha_1^2) \\
& + (\ln(\alpha_1^2))^2 + (\ln(\alpha_2^2))^2 + (\ln(\alpha_3^2))^2. \qquad (3.39)
\end{aligned}$$

Using identity (3.28) on the first term of line 1 and the second term of line 2 of (3.39) leaves the dilogarithm terms which subsequently cancel out. What then remains are only the logarithm terms, which also cancel out, leaving $\Gamma = 0$. This concludes the proof of the closure relation for H3.

2.3.5 $\mathbf{Q1}|_{\delta=0}$

By explicit form of the Lagrangians we have that

$$\begin{aligned}\Gamma &\equiv \Delta_1\mathcal{L}(u,u_2,u_3;\alpha_2,\alpha_3)+\Delta_2\mathcal{L}(u,u_3,u_1;\alpha_3,\alpha_1)+\Delta_3\mathcal{L}(u,u_1,u_2;\alpha_1,\alpha_2)\\
&= \alpha_1\ln(u_3-u_{31})-\alpha_2\ln(u_3-u_{23})-(\alpha_1-\alpha_2)\ln(u_{31}-u_{23})\\
&\quad -\alpha_1\ln(u-u_1)+\alpha_2\ln(u-u_2)+(\alpha_1-\alpha_2)\ln(u_1-u_2)\\
&\quad +\alpha_2\ln(u_1-u_{12})-\alpha_3\ln(u_1-u_{31})-(\alpha_2-\alpha_3)\ln(u_{12}-u_{31})\\
&\quad -\alpha_2\ln(u-u_2)+\alpha_3\ln(u-u_3)+(\alpha_2-\alpha_3)\ln(u_2-u_3)\\
&\quad +\alpha_3\ln(u_2-u_{23})-\alpha_1\ln(u_2-u_{12})-(\alpha_3-\alpha_1)\ln(u_{23}-u_{12})\\
&\quad -\alpha_3\ln(u-u_3)+\alpha_1\ln(u-u_1)+(\alpha_3-\alpha_1)\ln(u_3-u_1)\\
&= \alpha_1\ln\left(\frac{(u_3-u_{31})(u_{23}-u_{12})(u_1-u_2)}{(u_2-u_{12})(u_{31}-u_{23})(u_3-u_1)}\right)\\
&\quad +\alpha_2\ln\left(\frac{(u_1-u_{12})(u_{31}-u_{23})(u_2-u_3)}{(u_3-u_{23})(u_{12}-u_{31})(u_1-u_2)}\right)\\
&\quad +\alpha_3\ln\left(\frac{(u_2-u_{23})(u_{12}-u_{31})(u_3-u_1)}{(u_1-u_{31})(u_{23}-u_{12})(u_2-u_3)}\right).\end{aligned} \tag{3.40}$$

Using the equations to substitute in expressions for u_{12}, u_{23} and u_{31}, and defining the cyclic invariant

$$\begin{aligned}A \equiv\; &-\Big[\alpha_1(\alpha_2-\alpha_3)(u-u_2)(u-u_3)+\alpha_2(\alpha_3-\alpha_1)(u-u_1)(u-u_3)\\
&+\alpha_3(\alpha_1-\alpha_2)(u-u_1)(u-u_2)\Big].\Big[(\alpha_1(u-u_2)-\alpha_2(u-u_1))\\
&(\alpha_2(u-u_3)-\alpha_3(u-u_2))(\alpha_3(u-u_1)-\alpha_1(u-u_3))\Big]^{-1},\end{aligned} \tag{3.41}$$

we have

$$u_3-u_{31} = \frac{-\alpha_1(u-u_3)(u_3-u_1)}{\alpha_3(u-u_1)-\alpha_1(u-u_3)}, \tag{3.42}$$

$$u_2-u_{12} = \frac{-\alpha_1(u-u_2)(u_1-u_2)}{\alpha_1(u-u_2)-\alpha_2(u-u_1)}, \tag{3.43}$$

$$u_{23}-u_{12} = (u-u_2)[\alpha_3(u-u_1)-\alpha_1(u-u_3)]A, \tag{3.44}$$

and of course all cyclic permutations of these expressions. Substituting these in gives us

$$\begin{aligned}
\Gamma \;=\;& \alpha_1 \ln\left(\frac{-\alpha_1(u-u_3)(u_3-u_1)[\alpha_1(u-u_2)-\alpha_2(u-u_1)]}{-\alpha_1(u-u_2)(u_1-u_2)[\alpha_3(u-u_1)-\alpha_1(u-u_3)]}\right.\\
&\qquad\left.\cdot\frac{(u-u_2)[\alpha_3(u-u_1)-\alpha_1(u-u_3)]A(u_1-u_2)}{(u-u_3)[\alpha_1(u-u_2)-\alpha_2(u-u_1)]A(u_3-u_1)}\right)\\
&+\alpha_2 \ln\left(\frac{-\alpha_2(u-u_1)(u_1-u_2)[\alpha_2(u-u_3)-\alpha_3(u-u_2)]}{-\alpha_2(u-u_3)(u_2-u_3)[\alpha_1(u-u_2)-\alpha_2(u-u_1)]}\right.\\
&\qquad\left.\cdot\frac{(u-u_3)[\alpha_1(u-u_2)-\alpha_2(u-u_1)]A(u_2-u_3)}{(u-u_1)[\alpha_2(u-u_3)-\alpha_3(u-u_2)]A(u_1-u_2)}\right)\\
&+\alpha_3 \ln\left(\frac{-\alpha_3(u-u_2)(u_2-u_3)[\alpha_3(u-u_1)-\alpha_1(u-u_3)]}{-\alpha_3(u-u_1)(u_3-u_1)[\alpha_2(u-u_3)-\alpha_3(u-u_2)]}\right.\\
&\qquad\left.\cdot\frac{(u-u_1)[\alpha_2(u-u_3)-\alpha_3(u-u_2)]A(u_3-u_1)}{(u-u_2)[\alpha_3(u-u_1)-\alpha_1(u-u_3)]A(u_2-u_3)}\right)\\
\;=\;& 0. \qquad (3.45)
\end{aligned}$$

2.3.6 $\mathbf{Q1}|_{\delta\neq 0}$

By explicit form of the Lagrangians we have that

$$
\begin{aligned}
\Gamma \equiv\ & \Delta_1\mathcal{L}(u,u_2,u_3;\alpha_2,\alpha_3)+\Delta_2\mathcal{L}(u,u_3,u_1;\alpha_3,\alpha_1)+\Delta_3\mathcal{L}(u,u_1,u_2;\alpha_1,\alpha_2)\\
=\ & (u_3-u_{31}+\alpha_1\delta)\ln(u_3-u_{31}+\alpha_1\delta)-(u_3-u_{31}-\alpha_1\delta)\ln(u_3-u_{31}-\alpha_1\delta)\\
& -(u_3-u_{23}+\alpha_2\delta)\ln(u_3-u_{23}+\alpha_2\delta)+(u_3-u_{23}-\alpha_2\delta)\ln(u_3-u_{23}-\alpha_2\delta)\\
& -(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)\ln(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)\\
& +(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)\ln(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)\\
& -(u-u_1+\alpha_1\delta)\ln(u-u_1+\alpha_1\delta)+(u-u_1-\alpha_1\delta)\ln(u-u_1-\alpha_1\delta)\\
& +(u-u_2+\alpha_2\delta)\ln(u-u_2+\alpha_2\delta)-(u-u_2-\alpha_2\delta)\ln(u-u_2-\alpha_2\delta)\\
& +(u_1-u_2+\alpha_1\delta-\alpha_2\delta)\ln(u_1-u_2+\alpha_1\delta-\alpha_2\delta)\\
& -(u_1-u_2-\alpha_1\delta+\alpha_2\delta)\ln(u_1-u_2-\alpha_1\delta+\alpha_2\delta)\\
& +(u_1-u_{12}+\alpha_2\delta)\ln(u_1-u_{12}+\alpha_2\delta)-(u_1-u_{12}-\alpha_2\delta)\ln(u_1-u_{12}-\alpha_2\delta)\\
& -(u_1-u_{31}+\alpha_3\delta)\ln(u_1-u_{31}+\alpha_3\delta)+(u_1-u_{31}-\alpha_3\delta)\ln(u_1-u_{31}-\alpha_3\delta)\\
& -(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)\ln(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)\\
& +(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)\ln(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)\\
& -(u-u_2+\alpha_2\delta)\ln(u-u_2+\alpha_2\delta)+(u-u_2-\alpha_2\delta)\ln(u-u_2-\alpha_2\delta)\\
& +(u-u_3+\alpha_3\delta)\ln(u-u_3+\alpha_3\delta)-(u-u_3-\alpha_3\delta)\ln(u-u_3-\alpha_3\delta)\\
& +(u_2-u_3+\alpha_2\delta-\alpha_3\delta)\ln(u_2-u_3+\alpha_2\delta-\alpha_3\delta)\\
& -(u_2-u_3-\alpha_2\delta+\alpha_3\delta)\ln(u_2-u_3-\alpha_2\delta+\alpha_3\delta)\\
& +(u_2-u_{23}+\alpha_3\delta)\ln(u_2-u_{23}+\alpha_3\delta)-(u_2-u_{23}-\alpha_3\delta)\ln(u_2-u_{23}-\alpha_3\delta)\\
& -(u_2-u_{12}+\alpha_1\delta)\ln(u_2-u_{12}+\alpha_1\delta)+(u_2-u_{12}-\alpha_1\delta)\ln(u_2-u_{12}-\alpha_1\delta)\\
& -(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)\ln(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)\\
& +(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)\ln(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)\\
& -(u-u_3+\alpha_3\delta)\ln(u-u_3+\alpha_3\delta)+(u-u_3-\alpha_3\delta)\ln(u-u_3-\alpha_3\delta)\\
& +(u-u_1+\alpha_1\delta)\ln(u-u_1+\alpha_1\delta)-(u-u_1-\alpha_1\delta)\ln(u-u_1-\alpha_1\delta)\\
& +(u_3-u_1+\alpha_3\delta-\alpha_1\delta)\ln(u_3-u_1+\alpha_3\delta-\alpha_1\delta)\\
& -(u_3-u_1-\alpha_3\delta+\alpha_1\delta)\ln(u_3-u_1-\alpha_3\delta+\alpha_1\delta). \qquad (3.46)
\end{aligned}
$$

Rearranging, this gives

$$
\begin{aligned}
\Gamma \;=\;& u_{12}\ln\Bigg(\frac{(u_2-u_{12}+\alpha_1\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)(u_1-u_{12}-\alpha_2\delta)}{(u_2-u_{12}-\alpha_1\delta)(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)(u_1-u_{12}+\alpha_2\delta)}\\
&\cdot\frac{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}{(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)}\Bigg)\\
&+u_{23}\ln\Bigg(\frac{(u_3-u_{23}+\alpha_2\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)(u_2-u_{23}-\alpha_3\delta)}{(u_3-u_{23}-\alpha_2\delta)(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)(u_2-u_{23}+\alpha_3\delta)}\\
&\cdot\frac{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}{(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)}\Bigg)\\
&+u_{31}\ln\Bigg(\frac{(u_1-u_{31}+\alpha_3\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)(u_3-u_{31}-\alpha_1\delta)}{(u_1-u_{31}-\alpha_3\delta)(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)(u_3-u_{31}+\alpha_1\delta)}\\
&\cdot\frac{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}{(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)}\Bigg)\\
&+u_1\ln\Bigg(\frac{(u_1-u_2+\alpha_1\delta-\alpha_2\delta)(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_1-u_{12}+\alpha_2\delta)}{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)(u_1-u_{12}-\alpha_2\delta)}\\
&\cdot\frac{(u_1-u_{31}-\alpha_3\delta)}{(u_1-u_{31}+\alpha_3\delta)}\Bigg)\\
&+u_2\ln\Bigg(\frac{(u_2-u_3+\alpha_2\delta-\alpha_3\delta)(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_2-u_{23}+\alpha_3\delta)}{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)(u_2-u_{23}-\alpha_3\delta)}\\
&\cdot\frac{(u_2-u_{12}-\alpha_1\delta)}{(u_2-u_{12}+\alpha_1\delta)}\Bigg)\\
&+u_3\ln\Bigg(\frac{(u_3-u_1+\alpha_3\delta-\alpha_1\delta)(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_3-u_{31}+\alpha_1\delta)}{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)(u_3-u_{31}-\alpha_1\delta)}\\
&\cdot\frac{(u_3-u_{23}-\alpha_2\delta)}{(u_3-u_{23}+\alpha_2\delta)}\Bigg)\\
&+\alpha_1\delta\ln\Bigg(\frac{(u_3-u_{31}+\alpha_1\delta)(u_3-u_{31}-\alpha_1\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)}{(u_2-u_{12}+\alpha_1\delta)(u_2-u_{12}-\alpha_1\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)}\\
&\cdot\frac{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)}{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)}\\
&\cdot\frac{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}\Bigg)\\
&+\alpha_2\delta\ln\Bigg(\frac{(u_1-u_{12}+\alpha_2\delta)(u_1-u_{12}-\alpha_2\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)}{(u_3-u_{23}+\alpha_2\delta)(u_3-u_{23}-\alpha_2\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)}\\
&\cdot\frac{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)}{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)}\\
&\cdot\frac{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}\Bigg)\\
&+\alpha_3\delta\ln\Bigg(\frac{(u_2-u_{23}+\alpha_3\delta)(u_2-u_{23}-\alpha_3\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)}{(u_1-u_{31}+\alpha_3\delta)(u_1-u_{31}-\alpha_3\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)}\\
&\cdot\frac{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)}{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)}\\
&\cdot\frac{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}\Bigg).
\end{aligned}
\tag{3.47}
$$

The equation $Q1|_{\delta\neq 0}$ is

$$\left(\frac{u-u_1+\alpha_1\delta}{u-u_1-\alpha_1\delta}\right)\left(\frac{u-u_2-\alpha_2\delta}{u-u_2+\alpha_2\delta}\right)=\left(\frac{u-u_{1,2}+\alpha_1\delta-\alpha_2\delta}{u-u_{1,2}-\alpha_1\delta+\alpha_2\delta}\right). \tag{3.48}$$

Because of the D4 symmetries of the equation, we are able to make the transformation $u \leftrightarrow u_2$, $u_1 \leftrightarrow u_{12}$ to get

$$\left(\frac{u_2-u_{12}+\alpha_1\delta}{u_2-u_{12}-\alpha_1\delta}\right)\left(\frac{u-u_2+\alpha_2\delta}{u-u_2-\alpha_2\delta}\right)=\left(\frac{u_1-u_2-\alpha_1\delta+\alpha_2\delta}{u_1-u_2+\alpha_1\delta-\alpha_2\delta}\right), \tag{3.49a}$$

or alternatively we can transform (3.48) by $u \leftrightarrow u_1$, $u_2 \leftrightarrow u_{12}$ to get

$$\left(\frac{u-u_1-\alpha_1\delta}{u-u_1+\alpha_1\delta}\right)\left(\frac{u_1-u_{12}-\alpha_2\delta}{u_1-u_{12}+\alpha_2\delta}\right)=\left(\frac{u_1-u_2+\alpha_1\delta-\alpha_2\delta}{u_1-u_2-\alpha_1\delta+\alpha_2\delta}\right). \tag{3.49b}$$

Shifting these equations in the 3rd direction gives

$$\left(\frac{u_{23}-u_{123}+\alpha_1\delta}{u_{23}-u_{123}-\alpha_1\delta}\right)\left(\frac{u_3-u_{23}+\alpha_2\delta}{u_3-u_{23}-\alpha_2\delta}\right)=\left(\frac{u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta}{u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta}\right), \tag{3.50a}$$

$$\left(\frac{u_3-u_{31}-\alpha_1\delta}{u_3-u_{31}+\alpha_1\delta}\right)\left(\frac{u_{31}-u_{123}-\alpha_2\delta}{u_{31}-u_{123}+\alpha_2\delta}\right)=\left(\frac{u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta}{u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta}\right). \tag{3.50b}$$

Substituting these, along with their cyclic permutations, into (3.46) gives us

$$\begin{aligned}
\Gamma \;=\;& u_{12}\ln\Bigg(\frac{(u_2-u_{12}+\alpha_1\delta)(u_2-u_{12}-\alpha_1\delta)(u_{12}-u_{123}-\alpha_3\delta)}{(u_2-u_{12}-\alpha_1\delta)(u_2-u_{12}+\alpha_1\delta)(u_{12}-u_{123}+\alpha_3\delta)}\\
&\cdot\frac{(u_1-u_{12}-\alpha_2\delta)(u_1-u_{12}+\alpha_2\delta)(u_{12}-u_{123}+\alpha_3\delta)}{(u_1-u_{12}+\alpha_2\delta)(u_1-u_{12}-\alpha_2\delta)(u_{12}-u_{123}-\alpha_3\delta)}\Bigg)\\
&+u_{23}\ln\Bigg(\frac{(u_3-u_{23}+\alpha_2\delta)(u_3-u_{23}-\alpha_2\delta)(u_{23}-u_{231}-\alpha_1\delta)}{(u_3-u_{23}-\alpha_2\delta)(u_3-u_{23}+\alpha_2\delta)(u_{23}-u_{231}+\alpha_1\delta)}\\
&\cdot\frac{(u_2-u_{23}-\alpha_3\delta)(u_2-u_{23}+\alpha_3\delta)(u_{23}-u_{231}+\alpha_1\delta)}{(u_2-u_{23}+\alpha_3\delta)(u_2-u_{23}-\alpha_3\delta)(u_{23}-u_{231}-\alpha_1\delta)}\Bigg)\\
&+u_{31}\ln\Bigg(\frac{(u_1-u_{31}+\alpha_3\delta)(u_1-u_{31}-\alpha_3\delta)(u_{31}-u_{312}-\alpha_2\delta)}{(u_1-u_{31}-\alpha_3\delta)(u_1-u_{31}+\alpha_3\delta)(u_{31}-u_{312}+\alpha_2\delta)}\\
&\cdot\frac{(u_3-u_{31}-\alpha_1\delta)(u_3-u_{31}+\alpha_1\delta)(u_{31}-u_{312}+\alpha_2\delta)}{(u_3-u_{31}+\alpha_1\delta)(u_3-u_{31}-\alpha_1\delta)(u_{31}-u_{312}-\alpha_2\delta)}\Bigg)\\
&+u_1\ln\Bigg(\frac{(u-u_1-\alpha_1\delta)(u_1-u_{12}-\alpha_2\delta)(u-u_1+\alpha_1\delta)(u_1-u_{31}+\alpha_3\delta)}{(u-u_1+\alpha_1\delta)(u_1-u_{12}+\alpha_2\delta)(u-u_1-\alpha_1\delta)(u_1-u_{31}-\alpha_3\delta)}\\
&\cdot\frac{(u_1-u_{12}+\alpha_2\delta)(u_1-u_{31}-\alpha_3\delta)}{(u_1-u_{12}-\alpha_2\delta)(u_1-u_{31}+\alpha_3\delta)}\Bigg)\\
&+u_2\ln\Bigg(\frac{(u-u_2-\alpha_2\delta)(u_2-u_{23}-\alpha_3\delta)(u-u_2+\alpha_2\delta)(u_2-u_{12}+\alpha_1\delta)}{(u-u_2+\alpha_2\delta)(u_2-u_{23}+\alpha_3\delta)(u-u_2-\alpha_2\delta)(u_2-u_{12}-\alpha_1\delta)}\\
&\cdot\frac{(u_2-u_{23}+\alpha_3\delta)(u_2-u_{12}-\alpha_1\delta)}{(u_2-u_{23}-\alpha_3\delta)(u_2-u_{12}+\alpha_1\delta)}\Bigg)\\
&+u_3\ln\Bigg(\frac{(u-u_3-\alpha_3\delta)(u_3-u_{31}-\alpha_1\delta)(u-u_3+\alpha_3\delta)(u_3-u_{23}+\alpha_2\delta)}{(u-u_3+\alpha_3\delta)(u_3-u_{31}+\alpha_1\delta)(u-u_3-\alpha_3\delta)(u_3-u_{23}-\alpha_2\delta)}\\
&\cdot\frac{(u_3-u_{31}+\alpha_1\delta)(u_3-u_{23}-\alpha_2\delta)}{(u_3-u_{31}-\alpha_1\delta)(u_3-u_{23}+\alpha_2\delta)}\Bigg)\\
&+\alpha_1\delta\ln\Bigg(\frac{(u_3-u_{31}+\alpha_1\delta)(u_3-u_{31}-\alpha_1\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)}{(u_2-u_{12}+\alpha_1\delta)(u_2-u_{12}-\alpha_1\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)}\\
&\cdot\frac{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)}{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)}\\
&\cdot\frac{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}\Bigg)\\
&+\alpha_2\delta\ln\Bigg(\frac{(u_1-u_{12}+\alpha_2\delta)(u_1-u_{12}-\alpha_2\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)}{(u_3-u_{23}+\alpha_2\delta)(u_3-u_{23}-\alpha_2\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)}\\
&\cdot\frac{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)}{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)}\\
&\cdot\frac{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}\Bigg)\\
&+\alpha_3\delta\ln\Bigg(\frac{(u_2-u_{23}+\alpha_3\delta)(u_2-u_{23}-\alpha_3\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)}{(u_1-u_{31}+\alpha_3\delta)(u_1-u_{31}-\alpha_3\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)}\\
&\cdot\frac{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)}{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)}\\
&\cdot\frac{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}\Bigg),
\end{aligned}\tag{3.51}$$

which, on cancelling out terms, leaves

$$\begin{aligned}
\Gamma \;=\; & +\alpha_1\delta\ln\Bigg(\frac{(u_3-u_{31}+\alpha_1\delta)(u_3-u_{31}-\alpha_1\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)}{(u_2-u_{12}+\alpha_1\delta)(u_2-u_{12}-\alpha_1\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)} \\
& \quad\cdot\frac{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)}{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)} \\
& \quad\cdot\frac{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}\Bigg) \\
& +\alpha_2\delta\ln\Bigg(\frac{(u_1-u_{12}+\alpha_2\delta)(u_1-u_{12}-\alpha_2\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)}{(u_3-u_{23}+\alpha_2\delta)(u_3-u_{23}-\alpha_2\delta)(u_1-u_2+\alpha_1\delta-\alpha_2\delta)} \\
& \quad\cdot\frac{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_{31}-u_{23}+\alpha_1\delta-\alpha_2\delta)}{(u_1-u_2-\alpha_1\delta+\alpha_2\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)} \\
& \quad\cdot\frac{(u_{31}-u_{23}-\alpha_1\delta+\alpha_2\delta)}{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}\Bigg) \\
& +\alpha_3\delta\ln\Bigg(\frac{(u_2-u_{23}+\alpha_3\delta)(u_2-u_{23}-\alpha_3\delta)(u_3-u_1+\alpha_3\delta-\alpha_1\delta)}{(u_1-u_{31}+\alpha_3\delta)(u_1-u_{31}-\alpha_3\delta)(u_2-u_3+\alpha_2\delta-\alpha_3\delta)} \\
& \quad\cdot\frac{(u_3-u_1-\alpha_3\delta+\alpha_1\delta)(u_{12}-u_{31}+\alpha_2\delta-\alpha_3\delta)}{(u_2-u_3-\alpha_2\delta+\alpha_3\delta)(u_{23}-u_{12}+\alpha_3\delta-\alpha_1\delta)} \\
& \quad\cdot\frac{(u_{12}-u_{31}-\alpha_2\delta+\alpha_3\delta)}{(u_{23}-u_{12}-\alpha_3\delta+\alpha_1\delta)}\Bigg). \qquad (3.52)
\end{aligned}$$

Unfortunately it appears that there is no neat way in which to demonstrate these remaining terms vanish, but it can be verified through longer and more tedious calculations, by substituting in expressions for u_{12}, u_{23} and u_{31} in terms of u, u_1, u_2 and u_3 only, that Γ is indeed zero.

2.3.7 $\mathbf{Q3}|_{\delta=0}$

The Lagrangian is

$$\begin{aligned}\mathcal{L}_{\alpha_1\alpha_2} &\equiv \mathcal{L}(u,u_1,u_2;\alpha_1,\alpha_2)\\ &= -\mathrm{Li}_2\left(\frac{\alpha_1 u}{u_1}\right)+\mathrm{Li}_2\left(\frac{u}{\alpha_1 u_1}\right)+\mathrm{Li}_2\left(\frac{\alpha_2 u}{u_2}\right)-\mathrm{Li}_2\left(\frac{u}{\alpha_2 u_2}\right)\\ &\quad +\mathrm{Li}_2\left(\frac{\alpha_1 u_1}{\alpha_2 u_2}\right)-\mathrm{Li}_2\left(\frac{\alpha_2 u_1}{\alpha_1 u_2}\right)+\ln(\alpha_1^2)\ln\left(\frac{\alpha_2 u_1}{\alpha_1 u_2}\right). \qquad (3.53)\end{aligned}$$

We make a change of variables so that the computation is simpler and easier to follow. Let

$$\begin{aligned}A &= \frac{\alpha_1 u}{u_1},\\ B &= \frac{\alpha_2 u}{u_2},\\ C &= \frac{\alpha_3 u}{u_3}. \qquad (3.54)\end{aligned}$$

Then the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\alpha_1\alpha_2} &= -\mathrm{Li}_2\left(A\right)+\mathrm{Li}_2\left(\frac{A}{\alpha_1^2}\right)+\mathrm{Li}_2\left(B\right)-\mathrm{Li}_2\left(\frac{B}{\alpha_2^2}\right)\\ &\quad +\mathrm{Li}_2\left(\frac{\alpha_1^2 B}{\alpha_2^2 A}\right)-\mathrm{Li}_2\left(\frac{B}{A}\right)+\ln(\alpha_1^2)\ln\left(\frac{B}{A}\right). \qquad (3.55)\end{aligned}$$

The equations of motion, written in the variables A, B, C, are as follows:

$$\frac{\alpha_1^2}{\alpha_2^2}\frac{1-A}{1-B} = \frac{1-B_1}{1-A_2}, \qquad (3.56)$$

$$\frac{\alpha_2^2}{\alpha_3^2}\frac{1-B}{1-C} = \frac{1-C_2}{1-B_3}, \qquad (3.57)$$

$$\frac{\alpha_3^2}{\alpha_1^2}\frac{1-C}{1-A} = \frac{1-A_3}{1-C_1}, \qquad (3.58)$$

where again, for example, A_1 denotes A shifted in the n_1-direction.

From the explicit form of the Lagrangian $\mathcal{L}_{\alpha_1\alpha_2}$,

$$\begin{aligned}\Gamma &\equiv \Delta_3\mathcal{L}_{\alpha_1\alpha_2}+\Delta_1\mathcal{L}_{\alpha_2\alpha_3}+\Delta_2\mathcal{L}_{\alpha_3\alpha_1}\\
&= -\mathrm{Li}_2\left(A_3\right)+\mathrm{Li}_2\left(\frac{A_3}{\alpha_1^2}\right)+\mathrm{Li}_2\left(B_3\right)-\mathrm{Li}_2\left(\frac{B_3}{\alpha_2^2}\right)\\
&\quad +\mathrm{Li}_2\left(\frac{\alpha_1^2B_3}{\alpha_2^2A_3}\right)-\mathrm{Li}_2\left(\frac{B_3}{A_3}\right)-\mathrm{Li}_2\left(B_1\right)+\mathrm{Li}_2\left(\frac{B_1}{\alpha_2^2}\right)\\
&\quad +\mathrm{Li}_2\left(C_1\right)-\mathrm{Li}_2\left(\frac{C_1}{\alpha_3^2}\right)+\mathrm{Li}_2\left(\frac{\alpha_2^2C_1}{\alpha_3^2B_1}\right)-\mathrm{Li}_2\left(\frac{C_1}{B_1}\right)\\
&\quad -\mathrm{Li}_2\left(C_2\right)+\mathrm{Li}_2\left(\frac{C_2}{\alpha_3^2}\right)+\mathrm{Li}_2\left(A_2\right)-\mathrm{Li}_2\left(\frac{A_2}{\alpha_1^2}\right)\\
&\quad +\mathrm{Li}_2\left(\frac{\alpha_3^2A_2}{\alpha_1^2C_2}\right)-\mathrm{Li}_2\left(\frac{A_2}{C_2}\right)-\mathrm{Li}_2\left(\frac{\alpha_1^2B}{\alpha_2^2A}\right)+\mathrm{Li}_2\left(\frac{B}{A}\right)\\
&\quad -\mathrm{Li}_2\left(\frac{\alpha_2^2C}{\alpha_3^2B}\right)+\mathrm{Li}_2\left(\frac{C}{B}\right)-\mathrm{Li}_2\left(\frac{\alpha_3^2A}{\alpha_1^2C}\right)+\mathrm{Li}_2\left(\frac{A}{C}\right)\\
&\quad +\ln(\alpha_1^2)\ln\left(\frac{AB_3}{BA_3}\right)+\ln(\alpha_2^2)\ln\left(\frac{BC_1}{CB_1}\right)+\ln(\alpha_3^2)\ln\left(\frac{CA_2}{AC_2}\right).\end{aligned} \tag{3.59}$$

Note now that from the definitions of A, B, C we have the relations

$$\frac{B_1}{A_2} = \frac{B}{A}, \tag{3.60}$$

$$\frac{C_2}{B_3} = \frac{C}{B}, \tag{3.61}$$

$$\frac{A_3}{C_1} = \frac{A}{C}. \tag{3.62}$$

Combining these with the equations of motion give expressions for the appropriately shifted A's, B's and C's explicitly in terms of A, B, C. To write these in a simple way, define the function $H_{A,B} \equiv H(A, B; \alpha_1, \alpha_2)$ to be

$$H_{A,B} = \frac{\alpha_2^2(\alpha_1^2-1)A-\alpha_1^2(\alpha_2^2-1)B+(\alpha_2^2-\alpha_1^2)AB}{(\alpha_1^2-\alpha_2^2)+(\alpha_2^2-1)A-(\alpha_1^2-1)B}. \tag{3.63}$$

Then we have the following

$$\begin{aligned}A_3 &= \frac{H_{C,A}}{C}, \qquad B_3 = \frac{H_{B,C}}{C},\\
B_1 &= \frac{H_{A,B}}{A}, \qquad C_1 = \frac{H_{C,A}}{A},\\
C_2 &= \frac{H_{B,C}}{B}, \qquad A_2 = \frac{H_{A,B}}{B}.\end{aligned} \tag{3.64}$$

This means that (3.59) becomes

$$\begin{aligned}
\Gamma \;=\; & -\mathrm{Li}_2\left(\frac{H_{C,A}}{C}\right) + \mathrm{Li}_2\left(\frac{H_{C,A}}{\alpha_1^2 C}\right) + \mathrm{Li}_2\left(\frac{H_{B,C}}{C}\right) - \mathrm{Li}_2\left(\frac{H_{B,C}}{\alpha_2^2 C}\right) \\
& + \mathrm{Li}_2\left(\frac{\alpha_1^2 H_{B,C}}{\alpha_2^2 H_{C,A}}\right) - \mathrm{Li}_2\left(\frac{H_{B,C}}{H_{C,A}}\right) - \mathrm{Li}_2\left(\frac{H_{A,B}}{A}\right) + \mathrm{Li}_2\left(\frac{H_{A,B}}{\alpha_2^2 A}\right) \\
& + \mathrm{Li}_2\left(\frac{H_{C,A}}{A}\right) - \mathrm{Li}_2\left(\frac{H_{C,A}}{\alpha_3^2 A}\right) + \mathrm{Li}_2\left(\frac{\alpha_2^2 H_{C,A}}{\alpha_3^2 H_{A,B}}\right) - \mathrm{Li}_2\left(\frac{H_{C,A}}{H_{A,B}}\right) \\
& - \mathrm{Li}_2\left(\frac{H_{B,C}}{B}\right) + \mathrm{Li}_2\left(\frac{H_{B,C}}{\alpha_3^2 B}\right) + \mathrm{Li}_2\left(\frac{H_{A,B}}{B}\right) - \mathrm{Li}_2\left(\frac{H_{A,B}}{\alpha_1^2 B}\right) \\
& + \mathrm{Li}_2\left(\frac{\alpha_3^2 H_{A,B}}{\alpha_1^2 H_{B,C}}\right) - \mathrm{Li}_2\left(\frac{H_{A,B}}{H_{B,C}}\right) - \mathrm{Li}_2\left(\frac{\alpha_1^2 B}{\alpha_2^2 A}\right) + \mathrm{Li}_2\left(\frac{B}{A}\right) \\
& - \mathrm{Li}_2\left(\frac{\alpha_2^2 C}{\alpha_3^2 B}\right) + \mathrm{Li}_2\left(\frac{C}{B}\right) - \mathrm{Li}_2\left(\frac{\alpha_3^2 A}{\alpha_1^2 C}\right) + \mathrm{Li}_2\left(\frac{A}{C}\right) \\
& + \ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right) + \ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right) + \ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right).
\end{aligned} \tag{3.65}$$

To be able to manipulate this expression, we will need to use some dilogarithm identities. The three dilogarithm identities needed for this computation appear in the appendix; for easy reference they are reproduced below.

Identity I:

$$\begin{aligned}
\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2(xy) \;=\; & \mathrm{Li}_2\left(\frac{x-xy}{1-xy}\right) + \mathrm{Li}_2\left(\frac{y-xy}{1-xy}\right) \\
& + \ln\left(\frac{1-x}{1-xy}\right)\ln\left(\frac{1-y}{1-xy}\right).
\end{aligned} \tag{3.66}$$

Identity II:

$$\mathrm{Li}_2\left(\frac{1}{x}\right) + \mathrm{Li}_2(x) = -\frac{\pi^2}{6} - \frac{1}{2}\Big(\ln(-x)\Big)^2. \tag{3.67}$$

Identity III:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\Big(\ln(1-x)\Big)^2. \tag{3.68}$$

Now, the next step is simply to rearrange the terms in a way which suggests which identities to use where.

$$
\begin{aligned}
\Gamma \;=\; & +\mathrm{Li}_2\left(\frac{H_{A,B}}{B}\right)+\mathrm{Li}_2\left(\frac{B}{A}\right)-\mathrm{Li}_2\left(\frac{H_{A,B}}{A}\right) \\
& +\mathrm{Li}_2\left(\frac{H_{B,C}}{C}\right)+\mathrm{Li}_2\left(\frac{C}{B}\right)-\mathrm{Li}_2\left(\frac{H_{B,C}}{B}\right) \\
& +\mathrm{Li}_2\left(\frac{H_{C,A}}{A}\right)+\mathrm{Li}_2\left(\frac{A}{C}\right)-\mathrm{Li}_2\left(\frac{H_{C,A}}{C}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{A,B}}{\alpha_1^2 B}\right)-\mathrm{Li}_2\left(\frac{\alpha_1^2 B}{\alpha_2^2 A}\right)+\mathrm{Li}_2\left(\frac{H_{A,B}}{\alpha_2^2 A}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{B,C}}{\alpha_2^2 C}\right)-\mathrm{Li}_2\left(\frac{\alpha_2^2 C}{\alpha_3^2 B}\right)+\mathrm{Li}_2\left(\frac{H_{B,C}}{\alpha_3^2 B}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{C,A}}{\alpha_3^2 A}\right)-\mathrm{Li}_2\left(\frac{\alpha_3^2 A}{\alpha_1^2 C}\right)+\mathrm{Li}_2\left(\frac{H_{C,A}}{\alpha_1^2 C}\right) \\
& +\mathrm{Li}_2\left(\frac{\alpha_3^2 H_{A,B}}{\alpha_1^2 H_{B,C}}\right)+\mathrm{Li}_2\left(\frac{\alpha_2^2 H_{C,A}}{\alpha_3^2 H_{A,B}}\right)+\mathrm{Li}_2\left(\frac{\alpha_1^2 H_{B,C}}{\alpha_2^2 H_{C,A}}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{A,B}}{H_{B,C}}\right)-\mathrm{Li}_2\left(\frac{H_{B,C}}{H_{C,A}}\right)-\mathrm{Li}_2\left(\frac{H_{C,A}}{H_{A,B}}\right) \\
& +\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right).
\end{aligned}
\tag{3.69}
$$

On the first two terms of line 7 and the last term of line 8, use identity (3.67) to flip over the argument. The logarithm terms will always be collected towards the end of the expression for neatness.

$$
\begin{aligned}
\Gamma \;=\; & +\mathrm{Li}_2\left(\frac{H_{A,B}}{B}\right)+\mathrm{Li}_2\left(\frac{B}{A}\right)-\mathrm{Li}_2\left(\frac{H_{A,B}}{A}\right) \\
& +\mathrm{Li}_2\left(\frac{H_{B,C}}{C}\right)+\mathrm{Li}_2\left(\frac{C}{B}\right)-\mathrm{Li}_2\left(\frac{H_{B,C}}{B}\right) \\
& +\mathrm{Li}_2\left(\frac{H_{C,A}}{A}\right)+\mathrm{Li}_2\left(\frac{A}{C}\right)-\mathrm{Li}_2\left(\frac{H_{C,A}}{C}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{A,B}}{\alpha_1^2 B}\right)-\mathrm{Li}_2\left(\frac{\alpha_1^2 B}{\alpha_2^2 A}\right)+\mathrm{Li}_2\left(\frac{H_{A,B}}{\alpha_2^2 A}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{B,C}}{\alpha_2^2 C}\right)-\mathrm{Li}_2\left(\frac{\alpha_2^2 C}{\alpha_3^2 B}\right)+\mathrm{Li}_2\left(\frac{H_{B,C}}{\alpha_3^2 B}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{C,A}}{\alpha_3^2 A}\right)-\mathrm{Li}_2\left(\frac{\alpha_3^2 A}{\alpha_1^2 C}\right)+\mathrm{Li}_2\left(\frac{H_{C,A}}{\alpha_1^2 C}\right) \\
& -\mathrm{Li}_2\left(\frac{\alpha_1^2 H_{B,C}}{\alpha_3^2 H_{A,B}}\right)-\mathrm{Li}_2\left(\frac{\alpha_3^2 H_{A,B}}{\alpha_2^2 H_{C,A}}\right)+\mathrm{Li}_2\left(\frac{\alpha_1^2 H_{B,C}}{\alpha_2^2 H_{C,A}}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{A,B}}{H_{B,C}}\right)-\mathrm{Li}_2\left(\frac{H_{B,C}}{H_{C,A}}\right)+\mathrm{Li}_2\left(\frac{H_{A,B}}{H_{C,A}}\right) \\
& -\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2 H_{B,C}}{\alpha_3^2 H_{A,B}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2 H_{A,B}}{\alpha_2^2 H_{C,A}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{H_{A,B}}{H_{C,A}}\right)\right)^2
\end{aligned}
$$

$$-\frac{\pi^2}{6} + \ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right) + \ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right) + \ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right). \tag{3.70}$$

Use identity (3.66) on each of the lines 1 to 8 to convert the three dilogarithm terms into two.

$$\begin{aligned}
\Gamma \; = \; & +\mathrm{Li}_2\left(\frac{(B-A)H_{A,B}}{B(H_{A,B}-A)}\right) + \mathrm{Li}_2\left(\frac{H_{A,B}-B}{H_{A,B}-A}\right) \\
& +\mathrm{Li}_2\left(\frac{(C-B)H_{B,C}}{C(H_{B,C}-B)}\right) + \mathrm{Li}_2\left(\frac{H_{B,C}-C}{H_{B,C}-B}\right) \\
& +\mathrm{Li}_2\left(\frac{(A-C)H_{C,A}}{A(H_{C,A}-C)}\right) + \mathrm{Li}_2\left(\frac{H_{C,A}-A}{H_{C,A}-C}\right) \\
& -\mathrm{Li}_2\left(\frac{(\alpha_1^2B-\alpha_2^2A)H_{A,B}}{\alpha_1^2B(H_{A,B}-\alpha_2^2A)}\right) - \mathrm{Li}_2\left(\frac{H_{A,B}-\alpha_1^2B}{H_{A,B}-\alpha_2^2A}\right) \\
& -\mathrm{Li}_2\left(\frac{(\alpha_2^2C-\alpha_3^2B)H_{B,C}}{\alpha_2^2C(H_{B,C}-\alpha_3^2B)}\right) - \mathrm{Li}_2\left(\frac{H_{B,C}-\alpha_2^2C}{H_{B,C}-\alpha_3^2B}\right) \\
& -\mathrm{Li}_2\left(\frac{(\alpha_3^2A-\alpha_1^2C)H_{C,A}}{\alpha_3^2A(H_{C,A}-\alpha_1^2C)}\right) - \mathrm{Li}_2\left(\frac{H_{C,A}-\alpha_3^2A}{H_{C,A}-\alpha_1^2C}\right) \\
& -\mathrm{Li}_2\left(\frac{\alpha_1^2H_{\alpha_2\alpha_3(\alpha_3^2H_{A,B}-\alpha_2^2H_{C,A})}}{\alpha_3^2H_{A,B}(\alpha_1^2H_{B,C}-\alpha_2^2H_{C,A})}\right) - \mathrm{Li}_2\left(\frac{\alpha_3^2H_{A,B}-\alpha_1^2H_{B,C}}{\alpha_2^2H_{C,A}-\alpha_1^2H_{B,C}}\right) \\
& -\mathrm{Li}_2\left(\frac{H_{A,B}(H_{B,C}-H_{C,A})}{H_{B,C}(H_{A,B}-H_{C,A})}\right) - \mathrm{Li}_2\left(\frac{H_{B,C}-H_{A,B}}{H_{C,A}-H_{A,B}}\right) \\
& +\ln\left(\frac{A(H_{A,B}-B)}{B(H_{A,B}-A)}\right)\ln\left(\frac{B-A}{H_{A,B}-A}\right) \\
& +\ln\left(\frac{B(H_{B,C}-C)}{C(H_{B,C}-B)}\right)\ln\left(\frac{C-B}{H_{B,C}-B}\right) \\
& +\ln\left(\frac{C(H_{C,A}-A)}{A(H_{C,A}-C)}\right)\ln\left(\frac{A-C}{H_{C,A}-C}\right) \\
& -\ln\left(\frac{\alpha_2^2A(H_{A,B}-\alpha_1^2B)}{\alpha_1^2B(H_{A,B}-\alpha_2^2A)}\right)\ln\left(\frac{\alpha_1^2B-\alpha_2^2A}{H_{A,B}-\alpha_2^2A}\right) \\
& -\ln\left(\frac{\alpha_2^2B(H_{B,C}-\alpha_2^2C)}{\alpha_2^2C(H_{B,C}-\alpha_3^2B)}\right)\ln\left(\frac{\alpha_2^2C-\alpha_3^2B}{H_{B,C}-\alpha_3^2B}\right) \\
& -\ln\left(\frac{\alpha_1^2C(H_{C,A}-\alpha_3^2A)}{\alpha_3^2A(H_{C,A}-\alpha_1^2C)}\right)\ln\left(\frac{\alpha_3^2A-\alpha_1^2C}{H_{C,A}-\alpha_1^2C}\right) \\
& -\ln\left(\frac{\alpha_2^2H_{C,A}(\alpha_3^2H_{A,B}-\alpha_1^2H_{B,C})}{\alpha_3^2H_{A,B}(\alpha_2^2H_{C,A}-\alpha_1^2H_{B,C})}\right)\ln\left(\frac{\alpha_3^2H_{A,B}-\alpha_2^2H_{C,A}}{\alpha_1^2H_{B,C}-\alpha_2^2H_{C,A}}\right) \\
& -\ln\left(\frac{H_{C,A}(H_{B,C}-H_{A,B})}{H_{B,C}(H_{C,A}-H_{A,B})}\right)\ln\left(\frac{H_{B,C}-H_{C,A}}{H_{A,B}-H_{C,A}}\right) \\
& -\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2H_{B,C}}{\alpha_3^2H_{A,B}}\right)\right)^2 - \frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2H_{A,B}}{\alpha_2^2H_{C,A}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{H_{A,B}}{H_{C,A}}\right)\right)^2
\end{aligned}$$

$$-\frac{\pi^2}{6}+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right). \tag{3.71}$$

Here we use the definition of $H_{A,B}$. We shall write $U_{\alpha_1\alpha_2}$ for the numerator of $H_{A,B}$ and $L_{\alpha_1\alpha_2}$ for the denominator.

$$\begin{aligned}
\Gamma \;=\; & +\mathrm{Li}_2\left(\frac{U_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)+\mathrm{Li}_2\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(A-\alpha_1^2)}\right)\\
&+\mathrm{Li}_2\left(\frac{U_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)+\mathrm{Li}_2\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)}{(\alpha_3^2-1)(B-\alpha_2^2)}\right)\\
&+\mathrm{Li}_2\left(\frac{U_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)+\mathrm{Li}_2\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)}{(\alpha_1^2-1)(C-\alpha_3^2)}\right)\\
&-\mathrm{Li}_2\left(\frac{U_{\alpha_1\alpha_2}}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)-\mathrm{Li}_2\left(\frac{(\alpha_1^2-1)(B-1)}{(\alpha_2^2-1)(A-1)}\right)\\
&-\mathrm{Li}_2\left(\frac{U_{\alpha_2\alpha_3}}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)-\mathrm{Li}_2\left(\frac{(\alpha_2^2-1)(C-1)}{(\alpha_3^2-1)(B-1)}\right)\\
&-\mathrm{Li}_2\left(\frac{U_{\alpha_3\alpha_1}}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)-\mathrm{Li}_2\left(\frac{(\alpha_3^2-1)(A-1)}{(\alpha_1^2-1)(C-1)}\right)\\
&-\mathrm{Li}_2\left(-\frac{\alpha_1^2(A-1)U_{\alpha_2\alpha_3}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)-\mathrm{Li}_2\left(-\frac{(B-1)L_{\alpha_3\alpha_1}}{(C-1)L_{\alpha_1\alpha_2}}\right)\\
&-\mathrm{Li}_2\left(-\frac{(C-\alpha_3^2)U_{\alpha_1\alpha_2}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)-\mathrm{Li}_2\left(-\frac{(B-\alpha_2^2)L_{\alpha_3\alpha_1}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\\
&+\ln\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\right)\\
&+\ln\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\right)\\
&+\ln\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\right)\\
&-\ln\left(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\right)\\
&-\ln\left(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\right)\\
&-\ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right)\\
&-\ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right)\\
&-\ln\left(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\\
&-\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2U_{\alpha_2\alpha_3}L_{\alpha_1\alpha_2}}{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{\alpha_2^2U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2
\end{aligned}$$

$$
+\frac{1}{2}\left(\ln\left(-\frac{U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2
$$
$$
-\frac{\pi^2}{6}+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right). \tag{3.72}
$$

Rearrange the terms again to make it clear what to do next:

$$
\begin{aligned}
\Gamma \;=\; & +\mathrm{Li}_2\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(A-\alpha_1^2)}\right)+\mathrm{Li}_2\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)}{(\alpha_1^2-1)(C-\alpha_3^2)}\right)\\
& +\mathrm{Li}_2\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)}{(\alpha_3^2-1)(B-\alpha_2^2)}\right)\\
& -\mathrm{Li}_2\left(\frac{(\alpha_1^2-1)(B-1)}{(\alpha_2^2-1)(A-1)}\right)-\mathrm{Li}_2\left(\frac{(\alpha_2^2-1)(C-1)}{(\alpha_3^2-1)(B-1)}\right)\\
& -\mathrm{Li}_2\left(\frac{(\alpha_3^2-1)(A-1)}{(\alpha_1^2-1)(C-1)}\right)\\
& +\mathrm{Li}_2\left(\frac{U_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)-\mathrm{Li}_2\left(\frac{U_{\alpha_1\alpha_2}}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\\
& +\mathrm{Li}_2\left(\frac{U_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)-\mathrm{Li}_2\left(\frac{U_{\alpha_2\alpha_3}}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\\
& +\mathrm{Li}_2\left(\frac{U_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)-\mathrm{Li}_2\left(\frac{U_{\alpha_3\alpha_1}}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\\
& -\mathrm{Li}_2\left(-\frac{\alpha_1^2(A-1)U_{\alpha_2\alpha_3}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)-\mathrm{Li}_2\left(-\frac{(C-\alpha_3^2)U_{\alpha_1\alpha_2}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\\
& -\mathrm{Li}_2\left(-\frac{(B-1)L_{\alpha_3\alpha_1}}{(C-1)L_{\alpha_1\alpha_2}}\right)-\mathrm{Li}_2\left(-\frac{(B-\alpha_2^2)L_{\alpha_3\alpha_1}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\\
& +\ln\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\right)\\
& +\ln\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\right)\\
& +\ln\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\right)\\
& -\ln\left(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\right)\\
& -\ln\left(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\right)\\
& -\ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right)\\
& -\ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\ln\Big(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\Big)\ln\Big(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\Big)\\
&-\frac{1}{2}\Big(\ln\Big(-\frac{\alpha_1^2U_{\alpha_2\alpha_3}L_{\alpha_1\alpha_2}}{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}}\Big)\Big)^2-\frac{1}{2}\Big(\ln\Big(-\frac{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{\alpha_2^2U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\Big)\Big)^2\\
&+\frac{1}{2}\Big(\ln\Big(-\frac{U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\Big)\Big)^2\\
&-\frac{\pi^2}{6}+\ln(\alpha_1^2)\ln\Big(\frac{AH_{B,C}}{BH_{C,A}}\Big)+\ln(\alpha_2^2)\ln\Big(\frac{BH_{C,A}}{CH_{A,B}}\Big)+\ln(\alpha_3^2)\ln\Big(\frac{CH_{A,B}}{AH_{B,C}}\Big).
\end{aligned}
\tag{3.73}
$$

Use identity (3.67) to flip the arguments of the terms of lines 2,4,5,6 and 7.

$$
\begin{aligned}
\Gamma \;=\; &+\mathrm{Li}_2\Big(\frac{(\alpha_1^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(A-\alpha_1^2)}\Big)+\mathrm{Li}_2\Big(\frac{(\alpha_3^2-1)(A-\alpha_1^2)}{(\alpha_1^2-1)(C-\alpha_3^2)}\Big)\\
&-\mathrm{Li}_2\Big(\frac{(\alpha_3^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(C-\alpha_3^2)}\Big)\\
&-\mathrm{Li}_2\Big(\frac{(\alpha_1^2-1)(B-1)}{(\alpha_2^2-1)(A-1)}\Big)-\mathrm{Li}_2\Big(\frac{(\alpha_2^2-1)(C-1)}{(\alpha_3^2-1)(B-1)}\Big)\\
&+\mathrm{Li}_2\Big(\frac{(\alpha_1^2-1)(C-1)}{(\alpha_3^2-1)(A-1)}\Big)\\
&+\mathrm{Li}_2\Big(\frac{U_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)B}\Big)+\mathrm{Li}_2\Big(\frac{\alpha_1^2(\alpha_2^2-1)(A-1)B}{U_{\alpha_1\alpha_2}}\Big)\\
&+\mathrm{Li}_2\Big(\frac{U_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)C}\Big)+\mathrm{Li}_2\Big(\frac{\alpha_2^2(\alpha_3^2-1)(B-1)C}{U_{\alpha_2\alpha_3}}\Big)\\
&+\mathrm{Li}_2\Big(\frac{U_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)A}\Big)+\mathrm{Li}_2\Big(\frac{\alpha_3^2(\alpha_1^2-1)(C-1)A}{U_{\alpha_3\alpha_1}}\Big)\\
&-\mathrm{Li}_2\Big(-\frac{\alpha_1^2(A-1)U_{\alpha_2\alpha_3}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\Big)-\mathrm{Li}_2\Big(-\frac{(C-\alpha_3^2)U_{\alpha_1\alpha_2}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\Big)\\
&-\mathrm{Li}_2\Big(-\frac{(B-1)L_{\alpha_3\alpha_1}}{(C-1)L_{\alpha_1\alpha_2}}\Big)-\mathrm{Li}_2\Big(-\frac{(B-\alpha_2^2)L_{\alpha_3\alpha_1}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\Big)\\
&+\ln\Big(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\Big)\ln\Big(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\Big)\\
&+\ln\Big(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\Big)\ln\Big(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\Big)\\
&+\ln\Big(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\Big)\ln\Big(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\Big)\\
&-\ln\Big(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\Big)\ln\Big(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\Big)\\
&-\ln\Big(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\Big)\ln\Big(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right)\\
&-\ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right)\\
&-\ln\left(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\\
&-\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2U_{\alpha_2\alpha_3}L_{\alpha_1\alpha_2}}{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{\alpha_2^2U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(C-\alpha_3^2)}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)(C-1)}{(\alpha_3^2-1)(A-1)}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2(\alpha_2^2-1)(A-1)B}{U_{\alpha_1\alpha_2}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(\frac{\alpha_2^2(\alpha_3^2-1)(B-1)C}{U_{\alpha_2\alpha_3}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{\alpha_3^2(\alpha_1^2-1)(C-1)A}{U_{\alpha_3\alpha_1}}\right)\right)^2\\
&+\frac{\pi^2}{3}+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right).
\end{aligned}
\tag{3.74}
$$

Use identity (3.66) on lines 1&2, 3&4, 5,6,7 and 8.

$$
\begin{aligned}
\Gamma \;=\; &+\mathrm{Li}_2\left(-\frac{(B-\alpha_2^2)L_{\alpha_3\alpha_1}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)+\mathrm{Li}_2\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right)\\
&-\mathrm{Li}_2\left(-\frac{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)-\mathrm{Li}_2\left(-\frac{(C-1)L_{\alpha_1\alpha_2}}{(B-1)L_{\alpha_3\alpha_1}}\right)\\
&+\mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)}{A-\alpha_1^2}\right)+\mathrm{Li}_2\left(\frac{\alpha_2^2(B-1)}{(\alpha_2^2-1)B}\right)+\mathrm{Li}_2\left(-\frac{\alpha_1^2(A-1)(B-\alpha_2^2)}{U_{\alpha_1\alpha_2}}\right)\\
&+\mathrm{Li}_2\left(\frac{\alpha_2^2(B-1)}{B-\alpha_2^2}\right)+\mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)}{(\alpha_3^2-1)C}\right)+\mathrm{Li}_2\left(-\frac{\alpha_2^2(B-1)(C-\alpha_3^2)}{U_{\alpha_2\alpha_3}}\right)\\
&+\mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)}{C-\alpha_3^2}\right)+\mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)}{(\alpha_1^2-1)A}\right)+\mathrm{Li}_2\left(-\frac{\alpha_3^2(C-1)(A-\alpha_1^2)}{U_{\alpha_3\alpha_1}}\right)\\
&-\mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{\alpha_3^2(C-1)(A-\alpha_1^2)}\right)-\mathrm{Li}_2\left(-\frac{\alpha_1^2(A-1)(B-\alpha_2^2)}{U_{\alpha_1\alpha_2}}\right)\\
&-\mathrm{Li}_2\left(-\frac{\alpha_2^2(B-1)(C-\alpha_3^2)}{U_{\alpha_2\alpha_3}}\right)\\
&-\mathrm{Li}_2\left(-\frac{(B-1)L_{\alpha_3\alpha_1}}{(C-1)L_{\alpha_1\alpha_2}}\right)-\mathrm{Li}_2\left(-\frac{(B-\alpha_2^2)L_{\alpha_3\alpha_1}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\ln\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\right)\\
&+\ln\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\right)\\
&+\ln\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\right)\\
&-\ln\left(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\right)\\
&-\ln\left(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\right)\\
&-\ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right)\\
&-\ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right)\\
&-\ln\left(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\\
&+\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right)\\
&-\ln\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(B-1)L_{\alpha_3\alpha_1}}\right)\\
&+\ln\left(-\frac{B-\alpha_2^2}{(\alpha_2^2-1)B}\right)\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right)\\
&+\ln\left(-\frac{C-\alpha_3^2}{(\alpha_3^2-1)C}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right)\\
&+\ln\left(-\frac{A-\alpha_1^2}{(\alpha_1^2-1)A}\right)\ln\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{U_{\alpha_3\alpha_1}}\right)\\
&-\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right)\\
&-\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2U_{\alpha_2\alpha_3}L_{\alpha_1\alpha_2}}{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{\alpha_2^2U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(C-\alpha_3^2)}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)(C-1)}{(\alpha_3^2-1)(A-1)}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2(\alpha_2^2-1)(A-1)B}{U_{\alpha_1\alpha_2}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(\frac{\alpha_2^2(\alpha_3^2-1)(B-1)C}{U_{\alpha_2\alpha_3}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{\alpha_3^2(\alpha_1^2-1)(C-1)A}{U_{\alpha_3\alpha_1}}\right)\right)^2\\
&+\frac{\pi^2}{3}+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right).
\end{aligned}
\tag{3.75}
$$

Remove any terms which cancel out and rearrange once more.

$$
\begin{aligned}
\Gamma \;=\; & -\mathrm{Li}_2\left(-\frac{(C-1)L_{\alpha_1\alpha_2}}{(B-1)L_{\alpha_3\alpha_1}}\right) - \mathrm{Li}_2\left(-\frac{(B-1)L_{\alpha_3\alpha_1}}{(C-1)L_{\alpha_1\alpha_2}}\right) \\
& -\mathrm{Li}_2\left(-\frac{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right) + \mathrm{Li}_2\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right) \\
& +\mathrm{Li}_2\left(-\frac{\alpha_3^2(C-1)(A-\alpha_1^2)}{U_{\alpha_3\alpha_1}}\right) - \mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{\alpha_3^2(C-1)(A-\alpha_1^2)}\right) \\
& +\mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)}{A-\alpha_1^2}\right) + \mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)}{(\alpha_1^2-1)A}\right) \\
& +\mathrm{Li}_2\left(\frac{\alpha_2^2(B-1)}{B-\alpha_2^2}\right) + \mathrm{Li}_2\left(\frac{\alpha_2^2(B-1)}{(\alpha_2^2-1)B}\right) \\
& +\mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)}{C-\alpha_3^2}\right) + \mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)}{(\alpha_3^2-1)C}\right) \\
& + \ln\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\right) \\
& + \ln\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\right) \\
& + \ln\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\right) \\
& - \ln\left(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\right) \\
& - \ln\left(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\right) \\
& - \ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right) \\
& - \ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right) \\
& - \ln\left(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right) \\
& + \ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right) \\
& - \ln\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(B-1)L_{\alpha_3\alpha_1}}\right) \\
& + \ln\left(-\frac{B-\alpha_2^2}{(\alpha_2^2-1)B}\right)\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right) \\
& + \ln\left(-\frac{C-\alpha_3^2}{(\alpha_3^2-1)C}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right) \\
& + \ln\left(-\frac{A-\alpha_1^2}{(\alpha_1^2-1)A}\right)\ln\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{U_{\alpha_3\alpha_1}}\right) \\
& - \ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2 U_{\alpha_2\alpha_3} L_{\alpha_1\alpha_2}}{\alpha_3^2 U_{\alpha_1\alpha_2} L_{\alpha_2\alpha_3}}\right)\right)^2 - \frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2 U_{\alpha_1\alpha_2} L_{\alpha_3\alpha_1}}{\alpha_2^2 U_{\alpha_3\alpha_1} L_{\alpha_1\alpha_2}}\right)\right)^2 \\
&+\frac{1}{2}\left(\ln\left(-\frac{U_{\alpha_1\alpha_2} L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1} L_{\alpha_1\alpha_2}}\right)\right)^2 - \frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(C-\alpha_3^2)}\right)\right)^2 \\
&+\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)(C-1)}{(\alpha_3^2-1)(A-1)}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2(\alpha_2^2-1)(A-1)B}{U_{\alpha_1\alpha_2}}\right)\right)^2 \\
&+\frac{1}{2}\left(\ln\left(\frac{\alpha_2^2(\alpha_3^2-1)(B-1)C}{U_{\alpha_2\alpha_3}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(\frac{\alpha_3^2(\alpha_1^2-1)(C-1)A}{U_{\alpha_3\alpha_1}}\right)\right)^2 \\
&+\frac{\pi^2}{3} + \ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right) + \ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right) + \ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right).
\end{aligned} \tag{3.76}
$$

Use (3.68) on the first term of lines 2,3,4,5 and 6.

$$
\begin{aligned}
\Gamma = {} & -\mathrm{Li}_2\left(-\frac{(C-1)L_{\alpha_1\alpha_2}}{(B-1)L_{\alpha_3\alpha_1}}\right) - \mathrm{Li}_2\left(-\frac{(B-1)L_{\alpha_3\alpha_1}}{(C-1)L_{\alpha_1\alpha_2}}\right) \\
& +\mathrm{Li}_2\left(-\frac{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}\right) + \mathrm{Li}_2\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right) \\
& -\mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)(A-\alpha_1^2)}{\alpha_1^2(A-1)(C-\alpha_3^2)}\right) - \mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{\alpha_3^2(C-1)(A-\alpha_1^2)}\right) \\
& -\mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)}{(\alpha_1^2-1)A}\right) + \mathrm{Li}_2\left(\frac{\alpha_1^2(A-1)}{(\alpha_1^2-1)A}\right) \\
& -\mathrm{Li}_2\left(\frac{\alpha_2^2(B-1)}{(\alpha_2^2-1)B}\right) + \mathrm{Li}_2\left(\frac{\alpha_2^2(B-1)}{(\alpha_2^2-1)B}\right) \\
& -\mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)}{(\alpha_3^2-1)C}\right) + \mathrm{Li}_2\left(\frac{\alpha_3^2(C-1)}{(\alpha_3^2-1)C}\right) \\
& +\ln\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\right) \\
& +\ln\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\right) \\
& +\ln\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\right) \\
& -\ln\left(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\right) \\
& -\ln\left(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\right) \\
& -\ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right)\\
&-\ln\left(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\\
&+\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right)\\
&-\ln\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(B-1)L_{\alpha_3\alpha_1}}\right)\\
&+\ln\left(-\frac{B-\alpha_2^2}{(\alpha_2^2-1)B}\right)\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right)\\
&+\ln\left(-\frac{C-\alpha_3^2}{(\alpha_3^2-1)C}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right)\\
&+\ln\left(-\frac{A-\alpha_1^2}{(\alpha_1^2-1)A}\right)\ln\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{U_{\alpha_3\alpha_1}}\right)\\
&-\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right)\\
&-\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2U_{\alpha_2\alpha_3}L_{\alpha_1\alpha_2}}{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{\alpha_2^2U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(C-\alpha_3^2)}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)(C-1)}{(\alpha_3^2-1)(A-1)}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2(\alpha_2^2-1)(A-1)B}{U_{\alpha_1\alpha_2}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(\frac{\alpha_2^2(\alpha_3^2-1)(B-1)C}{U_{\alpha_2\alpha_3}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{\alpha_3^2(\alpha_1^2-1)(C-1)A}{U_{\alpha_3\alpha_1}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{U_{\alpha_3\alpha_1}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)A}{A-\alpha_1^2}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_2^2-1)B}{B-\alpha_2^2}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)C}{C-\alpha_3^2}\right)\right)^2\\
&+\frac{\pi^2}{3}+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right).
\end{aligned}
\tag{3.77}
$$

Clearly lines 4,5 and 6 disappear, and lines 1,2 and 3 will also go with the use of identity (3.67). So we are left with purely logarithm terms.

$$
\begin{aligned}
\Gamma \;=\; & +\ln\left(\frac{(\alpha_1^2-1)(B-\alpha_2^2)A}{(\alpha_2^2-1)(A-\alpha_1^2)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-\alpha_1^2)}\right) \\
& +\ln\left(\frac{(\alpha_2^2-1)(C-\alpha_3^2)B}{(\alpha_3^2-1)(B-\alpha_2^2)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-\alpha_2^2)}\right) \\
& +\ln\left(\frac{(\alpha_3^2-1)(A-\alpha_1^2)C}{(\alpha_1^2-1)(C-\alpha_3^2)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-\alpha_3^2)}\right) \\
& -\ln\left(\frac{\alpha_2^2(\alpha_1^2-1)(B-1)A}{\alpha_1^2(\alpha_2^2-1)(A-1)B}\right)\ln\left(\frac{L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)(A-1)}\right) \\
& -\ln\left(\frac{\alpha_3^2(\alpha_2^2-1)(C-1)B}{\alpha_2^2(\alpha_3^2-1)(B-1)C}\right)\ln\left(\frac{L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)(B-1)}\right) \\
& -\ln\left(\frac{\alpha_1^2(\alpha_3^2-1)(A-1)C}{\alpha_3^2(\alpha_1^2-1)(C-1)A}\right)\ln\left(\frac{L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)(C-1)}\right) \\
& -\ln\left(-\frac{\alpha_2^2(B-1)U_{\alpha_3\alpha_1}}{\alpha_3^2(C-1)U_{\alpha_1\alpha_2}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(C-1)L_{\alpha_1\alpha_2}}\right) \\
& -\ln\left(-\frac{(B-\alpha_2^2)U_{\alpha_3\alpha_1}}{(A-\alpha_1^2)U_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right) \\
& +\ln\left(-\frac{(C-\alpha_3^2)L_{\alpha_1\alpha_2}}{(A-\alpha_1^2)L_{\alpha_2\alpha_3}}\right)\ln\left(-\frac{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}\right) \\
& -\ln\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)\ln\left(-\frac{(A-1)L_{\alpha_2\alpha_3}}{(B-1)L_{\alpha_3\alpha_1}}\right) \\
& +\ln\left(-\frac{B-\alpha_2^2}{(\alpha_2^2-1)B}\right)\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right) \\
& +\ln\left(-\frac{C-\alpha_3^2}{(\alpha_3^2-1)C}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right) \\
& +\ln\left(-\frac{A-\alpha_1^2}{(\alpha_1^2-1)A}\right)\ln\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{U_{\alpha_3\alpha_1}}\right) \\
& -\ln\left(\frac{\alpha_2^2(B-1)(A-\alpha_1^2)}{U_{\alpha_1\alpha_2}}\right)\ln\left(\frac{\alpha_3^2(C-1)(B-\alpha_2^2)}{U_{\alpha_2\alpha_3}}\right) \\
& -\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2U_{\alpha_2\alpha_3}L_{\alpha_1\alpha_2}}{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{\alpha_3^2U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{\alpha_2^2U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(-\frac{U_{\alpha_1\alpha_2}L_{\alpha_3\alpha_1}}{U_{\alpha_3\alpha_1}L_{\alpha_1\alpha_2}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)(B-\alpha_2^2)}{(\alpha_2^2-1)(C-\alpha_3^2)}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)(C-1)}{(\alpha_3^2-1)(A-1)}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\alpha_1^2(\alpha_2^2-1)(A-1)B}{U_{\alpha_1\alpha_2}}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(\frac{\alpha_2^2(\alpha_3^2-1)(B-1)C}{U_{\alpha_2\alpha_3}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{\alpha_3^2(\alpha_1^2-1)(C-1)A}{U_{\alpha_3\alpha_1}}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}{(\alpha_2^2-1)L_{\alpha_3\alpha_1}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{\alpha_1^2(A-1)(C-\alpha_3^2)}{U_{\alpha_3\alpha_1}}\right)\right)^2 \\
& -\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)A}{A-\alpha_1^2}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_2^2-1)B}{B-\alpha_2^2}\right)\right)^2
\end{aligned}
$$

$$-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_3^2-1)C}{C-\alpha_3^2}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{(C-1)L_{\alpha_1\alpha_2}}{(B-1)L_{\alpha_3\alpha_1}}\right)\right)^2$$
$$-\frac{1}{2}\left(\ln\left(-\frac{(\alpha_1^2-1)L_{\alpha_2\alpha_3}}{(\alpha_3^2-1)L_{\alpha_1\alpha_2}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{\alpha_3^2(C-1)(A-\alpha_1^2)}{\alpha_1^2(A-1)(C-\alpha_3^2)}\right)\right)^2$$
$$+\frac{\pi^2}{2}+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right). \tag{3.78}$$

These simplify to

$$\begin{aligned}\Gamma &= +\ln(\alpha_1^2)\ln(H_{C,A})-\ln(\alpha_1^2)\ln(H_{B,C})-\ln(\alpha_1^2)\ln(A)+\ln(\alpha_1^2)\ln(B)\\ &\quad+\ln(\alpha_2^2)\ln(H_{A,B})-\ln(\alpha_2^2)\ln(H_{C,A})-\ln(\alpha_2^2)\ln(B)+\ln(\alpha_2^2)\ln(C)\\ &\quad+\ln(\alpha_3^2)\ln(H_{B,C})-\ln(\alpha_3^2)\ln(H_{A,B})-\ln(\alpha_3^2)\ln(C)+\ln(\alpha_3^2)\ln(A)\\ &\quad+\ln(\alpha_1^2)\ln\left(\frac{AH_{B,C}}{BH_{C,A}}\right)+\ln(\alpha_2^2)\ln\left(\frac{BH_{C,A}}{CH_{A,B}}\right)+\ln(\alpha_3^2)\ln\left(\frac{CH_{A,B}}{AH_{B,C}}\right)\\ &= 0.\end{aligned} \tag{3.79}$$

2.3.8 A1

The proof of closure is analogous to that for Q1, since A1 is related to Q1 by the transformation $u_1 \mapsto -u_1$, $u_2 \mapsto -u_2$.

2.3.9 A2

The proof of closure is analogous to that for $Q3|_{\delta=0}$, since A2 is related to $Q3|_{\delta=0}$ by the transformation $u_1 \mapsto 1/u_1$, $u_2 \mapsto 1/u_2$.

2.3.10 Remaining cases: Q2, $Q3|_{\delta\neq 0}$ and Q4

For the Lagrangians of the remaining equations in the ABS list (i.e., Q2, $Q3|_{\delta\neq 0}$ and Q4), the computations to check they satisfy (3.1) are more troublesome, as the structure is more implicit. Furthermore, in the case of Q4, if we wished to verify the closure relation by direct computation we would have to employ functional identities for an elliptic analogue of the dilogarithm function, functions of the type

$$f(z) = \int \ln \sigma(z) dz, \tag{3.80}$$

which as far as we know have not yet been developed. However, following on from the paper [65], Bobenko and Suris showed through a more indirect method [15] that the closure relation (3.1), which they refer to as *flip invariance*, does indeed hold for all members in the ABS list. Indeed, it has now emerged [125] that the ABS list admits a universal Lagrangian structure which is antisymmetric with respect to the interchange of the lattice directions, and obeys the closure relation (3.1) on solutions of the equation.

2.3.11 Linear case

For completeness, we will also present here an example of a linear case, the linearization of H1. Since zero is not a solution of (1.7), we make the transformation $u(n_1, n_2) \mapsto u(n_1, n_2) + n_1 p_1 + n_2 p_2$, where $-p_1^2 = \alpha_1$ and $-p_2^2 = \alpha_2$, to get the equation

$$(u - u_{1,2} - p_1 - p_2)(u_1 - u_2 + p_1 - p_2) + p_1^2 - p_2^2 = 0. \tag{3.81}$$

Now $u = 0$ is a solution of (3.81), and we can expand around this solution by considering $u(n_1, n_2) = \epsilon v(n_1, n_2)$ for small ϵ. To first order in ϵ, we get

$$(p_1 - p_2)(v - v_{1,2}) = (p_1 + p_2)(v_1 - v_2), \tag{3.82}$$

which can easily be shown to be consistent-around-a-cube. A Lagrangian for (3.82) which is antisymmetric with respect to the interchange of lattice directions is

$$\mathcal{L}(v, v_1, v_2; p_1, p_2) = (v_1 - v_2)v - \frac{1}{2}\left(\frac{p_1 + p_2}{p_1 - p_2}\right)(v_1 - v_2)^2, \tag{3.83}$$

and this Lagrangian can be seen to obey the closure relation (3.1):

$$
\begin{aligned}
\Gamma &\equiv \Delta_1\mathcal{L}(v,v_2,v_3;p_2,p_3)+\Delta_2\mathcal{L}(v,v_3,v_1;p_3,p_1)+\Delta_3\mathcal{L}(v,v_1,v_2;p_1,p_2)\\
&= (v_{12}-v_{31})v_1-\frac{1}{2}\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_{12}-v_{31})^2-(v_2-v_3)v\\
&\quad \frac{1}{2}\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)^2+(v_{23}-v_{12})v_2-\frac{1}{2}\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_{23}-v_{12})^2\\
&\quad -(v_3-v_1)v+\frac{1}{2}\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)^2+(v_{31}-v_{23})v_3\\
&\quad -\frac{1}{2}\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_{31}-v_{23})^2-(v_1-v_2)v+\frac{1}{2}\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)^2, \quad (3.84)
\end{aligned}
$$

which, after substituting in the equations, is

$$
\begin{aligned}
\Gamma &= \left[\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)-\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)\right]v_1+\frac{1}{2}\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)^2\\
&\quad -\frac{1}{2}\left(\frac{p_2+p_3}{p_2-p_3}\right)\left[\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)-\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)\right]^2\\
&\quad \left[\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)-\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)\right]v_2+\frac{1}{2}\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)^2\\
&\quad -\frac{1}{2}\left(\frac{p_3+p_1}{p_3-p_1}\right)\left[\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)-\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)\right]^2\\
&\quad \left[\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)-\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)\right]v_3+\frac{1}{2}\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)^2\\
&\quad -\frac{1}{2}\left(\frac{p_1+p_2}{p_1-p_2}\right)\left[\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)-\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)\right]^2, \quad (3.85)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Gamma \;=\; & -\frac{1}{2}\left(\frac{p_2+p_3}{p_2-p_3}\right)\left[1+\left(\frac{p_1+p_2}{p_1-p_2}\right)\left(\frac{p_2+p_3}{p_2-p_3}\right)\right. \\
& \qquad\qquad \left.+\left(\frac{p_2+p_3}{p_2-p_3}\right)\left(\frac{p_3+p_1}{p_3-p_1}\right)\right](v_2-v_3)^2 \\
& -\frac{1}{2}\left(\frac{p_3+p_1}{p_3-p_1}\right)\left[1+\left(\frac{p_2+p_3}{p_2-p_3}\right)\left(\frac{p_3+p_1}{p_3-p_1}\right)\right. \\
& \qquad\qquad \left.+\left(\frac{p_3+p_1}{p_3-p_1}\right)\left(\frac{p_1+p_2}{p_1-p_2}\right)\right](v_3-v_1)^2 \\
& -\frac{1}{2}\left(\frac{p_1+p_2}{p_1-p_2}\right)\left[1+\left(\frac{p_3+p_1}{p_3-p_1}\right)\left(\frac{p_1+p_2}{p_1-p_2}\right)\right. \\
& \qquad\qquad \left.+\left(\frac{p_1+p_2}{p_1-p_2}\right)\left(\frac{p_2+p_3}{p_2-p_3}\right)\right](v_1-v_2)^2 \\
& +\left(\frac{p_1+p_2}{p_1-p_2}\right)\left(\frac{p_2+p_3}{p_2-p_3}\right)\left(\frac{p_3+p_1}{p_3-p_1}\right)(v_3-v_1)(v_1-v_2) \\
& +\left(\frac{p_2+p_3}{p_2-p_3}\right)\left(\frac{p_3+p_1}{p_3-p_1}\right)\left(\frac{p_1+p_2}{p_1-p_2}\right)(v_1-v_2)(v_2-v_3) \\
& +\left(\frac{p_3+p_1}{p_3-p_1}\right)\left(\frac{p_1+p_2}{p_1-p_2}\right)\left(\frac{p_2+p_3}{p_2-p_3}\right)(v_2-v_3)(v_3-v_1), \qquad (3.86)
\end{aligned}
$$

and so

$$
\begin{aligned}
\Gamma \;=\; & \frac{1}{2}\left(\frac{p_3+p_1}{p_3-p_1}\right)\left(\frac{p_1+p_2}{p_1-p_2}\right)\left(\frac{p_2+p_3}{p_2-p_3}\right)\Big[(v_1-v_2)^2+(v_2-v_3)^2+(v_3-v_1)^2 \\
& \quad +2(v_3-v_1)(v_1-v_2)+2(v_1-v_2)(v_2-v_3)+2(v_2-v_3)(v_3-v_1)\Big] \\
=\; & 0. \qquad (3.87)
\end{aligned}
$$

The equation (3.82) is actually satisfied by the *plane wave factors*, i.e., discrete exponential functions,

$$
v=\prod_{i\in I}\left(\frac{p_i+k}{p_i-k}\right)^{n_i}, \qquad (3.88)
$$

where I is some index set. These appear in solutions of each of the lattice equations H1 through Q3 [82, 106].

2.4 Variational principle for Lagrangian 2-forms

In order to discuss the implications of the closure relation, we need to introduce some further notation. Let $\boldsymbol{e}_i$ denote the unit vector in the lattice direction labelled by i and let any point in the multidimensional lattice be specified by the vector $\boldsymbol{n}$ whose components are the coordinates $n_1, n_2, \ldots$ of the lattice, then elementary shifts in the lattice can be generated by the action $\boldsymbol{n} \to \boldsymbol{n} + \boldsymbol{e}_i$. If we select a lattice of finite dimensionality we could write the coordinates on the lattice as $\boldsymbol{n} = (n_1, n_2, \ldots)$, where the lattice directions are labelled according to the natural numbers. However in principle one could also have an infinite dimensional lattice and even a lattice labelled by an uncountable set.

Specifying an elementary oriented plaquette in this lattice requires the following data: the position $\boldsymbol{n}$ of one of its vertices in the lattice and the lattice directions given by the base vectors $\boldsymbol{e}_i, \boldsymbol{e}_j$. One way to characterize the oriented plaquette is by the ordered triplet $\sigma_{ij}(\boldsymbol{n}) = (\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i, \boldsymbol{n} + \boldsymbol{e}_j)$ (see Figure 2.8). Since the 3-

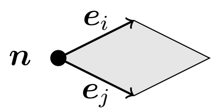

Figure 2.8: Elementary oriented plaquette.

point Lagrangians depend on two directions in the lattice, and when embedded in a multidimensional lattice at each point can be associated with an oriented plaquette $\sigma_{ij}(\boldsymbol{n})$, we can think of these Lagrangians as defining a discrete 2-form $\mathcal{L}_{ij}(\boldsymbol{n})$ whose evaluation on that plaquette is given by the Lagrangian function as follows

$$\mathcal{L}_{ij}(\boldsymbol{n}) = \mathcal{L}(u(\boldsymbol{n}), u(\boldsymbol{n} + \boldsymbol{e}_i), u(\boldsymbol{n} + \boldsymbol{e}_j); \alpha_i, \alpha_j). \tag{4.1}$$

Choosing now a surface σ in the multidimensional lattice consisting of a connected configuration of elementary plaquettes $\sigma_{ij}(\boldsymbol{n})$, such as illustrated in Figure 2.9 (which could be an infinite surface or a compact surface, with or without boundary) we can define an action on that surface simply by summing up the contributions

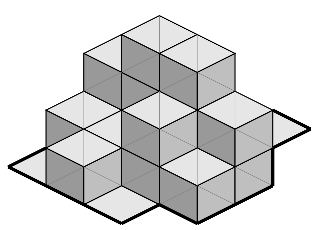

Figure 2.9: Example of a surface with boundary.

$\mathcal{L}_{ij}$ from each of the plaquettes on the surface, taking into account the directions associated with each face, i.e., we perform the sum:

$$S = S[u(\boldsymbol{n});\sigma] = \sum_{\sigma} \mathcal{L} = \sum_{\sigma_{ij}(\boldsymbol{n})\in\sigma} \mathcal{L}_{ij}(\boldsymbol{n}). \tag{4.2}$$

The sum in (4.2) is unambiguous for two reasons: first, because all the Lagrangians considered in the list (2.8a)-(2.15c) have the property of antisymmetry up to a constant with respect to transformations $i \leftrightarrow j$, i.e., $\mathcal{L}_{ij}(\boldsymbol{n}) = -\mathcal{L}_{ji}(\boldsymbol{n}) + \text{constant}$; second, we choose the base point $\boldsymbol{n}$ in such a way that $\mathcal{L}_{ij}(\boldsymbol{n})$, defined on $\sigma_{ij}(\boldsymbol{n})$, involves $u(\boldsymbol{n})$ along with its shifts only in the positive i and j directions. We choose not to use the abstract notation of difference forms, cf. e.g. [73], because we want to demonstrate on the basis of the examples given that all statements can be established through concrete computations.

It is obvious from (4.2) that the geometry of the surface σ forms an integral part of the action functional. The closure relation (3.1) implies the invariance of the action under local deformations of the surface σ while fixing its boundary. This we can easily see by considering an elementary variation of a locally flat surface at a single plaquette as illustrated by Figure 2.10. If S is the value of the action

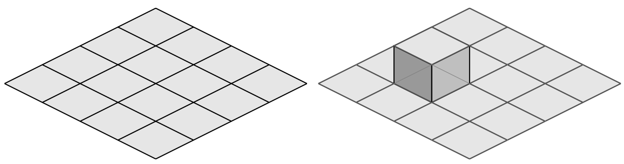

Figure 2.10: Local deformation of a discrete surface σ to a surface σ'.

functional for the undeformed surface in Figure 2.10, the value for the deformed surface in Figure 2.10 can be computed as follows

$$\begin{aligned} S' &= S - \mathcal{L}(u,u_i,u_j;\alpha_i,\alpha_j) + \mathcal{L}(u_k,u_{i,k},u_{j,k};\alpha_i,\alpha_j) + \mathcal{L}(u_i,u_{i,j},u_{i,k};\alpha_j,\alpha_k) \\ &\quad + \mathcal{L}(u_j,u_{j,k},u_{i,j};\alpha_k,\alpha_i) - \mathcal{L}(u,u_j,u_k;\alpha_j,\alpha_k) - \mathcal{L}(u,u_k,u_i;\alpha_k,\alpha_i), \end{aligned} \tag{4.3}$$

taking into account the orientation of the deformation $\sigma \to \sigma'$, defined as a transition between two collections of oriented plaquettes as indicated by Figure 2.10. From this argument it follows that the independence of the action under such a deformation is locally equivalent to the closure relation (3.1). We consider this invariance an essential aspect of the relevant variational principle underlying multidimensionally consistent lattice systems.

The aim of a Lagrangian multiform description over the usual scalar Lagrangian description is that it should provide us with not just one variational equation, but in principle an arbitrary number of compatible equations. We propose the following discrete variational principle for integrable lattice systems.

Discrete variational principle for integrable lattice systems: *The functions $u(\boldsymbol{n})$ solving an integrable multidimensional lattice system on each discrete quadrilateral surface σ are those for which the action $S[u(\boldsymbol{n});\sigma]$ of (4.2) is invariant under local deformations of the lattice, as described above, and for which the action attains an extremum under infinitesimal local deformations of the dependent variable $u(\boldsymbol{n})$.*

The mechanism that we propose is as follows.

1. Start with a surface as in Figure 2.9.

2. Define an action functional $S[u(\boldsymbol{n});\sigma]$ as in (4.2), and impose surface independence of this action.

3. This allows us to deform the surface σ as we choose, whilst keeping the boundary in place if there is a boundary. Thus, we can always render it into a locally flat surface away from the boundary, where we can choose any pair of local coordinates n_i, n_j.

4. In that part of the surface we can then apply the usual variational principle with respect to the field variables $u(\boldsymbol{n})$, leading in the usual manner to the Euler-Lagrange equations in those lattice directions.

If these equations subsequently imply the validity of the closure relation for the Lagrangian in terms of which the action is defined, this then ensures that the equations are consistent with invariance of the action under deformation of the surface, which in turn allowed the derivation of those equations in the first place. We view this circular mechanism as a manifestation of multidimensional consistency on the level of the Lagrangian.

2.5 Discussion

The discrete variational principle formulated in the previous section brings in the geometry as a variable of the action functional. This contrasts starkly with the usual variational principle (reproduced in the Introduction), where the Euler-Lagrange equations provide information rather on the parametrization of the underlying geometry than on the geometry itself. For instance, in the elementary case of a mechanical system with one degree of freedom the action

$$S[q(t)] = \int_0^T L(q, \dot{q}, t)dt \tag{5.1}$$

contains hardly any geometry at all, but the relevant Lagrange equation tells us how the one-dimensional motion is parametrized in a specific way according to the equations of motion. When we have more than one degree of freedom there is obviously room for nontrivial phase space geometry, but again the variational equations tell us more about how the geometry is parametrized rather than bringing in the geometry as a variational variable. Even in classical string theory [39], the geometry of the string trajectories (which sweep out a surface in configuration space) plays a role at the level of the dependent variables of the string action rather than of the independent variables which parametrize the surface. In contrast, our proposal involves the geometry of the space of independent variables which is somewhat reminiscent of the de Donder-Weyl formalism [51], although in this approach the connection to integrability is not evident. As far as we are aware, all Lagrangian descriptions of (continuous) integrable systems so far involve conventional scalar Lagrangians, even where an attempt is made to give a multi-Lagrangian description of integrable hierarchies, cf. [99, 100, 16].

This principle goes farther than just providing a variational derivation of equations of motion from a given Lagrangian, in that in some sense it also imposes conditions on the Lagrangian. It may even be possible to classify integrable systems on the level of the Lagrangian, using the closure property as a criterion. What is not clear at this stage is to what extent Lagrangians can be constructed by application of this principle, and we do not yet have a general proof that this procedure will automatically lead to multidimensionally consistent lattice equations.

Whilst there are many long and technical computations in the course of this thesis, it is the underlying idea of a fundamentally new variational principle for integrable systems that is the important message here.

2.6 Chapter summary

In this chapter we presented Lagrangians in terms of 3 points for equations in the ABS list, which are anti-symmetric with respect to interchange of the lattice directions, and showed that they obeyed a closure relation on solutions to the equations. On the basis of this we formulated a novel variational principle for integrable (in the sense of multidimensional consistency) systems, which brings in the geometry of the independent variables. Instead of scalar Lagrangians we now have Lagrangian multiforms, from which copies of the equation in all possible lattice directions can be derived.

Chapter 3

Lagrangian 2-form for a multi-component system

3.1 The lattice Gel'fand-Dikii hierarchy

The lattice Gel'fand-Dikii (GD) hierarchy first appeared in [86], where the direct linearization method was used to find a discrete analogue of the continuous GD hierarchy, which is a hierarchy of systems associated with higher order spectral problems [34, 35, 72, 28]. The first members in the hierarchy are the lattice KdV and lattice Boussinesq equations, higher order members are coupled systems of partial difference equations in terms of variables u, v_j, w_j, where $0 \leq j \leq N-2$, given by the following.

$$\widehat{v}_{j+1} - \widetilde{v}_{j+1} = (p - q + \widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j - p\widehat{v}_j + q\widetilde{v}_j, \tag{1.1a}$$

$$\widehat{w}_{j+1} - \widetilde{w}_{j+1} = -(p - q + \widehat{u} - \widetilde{u})w_j - q\widehat{w}_j + p\widetilde{w}_j, \tag{1.1b}$$

for $0 \leq j \leq N-3$, and

$$
\begin{aligned}
&(p-q+\widehat{u}-\widetilde{u})(\widehat{\widetilde{v}}_{N-2}-w_{N-2}) \\
&\quad = (p+q+u)\big[(p-q+\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_{N-3} - p\widehat{v}_{N-3} + q\widetilde{v}_{N-3}\big] \\
&\qquad + \sum_{j=0}^{N-3}\big[(-p)^{N-1-j}(\widetilde{v}_j - w_j) - (-q)^{N-1-j}(\widehat{v}_j - w_j) \\
&\qquad - w_j\big((-p)^{N-2-j}\widetilde{u} - (-q)^{N-2-j}\widehat{u}\big)\big] \\
&\qquad - \sum_{j=2}^{N-2}\sum_{i=0}^{N-1-j} w_i\big[(-p)^{N-1-j-i}\widetilde{v}_{j-1} - (-q)^{N-1-j-i}\widehat{v}_{j-1}\big],
\end{aligned} \tag{1.1c}
$$

identifying $v_0 = w_0 = u$. Because we are now dealing with a multi-component system there is a problem with notation, we already have subscripts denoting the components. In this chapter we use the notation we find most instructive: the dependent variables are u, v_j, w_j for $0 \leq j \leq N-2$, and we consider them to depend on two independent variables n, m. The symbol $\widetilde{\ }$ is used to denote shifts in the n-direction, and $\widehat{\ }$ denotes shifts in the m-direction, so that if $u = u(n, m)$, then $\widetilde{u} = u(n+1, m)$ and $\widehat{u} = u(n, m+1)$. The lattice parameters p, q are associated with the n, m-directions respectively. This is illustrated in Figure 3.1.

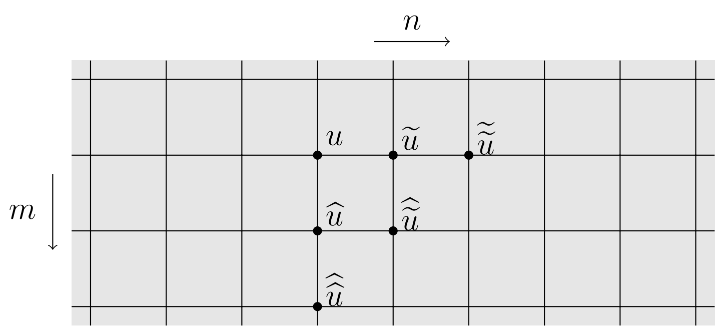

Figure 3.1: 2-d lattice.

The variables v_j, w_j are evaluated at the same lattice point as u, while $\widetilde{v}_j, \widetilde{w}_j$ are evaluated at the same lattice point as $\widetilde{u}$, and so on.

As noted in [86] the lattice GD hierarchy arises from a Zakharov-Shabat type of linear problem

$$
(p+\omega k)\widetilde{\varphi}_k = L_k \cdot \varphi_k \ , \ (q+\omega k)\widehat{\varphi}_k = M_k \cdot \varphi_k \ , \tag{1.2}
$$

in which

$$L_k = \begin{pmatrix} p-\widetilde{u} & 1 & & & \\ -\widetilde{v}_1 & p & 1 & & \\ \vdots & & \ddots & \ddots & \\ -\widetilde{v}_{N-2} & 0 & \cdots & p & 1 \\ k^N + * & w_{N-2} & \cdots & w_1 & p+u \end{pmatrix}, \tag{1.3}$$

and where the matrix M_k is a similar matrix obtained after the replacements $p \mapsto q$ and $\widetilde{\ } \mapsto \widehat{\ }$. The term $*$ in the lower left corner of the matrix L_k is such that the determinant $\det(L_k) = p^N - (-k)^N$, i.e., we have the expression

$$* = \sum_{j=0}^{N-2} (-p)^{N-1-j} (\widetilde{v}_j - w_j) - \sum_{j=1}^{N-1} \sum_{i=0}^{N-1-j} (-p)^{N-1-j-i} w_i \widetilde{v}_{j-1} . \tag{1.4}$$

Because the system is multidimensionally consistent [91, 14], we are free to impose copies of the equations in other lattice directions, with appropriate lattice parameters. The property of multidimensional consistency arises here essentially by the construction given in [86]. It was shown to hold explicitly for the Boussinesq equation in its scalar form in [123] considering an initial value problem on a 27-point cube, and as a coupled system of 3 equations in [117].

Suppose we have another lattice direction associated with parameter r, where shifts in this direction are denoted by $\bar{\ }$. Then copies of the equation (1.1a) can be imposed on each pair of lattice directions, giving

$$\widehat{v}_{j+1} - \widetilde{v}_{j+1} = (p - q + \widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j - p\widehat{v}_j + q\widetilde{v}_j, \tag{1.5a}$$

$$\overline{v}_{j+1} - \widehat{v}_{j+1} = (q - r + \overline{u} - \widehat{u})\widehat{\overline{v}}_j - q\overline{v}_j + r\widehat{v}_j, \tag{1.5b}$$

$$\widetilde{v}_{j+1} - \overline{v}_{j+1} = (r - p + \widetilde{u} - \overline{u})\widetilde{\overline{v}}_j - r\widetilde{v}_j + p\overline{v}_j, \tag{1.5c}$$

for $0 \le j \le N-3$.

In particular, summing equations (1.5a),(1.5b) and (1.5c) for $j = 0$ gives

$$0 = (p - q + \widehat{u} - \widetilde{u})(\widehat{\widetilde{u}} + r) + (q - r + \overline{u} - \widehat{u})(\widehat{\overline{u}} + p) + (r - p + \widetilde{u} - \overline{u})(\widetilde{\overline{u}} + q), \tag{1.6}$$

which is in fact the lattice Kadomtsev-Petviashvili (or lattice KP) equation [83].

That (1.6) holds is natural, since members of the lattice GD hierarchy can be viewed as a special type of periodic reduction of the lattice KP equation. Note that this is not the bilinear discrete KP equation considered in [66] and in Chapter 4, where a Lagrangian multiform structure for that system is given. A Lagrangian description for (1.6) has yet to be found, although one approach may be to consider the limit of the lattice GD hierarchy as $N \to \infty$.

3.2 Lagrangian structure for the lattice GD hierarchy

Lagrangians for the two lowest order members of the lattice GD hierarchy have already appeared in the literature, an action for the KdV lattice was first given in [20], and for the Boussinesq lattice in [86] (these systems arise from the lattice GD hierarchy by taking $N = 2$ or $N = 3$ respectively). In fact, as we show here, it is possible to write a Lagrangian for the generic member of the hierarchy.

Note first that if we define

$$\gamma_j(p,q) \equiv \frac{(-p)^{j+1} - (-q)^{j+1}}{-p+q} = (-1)^j(p^j + p^{j-1}q + \cdots + pq^{j-1} + q^j), \tag{2.1}$$

then equation (1.1c) can be written in a more convenient form as

$$\begin{aligned} \gamma_{N-1}(p,q) + \widehat{\widetilde{v}}_{N-2} - w_{N-2} \quad = \quad & \frac{(p-q)\gamma_{N-1}(p,q)}{p-q+\widehat{u}-\widetilde{u}} + \sum_{i=0}^{N-3}\sum_{j=0}^{N-3-i} \gamma_{N-3-i-j}\widehat{\widetilde{v}}_j w_i \\ & - \sum_{j=0}^{N-3} \gamma_{N-2-j}\left(\widehat{\widetilde{v}}_j - w_j\right), \end{aligned} \tag{2.2}$$

which enables us more easily to see that the following proposition holds.

Proposition 1: *The system consisting of equations* (1.1a),(1.1b) *and* (2.2) *solves the discrete Euler-Lagrange equations for the following Lagrangian*

$$\begin{aligned}\mathcal{L}_{pq} \equiv\ & (p-q)\gamma_{N-1}(p,q)\ln(p-q+\widehat{u}-\widetilde{u})-\gamma_{N-1}(p,q)(\widehat{u}-\widetilde{u})\\ & -\sum_{j=0}^{N-2}\gamma_{N-2-j}(p,q)(\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_j\\ & -\sum_{i=0}^{N-3}\sum_{j=1}^{N-2-i}\gamma_{N-2-i-j}(p,q)w_i[\widehat{v}_j-\widetilde{v}_j-(p-q+\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_{j-1}\\ & \qquad +p\widehat{v}_{j-1}-q\widetilde{v}_{j-1}],\end{aligned} \tag{2.3}$$

under independent variation of u, v and w.

Proof: Here we take the usual point of view and consider the action to be the sum of the Lagrangians over all n, m, i.e.

$$S=\sum_{n,m\in Z}\mathcal{L}_{pq}. \tag{2.4}$$

The discrete Euler-Lagrange equations arise as a consequence of the requirement that $\delta S=0$. We have

$$\begin{aligned}0 =\ & \delta S\\ =\ & \sum_{n,m\in Z}\Bigg\{\frac{(p-q)(\delta\widehat{u}-\delta\widetilde{u})\gamma_{N-1}(p,q)}{p-q+\widehat{u}-\widetilde{u}}-\gamma_{N-1}(p,q)(\delta\widehat{u}-\delta\widetilde{u})\\ & -\sum_{j=0}^{N-2}\gamma_{N-2-j}(p,q)\big\{(\delta\widehat{u}-\delta\widetilde{u})\widehat{\widetilde{v}}_j+(\widehat{u}-\widetilde{u})\delta\widehat{\widetilde{v}}_j\big\}\\ & -\sum_{i=0}^{N-3}\sum_{j=1}^{N-2-i}\gamma_{N-2-i-j}(p,q)\big\{\delta w_i[\widehat{v}_j-\widetilde{v}_j-(p-q+\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_{j-1}\\ & \qquad +p\widehat{v}_{j-1}-q\widetilde{v}_{j-1}]\\ & +w_i[\delta\widehat{v}_j-\delta\widetilde{v}_j-(\delta\widehat{u}-\delta\widetilde{u})\widehat{\widetilde{v}}_{j-1}-(p-q+\widehat{u}-\widetilde{u})\delta\widehat{\widetilde{v}}_{j-1}\\ & \qquad +p\delta\widehat{v}_{j-1}-q\delta\widetilde{v}_{j-1}]\big\}\Bigg\},\end{aligned} \tag{2.5}$$

and so

$$
\begin{aligned}
0 \quad = \quad \sum_{n,m\in Z} \Bigg\{ \Bigg(& \frac{(p-q)\gamma_{N-1}(p,q)}{p-q+\widehat{\widetilde{\widetilde{u}}}-\widetilde{\widetilde{u}}} - \frac{(p-q)\gamma_{N-1}(p,q)}{p-q+\widehat{\widehat{\widetilde{u}}}-\widehat{\widetilde{u}}} \\
& + \sum_{i=0}^{N-3} \sum_{j=0}^{N-3-i} \gamma_{N-3-i-j}(p,q)(\widehat{\widetilde{\widetilde{v}}}_j \widetilde{w}_i - \widehat{\widehat{\widetilde{v}}}_j \widehat{w}_i) \\
& - \sum_{j=0}^{N-3} \gamma_{N-2-j}(p,q)(\widehat{\widetilde{\widetilde{v}}}_j - \widehat{\widehat{\widetilde{v}}}_j - \widetilde{w}_j + \widehat{w}_j) \\
& - \widehat{\widetilde{\widetilde{v}}}_{N-2} + \widehat{\widehat{\widetilde{v}}}_{N-2} + \widetilde{w}_{N-2} - \widehat{w}_{N-2} \Bigg) \delta \widehat{\widetilde{u}} \\
- \sum_{i=0}^{N-3} \sum_{j=0}^{N-3-i} & \gamma_{N-3-i-j}(p,q)[\widehat{v}_{j+1} - \widetilde{v}_{j+1} - (p-q+\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_j \\
& \qquad + p\widehat{v}_j - q\widetilde{v}_j]\delta w_i \\
+ \sum_{i=0}^{N-3} \sum_{j=0}^{N-3-i} & \gamma_{N-3-i-j}(p,q)[\widehat{w}_{i+1} - \widetilde{w}_{i+1} + (p-q+\widehat{u}-\widetilde{u})w_i \\
& \qquad + q\widehat{w}_i - p\widetilde{w}_i]\delta \widehat{\widetilde{v}}_j \Bigg\}.
\end{aligned}
\tag{2.6}
$$

The coefficient of δw_i must be zero, which gives us equation (1.1a), the coefficient of $\delta \widehat{\widetilde{v}}_j$ must be zero, which gives us equation (1.1b), and the remaining term which multiplies $\delta \widehat{\widetilde{u}}$ is two shifted copies of equation (2.2). Hence the lattice Gel'fand-Dikii system of equations solve the Euler-Lagrange equations for the Lagrangian (2.3). ■

At this point we would like to make several remarks.

1. For clarity we have chosen to label the Lagrangian with the lattice parameters p, q to indicate it is defined on a plaquette in a 2-dimensional surface corresponding to the respective lattice directions.

2. The Lagrangian (2.3) is antisymmetric with respect to the interchange of the lattice directions associated with the parameters p, q, a property which will be important when we come to define the multiform structure.

3. Total derivative terms have been included which at first sight may appear superfluous. They do, however, prove necessary in the verification of the closure relation below. For the same reason, we do not have the freedom to

multiply the Lagrangian by constants involving the lattice parameters p, q, we may only multiply by true constants.

3.3 Closure relation and Lagrangian 2-form

Proposition 2: *The Lagrangian defined by* (2.3) *satisfies the following closure relation on solutions to the lattice GD hierarchy equations when embedded in a 3-dimensional lattice.*

$$\Delta_p \mathcal{L}_{qr} + \Delta_q \mathcal{L}_{rp} + \Delta_r \mathcal{L}_{pq} = 0, \tag{3.1}$$

where the difference operator Δ_r *acts on functions* f *of* $u = u(n_p, n_q, n_r)$ *by the formula* $\Delta_r f(u) = f(\overline{u}) - f(u)$*, and on a function* g *of* u *and its shifts by the formula* $\Delta_r g(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = g(\overline{u}, \widetilde{\overline{u}}, \widehat{\overline{u}}, \widehat{\widetilde{\overline{u}}}) - g(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}})$.

Proof: Firstly, on equation (1.1a) is is clear that the last term in the Lagrangian will disappear. This leaves us with

$$\begin{aligned}
\Gamma \equiv\ & \overline{\mathcal{L}}_{pq} + \widetilde{\mathcal{L}}_{qr} + \widehat{\mathcal{L}}_{rp} - \mathcal{L}_{pq} - \mathcal{L}_{qr} - \mathcal{L}_{rp} \\
=\ & (p-q)\gamma_{N-1}(p,q) \ln\left(\frac{p - q + \widehat{\overline{u}} - \widetilde{\overline{u}}}{p - q + \widehat{u} - \widetilde{u}}\right) \\
& +(q-r)\gamma_{N-1}(q,r) \ln\left(\frac{q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}}}{q - r + \overline{u} - \widehat{u}}\right) \\
& +(r-p)\gamma_{N-1}(r,p) \ln\left(\frac{r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}}}{r - p + \widetilde{u} - \overline{u}}\right) - \gamma_{N-1}(p,q)(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) \\
& -\gamma_{N-1}(q,r)(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) - \gamma_{N-1}(r,p)(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) \\
& - \sum_{j=0}^{N-2} \Big\{ \gamma_{N-2-j}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad +\gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u} - \widehat{u})\widehat{\overline{v}}_j] \\
& \qquad +\gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_j] \Big\},
\end{aligned} \tag{3.2}$$

which, on rearranging the terms, is

$$
\begin{aligned}
\Gamma \;=\; & -(-p)^N \ln\left(\left(\frac{p-q+\widehat{\overline{u}}-\widetilde{\overline{u}}}{p-q+\widehat{u}-\widetilde{u}}\right)\left(\frac{r-p+\widetilde{u}-\overline{u}}{r-p+\widehat{\widetilde{u}}-\widehat{\overline{u}}}\right)\right) \\
& -(-q)^N \ln\left(\left(\frac{q-r+\widetilde{\overline{u}}-\widehat{\widetilde{u}}}{q-r+\overline{u}-\widehat{u}}\right)\left(\frac{p-q+\widehat{u}-\widetilde{u}}{p-q+\widehat{\overline{u}}-\widetilde{\overline{u}}}\right)\right) \\
& -(-r)^N \ln\left(\left(\frac{r-p+\widehat{\widetilde{u}}-\widehat{\overline{u}}}{r-p+\widetilde{u}-\overline{u}}\right)\left(\frac{q-r+\overline{u}-\widehat{u}}{q-r+\widetilde{\overline{u}}-\widehat{\widetilde{u}}}\right)\right) - \gamma_{N-1}(p,q)(\widehat{\overline{u}}-\widetilde{\overline{u}}-\widehat{u}+\widetilde{u}) \\
& -\gamma_{N-1}(q,r)(\widetilde{\overline{u}}-\widehat{\widetilde{u}}-\overline{u}+\widehat{u}) - \gamma_{N-1}(r,p)(\widehat{\widetilde{u}}-\widehat{\overline{u}}-\widetilde{u}+\overline{u}) - (\widehat{\overline{u}}-\widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_{N-2} \\
& +(\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_{N-2} - (\widetilde{\overline{u}}-\widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_{N-2} + (\overline{u}-\widehat{u})\widehat{\overline{v}}_{N-2} - (\widehat{\widetilde{u}}-\widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_{N-2} + (\widetilde{u}-\overline{u})\widetilde{\overline{v}}_{N-2} \\
& - \sum_{j=0}^{N-3}\Big\{\gamma_{N-2-j}(p,q)[(\widehat{\overline{u}}-\widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad +\gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}}-\widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u}-\widehat{u})\widehat{\overline{v}}_j] \\
& \qquad +\gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}}-\widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u}-\overline{u})\widetilde{\overline{v}}_j]\Big\}. \qquad (3.3)
\end{aligned}
$$

We have already shown that the lattice KP equation (1.6) holds provided that equation (1.1a) holds. Using this fact, it is clear that the logarithm terms disappear. So we are left with

$$
\begin{aligned}
\Gamma \;=\; & -\gamma_{N-1}(p,q)(\widehat{\overline{u}}-\widetilde{\overline{u}}-\widehat{u}+\widetilde{u}) - \gamma_{N-1}(q,r)(\widetilde{\overline{u}}-\widehat{\widetilde{u}}-\overline{u}+\widehat{u}) \\
& -\gamma_{N-1}(r,p)(\widehat{\widetilde{u}}-\widehat{\overline{u}}-\widetilde{u}+\overline{u}) \\
& +\overline{u}(\widehat{\overline{v}}_{N-2}-\widetilde{\overline{v}}_{N-2}) + \widetilde{u}(\widetilde{\overline{v}}_{N-2}-\widehat{\widetilde{v}}_{N-2}) + \widehat{u}(\widehat{\widetilde{v}}_{N-2}-\widehat{\overline{v}}_{N-2}) \\
& - \sum_{j=0}^{N-3}\Big\{\gamma_{N-2-j}(p,q)[(\widehat{\overline{u}}-\widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad +\gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}}-\widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u}-\widehat{u})\widehat{\overline{v}}_j] \\
& \qquad +\gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}}-\widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u}-\overline{u})\widetilde{\overline{v}}_j]\Big\}. \qquad (3.4)
\end{aligned}
$$

Here it is helpful to introduce a new object

$$\epsilon_j \equiv \frac{1}{p-q}\big(\gamma_{j+1}(q,r) - \gamma_{j+1}(r,p)\big), \qquad (3.5)$$

which is invariant under cyclic permutations of p, q, r. This is not immediately apparent, but is due to the fact that γ_j obeys the relation

$$(p-q)\gamma_j(p,q) + (q-r)\gamma_j(q,r) + (r-p)\gamma_j(r,p) = 0, \qquad (3.6)$$

which allows us to write

$$\begin{aligned}\epsilon_j &= \frac{1}{p-q}\Big(\gamma_{j+1}(q,r)-\gamma_{j+1}(r,p)\Big)\\ &= \frac{1}{p-q}\Big(\gamma_{j+1}(q,r)+\frac{1}{r-p}\big((p-q)\gamma_{j+1}(p,q)+(q-r)\gamma_{j+1}(q,r)\big)\Big)\\ &= \frac{1}{r-p}\Big(\gamma_{j+1}(p,q)-\gamma_{j+1}(q,r)\Big), \qquad (3.7)\end{aligned}$$

and this is clearly (3.5) after a cyclic permutation of p, q and r. Hence ϵ_j is invariant under such cyclic permutations. The following identity for ϵ_j also holds

$$\epsilon_{j+1}+r\epsilon_j=\gamma_{j+1}(p,q), \qquad (3.8)$$

since

$$\begin{aligned}\gamma_{j+1}(q,r)+r\gamma_j(q,r) &= \frac{(-q)^{j+2}-(-r)^{j+2}}{-q+r}+r\frac{(-q)^{j+1}-(-r)^{j+1}}{-q+r}\\ &= (-q)^{j+1}, \qquad (3.9)\end{aligned}$$

and similarly $\gamma_{j+1}(r,p)+r\gamma_j(p,q)=(-p)^{j+1}$, so that

$$\begin{aligned}\epsilon_{j+1}+r\epsilon_j &= \frac{1}{p-q}\Big(\gamma_{j+2}(q,r)-\gamma_{j+2}(r,p)\Big)+\frac{r}{p-q}\Big(\gamma_{j+1}(q,r)-\gamma_{j+1}(r,p)\Big)\\ &= \frac{1}{p-q}\Big(\big(\gamma_{j+2}(q,r)+r\gamma_{j+1}(q,r)\big)-\big(\gamma_{j+2}(r,p)+r\gamma_{j+1}(r,p)\big)\Big)\\ &= \frac{(-q)^{j+2}-(-p)^{j+2}}{p-q}\\ &= \gamma_{j+1}(p,q). \qquad (3.10)\end{aligned}$$

From the definition, $\epsilon_0=1$, and so we can write

$$\begin{aligned}\Gamma &= -\gamma_{N-1}(p,q)(\widehat{\overline{u}}-\widetilde{\overline{u}}-\widehat{u}+\widetilde{u})-\gamma_{N-1}(q,r)(\widetilde{\overline{u}}-\widehat{\widetilde{u}}-\overline{u}+\widehat{u})\\ &\quad -\gamma_{N-1}(r,p)(\widehat{\widetilde{u}}-\widehat{\overline{u}}-\widetilde{u}+\overline{u})\\ &\quad +\epsilon_0\overline{u}(\widehat{\overline{v}}_{N-2}-\widetilde{\overline{v}}_{N-2})+\epsilon_0\widetilde{u}(\widetilde{\overline{v}}_{N-2}-\widehat{\widetilde{v}}_{N-2})+\epsilon_0\widehat{u}(\widehat{\widetilde{v}}_{N-2}-\widehat{\overline{v}}_{N-2})\\ &\quad -\sum_{j=0}^{N-3}\Big\{\gamma_{N-2-j}(p,q)[(\widehat{\overline{u}}-\widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j-(\widehat{u}-\widetilde{u})\widehat{\widetilde{v}}_j]\\ &\qquad +\gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}}-\widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j-(\overline{u}-\widehat{u})\widehat{\overline{v}}_j]\\ &\qquad +\gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}}-\widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j-(\widetilde{u}-\overline{u})\widetilde{\overline{v}}_j]\Big\}. \qquad (3.11)\end{aligned}$$

For any $0 \leq k \leq N-3$, the expression

$$
\begin{aligned}
\Lambda_k \quad \equiv \quad & \epsilon_k \overline{u}(\widehat{\overline{v}}_{N-2-k} - \widetilde{\overline{v}}_{N-2-k}) + \epsilon_k \widetilde{u}(\widetilde{\overline{v}}_{N-2-k} - \widehat{\widetilde{v}}_{N-2-k}) + \epsilon_k \widehat{u}(\widehat{\widetilde{v}}_{N-2-k} - \widehat{\overline{v}}_{N-2-k}) \\
& - \sum_{j=0}^{N-3-k} \Big\{ \gamma_{N-2-j}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad + \gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u} - \widehat{u})\widehat{\overline{v}}_j] \\
& \qquad + \gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_j] \Big\}
\end{aligned}
\tag{3.12}
$$

can be written as

$$
\begin{aligned}
\Lambda_k \quad = \quad & \epsilon_k \overline{u}((p - q + \widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_{N-3-k} - p\widehat{\overline{v}}_{N-3-k} + q\widetilde{\overline{v}}_{N-3-k}) \\
& + \epsilon_k \widetilde{u}((q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_{N-3-k} - q\widetilde{\overline{v}}_{N-3-k} + r\widehat{\widetilde{v}}_{N-3-k}) \\
& + \epsilon_k \widehat{u}((r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_{N-3-k} - r\widehat{\widetilde{v}}_{N-3-k} + p\widehat{\overline{v}}_{N-3-k}) \\
& - \gamma_{k+1}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_{N-3-k} - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_{N-3-k}] \\
& - \gamma_{k+1}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_{N-3-k} - (\overline{u} - \widehat{u})\widehat{\overline{v}}_{N-3-k}] \\
& - \gamma_{k+1}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_{N-3-k} - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_{N-3-k}] \\
& - \sum_{j=0}^{N-4-k} \Big\{ \gamma_{N-2-j}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad + \gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u} - \widehat{u})\widehat{\overline{v}}_j] \\
& \qquad + \gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_j] \Big\},
\end{aligned}
\tag{3.13}
$$

where we have made use of (1.1a) to eliminate the terms involving shifts of v_{N-2-k}. On rearranging,

$$\begin{aligned}
\Lambda_k \;=\; & [\epsilon_k \overline{u}(p - q + \widehat{\overline{u}} - \widetilde{\overline{u}}) + \epsilon_k \widetilde{u}(q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}}) + \epsilon_k \widehat{u}(r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}}) \\
& \quad -\gamma_{k+1}(p,q)(\widehat{\overline{u}} - \widetilde{\overline{u}}) - \gamma_{k+1}(q,r)(\widetilde{\overline{u}} - \widehat{\widetilde{u}}) - \gamma_{k+1}(r,p)(\widehat{\widetilde{u}} - \widehat{\overline{u}})]\widehat{\widetilde{\overline{v}}}_{N-3-k} \\
& +\widetilde{u}[(-q\epsilon_k + \gamma_{k+1}(r,p))\widetilde{\overline{v}}_{N-3-k} - (-r\epsilon_k + \gamma_{k+1}(p,q))\widehat{\widetilde{v}}_{N-3-k}] \\
& +\widehat{u}[(-r\epsilon_k + \gamma_{k+1}(p,q))\widehat{\widetilde{v}}_{N-3-k} - (-p\epsilon_k + \gamma_{k+1}(q,r))\widehat{\overline{v}}_{N-3-k}] \\
& +\overline{u}[(-p\epsilon_k + \gamma_{k+1}(q,r))\widehat{\overline{v}}_{N-3-k} - (-q\epsilon_k + \gamma_{k+1}(r,p))\widetilde{\overline{v}}_{N-3-k}] \\
& - \sum_{j=0}^{N-4-k} \Big\{ \gamma_{N-2-j}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad +\gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u} - \widehat{u})\widehat{\overline{v}}_j] \\
& \qquad +\gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_j] \Big\},
\end{aligned} \tag{3.14}$$

and then we can use (1.6) on the very top line to give

$$\begin{aligned}
\Lambda_k \;=\; & [(r\epsilon_k - \gamma_{k+1}(p,q))(\widehat{\overline{u}} - \widetilde{\overline{u}}) + (p\epsilon_k - \gamma_{k+1}(q,r))(\widetilde{\overline{u}} - \widehat{\widetilde{u}}) \\
& \quad +(p\epsilon_k - \gamma_{k+1}(r,p))(\widehat{\widetilde{u}} - \widehat{\overline{u}})]\widehat{\widetilde{\overline{v}}}_{N-3-k} \\
& +\widetilde{u}[(-q\epsilon_k + \gamma_{k+1}(r,p))\widetilde{\overline{v}}_{N-3-k} - (-r\epsilon_k + \gamma_{k+1}(p,q))\widehat{\widetilde{v}}_{N-3-k}] \\
& +\widehat{u}[(-r\epsilon_k + \gamma_{k+1}(p,q))\widehat{\widetilde{v}}_{N-3-k} - (-p\epsilon_k + \gamma_{k+1}(q,r))\widehat{\overline{v}}_{N-3-k}] \\
& +\overline{u}[(-p\epsilon_k + \gamma_{k+1}(q,r))\widehat{\overline{v}}_{N-3-k} - (-q\epsilon_k + \gamma_{k+1}(r,p))\widetilde{\overline{v}}_{N-3-k}] \\
& - \sum_{j=0}^{N-4-k} \Big\{ \gamma_{N-2-j}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j] \\
& \qquad +\gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u} - \widehat{u})\widehat{\overline{v}}_j] \\
& \qquad +\gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_j] \Big\}.
\end{aligned} \tag{3.15}$$

Here the identity (3.8) comes into play, to give

$$
\begin{aligned}
\Lambda_k &= \epsilon_{k+1}\overline{u}(\widehat{\overline{v}}_{N-3-k} - \widetilde{\overline{v}}_{N-3-k}) + \epsilon_{k+1}\widetilde{u}(\widetilde{\overline{v}}_{N-3-k} - \widehat{\widetilde{v}}_{N-3-k}) \\
&\quad + \epsilon_{k+1}\widehat{u}(\widehat{\widetilde{v}}_{N-3-k} - \widehat{\overline{v}}_{N-3-k}) \\
&\quad - \sum_{j=0}^{N-4-k} \Big\{ \gamma_{N-2-j}(p,q)[(\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widehat{u} - \widetilde{u})\widehat{\widetilde{v}}_j] \\
&\qquad + \gamma_{N-2-j}(q,r)[(\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{v}}}_j - (\overline{u} - \widehat{u})\widehat{\overline{v}}_j] \\
&\qquad + \gamma_{N-2-j}(r,p)[(\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{v}}}_j - (\widetilde{u} - \overline{u})\widetilde{\overline{v}}_j] \Big\} \\
&= \Lambda_{k+1},
\end{aligned} \tag{3.16}
$$

which means that for any $0 \leq j, k \leq N-2$ we have $\Lambda_j = \Lambda_k$. This allows us to greatly simplify Γ, since

$$
\begin{aligned}
\Gamma &= -\gamma_{N-1}(p,q)(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) - \gamma_{N-1}(q,r)(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) \\
&\quad - \gamma_{N-1}(r,p)(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) + \Lambda_0 \\
&= -\gamma_{N-1}(p,q)(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) - \gamma_{N-1}(q,r)(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) \\
&\quad - \gamma_{N-1}(r,p)(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) + \Lambda_{N-2} \\
&= -\gamma_{N-1}(p,q)(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) - \gamma_{N-1}(q,r)(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) \qquad (3.17) \\
&\quad - \gamma_{N-1}(r,p)(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) + \epsilon_{N-2}\overline{u}(\widehat{\overline{u}} - \widetilde{\overline{u}}) + \epsilon_{N-2}\widetilde{u}(\widetilde{\overline{u}} - \widehat{\widetilde{u}}) + \epsilon_{N-2}\widehat{u}(\widehat{\widetilde{u}} - \widehat{\overline{u}}).
\end{aligned}
$$

Using once again the equations (1.6) and then (3.8), this is

$$
\begin{aligned}
\Gamma &= -\gamma_{N-1}(p,q)(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) - \gamma_{N-1}(q,r)(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) \\
&\quad - \gamma_{N-1}(r,p)(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) + r\epsilon_{N-2}(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) \\
&\quad + p\epsilon_{N-2}(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) + q\epsilon_{N-2}(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) \\
&= (r\epsilon_{N-2} - \gamma_{N-1}(p,q))(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) \\
&\quad + (p\epsilon_{N-2} - \gamma_{N-1}(q,r))(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) \\
&\quad + (q\epsilon_{N-2} - \gamma_{N-1}(r,p))(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) \\
&= -\epsilon_{N-1}(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) - \epsilon_{N-1}(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) - \epsilon_{N-1}(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) \\
&= 0.
\end{aligned} \tag{3.18}
$$

Thus the closure relation is verified. ■

Note that in the above computation, the only equations used were the equation involving the v_j (1.1a), and the lattice KP equation (1.6), the latter was shown earlier to be a consequence of copies of (1.1a). It is not clear at this stage why the proof of the closure relation should rely only on these equations, especially since a Lagrangian description for (2.9) has, as yet, not been found.

3.4 Special cases: Lattice Boussinesq and KdV equations

The lattice Boussinesq equation deserves special mention as it has attracted much interest lately, for example with regard to the Pentagram map [103]. It is a particular case of the lattice GD hierarchy, taking $N = 3$, and as such can be written as a system of equations in the variables u, v_1, w_1. However, it is possible to eliminate v_1, w_1 and express the equation in terms of the variable u only, as follows.

$$\begin{aligned} \frac{p^3-q^3}{p-q+\widehat{\widetilde{u}}-\widetilde{\widetilde{u}}} - \frac{p^3-q^3}{p-q+\widehat{\widehat{u}}-\widehat{\widetilde{u}}} + (p+2q)(\widehat{\widetilde{\widetilde{u}}}+\widehat{u}) - (2p+q)(\widehat{\widehat{\widetilde{u}}}+\widetilde{u}) & \\ +(p-q+\widehat{\widehat{\widetilde{u}}}-\widehat{\widetilde{\widetilde{u}}})\widehat{\widehat{\widetilde{\widetilde{u}}}} + (p-q+\widehat{u}-\widetilde{u})u + \widetilde{u}\widehat{\widetilde{\widetilde{u}}} - \widehat{u}\widehat{\widehat{\widetilde{u}}} &= 0. \quad (4.1) \end{aligned}$$

As mentioned earlier, this was shown to be multidimensionally consistent in [123] by considering an initial value problem on a 27-point cube.

Starting from the action given in [86], we need to make only minor modifications in order to arrive at a Lagrangian which satisfies the closure relation (3.1) on solutions to the lattice KP equation (1.6),

$$\begin{aligned} \mathcal{L}_{pq} &= (p^3-q^3)\ln(p-q+\widehat{u}-\widetilde{u}) - (p^2+pq+q^2)(\widehat{u}-\widetilde{u}) + (p+q)(\widehat{u}-\widetilde{u})\widehat{\widetilde{u}} \\ &\quad +(p-q+\widehat{u}-\widetilde{u})u\widehat{\widetilde{u}} - pu\widehat{u} + qu\widetilde{u}. \quad (4.2) \end{aligned}$$

That the above Lagrangian satisfies the closure relation (3.1) can easily be verified by direct computation:

$$
\begin{aligned}
\Gamma &= \overline{\mathcal{L}}_{pq} + \widetilde{\mathcal{L}}_{qr} + \widehat{\mathcal{L}}_{rp} - \mathcal{L}_{pq} - \mathcal{L}_{qr} - \mathcal{L}_{rp} \\
&= (p^3 - q^3)\ln\left(\frac{p - q + \widehat{\overline{u}} - \widetilde{\overline{u}}}{p - q + \widehat{u} - \widetilde{u}}\right) - (p^2 + pq + q^2)(\widehat{\overline{u}} - \widetilde{\overline{u}} - \widehat{u} + \widetilde{u}) \\
&\quad + (p+q)\big((\widehat{\overline{u}} - \widetilde{\overline{u}})\widehat{\widetilde{\overline{u}}} - (\widehat{u} - \widetilde{u})\widehat{\widetilde{u}}\big) + (p - q + \widehat{\overline{u}} - \widetilde{\overline{u}})\overline{u}\widehat{\widetilde{\overline{u}}} \\
&\quad - (p - q + \widehat{u} - \widetilde{u})u\widehat{\widetilde{u}} - pu\widehat{\overline{u}} + q\overline{u}\widetilde{\overline{u}} + pu\widehat{u} - qu\widetilde{u} \\
&\quad + (q^3 - r^3)\ln\left(\frac{q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}}}{q - r + \overline{u} - \widehat{u}}\right) - (q^2 + qr + r^2)(\widetilde{\overline{u}} - \widehat{\widetilde{u}} - \overline{u} + \widehat{u}) \\
&\quad + (q+r)\big((\widetilde{\overline{u}} - \widehat{\widetilde{u}})\widehat{\widetilde{\overline{u}}} - (\overline{u} - \widehat{u})\widehat{\overline{u}}\big) + (q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}})\widetilde{u}\widehat{\widetilde{\overline{u}}} \\
&\quad - (q - r + \overline{u} - \widehat{u})u\widehat{\overline{u}} - qu\widetilde{\overline{u}} + r\widetilde{u}\widehat{\widetilde{u}} + qu\overline{u} - ru\widehat{u} \\
&\quad + (r^3 - p^3)\ln\left(\frac{r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}}}{r - p + \widetilde{u} - \overline{u}}\right) - (r^2 + rp + p^2)(\widehat{\widetilde{u}} - \widehat{\overline{u}} - \widetilde{u} + \overline{u}) \\
&\quad + (r+p)\big((\widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{\widetilde{\overline{u}}} - (\widetilde{u} - \overline{u})\widetilde{\overline{u}}\big) + (r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}})\widehat{u}\widehat{\widetilde{\overline{u}}} \\
&\quad - (r - p + \widetilde{u} - \overline{u})u\widetilde{\overline{u}} - ru\widehat{\widetilde{u}} + p\widehat{u}\widehat{\widetilde{u}} + ru\widetilde{u} - pu\overline{u} \\
&= p^3\ln\left(\frac{(p - q + \widehat{\overline{u}} - \widetilde{\overline{u}})(r - p + \widetilde{u} - \overline{u})}{(p - q + \widehat{u} - \widetilde{u})(r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}})}\right) \\
&\quad + q^3\ln\left(\frac{(q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}})(p - q + \widehat{u} - \widetilde{u})}{(q - r + \overline{u} - \widehat{u})(p - q + \widehat{\overline{u}} - \widetilde{\overline{u}})}\right) \\
&\quad + r^3\ln\left(\frac{(r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}})(q - r + \overline{u} - \widehat{u})}{(r - p + \widetilde{u} - \overline{u})(q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}})}\right) \\
&\quad + \widehat{\widetilde{\overline{u}}}\big\{(p - q + \widehat{\overline{u}} - \widetilde{\overline{u}})(\overline{u} + p + q) + (q - r + \widetilde{\overline{u}} - \widehat{\widetilde{u}})(\widetilde{u} + q + r) \\
&\qquad\qquad + (r - p + \widehat{\widetilde{u}} - \widehat{\overline{u}})(\widehat{u} + r + p)\big\} \\
&\quad + u\big\{(p - q + \widehat{u} - \widetilde{u})(p + q - \widehat{\widetilde{u}}) + (q - r + \overline{u} - \widehat{u})(q + r - \widehat{\overline{u}}) \\
&\qquad\qquad + (r - p + \widetilde{u} - \overline{u})(r + p - \widetilde{\overline{u}})\big\} \\
&\quad + \widehat{\widetilde{u}}\big\{-(p+q)(\widehat{u} - \widetilde{u}) - rp - p^2 + q^2 + qr - r\widehat{u} + r\widetilde{u}\big\} \\
&\quad + \widehat{\overline{u}}\big\{-(q+r)(\overline{u} - \widehat{u}) - pq - q^2 + r^2 + rp - p\overline{u} + p\widehat{u}\big\} \\
&\quad + \widetilde{\overline{u}}\big\{-(r+p)(\widetilde{u} - \overline{u}) - qr - r^2 + p^2 + pq - q\widetilde{u} + q\overline{u}\big\} \\
&\quad + (p^2 + pq + q^2)(\widehat{u} - \widetilde{u}) + (q^2 + qr + r^2)(\overline{u} - \widehat{u}) + (r^2 + rp + p^2)(\widetilde{u} - \overline{u}).
\end{aligned}
\tag{4.3}
$$

Using the lattice KP equation (1.6), we find that the first 5 lines disappear, leaving

$$\begin{aligned}\Gamma &= (p+q+r)\big\{(p-q+\widehat{u}-\widetilde{u})(p+q-\widehat{\widetilde{u}})+(q-r+\overline{u}-\widehat{u})(q+r-\widehat{\overline{u}}) \\ &\qquad +(r-p+\widetilde{u}-\overline{u})(r+p-\widetilde{\overline{u}})\big\} \\ &= 0,\end{aligned} \tag{4.4}$$

where once more we have used the lattice KP equation (1.6).

The lattice KdV equation, which is the member of the lattice GD hierarchy where $N = 2$, is a more degenerate case and needs to be treated separately. Here we do not have equations (1.1a) and (1.1b) as N is too small, we have only equation (2.2), which is

$$(p+q+u-\widehat{\widetilde{u}})(p-q+\widehat{u}-\widetilde{u}) = p^2-q^2. \tag{4.5}$$

A Lagrangian was first given in [20], which is equivalent to the following Lagrangian

$$\mathcal{L}_{pq} = -(p^2-q^2)\ln(p-q+\widehat{u}-\widetilde{u})+(\widehat{u}-\widetilde{u})(p+q-\widehat{\widetilde{u}}). \tag{4.6}$$

Again, the closure relation holds on solutions of the lattice KP equation (1.6), but we need to use copies of equation (4.5) in 3 lattice directions to show that (1.6) does indeed hold. The lattice KdV equation is of course a case already treated in Chapter 2; under its alternative name of H1 it is the simplest system in the ABS list.

3.5 Interpretation of results

Having established a Lagrangian for each member of the lattice GD hierarchy satisfying the closure relation (3.1), we now interpret this result in terms of a Lagrangian multiform structure. In fact, the existence of the closure relation allows us to develop the variational principle proposed in [65] (and in Chapter 2) for this class of systems. This comprises the following:

Noting that the Lagrangian (2.3) is defined on an elementary plaquette, we can define an action S for any given surface σ consisting of a connected configuration of elementary plaquettes σ_{pq} in the multidimensional lattice (where the labelling by the lattice parameters p, q indicates σ_{pq} lives on the sublattice corresponding to the respective lattice directions) by summing the Lagrangian contributions from each of the plaquettes in the surface σ, i.e.,

$$S[u, v_1, \ldots, v_{N-2}, w_1, \ldots, w_{N-2}; \sigma] = \sum_{\sigma_{pq} \in \sigma} \mathcal{L}_{pq}, \tag{5.1}$$

taking into account the orientation of the plaquette (as noted earlier, $\mathcal{L}_{pq}$ has the property of antisymmetry with respect to interchange of the two lattice directions, so this sum is well-defined). Once again, this action depends not only on the dependent variables, but also on the geometry of the independent variables. We follow the same line of reasoning as in the previous chapter. Imposing independence of the action under local variations of the surface, keeping the boundary fixed, requires the closure relation to hold. Furthermore, the surface independence allows us to locally deform the surface in any way we choose away from the boundary. In particular, away from the boundary we may render it locally flat so that we have here a regular 2-dimensional lattice on which we can apply the variational principle leading to the usual discrete Euler-Lagrange equations. It then follows from the proof of Proposition 2 in the previous section that for this specific Lagrangian, the equations of motion (1.1a),(1.1b) and (2.2) are compatible with the surface independence. This shows multidimensional consistency on the level of the Lagrangian.

3.6 Chapter summary

In this chapter we have presented a Lagrangian for the generic member of the lattice GD hierarchy and shown that it can be considered as a Lagrangian 2-form when embedded in a higher dimensional lattice, obeying a closure relation. Thus the multiform structure proposed in [65] (and in Chapter 2) has been extended to a multi-component system.

Chapter 4

Lagrangian 3-form

4.1 The bilinear discrete lattice KP system

Discrete equations of Kadomtsev-Petviashvili (KP) type have been studied extensively since the early 1980s (cf for example [23, 83]), following on from the famous "discrete analogue of a generalized Toda equation" (DAGTE) introduced by Hirota in [45] which is a bilinear form for the lattice KP equation. Hirota introduced his difference equation in a form equivalent to

$$\alpha\tau_i\tau_{\bar{\imath}} + \beta\tau_j\tau_{\bar{\jmath}} + \gamma\tau_k\tau_{\bar{k}} = 0, \tag{1.1}$$

where $\tau = \tau(n_i, n_j, n_k)$ is the dependent variable depending on three discrete independent variables n_i, n_j, n_k corresponding to lattice directions, and subscripts of τ, e.g. as in τ_i, denote shifts in the n_i-direction so that for example $\tau_i = \tau(n_i+1, n_j, n_k)$ and $\tau_{\bar{\jmath}} = \tau(n_i, n_j - 1, n_k)$. Here α, β, γ are constants satisfying $\alpha + \beta + \gamma = 0$. There are many possible reductions to 2-dimensional equations such as the KdV equation, modified KdV equation, sine-Gordon equation, nonlinear Klein-Gordon equation and the Benjamin-Ono equation; the details of these were all given in [45], along with soliton solutions and a Bäcklund transformation.

Other related KP-type lattice equations were introduced in [84], and appear in the Introduction. The equation we will refer to as the bilinear discrete KP equation, in order to distinguish it from equations that actually lead to the original KP equation in a continuum limit, is taken in the following form

$$A_{jk}\tau_i\tau_{jk} + A_{ki}\tau_j\tau_{ki} + A_{ij}\tau_k\tau_{ij} = 0, \tag{1.2}$$

where $A_{ij} = -A_{ji}$ are constants. These constants can be removed by a gauge transformation, but we find it more instructive to retain them. Miwa gave the connection between the KP hierarchy and Hirota's difference equation in [78], showing how solutions to the KP hierarchy can be transformed into solutions to (1.2), hence it is often called the *Hirota-Miwa* equation.

4.2 Lagrangian structure

It is a common feature of Lagrangians for equations of Korteweg-de Vries (KdV) and KP type (already in the continuous case) that these equations emerge as Euler-Lagrange equations by varying the action with respect to a dependent variable which obeys a potential (i.e., integrated) version of the equation. Hence, the variational equation is typically a "derived form" of the equation obeyed by this canonical variable, with respect to which the action is minimized. This we have seen already in Chapters 2 and 3. The same holds true in the case of a Lagrangian structure for the bilinear discrete KP system, where we will use the τ-function as the canonical variable. Thus, fixing three directions i, j, k, we introduce the following Lagrangian

$$\begin{aligned} &L(\tau_i, \tau_j, \tau_k, \tau_{ij}, \tau_{jk}, \tau_{ki}; A_{ij}, A_{jk}, A_{ki}) \\ &= \ln\left(\frac{\tau_k\tau_{ij}}{\tau_j\tau_{ki}}\right)\ln\left(-\frac{A_{ki}\tau_j}{A_{jk}\tau_i}\right) - \mathrm{Li}_2\left(-\frac{A_{ij}\tau_k\tau_{ij}}{A_{ki}\tau_j\tau_{ki}}\right) \quad =: \quad L_{ijk}, \end{aligned} \tag{2.1}$$

where Li_2 denotes the dilogarithm function defined as before by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-z)}{z} dz. \tag{2.2}$$

The Lagrangian (2.1) produces the following discrete Euler-Lagrange equation

$$\begin{aligned}\frac{\delta L}{\delta\tau} &= \Bigg\{\ln\left(-\frac{A_{ki}\tau_{j\bar{k}}\tau_i + A_{ij}\tau\tau_{ij\bar{k}}}{A_{jk}\tau_{i\bar{k}}\tau_j}\right) + \ln\left(-\frac{A_{ki}\tau_{\bar{j}k}\tau_{\bar{\imath}} + A_{ij}\tau\tau_{\bar{\imath}\bar{j}k}}{A_{jk}\tau_{\bar{\imath}k}\tau_j}\right) \\ &\quad - \ln\left(-\frac{A_{ki}\tau\tau_{i\bar{j}k} + A_{ij}\tau_{\bar{j}k}\tau_i}{A_{jk}\tau_{i\bar{j}}\tau_k}\right) - \ln\left(-\frac{A_{ki}\tau\tau_{\bar{\imath}j\bar{k}} + A_{ij}\tau_{j\bar{k}}\tau_{\bar{\imath}}}{A_{jk}\tau_{\bar{\imath}j}\tau_{\bar{k}}}\right)\Bigg\}\frac{1}{\tau} \\ &= 0, \end{aligned} \tag{2.3}$$

which is a consequence of (1.2) through the fact that it is a combination of 4 copies of the equation shifted in appropriate lattice directions.

Consequently the following functional of the lattice fields $\tau(n_i, n_j, n_k)$

$$S[\tau] = \sum_{n_i,n_j,n_k} L(\tau_i, \tau_j, \tau_k, \tau_{ij}, \tau_{jk}, \tau_{ki}; A_{ij}, A_{jk}, A_{ki}), \tag{2.4}$$

with L given by (2.1) can be considered to constitute an action for the lattice equation (2.3) as a derived equation of the bilinear discrete KP equation. However, we want to go further and take into account that the bilinear KP equation is part of a multidimensionally consistent system of equations, as has been recognized in recent years, cf e.g. [127, 7, 105]. In order to incorporate this multidimensionally consistent system of equations into a single Lagrangian framework we will now proceed to define the Lagrangian multiform structure for the lattice KP system.

The first step is to introduce a Lagrangian 3-form $\mathcal{L}_{ijk}$ where i, j, k denote any three distinct directions in a multidimensional lattice $\mathbf{\Lambda}$, whose vertices are labelled by integer vectors $\boldsymbol{n} = (n_i)_{i\in I}$ where I is an arbitrary set of labels, i, j, k taking values in I. The lattice 3-form $\mathcal{L}_{ijk}$ is based on the form of the Lagrangian (2.1), but we require it to be antisymmetric with respect to the interchange of any two indices, and we associate with it an elementary oriented cube σ_{ijk} spanned by unit vectors $\boldsymbol{e}_i$ which are associated with the corresponding lattice direction labelled by i in the multidimensional lattice $\mathbf{\Lambda}$. This leads us to define the following Lagrangian 3-form

$$\mathcal{L}_{ijk} = \frac{1}{2}\big(L_{ijk} + L_{jki} + L_{kij} - L_{ikj} - L_{jik} - L_{kji}\big),$$

which when written out explicitly and simplified is

$$
\begin{aligned}
\mathcal{L}_{ijk} \;=\;& \ln\left(\frac{\tau_k\tau_{ij}}{\tau_j\tau_{ki}}\right)\ln\left(-\frac{A_{ki}\tau_j}{A_{jk}\tau_i}\right) - \mathrm{Li}_2\left(-\frac{A_{ij}\tau_k\tau_{ij}}{A_{ki}\tau_j\tau_{ki}}\right)\\
&+\ln\left(\frac{\tau_i\tau_{jk}}{\tau_k\tau_{ij}}\right)\ln\left(-\frac{A_{ij}\tau_k}{A_{ki}\tau_j}\right) - \mathrm{Li}_2\left(-\frac{A_{jk}\tau_i\tau_{jk}}{A_{ij}\tau_k\tau_{ij}}\right)\\
&+\ln\left(\frac{\tau_j\tau_{ki}}{\tau_i\tau_{jk}}\right)\ln\left(-\frac{A_{jk}\tau_i}{A_{ij}\tau_k}\right) - \mathrm{Li}_2\left(-\frac{A_{ki}\tau_j\tau_{ki}}{A_{jk}\tau_i\tau_{jk}}\right)\\
&-\frac{1}{2}\Big(\big(\ln(\tau_{ij})\big)^2 + \big(\ln(\tau_{jk})\big)^2 + \big(\ln(\tau_{ki})\big)^2 - \big(\ln(\tau_i)\big)^2 - \big(\ln(\tau_j)\big)^2 - \big(\ln(\tau_k)\big)^2\\
&\quad - \ln(\tau_{ij})\ln(\tau_{jk}) - \ln(\tau_{jk})\ln(\tau_{ki}) - \ln(\tau_{ki})\ln(\tau_{ij}) + \ln(\tau_i)\ln(\tau_j)\\
&\quad + \ln(\tau_j)\ln(\tau_k) + \ln(\tau_k)\ln(\tau_i) + (\ln(A_{ij}))^2 + (\ln(A_{jk}))^2 + (\ln(A_{ki}))^2\\
&\quad - \ln(A_{ij})\ln(A_{jk}) - \ln(A_{jk})\ln(A_{ki}) - \ln(A_{ki})\ln(A_{ij}) + \frac{\pi^2}{2}\Big),
\end{aligned}
\tag{2.5}
$$

where the constant terms arise from dilogarithm identities given in Appendix A.

This Lagrangian is antisymmetric by construction. Considered as a usual scalar Lagrangian defined in the 3-dimensional sublattice of the directions i, j, k, the Euler-Lagrange equations of the corresponding action would yield an equation combining 12 shifted copies of the original bilinear equation (1.2), namely

$$
\begin{aligned}
\frac{\delta\mathcal{L}_{ijk}}{\delta\tau} \;=\;& \Bigg\{\ln\left(-\frac{A_{ki}\tau_{j\bar{k}}\tau_i + A_{ij}\tau\tau_{ij\bar{k}}}{A_{jk}\tau_{i\bar{k}}\tau_j}\right) + \ln\left(-\frac{A_{ki}\tau_{\bar{\jmath}k}\tau_{\bar{\imath}} + A_{ij}\tau\tau_{\bar{\imath}\bar{\jmath}k}}{A_{jk}\tau_{\bar{\imath}k}\tau_j}\right)\\
&-\ln\left(-\frac{A_{ki}\tau\tau_{i\bar{\jmath}k} + A_{ij}\tau_{\bar{\jmath}k}\tau_i}{A_{jk}\tau_{i\bar{\jmath}}\tau_k}\right) - \ln\left(-\frac{A_{ki}\tau\tau_{\bar{\imath}j\bar{k}} + A_{ij}\tau_{j\bar{k}}\tau_{\bar{\imath}}}{A_{jk}\tau_{\bar{\imath}j}\tau_{\bar{k}}}\right)\\
&+\ln\left(-\frac{A_{ij}\tau_{\bar{\imath}k}\tau_j + A_{jk}\tau\tau_{\bar{\imath}jk}}{A_{ki}\tau_{\bar{\imath}j}\tau_k}\right) + \ln\left(-\frac{A_{ij}\tau_{i\bar{k}}\tau_{\bar{\jmath}} + A_{jk}\tau\tau_{i\bar{\jmath}\bar{k}}}{A_{ki}\tau_{i\bar{\jmath}}\tau_{\bar{k}}}\right)\\
&-\ln\left(-\frac{A_{ij}\tau\tau_{ij\bar{k}} + A_{jk}\tau_{i\bar{k}}\tau_j}{A_{ki}\tau_{j\bar{k}}\tau_i}\right) - \ln\left(-\frac{A_{ij}\tau\tau_{\bar{\imath}\bar{\jmath}k} + A_{jk}\tau_{\bar{\imath}k}\tau_{\bar{\jmath}}}{A_{ki}\tau_{\bar{\jmath}k}\tau_{\bar{\imath}}}\right)\\
&+\ln\left(-\frac{A_{jk}\tau_{i\bar{\jmath}}\tau_k + A_{ki}\tau\tau_{i\bar{\jmath}k}}{A_{ij}\tau_{\bar{\jmath}k}\tau_i}\right) + \ln\left(-\frac{A_{jk}\tau_{\bar{\imath}j}\tau_{\bar{k}} + A_{ki}\tau\tau_{\bar{\imath}j\bar{k}}}{A_{ij}\tau_{j\bar{k}}\tau_{\bar{\imath}}}\right)\\
&-\ln\left(-\frac{A_{jk}\tau\tau_{\bar{\imath}jk} + A_{ki}\tau_{\bar{\imath}j}\tau_k}{A_{ij}\tau_{\bar{\imath}k}\tau_j}\right) - \ln\left(-\frac{A_{jk}\tau\tau_{i\bar{\jmath}\bar{k}} + A_{ki}\tau_{i\bar{\jmath}}\tau_{\bar{k}}}{A_{ij}\tau_{i\bar{k}}\tau_{\bar{\jmath}}}\right)\Bigg\}\frac{1}{\tau}\\
\;=\;& 0.
\end{aligned}
\tag{2.6}
$$

Equation (2.6) is actually a 19-point equation, i.e., an equation relating 19 points on the lattice, existing on a cube as in Figure 4.1. It comprises the 12 shifted copies of (1.2) as illustrated in Figure 4.2, where to each configuration of 6 points on an elementary cube correspond 2 copies of (1.2).

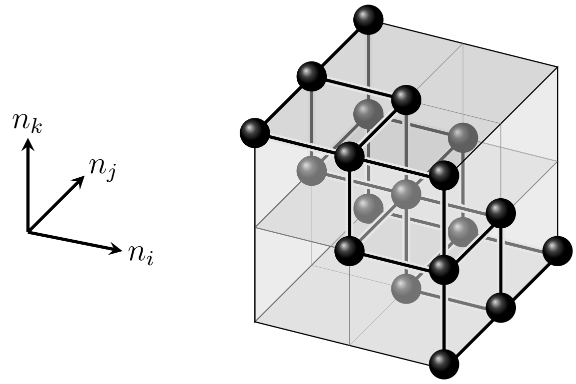

Figure 4.1: The 19-point equation.

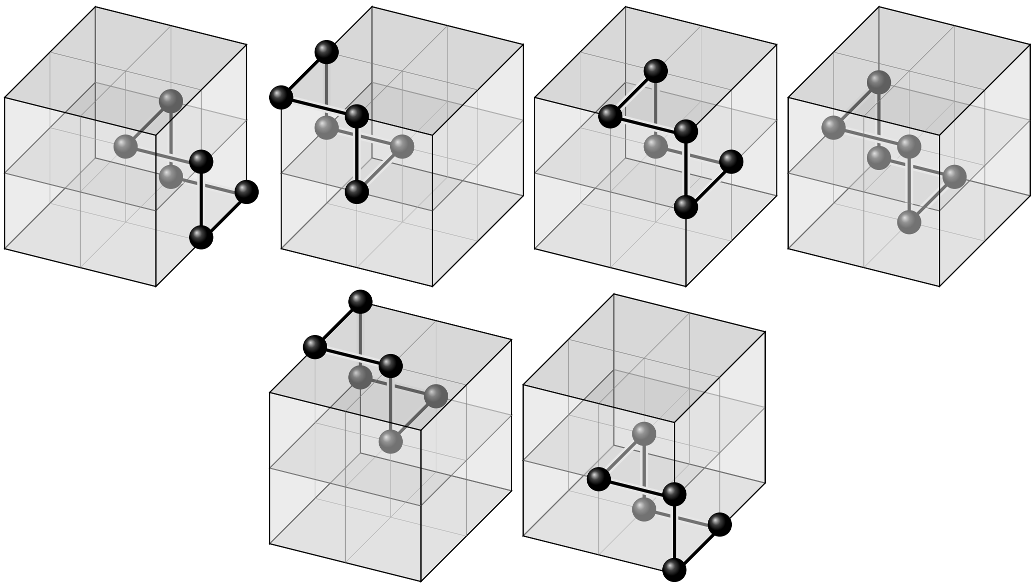

Figure 4.2: Copies of the 6-point equation.

4.3 Closure relation and Lagrangian 3-form

The main observation which allows the establishment of the multiform structure is that the Lagrangian 3-form defined in (2.5) is a closed form on the solution space of the original bilinear equation (1.2). In fact we have the following *closure property*

Proposition: *The Lagrangian defined by* (2.5) *satisfies the following closure relation on solutions to the equation* (1.2) *when embedded in a 4-dimensional lattice.*

$$\Delta_l \mathcal{L}_{ijk} - \Delta_i \mathcal{L}_{jkl} + \Delta_j \mathcal{L}_{kli} - \Delta_k \mathcal{L}_{lij} = 0, \tag{3.1}$$

where the difference operator Δ_i *acts on functions* f *of* $\tau = \tau(n_i, n_j, n_k, n_l)$ *by the formula* $\Delta_i f(\tau) = f(\tau_i) - f(\tau)$, *and on a function* g *of* τ *and its shifts by the formula* $\Delta_i g(\tau, \tau_j, \tau_k, \tau_l) = g(\tau_i, \tau_{ij}, \tau_{ik}, \tau_{il}) - g(\tau, \tau_j, \tau_k, \tau_l)$.

Proof: By explicit computation. The closure relation (3.1) holds on solutions of the original equation, so we need to make use of (1.2) and its shifted versions. If we add in a fourth lattice direction, we get the equations

$$A_{jk}\tau_i\tau_{jk} + A_{ki}\tau_j\tau_{ki} + A_{ij}\tau_k\tau_{ij} = 0, \tag{3.2a}$$

$$A_{kl}\tau_j\tau_{kl} - A_{jl}\tau_k\tau_{jl} + A_{jk}\tau_l\tau_{jk} = 0, \tag{3.2b}$$

$$A_{li}\tau_k\tau_{li} - A_{ki}\tau_l\tau_{ki} + A_{kl}\tau_i\tau_{kl} = 0, \tag{3.2c}$$

$$A_{ij}\tau_l\tau_{ij} + A_{jl}\tau_i\tau_{jl} + A_{li}\tau_j\tau_{li} = 0. \tag{3.2d}$$

When shifted, equations (3.2a) through (3.2d) become

$$A_{jk}\tau_{li}\tau_{jkl} + A_{ki}\tau_{jl}\tau_{kli} + A_{ij}\tau_{kl}\tau_{lij} = 0, \tag{3.2e}$$

$$A_{kl}\tau_{ij}\tau_{kli} - A_{jl}\tau_{ki}\tau_{lij} + A_{jk}\tau_{li}\tau_{ijk} = 0, \tag{3.2f}$$

$$A_{li}\tau_{jk}\tau_{lij} - A_{ki}\tau_{jl}\tau_{ijk} + A_{kl}\tau_{ij}\tau_{jkl} = 0, \tag{3.2g}$$

$$A_{ij}\tau_{kl}\tau_{ijk} + A_{jl}\tau_{ki}\tau_{jkl} + A_{li}\tau_{jk}\tau_{kli} = 0. \tag{3.2h}$$

We also need the following two key identities for the dilogarithm function

$$\begin{aligned} \mathrm{Li}_2(x) + \mathrm{Li}_2(y) &= \mathrm{Li}_2(xy) - \mathrm{Li}_2\left(\frac{x-xy}{x-1}\right) - \mathrm{Li}_2\left(\frac{y-xy}{y-1}\right) \\ &\quad -\frac{1}{2}\left(\ln\left(\frac{x-1}{y-1}\right)\right)^2, \end{aligned} \tag{3.3a}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{1}{2}\left(\ln(-x)\right)^2 - \frac{\pi^2}{6}. \tag{3.3b}$$

Proofs of these identities can be found in Appendix A. Equation (3.3a) is valid up to imaginary terms which can be chosen to cancel out in the course of the closure relation computation, whereas (3.3b) is valid for all real x. We will split the computation into two parts, considering the dilogarithm terms separately. Let

$$\Gamma = \Delta_l \mathcal{L}_{ijk} - \Delta_i \mathcal{L}_{jkl} + \Delta_j \mathcal{L}_{kli} - \Delta_k \mathcal{L}_{lij} \tag{3.4}$$

with $\mathcal{L}_{ijk}$ given by (2.5) and let $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is the part of Γ omitting dilogarithm terms from the Lagrangian, and Γ_2 consists of only the dilogarithm terms. We have

$$\begin{aligned}
\Gamma_1 \;=\; & \frac{1}{2}\big((\ln(\tau_{ijk})^2 - (\ln(\tau_{jkl}))^2 + (\ln(\tau_{kli}))^2 - (\ln(\tau_{lij}))^2 \\
& \quad +(\ln(\tau_i))^2 - (\ln(\tau_j))^2 + (\ln(\tau_k))^2 - (\ln(\tau_l))^2\big) \\
& - \ln(\tau_{ijk})\ln(\tau_{kli}) + \ln(\tau_{jkl})\ln(\tau_{lij}) - \ln(\tau_i)\ln(\tau_k) + \ln(\tau_j)\ln(\tau_l) \\
& + \ln(\tau_{ijk})\ln\left(-\frac{A_{ij}A_{jk}\tau_{kl}\tau_{li}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) + \ln(\tau_{jkl})\ln\left(\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{jk}A_{kl}\tau_{ij}\tau_{li}}\right) \\
& + \ln(\tau_{kli})\ln\left(-\frac{A_{kl}A_{li}\tau_{ij}\tau_{jk}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) + \ln(\tau_{lij})\ln\left(\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{ij}A_{li}\tau_{jk}\tau_{kl}}\right) \\
& + \ln(\tau_i)\ln\left(\frac{A_{jk}A_{kl}\tau_{jk}\tau_{kl}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) + \ln(\tau_j)\ln\left(-\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{kl}A_{li}\tau_{kl}\tau_{li}}\right) \\
& + \ln(\tau_k)\ln\left(\frac{A_{ij}A_{li}\tau_{ij}\tau_{li}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) + \ln(\tau_l)\ln\left(-\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{ij}A_{jk}\tau_{ij}\tau_{jk}}\right) \\
& + \ln(\tau_{ij})\ln\left(-\frac{A_{li}}{A_{jk}}\right) + \ln(\tau_{jk})\ln\left(-\frac{A_{kl}}{A_{ij}}\right) + \ln(\tau_{kl})\ln\left(-\frac{A_{jk}}{A_{li}}\right) \\
& + \ln(\tau_{li})\ln\left(-\frac{A_{ij}}{A_{kl}}\right) + \ln\left(\frac{\tau_{jl}}{\tau_{ki}}\right)\ln\left(\frac{A_{ij}A_{jk}A_{kl}A_{li}\tau_{ij}\tau_{jk}\tau_{kl}\tau_{li}}{A_{jl}^2A_{ki}^2\tau_{jl}^2\tau_{ki}^2}\right).
\end{aligned} \tag{3.5}$$

Now we consider the dilogarithm terms. The dilogarithm terms from Γ are

$$
\begin{aligned}
\Gamma_2 \;=\; & -\mathrm{Li}_2\left(-\frac{A_{ij}\tau_{kl}\tau_{lij}}{A_{ki}\tau_{jl}\tau_{kli}}\right) + \mathrm{Li}_2\left(-\frac{A_{ij}\tau_{k}\tau_{ij}}{A_{ki}\tau_{j}\tau_{ki}}\right) \quad - \mathrm{Li}_2\left(\frac{A_{kl}\tau_{ij}\tau_{jkl}}{A_{ki}\tau_{jl}\tau_{ijk}}\right) \\
& +\mathrm{Li}_2\left(\frac{A_{jk}\tau_{li}\tau_{ijk}}{A_{jl}\tau_{ki}\tau_{lij}}\right) - \mathrm{Li}_2\left(\frac{A_{jk}\tau_{l}\tau_{jk}}{A_{jl}\tau_{k}\tau_{jl}}\right) \quad + \mathrm{Li}_2\left(\frac{A_{kl}\tau_{i}\tau_{kl}}{A_{ki}\tau_{l}\tau_{ki}}\right) \\
& +\mathrm{Li}_2\left(-\frac{A_{li}\tau_{jk}\tau_{kli}}{A_{jl}\tau_{ki}\tau_{jkl}}\right) - \mathrm{Li}_2\left(-\frac{A_{li}\tau_{j}\tau_{li}}{A_{jl}\tau_{i}\tau_{jl}}\right) \quad + \mathrm{Li}_2\left(-\frac{A_{kl}\tau_{ij}\tau_{kli}}{A_{jk}\tau_{li}\tau_{ijk}}\right) \\
& -\mathrm{Li}_2\left(-\frac{A_{jk}\tau_{li}\tau_{jkl}}{A_{ij}\tau_{kl}\tau_{lij}}\right) + \mathrm{Li}_2\left(-\frac{A_{jk}\tau_{i}\tau_{jk}}{A_{ij}\tau_{k}\tau_{ij}}\right) \quad - \mathrm{Li}_2\left(-\frac{A_{kl}\tau_{j}\tau_{kl}}{A_{jk}\tau_{l}\tau_{jk}}\right) \\
& -\mathrm{Li}_2\left(-\frac{A_{li}\tau_{jk}\tau_{lij}}{A_{kl}\tau_{ij}\tau_{jkl}}\right) + \mathrm{Li}_2\left(-\frac{A_{li}\tau_{k}\tau_{li}}{A_{kl}\tau_{i}\tau_{kl}}\right) \quad - \mathrm{Li}_2\left(-\frac{A_{ki}\tau_{jl}\tau_{kli}}{A_{jk}\tau_{li}\tau_{jkl}}\right) \\
& +\mathrm{Li}_2\left(-\frac{A_{ij}\tau_{kl}\tau_{ijk}}{A_{li}\tau_{jk}\tau_{kli}}\right) - \mathrm{Li}_2\left(-\frac{A_{ij}\tau_{l}\tau_{ij}}{A_{li}\tau_{j}\tau_{li}}\right) \quad + \mathrm{Li}_2\left(-\frac{A_{ki}\tau_{j}\tau_{ki}}{A_{jk}\tau_{i}\tau_{jk}}\right) \\
& +\mathrm{Li}_2\left(\frac{A_{jl}\tau_{ki}\tau_{lij}}{A_{kl}\tau_{ij}\tau_{kli}}\right) - \mathrm{Li}_2\left(\frac{A_{jl}\tau_{k}\tau_{jl}}{A_{kl}\tau_{j}\tau_{kl}}\right) \quad + \mathrm{Li}_2\left(-\frac{A_{jl}\tau_{ki}\tau_{jkl}}{A_{ij}\tau_{kl}\tau_{ijk}}\right) \\
& -\mathrm{Li}_2\left(\frac{A_{ki}\tau_{jl}\tau_{ijk}}{A_{li}\tau_{jk}\tau_{lij}}\right) + \mathrm{Li}_2\left(\frac{A_{ki}\tau_{l}\tau_{ki}}{A_{li}\tau_{k}\tau_{li}}\right) \quad - \mathrm{Li}_2\left(-\frac{A_{jl}\tau_{i}\tau_{jl}}{A_{ij}\tau_{l}\tau_{ij}}\right)
\end{aligned}
\tag{3.6}
$$

Using (3.3b) on the terms in the dotted boxes, followed by (3.3a) on the terms in the solid boxes, we obtain the following expression. Here we have also made use of the equations (3.2a) through (3.2h).

$$
\begin{aligned}
\Gamma_2 \; = \; & +\mathrm{Li}_2\left(\frac{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}\right) - \mathrm{Li}_2\left(\frac{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}\right) - \mathrm{Li}_2\left(\frac{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}\right) - \mathrm{Li}_2\left(\frac{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}\right) - \mathrm{Li}_2\left(\frac{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}\right) - \mathrm{Li}_2\left(\frac{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}\right) \boxed{- \mathrm{Li}_2\left(\frac{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}\right)} \\
& +\mathrm{Li}_2\left(\frac{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}\right) \boxed{- \mathrm{Li}_2\left(\frac{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}\right)} - \mathrm{Li}_2\left(\frac{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}\right) - \mathrm{Li}_2\left(\frac{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}\right) - \mathrm{Li}_2\left(\frac{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}\right) \boxed{- \mathrm{Li}_2\left(\frac{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}\right)} - \mathrm{Li}_2\left(\frac{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}\right) - \mathrm{Li}_2\left(\frac{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}\right) \boxed{- \mathrm{Li}_2\left(\frac{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}\right)} \\
& +\mathrm{Li}_2\left(\frac{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}\right) \boxed{- \mathrm{Li}_2\left(\frac{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}\right)} \boxed{- \mathrm{Li}_2\left(\frac{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}\right)} \\
& +\mathrm{Li}_2\left(\frac{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}\right) - \mathrm{Li}_2\left(\frac{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}\right) - \mathrm{Li}_2\left(\frac{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}\right) - \mathrm{Li}_2\left(\frac{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}\right) - \mathrm{Li}_2\left(\frac{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}\right) - \mathrm{Li}_2\left(\frac{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}\right) - \mathrm{Li}_2\left(\frac{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}\right) \\
& +\mathrm{Li}_2\left(\frac{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}\right) - \mathrm{Li}_2\left(\frac{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}\right) - \mathrm{Li}_2\left(\frac{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}\right) \\
& +\frac{1}{2}\left(\ln\left(\frac{A_{ki}\tau_{jl}\tau_{kli}}{A_{ij}\tau_{kl}\tau_{lij}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{A_{jl}\tau_k\tau_{jl}}{A_{jk}\tau_l\tau_{jk}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{A_{ki}\tau_{jl}\tau_{ijk}}{A_{kl}\tau_{ij}\tau_{jkl}}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(\frac{A_{jl}\tau_i\tau_{jl}}{A_{li}\tau_j\tau_{li}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_{kl}\tau_{lij}}{A_{jk}\tau_{li}\tau_{jkl}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(\frac{A_{jk}\tau_l\tau_{jk}}{A_{kl}\tau_j\tau_{kl}}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(\frac{A_{kl}\tau_{ij}\tau_{jkl}}{A_{li}\tau_{jk}\tau_{lij}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(\frac{A_{li}\tau_j\tau_{li}}{A_{ij}\tau_l\tau_{ij}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(\frac{A_{jk}\tau_{li}\tau_{jkl}}{A_{ki}\tau_{jl}\tau_{kli}}\right)\right)^2 \\
& +\frac{1}{2}\left(\ln\left(-\frac{A_{kl}\tau_j\tau_{kl}}{A_{jl}\tau_k\tau_{jl}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(-\frac{A_{li}\tau_{jk}\tau_{lij}}{A_{ki}\tau_{jl}\tau_{ijk}}\right)\right)^2 + \frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_l\tau_{ij}}{A_{jl}\tau_i\tau_{jl}}\right)\right)^2 \\
& -\frac{1}{2}\left(\ln\left(\frac{A_{ki}\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}{A_{ij}\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}\right)\right)^2 - \frac{1}{2}\left(\ln\left(-\frac{A_{jk}\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}{A_{jl}\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}\right)\right)^2 \\
& -\frac{1}{2}\left(\ln\left(-\frac{A_{ki}\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}{A_{kl}\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}\right)\right)^2 - \frac{1}{2}\left(\ln\left(\frac{A_{li}\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}{A_{jl}\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}\right)\right)^2 \\
& -\frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}{A_{jk}\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}\right)\right)^2 - \frac{1}{2}\left(\ln\left(\frac{A_{kl}\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}{A_{jk}\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}\right)\right)^2
\end{aligned}
$$

$$-\frac{1}{2}\left(\ln\left(\frac{A_{kl}\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}{A_{li}\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}{A_{li}\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}\right)\right)^2$$
$$-\frac{1}{2}\left(\ln\left(\frac{A_{jk}\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}{A_{ki}\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{A_{jl}\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}{A_{kl}\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}\right)\right)^2$$
$$-\frac{1}{2}\left(\ln\left(-\frac{A_{li}\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}{A_{ki}\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{A_{jl}\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}{A_{ij}\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}\right)\right)^2+2\pi^2. \quad (3.7)$$

Using (3.3b) on all the terms in the dotted boxes, all the dilogarithm terms cancel out leaving only these logarithm terms

$$\begin{aligned}
\Gamma_2 \;=\; & +\frac{1}{2}\left(\ln\left(\frac{A_{ki}\tau_{jl}\tau_{kli}}{A_{ij}\tau_{kl}\tau_{lij}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{A_{jl}\tau_k\tau_{jl}}{A_{jk}\tau_l\tau_{jk}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{A_{ki}\tau_{jl}\tau_{ijk}}{A_{kl}\tau_{ij}\tau_{jkl}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(\frac{A_{jl}\tau_i\tau_{jl}}{A_{li}\tau_j\tau_{li}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_{kl}\tau_{lij}}{A_{jk}\tau_{li}\tau_{jkl}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{A_{jk}\tau_l\tau_{jk}}{A_{kl}\tau_j\tau_{kl}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(\frac{A_{kl}\tau_{ij}\tau_{jkl}}{A_{li}\tau_{jk}\tau_{lij}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{A_{li}\tau_j\tau_{li}}{A_{ij}\tau_l\tau_{ij}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{A_{jk}\tau_{li}\tau_{jkl}}{A_{ki}\tau_{jl}\tau_{kli}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{A_{kl}\tau_j\tau_{kl}}{A_{jl}\tau_k\tau_{jl}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{A_{li}\tau_{jk}\tau_{lij}}{A_{ki}\tau_{jl}\tau_{ijk}}\right)\right)^2+\frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_l\tau_{ij}}{A_{jl}\tau_i\tau_{jl}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(\frac{A_{ki}\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}{A_{ij}\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{A_{jk}\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}{A_{jl}\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(-\frac{A_{ki}\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}{A_{kl}\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{A_{li}\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}{A_{jl}\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}{A_{jk}\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{A_{kl}\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}{A_{jk}\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(\frac{A_{kl}\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}{A_{li}\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{A_{ij}\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}{A_{li}\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(\frac{A_{jk}\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}{A_{ki}\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}\right)\right)^2-\frac{1}{2}\left(\ln\left(-\frac{A_{jl}\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}{A_{kl}\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}\right)\right)^2\\
&-\frac{1}{2}\left(\ln\left(-\frac{A_{li}\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}{A_{ki}\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}\right)\right)^2-\frac{1}{2}\left(\ln\left(\frac{A_{jl}\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}{A_{ij}\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{\tau_k\tau_{ij}\tau_{li}\tau_{jkl}}{\tau_i\tau_{jk}\tau_{kl}\tau_{lij}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\tau_j\tau_{kl}\tau_{li}\tau_{ijk}}{\tau_l\tau_{ij}\tau_{jk}\tau_{kli}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{\tau_j\tau_{kl}\tau_{ki}\tau_{lij}}{\tau_k\tau_{ij}\tau_{jl}\tau_{kli}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\tau_l\tau_{jk}\tau_{ki}\tau_{lij}}{\tau_k\tau_{li}\tau_{jl}\tau_{ijk}}\right)\right)^2\\
&+\frac{1}{2}\left(\ln\left(-\frac{\tau_l\tau_{ij}\tau_{ki}\tau_{jkl}}{\tau_i\tau_{kl}\tau_{jl}\tau_{ijk}}\right)\right)^2+\frac{1}{2}\left(\ln\left(-\frac{\tau_i\tau_{jk}\tau_{jl}\tau_{kli}}{\tau_j\tau_{li}\tau_{ki}\tau_{jkl}}\right)\right)^2+3\pi^2. \quad (3.8)
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\Gamma_2 \;=\;& \frac{1}{2}\Big(-(\ln(\tau_{ijk}))^2 + (\ln(\tau_{jkl}))^2 - (\ln(\tau_{kli}))^2 + (\ln(\tau_{lij}))^2 \\
&\quad -(\ln(\tau_i))^2 + (\ln(\tau_j))^2 - (\ln(\tau_k))^2 + (\ln(\tau_l))^2\Big) \\
& + \ln(\tau_{ijk})\ln(\tau_{kli}) - \ln(\tau_{jkl})\ln(\tau_{lij}) + \ln(\tau_i)\ln(\tau_k) - \ln(\tau_j)\ln(\tau_l) \\
& + \ln(\tau_{ijk})\ln\left(-\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{ij}A_{jk}\tau_{kl}\tau_{li}}\right) + \ln(\tau_{jkl})\ln\left(\frac{A_{jk}A_{kl}\tau_{ij}\tau_{li}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) \\
& + \ln(\tau_{kli})\ln\left(-\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{kl}A_{li}\tau_{ij}\tau_{jk}}\right) + \ln(\tau_{lij})\ln\left(\frac{A_{ij}A_{li}\tau_{jk}\tau_{kl}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) \\
& + \ln(\tau_i)\ln\left(\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{jk}A_{kl}\tau_{jk}\tau_{kl}}\right) + \ln(\tau_j)\ln\left(-\frac{A_{kl}A_{li}\tau_{kl}\tau_{li}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) \\
& + \ln(\tau_k)\ln\left(\frac{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}{A_{ij}A_{li}\tau_{ij}\tau_{li}}\right) + \ln(\tau_l)\ln\left(-\frac{A_{ij}A_{jk}\tau_{ij}\tau_{jk}}{A_{jl}A_{ki}\tau_{jl}\tau_{ki}}\right) \\
& + \ln(\tau_{ij})\ln\left(-\frac{A_{jk}}{A_{li}}\right) + \ln(\tau_{jk})\ln\left(-\frac{A_{ij}}{A_{kl}}\right) + \ln(\tau_{kl})\ln\left(-\frac{A_{li}}{A_{jk}}\right) \\
& + \ln(\tau_{li})\ln\left(-\frac{A_{kl}}{A_{ij}}\right) + \ln\left(\frac{\tau_{ki}}{\tau_{jl}}\right)\ln\left(\frac{A_{ij}A_{jk}A_{kl}A_{li}\tau_{ij}\tau_{jk}\tau_{kl}\tau_{li}}{A_{jl}^2A_{ki}^2\tau_{jl}^2\tau_{ki}^2}\right). \qquad (3.9)
\end{aligned}$$

It can easily be checked that adding (3.9) to Γ_1 from (3.5) gives zero, verifying the closure relation. ■

The establishment of the closure property enables us to propose a novel variational principle for the multidimensionally consistent system of bilinear KP equations, along the same line as in the Chapters 2 and 3. Choosing a 3-dimensional hypersurface σ within a multidimensional lattice of dimension higher than 3, consisting of a connected configuration of elementary cubes σ_{ijk}, we can define an action S on this hypersurface by summing the contributions $\mathcal{L}_{ijk}$ from each of the cubes as follows

$$S[\tau;\sigma] = \sum_{\sigma_{ijk}\in\sigma} \mathcal{L}_{ijk}, \qquad (3.10)$$

taking into consideration the orientation of each elementary cube contributing to the surface. The antisymmetry of $\mathcal{L}_{ijk}$ guarantees that there is no ambiguity in how each discrete Lagrangian 3-form will contribute to the action. Furthermore, the closure relation (3.1) allows us to impose the independence of the action on local variations of the surface away from any boundary that the surface σ may possess. Thus, whilst keeping the boundary fixed we may locally deform σ in any way we choose, allowing us in particular to render it locally flat away from the boundary, such that we can specify a 3-dimensional hypersurface described in terms of three local coordinates n_i, n_j, n_k. There we can then apply the usual variational principle, taking the variational derivative with respect to τ, leading to the Euler-Lagrange equations (2.6). These equations of the motion are a consequence of the Hirota-Miwa equation (1.2), as are the closure relations that guarantee the surface independence of the action under local deformations. This interlinked scheme of variations with respect to the dependent variables as well as to the geometry of the independent variables is what constitutes the Lagrangian multiform structure of the lattice KP system.

4.4 Discussion

There are some remarks to make at this point:

First, one has to qualify what it means for a Lagrangian to be associated with a given equation, since as we have noted earlier the Euler-Lagrange equations rather than yielding the original bilinear KP equation only yield a derived equation comprising a combination of various copies of the original equation. Nevertheless we have taken the point of view that since the canonical variable is the τ-function, i.e., the τ-function is the variable involved in both systems and all solutions to the bilinear KP equation will automatically solve the derived equation, we consider this Lagrangian structure to be associated with the bilinear KP equation.

Second, the closure relation which is central to the Lagrangian multiform structure relies on the bilinear KP equation rather than on the Euler-Lagrange equations. It is not clear at this stage to what extent the closure property remains to be verified on all solutions of the Euler-Lagrange equations or only on a subvariety of solutions that obey the multidimensional systems of bilinear equations.

As far as KP-type systems are concerned, in some recent works in combinatorics 3-dimensional 6-point recurrence schemes have been studied from the point of view of the geometry of the octahedral lattice, cf e.g.[112, 42]. A classification of multidimensionally consistent 6-point equations has recently been done in [7], but this does not seem to yield any novel lattice equations (e.g. in comparison with the list in [84]). It would be of interest to see whether Lagrangian multiform structures can be established for all those equations, and whether these structures can be adapted to the octahedral lattice picture. Alternatively one can consider 3-dimensional lattice equations of BKP type, i.e. equations of the form

$$Q(\tau, \tau_i, \tau_j, \tau_k, \tau_{ij}, \tau_{jk}, \tau_{ki}, \tau_{ijk}) = 0, \tag{4.1}$$

but so far Lagrangian structures for such equations remain to be established. Lagrangian structures for 3d models have been considered in [111], and connections between the geometry of 3d lattices and 3d quantum systems was explored in [11].

4.5 Chapter summary

In this chapter we have presented a Lagrangian for the bilinear lattice KP system, and extended it to a Lagrangian 3-form. We have shown this Lagrangian 3-form obeys a 4-dimensional closure relation on solutions to the equation, and formulated the variational principle given in the previous chapters in the case of a 3-dimensional system.

Chapter 5

Continuous Lagrangian forms

So far, we have developed the idea of Lagrangian forms, and the corresponding new variational principle, for discrete systems. In fact, there is a continuous analogue to all of this, which appears in [65].

5.1 Linear example

At the continuous level, rather than systems of partial difference equations we are dealing with systems of partial differential equations. An interesting example of a linear system of PDEs which are mutually compatible is given by the following nonautonomous set of equations; this is the linearization of the system given in the next section. Here the independent variables are p_i and p_j, we have $w = w(p_i, p_j)$ as the dependent scalar variable and the n_i, n_j are a pair of parameters of the equation, where we associate the parameter n_i with the variable p_i, and the parameter n_j with the variable p_j.

The system of equations is

$$\partial_{p_i}\partial_{p_j}(p_i^2 - p_j^2)\partial_{p_i}\partial_{p_j} w = 4(n_j\partial_{p_i} - n_i\partial_{p_j})\frac{1}{p_i^2 - p_j^2}(n_j p_i^2 \partial_{p_i} - n_i p_j^2 \partial_{p_j})w, \tag{1.1}$$

where i, j run over some index set I. Each of these, for fixed labels i, j, arise as Euler-Lagrange equations from the Lagrange density

$$\mathcal{L}_{ij} = \frac{1}{n_j n_i}\left(\frac{1}{2}(p_i^2 - p_j^2)w_{p_i p_j}^2 + (n_j^2 w_{p_i}^2 - n_i^2 w_{p_j}^2) + \frac{p_i^2 + p_j^2}{p_i^2 - p_j^2}(n_j w_{p_i} - n_i w_{p_j})^2\right), \tag{1.2}$$

where notably the independent variables p_i, p_j are on an equal footing. It transpires that the system of PDEs (1.1), when the labels i, j are assumed to run over some index set of cardinality larger than 2, is multidimensionally consistent in a similar way as the lattice equations considered in Chapter 2. Furthermore, the Lagrangian (1.2) obeys a continuous analogue of the discrete closure relation; this closure relation is now expressed in terms of differential operators instead of difference operators, and has the following form

$$\partial_{p_i}\mathcal{L}_{jk} + \partial_{p_j}\mathcal{L}_{ki} + \partial_{p_k}\mathcal{L}_{ij} = 0. \tag{1.3}$$

It holds for (1.2) provided one or the other of the following two relations hold

$$w_{p_ip_jp_k} = -4\left(\frac{n_jn_kp_i^2w_{p_i}}{(p_k^2-p_i^2)(p_i^2-p_j^2)} + \frac{n_kn_ip_j^2w_{p_j}}{(p_i^2-p_j^2)(p_j^2-p_k^2)} + \frac{n_in_jp_k^2w_{p_k}}{(p_j^2-p_k^2)(p_k^2-p_i^2)}\right), \tag{1.4a}$$

or

$$\frac{(p_i^2-p_j^2)w_{p_ip_j}}{n_in_j} + \frac{(p_j^2-p_k^2)w_{p_jp_k}}{n_jn_k} + \frac{(p_k^2-p_i^2)w_{p_kp_i}}{n_kn_i} = 0. \tag{1.4b}$$

It seems somewhat artificial in this example to invoke the additional equations (1.4a) and (1.4b), the need for which is mainly due to the fact that we are dealing with higher order PDEs in terms of the derivatives. We note, however, that the PDEs (1.1), (1.4a) and (1.4b) all hold true on a large class of solutions given by the Fourier-type integral of the form

$$w = \int_C dk\ \ c(k)\prod_{i\in I}\left(\frac{p_i+k}{p_i-k}\right)^{n_i}, \tag{1.5}$$

over some suitably chosen curve C in the complex plane and suitably chosen coefficient function $c(k)$, where I denotes the index set as above. This example is inspired by the canonical form of the plane wave factors, i.e., discrete exponential functions, appearing in the solutions of the lattice equations [82, 106], which explains the use of the notation p_i as independent variables for historic reasons.

We define an action this time as being an integral of the Lagrangian over a surface, instead of a sum:

$$S[w;\sigma] = \int_\sigma \sum_{i,j\in I} \mathcal{L}_{ij} dp_i \wedge dp_j. \tag{1.6}$$

Again, as in the discrete case, the closure relation is implied by the action being independent of the surface σ on which it is defined; this can be understood using Stokes' theorem. Recall that Stokes' theorem states: if ω is an n-form with compact support on a (oriented, smooth) manifold M of dimension n+1, with boundary ∂M, then

$$\int_M d\omega = \oint_{\partial M} \omega. \tag{1.7}$$

In our case, "ω" is the Lagrangian 2-form, i.e.

$$\omega = \sum_{i,j\in I} \mathcal{L}_{ij} dp_i \wedge dp_j, \tag{1.8}$$

and

$$\begin{aligned} d\omega &= \sum_{i,j,k\in I} \partial_{p_k}\mathcal{L}_{ij} dp_k \wedge dp_i \wedge dp_j & (1.9)\\ &= \sum_{i<j<k} (\partial_{p_i}\mathcal{L}_{jk} + \partial_{p_j}\mathcal{L}_{ki} + \partial_{p_k}\mathcal{L}_{ij}) dp_i \wedge dp_j \wedge dp_k. & (1.10)\end{aligned}$$

If the action S does not depend on the surface, then taking the integral of ω over a closed surface σ_c will result in

$$\oint_{\sigma_c} \sum_{i,j\in I} \mathcal{L}_{ij} dp_i \wedge dp_j = 0, \tag{1.11}$$

and if this closed surface σ_c bounds a volume V, then

$$0 = \int_V \sum_{i<j<k} (\partial_{p_i}\mathcal{L}_{jk} + \partial_{p_j}\mathcal{L}_{ki} + \partial_{p_k}\mathcal{L}_{ij}) dp_i \wedge dp_j \wedge dp_k, \tag{1.12}$$

which, since the volume V is arbitrary, clearly implies the closure relation (1.3). Hence, the action being independent of the surface implies the closure relation holds.

Now, similar to the discrete case, we may choose a surface, on which we can define an action as in (1.6). Imposing surface independence, we may locally rectify the surface, i.e., deform it locally to a plane in terms of selected independent variables p_i, p_j, and then we can derive from the Euler-Lagrange equations in those variables the system of PDEs. Once again, if the system of PDEs subsequently implies that the closure relation for the Lagrangians holds, the mechanism is entirely consistent, and represents multidimensional consistency on the level of the Lagrangian.

5.2 Nonlinear example

The full nonlinear case analogous to (1.1) appeared first in [88], and it represents the full KdV hierarchy as a so-called *generating PDE* given as follows

$$\begin{aligned} U_{t_it_it_jt_j} &= U_{t_it_it_j}\left(\frac{1}{t_i-t_j}+\frac{U_{t_it_j}}{U_{t_i}}+\frac{U_{t_jt_j}}{U_{t_j}}\right)+U_{t_it_jt_j}\left(\frac{1}{t_j-t_i}+\frac{U_{t_it_j}}{U_{t_j}}+\frac{U_{t_it_i}}{U_{t_i}}\right) \\ &+U_{t_it_i}\left(\frac{n_i^2}{(t_i-t_j)^2}\frac{U_{t_j}^2}{U_{t_i}^2}-\frac{U_{t_it_j}^2}{U_{t_i}^2}-\frac{1}{t_i-t_j}\frac{U_{t_it_j}}{U_{t_i}}\right)-U_{t_it_j}\frac{U_{t_it_i}U_{t_jt_j}}{U_{t_i}U_{t_j}} \\ &+U_{t_jt_j}\left(\frac{n_j^2}{(t_i-t_j)^2}\frac{U_{t_i}^2}{U_{t_j}^2}-\frac{U_{t_it_j}^2}{U_{t_j}^2}-\frac{1}{t_j-t_i}\frac{U_{t_it_j}}{U_{t_j}}\right) \\ &+\frac{n_i^2}{2(t_i-t_j)^3}\frac{U_{t_j}}{U_{t_i}}(U_{t_i}+U_{t_j}+2(t_j-t_i)U_{t_it_j}) \\ &+\frac{n_j^2}{2(t_j-t_i)^3}\frac{U_{t_i}}{U_{t_j}}(U_{t_j}+U_{t_i}+2(t_i-t_j)U_{t_it_j}) \\ &+\frac{1}{2(t_i-t_j)}U_{t_it_j}^2\left(\frac{1}{U_{t_i}}-\frac{1}{U_{t_j}}\right), \end{aligned} \tag{2.1}$$

which represents a generalization of the Ernst-Weyl equation of general relativity as was shown in [118]. The variables t_i are closely related to the p_i of the previous linear example, namely by $t_i = p_i^2$. It was argued in [88] that (2.1) constitutes a multidimensionally consistent system in the same way as the linear equation. It is considered to be a generating PDE for the KdV hierarchy because upon expansion in the independent variables

$$\partial_{t_i} = -\frac{n_i}{\sqrt{t_i}}\sum_{k=1}^{\infty}\frac{1}{t_i^k}\partial_{s_k}, \tag{2.2a}$$

$$\partial_{t_j} = -\frac{n_j}{\sqrt{t_j}}\sum_{k=1}^{\infty}\frac{1}{t_j^k}\partial_{s_k}, \tag{2.2b}$$

where $s_1, s_2, \ldots$ is the infinite sequence of higher times associated with the KdV hierarchy, we recover the entire hierarchy of KdV equations. The Lagrangian for equation (2.1) is

$$\mathcal{L}_{ij} = \frac{1}{2}(t_i - t_j)\frac{U_{t_it_j}^2}{U_{t_i}U_{t_j}} + \frac{1}{2(t_i - t_j)}\left(n_j^2\frac{U_{t_i}}{U_{t_j}} + n_i^2\frac{U_{t_j}}{U_{t_i}}\right). \tag{2.3}$$

This satisfies the closure relation (1.3) provided again that one of two relations hold,

$$\begin{aligned} U_{t_it_jt_k} \quad = \quad & \frac{1}{2U_{t_i}U_{t_j}U_{t_k}}\left(U_{t_i}U_{t_jt_k}U_{t_j}U_{t_kt_i} + U_{t_j}U_{t_kt_i}U_{t_k}U_{t_it_j} + U_{t_k}U_{t_it_j}U_{t_i}U_{t_jt_k}\right) \\ & + \frac{n_i^2}{2(t_k - t_i)(t_i - t_j)U_{t_i}^2} + \frac{n_j^2}{2(t_i - t_j)(t_j - t_k)U_{t_j}^2} + \frac{n_k^2}{2(t_j - t_k)(t_k - t_i)U_{t_k}^2}, \end{aligned} \tag{2.4a}$$

or

$$(t_i - t_j)U_{t_k}U_{t_it_j} + (t_j - t_k)U_{t_i}U_{t_jt_k} + (t_k - t_i)U_{t_j}U_{t_kt_i} = 0. \tag{2.4b}$$

Once again the additional equations (2.4a) and (2.4b) are invoked solely because we are dealing with higher order PDEs in terms of the derivatives, which makes it difficult to verify by direct computation. Equations (2.4a) and (2.4b) are manifestations of the fact that 1+1-dimensional equations of KdV type can be embedded as dimensional reductions of 2+1-dimensional equations of KP type [104], which holds true both for the continuous as well as the discrete case. In fact, the continuous nonautonomous equation (2.4b) is remarkably similar to the fully discrete Hirota-Miwa equation and we believe that it plays the role of a generating PDE for the KP hierarchy. It would be interesting to see how this equation fits in with the results of [53]. All three equations (2.1), (2.4a) and (2.4b) hold on a large class of solutions of soliton type and hence they should certainly be compatible between themselves.

5.3 Generating PDEs for ABS equations

In [120], Xenitidis and Tsoubelis derived generating PDEs as coupled systems for all the ABS equations, through symmetry analysis. As an example, we can mention the following system corresponding to H1, H2 and Q1:

$$\frac{\partial u_i}{\partial \alpha_j} = \frac{u_i - u_j}{\alpha_i - \alpha_j}\left(n_j - (u_i - u_j)\frac{\partial u}{\partial \alpha_j}\right) + \delta^2(\alpha_i - \alpha_j)\frac{\partial u}{\partial \alpha_j}\,, \tag{3.1}$$

$$\frac{\partial^2 u}{\partial \alpha_i \partial \alpha_j} = 2\,\frac{u_i - u_j}{\alpha_i - \alpha_j}\frac{\partial u}{\partial \alpha_i}\frac{\partial u}{\partial \alpha_j} + \frac{n_i}{\alpha_i - \alpha_j}\frac{\partial u}{\partial \alpha_j} + \frac{n_j}{\alpha_j - \alpha_i}\frac{\partial u}{\partial \alpha_i}\,, \tag{3.2}$$

where u is the original dependent variable of the system, and u_i, u_j can be identified with the variable shifted in the i-, j-directions respectively. This system with $\delta = 0$ first appeared in [88] in relation to H1, and was derived from a reduction of the anti-self dual Yang-Mills equations in [118].

The Lagrange structure for the system reads in this case:

$$\begin{aligned}\mathcal{L}_{ij} = \; & \frac{\partial u}{\partial \alpha_i}\frac{\partial u_i}{\partial \alpha_j} - \frac{\partial u}{\partial \alpha_j}\frac{\partial u_j}{\partial \alpha_i} - \left(\delta^2(\alpha_i - \alpha_j) - \frac{(u_i - u_j)^2}{\alpha_i - \alpha_j}\right)\frac{\partial u}{\partial \alpha_i}\frac{\partial u}{\partial \alpha_j} \\ & - n_j\,\frac{u_i - u_j}{\alpha_i - \alpha_j}\,\frac{\partial u}{\partial \alpha_i} + n_i\,\frac{u_j - u_i}{a_j - a_i}\,\frac{\partial u}{\partial \alpha_j}.\end{aligned} \tag{3.3}$$

This obeys (on the EL equations) the closure relation

$$\partial_{\alpha_k}\mathcal{L}_{ij} + \partial_{\alpha_j}\mathcal{L}_{ki} + \partial_{\alpha_i}\mathcal{L}_{jk} = 0\,. \tag{3.4}$$

Both the continuous systems and their corresponding Lagrangians can be established for the entire ABS list, and these Lagrangians can be shown to obey the closure relation (3.4); the results in this section are due to appear in [125].

5.4 Chapter summary

We have shown that there is a natural continuous analogue to the theory of Lagrangian forms for multidimensionally consistent systems and the corresponding new variational principle proposed in the previous chapters. The Lagrangians in this case satisfy a continuous analogue of the closure relation, expressed in terms of differential operators instead of difference operators, which has an entirely analogous appearance. We have demonstrated that this holds in important concrete examples for the linear and non-linear generating PDEs for the KdV hierarchy.

Chapter 6

Conclusion

6.1 Discussion

The key idea put forward in this thesis is that of Lagrangian forms, and a corresponding new variational principle for multidimensionally consistent systems. This differs from the conventional Lagrangian description in that the Lagrangian is usually considered to be a scalar object, which through the Euler-Lagrange equations provides us with one single equation. When we consider systems with the property of multidimensional consistency, we have not just one equation but an infinite number of compatible equations all living together on a multidimensional lattice, and in fact one can consider the integrable system to be this entire family of equations. Then it should be possible to obtain the whole system from a variational principle, hence the idea of Lagrangian multiforms. The concept of Lagrangian forms themselves is not new, in fact it can be traced back to the work of Cartan[21] and Lepage[63], but there the role of the Lagrangian is that of a volume form, which through the Euler-Lagrange equations produces one equation per component of the dependent variable. The Lagrangian forms presented in this thesis are functions which can be evaluated on elementary plaquettes or cubes, in the case of 2- or 3-dimensional equations respectively, although the theory can be extended to equations in any number of dimensions. The important observation about these Lagrangian forms is that they are *closed*, i.e., they obey a *closure relation*, which prompted the proposal of a new variational principle, through which we can now derive any of the equations in the infinite family.

After an introductory chapter outlining the main results relevant to the work in this thesis in the areas of (discrete and continuous) integrable systems, variational principles and Lagrangian forms, new results are first presented in Chapter 2. This chapter is concerned with 2-dimensional systems on quad-graphs, in particular those given in the classification in [5], the so-called *ABS list*. Although this class of equations is subject to many restrictions (e.g. affine linearity, D_4 symmetry), it provides a useful number of equations to study. In this thesis are presented 3-point Lagrangians for systems in the ABS list, which are antisymmetric with respect to the interchange of the two lattice directions. The important observation was that these Lagrangians obey a closure relation on solutions to the equation, which, together with the antisymmetry, allows them to be interpreted as closed Lagrangian 2-forms. Choosing a surface consisting of a connected configuration of elementary plaquettes in a multidimensional lattice, an action on the surface can be defined by summing up the Lagrangian contribution from each elementary plaquette in the surface. The fact that the closure relation holding is equivalent to the action being independent of the surface on which it is defined led us to propose a new variational principle for multidimensionally consistent systems, which now not only involves variations with respect to the dependent variable, but also the geometry of the space of independent variables.

Proving that the closure relation holds by direct computation for some of the more difficult cases in the ABS list, namely H3 and $Q3|_{\delta=0}$, involves extensive use of the dilogarithm identities, in particular a 5-term identity. This suggests there may be connections with the pentagon equation, which appears in connection with topological quantum field theory [75, 49, 50]. This is certainly worthy of further investigation to find out what is the underlying structure behind these relations.

This variational principle is not a principle that applies solely to equations in the ABS list. Indeed, in Chapter 3 it is shown that 2-dimensional multi-component systems, those in the Gel'fand-Dikii hierarchy, have Lagrangians which obey a closure relation and hence fit in with this scheme. The Gel'fand-Dikii hierarchy is a particularly nice example of an integrable system since it has a universal Lagrangian structure, i.e., one can write down a Lagrangian for an arbitrary member of the hierarchy. It also contains the discrete potential KdV and discrete Boussinesq equations, both of which are important systems. A Lagrangian for the lattice modified Gel'fand-Dikii hierarchy is not yet known; a Lagrangian for the discrete modified KdV equation exists, and obeys a closure relation, but as yet not even a Lagrangian for the lattice modified Boussinesq equation is known. It would be an obvious next step to find a Lagrangian for the modified system and see if it can be shown to obey a closure relation (and hence fit in with the new variational principle), and also to see whether a closure relation holds for the continuous Gel'fand-Dikii hierarchy. The generating PDE for the Boussinesq hierarchy along with its Lagrangian appeared in [116, 117]. It can be expected that the Lagrangian structure would also obey a continuous analogue of the closure relation, in the same manner as the generating partial differential equation (PDE) for the KdV hierarchy [88] was shown to obey a continuous analogue of the closure relation in Chapter 5.

In Chapter 4 is presented a Lagrangian for the discrete bilinear KP equation, which is a 3-dimensional integrable system. The Lagrangian for this system is shown to obey a 4-dimensional closure relation, thereby extending the variational principle to higher dimensional systems. Once we are out of the realm of 2-dimensional systems it becomes more difficult to find examples of integrable systems. A Lagrangian for the lattice KP equation (1.9) still eludes us; it would be wonderful to find such a Lagrangian and to verify that it, too, satisfies the 4-dimensional closure relation. It is expected that a classification of equations in 3 dimensions, subject to some conditions as in the 2-dimensional classification, will soon appear [7]. No new examples are expected, at least on the level of single-component equations, but it would form an excellent testing-ground for our theory.

Chapter 5 provides a continuous analogue to the theory presented in the previous chapters, using the examples of generating PDEs. It is shown that Lagrangians for these equations also obey a closure relation, which is related to the surface independence of the action via Stokes' theorem.

6.2 Recent developments

There are two lines of development which have been progressing during the course of writing this thesis: one is in collaboration with P. Xenitidis and F.W. Nijhoff [125], the other with S. Yoo-Kong and F.W. Nijhoff [126]. Here follows a short summary of these results, due to appear in the near future.

6.2.1 Universal Lagrangian structure of affine linear quadrilateral equations

Recently we have gained a deeper insight into Lagrangian structures for affine-linear quadrilateral equations (such as those in the ABS list), and have now developed a universal Lagrangian structure [125], which can be systematically derived from some basic assumptions. When specializing to the ABS list, it can be shown that these Lagrangians obey the closure relation, so this gives a more generic description of Lagrangian structures for affine-linear quadrilateral lattices.

The equations under consideration are of a similar form to (1.1) of Chapter 2; they are of the form

$$Q(u, u_i, u_j, u_{ij}) = 0, \tag{2.1}$$

where u depends on an arbitrary number of independent discrete variables n_i, and, as in Chapter 2, u_j denotes u shifted in the n_j-direction. We assume a slightly more relaxed set of conditions than in the classification of [5]: we assume that

1. Q is affine linear and depends explicitly on all of the arguments;
2. Q is irreducible (i.e., it cannot be factorized into a product of two polynomials);
3. Q possesses Kleinian symmetry, i.e.,

$$Q(u, u_i, u_j, u_{ij}) \;=\; \epsilon\, Q(u_i, u, u_{ij}, u_j) \;=\; \sigma\, Q(u_j, u_{ij}, u, u_i)\,, \tag{2.2}$$

   where $\epsilon = \pm 1$ and $\sigma = \pm 1$.

We can define a 3-point Lagrangian for (2.1) by

$$\begin{aligned}\mathcal{L}(u, u_i, u_j) \;&=\; \int_{u^0}^{u}\!\int_{u_i^0}^{u_i} \frac{\mathrm{d}s\ \mathrm{d}t}{h(s,t)} + \int_{u^0}^{u}\!\int_{u_j^0}^{u_j} \frac{\mathrm{d}s\ \mathrm{d}t}{\bar{h}(s,t)} + \int_{u_i^0}^{u_i}\!\int_{u_j^0}^{u_j} \frac{\mathrm{d}s\ \mathrm{d}t}{H(s,t)} \\ &\quad - \int_{u_i^0}^{u_i} \mathrm{d}s \int_{u_j^0}^{w(s,u^0,u_{ij}^0)} \frac{\mathrm{d}t}{H(s,t)} - \int_{u_j^0}^{u_j} \mathrm{d}t \int_{u_i^0}^{z(t,u_{ij}^0,u^0)} \frac{\mathrm{d}s}{H(s,t)}\,,\end{aligned} \tag{2.3}$$

where u^0 and its shifts are arbitrary functions, w, z are solutions of the equations $Q(u^0, s, w, u^0_{ij}) = 0$, $Q(u^0, z, t, u^0_{ij}) = 0$ respectively, and h, H are symmetric bi-quadratic polynomials assigned to the edges and diagonals of the quadrilateral respectively:

$$\begin{aligned} h(u,u_i) &:= \mathcal{D}_{u_j,u_{ij}}(Q)\,, \quad h(u_j,u_{ij}) := \mathcal{D}_{u,u_i}(Q)\,, \\ \bar{h}(u,u_j) &:= \mathcal{D}_{u_i,u_{ij}}(Q)\,, \quad \bar{h}(u_i,u_{ij}) := \mathcal{D}_{u,u_j}(Q)\,, \\ H(u_i,u_j) &:= \mathcal{D}_{u,u_{ij}}(Q)\,, \quad H(u,u_{ij}) := \mathcal{D}_{u_i,u_j}(Q)\,. \end{aligned} \tag{2.4}$$

Here $\mathcal{D}$ is the following double-sided Wronskian operator

$$\mathcal{D}_{x,y}(f) := \begin{vmatrix} f & \partial_x f \\ \partial_y f & \partial_x \partial_y f \end{vmatrix}\,. \tag{2.5}$$

Further specializing to equations in the ABS list, making slight alterations to (2.3) it can be shown that the Lagrangian satisfies the closure relation on solutions to the ABS equation. Thus there is a Lagrangian structure for the generic member of the ABS list, which obeys the closure relation and so can be interpreted as a Lagrangian 2-form.

Also presented in [125] is the Lagrangian structure for the system of partial differential equations associated with the ABS list (the system derived in [120]), and proof that this obeys the continuous closure relation.

6.2.2 Lagrangian 1-form for the discrete-time Calogero-Moser system

So far in this thesis we have been concerned with Lagrangian 2-forms and 3-forms, so it is desirable to have an example of a Lagrangian 1-form. In [126] we present the Lagrangian 1-form structure for the discrete-time Caloger-Moser system; in the rational case the Lagrangian follows directly from the Lax equation, and that it satisfies a closure relation can be directly inferred from the compatibility of the Lax matrices.

The Calogero-Moser (CM) model [18, 19, 79] is a one-dimensional many-particle system with long-range interactions, originally a continuous system with pairwise inverse square interactions that has been generalized to the elliptic case and to a relativistic model (the Ruijsenaars-Schneider model [108, 109]). The model has been extensively studied in both the classical and quantum cases [101, 102], and is integrable on both of these levels. An interesting point to note is that there is a connection between the CM model and the KP system through pole solutions of the KP system [55, 122].

A discrete-time version of the (rational) CM model was presented by Nijhoff and Pang in [89], where it was also shown to be integrable. With dependent variables $x_1, x_2, \dots, x_N$ each depending on a discrete independent variable n, the equations take the form

$$\sum_{j=1}^{N}\left(\frac{1}{x_i-\widetilde{x}_j}+\frac{1}{x_i-\underset{\sim}{x}_j}\right)-2\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x_i-x_j}=0, \tag{2.6}$$

where $i=1,...,N$, $\widetilde{x}$ denotes x shifted forward in the n-direction, and $\underset{\sim}{x}$ denotes x shifted backward in the n-direction.

Let $\boldsymbol{x}=(x_1,x_2,\dots,x_N)$. Equation (2.6) can be computed from the variation of a discrete action given in [89] as

$$\mathcal{S}_p=\sum_n\mathcal{L}_p(\boldsymbol{x},\widetilde{\boldsymbol{x}})=\sum_n\left(-\sum_{i,j=1}^{N}\log|x_i-\widetilde{x}_j|+\sum_{i\neq j}^{N}\log|x_i-x_j|\right), \tag{2.7}$$

where the sum over n represents the sum over all discrete-time " $\widetilde{\ }$ " iterates. Remarkably, the action (2.7) can be obtained by considering the infinite product of the Lax matrix $\boldsymbol{M}$ in the following way

$$\mathcal{S}_p=\log|\det(...\boldsymbol{M}_{n-2}\boldsymbol{M}_{n-1}\boldsymbol{M}_n\boldsymbol{M}_{n+1}\boldsymbol{M}_{n+2}....)|\,. \tag{2.8}$$

Introducing a second lattice direction associated with a shift " $\widehat{\ }$," a parameter q and a Lax matrix N, the compatibility relation $\widehat{\boldsymbol{M}}\boldsymbol{N}=\widetilde{\boldsymbol{N}}\boldsymbol{M}$ can be rewritten as

$$\log\left|\det(\widehat{\boldsymbol{M}})\right|+\log|\det(\boldsymbol{N})|-\log\left|\det(\widetilde{\boldsymbol{N}})\right|-\log|\det(\boldsymbol{M})|=0. \tag{2.9}$$

The closure relation of the Lagrangian $\mathcal{L}_p(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$ in (2.7) takes the form

$$\widehat{\mathcal{L}_p(\boldsymbol{x}, \widetilde{\boldsymbol{x}})} - \mathcal{L}_p(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) - \widetilde{\mathcal{L}_q(\boldsymbol{x}, \widehat{\boldsymbol{x}})} + \mathcal{L}_q(\boldsymbol{x}, \widehat{\boldsymbol{x}}) = 0, \tag{2.10}$$

which in this case is equivalent to (2.9). Thus the compatibility of the Lax matrices implies the closure relation.

6.3 Future directions

An interesting question is what is the quantum analogue of this formalism, e.g. in the context of a path integral framework. The path integral approach to quantization involves the exponent of the action [24, 30, 31], and for multidimensionally consistent systems we now have actions which are invariant of the choice of surface on which they are defined. It has yet to be seen what significance this will have for the path integral formalism.

Another interesting question is whether it would be possible to classify integrable discrete and continuous systems on the level of the Lagrangians using the closure property. That the closure relation places such restrictions on the Lagrangian could be of relevance in the inverse problem of Lagrangian mechanics, a field of study which dates back to the 1880s [41], cf. [119] for a review. The new variational principle can be seen as a scheme which specifies not just the equations, but also the Lagrangians, since it necessarily picks out those which obey the closure relation. This would obviously be a major breakthrough in the area, and an ambitious undertaking.

Appendix A

The dilogarithm function

Many of the Lagrangians for systems covered in this thesis involve the dilogarithm function, and many computations rely on identities for this function. It is defined as follows:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-z)}{z} dz. \tag{0.1}$$

The function actually is useful in many areas of physics, such as quantum electrodynamics (e.g. vacuum polarization) and electrical network problems. It comes from multiple integration of certain rational forms of more than one variable [64], for example, integrating $\frac{a}{1-axy}$ for constant a with respect to x, then with respect to y, gives $\mathrm{Li}_2(axy)$. There are also connections with algebraic K-theory, representation theory of infinite dimensional algebras, and combinatorics [52].

Many functional relations involving dilogarithms are given in the book by Lewin [64], and in the review paper of Kirillov [52], which also covers some of the quantum analogues. The pivotal functional relation is the five-term identity as it appears in [64] is

$$\begin{aligned} \mathrm{Li}_2\left(\frac{x}{1-y}\frac{y}{1-x}\right) &= \mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) \\ &\quad - \ln(1-x)\ln(1-y), \quad x, y < 1. \end{aligned} \tag{0.2}$$

For the computations needed to prove that the closure relation holds for H3 and for $Q3|_{\delta=0}$ it is more convenient to write (0.2) in a slightly different form:

Identity I:

$$\begin{aligned}\mathrm{Li}_2(s)+\mathrm{Li}_2(t)-\mathrm{Li}_2(st) &= \mathrm{Li}_2\left(\frac{s-st}{1-st}\right)+\mathrm{Li}_2\left(\frac{t-st}{1-st}\right)\\ &\quad+\ln\left(\frac{1-s}{1-st}\right)\ln\left(\frac{1-t}{1-st}\right),\end{aligned} \tag{0.3}$$

for $s, t, st \neq 1$, is true up to imaginary constant terms.

Proof: It is a simple matter to prove this relation using differentiation. Firstly, differentiating the left-hand side with respect to s gives

$$\begin{aligned}\frac{\partial}{\partial s}\left\{\mathrm{Li}_2(s)+\mathrm{Li}_2(t)-\mathrm{Li}_2(st)\right\} &= -\frac{\ln(1-s)}{s}+t\frac{\ln(1-st)}{st}\\ &= -\frac{1}{s}\ln\left(\frac{1-s}{1-st}\right),\end{aligned} \tag{0.4}$$

and differentiating the right-hand side with respect to s gives

$$\begin{aligned}&\frac{\partial}{\partial s}\left\{\mathrm{Li}_2\left(\frac{s-st}{1-st}\right)+\mathrm{Li}_2\left(\frac{t-st}{1-st}\right)+\ln\left(\frac{1-s}{1-st}\right)\ln\left(\frac{1-t}{1-st}\right)\right\}\\ &= -\left\{\frac{1-t}{s-st}+\frac{t}{1-st}\right\}\ln\left(1-\frac{s-st}{1-st}\right)\\ &\quad -\left\{\frac{-t}{t-st}+\frac{t}{1-st}\right\}\ln\left(1-\frac{t-st}{1-st}\right)\\ &\quad +\left\{\frac{-1}{1-s}+\frac{t}{1-st}\right\}\ln\left(\frac{1-t}{1-st}\right)+\left\{\frac{t}{1-st}\right\}\ln\left(\frac{1-s}{1-st}\right)\\ &= -\frac{1}{s}\ln\left(\frac{1-s}{1-st}\right).\end{aligned} \tag{0.5}$$

Clearly (0.4) is equal to (0.5). Similarly, differentiating the left-hand side of (0.3) with respect to t gives

$$\frac{\partial}{\partial t}\left\{\mathrm{Li}_2(s)+\mathrm{Li}_2(t)-\mathrm{Li}_2(st)\right\} = -\frac{1}{t}\ln(1-t)+\frac{1}{t}\ln(1-st), \tag{0.6}$$

and differentiating the right-hand side with respect to t gives

$$\begin{aligned}&\frac{\partial}{\partial s}\left\{\mathrm{Li}_2\left(\frac{s-st}{1-st}\right)+\mathrm{Li}_2\left(\frac{t-st}{1-st}\right)+\ln\left(\frac{1-s}{1-st}\right)\ln\left(\frac{1-t}{1-st}\right)\right\}\\ &= -\frac{1}{t}\ln\left(\frac{1-t}{1-st}\right),\end{aligned} \tag{0.7}$$

and again (0.6) is equal to (0.7). Therefore relation (0.3) holds up to a constant term. By letting $t = 0$, and using the fact that $\mathrm{Li}_2(0) = 0$, we easily see that this constant term must be zero, at least up to imaginary terms coming from the multi-valuedness of the logarithm function.

An additional two identities needed are the following.

Identity II:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{1}{2}\left(\ln(-x)\right)^2 - \frac{\pi^2}{6}, \tag{0.8}$$

for $x \neq 0$.

Proof: By differentiating the left-hand side we have

$$\begin{aligned} \frac{d}{dx}\left\{\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right)\right\} &= -\frac{\ln(1-x)}{x} + \frac{1}{x}\ln\left(1 - \frac{1}{x}\right) \\ &= -\frac{\ln(1-x)}{x} + \frac{1}{x}\ln\left(\frac{1-x}{-x}\right), \end{aligned} \tag{0.9}$$

and differentiating the right-hand side we have

$$\frac{d}{dx}\left\{-\frac{1}{2}\left(\ln(-x)\right)^2 - \frac{\pi^2}{6}\right\} = \frac{\ln(-x)}{-x}. \tag{0.10}$$

Clearly these are equal, and so (0.8) holds up to a constant term. Letting $x = 1$, and using the fact that $\mathrm{Li}_2(1) = \pi^2/6$, we see that this constant term must be zero.

Identity III:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\left(\ln(1-x)\right)^2, \tag{0.11}$$

for $x \neq 1$, is true up to imaginary terms.

Proof: By differentiating the left-hand side we have

$$\begin{aligned} \frac{d}{dx}\left\{\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right)\right\} &= -\frac{\ln(1-x)}{x} - \left\{\frac{1}{x} - \frac{1}{x-1}\right\}\ln\left(1 - \frac{x}{x-1}\right) \\ &= -\frac{\ln(1-x)}{x} + \frac{1}{x(1-x)}\ln\left(\frac{1}{1-x}\right), \end{aligned} \tag{0.12}$$

and differentiating the right-hand side we have

$$\frac{d}{dx}\left\{-\frac{1}{2}\left(\ln(1-x)\right)^2\right\} = \frac{1}{1-x}\ln(1-x). \tag{0.13}$$

Clearly these are equal, and so (0.11) holds up to a constant term. Letting $x = 0$, we find that this constant term must be zero, at least up to imaginary terms coming from the multivaluedness of the logarithm function.

Equation (0.8) holds regardless of whether the arguments are positive or negative. Equations (0.2) and (0.11) require additional imaginary terms depending on the sign of the arguments; these however cancel out in the course of the closure relation calculations, so for the purposes in this thesis they are irrelevant.

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