Stochastic Evolution Equations in Banach Spaces and Applications to the Heath-Jarrow-Morton-Musiela Equation

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Abstract

The aim of this thesis is threefold. Firstly, we study the stochastic evolution equations (driven by an infinite dimensional cylindrical Wiener process) in a class of Banach spaces satisfying the so-called H-condition. In particular, we deal with the questions of the existence and uniqueness of solutions for such stochastic evolution equations. Moreover, we analyse the Markov property of the solution.

Secondly, we apply the abstract results obtained in the first part to the so-called Heath-Jarrow-Morton-Musiela (HJMM) equation. In particular, we prove the existence and uniqueness of solutions to the HJMM equation in a large class of function spaces, such as the weighted Lebesgue and Sobolev spaces.

Thirdly, we study the ergodic properties of the solution to the HJMM equation. In particular, we analyse the Markov property of the solution and we find a sufficient condition for the existence and uniqueness of an invariant measure for the Markov semigroup associated to the HJMM equation (when the coefficients are time independent) in the weighted Lebesgue spaces.

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Author's declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Introduction

The theory of stochastic integration in the class of the so called M-type 2 Banach spaces were initiated independently by Neidhardt [39] in 1978 and Dettweiler[22] in 1983. Using this stochastic integration theory, Brzeźniak developed the theory of stochastic evolution equations in M-type 2 Banach spaces, see [7] and [8]. In [7], Brzeźniak studied linear stochastic evolution equations (with the drift being an infinitesimal generator of an analytic semigroup and the coefficients of the stochastic part being linear operators) driven by a d-dimensional Wiener process. He proved the existence and uniqueness of solutions for such equations in some real interpolation spaces. In [8], he continued the line of research originated in [7]. He considered the stochastic evolution equations (with the linear part of the drift being an infinitesimal generator of an analytic semigroup and the coefficients satisfying Lipschitz conditions) driven by an infinite dimensional Wiener process. He proved the existence and uniqueness of solutions for corresponding equations in M-type 2 Banach spaces.

One of the aims of this thesis is to study the stochastic evolution equations driven by an infinite dimensional cylindrical Wiener process in Banach spaces. We continue the way of research originated in [7] and [8], however, we consider different assumptions on the coefficients. For example, we assume that the linear part of the drift is an infinitesimal generator of a C_0-semigroup. In particular, the corresponding semigroup has no smoothing properties and in consequence we have obtained different results. For instance, we prove the existence and uniqueness of continuous solutions for such equations in Banach spaces satisfying the so-called H-condition (these are spaces for which stochastic integral exists, see [10]), under stronger assumptions on the coefficients.

The notion of invariant measures is an important topic in the theory of stochastic dynamical systems. Many authors, see for instance [24], [25] and [47], have studied the questions of the existence and uniqueness of an invariant measure for the stochastic evolution equations in Hilbert spaces. In particular, under different conditions on the coefficients, they proved the existence and uniqueness of an invariant measure for stochastic evolution equations in Hilbert spaces. Recently, Brzeźniak, Long and Simão [15] paid attention to the theory of invariant measures for stochastic evolution equations in Banach spaces. They found some sufficient conditions about the existence and uniqueness of an invariant measure for stochastic evolution equations in Banach spaces. In this thesis, we introduce and use these conditions.

It is now a widely accepted fact that mathematics has a lot of interesting applications in finance. One of these applications appears to be the theory of stochastic evolution equations. In particular, the existence and uniqueness of solutions for some stochastic evolution equations in finance and their ergodic properties. The so called HJM model proposed by Heath-Jarrow-Morton (HJM) [36] is an example of a stochastic evolution equations in finance. This model contains the dynamics of interest rate, the so called forward rate. The HJM model was studied in the Hilbert spaces by many authors, see for instance [6], [26], [28], [31], [34], [35], [41],[55] and [59]. These authors proved the existence and uniqueness of solutions to the HJM model in some appropriate Hilbert spaces. Also some of these authors analysed ergodic properties of the solutions.

Another aim of this thesis is to apply abstract results from Chapter 3 to the HJM model. Thus, we focus on the questions of the existence and uniqueness of solutions to the HJM model in Banach spaces such as the weighted Lebesgue and Sobolev spaces. Also the study of the existence and uniqueness of an invariant measure for the HJM model in the weighted Lebesgue spaces is one of our aims.

Let us now describe briefly the content of this thesis. In Chapter 2, we provide all the necessary preliminaries so as to make it a well contained thesis. These preliminaries contain some definitions and results (obviously presented without proofs) about linear operators, semigroup theory, random variables and stochastic processes with values in Banach spaces, probability measures on Banach spaces, and stochastic integral in Banach spaces.

In Chapter 3, we give a brief introduction to some basic definitions in the theory of interest rates. We also mention the **arbitrage pricing theory**, but not in detail, as for this purpose, there are many excellent text books such as [3], [4], [42] and [45] in which details can be found. Next we introduce the dynamics of the forward rate processes proposed by Heath-Jarrow-Morton [36] driven by a Wiener process on a (possible infinite dimensional) Hilbert space. Using the Musiela parametrization, we construct stochastic partial differential equation, which is called the Heath-Jarrow-Morton-Musiela (HJMM) Equation, for the dynamics of forward rate processes. Finally, we present some known results about the existence and uniqueness of mild solutions for corresponding equation in Hilbert spaces, and ergodic properties of the solution.

In Chapter 4, we study the stochastic evolution equations (with the linear part of the drift being only an infinitesimal generator of a C_0-semigroup and the coefficients satisfying Lipschitz conditions) in the class of Banach spaces satisfying the so-called H-condition. Firstly, we prove the existence and uniqueness of solutions for corresponding equations with globally Lipschitz coefficients. Next using the previous result and approximation method, we prove the existence and uniqueness of solutions for corresponding equations with locally Lipschitz coefficients. We also analyse the Markov property of the solution. Finally, we introduce theorems proposed by [15] about the existence and uniqueness of an invariant measure for corresponding equations with time independent coefficients.

In Chapter 5, we apply the abstract results from the previous chapter to the HJMM equation. In particular, we prove the existence and uniqueness of solutions to the HJMM equation in the weighted Lebesgue and Sobolev spaces respectively. We also find a sufficient condition for the existence and uniqueness of an invariant measure for the Markov semigroup associated to the HJMM equation in the weighted Lebesgue spaces. An important feature of our results is that we are able to consider the HJMM equation driven by a cylindrical Wiener process on a (possibly infinite dimensional) Hilbert space. For this purpose, we use the characterizations of the so-called γ-radonifying operators from a Hilbert space to an L^p space and a Sobolev space $W^{1,p}$ found recently by Brzeźniak and Peszat in [10] and [12].

In Chapter 6, we consider the weighted Banach spaces $H^{1,p}_{\mathrm{w}}$, $p \geq 1$, which are natural generalization of the Hilbert space $H^{1,2}_{\mathrm{w}}$ used by Filipović [26] and prove some useful properties of them. This allows us to apply the abstract results from Chapter 4 to prove the existence of a unique continuous solution to the HJMM equation (driven by a standard d-dimensional Wiener process) in the spaces $H^{1,p}_{\mathrm{w}}$, $p \geq 2$.

In Chapter 7, we consider the fractional Sobolev spaces of 2π-periodic functions and prove some useful properties of them. Then using these properties, we apply the abstract results from Chapter 4 to prove the existence of a unique continuous solution to the

HJMM equation (driven by a standard d-dimensional Wiener process) in the fractional Sobolev spaces.

CHAPTER 2

PRELIMINARIES

2.1 Linear Bounded Operators

Throughout this section, we assume that X and Y are real vector spaces, in particular, real normed spaces endowed with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively.

2.1.1 Definitions and Properties

Definition 2.1. *A map A from X into Y is called* ***a linear operator*** *if $\mathscr{D}(A)$ is a subspace of X and for all $x_1, x_2 \in \mathscr{D}(A)$ and $\alpha, \beta \in \mathbb{R}$,*

$$A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2. \tag{2.1}$$

For a linear operator $A : X \to Y$, the notation Ax is usually used instead of $A(x)$. The set $\{y \in Y : \exists\, x \in \mathscr{D}(A) : y = Ax\}$ is called **the range of** A and denoted by $\mathscr{R}(A)$.

Definition 2.2. *A linear operator $A : X \to Y$ is called* ***bounded*** *if there exists a constant $C > 0$ such that for every $x \in \mathscr{D}(A)$,*

$$\|Ax\|_Y \le C\|x\|_X. \tag{2.2}$$

Definition 2.3. *A linear operator $A : X \to Y$, where $\mathscr{D}(A) = X$, is called* ***an isomorphism*** *if it is bijective and bounded, and its inverse $A^{-1} : Y \to X$ is bounded. If such a linear operator A exists, then $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are said to be* ***isomorphic spaces****.*

Definition 2.4. *A linear operator $A : X \to Y$ is called* ***an isometry or distance preserving*** *if for all $x \in X$,*

$$\|Ax\|_Y = \|x\|_X. \tag{2.3}$$

If A is, in addition, bijective, then A is called ***an isometric isomorphism****. If such a linear operator A exists, then $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are said to be* ***isometric spaces****.*

There is a relation given by the following theorem between two isometric spaces. We use this theorem to prove that some spaces we used in next chapters are Banach spaces. The proof of this theorem is obvious.

Theorem 2.1. *If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometric spaces, then $(X, \|\cdot\|_X)$ is a Banach space if and only if $(Y, \|\cdot\|_Y)$ is a Banach space.*

Definition 2.5. *If there exits an injective continuous map A from X into Y, then X is said to be* ***embeddable in Y****. Such a map A is called* ***the embedding****.*

Definition 2.6. *Assume that $X \subseteq Y$. Then X is called* ***continuously embedded*** *in Y if the inclusion (identity) map $i : X \to Y$ is continuous. In this case, we denote this embedding symbolically by $X \hookrightarrow Y$ and the map i is called* ***embedding map****.*

Theorem 2.2. *[38] If X is finite dimensional, then every linear operator from X into Y is bounded.*

Lemma 2.1. *[38] For the space of all bounded linear operators $A : X \to Y$, the following mapping*

$$\|A\| = \sup_{x \in \mathscr{D}(A) : \|x\|_X = 1} \|Ax\|_Y \tag{2.4}$$

defines a norm.

Theorem 2.3. *[38] Let $A : X \to Y$ be a linear operator. Then*
(i) A is continuous on $\mathscr{D}(A)$ iff A is bounded.
(ii) If A is continuous at a single point of $\mathscr{D}(A)$, then it is continuous on $\mathscr{D}(A)$.

The space of all bounded linear operators $A : X \to Y$, where $\mathscr{D}(A) = X$, is denoted by $\mathscr{L}(X,Y)$ (or $\mathscr{L}(X)$ if $Y = X$). By Lemma 2.1, $\mathscr{L}(X,Y)$ is a normed space with the norm

$$\|A\|_{\mathscr{L}(X,Y)} = \sup_{x \in X : \|x\|_X = 1} \|Ax\|_Y, \quad A \in \mathscr{L}(X,Y). \tag{2.5}$$

Theorem 2.4. *[38] If Y is a Banach space, then $\mathscr{L}(X,Y)$ is a Banach space with norm (2.5).*

Proposition 2.1. *[38] Let Z be a real normed space with a norm $\|\cdot\|_Z$. Assume that $A_1 : X \to Y$ and $A_2 : Y \to Z$ are bounded linear operators. Then the composition $A_2 \circ A_1$ is a bounded linear operator from X into Z. Moreover,*

$$\|A_2 \circ A_1\|_{\mathscr{L}(X,Z)} \leq \|A_2\|_{\mathscr{L}(Y,Z)} \|A_1\|_{\mathscr{L}(X,Y)}. \tag{2.6}$$

Definition 2.7. *A linear operator f from X into $\mathbb{R}$ is called **linear functional**.*

Definition 2.8. *The space of all bounded linear functionals $f : X \to \mathbb{R}$ is called **the dual space of** X and is denoted by X^*, i.e. $X^* = \mathscr{L}(X,\mathbb{R})$.*

Remark 2.1. *By Theorem 2.4, X^* is a Banach space with the norm*

$$\|x^*\|_{\mathscr{L}(X,\mathbb{R})} = \sup_{x \in X : \|x\|_X = 1} |x^*(x)|, \quad x^* \in X^*, \tag{2.7}$$

where $|\cdot|$ is the euclidean norm in $\mathbb{R}$.

2.1.2 Fréchet Derivative

Assume that Z is an open subset of X.

Definition 2.9. *A map $A : Z \to Y$ is said to be **Fréchet differentiable** at $z \in Z$ if there exists a linear operator $L \in \mathscr{L}(X,Y)$ such that*

$$\lim_{h \to 0} \frac{\|A(z+h) - Az - Lh\|_Y}{\|h\|_X} = 0.$$

*This operator L is called **the Fréchet derivative of** A at $z \in Z$ and denoted by $A'(z)$ or $d_z A$. If A is the Fréchet differentiable at each $z \in Z$, then A is said to be **Fréchet differentiable** on Z.*

Remark 2.2. *If A is Fréchet differentiable on Z, then it is continuous on Z.*

Definition 2.10. *Let $A : Z \to Y$ be a Fréchet differentiable map on Z. Then A is said to be **twice Fréchet differentiable** at $z \in Z$ if there exists a bounded bilinear operator $B : X \times X \to Y$, i.e $B \in \mathscr{L}(X \times X, Y)$ such that*

$$\lim_{h_2 \to 0} \frac{\|d_{z+h_2} A h_1 - d_z A h_1 - B(h_1,h_2)\|_Y}{\|h_2\|_X} = 0.$$

*This operator B is called **the second Fréchet derivative of** A at $z \in Z$, and denoted by $A''(z)$ or $d_z^2 A$. If A has the second Fréchet derivative at each $z \in Z$, then A is said to be **twice Fréchet differentiable** on Z.*

Remark 2.3. *If A is twice Fréchet differentiable on Z, then the map $A' : Z \to \mathscr{L}(X,Y)$ is continuous on Z.*

Definition 2.11. *Assume that $A : Z \to Y$ is a Fréchet differentiable map on Z. If the map $A' : Z \to \mathscr{L}(X,Y)$ is continuous on Z, then A is said to be **of** C^1 **class**.*

Definition 2.12. *Assume that $A : Z \to Y$ is a twice Fréchet differentiable map on Z. If the map $A'' : Z \to \mathscr{L}(X \times X, Y)$ is continuous on Z, then A is said to be **of** C^2 **class**.*

Lemma 2.2. *[19] If $A : Z \to Y$ is a linear operator, then A is twice Fréchet differentiable on Z such that $A'(z) = A$ and $A''(z) = 0$ for every $z \in Z$. Moreover, A is of C^2 class.*

2.2 C_0-Semigroup

Throughout this section, we assume that X is a Banach space endowed with a norm $\|\cdot\|_X$.

2.2.1 Definitions and Properties

Definition 2.13. *A function $S : [0,\infty) \ni t \mapsto S(t) \in \mathscr{L}(X)$, which is usually denoted by $S = \{S(t)\}_{t\geq 0}$, is called **a semigroup** on X if*
(i) $S(0) = I$, where I is the identity operator on X,
(ii) for all $t, s \geq 0$,

$$S(t+s) = S(t)S(s),$$

where $S(t)S(s)$ denotes the composition of the operators $S(t)$ and $S(s)$.

Definition 2.14. *Let S be a semigroup on X. If*

$$\lim_{t\to 0} \|S(t) - I\|_{\mathscr{L}(X)} = 0, \tag{2.8}$$

*then S is called **uniformly continuous**.*

Definition 2.15. *Let S be a semigroup on X. A linear operator A defined by*

$$\mathscr{D}(A) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \ \text{ exists} \right\} \tag{2.9}$$

and

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t} = \frac{dS(t)x}{dt}\Big|_{t=0}, \quad x \in \mathscr{D}(A) \tag{2.10}$$

*is called **the infinitesimal generator** of the semigroup S.*

Theorem 2.5. *[44] A linear operator A is the infinitesimal generator of a uniformly continuous semigroup iff A is a bounded linear operator.*

Theorem 2.6. *[44] Let S_1 and S_2 be uniformly continuous semigroups on X. If*

$$\lim_{t\to 0}\frac{S_1(t)-I}{t}=\lim_{t\to 0}\frac{S_2(t)-I}{t}, \tag{2.11}$$

then $S_1(t)=S_2(t)$ for each $t\geq 0$.

Theorem 2.7. *[44] Let S be a uniformly continuous semigroup on X. Then*
(i) there exists a constant $\omega\geq 0$ such that

$$\|S(t)\|_{\mathscr{L}(X)}\leq e^{\omega t},\quad t\geq 0,$$

(ii) there exists a unique bounded linear operator A such that

$$S(t)=e^{tA}=\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k,$$

(iii) the operator A in (ii) is the infinitesimal generator of S,
(iv) $S:[0,\infty)\ni t\mapsto S(t)\in\mathscr{L}(X)$ is differentiable in the norm of $\mathscr{L}(X)$ such that

$$\frac{dS(t)}{dt}=AS(t)=S(t)A.$$

Definition 2.16. *A semigroup S on X is called a C_0-**semigroup** (or **strongly continuous semigroup**) iff for each $x\in X$,*

$$\lim_{t\to 0}\|S(t)x-x\|_X=0. \tag{2.12}$$

Theorem 2.8. *[44] If S be a C_0-semigroup on X, then there exist constants $M\geq 1$ and $\beta\geq 0$ such that*

$$\|S(t)\|_{\mathscr{L}(X)}\leq Me^{\beta t},\quad t\geq 0. \tag{2.13}$$

Corollary 2.1. *[44] If S is a C_0-semigroup on X, then for each $x\in X$, the function $S(\cdot)(x):[0,\infty)\ni t\mapsto S(t)(x)\in X$ is continuous on $[0,\infty)$.*

Theorem 2.9. *[44] Let S be a C_0-semigroup on X with the infinitesimal generator A. Then*
(i) for each $x\in X$,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}S(s)x\,ds=S(t)x, \tag{2.14}$$

(ii) for each $x \in X$, $\int_0^t S(s)x ds \in \mathscr{D}(A)$ and

$$A\left(\int_0^t S(s)x ds\right) = S(t)x - x, \tag{2.15}$$

(iii) for each $x \in \mathscr{D}(A)$, $S(t)x \in \mathscr{D}(A)$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \tag{2.16}$$

(iv) for each $x \in \mathscr{D}(A)$,

$$S(t)x - S(s)x = \int_s^t S(\tau)Ax d\tau = \int_s^t AS(\tau)x d\tau. \tag{2.17}$$

Corollary 2.2. *[44] If A is the infinitesimal generator of a C_0 semigroup S on X, then $\mathscr{D}(A)$ is dense in X and A is a closed linear operator.*

Theorem 2.10. *[44] Let S_1 and S_2 be C_0-semigroups on X with the infinitesimal generators A and B respectively. If $A = B$, then $S_1(t) = S_2(t)$ for each $t \geq 0$.*

Definition 2.17. *A C_0-semigroup S on X is called **uniformly bounded** if there exists a constant $M \geq 1$ such that for each $t \geq 0$,*

$$\|S(t)\|_{\mathscr{L}(X)} \leq M. \tag{2.18}$$

Definition 2.18. *A C_0-semigroup S on X is called **contraction** if for each $t \geq 0$,*

$$\|S(t)\|_{\mathscr{L}(X)} \leq 1. \tag{2.19}$$

Definition 2.19. *A C_0-semigroup S on X is said to be **contraction type** if there exists $\beta \in \mathbb{R}$ such that*

$$\|S(t)\|_{\mathscr{L}(X)} \leq e^{\beta t}, \quad t \geq 0. \tag{2.20}$$

Remark 2.4. *If S is a contraction type C_0-semigroup on X with the infinitesimal generator A, then the family $T = \{T(t)\}_{t\geq 0}$ of operators $T(t)$ defined by $T(t) = e^{-\beta t}S(t)$ is a contraction C_0-semigroup on X. Moreover, the infinitesimal generator of T is defined by $A - \beta I$. On the other hand, If S is a contraction C_0-semigroup on X with the infinitesimal generator A, then the family $T = \{T(t)\}_{t\geq 0}$ of operators $T(t)$ defined by $T(t) = e^{\beta t}S(t)$ is a contraction type C_0-semigroup on X. Furthermore, the infinitesimal generator of T is defined by $A + \beta I$.*

2.2.2 The Hille-Yosida Theorem

This section is devoted to the characterization of the infinitesimal generator of a contraction C_0-semigroup on X.

Definition 2.20. *Let A be a linear (not necessary bounded) operator on X. Then the set of all real numbers λ for which $\lambda I - A$ is invertible, i.e, $\lambda I - A$ is bijective and $(\lambda I - A)^{-1}$ is a bounded linear operator on X, is called **the resolvent set of** A and it is denoted by $\rho(A)$. The family of these operators $(\lambda I - A)^{-1}$, $\lambda \in \rho(A)$, is called **the resolvent of** A and it is denoted by $R(\lambda : A)$.*

Lemma 2.3. *[44] Let a linear (unbounded) operator A be the infinitesimal generator of a contraction C_0-semigroup S on X. For each $\lambda > 0$ and $x \in X$, define $R(\lambda)x$ by*

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} S(t)x dt, \tag{2.21}$$

where the integral is in the sense of Bochner integral, which will be defined later. Then $R(\lambda)$ is the inverse of $\lambda I - A$ and

$$\|R(\lambda)\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda}. \tag{2.22}$$

Moreover, for every $\lambda > 0$ and $x \in X$, $R(\lambda)x \in \mathscr{D}(A)$ and

$$AR(\lambda) = \lambda R(\lambda) - I,$$

or

$$(\lambda I - A)R(\lambda) = I.$$

Theorem 2.11. ***(Hille-Yosida)****[44] A linear (unbounded) operator A is the infinitesimal generator of a contraction C_0-semigroup S on X if and only if*
(i) A is closed and $\overline{\mathscr{D}(A)} = X$,
(ii) the resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$,

$$\|(\lambda I - A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda}.$$

Corollary 2.3. *A linear (unbounded) operator A is the infinitesimal generator of a contraction type C_0-semigroup S on X if and only if*
(i) A is closed and $\overline{\mathscr{D}(A)} = X$,
(ii) the resolvent set $\rho(A)$ of A contains all $\lambda > \beta$ and for every $\lambda > \beta$,

$$\|(\lambda I - A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda - \beta}. \tag{2.23}$$

Lemma 2.4. *[44] Let A be a linear operator satisfying conditions (i) and (ii) of Theorem 2.11. Then for every $x \in X$,*

$$\lim_{\lambda\to\infty} \lambda(\lambda I - A)^{-1}x = x.$$

Definition 2.21. *A family of operators $A_\lambda : X \to X$ defined by*

$$A_\lambda(x) = \lambda A(\lambda I - A)^{-1}(x), \quad \lambda > 0, \quad x \in X \tag{2.24}$$

*is called **the Yosida approximation of** A and it is denoted by A_λ.*

Lemma 2.5. *[44] Let A be a linear operator on X satisfying conditions (i) and (ii) of Theorem 2.11. If A_λ is the Yosida approximation of A, then for every $x \in \mathscr{D}(A)$,*

$$\lim_{\lambda\to\infty} A_\lambda x = Ax. \tag{2.25}$$

Lemma 2.6. *[44] Let A be a linear operator on X satisfying conditions (i) and (ii) of Theorem 2.11. If A_λ is the Yosida approximation of A, then A_λ is the infinitesimal generator of a contraction uniformly continuous semigroup e^{tA_λ} on X . Moreover, we have, for each $\lambda_1, \lambda_2 > 0$,*

$$\left\| e^{tA_{\lambda_1}}x - e^{tA_{\lambda_2}}x \right\|_X \le t\|A_{\lambda_1}x - A_{\lambda_2}x\|_X, \quad t \ge 0, \quad x \in X. \tag{2.26}$$

Corollary 2.4. *[44] Let A be the infinitesimal generator of a contraction C_0 semigroup S on X. If A_λ is the Yosida approximation of A, then*

$$S(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x, \quad x \in X.$$

2.3 Random Variables with Values in Banach Spaces

In this section, we introduce some basic facts about random variables taking values in a separable Banach space. Throughout this section, we assume that X is a separable Banach space endowed with a norm $\|\cdot\|_X$ and Ω is a non empty set.

2.3.1 Definitions and Some Properties

Definition 2.22. *A family $\mathscr{F}$ of subsets of Ω is called **a σ-field** on Ω if*
(i) $\Omega \in \mathscr{F}$,
(ii) if $A \in \mathscr{F}$, then $\Omega \setminus A \in \mathscr{F}$,
(iii) if $A_1, A_2, \ldots$ is a sequence of sets in $\mathscr{F}$, then their union $A_1 \cup A_2 \cup \ldots$ also belongs to $\mathscr{F}$.
*The pair $(\Omega, \mathscr{F})$ is called **a measurable space**.*

Definition 2.23. *Let $(\Omega_1, \mathscr{F})$ and $(\Omega_2, \mathscr{G})$ be two measurable spaces. A map $\xi : \Omega_1 \to \Omega_2$ is said be $\mathscr{F}/\mathscr{G}$-**measurable** (or simply **measurable**) if for every $A \in \mathscr{G}$,*

$$\xi^{-1}(A) = \{\omega \in \Omega_1 : \xi(\omega) \in A\} \in \mathscr{F}.$$

*Such a measurable map is called **a random variable on** Ω_1.*

Definition 2.24. *Let $\mathscr{H}$ be a family of subsets of Ω. The smallest σ-field on Ω containing $\mathscr{H}$ is called **the σ-field generated by** $\mathscr{H}$ and it is denoted by $\sigma(\mathscr{H})$.*

Definition 2.25. *The smallest σ-field containing all closed (or open) subsets of X is called **the Borel σ-field** of X and it is denoted by $\mathscr{B}(X)$.*

Proposition 2.2. *[50] Let X^* be the dual space of X. Then $\mathscr{B}(X)$ is the smallest σ-field of X containing all sets of the form*

$$\{x \in X : \varphi(x) \leq \alpha\}, \quad \varphi \in X^*, \quad \alpha \in \mathbb{R}.$$

Definition 2.26. *Let $(\Omega, \mathscr{F})$ be a measurable space. A mapping $\xi : \Omega \to X$ is said to be **Borel measurable** if for each $A \in \mathscr{B}(X)$, $\xi^{-1}(A) \in \mathscr{F}$. Such a Borel measurable map is called **an X-valued random variable on** Ω.*

Definition 2.27. *Let $(\Omega, \mathscr{F})$ be a measurable space and ξ be an X-valued random variable on Ω. The smallest σ-field $\sigma(\xi)$ containing all sets $\xi^{-1}(A)$, $A \in \mathscr{B}(X)$, is called **the σ-field generated by** ξ. If $\{\xi_i\}_{i\in I}$ is a family of X-valued random variables on Ω, then the smallest σ-field $\sigma(\xi_i : i \in I)$ containing all sets $\xi_i^{-1}(A)$, $A \in \mathscr{B}(X)$, is called **the σ-field generated by random variables** ξ_i.*

Corollary 2.5. *It follows from Proposition 2.2 that a mapping $\xi : \Omega \to X$ is an X-valued random variable if and only if for any $\varphi \in X^*$, $\varphi(X)$ is a real-valued random variable on Ω.*

Proposition 2.3. *[50] Let $(\Omega, \mathscr{F})$ be a measurable space. Assume that ξ and ζ are X-valued random variables on Ω. Then*
(i) for $\alpha, \beta \in \mathbb{R}$, $\alpha\xi + \beta\zeta$ is an X-valued random variable on Ω,
(ii) the mapping $\Omega \ni \omega \mapsto \|\xi(\omega)\|_X$ is a real-valued random variable on Ω.

Definition 2.28. *Let $(\Omega, \mathscr{F})$ be a measurable space. An X-valued random variable ξ on Ω of the form*

$$\xi(\omega) = \sum_{i=1}^{k} \mathbb{1}_{A_i}(\omega) x_i, \quad \omega \in \Omega,$$

where $A_1, A_2, .., A_k$ are pairwise disjoint subspaces of Ω such that $\bigcup A_i = \Omega$ and $x_1, x_2, ..., x_k \in X$, is called a ***simple random variable****.*

Lemma 2.7. *[50] Let $(\Omega, \mathscr{F})$ be a measurable space. Assume that ξ is an X-valued random variable on Ω. Then there exists a sequence ξ_n of simple X-valued random variables such that*

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|_X = 0 \quad for\ every\ \omega \in \Omega.$$

Definition 2.29. *A family $\mathscr{H}$ of subsets of Ω is called* ***a π-system*** *if $\emptyset \in \mathscr{H}$ and if $A, B \in \mathscr{H}$, then $A \cap B \in \mathscr{H}$.*

Proposition 2.4. *[50] Assume that $\mathscr{H}$ is a π-system and $\mathscr{G}$ is the smallest family of subsets of Ω such that*
(i) $\mathscr{H} \subset \mathscr{G}$,
(ii) if $A \in \mathscr{G}$, then $A^c \in \mathscr{G}$,
(iii) if for all $i \in \mathbb{N}$, $A_i \in \mathscr{G}$ are such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i \in \mathscr{G}$.
Then $\mathscr{G} = \sigma(\mathscr{H})$.

Definition 2.30. *Let $(\Omega, \mathscr{F})$ be a measurable space. A map $\mu : \mathscr{F} \to \mathbb{R}$ is called* ***a measure*** *if $\mu(\emptyset) = 0$ and for all countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{F}$,*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),$$

which is called ***the σ-additive****. Moreover, if for all $A \in \mathscr{F}$, $\mu(A) \geqslant 0$, then μ is called* ***a non-negative measure****. The triple $(\Omega, \mathscr{F}, \mu)$ is called* ***a measure space****.*

Definition 2.31. *Let $(\Omega, \mathscr{F})$ be a measurable space. A non-negative measure $\mathbb{P}$ on $\mathscr{F}$ satisfying $\mathbb{P}(\Omega) = 1$ is called* ***a probability measure*** *and the triple $(\Omega, \mathscr{F}, \mathbb{P})$ is called* ***a probability space****.*

Definition 2.32. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A set $\overline{\mathscr{F}}$ defined by*

$$\overline{\mathscr{F}} = \{A \subset \Omega : \exists\, B, C \in \mathscr{F}; B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}$$

is a σ-field and is called ***the completion*** *of $\mathscr{F}$. If $\mathscr{F} = \overline{\mathscr{F}}$, then the probability measure $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be* ***complete****.*

Definition 2.33. *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and ξ be an X-valued random variable on Ω. A mapping $\mathscr{L}(\xi):\mathscr{B}(X)\to[0,1]$ defined by*

$$\mathscr{L}(\xi)(A)=\mathbb{P}\left(\xi^{-1}(A)\right)=\mathbb{P}(\{\omega\in\Omega:\xi(\omega)\in A\}),\quad A\in\mathscr{B}(X)$$

*is called **the distribution** (or **the law**) of ξ.*

Remark 2.5. *It is obvious that the map $\mathscr{L}(\xi)$ in the previous definition is a probability measure on $\mathscr{B}(X)$.*

2.3.2 Bochner Integral

Let $(\Omega,\mathscr{F},\mu)$ be a measure space. The Bochner integral generalizes the Lebesgue integral to functions taking values in a Banach space. Lemma 2.7 allows us to do this. We will not give the construction of the Bochner integral in this thesis, see [50] for detail. As in the Lebesgue integral, the Bochner integral of a function $\xi:\Omega\to X$ is denoted by

$$\int_\Omega \xi(\omega)\mu(d\omega),$$

or $\mathbb{E}(\xi)$ if μ is a probability measure.

Definition 2.34. *A function $\xi:\Omega\to X$ is called **Bochner measurable** if it is equal to μ almost everywhere to a function γ taking values in a separable subspace X_0 of X such that $\gamma^{-1}(A)\in\mathscr{F}$ for each open set $A\in X$.*

Chalk in his thesis [20] proved that if a function $\xi:\Omega\to X$ is Borel measurable if and only if it is Bochner measurable. Thus, we use Borel measurable instead of Bochner measurable. The Bochner integral satisfies many properties of the Lebesgue integral.

Theorem 2.12. *[50] A map $\xi:\Omega\to X$ is Bochner integrable iff it is Bochner measurable (or Borel measurable) and*

$$\int_\Omega \|\xi(\omega)\|_X\mu(d\omega)<\infty. \tag{2.27}$$

Proposition 2.5. *[50] If $\xi:\Omega\to X$ is a Bochner integrable map, then for all $B\in\mathscr{F}$,*

$$\left\|\int_B \xi(\omega)\mu(d\omega)\right\|_X\le\int_B\|\xi(\omega)\|_X\mu(d\omega).$$

Assume that $(\Omega_1,\mathscr{F}_1)$ and $(\Omega_2,\mathscr{F}_2)$ are two measurable spaces. Then we denote by $\mathscr{F}_1\otimes\mathscr{F}_2$ the smallest σ-filed of subsets of $\Omega_1\times\Omega_2$ containing all sets of the form $A_1\times A_2$ with $A_1\in\mathscr{F}_1$ and $A_2\in\mathscr{F}_2$. Moreover, μ_1 and μ_2 are measures on $(\Omega_1,\mathscr{F}_1)$ and $(\Omega_2,\mathscr{F}_2)$ respectively, then $\mu_1\times\mu_2$ is a measure on $(\Omega_1\times\Omega_2,\mathscr{F}_1\otimes\mathscr{F}_2)$ such that

$$(\mu_1\times\mu_2)(A_1\times A_2)=\mu_1(A_1)\mu_2(A_2),\quad A_1\in\mathscr{F}_1,\ A_2\in\mathscr{F}_2.$$

Theorem 2.13. ***(Fubini Theorem)****[50] If ξ is an X-valued random variable on $\Omega_1 \times \Omega_2$, then for all $\varphi \in X^*$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the functions $\varphi(\xi(\omega_1,\cdot)) : \Omega_2 \to \mathbb{R}$ and $\varphi(\xi(\cdot,\omega_2)) : \Omega_1 \to \mathbb{R}$ are Bochner (Borel) measurable. Moreover, if ξ is the Bochner integrable, i.e.*

$$\int_{\Omega_1 \times \Omega_2} \|\xi(\omega_1,\omega_2)\|_X (\mu_1 \times \mu_2)(d\omega_1 \times d\omega_2),$$

then

$$\begin{aligned}\int_{\Omega_1 \times \Omega_2} \xi(\omega_1,\omega_2)(\mu_1 \times \mu_2)(d\omega_1 \times d\omega_2) &= \int_{\Omega_2}\int_{\Omega_1} \xi(\omega_1,\omega_2)\mu_1(d\omega_1)\mu_2(d\omega_2)\\ &= \int_{\Omega_1}\int_{\Omega_2} \xi(\omega_1,\omega_2)\mu_2(d\omega_2)\mu_1(d\omega_1).\end{aligned}$$

Proposition 2.6. *[50] Assume that Y is a separable Banach space and $A : X \to Y$ is closed operator such that the domain $\mathscr{D}(A)$ of A is a Borel subset of X. If ξ is an X-valued random variable on Ω such that $\xi(\omega) \in \mathscr{D}(A)$ a.s, then $A\xi$ is a Y-valued random variable on Ω. Moreover, if*

$$\int_\Omega \|A\xi(\omega)\|_Y \mu(d\omega) < \infty, \tag{2.28}$$

then

$$A\int_\Omega \xi(\omega)\mu(d\omega) = \int_\Omega A\xi(\omega)\mu(d\omega). \tag{2.29}$$

Define $L^1(\Omega,\mathscr{F},\mu;X)$ to be the space of (equivalence classes of) all Bochner (Borel) measurable functions $\xi : \Omega \to X$ such that (2.27) holds. As in the Lebesgue spaces, one can show that the space $L^1(\Omega,\mathscr{F},\mu;X)$ is a separable Banach space with respect to the norm

$$\|\xi\|_1 = \int_\Omega \|\xi(\omega)\|_X \mu(d\omega).$$

Similarly, one can show that if we define by $L^p(\Omega,\mathscr{F},\mu;X)$ the space of (equivalence classes of) all Bochner (Borel) measurable functions $\xi : \Omega \to X$ such that

$$\int_\Omega \|\xi(\omega)\|_X^p \mu(d\omega) < \infty, \tag{2.30}$$

then for each $p \in [1,\infty)$, the space $L^p(\Omega,\mathscr{F},\mu;X)$ is a separable Banach space with respect to the norm

$$\|\xi\|_p = \left(\int_\Omega \|\xi(\omega)\|_X^p \mu(d\omega)\right)^{\frac{1}{p}}$$

and the space $L^\infty(\Omega,\mathscr{F},\mu;X)$ is a separable Banach space with respect to the norm

$$\|\xi\|_\infty = ess.\sup_{\omega\in\Omega}\|\xi(\omega)\|_X.$$

If Ω is an interval $[0,T]$, $\mathscr{F}=\mathscr{B}([0,T])$ and μ is the Lebesgue measure on $[0,T]$, then we use the notation $L^p(0,T;X)$ instead of $L^p(\Omega,\mathscr{F},\mu;X)$.

Proposition 2.7. *[50] Let $r \geq p \geq 1$ and $T>0$. Then the space $L^r(0,T;X)$ and the space $C(0,T;X)$ of all X-valued continuous functions on $[0,T]$ are Borel subsets of the space $L^p(0,T;X)$.*

2.3.3 Conditional Expectation

Definition 2.35. *Assume that $(\Omega,\mathscr{F},\mathbb{P})$ is a probability space. Let ξ be a Bochner integrable X-valued random variable on Ω and $\mathscr{G}$ be a σ-field contained in $\mathscr{F}$. A Bochner integrable X-valued random variable ζ on Ω is said to be **the conditional expectation** of ξ if*

$$\int_A \xi(\omega)\mathbb{P}(d\omega) = \int_A \zeta(\omega)\mathbb{P}(d\omega) \quad for\ every\ A\in\mathscr{G}. \tag{2.31}$$

This conditional expectation is denoted by $\mathbb{E}(\xi|\mathscr{G})$.

Remark 2.6. *By Proposition 1.10 of [50], such a random variable ζ exists and it is unique.*

2.4 Probability Measures on Banach Spaces

This section is devoted to basic facts about probability measures on Banach spaces. Throughout this section, we assume that X is a separable Banach space endowed with a norm $\|\cdot\|_X$. Recall that $\mathscr{B}(X)$ denotes the Borel σ-field of X.

2.4.1 Gaussian Probability Measures on Banach Spaces

Definition 2.36. *A probability measure μ on $\mathscr{B}(\mathbb{R})$ is called **Gaussian** if it has a density function defined by*

$$\mu_{m,q}(x) = \frac{1}{\sqrt{2q\pi}}e^{-\frac{(x-m)^2}{2q}}, \quad x\in\mathbb{R}, \tag{2.32}$$

*where $q>0$ and $m\in\mathbb{R}$. A Gaussian probability measure on $\mathscr{B}(\mathbb{R})$ is usually denoted by $\mathscr{N}(m,q)$. If $m=0$, then μ is called a **symmetric Gaussian probability measure**.*

Definition 2.37. *Let μ be a probability measure on $\mathscr{B}(X)$. A function $\hat{\mu}$ defined by*

$$\hat{\mu}(x^*) = \int_X e^{ix^*(x)}\mu(dx), \quad x^* \in X^*$$

*is called **the characteristic function of** μ.*

Remark 2.7. *If $X = \mathbb{R}^d$, more generally, X is a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, then the characteristic function $\hat{\mu}$ of μ is as follow*

$$\hat{\mu}(\lambda) = \int_X e^{i\langle\lambda,x\rangle}\mu(dx), \quad \lambda \in X.$$

Proposition 2.8. *[50] If μ is A Gaussian probability measure on $\mathscr{B}(\mathbb{R})$, then*

$$\int_{\mathbb{R}} x\mu(dx) = m, \quad \int_{\mathbb{R}} (x-m)^2\mu(dx) = q.$$

Moreover, for every $\lambda \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{i\lambda x}\mu(dx) = e^{im\lambda - \frac{1}{2}q\lambda^2}, \quad \int_{\mathbb{R}} e^{\lambda x}\mu(dx) = e^{m\lambda + \frac{1}{2}q\lambda^2}.$$

Definition 2.38. *A probability measure μ on $\mathscr{B}(X)$ is called a **Gaussian probability measure** if and only if the law of each $\varphi \in X^*$, considered as a real-valued random variable on X, is a Gaussian probability measure on $\mathscr{B}(\mathbb{R})$. Moreover, if the law of each $\varphi \in X^*$ is a symmetric (zero mean) Gaussian probability measure on $\mathscr{B}(\mathbb{R})$, then μ is called a **symmetric Gaussian probability measure**.*

2.4.2 Canonical Gaussian Probability Measures on Banach Spaces

Definition 2.39. *Let H be an n dimensional Hilbert space. A measure γ_H on $\mathscr{B}(H)$ is called a **Canonical Gaussian measure** if it has a density function defined by*

$$d\gamma_H(h) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\|h\|_H^2}, \quad h \in H.$$

Note that if H is infinite dimensional, then we can not define a Canonical Gaussian measure on $\mathscr{B}(H)$. However, we would like to find a measure which is a counterpart of the canonical Gaussian measure. For this aim, let $P(H)$ be the space of all finite dimensional Hilbert subspaces of H. For any $Y \in P(H)$, we denote the canonical Gaussian measure on $\mathscr{B}(Y)$ by γ_Y. Let

$$C(H) = \left\{A = \pi_Y^{-1}(B) : B \in \mathscr{B}(Y), Y \in P(H)\right\},$$

where $\pi_Y : H \to Y$ is an orthogonal projection. Each element of $C(H)$ is called a **cylinder set**.

Lemma 2.8. *[39] The following set*

$$\tilde{C}(H) = \left\{A = \psi^{-1}(B) \mid \exists\ \psi : H \to \mathbb{R}^n \text{ linear and bounded and } B \in \mathscr{B}(\mathbb{R}^n)\right\}$$

contains $C(H)$.

Proposition 2.9. *[39] $C(H)$ is an algebra, but not a σ-algebra.*

Define a function γ_H on $C(H)$ by

$$\gamma_H : C(H) \ni A \mapsto \gamma_H(A) = \gamma_Y(B) \in [0,1], \tag{2.33}$$

where $A = \pi_Y^{-1}(B)$, $Y \in P(H)$ and $B \in \mathscr{B}(Y)$.

Lemma 2.9. *[39] If $Y_1, Y_2 \in P(H)$ and $B_i \in \mathscr{B}(Y_i)$ such that*

$$\pi_{Y_1}^{-1}(B_1) = \pi_{Y_2}^{-1}(B_2),$$

then

$$\gamma_{Y_1}(B_1) = \gamma_{Y_2}(B_2).$$

Proposition 2.10. *[39] γ_H is well-defined and finitely additive, but it is not σ-additive.*

Set

$$\tilde{C}(X) = \{A = \psi^{-1}(B) \mid \exists\ \psi : X \to \mathbb{R}^n \text{ linear and bounded and } B \in \mathscr{B}(\mathbb{R}^n)\}. \tag{2.34}$$

Proposition 2.11. *[39] The smallest σ-field generated by $\tilde{C}(X)$ is contained in $\mathscr{B}(X)$.*

Lemma 2.10. *If $L : H \to X$ be a bounded linear operator, then for each $A \in \tilde{C}(X)$,*

$$L^{-1}(A) \in C(H).$$

Proof. Fix $A \in \tilde{C}(X)$. Then there exists a bounded linear map $\psi : X \to \mathbb{R}^n$ and $B \in \mathscr{B}(\mathbb{R}^n)$ such that $A = \psi^{-1}(B)$. Thus

$$L^{-1}(A) = L^{-1}(\psi^{-1}(B)) = (\psi \circ L)^{-1}(B).$$

Since ψ and L are linear bounded operators, $\psi \circ L : H \to \mathbb{R}^n$ is a linear bounded operator. Therefore by the definition of $\tilde{C}(H)$, we obtain $L^{-1}(A) \in \tilde{C}(H)$. Since $C(H) = \tilde{C}(H)$, $L^{-1}(A) \in C(H)$. This concludes the proof. ∎

Definition 2.40. *Let γ_H be a function defined in (2.33). A linear bounded operator $L : H \to X$ is called γ-**radonifying** if and only if a measure $L(\gamma_H)$ defined by*

$$L(\gamma_H)(A) = \gamma_H(L^{-1}(A)), \quad A \in \tilde{C}(X)$$

is σ-additive.

We denote by $\gamma(H,X)$ the space of all γ-radonifying operators from H into X.

Proposition 2.12. *[39] If $L : H \to X$ is a γ-radonifying operator, then there exists a Gaussian probability measure ν_L on $\mathscr{B}(X)$ which is an extension of $L(\gamma_H)$.*

Theorem 2.14. *[39] Let ν_L be the Gaussian probability measure on $\mathscr{B}(X)$ generated by $L \in \gamma(H,X)$. Then a mapping defined by*

$$\|L\|_{\gamma(H,X)} = \left(\int_X \|x\|_X^2 \, d\nu_L(x)\right)^{\frac{1}{2}}, \quad L \in \gamma(H,X)$$

defines a norm and $\gamma(H,X)$ is a separable Banach space with respect to this norm.

Theorem 2.15. *[2] Assume that H_1, H_2 are separable Hilbert spaces, and B_1, B_2 are separable Banach spaces. If $h : H_1 \to H_2$ and $b : B_2 \to B_1$ are linear bounded operators, and $k : H_2 \to B_2$ is a γ-radonifying operator, then $k \circ h$ and $b \circ k$ are also γ-radonifying operators.*

2.5 Stochastic Processes with Values in Banach Spaces

In this section, we introduce some definitions and facts about stochastic processes with values in Banach spaces. Throughout this section, we assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and X be a separable Banach space with a norm $\|\cdot\|_X$.

2.5.1 General Notions

Definition 2.41. *Let $I \subset \mathbb{R}$. A family $\{\xi(t)\}_{t\in I}$ of X-valued random variables $\xi(t)$, $t \in I$, defined on Ω is called an X-**valued stochastic process** on I. If $I = \{1, 2, ..\}$, then $\{\xi(t)\}_{t\in I}$ is said to be a stochastic process in discrete time. If I is an interval in $\mathbb{R}$ (usually $I = [0,T]$, $T > 0$), then $\{\xi(t)\}_{t\in I}$ is said to be a stochastic process in continuous time.*

We use the notation ξ in place of $\{\xi(t)\}_{t\in I}$ for simplicity. Throughout this section, unless otherwise specified, let $I=[0,T]$ and ξ be an X-valued stochastic process on $[0,T]$.

Definition 2.42. *For each $\omega\in\Omega$, a mapping defined by*

$$[0,T]\ni t\mapsto \xi(t,w)\in X$$

*is called **trajectory** (or **path**) of the process ξ.*

Definition 2.43. *The process ξ is called **continuous** if the trajectories of ξ are $\mathbb{P}$-a.s. continuous on $[0,T]$, i.e. there exists $\bar{\Omega}\in\mathscr{F}$ with $\mathbb{P}(\bar{\Omega})=1$ such that for each $\omega\in\bar{\Omega}$, the mapping $[0,T]\ni t\mapsto\xi(t,\omega)\in X$ is continuous.*

Definition 2.44. *An X-valued stochastic process ζ on $[0,T]$ is called **a modification** (or **version**) of the process ξ if*

$$\mathbb{P}(\{w\in\Omega\ :\ \xi(t,w)\neq\zeta(t,w)\})=0\quad for\ every\ t\in[0,T].$$

Definition 2.45. *The process ξ is called **measurable** if the following mapping*

$$\xi:[0,T]\times\Omega\to X$$

is $\mathscr{B}([0,T])\otimes\mathscr{F}$-measurable.

Definition 2.46. *The process ξ is called **stochastically continuous** at $t_0\in[0,T]$ if for every $\epsilon,\delta>0$, there exists $\rho>0$ such that for each $t\in[t_0-\rho,t_0+\rho]\cap[0,T]$,*

$$\mathbb{P}(\{\omega\in\Omega:\|\xi(t,\omega)-\xi(t_0,\omega)\|_X\geq\epsilon\})\leq\delta.$$

*If the process ξ is stochastically continuous for each $t_0\in[0,T]$, then it is said to be **stochastically continuous on** $[0,T]$.*

Proposition 2.13. *[50] If the process ξ is stochastically continuous on $[0,T]$, then it has a measurable modification.*

Definition 2.47. *The process ξ is called **stochastically uniformly continuous** on $[0,T]$ if for every $\epsilon,\delta>0$, there exists $\rho>0$ such that for every $t,s\in[0,T]$ with $|t-s|<\rho$,*

$$\mathbb{P}(\{w\in\Omega:\|\xi(t,\omega)-\xi(s,\omega)\|_X\geq\epsilon\})\leq\delta.$$

Proposition 2.14. *[50] If the process ξ is stochastically continuous on $[0,T]$, then it is also stochastically uniformly continuous on $[0,T]$.*

Definition 2.48. *The process ξ is called* ***mean square continuous*** *at $t_0 \in [0,T]$ if*

$$\lim_{t\to t_0} \mathbb{E}\left(\|\xi(t)-\xi(t_0)\|_X^2\right) = 0.$$

Furthermore, if the process ξ is mean square continuous at every point t_0 of $[0,T]$, then ξ is said to be ***mean square continuous on*** $[0,T]$.

Proposition 2.15. *[50] If the process ξ is mean square continuous on $[0,T]$, then it is stochastically continuous on $[0,T]$.*

2.5.2 Adapted Processes and Martingales

Definition 2.49. *A family $\{\mathscr{F}_t\}_{t\geq 0}$ of σ-fields $\mathscr{F}_t \subset \mathscr{F}$ is called a* ***filtration*** *if for any $0 \leq s \leq t < \infty$, $\mathscr{F}_s \subset \mathscr{F}_t$.*

Throughout this section, we assume that $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$ is a filtration.

Definition 2.50. *The process ξ is said to be* ***adapted to the filtration*** $\mathbb{F}$ *if for each $t \in [0,T]$, $\xi(t)$ is $\mathscr{F}_t$-measurable.*

Definition 2.51. *The process ξ is called* ***progressively measurable*** *if for each $t \in [0,T]$, the following mapping*

$$[0,t]\times\Omega \ni (s,w) \mapsto \xi(s,w) \in X$$

is $\mathscr{B}([0,t])\otimes\mathscr{F}_t$-measurable.

Remark 2.8. *If the process ξ is progressively measurable, then it is adapted to $\mathbb{F}$.*

Proposition 2.16. *[50] If the process ξ is stochastically continuous on $[0,T]$ and adapted to $\mathbb{F}$, then it has an $\mathbb{F}$-progressively measurable modification.*

Proposition 2.17. *[37] If the process ξ is measurable and adapted to $\mathbb{F}$, then it has an $\mathbb{F}$-progressively measurable modification.*

Proposition 2.18. *[37] If the process ξ is continuous and adapted to $\mathbb{F}$, then it is $\mathbb{F}$-progressively measurable.*

Definition 2.52. *The process ξ is said to be* ***integrable*** *if for each $t \in [0,T]$, $\xi(t)$ is Bochner intagrable, i.e. $\mathbb{E}(\|\xi(t)\|_X) < \infty$. Moreover, if for each $t \in [0,T]$, $\xi(t)$ is square integrable, i.e. $\mathbb{E}\left(\|\xi(t)\|_X^2\right) < \infty$, then the process ξ is called* ***square integrable***.

Definition 2.53. *The process ξ is called **martingale** if*
(i) ξ is adapted to $\mathbb{F}$,
(ii) for each $t \in [0,T]$, $\mathbb{E}(\|\xi(t)\|_X) < \infty$,
(iii) for each $t, s \in [0,T]$ with $t \geq s$,

$$\mathbb{E}(\xi(t)|\mathscr{F}_s) = \xi(s).$$

Proposition 2.19. *[50] Let $\mathscr{M}_T^2$ be the space of all X-valued, continuous and square integrable martingales ξ on $[0,T]$. Then $\mathscr{M}_T^2$ is a Banach space with respect to the following norm*

$$\|\xi\|_T = \left(\mathbb{E} \sup_{t\in[0,T]} \|\xi(t)\|_X^2\right)^{\frac{1}{2}}, \quad \xi \in \mathscr{M}_T^2.$$

Definition 2.54. *A process ζ is called a **step process** if it is of the form*

$$\zeta(t) = \sum_{i=0}^{n-1} \zeta_i \mathbb{1}_{[t_i,t_{i+1})}(t),$$

where for each $i = 0, 1, \ldots n-1$, $\zeta_i : \Omega \to X$ is $\mathscr{F}_{t_i}$-measurable random variable and $0 = t_0 < t_1 < \ldots < t_n = T$ is a partition of $[0,T]$.

Proposition 2.20. *[50] If ζ is a step process, then it is progressively measurable.*

Proposition 2.21. *[50] Let ζ be a step process given in Definition 2.54. If for each $i = 0, 1, \ldots n-1$, ζ_i is square integrable, then ζ belongs to the space $L^2([0,T] \times \Omega; X)$.*

2.5.3 Deterministic convolution

Recall that $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$, is a filtered probability space and X is a separable Banach space endowed with a norm $\|\cdot\|_X$. Assume that S is a C_0-semigroup on X, in particular, there exists $M > 0$ and $\beta \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathscr{L}(X)} \leq M e^{\beta t}, \quad t \geq 0. \tag{2.35}$$

Theorem 2.16. *Let f be an X valued stochastic process on $[0,T]$ such that the trajectories of f are $\mathbb{P}$-a.s Bochner integrable. For each $t \in [0,T]$, define an X-valued stochastic process u on $[0,t]$ by*

$$u(r) = S(t-r)f(r), \quad r \in [0,t].$$

Then the trajectories of u are $\mathbb{P}$-a.s Bochner integrable.

Proof. Since the trajectories of f are $\mathbb{P}$-a.s Bochner integrable, there exists $\bar{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for each $\omega \in \bar{\Omega}$, the mapping $f(\cdot,\omega) : [0,T] \to X$ is Borel measurable and

$$\int_0^T \|f(t,\omega)\|_X dt < \infty.$$

Fix $t \in [0,T]$. It is sufficient to show that for each $\omega \in \bar{\Omega}$, $u(\cdot,\omega) : [0,t] \to X$ is Bochner integrable, i.e. $u(\cdot,\omega) \in L(0,t;X)$. Fix $\omega \in \bar{\Omega}$. Define a function $g_\omega : [0,t] \to X$ by

$$g_\omega(r) = f(r,\omega), \quad r \in [0,t]$$

and so

$$u(r,\omega) = S(t-r)g_\omega(r), \quad r \in [0,t].$$

By [1], since $g_\omega \in L(0,t;X)$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions $g_n : [0,t] \to X$, i.e. $g_n \in C(0,t;X)$, such that

$$\lim_{n\to\infty} \|g_n - g_\omega\|_{L(0,t;X)} = 0.$$

For each $n \in \mathbb{N}$, define a function $u_n(\cdot,\omega) : [0,t] \to X$ by

$$u_n(r,\omega) = S(t-r)g_n(r), \quad r \in [0,t].$$

For each $n \in \mathbb{N}$, $u_n(\cdot,\omega)$ is continuous on $[0,t]$. Indeed, for a fixed $n \in \mathbb{N}$ and $r_0 \in [0,t]$, by the triangle inequality and inequality (2.35), we have

$$\begin{aligned}
\left\|u_n(r,\omega) - u_n(r_0,\omega)\right\|_X &= \big\|S(t-r)g_n(r) - S(t-r)g_n(r_0) \\
&\qquad + S(t-r)g_n(r_0) - S(t-r_0)g_n(r_0)\big\|_X \\
&\le \left\|S(t-r)(g_n(r) - g_n(r_0))\right\|_X + \left\|[S(t-r) - S(t-r_0)]g_n(r_0)\right\|_X \\
&\le \left\|S(t-r)\right\|_{\mathscr{L}(X)} \left\|g_n(r) - g_n(r_0)\right\|_X + \left\|S(t-r) - S(t-r_0)\right\|_{\mathscr{L}(X)} \left\|g_n(r_0)\right\|_X \\
&\le Me^{\beta(t-r)} \left\|g_n(r) - g_n(r_0)\right\|_X + M\left(e^{\beta(t-r)} + e^{\beta(t-r_0)}\right) \left\|g_n(r_0)\right\|_X.
\end{aligned}$$

If $r \in [0,t]$, then $t - r \in [0,t]$. So there exists a constant C such that for all $r \in [0,t]$,

$$Me^{\beta(t-r)} \le C. \tag{2.36}$$

Since g_n is continuous at $r_0 \in [0,t]$, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $r \in [0,t]$ with $|r - r_0| \le \delta$,

$$\|g_n(r) - g_n(r_0)\|_X \le \varepsilon.$$

Therefore we have, for all $r \in [0,t]$ with $|r-r_0| \le \delta$,

$$\|u_n(r,\omega)-u_n(r_0,\omega)\|_X \le C\varepsilon + 2C\|g_n(r_0)\|_X.$$

If we choose ε as

$$\varepsilon = \frac{\tilde{\varepsilon}}{C} - 2\|g_n(r_0)\|_X, \quad \tilde{\varepsilon} > 0,$$

then

$$\|u_n(r,\omega)-u_n(r_0,\omega)\|_X \le \tilde{\varepsilon}.$$

Thus, $u_n(\cdot,\omega)$ is continuous on $[0,t]$. Moreover, $u_n(\cdot,\omega)$ converges to $u(\cdot,\omega)$ in $L(0,t;X)$. Indeed by inequality (2.35), we have

$$\begin{aligned}
\|u_n(\cdot,\omega)-u(\cdot,\omega)\|_{L(0,t;X)} &= \int_0^t \|u_n(r,\omega)-u(r,\omega)\|_X dr \\
&= \int_0^t \|S(t-r)g_n(r)-S(t-r)g_\omega(r)\|_X dr \\
&= \int_0^t \|S(t-r)(g_n(r)-g_\omega(r))\|_X dr \\
&\le \int_0^t \|S(t-r)\|_{\mathscr{L}(X)}\|g_n(r)-g_\omega(r)\|_X dr \\
&\le \int_0^t Me^{\beta(t-r)}\|g_n(r)-g_\omega(r)\|_X dr.
\end{aligned}$$

It follows from inequality (2.36) that

$$\|u_n(\cdot,\omega)-u(\cdot,\omega)\|_{L(0,t;X)} \le C\|g_n - g_\omega\|_{L(0,t;X)} \to 0 \quad as\ n \to \infty.$$

Therefore $u(\cdot,\omega) \in L(0,t;X)$. ■

Corollary 2.6. *If f is an X valued stochastic process on $[0,T]$ such that the trajectories of f are $\mathbb{P}$-a.s Bochner integrable, then for each $t \in [0,T]$, the integral $\int_0^t S(t-r)f(r)dr$ exists $\mathbb{P}$-a.s. Therefore, a process ζ defined by*

$$\zeta(t) := \int_0^t S(t-r)f(r)dr, \quad t \in [0,T]$$

is an X-valued stochastic process on $[0,T]$ and is called ***the deterministic convolution****.*

Proposition 2.22. *The deterministic convolution ζ is continuous, i.e. the trajectories of ζ are $\mathbb{P}$-a.s continuous on $[0,T]$.*

Proof. We need to show that there exist $\bar{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for each $\omega \in \bar{\Omega}$, the function

$$\zeta(\cdot,\omega) : [0,T] \ni t \mapsto \zeta(t,\omega) = \int_0^t S(t-r)f(r,\omega)dr \in X$$

is continuous on $[0,T]$. Since the process f is continuous, there exists $\bar{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that the function

$$f(\cdot,\omega) : [0,T] \ni t \mapsto f(t,\omega) \in X$$

is continuous on $[0,T]$. Therefore, it is sufficient to show that for each $\omega \in \bar{\Omega}$, the mapping $\zeta(\cdot,\omega)$ is continuous on $[0,T]$. Fix $\omega \in \bar{\Omega}$. Define a function $g : [0,t] \to X$ by

$$g(r) = f(r,\omega) \quad r \in [0,t],$$

and

$$x(t) := \zeta(t,\omega) = \int_0^t S(t-r)g(r)dr, \quad t \in [0,T].$$

For each $n \in \mathbb{N}$, define a function x_n by

$$x_n(t) = \int_0^t S(t-r)g_n(r)dr, \quad t \in [0,T],$$

where $g_n \in C(0,t;X)$ such that $\|g_n - g\|_{L^1(0,t;X)} \to 0$. The sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly convergent to x. Indeed, for a fixed $t \in [0,T]$, by (2.36), we have

$$\begin{aligned}
\|x_n(t) - x(t)\|_X &= \left\| \int_0^t S(t-r)\big(g_n(r) - g(r)\big)dr \right\|_X \\
&\le \int_0^t \left\|S(t-r)\big(g_n(r) - g(r)\big)\right\|_X dr \\
&\le \int_0^t \|S(t-r)\|_{\mathscr{L}(X)} \|g_n(r) - g(r)\|_X dr \\
&\le C \int_0^t \|g_n(r) - g(r)\|_X dr \\
&\le C\|g_n - g\|_{L^1(0,t;X)} \to 0 \;\; as \; n \to \infty.
\end{aligned}$$

Similarly, one can show that for each $n \in \mathbb{N}$, x_n is continuous on $[0,T]$. Fix $n \in \mathbb{N}$ and $t_0 \in [0,T]$. Then since x_n is continuous at t_0, there exist $\delta > 0$ for a given $\frac{\varepsilon}{3}$ such that for all $t \in [0,T]$ with $|t - t_0| \le \delta$,

$$\|x_n(t) - x_n(t_0)\|_X \le \frac{\varepsilon}{3}. \tag{2.37}$$

Moreover, since $(x_n)_{n\in\mathbb{N}}$ is uniformly convergent to x, for a given $\frac{\varepsilon}{3}$, there exists $n_0 \in \mathbb{N}$ such that for all $t \in [0,T]$ and $n \geq n_0$,

$$\|x_n(t) - x(t)\|_X \leq \frac{\varepsilon}{3}, \tag{2.38}$$

in particular, for $t_0 \in [0,T]$,

$$\|x_n(t_0) - x(t_0)\|_X \leq \frac{\varepsilon}{3}. \tag{2.39}$$

Taking into account inequalities (2.37), (2.38) and (2.39), we infer that

$$\begin{aligned}\|x(t) - x(t_0)\|_X &= \|x(t) - x_n(t) + x_n(t) - x_n(t_0) + x_n(t_0)\|_X \\ &\leq \|x_n(t) - x(t)\|_X + \|x_n(t_0) - x(t_0)\|_X + \|x_n(t) - x_n(t_0)\|_X \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Therefore, x is continuous at $t_0 \in [0,T]$. Since t_0 was arbitrary, x is continuous on $[0,T]$. ■

Corollary 2.7. *The deterministic convolution ζ has an $\mathbb{F}$-progressively measurable modification.*

Remark 2.9. *All definitions and results in this section obtained for time interval $[0,T]$ can be generalized to intervals $[s,T]$, $s \in [0,T]$.*

2.6 Stochastic Integration in Banach Spaces

2.6.1 Stochastic Integral Preliminaries

Definition 2.55. *A Banach space X endowed with a norm $\|\cdot\|_X$ is called **martingale-type 2** if there exists a constant $C > 0$ depending only on X such that for any X-valued martingale $\{M_n\}_{n\in\mathbb{N}}$, the following inequality holds*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\|M_n\|_X^2 \leq C \sum_{n\in\mathbb{N}} \mathbb{E}\|M_n - M_{n-1}\|_X^2.$$

Definition 2.56. *Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$, be a filtered probability space and $(H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. A family $W = \{W(t)\}_{t\geq 0}$ of bounded linear operators $W(t)$, $t \geq 0$, from H into $L^2(\Omega, \mathscr{F}, \mathbb{P})$ is called an H-valued $\mathbb{F}$-cylindrical canonical Wiener process iff*

(i) for all $t \geq 0$ and $h_1, h_2 \in H$, $\mathbb{E}W(t)h_1W(t)h_2 = t\langle h_1, h_2\rangle_H$,

(ii) for any $h \in H$, $\{W(t)h\}_{t\geq 0}$ is a real-valued $\mathbb{F}$-adapted Wiener process.

Throughout this section, we assume that

- $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space,
- X is a martingale-type 2 Banach space endowed with a norm $\|\cdot\|_X$,
- $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space,
- W is an H-valued $\mathbb{F}$-cylindrical canonical Wiener process.

Recall that $\gamma(H,X)$ denotes the space of all γ-radonifying operators from H into X and it is a Banach space endowed with the norm

$$\|L\|_{\gamma(H,X)} = \left(\int_X \|x\|_X^2 d\nu_L(x)\right)^{\frac{1}{2}}, \quad L \in \gamma(H,X).$$

Theorem 2.17. *[11] Let $T > 0$. If ξ is a $\gamma(H,X)$-valued $\mathbb{F}$-progressively measurable process on $[0,T]$ such that*

$$\mathbb{E}\int_0^T \|\xi(t)\|_{\gamma(H,X)}^2 dt < \infty, \tag{2.40}$$

then the stochastic integral $\int_0^T \xi(t)dW(t)$ exists (it is an X-valued random variable on Ω) and there exists a constant $K > 0$ such that

$$\mathbb{E}\left(\left\|\int_0^T \xi(t)dW(t)\right\|_X^2\right) \le K\int_0^T \mathbb{E}\|\xi(t)\|_{\gamma(H,X)}^2 dt,$$

i.e $\int_0^T \xi(t)dW(t)$ belongs to the space $L^2(\Omega, \mathscr{F}, \mathbb{P}; X)$.

Corollary 2.8. *If ξ is a $\gamma(H,X)$-valued $\mathbb{F}$-progressively measurable process on $[0,T]$ such that*

$$\mathbb{E}\int_0^T \|\xi(t)\|_{\gamma(H,X)}^2 dt < \infty,$$

then for each $t \in [0,T]$, the stochastic integral $\int_0^t \xi(s)dW(s)$ exists. Therefore,

$$\phi(t) := \int_0^t \xi(s)dW(s), \quad t \in [0,T]$$

is an X-valued stochastic process on $[0,T]$.

Theorem 2.18. *[11] The process ϕ is a martingale and has a continuous modification. Moreover, there exists a constant $C > 0$ (independent of ξ) such that*

$$\mathbb{E}\left\|\phi(t)\right\|_X^2 \le C\int_0^t \mathbb{E}\|\xi(s)\|_{\gamma(H,X)}^2 ds, \quad t \ge 0. \tag{2.41}$$

Corollary 2.9. *The process ϕ is $\mathbb{F}$-progressively measurable.*

2.6.2 Stochastic Convolution

Let S be a C_0-semigroup on X. Assume that ξ is a $\gamma(H,X)$-valued $\mathbb{F}$-progressively measurable process on $[0,T]$ such that (2.40) holds. Define a process ζ by

$$\zeta(r) = S(T-r)\xi(r), \quad r \in [0,T].$$

It is obvious that ζ is a $\gamma(H,X)$-valued $\mathbb{F}$-progressively measurable process on $[0,T]$ such that

$$\mathbb{E}\int_0^T \|\zeta(r)\|^2_{\gamma(H,X)} dr < \infty.$$

Therefore, the following corollary follows from Theorems 2.17 and 2.18.

Corollary 2.10. *The stochastic integral $\int_0^T S(T-r)\xi(r)dW(r)$ exists. Moreover, the process defined by*

$$\chi(t) = \int_0^t S(t-r)\xi(r)dW(r), \quad t \in [0,T]$$

*is an X-valued martingale and has a continuous modification, i.e it is $\mathbb{F}$-progressively measurable. This process is called **the stochastic convolution**.*

Definition 2.57. *A Banach space X with a norm $\|\cdot\|_X$ is said to be satisfied **the H-condition** for some $q \geq 2$, if a function $\psi : X \to \mathbb{R}$ defined by*

$$\psi(x) = \|x\|_X^q, \quad x \in X \tag{2.42}$$

is of C^2 class on X (in the Fréchet derivative sense) and there exist constants $K_1(q), K_2(q) > 0$ depending on q such that

$$\left|d\psi(x)\right|_{\mathscr{L}(X,\mathbb{R})} \leq K_1(q)\|x\|_X^{q-1}, \quad x \in X \tag{2.43}$$

and

$$\left|d^2\psi(x)\right|_{\mathscr{L}(X\times X,\mathbb{R})} \leq K_2(q)\|x\|_X^{q-2}, \quad x \in X, \tag{2.44}$$

where $d\psi(x)$ and $d^2\psi(x)$ are the first and second Fréchet derivatives of ψ at $x \in X$ respectively.

Proposition 2.23. *[60] If X is a Banach space satisfying the H-condition, then X is a martingale-type 2 Banach space.*

Theorem 2.19. *[10] Let X be a Banach space satisfying the H-condition with the norm $\|\cdot\|_X$ and S be a contraction C_0-semigroup on X. If ξ is a $\gamma(H,X)$-valued $\mathbb{F}$-progressively measurable process on $[0,T]$ such that*

$$\mathbb{E}\int_0^T \|\xi(r)\|_X^2 dr < \infty, \tag{2.45}$$

then there exists a constant $K>0$ depending on H, X and $K_1(q)$,$K_2(q)$ appearing in the H-condition such that

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t S(t-r)\xi(r)dW(r)\right\|_X^2 \leq K\mathbb{E}\int_0^T \|\xi(t)\|_{\gamma(H,X)}^2 dt. \tag{2.46}$$

Recall that if S is a contraction type C_0-semigroup on X, i.e. there exists a constant $\beta\in\mathbb{R}$ such that

$$\|S(t)\|_{\mathscr{L}(X)} \leq e^{\beta t}, \quad t\geq 0,$$

then a family $T=\{T(t)\}_{t\geq 0}$ of operators defined by $T(t)=e^{\beta t}S(t)$ is a contraction C_0-semigroup on X. Therefore, the following corollary follows from the previous Theorem.

Corollary 2.11. *Let X be a Banach space satisfying the H-condition with a norm $\|\cdot\|_X$ and S be a contraction type C_0-semigroup on X. If ξ is a $\gamma(H,X)$-valued, $\mathbb{F}$-progressively measurable process on $[0,T]$ such that (2.45) is satisfied, then there exists a constant $K_T>0$ depending on $T>0$ and K appearing in (2.46) such that the following estimate holds*

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t S(t-r)\xi(r)dW(r)\right\|_X^2 \leq K_T\mathbb{E}\int_0^T \|\xi(t)\|_{\gamma(H,X)}^2 dt. \tag{2.47}$$

Remark 2.10. *All the results on stochastic integration obtained for the time interval $[0,T]$ can be generalized to intervals $[s,T]$, $s\in[0,T]$.*

CHAPTER 3

THE HEATH-JARROW-MORTON-MUSIELA (HJMM) EQUATION

In this chapter, we provide a brief introduction to some basic definitions in the theory of forward rates. We also mention the arbitrage pricing theory, but not in detail. For this purpose, there are many excellent text books such as [3], [4], [42] and [45]. Next we introduce the dynamics of the forward rate processes proposed by Heath-Jarrow-Morton [36], driven by a Wiener process on a (possible infinite dimensional) Hilbert space. Using the Musiela parametrization, we construct stochastic partial differential equation, which is known as the Heath-Jarrow-Morton-Musiela (HJMM) Equation, from the dynamics of the forward rate processes. Finally, we give some known results about the existence and uniqueness of solutions for the HJMM equation in Hilbert spaces, and ergodic properties of the solution.

3.1 Forward Rates

Definition 3.1. *The value of one Dollar at time t with the maturity date T is called* ***the zero-coupon bond*** *and it is denoted by $P(t,T)$. This is a contract which guarantees the holder one Dollar to be paid at the maturity date T. Thus, this is the most basic interest rate contract.*

Remark 3.1. *Because of some additional factors like changes of the economy in time, the value of one Dollar today is better than the value of one Dollar tomorrow and even the* *value of one Dollar next year. Therefore, the bond prices are unknown in advance. Thus, we assume that for each $T > 0$ and $t \in [0,T]$, $P(t,T)$ is an $\mathbb{R}$-valued random variable defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, i.e. for an arbitrary $T > 0$, the family $\{P(t,T)\}_{t\in[0,T]}$ of these random variables is an $\mathbb{R}$-valued stochastic process on $[0,T]$.*

Definition 3.2. *Assume that for each $t \in [0,T]$, $[0,\infty) \ni T \mapsto P(t,T)$ is a differentiable function. A function f defined by*

$$f(t,T) = -\frac{\partial}{\partial T} logP(t,T), \quad T > 0, \quad t \in [0,T]$$

*is called **the forward rate function**.*

Remark 3.2. *The forward rate function f contains all the original bond price information. Therefore, for each $T > 0$ and $t \in [0,T]$, $f(t,T)$ is an $\mathbb{R}$-valued random variable on $(\Omega,\mathscr{F},\mathbb{P})$ and it is called **the forward rate at time** t. Thus, for each $T > 0$, the family $\{f(t,T)\}_{t\in[0,T]}$ is an $\mathbb{R}$-valued stochastic process on $[0,T]$ and is called **the forward rate process**.*

Definition 3.3. *For each $t \in [0,T]$, the function $[0,\infty) \ni T \mapsto f(t,T)$ is called **the forward curve**.*

Remark 3.3. *We always assume the forward curves to be locally integrable. If in addition, we assume $P(T,T) = 1$, then we can write the following equality*

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

Definition 3.4. *A process $\mathscr{B}$ defined by*

$$\mathscr{B}(t) = e^{\int_0^t f(s,s)ds}, \quad t \geq 0$$

*is called **saving account**, see [26] for detail explanation.*

Definition 3.5. *For each $T > 0$, a process $D(\cdot,T)$ defined by*

$$D(t,T) = \frac{P(t,T)}{\mathscr{B}(t)}, \quad t \in [0,T]$$

*is called **the discounted bond price process**, see again [26] for detail explanation.*

For the definition of self financing trading strategy, see for instance chapter 10 of [42] or chapter 20 of [45]. An arbitrage opportunity is a strategy which leads to a risk-less benefit, see [3], [4], [42] and [45] for detail. Delbaen and Schachermayer [21] proposed the following theorem for the arbitrage opportunity.

Theorem 3.1. *Let $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ be a filtered probability space. Under the appropriate assumptions, the so called no arbitrage condition holds if and only if there exists a probability measure $\hat{\mathbb{P}}$, equivalent to $\mathbb{P}$, such that for each maturity dates $T>0$, the discounted bond price process $\{D(t,T)\}_{t\in[0,T]}$ is a local martingale on $(\Omega,\mathscr{F},\hat{\mathbb{P}})$.*

Definition 3.6. *The measure $\hat{\mathbb{P}}$ in the previous Theorem is called* ***an equivalent local martingale measure*** *or* ***risk neutral measure****.*

For the existence of such a measure $\hat{\mathbb{P}}$, see for instance [26], [36] and [45].

3.2 Derivation of the HJMM Equation

In the framework of Heath-Jarrow-Morton in [36], it is assumed that for an arbitrary but fixed $T>0$, the forward rate process $\{f(t,T)\}_{t\in[0,T]}$ satisfies the following stochastic differential equation

$$df(t,T)=\alpha(t,T)dt+\sum_{i}^{d}\sigma_i(t,T)dW_i(t),\quad t\in[0,T], \tag{3.1}$$

where $W=(W_1,W_2,...,W_d)$ is a standard Brownian motion in $\mathbb{R}^d$, and for each $i=1,2,...d$, $\sigma_i(\cdot,T)\in L^1_{loc}$ and $\alpha(\cdot,T)$ is progressively measurable such that $\alpha(\cdot,T)\in L^1$. By the famous HJM no-arbitrage drift condition which is given by

$$\alpha(t,T)=\sum_{i}^{d}\sigma_i(t,T)\int_t^T\sigma_i(t,u)du, \tag{3.2}$$

see [36] for detail, equation (3.1) is as follow

$$df(t,T)=\left(\sum_{i}^{d}\sigma_i(t,T)\int_t^T\sigma_i(t,u)du\right)dt+\sum_{i}^{d}\sigma_i(t,T)dW_i(t),\quad t\in[0,T]. \tag{3.3}$$

Filipovic [26] extended this approach by considering a Wiener process on a (possibly infinite dimensional) Hilbert space instead of a standard Brownian motion in $\mathbb{R}^d$. Thus, he assumed that for an arbitrary but fixed $T>0$, the forward rate process $\{f(t,T)\}_{t\in[0,T]}$ satisfies the following stochastic differential equation

$$df(t,T)=\alpha(t,T)dt+\langle\sigma(t,T),dW(t)\rangle_H,\quad t\in[0,T], \tag{3.4}$$

where W is a Wiener process on an infinite dimensional Hilbert space H endowed with an inner product $\langle\cdot,\cdot\rangle_H$, $\alpha(\cdot,T)$ is a real-valued stochastic process on $[0,T]$ and $\sigma(\cdot,T)$ is an H-valued stochastic process on $[0,T]$. The following lemma proposed in [26] gives the HJM no-arbitrage drift condition for equation (3.4).

Lemma 3.1. *For every $T > 0$, the discounted-price process $\{D(t,T)\}_{t\in[0,T]}$ is local-martingale if and only if for all $\theta \le T$ and $t \in [0,\theta]$, the following condition holds*

$$\alpha(t,\theta) = \left\langle \sigma(t,\theta), \int_t^\theta \sigma(t,v)dv \right\rangle_H, \quad \mathbb{P}-a.s.$$

The HJM no-arbitrage drift condition for equation (3.4) driven by a Lévy process is proposed in [45]. By the HJM no-arbitrary drift condition in the previous Lemma, equation (3.4) becomes as follow

$$df(t,T) = \left\langle \sigma(t,T), \int_t^T \sigma(t,v)dv \right\rangle_H dt + \langle \sigma(t,T), dW(t)\rangle_H, \quad t \in [0,T]. \tag{3.5}$$

By using the Musiela parametrization [41], we can write equation (3.5) as a stochastic partial differential equation. For this purpose, we define, for each $t \ge 0$,

$$r(t)(x) := f(t,t+x), \quad x \ge 0,$$

where T is replaced by $x+t$ and so $x = T-t$ which is called **the time to maturity**. Then for each $t \ge 0$, $r(t)$ is a random variable on $(\Omega,\mathscr{F},\mathbb{P})$ taking values in a space U of forward curves $f:[0,\infty) \to \mathbb{R}$. Thus $\{r(t)\}_{t\ge 0}$ is an U-valued stochastic process and it is called **the forward curve process**.

Remark 3.4. *Since we assume that the forward curves are locally integrable, U has to be a space of locally integrable functions.*

Let $S = \{S(t)\}_{t\ge 0}$ be the shift-semigroup on U, i.e.

$$S(t)f(x) = f(t+x), \quad f \in U, \quad x \in [0,\infty)$$

and let $\frac{\partial}{\partial x}$ be it's infinitesimal generator. For each $t \ge 0$, define

$$\zeta(t)(x) := \sigma(t,t+x), \quad x \ge 0,$$

where $\sigma(\cdot,T)$ is an H-valued stochastic process on $[0,T]$ in equation (3.5). Also for each $t \ge 0$, define

$$a(t)(x) := \left\langle \zeta(t)(x), \int_0^x \zeta(t)(y)dy \right\rangle_H, \quad x \ge 0$$

and

$$b(t)[h](x) := \langle \zeta(t)(x), h\rangle_H, \quad x \ge 0, \quad h \in H.$$

Here U is chosen such that $a(t)$ and $b(t)h$ are U-valued stochastic processes. Then

$$\begin{aligned}
f(t,t+x) &= f(0,t+x) + \int_0^t \left\langle \sigma(s,t+x), \int_s^{t+x} \sigma(s,v)dv \right\rangle_H ds \\
&\quad + \int_0^t \langle \sigma(s,t+x), dW(s)\rangle_H \\
&= r(0)(t+x) + \int_0^t \left\langle \zeta(s)(t-s+x), \int_s^{t+x} \zeta(s)(v-s)dv \right\rangle_H ds \\
&\quad + \int_0^t \langle \zeta(s)(t-s+x), dW(s)\rangle_H \\
&= r(0)(t+x) + \int_0^t \left\langle \zeta(s)(t-s+x), \int_0^{t+x-s} \zeta(s)(y)dy \right\rangle_H ds \\
&\quad + \int_0^t \langle \zeta(s)(t-s+x), dW(s)\rangle_H \\
&= S(t)r(0)(x) + \int_0^t S(t-s)a(s)(x)ds + \int_0^t S(t-s)b(s)(x)dW(s).
\end{aligned}$$

Therefore, we infer that

$$r(t)(x) = S(t)r(0)(x) + \int_0^t S(t-s)a(s)(x)ds + \int_0^t S(t-s)b(s)(x)dW(s), \quad t,x \geq 0.$$

This is a mild version of the following stochastic partial differential equation

$$dr(t)(x) = \left(\frac{\partial}{\partial x} r(t)(x) + \left\langle \zeta(t)(x), \int_0^x \zeta(t)(y)dy \right\rangle_H \right) dt + \langle \zeta(t)(x), dW(t)\rangle_H, \quad t,x \geq 0. \tag{3.6}$$

We assume that $g : [0,\infty) \times [0,\infty) \times \mathbb{R} \to H$ is a given locally integrable function with respect to the second variable and define the process ζ by

$$\zeta(t)(x) = g(t,x,r(t)(x)), \quad t,x \geq 0.$$

That is, the volatility structure depends on the forward curve process. Then the forward curve process becomes a mild version of the following stochastic partial differential equation

$$\begin{aligned}
dr(t)(x) &= \left(\frac{\partial}{\partial x} r(t)(x) + \left\langle g(t,x,r(t)(x)), \int_0^x g(t,y,r(t)(y))dy \right\rangle_H \right) dt \\
&\quad + \langle g(t,x,r(t)(x)), dW(t)\rangle_H, \quad t,x \geq 0.
\end{aligned} \tag{3.7}$$

This equation is known as **the Heath-Jarrow-Morton-Musiela (HJMM) equation**.

3.3 The Existence and Uniqueness of Solutions to the HJMM Equation in Hilbert Spaces

The existence and uniqueness of solutions to the HJMM equation (3.7) driven by a standard Wiener process in $\mathbb{R}^d$, and ergodic properties of the solution in the Hilbert spaces have been studied in [6], [26], [28], [31], [34], [35], [41],[55] and [59]. The HJMM equation driven by a Lévy process has been studied in [29], [45] and [54]. In this section, we introduce some known results by some authors, in particular, studied the HJMM equation driven by a standard Wiener process in $\mathbb{R}^d$. Let us begin by defining the Hilbert spaces considered by these authors. We prove all results of these spaces in chapters 5-6.

Let $\nu \in \mathbb{R}$. Define L^2_ν to be the space of all (of equivalence class) Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(x)|^2 e^{\nu x} dx < \infty.$$

For $\nu = 0$, L^2_0 coincides with the usual definition of L^2.

Lemma 3.2. *The space L^2_ν is a separable Hilbert space endowed with the inner product*

$$\langle f, g\rangle_\nu = \int_0^\infty f(x)g(x)e^{\nu x} dx, \quad f, g \in L^2_\nu.$$

The norm generated by the inner product $\langle\cdot,\cdot\rangle_\nu$ is as follow

$$\|f\|_\nu = \left(\int_0^\infty |f(x)|^2 e^{\nu x} dx\right)^{\frac{1}{2}}, \quad f \in L^2_\nu.$$

Proposition 3.1. *If $\nu > 0$, then the space L^2_ν is continuously embedded into the space L^1. In particular,*

$$\|f\|_1 \le \nu^{-\frac{1}{2}} \|f\|_\nu, \quad f \in L^2_\nu.$$

Lemma 3.3. *The shift-semigroup $\{S(t)\}_{t\ge 0}$ on L^2_ν is a contraction type C_0-semigroup, in particular,*

$$\|S(t)\|_{\mathscr{L}(L^2_\nu)} \le e^{\frac{-\nu t}{2}}, \quad t \ge 0.$$

Moreover, its infinitesimal generator A is characterized by

$$\mathscr{D}(A) = \left\{f \in L^2_\nu : Df \in L^2_\nu\right\}$$

and

$$Af = Df, \quad f \in \mathscr{D}(A),$$

where Df is the first weak derivative of f.

For each $\nu \in \mathbb{R}$, define $W_\nu^{1,2}$ to be the space of all functions $f \in L_\nu^2$ such that $Df \in L_\nu^2$, i.e

$$W_\nu^{1,2} = \{f \in L_\nu^2 : Df \in L_\nu^2\}.$$

For $\nu = 0$, $W_0^{1,2}$ coincides with the usual definition of $W^{1,2}$.

Lemma 3.4. *The space $W_\nu^{1,2}$ is a separable Hilbert space endowed with the inner product*

$$\langle f, g\rangle_{W_\nu^{1,2}} = \langle f, g\rangle_\nu + \langle Df, Dg\rangle_\nu, \quad f, g \in W_\nu^{1,2}.$$

The norm generated by the inner product $\langle \cdot, \cdot\rangle_{W_\nu^{1,2}}$ is as follow

$$\|f\|_{W_\nu^{1,2}} = \|f\|_\nu + \|Df\|_\nu, \quad f \in W_\nu^{1,2}.$$

Lemma 3.5. *The shift-semigroup on $W_\nu^{1,2}$ is a contraction type C_0-semigroup. Moreover, its infinitesimal generator A is characterized by*

$$\mathscr{D}(A) = \left\{f \in W_\nu^{1,2} : Df \in W_\nu^{1,2}\right\}$$

and

$$Af = Df, \quad f \in \mathscr{D}(A).$$

Now we present a weighted Hilbert space found by Filipović [26]. Let $\mathrm{w} : \mathbb{R}^+ \to [1, \infty)$ be a nondecreasing function such that

$$\int_0^\infty \frac{dx}{\mathrm{w}^{1/3}(x)} < \infty. \tag{3.8}$$

Define $H_\mathrm{w}^{1,2}$ to be the space of all functions $f \in L_{loc}^1$ such that the first weak derivative Df exists and

$$\int_0^\infty \left|Df(x)\right|^2 \mathrm{w}(x) dx < \infty.$$

Lemma 3.6. *The $H_\mathrm{w}^{1,2}$ is a separable Hilbert space endowed with the inner product*

$$(f, g)_{H_\mathrm{w}^{1,2}} = f(0)g(0) + \int_0^\infty Df(x)Dg(x)\mathrm{w}(x)dx, \quad f, g \in H_\mathrm{w}^{1,2},$$

which leads to the following norm

$$\|f\|_\mathrm{w} = \left(|f(0)|^2 + \int_0^\infty |Df(x)|^2 \mathrm{w}(x)dx\right)^{\frac{1}{2}}, \quad f \in H_\mathrm{w}^{1,2}.$$

Lemma 3.7. *The shift-semigroup on $H^{1,2}_{\mathrm{w}}$ is a contraction C_0-semigroup and its infinitesimal generator A is characterized by*

$$\mathscr{D}(A) = \{f \in H^{1,2}_{\mathrm{w}} : Df \in H^{1,2}_{\mathrm{w}}\}$$

and

$$Af = Df, \quad f \in \mathscr{D}(A).$$

Lemma 3.8. *For each $f \in H^{1,2}_{\mathrm{w}}$, $\lim_{x\to\infty} f(x)$ exists.*

Lemma 3.9. *The space defined by*

$$H^{1,2}_{0,\mathrm{w}} = \{f \in H^{1,2}_{\mathrm{w}} : \lim_{x\to\infty} f(x) = 0\}$$

is a closed subspace of $H^{1,2}_{\mathrm{w}}$.

We now present some results about the existence and uniqueness of mild solution to the HJMM equation (3.7) driven by a standard Wiener process in $\mathbb{R}^d$ and ergodic properties of the solution in the Hilbert spaces defined above. Let us start giving the following theorem proposed by [45].

Theorem 3.2. *Let $\nu > 0$ and W be a standard Wiener process in $\mathbb{R}^d$. Assume that $g : [0,\infty)\times[0,\infty)\times\mathbb{R} \to \mathbb{R}^d$ is a measurable function such that there exist $\bar{g} \in L^2_\nu$ and $\hat{g} \in L^2_\nu \cap L^\infty$ such that for all $t \in [0,\infty)$,*

$$|g(t,x,u)| \le |\bar{g}(x)|, \quad u \in \mathbb{R}, \quad x \in [0,\infty) \tag{3.9}$$

and

$$|g(t,x,u) - g(t,x,v)| \le |\hat{g}(x)|\,|u-v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty). \tag{3.10}$$

Then for each $r_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P},L^2_\nu)$, there exists a unique L^2_ν-valued mild solution to equation (3.7) with the initial value $r(0) = r_0$.

Define a function $G : [0,\infty)\times L^2_\nu \to L_{HS}(H,L^2_\nu)$, where $L_{HS}(\mathbb{R}^d,L^2_\nu)$ is the space of all Hilbert-Schmidt operators from $\mathbb{R}^d$ to L^2_ν, by

$$G(t,f)[z](x) = \langle g(t,x,f(x)), z\rangle, \quad f \in L^2_\nu, \quad x,t \in [0,\infty), \quad z \in \mathbb{R}^d$$

and $F : [0,\infty)\times L^2_\nu \to L^2_\nu$ by

$$F(t,f)(x) = \left\langle g(t,x,f(x)), \int_0^x g(t,y,f(y))dy \right\rangle, \quad f \in L^2_\nu, \quad x \in [0,\infty),$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product of $\mathbb{R}^d$. Then the abstract form of equation (3.7) in the space L^2_ν is as follow

$$dr(t) = \big(Ar(t)+F(t,r(t))\big)dt + G(t,r(t))dW(t), \quad t \geq 0, \tag{3.11}$$

where A is the infinitesimal generator of the shift semigroup on L^2_ν. Thus, in order to prove Theorem 3.2, it is sufficient to prove that equation (3.11) has a unique L^2_ν-valued mild solution. In view of Theorem 7.4 in [50], this proof follows from the following lemma, see [45] for the proof.

Lemma 3.10. *Assume that all the assumptions in Theorem 3.2 are satisfied. Then F and G are well-defined. Moreover,*
i) for each $t \geq 0$,

$$\|F(t,f)\|_\nu + \|G(t,f)\|^2_{L_{HS}(\mathbb{R}^d,L^2_\nu)} \leq \|\bar{g}\|_1^2\|\bar{g}\|_\nu^2 + \|\bar{g}\|_\nu^2, \quad f \in L^2_\nu,$$

ii) F and G are globally Lipschitz on L^2_ν.

In the next theorem, Peszat and Zabczyk [45] give a sufficient condition for the existence and uniqueness of an invariant measure for equation (3.11) (when the coefficients are time independent) in the space L^2_ν.

Theorem 3.3. *Let* $\nu > 0$. *Assume that all the assumptions of Theorem 3.2 are satisfied for the case when* $g : [0,\infty)\times\mathbb{R} \to \mathbb{R}^d$ *is a measurable map (independent of t). If*

$$\omega = \gamma - \|\hat{g}\|_\infty - 2\big(\|\bar{g}\|_\nu^2\|\hat{g}\|_\infty^2 + \|\bar{g}\|_1^2\|\hat{g}\|_\infty^2\big),$$

then there exist a unique invariant measure for equation (3.11) in the space L^2_ν. *This invariant measure is exponentially mixing with exponent* $\frac{\omega}{2}$.

Vargiolu [55] studied the HJMM equation (3.7) with additive noise, i.e. the volatility is independent of the forward curve, in the spaces $W^{1,2}_\nu$, $\nu \leq 0$. In the following theorem, he proved the existence and uniqueness of solutions for corresponding equation in the spaces $W^{1,2}_\nu$, $\nu < 0$.

Theorem 3.4. *Let* $\nu < 0$. *Assume W is a d-dimensional standard Brownian motion. Assume that for each* $k = 1,2,..,d$, $\tau_k : [0,\infty) \to \mathbb{R}$ *is a measurable map such that*

$$\sum_{k=1}^{\infty} \|\tau_k\|^2_{W^{1,2}_\nu} < \infty, \quad \sum_{k=1}^{\infty} \|\tau_k\|^4_{W^{1,2}} < \infty, \quad \sum_{k=1}^{\infty} \|\tau_k\|^4_{L^4_\nu} < \infty$$

and $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; W_\nu^{1,2})$*. Then the following equation*

$$\begin{cases} dr(t)(x) = \left(\frac{\partial}{\partial x} r(t)(x) + \sum_{k=1}^{d} \tau_k(x) \int_0^x \tau_k(y)dy\right) dt + \sum_{k=1}^{d} \tau_k(x) dW_k(t), \\ r(0) = r_0 \end{cases} \tag{3.12}$$

has a unique $W_\nu^{1,2}$*-valued mild solution given by*

$$r(t)(x) = r_0(x+t) + \sum_{k=1}^{\infty} \int_0^t \tau_k(x+t-s)\left(\int_0^{x+t-s} \tau_k(y)dy\right) ds + \int_0^t \tau_k(x+t-s) dW_k(s). \tag{3.13}$$

Moreover, the solution is a Gaussian process with mean

$$\mathbb{E}[r(t)(x)] = \mathbb{E}[r_0(x+t)] + \frac{1}{2}\sum_{k=1}^{\infty}\left(\left(\int_0^{x+t} \tau_k(y)dy\right)^2 - \left(\int_0^{x} \tau_k(y)dy\right)^2\right) \tag{3.14}$$

and covariance

$$cov(r(t)(x), r(s)(x)) = cov(r_0(t+x), r_0(s+x)) + \sum_{k=1}^{\infty} \int_0^t \tau_k(x+t-u)\tau_k(y+s-u) du. \tag{3.15}$$

In the next theorem, Vargiolu [55] analysed the existence of invariant measures for equation (3.12) in the spaces $W_\nu^{1,2}$, $\nu < 0$.

Theorem 3.5. *Let* $\nu < 0$*. If* $\sum_{k=1}^{\infty} \|\tau_k\|^2_{W^{1,2}} < \infty$ *and* $\sum_{k=1}^{\infty} \|\tau_k\|^4_{L^4_\nu} < \infty$ *, then there exists an infinite number of invariant measures for equation (3.12) in the space* $W_\nu^{1,2}$*.*

In the following theorems, Vargiolu [55] proved the existence and uniqueness of mild solutions to equation (3.12), and the existence and uniqueness of an invariant measure for the same equation in the space $W^{1,2}$.

Theorem 3.6. *If* $\sum_{k=1}^{\infty} \|\tau_k\|^2_{W^{1,2}} < \infty$*,* $\sum_{k=1}^{\infty} \|\tau_k\|^4_{L^4} < \infty$*, the functions* $\sqrt{x}\tau_k(x)$ *and* $\sqrt{x}\tau_k^{'}(x)$ *are uniformly bounded in* L^2 *and* $r_0 \in W^{1,2}$*, then equation (3.12) has a unique* $W^{1,2}$ *valued mild solution given by (3.13). Moreover, the solution is a Gaussian process with mean (3.14) and covariance (3.15).*

Theorem 3.7. *If* $\sum_{k=1}^{\infty} \|\tau_k\|^2_{W^{1,2}} < \infty$*,* $\sum_{k=1}^{\infty} \|\tau_k\|^4_{L^4} < \infty$*,* $\sum_{k=1}^{\infty} \|\sqrt{x}\tau_k(x)\|^2_{L^2} < \infty$*,* $\sum_{k=1}^{\infty} \|\sqrt{x}\tau_k^{'}(x)\|^2_{L^2} < \infty$*,* $\sum_{k=1}^{\infty} \|\tau_k\|^2_{L^1} < \infty$*, and the functions* $\int_\cdot^\infty \tau_k(u)du$ *are uniformly bounded in* L^2*, then there exists a unique invariant measure for equation (3.12) in the space* $W^{1,2}$*.*

As a conclusion of [55], there exists only one invariant measure for equation (3.12) in the space $W^{1,2}$, however, there exists an infinite number of invariant measures for the space

$W_\nu^{1,2}$. This implies that working on $W_\nu^{1,2}$, $\nu < 0$, gives better results than working on $W^{1,2}$ for a good financial interpretation.

Tehranchi [59] studied the HJMM equation (3.7) in the space $H_{\mathrm{w}}^{1,2}$ endowed with the equivalent norm to $\|\cdot\|_w$ defined by

$$\|f\|_{H_{\mathrm{w}}^{1,2}} = \left(|f(\infty)|^2 + \int_0^\infty |Df(x)|^2 \mathrm{w}(x)dx\right)^{\frac{1}{2}}.$$

He gave the following theorem for the existence and uniqueness of solutions to equation (3.7) (driven by a cylindrical Wiener process in $\mathbb{R}^\infty$) in the space $H_{\mathrm{w}}^{1,2}$.

Theorem 3.8. *Let W be a cylindrical Wiener process in $\mathbb{R}^\infty$. Assume that for each $t \geq 0$, $G(t,\cdot)$ is a function from H_{w}^2 into $\mathscr{L}_{HS}(\mathbb{R}^d, H_{0,\mathrm{w}}^{1,2})$ defined by*

$$G(t,r(r))(x) = \sum_{k=1}^{\infty} \tau_k(x)dW^k(t),$$

where for each $k = 1,2,\dots$, τ_k is a measurable map from $[0,\infty)$ into $\mathbb{R}$, such that there exists a constant $M > 0$ such that for each $t \geq 0$ and $f \in H_{\mathrm{w}}^{1,2}$,

$$\|G(t,f)\|_{\mathscr{L}_{HS}(\mathbb{R}^d, H_{0,\mathrm{w}}^{1,2})} \leq M \tag{3.16}$$

and there exists a constant $L > 0$ such that for each $t \geq 0$,

$$\|G(t,f_1) - G(t,f_2)\|_{\mathscr{L}_{HS}(\mathbb{R}^d, H_{0,\mathrm{w}}^{1,2})} \leq L\|f_1 - f_2\|_{\mathrm{w}}, \quad f_1, f_2 \in H_{\mathrm{w}}^{1,2}, \tag{3.17}$$

where $\mathscr{L}_{HS}(\mathbb{R}^d, H_{0,\mathrm{w}}^{1,2})$ is the space of all Hilbert-Schmidt operators from $\mathbb{R}^d$ into $H_{0,\mathrm{w}}^{1,2}$. Then for any $f_0 \in H_{\mathrm{w}}^{1,2}$, there exists a unique $H_{\mathrm{w}}^{1,2}$-valued mild solution to the following equation

$$\begin{cases} dr(t)(x) = \left(\frac{\partial}{\partial x} r(t)(x) + F(t,r(t))(x)\right)dt + G(t,r(t))(x)dW(t), \\ r(0) = f_0, \end{cases} \tag{3.18}$$

where

$$F(t,r(t))(x) = \sum_{k=1}^{\infty} \tau_k(t,r(t))(x) \int_0^x \tau_k(t,r(t))(y)dy.$$

In the following result, Tehranchi [59] proposed a sufficient condition for the existence of invariant measures for equation (3.18) (when the coefficients are time independent) in the space $H_{\mathrm{w}}^{1,2}$.

Theorem 3.9. *Assume that all the assumptions of the previous Theorem are satisfied. Moreover, assume that*

$$C_{\mathrm{w}} = \left(\int_0^\infty \frac{dx}{\mathrm{w}(x)^{1/3}} \right)^{\frac{3}{2}} \quad \textit{and} \quad \alpha_{\mathrm{w}} = \inf_{x \geq 0} \frac{\mathrm{w}'(x)}{\mathrm{w}(x)}.$$

If

$$L^2 + 2C_{\mathrm{w}} ML < \alpha_{\mathrm{w}},$$

then there exists an infinite number of invariant measures for equation (3.18) (when the coefficients are time independent) in the space H^2_{w}.

For the proof, Tehranchi used the well-known theorem, see [50], about the existence and uniqueness of an invariant measure for stochastic evolutions equations. These invariant measures have some properties, see [59] for detail.

Rusinek in [52] and [54] extended the results of Tehranchi [59] considering the same equation, but driven by an infinite dimensional Lévy process. In [54], he presented some sufficient conditions for the existence and uniqueness of solutions to the HJMM equation (3.7) with Lévy noise in the Hilbert space $H^{1,2}_{\mathrm{w}}$ and the space $\hat{H}_{\mathrm{w}}$ defined by

$$\hat{H}_{\mathrm{w}} = \{ f - c : f \in H^{1,2}_{\mathrm{w}} \quad c \in \mathbb{R} \}.$$

In [52], he derived a condition using Theorem 4.1 of [53] for the existence of invariant measures for corresponding equation in the Hilbert spaces L^2_ν, $\nu > 0$. Since Lévy process is not a part of our works, we do not give the results in this thesis, see [52] and [54] for detail.

CHAPTER 4

STOCHASTIC EVOLUTION EQUATIONS (SEE) IN BANACH SPACES

This chapter, we study the stochastic evolution equations (with the linear part of the drift being only an infinitesimal generator of a C_0-semigroup and the coefficients satisfying Lipschitz conditions) in Banach spaces satisfying the H-condition. Brzeźniak, Long and Simão considered such equations in [15]. They proved the existence and uniqueness of an invariant measure for corresponding equations (when the coefficients are time independent) in Banach spaces satisfying the H-condition. However, they did not prove the existence and uniqueness of solutions for corresponding equations in such Banach spaces. Therefore, in this chapter, we first prove the existence and uniqueness of solutions for corresponding equations with globally Lipschitz coefficients. Next, using the previous existence result and approximation, we prove the existence and uniqueness of solutions to the same equations with locally Lipschitz coefficients. Moreover, we analyse the Markov property of the solution. Finally, we present results found recently by [15] about the existence and uniqueness of an invariant measure for corresponding equations when the coefficients are the time independent.

4.1 Assumptions and Definition of Mild Solution

Throughout this chapter, we assume that

- $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$, is a filtered probability space.

- $(X, \|\cdot\|_X)$ is a separable Banach space satisfying the H-condition.
- S is a contraction type C_0-semigroup on X, that is, there exists a constant $\beta \in \mathbb{R}$ such that

$$\|S(t)\|_{L(X)} \leq e^{\beta t}, \quad t \geq 0. \tag{4.1}$$

- A is the infinitesimal generator of the C_0-semigroup S.
- $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space.
- W is an H-cylindrical canonical $\mathbb{F}$-Wiener process.
- F and G are mappings from $\mathbb{R}^+ \times X$ into X and $\gamma(H,X)$ respectively (We assume some sufficient conditions on them later).
- $u_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; X)$.

We consider the following stochastic evolution equation in X,

$$\begin{cases} du(t) = \big(Au(t) + F(t,u(t))\big)dt + G(t,u(t))dW(t), \quad t \geq 0 \\ u(0) = u_0. \end{cases} \tag{4.2}$$

Definition 4.1. *An X-valued $\mathbb{F}$-progressively measurable process $\{u(t)\}_{t\geq 0}$ is called **a mild solution** to equation (4.2) if for every $t \geq 0$,*

$$\mathbb{E}\int_0^t \|u(s)\|_X^2 ds < \infty \tag{4.3}$$

and $\mathbb{P}$-a.s.

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(r,u(r))dr + \int_0^t S(t-r)G(r,u(r))dW(r), \quad t \geq 0. \tag{4.4}$$

4.2 Existence and Uniqueness of Solutions to SEE with Globally Lipschitz Coefficients

In this section, we prove the existence and uniqueness of a mild solution to equation (4.2) when the coefficients F and G are globally Lipschitz.

Lemma 4.1. *Assume that the mappings F and G are globally Lipschitz on X, i.e. for each $T>0$, there exist constants $L_F>0$ and $L_G>0$ such that for all $t\in[0,T]$,*

$$\|F(t,x_1)-F(t,x_2)\|_X \le L_F\|x_1-x_2\|_X, \quad x_1,x_2\in X \tag{4.5}$$

and

$$\|G(t,x_1)-G(t,x_2)\|_{\gamma(H,X)} \le L_G\|x_1-x_2\|_X, \quad x_1,x_2\in X. \tag{4.6}$$

Moreover, we assume that for every $T>0$,

$$\sup_{t\in[0,T]}\big(\|F(t,0)\|_X+\|G(t,0)\|_{\gamma(H,X)}\big)<\infty. \tag{4.7}$$

Then F and G are of linear growth, i.e. for each $T>0$, there exist constants $\tilde{L}_F,\tilde{L}_G>0$ such that for all $t\in[0,T]$,

$$\|F(t,x)\|_X^2 \le \tilde{L}_F^2\big(1+\|x\|_X^2\big), \quad x\in X \tag{4.8}$$

and

$$\|G(t,x)\|_{\gamma(H,X)}^2 \le \tilde{L}_G^2\big(1+\|x\|_X^2\big), \quad x\in X. \tag{4.9}$$

Proof. Fix $T>0$. Then for each $t\in[0,T]$,

$$\|F(t,x)\|_X = \|F(t,x)-F(t,0)+F(t,0)\|_X \le \|F(t,x)-F(t,0)\|_X+\|F(t,0)\|_X, \quad x\in X.$$

It follows from inequalities (4.5) and (4.7) that for each $t\in[0,T]$,

$$\|F(t,x)\|_X \le L_F\|x\|_X + C \le \big(L_F+C\big)\big(1+\|x\|_X\big), \quad x\in X,$$

which implies that F is of linear growth. Similarly, one can show that G is of linear growth. ∎

For each $T>0$, define Z_T to be the space of all X-valued, continuous, $\mathbb{F}$-adapted processes u on $[0,T]$ such that

$$\|u\|_T = \left(\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_X^2\right)^{\frac{1}{2}}<\infty. \tag{4.10}$$

By Proposition 2.19, the space Z_T is a Banach space with respect to the norm $\|\cdot\|_T$ for each $T>0$.

Lemma 4.2. *Let $T > 0$. Assume that the mapping F is globally Lipschitz on X, i.e. (4.5) holds, and for every $T > 0$,*

$$\sup_{t\in[0,T]} \|F(t,0)\|_X < \infty. \tag{4.11}$$

Moreover, we assume that for every $x \in X$, the function $F(\cdot,x):[0,\infty) \to X$ is Borel measurable. Then a map $I_F : Z_T \to Z_T$ defined by

$$I_F(u)(t) = \int_0^t S(t-r)F(r,u(r))dr, \quad u \in Z_T, \quad t \in [0,T] \tag{4.12}$$

is well-defined. Moreover, it is of linear growth and globally Lipschitz on Z_T.

Proof. Fix $u \in Z_T$. We begin showing that for each $t \ge 0$, the integral $\int_0^t S(t-r)F(r,u(r))dr$ exists $\mathbb{P}$-a.s. Fix $t \ge 0$. Since u is continuous, there exists $\bar{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for all $\omega \in \bar{\Omega}$, the mapping $u(\cdot,\omega):[0,T] \to X$ is continuous. Consider the set $\bar{\Omega}$ and fix $\omega \in \bar{\Omega}$. Define a map $\eta_\omega : [0,t] \to X$ by

$$\eta_\omega(r) = F(r,u(r,\omega)), \quad r \in [0,t].$$

We assume that for each $x \in X$, the function $F(\cdot,x):[0,t] \to X$ is Borel measurable and for each $r \in [0,t]$, the function $F(r,\cdot): X \to X$ is continuous. Therefore, by Proposition 2.18, the function $F:[0,t]\times X \to X$ is Borel measurable. Moreover, since $u(\cdot,\omega)$ is continuous, the function $u_\omega : r \in [0,t] \mapsto u_\omega(r) = (r,u(r,\omega)) \in [0,t]\times X$ is continuous and thus, it is Borel measurable. Therefore, the map $\eta_\omega = F \circ u_\omega$ is Borel measurable. Furthermore, by inequalities (4.8) and (4.10), we obtain

$$\int_0^T \|F(t,u(t,\omega))\|_X dt \le \tilde{L}_F \int_0^T \big(1+\|u(t,\omega)\|_X\big)dt \le T\tilde{L}_F + T\tilde{L}_F \sup_{t\in[0,T]} \|u(t,\omega)\|_X < \infty.$$

Hence, we have showed that the map η_ω is Bochner integrable. Since we chose arbitrary $\omega \in \bar{\Omega}$, the trajectories of the process $F(r,u(r))$, $r \in [0,t]$, are $\mathbb{P}$-a.s. Bochner integrable. Therefore, by Theorem 2.16, the integral $\int_0^t S(t-r)F(r,u(r))dr$ exists $\mathbb{P}$-a.s.

Let us now show that I_F is well-defined. Fix $u \in Z_T$ and define

$$M_T := \sup_{t\in[0,T]} \|S(t)\|_{\mathscr{L}(X)}. \tag{4.13}$$

By Proposition 2.22, the process $I_F(u)$ is continuous and also by Corollary 2.7, it has an $\mathbb{F}$-progressively measurable modification. Moreover, by the Cauchy-Schwarz inequality and inequalities (4.8) and (4.13), we get

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\|I_F(u)(t)\|_X^2 &= \mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t S(t-r)F(r,u(r))dr\right\|_X^2\\
&\le T\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|S(t-r)F(r,u(r))\|_X^2 dr\\
&\le T\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|S(t-r)\|_{\mathscr{L}(X)}^2\|F(r,u(r))\|_X^2 dr\\
&\le TM_T^2\tilde{L}_F^2\mathbb{E}\sup_{t\in[0,T]}\int_0^t \left(1+\|u(r)\|_X^2\right)dr\\
&\le T^2M_T^2\tilde{L}_F^2 + T^2M_T^2\tilde{L}_F^2\mathbb{E}\sup_{t\in[0,T]}\|u(r)\|_X^2.
\end{aligned}$$

Thus $I_F(u)\in Z_T$. Therefore, I_F is well-defined. Moreover, it follows from the last inequality that

$$\|I_F(u)\|_T^2 \le T^2M_T^2\tilde{L}_F^2\left(1+\|u\|_T^2\right), \tag{4.14}$$

which implies that I_F is of linear growth.

Finally, we show that I_F is globally Lipschitz on Z_T. Fix $u_1,u_2\in Z_T$. Then by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|I_F(u_1)-I_F(u_2)\|_T^2 &= \mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t S(t-r)\left[F(r,u_1(r))-F(r,u_2(r))\right]dr\right\|_X^2\\
&\le T\mathbb{E}\sup_{t\in[0,T]}\int_0^t \left\|S(t-r)\left[F(r,u_1(r))-F(r,u_2(r))\right]\right\|_X^2 dr\\
&\le T\mathbb{E}\sup_{t\in[0,T]}\int_0^t \left\|S(t-r)\|_{\mathscr{L}(X)}\right\|F(r,u_1(r))-F(r,u_2(r))\big\|_X^2 dr.
\end{aligned}$$

It follows from inequalities (4.5) and (4.13) that

$$\begin{aligned}
\|I_F(u_1)-I_F(u_2)\|_T^2 &\le TM_T^2L_F^2\mathbb{E}\sup_{t\in[0,T]}\int_0^t \|u_1(r)-u_2(r)\|_X^2 dr\\
&\le T^2M_T^2L_F^2\mathbb{E}\sup_{r\in[0,T]}\|u_1(r)-u_2(r)\|_X^2.
\end{aligned}$$

Therefore, we have showed that

$$\|I_F(u_1)-I_F(u_2)\|_T \le M_TL_FT\|u_1-u_2\|_T. \tag{4.15}$$

Thus, I_F is globally Lipschitz on Z_T.

■

Lemma 4.3. *Let $T > 0$. Assume that the mapping G is globally Lipschitz on X, i.e. (4.6) holds, and for every $T > 0$,*

$$\sup_{t\in[0,T]} \|G(t,0)\|_{\gamma(H,X)} < \infty. \tag{4.16}$$

Moreover, we assume that for every $x \in X$, the function $G(\cdot,x) : [0,\infty) \to \gamma(H,X)$ are Borel measurable. Then the mapping $I_G : Z_T \to Z_T$ defined by

$$I_G(u)(t) = \int_0^t S(t-r)G(r,u(r))dW(r), \quad u \in Z_T, \quad t \in [0,T] \tag{4.17}$$

is well-defined. Moreover, it is of linear growth and globally Lipschitz on Z_T.

Proof. Fix $u \in Z_T$. We first show that for each $t \geq 0$, the integral $\int_0^t S(t-r)G(r,u(r))dW(r)$ exists. Fix $t \geq 0$. It is obvious from the definition of G that the process $G(r,u(r))$, $r \in [0,t]$, is $\gamma(H,X)$-valued. We assume that for each $x \in X$, the function $G(\cdot,x) : [0,t] \to \gamma(H,X)$ is Borel measurable and for each $r \in [0,t]$, $G(r,\cdot) : X \to \gamma(H,X)$ is continuous. Hence, by Proposition 2.18, the function $G : [0,t] \times X \to \gamma(H,X)$ is Borel measurable. Moreover, the function $\bar{u} : (r,\omega) \ni [0,t] \times \Omega \mapsto (r,u(r,\omega)) \in [0,t] \times X$ is $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$-measurable. Therefore, the function $G(\cdot,u(\cdot)) = G \circ \bar{u}$ is $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$-measurable. Thus, the process $G(r,u(r))$, $r \in [0,t]$, is $\mathbb{F}$-progressively measurable. Furthermore, by inequalities (4.9) and (4.10), we get

$$\mathbb{E}\int_0^t \|G(r,u(r))\|^2_{\gamma(H,X)}dr \leq \tilde{L}_G^2 \mathbb{E}\int_0^t \big(1 + \|u(r)\|_X^2\big)dr \leq t\tilde{L}_G^2 + t\tilde{L}_G^2 \mathbb{E} \sup_{r\in[0,t]} \|u(r)\|_X^2 < \infty.$$

Therefore, by Corollary 2.10, the stochastic integral $\int_0^t S(t-r)G(r,u(r))dW(r)$ exists.

Now we show that I_G is well-defined. Fix $u \in Z_T$. By Corollary 2.10, the process $I_G(u)$ has a continuous and $\mathbb{F}$-progressively modification. Moreover, by Corollary 2.11 and inequality (4.9), we obtain

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \|I_G(u)(t)\|_X^2 &\leq K_T \mathbb{E}\int_0^T \|G(t,u(t))\|^2_{\gamma(H,X)}dt \\
&\leq K_T\tilde{L}_G^2 \mathbb{E}\int_0^T \big(1 + \|u(t)\|_X^2\big)dt \\
&\leq TK_T\tilde{L}_G^2 + TK_T\tilde{L}_G^2 \mathbb{E} \sup_{t\in[0,T]} \|u(t)\|_X^2.
\end{aligned}$$

Hence $I_G(u) \in Z_T$. Therefore, I_G is well-defined, Furthermore, it follows from the last inequality that

$$\|I_G(u)\|_T^2 \leq TK_T\tilde{L}_G^2\big(1 + \|u\|_T^2\big), \tag{4.18}$$

which implies that I_G is of linear growth.

Finally, we show that $I_G(u)$ is globally Lipschitz on Z_T. Fix $u_1, u_2 \in Z_T$. Then by Corollary 2.11 and inequality (4.6), we get

$$\begin{aligned}
\|I_G(u_1) - I_G(u_2)\|_T^2 &= \mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t S(t-r)\big[G(r,u_1(r)) - G(r,u_2(r))\big]dW(r) \right\|_X^2 \\
&\le K_T \mathbb{E} \int_0^T \|G(t,u_1(r)) - G(t,u_2(r))\|_{\gamma(H,X)}^2 dr \\
&\le K_T L_G^2 \mathbb{E} \int_0^T \|u_1(r) - u_2(r)\|_X^2 dr \\
&\le K_T T L_G^2 \mathbb{E} \sup_{r\in[0,T]} \|u_1(r) - u_2(r)\|_X^2 .
\end{aligned}$$

Thus, we have showed that

$$\|I_G(u_1) - I_G(u_2)\|_T \le L_G \sqrt{K_T T}\, \|u_1 - u_2\|_T, \tag{4.19}$$

which implies that I_G is globally Lipschitz on Z_T. ■

Let us now present of the main result of this section with the proof.

Theorem 4.1. *Assume that the mappings F and G are globally Lipschitz on X, i.e. (4.5) and (4.6) hold respectively. Moreover, we assume that for every $x \in X$, the mappings $F(\cdot,x) : [0,\infty) \to X$ and $G(\cdot,x) : [0,\infty) \to \gamma(H,X)$ are Borel measurable and for every $T > 0$,*

$$\sup_{t\in[0,T]} \big(\|F(t,0)\|_X + \|G(t,0)\|_{\gamma(H,X)} \big) < \infty.$$

Then there exists a unique X-valued continuous mild solution to equation (4.2).

Proof. By the definition of the mild solution, it is sufficient to show that the integral equation (4.4) has a unique X-valued continuous solution. Define a function $\Phi : Z_T \to Z_T$ by

$$\Phi(u)(t) = S(t)u_0 + \int_0^t S(t-r)F(r,u(r))dr + \int_0^t S(t-r)G(r,u(r))dW(r), \quad u \in Z_T, \quad t \in [0,T]. \tag{4.20}$$

It is obvious that the process $S(\cdot)u_0$ is $\mathbb{F}$-adapted. Moreover, for every $\omega \in \Omega$, $S(\cdot)u_0(\omega) : [0,\infty) \to X$ is continuous and $\mathbb{E}\|u_0\|_X^2 < \infty$. Thus, the process $S(\cdot)u_0$ belongs to Z_T. Therefore, by Lemma 4.2 and Lemma 4.3, Φ is well-defined and of linear growth. Moreover, by inequalities (4.15) and (4.19), we infer that

$$\|\Phi(u_1) - \Phi(u_2)\|_T \le C(T)\|u_1 - u_2\|_T, \quad u_1, u_2 \in Z_T, \tag{4.21}$$

where $C(T) = M_T L_F T + L_G\sqrt{K_T T}$. Hence, Φ is globally Lipschitz on Z_T. If we choose T small enough, say T_0, such that $C(T_0) \leq \frac{1}{2}$, then by the Banach Fixed Point Theorem, there exists a unique process $u^1 \in Z_{T_0}$ such that $\Phi(u^1) = u^1$. Therefore, the integral equation (4.4) has a unique, X-valued, continous solution u^1 on $[0, T_0]$.
Define $Z_{(k-1)T_0,kT_0}$, $k = 1,2,3...$, to be the space of all X-valued, continuous, $\mathbb{F}$-adapted stochastic processes u on $[(k-1)T_0, kT_0]$ such that

$$\mathbb{E}\sup_{t\in[(k-1)T_0,kT_0]}\|u(t)\|_X^2 < \infty.$$

It is obvious that for each k, $Z_{(k-1)T_0,kT_0}$ is a Banach space endowed with the norm

$$\|u\|_{(k-1)T_0,kT_0} = \left(\mathbb{E}\sup_{t\in[(k-1)T_0,kT_0]}\|u(t)\|_X^2\right)^{\frac{1}{2}}.$$

As shown above, it can be easily shown that the following equation

$$\begin{aligned} u(t) = S(t-(k-1)T_0)u((k-1)T_0) + \int_{(k-1)T_0}^t S(t-r)F(r,u(r))dr \\ + \int_{(k-1)T_0}^t S(t-r)G(r,u(r))dW(r) \end{aligned} \tag{4.22}$$

has a unique, X-valued, continuous solution u^k on $[(k-1)T_0, kT_0]$ such that $u^k(kT_0) = u^{k+1}(kT_0)$. Consequently, we have a sequence $(u^k)_{k\in\mathbb{N}}$ of solutions. Define a process u by

$$u(t) = \sum_{k=1}^{\infty} u^k(t)1_{[(k-1)T_0,kT_0]}(t), \quad t \in [0,\infty). \tag{4.23}$$

This process is a unique X-valued, continuous solution to integral equation (4.4). Let us prove this. It is obvious that the process u is continuous and $\mathbb{F}$-adapted. Therefore, by Proposition 2.18, u is $\mathbb{F}$-progressively-measurable. Furthermore, for each $t \geq 0$,

$$\mathbb{E}\int_0^t \|u(r)\|_X^2 dr < \infty.$$

Thus, the integrals in integral equation (4.4) exist for the process u. Now we show that the process u solves integral equation (4.4). We have already proved that for $k = 1$, the process u on $[0, T_0]$ solves integral equation (4.4). By induction, we assume that the process u on $[0, kT_0]$ solves integral equation (4.4). Thus, we have

$$\begin{aligned} u(kT_0) = u^k(kT_0) = S(kT_0)u_0 + \int_0^{kT_0} S(kT_0-r)F\big(r,u(r)\big)dr \\ + \int_0^{kT_0} S(kT_0-r)G\big(r,u(r)\big)dW(r). \end{aligned} \tag{4.24}$$

We need to show that the process u on $[0,(k+1)T_0]$ solves equation (4.4). Since u^{k+1} on $[kT_0,(k+1)T_0]$ solves equation (4.22), we get

$$\begin{aligned} u^{k+1}(t) &= S(t-kT_0)u^{k+1}(kT_0) + \int_{kT_0}^t S(t-r)F\big(t,u^{k+1}(r)\big)dr \\ &\quad + \int_{kT_0}^t S(t-r)G\big(t,u^{k+1}(r)\big)dW(r), \quad t \in [kT_0,(k+1)T_0]. \end{aligned} \tag{4.25}$$

Since $u^k(kT_0) = u^{k+1}(kT_0)$ and by (4.24), we obtain

$$\begin{aligned} u(t) = S(t)u_0 &+ \int_0^{kT_0} S(t-r)F\big(r,u(r)\big)dr + \int_0^{kT_0} S(t-r)G\big(r,u(r)\big)dW(r) \\ &+ \int_{kT_0}^t S(t-r)F\big(t,u^{k+1}(r)\big)dr + \int_{kT_0}^t S(t-r)G\big(t,u^{k+1}(r)\big)dW(r) \\ &= S(t)u_0 + \int_0^t S(t-r)F(r,u(r))dr + \int_0^t S(t-r)G(r,u(r))dW(r), \quad t \geq 0. \end{aligned}$$

Therefore, the process u is an X-valued continuous solution to integral equation (4.4).

Uniqueness : In principle, the uniqueness of solutions follows from our proof via the Banach Fixed Point Theorem. However, for completeness and educational purposes, we present now our independent proof. Let u_1 and u_2 be two solutions to equation (4.4). Define a process z by

$$z(t) = u_1(t) - u_2(t), \quad t \geq 0.$$

Thus

$$z(t) = I_F(t) + I_G(t), \quad t \geq 0,$$

where

$$\begin{aligned} I_F(t) &:= \int_0^t S(t-r)\big[F(r,u_1(r)) - F(t,u_2(r))\big]dr, \quad t \geq 0 \\ I_G(t) &:= \int_0^t S(t-r)\big[G(r,u_1(r)) - G(t,u_2(r))\big]dW(r), \quad t \geq 0. \end{aligned}$$

Therefore, we have

$$\mathbb{E}\|z(t)\|_X^2 \leq 2\mathbb{E}\|I_F(t)\|_X^2 + 2\mathbb{E}\|I_G(t)\|_X^2, \quad t \geq 0.$$

Using the Cauchy-Schwarz inequality, (4.5) and (4.13), we have

$$\begin{aligned} \mathbb{E}\|I_F(t)\|_X^2 &\leq t\mathbb{E}\int_0^t \|S(t-r\|_{\mathscr{L}(X)}^2 \big\|F(r,u_1(r)) - F(r,u_2(r))\big\|_X^2 dr \\ &\leq tL_F^2 M_t^2 \int_0^t \mathbb{E}\|u_1(r) - u_2(r)\|_X^2 dr, \quad t \geq 0. \end{aligned}$$

Also by Theorem 2.18, and inequalities (4.6) and (4.13), we have

$$\begin{aligned}\mathbb{E}\|I_G(t)\|_X^2 &\le C\mathbb{E}\int_0^t \big\|S(t-r)\big[G(r,u_1(r))-G(r,u_2(r))\big]\big\|_{\gamma(H,X)}^2 dr\\ &\le C\mathbb{E}\int_0^t \|S(t-r)\|_{\mathscr{L}(X)}^2 \big\|G(r,u_1(r))-G(r,u_2(r))\big\|_{\gamma(H,X)}^2 dr\\ &\le CL_G^2M_t^2\int_0^t \mathbb{E}\|u_1(r)-u_2(r)\|_X^2 dr, \quad t\ge 0.\end{aligned}$$

Taking into account last two estimates, we infer that

$$\mathbb{E}\|z(t)\|_X^2 \le K(t)\int_0^t \mathbb{E}\|z(r)\|_X^2 dr, \quad t\ge 0,$$

where $K(t) = 2tL_F^2M_t^2 + 2CL_G^2M_t^2$. Define a function φ by

$$\varphi(t) = \mathbb{E}\|z(t)\|_X^2, \quad t\ge 0.$$

Therefore

$$\varphi(t) \le K(t)\int_0^t \varphi(r)dr, \quad t\ge 0.$$

Applying Gronwall's lemma to the function φ, we obtain that for all $t\ge 0$, $\varphi(t)=0$. Therefore, we infer that $z=0$ and so $u_1=u_2$. Thus, the process u is a unique X-valued continuous solution to integral equation (4.4). ■

Theorem 4.2. *For any $\zeta\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};X)$, let us denote the solution of equation (4.4) by $u(\cdot,\zeta)$. Then for every $T>0$, there exists a constant $C_T>0$ such that for all $t\in[0,T]$,*

$$\mathbb{E}\|u(t,\zeta)\|_X^2 \le C_T\big(1+\mathbb{E}\|\zeta\|_X^2\big), \quad \zeta\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};X) \tag{4.26}$$

and

$$\mathbb{E}\|u(t,\zeta)-u(t,\delta)\|_X^2 \le C_T\mathbb{E}\|\zeta-\delta\|_X^2, \quad \zeta,\delta\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};X). \tag{4.27}$$

Proof. Fix $T>0$ and $\zeta\in L^2(\Omega,\mathscr{F}_s,\mathbb{P};X)$. Then

$$\begin{aligned}\mathbb{E}\|u(t,\zeta)\|_X^2 \le 3\mathbb{E}\|S(t)\zeta\|_X^2 &+ 3\mathbb{E}\left\|\int_0^t S(t-r)F(r,u(r,\zeta))dr\right\|_X^2\\ &+3\mathbb{E}\left\|\int_0^t S(t-r)G(r,u(r,\zeta))dW(r)\right\|_X^2, \quad t\in[0,T].\end{aligned}$$

Define functions I_F and I_G by

$$I_F(t) = \mathbb{E}\left\|\int_0^t S(t-r)F(r,u(r,\zeta))dr\right\|_X^2, \quad t \in [0,T]$$
$$I_G(t) = \mathbb{E}\left\|\int_0^t S(t-r)G(r,u(r,\zeta))dW(r)\right\|_X^2, \quad t \in [0,T].$$

Then

$$\mathbb{E}\|u(t,\zeta)\|_X^2 \le 3\mathbb{E}\|S(t)\zeta\|_X^2 + 3\big(I_F(t) + I_G(t)\big), \quad t \in [0,T].$$

By the Cauch-Schwarz inequality, (4.8) and (4.13), we obtain

$$\begin{aligned} I_F(t) &\le T\mathbb{E}\int_0^t \|S(t-r)F(r,u(r,\zeta))\|_X^2 dr \\ &\le T\mathbb{E}\int_0^t \|S(t-r)\|_{\mathscr{L}(X)}^2 \|F(r,u(r,\zeta))\|_X^2 dr \\ &\le T\tilde{L}_F^2 M_T^2 \mathbb{E}\int_0^t \big(1 + \|u(r,\zeta)\|_X^2\big)dr \\ &= T^2\tilde{L}_F^2 M_T^2 + T\tilde{L}_F^2 M_T^2 \int_0^t \mathbb{E}\|u(r,\zeta)\|_X^2 dr, \quad t \in [0,T]. \end{aligned} \tag{4.28}$$

Similarly, using inequalities (2.41), (4.9) and (4.13), we get

$$\begin{aligned} I_G(t) &\le C\mathbb{E}\int_0^t \|S(t-r)G(r,u(r,\zeta))\|_{\gamma(H,X)}^2 dr \\ &\le C\mathbb{E}\int_0^t \|S(t-r)\|_{\mathscr{L}(X)}^2 \|G(r,u(r,\zeta))\|_{\gamma(H,X)}^2 dr \\ &\le C\tilde{L}_G^2 M_T^2 \mathbb{E}\int_0^t \big(1 + \|u(r,\zeta)\|_X^2\big)dr \\ &\le CT\tilde{L}_G^2 M_T^2 + C\tilde{L}_G^2 M_T^2 \int_0^t \mathbb{E}\|u(r,\zeta)\|_X^2 dr, \quad t \in [0,T]. \end{aligned} \tag{4.29}$$

Moreover, by (4.13), we have

$$\mathbb{E}\|S(t)\zeta\|_X^2 \le M_T^2 \mathbb{E}\|\zeta\|_X^2, \quad t \in [0,T]. \tag{4.30}$$

Taking into account estimates (4.28), (4.29) and (4.30), we infer that there exists constants $K_T, \tilde{K}_T > 0$ such that

$$\mathbb{E}\|u(t,\zeta)\|_X^2 \le K_T + \tilde{K}_T \int_0^t \mathbb{E}\|u(r,\zeta)\|_X^2 dr, \quad t \in [0,T].$$

Define a function φ by

$$\varphi(t) = \mathbb{E}\|u(t,\zeta)\|_X^2, \quad t \in [0,T]$$

and thus, we get

$$\varphi(t) \le K_T + \tilde{K}_T \int_0^t \varphi(r)dr, \quad t \in [0,T].$$

Applying Gronwall's lemma to the function φ, we obtain the desired result (4.26).

Let us now prove (4.27). Fix $\zeta, \delta \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; X)$. Then

$$\mathbb{E}\|u(t,\zeta) - u(t,\delta)\|_X^2 \le 3M_T^2 \mathbb{E}\|\zeta - \delta\|_X^2 + 3\big(I_F(t) + I_G(t)\big), \quad t \in [0,T],$$

where

$$\begin{aligned} I_F(t) &= \mathbb{E}\left\| \int_o^t S(t-r)\big[F(r,u(r,\zeta)) - F(r,u(r,\delta))\big]dr \right\|_X^2, \quad t \in [0,T] \\ I_G(t) &= \mathbb{E}\left\| \int_0^t S(t-r)\big[G(r,u(r,\zeta)) - G(r,u(r,\delta))\big]dW(r) \right\|_X^2, \quad t \in [0,T]. \end{aligned}$$

By the Cauchy-Schwarz inequality, (4.5) and (4.13), we obtain

$$\begin{aligned} I_F(t) &\le T\mathbb{E}\int_0^t \|S(t-r)\big[F(r,u(r,\zeta)) - F(r,u(r,\delta))\big]\|_X dr \\ &\le TM_T^2 \mathbb{E}\int_0^t \|F(r,u(r,\zeta)) - F(r,u(r,\delta))\|_X dr \\ &\le TM_T^2 L_F^2 \int_0^t \mathbb{E}\|u(r,\zeta) - u(r,\delta)\|_X dr, \quad t \in [0,T]. \end{aligned} \tag{4.31}$$

Similarly, using (2.41), (4.6) and (4.13), we get

$$\begin{aligned} I_G(t) &\le C\mathbb{E}\int_0^t \|S(t-r)\big[G(r,u(r,\zeta)) - G(r,u(r,\delta))\big]\|_{\gamma(H,X)}^2 dr \\ &\le CM_T^2 \mathbb{E}\int_0^t \|G(r,u(r,\zeta)) - G(r,u(r,\delta))\|_{\gamma(H,X)}^2 dr \\ &\le CM_T^2 L_G^2 \int_0^t \mathbb{E}\|u(r,\zeta) - u(r,\delta)\|_X^2 dr, \quad t \in [0,T]. \end{aligned} \tag{4.32}$$

Define a function ψ by

$$\psi(t) = \mathbb{E}\|u(t,\zeta) - u(t,\delta)\|_X^2, \quad t \in [0,T].$$

Taking into account estimates (4.31) and (4.32), we obtain

$$\psi(t) \le 3M_T^2 \mathbb{E}\|\zeta - \delta\|_X^2 + 3\big(TM_T^2 L_F^2 + CM_T^2 L_G^2\big)\int_0^t \psi(r)dr, \quad t \in [0,T].$$

Applying Gronwall's lemma to the function ψ, we infer the desired conclusion (4.27). ■

4.3 Existence and Uniqueness of Solutions to SEE with Locally Lipschitz Coefficients

In this section, using the existence result from the previous section and approximation, we prove the existence and uniqueness of a mild solution to equation (4.2) when the coefficients F and G are Lipschitz on balls.

Lemma 4.4. *Assume that the mapping $F : \mathbb{R}^+ \times X \to X$ is Lipschitz on balls, i.e. for each $T > 0$ and $R > 0$, there exists a constant $L_F(R) > 0$ such that if $t \in [0,T]$ and $\|x_1\|_X, \|x_2\|_X \le R$, then*

$$\|F(t,x_1) - F(t,x_2)\|_X \le L_F(R)\|x_1 - x_2\|_X. \tag{4.33}$$

Moreover, F is of linear growth (uniformly in t), i.e. for all $T > 0$, there exists a constant $\bar{L}_F > 0$ such that for all $t \in [0,T]$,

$$\|F(t,x)\|_X^2 \le \bar{L}_F^2\big(1 + \|x\|_X^2\big), \quad x \in X. \tag{4.34}$$

For each $n \in \mathbb{N}$, define a map $F^n : \mathbb{R}^+ \times X \to X$ by

$$F^n(\cdot,x) = \begin{cases} F(\cdot,x), & \|x\|_X \le n \\ F\left(\cdot, n\frac{x}{\|x\|_X}\right), & \|x\|_X > n. \end{cases} \tag{4.35}$$

Then for each $n \in \mathbb{N}$, F^n is globally Lipschitz on X with a constant $3L_F(n)$ (independent of t). Moreover, F^n is of linear growth, i.e. (4.34) holds.

Proof. See [13] for the proof of globally Lipschitz of F^n. Let us prove that F^n is of linear growth. Fix $n \in \mathbb{N}$ and $T > 0$.
Case 1 : Consider $x \in X$ such that $\|x\|_X \le n$. Thus $F^n(t,x) = F(t,x)$ and so using (4.34), we obtain, for each $t \in [0,T]$,

$$\left\|F^n(t,x)\right\|_X^2 = \|F(t,x)\|_X^2 \le \bar{L}_F^2\left(1 + \|x\|_X^2\right).$$

Case 2 : Consider $x \in X$ such that $\|x\|_X > n$. Thus, $F^n(t,x) = F\left(t, n\frac{x}{\|x\|_X}\right)$. Again by (4.34), we get, for each $t \in [0,T]$,

$$\left\|F^n(t,x)\right\|_X^2 \le \bar{L}_F^2\left(1 + n^2\left\|\frac{x}{\|x\|_X}\right\|_X^2\right) \le \bar{L}_F^2\left(1 + \|x\|_X^2\right).$$

Hence, we infer that for every $t \in [0,T]$,

$$\left\|F^n(t,x)\right\|_X^2 \le \bar{L}_F^2\left(1 + \|x\|_X^2\right), \quad x \in X. \tag{4.36}$$

Therefore, F^n is of linear growth. ■

Lemma 4.5. *Assume that the mapping $G : \mathbb{R}^+ \times X \to \gamma(H,X)$ is Lipschitz on balls, i.e. for each $T > 0$ and $R > 0$, there exists a constant $L_F(R) > 0$ such that if $t \in [0,T]$ and $\|x_1\|_X, \|x_2\|_X \le R$, then*

$$\|G(t,x_1) - G(t,x_2)\|_{\gamma(H,X)} \le L_G(R)\|x_1 - x_2\|_X. \tag{4.37}$$

Moreover, G is of linear growth (uniformly in t), i.e. for all $T > 0$, there exists a constant $\bar{L}_G > 0$ such that for all $t \in [0,T]$,

$$\|G(t,x)\|^2_{\gamma(H,X)} \le \bar{L}_G^2\big(1 + \|x\|_X^2\big), \quad x \in X. \tag{4.38}$$

For each $n \in \mathbb{N}$, define a map $G^n : \mathbb{R}^+ \times X \to \gamma(H,X)$ by

$$G^n(\cdot,x) = \begin{cases} G(\cdot,x), & \|x\|_X \le n, \\ G\left(\cdot, n\frac{x}{\|x\|_X}\right), & \|x\|_X > n. \end{cases} \tag{4.39}$$

Then for each $n \in \mathbb{N}$, G^n is globally Lipschitz on X with a constant $3L_G(n)$ (independent of t). Moreover, G^n is of linear growth, i.e.

$$\left\|G^n(t,x)\right\|^2_{\gamma(H,X)} \le \bar{L}_G^2\left(1 + \|x\|_X^2\right), \quad x \in X. \tag{4.40}$$

The proof is similar to the proof of the previous lemma. In view of Theorem 4.1, we have the following natural conclusion.

Corollary 4.1. *Assume that the mappings F and G are of linear growth and Lipschitz on balls. Moreover, we assume that for each $T > 0$ and $x \in X$, the functions $F(\cdot,x) : [0,T] \ni t \to F(t,x) \in X$ and $G(\cdot,x) : [0,T] \ni t \to G(t,x) \in \gamma(H,X)$ are Borel measurable. Then for each $n \in \mathbb{N}$, the following stochastic evolution equation*

$$\begin{cases} du^n(t) = \big(Au^n(t) + F^n(t,u^n(t))\big)dt + G^n(t,u^n(t))dW(t), \\ u^n(0) = u_0, \end{cases} \tag{4.41}$$

where F^n and G^n are mappings defined in (4.35) and (4.39) respectively, has a unique X-valued continuous mild solution.

Lemma 4.6. *Assume that u^n is the unique solution of equation (4.41). Then for every $T > 0$, there exists a constant $C(T) > 0$ (independent of n) such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|u^n(t)\|_X^2 \le C(T). \tag{4.42}$$

Proof. Fix $n \in \mathbb{N}$ and $T > 0$. Then by the definition of the mild solution, we have, for each $t \in [0,T]$,

$$u^n(t) = S(t)u_0 + \int_0^t S(t-r)F^n\big(r, u^n(r)\big)dr + \int_0^t S(t-r)G^n\big(r, u^n(r)\big)dW(r).$$

Thus, by the triangle inequality, we obtain, for each $s \in [0,T]$,

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,s]}\left\|u^n(t)\right\|_X^2 \le 3\mathbb{E}\sup_{t\in[0,s]}\|S(t)u_0\|_X^2 + 3\mathbb{E}\sup_{t\in[0,s]}\left\|\int_0^t S(t-r)F^n\big(r,u^n(r)\big)dr\right\|_X^2 \\ + 3\mathbb{E}\sup_{t\in[0,s]}\left\|\int_0^t S(t-r)G^n\big(r,u^n(r)\big)dW(r)\right\|_X^2.\end{aligned}$$

By the Cauchy-Schwarz inequality, (4.13) and (4.36), we obtain

$$\begin{aligned}\left\|\int_0^t S(t-r)F^n\left(r,u^n(r)\right)dr\right\|_X^2 &\le t\int_0^t \left\|S(t-r)F^n\left(r,u^n(r)\right)\right\|_X^2 dr \\ &\le t\int_0^t \left\|S(t-r)\right\|_{\mathscr{L}(X)}^2 \left\|F^n\left(r,u^n(r)\right)\right\|_X^2 dr \\ &\le tM_t^2\bar{L}_F^2\int_0^t \left(1+\left\|u^n(r)\right\|_X^2\right)dr \\ &= t^2M_t^2\bar{L}_F^2 + tM_t^2\bar{L}_F^2\int_0^t \left\|u^n(r)\right\|_X^2 dr \\ &\le t^2M_t^2\bar{L}_F^2 + tM_t^2\bar{L}_F^2\int_0^t \sup_{t\in[0,r]}\left\|u^n(t))\right\|_X^2 dr.\end{aligned}$$

Therefore, we deduce that for each $s \in [0,T]$,

$$\mathbb{E}\sup_{t\in[0,s]}\left\|\int_0^t S(t-r)F^n\left(r,u^n(r)\right)dr\right\|_X^2 \le T^2M_T^2\bar{L}_F^2 + TM_T^2\bar{L}_F^2\int_0^s \mathbb{E}\sup_{t\in[0,r]}\left\|u^n(t))\right\|_X^2 dr. \tag{4.43}$$

By Corollary 2.11 and inequality (4.40), we have, for each $s \in [0,T]$,

(4.44)
$$\begin{aligned}\mathbb{E}\sup_{t\in[0,s]}\left\|\int_0^t S(t-r)G^n\left(r,u^n(r)\right)dW(r)\right\|_X^2 &\le K_T\mathbb{E}\int_0^s \left\|G^n\left(r,u^n(r)\right)\right\|_{\gamma(H,X)}^2 dr \\ &\le K_T\bar{L}_G^2\mathbb{E}\int_0^s \left(1+\left\|u^n(r)\right\|_X^2\right)dr \\ &\le K_T T\bar{L}_G^2 + K_T\bar{L}_G^2\int_0^s \mathbb{E}\sup_{t\in[0,r]}\left\|u^n(t))\right\|_X^2 dr.\end{aligned}$$

Moreover, by (4.13), we get

$$\mathbb{E}\sup_{t\in[0,s]}\|S(t)u_0\|_X^2 \le M_T\|u_0\|_X^2, \quad s\in[0,T]. \tag{4.45}$$

Taking into account estimates (4.43), (4.44) and (4.45), we infer that there exist constants $M(T) > 0$ and $L(T) > 0$ such that

$$\mathbb{E} \sup_{t\in[0,s]} \|u^n(t)\|_X^2 \le M(T) + L(T) \int_0^s \mathbb{E} \sup_{t\in[0,r]} \|u^n(t)\|_X^2 dr, \quad s \in [0,T]. \tag{4.46}$$

Define a function ψ by

$$\psi(s) = \mathbb{E} \sup_{t\in[0,s]} \|u^n(t)\|_X^2, \quad s \in [0,T].$$

Then inequality (4.46) can be rewritten as follow

$$\psi(s) \le M(T) + L(T) \int_0^s \psi(r) dr.$$

Applying Gronwall's lemma to the function ψ, we deduce that

$$\psi(s) \le M(T) e^{L(T)T}, \quad s \in [0,T].$$

Therefore, we have showed that

$$\mathbb{E} \sup_{t\in[0,T]} \|u^n(t)\|_X^2 \le M(T) e^{L(T)T},$$

which gives the desired result (4.42). ∎

Lemma 4.7. *For each $n \in \mathbb{N}$, define a function $\tau_n : \Omega \to [0,\infty]$ by*

$$\tau_n(\omega) = \inf\{t \in [0,\infty) : \|u^n(t,\omega)\|_X \ge n\}, \quad \omega \in \Omega.$$

Then for each $n \in \mathbb{N}$, τ_n is a stopping time. Moreover, the sequence $(\tau_n)_{n\in\mathbb{N}}$ converges to ∞.

Proof. It was proven in [37] that for each $n \in \mathbb{N}$, τ_n is a stopping time. Thus, we only prove that the sequence $(\tau_n)_{n\in\mathbb{N}}$ converges to ∞. For this aim, we need to show that there exists $\bar{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for all $\omega \in \bar{\Omega}$, $\tau_n(\omega) \to \infty$, i.e. for every $T > 0$, there exists $k \in \mathbb{N}$ such that for all $n \ge k$, $\tau_n(\omega) \ge T$. Therefore, it is sufficient to show that for each $T > 0$,

$$\mathbb{P}\big(\{\omega \in \Omega \mid \exists\, k \in \mathbb{N} : \forall\, n \ge k\ \tau_n(\omega) \ge T\}\big) = 1. \tag{4.47}$$

Fix $T > 0$. For each $n \in \mathbb{N}$, set

$$A_n = \left\{\omega \in \Omega : \sup_{t\in[0,T]} \|u^n(t)\|_X \ge n\right\}.$$

Then by inequality (4.42) and the Chebyshev inequality, we have, for each $n \in \mathbb{N}$,

$$\mathbb{P}(A_n) \le C(T)\frac{1}{n^2}.$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

Thus, by the Borel Cantelli Lemma, we infer that $\mathbb{P}\left(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n\right) = 0$ and so

$$\mathbb{P}\left(\cup_{k=1}^{\infty} \cap_{n=k}^{\infty} (\Omega \setminus A_n)\right) = 1.$$

Choose $\bar{\Omega} = \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} (\Omega \setminus A_n)$ and fix $\omega \in \bar{\Omega}$. Then $\exists\, k \in \mathbb{N}$ such that $\omega \in \cap_{n=k}^{\infty}(\Omega \setminus A_n)$, i.e. $\forall\, n \ge k,\ \omega \in \Omega \setminus A_n$. Therefore, for all $n \ge k$,

$$\sup_{t \in [0,T]} \left\| u^n(t,\omega) \right\|_X < n.$$

Thus, $\exists\, k \in \mathbb{N}$ such that $\forall\, n \ge k$ and $t \in [0,T]$, $\|u^n(t,\omega)\|_X < n$, which implies that $\exists\, k \in \mathbb{N}$ such that $\forall\, n \ge k$, $\tau_n(\omega) > T$. This gives the desired conclusion. ∎

The main result of this section is the following theorem.

Theorem 4.3. *Assume that the mappings F and G are of linear growth (uniformly in t) and Lipschitz on balls, Moreover, we assume that for each $T > 0$ and $x \in X$, the functions $F(\cdot,x) : [0,T] \ni t \to F(t,x) \in X$ and $G(\cdot,x) : [0,T] \ni t \to G(t,x) \in \gamma(H,X)$ are Borel measurable. Then there exists a unique X-valued continuous mild solution to equation (4.2).*

Proof. Let u^n be the unique solution of equation (4.41). Consider the sequence $(\tau_n)_{n \in \mathbb{N}}$. Define a process u by

$$u(t) = u^n(t), \quad if \quad t \le \tau_n. \tag{4.48}$$

In view of Lemma 4.11 of [8], $(\tau_n)_{n \in \mathbb{N}}$ has the following properties
i) $\tau_n \le \tau_{n+1}$
ii) $u^n(t,\omega) = u^{n+1}(t,\omega), \quad t \le \tau_n(\omega), \quad \mathbb{P}-a.s.$
Therefore, the process u is well-defined. We claim that this process is a unique X-valued continuous mild solution to equation (4.2). Let us prove this. By the definition of the mild solution, it is sufficient to show that the process u is a unique X-valued continuous

solution to the integral equation (4.4). It is obvious that u is continuous and $\mathbb{F}$-adapted. Hence, by Proposition 2.18, u is $\mathbb{F}$- progressively measurable. Moreover, for each $t \geq 0$,

$$\mathbb{E}\int_0^t \|u(r)\|_X^2 dr < \infty.$$

Thus, the integrals in equation (4.4) exist for the process u. Now we show that the process u solves equation (4.4). Since u^n is the solution of equation (4.41), we have the following integral equation

$$\begin{aligned} u(t\wedge\tau_n) = u^n(t\wedge\tau_n) = S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)F^n\left(r, u^n(r)\right)dr \\ + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)G^n\left(r, u^n(r)\right)dW(r), \quad t \geq 0. \end{aligned} \tag{4.49}$$

However, we have a problem in this equation. The stochastic integral part of equation above is not well-defined because we integrate a process which is not adapted and hence not progressively measurable. To over come this problem, let us define a process I by

$$I(t) := \int_0^t S(t-r)G^n\left(r, u^n(r)\right)dW(r), \quad t \geq 0.$$

It is obvious that I is well-defined. Consider the following process

$$I_{\tau_n}(t) = \int_0^t S(t-r)\big(1_{[0,\tau_n)}(r)G^n(r\wedge\tau_n, u^n(r\wedge\tau_n))\big)dW(r), \quad t \geq 0.$$

It was shown in Lemma A.1 of [14], see also [18], that if the processes I and I_{τ_n} are continuous, then

$$S(t - t - \tau_n)I(t\wedge\tau_n) = I_{\tau_n}(t), \quad for\ all\ t \geq 0\ \mathbb{P}-a.s.$$

In particular,

$$I(t\wedge\tau_n) = I_{\tau_n}(t\wedge\tau_n), \quad for\ all\ t \geq 0\ \mathbb{P}-a.s.$$

Therefore, equation (4.49) can be rewritten as follow

$$u(t\wedge\tau_n) = S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)F^n\left(r, u^n(r)\right)dr + I_{\tau_n}(t\wedge\tau_n), \quad t \geq 0.$$

Since $r \leq \tau_n$, $\left\|u^n(r)\right\|_X \leq n$. Therefore, from the definition of F^n and G^n, we get, for every $r \leq \tau_n$,

$$F^n\left(r, u^n(r)\right) = F(r, u^n(r)) \quad and \quad G^n\left(r, u^n(r)\right) = G(r, u^n(r)).$$

Also by the definition of u, $u(r) = u^n(r)$ if $r \le \tau_n$. Hence, we obtain

$$u(t\wedge\tau_n) = S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)F(r,u(r))dr$$
$$+ \int_0^t S(t-r)\big(1_{[0,\tau_n)}(r)G(r\wedge\tau_n, u(r\wedge\tau_n)\big)dW(r), \quad t \ge 0.$$

We know that $\tau_n \to \infty$ and so $t\wedge\tau_n \to t$. Thus, $u(t\wedge\tau_n) \to u(t)$ and $S(t\wedge\tau_n) \to S(t)$. Therefore, we deduce that

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(r,u(r))dr + \int_0^t S(t-r)G(r,u(r))dW(r), \quad t \ge 0.$$

Thus, the process u solves the integral equation (4.4). The uniqueness of the solution follows from Theorem 4.1. ■

4.4 Markov Property

In this section, we are interested in the Markov property of the solution to equation (4.2). Let us recall that X is a separable Banach space satisfying H-condition and $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t\}_{t\ge 0}$, is a filtered probability space. Recall that we denote by $\mathscr{B}(X)$ the Borel σ-field of X. We denote by $B_b(X)$ the space of all bounded measurable functions from X into $\mathbb{R}$.

Definition 4.2. *An X-valued $\mathbb{F}$ adapted process $\{\xi(t)\}_{t\ge 0}$ is called* ***a Markov process*** *if for each $A \in \mathscr{B}(X)$ and $0 \le s \le t$,*

$$\mathbb{P}\big(\xi(t) \in A | \mathscr{F}_s\big) = \mathbb{P}\big(\xi(t) \in A | \sigma(\xi(s))\big), \quad \mathbb{P}-a.s. \tag{4.50}$$

where $\sigma(\xi(s))$ denotes the σ-field generated by $\xi(s)$.

Definition 4.3. *A family $\{P_t(x,\cdot)\}_{t\ge 0, x\in X}$ of probability measures $P_t(x,\cdot)$ on $\mathscr{B}(X)$ is called* ***the transition function*** *if*
(i) for each $x \in X$, $P_0(x,\cdot) = \delta_x$,
(ii) for all $A \in \mathscr{B}(X)$ and $t \ge 0$, the function $X \ni x \mapsto P_t(x,A) \in \mathbb{R}$ is measurable,
(iii) the family satisfies the Chapman-Kolmogorov equation, i.e.

$$P_{t+s}(x,A) = \int_X P_t(x,dy)P_s(y,A), \quad t,s \ge 0, \quad A \in \mathscr{B}(X), \quad x \in X.$$

Definition 4.4. *Let $\{P_t(x,\cdot)\}_{t\ge 0, x\in X}$ be a transition function on $\mathscr{B}(X)$. A family $\{P_t\}_{t\ge 0}$ of functions $P_t : B_b(X) \to B_b(X)$ defined by*

$$P_t\varphi(x) = \int_X P_t(x,dy)\varphi(y), \quad t \ge 0, \quad \varphi \in B_b(X), \quad x \in X$$

is called ***the transition semigroup****.*

Remark 4.1. *The semigroup property $P_t P_s = P_{t+s}$, $t,s \geq 0$, is a consequence of the Chapman-Kolmogorov equation.*

Definition 4.5. *Let $\{P_t(x,\cdot)\}_{t\geq 0, x\in X}$ be a transition function on $\mathscr{B}(X)$ and $\{P_t\}_{t\geq 0}$ be the corresponding transition semigroup. An X-valued $\mathbb{F}$ adapted process $\{\xi(t)\}_{t\geq 0}$ is called **a Markov process with respect to** $\{P_t\}_{t\geq 0}$ if for each $\varphi \in B_b(X)$ and $t,h\geq 0$,*

$$\mathbb{E}\big(\varphi(\xi(t+h))|\mathscr{F}_t\big) = P_h\varphi(\xi(t)), \quad \mathbb{P}-a.s. \tag{4.51}$$

In the previous sections, we proved that equation (4.2) under the Lipschitz assumptions on the coefficients has a unique X-valued continuous mild solution. Similarly, one can show that for each $s \geq 0$ and $u_s \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; X)$, the same equation with the initial value $u(s) = u_s$ has a unique X-valued continuous mild solution $\{u(t)\}_{t\in[s,\infty)}$. This solution is denoted by $u(\cdot, s, u_s)$. If $u_s = x \in X$, then it is denoted by $u(\cdot, s, x)$.

Definition 4.6. *A family $\{P_{s,t}\}_{0\leq s\leq t}$ of functions $P_{s,t} : B_b(X) \mapsto B_b(X)$ defined by*

$$P_{s,t}\varphi(x) := \mathbb{E}\varphi(u(t,s,x)), \quad 0 \leq s \leq t, \quad \varphi \in B_b(X), \quad x \in X$$

*is called **the transition semigroup corresponding to equation (4.2)**.*

Definition 4.7. *For each $0 \leq s \leq t$ and $x \in X$, a family $\{P_{s,t}(x,\cdot)\}_{s\leq t, x\in X}$ of functions $P_{s,t}(x,\cdot)$ defined by*

$$P_{s,t}(x,A) := P_{s,t}\mathbb{1}_A(x) = \mathbb{P}(u(t,s,x) \in A), \quad A \in \mathscr{B}(X)$$

*is called **the transition function corresponding to equation (4.2)**.*

The proof of the following theorem is similar to the proof of Theorem 9.30 in [46]. Therefore, we do not prove this theorem here.

Theorem 4.4. *Assume that all the assumptions of Theorem 4.1 hold. Then for all $0 \leq s \leq r \leq t$, $\varphi \in B_b(X)$ and $u_s \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; X)$,*

$$\mathbb{E}\big(\varphi(u(t,s,u_s)|\mathscr{F}_r\big) = P_{r,t}\varphi(u(r,s,u_s)), \quad \mathbb{P}-a.s. \tag{4.52}$$

The following propositions are consequence of Theorem 4.4.

Proposition 4.1. *For each $\varphi \in B_b(X)$, $A \in \mathscr{B}(X)$, $0 \leq s \leq r \leq t$ and $x \in X$,*

$$P_{s,r}\big(P_{r,t}\varphi\big)(x) = P_{s,t}\varphi(x) \tag{4.53}$$

and

$$P_{s,t}(x,A) = \int_X P_{s,r}(x,dy)P_{r,t}(y,A). \tag{4.54}$$

Proof. Fix $\varphi \in B_b(X)$, $0 \le s \le r \le t$ and $x \in X$. Then by (4.52), we obtain

$$P_{s,t}\varphi(x) = \mathbb{E}\varphi(u(t,s,x)) = \mathbb{E}\mathbb{E}\big(\varphi(u(t,s,x))|\mathscr{F}_r\big) = \mathbb{E}P_{r,t}\varphi(u(r,s,x)) = P_{s,r}\big(P_{r,t}\varphi\big)(x),$$

which gives the desired result (4.53). If we take $\varphi = \mathbb{1}_A$ for a given $A \in \mathscr{B}(X)$, then (4.54) follows from (4.53). ■

Proposition 4.2. ***(Proposition 9.32, pp 169, [46])*** *For each $0 \le s \le t$, $P_{s,t} = P_{0,t-s}$. In particular,*

$$P_{s,t}(x,A) = P_{0,t-s}(x,A), \quad x \in X, \quad A \in \mathscr{B}(X). \tag{4.55}$$

Remark 4.2. *Let us write $u(t,x)$ for $u(t,0,x)$, P_t for $P_{0,t}$ and $P_t(x,A)$ ifor $P_{0,t}(x,A)$. It can be easily seen that $\{P_t(x,\cdot)\}_{t\ge 0, x\in X}$ is a transition function and $\{P_t\}_{t\ge 0}$ is the corresponding transition semigroup (According definitions 4.3 and 4.4).*

Remark 4.3. *By Theorem 4.4, for each $u_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P};X)$, $u(\cdot,u_0)$ is a Markov process with respect to the transition semigroup $\{P_t\}_{t\ge 0}$. By Theorem 4.2, the transition semigroup $\{P_t\}_{t\ge 0}$ is Feller, that is, for any $t \ge 0$, $P_t : C_b(X) \to C_b(X)$, where $C_b(X)$ is the space of all continuous bounded functions from X into $\mathbb{R}$, is well-defined. In other words, for each $u_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P};X)$, $u(\cdot,u_0)$ is a Markov process satisfying the Feller property.*

4.5 Existence and Uniqueness of Invariant Measures for SEE

Brzeźniak, Long and Simão in [15] found some sufficient conditions for the existence and uniqueness of an invariant measure for equation (4.2) with the time independent coefficients. Before we introduce these conditions, let us start presenting the following two natural consequences of Theorems 4.1 and 4.3 respectively.

Corollary 4.2. *Assume that the maps $F : X \to X$ and $G : X \to \gamma(H,X)$ are globally Lipschitz on X, i.e. there exist constants $L_F > 0$ and $L_G > 0$ such that*

$$\|F(x_1) - F(x_2)\|_X \le L_F \|x_1 - x_2\|_X, \quad x_1, x_2 \in X \tag{4.56}$$

and

$$\|G(x_1) - G(x_2)\|_{\gamma(H,X)} \le L_G \|x_1 - x_2\|_X, \quad x_1, x_2 \in X. \tag{4.57}$$

Moreover,

$$\|F(0)\|_X + \|G(0)\|_{\gamma(H,X)} < \infty. \tag{4.58}$$

Then there exists a unique X-valued continuous mild solution to the following equation

$$\begin{cases} du(t) = (Au(t) + F(u(t)))dt + G(u(t))dW(t), \quad t \geq 0, \\ u(0) = u_0. \end{cases} \tag{4.59}$$

Corollary 4.3. *Assume that the maps $F : X \to X$ and $G : X \to \gamma(H,X)$ are Lipschitz on balls, i.e. for each $R > 0$, there exist constants $L_F(R) > 0$ and $L_G(R) > 0$ such that if $\|x_1\|_X, \|x_2\|_X \leq R$, then*

$$\|F(x_1) - F(x_2)\|_X \leq L_F(R)\|x_1 - x_2\|_X, \tag{4.60}$$

and

$$\|G(x_1) - G(x_2)\|_{\gamma(H,X)} \leq L_G(R)\|x_1 - x_2\|_X. \tag{4.61}$$

Moreover, we assume that F and G are of linear growth, i.e. there exist constants $\bar{L}_F > 0$ and $\bar{L}_G > 0$ such that

$$\|F(x)\|_X^2 \leq \bar{L}_F^2 \left(1 + \|x\|_X^2\right), \quad x \in X, \tag{4.62}$$

$$\|G(x)\|_{\gamma(H,X)}^2 \leq \bar{L}_G^2 \left(1 + \|x\|_X^2\right), \quad x \in X. \tag{4.63}$$

Then there exists a unique X-valued continuous mild solution to equation (4.59).

Definition 4.8. *Let $\{P_t(x,\cdot)\}_{t\geq 0, x\in X}$ be the transition function of an X-valued Markov process ξ. A probability measure μ on $\mathscr{B}(X)$ is called* ***invariant with respect to transition function*** $\{P_t(x,\cdot)\}_{t\geq 0, x\in X}$ ***(or invariant for*** *ξ) if for every $A \in \mathscr{B}(X)$ and $t \geq 0$,*

$$\mu(A) = \int_X P_t(x,A)\mu(dx).$$

Let us now present the main results of this section, see Theorem 3.7 and Theorem 4.6 of [15] for the proofs.

Theorem 4.5. *Assume that all the assumptions of Corollary 4.2 are satisfied. If there exist constants $\omega > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x_1, x_2 \in X$,*

$$\left[A_n(x_1 - x_2) + F(x_1) - F(x_2), x_1 - x_2\right]_X + \frac{K_2(q)}{q}\|G(x_1) - G(x_2)\|_{\gamma(H,X)}^2 \leq -\omega\|x_1 - x_2\|_X^2, \tag{4.64}$$

where A_n is the Yosida approximation of A, $K_2(q)$ is a constant appearing in the H-condition and $[\cdot,\cdot]_X$ is the semi-inner product on X (see Definition 4.9 below), then there exists a unique invariant probabilty measure for equation (4.59).

Theorem 4.6. *Assume that all the assumptions of Corollary 4.3 are satisfied. If there exist constants $\omega > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x_1, x_2 \in X$,*

$$[A_n(x_1-x_2)+F(x_1)-F(x_2), x_1-x_2]_X + \frac{K_2(q)}{q}\|G(x_1)-G(x_2)\|^2_{\gamma(H,X)} \leq -\omega\|x_1-x_2\|^2_X, \tag{4.65}$$

then there exists a unique invariant probability measure for equation (4.59).

Definition 4.9. *A semi-inner product on a complex or real vector space V is a mapping $[\cdot,\cdot]_V : V \times V \to \mathbb{C}$ (or $\mathbb{R}$) such that*
(i) $[x+y,z]_V = [x,z]_V + [y,z]_V, \quad x,y,z \in V,$
(ii) $[\lambda x, y]_V = \lambda[x,y]_V, \quad x,y \in V, \quad \lambda \in \mathbb{C}$ *(or $\mathbb{R}$),*
(iii) $[x,x]_V > 0, \quad$ *for $x \neq 0$,*
(iv) $\left|[x,y]_V\right|^2 \leq [x,x]_V[y,y]_V, \quad x,y \in V.$
Such a vector space V with the semi inner product $[\cdot,\cdot]_V$ is called ***a semi-inner product space***.

Lemma 4.8. *[15] The following mapping on $X \times X$ defined by*

$$[x,y]_X = \langle x, y^* \rangle, \quad x,y \in X,$$

where $y^* \in X^*$ such that $\|y^*\| = \|y\|_X$ and $\langle y, y^* \rangle = \|y\|_X$, is a semi-inner product. Such a $y^* \in X^*$ exists by the Hahn-Banach theorem.

Remark 4.4. *Invariant measure is a subject related to semigroups and a SDE generates a semigroup only when the coefficients are time independent. However, there are some papers considering a generalization of an invariant measure for time dependent equations, e.q. [43], [48] and [49]. We plan to investigate this concept in relation to the HJMM equation in the future.*

CHAPTER 5

APPLICATION TO THE HJMM EQUATION

In this chapter, we apply the abstract results from the previous chapter to the HJMM equation (3.7). In particular, we prove the existence and uniqueness of solutions to equation (3.7) in the weighted Lebesgue L^p and Sobolev $W^{1,p}$, $p \geq 2$, spaces respectively. We also find a sufficient condition for the existence and uniqueness of an invariant measure for the Markov semigroup associated to equation (3.7) (when the coefficients are time independent) in the weighted Lebesgue spaces. The HJMM equation driven by a standard d-dimensional Brownian motion has been already studied in the weighted L^2 and $W^{1,2}$ spaces, see Chapter 3. Under appropriate choice of the weight, for each $p \geq 2$, the weighted Lebesgue L^p and Sobolev $W^{1,p}$ spaces are subspaces of the weighted Lebesgue L^2 and Sobolev $W^{1,2}$ spaces respectively. Therefore, an important feature of our results is that we are able to prove that the HJMM equation has a unique solution and an invariant measure in smaller spaces. Another important feature of our results is that we are able to consider the HJMM equation driven by a cylindrical Wiener process on a (possibly infinite dimensional) Hilbert space. For this purpose, we use the characterizations of γ-radonifying operators with values in Lebesgue L^p and Sobolev $W^{1,p}$ spaces found recently by Brzeźniak and Peszat in [9] and [12].

5.1 Existence and Uniqueness of Solutions to the HJMM Equation in the Weighted Lebesgue Spaces

In this section, we first introduce the weighted Lebesgue spaces and some useful properties of them which allow us to apply the abstract results from the previous chapter to prove the existence and uniqueness of solutions to equation (3.7) in the weighted Lebesgue spaces. Then we present the main result of this section.

For each $\nu \in \mathbb{R}$ and $p \geq 1$, define L^p_ν to be the space of all (equivalence classes of) Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(x)|^p e^{\nu x} dx < \infty.$$

For each $\nu \in \mathbb{R}$ and $p \geq 1$, the space L^p_ν is called **the weighted Lebesgue space**.

Lemma 5.1. *For each $\nu \in \mathbb{R}$ and $p \geq 1$, L^p_ν is a separable Banach space endowed with the norm*

$$\|f\|_{\nu,p} = \left(\int_0^\infty |f(x)|^p e^{\nu x} dx\right)^{\frac{1}{p}}.$$

Proof. Fix $\nu \in \mathbb{R}$ and $p \geq 1$. It is well-known that the space L^p of all (equivalence classes of) Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}} < \infty$$

is a separable Banach space with respect to the norm $\|\cdot\|_p$. Define a linear operator $T : L^p_\nu \to L^p$ by

$$Tf(x) = f(x)e^{\frac{\nu}{p}x}, \quad f \in L^p_\nu \quad x \in [0,\infty).$$

The map T is well-defined. Indeed, for a fixed $f \in L^p_\nu$,

$$\int_0^\infty |Tf(x)|^p dx = \int_0^\infty |f(x)|^p e^{\nu x} dx < \infty. \tag{5.1}$$

Thus, $Tf \in L^p$ and hence, T is well-defined. It is obvious that the map T is bijective. Moreover, by (5.1), we have

$$\|Tf\|_p = \|f\|_{\nu,p}, \quad f \in L^p_\nu,$$

which implies that T is an isometric isomorphism. Hence, $\left(L^p, \|\cdot\|_p\right)$ and $\left(L^p_\nu, \|\cdot\|_{\nu,p}\right)$ are isometric spaces. Thus, by Theorem 2.1, L^p_ν is a separable Banach space with respect to the norm $\|\cdot\|_{\nu,p}$. ∎

Lemma 5.2. *For each $\nu \in \mathbb{R}$ and $p \geq 2$, the space L^p_ν satisfies the H-condition. In particular, if ψ is a function defined by*

$$\psi : L^p_\nu \ni f \mapsto \psi(f) = \|f\|^p_{\nu,p} \in \mathbb{R},$$

then

$$\|\psi'(f)\| \leq p\|f\|^{p-1}_{\nu,p} \quad and \quad \|\psi''(f)\| \leq p(p-1)\|f\|^{p-2}_{\nu,p}, \quad f \in L^p_\nu, \tag{5.2}$$

where ψ' and ψ'' are the first and second Fréchet derivatives of ψ at $f \in L^p_\nu$ respectively.

Proof. We know from Lemma 2.2 that any linear operator A from a normed space X into a normed space Y is twice Fréchet differentiable on X and for each $f \in X$, $A'(f) = A$ and $A''(f) = 0$. Consider the operator T from the proof of the previous Lemma. Since T is linear, $T'(f) = T$ and $T''(f) = 0$ for any $f \in L^p_\nu$. Moreover, since $\|Tf\|_p = \|f\|_{\nu,p}$, we have

$$\|T\|_{\mathscr{L}(L^p_\nu, L^p)} \leq 1. \tag{5.3}$$

By Proposition 2.1 of [12], for every $p \geq 2$, the space L^p satisfies the H-condition with any $q \geq p$, i.e. for some $q \geq p$ (in particular, $q = p$), the map $\phi : L^p \ni f \mapsto \phi(f) = \|f\|^p_p \in \mathbb{R}$ is of C^2 class and

$$\|\phi'(f)\| \leq p\|f\|^{p-1}_{p} \quad and \quad \|\phi''(f)\| \leq p(p-1)\|f\|^{p-2}_{p}, \quad f \in L^p. \tag{5.4}$$

It is obvious that we can write $\psi = \phi \circ T$. Since T and ϕ are of C^2 class, ψ is of C^2 class. Moreover, by inequalities (5.3) and (5.4), we infer that

$$\|\psi'(f)\| = \|T\phi'(Tf)\| \leq \|\phi'(Tf)\| \leq p\|Tf\|^{p-1}_p = p\|f\|^{p-1}_{\nu,p}, \quad f \in L^p_\nu.$$

Similarly, we can show that

$$\|\psi''(f)\| \leq p(p-1)\|f\|^{p-2}_{\nu,p}, \quad f \in L^p_\nu,$$

which finishes the proof. ∎

Proposition 5.1. *For each $\nu > 0$ and $p \geq 1$, the space L^p_ν is continuously embedded into the space L^1. In particular,*

$$\|f\|_1 \leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|f\|_{\nu,p}, \quad f \in L^p_\nu,$$

where $q \in [1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Fix $\nu > 0$, $p \geq 1$ and $f \in L^p_\nu$. Then by the Hölder inequality, we get

$$\begin{aligned}
\|f\|_1 &= \int_0^\infty |f(x)|dx = \int_0^\infty |f(x)|e^{\frac{\nu x}{p}} e^{-\frac{\nu x}{p}} dx \\
&\leq \left(\int_0^\infty |f(x)|^p e^{\nu x} dx\right)^{\frac{1}{p}} \left(\int_0^\infty e^{-\frac{\nu q x}{p}} dx\right)^{\frac{1}{q}} \\
&= \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|f\|_{\nu,p},
\end{aligned}$$

which gives the desired conclusion. ∎

Lemma 5.3. *Assume that $\nu > 0$ and $p \geq 1$. The shift semigroup $\{S(t)\}_{t\geq 0}$ on L^p_ν is a contraction type C_0-semigroup on L^p_ν, in particular,*

$$\|S(t)\|_{\mathscr{L}(L^p_\nu)} \leq e^{\frac{-\nu t}{p}}, \quad t \geq 0.$$

Lemma 5.4. *[32] Let $C^1_c(\mathbb{R}^+)$ be the space of all continuously differentiable functions $f : [0,\infty) \to \mathbb{R}$ with compact support. Then $C^1_c(\mathbb{R}^+)$ is a dense subspace of L^p_ν.*

Lemma 5.5. *The shift-semigroup on $C^1_c(\mathbb{R}^+)$ is strongly continuous in the norm of L^p_ν.*

Proof. Fix $g \in C^1_c(\mathbb{R}^+)$. It is obvious that for each $t \geq 0$, $S(t) : C^1_c(\mathbb{R}^+) \to C^1_c(\mathbb{R}^+)$ is well-defined, linear and bounded. Since g is uniformly continuous, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in [0,\infty)$ with $|x_1 - x_2| \leq \delta$, $|g(x_1) - g(x_2)| \leq \varepsilon$. Our aim is to show that $\|S(t)g - g\|_{\nu,p} \to 0$ as $t \to 0$. Since g has compact support, there exists $a > 0$ such that for all $x \in [a,\infty)$, $g(x) = 0$, i.e. $g(x) = 0$ if $x \notin [0,a)$. Note that if $x \notin [0,a)$, then $x + t \notin [0,a)$ and so $g(x+t) = 0$ if $x \notin [0,a)$. Thus, we get, for all $t \geq 0$,

$$\|S(t)g - g\|^p_{\nu,p} = \int_0^\infty |g(t+x) - g(x)|^p e^{\nu x} dx = \int_0^a |g(t+x) - g(x)|^p e^{\nu x} dx.$$

Put $x_1 = x + t$ and $x_2 = x$. Then by the uniformly continuity of g, we obtain, for all $t < \delta$,

$$\|S(t)g - g\|^p_{\nu,p} = \int_0^a |g(x_1) - g(x_2)|^p e^{\nu x} dx \leq \frac{1}{\nu}\left(e^{\nu a} - 1\right)\varepsilon^p.$$

If we choose $\varepsilon = \left(\frac{\nu}{e^{\nu a}-1}\right)^{\frac{1}{p}} \bar{\varepsilon}$, $\bar{\varepsilon} > 0$, then we infer that

$$\|S(t)g - g\|_{\nu,p} \leq \bar{\varepsilon}. \tag{5.5}$$

Thus, we have showed that for every $\bar{\varepsilon} > 0$, there exists $\bar{\delta} = \delta > 0$ such that for all $t \leq \bar{\delta}$, $\|S(t)g - g\|_{\nu,p} \leq \bar{\varepsilon}$. This gives the desired conclusion. ∎

Proof of Lemma 5.3. We first show that for each $t \ge 0$, $S(t) : L^p_\nu \to L^p_\nu$ is well-defined, linear and bounded. Fix $t \ge 0$ and $f \in L^p_\nu$. Since f is Lebesgue measurable, for each $A \in \mathscr{B}(\mathbb{R})$, set E defined by

$$E := f^{-1}(A) = \{y \in \mathbb{R} : f(y) \in A\}$$

is Lebesgue measurable. Thus, we get

$$(S(t)f)^{-1}(A) = \{x \in \mathbb{R} : f(t+x) \in A\} = \{y - t : y \in E\} = E - t.$$

We know from [51] that if a set E is Lebesgue measurable, then the set $E - t$ is also Lebesgue measurable. Therefore, $S(t)f$ is Lebesgue measurable. Moreover, by the change of variable, put $y = t + x$ and so $dy = dx$, we have

$$\begin{aligned}\int_0^\infty |S(t)f(x)|^p e^{\nu x} dx &= \int_0^\infty |f(t+x)|^p e^{\nu x} dx \\ &= \int_t^\infty |f(y)|^p e^{\nu y} e^{-\nu t} dy \\ &\le e^{-\nu t} \int_0^\infty |f(y)|^p e^{\nu y} dy \\ &= e^{-\nu t} \|f\|^p_{\nu,p}.\end{aligned}$$

Hence $S(t)f \in L^p_\nu$ and thus, $S(t)$ is well-defined. The linearity of $S(t)$ is obvious and it follows from the last inequality that

$$\|S(t)f\|_{\nu,p} \le e^{-\frac{\nu}{p}t} \|f\|_{\nu,p}, \tag{5.6}$$

which implies that $S(t)$ is bounded.

Now we show that $S = \{S(t)\}_{t\ge0}$ is a contraction type C_0-semigroup on L^p_ν. It is clear that $S(0) = I$. Fix $t, s \ge 0$. Then

$$\begin{aligned}S(t)f(x) &= f(x+t), \quad f \in L^p_\nu, \quad x \in [0,\infty) \\ S(s)f(x) &= f(x+s), \quad f \in L^p_\nu, \quad x \in [0,\infty)\end{aligned}$$

and so

$$S(t)(S(s)f(x)) = S(t)f(x+s) = f(x+t+s) = S(t+s)f(x), \quad f \in L^p_\nu, \quad x \in [0,\infty).$$

Thus $S(t)S(s) = S(t+s)$ and hence, S is a semigroup on L^p_ν. Let us now show the strong continuity of S on L^p_ν. Fix $f \in L^p_\nu$. Since $C^1_c(\mathbb{R}^+)$ is dense in L^p_ν, for every $\frac{\varepsilon}{3} > 0$, there exists $g \in C^1_c(\mathbb{R}^+)$ such that

$$\|f - g\|_{\nu,p} \le \frac{\varepsilon}{3}. \tag{5.7}$$

Moreover, since S is strongly continuous on $C_c^1(\mathbb{R}^+)$, in particular, at $g \in C_c^1(\mathbb{R}^+)$, for every $\frac{\varepsilon}{3} > 0$,

$$\|S(t)g - g\|_{\nu,p} \leq \frac{\varepsilon}{3}. \tag{5.8}$$

Using inequalities (5.6), (5.7) and (5.8), we deduce that

$$\begin{aligned}
\|S(t)f - f\|_{\nu,p} &= \|S(t)f - S(t)g + S(t)g - g + g - f\|_{\nu,p} \\
&\leq \|S(t)(f-g)\|_{\nu,p} + \|S(t)g - g\|_{\nu,p} + \|f - g\|_{\nu,p} \\
&\leq \|f - g\|_{\nu,p} + \|S(t)g - g\|_{\nu,p} + \|f - g\|_{\nu,p} \leq \varepsilon.
\end{aligned}$$

Therefore, S is strongly continuous on L_ν^p. Moreover, it follows from inequality (5.6) that

$$\|S(t)\|_{\mathscr{L}(L_\nu^p)} \leq e^{\frac{-\nu t}{p}}, \quad t \geq 0,$$

which implies that S is a contraction type C_0-semigroup on L_ν^p. This completes the proof.

Lemma 5.6. *Assume that $\nu > 0$ and $p \geq 1$. Let A be the infinitesimal generator of the shift-semigroup on L_ν^p. Then for each $\lambda > -\frac{\nu}{p}$, the resolvent $(\lambda I - A)^{-1}$ exists and is defined by*

$$R(\lambda)f := (\lambda I - A)^{-1} f = \int_0^\infty e^{-\lambda t} S(t) f \, dt, \quad f \in L_\nu^p. \tag{5.9}$$

In particular,

$$\|R(\lambda)f\|_{\nu,p} \leq \frac{p}{\nu + \lambda p} \|f\|_{\nu,p}, \quad f \in L_\nu^p \tag{5.10}$$

and for each $f \in L_\nu^p$, $R(\lambda)f \in \mathscr{D}(A)$, i.e. $\mathscr{R}(R(\lambda)) = \mathscr{D}(A)$, and

$$AR(\lambda) = \lambda R(\lambda) - I.$$

Proof. Fix $f \in L_\nu^p$. Since $t \mapsto S(t)f \in L_\nu^p$ is continuous, $t \mapsto e^{-\lambda t} S(t) f \in L_\nu^p$ is measurable. Moreover, by inequality (5.6), we have

$$\int_0^\infty \left\| e^{-\lambda t} S(t) f \right\|_{\nu,p} dt \leq \|f\|_{\nu,p} \int_0^\infty e^{\left(-\lambda - \frac{\nu}{p}\right)t} dt = \frac{p}{\nu + \lambda p} \|f\|_{\nu,p}.$$

Thus, the integral exists in the Bochner integral sense and $R(\lambda)f \in L_\nu^p$. It is obvious that $R(\lambda)$ is linear and inequality (5.10) follows from the last inequality. Hence, $R(\lambda)$ is a bounded linear operator from L^p_ν into itself. Let us now show that for every $\lambda > -\frac{\nu}{p}$ and $f \in L^p_\nu$, $R(\lambda)f \in \mathscr{D}(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$. For $h > 0$, we have

$$\begin{aligned}\frac{S(h)-I}{h} R(\lambda)f &= \frac{1}{h}\int_0^\infty e^{-\lambda t}\big(S(t+h)f - S(t)f\big)dt \\ &= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t} S(t)f\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)f\,dt.\end{aligned}$$

By Theorem 2.9 (i), we get

$$\frac{1}{h}\int_0^h e^{-\lambda t} S(t)f\,dt \to f \quad as \quad h \to 0$$

and

$$\frac{e^{\lambda h}-1}{h} \to \lambda \; as \quad h \to 0.$$

Therefore

$$\frac{S(h)-I}{h} R(\lambda)f \to \lambda R(\lambda)f - f \; as \quad h \to 0.$$

This implies that for every $\lambda > -\frac{\nu}{p}$ and $f \in L^p_\nu$, $R(\lambda)f \in \mathscr{D}(A)$ and

$$AR(\lambda) = \lambda R(\lambda) - I,$$

i.e.

$$(\lambda I - A)R(\lambda) = I. \tag{5.11}$$

Finally, we prove that $R(\lambda)$ is the inverse of $(\lambda I - A)$. By Theorem 2.9 (iii) and closedness of A, we get, for $f \in \mathscr{D}(A)$,

$$\begin{aligned}R(\lambda)Af &= \int_0^\infty e^{-\lambda t} S(t)Af\,dt \\ &= \int_0^\infty e^{-\lambda t} AS(t)f\,dt \\ &= A\left(\int_0^\infty e^{-\lambda t} S(t)f\,dt\right) \\ &= AR(\lambda)f.\end{aligned} \tag{5.12}$$

Taking into account (5.11) and (5.12), we infer that

$$R(\lambda)(\lambda I - A)f = f \quad f \in \mathscr{D}(A).$$

Therefore $R(\lambda) = (\lambda I - A)^{-1}$. ■

Lemma 5.7. *Assume that $\nu > 0$ and $p \geq 1$. For a fixed $f \in L^p_\nu$ and each $\lambda > -\frac{\nu}{p}$, define a function g_λ by*

$$g_\lambda(x) = \lambda \int_0^\infty e^{-\lambda t} S(t) f(x) dt, \quad x \in [0,\infty). \tag{5.13}$$

Then g_λ is weakly differentiable on $(0,\infty)$. In particular,

$$D g_\lambda = -\lambda f + \lambda g_\lambda. \tag{5.14}$$

Proof. By the previous lemma, for each $\lambda > -\frac{\nu}{p}$, g_λ is well-defined and $g_\lambda \in L^p_\nu$. Fix $\lambda > -\frac{\nu}{p}$. Recall that $Df \in L^1_{loc}(\mathbb{R}^+)$ is the first weak derivative of $f \in L^1_{loc}(\mathbb{R}^+)$ if and only if for all $\varphi \in C^1_c\big((0,\infty)\big)$,

$$\int_0^\infty f(x)\varphi'(x)dx = -\int_0^\infty Df(x)\varphi(x)dx. \tag{5.15}$$

Therefore, in order to prove (5.14), it is sufficient to show that for each $\varphi \in C^1_c\big((0,\infty)\big)$,

$$\int_0^\infty g_\lambda(x)\varphi'(x)dx = \lambda \int_0^\infty f(x)\varphi(x)dx - \lambda \int_0^\infty g_\lambda(x)\varphi(x)dx. \tag{5.16}$$

By the change of variable, we can write g_λ as follow

$$g_\lambda(x) = \lambda \int_0^\infty 1_{(0,\infty)}(t-x) e^{-\lambda(t-x)} f(t) dt. \tag{5.17}$$

Define a function ϕ by

$$\phi(t,x) = 1_{(0,\infty)}(t-x) e^{-\lambda(t-x)} f(t) \varphi'(x), \quad t,x \in [0,\infty).$$

Since all products in the definition of ϕ are measurable, ϕ is measurable. Moreover, ϕ is integrable, i.e.

$$\int_0^\infty \int_0^\infty |\phi(t,x)| dt dx < \infty.$$

Thus, we can apply the Fubini Theorem to the function ϕ. Therefore, we obtain

$$\begin{aligned}
\int_0^\infty g_\lambda(x)\varphi'(x)dx &= \lambda \int_0^\infty \int_0^\infty 1_{(0,\infty)}(t-x) e^{-\lambda(t-x)} f(t)\varphi'(x) dt dx \\
&= \lambda \int_0^\infty f(t) e^{-\lambda t} \left[\int_0^\infty 1_{(0,\infty)}(t-x) e^{\lambda x} \varphi'(x) dx \right] dt \\
&= \lambda \int_0^\infty f(t) e^{-\lambda t} \left[\int_0^t e^{\lambda x} \varphi'(x) dx \right] dt.
\end{aligned}$$

Since φ has compact support, $\varphi(0) = 0$. Therefore, by integration by part, we have

$$\int_0^t e^{\lambda x} \varphi'(x) dx = e^{\lambda t}\varphi(t) - \lambda \int_0^t e^{\lambda x} \varphi(x) dx. \tag{5.18}$$

Therefore, by the last inequality, we infer that

$$\begin{aligned}\int_0^\infty g_\lambda(x)\varphi'(x)dx &= \lambda\int_0^\infty f(t)e^{-\lambda t}\left[e^{\lambda t}\varphi(t)-\lambda\int_0^t e^{\lambda x}\varphi(x)dx\right]dt\\ &= \lambda\int_0^\infty f(t)\varphi(t)dt-\lambda^2\int_0^\infty f(t)\left[\int_0^t e^{-\lambda(t-x)}\varphi(x)dx\right]dt\\ &= \lambda\int_0^\infty f(t)\varphi(t)dt-\lambda\Phi,\end{aligned}$$

where

$$\Phi := \lambda\int_0^\infty f(t)\left[\int_0^t e^{-\lambda(t-x)}\varphi(x)dx\right]dt.$$

Again by the Fubini theorem, we get

$$\begin{aligned}\Phi &= \lambda\int_0^\infty f(t)\left[\int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}\varphi(x)dx\right]dt\\ &= \int_0^\infty \left[\lambda\int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}f(t)dt\right]\varphi(x)dx\\ &= \int_0^\infty g_\lambda(x)\varphi(x)dx.\end{aligned}$$

Taking into account the last inequality, we infer that

$$\int_0^\infty g_\lambda(x)\varphi'(x)dx = \lambda\int_0^\infty f(t)\varphi(t)dt-\lambda\int_0^\infty g_\lambda(x)\varphi(x)dx,$$

which gives the desired result. ■

Lemma 5.8. *Assume that $\nu > 0$ and $p \geq 1$. Then the infinitesimal generator A of the shift-semigroup on L^p_ν is characterized by*

$$\mathscr{D}(A) = \{f \in L^p_\nu : Df \in L^p_\nu\} \tag{5.19}$$

and

$$Af = Df, \quad f \in \mathscr{D}(A). \tag{5.20}$$

Proof. Define, for each $\lambda > -\frac{\nu}{p}$, a function $J_\lambda : L^p_\nu \to L^p_\nu$ by

$$J_\lambda = (I-\lambda^{-1}A)^{-1} = \lambda R(\lambda). \tag{5.21}$$

By Lemma 5.6, J_λ is well-defined, $\mathscr{R}(J_\lambda) = \mathscr{D}(A)$ and

$$AJ_\lambda = \lambda(J_\lambda - I). \tag{5.22}$$

For each $f \in L^p_\nu$, consider the function g_λ from the previous lemma. It is obvious that $g_\lambda = J_\lambda f \in \mathscr{R}(J_\lambda)$. Thus, we obtain $g_\lambda \in \mathscr{D}(A)$ since $\mathscr{R}(J_\lambda) = \mathscr{D}(A)$. Hence, in order to show (5.20), it is sufficient to show $Ag_\lambda = Dg_\lambda$. By (5.22), we obtain

$$Ag_\lambda = AJ_\lambda f = \lambda(J_\lambda - I)f = \lambda g_\lambda - \lambda f.$$

We know from the previous Lemma that $Dg_\lambda = \lambda g_\lambda - \lambda f$. Therefore $Ag_\lambda = Dg_\lambda$. This finishes the proof of (5.20).

Let us now prove (5.19). Set

$$V = \{f \in L^p_\nu : Df \in L^p_\nu\}.$$

We need to show that $\mathscr{D}(A) \subset V$ and $V \subset \mathscr{D}(A)$. We first show $\mathscr{D}(A) \subset V$. Fix $f \in \mathscr{D}(A)$. Then $f \in \mathscr{R}(J_\lambda)$, i.e., there exist $h \in L^p_\nu$ such that $f = J_\lambda h \in L^p_\nu$. By Lemma 5.6, $Df \in L^p_\nu$ and so $f \in V$. Therefore $\mathscr{D}(A) \subset V$. Now we show $V \subset \mathscr{D}(A)$. Fix $f \in V$. Then $f \in L^p_\nu$ and $Df \in L^p_\nu$. Let us choose and fix an arbitrary $\lambda > -\frac{\nu}{p}$. Define a function $u \in L^p_\nu$ by

$$u = f - \frac{1}{\lambda} Df$$

and so

$$Df = \lambda f - \lambda u. \tag{5.23}$$

Define a function g_λ by

$$g_\lambda = J_\lambda u.$$

Then by Lemma 5.7, we have

$$Dg_\lambda = -\lambda u + \lambda g_\lambda. \tag{5.24}$$

Let $L^p_\nu \ni w = f - g_\lambda$. Then by (5.23) and (5.24), we obtain

$$Dw = Df - Dg_\lambda = \lambda f - \lambda u + \lambda u - \lambda g_\lambda = \lambda f - \lambda g_\lambda = \lambda w. \tag{5.25}$$

Define a function z by

$$z(x) = e^{-\lambda x} w(x), \quad x \in [0,\infty).$$

It is obvious that $z \in L^p_\nu$ and by (5.25), we get

$$Dz = D(e^{-\lambda} w) = -\lambda e^{-\lambda} w + e^{-\lambda} \lambda w = 0.$$

Thus, we have proved that $Dz = 0$, which implies that for each $\varphi \in C_c^1\big((0,\infty)\big)$,

$$\int_0^\infty z(x)\varphi'(x)dx = 0.$$

Since e^λ and w are continuous on $[0,\infty)$, z is continuous on $[0,\infty)$. Thus by Lemma 2 in [33], we infer that $z = C$, where C is a constant. Therefore, we have

$$w(x) = Ce^{\lambda x}, \quad x \in [0,\infty).$$

Note that $w \in L_\nu^p$ if and only if $w = 0$. Indeed, if $w \in L_\nu^p$, then since $\lambda p + \nu > 0$, we have

$$\int_0^\infty |w(x)|^p e^{\nu x}dx = |C|^p \int_0^\infty e^{(p\lambda+\nu)x}dx = \infty.$$

Hence, w has to be 0. Therefore $f = g_\lambda$ and so $f \in \mathscr{D}(A)$. Thus $V \subset \mathscr{D}(A)$. ∎

Let H be a Hilbert space with respect to an inner product $\langle\cdot,\cdot\rangle_H$. Define $L_\nu^p(H)$ to be the space of all (equivalence classes of) Borel measurable functions $f : [0,\infty) \to H$ such that

$$\|f\|_{L_\nu^p(H)} := \left(\int_0^\infty \|f(x)\|_H^p e^{\nu x}dx\right)^{\frac{1}{p}} < \infty.$$

All the results above of L_ν^p can be generalized for the space $L_\nu^p(H)$. Thus, $L_\nu^p(H)$ is a separable Banach space endowed with the norm $\|\cdot\|_{L_\nu^p(H)}$. The following proposition gives a sufficient condition under which an L_ν^p-valued operator K defined on H is γ-radonifying.

Proposition 5.2. *Let $\nu \in \mathbb{R}$ and $p \geq 2$. Then for every $\kappa \in L_\nu^p(H)$, a bounded linear operator $K : H \to L_\nu^p$ defined by*

$$K[h](x) = \langle\kappa(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is γ-radonifying, i.e. $K \in \gamma(H, L_\nu^p)$, and there exists a constant $N > 0$ independent of κ such that

$$\|K\|_{\gamma(H,L_\nu^p)} \leq N\|\kappa\|_{L_\nu^p(H)}.$$

Proof. Define a linear operator $V : L_\nu^p(H) \to L^p(H)$ by

$$Vf = fe^{\frac{\nu}{p}}, \quad f \in L_\nu^p(H).$$

As in the proof of Lemma 5.1, it can easily be shown that V is well-defined, bijective and isometry, i.e.

$$\|Vf\|_{L^p(H)} = \|f\|_{L_\nu^p(H)}, \quad f \in L_\nu^p(H). \tag{5.26}$$

Fix $\kappa \in L^p_\nu(H)$. Then $V\kappa \in L^p(H)$. It was proven in Proposition 2.1 of [9] that for every $\phi \in L^p(H)$, a bounded linear operator $M : H \to L^p$ defined by

$$M[h](x) = \langle \phi(x), h \rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is γ-radonifying and for a constant $N > 0$ independent of ϕ,

$$\|M\|_{\gamma(H,L^p)} \le N \|\phi\|_{L^p(H)}.$$

Therefore, the following operator $\bar{K}$ defined by

$$\bar{K}[h](x) = \langle V\kappa(x), h \rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is γ-radonifying from H into L^p and

$$\|\bar{K}\|_{\gamma(H,L^p)} \le N \|V\kappa\|_{L^p(H)}.$$

Moreover, it follows from (5.26) that

$$\|\bar{K}\|_{\gamma(H,L^p)} \le N \|\kappa\|_{L^p_\nu(H)}. \tag{5.27}$$

Consider the operator T from the proof of Lemma 5.1. Then K can be rewritten as $T^{-1} \circ \bar{K}$. Thus, by Theorem 2.15, since $T^{-1} : L^p \to L^p_\nu$ is linear bounded and $\bar{K} : H \to L^p$ is γ-radonifying, the map K is γ-radonifying. Moreover, since $\|T^{-1}\| \le 1$ and by inequality (5.27), we infer that

$$\|K\|_{\gamma(H,L^p_\nu)} \le \|T^{-1}\| \|\bar{K}\|_{\gamma(H,L^p)} \le N \|\kappa\|_{L^p_\nu(H)},$$

which completes the proof. ■

Lemma 5.9. *Let $\nu > 0$ and $p \ge 1$. Assume that $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space and $g : [0,\infty) \times [0,\infty) \times \mathbb{R} \to H$ is a measurable function w.r.t the second variable such that there exist functions $\bar{g} \in L^p_\nu$ and $\hat{g} \in L^p_\nu \cap L^\infty$ such that for all $t \in [0,\infty)$,*

$$\|g(t,x,u)\|_H \le |\bar{g}(x)|, \quad u \in \mathbb{R}, \quad x \in [0,\infty) \tag{5.28}$$

and

$$\|g(t,x,u) - g(t,x,v)\|_H \le |\hat{g}(x)||u - v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty). \tag{5.29}$$

Define a function $F : [0,\infty) \times L^p_\nu \to L^p_\nu$ by

$$F(t,f)(x) = \left\langle g(t,x,f(x)), \int_0^x g(t,y,f(y))dy \right\rangle_H, \quad f \in L^p_\nu, \quad x,t \in [0,\infty). \tag{5.30}$$

Then F is well-defined. Moreover,
(i) for all $t \in [0,\infty)$ and $f \in L^p_\nu$,

$$\|F(t,f)\|_{\nu,p} \le \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|^2_{\nu,p} \tag{5.31}$$

(ii) F is globally Lipschitz on L^p_ν with Lipschitz constants independent of time t.

Proof. Fix $t \ge 0$ and $f \in L^p_\nu$. Since $g(t,\cdot,f(\cdot))$ is measurable and by inequality (5.28),

$$\int_0^x |g(t,y,f(y))|dy \le \int_0^x |\bar{g}(y)|dy < \infty,$$

the integral $\int_0^x g(t,y,f(y))dy$ exists. Define a function h by

$$h(x) = \int_0^x g(t,y,f(y))dy, \quad x \in [0,\infty).$$

This function h is continuous. Indeed, for every sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ converging to $x \in [0,\infty)$, by inequality (5.28), we have

$$|h(x_n) - h(x)| \le \int_x^{x_n} |g(t,y,f(y))|dy \le \int_0^\infty \mathbb{1}_{[x,x_n]}(y)|\bar{g}(y)|dy \to 0 \quad as\ n \to \infty.$$

Thus, h is continuous and so, h is measurable. Define a function $K : H \times H \to \mathbb{R}$ by

$$K(h_1,h_2) = \langle h_1,h_2\rangle_H, \quad h_1,h_2 \in H.$$

We can write $F(t,f)$ as follow

$$F(t,f)(x) = K\left(g(t,x,f(x)), \int_0^x g(t,y,f(y))dy\right), \quad x \in [0,\infty).$$

It is obvious that K is continuous. Therefore, $F(t,f)$ is measurable. Moreover, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|F(t,f)(x)\right| &\le \left\|g(t,x,f(x)\right\|_H \left\|\int_0^x g(t,y,f(y))dy\right\|_H \\ &\le \left\|g(t,x,f(x)\right\|_H \int_0^\infty \left\|g(t,y,f(y))\right\|_H dy, \quad x \in [0,\infty).\end{aligned}$$

It follows from Proposition 5.1 and inequality (5.28) that

$$\left|F(t,f)(x)\right| \le |\bar{g}(x)| \int_0^\infty |\bar{g}(y)|dy \le \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p}|\bar{g}(x)|, \quad x \in [0,\infty).$$

Taking into account the last inequlity, we deduce that

$$\int_0^\infty \left|F(t,f)(x)\right|^p e^{\nu x} dx \le \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^p \int_0^\infty |\bar{g}(x)|^p e^{\nu x} dx = \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^{2p}.$$

Therefore $F(t,f) \in L_\nu^p$ and thus, F is well-defined. Also last inequality gives the desired conclusion (5.31). Finally, we prove (ii). Fix $t \ge 0$ and $f_1, f_2 \in L_\nu^p$. Define a function I by

$$I(x) = F(t,f_1)(x) - F(t,f_2)(x), \quad x \in [0,\infty).$$

Using the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left|I(x)\right| &\le \left|\left\langle g(t,x,f_1(x)), \int_0^x (g(t,y,f_1(y)) - g(t,y,f_2(y)))dy \right\rangle_H\right| \\
&+ \left|\left\langle g(t,x,f_1(x)) - g(t,x,f_2(x)), \int_0^x g(t,y,f_2(y))dy \right\rangle_H\right| \\
&\le \left\|g(t,x,f_1(x))\right\|_H \left\|\int_0^x \big(g(t,y,f_1(y)) - g(t,y,f_2(y))\big)dx\right\|_H \\
&+ \left\|g(t,x,f_1(x)) - g(t,x,f_2(x))\right\|_H \left\|\int_0^x \big(g(t,y,f_2(y))\big)dx\right\|_H \\
&\le \left\|g(t,x,f_1(x))\right\|_H \int_0^\infty \left\|g(t,x,f_1(x)) - g(t,x,f_2(x))\right\|_H dx \\
&+ \left\|g(t,x,f_1(x)) - g(t,x,f_2(x))\right\|_H \int_0^\infty \left\|g(t,x,f_2(x))\right\|_H dx, \quad x \in [0,\infty).
\end{aligned}$$

It follows from inequalities (5.28) and (5.29), and Proposition 5.1 that

$$\begin{aligned}
\left|I(x)\right| &\le |\bar{g}(x)| \int_0^\infty |\hat{g}(x)||f_1(x) - f_2(x)|dx + |\hat{g}(x)||f_1(x) - f_2(x)| \int_0^\infty |\bar{g}(x)|dx \\
&\le \|\hat{g}\|_\infty |\bar{g}(x)| \int_0^\infty |f_1(x) - f_2(x)|dx + |\hat{g}(x)||f_1(x) - f_2(x)| \int_0^\infty |\bar{g}(x)|dx \\
&\le \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\hat{g}\|_\infty \|f_1 - f_2\|_{\nu,p} |\bar{g}(x)| + \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p} |\hat{g}(x)|\ |f_1(x) - f_2(x)|, \quad x \in [0,\infty).
\end{aligned}$$

Taking into account the last estimate, we infer that

$$\begin{aligned}
\left\|I\right\|_{\nu,p}^p &\le 2^{p-1} \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\hat{g}\|_\infty^p \|f_1 - f_2\|_{\nu,p}^p \int_0^\infty |\bar{g}(x)|^p e^{\nu x} dx \\
&+ 2^{p-1} \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^p \int_0^\infty |\hat{g}(x)|^p |f_1(x) - f_2(x)|^p e^{\nu x} dx \\
&\le 2^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\hat{g}\|_\infty^p \|\bar{g}\|_{\nu,p}^p \|f_1 - f_2\|_{\nu,p}^p.
\end{aligned}$$

Thus, we have showed that

$$\|F(t,f_1) - F(t,f_2)\|_{\nu,p} \le 2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\hat{g}\|_\infty \|\bar{g}\|_{\nu,p} \|f_1 - f_2\|_{\nu,p}, \tag{5.32}$$

which implies that F is globally Lipschitz on L^p_ν.

■

Lemma 5.10. *Assume that all the assumptions of Lemma 5.9 are satisfied. Define a function* $G:[0,\infty)\times L^p_\nu \to \gamma(H,L^p_\nu)$ *by*

$$G(t,f)[h](x) = \langle g(t,x,f(x)),h\rangle_H, \quad f\in L^p_\nu, \quad h\in H, \quad x,t\in[0,\infty). \tag{5.33}$$

Then G is well-defined. Moreover,
(i) for all $t\in[0,\infty)$ and $f\in L^p_\nu$, there exists a constant $N>0$ such that

$$\|G(t,f)\|_{\gamma(H,L^p_\nu)} \le N\|\bar g\|_{\nu,p} \tag{5.34}$$

(ii) G is globally Lipschitz on L^p_ν with Lipschitz constants independent of time t.

Proof. Fix $t\ge 0$ and $f\in L^p_\nu$. Define a function $\kappa:[0,\infty)\to H$ by

$$\kappa(x) = g(t,x,f(x)), \quad x\in[0,\infty).$$

Then $G(t,f)$ can be rewritten as follow

$$G(t,f)[h](x) = \langle\kappa(x),h\rangle_H, \quad h\in H, \quad x\in[0,\infty).$$

In order to show that $G(t,f)\in\gamma(H,L^p_\nu)$, by Proposition 5.2, it is sufficient to show $\kappa\in L^p_\nu(H)$. Since we assume that $g(t,\cdot,f(\cdot))$ is measurable, κ is measurable. Moreover, by inequality (5.28), we have

$$\int_0^\infty \|\kappa(x)\|^p_H e^{\nu x}dx = \int_0^\infty \|g(t,x,f(x))\|^p_H e^{\nu x}dx \le \int_0^\infty |\bar g(x)|^p e^{\nu x}dx = \|\bar g\|^p_{\nu,p}. \tag{5.35}$$

Thus $\kappa\in L^p_\nu(H)$. Therefore, G is well-defined. Again by Proposition 5.2, we have, for a constant $N>0$ independent of κ,

$$\|G(t,f)\|_{\gamma(H,L^p_\nu)} \le N\left(\int_0^\infty \|\kappa(x)\|^p_H e^{\nu x}dx\right)^{\frac{1}{p}}.$$

It follows from inequality (5.35) that

$$\|G(t,f)\|_{\gamma(H,L^p_\nu)} \le N\|\bar g\|_{\nu,p}, \tag{5.36}$$

which gives the desired result (5.34). Let us now prove (ii). Fix $t\ge 0$ and $f_1,f_2\in L^p_\nu$. Define a function λ by

$$\lambda(x) = g(t,x,f_1(x)) - g(t,x,f_2(x)), \quad x\in[0,\infty).$$

Then $G(t,f_1)-G(t,f_2)$ can be rewritten as follow

$$\big(G(t,f_1)-G(t,f_2)\big)[h](x) = \langle \lambda(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty).$$

Using inequality (5.29), we get

$$\int_0^\infty \|\lambda(x)\|_H^p e^{\nu x} dx \le \int_0^\infty |\hat{g}(x)|^p |f_1(x)-f_2(x)|^p e^{\nu x} dx \le \|\hat{g}\|_\infty^p \|f_1-f_2\|_{\nu,p}^p. \tag{5.37}$$

Therefore, by Proposition 5.2 and estimate (5.37), we deduce that for a constant $N>0$ independent λ,

$$\big\|G(t,f_1)-G(t,f_2)\big\|_{\gamma(H,L_\nu^p)} \le N\|\hat{g}\|_\infty \|f_1-f_2\|_{\nu,p}, \tag{5.38}$$

which implies that $G(t,\cdot)$ is globally Lipschitz on L_ν^p. ■

Now we introduce main result of this section.

Theorem 5.1. *Let $\nu>0$ and $p\ge 2$. Let $(H,\langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. Assume that $g:[0,\infty)\times[0,\infty)\times\mathbb{R}\to H$ is a measurable function w.r.t the second variable such that there exist functions $\bar{g}\in L_\nu^p$ and $\hat{g}\in L_\nu^p\cap L^\infty$ such that for all $t\in[0,\infty)$,*

$$\|g(t,x,u)\|_H \le |\bar{g}(x)|, \quad u\in\mathbb{R}, \quad x\in[0,\infty)$$

and

$$\|g(t,x,u)-g(t,x,v)\|_H \le |\hat{g}(x)||u-v|, \quad u,v\in\mathbb{R}, \quad x\in[0,\infty).$$

then for each $r_0\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};L_\nu^p)$, there exists a unique L_ν^p-valued continuous mild solution r to equation (3.7) with the initial value $r(0)=r_0$. Moreover, the solution is a Markov process.

Proof. The abstract form of equation (3.7) in the space L_ν^p is as follows

$$dr(t) = \big(Ar(t)+F(t,r(t))\big)dt + G(t,r(t))dW(t), \quad t\ge 0, \tag{5.39}$$

where A is the infinitesimal generator of the shift-semigroup on L_ν^p (see Lemma 5.8), and F and G are functions defined in Lemma 5.9 and 5.10 respectively. Now equation (3.7) is the form of equation (4.2). In lemma 5.2, we showed that L_ν^p is a Banach space satisfying H-condition. Also in Lemma 5.3, we showed that the shift-semigroup on L_ν^p is a contraction type C_0-semigroup. Moreover, in Lemma 5.9 and Lemma 5.10, we showed that F and G satisfy the conditions of Theorem 4.1. Therefore, for each $r_0\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};L_\nu^p)$, equation (5.39) has a unique L_ν^p-valued continuous mild solution with the initial value r_0. It follows from Theorem 4.4 that the solution is a Markov process. ■

Example 5.1. *We consider the HJMM equation (driven by a one dimensional Brownian motion) having the volatility*

$$g(x) = \sigma e^{-\alpha x}, \quad x \in [0,\infty),$$

where σ is a constant and $\alpha > 0$. Therefore, the forward rate curve process $\{r(t)\}_{t\geq 0}$ satisfies the following stochastic differential equation

$$dr(t)(x) = \left(\frac{\partial}{\partial x} r(t)(x) + \frac{\sigma^2}{\alpha}\left(e^{-\alpha x} - e^{-2\alpha x}\right)\right)dt + \sigma e^{-\alpha x} dW(t), \quad t,x \geq 0. \tag{5.40}$$

One can be easily shown that the function g satisfies all the assumption of Theorem 5.1. Therefore, for each $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; L^p_\nu)$, equation (5.40) has a unique L^p_ν-valued continuous mild solution given by

$$r(t)(x) = r_0(x+t) + \frac{\sigma^2}{\alpha}\int_0^t \left(e^{-\alpha(x+t)} - e^{-2\alpha(x+t)}\right)dt + \sigma \int_0^t e^{-\alpha(x+t)} dW(t), \quad t,x \geq 0. \tag{5.41}$$

We can also consider the HJMM equation having the following volatilities

$$g(x) = e^{-\alpha x}\cos(\beta x), \quad x \in [0,\infty),$$

or

$$g(x) = e^{-\alpha x}\sin(\beta x), \quad x \in [0,\infty),$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$.

Let us now give an example for the HJMM equation with the volatility depending on the forward curve process $\{r(t)\}_{t\geq 0}$.

Example 5.2. *Define a function $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$g(x,u) = e^{-\alpha x}\sin(u), \quad x \in [0,\infty), \quad u \in \mathbb{R},$$

where $\alpha > 0$. We consider a HJMM equation (driven by a one dimensional Wiener process) having the volatility $g(x, r(t)(x))$. Thus, the forward curve process $\{r(t)\}_{t\geq 0}$ satisfies the following stochastic differential equation

$$dr(t)(x) = \left(\frac{\partial}{\partial x} r(t)(x) + e^{-\alpha x}\sin\left(r(t)(x)\right)\int_0^x e^{-\alpha y}\sin\left(r(t)(y)\right)dy\right)dt + e^{-\alpha x}\sin\left(r(t)(x)\right)dW(t). \tag{5.42}$$

It is obvious that g is measurable with respect to the first variable. Since $|\sin(u)| \leq 1$ for every $u \in \mathbb{R}$, we have

$$|g(x,u)| \leq e^{-\alpha x}.$$

Moreover, since $\sin(u)$ *is Lipschitz continuous, there exists a constant* $C > 0$ *such that*

$$|g(x,u) - g(x,v)| \le Ce^{-\alpha x}|u - v|, \quad u, v \in \mathbb{R}.$$

Hence, g satisfies all the assumptions of Theorem 5.1. Thus, for each $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; L^p_\nu)$*, equation (5.42) has a unique* L^p_ν*-valued continuous mild solution with* $r(0) = r_0$.

5.2 Existence and Uniqueness of Solutions to the HJMM Equation in the Weighted Sobolev Spaces

In this section, we first present the weighted Sobolev spaces and some useful properties of them which allows to apply Theorem 4.3 to prove the existence and uniqueness of solutions to the HJMM equation in the weighted Sobolev space. Then we give main result of this section. This section is very similar to the previous section. We explain our main goal for this section in the end of section.

For each $\nu \in \mathbb{R}$ and $p \ge 1$, define $W^{1,p}_\nu$ to be the space of all functions $f \in L^p_\nu$ such that $Df \in L^p_\nu$, i.e.

$$W^{1,p}_\nu = \{f \in L^p_\nu \ : \ Df \in L^p_\nu\}.$$

For each $\nu \in \mathbb{R}$ and $p \ge 1$, the space $W^{1,p}_\nu$ is called **the weighted Sobolev space**.

Lemma 5.11. *For each* $\nu \in \mathbb{R}$ *and* $p \ge 1$*, the space* $W^{1,p}_\nu$ *is a separable Banach space endowed with the norm*

$$\|f\|_{W^{1,p}_\nu} = \|f\|_{\nu,p} + \|Df\|_{\nu,p}.$$

Proof. Fix $\nu \in \mathbb{R}$ and $p \ge 1$. It is well known that the space $W^{1,p}$ of all functions $f \in L^p$ such that $Df \in L^p$ is a separable Banach space with respect to the norm

$$\|f\|_{W^{1,p}} = \|f\|_p + \|Df\|_p, \quad f \in W^{1,p}.$$

Define a linear operator $T : W^{1,p}_\nu \to W^{1,p}$ given by

$$Tf = fe^{\frac{\nu}{p}}, \quad f \in W^{1,p}_\nu.$$

As in the proof of Lemma 5.1, it can be easily shown that the map T is well-defined, bijective and isometry. Thus, $W^{1,p}_\nu$ and $W^{1,p}$ are isometric spaces and hence, by Theorem 2.1, $W^{1,p}_\nu$ is a separable Banach space with respect to the norm $\|\cdot\|_{W^{1,p}_\nu}$. ■

Lemma 5.12. *For each $\nu \in \mathbb{R}$ and $p \geq 2$, the space $W^{1,p}_\nu$ satisfies the H-condition.*

Proof. The proof is similar to the proof of Lemma 5.2. Consider the operator T from the previous Lemma. For each $p \geq 2$, the space $W^{1,p}$ satisfies the H-condition, see [12]. Therefore, $W^{1,p}_\nu$ satisfies the H-condition. ■

Proposition 5.3. *For each $\nu \geq 0$ and $p \geq 1$, the space $W^{1,p}_\nu$ is continuously embedded into the space L^∞. In particular, there exists a constant $C > 0$ depending on ν and p such that*

$$\sup_{x\in[0,\infty)} e^{\nu x}|f(x)|^p \leq C^p \|f\|^p_{W^{1,p}_\nu}, \quad f \in W^{1,p}_\nu. \tag{5.43}$$

Proof. Fix $\nu > 0$, $p \geq 1$ and $f \in W^{1,p}_\nu$. Let $\varepsilon > 0$. Since

$$\int_0^\infty |f(x)|^p e^{\nu x} dx < \infty,$$

there exists $x_0 \in [0,\infty)$ such that $e^{\nu x_0}|f(x_0)|^p < \varepsilon$. Consider $x \in [x_0,\infty)$. Then

$$\begin{aligned} e^{\nu x}|f(x)|^p &= e^{\nu x_0}|f(x_0)|^p + \int_{x_0}^x D\big(|f(y)|^p e^{\nu y}\big)dy \\ &= e^{\nu x_0}|f(x_0)|^p + p\int_{x_0}^x |f(y)|^{p-1} Df(y) e^{\nu y} dy \\ &+ \nu \int_{x_0}^x |f(y)|^p e^{\nu y} dy. \end{aligned}$$

Therefore

$$\sup_{x\in[x_0,\infty)} e^{\nu x}|f(x)|^p \leq \varepsilon + p\int_{x_0}^\infty |f(x)|^{p-1} Df(x) e^{\nu x} dx + \nu \|f\|^p_{\nu,p}.$$

Using the Hölder inequality and the Young inequality, we get

$$\begin{aligned} \int_{x_0}^\infty |f(x)|^{p-1} Df(x) e^{\nu x} dx &= \int_{x_0}^\infty |f(x)|^{p-1} Df(x) e^{\frac{(p-1)\nu x}{p}} e^{\frac{\nu x}{p}} dx \\ &\leq \left(\int_{x_0}^\infty |f(x)|^p e^{\nu x} dx\right)^{\frac{p-1}{p}} \left(\int_{x_0}^\infty |Df(x)|^p e^{\nu x} dx\right)^{\frac{1}{p}} \\ &\leq \frac{p-1}{p}\|f\|^p_{\nu,p} + \frac{1}{p}\|Df\|^p_{\nu,p}. \end{aligned}$$

Taking into account the last inequality, we infer that

$$\sup_{x\in[x_0,\infty)} e^{\nu x}|f(x)|^p \leq \varepsilon + (p-1)\|f\|^p_{\nu,p} + \|Df\|^p_{\nu,p} + \nu\|f\|^p_{\nu,p}.$$

Similarly, we can prove the above inequality for $x \in [0, x_0)$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\sup_{x\in[0,\infty)} e^{\nu x}|f(x)|^p \leq (p-1)\|f\|^p_{\nu,p} + \|Df\|^p_{\nu,p} + \nu\|f\|^p_{\nu,p},$$

which concludes the proof. ■

Remark 5.1. *In fact, if $f \in W_\nu^{1,p}$, then by inequality (5.43),*

$$\lim_{x\to\infty} |f(x)| = 0.$$

Proposition 5.4. *For each $\nu > 0$ and $p \geq 1$, the following inequality holds.*

$$\sup_{x\in[0,\infty)} |f(x)| \leq \|Df\|_1, \quad f \in W_\nu^{1,p}. \tag{5.44}$$

Proof. Fix $\nu > 0$ and $p \geq 1$. Let $\varepsilon > 0$. For any $f \in W_\nu^{1,p}$, $f \in L^1$. Thus

$$\int_0^\infty |f(x)|dx < \infty$$

and hence, there exists $x_0 \in [0,\infty)$ such that $|f(x_0)| < \varepsilon$. Consider $x \in [x_0,\infty)$. Then

$$|f(x) - f(x_0)| = \left|\int_{x_0}^x Df(x)dx\right| \leq \int_{x_0}^x |Df(x)|\,dx$$

and so

$$|f(x)| \leq |f(x_0)| + |f(x) - f(x_0)| \leq \varepsilon + \int_{x_0}^x |Df(x)|\,dx.$$

Therefore, we obtain

$$\sup_{x\in[x_0,\infty)} |f(x)| \leq \varepsilon + \int_0^\infty |Df(x)|\,dx.$$

It follows from Proposition 5.1 that

$$\sup_{x\in[x_0,\infty)} |f(x)| \leq \varepsilon + \|Df\|_1.$$

Similarly, we can prove the above inequality for $x \in [0,x_0)$. Since ε is arbitrary, this gives the desired conclusion. ■

Lemma 5.13. *Assume that $\nu \in \mathbb{R}$ and $p \geq 1$. Then the shift-semigroup is a contraction type C_0-semigroup on $W_\nu^{1,p}$.*

Lemma 5.14. *[32] Let $C_c^2(\mathbb{R}^+)$ be the space of all twice continuously differentiable functions $f : [0,\infty) \to \mathbb{R}$ with compact support. Then $C_c^2(\mathbb{R}^+)$ is a dense subspace of $W_\nu^{1,p}$.*

Lemma 5.15. *The shift-semigroup on $C_c^2(\mathbb{R}^+)$ is strongly continuous in the norm of $W_\nu^{1,p}$.*

Proof. Fix $g \in C_c^2(\mathbb{R}^+)$. Then $g, Dg \in C_c^1(\mathbb{R}^+)$. In Lemma 5.5, we showed that for $g \in C_c^1(\mathbb{R}^+)$, $\|S(t)g - g\|_p \to 0$ as $t \to 0$. Similarly, we can show that $Dg \in C_c^1(\mathbb{R}^+)$, $\|S(t)Dg - Dg\|_p \to 0$ as $t \to 0$. Thus, $\|S(t)g - g\|_{W_\nu^{1,p}} \to 0$ as $t \to 0$. Therefore, S is strongly continuous on $C_c^2(\mathbb{R}^+)$. ■

Proof of Lemma 5.13. First we show that for each $t \ge 0$, $S(t): W_\nu^{1,p} \ni f \mapsto f(\cdot + t) \in W_\nu^{1,p}$ is well-defined, linear and bounded. Fix $t \ge 0$ and $f \in W_\nu^{1,p}$. In Lemma 5.3, we showed that $S(t)f \in L_\nu^p$. Let us now show $DS(t)f \in L_\nu^p$. Recall that Df is the first weak derivative of $f \in L_\nu^p$ iff for each $\varphi \in C_c^1\big((0,\infty)\big)$,

$$\int_0^\infty f(x)\varphi'(x)dx = -\int_0^\infty Df(x)\varphi(x)dx.$$

Fix $\varphi \in C_c^1\big((0,\infty)\big)$. Since $\varphi(\cdot - t) \in C_c^1\big((0,\infty)\big)$, we have

$$\begin{aligned}\int_0^\infty S(t)f(x)\varphi'(x)dx &= \int_0^\infty f(x+t)\varphi'(x)dx\\ &= \int_t^\infty f(x)\varphi'(x-t)dx\\ &= -\int_t^\infty Df(x)\varphi(x-t)dx\\ &= -\int_0^\infty Df(x+t)\varphi(x)dx\\ &= -\int_0^\infty S(t)Df(x)\varphi(x)dx.\end{aligned}$$

Therefore, the first weak derivative of $S(t)f$ exists and equal to $S(t)Df$. Moreover, we have

$$\begin{aligned}\int_0^\infty |DS(t)f(x)|^p e^{\nu x}dx &= \int_0^\infty |Df(x+t)|^p e^{\nu x}dx\\ &= e^{-\nu t}\int_t^\infty |Df(u)|^p e^{\nu u}du\\ &\le e^{-\nu t}\|Df\|_{\nu,p}^p.\end{aligned} \tag{5.45}$$

Hence, $S(t)$ is well-defined. It is obvious that $S(t)$ is linear and it follows from estimates (5.6) and (5.45) that

$$\|S(t)f\|_{W_\nu^{1,p}} \le e^{-\frac{\nu}{p}t}\|f\|_{W_\nu^{1,p}}, \quad t \ge 0, \tag{5.46}$$

which implies that $S(t)$ is bounded.

Now we show that S is a contraction type C_0-semigroup on $W_\nu^{1,p}$. It is obvious that S is a semigroup on $W_\nu^{1,p}$. Let us prove that S is strongly continuous on $W_\nu^{1,p}$. Fix $f \in W_\nu^{1,p}$. Since $C_c^2(\mathbb{R}^+)$ is dense in $W_\nu^{1,p}$, for every $\frac{\varepsilon}{3} > 0$, there exists $g \in C_c^2(\mathbb{R}^+)$ such that

$$\|f - g\|_{W_\nu^{1,p}} \le \frac{\varepsilon}{3}. \tag{5.47}$$

Also by the previous Lemma, we have, for every $\frac{\varepsilon}{3} > 0$,

$$\|S(t)g - g\|_{W_\nu^{1,p}} \le \frac{\varepsilon}{3}. \tag{5.48}$$

Taking into account inequalities (5.46), (5.47) and (5.48), we deduce that

$$\begin{aligned}\|S(t)f-f\|_{W_\nu^{1,p}} &= \|S(t)f-S(t)g+S(t)g-g+g-f\|_{W_\nu^{1,p}}\\ &\le \|S(t)(f-g)\|_{W_\nu^{1,p}}+\|S(t)g-g\|_{W_\nu^{1,p}}+\|f-g\|_{W_\nu^{1,p}}\\ &\le e^{\frac{\nu}{p}t}\|f-g\|_{W_\nu^{1,p}}+\|S(t)g-g\|_{W_\nu^{1,p}}+\|f-g\|_{W_\nu^{1,p}}\le\varepsilon,\end{aligned}$$

which implies that S is strongly continuous on $W_\nu^{1,p}$. Moreover, it follows from inequality (5.46) that

$$\|S(t)\|_{\mathscr{L}(W_\nu^{1,p})}\le e^{-\frac{\nu}{p}t},\quad t\ge 0.$$

Thus, S is a contraction type C_0-semigroup on $W_\nu^{1,p}$.

Lemma 5.16. *The infinitesimal generator A of the shift-semigroup on $W_\nu^{1,p}$ is characterized by*

$$\mathscr{D}(A)=\{f\in W_\nu^{1,p}:Df\in W_\nu^{1,p}\}$$

and

$$Af=Df,\quad f\in\mathscr{D}(A).$$

The proof of this lemma is similar to the proof of Lemma 5.8.

Let H be a Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle_H$. Define $W_\nu^{1,p}(H)$ to be the space of all functions $f\in L^p(H)$ such that $Df\in L^p(H)$. All the results above of the spaces $W_\nu^{1,p}$ can be generalized for the spaces $W_\nu^{1,p}(H)$. Therefore, $W_\nu^{1,p}(H)$ is a separable Banach space with respect to the norm

$$\|f\|_{W_\nu^{1,p}(H)}=\|f\|_{L_\nu^p(H)}+\|Df\|_{L_\nu^p(H)}\quad f\in W_\nu^{1,p}(H).$$

The following proposition gives a sufficient condition under which a $W_\nu^{1,p}$-valued operator K defined on H is γ-radonifying.

Proposition 5.5. *Let $p\ge 2$. If κ is a function in $W_\nu^{1,p}(H)$, then a bounded linear operator $K:H\to W_\nu^{1,p}$ defined by*

$$K[h](x)=\langle\kappa(x),h\rangle_H,\quad h\in H,\quad x\in[0,\infty),$$

is γ radonifying, i.e. $K\in\gamma(H,W_\nu^{1,p})$. Moreover, there exists a constant $N>0$ independent of κ such that

$$\|K\|_{\gamma(H,W_\nu^{1,p})}\le N\|\kappa\|_{W_\nu^{1,p}(H)}.$$

Proof. Define a linear operator $V : W^{1,p}_\nu(H) \to W^{1,p}(H)$ by

$$Vf = fe^{\frac{\nu}{p}}, \quad f \in W^{1,p}_\nu(H).$$

As in the proof of Lemma 5.1, it can be easily seen that the operator V is well-defined, bijective and isometry, i.e.

$$\|Vf\|_{W^{1,p}(H)} = \|f\|_{W^{1,p}_\nu(H)} \quad f \in W^{1,p}_\nu(H). \tag{5.49}$$

Fix $\kappa \in W^{1,p}_\nu(H)$. Then $V\kappa \in W^{1,p}(H)$. It was proven in Theorem 4.1 of [12] that for every $\phi \in W^{1,p}(H)$, a bounded linear operator $M : H \to W^{1,p}$ defined by

$$M[h](x) = \langle \phi(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is γ radonifying, i.e. $M \in \gamma(H, W^{1,p})$, and there exists a constant $N > 0$ independent of ϕ such that

$$\|M\|_{\gamma(H,W^{1,p})} \le N\|\phi\|_{W^{1,p}(H)}.$$

Therefore, the following bounded linear operator $\bar{K}$ defined by

$$\bar{K}[h](x) = \langle V\kappa(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is a γ radonifying operator from H into $W^{1,p}$ and for constant $N > 0$,

$$\|\bar{K}\|_{\gamma(H,W^{1,p})} \le N\|V\kappa\|_{W^{1,p}(H)}.$$

It follows from (5.49) that

$$\|\bar{K}\|_{\gamma(H,W^{1,p})} \le N\|\kappa\|_{W^{1,p}_\nu(H)}. \tag{5.50}$$

Define an operator $T : W^{1,p} \to W^{1,p}_\nu$ by

$$Tf = fe^{-\frac{\nu}{p}}, \quad f \in W^{1,p}.$$

As in the proof of Lemma 5.1, it can be easily showed that the map T is well-defined, bijective and isometry, i.e.

$$\|Tf\|_{W^{1,p}_\nu} = \|f\|_{W^{1,p}}, \quad f \in W^{1,p},$$

and so $\|T\| \le 1$. Therefore, we can rewrite K as $T \circ \bar{K}$. By Theorem 2.15, since T is linear bounded and $\bar{K}$ is γ radonifying, the map K is γ radonifying. Moreover, since $\|T\| \le 1$ and (5.50), we infer that

$$\|K\|_{\gamma(H,W^{1,p}_\nu)} \le \|T\|\|\bar{K}\|_{\gamma(H,W^{1,p})} \le N\|\kappa\|_{W^{1,p}_\nu(H)},$$

which completes the proof. ■

Lemma 5.17. *Let $\nu > 0$ and $p \geq 2$. Let H be a separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_H$. Assume that $g : [0,\infty) \times [0,\infty) \times \mathbb{R} \to H$ is a continuously weakly differentiable mapping with respect to the second and third variables such that there exist functions $\bar{g}, \hat{g} \in W^{1,p}_\nu$ such that*
(i) for all $t \in [0,\infty)$,

$$\|g(t,x,u)\|_H \leq |\bar{g}(x)|, \quad u \in \mathbb{R}, \quad x \in [0,\infty), \tag{5.51}$$

(ii) for all $t \in [0,\infty)$,

$$\|g(t,x,u) - g(t,x,v)\|_H \leq |\hat{g}(x)|\ |u - v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty), \tag{5.52}$$

(iii) for all $t \in [0,\infty)$,

$$\|D_x g(t,x,u)\|_H \leq |D\bar{g}(x)|, \quad u \in \mathbb{R}, \quad x \in [0,\infty), \tag{5.53}$$

(iv) for all $t \in [0,\infty)$,

$$\|D_x g(t,x,u) - D_x g(t,x,v)\|_H \leq |D\hat{g}(x)|\ |u - v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty), \tag{5.54}$$

(v) there exists a constant $K_1 > 0$ such that for all $t \in [0,\infty)$,

$$\|D_u g(t,x,u)\|_H \leq K_1, \quad u \in \mathbb{R}, \quad x \in [0,\infty), \tag{5.55}$$

(vi) there exists a constant $K_2 > 0$ such that for all $t \in [0,\infty)$,

$$\|D_u g(t,x,u) - D_v g(t,x,v)\|_H \leq K_2 |u - v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty), \tag{5.56}$$

where $D_x g(t,x,u)$ is the first weak derivative of function $[0,\infty) \ni x \mapsto g(t,x,u)$ when t and u are fixed. Similarly, $D_u g(t,x,u)$ is the first weak derivative of function $\mathbb{R} \ni u \mapsto g(t,x,u)$ when t and x are fixed. Define a function $F : [0,\infty) \times W^{1,p}_\nu \to W^{1,p}_\nu$ by

$$F(t,f)(x) = \left\langle g(t,x,f(x)), \int_0^x g(t,y,f(y))dy \right\rangle_H, \quad f \in W^{1,p}_\nu, \quad x,t \in [0,\infty). \tag{5.57}$$

Then F is well-defined. Moreover,
(i) for every $t \in [0,\infty)$,

$$\begin{aligned} \|F(t,f)\|_{W^{1,p}_\nu} \leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|^2_{\nu,p} + 3\left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p} \left(\|D\bar{g}\|_{\nu,p} + K_1 \|Df\|_{\nu,p}\right) \\ + 3C \|\bar{g}\|_{W^{1,p}_\nu} \|\bar{g}\|_{\nu,p}, \quad f \in W^{1,p}_\nu \end{aligned} \tag{5.58}$$

(ii) F is Lipschitz on balls with Lipschitz constant independent of time t.

Proof. Fix $t \geq 0$ and $f \in W_\nu^{1,p}$. In Lemma 5.9, we have already showed $F(t,f) \in L_\nu^p$. Let us show that $DF(t,f) \in L_\nu^p$. By the chain rule, we have

$$\begin{aligned} DF(t,f)(x) &= \left\langle Dg(t,x,f(x)), \int_0^x g(t,y,f(y))dy \right\rangle_H \\ &\quad + \langle g(t,x,f(x)), g(t,x,f(x))\rangle_H, \quad x \in [0,\infty). \end{aligned}$$

It is obvious that $DF(t,f)$ is measurable. Moreover, using the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left|DF(t,f)(x)\right| &\leq \left\|Dg(t,x,f(x))\right\|_H \int_0^\infty \left\|g(t,x,f(x))\right\|_H dx \\ &\quad + \|g(t,x,f(x))\|_H \|g(t,x,f(x))\|_H, \quad x \in [0,\infty). \end{aligned}$$

The first weak derivative $Dg(t,\cdot,f(\cdot))$ of $g(t,\cdot,f(\cdot))$ is given by

$$Dg(t,x,f(x)) = D_x g(t,x,f(x)) + D_u g(t,x,f(x)) Df(x), \quad x \in [0,\infty).$$

By the triangle inequality, and inequalities (5.53) and (5.55), we get

$$\begin{aligned} \|Dg(t,x,f(x))\|_H &\leq \|D_x g(t,x,f(x))\|_H + \|D_u g(t,x,f(x))\|_H \,|Df(x)| \\ &\leq |D\bar{g}(x)| + K_1 |Df(x)|, \quad x \in [0,\infty). \end{aligned} \tag{5.59}$$

Therefore, by Proposition 5.1, and inequalities (5.51) and (5.59), we have, for every $x \in [0,\infty)$,

$$\left|DF(t,f)(x)\right| \leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p} \left|D\bar{g}(x)\right| + K_1 \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p} \left|Df(x)\right| + |\bar{g}(x)|^2.$$

From the last estimate, we obtain

$$\begin{aligned} \int_0^\infty \left|DF(t,f)(x)\right|^p e^{\nu x} dx &\leq 3^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^p \int_0^\infty |D\bar{g}(x)|^p e^{\nu x} dx \\ &\quad + 3^p K_1^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^p \int_0^\infty \left|Df(x)\right|^p e^{\nu x} dx \\ &\quad + 3^p \int_0^\infty |\bar{g}(x)|^p |\bar{g}(x)|^p e^{\nu x} dx. \end{aligned}$$

It follows from Proposition 5.3 that

$$\begin{aligned} \int_0^\infty \left|DF(t,f)\right|^p e^{\nu x} dx &\leq 3^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^p \left\|D\bar{g}\right\|_{\nu,p}^p \\ &\quad + 3^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^p K_1^p \left\|Df\right\|_{\nu,p}^p + 3^p C^p \|\bar{g}\|_{\nu.p}^p \|\bar{g}\|_{W_\nu^{1.p}}^p. \end{aligned} \tag{5.60}$$

Thus $DF(t,f) \in L^P_\nu$ and so $F(t,f) \in W^{1,p}_\nu$. Hence, F is well-defined. Moreover, it follows from estimates (5.31) and (5.60) that

$$\|F(t,f)\|_{W^{1,p}_\nu} \leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|^2_{\nu,p} + 3\left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p} \left(\|D\bar{g}\|_{\nu,p} + K_1\|Df\|_{\nu,p}\right)$$
$$+ 3C\|\bar{g}\|_{W^{1,p}_\nu}\|\bar{g}\|_{\nu,p},$$

which gives the desired conclusion (5.58).

Let us now prove that F is locally Lipschitz. Fix $t \geq 0$, $R > 0$ and $f_1, f_2 \in W^{1,p}_\nu$ such that $\|f_1\|_{W^{1,p}_\nu} \leq R$ and $\|f_2\|_{W^{1,p}_\nu} \leq R$. Define a function I by

$$I(x) = F(t,f_1)(x) - F(t,f_2)(x), \quad x \in [0,\infty).$$

Then by the triangle inequality and the Cauchy-Schwarz inequality, we have, for each $x \in [0,\infty)$,

$$\begin{aligned}
|I(x)| &\leq \left|\left\langle g(t,x,f_1(x)) - g(t,x,f_2(x)), \int_0^x g(t,y,f_2(y))dy\right\rangle_H\right| \\
&+ \left|\left\langle g(t,x,f_1(x)), \int_0^x [g(t,y,f_1(y)) - g(t,y,f_2(y))]dy\right\rangle_H\right| \\
&\leq \left\|g(t,x,f_1(x)) - g(t,x,f_2(x))\right\|_H \int_0^\infty \left\|g(t,x,f_2(x))\right\|_H dx \\
&+ \left\|g(t,x,f_1(x))\right\|_H \int_0^\infty \left\|g(t,x,f_1(x)) - g(t,x,f_2(x))\right\| dx.
\end{aligned}$$

Using inequalities (5.51) and (5.52), we obtain, for each $x \in [0,\infty)$,

$$|I(x)| \leq |\hat{g}(x)|\,|f_1(x) - f_2(x)| \int_0^\infty |\bar{g}(x)|\,dx + |\bar{g}(x)| \int_0^\infty |\hat{g}(x)|\,|f_1(x) - f_2(x)|\,dx.$$

It follows from Proposition 5.1 and Proposition 5.4 that for each $x \in [0,\infty)$,

$$|I(x)| \leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\bar{g}\|_{\nu,p}\,|\hat{g}(x)|\,|f_1(x) - f_2(x)| + \left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|f_1 - f_1\|_{\nu,p}\,\|D\hat{g}\|_1\,|\bar{g}(x)|.$$

Therefore, taking into account the last inequality, we get

$$\|I\|^p_{\nu,p} \leq 2^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|^p_{\nu,p} \int_0^\infty |\hat{g}(x)|^p\,|f_1(x) - f_2(x)|^p\,e^{\nu x}dx$$
$$+ 2^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|f_1 - f_1\|^p_{\nu,p}\,\|D\hat{g}\|^p_1 \int_0^\infty |\bar{g}(x)|^p\,e^{\nu x}dx.$$

It follows from Proposition 5.3 that

$$\|I\|^p_{\nu,p} \leq 2^p C^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|^p_{\nu,p}\|\hat{g}\|^p_{W^{1,p}_\nu}\|f_1 - f_2\|^p_{\nu,p}$$
$$+ 2^p \left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|f_1 - f_1\|^p_{\nu,p}\left\|D\hat{g}\right\|^p_1\|\bar{g}\|^p_{\nu,p}.$$

Thus, we have showed that

$$\begin{aligned}\|F(t,f_1)-F(t,f_2)\|_{\nu,p} &\le 2C\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\bar{g}\|_{\nu,p}\|\hat{g}\|_{W^{1,p}_\nu}\|f_1-f_2\|_{\nu,p}\\ &\quad+2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|f_1-f_1\|_{\nu,p}\left\|D\hat{g}\right\|_1\|\bar{g}\|_{\nu,p}.\end{aligned} \tag{5.61}$$

By the chain rule, we get

$$\begin{aligned}DI(x)&=\left\langle Dg(t,x,f_1(x)),\int_0^x\big(g(t,y,f_1(y))-g(t,y,f_2(y))\big)dy\right\rangle_H\\ &\quad+\left\langle Dg(t,x,f_1(x))-Dg(t,x,f_2(x)),\int_0^x g(t,y,f_2(y))dx\right\rangle_H\\ &\quad+\left\langle g(t,x,f_1(x)),g(t,x,f_1(x))-g(t,x,f_2(x))\right\rangle_H\\ &\quad+\left\langle g(t,x,f_1(x))-g(t,x,f_2(x)),g(t,x,f_2(x))\right\rangle_H,\quad x\in[0,\infty).\end{aligned}$$

Using the triangle inequality and the Cauchy-Schwarz inequality we get

$$\begin{aligned}|DI(x)|&\le\left\|Dg(t,x,f_1(x))\right\|_H\int_0^\infty\left\|g(t,x,f_1(x))-g(t,x,f_2(x))\right\|_H dx\\ &\quad+\left\|Dg(t,x,f_1(x))-Dg(t,x,f_2(x)\right\|_H\int_0^\infty\left\|g(t,x,f_2(x))\right\|_H dx\\ &\quad+\left\|g(t,x,f_1(x))\right\|_H\left\|g(t,x,f_1(x))-g(t,x,f_2(x))\right\|_H\\ &\quad+\left\|g(t,x,f_2(x))\right\|_H\left\|g(t,x,f_1(x))-g(t,x,f_2(x))\right\|_H,\quad x\in[0,\infty).\end{aligned}$$

By inequalities (5.51) and (5.52), we obtain

$$\begin{aligned}\left|DI(x)\right|&\le\left\|Dg(t,x,f_1(x))\right\|_H\int_0^\infty|\hat{g}(x)|\,|f_1(x)-f_2(x)|\,dx\\ &\quad+\left\|Dg(t,x,f_1(x))-Dg(t,x,f_2(x)\right\|_H\int_0^\infty|\bar{g}(x)|\,dx\\ &\quad+2\,|\bar{g}(x)|\,|\hat{g}(x)|\,|f_1(x)-f_2(x)|,\quad x\in[0,\infty).\end{aligned}$$

Using Proposition 5.1 and Proposition 5.4 we get

$$\begin{aligned}\left|DI(x)\right|&\le\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|D\hat{g}\|_1\,\|f_1-f_2\|_{\nu,p}\,\|Dg(t,x,f_1(x))\|_H\\ &\quad+\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\bar{g}\|_{\nu,p}\,\|Dg(t,x,f_1(x))-Dg(t,x,f_2(x)\|_H\\ &\quad+2\,|\bar{g}(x)|\,|\hat{g}(x)|\,|f_1(x)-f_2(x)|,\quad x\in[0,\infty).\end{aligned}$$

Note that

$$\begin{aligned}Dg(t,x,f_1(x))&=D_xg(t,x,f_1(x))+D_ug(t,x,f_1(x))Df_1(x),\quad x\in[0,\infty),\\ Dg(t,x,f_2(x))&=D_xg(t,x,f_2(x))+D_vg(t,x,f_2(x))Df_2(x),\quad x\in[0,\infty).\end{aligned} \tag{5.62}$$

Thus by the triangle inequality, we obtain

$$\begin{aligned}\|Dg(t,x,f_1(x))-Dg(t,x,f_2(x)\|_H &\le \|D_xg(t,x,f_1(x))-D_xg(t,x,f_2(x))\|_H\\ &+\|D_ug(t,x,f_1(x))\|_H\left|Df_1(x)-Df_2(x)\right|\\ &+\|D_ug(t,x,f_1(x))-D_vg(t,x,f_2(x))\|_H\left|Df_2(x)\right|,\quad x\in[0,\infty).\end{aligned}$$

It follows from inequalities (5.54), (5.55) and (5.56) that

$$\begin{aligned}\|Dg(t,x,f_1(x))-Dg(t,x,f_2(x)\|_H &\le \left|D\hat{g}(x)\right||f_1(x)-f_2(x)|+K_1\left|Df_1(x)-Df_2(x)\right|\\ &+K_2\,|f_1(x)-f_2(x)|\left|Df_2(x)\right|,\quad x\in[0,\infty).\end{aligned}\tag{5.63}$$

Thus, by inequalities (5.59) and (5.63), we get

$$\begin{aligned}\left|DI(x)\right| \le &\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|D\hat{g}\|_1\,\|f_1-f_2\|_{\nu,p}\,|D\bar{g}(x)|\\ &+K_1\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|D\hat{g}\|_1\,\|f_1-f_2\|_{\nu,p}\,|Df_1(x)|\\ &+\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\bar{g}\|_{\nu,p}\,|D\hat{g}(x)|\,|f_1(x)-f_2(x)|\\ &+K_1\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\bar{g}\|_{\nu,p}\,|Df_1(x)-Df_2(x)|\\ &+K_2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\bar{g}\|_{\nu,p}\,|f_1(x)-f_2(x)|\,|Df_2(x)|\\ &+2\,|\bar{g}(x)|\,|\hat{g}(x)|\,|f_1(x)-f_2(x)|,\quad x\in[0,\infty).\end{aligned}$$

Therefore, from the last inequality, we obtain

$$\begin{aligned}\left\|DI\right\|_{\nu,p}^p \le\, &6^p\left(\frac{p}{\nu q}\right)^{\frac{p}{q}}\left\|D\hat{g}\right\|_1^p\|f_1-f_2\|_{\nu,p}^p\int_0^\infty|D\bar{g}(x)|^p\,e^{\nu x}dx\\ &+6^pK_1^p\left(\frac{p}{\nu q}\right)^{\frac{p}{q}}\left\|D\hat{g}\right\|_1^p\|f_1-f_2\|_{\nu,p}^p\int_0^\infty|Df_1(x)|^p\,e^{\nu x}dx\\ &+6^p\left(\frac{p}{\nu q}\right)^{\frac{p}{q}}\|\bar{g}\|_{\nu,p}^p\int_0^\infty\left|D\hat{g}(x)\right|^p|f_1(x)-f_2(x)|^p e^{\nu x}dx\\ &+6^pK_1^p\left(\frac{p}{\nu q}\right)^{\frac{p}{q}}\left\|\bar{g}\right\|_{\nu,p}^p\int_0^\infty\left|Df_1(x)-Df_2(x)\right|^p e^{\nu x}dx\\ &+6^pK_2^p\left(\frac{p}{\nu q}\right)^{\frac{p}{q}}\|\bar{g}\|_{\nu,p}^p\int_0^\infty|f_1(x)-f_2(x)|^p\left|Df_2(x)\right|^p e^{\nu x}dx\\ &+6^p\int_0^\infty|\bar{g}(x)|^p\,|\hat{g}(x)|^p\,|f_1(x)-f_2(x)|^p\,e^{\nu x}dx.\end{aligned}$$

It follows from Proposition 5.3 that

$$
\begin{aligned}
\|DI\|_{\nu,p}^{p} \le\; & 6^{p}\left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|D\hat{g}\|_{1}^{p}\,\|f_1-f_2\|_{\nu,p}^{p}\,\|D\bar{g}\|_{\nu,p}^{p} \\
&+6^{p}K_1^{p}\left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|D\hat{g}\|_{1}^{p}\,\|f_1-f_2\|_{\nu,p}^{p}\,\|Df_1\|_{\nu,p}^{p} \\
&+6^{p}C^{p}\left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^{p}\,\|D\hat{g}\|_{\nu,p}^{p}\,\|f_1-f_2\|_{W_\nu^{1,p}}^{p} \\
&+6^{p}K_1^{p}\left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}\,\|Df_1-Df_2\|_{\nu,p}^{p} \\
&+6^{p}C^{p}K_2^{p}\left(\frac{p}{\nu q}\right)^{\frac{p}{q}} \|\bar{g}\|_{\nu,p}^{p}\,\|f_1-f_2\|_{W_\nu^{1,p}}^{p}\,\|Df_2\|_{\nu,p}^{p} \\
&+6^{p}C^{2p}\,\|f_1-f_2\|_{W_\nu^{1,p}}^{p}\,\|\hat{g}\|_{W_\nu^{1,p}}^{p}\,\|\hat{g}\|_{\nu,p}^{p}.
\end{aligned}
\tag{5.64}
$$

Taking into account estimates (5.61) and (5.64), we infer that there exists a constant $C(R)>0$ such that

$$\|F(t,f_1)-F(t,f_2)\|_{W_\nu^{1,p}} \le C(R)\|f_1-f_2\|_{W_\nu^{1,p}},$$

which implies that F is Lipschitz on balls. ∎

Lemma 5.18. *Assume that all the assumption of the previous Lemma are satisfied. Define a function $G:[0,\infty)\times W_\nu^{1,p}\to\gamma(H,W_\nu^{1,p})$ by*

$$G(t,f)[h](x)=\langle g(t,x,f(x)),h\rangle_H,\quad f\in W_\nu^{1,p},\quad h\in H,\quad x,t\in[0,\infty). \tag{5.65}$$

Then G is well-defined. Moreover,
(i) for every $t\in[0,\infty)$,

$$\|G(t,f)\|_{\gamma\left(H,W_\nu^{1,p}\right)} \le N\left(\|\bar{g}\|_{\nu,p}+2\|D\bar{g}\|_{\nu,p}+2K_1\|Df\|_{\nu,p}\right),\quad f\in W_\nu^{1,p} \tag{5.66}$$

(ii) G is Lipschitz on balls with Lipschitz constant independent of time t.

Proof. Fix $t\ge 0$ and $f\in W_\nu^{1,p}$. To show that G is well-defined, we have to show that $G(t,f)\in\gamma(H,W_\nu^{1,p})$. Define a function $\kappa:[0,\infty)\to H$ by

$$\kappa(x)=g(t,x,f(x)),\quad x\in[0,\infty).$$

Then we can rewrite $G(t,f)$ as follow

$$G(t,f)[h](x)=\langle\kappa(x),h\rangle_H,\quad h\in H,\quad x\in[0,\infty).$$

By Proposition 5.5, it is sufficient to show $\kappa \in W_\nu^{1,p}(H)$. In Lemma (5.3), we have already showed $\kappa \in L_\nu^p(H)$. Thus, we need to show $D\kappa \in L_\nu^p(H)$. It is obvious that $D\kappa$ is measurable. Moreover, by inequality (5.59), we have

$$\begin{aligned}\|D\kappa(x)\|_H &\le \|D_x g(t,x,f(x))\|_H + \|D_u g(t,x,f(x))\|_H \, |Df(x)| \\ &\le |D\bar{g}(x)| + K_1 \, |Df(x)|, \quad x \in [0,\infty).\end{aligned} \tag{5.67}$$

From the last inequality, we infer that

$$\begin{aligned}\int_0^\infty \|D\kappa(x)\|_H^p e^{\nu x} dx &\le 2^p \int_0^\infty |D\bar{g}(x)|^p \, e^{\nu x} dx + 2^p K_1^p \int_0^\infty |Df(x)|^p \, e^{\nu x} dx \\ &= 2^p \, \|D\bar{g}\|_{\nu,p}^p + 2^p K_1^p \, \|Df\|_{\nu,p}^p .\end{aligned} \tag{5.68}$$

Hence $D\kappa \in L_\nu^p(H)$. Thus, $G(t,f)$ is a γ-radonifying operator from H into $W_\nu^{1,p}$. Therefore, G is well-defined. Again by Proposition 5.5, we have, for a constant $N > 0$ independent of κ,

$$\|G(t,f)\|_{\gamma\left(H, W_\nu^{1,p}\right)} \le N \|\kappa\|_{W_\nu^{1,p}(H)}.$$

It follows from estimates (5.35) and (5.68) that

$$\|G(t,f)\|_{\gamma\left(H, W_\nu^{1,p}\right)} \le N \left(\|\bar{g}\|_{\nu,p} + 2 \, \|D\bar{g}\|_{\nu,p} + 2K_1 \, \|Df\|_{\nu,p} \right), \tag{5.69}$$

which gives the desired result (5.66).

Let us now prove that G is locally Lipschitz. For this aim, we again use Proposition 5.5. Fix $t \ge 0$, $R > 0$ and $f_1, f_2 \in W_\nu^{1,p}$ such that $\|f_1\|_{W_\nu^{1,p}} \le R$ and $\|f_2\|_{W_\nu^{1,p}} \le R$. Define a function $\lambda : [0,\infty) \to H$ by

$$\lambda(x) = g(t,x,f_1(x)) - g(t,x,f_2(x)), \quad x \in [0,\infty).$$

Then

$$\big(G(t,f_1) - G(t,f_1)\big)[h](x) = \langle \lambda(x), h \rangle_H, \quad x \in [0,\infty), \quad h \in H.$$

By inequality (5.52) and Proposition 5.3, we get

$$\begin{aligned}\int_0^\infty \|\lambda(x)\|_H^p e^{\nu x} dx &\le \int_0^\infty |\hat{g}(x)|^p \, |f_1(x) - f_2(x)|^p \, e^{\nu x} dx \\ &\le \sup_{x \in [0,\infty)} |f_1(x) - f_2(x)|^p \int_0^\infty |\hat{g}(x)|^p \, e^{\nu x} dx \\ &\le C^p \, \|\hat{g}\|_{\nu,p}^p \|f_1 - f_2\|_{W_\nu^{1,p}}^p .\end{aligned} \tag{5.70}$$

Moreover, by inequality (5.63), we have

$$
\begin{aligned}
\|D\lambda(x)\|_H \le \left|D\hat{g}(x)\right| |f_1(x)-f_2(x)| + K_1\left|Df_1(x)-Df_2(x)\right| \\
+K_2|f_1(x)-f_2(x)|\left|Df_2(x)\right|, \quad x\in[0,\infty).
\end{aligned}
\tag{5.71}
$$

Therefore

$$
\begin{aligned}
\int_0^\infty \|D\lambda(x)\|_H^p e^{\nu x}dx \le 3^p \int_0^\infty &\left|D\hat{g}(x)\right|^p |f_1(x)-f_2(x)|^p e^{\nu x}dx \\
&+3^pK_1^p\int_0^\infty \left|Df_1(x)-Df_2(x)\right|^p e^{\nu x}dx \\
&+3^pK_2^p\int_0^\infty |f_1(x)-f_2(x)|^p\left|Df_2(x)\right|^p e^{\nu x}dx.
\end{aligned}
$$

Using Proposition 5.3, we infer that

$$
\begin{aligned}
\int_0^\infty \|D\lambda(x)\|_H^p e^{\nu x}dx \le 3^pC^p\|f_1-f_2\|^p_{W^{1,p}_\nu}\left\|D\hat{g}\right\|^p_{\nu,p} + 3^pK_1^p\left\|Df_1-Df_2\right\|^p_{\nu,p} \\
+3^pK_2^pC^p\|f_1-f_2\|^p_{W^{1,p}_\nu}\left\|Df_2\right\|^p_{\nu,p}.
\end{aligned}
\tag{5.72}
$$

By Proposition 5.5, we have, for a constant $N>0$ independent λ,

$$
\left\|G(t,f_1)-G(t,f_1)\right\|_{\gamma(H,W^{1,p}_\nu)} \le N\|\lambda\|_{W^{1,p}_\nu(H)}.
$$

It follows from estimates (5.70) and (5.72) that there exists a constant $C(R)$ such that

$$
\left\|G(t,f_1)-G(t,f_1)\right\|_{\gamma(H,W^{1,p}_\nu)} \le C(R)\|f_1-f_2\|_{W^{1,p}_\nu},
\tag{5.73}
$$

which completes the proof. ■

Let us now introduce the main result of this section by the following theorem.

Theorem 5.2. *Let $\nu>0$ and $p\ge 2$. Let H be a separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_H$. Assume that $g:[0,\infty)\times[0,\infty)\times\mathbb{R}\to H$ is a continuously weakly differentiable mapping with respect to the second and third variables such that there exist functions $\bar{g},\hat{g}\in W^{1,p}_\nu$ such that*
(i) for all $t\in[0,\infty)$,

$$
\|g(t,x,u)\|_H \le |\bar{g}(x)|, \quad u\in\mathbb{R}, \quad x\in[0,\infty),
$$

(ii) for all $t\in[0,\infty)$,

$$
\|g(t,x,u)-g(t,x,v)\|_H \le |\hat{g}(x)|\;|u-v|, \quad u,v\in\mathbb{R}, \quad x\in[0,\infty),
$$

(iii) for all $t \in [0,\infty)$*,*

$$\|D_x g(t,x,u)\|_H \le |D\bar{g}(x)|, \quad u \in \mathbb{R}, \quad x \in [0,\infty),$$

(iv) for all $t \in [0,\infty)$*,*

$$\|D_x g(t,x,u) - D_x g(t,x,v)\|_H \le |D\hat{g}(x)| \; |u-v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty),$$

(v) there exists a constant $K_1 > 0$ *such that for all* $t \in [0,\infty)$*,*

$$\|D_u g(t,x,u)\|_H \le K_1, \quad u \in \mathbb{R}, \quad x \in [0,\infty),$$

(vi) there exists a constant $K_2 > 0$ *such that for all* $t \in [0,\infty)$*,*

$$\|D_u g(t,x,u) - D_v g(t,x,v)\|_H \le K_2|u-v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty).$$

Then for each $r_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P};W_\nu^{1,p})$*, there exists a unique* $W_\nu^{1,p}$*-valued continuous mild solution* r *to equation (3.7) with the initial value* r_0*.*

Proof. The abstract form of equation (3.7) in the space $W_\nu^{1,p}$ is as follows

$$dr(t) = \big(Ar(t) + F(t,r(t))\big)dt + G(t,r(t))dW(t), \quad t \ge 0, \tag{5.74}$$

where F and G are functions defined in Lemma 5.17 and 5.18 respectively, and A is the infinitesimal generator of the shift semigroup on $W_\nu^{1,p}$ (see Lemma 5.16). Now equation (3.7) is the form of equation (4.2). In lemma 5.12, we showed that $W_\nu^{1,p}$ is a Banach space satisfying H-condition. Also in Lemma 5.13, we showed that the shift semigroup on $W_\nu^{1,p}$ is a contraction type C_0-semigroup. Moreover, in Lemma 5.17 and 5.18 , we showed that F and G satisfy the conditions of Theorem 4.3. Therefore, for each $r_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P};L_\nu^p)$, equation (5.74) has a unique $W_\nu^{1,p}$-valued continuous mild solution with the initial value $r(0) = r_0$. ■

Example 5.3. *For each* $r_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P};L_\nu^p)$*, the HJMM equation in Example 5.1 has a unique* $W_\nu^{1,p}$*-valued continuous mild solution given by (5.41) since the function* g *in Example 5.1 satisfies all the assumptions of Theorem 5.2.*

Example 5.4. *Consider Example 5.2. One can easily check that the function* g *in Example 5.2 satisfies all the assumptions of Theorem 5.2. Thus, for each* $r_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P};L_\nu^p)$*, the HJMM equation in Example 5.2 has a unique* $W_\nu^{1,p}$*-valued continuous mild solution.*

Remark 5.2. *Elements of $W_\nu^{1,p}$ are α-Hölder continuous functions for $\alpha < 1-\frac{1}{p}$ and hence, for each $p \geq 2$, the solution to the HJMM equation in the space $W_\nu^{1,p}$ is more regular than the solution in the space $W_\nu^{1,2}$.*

Remark 5.3. *In the spaces C^α of α-Hölder continuous functions, one can not define an Itô integral and hence, these spaces are not suitable for our purpose.*

5.3 Existence and Uniqueness of Invariant Measures for the HJMM Equation in the Weighted Lebesgue Spaces

In this section, we find a sufficient condition, using Theorem 4.5, for the existence and uniqueness of an invariant measure to equation (3.7) (when the coefficients are time independent) in the weighted Lebesgue spaces. Let us begin presenting the following natural consequence of Theorem 5.1.

Corollary 5.1. *Assume that $\nu > 0$ and $p \geq 2$. Let $g : [0,\infty) \times \mathbb{R} \to H$ be a measurable function with respect to first variable such that there exist functions $\bar{g} \in L_\nu^p$ and $\hat{g} \in L_\nu^p \cap L^\infty$ such that*

$$\|g(x,u)\|_H \leq |\bar{g}(x)|, \quad u \in \mathbb{R}, \quad x \in [0,\infty) \tag{5.75}$$

and

$$\|g(x,u) - g(x,v)\|_H \leq |\hat{g}(x)|\ |u-v|, \quad u,v \in \mathbb{R}, \quad x \in [0,\infty). \tag{5.76}$$

Then for each $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; L_\nu^p)$, there exists a unique L_ν^p-valued continuous mild solution r with the initial value $r(0) = r_0$ to the following equation

$$\begin{aligned} dr(t)(x) &= \left(\frac{\partial}{\partial x} r(t)(x) + \left\langle g(x, r(t)(x)), \int_0^x g(y, r(t)(y))dy \right\rangle_H \right) dt \\ &\quad + \langle g(x, r(t)(x)), dW(t)\rangle_H, \quad t, x \in [0,\infty). \end{aligned} \tag{5.77}$$

Moreover, the solution is a Markov process.

Let us now present our main result for this section.

Theorem 5.3. *Assume that $\nu > 0$ and $p \geq 2$. Let all the assumptions of Corollary 5.1 be satisfied. If*

$$2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}} \|\hat{g}\|_{\infty}\|\bar{g}\|_{\nu,p} + (p-1)N^2\|\hat{g}\|_{\infty}^2 < \frac{\nu}{2}, \tag{5.78}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and N is a constant appearing in inequality (5.38), then equation (5.77) has a unique invariant probability measure in L^p_ν.

Proof. Assume that G and F are functions defined by

$$F(f)(x) = \left\langle g(x, f(x)), \int_0^x g(y, f(y))dy \right\rangle_H, \quad f \in L^p_\nu, \quad x \in [0,\infty). \tag{5.79}$$

and

$$G(f)[h](x) = \langle g(x, f(x)), h\rangle_H, \quad f \in L^p_\nu, \quad x \in [0,\infty), \quad h \in H. \tag{5.80}$$

Then abstract form of equation (5.77) in L^p_ν is as follows

$$dr(t) = \big(Ar(t) + F(r(t))\big)dt + G(r(t))dW(t), \quad t \geq 0, \tag{5.81}$$

where A is the infinitesimal generator of the shift-semigroup on L^p_ν. In order to show that equation (5.81) has a unique invariant measure, by Theorem 4.5, it is sufficient to show that there exist a constant $\omega > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $f_1, f_2 \in L^p_\nu$,

$$\big[A_n(f_1 - f_2) + F(f_1) - F(f_2), f_1 - f_2\big] + \frac{K_2(p)}{p}\|G(f_1) - G(f_2)\|^2_{\gamma(H,L^p_\nu)} \leq -\omega\|f_1 - f_2\|^2_{\nu,p}, \tag{5.82}$$

where $[\cdot,\cdot]$ is the semi-inner product on L^p_ν which is given, see [15], by

$$[f,g] = \|g\|^{2-p}_{\nu,p}\int_0^\infty f(x)g(x)|g(x)|^{p-2}e^{\nu x}dx, \quad f,g \in L^p_\nu,$$

and A_n is the Yosida approximation of A. Moreover, $K_2(p)$ is a constant appearing in Lemma 5.2. We prove (5.82) in the following few steps.

Step 1 : Fix $f_1, f_2 \in L^p_\nu$. Then by the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\big|F(f_1)(x) - F(f_2)(x)\big| \leq{}& \big\|g(x, f_1(x))\big\|_H \int_0^\infty \big\|g(x, f_1(x)) - g(x, f_2(x))\big\|_H dx \\ &+ \big\|g(x, f_1(x)) - g(x, f_2(x))\big\|_H \int_0^\infty \big\|g(x, f_2(x))\big\|_H dx, \quad x \in [0,\infty).\end{aligned}$$

Using inequalities (5.75), (5.76) and Proposition 5.1, we have

$$\begin{aligned}\left|F(f_1)(x)-F(f_2)(x)\right| &\leq |\bar{g}(x)|\int_0^\infty |\hat{g}(x)|\,|f_1(x)-f_2(x)|dx\\ &\quad+|\hat{g}(x)|\,|f_1(x)-f_2(x)|\int_0^\infty |\bar{g}(x)|dx\\ &\leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\hat{g}|_\infty\|f_1-f_2\|_{\nu,p}|\bar{g}(x)|\\ &\quad+\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\bar{g}\|_{\nu,p}|\hat{g}(x)|\,|f_1(x)-f_2(x)|,\quad x\in[0,\infty).\end{aligned}$$

Put

$$I:=\left[F(f_1)-F(f_2),f_1-f_2\right].$$

Taking into account last inequality, we obtain

$$\begin{aligned}I &\leq \|f_1-f_2\|_{\nu,p}^{2-p}\int_0^\infty \left|F(f_1)(x)-F(f_2)(x)\right||f_1(x)-f_2(x)|^{p-1}e^{\nu x}dx\\ &\leq \left(\frac{p}{\nu q}\right)^{\frac{1}{q}}||\hat{g}||_\infty||f_1-f_2\|_{\nu,p}^{3-p}\int_0^\infty |\bar{g}(x)|\,|f_1(x)-f_2(x)|^{p-1}e^{\nu x}dx\\ &\quad+\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}||\bar{g}||_{\nu,p}\|f_1-f_2\|_{\nu,p}^{2-p}\int_0^\infty |\hat{g}(x)|\,|f_1(x)-f_2(x)|^p e^{\nu x}dx.\end{aligned}$$

Using the Hölder inequality, we get

$$\begin{aligned}\int_0^\infty |\bar{g}(x)||f_1(x)-f_2(x)|^{p-1}e^{\nu x}dx &= \int_0^\infty |\bar{g}(x)|\,|f_1(x)-f_2(x)|^{p-1}e^{\nu x\left(\frac{p-1}{p}\right)}e^{\frac{\nu x}{p}}dx\\ &\leq \left(\int_0^\infty |\bar{g}(x)|^p e^{\nu x}dx\right)^{\frac{1}{p}}\left(\int_0^\infty |f_1(x)-f_2(x)|^p e^{\nu x}dx\right)^{\frac{p-1}{p}}\\ &= \|\bar{g}\|_{\nu,p}\|f_1-f_2\|_{\nu,p}^{p-1}.\end{aligned}\tag{5.83}$$

Also we have

$$\int_0^\infty |\hat{g}(x)|\,|f_1(x)-f_2(x)|^p e^{\nu x}dx \leq \|\hat{g}\|_\infty\|f_1-f_2\|_{\nu,p}^p.\tag{5.84}$$

Taking into account inequalities (5.83) and (5.84), we deduce that

$$I\leq 2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\hat{g}\|_\infty\|\bar{g}\|_{\nu,p}\|f_1-f_2\|_{\nu,p}^2.\tag{5.85}$$

Step 2 : For each $t\geq 0$, define an operator $P(t):L_\nu^p\to L_\nu^p$ by

$$P(t)f=e^{\frac{\nu t}{p}}S(t)f,\quad f\in L_\nu^p,$$

where $\{S(t)\}_{t\geq 0}$ is the shift-semigroup on L^p_ν in Lemma 5.3. Recall that the family $\{P(t)\}_{t\geq 0}$ of these operators is a contraction C_0-semigroup on L^p_ν and its infinitesimal generator is given by

$$B = \frac{\nu I}{p} + A.$$

By Theorem 4.3 (b) in [44], B is dissipative and so for all $f \in \mathscr{D}(B)$ and $f^* \in F(f)$, $\langle Bf, f^* \rangle \leq 0$, where

$$F(f) = \{f^* \in \left(L^p_\nu\right)^* \ : \ \langle f, f^* \rangle = \|f\|^2 = \|f^*\|^2\},$$

where $\left(L^p_\nu\right)^*$ is the dual space of L^p_ν and by [15], $\left(L^p_\nu\right)^* = L^q_\nu$. By the definition of the semi-inner product $[\cdot,\cdot]$, see Definition 4.9, we have

$$[f,g] = \langle f, g^* \rangle, \quad g^* \in \left(L^p_\nu\right)^*$$

and $\langle g, g^* \rangle = \|g\|^2$, i.e. $g^* \in \left(L^p_\nu\right)^*$. Therefore, $\langle Bf, f^* \rangle = [Bf, f]$. Hence $[Bf, f] \leq 0$. Note that

$$A_n = nA(nI - A)^{-1} = n\left(B - \frac{\nu}{p}I\right)\left(nI + \frac{\nu}{p}I - B\right)^{-1}.$$

Let $\omega_2 := \frac{-\nu}{p}$ and $k := n - \omega_2$. Then we obtain

$$A_n = (\omega^2 + k\omega_2)(kI - B)^{-1} + \left(1 + \frac{\omega_2}{k}\right)B_k.$$

Therefore, we get

$$[A_n f, f] \leq (\omega_2^2 + k\omega_2)\|(kI - B)^{-1}\|\|f\|^2_{\nu,p}.$$

By Theorem 2.11, we know that $\|(kI - B)^{-1}\| \leq \frac{1}{k}$. Hence

$$[A_n f, f] \leq \frac{\omega_2^2 + k\omega_2}{k}\|f\|^2_{\nu,p}.$$

Therefore, we infer that

$$[A_n(f_1 - f_2), f_1 - f_2] \leq \frac{-\nu n}{np + \nu}\|f_1 - f_2\|^2_{\nu,p}, \quad f_1, f_2 \in L^p_\nu. \tag{5.86}$$

Step 3 : By inequalities (5.2) and (5.38), we get

$$\frac{K_2(p)}{p}\|G(f_1) - G(f_2)\|^2_{\gamma(H,L^p_\nu)} \leq (p-1)N^2\,\|\hat{g}\|^2_{L^\infty}\,\|f_1 - f_2\|^2_{\nu,p}, \tag{5.87}$$

Taking into account estimates (5.85), (5.86) and (5.87), we obtain

$$\left[A_n(f_1-f_2)+F(f_1)-F(f_2), f_1-f_2\right] + \frac{K_p}{p}\|G(f_1)-G(f_2)\|^2_{\gamma(H,L^p_\nu)} \le C_n\|f_1-f_2\|^2_{\nu,p},$$

where $C_n = 2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\hat{g}\|_\infty\|\bar{g}\|_{\nu,p}+(p-1)N^2\|\hat{g}\|^2_\infty+\frac{-\nu n}{np+\nu}$. Since by (5.78),

$$C_n \to 2\left(\frac{p}{\nu q}\right)^{\frac{1}{q}}\|\hat{g}\|_\infty\|\bar{g}\|_{\nu,p}+(p-1)N^2\|\hat{g}\|^2_\infty-\frac{\nu}{p} = C < 0,$$

there exists $n_0 \in \mathbb{N}$ such that for all $\omega \in (0,-C)$,

$$C_n \le -\omega, \quad n \ge n_0.$$

Therefore, (5.82) holds for any $\omega \in (0,-C)$. ■

Remark 5.4. *Tehranchi [59] proved that for each $c \in \mathbb{R}$, there exists an invariant probability measure μ^c supported on an affine subspace*

$$H^{1,2}_{c,\mathrm{w}} = \left\{f \in H^{1,2}_{\mathrm{w}} : \lim_{x\to\infty} f(x) = c\right\}.$$

Since for $c \neq 0$ and $\nu > 0$, $H^{1,2}_{c,\mathrm{w}} \cap L^p_\nu = \emptyset$, the measure μ^c is obviously not an invariant probability measure on L^p_ν. Moreover, for $c = 0$, and appropriate choice of the weight w, *$H^{1,2}_{0,\mathrm{w}} \subset L^p_\nu$ but as Marinelli [40] proved that there exists a unique invariant probability measure supported on $H^{1,2}_{0,\mathrm{w}}$.*

Remark 5.5. *One can generalise the results from Tehranchi and Marinelli to the framework in chapter 6.*

Remark 5.6. *The existence and uniqueness of an invariant measure for the HJMM equation in the weighted Sobolev space $W^{1,p}_\nu$ is an open problem.*

Chapter 6

The HJMM Equation in the Spaces $H_{\mathrm{w}}^{1,p}$

In this chapter, we consider a weighted Banach space $H_{\mathrm{w}}^{1,p}$, $p \geq 1$, which is a natural generalization of the Hilbert space $H_{\mathrm{w}}^{1,2}$ used by Filipović [26] and prove some useful properties of it. This allows us to apply the abstract results from Chapter 4 to prove the existence of a unique continuous solution to the HJMM equation (driven by a standard d-dimensional Wiener process) in the spaces $H_{\mathrm{w}}^{1,p}$, $p \geq 2$. The HJMM equation has been already studied in the space $H_{\mathrm{w}}^{1,2}$, see Chapter 3 for detail. Under appropriate choice of the weight, for each $p \geq 2$, the space $H_{\mathrm{w}}^{1,p}$ is a subspace of the space $H_{\mathrm{w}}^{1,2}$. Therefore, as in Chapter 5, an important feature of our results is that we are able to prove that the HJMM equation has a unique solution in smaller space $H_{\mathrm{w}}^{1,p}$ than the space $H_{\mathrm{w}}^{1,2}$. Moreover, elements of $H_{\mathrm{w}}^{1,p}$ are α-Hölder continuous functions for $\alpha < 1 - \frac{1}{p}$ and hence, for each $p \geq 2$, the solution to the HJMM equation in the space $H_{\mathrm{w}}^{1,p}$ is more regular than the solution in the space $H_{\mathrm{w}}^{1,2}$. This is a reason why we study the HJMM equation in the spaces $H_{\mathrm{w}}^{1,p}$, $p \geq 2$.

6.1 Definition and Some Properties of the Space $H_{\mathrm{w}}^{1,p}$

Suppose that $p \geq 1$ and the weight function $\mathrm{w} : [0,\infty) \to [1,\infty)$ is continuous such that

$$\int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{p-1}} dx < \infty,$$

i.e. $\mathrm{w}^{-\frac{1}{p-1}} \in L^1$. Define L^p_{w} to be the space of all (equivalence classes of) Lebesgue measurable functions $f:[0,\infty)\to\mathbb{R}$ such that

$$\int_0^\infty |f(x)|^p \mathrm{w}(x)dx < \infty.$$

Theorem 6.1. *The space L^p_{w} is a separable Banach space with respect to the norm*

$$\|f\|_{L^p_{\mathrm{w}}} = \left(\int_0^\infty |f(x)|^p \mathrm{w}(x)dx\right)^{\frac{1}{p}}, \quad f \in L^p_{\mathrm{w}}.$$

Proof. Define a linear operator $T: L^p_{\mathrm{w}} \to L^p$ by

$$Tf = f\mathrm{w}^{\frac{1}{p}}, \quad f \in L^p_{\mathrm{w}}.$$

The map T is well-defined. Indeed, for a fixed $f \in L^p_{\mathrm{w}}$, since w is continuous, the product $f\mathrm{w}^{\frac{1}{p}}$ is measurable. Moreover,

$$\int_0^\infty \left|f(x)\mathrm{w}^{\frac{1}{p}}(x)\right|^p dx = \int_0^\infty |f(x)|^p \mathrm{w}(x)dx < \infty. \tag{6.1}$$

Thus $Tf \in L^p$ and hence, T is well-defined. It is obvious that T is bijective. Furthermore, by (6.1), we have

$$\|Tf\|_{L^p} = \|f\|_{L^p_{\mathrm{w}}}, \quad f \in L^p_{\mathrm{w}}.$$

Therefore, T is an isometric isomorphism. Hence, the spaces $(L^p, \|\cdot\|_{L^p})$ and $\left(L^p_{\mathrm{w}}, \|\cdot\|_{L^p_{\mathrm{w}}}\right)$ are isometric. Thus, by Theorem 2.1, L^p_{w} is a separable Banach space with respect to the norm $\|\cdot\|_{L^p_{\mathrm{w}}}$ because $(L^p, \|\cdot\|_{L^p})$ is a separable Banach space. ■

Proposition 6.1. *The space L^p_{w} is continuously embedded into the space L^1. In particular,*

$$\|f\|_1 \le \left\|\mathrm{w}^{-\frac{1}{p-1}}\right\|_1^{\frac{p-1}{p}} \|f\|_{L^p_{\mathrm{w}}}, \quad f \in L^p_{\mathrm{w}}. \tag{6.2}$$

Proof. Fix $f \in L^p_{\mathrm{w}}$. Then by the Hölder inequality, we have

$$\begin{aligned}
\|f\|_1 &= \int_0^\infty |f(x)|\mathrm{w}^{\frac{1}{p}}(x)\mathrm{w}^{-\frac{1}{p}}(x)dx \\
&\le \left(\int_0^\infty |f(x)|^p \mathrm{w}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{p-1}}dx\right)^{\frac{p-1}{p}} \\
&= \left\|\mathrm{w}^{-\frac{1}{p-1}}\right\|_1^{\frac{p-1}{p}} \|f\|_{L^p_{\mathrm{w}}}.
\end{aligned}$$

This gives the desired conclusion. ■

Suppose that $p \geq 1$ and the weight function $\mathrm{w} : [0,\infty) \to [1,\infty)$ is an increasing, continuously differentiable function such that

$$\int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{2p-1}} dx < \infty, \tag{6.3}$$

i.e. $\mathrm{w}^{-\frac{1}{2p-1}} \in L^1$. If $\mathrm{w}^{-\frac{1}{2p-1}} \in L^1$, then $\mathrm{w}^{-\frac{1}{p-1}} \in L^1$. Indeed, since $\mathrm{w}(x) \geq 1$, we obtain

$$\int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{p-1}} dx = \int_0^\infty |\mathrm{w}(x)|^{-\frac{p}{(2p-1)(p-1)}} |\mathrm{w}(x)|^{-\frac{1}{2p-1}} dx \leq \int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{2p-1}} dx.$$

Thus $\mathrm{w}^{-\frac{1}{p-1}} \in L^1$.

Lemma 6.1. *Assume that $f \in L^1_{loc}$ such that the first weak derivative Df exists. If*

$$\int_0^\infty \left|Df(x)\right|^p \mathrm{w}(x) dx < \infty, \tag{6.4}$$

then the limit $\lim_{x\to\infty} f(x)$ *exists.*

Proof. It is sufficient to prove that for each $\varepsilon > 0$, there exists $R > 0$ such that for all $x_1, x_2 \geq R$,

$$|f(x_2) - f(x_1)| \leq \varepsilon.$$

Fix $\varepsilon > 0$. By the Hölder inequality, we obtain

$$\begin{aligned} |f(x_2) - f(x_1)| &\leq \int_{x_1}^{x_2} |Df(x)| dx = \int_{x_1}^{x_2} |Df(x)| \mathrm{w}^{\frac{1}{p}}(x) \mathrm{w}^{-\frac{1}{p}}(x) dx \\ &\leq \left(\int_{x_1}^{x_2} |Df(x)|^p \mathrm{w}(x) dx \right)^{\frac{1}{p}} \left(\int_{x_1}^{x_2} |\mathrm{w}(x)|^{-\frac{1}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\leq \left\| \mathrm{w}^{-\frac{1}{p-1}} \right\|_1^{\frac{p-1}{p}} \left(\int_{x_1}^{x_2} |Df(x)|^p \mathrm{w}(x) dx \right)^{\frac{1}{p}}. \end{aligned}$$

Define a function g by

$$g(x) = |Df(x)|^p \mathrm{w}(x), \quad x \in [0,\infty).$$

Define a measure μ on $\mathscr{B}([0,\infty))$ by

$$\mu(A) = \int_A g(x) dx, \quad A \in \mathscr{B}([0,\infty)).$$

Since by (6.4)

$$\int_0^\infty g(x) dx < \infty,$$

we have, $\mu([0,\infty)) < \infty$. Since $\bigcap_{n=1}^{\infty}[n,\infty) = \emptyset$, by the σ-additivity of μ, we obtain

$$\mu([n,\infty)) \to 0 \ \ as \ \ n \to \infty,$$

which implies that

$$\int_n^{\infty} g(x)dx \to 0 \ \ as \ \ n \to \infty.$$

Thus, we obtain

$$\int_x^{\infty} g(y)dy \to 0 \ \ as \ \ x \to \infty.$$

Define a function G by

$$G(x) = \int_0^x g(y)dy, \quad x \in [0,\infty).$$

If $\varepsilon > 0$, then for each $x_1 < x_2$,

$$|G(x_2) - G(x_1)| \le \int_{x_1}^{x_2} g(y)dy \le \int_{x_1}^{\infty} g(y)dy \le \varepsilon$$

provided $x_1 \ge R$ where $R = R_\varepsilon$ is chosen so that

$$\int_x^{\infty} g(y)dy \le \varepsilon, \quad if \ \ x \ge R.$$

Thus, we have proved that for each $\varepsilon > 0$, there exists $R > 0$ such that for all $x_1, x_2 \ge R$,

$$|G(x_2) - G(x_1)| \le \varepsilon,$$

i.e.

$$\int_{x_1}^{x_2} |Df(x)|^p \mathrm{w}(x)dx \le \varepsilon. \tag{6.5}$$

Taking into account the last estimate, we obtain

$$|f(x_2) - f(x_1)| \le \left\| \mathrm{w}^{-\frac{1}{p-1}} \right\|_1^{\frac{p-1}{p}} (\varepsilon)^{\frac{1}{p}}.$$

If we choose $\varepsilon = \frac{(\varepsilon')^p}{\left\| \mathrm{w}^{-\frac{1}{p-1}} \right\|_1^{p-1}}$, where $\varepsilon' > 0$, then there exists $R_{\varepsilon'} > 0$ such that for every $x_1, x_2 \ge R_{\varepsilon'}$,

$$|f(x_2) - f(x_1)| \le \varepsilon',$$

which completes the proof. ∎

Define $H_{\mathrm{w}}^{1,p}$ to be the space of all functions $f \in L_{loc}^1$ such that the first weak derivative Df exists and

$$\|f\|_{\mathrm{w},p}^p := |f(\infty)|^p + \int_0^\infty \left|Df(x)\right|^p \mathrm{w}(x)dx < \infty,$$

where

$$f(\infty) := \lim_{x\to\infty} f(x).$$

Theorem 6.2. *The space $H_{\mathrm{w}}^{1,p}$ is a separable Banach space with respect to the norm $\|\cdot\|_{\mathrm{w},p}$.*

Proof. Consider the space $\mathbb{R}\times L^p$ which is a separable Banach endowed with the norm

$$\|\cdot\|_{\mathbb{R}\times L^p}^p = |\cdot|^p + \|\cdot\|_p^p.$$

Define a linear map $T : H_{\mathrm{w}}^{1,p} \to \mathbb{R}\times L^p$ by

$$Tf = \left(f(\infty), \mathrm{w}^{\frac{1}{p}} Df\right), \quad f \in H_{\mathrm{w}}^{1,p}.$$

The map T is well-defined. Indeed, for a fixed $f \in H_{\mathrm{w}}^{1,p}$, $\mathbb{R} \ni f(\infty)$ exists and since

$$\int_0^\infty \left|Df(x)\mathrm{w}^{\frac{1}{p}}(x)\right|^p dx = \int_0^\infty |Df(x)|^p \mathrm{w}(x)dx < \infty,$$

$Df\mathrm{w}^{\frac{1}{p}} \in L^p$. Thus, T is well-defined. It is obvious that the map T is injective. Let us show the surjectivity of T. For this, fix $(r,g) \in \mathbb{R}\times L^p$. We need to find $f \in H_{\mathrm{w}}^{1,p}$ such that

$$\left(f(\infty), \mathrm{w}^{\frac{1}{p}} Df\right) = (r,g).$$

Suppose that f is a function such that $f(\infty) = r$ and $g = \mathrm{w}^{\frac{1}{p}} Df$. Hence

$$Df(x) = g(x)\mathrm{w}^{-\frac{1}{p}}(x), \quad x \in [0,\infty)$$

and thus, we get

$$f(x) = r - \int_x^\infty g(x)\mathrm{w}^{-\frac{1}{p}}(x)dx, \quad x \in [0,\infty) \tag{6.6}$$

because $g\mathrm{w}^{-\frac{1}{p}} \in L_{loc}^1$. Now we show that the function f defined in (6.6) is the one we look for. By the Hölder inequality, we get

$$\int_0^\infty |g(x)|\mathrm{w}^{-\frac{1}{p}}(x)dx \le \left(\int_0^\infty |g(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{p-1}} dx\right)^{\frac{p-1}{p}} = \left\|\mathrm{w}^{-\frac{1}{p-1}}\right\|_1^{\frac{p-1}{p}} \|g\|_{L_{\mathrm{w}}^p}.$$

Thus, we have, for each $T > 0$,

$$\int_0^T |f(x)|dx < \infty$$

and so $f \in L^1_{loc}$. Moreover, we have

$$\int_0^\infty |Df(x)|^p \mathrm{w}(x)dx = \int_0^\infty |g(x)|^p dx = \|g\|^p_{L^p_{\mathrm{w}}}.$$

Hence $f \in H^{1,p}_{\mathrm{w}}$. Thus, we have showed that the map T is also surjective and hence, it is bijective. Moreover, for a fixed $f \in H^{1,p}_{\mathrm{w}}$, we obtain

$$\|Tf\|^p_{\mathbb{R}\times L^p} = |f(\infty)|^p + \int_0^\infty \left|Df(x)\mathrm{w}^{\frac{1}{p}}(x)\right|^p dx = \|f\|^p_{\mathrm{w},p},$$

which implies that the map T is an isometric isomorphism. Hence, the spaces $\left(H^{1,p}_{\mathrm{w}}, \|\cdot\|_{\mathrm{w},p}\right)$ and $(\mathbb{R} \times L^p, \|\cdot\|_{\mathbb{R}\times L^p})$ are isometric. TTherefore, by Theorem 2.1, the space $H^{1,p}_{\mathrm{w}}$ is a separable Banach space with respect to the norm $\|\cdot\|_{\mathrm{w},p}$. ■

Proposition 6.2. *If $p \geq 2$, then the space $H^{1,p}_{\mathrm{w}}$ satisfies the H-condition.*

Proof. It is obvious that $\mathbb{R}$ satisfies the H-condition. Also for every $p \geq 2$, L^p satisfies the H-condition, see [12]. One can easily show that the product $\mathbb{R} \times L^p$ satisfies the H-condition, i.e the map $\psi : \mathbb{R} \times L^p \to \mathbb{R}$ defined by

$$\psi(r, f) = |r|^p + \|f\|^p_p, \quad (r, f) \in \mathbb{R} \times L^p$$

is of C^2 class (in the Fréchet derivative sense) and

$$\|\psi'(r, f)\| \leq p|r|^{p-1} + p\,\|f\|^{p-1}_p, \quad (r, f) \in \mathbb{R} \times L^p \tag{6.7}$$

and

$$\|\psi''(r, f)\| \leq p(p-1)|r|^{p-2} + p(p-1)\,\|f\|^{p-2}_p, \quad (r, f) \in \mathbb{R} \times L^p \tag{6.8}$$

where $\psi'(r, f)$ and $\psi''(r, f)$ are the first and second Fréchet derivatives of ψ at $(r, f) \in \mathbb{R} \times L^p$ respectively. Consider the operator T from the previous Lemma. By Lemma 2.2, since the map T is linear, it is twice Fréchet differentiable such that $T'(f) = T$ and $T''(f) = 0$ for every $f \in H^{1,p}_{\mathrm{w}}$. It is obvious that the map $\varphi = \psi \circ T : H^{1,p}_{\mathrm{w}} \to \mathbb{R}$ is well-defined and, since $\|Tf\|_{\mathbb{R}\times L^p} = \|f\|_{\mathrm{w},p}$, we can write φ as

$$\varphi(f) = \|f\|^p_{\mathrm{w},p}, \quad f \in H^{1,p}_{\mathrm{w}}.$$

Since ψ and T are of C^2 class, φ is of C^2 class. Moreover, since $T'(f) = T$, $\|T\|_{\mathscr{L}(H^{1,p}_{\mathrm{w}},\mathbb{R}\times L^p)} \le 1$ and inequality (6.7), we infer that

$$\begin{aligned}\|\varphi'(f)\| = \|T'(f)\psi'(Tf)\| &\le \|T\|_{\mathscr{L}(H^{1,p}_{\mathrm{w}},\mathbb{R}\times L^p)}\|\psi'(Tf)\| \\ &\le p|f(\infty)|^{p-1} + p\left\|Df\mathrm{w}^{\frac{1}{p}}\right\|_p^{p-1} \le p\|f\|^{p-1}_{\mathrm{w},p}, \quad f \in H^{1,p}_{\mathrm{w}}.\end{aligned}$$

Similarly, we can show that

$$\|\varphi''(f)\| \le p(p-1)\|f\|^{p-2}_{\mathrm{w},p}, \quad f \in H^{1,p}_{\mathrm{w}}.$$

Thus, the space $H^{1,p}_{\mathrm{w}}$ satisfies the H-condition. ■

Proposition 6.3. *The space $H^{1,p}_{\mathrm{w}}$ is continuously embedded into the space L^∞. In particular, there exists a constant C_1 depending on* w *and p such that*

$$\sup_{x\in[0,\infty)} |f(x)| \le C_1\|f\|_{\mathrm{w},p}, \quad f \in H^{1,p}_{\mathrm{w}}. \tag{6.9}$$

Proof. Fix $f \in H^{1,p}_{\mathrm{w}}$. Since $f(\infty)$ exists, we can write f as

$$f(x) = f(\infty) - \int_x^\infty Df(y)dy.$$

Thus

$$\sup_{x\in[0,\infty)} |f(x)| \le |f(\infty)| + \int_0^\infty \left|Df(x)\right|dx.$$

By the Hölder inequality, we obtain

$$\begin{aligned}\int_0^\infty \left|Df(x)\right|dx &= \int_0^\infty \left|Df(x)\right|\mathrm{w}^{\frac{1}{p}}(x)\mathrm{w}^{-\frac{1}{p}}(x)dx \\ &\le \left(\int_0^\infty \left|Df(x)\right|^p\mathrm{w}(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty |\mathrm{w}(x)|^{-\frac{1}{p-1}}dx\right)^{\frac{p-1}{p}} \\ &\le \left\|\mathrm{w}^{-\frac{1}{p-1}}\right\|_1^{\frac{p-1}{p}}\|f\|_{\mathrm{w},p}.\end{aligned} \tag{6.10}$$

Taking into account the last estimate, we deduce that

$$\sup_{x\in[0,\infty)} |f(x)| \le \left(1 + \left\|\mathrm{w}^{-\frac{1}{p-1}}\right\|_1^{\frac{p-1}{p}}\right)\|f\|_{\mathrm{w},p},$$

which gives the desired conclusion. ■

Proposition 6.4. *There exists a constant C_2 depending on* w *and* p *such that the following inequalities hold*

$$\|f-f(\infty)\|_1 \le C_2\|f\|_{\mathrm{w},p}, \quad f\in H^{1,p}_{\mathrm{w}} \tag{6.11}$$

and

$$\|(f-f(\infty))^{2p}\mathrm{w}\|_1 \le C_2^p\|f\|^{2p}_{\mathrm{w},p}, \quad f\in H^{1,p}_{\mathrm{w}}. \tag{6.12}$$

Proof. Define a function Π by

$$\Pi(x)=\int_x^\infty \mathrm{w}^{-\frac{1}{p-1}}(x)dx, \quad x\in[0,\infty).$$

Since w is increasing, we have

$$\Pi(x)\le \mathrm{w}^{-\frac{p}{(2p-1)(p-1)}}(x)\int_x^\infty \mathrm{w}^{-\frac{1}{2p-1}}(x)dx \le \mathrm{w}^{-\frac{p}{(2p-1)(p-1)}}(x)\left\|\mathrm{w}^{-\frac{1}{2p-1}}\right\|_1.$$

Thus

$$\left\|\Pi^{\frac{(2p-1)(p-1)}{p}}\mathrm{w}\right\|_\infty \le \left\|\mathrm{w}^{-\frac{1}{2p-1}}\right\|_1^{\frac{(2p-1)(p-1)}{p}} \tag{6.13}$$

and

$$\left\|\Pi^{\frac{p-1}{p}}\right\|_1 \le \left\|\mathrm{w}^{-\frac{1}{2p-1}}\right\|_1^{\frac{2p-1}{p}}. \tag{6.14}$$

Fix $f\in H^{1,p}_{\mathrm{w}}$. By the Hölder inequality, we get

$$\begin{aligned}
|f(x)-f(\infty)| &= \left|\int_x^\infty Df(y)dy\right| \le \int_x^\infty |Df(x)|dx \\
&= \int_x^\infty |Df(x)|\mathrm{w}^{\frac{1}{p}}(x)\mathrm{w}^{-\frac{1}{p}}(x)dx \\
&\le \left(\int_0^\infty |Df(x)|^p\mathrm{w}(x)dx\right)^{\frac{1}{p}}\Pi^{\frac{p-1}{p}}(x) \\
&\le \|f\|_{\mathrm{w},p}\Pi^{\frac{p-1}{p}}(x), \quad x\in[0,\infty).
\end{aligned} \tag{6.15}$$

Therefore, by inequality (6.14), we infer that

$$\int_0^\infty |f(x)-f(\infty)|dx \le \left\|\mathrm{w}^{-\frac{1}{2p-1}}\right\|_1^{\frac{2p-1}{p}}\|f\|_{\mathrm{w},p},$$

which gives the desired result (6.11). Similarly, by estimate (6.15), we have

$$\begin{aligned}
\int_0^\infty |f(x)-f(\infty)|^{2p}\mathrm{w}(x)dx &\le \|f\|^{2p}_{\mathrm{w},p}\int_0^\infty \Pi^{2p-2}(x)\mathrm{w}(x)dx \\
&\le \|f\|^{2p}_{\mathrm{w},p}\int_0^\infty \Pi^{\frac{(2p-1)(p-1)}{p}}(x)\Pi^{\frac{p-1}{p}}(x)\mathrm{w}(x)dx \\
&\le \|f\|^{2p}_{\mathrm{w},p}\left\|\Pi^{\frac{(2p-1)(p-1)}{p}}\mathrm{w}\right\|_\infty\left\|\Pi^{\frac{p-1}{p}}\right\|_1.
\end{aligned}$$

It follows from inequalities (6.13) and (6.14) that

$$\int_0^\infty |f(x)-f(\infty)|^{2p}\mathrm{w}(x)dx \le \left\|\mathrm{w}^{-\frac{1}{2p-1}}\right\|_1^{2p-1} \|f\|_{\mathrm{w},p}^{2p},$$

which gives the desired conclusion (6.12). Thus, the proof is complete. ■

Lemma 6.2. *The shift-semigroup S is a contraction C_0-semigroup on $H^{1,p}_{\mathrm{w}}$.*

Lemma 6.3. *Set*

$$D_0 = \{f \in C^2(\mathbb{R}^+) \mid Df \in C^1_c(\mathbb{R}^+)\}.$$

Then D_0 is a dense subspace of $H^{1,p}_{\mathrm{w}}$.

Proof. First we show that D_0 is a subspace of $H^{1,p}_{\mathrm{w}}$. Fix $f \in D_0$. Then f is measurable and for each $T > 0$,

$$\int_0^T |f(x)|dx \le \sup_{x\in[0,T]} |f(x)| \int_0^T dx = T \sup_{x\in[0,T]} |f(x)| < \infty.$$

Therefore $f \in L^1_{loc}$. Moreover, since Df has compact support, i.e. there exists $a > 0$ such that for all $x \in [a,\infty)$, $Df(x) = 0$, we have

$$\int_0^\infty |Df(x)|^p\mathrm{w}(x)dx = \int_0^a |Df(x)|^p\mathrm{w}(x)dx \le \sup_{x\in[0,a]} |Df(x)|^p \int_0^a \mathrm{w}(x)dx < \infty.$$

Thus $f \in H^{1,p}_{\mathrm{w}}$ and hence, D_0 is a subspace of $H^{1,p}_{\mathrm{w}}$.

Let us now show that D_0 is dense. For this aim, it is enough to show that for each $f \in H^{1,p}_{\mathrm{w}}$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of D_0 such that $h_n \to f$. Fix $f \in H^{1,p}_{\mathrm{w}}$. Then $Df\mathrm{w}^{\frac{1}{p}} \in L^p$. Indeed,

$$\int_0^\infty \left|Df(x)\mathrm{w}^{\frac{1}{p}}(x)\right|^p dx = \int_0^\infty \left|Df(x)\right|^p \mathrm{w}(x)dx < \infty.$$

Since $C^1_c(\mathbb{R}^+)$ is a dense in L^p (see [32]), there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of $C^1_c(\mathbb{R}^+)$ such that $g_n \to Df\mathrm{w}^{\frac{1}{p}}$. Consider the operator T defined in the proof of Theorem 6.2. Let $(u,h) \in \mathbb{R}\times L^p$ such that

$$Tf(x) = \left(f(\infty), Df(x)\mathrm{w}^{\frac{1}{p}}(x)\right) = (u,h(x))$$

Then $f(\infty) = u$ and $Df(x)\mathrm{w}^{\frac{1}{p}}(x) = h(x)$. Thus

$$\int_x^\infty Df(y)dy = u - f(x) = \int_x^\infty h(y)\mathrm{w}^{-\frac{1}{p}}(y)dy.$$

Therefore, we get

$$T^{-1}(u,h)(x) = u - \int_x^\infty h(y)\mathrm{w}^{-\frac{1}{p}}(y)dy, \quad (u,h) \in \mathbb{R} \times L^p.$$

For each $n \in \mathbb{N}$, define a function by

$$h_n(x) = f(\infty) - \int_x^\infty g_n(y)\mathrm{w}^{-\frac{1}{p}}(y)dy, \quad x \in [0,\infty).$$

For each $n \in \mathbb{N}$, $h_n \in D_0$. Indeed, for a fixed $n \in \mathbb{N}$, we have

$$Dh_n(x) = g_n(x)\mathrm{w}^{-\frac{1}{p}}(x), \quad x \in [0,\infty)$$

and

$$D^2 h_n(x) = D\big(g_n(x)\big)\mathrm{w}^{-\frac{1}{p}}(x) - \frac{1}{p} g_n(x)\mathrm{w}^{-\frac{p+1}{p}}(x), \quad x \in [0,\infty).$$

Thus, $h_n \in C^2(\mathbb{R}^+)$. Also, since g_n has compact support, Dh_n has compact support. Hence, $Dh_n \in C^1_c(\mathbb{R}^+)$. Moreover, $h_n \to f$. Indeed,

$$\begin{aligned}
\|h_n - f\|^p_{\mathrm{w},p} &= |h_n(\infty) - f(\infty)|^p + \int_0^\infty |Dh_n(x) - Df(x)|^p \mathrm{w}(x)dx \\
&= \int_0^\infty \left| g_n(x)\mathrm{w}^{-\frac{1}{p}}(x) - Df(x)\right|^p \mathrm{w}(x)dx \\
&= \int_0^\infty \left| g_n(x) - Df(x)\mathrm{w}^{-\frac{1}{p}}(x)\right|^p dx \\
&= \left\| g_n - Df\mathrm{w}^{\frac{1}{p}} \right\|_p \to 0 \quad as \;\; n \to \infty.
\end{aligned}$$

Therefore, D_0 is a dense subspace of $H^{1,p}_{\mathrm{w}}$. ∎

Lemma 6.4. *The shift-semigroup on D_0 is strongly continuous in the norm of $H^{1,p}_{\mathrm{w}}$.*

Proof. For any $f \in H^{1,p}_{\mathrm{w}}$, we can write the following equality

$$f(x+t) - f(x) = t\int_0^1 Df(x+ts)ds, \quad x \in [0,\infty).$$

By the Hölder inequality, we get

$$|f(x+t) - f(x)| \le t\int_0^1 \big|Df(x+ts)\big|ds \le t\left(\int_0^1 \big|Df(x+ts)\big|^p ds\right)^{\frac{1}{p}}.$$

Therefore, we obtain

$$|f(x+t) - f(x)|^p \le t^p \int_0^1 \big|Df(x+ts)\big|^p ds. \tag{6.16}$$

Fix $g \in D_0$. Then $Dg \in H^{1,p}_{\mathrm{w}}$. Using inequaliy (6.16) and the Fubini Theorem, we obtain

$$\begin{aligned}
\|S(t)g - g\|^p_{\mathrm{w},p} &= \int_0^\infty \left|Dg(x+t) - Dg(x)\right|^p \mathrm{w}(x)dx \\
&\le t^p \int_0^\infty \int_0^1 \left|D^2 g(x+ts)\right|^p \mathrm{w}(x) ds dx \\
&= t^p \int_0^1 \int_0^\infty \left|D^2 g(x+ts)\right|^p \mathrm{w}(x) dx ds \\
&\le t^p \int_0^1 \left\|S(st)Dg\right\|^p_{\mathrm{w},p} ds \\
&\le t^p \left\|Dg\right\|^p_{\mathrm{w},p} \to 0 \quad as \;\; t \to 0,
\end{aligned}$$

which gives the desired result. ∎

Proof of Lemma 6.2. First we show that for each $t \ge 0$, $S(t) : H^{1,p}_{\mathrm{w}} \ni f \mapsto f(t+\cdot) \in H^{1,p}_{\mathrm{w}}$ is well-defined, linear and bounded. Fix $t \ge 0$ and $f \in H^{1,p}_{\mathrm{w}}$. In lemma 5.3, we showed that $S(t)f$ is measurable when f is measurable. Moreover, for each $T > 0$,

$$\int_0^T |S(t)f(x)|dx = \int_0^T |f(x+t)|dx = \int_t^{T+t} |f(u)|du < \infty$$

and hence $S(t)f \in L^1_{loc}$. In lemma 5.13, we showed that the first weak derivative of $S(t)f$ exists and equal to $S(t)Df$ when f is weakly differentiable. Furthermore, we have

$$S(t)f(\infty) = \lim_{x\to\infty} S(t)f(x) = \lim_{x\to\infty} f(x+t) = \lim_{x\to\infty} f(x) = f(\infty).$$

Thus

$$\begin{aligned}
\|S(t)f\|^p_{\mathrm{w},p} &= |S(t)f(\infty)|^p + \int_0^\infty \left|DS(t)f(x)\right|^p \mathrm{w}(x)dx \\
&= |f(\infty)|^p + \int_0^\infty \left|Df(x+t))\right|^p \mathrm{w}(x)dx \\
&= |f(\infty)|^p + \int_t^\infty \left|Df(x))\right|^p \mathrm{w}(x-t)dx.
\end{aligned}$$

It follows from the monotonicity of w that

$$\|S(t)f\|^p_{\mathrm{w},p} \le |f(\infty)|^p + \int_0^\infty \left|Df(x))\right|^p \mathrm{w}(x)dx = \|f\|^p_{\mathrm{w},p}. \tag{6.17}$$

Therefore $S(t)f \in H^{1,p}_{\mathrm{w}}$ and thus, $S(t)$ is a well-defined operator from $H^{1,p}_{\mathrm{w}}$ into $H^{1,p}_{\mathrm{w}}$. The linearity of $S(t)$ is obvious and the boundedness of $S(t)$ follows from (6.17).

Now we show that S is a contraction C_0-semigroup on $H^{1,p}_{\mathrm{w}}$. It can be easily shown as Lemma 5.3 that S is a semigroup on $H^{1,p}_{\mathrm{w}}$. Let us show the strong continuity of S on

$H^{1,p}_{\mathrm{w}}$. Fix $f \in H^{1,p}_{\mathrm{w}}$. Since D_0 is a dense in $H^{1,p}_{\mathrm{w}}$, for every $\frac{\varepsilon}{3} > 0$, there exists $g \in D_0$ such that

$$\|f - g\|_{\mathrm{w},p} \le \frac{\varepsilon}{3}. \tag{6.18}$$

By Lemma 6.4, we have, for every $\frac{\varepsilon}{3} > 0$,

$$\|S(t)g - g\|_{\mathrm{w},p} \le \frac{\varepsilon}{3}. \tag{6.19}$$

Taking into account inequalities (6.17), (6.18) and (6.19), we infer that

$$\begin{aligned}
\|S(t)f - f\|_{\mathrm{w},p} &= \|S(t)f - S(t)g + S(t)g - f + g - g\|_{\mathrm{w},p} \\
&\le \|S(t)(f-g)\|_{\mathrm{w},p} + \|S(t)g - g\|_{\mathrm{w},p} + \|f - g\|_{\mathrm{w},p} \\
&\le \|f-g\|_{\mathrm{w},p} + \|S(t)g - g\|_{\mathrm{w},p} + \|f - g\|_{\mathrm{w},p} \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

Which implies that S is strongly continuous on $H^{1,p}_{\mathrm{w}}$. Moreover, by inequality (6.17), we obtain

$$\|S(t)\|_{\mathscr{L}(H^{1,p}_{\mathrm{w}})} \le 1, \quad t \ge 0.$$

Therefore, S is a contraction C_0-semigroup on $H^{1,p}_{\mathrm{w}}$. This completes the proof.

Lemma 6.5. *The infinitesimal generator A of the shift-semigroup S is characterized by*

$$\mathscr{D}(A) = \{f \in H^{1,p}_{\omega} \mid Df \in H^{1,p}_{\mathrm{w}}\} \tag{6.20}$$

and

$$Af = Df, \quad f \in \mathscr{D}(A). \tag{6.21}$$

The proof is similar to the proof of Lemma 5.8.

Let H be a Hilbert space with respect to an inner product $\langle \cdot, \cdot \rangle_H$. Define $H^{1,p}_{\mathrm{w}}(H)$ to be the space of all (equivalence classes of) Borel measurable functions $f \in L^1_{loc}(H)$ such that the first weak derivative Df exists and

$$\|f\|^p_{H^{1,p}_{\mathrm{w}}(H)} := \|f(\infty)\|^p_H + \int_0^\infty \|Df(x)\|^p_H \mathrm{w}(x)dx < \infty.$$

All the results above of the space $H^{1,p}_{\mathrm{w}}$ can be generalized for the space $H^{1,p}_{\mathrm{w}}(H)$. Thus, $H^{1,p}_{\mathrm{w}}(H)$ is a separable Banach space endowed with the norm $\|\cdot\|_{H^{1,p}_{\mathrm{w}}(H)}$. The following proposition gives a sufficient condition under which an $H^{1,p}_{\mathrm{w}}$-valued operator K defined on H is the γ-radonifying.

Proposition 6.5. *If $p \geq 2$, then for every $\kappa \in H_{\mathrm{w}}^{1,p}(H)$, a linear map $K : H \to H_{\mathrm{w}}^{1,p}$ defined by*

$$K[h](x) = \langle \kappa(x), h \rangle_H, \quad h \in H, \quad x \in [0,\infty), \tag{6.22}$$

is the γ-radonifying, i.e. $K \in \gamma(H, H_{\mathrm{w}}^{1,p})$, and for a constant $N > 0$ independent of κ,

$$\|K\|_{\gamma(H,H_{\mathrm{w}}^{1,p})} \leq N \|\kappa\|_{\mathrm{w},p}. \tag{6.23}$$

Proof. Fix $a \notin \mathbb{R}^+$ and set $A_a = \{a\} \cup \mathbb{R}^+$. Define a measure μ on A_a by

$$\mu(A) = \delta_a(A \cap \{a\}) + Leb(A \cap \mathbb{R}^+), \quad A \in \mathscr{B}(A_a).$$

Define $L^p(A_a)$ to be the space of all (equivalence classes of) measurable functions $f : A_a \to \mathbb{R}$ such that

$$\|f\|_{L^p(A_a)} := \left(\int_{A_a} |f(x)|^p \mu(dx) \right)^{\frac{1}{p}} < \infty.$$

Consider the Banach space $\mathbb{R} \times L^p$ with the norm $\|\cdot\|_{\mathbb{R}\times L^p} = \left(|\cdot|^p + \|f\|_p^p\right)^{\frac{1}{p}}$. The space $L^p(A_a)$ is equivalent to the space $\mathbb{R} \times L^p$ with respect to the norm $\|\cdot\|_{L^p(A_a)}$. Let us prove this. Define a linear map $S : \mathbb{R} \times L^p \to L^p(A_a)$ by

$$S(r,f) = r1_{\{a\}} + f1_{\{\mathbb{R}^+\}}, \quad (r,f) \in \mathbb{R} \times L^p.$$

The map is well-defined. Indeed, for a fixed $(r,f) \in \mathbb{R} \times L^p$,

$$\begin{aligned}
\|r1_{\{a\}} + f1_{\{\mathbb{R}^+\}}\|_{L^p(A_a)}^p &= \int_{\{a\}\cup\mathbb{R}^+} |r1_{\{a\}}(x) + f1_{\{\mathbb{R}^+\}}(x)|^p \mu(dx) \\
&= \int_{\{a\}} |r1_{\{a\}}(x) + f1_{\{\mathbb{R}^+\}}(x)|^p d\delta_a(dx) \\
&+ \int_{\mathbb{R}^+} |r1_{\{a\}}(x) + f1_{\{\mathbb{R}^+\}}(x)|^p dLeb(dx) \\
&= |r|^p + \|f\|_p^p.
\end{aligned} \tag{6.24}$$

Thus, the map S is well-defined. It is obvious that S is bijective and it follows from the last inequality that S is an isometry. Therefore, $L^p(A_a) \cong \mathbb{R} \times L^p$. Let $L^p(A_a;H)$ be the space of all (equivalence classes of) measurable functions $f : A_a \to H$ such that

$$\|f\|_{L^p(A_a;H)} := \left(\int_{A_a} \|f(x)\|_H^p \mu(dx) \right)^{\frac{1}{p}} < \infty.$$

and consider the Banach space $H \times L^p(H)$ with the norm

$$\|\cdot\|_{H\times L^p(H)} = (\|\cdot\|_H + \|\cdot\|_{L^p(H)})^{\frac{1}{p}}.$$

Assume that $T : H^{1,p}_{\mathrm{w}} \to \mathbb{R} \times L^p$ is a map defined in the proof of Theorem 6.2. We showed that the map T is bijective and isometry. Similarly, we can prove that the same map T, but defined from $H^{1,p}_{\mathrm{w}}(H)$ into $H \times L^p(H)$, is bijective and isometry. Also, we can show that above map S, but defined from $H \times L^p(H)$ into $L^p(A_a;H)$, is bijective and isometry. Therefore, the linear map $V = S \circ T : H^{1,p}_{\mathrm{w}}(H) \to L^p(A_a;H)$ is bijective and isometry, i.e.

$$\|Vf\|_{L^p(A_a,H)} = \|f\|_{H^{1,p}_{\mathrm{w}}(H)}, \quad f \in H^{1,p}_{\mathrm{w}}(H). \tag{6.25}$$

Fix $\kappa \in^{1,p}_{\mathrm{w}} (H)$ then $V\kappa \in L^p(A_a;H)$ and

$$\|V\kappa\|_{L^p(A_a,H)} = \|\kappa\|_{H^{1,p}_{\mathrm{w}}(H)}. \tag{6.26}$$

It was proven in Proposition 2.1 of [9] that for every $\phi \in L^p(A_a;H)$, a bounded linear operator $M : H \to L^p(A_a)$ defined by

$$M[h](x) = \langle \phi(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is γ-radonifying and for a constant $N > 0$ independent of ϕ,

$$\|M\|_{\gamma(H,L^p(A_a))} \le N\|\phi\|_{L^p((A_a;H)}.$$

Therefore, a map $\tilde{M}$ defined by

$$\tilde{M}[h](x) = \langle V\kappa(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty)$$

is γ-radonifying and for a constant $N > 0$ independent of $V\kappa$,

$$\|\tilde{M}\|_{\gamma(H,L^p(A_a))} \le N\|V\kappa\|_{L^p((A_a;H)}.$$

It follows from (6.26) that

$$\|\tilde{M}\|_{\gamma(H,L^p(A_a))} \le N\|\kappa\|_{H^{1,p}_{\mathrm{w}}(H)}. \tag{6.27}$$

It is obvious that we can write K as $V^{-1} \circ \tilde{M}$, where $V^{-1} : L^p(A_a) \to H^{1,p}_{\mathrm{w}}$ is the inverse of the map $V = S \circ T : H^{1,p}_{\mathrm{w}} \to L^p(A_a)$ defined above. Thus, by Theorem 2.15, since $V^{-1} : L^p(A_a) \to H^{1,p}_{\mathrm{w}}$ is linear bounded and $\tilde{M} : H \to L^p(A_a)$ is γ-radonifying, K is γ-radonifying. Moreover, since $\|V^{-1}\|_{\mathscr{L}(L^p(A_a),H^{1,p}_{\mathrm{w}})} \le 1$ and (6.27), we infer that

$$\|K\|_{\gamma(H,H^{1,p}_{\mathrm{w}})} \le \|\tilde{M}\|_{\gamma(H,L^p(A_a))}\|V^{-1}\|_{\mathscr{L}(L^p(A_a),H^{1,p}_{\mathrm{w}})} \le N\|\kappa\|_{H^{1,p}_{\mathrm{w}}(H)},$$

which gives the desired conclusion. ∎

6.2 Existence and Uniqueness of Solutions to the HJMM Equation in the space $H^{1,p}_{\mathrm{w}}$

In this section, we prove the existence and uniqueness of the HJMM equation (3.7) (driven by standard d-dimensional Wiener process) in the space $H^{1,p}_{\mathrm{w}}$, $p \geq 2$. Thus, the following theorem gives the main result of this chapter.

Theorem 6.3. *Let $p \geq 2$. Assume that for each $i = 1,2,...d$, $g_i : [0,\infty) \times [0,\infty) \times H^{1,p}_{\mathrm{w}} \to \mathbb{R}$ is a continuously weakly differentiable function with respect to the second variable such that for every $t \geq 0$ and $f \in H^{1,p}_{\mathrm{w}}$,*

$$\lim_{x\to\infty} g_i(t,x,f) = 0. \tag{6.28}$$

Moreover, there exist constants $M_i, L_i > 0$ such that for each $t \geq 0$,

$$|D_x g_i(t,x,f)| \leq M_i |Df(x)|, \quad f \in H^{1,p}_{\mathrm{w}}, \quad x \in [0,\infty) \tag{6.29}$$

and

$$|D_x g_i(t,x,f_1) - D_x g_i(t,x,f_2)| \leq L_i |Df_1(x) - Df_2(x)|, \;\; f_1, f_2 \in H^{1,p}_{\mathrm{w}}, \;\; x \in [0,\infty) \tag{6.30}$$

where $D_x g_i$ is the first partial weak derivative of g_i with respect to the second variable (when the first and third variables are fixed). Furthermore, we assume that there exist $\bar{f}_i, \hat{f}_i \in H^{1,p}_{\mathrm{w}}$ with $\bar{f}_i(\infty) = 0$, $i = 1,2,...,d$, such that for all $t \geq 0$ and $f \in H^{1,p}_{\mathrm{w}}$,

$$|g_i(t,x,f)| \leq |\bar{f}_i(x)|, \quad x \in [0,\infty) \tag{6.31}$$

and

$$|g_i(t,x,f_1) - g_i(t,x,f_2)| \leq \left|\hat{f}_i(x)\right| |f_1(x) - f_2(x)|, \quad f_1, f_2 \in H^{1,p}_{\mathrm{w}}, \quad x \in [0,\infty). \tag{6.32}$$

Then for each $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; H^{1,p}_{\mathrm{w}})$, the following stochastic differential equation

$$\begin{aligned} dr(t)(x) = &\left(\frac{\partial}{\partial x} r(t)(x) + \sum_{i=1}^{d} g_i(t,x,r(t)) \int_0^x g_i(t,y,r(t))dy \right) dt \\ &+ \sum_{i=1}^{d} g_i(t,x,r(t)) dW_i(t), \quad t \geq 0, \end{aligned} \tag{6.33}$$

where $W(t) = (W_1(t), W_2(t), ..W_d(t))$ is a d-dimensional Brownian motion, has a unique $H^{1,p}_{\mathrm{w}}$-valued continuous mild solution with the initial value $r(0) = r_0$.

Lemma 6.6. *Assume that all the assumptions of the previous Theorem hold. Define a function* $F : [0,\infty) \times H^{1,p}_{\mathrm{w}} \to H^{1,p}_{\mathrm{w}}$ *by*

$$F(t,f)(x) = \sum_{i=1}^{d} g_i(t,x,f) \int_0^x g_i(t,y,f)dy, \quad f \in H^{1,p}_{\mathrm{w}}, \quad x,t \in [0,\infty).$$

Then F is well-defined. Moreover, it is of linear growth and locally Lipschitz on $H^{1,p}_{\mathrm{w}}$ with Lipschitz constants independent of time t.

Proof. For each $i = 1,2,3...d$, define a function $F_i : [0,\infty) \times H^{1,p}_{\mathrm{w}} \to H^{1,p}_{\mathrm{w}}$ by

$$F_i(t,f)(x) = g_i(t,x,f) \int_0^x g_i(t,y,f)dy, \quad f \in H^{1,p}_{\mathrm{w}}, \quad x,t \in [0,\infty).$$

Then F can be written as follows

$$F(t,f)(x) = \sum_{i=1}^{d} F_i(t,f)(x), \quad f \in H^{1,p}_{\mathrm{w}}, \quad x,t \in [0,\infty).$$

Thus, it is sufficient to show that for each $i = 1,2,3...d$, F_i is well-defined, of linear growth and locally Lipschitz on $H^{1,p}_{\mathrm{w}}$.
Fix $i \in \{1,2,..,d\}$, $t \geq 0$ and $f \in H^{1,p}_{\mathrm{w}}$. The function $g_i(t,\cdot,f)$ is continuous. Moreover, by inequality (6.31), we have

$$\int_0^x |g_i(t,y,f)|dy \leq \int_0^\infty |\bar{f}_i(y)|dy.$$

It follows from inequality (6.11) that for some constant $C_2 > 0$ independent of $\bar{f}_i$,

$$\int_0^x |g_i(t,y,f)|dy \leq C_2 \|\bar{f}_i\|_{\mathrm{w},p}$$

since $\bar{f}_i(\infty) = 0$. Thus, the integral $\int_0^x g_i(t,y,f)dy$ exists and it is continuous. Therefore, $F_i(t,f)$ is measurable. Furthermore by the last inequality, we have, for every $T > 0$,

$$\begin{aligned}\int_0^T |F_i(t,f)(x)|^p \, dx &= \int_0^T |g_i(t,x,f)|^p \left| \int_0^x g_i(t,y,f)dy \right|^p dx \\ &\leq C_2^p \|\bar{f}_i\|^p_{\mathrm{w},p} \int_0^T |g_i(t,x,f)|^p dx.\end{aligned}$$

Using Proposition 6.3 and inequality (6.31), we get

$$\int_0^T |g_i(t,x,f)|^p dx \leq \int_0^T |\bar{f}_i(x)|^p dx \leq T C_1^p \|f\|^p_{\mathrm{w},p}.$$

Taking into account the last inequality, we infer that

$$\int_0^T |F_i(t,f)(x)|^p\,dx \le TC_1^p C_2^p \|\bar f_i\|^{2p}_{\mathrm{w},p},$$

which implies that $F_i(t,f)\in L^1_{loc}$. Let us now show that $DF(t,f)\in L^p_{\mathrm{w}}$. Note that

$$DF_i(t,f)(x) = D_x g_i(t,x,f)\int_0^x g_i(t,y,f)dy + g_i^2(t,x,f), \quad x\in[0,\infty).$$

Using inequalities (6.29) and (6.31), we get

$$\begin{aligned}\left|DF_i(t,f)(x)\right| &\le \left|D_x g_i(t,x,f)\right|\left|\int_0^x g_i(t,y,f)dy\right| + \left|\zeta_i(t,x,f)\right|^2\\ &\le M_i\left|Df(x)\right|\int_0^\infty \left|\bar f_i(x)\right|dx + \left|\bar f_i(x)\right|^2, \quad x\in[0,\infty).\end{aligned}$$

By (6.11), we have , for some constant $C_2>0$ independent of $\bar f_i$,

$$\int_0^\infty \left|\bar f_i(x)\right|dx \le C_2\|\bar f_i\|_{\mathrm{w},p}.$$

Thus

$$\left|DF_i(t,f)(x)\right| \le M_i C_2 \left\|\bar f_i\right\|_{\mathrm{w},p}\left|Df(x)\right| + \left|\bar f_i(x)\right|^2, \quad x\in[0,\infty).$$

It follows from the last inequality that

$$\begin{aligned}\int_0^\infty \left|DF_i(t,f)(x)\right|^p \mathrm{w}(x)dx &\le 2^p M_i^p C_2^p \|\bar f_i\|^p_{\mathrm{w},p}\int_0^\infty \left|Df(x)\right|^p \mathrm{w}(x)dx + 2^p\int_0^\infty \left|\bar f_i(x)\right|^{2p}\mathrm{w}(x)dx\\ &\le 2^p M_i^p C_2^p \|\bar f_i\|^p_{\mathrm{w},p}\|f\|^p_{\mathrm{w},p} + 2^p\int_0^\infty \left|\bar f_i(x)\right|^{2p}\mathrm{w}(x)dx.\end{aligned}$$

Using (6.12), since $\bar f_i(\infty)=0$, we have

$$\int_0^\infty \left|\bar f_i(x)\right|^{2p}\mathrm{w}(x)dx \le C_2^p\|\bar f_i\|^{2p}_{\mathrm{w},p}.$$

From the last inequality, we deduce that

$$\int_0^\infty \left|D_x F_i(t,f)(x)\right|^p \mathrm{w}(x)dx \le 2^p M_i^p C_2^p \|\bar f_i\|^p_{\mathrm{w},p}\|f\|^p_{\mathrm{w},p} + 2^p C_2^p \|\bar f_i\|^{2p}_{\mathrm{w},p}. \tag{6.34}$$

Thus $DF(t,f)\in L^p_{\mathrm{w}}$. By Lemma 6.1, the limit of $F_i(t,f)$ at ∞ exists such that by (6.28) ,

$$\lim_{x\to\infty} F_i(t,f)(x) = 0.$$

Hence $F_i(t,f)(\infty)=0$. Thus, we showed that $F_i(t,f)\in H^{1,p}_{\mathrm{w}}$. Moreover, it follows from the estimate (6.34) that there exists a constant C depending on w, p and $\bar f_i$ such that

$$\left\|F_i(t,f)\right\|_{\mathrm{w},p} \le C(1+\|f\|_{\mathrm{w},p}), \tag{6.35}$$

which implies that F_i is of linear growth.

Finally, we prove that F_i is locally Lipschitz. Fix $t \geq 0$ and $f_1, f_2 \in H_{\mathrm{w}}^{1,p}$. Define a function I_i by

$$I_i(x) = DF_i(t,f_1)(x) - DF_i(t,f_2)(x), \quad x \in [0,\infty).$$

Then

$$\begin{aligned}
I_i(x) &= D_x g_i(t,x,f_1)\int_0^x g_i(t,y,f_1)dy + g_i^2(t,x,f_1) \\
&\quad - D_x g_i(t,x,f_2)\int_0^x g_i(t,y,f_2)dy - g_i^2(t,x,f_2) \\
&= D_x g_i(t,x,f_1)\int_0^x \big(g_i(t,y,f_1) - g_i(t,y,f_2)\big)dy \\
&\quad + \big(D_x g_i(t,x,f_1) - D_x g_i(t,x,f_2)\big)\int_0^x g_i(t,y,f_2)dy \\
&\quad + g_i^2(t,x,f_1) - g_i^2(t,x,f_2), \quad x \in [0,\infty).
\end{aligned}$$

By inequalities (6.29), (6.30), (6.31) and (6.32), we get, for each $x \in [0,\infty)$,

$$\begin{aligned}
\big|I_i(x)\big| &\leq \big|D_x g_i(t,x,f_1)\big|\int_0^\infty \big|g_i(t,x,f_1) - g_i(t,x,f_2)\big|dx \\
&\quad + \big|D_x g_i(t,x,f_1) - D_x g_i(t,x,f_2)\big|\int_0^\infty \big|g_i(t,x,f_2)\big|dx \\
&\quad + \big(|g_i(t,x,f_1)| + |g_i(t,x,f_2)|\big)\big|g_i(t,x,f_1) - g_i(t,x,f_2)\big| \\
&\leq M_i\big|Df_1(x)\big|\int_0^\infty \big|\hat{f}_i(x)\big|\big|f_1(x) - f_2(x)\big|dx \\
&\quad + L_i\big|Df_1(x) - Df_2(x)\big|\int_0^\infty \big|\bar{f}_i(x)\big|dx + 2\big|\bar{f}_i(x)\big|\big|\hat{f}_i(x)\big||f_1(x) - f_2(x)|.
\end{aligned}$$

It follows from inequalities (6.9) and (6.11) that, for each $x \in [0,\infty)$,

$$\begin{aligned}
\big|I_i(x)\big| \leq M_i C_1 C_2 \|\hat{f}_i\|_{\mathrm{w},p}\|f_1 - f_2\|_{\mathrm{w},p}\big|Df_1(x)\big| + L_i C_2 \|\bar{f}_i\|_{\mathrm{w},p}\big|Df_1(x) - Df_2(x)\big| \\
+ 2\big|\bar{f}_i(x)\big|\big|\hat{f}_i(x)\big||f_1(x) - f_2(x)|.
\end{aligned}$$

Therefore, from the last inequality and inequality (6.9), we obtain

$$\begin{aligned}
\int_0^\infty \big|I_i(x)\big|^p \mathrm{w}(x)dx &\leq 3^p M_i^p C_1^p C_2^p \|\hat{f}_i\|_{\mathrm{w},p}^p \|f_1 - f_2\|_{\mathrm{w},p}^p \int_0^\infty \big|Df_1(x)\big|\mathrm{w}(x)dx \\
&\quad + 3^p L_i^p C_2^p \|\bar{f}_i\|_{\mathrm{w},p}^p \int_0^\infty \big|Df_1(x) - Df_2(x)\big|^p \mathrm{w}(x)dx \\
&\quad + 3^p \int_0^\infty \big|\bar{f}_i(x)\big|^p\big|\hat{f}_i(x)\big|^p|f_1(x) - f_2(x)|^p \mathrm{w}(x)dx \\
&\leq 3^p M_i^p C_1^p C_2^p \|\hat{f}_i\|_{\mathrm{w},p}^p \|f_1\|_{\mathrm{w},p}^p \|f_1 - f_2\|_{\mathrm{w},p}^p + 3^p L_i^p C_2^p \|\bar{f}_i\|_{\mathrm{w},p}^p \|f_1 - f_2\|_{\mathrm{w},p}^p \\
&\quad + 3^p C_1^p \|f_1 - f_2\|_{\mathrm{w},p}^p \int_0^\infty \big|\bar{f}_i(x)\big|^p\big|\hat{f}_i(x)\big|^p \mathrm{w}(x)dx.
\end{aligned}$$

Using the Hölder inequality and inequality (6.12), we get

$$
\begin{aligned}
\int_0^\infty \left|\bar f_i(x)\right|^p \left|\hat f_i(x)\right|^p \mathrm{w}(x)dx &= \int_0^\infty \left|\bar f_i(x)\right|^p \left|\hat f_i(x)\right|^p \mathrm{w}^{\frac12}(x)\mathrm{w}^{\frac12}(x)dx \\
&\le \left(\int_0^\infty \left|\bar f_i(x)\right|^{2p}\mathrm{w}(x)dx\right)^{\frac12}\left(\int_0^\infty \left|\hat f_i(x)\right|^{2p}\mathrm{w}(x)dx\right)^{\frac12} \\
&\le C_2^p \|\bar f_i\|^p_{\mathrm{w},p}\|\hat f_i\|^p_{\mathrm{w},p}.
\end{aligned}
\tag{6.36}
$$

Taking into account (6.36), we infer that

$$
\begin{aligned}
\int_0^\infty \left|I_i(x)\right|^p \mathrm{w}(x)dx &\le 3^p M_i^p C_1^p C_2^p \|\hat f_i\|^p_{\mathrm{w},p}\|f_1\|^p_{\mathrm{w},p}\|f_1-f_2\|^p_{\mathrm{w},p} \\
&+3^p L_i^p C_2^p\|\bar f_i\|^p_{\mathrm{w},p}\|f_1-f_2\|^p_{\mathrm{w},p}+3^pC_1^pC_2^p\|\bar f_i\|^p_{\mathrm{w},p}\|\hat f_i\|^p_{\mathrm{w},p}\|f_1-f_2\|^p_{\mathrm{w},p}.
\end{aligned}
\tag{6.37}
$$

Since, by (6.28),

$$
\lim_{x\to\infty} F_i(t,f_1)(x) = \lim_{x\to\infty} F_i(t,f_2)(x) = 0,
$$

we obtain

$$
\left|F_i(t,f_1)(\infty) - F_i(t,f_2)(\infty)\right| = 0.
$$

Therefore, taking into account estimate (6.37), we conclude that

$$
\begin{aligned}
\left\|F_i(t,f_1) - F_i(t,f_2)\right\|_{\mathrm{w},p} &\le 3M_iC_1C_2\|\hat f_i\|_{\mathrm{w},p}\|f_1\|_{\mathrm{w},p}\|f_1-f_2\|_{\mathrm{w},p} \\
&+3L_iC_2\|\bar f_i\|_{\mathrm{w},p}\|f_1-f_2\|_{\mathrm{w},p}+3C_1C_2\|\bar f_i\|_{\mathrm{w},p}\|\hat f_i\|_{\mathrm{w},p}\|f_1-f_2\|_{\mathrm{w},p},
\end{aligned}
$$

which implies that F_i is locally Lipschitz. This completes the proof. ∎

Lemma 6.7. *Assume that all the assumptions of Theorem 6.3 are satisfied. Define a function* $G:[0,\infty)\times H^{1,p}_{\mathrm{w}} \to \gamma\big(\mathbb{R}^d, H^{1,p}_{\mathrm{w}}\big)$ *by*

$$
G(t,f)[z](x) = \sum_{i=1}^d g_i(t,x,f)z_i, \quad f\in H^{1,p}_{\mathrm{w}}, \quad z=(z_1,z_2,\dots,z_d)\in\mathbb{R}^d, \quad t,x\in[0,\infty).
$$

Then $G:[0,\infty)\times H^{1,p}_{\mathrm{w}} \ni (t,f)\mapsto \{\mathbb{R}^d\ni z\mapsto G(t,f)[z]\in H^{1,p}_{\mathrm{w}}\}\in\gamma\big(\mathbb{R}^d,H^{1,p}_{\mathrm{w}}\big)$ *is well-defined. Moreover, it is of linear growth and globally Lipschitz on* $H^{1,p}_{\mathrm{w}}$ *with Lipschitz constants independent of time* t.

Proof. In order to show that G is well-defined, we have to show that for each $t\ge 0$ and $f\in H^{1,p}_{\mathrm{w}}$, $G(t,f)\in\gamma\big(\mathbb{R}^d,H^{1,p}_{\mathrm{w}}\big)$. Let $\|\cdot\|$ be the norm on $\mathbb{R}^d$ generated the inner product of $\mathbb{R}^d$. Fix $t\ge0$ and $f\in H^{1,p}_{\mathrm{w}}$. Define a function $\kappa:[0,\infty)\to\mathbb{R}^d$ by

$$
\kappa(x) = \big(g_1(t,x,f), g_2(t,x,f),\dots,g_d(t,x,f)\big), \quad x\in[0,\infty).
$$

Then $G(t,f)$ can be written as follows

$$G(t,f)[z](x) = \langle \kappa(x), z\rangle, \quad z \in \mathbb{R}^d, \quad x \in [0,\infty),$$

where $\langle \cdot,\cdot\rangle$ is usual inner product of $\mathbb{R}^d$. In order to show that $G(t,f) \in \gamma\big(\mathbb{R}^d, H^{1,p}_{\mathrm{w}}\big)$, by Proposition 6.5, it is sufficient to show that $\kappa \in H^{1,p}_{\mathrm{w}}(\mathbb{R}^d)$. Since for each $i = 1,2,..,d$, the function $g_i(t,\cdot,f)$ is measurable, κ is measurable. By inequality (6.31), we obtain

$$\|\kappa(x)\| \le |g_1(t,x,f)| + |g_2(t,x,f)| + \ldots + |g_d(t,x,f)| \le d|\bar{f}_i(x)|.$$

Using Proposition 6.3 and the last inequality, we infer that for every $T > 0$,

$$\int_0^T \|\kappa(x)\| dx \le d\int_0^T |\bar{f}_i(x)| \le dTC_1\|\bar{f}_i\|_{\mathrm{w},p}.$$

Therefore, $\kappa \in L^1_{loc}(H)$. Let us now show that $D\kappa \in L^p_{\mathrm{w}}(H)$. Note that

$$D\kappa(x) = \big(D_x g_1(t,x,f), D_x g_2(t,x,f), \ldots, D_x g_d(t,x,f)\big), \quad x \in [0,\infty).$$

Then

$$\|D\kappa(x)\| \le |D_x g_1(t,x,f)| + |D_x g_2(t,x,f)| + \ldots + |D_x g_d(t,x,f)|, \quad x \in [0,\infty).$$

It follows from inequality (6.29) that

$$\|D\kappa(x)\| \le M|Df(x)|, \quad x \in [0,\infty),$$

where $M = M_1 + M_2 + \ldots + M_d$. Taking into account the last estimate, we obtain

$$\int_0^\infty \|D\kappa(x)\|^p \mathrm{w}(x) dx \le M^p \int_0^\infty |Df(x)|^p \mathrm{w}(x) dx \le M^p \|f\|^p_{\mathrm{w},p}. \tag{6.38}$$

Moreover, by (6.28), $\kappa(\infty) = 0$. Therefore $\kappa \in H^{1,p}_{\mathrm{w}}(\mathbb{R}^d)$ and so $G(t,f) \in \gamma(H, H^{1,p}_{\mathrm{w}})$. Thus, we have showed that G is well-defined. Moreover, by Proposition 6.5, we have, for a constant $N > 0$ independent κ,

$$\|G(t,f)\|_{\gamma(H,H^{1,p}_{\mathrm{w}})} \le N\|\kappa\|_{\mathrm{w},p}. \tag{6.39}$$

It follows from inequality (6.38) that there exists a constant $C > 0$ such that

$$\|G(t,f)\|_{\gamma(H,H^{1,p}_{\mathrm{w}})} \le C\|f\|_{\mathrm{w},p}, \tag{6.40}$$

which implies that G is of linear growth.

Finally, let us prove that G is locally Lipschitz. Fix $t \geq 0$ and $f_1, f_2 \in H^{1,p}_{\mathrm{w}}$. Define a function $\lambda : [0,\infty) \to \mathbb{R}^d$ by

$$\lambda(x) = \big(g_1(t,x,f_1) - g_1(t,x,f_2), \ldots, g_d(t,x,f_1) - g_d(t,x,f_2)\big), \quad x \in [0,\infty)$$

Therefore

$$G(t,f_1)[z](x) - G(t,f_2)[z](x) = \langle \lambda(x), z \rangle, \quad z \in \mathbb{R}^d, \quad x \in [0,\infty).$$

By inequality (6.30), we get

$$\begin{aligned}\|D\lambda(x)\| &\leq \big|D_x g_1(t,x,f_1) - D_x g_1(t,x,f_2)\big| + \ldots + \big|D_x g_d(t,x,f_1) - D_x g_d(t,x,f_2)\big| \\ &\leq L_1 |Df_1(x) - Df_2(x)| + \ldots + L_d |Df_1(x) - Df_2(x)|.\end{aligned}$$

Using the last estimate, we obtain

$$\int_0^\infty \|D\lambda(x)\|^p \mathrm{w}(x) dx \leq L^p \|f_1 - f_2\|^p_{\mathrm{w},p}, \tag{6.41}$$

where $L = L_1 + \ldots + L_d$. Moreover, it is obvious from (6.28) that $\lambda(\infty) = 0$. Therefore, using Proposition 6.5 and estimate (6.41), we conclude that for a constant $C > 0$,

$$\|G(t,f_1) - G(t,f_2)\|_{\gamma(H, H^{1,p}_{\mathrm{w}})} \leq C \|f_1 - f_2\|_{\mathrm{w},p}, \tag{6.42}$$

which implies that G is globally Lipschitz. ■

Proof of Theorem 6.3. The abstract form equation (6.33) in $H^{1,p}_{\mathrm{w}}$ is as follow

$$dr(t) = \big(Ar(t) + F(t, r(t))\big)dt + G(t, r(t))dW(t), \quad t \geq 0, \tag{6.43}$$

where F and G are functions defined in Lemma 6.6 and 6.7 respectively, and A is the infinitesimal generator of the shift semigroup on $H^{1,p}_{\mathrm{w}}$. Therefore, equation (6.33) is the form of equation (4.2). In Proposition 6.2, we showed that the space $H^{1,p}_{\mathrm{w}}$ is a separable Banach space satisfying the H-condition and in Lemma 6.2, we proved that the shift-semigroup on $H^{1,p}_{\mathrm{w}}$ is a contraction C_0-semigroup. Moreover, in lemma 6.6 and 6.7, we proved that F and G satisfies the conditions of Theorem 4.3. Therefore, for every $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; H^{1,p}_{\mathrm{w}})$, equation (6.43) has a unique $H^{1,p}_{\mathrm{w}}$-valued continuous mild solution with the initial value $r(0) = r_0$. This finishes the proof.

Remark 6.1. *As we mentioned in Chapter 5, one can prove the existence of invariant measures for the HJMM equation in the spaces $H^{1,p}_{\mathrm{w}}$, $p \geq 2$, using the results from Tehranchi [59] and Marinelli [40].*

CHAPTER 7

THE HJMM EQUATION IN THE FRACTIONAL SOBOLEV SPACES

In this chapter, we consider the fractional Sobolev spaces of 2π-periodic functions and prove some useful properties of them. Then using these properties, we apply the abstract results from Chapter 4 to prove the existence and uniqueness of a solution to the HJMM equation (driven by a standard d-dimensional Wiener process) in these spaces. Let us start introducing the fractional Sobolev spaces and some of their properties with proofs. Then we give the main result of this chapter with the proof.

7.1 Fractional Spaces

Let $p \geq 1$. Define L^p to be the space of all (equivalence classes of) Lebesgue measurable functions $f : [0, 2\pi] \to \mathbb{R}$ such that

$$\int_0^{2\pi} |f(x)|^p \, dx < \infty.$$

It is well-known that L^p is a separable Banach space with respect to the norm

$$\|f\|_p := \left(\int_0^{2\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad f \in L^p.$$

Proposition 7.1. *The space L^p is continuously embedded into the space L^1. In particular,*

$$\|f\|_1 \leq (2\pi)^{\frac{p-1}{p}} \|f\|_p, \quad f \in L^p.$$

Proof. Fix $f \in L^p$. Then by the Hölder inequality, we have

$$\|f\|_1 = \int_0^{2\pi} |f(x)|dx \le \left(\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^{2\pi} dx\right)^{\frac{p-1}{p}} = (2\pi)^{\frac{p-1}{p}} \|f\|_p,$$

which gives the desired conclusion. ■

Define L_0^p to be the space of all functions $f \in L^p$ such that

$$\int_0^{2\pi} f(x)dx = 0.$$

Since the space L_0^p is a closed subspace of L^p, it is a separable Banach space with the norm $\|\cdot\|_p$.

For each $\theta \in (0,1)$ and $p \ge 1$, define $W^{\theta,p}$ to be the space of all functions $f \in L^p$ such that

$$[f]_{\theta,p} := \left(\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\theta p}} dxdy\right)^{\frac{1}{p}} < \infty.$$

For each $\theta \in (0,1)$ and $p \ge 1$, $W^{\theta,p}$ is called **the fractional Sobolev space** and it is well-known that $W^{\theta,p}$ is a separable Banach space endowed with the norm

$$\|f\|_{\theta,p} = \|f\|_p + [f]_{\theta,p}, \quad f \in W^{\theta,p}.$$

Proposition 7.2. *[58] If $\theta > \frac{1}{p}$, then the space $W^{\theta,p}$ is continuously embedded into the space L^∞, i.e. for a constant $C > 0$,*

$$\|f\|_\infty \le C\|f\|_{\theta,p}, \quad f \in W^{\theta,p}. \tag{7.1}$$

Lemma 7.1. *If $\theta > \frac{1}{p}$, then a map $J : W^{\theta,p} \to W^{\theta,p}$ defined by*

$$Jf(x) = \int_0^x f(u)du, \quad f \in W^{\theta,p}, \quad x \in [0,2\pi]$$

is well-defined, linear and bounded.

Proof. Fix $f \in W^{\theta,p}$. Since f is measurable, Jf is measurable. Moreover, by the Hölder inequality, we get

$$\begin{aligned}
\int_0^{2\pi} |Jf(x)|^p dx &= \int_0^{2\pi} \left|\int_0^x f(u)du\right|^p dx \\
&\le \int_0^{2\pi} \left(\int_0^x |f(u)|^p du\right)\left(\int_0^x du\right)^{p-1} dx \\
&\le (2\pi)^{2p-1}\|f\|_p^p.
\end{aligned} \tag{7.2}$$

Hence $Jf \in L^p$. Furthermore, by Proposition 7.2, we obtain

$$\begin{aligned}
[Jf]^p_{\theta,p} &= \int_0^{2\pi}\int_0^{2\pi} \frac{\left|\int_0^x f(u)du - \int_0^y f(u)du\right|^p}{|x-y|^{1+\theta p}} dxdy \\
&= \int_0^{2\pi}\int_0^{2\pi} \frac{\left|\int_y^x f(u)du\right|^p}{|x-y|^{1+\theta p}} dxdy \\
&\le C^p \|f\|^p_{\theta,p} \int_0^{2\pi}\int_0^{2\pi} \frac{|x-y|^p}{|x-y|^{1+\theta p}} dxdy \\
&\le C^p \frac{(2\pi)^{p-\theta p+1}}{(p-\theta p)(p-\theta p+1)} \|f\|^p_{\theta,p}
\end{aligned} \tag{7.3}$$

.

Therefore $Jf \in W^{\theta,p}$ and thus, J is well-defined. The linearity of J is obvious. Taking into account estimates (7.2) and (7.3), we infer that there exists a constant $C_{\theta,p}$ depending on θ and p such that

$$\|Jf\|_{\theta,p} \le C_{\theta,p} \|f\|_{\theta,p}, \tag{7.4}$$

which implies that J is bounded. This completes the proof. ■

Lemma 7.2. *If $\theta \in (0,1)$ and $p \ge 1$, then*

$$\|fg\|_{\theta,p} \le 2\big[\|f\|_\infty \|g\|_{\theta,p} + \|f\|_{\theta,p}\|g\|_\infty\big], \quad f,g \in W^{\theta,p}.$$

In particular, if $\theta > \frac{1}{p}$, then $W^{\theta,p}$ is an algebra and for a constant $K > 0$,

$$\|fg\|_{\theta,p} \le K \|f\|_{\theta,p}\|g\|_{\theta,p}, \quad f \in W^{\theta,p}.$$

Proof. Fix $f,g \in W^{\theta,p}$. Then we have

$$\int_0^{2\pi} |f(x)g(x)|^p dx \le \|f\|^p_\infty \int_0^{2\pi} |g(x)|^p dx = \|f\|^p_\infty \|g\|^p_p. \tag{7.5}$$

Moreover, we have

$$
\begin{aligned}
[fg]^p_{\theta,p} &= \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)g(x)-f(y)g(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&= \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&= \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)(g(x)-g(y))+(f(x)-f(y))g(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&\le 2^{p-1}\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)|^p|g(x)-g(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&+ 2^{p-1}\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)-f(y)|^p|g(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&\le 2^{p-1}\|f\|^p_\infty\int_0^{2\pi}\int_0^{2\pi} \frac{|g(x)-g(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&+ 2^{p-1}\|g\|^p_\infty\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&\le 2^{p-1}\|f\|^p_\infty\|g\|^p_{\theta,p} + 2^{p-1}\|g\|^p_\infty\|f\|^p_{\theta,p}.
\end{aligned}
\tag{7.6}
$$

Taking into account estimates (7.5) and (7.6), we deduce that

$$
\|fg\|_{\theta,p} \le 2\big[\|f\|_\infty\|g\|_{\theta,p} + \|f\|_{\theta,p}\|g\|_\infty\big].
$$

In particular, if $\theta > \frac{1}{p}$, then by Proposition 7.2, there exists a constant $K > 0$ such that

$$
\|fg\|_{\theta,p} \le K\|f\|_{\theta,p}\|g\|_{\theta,p}.
$$

Thus, the proof is complete. ∎

Lemma 7.3. *If $\theta > \frac{1}{p}$, then a function $F : W^{\theta,p} \to W^{\theta,p}$ defined by*

$$
F(f) = fJf, \quad f \in W^{\theta,p},
$$

where J is a map defined in Lemma 7.1, is well-defined, quadratic growth and locally Lipschitz.

Proof. Fix $f \in W^{\theta,p}$. By Lemma 7.1 , $Jf \in W^{\theta,p}$. Thus by Lemma 7.2, $F(f) \in W^{\theta,p}$ and hence, F is well-defined. Moreover, by Lemmata 7.1 and 7.2, we infer that for a constant $M > 0$,

$$
\|F(f)\|_{\theta,p} \le M\|f\|^2_{\theta,p},
$$

which implies that F is quadratic growth.

Finally, Let us show that F is locally Lipschitz. Fix $f_1, f_2 \in W^{\theta,p}$. Then by the linearity of J, we have

$$\begin{aligned}
\|F(f_1)-F(f_2)\|_{\theta,p} &= \|f_1J(f_1)-f_2J(f_2)\|_{\theta,p}\\
&= \|f_1J(f_1)-f_1J(f_2)+f_1J(f_2)-f_2J(f_2)\|_{\theta,p}\\
&= \|f_1(J(f_1)-J(f_2))+(f_1-f_2)J(f_2)\|_{\theta,p}\\
&= \|f_1J(f_1-f_2)+(f_1-f_2)J(f_2)\|_{\theta,p}\\
&\le \|f_1J(f_1-f_2)\|_{\theta,p}+\|(f_1-f_2)J(f_2)\|_{\theta,p}.
\end{aligned}$$

It follows Lemmata 7.1 and 7.2 that

$$\begin{aligned}
\|F(f_1)-F(f_2)\|_{\theta,p} &\le K\|f_1\|_{\theta,p}\|J(f_1-f_2)\|_{\theta,p}+K\|f_1-f_2\|_{\theta,p}\|J(f_2)\|_{\theta,p}\\
&\le C_{\theta,p}K\|f_1\|_{\theta,p}\|f_1-f_2\|_{\theta,p}+C_{\theta,p}K\|f_1-f_2\|_{\theta,p}\|f_2\|_{\theta,p}\\
&\le C_{\theta,p}K\big(\|f_1\|_{\theta,p}+\|f_2\|_{\theta,p}\big)\|f_1-f_2\|_{\theta,p},
\end{aligned}$$

which implies that F is locally Lipschitz. Thus, the proof is complete. ■

Define $W^{\theta,p}_{0,per}$ to be the space of all functions $f \in W^{\theta,p} \cap L^p_0$ such that f is 2π-periodic i.e. $f(0) = f(2\pi)$. Since the space $W^{\theta,p}_{0,per}$ is a closed subspace of $W^{\theta,p}$, it is a separable Banach space with the norm $\|\cdot\|_{\theta,p}$.

Remark 7.1. *It is obvious that for each $f \in W^{\theta,p}_{0,per}$, there exists a unique, continuous, 2π-periodic function $\tilde{f} : [0,\infty) \to \mathbb{R}$, i.e. $\tilde{f} \in C_{per}$ such that $\tilde{f}\big|_{[0,2\pi]} = f$.*

Lemma 7.4. *A family $S = \{S(t)\}_{t\ge 0}$ of operators defined by*

$$S(t)f(x) = \tilde{f}(t+x), \quad f \in W^{\theta,p}_{0,per}, \quad x \in [0,2\pi], \quad t \ge 0,$$

where $\tilde{f}$ is a function introduced in the previous Remark, is a contraction C_0 semigroup on $W^{\theta,p}_{0,per}$.

Lemma 7.5. *[23] A set defined by*

$$C^\infty_{0,per} := \left\{ f = \tilde{f}\big|_{[0,2\pi]} \;:\; \tilde{f} \in C^\infty(\mathbb{R}), \textit{periodic and } \int_0^{2\pi} \tilde{f}(x)dx = 0 \right\}.$$

is a dense subspace of the space $W^{\theta,p}_{0,per}$.

Lemma 7.6. *The semigroup S defined in Lemma 7.4 is strongly continuous on $C^\infty_{0,per}$ in the norm of $W^{\theta,p}_{0,per}$.*

Proof. Fix $f \in C^{\infty}_{0,per}$. Then there exists $\tilde{f} \in C^{\infty}(\mathbb{R})$. Thus, $\tilde{f}$ (and $D\tilde{f}$) is uniformly continuous, i.e. for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in [0,\infty)$ with $|x_1 - x_2| \le \delta$,

$$|\tilde{f}(x_1) - \tilde{f}(x_2)| \le \varepsilon, \quad (and\ |D\tilde{f}(x_1) - D\tilde{f}(x_2)| \le \varepsilon).$$

Choose $x_1 = x + t$ and $x_2 = x$ and so $|x_1 - x_2| = t \le \delta$. Then by the uniformly continuity of $\tilde{f}$, we get

$$\|S(t)f - f\|_p^p = \int_0^{2\pi} |\tilde{f}(x+t) - \tilde{f}(x)|^p dx = \int_0^{2\pi} |\tilde{f}(x_1) - \tilde{f}(x_2)|^p dx \le 2\pi\varepsilon^p. \tag{7.7}$$

Similarly, if we choose $x_1 = u + t$ and $x_2 = u$, then by the uniformly continuity of $D\tilde{f}$, we obtain

$$\begin{aligned} |\tilde{f}(x+t) - \tilde{f}(x) - \tilde{f}(y+t) + \tilde{f}(y)| &= \left| \int_y^x \big(D\tilde{f}(u+t) - D\tilde{f}(u)\big) du \right| \\ &\le \int_y^x \big|D\tilde{f}(u+t) - D\tilde{f}(u)\big| du \\ &= \int_y^x \big|D\tilde{f}(x_1) - D\tilde{f}(x_2)\big| du \\ &\le \varepsilon \int_x^y du = \varepsilon |x - y|. \end{aligned} \tag{7.8}$$

It follows from inequality (7.8) that

$$\begin{aligned} [S(t)f - f]_{\theta,p}^p &= \int_0^{2\pi}\int_0^{2\pi} \frac{|\tilde{f}(x+t) - \tilde{f}(x) - \tilde{f}(y+t) + \tilde{f}(y)|^p}{|x-y|^{1+\theta p}} dx dy \\ &\le \varepsilon^p \int_0^{2\pi}\int_0^{2\pi} |x-y|^{p-\theta p - 1} dx dy = \frac{\varepsilon^p (2\pi)^{p-\theta p+1}}{(p-\theta p)(p - \theta p + 1)}. \end{aligned} \tag{7.9}$$

Taking into account inequality (7.7) and (7.9), we infer that there exists a constant $C > 0$ depending on θ, p such that

$$\|S(t)f - f\|_{\theta,p} \le C\varepsilon.$$

If we choose $\varepsilon = \frac{\tilde{\varepsilon}}{C}$, $\tilde{\varepsilon} > 0$, then we obtain

$$\|S(t)f - f\|_{\theta,p} \le \tilde{\varepsilon}. \tag{7.10}$$

Thus, we have proved that for each $\tilde{\varepsilon} > 0$, there exists $\delta > 0$ such that for all $t \le \delta$, (7.10) holds. Thus, S is strongly continuous on $C^{\infty}_{0,per}$. ■

Proof of Lemma 7.4: We first show that for each $t \geq 0$, $S(t)$ is a well-defined, linear and bounded operator from $W^{\theta,p}_{0,per}$ into itself. For this aim, fix $t \geq 0$ and $f \in W^{\theta,p}_{0,per}$. Since $\tilde{f}$ is Lebesgue measurable, for each Lebesgue measurable set A,

$$\tilde{f}^{-1}(A) = \{y \in \mathbb{R} : \tilde{f}(y) \in A\} = E \in \mathscr{B}([0,\infty)).$$

Thus, we get

$$(S(t)f)^{-1}(A) = \{x \in \mathbb{R} : \tilde{f}(t+x) \in A\} = \{y - t : y \in E\} = E - t.$$

We know from [51] that if E is a Lebesgue measurable set, then the set $E - t$ is also Lebesgue measurable. Therefore $(S(t)f)^{-1}(A) \in \mathscr{B}([0,\infty))$ and thus, $S(t)f$ is measurable. Moreover, since $\tilde{f}$ is 2π-periodic, we obtain

$$\begin{aligned}
\int_0^{2\pi} |S(t)f(x)|^p dx &= \int_0^{2\pi} \left|\tilde{f}(x+t)\right|^p dx = \int_t^{2\pi+t} \left|\tilde{f}(u)\right|^p du \\
&= \int_t^{2\pi} \left|\tilde{f}(u)\right|^p du + \int_{2\pi}^{2\pi+t} \left|\tilde{f}(u)\right|^p du \\
&= \int_t^{2\pi} \left|\tilde{f}(u)\right|^p du + \int_0^{t} \left|\tilde{f}(v+2\pi)\right|^p dv \\
&= \int_t^{2\pi} \left|\tilde{f}(u)\right|^p du + \int_0^{t} \left|\tilde{f}(v)\right|^p dv \\
&= \int_0^{2\pi} \left|\tilde{f}(u)\right|^p du = \int_0^{2\pi} |f(u)|^p du = \|f\|_p^p.
\end{aligned} \tag{7.11}$$

Therefore $S(t)f \in L^p$. By the change of variable, put $u = x+t$ and $v = y+t$, we get

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi} \frac{|\tilde{f}(x+t) - \tilde{f}(y+t)|^p}{|x-y|^{1+\theta p}} dxdy &= \int_t^{2\pi+t}\int_t^{2\pi+t} \frac{|\tilde{f}(u) - \tilde{f}(v)|^p}{|u-v|^{1+\theta p}} dudv \\
\int_t^{2\pi}\int_t^{2\pi} \frac{|\tilde{f}(u) - \tilde{f}(v)|^p}{|u-v|^{1+\theta p}} dudv &+ \int_{2\pi}^{2\pi+t}\int_{2\pi}^{2\pi+t} \frac{|\tilde{f}(u) - \tilde{f}(v)|^p}{|u-v|^{1+\theta p}} dudv.
\end{aligned}$$

Again by the change of variable, put $u = x+2\pi$ and $v = y+2\pi$, we obtain

$$\begin{aligned}
\int_{2\pi}^{2\pi+t}\int_{2\pi}^{t+2\pi} \frac{|\tilde{f}(u) - \tilde{f}(v)|^p}{|u-v|^{1+\theta p}} dudv &= \int_0^t\int_0^t \frac{|\tilde{f}(x+2\pi) - \tilde{f}(y+2\pi)|^p}{|x-y|^{1+\theta p}} dxdy \\
&= \int_0^t\int_0^t \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x-y|^{1+\theta p}} dxdy.
\end{aligned}$$

Taking into account the last inequality, we infer that

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi} \frac{|\tilde{f}(x+t) - \tilde{f}(y+t)|^p}{|x-y|^{1+\theta p}} dxdy &= \int_0^{2\pi}\int_0^{2\pi} \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x-y|^{1+\theta p}} dxdy \\
&= \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x) - f(y)|^p}{|x-y|^{1+\theta p}} dxdy = [f]^p_{\theta,p}.
\end{aligned} \tag{7.12}$$

Hence $S(t)f \in W^{\theta,p}$. Let us now show that $S(t)f$ is 2π-periodic and

$$\int_0^{2\pi} S(t)f(x) = 0.$$

Since $\tilde{f}$ is 2π-periodic, we get

$$S(t)f(0) = \tilde{f}(t) = \tilde{f}(t+2\pi) = S(t)f(2\pi)$$

and thus, $S(t)f$ is 2π-periodic. Since $\int_0^{2\pi} f(x)dx = 0$ and $\tilde{f}$ is 2π-periodic, we have

$$\begin{aligned}
\int_0^{2\pi} S(t)f(x)dx &= \int_0^{2\pi} \tilde{f}(t+x)dx = \int_t^{t+2\pi} \tilde{f}(u)du \\
&= \int_t^{2\pi} \tilde{f}(u)du + \int_{2\pi}^{t+2\pi} \tilde{f}(u)du \\
&= \int_t^{2\pi} \tilde{f}(u)du + \int_0^{t} \tilde{f}(x+2\pi)dx \\
&= \int_t^{2\pi} \tilde{f}(u)du + \int_0^{t} \tilde{f}(x)dx \\
&= \int_0^{2\pi} \tilde{f}(x)dx = \int_0^{2\pi} f(x)dx = 0.
\end{aligned}$$

Therefore, we have showed that $S(t)f \in W^{\theta,p}_{0,per}$ and hence, $S(t)$ is well-defined. The linearity of $S(t)$ is obvious and the boundedness of $S(t)$ follows from estimates (7.11) and (7.12).

We now show that $S = \{S(t)\}_{t\geq 0}$ is a contraction C_0-semigroup on $W^{\theta,p}_{0,per}$. It is clear that S is a semigroup on $W^{\theta,p}_{0,per}$. Let us show that S is strongly continuous on $W^{\theta,p}_{0,per}$. Fix $f \in W^{\theta,p}_{0,per}$. Since $C^{\infty}_{0,per}$ is dense in $W^{\theta,p}_{0,per}$, for every $\frac{\varepsilon}{3} > 0$, there exists $g \in C^{\infty}_{0,per}$ such that

$$\|f-g\|_{\theta,p} \leq \frac{\varepsilon}{3}. \tag{7.13}$$

Moreover, by the strong continuity of S on $C^{\infty}_{0,per}$, we have

$$\|S(t)g-g\|_{\theta,p} \leq \frac{\varepsilon}{3}. \tag{7.14}$$

Taking into account last two inequalities, we infer that

$$\begin{aligned}
\|S(t)f-f\|_{\theta,p} &= \|S(t)f+g-g+S(t)g-S(t)g-f\|_{\theta,p} \\
&\leq \|S(t)f-S(t)g\|_{\theta,p} + \|S(t)g-g\|_{\theta,p} + \|f-g\|_{\theta,p} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which implies that S is strongly continuous on $W^{\theta,p}_{0,per}$. Moreover, it follows from estimates (7.11) and (7.12) that

$$\|S(t)\|_{\mathscr{L}(W^{1,p}_{per})} \le 1, \quad t \ge 0.$$

Therefore, S is a contraction C_0 semigroup on $W^{\theta,p}_{0,per}$. The proof is complete.

Lemma 7.7. *The infinitesimal generator of S is characterised by*

$$\mathscr{D}(A) = \left\{ f \in W^{\theta,p}_{0,per} : Df \in W^{\theta,p}_{0,per} \right\}, \tag{7.15}$$

and

$$Af = Df, \quad f \in \mathscr{D}(A). \tag{7.16}$$

The proof is similar to the proof of Lemma 5.8.

Lemma 7.8. *If $p \ge 2$, then the space $W^{\theta,p}_{0,per}$ satisfies the H-condition.*

Proof. We know from [12] that for each $p \ge 2$, the space $W^{\theta,p}$ satisfies the H-condition, i.e. the map $\psi : W^{\theta,p} \ni f \mapsto \psi(f) = \|f\|^p_{\theta,p} \in \mathbb{R}$ is of C^2 class (in the Fréchet derivate sense) and

$$\|\psi'(f)\| \le p\|f\|^{p-1}_{\theta,p}, \quad f \in W^{\theta,p}, \tag{7.17}$$

and

$$\|\psi''(f)\| \le p(p-1)\|f\|^{p-2}_{\theta,p}, \quad f \in W^{\theta,p}, \tag{7.18}$$

where $\psi'(f)$ and $\psi''(f)$ are the first and second Fréchet derivatives of ψ at $f \in W^{\theta,p}$. We need to show that the map $\varphi : W^{\theta,p}_{0,per} \ni f \mapsto \varphi(f) = \|f\|^p_{\theta,p} \in \mathbb{R}$ is of C^2 class and

$$\|\varphi'(f)\| \le p\|f\|^{p-1}_{\theta,p}, \quad f \in W^{\theta,p}_{0,per},$$

and

$$\|\varphi''(f)\| \le p(p-1)\|f\|^{p-2}_{\theta,p}, \quad f \in W^{\theta,p}_{0,per}.$$

Note that we can write the map φ as $\psi \circ i$, where $i : W^{\theta,p}_{0,per} \hookrightarrow W^{\theta,p}$ is the inclusion map. Since i and ψ are of C^2 class, φ is of C^2 class, see [19]. Moreover, since $i'(f) = i$ and $\|i\| \le 1$, we get

$$\|\varphi'(f)\| = \|i'(f)\psi'(i(f))\| \le \|i\|\|\psi'(f)\| \le \|\psi'(f)\|.$$

It follows from (7.17) that

$$\|\varphi'(f)\| \le p\|f\|_{\theta,p}^{p-1}.$$

Similarly, we can show that

$$\|\varphi''(f)\| \le p(p-1)\|f\|_{\theta,p}^{p-2}.$$

This completes the proof. ■

Let H be a separable Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle_H$. Define $W^{\theta,p}(H)$ to be the space of all functions $f \in L^p(H)$ such that

$$[f]_{W^{\theta,p}(H)} := \left(\int_0^{2\pi}\int_0^{2\pi} \frac{\|f(x)-f(y)\|_H^p}{|x-y|^{1+\theta p}} dxdy\right)^{\frac{1}{p}} < \infty.$$

Moreover, define $W^{\theta,p}_{0,per}(H)$ to be the space of all function $f \in W^{\theta,p}(H)$ such that f is 2π-periodic and

$$\int_0^{2\pi} f(x)dx = 0.$$

All results of the spaces $W^{\theta,p}$ and $W^{\theta,p}_{0,per}$ can be generalized to the spaces $W^{\theta,p}(H)$ and $W^{\theta,p}_{0,per}(H)$. Therefore, $W^{\theta,p}(H)$ is a separable Banach space with respect to the norm $\|\cdot\|_{W^{\theta,p}(H)} = \|\cdot\|_{L^p(H)} + [\cdot]_{W^{\theta,p}(H)}$. Furthermore, the space $W^{\theta,p}_{0,per}(H)$ is a closed subspace of $W^{\theta,p}(H)$ and thus, $W^{\theta,p}_{0,per}(H)$ is a separable Banach space with respect to the same norm $\|\cdot\|_{W^{\theta,p}(H)}$. The following proposition gives a sufficient condition under which a $W^{\theta,p}_{0,per}$-valued operator K is γ-radonifying, see [12] for the proof.

Proposition 7.3. *If $p \ge 2$, then for every $\kappa \in W^{\theta,p}_{0,per}(H)$, a linear operator $K : H \to W^{\theta,p}_{0,per}$ defined by*

$$K[h](x) = \langle\kappa(x), h\rangle_H, \quad h \in H, \quad x \in [0,\infty), \tag{7.19}$$

is γ-radonifying and for a constant $N > 0$ independent of κ,

$$\|K\|_{\gamma(H,W^{\theta,p}_{0,per})} \le N\|\kappa\|_{\theta,p}. \tag{7.20}$$

7.2 Existence and Uniqueness of Solution to the HJMM Equation in the Fractional Sobolev Spaces

In this section, we prove the existence and uniqueness of a mild solution to the HJMM equation (driven by a standard d-dimensional Wiener process) in the fractional Sobolev spaces defined in the previous section. Therefore, the main result of this chapter is the following theorem.

Theorem 7.1. *Assume that $\theta > \frac{1}{p}$. For each $i = 1,2,3...d$, let $g_i : [0,\infty) \times [0,2\pi] \times W^{\theta,p} \to \mathbb{R}$ be a measurable and 2π-periodic function with respect to the second variable such that for every $t \geq 0$ and $f \in W^{\theta,p}$,*

$$\int_0^{2\pi} g_i(t,x,f)dx = 0. \tag{7.21}$$

Also let g_i be linear with respect to the third variable, i.e. for every $f_1, f_2 \in W^{\theta,p}$ and $\alpha, \beta \in \mathbb{R}$,

$$g_i(t,x,\alpha f_1 + \beta f_2) = \alpha g_i(t,x,f_1) + \beta g_i(t,x,f_2).$$

Moreover, there exist constants $C_i, M_i > 0$ such that for every $t \geq 0$,

$$|g_i(t,x,f)| \leq C_i |f(x)|, \quad f \in W^{\theta,p}, \quad x \in [0,2\pi] \tag{7.22}$$

and

$$|g_i(t,x,f) - g_i(t,y,f)| \leq M_i |f(x) - f(y)|, \quad f \in W^{\theta,p}, \quad x,y \in [0,2\pi]. \tag{7.23}$$

Then for each $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; W^{\theta,p}_{0,per})$, the following stochastic differential equation

$$\begin{aligned} dr(t)(x) &= \left(\frac{\partial}{\partial x} r(t)(x) + \sum_{i=1}^{d} g_i(t,x,r(t)) \int_0^x g_i(t,y,r(t))dy \right) dt \\ &\quad + \sum_{i=1}^{d} g_i(t,x,r(t)) dW_i(t), \quad t,x \geq 0, \end{aligned} \tag{7.24}$$

where $W(t) = (W_1(t), W_2(t), ..., W_d(t))$ is a d-dimensional standard Wiener process, has a unique $W^{\theta,p}_{0,per}$-valued continuous mild solution.

Lemma 7.9. *Assume that all the assumptions of Theorem 7.1 are satisfied. Then a function* $J_i : [0,\infty) \times W^{\theta,p} \to W^{\theta,p}$ *defined by*

$$J_i(t,f)(x) = \int_0^x g_i(t,y,f)dy, \quad f \in W^{\theta,p}, \quad t \geq 0, \quad x \in [0,2\pi].$$

is well-defined, linear with respect to the second variable and of linear growth.

Proof. Fix $i \in \{1,2,..d\}$. First we will show that J_i is well-defined. Fix $t \geq 0$ and $f \in W^{\theta,p}$. Since we assume that $g_i(t,\cdot,f)$ is locally integrable, $J_i(t,f)$ is measurable. By Proposition 7.1 and inequality (7.22), we have, for all $x \in [0,2\pi]$,

$$\left|J_i(t,f)(x)\right| \leq \int_0^{2\pi} |g_i(t,y,f)|dy \leq C_i \int_0^{2\pi} |f(y)|dy \leq C_i(2\pi)^{\frac{p-1}{p}} \|f\|_p.$$

Using the last inequality, we obtain

$$\int_0^{2\pi} \left|J_i(t,f)(x)\right|^p dx \leq C_i^p (2\pi)^{p-1} \|f\|_p^p \int_0^{2\pi} dx = C_i^p (2\pi)^p \|f\|_p^p. \tag{7.25}$$

Thus $J_i(t,f) \in L^p$, Also by Proposition 7.2 and inequality (7.22), we get, for all $x, y \in [0,2\pi]$,

$$\left|J_i(t,f)(x) - J_i(t,f)(y)\right| \leq \int_y^x |g_i(t,u,f)|du \leq C_i \int_y^x |f(u)|du \leq C_i C \|f\|_{\theta,p} |x-y|.$$

By the last inequality, we deduce that

$$\begin{aligned} [J_i(t,f)]_{\theta,p}^p &= \int_0^{2\pi}\int_0^{2\pi} \frac{\left|J_i(t,f)(x) - J_i(t,f)(x)\right|^p}{|x-y|^{1+\theta p}} dxdy \\ &\leq C_i^p C^p \|f\|_{\theta,p}^p \int_0^{2\pi}\int_0^{2\pi} \frac{\|x-y|^p}{|x-y|^{1+\theta p}} dxdy \\ &= C_i^p C^p \frac{(2\pi)^{p-\theta p+1}}{(p-\theta p)(p-\theta p+1)} \|f\|_{\theta,p}^p. \end{aligned} \tag{7.26}$$

Hence $J_i(t,f) \in W^{\theta,p}$ and thus, J_i is well-defined. Since g_i is linear with respect to the third variable, J_i is linear. Taking into account estimates (7.25) and (7.26), we infer that there exists a constant K depending on θ, p such that

$$\|J_i(t,f)\|_{\theta,p} \leq K \|f\|_{\theta,p}. \tag{7.27}$$

Thus, J_i is of linear growth. The proof is complete. ∎

Lemma 7.10. *Assume that all the assumptions of Theorem 7.1 are satisfied. Define a function* $F:[0,\infty)\times W^{\theta,p}_{0,per}\to W^{\theta,p}_{0,per}$ *by*

$$F(t,f)(x)=\sum_{i=1}^{d} g_i(t,x,f)\int_0^x g_i(t,y,f)dy,\quad f\in W^{\theta,p}_{0,per}\quad t\geq 0,\quad x\in[0,2\pi].$$

Then F is well-defined. Moreover, it is quadratic growth and Lipschitz on balls with Lipschitz constant independent of time t.

Proof. For each $i=1,2,..d$, define a function $F_i:[0,\infty)\times W^{\theta,p}_{0,per}\to W^{\theta,p}_{0,per}$ by

$$F_i(t,f)=g_i(t,\cdot,f)J_i(t,f),\quad f\in W^{\theta,p}_{0,per},\quad t\geq 0,$$

where J_i is a map defined in the previous Lemma. Then we can rewrite F as follow

$$F(t,f)=\sum_{i=1}^{d}F_i(t,f),\quad f\in W^{\theta,p}_{0,per},\quad t\geq 0.$$

Therefore, it is sufficient to show that for each $i=1,2,..d$, F_i is well-defined map, quadratic growth and Lipschitz on balls with Lipschitz constant independent of time t.

Fix $i\in\{1,2,..,d\}$, $t\geq 0$ and $f\in W^{\theta,p}_{0,per}$. Since $g_i(t,\cdot,f)$ is measurable and by inequality (7.22),

$$\int_0^{2\pi}|g_i(t,x,f)|^p dx\leq C_i^p\int_0^{2\pi}|f(x)|^p dx=C_i^p\|f\|_p^p, \tag{7.28}$$

$g_i(t,\cdot,f)\in L^p$. Moreover, using inequality (7.23), we obtain

$$\begin{aligned}[g_i(t,f)]^p_{\theta,p}&=\int_0^{2\pi}\int_0^{2\pi}\frac{|g_i(t,f)(x)-g_i(t,f)(y)|^p}{|x-y|^{1+\theta p}}dxdy\\ &\leq M_i^p\int_0^{2\pi}\int_0^{2\pi}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\theta p}}dxdy=M_i^p[f]^p_{\theta,p}.\end{aligned} \tag{7.29}$$

Thus $g_i(t,\cdot,f)\in W^{\theta,p}$. Furthermore, it follows from estimates (7.28) and (7.29) that there exists a constant $L_i>0$ such that

$$\|g_i(t,\cdot,f)\|_{\theta,p}\leq L_i\|f\|_{\theta,p}. \tag{7.30}$$

We know from Lemma 7.9 that $J_i(t,f)\in W^{\theta,p}$. Therefore, by Lemma 7.2, $F_i(t,f)\in W^{\theta,p}$. Let us now show that $F_i(t,f)$ is 2π-periodic and

$$\int_0^{2\pi}F_i(t,f)(y)dy=0. \tag{7.31}$$

Since $F_i(t,f)(0)=0$ and by (7.21),

$$F_i(t,f)(2\pi)=g_i(t,2\pi,f)\int_0^{2\pi}g_i(t,y,f)dy=0,$$

$F_i(t,f)$ is 2π-periodic. Note that

$$\frac{\partial}{\partial x}[J_i(t,f)](x)=g_i(t,x,f),\quad x\in[0,2\pi].$$

Therefore, we can rewrite $F_i(t,f)$ as

$$F_i(t,f)=J_i(t,f)\frac{\partial}{\partial x}[J_i(t,f)](x).$$

Since $J_i(t,f)(0)=J_i(t,f)(2\pi)=0$, we obtain

$$\begin{aligned}\int_0^{2\pi}F_i(t,f)(x)dx&=\int_0^{2\pi}J_i(t,f)(x)\frac{\partial}{\partial x}[J_i(t,f)](x)dx\\&=\frac{1}{2}\int_0^{2\pi}\frac{\partial}{\partial x}\left[J_i^2(t,f)\right](x)dx\\&=\frac{1}{2}\left[J_i^2(t,f)(2\pi)-J_i^2(t,f)(0)\right]=0,\end{aligned}$$

which gives the desired reesult (7.31). Therefore, we have showed $F_i(t,f)\in W^{\theta,p}_{0,per}$ and thus, $F_i(t,\cdot)$ is well-defined. Moreover, taking into account Lemma 7.2 and estimates (7.27) and (7.30), we infer that

$$\|F_i(t,f)\|_{\theta,p}\le K_{\theta,p}L_iK\|f\|^2_{\theta,p}, \tag{7.32}$$

which implies that $F_i(t,\cdot)$ is quadratic growth.

Finally, we will show that F_i is locally Lipschitz. Fix $t\ge0$ and $f_1,f_2\in W^{\theta,p}_{0,per}$. Then by the linearity of g_i, we obtain

$$\begin{aligned}&\|F_i(t,f_1)-F_i(t,f_2)\|_{\theta,p}=\|g_i(t,\cdot,f_1)J(t,f_1)-g_i(t,\cdot,f_2)J(t,f_2)\|_{\theta,p}\\&=\|g_i(t,\cdot,f_1)J(t,f_1)-g_i(t,\cdot,f_1)J(t,f_2)+g_i(t,\cdot,f_1)J(t,f_2)-g_i(t,\cdot,f_2)J(t,f_2)\|_{\theta,p}\\&=\|g_i(t,\cdot,f_1)(J(t,f_1)-J(t,f_2))+(g_i(t,\cdot,f_1)-g_i(t,\cdot,f_2))J(t,f_2)\|_{\theta,p}\\&=\|g_i(t,\cdot,f_1)J(t,f_1-f_2)+g_i(t,\cdot,f_1-f_2)J(t,f_2)\|_{\theta,p}\\&\le\|g_i(t,\cdot,f_1)J(t,f_1-f_2)\|_{\theta,p}+\|g_i(t,\cdot,f_1-f_2)J(t,f_2)\|_{\theta,p}.\end{aligned}$$

Using Lemma 7.2 and estimates (7.27) and (7.30), we deduce that

$$\begin{aligned}\|F_i(t,f_1)-F_i(t,f_2)\|_{\theta,p} &\le K_{\theta,p}\|g_i(t,\cdot,f_1)\|_{\theta,p}\|J(t,f_1-f_2)\|_{\theta,p}\\ &\quad+K_{\theta,p}\|g_i(t,\cdot,f_1-f_2)\|_{\theta,p}\|J(t,f_2)\|_{\theta,p}\\ &\le K_{\theta,p}KL_i\|f_1\|_{\theta,p}\|f_1-f_2\|_{\theta,p}\\ &\quad+K_{\theta,p}KL_i\|f_2\|_{\theta,p}\|f_1-f_2\|_{\theta,p}\\ &=K_{\theta,p}KL_i\big(\|f_1\|_{\theta,p}+\|f_2\|_{\theta,p}\big)\|f_1-f_2\|_{\theta,p},\end{aligned}$$

which implies that F_i is locally Lipschitz. ∎

Lemma 7.11. *Assume that all the assumptions of Theorem 7.1 hold. Define a function* $G:[0,\infty)\times W^{\theta,p}_{0,per}\to\gamma\big(\mathbb{R}^d,W^{\theta,p}_{0,per}\big)$ *by*

$$G(t,f)[z](x)=\sum_{i=1}^{d}g_i(t,f)(x)z_i,\quad f\in W^{\theta,p}_{0,per},\quad z=(z_1,z_2,\dots,z_d)\in\mathbb{R}^d,\quad t\ge0,\quad x\in[0,2\pi].$$

Then $G:[0,\infty)\times W^{\theta,p}_{0,per}\ni(t,f)\mapsto\{\mathbb{R}^d\ni z\mapsto G(t,f)[z]\in W^{\theta,p}_{0,per}\}\in\gamma\big(\mathbb{R}^d,W^{\theta,p}_{0,per}\big)$ *is well-defined. Moreover, it is of linear growth and globally Lipschitz on* $W^{\theta,p}_{0,per}$ *with Lipschitz constant independent of time* t.

Proof. Let $\|\cdot\|$ be the norm on $\mathbb{R}^d$ generated by the usual inner product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^d$. Fix $t\ge0$ and $f\in W^{\theta,p}_{0,per}$. Define a function $\kappa:[0,\infty)\to\mathbb{R}^d$ by

$$\kappa(x)=\big(g_1(t,x,f),g_2(t,x,f),\dots,g_d(t,x,f)\big),\quad x\in[0,2\pi].$$

Then $G(t,f)$ can be written as follow

$$G(t,f)[z](x)=\langle\kappa(x),z\rangle,\quad z\in\mathbb{R}^d,\quad x\in[0,2\pi].$$

In order to show $G(t,f)\in\gamma\big(\mathbb{R}^d,W^{\theta,p}_{0,per}\big)$, by Proposition 7.3, it is sufficient to show $\kappa\in W^{\theta,p}_{0,per}(\mathbb{R}^d)$. Let us first show $\kappa\in W^{\theta,p}(\mathbb{R}^d)$. Since $g_i(t,\cdot,f)$ is measurable for each $i=1,2,..d$, κ is measurable. Moreover, using inequality (7.22), we get

$$\|\kappa(x)\|\le|g_1(t,x,f)|+|g_2(t,x,f)|+\dots+|g_d(t,x,f)|\le C|f(x)|,\quad x\in[0,2\pi],$$

where $C=C_1+\dots+C_d$. By the last inequality, we infer that

$$\int_0^{2\pi}\|\kappa(x)\|^p\,dx\le C^p\int_0^{2\pi}|f(x)|^p\,dx=C^p\|f\|_p^p. \tag{7.33}$$

Thus $\kappa \in L^p(\mathbb{R}^d)$. Furthermore, by inequality (7.23), we have

$$\begin{aligned}\|\kappa(x)-\kappa(y)\| &\le |g_1(t,x,f)-g_1(t,y,f)|+\ldots+|g_d(t,x,f)-g_d(t,y,f)|\\ &\le M_1|f(x)-f(y)|+\ldots+M_d|f(x)-f(y)|\\ &= M|f(x)-f(y)|, \quad x,y\in[0,2\pi],\end{aligned}$$

where $M = M_1+\ldots+M_d$. Using the last estimate, we obtain

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\|\kappa(x)-\kappa(y)\|^p}{|x-y|^{1+\theta p}}dxdy \le M^p\int_0^{2\pi}\int_0^{2\pi}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\theta p}}dxdy = M^p[f]_p^p. \tag{7.34}$$

Thus $\kappa \in W^{\theta,p}(\mathbb{R}^d)$. Since for each $i=1,2,..d$, $g_i(t,\cdot,f)$ is 2π-periodic, κ is 2π-periodic. Moreover, since for each $i=1,2,..d$, $\int_0^{2\pi} g_i(t,x,f)dx = 0$, we have

$$\int_0^{2\pi}\kappa(x)dx = 0.$$

Therefore, we have showed $\kappa \in W^{\theta,p}_{0,per}(\mathbb{R}^d)$ and so $G(t,f)\in\gamma\big(\mathbb{R}^d, W^{\theta,p}_{0,per}\big)$. Thus, G is well-defined. Moreover, by Proposition 7.3 and estimates (7.33) and (7.34), we conclude that there exists a constant $K>0$ such that

$$\|G(t,f)\|_{\gamma(\mathbb{R}^d,W^{\theta,p}_{per})} \le K\|f\|_{\theta,p}, \tag{7.35}$$

which implies that $G(t,\cdot)$ is of linear growth.

Last we show that G is globably Lipschitz on $W^{\theta,p}_{0,per}$. Fix $t\ge 0$ and $f_1,f_2\in W^{\theta,p}_{per}$. Define a function $\lambda:[0,\infty)\to\mathbb{R}^d$ by

$$\lambda(x) = \big(g_1(t,x,f_1)-g_1(t,x,f_2),\ldots,g_d(t,x,f_1)-g_d(t,x,f_2)\big), \quad x\in[0,2\pi].$$

Then

$$\big(G(t,f_1)-G(t,f_2)\big)[z](x) = \langle\lambda(x),z\rangle, \quad z\in\mathbb{R}^d, \quad x\in[0,2\pi].$$

By linearity of g_i and inequality (7.23), we get

$$\begin{aligned}\|\lambda(x)\| &\le |g_1(t,x,f_1)-g_1(t,x,f_2)|+\ldots+|\zeta_d(t,x,f_1)-\zeta_d(t,x,f_2)|\\ &= |g_1(t,x,f_1-f_2)|+\ldots+|g_d(t,x,f_1-f_2)|\\ &\le M_1|f_1(x)-f_2(x)|+\ldots+M_d|f_1(x)-f_2(x)|\\ &= M|f_1(x)-f_2(x)|, \quad x\in[0,2\pi],\end{aligned}$$

where $M = M_1 + ... + M_d$. From the last inequality, we obtain

$$\int_0^{2\pi} \|\lambda(x)\|^p dx \le M^p \int_0^{2\pi} |f_1(x) - f_2(x)|^p dx = M^p \|f_1 - f_2\|_p^p. \tag{7.36}$$

By linearity of $g_i(t,\cdot,f)$, we have

$$\begin{aligned}\lambda(x) - \lambda(y) = \big(&g_1(t,x,f_1-f_2) - g_1(t,y,f_1-f_2), ... \\ &+ g_d(t,x,f_1-f_2) - g_d(t,y,f_1-f_2)\big), \quad x,y \in [0,2\pi].\end{aligned}$$

It follows from inequality (7.23) that

$$\begin{aligned}\|\lambda(x) - \lambda(y)\| &\le \big|g_1(t,x,f_1-f_2) - g_1(t,y,f_1-f_2)\big| + ... \\ &\quad + \big|g_d(t,x,f_1-f_2) - g_d(t,y,f_1-f_2)\big| \\ &\le M_1\big|(f_1-f_2)(x) - (f_1-f_2)(y)\big| + ... \\ &\quad + M_1\big|(f_1-f_2)(x) - (f_1-f_2)(y)\big| \\ &= M\big|(f_1-f_2)(x) - (f_1-f_2)(y)\big|, \quad x,y \in [0,2\pi],\end{aligned}$$

where $M = M_1 + ...M_d$. By the last inequality, we deduce that

$$[\lambda]_p^p \le M^p \int_0^{2\pi}\int_0^{2\pi} \frac{\big|(f_1-f_2)(x) - (f_1-f_2)(y)\big|^p}{|x-y|^p} dxdy = M^p [f_1 - f_2]_p^p. \tag{7.37}$$

Taking into account Proposition 7.3 and estimates (7.36) and (7.37), we conclude that there exists a constant $K > 0$ such that

$$\|G(t,f_1) - G(t,f_2)\|_{\gamma(\mathbb{R}^d, W^{\theta,p}_{per})} \le K \|f_1 - f_2\|_{\theta,p}.$$

Thus, $G(t,\cdot)$ is globally Lipschitz on $W^{\theta,p}_{0,per}$. ■

Proof of Theorem 7.1. Recall that the shift semigroup on $W^{\theta,p}_{0,per}$ is a contraction C_0-semigroup with the infinitesimal generator A which is equal to the first weak derivative. Therefore, the abstract form equation (7.24) in the space $W^{\theta,p}_{0,per}$ is as follows

$$dr(t) = \big(Ar(t) + F(t,r(t))\big)dt + G(t,r(t))dW(t), \quad t \ge 0, \tag{7.38}$$

where F and G are functions defined in Lemma 7.10 and 7.11 respectively. Now equation (7.24) is the form of equation (4.2). In Lemma 7.8, we showed that the space $W^{\theta,p}_{0,per}$ is a separable Banach space satisfying the H-condition. Moreover, in lemma 7.10 and 7.11, we proved that F and G satisfies the conditions of Theorem 4.3. Therefore, for every $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}; W^{\theta,p}_{0,per})$, equation (6.33) has a unique $W^{\theta,p}_{0,per}$-valued continuous mild solution with the initial value $r(0) = r_0$.

Remark 7.2. *Similarly, one can prove the existence and uniqueness of solutions to problem (7.24) in the fractional Sobolev spaces* $W^{\theta,p}([0,\infty))$*,* $p \geq 2$*.*

Open Problems:
1. The existence and uniqueness of solutions to the HJMM equation in the weighted fractional Sobolev spaces.
2. The existence and uniqueness of an invariant measure for the HJMM equation in the fractional Sobolev spaces, in particular, in the weighted fractional Sobolev spaces.

Chapter 8

Conclusions

In the first part of this thesis, using the Banach Fixed Point Theorem, we proved the existence and uniqueness of solutions to the stochastic evolution equations (where the linear part of the drift is an infinitesimal generator of a contraction type C_0 semigroup and coefficients satisfy globably Lipschitz conditions) in Banach spaces satisfying the H-condition. Then using this existence result and approximation, we proved the existence and uniqueness of solutions for corresponding equations (with the coefficients satisfying locally Lipschitz condition) in such Banach spaces. Furthermore, we analysed the Markov property of the solution and presented results found recently by [15] about the existence and uniqueness of an invariant measure for corresponding equations when the coefficients are the time independent.

In the second part, we applied abstract results from the first part to the HJMM equation. We proved the existence and uniqueness of solutions to the HJMM equation (driven by a cylindrical Wiener process on an infinite dimensional Hilbert space) in the weighted Lebesgue and Sobolev spaces. We also found a sufficient condition for the existence and uniqueness of an invariant measure for the Markov semigroup associated to the HJMM equation (when the coefficients are time independent) in the weighted Lebesgue spaces. Moreover, we proved the existence and uniqueness of solutions to the HJMM equation (driven by a standard d-dimensional Wiener process) in the spaces $H^{1,p}_{\mathrm{w}}$, $p \geq 2$, which are natural generalizations of the Hilbert space $H^{1,2}_{\mathrm{w}}$ found by Filipović [26]. Furthermore, we proved the existence and uniqueness of solutions to the HJMM equation in the fractional Sobolev spaces of 2π-periodic functions. The HJMM equation has been not studied even in the Hilbert space $W^{\theta,2}$ before. Therefore, the proof of the existence of a unique solution to the HJMM equation in the fractional Sobolev spaces $W^{\theta,p}$, $p \geq 2$, was a new result for the HJMM equation.

There are three important features of our results. First of all, we were able to prove that the HJMM equation has a unique solution and an invariant measure in smaller spaces. Secondly, we were able to consider the HJMM equation driven by a cylindrical Wiener process on a (possibly infinite dimensional) Hilbert space. Lastly, elements of $W_\nu^{1,p}$ (or $H_{\mathrm{w}}^{1,p}$) are α-Hölder continuous functions for $\alpha < 1 - \frac{1}{p}$ and hence, for each $p \geq 2$, the solution to the HJMM equation in the space $W_\nu^{1,p}$ (or $H_{\mathrm{w}}^{1,p}$) is more regular than the solution in the space $W_\nu^{1,2}$. In the spaces C^α, $\alpha > 0$, of α-Hölder continuous functions, one can not define an It$\hat{o}$ integral and hence, these spaces are not suitable for our purpose. This is another important feature of our results.

Future Research

1. Existence of finite dimensional realizations for the HJMM equation in Banach spaces
Let X be a separable Banach space of real-valued functions on $[0,\infty)$ (satisfying H-condition) such that the shift semigroup on X is a contraction (or contraction type) C_0-semigroup and its infinitesimal generator is the first weak derivative. We consider an HJMM equation (driven by a standard d-dimension Brownian motion) in X having the forward rate structure $\sigma(r(t)) = (\sigma_1(r(t)), \sigma_2(r(t)), ..., \sigma_d(r(t)))$ where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_d) : X \to \mathbb{R}^d$ satisfy Lipschitz condition. Therefore

$$\begin{cases} dr(t) = \big(Ar(t) + \alpha_{HJM}(r(t))\big)dt + \sum_{j=1}^{d} \sigma_j(r(t))dW_j(t), \quad t \geq 0 \\ r(0) = r_0 \in X. \end{cases} \tag{8.1}$$

The existence of finite dimensional realizations for such HJMM equation in separable Hilbert spaces was studied in [6], [17], [27] and [28]. We would like to prove the existence of finite dimensional realizations for such HJMM equation in such Banach space X. There are several reasons why one is interested in the existence of finite dimensional realizations for such HJMM equation, see [6], [17], [26] and [28] for detail.

2. Lognormality of the HJM model in Banach spaces
Goldys, Musiela and Sondermann [34] considered forward rates defined by

$$j(t,x) = e^{r(t,x)} - 1, \quad t, x \geq 0.$$

They assumed that these forward rates satisfy the following stochastic partial differential equation

$$\begin{cases} dj(t,x) = \Big[\frac{\partial}{\partial x} j(t,x) + \Big\langle \vartheta(t,x)j(t,x), \int_0^x \frac{j(t,x)}{1+j(t,x)} \vartheta(t,y)dy \Big\rangle + \frac{|\vartheta(t,x)|^2}{2} \frac{j^2(t,x)}{(1+j(t,x))}\Big] dt \\ \qquad\qquad + \big\langle \vartheta(t,x)j(t,x), dW(t)\big\rangle, \quad t, x \geq 0, \end{cases} \tag{8.2}$$

where W denotes a standard d-dimensional Brownian motion, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$ and ϑ is deterministic. They proved that equation (8.2), under some sufficient conditions on ϑ, has a unique nonnegative solution in the weighted Lebesgue L^2 space. Moreover, they found a sufficient condition for the existence of an invariant measure to equation (8.2) in the weighted Lebesgue L^2 space. We would like to prove, under some sufficient conditions on ϑ, the existence of a unique nonnegative solution to equation (8.2) (driven by a cylindrical Wiener process on a separable Hilbert space) in the weighted Lebesgue L^p, $p \geq 2$, spaces. Moreover, we would like to find a sufficient condition for the existence of an invariant measure to equation (8.2) in the weighted Lebesgue spaces.

3. Forward mortality rates

Tappe introduced a family of processes called forward mortality rates in [57]. Using the ideas from the theory of the Heath-Jarrow-Morton-Musiela (HJMM) model, he proposed dynamics of the forward mortality rates. He proved, under some consistency conditions, that these dynamics satisfy the HJMM-drift condition. However, the results in [57] are not sufficient for modelling mortality rates. This is because Tappe imposed strong restrictions on the class of models and the paper does address the issue of stability of the models proposed. Therefore, we need to show that the SPDE for the forward mortality rates is well posed in a sufficiently large class of function spaces and it possesses (possibly unique) invariant measure. Hence, we would like to prove the existence and uniqueness of solutions to stochastic partial differential equations describing the dynamics of forward mortality rates in the weighted Lebesgue and Sobolev spaces. Moreover, we would like to prove, under sufficient conditions, the existence of a unique invariant measure for corresponding stochastic equations in weighted Lebesgue spaces.

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