Toeplitz and Hankel operators on Hardy spaces of complex domains

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Abstract

The major focus is on the Hardy spaces of the annulus \( \{ z : s < |z| < 1 \} \), with the measure on the boundary being Lebesgue measure normalised such that each boundary has weight 1. There is also consideration of higher order annuli, the Bergmann spaces and slit domains. The focus was on considering analogues of classical problems in the disc in multiply connected regions.

Firstly, a few factorisation results are established that will assist in later chapters. The Douglas-Rudin type factorisation is an analogue of factorisation in the disc, and the factorisation of \( H^1 \) into \( H^2 \) functions are analogues of factorisation in the disc, whereas the multiplicative factorisation is specific to multiply connected domains. The Douglas-Rudin type factorisation is a classical result for the Hardy space of the disc, here it is shown for the domain \( \{ z : s < |z| < 1 \} \). A previous factorisation for \( H^1 \) into \( H^2 \) functions exists in [4], an improved constant not depending on \( s \) is found here.

We proceed to investigate real-valued Toeplitz operators in the annulus, focusing on eigenvalues and eigenfunctions, including for higher order annuli, and amongst other results the general form of an eigenfunction is determined. A paper of Broschinski [10] details the same approach for the annulus \( \{ z : s < |z| < 1 \} \) as here, but does not consider higher genus settings. There exists work such as in [6] and [5] detailing an alternative approach to eigenvalues in a general setting, using theta-functions, and does not detail the eigenfunctions.

After this, kernels of a more general symbol are considered, compared to the disc, and Dyakanov’s theorem from the disc is extended for the annulus.

Hankel operators are also considered, in particular with regards to optimal symbols. Finally, analogues of results from previous chapters are considered in the Bergman space, and the Hardy space of a slit annulus.
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Chapter 1

Introduction

This thesis will focus mainly on the Hardy spaces $H^p$ for the annulus, mostly in comparison to the corresponding spaces in the disc.

1.1 Literature Review

Starting with the background on Hardy spaces in the disc, we have Duren’s book [18] details the $H^p$ spaces for a simply connected region, together with some of the important related spaces, and details some of the important function theory in this area, such as the construction of Blaschke products and existance of boundary values for the functions. The book also details how the $H^p$ spaces can be constructed in the case of multiply connected regions, though it does not proceed further in these spaces. The book [24] is also a useful reference for the $H^p$ spaces for the disc, and as well as detailing the basic function theory, it also details the shift operator in this space.

For the operators on these spaces, the book [17] details Toeplitz operators on the Hardy spaces of simply connected regions quite well, especially with regards to the spectrum, and Nikolski’s book [30] also covers Toeplitz operators, as well as providing a useful reference for Hankel operators, including discussion on the Nehari problem.
In moving into the spaces for multiply connected regions, the paper [2] by Abrahamse gives a good deal of information on the $H^p$ spaces here, in particular, the inner and outer factorisation for these spaces is covered, and the associated spaces (of no longer singly valued functions) are described.

Topelitz operators on these spaces are also discussed here, with Abrahamse detailing some of the theory of the essential spectra for these, and the failure of Copburn’s lemma (and thus the possibility for self-adjoint Toeplitz operators to have eigenvalues) is shown. The most important result with regards to the essential spectra here appears to be Abrahamse’s reduction theory, which essentially reduces problems regarding the essential spectra in multiply connected domains back to the disc.

The older work of Abrahamse [1] covers some of the ideas in [2], but the later paper has more content, and covers everything in the older paper.

Sarason’s work in [38] is also a good reference for the function theory in the $H^p$ spaces of the annulus, in particular it constructs the formula for the character of an outer function with given boundary values, and another for the character of a given inner function.

The book [16] also mentions Toeplitz operators defined on multiply connected regions, mostly following along the same lines as the work in Abrahamse, though a few results are extended slightly, concerning the algebra generated by Toeplitz operators with continuous symbol.

For discussion of the eigenvalues of self-adjoint Toeplitz operators on multiply connected domains, [40] section 4 illustrates the diagonalisation of the Toeplitz operator on the annulus $\{z : s < |z| < 1\}$ whose symbol is the indicator function for the boundary $\{z : |z| = 1\}$, giving the eigenvalues and eigenfunctions for this.

The recent paper [10], and the author’s recent (2014) thesis [11] detail the eigenvalues and eigenfunctions for a general self-adjoint Toeplitz operator on this annulus, using the same methods that were arrived at here (though differing in the approach taken to showing that the eigenfunctions will be outer). The thesis [11] also mentions more general multiply connected regions, however only symbols that have constant sign on
each boundary are considered, and the general approach here is in looking for characters such that an eigenvalue exists for the Toeplitz operator applied to the space of modulus automorphic functions with character $\alpha$, which is a different problem than here where the character is fixed (as $0$).

[5] [6] also consider eigenvalues for Toeplitz operators in more general regions, with the approach here considering the resolvent operator, with the eigenvalues resulting from the zeros of certain theta functions. The actual eigenfunctions are not considered in these papers, however, and the eigenvalues are not explicitly calculated, though conditions for infinite accumulation in an interval are given.

There appear to be few papers on Hankel operators in an annulus, and most of these appear to deal with Hankel operators on the Bergman space of the annulus, whereas in this thesis the Hankel operators are only considered with regards to the Hardy space on the annulus. The paper [4] does focus on Hankel operators for the Hardy space of the annulus, and includes one of the factorisations that will be presented in the second factor, though the result presented here will improve on the norms. This paper also does not consider optimal symbols, but instead uses the factorisation in consideration of the boundedness of Hankel operators.

With regards to Bergman spaces, the book [23] provides the background for the space. As with the Hardy space, there has been considerable interest in Toeplitz operators here. The survey [33] mentions some of the important problems in this area, such as the still open problem of when a Toeplitz operator is open on the space, and the solved problem of compactness in terms of Berezin transforms. The approach to these for radial Toeplitz operators is also mentioned, pointing to the work of the paper [20], detailing compactness and boundedness of radial Toeplitz operators in the disc.

The paper [22] also considers Toeplitz operators on Bergman spaces, the main results in this connect the reproducing kernels for the Bergman space with Schatten class norms of Toeplitz operators and Carleson measures.

Generalisation of Toeplitz operators on the Bergman space of the disc appears mostly
focused on weighted Bergman spaces and polydiscs, rather than to multiply connected regions. However, a few papers do exists in the area, mostly focusing on when a Toeplitz operator is compact, with the strongest result being from the paper [25], which extends the solution of the problem of finding when a Toeplitz operator is compact from the Bergman space of the disc to the Bergman space of multiply connected domains.

For the slit domain, the book [3] provides a detailed study of the invariant and nearly invariant subspaces of the slit disc, with the central results being a complete classification of the invariant and nearly invariant subspaces under the shift.

### 1.2 Background

The description of the $H^p$ spaces for the disc from in [18], [34], will be outlined here.

**Definition 1.2.1** Take $\Delta$ to be the interior of the unit disc- $\Delta = \{z : |z| < 1\}$.

Let $p \in [1, \infty]$. $H^p(\Delta)$ is defined as the set of complex valued functions $f$ defined on $\Delta$ such that $f$ is analytic inside $\Delta$, and for which $\|f\|_p = \lim_{r \to 1} M_p(r, f) < \infty$, where $M_p(r, f) = (\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}}$ for $1 \leq p < \infty$, and for $p = \infty$, $M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$.

These spaces will be Banach spaces for $1 \leq p < \infty$, and a Hilbert space when $p = 2$, as shown in [24] and [18].

It can be shown (see [24], p.51) that for any $f \in H^p$ for $p \in [1, \infty]$, well-defined radial limits exist almost everywhere on the boundary, and the behaviour of any $H^p$ function can be recovered from its boundary values. Thus, we have a natural embedding of $H^p$ into $L^p$.

The inner product for $H^2$ is $\langle f, g \rangle = \int_{|z|=1} f \bar{g}$, taking the integral with respect to uniform Lebesgue measure on the boundary of the disc.

$H^p$ can be identified with the subset of $L^p$ in which all negative Fourier coefficients vanish.
As a result, \((z^n)_{n=0}^\infty\) will be a basis for these spaces for \(1 < p < \infty\).
Since \(z^n\) are all bounded functions, it follows immediately from this that \(H^\infty\) is dense in \(H^p\) for all \(1 \leq p < \infty\). As well as \(H^p\), there also exists the Nevalinna space \(N(\Delta)\).

**Definition 1.2.2** \(N(\Delta)\) is the space of functions \(f\) holomorphic on \(\Delta\) with

\[
\lim_{r \to \infty} \int_{|z|=r} \log^+ |f(z)||dz| < \infty.
\]

These are quite important in Hardy space theory. Firstly, we have that \(H^p \subset N\) for all \(p\) (see [15] p 273), and thus every \(f\) in \(H^p\) is log-integrable. We also have that for any \(f \in N\), there exist \(g_1, g_2 \in H^\infty\) with \(f = g_1 g_2\).

It follows from these (see [15] theorem 2.11) that we have \(\log |f| \in L^1\) whenever \(f \in H^p\).

### 1.2.1 Inner-outer factorisation

An important result in the theory of \(H^p\) spaces is the existence of a unique inner-outer factorisation for functions in these spaces. Firstly, note that we have the following restriction on accumulation of zeros at the boundary ([18] p.18):

**Theorem 1.2.3** Let \(f\) be holomorphic in the disc, with \(f \neq 0\), and let \((z_n)\) be the zeros of \(f\), repeated according to multiplicity. Then \(\lim_{r \to \infty} \int_0^{2\pi} \log |f(re^{i\theta})|d\theta\) is bounded only if \(\sum_{n=1}^\infty (1 - |z_n|) < \infty\).

Since \(\int_0^{2\pi} \log |f|d\theta < \infty\) was seen to be a necessary condition for \(f \in H^p\), this is a necessary condition on the zero set of an \(H^p\) function.

Furthermore, given any set of \(a_n\) satisfying the condition \(\sum (1 - |a_n|) < \infty\), we have that there exists \(B(z) \in H^\infty\) with \(B(a_n) = 0 \forall n\), and \(|B(z)| = 1\) a.e. on \(|z| = 1\). The construction is given in [18] p.19, with \(B(z) = \prod_{n=1}^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z}\).

This is termed a Blaschke product. For any \(f \in H^p\), if we set \((a_n)\) to be the zero set of \(f\),
and $B$ to be the Blaschke product with zero set $(a_n)$, we have that $f/B \in H^p$ (see [18] p.20), and $f/B \neq 0$ inside $\Delta$. Furthermore, $\|f/B\|_p = \|f\|_p$, since $|B| = 1$ a.e. on $\partial \Delta$.

It follows from the previous theorem that $\sum_{z \in A} (1 - |z|) < \infty$ is a necessary condition for $A$ to be the zero set of some $H^p$ function. Since we can construct suitable Blaschke products when this condition holds, it is also sufficient.

We can proceed further and factorise what is left into what is called an outer function and singular inner function.

Definition 1.2.4 [24] p.61

An outer function is defined to be $F : \Delta \rightarrow \mathbb{C}$ such that

$$F(z) = \lambda \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log f(e^{i\theta}) d\theta\right).$$

Where $\lambda \in \mathbb{C}$ a constant, and $f$ is a positive real-valued integrable function.

It follows from the definition that an outer function will have no zeros inside the disc.

The following results on outer functions will also be important, a proof of which can be found in [37] 17.16:

Theorem 1.2.5 Let $F$ be an outer function as defined in 1.2.4 with respect to $f$, $\log f \in L^1$.

1. $\lim_{r \to 1} |F(re^{i\theta})| = f(e^{i\theta})$ a.e.

2. $F \in H^p(\Delta)$ if and only if $f \in L^p(\partial \Delta)$.

We also have that $|F(z)| = |f(z)|$ almost everywhere on the boundary, and that $F \in H^p$ if and only if $f \in L^p$. 
A singular inner function is also defined by a boundary integral, this time against a singular measure on the boundary. The following is, once again, from [24]:

**Definition 1.2.6** A function \( f \) is said to be a singular inner function if there exists a singular, positive, measure \( \mu \) on \( \partial \Delta \) such that:

\[
f(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right).
\]

The factorisation theorem for the disc is as follows:

**Theorem 1.2.7** Let \( f \neq 0 \in H^1 \). Then there exists a unique factorisation \( f = BSF \) where \( B \) is a Blaschke product, \( S \) is singular inner, and \( F \) is an outer function.

Again, this comes from [24] p.67-68.

### 1.2.2 Toeplitz Operators

Using the equivalence between \( H^2 \) functions and their boundary values, and that \( H^2 \) can thus be embedded as a closed subspace of \( L^2 \), there exists orthogonal projection \( P \) from \( L^2 \) onto \( H^2 \). Thus, given \( \phi \in L^\infty \), we can define a Toeplitz operator \( T_\phi \) as follows:

**Definition 1.2.8** \( T_\phi f = P(\phi f) \).

In the disc, it can be shown that if \( \phi \) a non-constant real-valued function on the boundary, then the set of eigenvalues for \( T_\phi \) is empty- this follows from Coburn’s theorem (see [29])- we will have that either \( T \) or \( T^* \) is injective, if \( T \neq 0 \). Since \( T = T^* \), it must therefore have no eigenvalues.

It is interesting to consider the matrix with respect to our basis \((z^n)\) for a Toeplitz operator. Since we have \( \langle T_\phi z^n, z^m \rangle = \langle \phi z^n, z^m \rangle = \langle \phi z^{n+1}, z^{m+1} \rangle \), we will have that \( a_{n,m} \) is constant on the diagonals of fixed \( n - m \), which will be the infinite diagonals in our
matrix. As a consequence, we immediately have that no non-trivial compact Toeplitz operators exist.

We can also note that $T \phi f = 0$ implies that $\phi$ is log-integrable.

**Lemma 1.2.9** Let $f \neq 0$, $\phi \in L^\infty(\partial \Delta)$, and $T \phi f = 0$. Then $\log |\phi| \in L^1$.

**Proof**

In the disc, we have that $H^2 = \overline{z}H^2$, and so we have that all functions in $H^2$ have log-integrable modulus on the boundary.

Now, if $T \phi f = 0$, we have that $\phi f \in H^2$. Since $|f|$ and $|\phi f|$ are log-integrable, $|\phi|$ must also be log-integrable. $\square$

### 1.2.3 Spectra

This thesis will spend some time on discussion of spectra of operators, so these will be mentioned here for bounded operators. For $T$ a bounded operator $\sigma(T)$ is said to be the spectrum of $T$, where:

**Definition 1.2.10** $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{has no bounded inverse}\}$.

The spectrum has several subsets that are important in operator theory, generally classified based on why $(T - \lambda I)$ fails to be invertible.

We have the point spectrum $\sigma_p(T) = \{\lambda : (T - \lambda I) \text{ not injective}\}$.

If $(T - \lambda I)$ is not injective, it follows that $\lambda$ must be an eigenvalue for $T$, and so the point spectrum corresponds to the set of eigenvalues.

The essential spectrum of operators will also be considered here. This is taken to be $\sigma_e = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ not Fredholm}\}$, where a Fredholm operator is an operator which
is invertible in the space of operators quotiented out by the compacts.

An introduction to Fredholm operators can be found in [39] chapter 5. This also mentions essential spectra, briefly.

Some important properties are that $T$ a Fredholm operator is equivalent to $T$ having finite dimensional kernel, finite dimensional co-kernel, and closed range.

If we define the index $\text{Ind}(T) = \dim(\ker(T)) - \dim(\text{coker}(T))$ on the Fredholm operators, we will also have $\text{Ind}(T_1T_2) = \text{Ind}(T_1) + \text{Ind}(T_2)$.

We do not necessarily have that $\sigma_e$ and $\sigma_p$ are disjoint here- if we have $\lambda$ is an eigenvalue of infinite multiplicity, then $\lambda \in \sigma_p$ is immediate. $\lambda \in \sigma_e$ will also follow since $T - \lambda I$ has infinite dimensional kernel, and thus is not Fredholm.

For $T$ self-adjoint (and so kernel and co-kernel are both equal in dimension), we have that $\sigma_p(T) \cup \sigma_e(T) = \sigma(T)$, since suppose $0 \notin \sigma_e(T), 0 \notin \sigma_p(T)$.

Then, we have that $T$ has zero kernel and cokernel, and closed range. Thus, $T$ is a $1 - 1$ mapping from our Hilbert space into itself, and it follows that it has bounded inverse.

However, this is not necessarily the case for non self-adjoints. Considering the shift operator on the Hardy space of the disc $T_z$ it is immediate that $0$ is not an eigenvalue for $T_z$. We have that $T_zH^2 = zH^2$, is a closed subspace of codimension 1. Thus $T_z$ is Fredholm. Thus $0 \notin \sigma_p(T_z)$, and $0 \notin \sigma_e(T_z)$. Yet $0 \in \sigma(T_z)$ follows as $T_z^*$ has a zero-eigenvalue, and is thus not invertible.

There exist further subsets of the spectrum which are of interest in various parts of analysis, such as the residual and continuous spectrum, however only the essential and point spectra are considered in this thesis.

Spectral properties of Toeplitz operators in the disc have long been studied. It is already known, for instance, that $\sigma_p(T_{\phi})$ is empty for $\phi$ a real-valued non-trivial symbol, and that $\sigma_e(T_{\phi})$ is the range of $\phi$ for a continuous symbol $\phi$, with the Fredholm index given by the winding number of $\phi$ (see for example [17] 7.26).
1.2.4 Hardy space of the annulus

Sarason’s work [38], as well as [2] provide useful references for the Hardy spaces in multiply connected regions.

We will take the annulus $\mathbb{A}$ to be the set $\{z : s < |z| < 1\}$ where $0 < s < 1$ is some constant.

In this thesis, $D$ will be used to represent a bounded multiply connected region in the plane. We will consider only $D$ for which $D$ is an open, connected domain in the plane bounded by at most finitely many analytic Jordan curves.

In both cases, we take $\Gamma_i$ to be the connected components of the boundary of our region, with $\Gamma_0$ chosen to correspond to the outermost boundary of our region when possible (so in the case of $\mathbb{A}$, $\Gamma_0 = \{z : |z| = 1\}$).

There are a couple of different ways to define the hardy spaces on an annulus, both of which will be detailed for arbitrary multiply connected regions in the plane. In a domain $D$ whose boundaries consist of finitely many analytic Jordan curves, these definitions will in fact be equivalent.

Firstly, we can take the definition of $H^p$ from [18] p179-183, which defines the space as follows:

**Definition 1.2.11** For $D$ an arbitrary domain in the complex plane, $H^p(D)$ is the set of analytic functions $f$ such that $|f(z)|^p$ has a harmonic majorant in $D$.

In [18], the space is then given norm based on the harmonic majorant- here, we have $\|f\|_p = \inf(u(z_0))^{1/p}$, where $z_0$ is some fixed arbitrarily chosen point in $D$, and taking infimum over $u$ harmonic majorants for $|f|^p$. Our inner product for the $p = 2$ case will be $\langle f, g \rangle = \int_{\partial D} f\overline{g}dm$ for some measure on the boundary of $D$. An explicit calculation of this measure can be found in [2] p263.
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To see that this construction gives a Banach space, one can consult [36] p49-50.

For the alternative definition, we can define \( E^p(D) \) as the set of functions \( f \) which are analytic inside \( D \), and for which there exists a sequence of domains with boundaries \( \{ \Gamma_n \} \), consisting of a finite number of rectifiable Jordan curves, which eventually enclose each compact subset of \( D \), have bounded length, and \( \limsup_{n \to \infty} \int_{\Gamma_n} |f(z)|^p|dz| < \infty \).

It is shown in [18] that these definitions are in fact equivalent for finitely connected domains whose boundary curves are analytic. The approach taken there is that if \( C_n \) is the domain enclosed by \( \Gamma_n \), we have that for every \( f \in H^p(D) \), one has \( f = f_1 + \ldots + f_n \), with \( f_i \in H^p(D_k) \), for either of the definitions, and the same decomposition can be shown for \( f \in E^p(D) \) also. The equivalence then follows from the equivalence of \( E^p \) and \( H^p \) for simply connected domains- see, for example, [37].

Since defining \( H^p \) by means of harmonic majorants and by integrals with respect to uniform Lebesgue measure are equivalent, it follows that the harmonic measure \( dm \) will be equivalent to the uniform measure \( |dz| \). More detail on the relationship can be found in [2] p263.

Whilst normally the space will be considered with respect to the uniform Lebesgue measure, at stated points the harmonic measure will be used. In this case, we will use \( P^h \) to denote orthogonal projections with respect to the harmonic measure, and \( \langle ., . \rangle_h \) to denote an inner-product with respect to harmonic measure.

It is useful to consider a covering map from the disc to the annulus, \( \theta \). For the annulus \( \{ z : s < |z| < 1 \} \), one can define this as follows:

First, let \( f_1 : z \to \frac{z+1}{1-z} \). This is a conformal equivalence from the disc to the right half plane.

Let \( f_2 : z \to -\log s \frac{i \log z + -\log s}{2} \).

This is a conformal equivalence between the right half plane and the strip \( \{ 0 < Re(z) < - \log s \} \).

Let \( f_3 : z \to e^{-z} \). This maps the previous strip to the set \( \{ s < |z| < 1 \} \), i.e. to an annulus.
So $\theta = f_3 \circ f_2 \circ f_1$ gives us a covering map from the disc to $A$.

Since $f_2$ and $f_1$ are conformal equivalences, and $f_3$ is $\infty$ to $1$ with the identity $f_3(z + 2\pi i) = f_3(z)$ generating the automorphism group, we see that there exists a conformal equivalence $\psi$ on the disc generating the group of identities for the map $\theta$, with $\psi(z) = f_1^{-1}(f_2^{-1}(f_1(z)) + 2\pi i)$.

It is useful to consider how our $\theta$ acts on the boundary.

Firstly, note that moving from the disc to the annulus has some pathological points—we have that $\theta$ does not extend to the boundary at $\pm 1$. This results from the behaviour of $f_3$ around the boundary of the strip at $\pm \infty i$ (which $f_2 f_1$ sends $\pm 1$ to). In fact, considering $f_3$ we can see that $\{ z : 0 < \text{Re}(z) < -\log s, 2\pi n < \text{Im}(z) < 2\pi(n + 1) \}$ is mapped $1 - 1$ into $A$. So any relatively open region of the boundary of the strip about $\pm \infty i$ is mapped in an $\infty - 1$ manner to the entire boundary of $A$. It follows therefore that on any arc of $\partial \Delta$ relatively open and containing $\pm 1$, $\theta$ is an $\infty - 1$ map onto the boundary of the annulus. These are the only such points in $\partial \Delta$, since $\log(z)$ extends to the boundary of the strip everywhere except at $\pm \infty i$.

This can also be seen from the fact that $\psi^{\pm n}(z) \to \pm 1$ as $n \to \infty$ (since we have $\psi$ can be represented as mapping to the strip, adding $2\pi i$, then mapping back to the disc, and so accumulates at the points in the disc corresponding to $\pm \infty i$ in the strip.

If we take $t$ with $t \neq 0, t \neq \pi$, consider the effect of $\theta$ on $e^{it}$.

We have $f_1(e^{it}) = \frac{e^{it} + 1}{e^{it} - 1}$.

$f_2$ takes a log of this, then multiplies and shifts by a constant, so we have

$f_2(f_1(e^{it})) = i(\log(\frac{2\sin(t)}{1-e^{it}}))\frac{-\log s}{\pi} + 0$ when $0 < t < \pi$.

$f_2(f_1(e^{it})) = i(\log(\frac{2\sin(t)}{1-e^{it}}))\frac{-\log s}{\pi} + (-\log s)$ for $\pi < t < 2\pi$. 

Finally, $f_3$ is exponentiation, so $\theta(e^{it}) = e^{-i(\log(\frac{2\sin(t)}{1-e^{it}}))\frac{-\log s}{\pi}}$ when $0 < t < \pi$, and $\theta(e^{it}) = se^{-i(\log(\frac{2\sin(-t)}{1-e^{it}}))\frac{-\log s}{\pi}}$ for $\pi < t < 2\pi$.

So we can see once more that as $t$ approaches $0$ or $\pi$, we are wrapping around the appropriate boundary of the annulus infinitely many times.

It is also clear that $\Gamma_0$ lifts to the set of points on the boundary of the disc with positive
imaginary part, and \( \Gamma \) to those with negative imaginary part.

If we take a lift \( \tilde{f} \) of \( f \) an \( H^p(D) \) function back to the disc under a covering map \( \theta_D \) from \( \Delta \) onto \( D \), we have that \( \tilde{f} \in H^p(\Delta) \). This is trivial in the case \( p = \infty \). It is proven in [2] p267 that (for the space defined with respect to Harmonic majorants) that if \( \tilde{f} \) is the lift to the disc of \( f \in H^p(D) \), we have \( \| \tilde{f} \|_p = \| f \|_p \) (the proof is to show that if \( u \) is a harmonic majorant for \( |f|^p \), then lifting \( u \) gives a suitable majorant for the disc).

Now, since we are working with uniform measure on the boundary, we will not have that the norms remain equal. However, since the harmonic and lebesgue norms are equivalent, we will have that \( \| \tilde{f} \|_p < \infty \Leftrightarrow \| f \|_p < \infty \).

Abrahamse in [2] in fact proves more than that just the lifts of \( H^p(D) \) functions are in \( H^p(\Delta) \). We in fact have that \( L^p(\partial D) \) functions lift to \( L^p(\partial \Delta) \) functions.

It follows, for instance, that the functions in \( H^p(\hat{A}) \) will have well-defined radial limits on the boundary of the annulus- one simply has to show that a non-tangential line to the boundary of the annulus is mapped to a set of non-tangential lines on the boundary of the disc under \( \theta^{-1} \).

Let \( A : [0, 1] \to S \) be our path approaching the boundary of the annulus non-tangentially, and we can assume w.l.o.g. \( A(1) = 1 \).

Since we approach the boundary of the annulus non-tangentially, we must have that our path is of finite length, and so it can only wind about the annulus at most finitely many times. Since \( f_1 \) and \( f_2 \) are conformal equivalences, we need only consider \( f_3 \). By taking \( A'(x) = A(t + (1 - t)x) \) for \( t \) suitably close to 1, we can assume that \( |(A(x) - 1| < \epsilon \), where \( \epsilon \) is taken so that \( \{z : |z - 1| < \epsilon \} \cap \hat{A} \) is simply connected. Now, we have that \( f_3 \) restricted to a suitable set is a conformal equivalence between \( \{z : |z - 1| < \epsilon \} \cap \hat{A} \) and the appropriate pre-image, so the pre-images of \( A \) must approach the boundary non-tangentially.

Since we have that boundary values now exist almost everywhere, we have that \( H^2 \) can
be considered as a subspace of the $L^2$ space of the boundary of the annulus, and it must be closed since $H^2$ is complete. From this, we can define orthogonal projection onto $H^2$, and thus a Toeplitz operator can be defined on the annulus by $T_\phi f = P_{H^2(\mathbb{A})}(\phi f)$. When working with the harmonic measure, we will denote the associated Toeplitz operator $T_\phi$, with $T_\phi f = P_h(\phi f)$.

1.2.5 Basis for $H^p(\mathbb{A})$

In the disc, we have that $(z^n)_{n=0}^\infty$ provide an orthonormal basis for $H^2(\Delta)$. For the annulus $\mathbb{A}$, with normalized lebesgue measure on the boundary, we have a very similar result.

**Theorem 1.2.12** $(\frac{z^n}{||z^n||_2} : -\infty < n < \infty)$ is an orthonormal basis for $H^2(\mathbb{A})$.

**Proof**

It is immediate that $z^n \perp z^m$ when $n \neq m$, since we have that $\langle z^n, z^m \rangle = \int_0^1 e^{2i\pi(n-m)t} dt + \int_0^1 s^{n+m}e^{2i\pi(n-m)t} dt$.

Now, given any $f$ holomorphic in $\mathbb{A}$, we have a Laurent series $f = \sum_{-\infty}^{+\infty} a_n z^n$.

So, $\int_{|z|=r} |f|^2 = \sum_{n=-\infty}^{+\infty} |a_n|^2 r^{2n}$.

So, for $f \in H^2(\mathbb{A})$, we must have $\sum |a_n|^2(1 + s^{2n}) < \infty$. However, $||z^n||_2^2 = 1 + s^{2n}$, and thus $\sum_{n=-N}^{N} a_n z^n$ is convergent in $H^2$, and thus converges to $f$.

It follows that $(\frac{z^n}{||z^n||_2})$ for $-\infty < n < \infty$ gives a basis for $H^2(\mathbb{A})$. □

From this, for instance, it follows that $H^\infty$ is dense in $H^p$ for $1 < p \leq \infty$.

For a more general annulus $D$ of genus $g$, constructing a basis for $H^2(D)$ is somewhat harder. However, we can use the additive decomposition (previously used in showing the two alternate definitions for $H^p$ were equivalent) to show that $\{z^n, (z - a_i)^{-n} : 0 \leq n < \infty\}$ has dense span in $H^2(D)$, where $a_i \in \mathbb{C}$ are chosen for
1 ≤ a_i ≤ g such that a_i is enclosed by Γ_i, and so it will follow that H^\infty is dense in H^p.

However, even our \((z^n)\) need not be an orthogonal series in H^2(D).

1.2.6 Decomposition of L^2

In the disc, we have \(L^2(\partial \Delta) = H^2(\Delta) \oplus \overline{zH^2(\Delta)}\), and so \(H^{2\perp} = \overline{zH^2}\).

In the case of \(\mathbb{A}\), if \(\omega\) is the function with \(\omega = 1\) on \(\Gamma_0\), and \(\omega = -1\) on \(\Gamma_1\), it is easy to see that \(\omega H^{2\perp} \perp H^2\), since \(\langle f, \omega g \rangle = \langle fg, \omega \rangle = 0\).

We also have that \(H^2 \oplus \omega H^2 = L^2\). To do so, note that \(z^n + s^{2n} \omega z^{-n} = (1 + s^{2n})e^{in\theta}\) on \(\Gamma_0\), and is 0 on \(\Gamma_1\). \(z^n - \omega z^{-n}\) is \((1 + s^{-n})e^{in\theta}\) on \(\Gamma_1\) and 0 on \(\Gamma_0\).

Thus, we have that \(H^2 \oplus \omega H^2\) is dense in \(L^2\), and since the space is closed, it is \(L^2\). It follows that \(H^{2\perp} = \omega H^2\).

We may also wish to consider a similar result in an arbitrary annulus \(D\).

We have from [2] Theorem 1.7 that for the Hardy space on \(D\) (with Harmonic measure), we have \(H^{2\perp} = \nu^{-1}H^2\) for some function \(\nu\), meromorphic on an open set about our region \(D\), with zeros and poles strictly inside \(D\). We have that \(\nu^{-1}\) is thus continuous, bounded and non-zero on \(\partial D\). Thus, if we are working with uniform Lebesgue measure, we have \(H^{2\perp} = \nu^{-1}\frac{|dz|}{dm}\), where \(dm\) was the previous Harmonic measure.

We know that \(|dz|\) and \(dm\) are equivalent measures [2], so \(\frac{|dz|}{dm}\) will be bounded away from 0 and \(\infty\). It is also shown that this is strictly positive in [2].

One consequence is that \(H^{2\perp}\) will consist of functions whose boundary values have log-integrable modulus, since \(|\nu|\) is log-integrable on the boundary, as is our change of measure \(\frac{|dz|}{dm}\) and \(H^2\) (\(H^2\) having log-integrable boundary values is in [2] Theorem 1.18).

It follows that a necessary condition for \(T_\phi f = 0\) will be that \(|\phi|\) is log-integrable.

Another decomposition shown in Abrahamse is that \(L^2 = H^2 \oplus \overline{H^2} \oplus N\), with \(N\) a \(g\)-dimensional set, where \(g\) is the genus of \(D\).

Again, this is shown with respect to harmonic norm. If we instead use uniform Lebesgue
measure, we still have $H^2 + \overline{H}^2 + N = L^2$, however orthogonality is no longer preserved.

### 1.2.7 Inner-outer factorisation in the annulus

Inner-outer factorisation in an annulus works somewhat differently than in the disc. By taking lifts back to the disc, we can produce an inner-outer factorisation in the disc of the lifted function, however there is no guarantee that the new functions from factorisation can be passed back down to the disc (since our covering map from the disc will be an $\infty$-1 mapping, and so we cannot send a general $H^p(\Delta)$ function back to $D$).

Considering the case for an annulus first, we have by a result of [38] that if $f$ is an outer function, and $|f|$ is $\psi$-invariant on the boundary, then $f$ will be a modulus automorphic function, defined as follows:

**Definition 1.2.13** $f \in H^p(\Delta)$ is a modulus automorphic function, and $k \in (0, 1]$ its character if we have that $f(\psi \circ z) = e^{2\pi ik}f(z)$ a.e. on $\partial\Delta$.

We can take an inner-outer factorisation in the disc of $\tilde{f} = \tilde{f}_i\tilde{f}_o$ for an arbitrary function $f \in H^p(\mathbb{A})$, and we will have that $\tilde{f}_o$ will be a modulus invariant function, and thus $\tilde{f}_i$ will also be modulus invariant, with complementary character.

When considering the disc as a covering space of more general regions, we have that the identity group of our covering map $\theta_D$ is no longer generated by a single element.

We can still, however, define modulus automorphic functions, as is done in [2] p267.

**Definition 1.2.14** If $G$ is the group of disc automorphisms such that $\theta_D \circ G = \theta_D$, we have that $f$ is a modulus automorphic function if we have $|f \circ S| = |f|$ for all $S \in G$.

Now, [2] states that if $f$ is modulus automorphic, then we have $f \circ S = \alpha(S)f$ for some $\alpha \in \hat{G}$ the dual group of $G$. We say that $\alpha$ is the character of $f$. 
If we have $f$ a many-valued function in $H^p(D)$ that lifts to a modulus automorphic function in $H^p(\Delta)$, the character of $f$ is taken to be the character of its lift to $\Delta$.

There are two approaches to defining inner and outer functions in the annulus. One is to accept many-valued functions that lift to modulus automorphic functions for the inner and outer factorisation, and proceed as before.

The second is to only accept single-valued functions, and instead cancel out the modulus automorphism of the lifts of each part by multiplying by $g$ and $g^{-1}$, with $g$ an outer (by the former definition) function chosen to have constant modulus on each boundary, and to have suitable character (in the case of $A$, these will be $z^\alpha$, with $\alpha$ corresponding to the character of the previous inner-outer factorisations). The requirement for inner functions to have $|f_i| = 1$ on the boundary must then be reduced instead to the requirement for $|f|$ to be constant on each boundary.

In this nomenclature, the functions $z^n$ are both inner and outer for the $H^p$ spaces of $A$. Functions that are both inner and outer for an annulus are referred to as units.

In this thesis, the first of these approaches will be taken unless stated otherwise- inner and outer components of a function may be considered as multivalued. Considering multiply valued functions whose lifts back to the disc are modulus automorphic in fact generates a family of spaces related to our $H^p$ spaces:

**Definition 1.2.15** $H^p_\alpha(D)$ is the set of modulus-automorphic functions in $\Delta$ with character $\alpha$.

Further details can be found in [2] p266-267, or in [38]. That these are essentially copies of $H^p(D)$ can be seen from the result $H^p_\alpha(D) = \tilde{s}_\alpha H^p_0(D)$, where $s_\alpha$ is an outer function, with modulus bound away from 0 and $\infty$ on each boundary, and such that $\tilde{s}_\alpha$ has character $\alpha$.

The result follows from the observation that multiplying two modulus automorphic functions will result in a function whose character is the sum of the individual characters (shown in, for example, [2]).
**Definition 1.2.16** Let \( \tilde{f} \in H^1_{\alpha} \). Then \( f \) is said to be outer/Blaschke/singular inner for \( D \) if \( \tilde{f} \) is outer/Blaschke/singular inner for \( H^1(\Delta) \), where \( \tilde{\cdot} \) denotes the lift to the disc: 
\[
\tilde{f}(z) = f(\theta(z)),
\]
where \( \theta \) is our covering map from \( \Delta \) to \( D \).

**Theorem 1.2.17** Let \( f \in H^1(D) \). Then there exists a unique factorisation 
\[
f = BSG,
\]
where \( B \) is a Blaschke product for \( D \), \( S \) is a singular inner function for \( D \), and \( G \) is outer for \( D \).

If \( \alpha, \beta, \gamma \) are the corresponding characters for \( B, S, G \), then 
\[
\alpha + \beta + \gamma \in \mathbb{Z}.
\]

This is from [2] p268.

For an outer function with given modulus on the boundary, [38] p.35, Theorem 6, gives the following technique to calculate its character (modified by a constant to take characters in the range \([0, 1)\).

**Theorem 1.2.18** Let \( f \) be a multiply-valued function such that \( \tilde{f} \in H^p_{\alpha_f}(\Delta) \). We have that 
\[
\alpha_f = \frac{\int_{\Gamma_0} \log |f| - \int_{\Gamma_1} \log |f|}{\log s} \pmod{1}.
\]

**Corollary 1.2.19** \( z^x \) as a multi-valued function in \( \mathbb{A} \) has character \( x \pmod{1} \).

### 1.2.8 Blaschke products

In \( \mathbb{A} \), similar conditions on the zeros accumulating at the boundaries hold as to in the disc, though the condition instead becomes 
\[
\sum_{z : f(z) = 0} \min \{1 - |z|, |z| - s\} < \infty.
\]

This can be seen by taking lifts back to the disc. If \( f \in H^p(\mathbb{A}) \), we have that \( \tilde{f} \in H^p(\Delta) \), and so we have that 
\[
\sum_{z : f(z) = 0} (1 - |\tilde{f}(z)|) < \infty.
\]

Now, consider some region of \( \Delta \) on which \( \theta \) gives a \( 1 \) \( 1 \) mapping from the annulus. Such a region will have a boundary consisting of two arcs on the boundary of the disc,
connected by two paths in the interior of the disc. On this region, our lift \( \theta \) will only distort distances between the respective boundaries by a finite factor. So, if we restrict to a sequence of zeros accumulating on the outer boundary of the annulus, we must have
\[
\sum (1 - |z|) < \infty
\]
on this sequence.

To show the result for a sequence of zeroes approaching other boundary, apply the conformal equivalence \( z \to sz^{-1} \) to \( \mathbb{A} \) to swap the interior and outer boundaries.

We can define Blaschke products in the annulus to be those functions whose lifts back to the disc are Blaschke products (all such lifts will be infinite Blaschke products in the disc, since \( \theta \) is infinity to one).

Consider the Blaschke product in the annulus with a single zero at \( \omega \), \( B_w \). It is useful to know what the character of this function is. In order to do so, note that \( (z - w) = B_w f_o \), where \( f_o \) is an outer function with \( |f_o| = |z - w| \) almost everywhere on \( \partial \mathbb{A} \). (This follows since we know the zero set of \( z - w \) and there is no singular inner component since \( |z - w| \) does not decay on any radial limit approaching the boundary.)

Now, since \( (z - w) \) is single-valued, \( B_w \) has character complementary to \( f_o \), which by Theorem 6 of [38] has character
\[
\frac{\int_{\Gamma_0} \log |z - w| - \int_{\Gamma_1} \log |z - w|}{\log s}.
\]
Alternatively, [38] provides a construction of the character of an inner function directly.

There are two things to note:

1. The character of \( B_w \) depends only on \( |w| \).

2. The character of \( B_w \) lies in the range \( (0, \frac{1}{2}] \), attaining \( \frac{1}{2} \) only when \( |w| = \sqrt{s} \).

So a single-valued Blaschke-product must have at least 2 zeros (with multiplicity) in the annulus, and in the case of a single-valued Blaschke with two zeros, both must lie on the line \( |z| = \sqrt{s} \).

It is useful to extend here a classical result from the Hardy space of the disc, that for \( g \) outer we have \( g H^2 \) is dense in \( H^2 \).

**Theorem 1.2.20** Let \( g \in H^2(\mathbb{A}) \) be outer, and single-valued. Then \( g H^\infty \) is dense in \( H^2 \).
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Proof
This follows the same way as it does in the disc. We have that Cl(gH∞) is invariant under $M_z$ and $M_{z^{-1}}$, and closed by definition. Now, Beurling’s theorem has been extended to a general multiply connected region in [35] p161, and thus we have that $Cl(gH∞) = ΘH^2$ for some (single-valued, constant modulus on each boundary) inner function Θ. $g ∈ ΘH^2$, so we have Θ divides $g$. If $g$ is outer, it follows that Θ must be a unit, since the lift of Θ divides an outer function, it is outer, and we have from before that Θ has constant modulus on each boundary). However, any outer function with constant modulus on each boundary is invertible with inverse in $H^∞$, and so $ΘH^2 = H^2 = Cl(gH∞)$.

1.2.9 Reproducing kernels

A Hilbert $H$ consisting of functions on a set $X$ is said to be a reproducing kernel space if, for all $λ ∈ X$, we have there exists $k_λ ∈ H$ for which $⟨f, k_λ⟩ = f(λ)$ for all $f ∈ H$. $k_λ$ is referred to as a reproducing kernel for $λ$.

These can be a useful tool for the study of our space, and it happens that our Hardy spaces are reproducing kernel spaces.

For seperable Hilbert spaces, it is easy to construct our $k_λ$ in terms of a given orthonormal basis by the following well known lemma:

**Lemma 1.2.21** Let $H$ be a separable reproducing kernel Hilbert space of functions on $X$, $λ ∈ X$, and $(e_n)$ an orthonormal basis for $H$. Then $k_λ = \sum_{i=1}^{∞} \overline{e_n}(λ)e_n$

**Proof**
Let $f ∈ H$, then $f = \sum_n a_ne_n$ for some square summable sequence $(a_n)$. 
Now, we have that 
\[ \langle f, \sum_{n=1}^{\infty} e_n(\lambda)e_n \rangle = \langle \sum_{m=0}^{\infty} a_m e_m, \sum_{n=0}^{\infty} e_n(\lambda)e_n \rangle. \]
\[ = \sum_{n=0}^{\infty} a_n e_n(\lambda) = f(\lambda). \]
Thus, \( k_\lambda \) will be a reproducing kernel for \( H \).

\[ \square \]

Applying this lemma, we can construct the well-known reproducing kernels for the disc:

**Theorem 1.2.22** For \( H^2(\Delta) \), we have \( k_\lambda(z) = \frac{1}{1-\lambda z} \).

**Proof**
From Lemma 1.2.21 applied to our basis \( z^n \), we have that \( k_\lambda(z) = \sum_{n=0}^{\infty} \lambda^n z^n = \frac{1}{1-\lambda z} \).

\[ \square \]

In the case of \( H^2(\mathbb{A}) \), we can apply lemma 1.2.21 with respect to our basis \( \frac{z^n}{1+z^n} \) to obtain
\[ k_\lambda = \frac{\sum_{n=0}^{\infty} \lambda^n (\frac{z^n}{1+z^n})}{1+z^n}. \]
This does not have such a pleasing simplification as in the disc, but we can see that the series converges for all \( \lambda \in \mathbb{A} \), and so we are once again in a reproducing kernel space.
For a more general annulus \( D \), the reproducing kernels are somewhat harder to construct, and typically theta functions are used to do so, however we do know that we have a reproducing kernel space, and that \( k_\lambda \) are \( H^\infty(D) \) functions (see [8] for instance).

### 1.2.10 Vector valued Hardy Spaces

The vector valued Hardy Spaces are a classical extension of the Hardy space, whose elements are vector valued functions on the disc. An introduction to these can be found in [30] 3.11. In this thesis, only finite dimensional vectors will be considered, so one can consider these spaces to be defined as follows:

**Definition 1.2.23** For \( n \in \mathbb{Z}, n \geq 1 \), and taking \( \| f \|_p = \sup_{0 < r < 1} (\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{1/p} \) for \( f \in Hol(\Delta, \mathbb{C}^n) \), we define the vector valued Hardy spaces as
$H^p(\Delta, \mathbb{C}^n) = \{ f : f \in Hol(\Delta, \mathbb{C}^n), \|f\|_p < \infty \}.$

We will have once again that boundary values exist almost everywhere, and that $f$ tends to its boundary values a.e. along non-tangential paths. We will thus be able to define Toeplitz operators, whose symbols in this setting will be matrices as opposed to scalars in the scalar valued case.

In order to extend this to a multiply-connected region $D$, it helps to consider defining this space in terms of harmonic majorants instead. We have from [27] p3 that we can define our $H^2$ space as follows:

**Definition 1.2.24**

$$H^2(\Delta, \mathbb{C}^n) = \{ f : f \in Hol(D, \mathbb{C}^n), \|f\|_2 < \infty \},$$

where $\|f\|_2 = \inf \{ \nu(z_0)^{\frac{1}{2}} : \nu$ a harmonic majorant for $\|f\|^2 \}$, and $z_0$ some arbitrarily chosen point in $D$.

We will once again be able to lift back to the disc under a suitable covering map. In particular, since we already showed that the lifts under $\theta$ of non-tangential paths approaching the boundary of $A$ we will have that limits as we approach the boundary are defined a.e. and will lie in $L^2$, so we can discuss Toeplitz operators for the vector-valued Hardy space on $A$.

### 1.2.11 Bergman spaces

As well as the Hardy spaces, we can also define the Bergman spaces on $\Delta$, $A^p(\Delta)$.

**Definition 1.2.25** For $1 < p < \infty$, we have $A^p(\Delta) = \{ f : f \in Hol(\Delta, \mathbb{C}) : \|f\|_{A^p} = \int_\Delta |f(z)|^2 dA(z) < \infty \}$, where $A$ is Lebesgue area measure.
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[23] provides a good reference to these spaces. Now, we can no longer recover a function in this space by its boundary values, but instead we have that \( A^p(\Delta) \) a closed subspace of \( L^p(\Delta) \), (the limit of any \( A^p \) function is in \( L^p \) by definition, and holomorphic in the interior of \( \Delta \) since we have almost uniform convergence), so we can define Toeplitz operators on the Bergman spaces with symbols a function in \( L^\infty(\Delta) \).

One key difference is the lack of inner-outer factorisations. Blaschke products fail to be so useful in Bergman spaces since we can no longer remove a Blaschke product whilst preserving norm. Moreover, we no longer have the same conditions on how zeros may accumulate on the boundary, and an analogue of the necessary and sufficient conditions that we have in the annulus has yet to be found. In the Bergman space, for instance, if we have a set \( z_n \) of zeros of a Bergman space function lie on some radial line in the disc, we have \( \sum 1 - |z_n| < \infty \) as in the case of the Hardy space. However, it is quite possible for \( \sum 1 - |z_n| > \infty \) for the set of zeros of a Bergman space function. (See [23], chapter 4).

We have that \( A^2 \) is a reproducing kernel space, with \( k_\lambda(z) = \frac{1}{(1-\lambda z)^2} \).

1.2.12 Slit disc

We have \( G \) the slit disc is the set \( G = \{ z : |z| < 1, z \notin [0,1) \} \).

A full description of how to define the Hardy spaces on \( G \) can be found in [3]. The Hardy space here will be taken as what is referred to as \( E^2 \) and the Hardy-Smirnov class in [3] in slit domains, we no longer necessarily have that this is equivalent to the space one obtains when dealing with harmonic majorants.

As in [3] p11, \( E^2(G) \) is the spaces for which \( \sup_n \int_{\gamma_n} |f|^2 ds < \infty \), with \( ds \) arc-length measure, and \( (\gamma_n) \) a sequence of rectifiable Jordan curves, which eventually contain every compact subset of our domain.

An application of a theorem originally attributed to Keldysh and Lavarentiv, which can be found in [3] page 11, details that if \( \phi_G \) is a conformal map from the disc to \( G \), then \( f \in E^2 \) if and only if \( \sup_{0<r<1} \int_{\phi(\{|z|=r\})} |f|^2 ds < \infty \).
We can thus take $\|f\|_{E^2(G)} = \left( \sup_{0<r<1} \int_{\{ |z|=r \}} |f|^2 ds \right)^{\frac{1}{2}}$.
It will follow from this that $f \in E^2(G) \Leftrightarrow (f \circ \phi_G)(\phi_G')^{\frac{1}{2}} \in H^2(\Delta)$.

We can also see from this, (and since $\phi_G'$ extends a.e. to the boundary, as will be shown in chapter 5) that $f \in E^2(G)$ has boundary values a.e. (considering limits from above and below on the slit separately), and $\|f\|_{E^2(G)} = \int_{\partial \Delta} |f \circ \phi_G|^2 |\phi_G'| = \int_{\partial G} |f|^2 |dz|$, (Where we take the integral over the boundary to cover the slit twice, once with limits from above, once with limits from below) for $f \in E^2$, and the inner product can be written similarly.

### 1.2.13 Hankel operators

Hankel operators have long been studied in the disc, defined as follows:

**Definition 1.2.26** For $\phi \in L^\infty$, $\Gamma_\phi$ is defined to be the operator from $H^2 \to \overline{H}^2$ defined as $\Gamma_\phi f = P_{\overline{H}^2} \phi f$.

[31] provides a useful introduction to the Hankel operators, and covers important results such as Nehari’s theorem, the Nehari extension problem, and Nevalinna-pick interpolation. A Hankel operator will, if we represent it as a matrix with respect to our $z^n$ basis, have constant value along the finite length diagonals, $a_{n,m}$ with $n + m$ constant, in contrast to the result with Toeplitz operators constant along diagonals with $n - m$ constant (shown in [31] p30). Now, as opposed to Toeplitz operators, we no longer immediately have $\|\Gamma_\phi\| = \|\phi\|_\infty$, nor do we even have that the symbol of the operator is unique, for it is immediate that $\Gamma_f = 0$ whenever $f$ is an analytic function.

The problem of finding a symbol of optimal norm is a classic problem in the Hardy space of the disc. It turns out that there will always exist an optimal symbol $\phi$ for which $\|\phi\|_\infty = \|\Gamma_\phi\|$, and this is easy to construct for a Toeplitz operator achieving its norm:
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Theorem 1.2.27 Sarason’s solution to the Nehari problem

Suppose that $\Gamma$ is a Hankel operator on the disc, with $\|\Gamma f\|_2 = \|\Gamma\|_\infty \|f\|_2$. Then we have that $\phi = \overline{\Gamma f}$ is an optimal symbol for $\Gamma$ with $\|\phi\|_\infty = \|\Gamma\|$.

This can be seen in [31]. The theorem is quite powerful—while a Hankel operator need not achieve its norm on $H^2$, there are interesting classes of Hankel operators which will. Hankel operators with a symbol polynomial in $\bar{z}$, for instance—representing these as a matrix, only finitely many entries are non-zero. So, using finite dimensional linear algebra, it is easy to construct a function where our Hankel operator achieves its norm.

For multiply connected regions, there are two different ways that a Hankel operator can be defined, as can be found in [30].

One is to take $\Gamma \phi f = P_{H^2_\perp} \phi f$, which is the definition that will be considered in this thesis. The other is to take $\Gamma \phi f = P_{H^2} \phi \bar{f}$.

These parallel the two approaches that can be taken for Hankel operators defined on the Bergman spaces, which are referred to as the big and little Hankel operators. These act rather differently since the $L^2$ for Bergman spaces is a far larger space than $A^2$.

In comparison, for the Hardy spaces, $L^2 = H^2 + \overline{H^2} + N$ where $N$ has dimension equal to the genus of our annulus, as mentioned earlier.
Chapter 2

Factorisations

This section will discuss a few factorisation results in the annulus. One of the results will be an analogue of a factorisation result from the disc—the result that an $L^1$ function can be factorised as $far{g}$ with $f, g \in H^2$ whenever it is log-integrable, which comes from [9]. The next will be a multiplicative analogue of the standard additive decomposition, and the final one an extension of the $H^1 = H^2H^2$ factorisation in the disc.

2.1 Douglas-Rudin in the annulus

The following result is known in the Hardy space of the unit disc:

**Theorem 2.1.1** Let $f \in L^\infty(\partial\Delta)$, $log|f| \in L^1(\partial\Delta)$, and $\epsilon > 0$. Then there exist $g, h \in H^\infty(\Delta)$ such that $f = g\bar{h}$, with

$$\|f\|_\infty \leq \|g\|_\infty \|h\|_\infty < (1 + \epsilon)\|f\|_\infty.$$
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The proof of this, which will be outlined here, is that shown in [14]. This section shall focus on showing that a version of this also holds in the annulus, and the proof will be given afterwards.

A few preliminary results are shown in [14] with the theme of approximating functions with the argument of the ratio of two Blaschke products. The first important one is Theorem 3.2.5. of [14].

Lemma 2.1.2 Let $\epsilon, \theta, \delta \in (0, \pi)$, $\eta \in (0, \theta/\pi)$, $\alpha \in (\epsilon - \pi, \pi - \epsilon)$, and

$\Omega = \{e^{iw} : -\theta < w < \theta\}$.

Then there exist finite Blaschke products $B_1$ and $B_2$ with simple zeroes such that:

$$m(\{z \in \Omega : |\alpha - \arg(B_1(z)/B_2(z))| > \epsilon\}) < \eta,$$

with $-\delta < \arg(B_1(z)/B_2(z)) < \delta$ for all $z \in \partial\Delta \setminus \Omega$,

and $\sum_{z:B_1(z)B_2(z)=0}(1 - |z|) \leq 2m(\Omega) \log\left(\frac{4\pi}{\epsilon}\right)$.

This is then used to show that one can approximate measurable functions with such ratios:

Lemma 2.1.3 Let $\epsilon, \theta, \delta \in (0, \pi)$, $\nu \in (0, 1)$, $\Omega$ a relatively open subset of $\partial\Delta$, and $\psi : \Omega \to (-\pi, \pi]$ be a measurable function. Then there exist finite Blaschke products $B_1$ and $B_2$, with simple zeroes, such that:

$$m(\{z \in \Omega : |\psi(z) - \arg(B_1(z)/B_2(z))| > \epsilon\}) < \eta,$$

while $-\delta < \arg(B_1(z)/B_2(z)) < \delta$ for all $z \in \partial\Delta \setminus \Omega$,

and

$$\sum_{\{z:B_1(z)B_2(z)=0\}}(1 - |z|) \leq 2m(\Omega) \log\left(\frac{12\pi}{\epsilon}\right).$$

Lemma 2.1.3 is then strengthened to the following:
Lemma 2.1.4 Let $\epsilon \in (0, \pi]$, $A \subset \partial \Delta$ measurable, and $\phi : \partial \Delta \to (-\pi, \pi]$ a measurable function vanishing outside $A$. Then there exist Blaschke products $B_1$ and $B_2$ such that

$$\| \phi - \text{arg}(B_1/B_2) \|_\infty < \epsilon,$$

with $\sum \{ z : B_1(z)B_2(z) = 0 \} (1 - |z|) \leq 2m(A) \log(\frac{50\pi}{\epsilon})$.

The approximations are then used to show Theorem 3.2.8 in [14], namely,

Theorem 2.1.5 Let $\phi \in L^\infty(\partial \Delta, (-\pi, \pi])$ and $\epsilon > 0$. Then there exist Blaschke products $B_1$ and $B_2$ such that

$$\| H(\phi - \text{arg}(B_1/B_2)) \|_\infty < \epsilon,$$

$H$ denoting Hilbert transform.

The Douglas-Rudin factorisation then appears from Theorem 2.1.5 as Corollary 3.2.9 of [14], as follows:

**Proof**

Let $v$ be the outer function with

$$v(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{it})| \frac{e^{it} + z}{e^{it} - z} dt\right),$$

i.e. $|v| = |f|$ a.e.

Then $v \in H^\infty$, and $f/v$ is unimodular. Let $f/v = e^{it}$, $\phi \in L^\infty(\partial \Delta, (-\pi, \pi])$.

Theorem 2.1.5 shows that given any $\phi$, we have Blaschke products in the disc, $B_1$ and $B_2$, and $\gamma \in L_\infty$ such that $\gamma = \phi - \text{arg}(B_1/B_2)$, and $\| H(\gamma) \|_\infty < \log(1 + \epsilon)$, where $H$ is the Hilbert transform.

Thus, letting $F = \frac{1}{2}(\gamma + iH(\gamma))$, then $F \in H^\infty$, and $\exp(i\phi) = B_1 \exp(iF) \overline{B_2} \exp(-iF)$.

So if $g = vB_1 \exp(iF)$, $h = B_2 \exp(-iF)$, then $f = gh$, $\| g \|_\infty < \| f \|_\infty \sqrt{1 + \epsilon}$, and $\| h \|_\infty < \sqrt{1 + \epsilon}$. □
Adaptation to the annulus

The following analogue of the result can be shown in the annulus:

**Theorem 2.1.6** Let \( f \in L^\infty(\partial \mathbb{H}) \) be such that \( \log |f| \in L^1(\partial \mathbb{H}) \). Then, there exists \( g, h \in H^\infty(\mathbb{H}) \) with \( f = g \bar{h} \).

The proof of this will follow most of the method used in [14] for the disc, but several adaptations need to be made when working with the Hardy spaces for an annulus.

The following preliminary result is needed on outer functions with modulus close to 1.

**Lemma 2.1.7** Suppose that \( f \in H^2(\mathbb{H}) \) is outer, \( 1 - \epsilon < |f| < 1 \) on \( \partial \mathbb{H} \), and \( \tilde{f}(0) > 0 \), with \( \tilde{f} \) being the lift to \( H^2(\Delta) \) under \( \theta \) (\( \theta \) defined in section 1.12).

Then \( \|1 - f\|^2 < 2\epsilon \).

**Proof**

Since taking lifts back to \( \Delta \) preserves the given conditions, it is enough to show this is true for \( f \in H^2(\Delta) \), as \( H^2(\mathbb{H}) \) is equivalent to the subspace of lifts in \( H^2(\Delta) \).

Since \( f \) is outer, and \( 1 - \epsilon < |f| < 1 \) on \( \partial \mathbb{H} \), we have that \( 1 - \epsilon < |f(z)| < 1 \) \( \forall z \in \Delta \).

Now, \( \langle 1 - f, 1 - f \rangle = 1 + \|f\|^2 - 2Re(\langle f, 1 \rangle) = 1 + \|f\|^2 - 2Re(f(0)) \leq 1 + 1 - 2(1 - \epsilon) \leq 2\epsilon \).

We start with the following lemma:

**Lemma 2.1.8** Let

\[ \epsilon, \eta, \delta, \alpha, \nu, \nu_0, \eta_0 \in (0, \pi), \alpha \in (\epsilon - \pi, \pi - \epsilon), \Omega = \{e^{i\omega} : -t < \omega < t\}. \]

Then we have finite Blaschke products \( B_1, B_2 \in H^2(\mathbb{H}) \) with simple zeroes, such that

\[ m(\{z \in \Omega : |\alpha - \arg(B_1/B_2)| > \nu \}) < \nu, -\delta < \arg(B_1/B_2) < \delta \forall z \in \partial \mathbb{H} \setminus \Omega \text{ except} \]
on a set of measure at most $\eta$.

Also, $\sum_{z:B_1(z)B_2(z)=0}(1-|z|) \leq 2m(\Omega) \log(\frac{\omega}{\epsilon})$.

**Proof**

Let $B'_1$ and $B'_2$ be the restriction to the annulus of the Blaschke products for the corresponding theorem for the disc (Lemma 2.1.7). These will not be Blaschke products in $H^2(\mathbb{D})$— their modulus will be strictly less than 1 on $\Gamma_1$.

$B'_1/B'_2$ takes values as required on $\Gamma_0$, by construction. Consider $B'_1/B'_2$ on $\Gamma_1$.

We have that $B'_1$ is a Blaschke product in the disc with zeros at $z \in Z_1$, where $Z_1 = \{r^{1/N}e^{2\pi ik/N} : 0 \leq k \leq N-1\} \cap \{r^{1/N}e^{iw} : -\beta < w < \beta\}$ for some constants $r, N, \beta$, as in [14].

Now $B'_2$ has zero set $Z_2 = \{e^{i\alpha/N}z : z \in Z_1\}$.

Then $|\frac{e^{i\alpha/N}(w-z)(1-w^{-1})}{(1-wz)(w^{-1}e^{i\alpha/N})z} - 1| = |\frac{e^{i\alpha/N}(w-z)(1-w^{-1})}{(1-wz)(w^{-1}e^{i\alpha/N})z} - 1| = |s|^2 |\frac{(1-|z|^2)(1-e^{i\alpha/N})}{(1-s^2)(1-\omega z)(1-\omega z^* e^{-i\alpha/N})}|$

Now, we have that $(1-|w|^2) = (1-r^{2/N}) \leq 2N^{-1}\log(1/r)$, $|1-e^{i\alpha/N}| \leq \alpha/N$, as was the case in the disc. Also, we have that $|z-s^2\omega| \geq (s-s^2)$ and $|z-e^{i\alpha/N}| \geq (1-s)$, for $z \in \Gamma_1$.

Thus, we have that $|\frac{e^{i\alpha/N}(w-z)(1-w^{-1})}{(1-wz)(w^{-1}e^{i\alpha/N})z} - 1| \leq A\alpha N^{-2}\log(1/r)$ Since $B'_1/B'_2$ is the product of $N$ such terms, we have that on these boundaries, $|B'_1/B'_2 - 1| \leq (1+AN^{-2}\log(1/r))^{N-1} = O(1/N)$ for large $N$. However, in this construction, $N$ can be chosen to be arbitrarily large, so we can make $B'_1/B'_2$ arbitrarily close to 1 on $\Gamma_1$.

Since a bound on $\sum_{z\in Z_0}(1-|z|)$ independent of $N$ from the proof in [14] was used, this does not affect our bound on for $\sum_{B_1(z)B_2(z)=0}(1-|z|)$. Now, the $B'_1, B'_2$ constructed are not inner functions for the annulus, however, letting $B'_1 = f_1f_0, B'_2 = g_1g_0$, inner-outer factorisations into (potentially multiply-valued) functions, we have that $B'_1/B'_2 = f_1f_0/(g_1g_0) = (f_0/g_0)f_1g_1$.

Since $|B'_1/B'_2| - 1$ can be made arbitrarily small, we have that $|f_0/g_0| - 1$ can be made arbitrarily small. However, since $|g_0| = |B'_2|$ we have that $|g_0|$ is continuous, bounded
away from 0 and infinity, thus \( f_o/g_o \in H^2 \) and is another outer function.

Since \(|f_o/g_o|\) is close to 1 everywhere, we have from Lemma 2.1.7 that \( \|f_o/g_o - 1\| \) can be made arbitrarily close to 0, and thus have argument close to 0 except on a set of small measure. We must choose \( N \) sufficiently large that this set has measure less than \( \eta \).

(Since we do not necessarily have approximation in \( \infty \)-norm, we no longer have \( B_1/B_2 \) has small argument outside \( \Omega \) in our final result, as opposed to the case for the disc in [14]).

Now, \( f_i \) and \( g_i \) obtained need not be single valued, however this can be avoided— the zero sets of \( f_i \) and \( g_i \) coincide with those of \( B'_1 \) and \( B'_2 \) by construction, and the zero set of \( B'_2 \) is a rotation of the zero set of \( B'_1 \).

In \( H^2(\mathbb{A}) \), the character of a Blaschke product with zero at \( z \) depends only on \(|z|\), and so \( f_i \) and \( g_i \) will have the same character, call it \( \kappa \).

Then, let \( B_1 = f_iK \), and \( B_2 = g_iK \), where \( K \) is a finite Blaschke product with character \(-\kappa\) (the existence of such a \( K \) is proven in [2] p269). Then \( B_1\overline{B_2} = f_i\overline{g_i} \), and both have character 0, i.e. they are single-valued.

\( \square \)

From this, we proceed to the following refinement, which is an analogue of Lemma 3.2.6 from [14]:

**Lemma 2.1.9** Let \( \epsilon, \delta \in (0, \pi) \), \( \nu, \eta \in (0, 1) \), \( \Omega \) relatively open, and \( \Omega \subset \theta^{-1}(\Gamma_0) \) and \( \phi : \partial \Delta \to (-\pi, \pi] \) a \( \psi \)-invariant measurable function with support in \( \Omega \). Then there exist finite, \( \psi \)-invariant Blaschke products with \( m(\{z \in \Omega : |\phi - \arg(B_1/B_2)| > \epsilon\}) < \nu \), with \(-\delta < \arg(B_1/B_2) < \delta\) outside \( \Omega \) except on a set of measure \( \eta \), and \( \sum z:B_1(z)\overline{B_2(z)} = 0 \) \((1 - |z|) \leq 2m(\Omega) \log \frac{12\pi}{\epsilon}\).

Essentially, this says that if one takes a function on the outer boundary of the annulus and lift back to the disc, then one can approximate with Blaschke products that are also lifts of Blaschke products from the annulus. Unlike in the case of the disc, we no longer
have \( \arg(B_1/B_2) \) small everywhere outside \( \Omega \), since our previous approximation in 2.1.8 no longer had this property (as a result of having to multiply by an outer function lying close to 1 in norm).

**Proof**

As in [14], simply take an appropriate step function approximating the function that we wish to target (using lifts of the functions in 2.1.8), and factorise this as a product of functions of the type dealt with in Lemma 2.1.8. The result then follows. \( \square \)

Now, since this holds for the outer boundary, it does so after application of conformal equivalences. So, for \( \Gamma_i \), we can apply a conformal equivalence to swap \( \Gamma_i \) and \( \Gamma_0 \), apply the result, then move back via the conformal equivalence. Thus we can drop the requirement that \( \Omega \) has support on \( \theta^{-1}(\Gamma_0) \).

So, we have

**Lemma 2.1.10** Let \( \epsilon, \delta \in (0, \pi), \nu, \eta \in (0, 1), \Omega \) relatively open and \( \phi : \partial \Delta \to (-\pi, \pi] \) a \( \psi \)-invariant measurable function. Then there exist finite, \( \psi \)-invariant Blaschke products with \( m(\{z \in \Omega : |\phi - \arg(B_1/B_2)| > \epsilon\}) < \nu \), while \( -\delta < \arg(B_1/B_2) < \delta \) outside \( \Omega \) except on a set of measure \( \eta \), and

\[
\sum_{z : B_1(z)B_2(z)=0} (1 - |z|) \leq 2m(\Omega) \log \frac{12\pi}{\epsilon}.
\]

From this lemma, we can produce the following analogue of [14] Theorem 3.2.7:

**Lemma 2.1.11** Let \( \epsilon \in (0, \pi], \Omega \subset \partial \Delta \) and \( \phi \) a \( \psi \)-invariant measurable function with support \( \Omega \). Then there exist \( \psi \)-invariant Blaschke products \( B_1 \) and \( B_2 \) with \( \|\phi - \arg(B_1/B_2)\|_\infty < \epsilon \), \( \sum_{z : B_1(z)B_2(z)=0} (1 - |z|) < 2m(\Omega) \log \frac{50\pi}{\epsilon} \).
The proof is mostly the same as in the disc from [14], except for use of Borel-Cantelli since our approximations are weaker than in [14].

**Proof**

Let \((\nu_k)\) and \((\delta_k)\) be sequences with \(2\sum_{k=1}^{\infty} \nu_k \leq m(\Omega) \log\left(\frac{26}{21}\right)\) and \(\sum_{k=1}^{\infty} \delta_k < \epsilon/4\).

Let \(\Xi_1\) be an open set (on the boundary) containing \(\Omega\), \(m(\Xi_1) < m(\Omega) + \nu_0\), and let \(\psi_1\) be defined by:

\[
\psi_1(z) = \phi(z) - \epsilon/2 \quad \text{if} \quad \phi(z) > 0,
\]

\[
\psi_1(z) = \phi(z) + \epsilon/2 \quad \text{for} \quad \phi(z) \leq 0.
\]

Then, we construct inductively a sequence of open sets \(\Xi_k\), \(\Omega_k\), measurable functions \(\psi_k\), and Blaschke products \(B_1^{(k)}\), \(B_2^{(k)}\) such that the following hold:

1. 
\[
\sum_{\{z: B_1^{(k)} B_2^{(k)} = 0\}} (1 - |z|) \leq 2m(\Xi_k) \log\left(\frac{48\pi}{\epsilon}\right).
\]

2. 
\[
m(\{z \in \Xi^c_k : |\arg(B_1^{(k)} / B_2^{(k)})| > \delta_k\}) < \nu_k.
\]

3. \(\Xi_k\) contains
\[
\Omega_{k+1} = \{z \in \Omega_k : |\psi_k(z) - \arg(B_1^{(k)} / B_2^{(k)}(z))| > \epsilon/4\} \bigcup \{z \in \Omega_k : |\arg(B_1 / B_2)| > \delta_k\}.
\]

and also
\[
m(\Xi_k) < m(\Omega_k) + \nu_k < 2\nu_k.
\]

4. 
\[
\psi_{k+1} = \phi(z) - \epsilon/2 - \sum_{j=1}^{k} \arg(B_1^{(j)}) - \arg(B_2^{(j)}) \quad \text{for} \quad \phi(z) > 0.
\]

\[
= \phi(z) + \epsilon/2 - \sum_{j=1}^{k} \arg(B_1^{(j)}) - \arg(B_2^{(j)}) \quad \text{for} \quad \phi(z) \leq 0.
\]
If we let \( l \) be a positive integer, and suppose we have found \( \Xi_k^{+1}, \psi_k^{+1}, B_1^{(k)}, B_2^{(k)} \) for \( k < l \).

Then, applying Lemma 2.1.10 with \( \nu = \nu_l, \delta = \delta_l, \Omega = \Xi_l \) and \( \psi = \psi_l|_\Omega, \epsilon = \epsilon/4 \), we obtain \( B_1 \) and \( B_2 \) \( \psi \)-invariant Blaschke products with

\[
m(\{z \in \Xi_l : |\psi(z) - \arg(B_1^{(l)}/B_2^{(l)})| > \epsilon/4\}) < \nu_l, \text{ and } |\arg(B_1^{(l)}/B_2^{(l)})| < \delta
\]

outside \( \Omega_k \) except on a set of measure \(< \nu\)

and \( \sum_{B_{1(z)}B_{2(z)}=0}(1 - |z|) \leq 2m(\Omega) \log(\frac{48\pi}{\epsilon}) \).

Now, we have that 1, 2, 3, 4 are all satisfied for \( k = l \), hence by induction we can make the construction for all integers.

We have that \( \sum_{k=1}^{\infty} \sum_{B_1^{(z)}B_2^{(z)}=0}(1 - |z|) \leq 2m(\Xi_k) \log(\frac{48\pi}{\epsilon}) \).

Since \( m(\Xi_k) < 2\nu_k \), this series is summable, so \( B_1 = \Pi_{k=1}^{\infty} B_1^{(k)} \) and \( B_2 = \Pi_{k=1}^{\infty} B_2^{(k)} \) exist, and are Blaschke products, with \( \sum_{B_{1(z)}B_{2(z)}=0}d(z, \partial \Delta) \leq 2m(\Omega) \log(\frac{50\pi}{\epsilon}) \).

To show that we have \( \|\phi - \arg(B_1/B_2)\|_\infty < \epsilon \), note that as \( m(\Omega_k) \) is summable, we have that \( m\{z : z \in \Omega_k \text{ infinitely often}\} = 0 \) by the Borel-Cantelli Lemma.

Thus, we have that, aside from a null set, for each \( z \in \partial \Delta \) we have there exists an \( l \) for which \( z \in \Omega_l \) and \( z \notin \Omega_k \forall k > l \).

Thus, \( \psi_{l+1}(z) \leq \epsilon/4 \), and since \( |\arg(B_1^{k}/B_2^{k})| < \delta_k \) all \( k > l \), and \( \sum \delta_k \leq \epsilon/4 \), we have that \( \psi_k(z) \leq \epsilon/2\forall k > l \).

It thus follows that \( \|\phi - \arg(B_1/B_2)\|_\infty < \epsilon \), and \( \sum(1 - |z|) < 2m(\Omega) \log(\frac{50\pi}{\epsilon}) \).

We then have that the following analogue of [14] Theorem 3.2.8 holds:

**Theorem 2.1.12** Let \( \phi \in L^\infty(\partial \Delta) \) be \( \psi \)-invariant, and \( \epsilon > 0 \).

Then there exist \( \psi \)-invariant Blaschke products \( B_1 \) and \( B_2 \) such that

\[
\|H(\phi - \arg(B_1/B_2))\|_\infty < \epsilon.
\]

where \( H \) is the Hilbert transform.
Proof

Let $\delta$ be constant depending on $\epsilon$, $\phi_0 = \phi$, $h_1 = 0$. Then, by the previous lemma, we have $\phi_1$, and Blaschke products $B_1^{(1)}$, $B_2^{(1)}$, $\|\phi_1\|_\infty \leq \frac{\delta}{2}$ and $\phi_1 = \phi_0 - \text{Re}(h_1) - \text{arg}(B_1^{(1)}(z)B_2^{(1)}(z)=0(1-|z|) \leq \log(\frac{100\pi}{\delta})$.

Suppose for $k \geq 1$, we have $\phi_k$ $\psi$-invariant, with $\|\phi_k\|_\infty \leq 2^{-k}\delta$.

Then, by Corollary 3.2.4 from [14], we have that there exists $h_{k+1}$ with

$$\|\phi_k - \text{Re}(h_{k+1})\|_2 \leq 2^{-k}\|\phi_k\|_\infty \leq 2^{-2k}\delta.$$  

and $\|h_{k+1}\|_\infty \leq c\log(2^k)\|\phi_k\|_\infty \leq \frac{ck\log 2}{2^k}\delta$.

We need to adapt this so that the $h_k$ obtained is modulus-invariant. Let $f$ be a real-valued bounded $\psi$-invariant function, and $g_1$ and $g_2$ analytic functions such that $f = g_1 + \overline{g}_2$.

Since $f \in H^\infty$, these will be BMOA functions, and thus lie in $H^1$.

Then we must have that $g_1=g_2+A$, where $A$ a constant, since $f = \bar{f}$ thus $g_1+\overline{g}_2 = \overline{g}_1+g_2$,  

thus $g_1 - g_2 = g_1 - g_2$, and so $g_1 - g_2 = A$.

So, $(f - A) = g + \bar{g}$. Since the left hand-side is $\psi$-invariant, we have that $\text{Re}(g)$ is invariant, and thus $g$ must be $\psi$-invariant (otherwise, $g - \psi \circ g$ would be in $H^p$ and have vanishing real part on the boundary).

Now, applying [14] Lemma 3.2.2 to deal with boundedness, the construction involves multiplying by an outer function whose modulus on the boundary will be invariant, and thus this yields a modulus-invariant function $h_{k+1}$. Suppose this has character $-\beta_{k+1}$, and since $h_k$ approximates an invariant function arbitrarily well for large $k$, we must have that the $\beta_k$ are tending to 0. So, replacing $h_{k+1}$ with $e_{\alpha_{k+1}}h_{k+1}$, with $e_{\alpha_k}$ the lift to the disc of the function $z^{\alpha_k}$ in the annulus, where $\alpha_k = \beta_k + m$, with $m \in \mathbb{Z}$ chosen so that $-\frac{1}{2} \leq \alpha_k \leq \frac{1}{2}$, we recover an invariant function. Since $-\frac{1}{2} \leq \alpha_k \leq \frac{1}{2}$, we at most increase by a constant multiple of $s^{-1}$ for $\|e_{\alpha_{k+1}}h_{k+1}\|_\infty$. We need the following to show that we still approximate in $L_2$ as well:

**Lemma 2.1.13** $\|\phi_k - \text{Re}(h_{k+1}e_{\alpha_{k+1}})\|_2 \leq 2^{-2k}E\delta$ for $E$ some constant depending only on the inner radius of our annulus $A$. 


**Proof**

Since we have that \( \|h_{k+1} - e_{\alpha_{k+1}}h_{k+1}\|_2 = \|h_{k+1}(1 - e_{\alpha_{k+1}})\|_2 \leq \|h_{k+1}\|_\infty \|1 - e_{\alpha_{k+1}}\|_2 \), and calculate the integral on the strip \( S = \{ z : 0 < Re(z) < -\log s \} \) (choosing the map from disc to strip so that composition with \( e^{-z} \) will return our map from the disc to annulus \( \theta \)) with measure on the boundary \( |dz| \frac{1}{\cosh(Im(z))} \) (which is an equivalent norm to lebesgue measure in the disc on applying our conformal equivalence, by [7]. Since \( z \rightarrow e^{-z} \) is our map from the strip to the annulus, we have that our \( e_{\alpha_k} \) lifts to \( e^{-z\alpha_k} \).

So \( \|1 - e_{\alpha_{k+1}}\|_2^2 \leq \sum_{n=1}^{\infty}(\sup_{2n\pi \leq \theta \leq 2(n+1)\pi} |1 - e^{i\alpha_k\theta}|^2 + |1 - s^{\alpha_k}e^{i\alpha_k\theta}|^2)\mu(A_n) \), where \( A_n \) is the set \( \{ it : 2n\pi < t < 2(n+1)\pi \} \cup \{-\log s + it : 2n\pi < t < 2(n+1)\pi \} \).

Now, \( |1 - s^{\alpha_k}e^{i\alpha_k\theta}|^2 \leq |1 - e^{i\alpha_k\theta}|^2 + |1 - s^{\alpha_k}|^2 \), so we have that:

\[
\|1 - e_{\alpha_{k+1}}\|_2^2 \leq 2 \sum_{n=1}^{\infty}(\sup_{2n\pi \leq \theta \leq 2(n+1)\pi} |1 - e^{i\alpha_k\theta}|^2\mu(A_n)) + (1 - s^{\alpha_k})^2.
\]

Using the approximation \( |1 - e^{ix}| \leq |x| \) (which follows since the length of the chord between 1 and \( e^{ix} \) on the unit disc is bounded by the length of the arc between them):

\[
\|1 - e_{\alpha_{k+1}}\|_2^2 \leq \sum_{n=1}^{\infty} \pi^2(s + 1)^2\alpha_k^2 \frac{1}{\cosh(2\pi n)} + |1 - s^{\alpha_k}|^2.
\]

Since \( \sum_{n=1}^{\infty}(2\pi(n + 1))^2 \frac{1}{\cosh(2\pi n)} = D \) for \( D \) some real constant, it follows that

\[
\|1 - z^{\alpha_k}\|_2^2 \leq \alpha_k^2 D_1 + (1 - s^{\alpha_k})^2 \leq \alpha_k^2 C \text{ for some real constant } C \text{ depending only on our inner radius.}
\]

To get bounds for \( \alpha_k \), suppose we have that \( s_k \) is the \( H^2(\Delta) \) function such that \( \phi_k = s_k + \bar{s}_k \) and \( s_k(0) \) is real.

Then (see [14] Lemma 3.2.1 and corollary 3.2.4), we obtain \( h_k \) by multiplying \( s_k \) by the outer function \( \tilde{G}_k \), the lift of \( G_k \in H^2(\mathbb{D}) \) to the disc where \( |G_k| = 1 \) on \( F_{2k} = \{ z : z \in \partial \mathbb{D}, |s_k(\theta^{-1}(z))| \leq 2^k \} \), and \( |G_k| = \frac{2^k}{|s_k|} \) on \( E_{2k} = F_{2k}.c \).

Since \( s_k \) was invariant, we only need to know the character of \( G_k \), and \( \alpha_k = \frac{(\int_{|s_k|^{\log s}} \log |G_k| - \int_{|s_k|^{\log s}} \log |G_k|)}{\log s} \).

Thus \( |\alpha_k| \leq \frac{1}{\log s} \int_{|s_k|^{\log 2^k}} \log \left( \frac{|G_k|}{|s_k|} \right) \leq \frac{1}{\log s} \int_{|s_k|^{\log 2^k}} \log (|s_k|) \leq \frac{1}{\log s} \sum_{k \leq t < \infty} (t + 1) \mu(\{ z : 2^t < |s_k(z)| < 2^{t+1} \}) \).

Now, we have \( \mu(z : 2^t < |s_k(z)| < 2^{t+1}) \leq \frac{1}{2\pi} \|s_k\|_2^2 \).
Thus $|\alpha_k| \leq \frac{1}{-2\pi \log s} \|s_k\|_2^2 \sum_{t=k}^{\infty} \frac{(t+1)}{2^t}$. Now, since $\|s_k\|_2^2 \leq \|\phi_k\|_2^2 \leq \|\phi_k\|_\infty^2$ (For the first inequality, if $\phi_k = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n \overline{z}^n$, we have $s_k = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n$, and so the result follows), and the series $\sum_{t=k}^{\infty} \frac{(t+1)}{2^t}$ has a bound independent of $k$, we have that $|\alpha_k| \leq E(2^{-2k})\delta$, where $E$ is a constant depending only on our inner radius.

It will follow that $\|\phi_k - Re(h_{k+1}e_{k+1})\|_2 \leq \|\phi_k - Re(h_{k+1})\|_2 + \|h_{k+1} - h_{k+1}e_{k+1}\|_2 \leq E2^{-2k}\delta$ for some constant $E$ depending only on inner radius (since $\delta$ will later be chosen to be small, we can assume $\delta < 1$ and so $\delta^2 < \delta$). □

The rest of the proof of this theorem proceeds without further need for modification from that in [14] Theorem 3.2.8.

Letting $g_{k+1} = h_{k+1}z^{\alpha_{k+1}}$, we have that:

$$\|\phi_k - Re(g_{k+1})\|_2 \leq 2^{-2k}E\delta$$

$$\|g_{k+1}\|_\infty \leq \frac{ck \log 2}{2^k} \delta.$$ 

Letting $\Delta_k = \{z \in \partial \Delta : |\phi_k - Re(g_{k+1})| > 2^{-(k+2)}\delta\}$, we have that this will be $\psi$-invariant, and thus $f_k = (\phi_k - Re(h_{k+1}z^{\alpha_{k+1}}))\chi_{\Delta_k}$ will be $\psi$-invariant, and $m(\Delta_k) \leq \frac{(2^{-2k}E\delta)^2}{2^{-2(k+2)}\delta^2} \leq \frac{16E^2 k^2}{2^{2k}}$.

We can, by lemma 2.1.11 construct $B_1^{k+1}/B_2^{k+1}$ $\psi$-invariant, with

$$\|f_k - \arg(B_1^{k+1}/B_2^{k+2})\|_\infty < 2^{-(k+2)},$$

$$\sum_{B_1(z)B_2(z)=0} (1 - |z|) \leq 2m(\Delta_k) \log \left(\frac{50\pi}{2^{-(k+2)}\delta}\right).$$

Take $\phi_{k+1} = \phi_k - Re(g_{k+1}) - \arg(B_1^{k+1}/B_2^{k+1})$.

Then, by evaluating on $\Delta_k$ and $\Delta_\epsilon$ separately, we can see that $\|\phi_{k+1}\|_\infty \leq 2^{-(k+1)}\delta$.

We have that $\sum_{k=0}^{\infty} \|\phi_k - \arg(B_1^{k+1}/B_2^{k+2}) - \phi_{k+1}\|_\infty = \sum_{k=0}^{\infty} \|Re(g_{k+1})\|_\infty \leq \sum_{k=0}^{\infty} \|g_{k+1}\|_\infty \leq \epsilon$ for suitably small $\delta$.

Thus $\sum_{k=0}^{\infty} \phi_k - \arg(B_1^{k+1}/B_2^{k+1}) - \phi_{k+1} = \phi_0 - \arg(\Pi_{k=0}^{\infty}B_1^{k+1}/B_2^{k+1})$.

Also, $\|H(\phi_0 - \arg(\Pi_{k=0}^{\infty}B_1^{k+1}/B_2^{k+1}))\|_\infty = \|H(\sum_{k=0}^{\infty} \phi_k - \arg(B_1^{k+1}/B_2^{k+1}) -$
\[ \phi_{k+1} \|_{\infty} = \| H(\sum Re(g_{k+1})) \|_{\infty} \leq \sum \| Im(g_{k+1}) \|_{\infty} \leq \sum \| g_{k+1} \|_{\infty} < \epsilon. \] And, we also have \[ \sum_{k=0}^{\infty} \sum_{B_1^{k+1}(z)B_2^{k+1}(z)=0} (1 - |z|) < \infty. \] Thus the products of the \( \psi \)-invariant Blaschke factors converge to suitable \( \psi \)-invariant Blaschke products \( B_1 \) and \( B_2 \).

From this, we can show the main theorem, which now proceeds in the same way as the the proof in [14] follows from Theorem 3.2.8. in that book:

**Theorem 2.1.14** Whenever \( f \in L^\infty(\partial \mathbb{A}) \) with \( \log |f| \in L^1 \), we have \( f = gh \) for some \( g, h \in H^\infty(\mathbb{A}) \).

**Proof**

First, lift to the disc so that we have a \( \psi \)-invariant boundary function in the disc, factor out an (invariant) outer function so that we can restrict to the case of factorising a unimodular function (to deal with this outer function having non-integer character, use the fact that \( z^\alpha s_i \bar{s}_i = |z|^{2\alpha} \) in the annulus and taking lifts back to the disc, we can adjust for this (where \( s_i \) is some inner function with the right character to make \( z^\alpha s_i \) single valued).

So \( f = e^{i\phi} \), with \( \phi \) real valued. Lifting to the disc, we can produce \( B_1 \) and \( B_2 \) invariant such that \( \| H(\phi - \arg(B_1/B_2)) \|_{\infty} < \log(1 + \epsilon) \).

Then, have that \( F = \frac{1}{2}(\gamma + iH(\gamma)) \in H^\infty \) where \( \gamma = \phi - \arg(B_1/B_2) \), and \( \exp(i\phi) = B_1 \exp(iF) \bar{B}_2 \exp(-iF) \).

Since \( \gamma \) is \( \psi \)-invariant, so is \( F \) (since otherwise, \( F - F \circ \psi \) would be non-constant, analytic with constant real part) and thus the decomposition is in terms of invariant functions, which can then be passed down to the annulus to obtain our factorisation.

The proof here does not seem to extend to a higher genus annulus, the main problem being that Lemma 2.1.8 appears difficult to adapt to have \( B_i \) single-valued for the higher genus annulus.
2.2 Multiplicative factorisation

There is a well known additive decomposition for $H^2(\mathbb{A})$ into $H^2(\Delta) \oplus H^2((s\Delta)^c)$.

If $f = \sum_{n=0}^{-\infty} a_n z^n$, then $f = f_1 + f_2$ with $f_1 = \sum_{n=0}^{\infty} a_n z^n$, and $f_2 = \sum_{n=\infty}^{-1} a_n z^n$.

It is immediate that $f_1 \in H^2(\Delta)$, $f_2 \in H^2((s\Delta)^c)$, and $f_1 \perp f_2$.

The following gives a multiplicative decomposition which somewhat parallels this.

**Theorem 2.2.1** Let $f \in H^p(\mathbb{A}), p \in [1, \infty]$. Then there exists $f_1, f_2$ with $f = f_1 f_2$, $f_1 \in H^p(\Delta)$, and $f_2 \in H^p((s\Delta)^c)$, such that $f = f_1 f_2$.

**Proof**

First, consider the case when $|f| > 0$ in $\mathbb{A}$. Since $f$ is analytic and non-zero on $\mathbb{A}$, it has well-defined winding number about 0. Call this $n_f$. Then, letting $f' = fz^{-n_f}$, $f'$ has winding number 0 about 0, and thus we have a well-defined logarithm in $\mathbb{A}$.

Let $\log f' = \sum_{n=1}^{\infty} a_n z^n$ be the power representation of this function, and set $g_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $g_2(z) = \sum_{n=-\infty}^{1} a_n z^n$. Since $\log f'$ is bounded on $\{z : (s + \epsilon) \leq |z| < 1 - \epsilon\}$ for all $\epsilon > 0$, it follows that $g_1$ is bounded on $\{z : |z| < 1 - \epsilon\}$, and $g_2$ bounded on $\{z : |z| > s + \epsilon\}$ for all $\epsilon > 0$.

We have that $f = z^n e^{g_1 + g_2} = z^n e^{g_1} e^{g_2}$, so we have only to verify that $e^{g_1} \in H^p(\Delta)$, and $e^{g_2} \in H^p((s\Delta)^c)$.

$g_2$ is analytic and non-zero on $\{|z| > s + \epsilon\}$, and hence $|e^{g_2}|$ has non-zero infimum.

Letting $A_2 = \inf_{|z|=1} |e^{g_2}| \neq 0$, we have that $\int_{|z|=1} |e^{g_2}|^p \leq \int_{|z|=1} |f'|^p A_2^{-p} < \infty$.

Hence $e^{g_1} \in H^p(\Delta)$ (since we have a power series consisting of only positive powers of $z$, we know that that the integral on $|z| = \rho$ is strictly increasing with $\rho$). By a symmetric argument, $e^{g_2} \in H^p((s\Delta)^c)$.

Thus a function without zeros can be factorised. However, for an arbitrary function $f$, we can remove a Blaschke factor to deal with the zeros— let $B_1$ be a Blaschke product in $H^p(\Delta)$ with $B_1(z) = 0 \iff \{f(z) = 0, |z| \geq s_1\}$, and $B_2$ a Blaschke product in...
$H^p((s\Delta)^c)$ with $B_2(z) = 0 \Leftrightarrow \{f(z) = 0, |z| < s^{\frac{1}{2}}\}$.

Then we have that $f = B_1B_2f'$ with $f' \in H^p, |f'| > 0$ in $A$. Letting $f' = f_1f_2$ from before, we have that $f = (f_1B_1)(f_2B_2)$ factorises the function as required. □

### 2.2.1 Non-uniqueness

This factorisation fails to be unique— if one takes $f$ with finitely many zeros, the Blaschke product factor of $f$ can be included it in either factor, and each will yield a different factorisation (since the method for factorising non-zero functions produces non-zero functions in their respective domains).

Consider the following example:

Take $p(z) = z - \frac{1}{2}$, assuming that $s < \frac{1}{2}$.

To factorise this, we can take a Blaschke product for $\Delta$, zero at $\frac{1}{2}$, or for $(s\Delta)^c$.

The first choice gives the factorisation $p(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z}(1 - \frac{1}{2}z)$, we observe that the second factor already lies in $H^2(\Delta)$, so we are done.

Alternatively, we can take the Blaschke product for $(s\Delta)^c$ with zero at $\frac{1}{2}$ ($= \frac{1}{2}z$).

So, we have $z - \frac{1}{2} = \frac{\frac{1}{2} - 2z}{s - \frac{1}{2}z}(s - \frac{1}{2}z)$.

$s - \frac{1}{2}z$ winds about 0 on any loop in our annulus, so take out a factor $z$. $s - \frac{1}{2}z = -z(\frac{1}{2}z - sz^{-1})$. But $\frac{1}{2}z - sz^{-1}$ is already in $H^2((s\Delta)^c)$, so we are done. Simplifying each product we have, $z - \frac{1}{2} = (z - \frac{1}{2})(1 = z)(1 - \frac{1}{2}z)$ are the two different ways of writing the product as $H^2(\Delta)H^2((s\Delta)^c)$, depending on which side the zero goes. These factorisations differ by multiplication of meromorphic functions, and we can in fact show that factorisation is unique up to meromorphic functions.

**Theorem 2.2.2** Suppose that we have $f, f_1, f_2, f_3, f_4$ with $f \in H^2(\Delta)$, $f_1, f_3 \in H^2(\Delta)$, $f_2, f_4 \in H^2((s\Delta)^c)$, and $f = f_1f_2 = f_3f_4$.

Then we have that $f_3/f_1$ is meromorphic in $\mathbb{C}$. 

Proof

Eugene Shargorodsky suggested the following proof, which is simpler than the original.

We have that $\frac{f}{f_4} = \frac{f_2}{f_4} = \frac{f_3}{f_1}$ on $\mathbb{A}$. Now, we have that $\frac{f_2}{f_4}$ is meromorphic in $(s\Delta)^c$, as it is the ratio of analytic functions in that domain. We also have that $\frac{f_3}{f_1}$ is meromorphic in $\Delta$. Since $\frac{f_2}{f_4} = \frac{f_3}{f_1}$, we can extend to a meromorphic function on $\mathbb{C}$, as required. □

2.2.2 Extension to higher genus setting

We can extend this factorisation as follows:

**Theorem 2.2.3** Let $D$ be a general genus $g$ annulus, consisting of the unit disc $\Delta$ with $g$ components $A_i$ removed, with each $A_i$ conformally equivalent to the unit disc. Let $f \in H^p(D)$. Then there exists $f_i$, $0 \leq i \leq g$, such that $f_0 \in H^p(\Delta)$, and $f_i \in H^p(A_i^c)$.

**Proof**

We can prove this by induction. Firstly, note that if we are in the case $g = 1$, we can apply a conformal equivalence to ensure that the inner disc is centred about 0, and then apply the case for $\{z : s < |z| < 1\}$.

Now, suppose $D$ is an annulus of genus $n + 1$ constructed as in the theorem. Let $D' = \{z : s_1 < |z| < 1\}$, where $s_1$ is chosen so that $D'$ is contained in $D$. Now, if we have $f \in H^2(D)$, we also have that $f \in H^p(D')$, and applying the $g = 1$ case, we have that $f = f_0f_1$ with $f_0 \in H^p(\Delta)$, and $f_1 \in H^p(\{z : |z| > s_1\})$.

We can extend $f_1$ analytically by $f_1 = f/f_0$ to yield $f_1 \in H^p(D \cup \Delta^c)$, with $f = f_0f_1$.

Now, $D \cup \Delta^c$ is a genus-$n$ annulus. Through application of a conformal equivalence, we can send this to an annulus consisting of the unit disc with $g - 1$ removed subdiscs, and thus $f_1$ can be factorised completely. Moving back to the original domain with a conformal equivalence, we will have that
\[ f = f_0 f_1 \ldots f_g. \]

With \( f_i \in H^p(A_i^e), f_0 \in H^p(\Delta). \) □

### 2.3 Factorisation of \( H^1 \) function into \( H^2 \) functions

In the disc, the following result is a well known result following from the inner-outer decomposition of functions:

**Theorem 2.3.1** Let \( f \in H^1(\Delta) \), then there exist \( g_1, g_2 \) with \( f = g_1 g_2 \), \( g_i \in H^2(\Delta) \), and
\[ \| f \|_1 = \| g_1 \|_2 \| g_2 \|_2. \]

In the annulus, whilst an \( H^1 \) function can still be factorised as a product of \( H^2 \) functions, the norm of the factorisation is no longer preserved— which is quite important with later consideration of Hankel operators.

**Theorem 2.3.2** Let \( f \in H^1(\mathbb{A}) \), then there exist \( g_1, g_2 \), \( f = g_1 g_2 \), \( g_i \in H^2(\mathbb{A}) \),
\[ \| g_1 \|_2 \| g_2 \|_2 \leq A \| f \|_1, \] where \( A \in \mathbb{R} \) is a constant.

This has been shown already in [4] for arbitrary multiply connected regions, with the constant of factorisation depending on the region in question. In the case of an annulus of the form \( \{ z : s < |z| < 1 \} \), this will be improved upon to show that a constant of factorisation independent of \( s \) can be obtained.

A proof of the theorem with the constant \( A = \frac{1}{s} \) depending on the inner radius of \( \mathbb{A}, s \), will first be given. Later, this constant will be made independent of the inner radius of the annulus \( \mathbb{A} \). The method is akin to that in the disc, with a small alteration.

**Proof**
Let \( f = f_i f_o \) be the inner outer factorisation of \( f \).

Then let \( g_1 = (f_i f_0^{1/2} z^{\alpha/2}) \), and \( g_2 = (f_0^{1/2} z^{-\alpha/2}) \), where \( \alpha \in (0, 1) \) is the character of \( f_o \).

(The \( z^{\alpha/2} \) term appears to ensure that our \( g_i \) are still single-valued functions, as \( f_0^{1/2} \) will have character \( \frac{\alpha}{2} \), thus \( f_i f_0^{1/2} \) would have character \( -\frac{\alpha}{2} \)).

then \( f = g_1 g_2 \), and \( \|g_1\|_2 \|g_2\|_2 \leq s^{-1} \|f\|_1 \).

This illustrates the difficulty that arises for the annulus— in taking square roots for our outer part, we are altering the characters, and so the inclusion of \( z^{\frac{\alpha}{2}} \) is needed to counter this.

The constant in the following proof comes out as \( (1+e)(1+2e))^\frac{1}{2} \), which is independent of our annulus inner radius.

**Proof**

Assume w.l.o.g. that \( \int_{\Gamma_1} |f| = 1 \), and \( \int_{\Gamma_0} |f| \geq 1 \) (transform to swap boundaries if necessary, then scale). Let \( \int_{\Gamma_0} |f| = B \).

Let \( f = f_o f_i \) be the inner-outer factorisation of \( f \), and factorise as \( f = (g_1 f_i z^x)(g_2 z^{-x}) \),

where we have that:

- \( g_1 \) is an outer function,
- \( |g_1| = |f| \frac{1}{2} \) whenever \( |f| > 1 \),
- \( |g_1| = |f| \) when \( |f| \leq 1 \),

and \( x \) is chosen so that \( g_1 f_i z^x \) is a single valued function.

We have from [38] that the character of an outer function is given by \( \frac{\int_{\Gamma_0} \log |f| - \int_{\Gamma_1} \log |f|}{\log s} \)
(mod 1).

It thus follows that the difference between the characters of \( g_1 \) and \( f_o \) is equal to \( \frac{\int_{\Gamma_0} \frac{1}{2} \log^+ |f| - \int_{\Gamma_1} \frac{1}{2} \log^+ |f|}{\log s} \).

Now, since \( \int_{\Gamma_1} |f| = 1 \), we have that \( \int_{\Gamma_1} \log^+ |f| \leq 1 \) (since \( \log^+ |f| \leq |f| \)).

We also have that \( \int_{\Gamma_0} \log^+ |f| \leq 1 + \log B \). To see this, consider the problem of finding a function \( g \in L^2(\Gamma_0, \mathbb{R}) \) to maximise \( \int_{\Gamma_0} \log^+ |g| \) subject to the constraint \( \int |g| = B \)

It is immediate that a maximal function will not take values in the set \((0, 1)\)- otherwise, we can create a function \( g' \) with \( g' = 0 \) where \( g < 1 \), which would decrease \( \int g' \) without
changing $\int \log^+ g'$.

Secondly, note that a maximal function will be constant on the set $g > 1$— otherwise, choose $g'$ such that $g' = 1_{\{z : g > 1\}} \left( \int_{g > 1} \frac{g}{\mu(\{z : g > 1\})} \right)$. Then, we will have $\int g = \int g'$, and by Jensen’s inequality (and noting that $\log^+ g = \log g$ on $g > 1$), we have that $\int \log^+ g' > \int \log^+ g$.

Thus, we have that the maximal $g$ will be such that $g = 0$ on a set of measure $1 - x$, and $g = B/x$ on a set of measure $x$. The case $x = 0$ is trivial, and optimising over $x$ we see that $\int \log^+ g' > \int \log^+ g$.

This is maximised as $x$ approaches 1 for $B > e$, giving an upper bound of $\log B$ in this case. For $1 < B < e$, we can see that 2 is an upper bound, so $1 + \log B$ is an upper bound for all $A > 1$.

Since the character of $z^x$ is $x$, we have that $|x| \leq \frac{1 + \log B}{2\log s}$, and so $s^{2x} \leq s^{\frac{1 + \log B}{\log s}} \leq eB$.

Now, we have that $\|f\|_1 = (B + 1)$.

Also, $\|g_1 f_iz^x\|_2^2 \leq (B + eB)$ (since $|g_1|^2 \leq |f|$), and $\|g_2 z^{-x}\| \leq ((B + 1) + (1 + 1)(eB))$ (since $|g_2|^2 \leq \max\{1, |f|\}$).

So $\frac{\|g_1 f_iz^x\|_2^2 \|g_2 z^{-x}\|}{\|f\|_1}^2 \leq \frac{(1 + eB)(1 + 2eB)}{(1 + B)^2} \leq ((1 + e)(1 + 2e))$. $\square$

The constant that arises in these factorisations is entirely due to characters— if you permit factorisations of the type $f = g_1 g_2$ with $\tilde{g}_1 \in H^2_\alpha$, $\tilde{g}_2 \in H^2_{-\alpha}$, $\alpha \in [0, 2\pi)$ and $H^2_\alpha$ the set of analytic functions in the annulus of character $\alpha$ (see, e.g. [38]), one can factorise with $\|g_1\|_2 \|g_2\|_2 = \|f\|_1$— we can simply take square roots of the outer component of $f$, now that we are no longer restricted by characters. The same is also true in a higher order annulus, with $\alpha \in \hat{G}$:

**Theorem 2.3.3** Let $f \in H^1(D)$, then there exists $g_1, g_2$, with $\tilde{g}_1 \in H^2_\alpha$, $\tilde{g}_2 \in H^2_{-\alpha}$ for some $\alpha$ some character, and with $\|g_1\|_2 \|g_2\|_2 = \|f\|_1$.

**Proof**

This follows straightforwardly from the inner-outer factorisation. Let $f = f_if_o$. Then
take \( g_1 = f_1^{-\frac{1}{2}} \), and \( g_2 = f_2^{\frac{1}{2}} \), taking \( f_2^{\frac{1}{2}} \) to be the outer function with modulus on the boundary to be the root of the modulus of \( f_o \).

It is immediate that \( f = g_1 g_2 \), and that \( \|g_1\|_2 = \|g_2\|_2 = \|f\|^{\frac{1}{2}}_1 \). If \( f_o \) has character \( \kappa \), \( f_1 f_2^{\frac{1}{2}} \) will have character \(-\frac{\kappa}{2}\), so these are no longer single-valued functions in general. \( \square \)
Chapter 3

Toeplitz operators on the annulus

3.1 Toeplitz Operators with real-valued symbol

3.1.1 Eigenvalues/eigenvectors of Toeplitz operators with real-valued symbols

The initial motivation at the start of this project was to consider whether a self-adjoint (and thus real valued) Toeplitz operator on the Hardy space of the annulus could have eigenvalues, and as to what they would be. The solution to this problem followed quickly from some of the results in Abrahamse’s paper on Toeplitz operators in multiply connected domains [2]. Research in a paper of Broschinski also came to the same conclusions with similar method on the eigenvalues for self-adjoint Toeplitz operators on the annulus in [10], though they did not consider higher order annuli. An older paper by Aryana and Clancey [6] describes the eigenvalues of Toeplitz operators in a general annulus of arbitrary genus, in terms of theta-functions, and a recent paper by Aryana [5] gives more details in the genus-1 case, giving the same theorems on when infinitely many eigenvalues accumulate in the spectrum, but not giving explicit calculation of the eigenvalues. The methods in the Aryana paper were focused on the resolvent of the
operators in question, as opposed to the methods here. Whilst Brochinski uses much the same methods, he does not continue to higher genus annuli in his paper. In his recent (2014) thesis [11], he mentions Toeplitz on a more general multiply connected domain, but the results stated are in terms of Toeplitz operators on $H^2_\alpha$, looking for $\alpha$ that give an eigenvalue given a Toeplitz operator, and so do not immediately provide information on $H^2$. The results on Toeplitz operators in [11] are also limited to real-valued symbol with constant sign on each boundary component, and properties of the eigenfunctions are also not considered in Brochinski’s work on more general domains.

In this chapter, $\mathbb{A}$ will be used to refer to the annulus $\{ z : s < |z| < 1 \}$, and $D$ will be used to refer to an arbitrary finitely-connected domain contained in the disc, whose boundary curves are analytic Jordan curves.

We will take $\mathbb{A}$ and $D$ to have uniform Lebesgue measure on the boundary, unless stated otherwise.

**Lemma 3.1.1** Let $\phi$ be a bounded real-valued symbol of a Toeplitz operator on the Hardy space of the annulus $\mathbb{A} := \{ z : s < |z| < 1 \}$.

Let $\Gamma_0 = \{ z : |z| = 1 \}$, and $\Gamma_1 = \{ z : |z| = s \}$.

Let $\text{ess inf} \phi|_{\Gamma_0} = a$; $\text{ess sup} \phi|_{\Gamma_0} = b$.

Let $\text{ess inf} \phi|_{\Gamma_1} = c$; $\text{ess sup} \phi|_{\Gamma_1} = d$.

Then we have that:

1. $\sigma_e(T_\phi) = [a, b] \cup [c, d]$.
2. If $a \leq b \leq c \leq d$, then $\sigma_p \subset [b, c]$, and if $c \leq d \leq a \leq b$, then $\sigma_p \subset [d, a]$.

otherwise, $\sigma_p = \emptyset$.

**Proof**

(i) follows from a corollary of Abrahamse’s reduction theorem, Corollary 3.2 of [2].

**Theorem 3.1.2** (Abrahamse) The operator $T_\phi$ is Fredholm if and only if $T_{\phi_i}$ is Fredholm for $i = 0, \ldots, n$, and so $\sigma_e(T_\phi) = \bigcup_{i=0}^n \sigma_e(T_{\phi_i})$. 
(\(T_{\phi_i}\) refers to the Toeplitz operator on the Hardy space of the complement of the area enclosed by \(\Gamma_i\) for \(i \geq 1\), and on the disc for \(i = 0\), with \(\phi_i\) being the restriction of \(\phi\) to \(\Gamma_i\)).

The essential spectrum of a self-adjoint Toeplitz operator on a disc, or any other connected space, is the essential range of its symbol. Thus (i) follows. □

To prove (ii), we first need the following lemma on the eigenfunctions with eigenvalue 0 of a self-adjoint Toeplitz operator:

**Lemma 3.1.3** If \(\phi\) is a real-valued \(L^\infty\) function with \(T_{\phi}f = 0\) then \(\phi|f|^2 \in H_\infty^\perp \cap H_\infty^\perp \).

**Proof**

\(T_{\phi}f = 0 \Rightarrow \langle \phi f, g \rangle = 0\) for all \(g \in H^2\).

Thus \(\langle \phi f, g \rangle = 0\) for all \(g \in H^\infty\), and so \(\langle \phi f \bar{f}, g \rangle = 0\) for all \(g \in H^\infty\).

Thus we have that \(\phi|f|^2 \in H_\infty^\perp\).

However, \(\phi|f|^2\) is real-valued, so on taking conjugates we have orthogonality to \(H_\infty^\perp\) also. □

Note that this result will work on the Hardy space of any domain. In particular, if we consider the Hardy space of the disc, this shows that \(\phi|f|^2\) is perpendicular to both the analytic and anti-analytic functions, and thus we have no eigenvalues when \(\phi\) is non-zero.

Whilst it is not immediate that \(H_\infty^\perp\) is contained in \(L^1\) (as \(L_\infty^\perp\) is not \(L^1\) in general), \(\phi|f|^2\) will always be in \(L^1\), so we can take intersection with \(L^1\) in any case.

Now, for the Hardy space of analytic functions on the annulus under consideration, we have that the space of functions orthogonal to both analytics and anti-analytic functions is a 1-dimensional space, spanned by the function \(\omega\), where \(\omega|_{\Gamma_0} = 1\), and \(\omega|_{\Gamma_1} = -1\).

**Lemma 3.1.4** If \(h\) is such that \(\langle h, f \rangle = 0\) for all \(f \in H^\infty\), and \(\langle h, \bar{f} \rangle = 0\) for all \(f \in H^\infty\), then \(h = k\omega\) for some \(k \in \mathbb{R}\).
Proof

Considering orthogonality against $z^n$ and $\bar{z}^{-n}$ respectively:

\[
\int_{\Gamma_0} h(e^{i\theta})e^{in\theta} + s^n \int_{\Gamma_1} h(se^{i\theta})e^{in\theta} = 0
\]

and

\[
\int_{\Gamma_0} h(e^{i\theta})e^{in\theta} + s^{-n} \int_{\Gamma_1} h(se^{i\theta})e^{in\theta} = 0.
\]

Taking the difference, when $n \neq 0$, we have that all Fourier coefficient apart from the zeroth are 0 on $\Gamma_1$, i.e. $h$ is a constant on $\Gamma_1$, and the same follows for the Fourier coefficients on $\Gamma_0$. Thus $h = k$ on $\Gamma_0$, and $h = l$ on $\Gamma_1$, where $k$ and $l$ are constants.

Since $\langle h, 1 \rangle = 0$, we must have that $l = -k$, i.e. $h = k\omega$. $\Box$

Combining these two lemmas, we have that $T_\phi f = 0 \Rightarrow \phi|f|^2 = \lambda\omega$.

This can be improved upon as follows:

**Theorem 3.1.5** If $\phi \in L^\infty(\partial A)$ is real-valued, and $f \in H^2$, $f \neq 0$, then $T_\phi f = 0 \iff f^{-1} \in H^2$, and $\phi|f|^2 = k\omega$, with $k \in \mathbb{C}$.

Proof

To show the $\Rightarrow$ direction of the statement, we have from the previous that $\phi|f|^2 = k\omega$ on the boundary of the annulus. Thus we have $|f|$ is bounded away from 0 (since $\phi$ is bounded), and so $f^{-1} \in L_2$.

We have that $L_2(\mathbb{A}) = H^2 \oplus \bar{H}^2 \oplus \langle \omega \rangle$, and so $f^{-1} = h_1 + \bar{h}_2 + \mu\omega$, $\mu \in \mathbb{C}$ (where $H^2$ is taken to be the space quotiented out by the constant functions).

By assumption $\langle T_\phi f, g \rangle = 0 \forall g \in H^2$.

Then

\[
\langle T_\phi f, g \rangle = \langle \phi f \bar{f}^{-1}, g \rangle = \langle k\omega, (h_1 + \bar{h}_2 + \mu\omega)g \rangle.
\]

We have that $k \neq 0$ if $\phi \neq 0$. Taking $g = 1$, we see that $\mu = 0$, so $f^{-1} = h_1 + \bar{h}_2$.

We have that $\langle \omega, h_1 g \rangle = 0 \forall g \in H^2$ (since $\omega$ is orthogonal to all $H^1$ functions), and so
\[ \langle k \omega, \overline{h_2}g \rangle = 0 \forall g. \]

Dropping the \( k \) (\( k \neq 0 \) unless \( \phi = 0 \), the trivial case), we have that \( \langle \omega h_2, g \rangle = 0 \forall g \in H^2 \), that is to say, \( T_\omega h_2 = 0 \).

Noting that \( T_\omega = -2(T_{1_{r_1}} - \frac{1}{2} I) \), we have that \( T_{1_{r_1}} h_2 = -\frac{1}{2} h_2 \).

However, the eigenvalues and eigenfunctions for \( T_{1_{r_1}} \) are described in [40], and we thus have that \( h_2 = A \) for some constant \( A \). However, we quotient out by the constant functions, so \( A = 0 \). Thus \( f^{-1} = h_1 \), i.e. \( f^{-1} \in H^2 \).

For the \( \Leftarrow \) direction of the statement:

If \( f \) is invertible in \( H^2 \), and \( \phi|f|^2 = k \omega \), then \( f^{-1} \in H^\infty \), since \( \phi \) is a bounded function.

Now \( \langle \phi f, g \rangle = \langle \phi f \overline{f}, f^{-1} g \rangle = 0 \forall g \in H^\infty \), since multiplication by \( H^\infty \) functions maps \( H^\infty \) to itself, and \( \phi f \overline{f} \) is orthogonal to the analytics by assumption.

Since \( H^\infty \) is dense in \( H^2 \), we thus have \( T_\phi f = 0, \Box \)

Since \( T_1 = I \), we thus have the following:

**Theorem 3.1.6**

For \( f \neq 0, \phi \neq 0, T_\phi f = \lambda f \iff f^{-1} \in H^2 \text{ and } |f|^2(\phi - \lambda) = k \omega \text{ some } k \in \mathbb{R}. \)

Thus, the problem reduces to finding when \( f \) exists with the appropriate modulus on the boundary. Since \( f \) invertible immediately implies that \( f \) is an outer function, and outer functions with modulus bounded away from 0 are invertible, we can restrict our attention to the problem of finding outer functions with the appropriate modulus on the boundary.

Considering the space of modulus automorphic functions, there will exist a modulus automorphic function with given boundary values (provided that \( \int_{\partial A} \log |f| \) is well-defined and bounded), and its character, \( \kappa \), is given by

\[
\kappa = \frac{1}{2\pi \log s} \left[ \int_0^{2\pi} \log |f(e^{it})| dt - \int_0^{2\pi} \log |f(se^{it})| dt \right].
\]
This is proven in [38], p35 (Since here we are consider character in \([0, 1]\), the extra \(2\pi\) factor appears).

So, we have:

**Lemma 3.1.7** \(\lambda\) is an eigenvalue for \(T_\phi\) if and only if \(\kappa_\lambda \in \mathbb{Z}\), where

\[
\kappa_\lambda = \frac{1}{2\pi \log s} \int_0^{2\pi} \log \left| \frac{1}{\phi(e^{it}) - \lambda} - \log \left| \frac{1}{\lambda - \phi(se^{it})} \right| \right| dt
\]

\[
= \frac{1}{4\pi \log s} \int_0^{2\pi} - \log |(\phi(e^{it}) - \lambda)| + \log |(\lambda - \phi(se^{it}))| dt.
\]

In this case the eigenfunction is given by the unique (up to a constant factor) outer function \(f\) in the annulus with boundary values given by \(|f|^2 = \pm \frac{\omega}{\phi - \lambda}\), choosing sign appropriately.

Such an eigenfunction can be constructed by constructing the outer function in the strip \(\{0 < \text{Re}(z) < 1\}\) (or disc) with appropriate modulus on the boundary, given by an appropriate integration on the boundary. The outer functions whose modulus on the boundary is invariant are all modulus invariant inside the disc/strip. When the conditions are met, they will be actually invariant, rather than merely modulus invariant, and so can be passed back to the annulus, giving the eigenfunction with eigenvalue \(\lambda\) by the previous results.

So, combining the results, we can describe the eigenvalues and eigenfunctions for the self-adjoint Toeplitz operators on the annulus fully as follows:

**Theorem 3.1.8** For \(\phi\), a real-valued symbol, we have \(\lambda \in \sigma_p(T_\phi)\) if and only if

\[
\sup \phi|_{\Gamma_0} \leq \lambda \leq \inf \phi|_{\Gamma_1} \text{ (or } \sup \phi|_{\Gamma_1} \leq \lambda \leq \inf \phi|_{\Gamma_0}) \text{ and } \kappa_\lambda \in 2\pi \mathbb{Z}.
\]

Furthermore, if these conditions hold, \(\lambda\) is an eigenvalue of multiplicity 1, \(f\) is the associated eigenfunction if \(f\) is a constant multiple of the unique outer function such that \(|f|^2(\phi - \lambda) = \pm \omega\), choosing sign appropriately.
Proof

If we are not in either of the cases $\phi|_{\Gamma_0} \leq \lambda \leq \phi|_{\Gamma_1}$ (or $\phi|_{\Gamma_1} \leq \lambda \leq \phi|_{\Gamma_0}$) then we have that the sign of $\phi - \lambda$ is not constant on some boundary, and thus $(\phi - \lambda)|f|^2 \neq k\omega$ for any $k$, a contradiction by Theorem 3.1.6.

Supposing now this condition does hold, then $T_\phi f = \lambda f$ if and only if $f$ is an outer function with $|f| = \sqrt{k\omega \phi - \lambda}$.

Since the outer function with given modulus on the boundary is unique up to constant multiples, we have that $f$ is uniquely determined up to multiplication by constants, i.e. the eigenvalue has multiplicity one.

Finally, by the result of [38] p.35, we have that such an $f$ is a single-valued function on the annulus if and only if $\kappa\lambda$ is an integer, since otherwise the appropriate modulus automorphic function will not correspond to a single valued function in the annulus. □

Theorem 3.1.9 Let $\phi$ be a real-valued bounded function on our boundary, as before. Assuming that $\sup \phi|_{\Gamma_0} < \inf \phi|_{\Gamma_1}$, we have that infinitely many eigenvalues accumulate at $\sup \phi|_{\Gamma_0}$ if and only if $\int_{\Gamma_0} \log |(\sup \phi|_{\Gamma_0} - \phi)| = \infty$, and infinitely many accumulate at $\inf \phi|_{\Gamma_1}$ if and only if $\int_{\Gamma_1} \log |(\phi - \inf \phi|_{\Gamma_1})| = \infty$.

Proof

Note that $\kappa\lambda$ is a strictly increasing (decreasing) function in the case $a \leq b \leq c \leq d$ ($c \leq d \leq a \leq b$), for $b \leq \lambda \leq c$ Assuming we are in the case $a \leq b \leq c \leq d$, we thus have that if $\log |\phi(e^{it}) - b|$ is not integrable, $b \neq c$, then one must have infinitely many eigenvalues accumulating at $b$, since $-\int_{\Gamma_0} \log |\phi(e^{it}) - \lambda| \to \infty$ as $\lambda \downarrow b$, and so $\kappa\lambda \to -\infty$ thus there are infinitely many $\kappa\lambda \in 2\pi\mathbb{Z}$ by continuity in $\lambda$.

In this case, $b$ can not be an eigenvalue since any $f \in H^2$ has $\int_{\partial\mathbb{A}} \log |f| < \infty$. If $f$ were to be an eigenfunction with eigenvalue $b$, then we would have $|f|^2(\phi - b) = C\omega$, and thus $|f|(z) = C|\phi(z) - b|^{-\frac{1}{2}}$ for $z \in \partial\mathbb{A}$, contradicting the requirement of log integrability.
If \( \log |\phi(e^{it}) - b| \) is an integrable function, we cannot have eigenvalues accumulating at this point since \( \kappa_\lambda \) is strictly decreasing and bounded below by its value at \( b \), so it can pass through only finitely many integer values between \( b \) and \( k \), for any \( b < k < c \) (likewise eigenvalues accumulate at \( c \) if and only if \( \log(c - \phi(se^{it})) \) is not integrable, regardless of the behaviour at \( b \).

If the log integral at \( \lambda = b \) does exist, it is possible for \( b \) to be an eigenvalue if and only if \( \kappa_b \) is well-defined and an integer (even in the case \( b = c \), provided the log integrals on both boundaries are finite as well as \( \kappa_b \) being an integer. As, for example, occurs at 0 in the case of \( \phi|_{\Gamma_0} = -\phi|_{\Gamma_1} \) for any \( \phi \) such that \( \phi|_{\Gamma_0} \geq 0 \), \( \log \phi|_{\Gamma_0} \) integrable, and \( \text{ess inf} \phi|_{\Gamma_0} = 0 \).

Thus one has infinitely many eigenvalues in the case where either \( (b - \phi|_{\Gamma_0}) \) or \( (\phi|_{\Gamma_1} - c) \) are not log integrable, and \( b \neq c \).

By using the conformal equivalence \( z \to sz^{-1} \), or by repeating the above argument with some relabelling, it can be seen that in the case that \( c \leq d \leq a \leq b \), we instead have accumulation of eigenvalues at \( \sup \phi|_{\Gamma_1} \) if and only if \( \int_{\Gamma_1} (\sup \phi|_{\Gamma_1} - \phi) = \infty \), and similarly for accumulation at \( \inf \phi|_{\Gamma_0} \).

### 3.1.2 Toeplitz operators on vector valued Hardy spaces

The analogue of a Toeplitz operator on the space of vector valued analytic function on the disc is multiplication by a matrix followed by compression to the Hardy space.

In this situation, the self-adjoint operators will be those for which the representing matrix is Hermitian.

In this setting we have that an extension of Lemma 3.1.3 still holds:

**Lemma 3.1.10** If \( \phi \) is a bounded Hermitian symbol, \( f \neq 0 \), \( \phi \neq 0 \), then \( T_\phi f = 0 \Rightarrow f^\dagger \phi f \in W \), where \( W = H_\infty^\perp \cap \overline{H_\infty^{-1}} \), and \( \dagger \) denotes the conjugate transpose.
Proof

Let \( g \in H^\infty \), and \( f \) be such that \( T_\phi f = 0 \).

Then \( \langle f^\dagger \phi f, g \rangle = \langle \sum_j \bar{f}_j (\phi f)_j, g \rangle = \sum_j \langle (\phi f)_j, g f_j \rangle = 0 \) (since \( (\phi f)_j \in H^2_2 \)).

Now \( \langle f^\dagger \phi f, \bar{g} \rangle = 0 \) follows since \( \phi \) is Hermitian. \( \square \)

However, this result is not so useful in the context of vector valued Toeplitz operators. Whilst a necessary condition, there appears to be no obvious way to make this sufficient.

Consider for example \( \phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

Here \( 0 \notin \sigma_p(T_\phi) \), yet for any \( f \) of the form \( \begin{pmatrix} ig \\ g \end{pmatrix} \) for \( g \in H^2 \), the conditions of the lemma are met.

Whilst these difficulties occur in attempting to approach the point spectra of operators here, the essential spectrum is easier to deduce from the results attained in the scalar case, in the case where we have the components of the symbol to be continuous functions. The following result is similar to one proven for the polydisc in [16] p44.

**Theorem 3.1.11** Let \( \phi \) be a continuous, hermitian symbol. Then we have:

\[ \lambda \in \sigma_e(T_\phi) \iff 0 \in \sigma_e(T_{\det(\phi - \lambda I)}). \]

**Proof**

Let \( A \cong B \) be taken to mean \( A = B + K \), with \( K \) a compact operator, i.e. \( \cong \) is equality modulo the compacts.

We require the following well known lemma on congruence of Toeplitz operators, which can be found in [16] p38 for (scalar) Toeplitz operators on a multiply connected domain.

**Lemma 3.1.12** \( T_{\phi_1} T_{\phi_2} \cong T_{\phi_1 \phi_2} \) for \( \phi_1 \) and \( \phi_2 \) continuous.
It suffices prove the theorem in the case $\lambda = 0$.

The result follows from [28] Theorem 1.1, which shows invertibility of a matrix is equivalent to invertibility of the determinant under certain conditions:

**Theorem 3.1.13** Let $K$ be some ring, $a_{mk} \in K$ (for $1 \leq m \leq n, 1 \leq k \leq n$), and suppose $[a_{mk}, a_{pq}] = 0$ all $m, k, p, q$, where $[a, b]$ denotes the commutator $[a, b] = ab - ba$. Then, for $a = [a_{mk}] \in K^{n \times n}$, we have $a$ is invertible if and only if $\det a$ is invertible.

Now, to apply this theorem here, we take $K$ to be the Calkin algebra for $H^2(D)$, and apply to the matrix $a_{mk}$, where $a_{mk}$ is $T_{mk}$ modulo the compact operators.

We have that the condition on the commutators is met, by application of Lemma 3.1.12. The result follows.

From this we can deduce when a given vector-valued Toeplitz operator will be diagonalizable, as this condition is equivalent to the essential spectrum consisting of discrete points for a self-adjoint operator on a separable Hilbert space.

Restricting to the case where $n = 2$, and supposing we have that $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \bar{\phi}_{12} & \phi_{22} \end{pmatrix}$, then we have $\det(\phi - \lambda I) = \lambda^2 - (\phi_{11} + \phi_{22})\lambda + \phi_{11}\phi_{22} - |\phi_{12}|^2$.

The essential spectrum for the Toeplitz operator with this symbol consists of the $\lambda$ for which this is zero at some point on $\partial D$, that is

$$\lambda_{\pm} = ((\phi_{11} + \phi_{22}) \pm \sqrt{((\phi_{11} + \phi_{22})^2 + 4|\phi_{12}|^2 - 4\phi_{11}\phi_{22})})/2$$

is the condition for $\lambda$ to be in the essential spectrum.

So a symbol with continuous values is diagonalizable if and only if $\lambda_{\pm}$ are both constant on each boundary of $D$. Thus $(\phi_{11} + \phi_{22})$ is constant on each boundary, as is $(\phi_{11} - \phi_{22})^2 + 4|\phi_{12}|^2$. 

As an example, we have that in the case where \( D \) is the unit disc, \( T_\phi \), with \( \phi = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \) is diagonalizable, since it has essential spectrum \( \{0, 1\} \).

Theorem 3.1.11 cannot be extended to the case of a Toeplitz operator with symbol with discontinuous boundary values.

Consider the operator with symbol \( \phi = \begin{pmatrix} 1_A & 0 \\ 0 & 1_{A^C} \end{pmatrix} \) where \( A \) is a subset of one component of the boundary, with \( 0 < m(A) < 1 \).

This has \( \det(\phi - \lambda I) = \lambda^2 - \lambda \), and 0 is in the essential spectrum of \( T_{\lambda^2 - \lambda} \) if and only if \( \lambda = 0 \) or 1.

However, \( T_\phi \) has essential spectrum \([0, 1]\) (since its symbol is diagonal, simply consider the essential spectrum of \( T_{1_A} \) and \( T_{1_{A^C}} \), which is known to be \([0, 1]\) for Toeplitz operators on the disc). Thus the result cannot be extended to operators with discontinuous symbols.

### 3.1.3 Toeplitz operators on higher genus annuli

This section will focus on scalar-valued functions once again, this time considering how these results change when rather than considering Toeplitz operators on the boundary of the set \( \{ z : s \leq |z| \leq 1 \} \), instead considering a region \( D \) with \( g \) removed components.

The result of Lemma 3.1.3 still holds in this setting, however the space \( H_\infty^\perp \cap H_\infty^\perp \) is larger—for an annulus of genus \( g \), this space will be \( g \) dimensional (shown by Abrahamse in [2], which also gives a basis for this space in the case of harmonic, rather than uniform measure— Theorem 1.6 of [2]).

One consequence is that our eigenfunctions need no longer be invertible, instead we have the following:

**Lemma 3.1.14** Let \( D \) be our genus-\( g \) annulus. Suppose \( \phi \) is a bounded, real valued function defined on \( \partial D \), \( 0 \notin \sigma_e(T_\phi) \), \( T_\phi f = 0 \) for some \( f \in H^2(D) \).

Then \( f \) has at most \( g - 1 \) zeros in the interior of \( D \).
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Proof

We need the following lemma about the Fredholm index of a Toeplitz operator:

**Lemma 3.1.15** Suppose that $T_\phi$ is Fredholm, with Fredholm index $n$. Then $T_{(z-a)\phi}$ has Fredholm index $n - 1$ when $a \in \mathbb{A}$.

**Proof**

We have that $T_{z-a}$ is Fredholm (by Abrahamse’s reduction theorem, for instance, and using continuity on the boundary), with index $-1$ when $a \in D$ (the Fredholm index for Toeplitz operator with bounded analytic symbol is given in [2] p278 as the number of zeros inside $D$).

Now, $T_{(z-a)\phi} = T_{z-a}T_\phi$, so $\text{Ind}(T_{(z-a)\phi}) = \text{Ind}(T_{z-a}) + \text{Ind}(T_\phi) = n - 1$. □

Now, suppose we have a real-valued symbol $\phi$, $f$ and $a_i \in D$ such that $T_\phi f = 0$, and $f = (z - a_1) \ldots (z - a_g)h$.

Then we have that $T_\phi(z-a_1)\ldots(z-a_g)h = 0$, and this operator will have Fredholm index $-n$ (since $T_\phi$ has Fredholm index 0 from its self-adjointness).

Thus $T_{\phi(z-a_1)\ldots(z-a_g)}$ has kernel of dimension at least $n + 1$.

However, it is an old result from Abrahamse [2] that if $T_\phi f = 0$ and $T_\phi^* h = 0$, then $\phi f \bar{h} \in W$, with $W$ the space orthogonal to the bounded analytics and anti-analytic functions in $D$.

Since $W$ is $g$-dimensional (from [2]), we must therefore have $n \leq g - 1$.

□

**Lemma 3.1.16** Let $D$ be a genus-2 annulus. For all $a$ in $D$ there exists a Toeplitz operator $T_\phi$, such that $(z - a)$ lies in the kernel of $T_\phi$.

We first need to establish a preliminary condition:
Lemma 3.1.17 Suppose that \( f = (z - a)h \), where \( h \) is invertible in \( H^2 \), and \( \phi|f|^2 \in W \). Then we have that \( T_\phi f = 0 \iff \langle \phi f, 1 \rangle = 0 \).

Proof
We have that \( T_\phi f = 0 \iff \langle \phi f, g \rangle = 0 \forall g \in H^\infty \) (since \( H^\infty \) dense in \( H^2 \)).

Since \( f = (z - a)h \), and \( \phi|f|^2 \in W \), we have that \( \langle \phi f, (z - a)g \rangle = \langle \phi|f|^2, gh^{-1} \rangle = 0 \forall g \in H^\infty \).

If \( \langle \phi f, 1 \rangle = 0 \), then \( \langle \phi f, g \rangle = 0 \forall g \in H^\infty \), since \( g = \frac{g-\varphi(a)}{z-a}(z-a) + \varphi(a) \), and so:

\[
\langle \phi f, g(a) \rangle = g(a)\langle \phi f, 1 \rangle = 0, \quad \text{and} \quad \langle \phi f, \frac{g-\varphi(a)}{z-a}(z-a) \rangle = \langle \phi f, (z-a)h \rangle = 0 \text{ since } h = \frac{g-\varphi(a)}{z-a} \in H^\infty.
\]

Now, we can prove the initial lemma:

Proof
Let \( w_1 \) and \( w_2 \) be two basis vectors for our space \( W \), and consider \( \phi = \frac{xw_1 + yw_2}{|z-a|^2} \),

with \( x \) and \( y \) to be determined later.

Then it follows immediately that \( \phi|z-a|^2 \in W \).

Given \( \phi = \frac{xw_1 + yw_2}{|z-a|^2} \) we have two linear degrees of freedom in choosing \( \phi \), and the requirement \( \langle \phi(z-a), 1 \rangle = 0 \) is a linear function of \( \phi \), we can always find non-trivial choice of \( x \) and \( y \) such that the condition \( \langle \phi f, 1 \rangle = 0 \) is met.

\( \square \)

It can thus be seen that eigenfunctions exist which are no longer invertible.

It will be proven later that if some function in the kernel of a (not-necessarily self-adjoint) Toeplitz operator has non-trivial singular inner component, then this kernel must be infinite-dimensional, and so for the \( g \)-dimensional annulus we can restrict our attention to functions \((z - a_1)\ldots(z - a_i)h \) with \( h \) invertible and \( i \leq g \).
A special case

Whilst the description of the space $W$ is considerably more complicated in even the $g = 2$ case than it was for the simple annulus with both circles centred at 0, some of the behaviour of the eigenvalues can be worked out quite easily for a particular operator on a genus-2 annulus with certain symmetry, and assuming we define our norm by integrals with respect to the harmonic measure. So, recall, we will denote our Toeplitz operators in this setting by $T$.

Consider an annulus of genus 2, with inner radii of removed circles $s$, centres at $\pm r$.

Then, we have $W = \langle \omega_1, \omega_2 \rangle$, with $\omega_i \in L^\infty(D)$ defined to be the derivative with respect to the normal of the harmonic extension to $D$ of the indicator function for $\Gamma_i$ (as in [2]) for $i = 0, 1, 2$. ($\omega_0$ is not needed in the basis for $W$ since $\omega_0 + \omega_1 + \omega_2 = 0$ follows since this will be the derivative of the harmonic extension of the constant function on the boundary of $D$, which is clearly 0, since the harmonic extension of a constant is constant).

By symmetry under the map $z \to -z$, which maps $\Gamma_1$ to $\Gamma_2$ and vice-versa, we must have that $\omega_1(z) = \omega_2(-z)$, as well as $\omega_0(z) = \omega_0(-z)$.

So, if $\int_{\Gamma_0} \omega_0 = A$, then $\int_{\Gamma_0} \omega_1 = \int_{\Gamma_0} \omega_2 = -A/2$,

and $\int_{\Gamma_1} \omega_0 = \int_{\Gamma_2} \omega_0 = -A/2$, and letting $\int_{\Gamma_1} \omega_1 = B$, we have $\int_{\Gamma_1} \omega_2 = A/2 - B$.

Lemma 3.1.18  Given real-valued $\phi \in L^\infty$, with $\phi$ bounded away from 0, there exists invertible $f \in H^2$ such that $T_\phi f = 0$ if and only if there exist $a_0, a_1, a_2 \in \mathbb{C}$ such that:

$$\int_{\Gamma_j} \frac{\partial}{\partial n} \left( \frac{1}{2} \log \left( \sum a_i \omega_i \right) \right) - \frac{\partial}{\partial n} \left( \frac{1}{2} \log \phi \right) |dz| \in 2\pi \mathbb{Z}$$

for $0 \leq j \leq 2$,

with $\hat{s}$ denoting the harmonic extension of a boundary function $s$, and $\Gamma'_j$ a loop homotopic to $\Gamma_j$. If so, $f$ is the outer function such that $|\phi| |f|^2 = \sum a_i \omega_i$.

If one has suitable convergence at the boundary, then we can simply consider $\Gamma_j$ instead.
of $\Gamma_j$.

A single redundant condition exists since $\Gamma_0$ is homotopic to $\sum_{i=1}^2 -\Gamma_i$ (since the integrals come from [26] and correspond to the change in argument of our multi-valued function when traversing the contour, and so are constant when distorting the contour by a continuous deformation.)

Proof

We have that $f$ is a 0-eigenfunction for $T_{\phi}$ if $\phi|f|^2 \in W$ and $f$ is invertible in $H^2$:

$$\langle T_{\phi} f, g \rangle_h = \langle \phi f, g \rangle_h = \langle \phi |f|^2, fg \rangle = 0$$

for all $g \in H^\infty$ (since $\phi|f|^2$ orthogonal to the analytics), and we have $f H^\infty = H^\infty$ (since $f$ invertible).

So, it is enough to seek for $\phi|f|^2 = \sum_0\omega_i$ for some $\omega_i$ with $f$ an outer function. If we take $u$ to be the harmonic extension of $\log((\sum_0\omega_i))^{1/2}$ from the boundary to $D$, $v$ to be its conjugate, and $f = e^{u+iv}$, we want $f$ to be single-valued, and so $v$ must have period conjugate to $0$ mod $2\pi$ about each component of boundary. From Khavinson’s work on conjugate functions in multiply connected domains [26], if $u$ is a harmonic function in $D$ with harmonic conjugate $v$, $v$ has period around $\Gamma_j$ of $\Delta \Gamma_j = -\int_{\Gamma_j} \frac{\partial}{\partial n}\hat{u}|dz|$ (with the derivative normal to the boundary). The result follows. □

Now, taking our symbol $\phi = 1_{\Gamma_0}$, it follows from the definition of $\omega_i$ that for $0 < \lambda < 1$,

$$\frac{\partial}{\partial n} \log(\phi - \lambda) = \log(1 - \lambda)\omega_0 + \log(\lambda)\omega_1 + \log(\lambda)\omega_2 + i\pi(\omega_1 + \omega_2).$$

Taking the result of the lemma, and evaluating on $\Gamma_0$ and $\Gamma_1$, we have:

$$(\int_{\Gamma_0} \frac{\partial}{\partial n}(\frac{1}{2} \log(\sum a_{i\omega_i}))|dz| - \frac{1}{2} (\log(1 - \lambda)A - (\log(\lambda)A) + i\pi A) \in 2\pi \mathbb{Z},$$

$$(\int_{\Gamma_1} \frac{\partial}{\partial n}(\frac{1}{2} \log(\sum a_{i\omega_i}))|dz| - \frac{1}{2} (\frac{-A}{2} \log(1 - \lambda) + (\log(\lambda)B + (\frac{A}{2} - B) \log \lambda) + i\pi \frac{A}{2}) \in 2\pi \mathbb{Z}.)$$

Let $C = \int_{\Gamma_0} \frac{\partial}{\partial n}(\frac{1}{2} \log(\sum a_{i\omega_i}))|dz|, D = \int_{\Gamma_1} \frac{\partial}{\partial n}(\frac{1}{2} \log(\sum a_{i\omega_i}))|dz|$, then:

$$(C - i\pi A) - A \log \frac{1 - \lambda}{\lambda} \in 2\pi \mathbb{Z},$$

$$(D - \frac{A}{2} \log \frac{1 - \lambda}{\lambda}) \in 2\pi \mathbb{Z}.$$
So, we have that $C + 2D \in 2\pi\mathbb{Z}$, and $C$ and $D$ depend only on $a_i$ in $\omega|f|^2 = \sum a_i\omega_i$.

Since $\omega_0 + \omega_1 + \omega_2 = 0$, we can assume that $a_2 = 0$, and since scaling $f$ by a constant has trivial effect, we can assume $|a_0|^2 + |a_1|^2 = 1$, and $a_1 \in \mathbb{R}$ and can consider $a_0$ as our only independent complex variable. Thus if we fix an $n$ and seek $a_i$ such that $C + 2D = 2\pi n$, then we have a finite set of $a_i$ which solve this. Taking one such set of $a_i$, and letting $E = (C + i\pi A)$, then eigenvalues corresponding to outer eigenfunctions exist when $E - A\log(\frac{1-\lambda}{\lambda}) = 2\pi m$, $m \in \mathbb{Z}$. So, we have that

$$\lambda_m = \frac{1}{1 + e^{\frac{E - 2\pi m}{A}}}$$

parametrises the set of solutions with the chosen $a_i$.

In comparison, the eigenvalues in the one-holed annulus for $\phi$ the indicator of one component of the boundary are given in [40] to be $\frac{s^{2n}}{(1+s^{2n})}$.

### 3.2 Kernels of Toeplitz operators

We will consider once again the Hardy space, inner product and associated projections with respect to uniform Lebesgue measure. When considering Toeplitz operators with more general, not necessarily real-valued, symbols, some results were found when considering the kernel of a Toeplitz operator on the annulus.

Once again our annulus $A$ is taken to be $\{z : s < |z| < 1\}$.

**Theorem 3.2.1** Let $f$ in $H^2$ be continuous and non-zero on the boundary of $A$, $\phi$ be continuous and non-zero on the boundary of $A$, and $f$ in the kernel of $T_\phi$.

Then we have that $\dim \ker(T_\phi) \geq \max(n, 1)$, where $n$ is the number of zeros of $f$ inside of $A$. 
Proof

We have that $T_\phi$ is Fredholm, and [38] gives the Fredholm index to be $i(T_\phi) = n(\phi|_{\Gamma_1}, 0) - n(\phi|_{\Gamma_0}, 0)$, that is, the difference of the winding number about 0 on the inner and outer boundaries.

By definition, $i(T_\phi) = \dim \ker T_\phi - \dim \ker T_\phi^*$.

If $f$ is in the kernel of $T$, and $g$ in the kernel of $T^*$, then $\phi f \bar{g} = \omega$, (shown in [2]), and thus both kernels can only be non-zero if they are one-dimensional (since $W$ is one-dimensional), and in this case we have index 0.

Thus, if $\ker T_\phi \neq 0$, then $i(T_\phi) \geq 0$, and if $i(T_\phi) > 0$, we have $i(T_\phi) = \dim \ker(T_\phi)$. If $T_\phi f = 0$, we have that $T_\phi f 1 = 0$, hence $i(T_\phi f) \geq 0$, however,

$$i(T_{\phi f}) = n(\phi f|_{\Gamma_1}, 0) - n(\phi f|_{\Gamma_0}, 0) = n(\phi|_{\Gamma_1}, 0) - n(\phi|_{\Gamma_0}, 0) + n(f|_{\Gamma_1}, 0) - n(f|_{\Gamma_0}, 0)$$

$$= i(T_\phi) - n.$$

Thus $i(T_\phi) \geq n$, i.e. we have a kernel with at least dimension $n$. \(\square\)

If $f$ is non-zero and continuous on the outer boundary, we can construct a Toeplitz operator whose kernel contains $f$ and has dimension $n$. Considering $\phi = \frac{\omega}{f}$, we have by computing winding numbers, $i(T_{\omega/f}) = n$, thus our kernel has dimension $n$ if $n > 0$, dimension 1 if $n = 0$, and $f$ lies in the kernel since $T_{\omega/f}f = P_{H^2}\omega = 0$.

Infinite-dimensional kernel given singular inner factors

In the disc, it is trivial to show that if $T_{\bar{f}}$ is an anti-analytic Toeplitz operator, with $f$ containing a non-trivial singular inner factor, we have that $\ker T_{\bar{f}}$ is infinite dimensional.

In the setting of the annulus, it appears considerably more difficult to prove—the complication is caused by the fact that we can no longer take arbitrary powers of a singular inner function and still obtain single valued functions.

The following proof works for an arbitrary annulus, not just the single-holed case.
**Theorem 3.2.2** Let \( f \in H^2(D) \) be a function with \( f = f_if'_i \) where \( f_i \) is a singular inner function, \( f_i \neq 1 \), and \( f' \in H^2 \).

Then \( \ker T_f \) is infinite-dimensional.

**Proof**

Let \( s_o \) be an outer function with constant modulus on each boundary, and complementary character to \( f_i \) (existence is given by [2], p269 proposition 1.15. In the case of the annulus \( \{s < |z| < 1\} \) we can take \( s_o = z^\alpha \) some \( \alpha \in [0,1) \)).

Since \( T_f = T_{\overline{f} s_o} T_{\overline{f} s_o} \), it is enough to show that \( \ker T_{\overline{f} s_o} \) has infinite dimension. To do so, first we show that \( 0 \in \sigma_e(T_{\overline{f} s_o}) \). Use the multiplicative factorisation to factor \( f_is_o = k_1k_2 \) with \( k_1 \in H^\infty(\Delta) \), and \( k_2 \in H^\infty(D \cup \Delta^c) \).

Proceed using Abrahamse’s reduction theorem ([2] p.282)– \( \sigma_e(T_o) = \bigcup \sigma_e(T_{\phi|_{r}}) \). We have that \( T_{\overline{f} s_o}|_{\Gamma_0} = T_{k_1|_{\Gamma_0}} T_{k_2|_{\Gamma_0}} \). Now, since \( k_2 \) is analytic on an open set around \( \Gamma_0 \), it is continuous on \( \Gamma_0 \), and since \( |f_is_o| = 1 = |k_1||k_2| \), with \( |k_1| \) bounded, we have \( k_2 \) is non-zero, and thus \( T_{k_2|_{\Gamma_0}} \) is Fredholm invertible. Since we have that \( T_{k_1} \) is an anti-analytic Toeplitz operator with non-trivial singular inner component, it is well known that the kernel must be infinite-dimensional in the Hardy space on the unit disc, and so is non-Fredholm. Since the product of a Fredholm operator with a non-Fredholm operator is non-Fredholm, we have that \( 0 \in \sigma_e(T_{\overline{f} s_o}) \).

Now, if an operator has finite-dimensional kernel, finite-dimensional co-kernel, and closed range, it must be Fredholm, thus if \( T_{\overline{f} s_o} \) has finite-dimensional co-kernel and closed range, it must then have infinite-dimensional kernel.

Being an anti-analytic Toeplitz operator, the co-kernel must be trivial, so it remains only to show the range is closed. The range will in fact be \( H^2 \):

We have that \( T_{s_o f_i}(s_of_i)g = P_{H^2}(|s_o|^2g) = T_{|s_o|^2}g \).

Now, we have by our choice of \( s_o \) that \( |s_o|^2 \) is a strictly positive constant on each boundary. Thus, by the result on essential spectrum, \( T_{|s_o|^2} \) will be a Toeplitz operator whose essential spectrum consists of discrete points, all of which are positive. Furthermore,
its eigenvalues will be strictly positive. So, it is an invertible operator, and thus given any
\( h \), we can find \( g \) such that \( T|_{s_0}|^2 g = h \), and so \( T_{s_0} f_i (s_0 f_i g) = h \). Thus the range is closed,
and therefore \( T_{s_0} f_i \) has infinite-dimensional kernel, as does \( T f_i \).
\[ \square \]

The only results used here were our factorisation (which still holds on changing to an equivalent norm), and the reduction theorem, which also remains true changing to equivalent norms, so this will still hold taking Toeplitz operators defined on our space with equivalent norms, i.e. it will hold true with harmonic norm on our boundary.

We can proceed from here to show a further result on kernels of self-adjoint Toeplitz operators. The proof requires use of harmonic measure on the boundary of our domain \( D \).

**Theorem 3.2.3** In the Hardy space of the annulus, with norms and inner product given by integration with respect to a harmonic measure (as is the case in the construction of \( H^2(D) \) by harmonic majorants), if \( \phi \) is real-valued, \( \phi \in L^\infty(\partial D) \), and \( T_\phi f = 0 \), then \( f \) has trivial singular inner component.

We first need to establish a preliminary result on the kernel that was proven to be infinite dimensional in Theorem 3.2.2

**Lemma 3.2.4** Let \( T_\phi \) be a Toeplitz operator where \( \phi \) is anti-analytic with non-trivial singular inner component. Then there exists a sequence \( (s_j)^\infty_{j=1} \) such that \( s_j \perp s_k \) for \( j \neq k \), \( s_j \in H^\infty \), and \( T_\phi s_j = 0 \).

Once this lemma is established, we can prove our theorem as follows:

**Proof**

Since we are using harmonic measure here, there exists \( \nu \) such that \( H^2_2 = \overline{\nu}^{-1} \overline{H^2} \), and \( \nu \) is meromorphic on some open set containing the closure of \( \Delta \), with \( n \) zeros and 1 pole ([2] p263). Thus \( \nu = \frac{(z-a_1)\ldots(z-a_n)}{(z-b_1)} v_o \), where \( v_o \in H^\infty, v_o^{-1} \in H^\infty \).
Suppose $T_\phi f = 0$ with $\phi$ real-valued and $f$ having non-trivial singular inner component. Suppose $f = f'g_o$, where $f'$ has constant modulus on each boundary, and $g_o$ outer, and suppose $f = f_i f_o$ is an inner-outer factorisation of $f$. Let $(s_j)$ be the bounded sequence of orthogonal functions from Lemma 3.2.4, $T_j f s_j = 0$. Now, we have that $\bar{f}' s_j \in \nu^{-1} H^2$.

We have that $\phi f \in \nu^{-1} H^2$, thus $\phi f \bar{f}_j \in \nu^{-1} H^2$.

Now, $f \bar{f}_j = f \nu f' s_j = f_i f_o g_o \nu \bar{f}_i f_o s_j = f_i f_o |f_o|^2 |(z - a_i) \ldots (z - a_n)\rangle_f (z - b_1) s_j.$

So we have that

$$P_{H^2}^h(\phi | f_o|^2 (z - a_1) \ldots (z - a_n) f_o g_o \frac{1}{(z - b_1)} s_j v_0)$$

using $\frac{1}{z - b_1} = \frac{z - b_1}{|z - b_1|^2}$, and the identities $T_h T_\phi = T_{\phi h}$ for $h \in H^\infty$. By our choice of $s_j$, the $f_o g_o (z - b_1)s_j$ are a linear independent sequence. Since $T_h$ has 0-kernel for $h$ an outer function (follows simply from $h H^\infty$ dense for $h$ outer), and $T_{(z - a_1) \ldots (z - a_n)}$ has $n$-dimensional kernel, it follows that $T_{\phi | f_o|^2} \frac{1}{|z - b_1|^2}$ has infinite dimensional kernel. However, $T_{\phi | f_o|^2} \frac{1}{|z - b_1|^2}$ is a self-adjoint Toeplitz operator, and thus has finite dimensional kernel. Thus we arrive at a contradiction from our assumptions, thus any $f$ in the kernel of a self-adjoint Toeplitz operator $T$ has trivial singular inner component.

\[ \square \]

It remains to prove the lemma:

**Proof**

Let $f = u f''$, where $u \in H^\infty(D)$ has non-trivial singular inner component, $|u|$ bounded away from 0 and $\infty$ on each boundary, $u \neq 0$ in $D$, and $f'' \in H^2(D)$. Let $u = u_i u_o$ be the inner-outer factorisation of $u$. Since $T_f = T_{f''} T_u$, it is enough to establish the lemma for $T_u$.

We have that ker $T_u = K_u = H^2 \ominus u H^2$.

To find the required sequence, we must consider the reproducing kernels for our spaces. Let $k_\lambda$ be the reproducing kernels for $H^2(D)$, $f_\lambda$ for $u H^2$ and $g_\lambda$ for $K_u$. If we can show
that \( g_\lambda \) are bounded, the lemma will follow. We have that \( k_\lambda \in H^\infty(\mathbb{D}) \) for all \( \lambda \)— see, for example, Ball and Clancey paper on reproducing kernels [8]. Thus, we have that \( g_\lambda \) bounded follows from \( f_\lambda \) being bounded.

Suppose that we have some \( \lambda \) for which \( f_\lambda \) is unbounded, let \( f = f_\lambda \). Let \( (z_n) \) be a sequence for which \( f(z_n) \to \infty \).

Since \( u \) is a bounded function, we have \( |(f/u)(z_n)| \to \infty \) also.

Let \( B_1 \) be a finite Blaschke product such that \( B_1 \) has the same character as \( u_i \), and that \( B_1(\lambda) \neq 0 \).

We have that \( |(fz/\lambda uB_1/u)(z_n)| \to \infty \) still, and thus \( |\langle \frac{fz/uB_1}{u}, k_{zn} \rangle_h| \to \infty \).

We have that \( |u_i/B_1| = 1 \) so \( |\langle u_i / B_1, k_{zn} u_i \rangle_h| \to \infty \).

Thus \( |\langle f_\lambda, k_{zn} \rangle_h| \to \infty \).

Now, were we to have that \( B_1 \) was a factor of \( k_{zn} \), we would have \( k_{zn} \frac{u_i}{B_1} \in uH^2 \), and so a contradiction would follow since \( f_\lambda \) is a reproducing kernel there— we would have:

\[
\langle k_{zn} \frac{u_i}{B_1}, f_\lambda \rangle_h = k_{zn}(\lambda) \frac{u_i}{B_1}(\lambda) = k_\lambda(z_n) \frac{u_i}{B_1}(\lambda),
\]

and we have already that \( k_\lambda \) is bounded, giving a contradiction.

Since we cannot assume this, suppose that \( B_1 \) has zeroes at \( \omega_1, \omega_2, \ldots, \omega_m \), and that these are distinct points.

We seek to find \( z_n a_1, \ldots, z_n a_m \) such that \( k_{zn}(\omega_i) u_i(\omega_i) + \sum_{j=1}^{m} z_n a_j u_j(\omega_i) = 0 \) for each \( i \).

This is the problem, given \( b_i = k_{zn}(\omega_i) u_i(\omega_i) \), of finding a polynomial \( P_m \) which takes the values \( b_i \) at the points \( u(\omega_i) \).

Now, we require \( u(\omega_i) \) to be a distinct set. Supposing they are not, then, we can replace the factorisation \( f = uf'' \) with \( f = (u(z-t)^N)(f''(z-t)^{-N}) \), where \( t \notin D \), and \( N \in \mathbb{Z} \).

\( u(z-t)^N \) will still be single-valued, \( H^\infty \), non-zero in \( D \) and the character of \( u_i \) is unchanged, and \( f''(z-t)^{-N} \) also in \( H^\infty \), so we can apply the previous parts of the proof, with \( \omega_i \) are unchanged, and there will always exist a choice of \( N \) and \( t \) such that \( u(\omega_i)(\omega_i - t)^N \) are distinct points (for instance, if we choose \( t \) such that \( |\omega_i - t| \) are unique, we can then choose \( N \) such that \( |(\omega_i - t)^N u(\omega_i)| \) are all unique, and it follows
that \((\omega_i - t)^N u(\omega_i)\) are unique).

Thus, we can assume without loss of generality that \(u(\omega_i)\) are distinct points.

Then, since \(\omega_i\) are fixed, and the \(b_i\) lie in a bounded set (since \(k_z(\omega) = k_\omega(z)\) is bounded over all \(z\) for fixed \(\omega\)), we can solve this with a universal bound on \(z_n a_i\) independent of \(z_n\).

We thus have that \(\langle \frac{k_z u_i}{B_1}, f, \lambda \rangle_{h} = \langle \frac{k_z u_i - \sum z_n a_j u^j}{B_1} + \frac{\sum z_n a_j u^j}{B_1}, f, \lambda \rangle_{h}\).

Since the \(z_n a_j\) are bounded over all \(z_n\), we have

\[
\| \sum_{j=1}^{m} z_n a_j u^j \|_2 \leq \| \sum_{j=1}^{m} z_n a_j u^j \|_\infty < A_1
\]

is a universal bound over all \(z_n\).

We also have that \(\frac{k_z u_i - \sum z_n a_j u^j}{B_1} \in uH^2\), so that

\[
\langle \frac{k_z u_i - \sum z_n a_j u^j}{B_1}, f, \lambda \rangle_{h} = \langle \frac{k_z u_i - \sum z_n a_j u^j}{B_1}, f, \lambda \rangle(\lambda),
\]

which is bounded over all \(z_n\) since \(k_z(\lambda)\) and \(z_n a_j\) are.

Thus, we have that \(f, \lambda(z_n)\) must be bounded.

Since \(f, \lambda \in H^\infty\), we have \(g, \lambda \in H^\infty\).

Since the reproducing kernels have dense closed span, and we have already established that the space is infinite dimensional, we can pick out a sequence with no linear dependences. Applying Gram-Schmidt we can produce a suitable orthonormal sequence.

\[\Box\]

We would like to show this result also holds for Toeplitz operators defined on \(H^2(D)\) with respect to uniform Lebesgue measure on \(D\).

In fact, this follows directly from the result with harmonic measure.

Note that if we consider \(T, \phi\) a Toeplitz operator symbol \(\phi\) with respect to projection defined on the space with uniform Lebesgue measure, and take \(T, \phi\) to be a Toeplitz operator with respect to harmonic measure, we have that \(T, \phi \frac{dm}{|dz|} f = T, \phi f\) for all functions \(f \in H^2\). \(\frac{dm}{|dz|}\) is bounded since our measures are equivalent, and if \(\langle \cdot, \cdot \rangle\) is the inner product with respect to uniform Lebesgue measure, and \(\langle \cdot, \cdot \rangle_{h}\) with respect to harmonic measure, we have:
\[ \langle T_{\phi} \frac{dm}{|z|}, f, g \rangle = \int_{\partial D} \phi \frac{dm}{|z|} f \bar{g} |dz| = \int_{\partial D} \phi f \bar{g} dm = \langle \phi f \bar{g} dm \rangle_h = \langle T_{\phi}, g \rangle_h. \]

Now, by [2] p263 (3), we have that \( \frac{dm}{|z|} \) is positive, and since our measures are absolutely continuous, it is bounded away from 0 and \( \infty \).

It follows that we can extend the previous theorem to the space with Lebesgue measure.

**Theorem 3.2.5**  Considering the norm and projections on \( H^2 \) with respect to uniform lebesgue measure, we have that if \( \phi \in L^\infty(\partial D) \) real-valued, \( T_{\phi} f = 0 \), then \( f \) has trivial singular inner component.

**Proof**

Let \( \phi' = \phi \frac{|dz|}{dm} \). We will have that \( \phi' \) is real-valued, and bounded, and \( T_{\phi} f = 0 \) if and only if \( T_{\phi'} f = 0 \). The result thus follows from the case with respect to harmonic measure. □

We shall consider an annulus to have uniform Lebesgue norm on the boundary for the remainder of this thesis. Combining this result, Lemma 3.1.14, and Lemma 3.1.3 we can state the following on the eigenfunctions of Toeplitz operators with real valued symbol:

**Theorem 3.2.6**  Considering Toeplitz operators in \( H^2(D) \) with respect to uniform lebesgue measure, let \( \phi \in L^\infty(D) \) be a real-valued function, and suppose we have \( f \in H^2(D) \) such that \( T_{\phi} f = \lambda f \) some \( \lambda \in \mathbb{R} \). Then we have that \( f = f_o B \) where \( B \) is a finite Blaschke product, of order at most \( g - 1 \), where \( g \) is the genus of the annulus \( D \), and \( f_o \) is an outer function with \( \phi |f|^2 \in H^\perp \cap \overline{H^\perp} \).

### 3.2.1 Minimal kernels

The paper [12] discusses minimal kernels of Toeplitz operators containing a given function in the setting of the Hardy space of the disc, and the following results concern attempts to reproduce some of these results in the annulus.
Theorem 3.2.7 If \( f = f_if_i \) is the inner outer factorisation of \( f \in H^2(\mathbb{A}) \), then the symbol of a Toeplitz operator with minimal kernel containing \( f \) is \( \psi = \omega j_i f_i f_i f_i z^{-\alpha} \), with \( \alpha \) being the character of \( f_o \).

(In comparison, the corresponding symbol for an operator with minimal kernel in the setting of the disc is known to be \( j_i f_i f_i f_i z^{-\alpha} \).)

Proof

Let \( f = f_if_o \) be an \( H^2 \) function with \( \alpha \) the character of \( f_o \), and \( \phi \) some symbol such that \( T_\phi f_if_o = 0 \).

Then, we must have that \( \phi f_if_o = \omega g_i g_o \).

Let \( v \) be an outer function that \( |\phi| = v \) on \( \Gamma_0 \), and \( |\phi| = Av \) on \( \Gamma_1 \). Then we have that \( T_\phi = T_v T_\phi' \), with \( |\phi'| = 1 \) on \( \Gamma_0 \), \( |\phi'| = A \) on \( \Gamma_1 \) Since \( \ker T_v = \{0\} \), we have \( \ker T_\phi = \ker T_\phi' \), so we may assume w.l.o.g.that \( |\phi| = 1 \) on \( \Gamma_0 \), and \( |\phi| = A \) on \( \Gamma_1 \) for some \( A \).

So \( |g_o| = |f_o| \) on \( \Gamma_0 \), \( |g_o| = A|f_o| \) on \( \Gamma_1 \), that is, we have \( \beta \in \mathbb{R} \) such that \( |f_o z^\beta| = |g_o| \) (on the boundary).

Since modulus on the boundary determines an outer function up to constant factors, we can take \( g_o = f_o z^\beta \). We will thus have the character of \( g_0 \) is the character of \( f_o \) plus \( \beta \), i.e. \( \alpha + \beta \).

So, we have that \( \phi = \omega j_i f_i f_i f_i z^\beta \).

Now, suppose that \( h \) lies in the kernel of \( T_\psi \).

Then \( \omega j_i f_i f_i z^{-\alpha} h = \omega l_o j_i \). Thus \( h = l_o j_i f_i f_i z^\alpha \), and so \( \phi h = l_o j_i f_i f_i z^\alpha \omega j_i f_i f_i z^\beta = \omega l_o j_i z^{\beta + \alpha} \). Since \( g_o \) has character \( \alpha + \beta \), \( g_i \) must have character \( (-(\alpha + \beta)) \), and so \( l_o j_i z^{\beta + \alpha} \) is single valued.

It follows that \( \phi h \in \omega H^2 \), thus \( T_\phi h = 0 \).

Hence, \( \ker T_\phi \supset \ker T_\psi \), so \( \psi \) is the symbol with minimal kernel. \( \square \)
This can be extended to cover an arbitrary $g$-holed annulus.

**Lemma 3.2.8** If we have $\phi$ and $f$ such that $T_\phi f = 0$, then $\ker T_\phi \supset \ker T_\psi$, where $\psi$ is the function $\psi = \frac{T_\phi f}{f_\circ l_f} \bar{l_f} \bar{\nu}$, with $l_f$ the unit, i.e. outer function with constant modulus on each boundary, such that $\text{character}(l_f) = -\text{character}(f_\circ)$, and $\nu$ the measure such that $H^\perp_2 = \bar{\nu} \bar{H}^2$.

**Proof**

First, given any $\phi$, if $\phi$ is not log integrable, it has zero kernel, and the result is trivial (since $\phi f$ is not log integrable, whereas all of $H^\perp_2$ is). Otherwise, we have that there exists an outer function $s$ such that $|\phi/s|$ is constant on each boundary (the existence of such an $s$ follows from the existence of a (multiple-valued) outer function with given boundary values, and the work in [2]).

Then, we have that $\phi = s \phi'$, and so $T_\phi = T_s T_{\phi'}$. Since $T_s$ has zero kernel, we can restrict attention to $T_{\phi'}$, i.e. we can assume without loss of generality that our symbol has constant modulus on each boundary. We can then proceed as follows:

$$\phi f_i f_\circ = \bar{\nu} \bar{g_i} g_\circ, \text{ with } |g_\circ| = |\phi| \frac{|f_\circ|}{|\nu|}.$$  

So, $g_\circ = f_\circ l_\phi \mu$, where $\mu$ is the outer function such that $|\mu| = |\nu|^{-1}$, and $l_\phi$ is the outer function with modulus equal to that of $\phi$ on each boundary (which will be constant and non-zero on each boundary).

Substituting this into the formula for $\phi$, we obtain

$$\phi = \bar{\nu} \bar{g_i} f_i l_\phi \mu \frac{f_\circ}{f_\circ}.$$
Suppose that we have \( h \) such that \( T_\psi h = 0 \), then we must show that \( T_\phi h = 0 \).

Since \( T_\psi h = 0 \), we have that \( \psi h_i h_o = \bar{\nu} \bar{m} \bar{m} \bar{\mu} \), so:

\[
\phi h_i h_o = \frac{\nu g_i l_i \bar{\phi} f_i h_o}{f_o h_i h_o} = \psi l_f^{-1} \bar{g}_i l_i h_o = \bar{\nu} \bar{m} \bar{m} \bar{l}_f^{-1} \bar{\phi} f_i h_o.
\]

Since \( l_f \) is outer, with constant non-zero modulus on each boundary, we have that \( l_f^{-1} \) is also \( H^\infty \), and so \( T_\phi h = 0 \), as required. \( \square \)

The next problem to consider is what the minimal kernel is for a Toeplitz operator containing a set of functions, which is a harder problem.

The following is a start to showing whether the existence of a non-trivial kernel containing a pair of functions exists reduces to showing this in the disc.

**Lemma 3.2.9** For \( f_1, f_2 \in H^2(\mathbb{A}) \), there exists non-trivial \( \phi \) such that \( T_\phi f_1 = 0, T_\phi f_2 = 0 \) only if there exists \( \phi' \in L^\infty(\partial \Delta) \) such that \( T' \bar{f}_i = 0 \) for each \( i \), where tildes denote lifts back to the disc, and \( T' \) indicates that the Toeplitz operator acts on the Hardy space for the disc.

**Proof**

By assumption, we have \( \phi f_1 = \omega \bar{g}_1 \) and \( \phi f_2 = \omega \bar{g}_2 \) for some \( g_1, g_2 \).

We have that \( \bar{z}(\bar{\omega} \bar{\phi}) \bar{f}_1 = \bar{z} \bar{g}_1 \), and the same for \( \bar{f}_2 \), so we have \( \bar{z}(\bar{\omega} \bar{\phi}) \) is a symbol for a Toeplitz operator whose kernel contains \( \bar{f}_1 \) and \( \bar{f}_2 \). \( \square \)

So, if \( f_1 \) and \( f_2 \) lie in a non-trivial Toeplitz kernel in the annulus, their exists a non-trivial Toeplitz kernel in the disc containing both \( \bar{f}_1 \) and \( \bar{f}_2 \). Thus we have a necessary condition for two functions in the Hardy space on the annulus to lie in a non-trivial kernel.
In fact, the same will work if, rather than a pair, we instead have an arbitrary set of functions lying in a non-trivial Toeplitz kernel in the annulus. Sufficiency has yet to be ascertained.

3.2.2 Dyakonov’s theorem in the annulus

In the paper of Dyakonov [19], the following theorem is proven:

**Theorem 3.2.10** Given $\phi \in L^\infty(\Delta)$, there exists $B, b, g$ with $B, b$ Blaschke products and $g$ with $g, g^{-1} \in H^\infty$, such that

$$\ker T_\phi = \frac{g}{b}(K_B \cap BH^2)$$

($K_B$ denoting the model space $H^2 \ominus BH^2$).

In the annulus, a similar result holds, though care has to be taken regarding the characters of functions in inner-outer factorisations, since these factorisations are no longer single-valued in the annulus. The theorem in the annulus becomes:

**Theorem 3.2.11** Given $\phi \in L^\infty(\mathbb{A})$, there exists $B, b, g, \alpha, \beta$ with $B, b$ Blaschke products, $g$ with $g, g^{-1} \in H^\infty$, $\alpha, \beta \in [0, 1]$ such that:

$$\ker T_\phi = \frac{gz^{-\alpha}}{b}(bz^{\alpha-\beta}H^2 \cap K_{\omega z^\beta B}).$$

($\alpha$ and $\beta$ will be chosen such that $\frac{gz^{-\alpha}}{b}, bz^{\alpha-\beta}, \omega z^\beta B$ are all single-valued.

**Proof**

First assume that $\phi$ is log-integrable (else the kernel is zero), and that $|\phi| = 1$ on $\Gamma_0$, $|\phi| = A$ on $\Gamma_1$ (by writing $\phi = \bar{g}\phi'$ for $g$ a suitable outer function).

Then, one has that $\omega \phi = \frac{\bar{g}}{g}z^\alpha b\bar{B}$, with $g, g^{-1} \in H^\infty$, and $B, b$ Blaschke products. (This
follows from Theorem 2.1.6).

Letting $\beta$ be the character of $g$, then $B$ has character $-\beta$, and $b$ will have character $\beta - \alpha$.

If $T_\phi f = 0$, we have that $\phi f \in \omega \bar{H}^2$, and thus $\omega \phi f \in \bar{H}^2$.

Then, $\frac{g}{\bar{g}} z^\alpha b \bar{B} f = \bar{s}$, and so $f z^\alpha \frac{1}{g} = \bar{s} B \frac{1}{g}$.

It follows that:

$$\frac{z^\alpha b}{g} f \in b g^{-1} z^\alpha H^2 \cap \frac{B}{g} \bar{H}^2.$$  

We have that $\frac{1}{g} = \frac{1}{z^\alpha} \frac{z^\beta}{g}$, where $\frac{z^\beta}{g}$ is a single-valued analytic and outer. Since $z^{-\beta} g$ and $z^\beta g^{-1}$ are single-valued, lie in $H^\infty$ and have inverse in $H^\infty$, we have $z^{-\beta} g H^\infty = H^\infty = z^\beta g^{-1} H^\infty$, the previous becomes

$$\frac{z^\alpha b}{g} f \in b z^{-\beta} H^2 \cap \frac{B}{z^\beta} \bar{H}^2.$$  

**Lemma 3.2.12** $\frac{B}{z^\beta} \bar{H}^2 = K_{\omega B z^\beta}.$

**Proof**

Let $f \in B z^{-\beta} \bar{H}^2$, i.e. $f = B z^{-\beta} \bar{g}$.

Then $\langle f, \omega B z^\beta s \rangle = \langle B z^{-\beta} \bar{g}, \omega B z^\beta s \rangle = \langle \bar{g}, \omega s \rangle = 0 \forall s \in H^2$.

Conversely, if $f \in H^2 \ominus \omega B z^\beta H^2$, then $\langle f, \omega B z^\beta s \rangle = 0 \forall s \in H^2$, so $\langle \omega B \bar{z}^\beta f, s \rangle = 0 \forall s$,

hence $\omega B \bar{z}^\beta f = \omega \bar{g}$ for some $g$, hence $f = B \bar{z}^{-\beta} \bar{g}$. Thus $\frac{B}{z^\beta} \bar{H}^2 = K_{\omega B z^\beta}$. □

Thus we have that

$$f \in \frac{g z^{-\alpha}}{b}(b z^{-\beta} H^2 \cap K_{\omega z^\beta B}).$$  

□
3.2.3 Intertwining

On the Hardy space of the disc, one can characterise the Toeplitz operators as being those which fulfil a particular intertwining relationship- that $T$ is a Toeplitz operator if $S^*TS = T$, where $S$ is the shift operator (i.e. $M_z$) on the Hardy space for the disc.

It seems natural to ask if there exists any operator for which the same holds in the annulus, however this is not the case.

**Theorem 3.2.13** There does not exist an operator $S$ such that:

$S^*TS = T$ is equivalent to $T$ being a Toeplitz operator on the Hardy space of the annulus $\mathbb{A}$.

**Proof**

To start the proof, we first show that no multiplication operator $S$ has this property:

**Lemma 3.2.14** $\forall \mu \in H^\infty(\mathbb{A})$, \{ $T : M\bar{\mu}TM\mu = T$ \} \neq \{ $T : T$ is Toeplitz \}

**Proof**

Suppose that $\mu$ is such that \{ $T : M\bar{\mu}TM\mu = T$ \} = \{ $T : T$ is Toeplitz \}. Then, applying to $T_1 = I$, we have that $M\bar{\mu}M\mu = I$, and so $|\mu|^2 = 1$ almost everywhere.

Thus $\mu$ is an inner function. It is then clear that $M\bar{\mu}TM\mu = T$ for all Toeplitz operators $T$, to prove a contradiction we must show that there exists a non-Toeplitz operator such that this holds.

Define $A$ an operator on the subspace generated by $\mu$ in $H^2(\mathbb{A})$ by $A(1) = 1$, $A(\mu^n) = \mu^n$.

Then $A$ satisfies the required conditions on the subspace generated by the closed span of $\mu^n$.

However, if $A$ is a Toeplitz operator, we must have that $A$ is uniquely determined—otherwise, we would have a Toeplitz operator $T_\phi$ with $\phi \neq 0$, and $T_\phi\mu^n = 0$ for all $n$,
leading to a contradiction as follows:
Suppose \( T_\phi \mu^n = 0 \), then \( T_\phi 1 = 0 \), and have that \( \omega \phi = \bar{s} \) with \( s \) analytic.

However, \( T_\phi \mu^n = 0 \), so \( \bar{s} \mu^n = t_n \) for some analytic functions \( t_n \), and so \( s \) is divided by arbitrarily large factors of \( \mu \). However, this is impossible for \( s \) and \( \mu \) non-zero.

So, we have that the closed span of \( \mu^n \) for \( n \geq 0 \) is \( H^2 \), as otherwise we could extend \( A \) to \( H^2 \) whilst preserving \( M_\mu A M_\mu = A \) in at least two different ways—firstly by defining \( A \) to be 0 on the orthogonal complement of the span of \( \mu \), and secondly by defining \( A \) to be the identity on the orthogonal complement of their span.

Now, if \( \mu \) has singular inner factor or multiple zeros, this span cannot equal \( H^2 \), yet there are no single valued inner functions in \( H^2(\mathbb{D}) \) with only a single zero.

Thus there is no function \( \mu \) with the required property in \( H^\infty(\mathbb{D}) \).

To extend this to the set of all operators on our Hardy space, note that \( S^* IS = I \), so \( S^*S = I \), and thus we have that \( S^*SS = S \), so \( S \) must be a Toeplitz operator if the intertwining defines the Toeplitz operators. To complete the result, we must show it to be analytic.

Given that \( S \) is a Toeplitz operator, and letting \( \mu \) be its symbol, we have from \( S^*S = I \) that it must be an isometry (Since \( T_\mu T_\mu = I \)).

We must thus have \( |\mu| \leq 1 \) a.e., since \( ||\mu||_\infty = ||T_\mu|| = 1 \) ([2] p276, theorem 2.11).

Now, \( ||T_\mu f||_2 \leq ||\mu f||_2 \) with equality if and only if \( \mu f \) is analytic. Thus, we have \( \mu f \in H^2 \) for all \( f \in H^2 \), and thus \( \mu \) must be an analytic function. Combined with the lemma, no such function can exist. \( \Box \)
Chapter 4

Hankel Operators

4.1 Nehari for the annulus

This section will outline Hankel operators on multiply-connected domains, and how they differ from the Hankel operators on the disc.

We take our Hardy space $H^2(D)$ as defined in chapter 1, with uniform lebesgue measure on the boundary.

We define a Hankel operator $\Gamma$ with symbol $\phi$ on $H^2(D)$ by $\Gamma f = P_{\omega H^2} \phi f$.

$\phi$ is said to be an optimal symbol for $\Gamma$ if $\|\phi\|_{\infty} \leq \|\psi\|_{\infty}$ for any $\psi$ a symbol for $\Gamma$.

**Theorem 4.1.1** Let $\Gamma$ be a Hankel operator, $\|\Gamma\| < \infty$. Then we have that there exists optimal symbol $\phi$ for $\Gamma$, with $\|\Gamma\| \leq \|\phi\|_{\infty} \leq A\|\Gamma\|$, where $A$ is the constant in Theorem 2.3.2 for the factorisation of $H^1$ functions as a product of $H^2$ functions.

**Proof**

The proof parallels the proof in the case of the disc, with the major difference being that the $H^1 = H^2 H^2$ factorisation now has a constant introduced.

Let $\phi = \sum_{-\infty}^{\infty} a_n z^n + \sum_{-\infty}^{\infty} b_n \omega z^n$ be a symbol for our operator, then we have that
\begin{align*}
\langle z^n, \omega z^m \rangle = \langle \phi z^n, \omega z^m \rangle &= \langle \phi z^{n+m}, \omega \rangle = b_{n+m}(1 + s^{2(m+n)}).
\end{align*}

It follows that we can produce a linear functional \( \alpha \) on products of polynomials by \( \alpha(f_1f_2) = \langle \Gamma(f_1f_2), \omega \rangle = \langle \Gamma(f_2), \omega f_1 \rangle \), since we have that

\[ \|\alpha\| \leq \|\Gamma\| \|f_1\|_2 \|f_2\|_2 \leq A \|f_1f_2\|_1 \Gamma. \]

We will have that \( \|\alpha\| \leq A \|\Gamma\| \).

Since our operator is bounded, we can extend to \( H^1 \) by continuity, and then to \( L^1 \) by Hahn-Banach, without increasing norm.

We thus have that there exists \( h \) such that \( \alpha(g) = \int_{\partial A} g(z)h(z) \).

Taking the expansion \( \omega h = \sum_{-\infty}^{\infty} j_i z^i + \sum_{-\infty}^{\infty} k_i \omega \bar{z}^i \) for some \( (j_i), (k_i) \) square summable sequences, we have that

\[ k_i(1 + s^{2i}) = \langle \omega h, \omega \bar{z}^i \rangle = \int h(z)z^i = \alpha(z^i) = b_i(1 + s^{2i}). \]

So \( \omega h \) and \( \phi \) differ by an analytic function, hence they are both symbols for the same Hankel operator.

From the integral representation of \( \alpha \), we have that \( \|\omega h\|_\infty = \|h\|_\infty = \|\alpha\| \leq C \|\Gamma\| \).

\[ \|h\|_\infty \geq \|\Gamma\| \] follows from being a symbol for \( \Gamma \). To show that this is optimal, we must show that for any symbol \( \phi \) of \( \Gamma \), we have \( \|\phi\| \geq \|\alpha\| \).

For \( f \in H^\infty \), we have \( \|\alpha(f)\| = \|\langle \Gamma(f), \omega \rangle\| = \|\langle \phi f, \omega \rangle\| \leq \|\phi\|_\infty \|f, \omega\| \leq \|\phi\|_\infty \|f\|_1 \).

Thus, \( \|\alpha\| \leq \|\phi\| \) for all symbols \( \phi \) of \( \Gamma \).

\[ \square \]

This gives us upper and lower bounds for the optimal symbol, and shows that a (not necessarily unique) optimal symbol exists.

It is easy to demonstrate a symbol for which the optimal symbol no longer has the same norm as the Hankel operator, in contrast to the case in the disc where an optimal symbol with the same norm always exists.

**Theorem 4.1.2** There exists Hankel operator \( \Gamma \), such that for \( h \) an optimal symbol, \( \|h\| \neq \|\Gamma\| \).

**Proof**
Consider $\Gamma$ defined by the symbol $\phi = \frac{s}{z}$.

We have that $\langle \Gamma z^n, \omega \bar{z}^m \rangle = \langle \omega z^{n-1}, \omega \bar{z}^m \rangle = \langle z^{n-1}, z \bar{m} \rangle = \langle z^{n+1-m}, 1 \rangle$.

Thus $\Gamma z^n = \omega a_n z^{1-n}$, with $a_n = \frac{2}{1 + s^2 + 2n}$.

Letting $e_n = \frac{z^n}{\|z^n\|_2}$ be the orthonormal basis,

$$\Gamma \phi e_n = b_n \omega e_{1-n} = \frac{2}{(1 + s^2 - 2n)0.5(1 + s^2)^{0.5}} \omega e_{1-n}.$$  

Since the $e_n$ and $\omega e_n$ are orthonormal basis for $H^2$ and $\omega H^2$, we see that $\|\Gamma\| = \sup_n b_n$.

$b_n$ achieve their maximum at 0 and 1, so $\|\Gamma\| = \frac{\sqrt{2}}{\sqrt{1 + s^2}}$.

Suppose that we have symbol $\Phi$ for which $\|\Phi\|_\infty = \|\Gamma\|$.

Since we have that $\Gamma$ achieves its norm at $f$, where $f(z) = z$, we have that, as is the case in this situation in the disc, that:

$$\|\Gamma\| \|f\|_2 = \|\Gamma \phi f\|_2 \leq \|\Phi f\|_2 \leq \|\Phi\|_\infty \|f\|_2.$$  

As is the case in this situation on the disc, when $\|\Gamma\| = \|\Phi\|$, we have equality throughout.

So, $\Gamma f = \Phi f$, i.e. $\Phi = \frac{\Gamma f}{f} = \frac{s}{z}$.

However, $\|\frac{s}{z}\|_\infty = s^{-1} > \frac{\sqrt{2}}{\sqrt{1 + s^2}}$, contradicting our assumption.

Thus, there cannot exist a symbol which achieves the norm of our operator. \(\square\)

### 4.1.1 Finding optimal symbols

Even in the case of $\Gamma$ which attain their norm at some $f \in H^2$, we no longer have that $\frac{\Gamma f}{f}$ gives an optimal symbol for our $\Gamma$— since we no longer necessarily have that the optimal symbol matches the norm of the operator (though, in the case where they do agree, this argument works as it does in the disc).
By considering \( \Gamma \) as an operator on each member of the family of spaces \( H^2_\alpha \) (the sets of modulus automorphic functions with character \( \alpha \)), some of this difficulty can be averted.

Let \( \Gamma_{\alpha,\tilde{f}} = P_{(H^2_\alpha)\perp}(\tilde{\phi} \tilde{f}) \) be the extension onto \( H^2_\alpha \).

Then, attempting the Nehari theorem once more, letting \( \beta(f_1f_2) = \langle \Gamma f_1f_2, \omega \rangle \), we have that \( |\beta(f)| = \langle \Gamma f_1, \omega \hat{f}_2 \rangle \leq \|\Gamma\| \|f_1\|_2 \|f_2\|_2 \).

Since \( f_1 \) and \( f_2 \) can now be chosen from \( H^2_\alpha \) rather than merely \( H^2 \), we can factorise while preserving norms (as in 2.3.3).

Thus \( \|\beta\| \leq \sup_{\alpha \in [0,1]} \|\Gamma\| \).

Thus, for the \( \alpha \) giving the maximal value of \( \|\Gamma\| \), if we have that \( \Gamma_{\alpha} \) takes its norm at some function \( f \in H^2_\alpha \), we have that \( \Gamma_{\alpha}f \) is an optimal symbol for \( \Gamma \), as in the case of the disc (Sarason’s solution of the Nehari problem).

However, the problem of finding which \( \alpha \) to consider, and finding a function \( f \) at which the symbol achieves its norm, appear difficult.

Whilst it can be shown \( \Gamma_k \) is compact on \( H^2_k \) for arbitrary \( k \) if and only if \( \Gamma_{\alpha} \) is compact on \( H^2_\alpha \) for all \( \alpha \), and thus we will still have that \( \Gamma_{\alpha} \) achieves its norm on all these if our original Hankel operator is compact, the problem is to find where it achieves its norm.

In the disc, it is interesting to consider Hankel operators with a finite polynomial symbol. The point at which the norm is attained can then be found easily since the associated Hankel matrix will have only a finite number of non-zero entries, and thus can be diagonalized with elementary techniques to find where the norm is achieved.

However, in the case of the annulus, any non-zero Hankel matrix will have an infinite number of non-zero entries. If one considers the matrix for a Hankel operator, symbol \( \psi \), with respect to the basis \( e_n \) and \( \omega e_n \) defined earlier, we have that \( A_{n,m} = \langle \psi k_n z^n, \omega z^m k_m \rangle \), with \( k_n = \|z^n\|_2^{-1} \).

Thus, \( A_{n,m} = k_n k_m \hat{\psi}(n + m) \). Whilst not constant along backwards diagonals due to the \( k_nk_m \) term, if any term on a given backwards diagonal is non-zero, all of them are non-zero. In the case of the disc, we would have that \( n, m \) range from 0 to \( \infty \), and so these diagonals are finite, but in the annulus they range from \( -\infty \) to \( +\infty \), so there are...
infinitely many terms on these diagonals. It seems that finding the optimal symbol in this setting is a more difficult problem.

### 4.2 Compactness

In the case of Hankel operators on the disc, it is known that the compact Hankel operators are those with symbol in \( H^\infty + C \), where \( C \) denotes the space of continuous functions on the boundary. The result can be found with proof in [32] Chapter 1, section 5.

We can show the same is true in the annulus.

First, note that, as in the case of the disc, symbols of a rational function will correspond to compact operators.

**Lemma 4.2.1** If \( \phi \) is a rational function, then \( \Gamma_\phi \) is of finite rank.

**Proof**

Since we have that \( \Gamma_\frac{1}{z} = \Gamma_\frac{1}{z} M_f \) and \( \Gamma_{\phi_1 + \phi_2} = \Gamma_{\phi_1} + \Gamma_{\phi_2} \), we have that by dealing with partial fractions, it remains only to show that \( \Gamma_{\frac{1}{(z-a)^n}} \) is compact for \( a \in A \).

Now, we have that \( H^2(A) = (z - a)^n H^2 \bigoplus ((z - a)^n H^2)^\perp \). \( \Gamma_{(z-a)^{-n}} \) is the zero operator on restriction to the first space in the decomposition, and the latter space is finite dimensional. Thus, \( \Gamma \) is a finite rank operator. □

In order to show that the whole of \( C + H^\infty \) gives compact operators, a little more is needed.

**Lemma 4.2.2** If \( \phi \) is of form \( \phi = \omega q \), where \( q \) is a rational function, then \( \Gamma_\phi \) is compact.

**Proof**

As before, we can split by partial fractions, and it remains only to consider symbols of
Chapter 4. Hankel Operators

Decomposing $H^2$ as before, we have that $\Gamma_\phi$ is no longer a zero operator on $(z - a)^n H^2$, however, we have on this space that $\Gamma_{\frac{\omega}{(z-a)^n}} = \Gamma_\omega \circ M_{(z-a)^{-n}}$.

We have that $\Gamma_\omega$ is a compact operator 
(on $H^2$ we have that $\Gamma_{\frac{\omega}{(z-a)^n}} = \frac{1}{(1+s^{-2n})(1+s^{2n})} \frac{\omega z^{-n}}{\|\omega z^{-n}\|}$.)

Thus, it follows that $\Gamma_{\frac{\omega}{(z-a)^n}}$ is compact (but will not be of finite rank).

By considering $\Gamma_{q+\omega q}$ where $q$ is a rational function, we can approximate in norm functions whose restriction to the outer boundary is a continuous function, and zero on the inner boundary, and thus Hankel operators with such a symbol are compact. The same holds with the boundaries reversed, and so we have that any continuous function is compact. Since the Hankel operator with analytic symbol is equivalent to the zero operator, we thus have that all functions in $H^\infty + \mathbb{C}$ are symbols for a compact Hankel operator. \(\Box\)

Proving the reverse, that any compact operator has symbol in $H^\infty + \mathbb{C}$, is a little harder.

**Lemma 4.2.3** If $\Gamma_\phi$ is compact, then $\phi \in Cl(H^\infty + \mathbb{C})$, with $Cl$ denoting closure in norm (the space will be shown later to in fact be closed.)

**Proof**

First, note that $\|\Gamma_\phi M_B\| \to 0$ if $B$ is a Blaschke product with finitely many zeroes.

Suppose otherwise, then we have $f_n$ with $\|f_n\| = 1$ and $\|\Gamma_\phi B^n f_n\| \geq \epsilon$ infinitely often for some $\epsilon$.

From compactness of $\Gamma$, for some subsequence $n_i$ we have $\Gamma_\phi B^{n_i} f_{n_i} \to \omega g$, for some $g$ in $H^2$.

By taking our subsequence to include only $n$ with $\|\Gamma_\phi B^n f_n\| \geq \epsilon$, we have $g \neq 0$.

To show a contradiction, we need to show that $B^{n_i} f_{n_i}$ is tending to 0 weakly. Suppose this is not the case, that we have $g \in H^2$ with $|\langle B^{n_i} f_{n_i}, g \rangle| \geq \epsilon$ for some subsequence of
Letting $f'$ denote the lifts back to the disc, and letting $\kappa$ represent the appropriate change of measure, we have $|\langle B'^m f'_m, g' \kappa \rangle_D| \geq \epsilon \forall i$.

Letting $h = P_{H^2} (\kappa g')$, we have $|\langle B'^m f'_m, h \rangle_D| \geq \epsilon \forall i$.

However, this cannot be as in the Hardy space of the disc, $B^m f_n$ tends to $0$ in the weak topology when $B$ is a Blaschke product and $\|f_n\|$ bounded, (this follows from [13] lemma 3.3, which shows that $T_{B^m} g$ tends to $0$ in norm for any $g$).

We thus have $\langle B^m f_n, \phi \bar{\omega} g \rangle \to 0$, and so $\langle \Gamma \phi B^m f_n, \omega \bar{g} \rangle \to 0$, contradicting the assumption $\Gamma \phi B^n f_n \to \omega g$ for $g \neq 0$.

With $\|\Gamma \phi M_{B^n}\| \to 0$, the rest proceeds as the proof does for compactness in the disc:

We have that $\phi B^n = s_n + k_n$, with $\|s_n\| \to 0$, and $k_n \in H^\infty$.

Thus $\phi = s_n \bar{B}^n + (k_n / B^n)$.

For $B_n$ a finite Blaschke product, we have that $k_n / B^n \in H^\infty + C$, and thus $\phi \in \overline{H^\infty + C}$.

If we can show that $H^\infty + C$ is closed, then the result is complete. □

However, it is an old result that this space is closed in $A$. It is proven, for example, in [2] Theorem 1.22. An alternative proof will be given here.

**Closedness of $H^\infty + C$**

**Theorem 4.2.4** $H^\infty + C$ is a closed subspace of $L^\infty$.

**Proof**

Suppose we have $f_n$ is a Cauchy sequence in $H^\infty + C$, with $f_n = q_n + c_n$, $q_n \in H^\infty$, $c_n \in C$.

Let $F$ denote the limit function in $L^\infty(A)$, Now, use additive decomposition of $H^\infty(A)$, to have $q_n = r_n + s_n$, and $c_n = d_n + e_n$, where $r_n \in H^\infty(\Delta)$, $s_n \in H^\infty((s\Delta)^c)$, $d_n = c_n|_{\Gamma_0}$, $e_n = c_n|_{\Gamma_1}$.

Now, consider $f_n$ on $\Gamma_0$. $f_n|_{\Gamma_0}$ will still be Cauchy, and we have that
\( f_n|_{\Gamma_0} = r_n|_{\Gamma_0} + s_n|_{\Gamma_0} + d_n|_{\Gamma_0}. \)

Now, since \( s_n \in H^\infty((s\Delta)^c) \), we have that \( s_n \) is continuous on \( \Gamma_0 \).

Thus \( f_n|_{\Gamma_0} \) is a sequence in \( H^\infty(\Delta) + C \) converging to \( F|_{\Gamma_0} \).

This space is closed in the disc, so \( F|_{\Gamma_0} \) also belongs to it.

Therefore we have \( R \) and \( D \) with \( R \in H^\infty(\Delta), D \in C(\partial\Delta), \) with \( F|_{\Gamma_0} = R + D. \)

We can do the same on \( \Gamma_1 \) to attain \( S \) and \( E, S \in H^\infty((s\Delta)^c), E \in C(\partial s\Delta), \) with \( F|_{\Gamma_1} = S + E. \)

So \( F = R + D + S + E - R|_{\Gamma_1} - S|_{\Gamma_0}. \)

Now, we have that \( R|_{\Gamma_1} \) is continuous, as is the restriction of \( S \) to \( \Gamma_0. \)

Thus \( F \in H^\infty + C(\partial\Delta). \) \( \square \)
Chapter 5

The Bergman space and the slit disc

Some similar results to those from previous chapters can be extended to other settings, in particular that of the Bergman space on the annulus.

5.1 The Bergman space

Letting $A$ represent the annulus as before, define the Bergman space of the annulus $A^2(A)$ to be the set of analytic functions on the disc for which $\int_A |f|^2 dA(z) < \infty$, with $A$ the Lebesgue area measure, and norm $\|f\|_2 = \sqrt{\int_A |f|^2 dA(z)}$.

They are described in [23], for instance.

First, note that we have a factorisation result akin to Theorem 2.2.1-

**Theorem 5.1.1** If $f \in A^2(A)$, we have $f = f_1f_2$ with $f_1 \in A^2(\Delta)$, and $f_2 \in A^2((s\Delta)^c)$.

**Proof**

The proof is akin to that in the Hardy space. Suppose first that $f$ has no zeros in $A$. Then $f = z^n f'$, where $f'$ has 0 winding number about 0, and has no zeros in the annulus, and

thus we can take logarithms.

Say, \( \log f' = \sum_{-\infty}^{\infty} a_n z^n = \sum_{0}^{\infty} a_n z^n + \sum_{-\infty}^{-1} a_n z^n = g_1 + g_2. \)

We thus have that \( f = e^{\log f'} = e^{g_1} e^{g_2} = f_1 f_2. \)

Now, we have that \( g_1 \) and thus \( f_1 \) is analytic in \( \Delta \), similarly \( f_2 \) is analytic in \( (s\Delta)^c \). We must prove that the norms are finite also.

Since \( f_1 \) and \( \frac{1}{f_1} \) are continuous and thus bounded in an open set about \( \Gamma_1 \), \( f = f_1 f_2 \), and \( f \) is square integrable, we have that \( f_2 \) has bounded square integral in some open set about \( \Gamma_1 \). By continuity, on the complement of any such set in \( (s\Delta)^c \), \( f_2 \) has bounded square integral, and thus \( f_2 \) has bounded square integral in \( s(\Delta)^c \), and so \( f_2 \in A^2((s\Delta)^c) \). Similarly \( f_1 \in A^2(\Delta) \).

Now, let \( f \) be an arbitrary \( A^2(\mathbb{H}) \) function. We can assume w.l.o.g that all zeros of \( f \) lie in \( \{1 > |z| > 1 - \epsilon\} \cup \{s < |z| < s + \epsilon\} \) for some \( \epsilon \) such that these are disjoint sets, since we have finitely many zeros inside any compact set, and if we can factor \( f/g \) for \( g \) a polynomial, then we can factor \( f \).

Now, we have that \( f \) has no zeros on the set \( A_2 = \{1 - \epsilon > |z| > s + \epsilon\} \), thus we can apply the previous factorisation to obtain:

\[
f = f_1 f_2 \text{ with } f_1 \in A_2(\{z : |z| < 1 - \epsilon\}), \text{ and } f_2 \in A_2(\{z : |z| > s + \epsilon\}).
\]

By the domain of \( f_2 \), we have at most finitely many zeros in the region \( \{1 > |z| > 1 - 2\epsilon\} \).

Thus, we can extend \( f_1 \) across this region by \( f_1 = f/f_2 \), and the resultant function will have at most finitely many poles. Similarly, we can extend \( f_2 \) to the boundary \( |z| = r \) by \( f_2 = f/f_1 \), and will have at most finitely many poles.

Finally, since \( f = f_1 f_2 \), and \( f \) has no poles in the region, we can remove the poles by factorising instead as \( f = \frac{f_1 \Pi(z-a_i)}{\Pi(z-b_i)} \frac{f_2 \Pi(z-b_i)}{\Pi(z-a_i)} = g_1 g_2 \), where \( a_i \) are the poles of \( f_1 \), and \( b_i \) poles of \( f_2 \).

Now, \( g_1 \) is analytic in \( \Delta \), \( g_2 \) analytic in \( (s\Delta)^c \), and by repeating the argument for the case of no zeros, we have that these belong to the appropriate Bergman spaces. \( \square \)
5.1.1 Toeplitz operators in the Bergman space of the annulus

We have that \( A^2(\mathbb{A}) \) is a closed subspace of \( L^2(\mathbb{A}) \), thus we can define a projection operator \( P \) from \( L^2 \) to \( A^2 \), and so define a Toeplitz operator \( T_\phi \) by \( PM_\phi \) for \( \phi \) measurable functions.

In the Bergman space for the disc, it is known that for symbols continuous on the closure of the disc, up to compact operators only behaviour of \( \phi \) on the boundary matters, and the essential spectrum will be the same of that of a Toeplitz operator on the Hardy space with the same boundary values- for example, [21] p16-17.

We can in fact extend this to \( \mathbb{A} \):

**Theorem 5.1.2** If \( U \) is the isometry from \( A^2(\mathbb{A}) \) to \( H^2(\mathbb{A}) \) defined by \( U(z^n/\|z^n\|) = z^n/\|z^n\|', \) where \( \cdot/\|\cdot\| \) is the Hardy norm and \( \cdot/\|\cdot\| \) the Bergman space norm, then we have that for any \( \phi \) symbol continuous on the closure of \( \mathbb{A} \), we have:

\[
UT_\phi U^* = T'_{\phi|_{\partial \mathbb{A}}} + K, \text{ where } K \text{ is a compact operator fixed by } \phi, \text{ } T' \text{ denotes a Toeplitz operator on } H^2(\mathbb{A}), \text{ and } T \text{ denotes a Toeplitz operator on } A^2(\mathbb{A}).
\]

**Proof**

We proceed by first proving the result holds for \( T_z \) and \( T_{z^{-1}} \). From here, we can then extend to symbols in the closure of the set of polynomials in \( z, \bar{z}, z^{-1}, \bar{z}^{-1} \), since if the lemma holds for \( z \), it must also hold for \( \bar{z} \) by taking adjoints, and by taking powers of each side we have it for \( \bar{z}^n \) and \( z^n \) all \( n \geq 0 \). Similarly \( \phi = z^{-1} \) will give the result for \( z^n \) and \( \bar{z}^n \) for all \( n < 0 \), and if it holds for \( \phi_1 \) and \( \phi_2 \) it holds for \( \phi_1 + \phi_2 \), etc.

Since the closure of these polynomials generate the continuous functions, we have the required set.

Letting \( e_n = z^n/\|z^n\| \), and \( e'_n = z^n/\|z^n\|' \), we have that \( UT_z U^* e'_n = U^* z e_n = \|z^{n+1}\|/\|z^n\| e'_{n+1} \) and \( T_z e'_n = \|z^{n+1}\|'/\|z^n\|' e'_{n+1} \).

Thus, \( \|UT_z U^* e'_n - T_z e'_n\| = \|z^{n+1}\|/\|z^n\| - \|z^{n+1}\|'/\|z^n\|' \).
Now, we have that $\|z^n\|^2 = \int_s^1 r^{2n} r dr = \frac{1-s^{2n+2}}{2n+2}$, and $\|z^n\|'^2 = 1 + s^{2n}$.

Letting $n \to +\infty$, we have that $\|z^{n+1}\|/\|z^n\| \to 1$, $\|z^{n+1}\|'/\|z^n\|' \to 1$.

Letting $n \to -\infty$, we have that $\|z^{n+1}\|/\|z^n\| \to s$, and $\|z^{n+1}\|'/\|z^n\|' \to s$.

Thus, we have that $\|UTz^*e_n' - Tz'e_n'\| \geq \epsilon$ at most a finite number of times. Applying the same to $z^{-1}$ works similarly.

Now, we have that $UTz^nU^* = (UTzU^*)^n = (T_{z|\partial A} + K)^n = T_{z^n|\partial A} + S$, where $K$ is the compact operator evaluated before, and $S$ is compact as it is a finite sum of products of a compact operator with bounded operators.

Thus, we can extend to $z^n$. We can extend from $z^{-1}$ to $z^{-n}$ in the same manner, and taking adjoints we can show this for $\bar{z}^{-n}$ and $\bar{z}^{-1}$.

It follows immediately that the result holds when $\phi$ can be written as a finite polynomial in $z, z^{-1}, \bar{z}, \bar{z}^{-1}$.

We can also extend the result to $\phi$ which can be uniformly approximated by polynomials in these (which, by the Stone-Weierstrass theorem, is the space of continuous functions on the closure of $A$):

Let $p_n$ be a sequence of polynomials uniformly approximating $\phi$ in $\infty$-norm on $L_\infty(Cl(A))$ (with $Cl$ denoting closure).

Then, we have that $T_{p_n} \to T_{\phi}$ in operator norm, and also $T'_{p_n|A} \to T'_{\phi|A}$ in operator norm.

Thus, $UT_{p_n}U^* - T'_{p_n|A} = \lim_{n \to \infty} K_n$, where $K_n = UT_{p_n}U^* - T'_{p_n|A}$ is compact, and the limit is in operator norm. Thus $K$ will be compact here, and so the result holds for all $\phi$ continuous on the closure of $A$. $\square$
Radial Toeplitz operators

We have in the case of a Toeplitz operator on the Bergman space of the disc that if $\phi$ is a radial symbol, then we have $T_\phi z^n = A_n z^n$ for some $A_n \in \mathbb{C}$, since

$$\langle T_\phi z^n, z^m \rangle = \langle \phi z^n, z^m \rangle = \int_0^1 \int_0^{2\pi} \phi(r, \theta) r^{n+m} re^{i(n-m)\theta} \, dr \, d\theta$$

$$= \int_0^1 r^{n+m+1} \phi(r) \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \, dr.$$

Here, the $\theta$-integral is 0 when $n \neq m$.

The same occurs in the case of the annulus, since:

$$\langle T_\phi z^n, z^m \rangle = \langle \phi z^n, z^m \rangle = \int_s^1 \int_0^{2\pi} \phi(r, \theta) r^{n+m} re^{i(n-m)\theta} \, dr \, d\theta$$

$$= \int_s^1 r^{n+m+1} \phi(r) \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \, dr = 0 \text{ for } n \neq m,$$

where $s$ is the inner radius of the annulus.

The paper by S. Grudsky, A. Karapetyants, and N. Vasilevski [20] describes bounded and even compact Toeplitz operators with unbounded radial symbols, and gives some conditions for an operator to be as such.

In the annulus, the condition for the radial Toeplitz operators to be bounded is quite similar to the case in the disc:

**Theorem 5.1.3** If $\phi$ is a radial symbol for a Toeplitz operator on the Bergman space of the annulus, $T_\phi$ is bounded if and only if there exists a bounded Toeplitz operator on the Bergman space of the disc whose symbol on the disc whose symbol $\phi'$ is equal to $\phi$ in some neighbourhood of 1. The same holds replacing bounded with compact. In other words, the condition to be bounded/compact is simply that the behaviour at each boundary is that of a bounded/compact radial Toeplitz operator on the disc.

**Proof**

Let $\phi$ be a radial symbol. It is immediate that $\langle \phi z^n, z^m \rangle = 0$ for $n \neq m$, thus the operator is diagonal. So, letting $a_n$ be such that $T_\phi z^n = a_n z^n$, we have that $T_\phi$ is bounded if $\{a_n\}$
is bounded, and compact if \( a_n \to 0 \) as \( n \to \pm \infty \).

Suppose first that \( \phi \) is continuous at \( s \), and that \( \phi' \) is a symbol for a bounded Toeplitz operator on \( A^2(\Delta) \) such that \( \phi = \phi' |_{[s,1]} \). We show that \( T_\phi \) must now be bounded.

We have that \( a_n = \langle \phi' z^n, z^n \rangle / \langle z^n, z^n \rangle \) are a bounded sequence by assumption.

Looking at the limiting behaviour as \( n \to +\infty \), we have that
\[
\int_0^1 \phi'(r) r^{2n+1} dr (2n + 2)
\]
is bounded.

Now, looking at \( b_n \) the \( n \)th eigenvalue for the Toeplitz operator on the Bergman space of the annulus, we have that:
\[
b_n = \langle \phi z^n, z^n \rangle / \langle z^n, z^n \rangle, \quad \text{and} \quad \langle z^n, z^n \rangle = \frac{1}{2n+2} (1 - s^{2n+2}) > \frac{1}{2(2n+2)} \text{ for all sufficiently large } n.
\]

We thus have that
\[
|b_n| = |\langle \phi r^{2n+1} dr / \langle z^n, z^n \rangle| \leq 2(2n + 2) |\int_0^1 \phi r^{2n+1} dr| \leq 2|a_n| + 2(2n + 2) |\int_0^1 \phi' r^{2n+1} dr|.
\]
However, \( |a_n| \) is bounded and the latter integral clearly converges to 0 as \( n \to \infty \).

Thus, we have that the eigenvalues corresponding to positive values of \( n \) behave appropriately.

Now, since we have that \( \phi \) is approximating \( C \) for some constant close to the inner boundary, and we have that the contribution of \( z^{-n} \|z^{-n}\|_2 \) on the outer boundary falls exponentially as \( n \) tends to \( \infty \), we have that \( b_{-n} \to C \) as \( n \to \infty \).

Thus, we have that \( T_\phi \) is a bounded operator for the Bergman space of the annulus if it is for the disc and it is continuous about \( s \).

In the case \( C = 0 \), the same argument shows that \( T_\phi \) is compact if \( T'_{\phi} \) is compact. In order to deal with a general symbol, we simply need to show that the corresponding result holds for a symbol continuous at 1 when considering the symbol as the restriction of a bounded symbol on \( A^2((s\Delta)^C) \).

From this, the result follows for arbitrary symbols since we can decompose the symbol as \( \phi = \phi_1 + \phi_2 \) with \( \phi_1 \) continuous at \( s \), and \( \phi_2 \) continuous at 1.

Considering \( n \) negative, we have that
\[
b_n = \frac{\int_0^1 \phi r^{2n+1} dr}{\int_s^r r^{2n+1} dr}.
\]
Making the substitution \( q = s / r \), this becomes
\[
b_n = \frac{\int_s^1 \phi(q) q^{-(2n+3)} dq}{\int_s^1 q^{-(2n+3)} dq}.
\]
Chapter 5. The Bergman space and the slit disc

Hence the boundedness of $b_n$ for negative $n$ translates to the question of boundedness of $b_n$ for positive $n$ for the symbol $\tilde{\phi}$ with $\tilde{\phi}(r) = \phi(s/r)$. □

In addition to boundedness/compactness, a result in a similar vein can be shown on the Hilbert-Schmidt norms of the diagonal Toeplitz operators (which all radial Toeplitz operators are included in).

A paper of Harper-Smith [22] describes the relation between Hilbert-Schmidt norms for Toeplitz operators on the Bergman space of the disc, and the following shows that the same hold in the annulus.

**Theorem 5.1.4** Suppose that $T$ is a diagonal operator from the Bergman space of the annulus into an arbitrary Hilbert Space.

Let $T \in B(A^2(\mathbb{A}), \mathcal{H})$. Then we have \( \|T\|_{S^2}^2 = \int_0^1 \int_0^{2\pi} \|Tk_{e^{i\theta}}\|^2 rdrd\theta \), where $k_z$ is the reproducing kernel at $z$ (for the Bergman space here).

**Proof**

The proof is almost identical to that in the Harper-Smith paper. Let $e_n$ be an arbitrary orthonormal basis. Then we have that:

\[
\int_0^1 \int_0^{2\pi} \|Tk_{re^{i\theta}}\|^2 rdrd\theta = \sum_{n=0}^{\infty} \int \int \|T^* e_n\|^2 rdrd\theta,
\]

\[
= \sum_n \|T^* e_n\|^2 = \|T\|_{S^2}^2 = \|T\|_{S^2}^2,
\]

□

The Harper-Smith paper [22] describes the result in general Schatten classes, and the following will show the proof works in the same way on the annulus. However, the results have also been proven for Bergman spaces on multiply connected domains in [41] by means of conformal equivalences.

**Theorem 5.1.5** If $p > 2$ and $T \in S_p(A^2(\mathbb{A}), \mathcal{H})$, then we have:

\[
\int_{\mathbb{A}} \|Tk_z\|^p dm(z) \leq \|T\|_{S_p}^p,
\]
where \( dm(z) = dA(z)\|k_z\|^2 \), and \( \tilde{k}_z = \frac{k_z}{\|k_z\|} \).

If \( 1 < p \leq 2 \) and \( \int_A \|T\tilde{k}_z\|^p dm(z) < \infty \), we have that \( T \in S_p \) and \( \int_A \|T\tilde{k}_z\|^p dm(z) \geq \|T\|_{S_p}^p \).

**Proof**

We have that if \( A \) is a positive operator, then \( Tr(A) = \int_A \langle A\tilde{k}_z,\tilde{k}_z \rangle dm(z) \), with the inner product taken over our Bergman space on \( A \).

This follows identically to the proof in [42] p116:

\[
tr(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle = \sum_{n=1}^{\infty} \int_A \langle Ae_n(z)\bar{e_n(z)}dA(z) \\
= \sum_{n=1}^{\infty} \int_A \langle Ae_n, k_z \rangle \bar{e_n(z)}dA(z) \\
= \int_A \sum_{n=1}^{\infty} e_n \bar{e_n(z)}(z)k_z dA(z) = \int_A \langle Ak_z, k_z \rangle dA(z) \\
= \int_A \langle \tilde{A\tilde{k}_z}, k_z \rangle dm(z).
\]

From this, and using the fact that for \( A \) a positive operator, and \( f \) of unit norm, \( \langle A^p f, f \rangle \geq \langle Af, f \rangle^p \) for \( p \geq 1 \), and \( \langle A^p f, f \rangle \leq \langle Af, f \rangle^p \) for \( p \leq 1 \), we have that:

\[
\|T\|_{S_p}^p = Tr((T^*T)^{p/2}) = \int_A \langle (T^*T)^{p/2}\tilde{k}_z, k_z \rangle dm(z) \\
\geq \int_A \langle (T^*T\tilde{k}_z, k_z) \rangle^{p/2} dm(z), \\
= \int_A \|T\tilde{k}_z\|^p dm(z).
\]

For the case \( 1 < p \leq 2 \), the reverse inequality holds since the direction of inequality for \( \langle A^p f, f \rangle \) is reversed. \( \Box \)
5.2 Hardy Space of the slit disc

Consider the domain of the slit disc $G$, the unit disc with the interval $[0, 1)$ removed. We will work in the space that was defined in the introduction as $E^2(G)$, the Hardy-Smirnov space. The existence of boundary values, and thus toepplitz operators were covered in the introduction.

There exists a conformal mapping $\phi_G$ between the disc and the slit disc, given in [3] in the appendix, defined as $\phi_G(z) = (w_3(w_2(w_1(-iz))))^2$, where $w_3(z) = \frac{z+1}{z+1}, w_2(z) = \sqrt{z}, w_1(z) = i\frac{z+1}{1-z}$.

The book [3] also describes the effect on the boundary here—the arc $\{e^{i\theta} : 0 < \theta < \pi/2\}$ is mapped to the top half of the slit, and the arc $\{e^{i\theta} : -\pi/2 < 0 < \theta\}$ is mapped to the lower half of the slit.

It is useful to know the poles and zeros on the boundary of the derivative of our mapping, $\phi'_G$.

We have that $\phi'_G(z) = 2(w_3w_2w_1(-iz))w'_3(w_2w_1(-iz))w'_2(w_1(-iz))w'_1(-iz)$.

Considering each factor in turn, $w_3(w_2(w_1(-iz)))$ has no poles on the disc, and has a zero at $z = 1$.

Then $w'_3(z) = \frac{2}{(z+1)^2}$, so $w'_3(w_2(w_1(-iz)))$ has no poles but has a zero of order 1 at $z = i$.

Next, $w'_2(z) = \frac{1}{2}z^{-1/2}$, so $w'_2(w_1(-iz))$ has singularity at $z = -i$, and decays at $z = i$, with the local growth/decay that of $z^{-\frac{1}{2}}$ and $z^{\frac{1}{2}}$ respectively.

Finally, $w'_1(-iz) = \frac{2}{(1+iz)^2}$ has an order 2 pole at $z = i$, and no zeros in the range we are interested in. Thus, putting this all together, we have that $\phi'_G$ has a zero at $z = 1$ with $\phi'_G(1-z) = O(z)$ locally, and we have singularities at $\pm i$ with $\phi'_G(z \pm i) = O(z^{-1/2})$.

The function $\phi'_G$ is important as it is used to normalize the measure when transforming between integrals on the boundary of the slit disc and on the boundary of the disc under the map $\phi_G$. 
Isomorphism between $E^2(G)$ and $H^2(\Delta)$, and Toeplitz operators

**Proposition 5.2.1** Let $\mu$ be the isomorphism between $E^2(G)$ and $H^2(\Delta)$ considered in the introduction, $\mu : f \to F$, with $F(z) = f(\phi_G(z))(\frac{d\phi_G}{dz})^{\frac{1}{2}}(z)$.

We have that if $T'_\psi$ is a Toeplitz operator on the Hardy space of the slit disc, with symbol $\psi$, then we have that $T'_\psi(\mu f) = T_{\tilde{\psi}}f$, where $\tilde{\psi}(z) = \psi(\phi_G(z))$.

**Proof**

This is simply a consequence of the definition of our space and the toeplitz operators on it.

Consider the inner product of a Toeplitz operator applied to $\mu f$ with $\mu g$ in the disc.

\[
\langle \tilde{\psi} \mu f, \mu g \rangle_{\Delta} = \int_{\partial \Delta} \psi(\phi_G(z))f((\phi_G(z)))g(\phi_G(z))|\phi'_G(z)||dz|
\]

\[
= \int_{\partial G} f(z)g(z)(\bar{\psi}(z))|\phi'(\phi^{-1}_G(z))||\phi'(\phi^{-1}_G(z))|^{-1}|dz| = \langle \psi f, g \rangle_G = \langle T_{\psi}f, g \rangle_G.
\]

Thus, we have that $\langle T'_\psi \mu f, \mu g \rangle_{\Delta} = \langle T_{\psi}f, g \rangle_G$ for all $g \in H^2(\Delta)$, and so the result follows.

\[\square\]

Results about Toeplitz operators in the slit disc can thus be deduced from those in the disc, and vice-versa.
Bibliography


